

# 1 Logistic Map

To begin understand the scaling properties in period-doubling systems, I looked at the familiar iterative map defined by

$$f_r(x) = rx(1 - x).$$

As shown in `IterMap.java`, I used Newton's method to find out when  $f_r^{(2^k)}(x_m) = x_m$ . Finding the  $\tilde{r}_k$  that satisfy this condition for each  $k$  yields a sequence that must converge to the same value,  $r_\infty$ , as the  $r_k$  which are the bifurcation points of  $f$ . To understand how  $r_k \rightarrow r_\infty$ , the ratio  $\delta$  can be calculated as a scaling parameter. A similar parameter  $\alpha$  can also be calculated, which describes the scaling of  $f \rightarrow f_{r_{inf}}$  by tracking the parameter. The calculated results are shown in the following table:

$k$	$r_{sk}$	$\delta$	$y$	$\alpha$
01	2.0000	—	0.5000	—
02	3.2361	—	0.8090	—
03	3.4986	4.7089	0.3836	0.7264
04	3.5546	4.6808	0.5460	2.6199
05	3.5667	4.6630	0.4817	2.5252
06	3.5692	4.6684	0.5073	2.5074
07	3.5698	4.6690	0.4971	2.5038
08	3.5699	4.6692	0.5012	2.5031
09	3.5699	4.6692	0.4995	2.5029
10	3.5699	4.6692	0.5002	2.5029

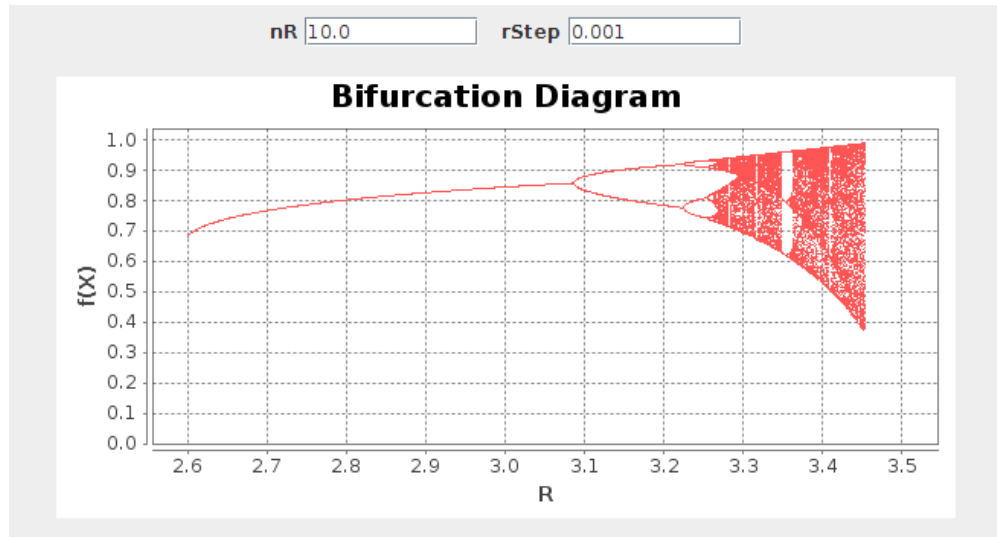
# 2 Universality

When the function is changed, there is no *a priori* reason to assume that the behavior of  $r$  and  $f$  will be the same as the logistic map. Using the function

$$G_r(x) = rx^2\sqrt{1-x}$$

I performed the same analysis of  $\delta$  and  $\alpha$ . That these two numbers converge to the same values as for the iterative map should in fact not be surprising. The derivation of these scaling parameters revealed that for any function “like” the logistic map (single quadratic maximum,  $f(0) = 0$ ) should scale like the logistic map. Thus choice of function became an irrelevant parameter. The table of  $\delta$  and  $\alpha$  are presented here, along with the bifurcation diagram for  $G_r$ . The code used to calculate these was `IterMap2.java`.

$k$	$\delta$	$\alpha$
01	—	—
02	—	—
03	4.3977	0.6785
04	4.6123	2.2768
05	4.6578	2.6430
06	4.6668	2.4575
07	4.6687	2.5228
08	4.6691	2.4953
09	4.6692	2.5060
10	4.6696	2.5017



### 3 Renormalization Groups

The emergent universality in the scaling of iterated maps begs the question of how odd our function can be and still behave similarly. To check this, I used a function with a  $3/2$  power cusp defined by

$$f(x) = r \left( \left( \frac{1}{2} \right)^{3/2} - \left| x - \frac{1}{2} \right|^{3/2} \right).$$

Performing the same calculations as below yielded the table and bifurcation diagram below (see `IterMap3.java`):

$k$	$\delta$	$\alpha$
01	—	—
02	—	—
03	3.8261	0.7856
04	3.7812	3.6168
05	3.7918	3.4416
06	3.7982	3.4019
07	3.7999	3.3921
08	3.8004	3.3896
09	3.8005	3.3889
10	3.8005	3.3887

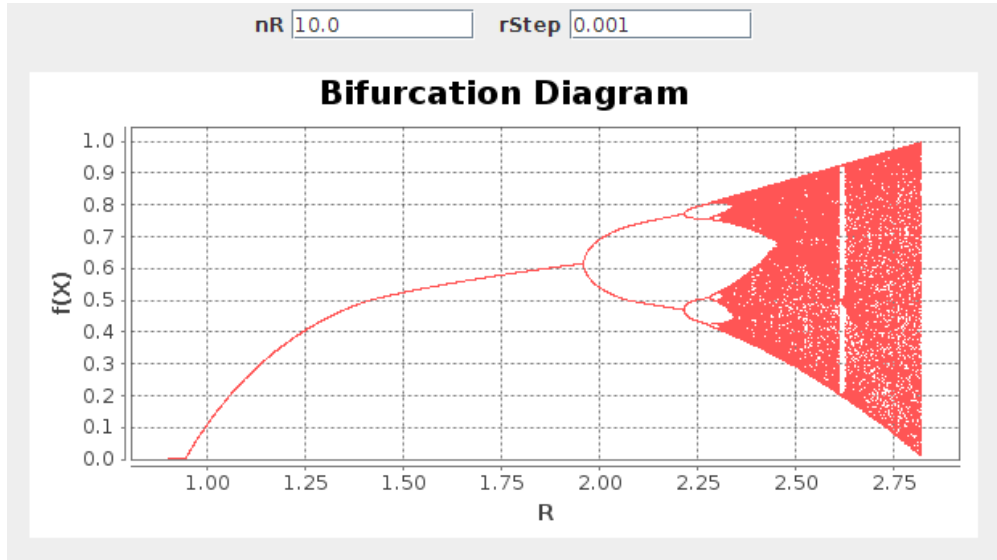
The surprising result here is that the power laws governing approach to  $r_\infty$  are have different coefficients than for the logistic map! However, this may not be as crazy as it seems. The derivation of  $g(r)$  is the same for all functions  $f$  with a single maximum, however, there is an assumption that we used in the estimation of the scaling exponents which is no longer valid. In it, we approximated that  $g(z)$  the limiting function of  $f$  as  $r \rightarrow r_\infty$  could be approximated as

$$g(z) \approx A + Bz^2$$

near  $z = 0$  because of the quadratic maximum. However, this expansion is not the right expansion for  $g$  when  $f$  is the cusped function above. In this case, we must approximate

$$g(x) \approx A + B|z|^{3/2}.$$

All other assumptions that led to the Feigenbaum equation were valid, and thus we can use this minor adjustment to reevaluate our expectations for  $\alpha$  in this new system.



Feigenbaum can take the form

$$g(z) = -\alpha g(g(-z/\alpha)).$$

With this equation and the approximation, we can write:

$$\begin{aligned} g(z) &= A - B|z|^{3/2} \\ g(g(z)) &= A - B|A - B|z|^{3/2}|^{3/2} \\ &= A - BA^{3/2}|1 - \frac{B}{A}|z|^{3/2} \end{aligned}$$

At this point, we can exploit the fact that  $z$  is small, and expand the term inside the absolute values as  $(1 + \epsilon)^{3/2} \approx 1 + \frac{3}{2}\epsilon$ . Doing this and substituting into the Feigenbaum eq. we find that

$$A - B|z|^{3/2} = -\alpha A + \alpha BA^{3/2}(1 - \frac{3B}{2A\alpha^{3/2}}|z|^{3/2})$$

Equating the coefficients of like terms, we find that

$$\begin{aligned} A &= -\alpha A + \alpha BA^{3/2} \\ \Rightarrow BA^{1/2} &= \frac{1 + \alpha}{\alpha} \\ B &= \frac{3B^2 A^{1/2}}{2\alpha^{1/2}} \\ \Rightarrow 2\alpha^{1/2} &= 3\frac{1 + \alpha}{\alpha} \\ \therefore 0 &= 9(1 + \alpha)^2 - 4\alpha^3 \end{aligned}$$

Using the previously discovered value of  $\alpha$  and `NSolve.java`, I numerically solved for the root of this final equation and calculated an estimate of

$$\alpha \approx 3.65.$$

Comparing this to the value attained by iteration, we find a discrepancy of less than 10%. To improve the agreement, the approximate Feigenbaum equation could include higher order terms.

Regardless of numerical precision, the existence of a second scaling behavior supports the existence of Universality classes, which can broadly describe the behavior of certain functions that meet requirements specific to the class. Here we have seen a new universality class of functions with  $3/2$  power law cusps. While the period doubling and convergence to  $r_{\text{inf}}$  behavior are the same as for functions with quadratic maxima, the manner in which they converge differs significantly.

## 4 Code

../IterMap.java

```
1 // IterMap.java
2 // Study the convergence of r_k in the logistic map
3
4 import javax.swing.*;
5 import P251.*;
6
7 public class IterMap extends P251Applet {
8
9     /**
10      * ***** VARIABLES *****
11      */
12
13     double tol = 1e-8; // tolerance for N-R convergence
14     int iterMax = 1000000; // max number of iterations in N-R
15     double epsilon = 1e-6; // delta for use in numerical deriv
16
17     int nR; // number of rsk to look for
18     double delta; // scaling parameter for r
19     double alpha; // scaling parameter for y
20
21     double [] roots; // discovered roots
22     double [] y; // value of iterf half way through a 2^k cycle
23
24     double xMax = .5; // x where maximum occurs for logistic map
25
26     inputPanel ip1;
27
28     /**
29      * ***** METHODS *****
30      */
31
32     /** Custom Math Functions */
33
34     double f (double x, double r) {
35         // logistic map
36         return r * x * (1 - x);
37     }
38
39     double iterf (double x, double r, int n) {
40         // return n-th iterate of f
41         if (n>1) return f(iterf(x,r,n-1), r);
42         else return f(x,r);
43     }
44
45     double F (double r, int k) {
46         // function used for optimization of r when x=xMax
47         //
48         double ans = iterf(.5, r, (int) Math.pow(2,k)) - xMax;
49         for (int i=0; i<nR; i++){
50             ans /= (r-roots[i]);
51         }
52 }
```

```
53     return ans;
54
55 }
56
57 double dFdr (double r, int k){
58     // take the derivative of F WRT r
59     return (F(r+epsilon, k) - F(r, k)) / epsilon;
60 }
61
62 double rootR (double rg, int k){
63     // use Newton's method to find r_{si}
64
65     double r = rg;
66     double delta;
67     int i = 0;
68
69     do {
70         // find derivative towards zero
71         delta = -F(r,k) / dFdr(r,k);
72         // update r
73         r += delta;
74
75         if (i>=iterMax) break;
76         i++;
77     } while(Math.abs(delta)>tol);
78
79     // if we timed out, then return NaN
80     if (i==iterMax) {
81         r = Double.NaN;
82         System.out.println("Tried too hard to find rootR");
83     }
84
85
86     return r;
87 }
88
89 /***** P251Applet Methods *****/
90 public void fillPanels() {
91     // define the panels for human interaction
92     ip1 = new inputPanel();
93
94     ip1.addField("nR", 10);
95     addPanel(ip1);
96     initValues();
97
98 }
99
100 public void initValues() {
101     // set up initial values
102
103     nR = 10;
104     roots = new double[nR];
105     y = new double[nR];
106     // set roots to 1 so as not to cause overflow on divide
107     for (int i=0; i<nR; i++) roots[i] = 1;
108 }
```

```
109     }
110
111     public void readValues() {
112         // read input panel values
113         nR = (int) ip1.getValue(0);
114         roots = new double[nR];
115         y = new double[nR];
116         // set roots to 1 so as not to cause overflow on divide
117         for (int i=0; i<nR; i++) roots[i] = 1;
118
119     }
120
121     public void compute() {
122
123         double rg; // r guess for root finding
124
125         // solved for first two analytically
126         roots[0] = 2;
127         roots[1] = 1 + Math.sqrt(5);
128
129         System.out.println("\nk\t r-sk\t d\t y\t a");
130
131         // solve details of k=0,1 cases outside of the loop
132         int k = 0;
133         y[k] = iterf(.5, roots[k], (int) Math.pow(2,k-1));
134         System.out.println(String.format("%02d\t %4.4f\t --\t %4.4f\t --", k+1, roots[k],
135             y[k]));
136
137         k = 1;
138         y[k] = iterf(.5, roots[k], (int) Math.pow(2,k-1));
139         System.out.println(String.format("%02d\t %4.4f\t --\t %4.4f\t --", k+1, roots[k],
140             y[k]));
141
142         // for each k, figure out r-sk, delta, y, alpha
143         for (k=2; k<nR; k++) {
144             rg = roots[k-1] + .1 * (roots[k-1] - roots[k-2]);
145             roots[k] = rootR(rg, k);
146             delta = (roots[k-1] - roots[k-2]) / (roots[k]-roots[k-1]);
147             y[k] = iterf(.5, roots[k], (int) Math.pow(2,k-1));
148             alpha = - (y[k-1] - y[k-2]) / (y[k] - y[k-1]);
149             System.out.println(String.format("%02d\t %4.4f\t %4.4f\t %4.4f\t %4.4f", k+1,
150                 roots[k], delta, y[k], alpha));
151
152             if (Thread.interrupted()) return;
153         }
154     }
155 }
156
157 }
```

../IterMap2.java

```
1 // IterMap2.java
2 // Study the convergence of r_k in an iterative map
3
4 import javax.swing.*;
5 import P251.*;
6
7 public class IterMap2 extends P251Applet {
8
9     /**
10      * ***** VARIABLES *****
11      */
12
13     double tol = 1e-8; // tolerance for N-R convergence
14     int iterMax = 1000000; // max number of iterations in N-R
15     double epsilon = 1e-6; // delta for use in numerical deriv
16
17     int nR; // number of rsk to look for
18     double delta; // scaling parameter for r
19     double alpha; // scaling parameter for y
20
21     double [] roots; // discovered roots
22     double [] y; // value of iterf half way through a 2^k cycle
23
24     double X0 = .8; // initialize X0 for iteration at startup
25
26     double xMax = .8; // x where maximum occurs for f
27     double rMin = 2.6; // just before the first bifurcation
28     double rMax = 3.5; // once f leaves the range of interest
29     double rStep; // how finely to look at r in the bifurcation map
30
31     inputPanel ip1;
32     graphPanel gpl;
33
34     /**
35      * ***** METHODS *****
36      */
37
38     /**
39      * ***** Custom Math Functions *****
40      */
41
42     double f (double x, double r) {
43         // Gr (RP3.8)
44         return r * x*x * Math.sqrt(1 - x);
45     }
46
47     double iterf (double x, double r, int n) {
48         // return n-th iterate of f
49         if (n>1) return f(iterf(x,r,n-1), r);
50         else return f(x,r);
51     }
52
53     double F (double r, int k) {
54         // function used for optimization of r when x=xMax
55         double ans = iterf(xMax, r, (int) Math.pow(2,k)) - xMax;
```



```
56     for (int i=0; i<nR; i++){
57         ans /= (r-roots[i]);
58     }
59     return ans;
60 }
61
62 double dFdr (double r, int k){
63     // take the derivative of F WRT r
64     return (F(r+epsilon, k) - F(r, k)) / epsilon;
65 }
66
67 double rootR (double rg, int k){
68     // use Newton's method to find r_{si}
69
70     double r = rg;
71     double delta;
72     int i = 0;
73
74     do {
75         // find derivative towards zero
76         delta = -F(r,k) / dFdr(r,k);
77         // update r
78         r += delta;
79
80         if (i>=iterMax) break;
81         i++;
82     } while(Math.abs(delta)>tol);
83
84     // if we timed out, then return NaN
85     if (i==iterMax) {
86         r = Double.NaN;
87         System.out.println("Tried too hard to find rootR");
88     }
89
90
91     return r;
92 }
93
94 double [] getLastValues(double r, int num, int nit) {
95     // get the last num values of f
96     // after iterating nit times
97     double [] output = new double[num];
98     double X = X0;
99     for (int i=0; i<nit; i++) {
100         X = f(X, r);
101         if (i>nit-num-1) {
102             output[i + num - nit] = X;
103         }
104     }
105     return output;
106 }
107
108
109 /***** Plotting Methods *****/
110 void plotBifurcations(double rStep, graphPanel gp, int nKeep, int nIter){
111     gp.clear();
```

```
112 double [] RR = new double[nKeep];
113 double [] XX = new double[nKeep];
114
115 double R = rMin;
116 while (R<=rMax) {
117     XX = getLastValues(R, nKeep, nIter);
118     for (int i = 0; i<nKeep; i++) RR[i] = R;
119
120     gp.addData(RR, XX, "bifurcation");
121     R += rStep;
122     if (Thread.interrupted()) return;
123 }
124 }
125
126 /***** P251Applet Methods *****/
127 public void fillPanels() {
128     // define the panels for human interaction
129     ip1 = new inputPanel();
130
131     ip1.addField("nR", 10);
132     ip1.addField("rStep", .001);
133     addPanel(ip1);
134     initValues();
135
136     gp1 = new graphPanel(600, 300, false);
137
138     gp1.setXLabel("R");
139     gp1.setYLabel("f(X)");
140     gp1.setTitle("Bifurcation Diagram");
141     addPanel(gp1);
142
143 }
144
145 public void initValues() {
146     // set up initial values
147
148     nR = 10;
149     rStep = .001;
150     roots = new double[nR];
151     y = new double[nR];
152     // set roots to 1 so as not to cause overflow on divide
153     for (int i=0; i<nR; i++) roots[i] = 1;
154
155 }
156
157 public void readValues() {
158     // read input panel values
159     nR = (int) ip1.getValue(0);
160     rStep = ip1.getValue(1);
161     roots = new double[nR];
162     y = new double[nR];
163     // set roots to 1 so as not to cause overflow on divide
164     for (int i=0; i<nR; i++) roots[i] = 1;
165
166 }
167
```

```
168 public void compute() {
169
170     double rg; // r guess for root finding
171
172     // solved for first two analytically
173     roots[0] = 2.79508;
174     roots[1] = 3.15783;
175
176     System.out.println("\nk\td\ta");
177
178     // solve details of k=0,1 cases outside of the loop
179     int k = 0;
180     y[k] = iterf(xMax, roots[k], (int) Math.pow(2,k-1));
181     System.out.println(String.format("%02d\t--\t--", k+1));
182
183     k = 1;
184     y[k] = iterf(xMax, roots[k], (int) Math.pow(2,k-1));
185     System.out.println(String.format("%02d\t--\t--", k+1));
186
187     // for each k, figure out r_sk, delta, y, alpha
188     for (k=2; k<nR; k++) {
189         rg = roots[k-1] + .1 * (roots[k-1] - roots[k-2]);
190         roots[k] = rootR(rg, k);
191         delta = (roots[k-1] - roots[k-2]) / (roots[k]-roots[k-1]);
192         y[k] = iterf(xMax, roots[k], (int) Math.pow(2,k-1));
193         alpha = - (y[k-1] - y[k-2]) / (y[k] - y[k-1]);
194         System.out.println(String.format("%02d\t%4.4f\t%4.4f", k+1, delta, alpha));
195
196         if (Thread.interrupted()) return;
197     }
198
199     plotBifurcations(rStep, gpl, 100, 1000);
200
201
202
203
204
205 }
206
207 }
```

../IterMap3.java

```
1 // IterMap3.java
2 // Study the convergence of r_k in the cusped map
3
4 import javax.swing.*;
5 import P251.*;
6
7 public class IterMap3 extends P251Applet {
8
9     /**
10      * ***** VARIABLES *****
11      */
12
13     double tol = 1e-8; // tolerance for N-R convergence
14     int iterMax = 1000000; // max number of iterations in N-R
15     double epsilon = 1e-6; // delta for use in numerical deriv
16
17     int nR; // number of rsk to look for
18     double delta; // scaling parameter for r
19     double alpha; // scaling parameter for y
20
21     double [] roots; // discovered roots
22     double [] y; // value of iterf half way through a 2^k cycle
23
24     double X0 = .5; // initialize X0 for iteration at startup
25
26     double xMax = .5; // x where maximum occurs for f
27     double rMin = .9; // just before the first bifurcation
28     double rMax = 2.82; // once f leaves the range of interest
29     double rStep; // how finely to look at r in the bifurcation map
30
31     inputPanel ip1;
32     graphPanel gpl;
33
34
35     /**
36      * ***** METHODS *****
37      */
38
39     /** Custom Math Functions */
40
41     double f (double x, double r) {
42         // cusp map
43         return r * ( Math.pow(.5, 1.5) - Math.pow(Math.abs(x-.5), 1.5) );
44     }
45
46     double iterf (double x, double r, int n) {
47         // return n-th iterate of f
48         if (n>1) return f(iterf(x,r,n-1), r);
49         else return f(x,r);
50     }
51
52     double F (double r, int k) {
53         // function used for optimization of r when x=xMax
54         //
55         double ans = iterf(xMax, r, (int) Math.pow(2,k)) - xMax;
```

```
56     for (int i=0; i<nR; i++){
57         ans /= (r-roots[i]);
58     }
59     return ans;
60 }
61
62 double dFdr (double r, int k){
63     // take the derivative of F WRT r
64     return (F(r+epsilon, k) - F(r, k)) / epsilon;
65 }
66
67 double rootR (double rg, int k){
68     // use Newton's method to find r_{si}
69
70     double r = rg;
71     double delta;
72     int i = 0;
73
74     do {
75         // find derivative towards zero
76         delta = -F(r,k) / dFdr(r,k);
77         // update r
78         r += delta;
79
80         if (i>=iterMax) break;
81         i++;
82     } while(Math.abs(delta)>tol);
83
84     // if we timed out, then return NaN
85     if (i==iterMax) {
86         r = Double.NaN;
87         System.out.println("Tried too hard to find rootR");
88     }
89
90
91     return r;
92 }
93
94 double [] getLastValues(double r, int num, int nit) {
95     // get the last num values of f
96     // after iterating nit times
97     double [] output = new double[num];
98     double X = X0;
99     for (int i=0; i<nit; i++) {
100         X = f(X, r);
101         if (i>nit-num-1) {
102             output[i + num - nit] = X;
103         }
104     }
105     return output;
106 }
107
108
109 /***** Plotting Methods *****/
110 void plotBifurcations(double rStep, graphPanel gp, int nKeep, int nIter){
111     gp.clear();
```

```
112 double [] RR = new double[nKeep];
113 double [] XX = new double[nKeep];
114
115 double R = rMin;
116 while (R<=rMax) {
117     XX = getLastValues(R, nKeep, nIter);
118     for (int i = 0; i<nKeep; i++) RR[i] = R;
119
120     gp.addData(RR, XX, "bifurcation");
121     R += rStep;
122     if (Thread.interrupted()) return;
123 }
124 }
125
126 /***** P251Applet Methods *****/
127 public void fillPanels() {
128     // define the panels for human interaction
129     ip1 = new inputPanel();
130
131     ip1.addField("nR", 10);
132     ip1.addField("rStep", .001);
133     addPanel(ip1);
134     initValues();
135
136     gp1 = new graphPanel(600, 300, false);
137
138     gp1.setXLabel("R");
139     gp1.setYLabel("f(X)");
140     gp1.setTitle("Bifurcation Diagram");
141     addPanel(gp1);
142
143 }
144
145 public void initValues() {
146     // set up initial values
147
148     nR = 10;
149     rStep = .001;
150     roots = new double[nR];
151     y = new double[nR];
152     // set roots to 1 so as not to cause overflow on divide
153     for (int i=0; i<nR; i++) roots[i] = 1;
154
155 }
156
157 public void readValues() {
158     // read input panel values
159     nR = (int) ip1.getValue(0);
160     rStep = ip1.getValue(1);
161     roots = new double[nR];
162     y = new double[nR];
163     // set roots to 1 so as not to cause overflow on divide
164     for (int i=0; i<nR; i++) roots[i] = 1;
165
166 }
167
```

```
168 public void compute() {
169
170 double rg; // r guess for root finding
171
172 // solved for first analytically
173 roots[0] = Math.pow(2, .5);
174
175 System.out.println("\nk\td\ta");
176
177 // solve details of k=0,1 cases outside of the loop
178 int k = 0;
179 y[k] = iterf(xMax, roots[k], (int) Math.pow(2,k-1));
180 System.out.println(String.format("%02d\t--\t—", k+1));
181
182 k = 1;
183 rg = roots[k-1] * 1.1;
184 roots[k] = rootR(rg, k);
185 y[k] = iterf(xMax, roots[k], (int) Math.pow(2,k-1));
186 System.out.println(String.format("%02d\t--\t—", k+1));
187
188 // for each k, figure out r_sk, delta, y, alpha
189 for (k=2; k<=nR; k++) {
190     rg = roots[k-1] + .1 * (roots[k-1] - roots[k-2]);
191     roots[k] = rootR(rg, k);
192     delta = (roots[k-1] - roots[k-2]) / (roots[k]-roots[k-1]);
193     y[k] = iterf(xMax, roots[k], (int) Math.pow(2,k-1));
194     alpha = - (y[k-1] - y[k-2]) / (y[k] - y[k-1]);
195     System.out.println(String.format("%02d\t%4.4f\t%4.4f", k+1, delta, alpha));
196
197     if (Thread.interrupted()) return;
198 }
199
200 plotBifurcations(rStep, gpl, 100, 1000);
201
202
203
204
205 }
206
207
208 }
```

../NSolve.java

```
1 // NSolve.java
2 // Newton's Method Numerical Solver
3 // for one dimensional functions, looks for single root at a time
4 // just change f and xguess and recompile for different functions
5
6 import javax.swing.*;
7 import P251.*;
8
9 public class NSolve extends P251Applet {
10
11     // Global Variables
12
13     double defaultTolerance = 1e-9; // tolerance for N-R convergence
14     double derivDelta = 1e-6;      // delta for use in numerical deriv
15     int iterMax = 1000000;         // max number of iterations in N-R
16
17     double xg; // x guess for solver
18
19     private inputPanel ip1;
20
21     // functions
22
23
24     /** Customize Function Here **/
25
26     double F(double x) {
27         // function to solve for root
28
29         return 4 * Math.pow(x,3) - 9 * ( Math.pow(x,2) + 2 * x + 1 );
30
31     }
32
33     /**
34
35
36     double dFdx (double x) {
37         // numerical first derivative of F
38         return (F(x+derivDelta) - F(x)) / derivDelta;
39     }
40
41     double root (double x0) {
42         // Newton's method for root finding
43         double x = x0;
44         double delta;
45         int i = 0;
46
47         do {
48             delta = - F(x) / dFdx(x);
49             x += delta;
50
51             if (i>=iterMax) break;
52             i++;
53         } while (Math.abs(delta)>defaultTolerance);
54
55         if (i==iterMax) x = Double.NaN;
```



```
56
57     return x;
58 }
59
60
61
62 // Applet Methods
63
64 public void fillPanels() {
65     ip1 = new inputPanel();
66     ip1.addField("xg", 0);
67
68     addPanel(ip1);
69
70 }
71
72 public void initValues() {
73     xg = 0;
74 }
75
76 public void readValues() {
77     xg = ip1.getValue(0);
78 }
79
80 public void compute() {
81     double r = root(xg);
82     System.out.println("\nroot: " + r);
83 }
84
85
86 }
```