

RECOVERY PROBLEM OF PARAMETRIZATIONS FROM LEGENDRE DATA

C. MUÑOZ-CABELLO, T. NISHIMURA, AND R. OSET SINHA

ABSTRACT. The problem of recovery of parametrizations from Legendre data is a very important inverse problem. In this paper, we provide a systematic and widely-applicable method to recover parametrizations $f : U_n \rightarrow \mathbb{R}^{n+1}$ from Legendre data where U_n is an open subset of \mathbb{R}^n . Namely, for a dense subset of the space of real-analytic parametrizations from U_n into \mathbb{R}^{n+1} , we show how to recover the parametrization from the Gauss mapping and the height function. Moreover, in order to assist readers to apply results of this paper, many concrete examples are given.

1. INTRODUCTION

Throughout this paper, unless otherwise stated, n is a positive integer and U_n is an open subset of \mathbb{R}^n . Moreover, all functions and mappings are real-analytic.

Recovery problems are important inverse problems since the recovery of unprocessed raw data from processed data often gives a breakthrough in Science and Engineering.

The Legendre transform is one example of such a recovery. A brief explanation of the Legendre transform is as follows. For instance, let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = x^2$ and consider f to be the unprocessed data. Draw the graph of f in the (X, Y) -plane. The tangent line to the graph of f at $x = x_0$ is defined by

$$\begin{aligned} Y &= \frac{df}{dx}(x_0)(X - x_0) + x_0^2 \\ &= 2x_0X - x_0^2. \end{aligned}$$

Set $p = \frac{df}{dx}(x_0) = 2x_0$ and $q = -(-x_0^2) = x_0^2$. Then, we can obtain the new function

$$q(p) = \left(\frac{p}{2}\right)^2.$$

Namely, the Y -intercept $-q$ of the tangent line to the graph of f is a function of the slope p of the tangent line to the graph of f . The function $q(p)$ is a processed data of $f(x)$ and the process $\mathcal{L}(f(x)) = q(p)$ is called the *Legendre transform*. The recovery problem in this case is to obtain the original raw data $f(x)$ from the processed data $q(p)$. It is well-known that the same process gives the perfect solution of this recovery problem, that is, $\mathcal{L}(q(p)) = f(x)$. Nowadays, Legendre transform has been widely extended and has important applications in Mathematical Physics, Thermodynamics, Convex analysis etc. For more details on the Legendre Transform, see for instance [2, 10, 19].

On the other hand, consider for example the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = x^3$. Draw the graph of g in the (X, Y) -plane. The tangent line to the graph of g at $x = x_0$ is defined by

$$\begin{aligned} Y &= \frac{dg}{dx}(x_0)(X - x_0) + x_0^3 \\ &= 3x_0^2X - 2x_0^3. \end{aligned}$$

If we set $p = \frac{dg}{dx}(x_0) = 3x_0^2$, the Y -intercept $-q = -2x_0^3$ is not a function of p . Thus, it is impossible to apply the Legendre transform to obtain the original unprocessed data $g(x) = x^3$ from the processed data $\{(p(x), q(x))\}_{x \in \mathbb{R}} = \{(3x^2, 2x^3)\}_{x \in \mathbb{R}}$. Nevertheless, very recently, in [16, 17], by replacing the processed data $\{(p(x), q(x))\}_{x \in \mathbb{R}} = \{(3x^2, 2x^3)\}_{x \in \mathbb{R}}$ with the processed data in the sense of Legendre (called *Legendre data*) $\{(\nu(x), a(x))\}_{x \in \mathbb{R}}$, the second author provides a new method to solve the recovery problem. A brief explanation of the new method is as follows. For the given $g(x) = x^3$, let $\nu : \mathbb{R} \rightarrow S^1$ be the mapping defined by $\nu(x) = \frac{1}{\sqrt{1+9x^4}}(-3x^2, 1)$. Then, $\nu(x_0)$ is a unit normal vector to the tangent

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line to the graph of $g(x) = x^3$ at $x = x_0$ and the mapping $\nu : \mathbb{R} \rightarrow S^1$ is called the *Gauss mapping*. Set $a(x) = (x, g(x)) \cdot \nu(x) = \frac{-2x^3}{\sqrt{1+9x^4}}$, where the dot in the center stands for the standard scalar product of 2-dimensional vectors. Then, $a(x_0)$ is the height of the tangent line to the graph of $g(x) = x^3$ at $x = x_0$ relative to the origin $(0, 0)$ and the function $a : \mathbb{R} \rightarrow \mathbb{R}$ is called the *height function*. Thus, the Legendre data in this case is $\left\{ \frac{1}{\sqrt{1+9x^4}} (-3x^2, 1), \frac{-2x^3}{\sqrt{1+9x^4}} \right\}_{x \in \mathbb{R}}$. Next, set $\nu(x) = (\cos \theta(x), \sin \theta(x))$. It has been shown in [16, 17] that $\frac{da}{dx}(x)$ is divided by $\frac{d\theta}{dx}(x)$ and by setting $b(x) = \frac{\frac{da}{dx}(x)}{\frac{d\theta}{dx}(x)}$, the original unprocessed data $g(x)$ can be actually recovered as follows.

$$(x, g(x)) = a(x) (\cos \theta(x), \sin \theta(x)) + b(x) (-\sin \theta(x), \cos \theta(x)).$$

In the case that $g(x) = x^3$, by elementary calculation, it follows that $b(x) = \frac{-x-3x^5}{\sqrt{1+9x^4}}$. Thus, as desired, we certainly have

$$\begin{aligned} & a(x) (\cos \theta(x), \sin \theta(x)) + b(x) (-\sin \theta(x), \cos \theta(x)) \\ = & \frac{-2x^3}{\sqrt{1+9x^4}} \left(\frac{-3x^2}{\sqrt{1+9x^4}}, \frac{1}{\sqrt{1+9x^4}} \right) + \frac{-x-3x^5}{\sqrt{1+9x^4}} \left(\frac{-1}{\sqrt{1+9x^4}}, \frac{-3x^2}{\sqrt{1+9x^4}} \right) \\ = & \left(\frac{6x^5}{1+9x^4}, \frac{-2x^3}{1+9x^4} \right) + \left(\frac{x+3x^5}{1+9x^4}, \frac{3x^3+9x^7}{1+9x^4} \right) \\ = & \left(\frac{x+9x^5}{1+9x^4}, \frac{x^3+9x^7}{1+9x^4} \right) \\ = & (x, x^3) = (x, g(x)). \end{aligned}$$

The new method developed in [16, 17] is based on the study of anti-orthotomics of frontals ([13], see also [1] in which the usefulness of anti-orthotomics for the recovery problem of wave fronts has been already clarified). Moreover, the new method may be regarded as a natural generalization of the outstanding Cahn-Hoffman vector formula (see [9]. see also [15] in which hedgehogs were defined as natural geometric objects whose parametrizations can be described by using the Cahn-Hoffman vector formula).

In this paper, we sublimate the new method in [16, 17] to solve the recovery problem even in higher dimensional cases. As it turns out, our method solves the recovery problem for any real-analytic parametrisation $f : U_n \rightarrow \mathbb{R}^{n+1}$ from Legendre data (Theorem 3). Thus, our sublimation is much stronger than expected. More precisely, we first define the notion of regular frontal and we show the recovery problem for any regular frontal can be directly solved by our sublimation (Theorem 1). Secondly, we generalize the notion of regular frontal to the notion of pseudo regular frontal, for which the sublimation can be directly applied as well (the assertion (1) of Theorem 2). Moreover, we show the space of pseudo regular frontals is dense in the space of all real-analytic mappings (the assertion (2) of Theorem 2), and thus, for any non-pseudo regular frontal, we can recover the parametrisation as the limit of a sequence of pseudo regular frontals (Theorem 3).

The paper is organized as follows. In Section 2 the main results (Theorem 1, Theorem 2, Theorem 3) shall be stated. Theorem 1, Theorem 2 and Theorem 3 shall be proved in Section 3, Section 4 and Section 5, respectively. Finally, in Section 6, many concrete examples with precise calculations shall be given.

2. MAIN RESULTS

Definition 1. A real-analytic mapping $f : U_n \rightarrow \mathbb{R}^{n+1}$ is called a *frontal* if there exists a real-analytic mapping $\nu : U_n \rightarrow S^n$ such that $df_{\mathbf{x}}(\mathbf{v}) \cdot \nu(\mathbf{x}) = 0$ for any $\mathbf{x} \in U_n$ and any $\mathbf{v} \in T_{\mathbf{x}}U_n$. Here, S^n is the unit n -dimensional sphere in \mathbb{R}^{n+1} .

For a frontal $f : U_n \rightarrow \mathbb{R}^{n+1}$, the mapping $\nu : U_n \rightarrow S^n$ satisfying Definition 1 (resp., the function $a : U_n \rightarrow \mathbb{R}$ defined by $a(\mathbf{x}) = f(\mathbf{x}) \cdot \nu(\mathbf{x})$) is called the *Gauss mapping* (resp., the *height function*) of f . For readers who are not familiar with frontals, [11, 12] are recommended as excellent overviews with clear explanations.

Given a frontal $f : U_n \rightarrow \mathbb{R}^{n+1}$, by the Gauss mapping $\nu : U_n \rightarrow S^n$ and the height function $a : U_n \rightarrow \mathbb{R}$, the hyperplane family $\mathcal{H}_{(\nu,a)}$ can be naturally defined as follows.

$$\begin{aligned} H_{(\nu(\mathbf{x}),a(\mathbf{x}))} &= \{ \mathbf{X} \in \mathbb{R}^{n+1} \mid \mathbf{X} \cdot \nu(\mathbf{x}) = a(\mathbf{x}) \} \\ \mathcal{H}_{(\nu,a)} &= \{ H_{(\nu(\mathbf{x}),a(\mathbf{x}))} \}_{\mathbf{x} \in U_n}. \end{aligned}$$

Hence, the data set $\{\nu(\mathbf{x}) \in S^n, a(\mathbf{x}) \in \mathbb{R}\}_{\mathbf{x} \in U_n}$ is called the *Legendre data* of the frontal f .

Example 1. Let $\nu : \mathbb{R} \rightarrow S^1$ be the constant mapping given by $\nu(x) = (0, 1) \in S^1$ and $a : \mathbb{R} \rightarrow \mathbb{R}$ be the function $a(x) = x$. Then, it is clear that there is no frontal f such that $\{(\nu(x), a(x))\}_{x \in \mathbb{R}}$ is the Legendre data of f .

By Example 1, not all data sets $\{\nu(\mathbf{x}) \in S^n, a(\mathbf{x}) \in \mathbb{R}\}_{\mathbf{x} \in U_n}$ can become the Legendre data of a certain frontal f .

Definition 2. Let $\mathcal{H}_{(\nu,a)}$ be a hyperplane family. A real-analytic mapping $f : U_n \rightarrow \mathbb{R}^{n+1}$ is called an *envelope created by $\mathcal{H}_{(\nu,a)}$* if the following two conditions are satisfied.

- (a) $f(\mathbf{x}) \in H_{(\nu(\mathbf{x}),a(\mathbf{x}))}$ for any $\mathbf{x} \in U_n$.
- (b) $df_{\mathbf{x}}(\mathbf{v}) \cdot \nu(\mathbf{x}) = 0$ for any $\mathbf{x} \in U_n$ and any $\mathbf{v} \in T_{\mathbf{x}}U_n$.

In other words, an envelope created by $\mathcal{H}_{(\nu,a)}$ is a mapping $f : U_n \rightarrow \mathbb{R}^{n+1}$ giving a solution of the following system of first order differential equations with one constraint condition, where $(U, (x_1, \dots, x_n))$ is an arbitrary coordinate neighborhood of U_n such that $\mathbf{x} \in U \subset U_n$.

$$\begin{cases} \frac{\partial f}{\partial x_1}(\mathbf{x}) \cdot \nu(\mathbf{x}) = 0, \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \cdot \nu(\mathbf{x}) = 0, \\ f(\mathbf{x}) \cdot \nu(\mathbf{x}) = a(\mathbf{x}). \end{cases}$$

For details on envelopes created by families of plane regular curves, refer to the excellent book [7]. By definition, any frontal $f : U_n \rightarrow \mathbb{R}^{n+1}$ is an envelope created by a hyperplane family $\mathcal{H}_{(\nu,a)}$ where $\nu : U_n \rightarrow S^n$ (resp., $a : U_n \rightarrow \mathbb{R}$) is the Gauss mapping (resp., the height function) of f . Conversely, again by definition, any envelope $f : U_n \rightarrow \mathbb{R}^{n+1}$ created by a hyperplane family $\mathcal{H}_{(\nu,a)}$ is a frontal with the Gauss mapping $\nu : U_n \rightarrow S^n$ and the height function $a : U_n \rightarrow \mathbb{R}$. Thus, frontals and envelopes of hyperplane families are exactly the same notion. Nevertheless, we prefer to study the notion of envelopes created by hyperplane families more because it seems that envelopes created by hyperplane families are more applicable (for instance, see [18]). As in Example 1, not all hyperplane families create envelopes. Thus, it is important to solve the following item (1) of Problem 1. In addition, since envelopes are real-analytic mappings by definition, especially for applications of envelopes created by hyperplane families, solving the following item (2) of Problem 1 is important as well.

Problem 1. Let $\nu : U_n \rightarrow S^n$ and $a : U_n \rightarrow \mathbb{R}$ be a real-analytic mapping and a real-analytic function respectively.

- (1) When and only when does the hyperplane family $\mathcal{H}_{(\nu,a)}$ create an envelope?
- (2) Suppose that the hyperplane family $\mathcal{H}_{(\nu,a)}$ creates an envelope $f : U_n \rightarrow \mathbb{R}^{n+1}$. Then, describe f in terms of ν, a .

Both problems in Problem 1 have been solved as follows.

Fact 1 ([16]). *Let $\nu : U_n \rightarrow S^n$ and $a : U_n \rightarrow \mathbb{R}$ be a real-analytic mapping and a real-analytic function respectively.*

- (1) *The hyperplane family $\mathcal{H}_{(\nu,a)}$ creates an envelope if and only if there exists a 1-form Ω along ν such that for any $\mathbf{x}_0 \in U_n$ by using of a coordinate neighborhood $(U, (x_1, \dots, x_n))$ of U_n at \mathbf{x}_0 and a normal coordinate neighborhood $(V, (\Theta_1, \dots, \Theta_n))$ of S^n at $\nu(\mathbf{x}_0)$, the 1-form germ $d\gamma$ at \mathbf{x}_0 is expressed as follows.*

$$da = \sum_{i=1}^n \left(\omega(\mathbf{x}) \left(\Pi_{(\nu(\mathbf{x}), \nu(\mathbf{x}_0))} \left(\frac{\partial}{\partial \Theta_i} \right) \right) \right) d(\Theta_i \circ \nu),$$

where a normal coordinate neighborhood $(V, (\Theta_1, \dots, \Theta_n))$ is a nothing but a local inverse mapping of the exponential mapping at $\nu(\mathbf{x}_0)$ and $\Pi_{(\nu(\mathbf{x}), \nu(\mathbf{x}_0))} : T_{\nu(\mathbf{x}_0)}S^n \rightarrow T_{\nu(\mathbf{x})}S^n$ is the Levi-Civita translation. Notice that since the unit sphere S^n with metric inherited from \mathbb{R}^{n+1} , the Levi-Civita translation $\Pi_{(\nu(\mathbf{x}), \nu(\mathbf{x}_0))}$ is merely the restriction of the rotation $R : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ satisfying $R(\nu(\mathbf{x}_0)) = \nu(\mathbf{x})$ to the tangent space $T_{\nu(\mathbf{x}_0)}S^n$. In particular, in the case $n = 1$, a normal coordinate Θ at $\nu(x)$ is just the radian (or, its negative) between two unit vectors $\nu(\mathbf{x}_0)$ and $\nu(\mathbf{x})$ and the Levi-Civita translation $\Pi_{(\nu(\mathbf{x}), \tilde{\nu}(\mathbf{x}_0))}$ is nothing but the restriction of the plane rotation through Θ to the tangent space $T_{\tilde{\nu}(\mathbf{x}_0)}S^1$.

- (2) Suppose that the hyperplane family $\mathcal{H}_{(\nu, a)}$ creates an envelope $f : U_n \rightarrow \mathbb{R}^{n+1}$. Then, for any $\mathbf{x} \in U_n$, under the canonical identifications $T_{\nu(\mathbf{x})}^*S^n \cong T_{\nu(\mathbf{x})}S^n \subset T_{\nu(\mathbf{x})}\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$, the $(n+1)$ -dimensional vector $f(\mathbf{x})$ is represented as follows.

$$f(\mathbf{x}) = \omega(\mathbf{x}) + a(\mathbf{x})\nu(\mathbf{x}),$$

where the $(n+1)$ -dimensional vector $\omega(\mathbf{x})$ is identified with the corresponding n -dimensional cotangent vector $\omega(\mathbf{x})$ under these identifications.

For readers who are not familiar with normal coordinate neighborhoods, exponential mappings and Levi-Civita translations, [14] is recommended as one of excellent references.

The recovery problem from Legendre data which we deal with in this paper arises as a hard problem when we try to do concrete computations by using the assertion (2) of Fact 1 in the case $n \geq 2$. In the case $n = 1$, there is no problem on doing concrete computations as in Section 1. On the other hand, in the case $n \geq 2$, even by using symbolic algebras, it seems that local concrete computations on the one form $\omega(\mathbf{x})$ and the Levi-Civita translation $\Pi_{(\nu(\mathbf{x}), \nu(\mathbf{x}_0))} : T_{\nu(\mathbf{x}_0)}S^n \rightarrow T_{\nu(\mathbf{x})}S^n$ are almost impossible. Thus, for the recovery of the concrete form of the parametrization f from the Legendre data $\{\nu(\mathbf{x}), a(\mathbf{x})\}_{\mathbf{x} \in U_n}$ in the case $n \geq 2$, we had to exploit a new technique instead of assertion (2) of Fact 1.

Notice that any non-singular mapping $f : U_n \rightarrow \mathbb{R}^{n+1}$ is a frontal. It is T. Banchoff, T. Gaffney and C. McCrory who started to study the Gauss mappings $\nu : U_2 \rightarrow S^2$ (see [5]). Their study was concentrated on cusps of Gauss mappings for non-singular surfaces $f : U_2 \rightarrow \mathbb{R}^3$. On the other hand, as in the following Example 2, there are many singular frontals. Notice that the Gauss mappings for singular frontals $f : U_n \rightarrow \mathbb{R}^{n+1}$ have not been studied in depth so far. One of the purposes of this paper is to provide an essential contribution on studying Gauss mappings for singular frontals.

Definition 3. A frontal $f : U_n \rightarrow \mathbb{R}^{n+1}$ is called a *regular frontal* if for the Gauss mapping $\nu : U_n \rightarrow S^n$ of f , the set

$$Reg(\nu) = \{\mathbf{x} = (x_1, \dots, x_n) \in U_n \mid \mathbf{x} \text{ is a regular point of } \nu\}$$

is dense in U_n .

Notice that $Reg(\nu)$ is always open since $Reg(\nu)$ may be characterized as the inverse image of an open subset $\mathbb{R} - \{0\}$ (of \mathbb{R}) by the Jacobian determinant of ν which is a continuous function $U_n \rightarrow \mathbb{R}$.

Definition 4. A real-analytic mapping $f : U_n \rightarrow \mathbb{R}^{n+1}$ is called a *pseudo regular frontal* if the following (1), (2) are satisfied.

- (1) The set

$$Reg(f) = \{\mathbf{x} = (x_1, \dots, x_n) \in U_n \mid \mathbf{x} \text{ is a regular point of } f\}$$

is dense in U_n .

- (2) For a Gauss mapping $\nu : Reg(f) \rightarrow S^n$ of $f|_{Reg(f)}$, the set

$$Reg(\nu) = \{\mathbf{x} = (x_1, \dots, x_n) \in Reg(f) \mid \mathbf{x} \text{ is a regular point of } \nu\}$$

is dense in $Reg(f)$.

Notice that similarly as $Reg(\nu)$, the set $Reg(f)$ is always open.

Note 1. (1) Any regular frontal is a frontal, but its converse is not true in general (see (2), (4) of Example 2 below).

- (2) Any regular frontal is a pseudo regular frontal, but its converse is not true in general (see (5) of Example 2 below).

Example 2. For detailed analysis of the following examples, see Section 6.

- (1) The mapping $f : U_1 \rightarrow \mathbb{R}^2$ defined by

$$f(x) = (x^2, x^3)$$

is called the *standard normal form of the cusp*, and it is an example of frontal and of regular frontal as well.

- (2) The mapping $f : U_2 \rightarrow \mathbb{R}^3$ defined by

$$f(x, y) = (x, y^2, y^3)$$

is called the *standard normal form of the cuspidal edge*, and it is an example of frontal but not a regular frontal. Any f equivalent under diffeomorphisms in source and target to f is called a *cuspidal edge*. The mapping $f : U_2 \rightarrow \mathbb{R}^3$ defined by

$$f(x, y) = (x, y^2 + x^2, y^3)$$

is a cuspidal edge which is a frontal and a regular frontal as well.

- (3) The mapping $f : U_2 \rightarrow \mathbb{R}^3$ defined by

$$f(x, y) = (x, y^2, xy^3)$$

is called the *standard normal form of the cuspidal crosscap*, and it is an example of frontal and of regular frontal as well.

- (4) The mapping $f : U_2 \rightarrow \mathbb{R}^3$ defined by

$$f(x, y) = (x, 4y^3 + 2xy, 3y^4 + xy^2)$$

is called the *standard normal form of the swallowtail*, and it is an example of frontal but not a regular frontal.

- (5) The mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$f(x, y) = (x, y^2, xy)$$

is called the *standard normal form of the crosscap*, and it is not a frontal. Set $U_2 = \mathbb{R}^2 - \{(0, 0)\}$. Then, $f|_{U_2}$ is non-singular. It is easily seen that $f|_{U_2}$ is a regular frontal as well. Thus, f is a pseudo regular frontal.

The image of these mappings can be found in Figure 1.

Definition 5. Given a pseudo regular frontal $f : U_n \rightarrow \mathbb{R}^{n+1}$, the set $\{\nu(\mathbf{x}), a(\mathbf{x})\}_{\mathbf{x} \in \text{Reg}(\nu)}$ is called the *Legendre data* of f , where $\nu : \text{Reg}(f) \rightarrow S^n$ is a Gauss mapping of $f|_{\text{Reg}(f)}$, $a : \text{Reg}(f) \rightarrow \mathbb{R}$ is the height function of $f|_{\text{Reg}(f)}$ and $\text{Reg}(\nu)$ is the set of regular points of ν .

Theorem 1. Let $f : U_n \rightarrow \mathbb{R}^{n+1}$ be a regular frontal. Then, f is recovered from its Legendre data $\{\nu(\mathbf{x}), a(\mathbf{x})\}_{\mathbf{x} \in \text{Reg}(\nu)}$. Namely, for any $\mathbf{x} \in U_n$, $f(\mathbf{x})$ can be concretely described in terms of $\nu(\mathbf{x}_i), a(\mathbf{x}_i)$ and $\lim_{i \rightarrow \infty}$ where $\{\mathbf{x}_i\}_{i=1,2,\dots}$ is any sequence of regular points of $\nu : U_n \rightarrow S^n$ such that $\lim_{i \rightarrow \infty} \mathbf{x}_i = \mathbf{x}$.

Let $A(U_n, \mathbb{R}^{n+1})$ be the topological space of real-analytic mappings $f : U_n \rightarrow \mathbb{R}^{n+1}$ endowed with the Whitney C^∞ topology. Let $\text{PRF}(U_n, \mathbb{R}^{n+1})$ be the topological subspace of $A(U_n, \mathbb{R}^{n+1})$, consisting of pseudo regular frontals.

Theorem 2. As for the subspace of pseudo regular frontals $\text{PRF}(U_n, \mathbb{R}^{n+1})$, the following two hold.

- (1) Any $f \in \text{PRF}(U_n, \mathbb{R}^{n+1})$ can be recovered from the Legendre data $\{\nu(\mathbf{x}), a(\mathbf{x})\}_{\mathbf{x} \in \text{Reg}(\nu)}$. Namely, for any $\mathbf{x} \in U_n$, $f(\mathbf{x})$ can be concretely described in terms of $\nu(\mathbf{x}_i), a(\mathbf{x}_i)$ and $\lim_{i \rightarrow \infty}$ where $\{\mathbf{x}_i\}_{i=1,2,\dots}$ is any sequence of regular points of $\nu : \text{Reg}(f) \rightarrow S^n$ such that $\lim_{i \rightarrow \infty} \mathbf{x}_i = \mathbf{x}$.
- (2) The subspace $\text{PRF}(U_n, \mathbb{R}^{n+1})$ is dense in $A(U_n, \mathbb{R}^{n+1})$.

Theorem 3. For any real-analytic mapping $f : U_n \rightarrow \mathbb{R}^{n+1}$, let $\{f_1, f_2, \dots\}$ be a sequence of pseudo regular frontals such that $\lim_{i \rightarrow \infty} f_i = f$. Then, f can be recovered from the sequence of Legendre data $\left\{ \{\nu_i(\mathbf{x}), a_i(\mathbf{x})\}_{\mathbf{x} \in \text{Reg}(\nu_i)} \right\}_{i=1,2,\dots}$ of f_i . Namely, for any $\mathbf{x} \in U_n$, $f(\mathbf{x})$ can be concretely described in terms of $\nu_i(\mathbf{x}_j), a_i(\mathbf{x}_j)$, $\lim_{i \rightarrow \infty}$ and $\lim_{j \rightarrow \infty}$ where $\{\mathbf{x}_j\}_{j=1,2,\dots}$ is any sequence of regular points of $\nu_i : \text{Reg}(f_i) \rightarrow S^n$ such that $\lim_{j \rightarrow \infty} \mathbf{x}_j = \mathbf{x}$.

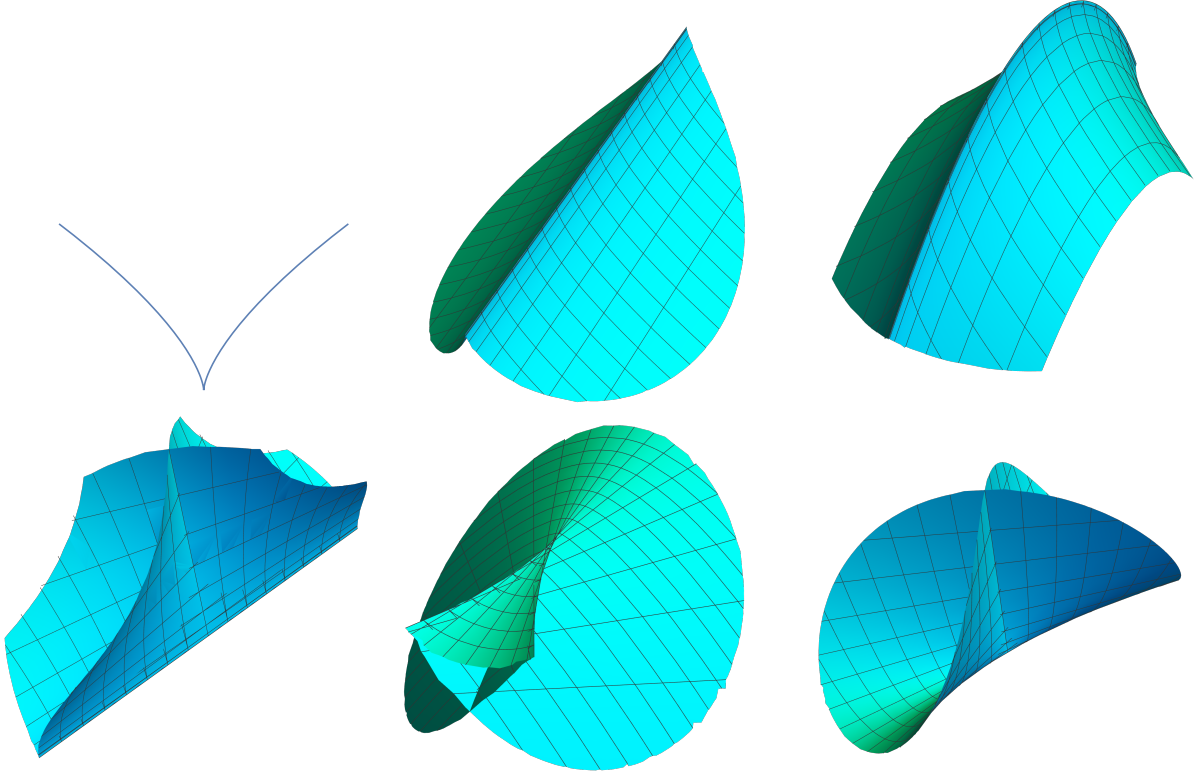


FIGURE 1. Images of the mappings from Example 2, in order of appearance.

3. PROOF OF THEOREM 1

Let $\nu^n: W = \underbrace{]-\pi, \pi[\times \cdots \times]-\pi, \pi[}_{n\text{-tuples}} \rightarrow S^n$ be the mapping defined by

$$\nu^n(\theta) = \left(\prod_{i=1}^n \cos \theta_i, \sin \theta_n \prod_{i=1}^{n-1} \cos \theta_i, \sin \theta_{n-1} \prod_{i=1}^{n-2} \cos \theta_i, \dots, \sin \theta_2 \cos \theta_1, \sin \theta_1 \right).$$

This mapping is a homeomorphism on its image, thus defining a parametrisation of the n -sphere. If we identify \mathbb{R}^{n+1} with $T_{\nu^n(\theta)}\mathbb{R}^{n+1}$ for some $\theta \in W$, it follows from the properties of S^n that $\nu^n(\theta)$ is orthogonal to $T_{\nu^n(\theta)}S^n$. We also define the vector fields $\tilde{\mu}_1, \dots, \tilde{\mu}_n: W \rightarrow TS^n$ along ν^n given by

$$\tilde{\mu}_i(\theta) = d\nu_{\theta}^n \left(\frac{\partial}{\partial \theta_i} \right) = \frac{\partial \nu^n}{\partial \theta_i}.$$

Note that since ν^n is a unit vector field,

$$1 = |\nu^n(\theta)|^2 = \nu^n(\theta) \cdot \nu^n(\theta) \implies 0 = 2 \frac{\partial \nu^n}{\partial \theta_i}(\theta) \cdot \nu^n(\theta) = 2 \tilde{\mu}_i(\theta) \cdot \nu^n(\theta)$$

and $\tilde{\mu}_i(\theta)$ are orthogonal to $\nu^n(\theta)$.

Lemma 3.1. *For $1 \leq i < j \leq n$ and $\theta \in \text{Reg}(\nu)$, $\tilde{\mu}_i(\theta) \cdot \tilde{\mu}_j(\theta) = 0$. Therefore, the set $\{\nu^n(\theta), \tilde{\mu}_1(\theta), \dots, \tilde{\mu}_n(\theta)\}$ is an orthogonal basis for \mathbb{R}^{n+1} . Moreover,*

$$|\tilde{\mu}_1(\theta)| = 1; \quad |\tilde{\mu}_2(\theta)| = \cos \theta_1; \quad \dots \quad |\tilde{\mu}_n(\theta)| = \cos \theta_1 \cdots \cos \theta_{n-1}.$$

Proof. We first note that the vector fields ν^n can be written recursively as

$$\nu^{n+1}(\theta, \phi) = \nu_1^n(\theta)(\cos \phi, \sin \phi, 0 \dots, 0) + (0, 0, \nu_2^n(\theta), \dots, \nu_n^n(\theta)),$$

where $\theta \in \mathbb{R}^n$ and $\phi \in \mathbb{R}$. We then proceed by induction over n .

For $n = 2$, we have

$$\nu^2(\theta_1, \theta_2) = (\cos \theta_1 \cos \theta_2, \cos \theta_1 \sin \theta_2, \sin \theta_1),$$

from which it follows that

$$\begin{aligned}\tilde{\mu}_1(\theta_1, \theta_2) &= \frac{\partial \nu^2}{\partial \theta_1}(\theta_1, \theta_2) = (-\sin \theta_1 \cos \theta_2, -\sin \theta_1 \sin \theta_2, \cos \theta_1); \\ \tilde{\mu}_2(\theta_1, \theta_2) &= \frac{\partial \nu^2}{\partial \theta_2}(\theta_1, \theta_2) = (-\cos \theta_1 \sin \theta_2, \cos \theta_1 \cos \theta_2, 0).\end{aligned}$$

We then compute their modules and the dot products among them:

$$\begin{aligned}|\tilde{\mu}_1| &= \sqrt{\sin^2 \theta_1 \cos^2 \theta_2 + \sin^2 \theta_1 \sin^2 \theta_2 + \cos^2 \theta_1} = \sqrt{\sin^2 \theta_1 + \cos^2 \theta_1} = 1; \\ |\tilde{\mu}_2| &= \sqrt{\cos^2 \theta_1 \sin^2 \theta_2 + \cos^2 \theta_1 \cos^2 \theta_2} = \cos \theta_1; \\ \tilde{\mu}_1 \cdot \tilde{\mu}_2 &= \sin \theta_1 \cos \theta_1 \sin \theta_2 \cos \theta_2 - \sin \theta_1 \cos \theta_1 \sin \theta_2 \cos \theta_2 = 0.\end{aligned}$$

Now assume the statement to be true for some $n > 1$. We have that for $i = 1, \dots, n$,

$$\begin{aligned}\tilde{\mu}_i(\boldsymbol{\theta}, \phi) &= \frac{\partial \nu^{n+1}}{\partial \theta_i}(\boldsymbol{\theta}, \phi) = \frac{\partial \nu_1^n}{\partial \theta_i}(\boldsymbol{\theta})(\cos \phi, \sin \phi, 0, \dots, 0) + \left(0, 0, \frac{\partial \nu_2^n}{\partial \theta_i}(\boldsymbol{\theta}), \dots, \frac{\partial \nu_n^n}{\partial \theta_i}(\boldsymbol{\theta})\right); \\ \tilde{\mu}_{n+1}(\boldsymbol{\theta}, \phi) &= \frac{\partial \nu^{n+1}}{\partial \phi}(\boldsymbol{\theta}, \phi) = \nu_1^n(\boldsymbol{\theta})(-\sin \phi, \cos \phi, 0, \dots, 0).\end{aligned}$$

Computing their modules and dot products gives us

$$\begin{aligned}|\tilde{\mu}_1| &= \sqrt{\left(\frac{\partial \nu_1^n}{\partial \theta_1}\right)^2 (\cos^2 \phi + \sin^2 \phi) + \left(\frac{\partial \nu_2^n}{\partial \theta_1}\right)^2 + \dots + \left(\frac{\partial \nu_n^n}{\partial \theta_1}\right)^2} = \left|\frac{\partial \nu^n}{\partial \theta_1}\right| = 1; \\ |\tilde{\mu}_i| &= \sqrt{\left(\frac{\partial \nu_1^n}{\partial \theta_i}\right)^2 (\cos^2 \phi + \sin^2 \phi) + \left(\frac{\partial \nu_2^n}{\partial \theta_i}\right)^2 + \dots + \left(\frac{\partial \nu_n^n}{\partial \theta_i}\right)^2} = \\ &= \left|\frac{\partial \nu^n}{\partial \theta_i}\right| = \cos \theta_1 \dots \cos \theta_{i-1}; \\ |\tilde{\mu}_{n+1}| &= \sqrt{\nu_1^n(\boldsymbol{\theta})^2 (\sin^2 \phi + \cos^2 \phi)} = \nu_1^n = \cos \theta_1 \dots \cos \theta_n; \\ \tilde{\mu}_i \cdot \tilde{\mu}_j &= \frac{\partial \nu_1^n}{\partial \theta_i} \frac{\partial \nu_1^n}{\partial \theta_j} (\cos^2 \phi + \sin^2 \phi) + \frac{\partial \nu_2^n}{\partial \theta_i} \frac{\partial \nu_2^n}{\partial \theta_j} + \dots + \frac{\partial \nu_n^n}{\partial \theta_i} \frac{\partial \nu_n^n}{\partial \theta_j} = \frac{\partial \nu^n}{\partial \theta_i} \cdot \frac{\partial \nu^n}{\partial \theta_j} = 0 \\ \tilde{\mu}_{n+1} \cdot \tilde{\mu}_j &= \frac{\partial \nu_1^n}{\partial \theta_j} \nu_1^n (\sin \phi \cos \phi - \cos \phi \sin \phi) = 0.\end{aligned}$$

The statement then follows. \square

From all this, we get the following

Proposition 1. *The set $\{\nu^n(\boldsymbol{\theta}), \hat{\mu}_1(\boldsymbol{\theta}), \dots, \hat{\mu}_n(\boldsymbol{\theta})\}$ is an orthonormal basis for $T_{\nu(\boldsymbol{\theta})}\mathbb{R}^{n+1}$, where $\hat{\mu}_i = \tilde{\mu}_i/|\tilde{\mu}_i|$ for $i = 1, \dots, n$.*

Proof of Theorem 1. Given $\mathbf{x} \in U_n$, we identify \mathbb{R}^{n+1} with $T_{f(\mathbf{x})}\mathbb{R}^{n+1}$. Let $\nu: \text{Reg}(f) \rightarrow S^n$ be the unit vector field along f and $a = f \cdot \nu$. Since $|\nu(\mathbf{x})| = 1$ for all $\mathbf{x} \in U_n$, we can choose $\theta_1, \dots, \theta_n: \text{Reg}(f) \rightarrow W$ such that

$$\nu = \nu \circ (\theta_1, \dots, \theta_n).$$

Moreover, ν verifies the identity $\nu(\mathbf{x}) \cdot df_{\mathbf{x}}(\mathbf{v}) = 0$ for all $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n$ and $\mathbf{x} \in \text{Reg}(f)$, hence

$$\frac{\partial a}{\partial x_i}(\mathbf{x}) = \frac{\partial f}{\partial x_i}(\mathbf{x}) \cdot \nu(\mathbf{x}) + f(\mathbf{x}) \cdot \frac{\partial \nu}{\partial x_i}(\mathbf{x}) = f(\mathbf{x}) \cdot \frac{\partial \nu}{\partial x_i}(\mathbf{x}).$$

Using the chain rule, we obtain the following system of differential equations:

$$\frac{\partial a}{\partial x_i} = f \cdot \frac{\partial \nu^n}{\partial x_i} = f \cdot \frac{\partial \nu^n}{\partial \theta_1} \frac{\partial \theta_1}{\partial x_i} + \dots + f \cdot \frac{\partial \nu^n}{\partial \theta_n} \frac{\partial \theta_n}{\partial x_i} = b_1 \frac{\partial \theta_1}{\partial x_i} + \dots + b_n \frac{\partial \theta_n}{\partial x_i},$$

which can be written in matrix form as

$$(1) \quad \begin{pmatrix} \frac{\partial a}{\partial x_1} \\ \vdots \\ \frac{\partial a}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial \theta_1}{\partial x_1} & \cdots & \frac{\partial \theta_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \theta_1}{\partial x_n} & \cdots & \frac{\partial \theta_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Notice that the mapping $d\nu_{\mathbf{x}}: T_{\mathbf{x}} \text{Reg}(f) \rightarrow T_{\nu(\mathbf{x})} S^n$ is a monomorphism for any $\mathbf{x} \in \text{Reg}(\nu)$. By the chain rule,

$$d\nu_{\mathbf{x}} = d\nu_{\theta(\mathbf{x})}^n \circ d\theta_{\mathbf{x}},$$

hence $d\theta_{\mathbf{x}}: T_{\mathbf{x}} \text{Reg}(\nu) \rightarrow T_{\theta(\mathbf{x})} W$ is also a monomorphism. Since $\dim T_{\mathbf{x}} \text{Reg}(\nu) = \dim T_{\theta(\mathbf{x})} W$, the Grassman formula implies that $\text{rk } d\theta_{\mathbf{x}} = \dim T_{\mathbf{x}} U$, so $d\theta_{\mathbf{x}}$ is an isomorphism and the coefficient matrix of Equation (1) is invertible at \mathbf{x} . It follows that the system of equations has a unique solution $b_1, \dots, b_n: \text{Reg}(\nu) \rightarrow \mathbb{R}$.

If we set $\mu_i = \hat{\mu}_i \circ \theta$ for $i = 1, \dots, n$, it follows from Proposition 1 that we can write

$$f(\mathbf{x}) = f(\mathbf{x}) \cdot \nu(\mathbf{x}) \nu(\mathbf{x}) + f(\mathbf{x}) \cdot \mu_1(\mathbf{x}) \mu_1(\mathbf{x}) + \cdots + f(\mathbf{x}) \cdot \mu_n(\mathbf{x}) \mu_n(\mathbf{x}).$$

On the one hand, we know that $f(\mathbf{x}) \cdot \nu(\mathbf{x}) = a(\mathbf{x})$ by definition of height function. On the other hand, for $j = 1, \dots, n$,

$$f(\mathbf{x}) \cdot \mu_j(\mathbf{x}) = f(\mathbf{x}) \cdot \hat{\mu}_j(\theta(\mathbf{x})) = f(\mathbf{x}) \cdot \frac{\tilde{\mu}_j(\theta(\mathbf{x}))}{|\tilde{\mu}_j(\theta(\mathbf{x}))|} = \frac{f(\mathbf{x})}{|\tilde{\mu}_j(\theta(\mathbf{x}))|} \cdot \frac{\partial \nu^n}{\partial \theta_j}(\theta(\mathbf{x})) = \frac{b_j(\mathbf{x})}{|\tilde{\mu}_j(\theta(\mathbf{x}))|}.$$

Therefore,

$$(2) \quad f(\mathbf{x}) = a(\mathbf{x}) \nu(\mathbf{x}) + \frac{b_1(\mathbf{x})}{|\tilde{\mu}_1(\theta(\mathbf{x}))|} \mu_1(\mathbf{x}) + \cdots + \frac{b_n(\mathbf{x})}{|\tilde{\mu}_n(\theta(\mathbf{x}))|} \mu_n(\mathbf{x})$$

for any $\mathbf{x} \in \text{Reg}(\nu)$. Note that every function on the right-hand side of this identity can be computed entirely in terms of a and ν .

The set $\text{Reg}(\nu)$ is open and dense in $\text{Reg}(f)$ by definition of pseudo regular frontal, and since $\text{Reg}(f)$ is open and dense in U_n , $\text{Reg}(\nu)$ is open and dense in U_n . Since they are continuous, we can uniquely extend the mapping $\theta: \text{Reg}(f) \rightarrow W$ and the functions $b_1, \dots, b_n: \text{Reg}(\nu) \rightarrow \mathbb{R}$ to U_n , which is the domain of f . We conclude that the identity (2) holds in U_n , so it can be written entirely in terms of the Legendre data, as stated. \square

Remark 1. Most of the arguments in the proof above can be carried out for any $f: U_n \rightarrow \mathbb{R}^{n+1}$ regardless of whether $\text{Reg}(\nu)$ is dense in $\text{Reg}(f)$ or not. The only obstruction to this argument lies when one wishes to find the functions b_1, \dots, b_n that verify the equation

$$da = b_1 d\theta_1 + \cdots + b_n d\theta_n,$$

as the solution is only guaranteed to be unique in $\text{Reg}(\nu)$. If we denote by $\Sigma^\ell(\nu)$ the set of points $\mathbf{x} \in \text{Reg}(f)$ where $d\nu_{\mathbf{x}}$ has corank ℓ , then $d\theta_{\mathbf{x}}$ has corank ℓ and there are indices $1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$ such that $b_{i_1}, \dots, b_{i_\ell}: \Sigma^\ell(\nu) \rightarrow \mathbb{R}$ can be chosen to be any function. Being able to recover f then depends on being able to choose the appropriate functions $b_{i_1}, \dots, b_{i_\ell}$. In other words, for non pseudo regular frontals there might be infinitely many different mappings sharing the same Legendre data.

There are two ways to solve this conundrum: on the one hand, Theorem 2 claims that the set of pseudo regular frontals $U_n \rightarrow \mathbb{R}^{n+1}$ are dense in $A(U_n, \mathbb{R}^{n+1})$, meaning that one can choose a sequence of pseudo regular frontals $f_k: U_n \rightarrow \mathbb{R}^{n+1}$ which converge to f pointwise when k goes to infinity (see §5 below). On the other hand, one can use additional information about f , such as the lowest rank df_x attains in U_n (e.g. the cuspidal edge from Example 2 Item 3 has at least rank 1, so it can be written in the form

$$(x, y) \mapsto (x, p(x, y), q(x, y))$$

by taking suitable coordinates in the source and target).

4. PROOF OF THEOREM 2

4.1. Proof of the assertion (1) of Theorem 2. The proof of Theorem 1 given in Section 3 works well even for pseudo regular frontals. This completes the proof. \square

4.2. Proof of the assertion (2) of Theorem 2. Let $RM(U_n, \mathbb{R}^{n+1})$ be the set consisting of real-analytic mappings $f : U_n \rightarrow \mathbb{R}^{n+1}$ such that $Reg(f)$ is dense in U_n . The set $RM(U_n, \mathbb{R}^{n+1})$ is endowed with the Whitney C^∞ topology. Then, $RM(U_n, \mathbb{R}^{n+1})$ is a topological subspace of $A(U_n, \mathbb{R}^{n+1})$ and $PRF(U_n, \mathbb{R}^{n+1})$ is a topological subspace of $RM(U_n, \mathbb{R}^{n+1})$. Namely, the following holds.

$$PRF(U_n, \mathbb{R}^{n+1}) \subset RM(U_n, \mathbb{R}^{n+1}) \subset A(U_n, \mathbb{R}^{n+1}).$$

It is clear that the following two lemmas show the assertion (2) of Theorem 2.

Lemma 4.1. *$RM(U_n, \mathbb{R}^{n+1})$ is dense in $A(U_n, \mathbb{R}^{n+1})$.*

Lemma 4.2. *$PRF(U_n, \mathbb{R}^{n+1})$ is dense in $RM(U_n, \mathbb{R}^{n+1})$.*

Before proving Lemma 4.1 and Lemma 4.2, several preparations are given.

Let $C^\infty(U_n, \mathbb{R}^{n+1})$ be the set consisting of C^∞ mappings $f : U_n \rightarrow \mathbb{R}^{n+1}$. The Whitney C^∞ topology is the standard topology on $C^\infty(U_n, \mathbb{R}^{n+1})$, which is a natural smooth generalisation of the compact open topology on the space of continuous mappings $U_n \rightarrow \mathbb{R}^{n+1}$ (see [8], §3 for details). Since any real-analytic mapping $f : U_n \rightarrow \mathbb{R}^{n+1}$ is a C^∞ mapping, it follows that $A(U_n, \mathbb{R}^{n+1})$ is a topological subspace of $C^\infty(U_n, \mathbb{R}^{n+1})$.

Fact 2 (Identity Theorem). *Let p be a positive integer, V be a connected open subset of \mathbb{R}^p and let $F : V \rightarrow \mathbb{R}$ be a real-analytic function. Suppose that the fiber $F^{-1}(0)$ has an interior point. Then, $F(\mathbf{x}) \equiv 0$ for any $\mathbf{x} \in V$.*

Fact 2 is a beautiful and strong fact. By using Fact 2, Lemma 4.1 and Lemma 4.2 can be proved in a clear and easy-to-understand manner. Notice that the real-analytic assumption for the function F cannot be generalized to the C^∞ assumption. For details on this matter, see for instance §3 of [6]. Nevertheless, by using common notions in Singularity Theory of Differentiable Mappings such as Jet Spaces, Thom's Jet Transversality Theorem, it is possible to prove Lemma 4.1 for C^∞ mappings. However, even if notions and results in Singularity Theory of Differentiable Mappings found in standard textbooks such as [4, 8] are fully incorporated, it seems that the proof of the C^∞ version of Lemma 4.2 remains wrapped in mystery. The biggest obstruction for the proof of the C^∞ version of Lemma 4.2 is that the source space $Reg(f)$ of the Gauss mapping $\nu : Reg(f) \rightarrow S^n$ varies depending on parametrizations $f : U_n \rightarrow \mathbb{R}^{n+1}$. Then, we pose the following problem here.

Problem 2. Does the C^∞ version of Lemma 4.2 hold?

On the other hand, even if we restrict ourselves to real-analytic mappings, in practical terms, there are almost no problems when applying Theorem 1, Theorem 2 and Theorem 3. Therefore, in this paper, we assume that all functions and mappings are real-analytic unless otherwise stated.

Proof of Lemma 4.1. Now, we start to give a proof of Lemma 4.1. Let $f : U_n \rightarrow \mathbb{R}^{n+1}$ be a real-analytic mapping. For any $\mathbf{x} \in U_n$, set

$$\mathbf{x} = (x_1, \dots, x_n).$$

Namely, x_i ($1 \leq i \leq n$) is the i -th coordinate of $\mathbf{x} \in U_n \subset \mathbb{R}^n = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{n\text{-tuples}}$. Set $f_j = X_j \circ f$ ($1 \leq j \leq n+1$) where $X_j : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ stands for the j -th coordinate function of $\mathbb{R}^{n+1} = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{(n+1)\text{-tuples}}$. Thus, we have set

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_{n+1}(\mathbf{x})).$$

Consider the Jacobian matrix

$$Jf(\mathbf{x}) = \left(\frac{\partial f_j}{\partial x_i}(\mathbf{x}) \right)_{1 \leq i \leq n, 1 \leq j \leq n+1}$$

of f . For any k ($1 \leq k \leq n+1$), let $M_{f,k}(\mathbf{x})$ be the determinant of the $n \times n$ matrix given by removing the k -th row ($1 \leq k \leq n+1$) from $Jf(\mathbf{x})$. That is to say,

$$M_{f,k}(\mathbf{x}) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \vdots \\ \widehat{\frac{\partial f_k}{\partial x_1}(\mathbf{x})} & \widehat{\cdots} & \widehat{\frac{\partial f_k}{\partial x_n}(\mathbf{x})} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_{n+1}}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial f_{n+1}}{\partial x_n}(\mathbf{x}) \end{vmatrix},$$

where the $\widehat{\cdots}$ denotes deleting \cdots . Let $\Phi_f : U_n \rightarrow \mathbb{R}$ be the real-analytic function defined by

$$\Phi_f(\mathbf{x}) = \sum_{k=1}^{n+1} (M_{f,k}(\mathbf{x}))^2.$$

Recall that a point $\mathbf{x} \in U_n$ is singular if $df_{\mathbf{x}} : T_{\mathbf{x}}U_n \rightarrow T_{f(\mathbf{x})}\mathbb{R}^{n+1}$ is not injective. This happens at and only at points \mathbf{x} where all of the n -minors $M_{f,k}(\mathbf{x})$ of the Jacobian matrix $Jf(\mathbf{x})$ vanish, which is equivalent to saying that this happens at and only at points \mathbf{x} satisfying

$$\Phi_f(\mathbf{x}) = 0.$$

Suppose that U_n is connected. Suppose moreover that $f \notin RM(U_n, \mathbb{R}^{n+1})$. The second supposition is equivalent to assume that the closed set

$$Ker(\Phi_f) = \{\mathbf{x} \in U_n \mid \Phi_f(\mathbf{x}) = 0\}$$

has an interior point. Then, by Fact 2 (the Identity Theorem) and by the first supposition, it follows

$$\Phi_f(\mathbf{x}) \equiv 0 \quad (\text{for any } \mathbf{x} \in U_n).$$

Next, we consider linear perturbations of f . For any j ($1 \leq j \leq n+1$), let $\mathbf{c}_j \in \mathbb{R}^n$ be an n -dimensional constant vector.

$$\mathbf{c}_j = (c_{(j,1)}, \dots, c_{(j,n)}) \in \mathbb{R}^n.$$

For the $f \in A(U_n, \mathbb{R}^{n+1})$, let $f + \mathbf{C} : U_n \rightarrow \mathbb{R}^{n+1}$ be a mapping of the following type.

$$(f + \mathbf{C})(\mathbf{x}) = (f_1(\mathbf{x}) + \mathbf{c}_1 \cdot \mathbf{x}, \dots, f_{n+1}(\mathbf{x}) + \mathbf{c}_{n+1} \cdot \mathbf{x})$$

Thus, $f + \mathbf{C}$ is a linear perturbation of f . Take one point \mathbf{x}_0 of U_n and fix it. For any $\mathbf{C} \in \mathbb{R}^{n(n+1)}$, set

$$\Psi_{f,\mathbf{x}_0}(\mathbf{C}) = \Phi_{f+\mathbf{C}}(\mathbf{x}_0).$$

If we regard $c_{(j,i)}$ ($1 \leq i \leq n, 1 \leq j \leq n+1$) as variables, then the function $\Psi_{f,\mathbf{x}_0} : \mathbb{R}^{n(n+1)} \rightarrow \mathbb{R}$ is a monic polynomial function with degree $2n$, in particular not identically zero. Hence we have the following.

Sublemma 4.1. *The following subset of $\mathbb{R}^{n(n+1)}$ is of Lebesgue measure zero and closed.*

$$(4.1) \quad Ker(\Psi_{f,\mathbf{x}_0}) = \left\{ \mathbf{C} \in \mathbb{R}^{n(n+1)} \mid \Psi_{f,\mathbf{x}_0}(\mathbf{C}) = 0 \right\}.$$

By Fact 2 and Sublemma 4.1, for any $\mathbf{C} \in \mathbb{R}^{n(n+1)}$ satisfying $\Psi_{f,\mathbf{x}_0}(\mathbf{C}) \neq 0$, the linear perturbation $f + \mathbf{C}$ must be contained in $RM(U_n, \mathbb{R}^{n+1})$. Moreover, again by Sublemma 4.1, for any positive integer ℓ there exists $\mathbf{C} \in \mathbb{R}^{n(n+1)}$ such that the linear perturbation $f + \mathbf{C}$ satisfies the following condition.

$$(4.2) \quad \sum_{j=1}^{n+1} \left(\sum_{i=1}^n c_{(j,i)}^2 \right) < \frac{1}{\ell^2}.$$

For any positive integer ℓ , take one constant vector $\mathbf{C}_0[\ell] \in \mathbb{R}^{n(n+1)}$ satisfying (4.1) and (4.2) and fix it. Then, it is clear that

$$(4.3) \quad f + \mathbf{C}_0[\ell] \in RM(U_n, \mathbb{R}^{n+1}) \quad (\text{for any } \ell \in \mathbb{N})$$

and

$$(4.4) \quad \lim_{\ell \rightarrow \infty} (f + \mathbf{C}_0[\ell]) = f.$$

Suppose that U_n is not connected. Then, since \mathbb{R}^n satisfies the axiom of second countability, the number of connected components of U_n is countable. Since the union of countably many Lebesgue measure zero subsets in $\mathbb{R}^{n(n+1)}$ is a subset of Lebesgue measure zero, we may again choose one constant vector $\mathbf{c}_0[\ell] \in \mathbb{R}^{n(n+1)}$ satisfying (4.1) and (4.2) for any $\ell \in \mathbb{N}$. Thus, we can again obtain (4.3) and (4.4). This completes the proof. \square

Proof of Lemma 4.2. Suppose that U_n is connected. Let f be a mapping contained in $RM(U_n, \mathbb{R}^{n+1})$. There is a $k \in \{1, \dots, n+1\}$ such that $M_{f,k}^{-1}(0)$ has no interior points. If this was not the case, we would have by Fact 2 that $M_{f,1}, \dots, M_{f,n+1}$ all vanish on U_n , meaning that

$$\Phi_f(\mathbf{x}) = \sum_{k=1}^{n+1} (M_{f,k}(\mathbf{x}))^2 \equiv 0$$

and $\text{Reg}(f) = U_n \setminus \ker(\Phi_f) = \emptyset$, hence f would not be in $RM(U_n, \mathbb{R}^{n+1})$. We can thus assume, by permutating coordinates (X_1, \dots, X_{n+1}) of \mathbb{R}^{n+1} in advance if necessary, that $M_{f,n+1}^{-1}(0)$ has no interior points, hence $\text{Reg}(F)$ is dense in U_n , where the mapping $F : U_n \rightarrow \mathbb{R}^n$ is defined by

$$F(\mathbf{x}) = (X_1 \circ f(\mathbf{x}), \dots, X_n \circ f(\mathbf{x})).$$

We are going to construct a sequence $\{f_\ell : U_n \rightarrow \mathbb{R}^{n+1}\}_{\ell \in \mathbb{N}}$ such that the following three are satisfied.

- (4.A) For any $\ell \in \mathbb{N}$, $\text{Reg}(f_\ell)$ is dense in U_n .
- (4.B) For any $\ell \in \mathbb{N}$, $\text{Reg}(\nu_{f_\ell} : \text{Reg}(f_\ell) \rightarrow S^n)$ is dense in $\text{Reg}(f_\ell)$.
- (4.C) $\lim_{\ell \rightarrow \infty} f_\ell = f$.

Set

$$X_i \circ f_\ell = X_i \circ f = X_i \circ F \quad (1 \leq i \leq n).$$

Then, it is clear that

$$\text{Reg}(f_\ell) \supset \text{Reg}(F).$$

Hence, from the assumption above, (4.A) follows. From now on, for any $\ell \in \mathbb{N}$, we construct $X_{n+1} \circ f_\ell$ satisfying (4.B) and (4.C). For any sufficiently small positive number ε and any $\mathbf{x}_0 \in U_n$ let $D(\mathbf{x}_0, \varepsilon)$ be the open disc centered at \mathbf{x}_0 with radius ε . Namely,

$$D(\mathbf{x}_0, \varepsilon) = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) < \varepsilon^2\}.$$

For any $\mathbf{x} \in D(\mathbf{x}_0, \varepsilon)$, define the n -dimensional vector $F^2(\mathbf{x}) \in \mathbb{R}^n$ by

$$F^2(\mathbf{x}) = \left((X_1 \circ F(\mathbf{x}) - X_1 \circ F(\mathbf{x}_0))^2, \dots, (X_n \circ F(\mathbf{x}) - X_n \circ F(\mathbf{x}_0))^2 \right).$$

Then define the function $X_{n+1} \circ f_\ell : U_n \rightarrow \mathbb{R}$ by

$$X_{n+1} \circ f_\ell(\mathbf{x}) = X_{n+1} \circ f(\mathbf{x}) + \frac{1}{2} (\mathbf{c}_\ell \cdot F^2(\mathbf{x})),$$

where $\mathbf{c}_\ell = (c_{(\ell,1)}, \dots, c_{(\ell,n)})$ is an n -dimensional constant vector in \mathbb{R}^n . By Fact 2, (4.B) is equivalent to the following (4.B.1).

- (4.B.1) For any $\ell \in \mathbb{N}$, any sufficiently small positive number $\varepsilon > 0$ and any $\mathbf{x}_0 \in \text{Reg}(F)$ satisfying $D(\mathbf{x}_0, \varepsilon) \subset \text{Reg}(F)$, the set

$$\text{Reg}(\nu_{f_\ell}|_{D(\mathbf{x}_0, \varepsilon)} : D(\mathbf{x}_0, \varepsilon) \rightarrow S^n) = \text{Reg}(\nu_{f_\ell}) \cap D(\mathbf{x}_0, \varepsilon)$$

is dense in $D(\mathbf{x}_0, \varepsilon)$.

The argument is as follows: if $\text{Reg}(\nu_{f_\ell})$ is not dense in $\text{Reg}(f_\ell)$, then $\ker(\Phi_{\nu_{f_\ell}})$ must have an interior point and $\text{Reg}(\nu_{f_\ell})$ is empty by Fact 2, thus $\text{Reg}(\nu_{f_\ell}|_{D(\mathbf{x}_0, \varepsilon)})$ is empty for all $\mathbf{x}_0 \in \text{Reg}(f_\ell)$. Conversely, if there is an $\mathbf{x}_0 \in \text{Reg}(f)$ such that $\text{Reg}(\nu_{f_\ell}|_{D(\mathbf{x}_0, \varepsilon)})$ is not dense in $D(\mathbf{x}_0, \varepsilon)$, then $\ker(\Phi_{\nu_{f_\ell}})$ has an interior point in $D(\mathbf{x}_0, \varepsilon)$ and $\text{Reg}(\nu_{f_\ell}|_{D(\mathbf{x}_0, \varepsilon)})$ is empty by Fact 2, thus $\text{Reg}(\nu_{f_\ell})$ is not dense in $\text{Reg}(f_\ell)$.

For any i ($1 \leq i \leq n$), let $\tilde{x}_i : D(\mathbf{x}_0, \varepsilon) \rightarrow \mathbb{R}$ be the function defined by

$$\tilde{x}_i = X_i \circ (F|_{D(\mathbf{x}_0, \varepsilon)}).$$

Since $F|_{D(\mathbf{x}_0, \varepsilon)} : D(\mathbf{x}_0, \varepsilon) \rightarrow F(D(\mathbf{x}_0, \varepsilon))$ is an analytic diffeomorphism, $(D(\mathbf{x}_0, \varepsilon), (\tilde{x}_1, \dots, \tilde{x}_n))$ is regarded as a coordinate neighborhood of \mathbb{R}^n . Set $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)$. Then, the mapping $f_\ell|_{D(\mathbf{x}_0, \varepsilon)} : D(\mathbf{x}_0, \varepsilon) \rightarrow \mathbb{R}^{n+1}$ may be described with respect to coordinates $(\tilde{x}_1, \dots, \tilde{x}_n)$ as follows.

$$\begin{aligned} X_i \circ f_\ell(\tilde{\mathbf{x}}) &= \tilde{x}_i \quad (1 \leq i \leq n), \\ X_{n+1} \circ f_\ell(\tilde{\mathbf{x}}) &= X_{n+1} \circ f(\tilde{\mathbf{x}}) + \frac{1}{2} (\mathbf{c}_\ell \cdot F^2(\tilde{\mathbf{x}})) \\ &= X_{n+1} \circ f(\tilde{\mathbf{x}}) + \frac{1}{2} \sum_{i=1}^n c_{(\ell, i)} (\tilde{x}_i - \tilde{x}_{(0, i)})^2, \end{aligned}$$

where $\tilde{\mathbf{x}}(\mathbf{x}_0) = (\tilde{x}_{(0, 1)}, \dots, \tilde{x}_{(0, n)})$. For any $\mathbf{x}_0 \in \mathbb{R}^n$, let $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the parallel translation in \mathbb{R}^n defined by

$$h(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0.$$

For any $\ell \in \mathbb{N}$, any sufficiently small positive $\varepsilon > 0$ and any $\mathbf{x}_0 \in \text{Reg}(F)$, let $H_\ell : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be the affine transformation of \mathbb{R}^{n+1} defined by

$$\begin{aligned} &H_\ell(X_1, \dots, X_n, X_{n+1}) \\ &= \left(X_1 - \tilde{x}_{(0, 1)}, \dots, X_n - \tilde{x}_{(0, n)}, X_{n+1} - X_{n+1} \circ f(\tilde{\mathbf{x}}(\mathbf{x}_0)) - \sum_{i=1}^n \frac{\partial (X_{n+1} \circ f)}{\partial \tilde{x}_i}(\tilde{\mathbf{x}}(\mathbf{x}_0)) (X_i - \tilde{x}_{(0, i)}) \right). \end{aligned}$$

For any sufficiently small $\varepsilon > 0$, any $\mathbf{x}_0 \in \text{Reg}(F)$ and any $\ell \in \mathbb{N}$, define the mapping $\tilde{f}_\ell : D(\mathbf{0}, \varepsilon) \rightarrow \mathbb{R}^{n+1}$ by

$$\tilde{f}_\ell = H_\ell \circ f_\ell \circ (h|_{D(\mathbf{0}, \varepsilon)}).$$

Then, the convergent power series of $X_i \circ \tilde{f}_\ell$ ($1 \leq i \leq n$) around the origin of \mathbb{R}^n has the following form where $\tilde{y}_i = \tilde{x}_i - \tilde{x}_{(0, i)}$.

$$\begin{aligned} &X_i \circ \tilde{f}_\ell(\tilde{y}_1, \dots, \tilde{y}_n) \\ &= X_i \circ H_\ell \circ f_\ell \circ h(\tilde{y}_1, \dots, \tilde{y}_n) \\ &= X_i \circ H_\ell \circ f_\ell(\tilde{y}_1 + \tilde{x}_{(0, 1)}, \dots, \tilde{y}_n + \tilde{x}_{(0, n)}) \\ &= X_i \circ H_\ell(\tilde{y}_1 + \tilde{x}_{(0, 1)}, \dots, \tilde{y}_n + \tilde{x}_{(0, n)}, X_{n+1} \circ f_\ell(\tilde{y}_1 + \tilde{x}_{(0, 1)}, \dots, \tilde{y}_n + \tilde{x}_{(0, n)})) \\ &= X_i(\tilde{y}_1, \dots, \tilde{y}_n, X_{n+1} \circ H_\ell \circ f_\ell(\tilde{y}_1 + \tilde{x}_{(0, 1)}, \dots, \tilde{y}_n + \tilde{x}_{(0, n)})) \\ &= \tilde{y}_i. \end{aligned}$$

On the other hand, from the construction of the affine transformation H_ℓ , it is easily seen that the following holds.

Sublemma 4.2. *For any $\ell \in \mathbb{N}$, the convergent power series of $X_{n+1} \circ \tilde{f}_\ell$ around the origin of \mathbb{R}^n starts from the quadratic terms. Namely, the following two holds for any $\ell \in \mathbb{N}$.*

- (1) $X_{n+1} \circ \tilde{f}_\ell(\mathbf{0}) = 0$.
- (2) For any i ($1 \leq i \leq n$), we have the following.

$$\frac{\partial (X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_i}(\mathbf{0}) = 0.$$

Notice that $\tilde{f}_\ell(\mathbf{0}) = \mathbf{0}$ and that by Sublemma 4.2, the power series expression of $X_{n+1} \circ \tilde{f}_\ell$ around the origin has no linear terms for any $\ell \in \mathbb{N}$. Notice moreover that since h is a homeomorphism, (4.B.1) is equivalent to the following (4.B.2)

(4.B.2) For any $\ell \in \mathbb{N}$, any sufficiently small positive number $\varepsilon > 0$ and any $\mathbf{x}_0 \in \text{Reg}(F)$ satisfying $D(\mathbf{x}_0, \varepsilon) \in \text{Reg}(F)$, the set

$$\text{Reg}(\nu_{\tilde{f}_\ell} : D(\mathbf{0}, \varepsilon) \rightarrow S^n)$$

is dense in $D(\mathbf{0}, \varepsilon)$.

The Jacobian matrix of \tilde{f}_ℓ with respect to partial derivatives of $X_j \circ \tilde{f}_\ell$ ($1 \leq j \leq n+1$) by \tilde{y}_i ($1 \leq i \leq n$) is as follows.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_1}(\tilde{\mathbf{y}}) & \frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_2}(\tilde{\mathbf{y}}) & \cdots & \frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_n}(\tilde{\mathbf{y}}) \end{pmatrix}.$$

Thus, the Gauss mapping $\nu_{\tilde{f}_\ell} : D(\mathbf{0}, \varepsilon) \rightarrow S^n$ must be one of the following two.

$$\begin{aligned} D(\mathbf{0}, \varepsilon) \ni \tilde{\mathbf{y}} &\mapsto \frac{1}{\sqrt{\sum_{i=1}^n \left(\frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_i}(\tilde{\mathbf{y}}) \right)^2 + 1}} \left(-\frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_1}(\tilde{\mathbf{y}}), \dots, -\frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_n}(\tilde{\mathbf{y}}), 1 \right) \in S^n \\ D(\mathbf{0}, \varepsilon) \ni \tilde{\mathbf{y}} &\mapsto \frac{1}{\sqrt{\sum_{i=1}^n \left(\frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_i}(\tilde{\mathbf{y}}) \right)^2 + 1}} \left(\frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_1}(\tilde{\mathbf{y}}), \dots, \frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_n}(\tilde{\mathbf{y}}), -1 \right) \in S^n. \end{aligned}$$

These expressions can be obtained by computing the cross product of the columns of the Jacobian matrix above. Without loss of generality, we may assume

$$\nu_{\tilde{f}_\ell}(\tilde{\mathbf{y}}) = \frac{1}{\sqrt{\sum_{i=1}^n \left(\frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_i}(\tilde{\mathbf{y}}) \right)^2 + 1}} \left(-\frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_1}(\tilde{\mathbf{y}}), \dots, -\frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_n}(\tilde{\mathbf{y}}), 1 \right).$$

Since $X_{n+1} \circ \tilde{f}_\ell$ does not have linear terms for any $\ell \in \mathbb{N}$, it follows

$$\nu_{\tilde{f}_\ell}(\mathbf{0}) = (0, \dots, 0, 1) \in S^n.$$

Set $\mathbf{e}_{n+1} = (0, \dots, 0, 1) \in S^n$. Since \mathbb{R}^{n+1} is a real affine space and the n -dimensional unit sphere S^n is canonically embedded in \mathbb{R}^{n+1} , $T_{\mathbf{e}_{n+1}}S^n$ is naturally identified with $\mathbb{R}^n \times \{1\}$. Under this identification, for any $\ell \in \mathbb{N}$ define the mapping $N_{\tilde{f}_\ell} : D(\mathbf{0}, \varepsilon) \rightarrow T_{\mathbf{e}_{n+1}}S^n = \mathbb{R}^n \times \{1\}$ by

$$N_{\tilde{f}_\ell}(\tilde{\mathbf{y}}) = \left(-\frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_1}(\tilde{\mathbf{y}}), \dots, -\frac{\partial(X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_n}(\tilde{\mathbf{y}}), 1 \right).$$

Then, since any normal coordinates around the point \mathbf{e}_{n+1} is nothing but a local inverse of the exponential mapping from $T_{\mathbf{e}_{n+1}}S^n$ to S^n , we have the following.

Sublemma 4.3. *Let (V, ξ) be a normal coordinate neighborhood of S^n at \mathbf{e}_{n+1} . Then, there exists a real-analytic diffeomorphism $\varphi : T_{\mathbf{e}_{n+1}}S^n \rightarrow T_{\mathbf{e}_{n+1}}S^n$ such that*

$$\varphi \circ \xi \circ \nu_{\tilde{f}_\ell}(\tilde{\mathbf{y}}) = N_{\tilde{f}_\ell}(\tilde{\mathbf{y}})$$

for any $\ell \in \mathbb{N}$ and any $\tilde{\mathbf{y}} \in D(\mathbf{0}, \varepsilon)$.

By Sublemma 4.3, it follows that (4.B.2) is equivalent to the following (4.B.3).

(4.B.3) For any $\ell \in \mathbb{N}$, any sufficiently small positive number $\varepsilon > 0$ and any $\mathbf{x}_0 \in \text{Reg}(F)$ satisfying $D(\mathbf{x}_0, \varepsilon) \subset \text{Reg}(F)$, there exists a sufficiently small positive number δ ($0 < \delta \leq \varepsilon$) such that the set

$$\text{Reg}\left(N_{\tilde{f}_\ell}|_{D(\mathbf{0}, \delta)} : D(\mathbf{0}, \delta) \rightarrow T_{\mathbf{e}_{n+1}}S^n\right)$$

is dense in $D(\mathbf{0}, \delta)$.

Recall that $X_{n+1} \circ \tilde{f}_\ell$ is a quadratic perturbation of $X_{n+1} \circ f$ having the following form and also that h is merely a parallel transformation of \mathbb{R}^n .

$$X_{n+1} \circ \tilde{f}_\ell(\mathbf{x}) = X_{n+1} \circ f(\mathbf{x}) + \frac{1}{2}(\mathbf{c}_\ell \cdot F^2(\mathbf{x})).$$

Hence the second derivative of $X_{n+1} \circ f_\ell$ with respect to $(\tilde{y}_1, \dots, \tilde{y}_n)$ has the following form.

$$(4.5) \quad \frac{\partial^2 (X_{n+1} \circ \tilde{f}_\ell)}{\partial \tilde{y}_i \partial \tilde{y}_j}(\tilde{\mathbf{y}}) = \begin{cases} \frac{\partial^2 (X_{n+1} \circ f)}{\partial \tilde{x}_i^2}(\tilde{\mathbf{x}}) + c_{(\ell, i)} & (\text{if } i = j) \\ \frac{\partial^2 (X_{n+1} \circ f)}{\partial \tilde{x}_i \partial \tilde{x}_j}(\tilde{\mathbf{x}}) & (\text{if } i \neq j). \end{cases}$$

Denote by $\Psi_{\tilde{f}_\ell, \mathbf{0}}(\mathbf{c}_\ell)$ the Jacobian determinant of $N_{\tilde{f}_\ell}|_{D(\mathbf{0}, \delta)}$ in S^n at $\mathbf{0}$. By (4.5), if we regard $c_{(\ell, i)}$ ($1 \leq i \leq n$) as variables, then the Jacobian determinant of $N_{\tilde{f}_\ell}|_{D(\mathbf{0}, \delta)}$ at $\mathbf{0}$ is a monic polynomial with degree n . Thus, by Fact 2, we have the following sublemma.

Sublemma 4.4. *For any $\ell \in \mathbb{N}$ and any $\mathbf{x}_0 \in \text{Reg}(F)$, the following subset of \mathbb{R}^n is of Lebesgue measure zero and closed.*

$$\text{Ker}(\Psi_{\tilde{f}_\ell, \mathbf{0}}) = \left\{ \mathbf{c}_\ell \in \mathbb{R}^n \mid \Psi_{\tilde{f}_\ell, \mathbf{0}}(\mathbf{c}_\ell) = 0 \right\}.$$

By Sublemma 4.4, for any $\ell \in \mathbb{N}$, any \mathbf{x}_0 contained in $\text{Reg}(F)$ and almost all $\mathbf{c}_\ell \in \mathbb{R}^n$ in the sense of Lebesgue measure, (4.B.3) is satisfied. Moreover, again by Sublemma 4.4, for any $\ell \in \mathbb{N}$ and almost all $\mathbf{c}_\ell \in \mathbb{R}^n$ (in the sense of Lebesgue measure) satisfying

$$(4.6) \quad \sum_{i=1}^n c_{(\ell, i)}^2 < \frac{1}{\ell^2},$$

$\mathbf{c}_\ell \in \mathbb{R}^n$ does not belong to $\text{Ker}(\Psi_{\tilde{f}_\ell, \mathbf{0}})$. The inequality (4.6) implies that we can construct a sequence of quadratic perturbations $\{f_\ell : U_n \rightarrow \mathbb{R}^{n+1}\}_{\ell \in \mathbb{N}}$ satisfying even (4.C) as well.

Suppose that U_n is not connected. Then, for each connected component of U_n , the proof in the case that U_n is connected works well. Since \mathbb{R}^n satisfies the axiom of second countability, the number of connected components of U_n is countable. Since the union of countably many Lebesgue measure zero subsets in \mathbb{R}^n is a subset of Lebesgue measure zero, we may again construct a sequence of quadratic perturbations $\{f_\ell : U_n \rightarrow \mathbb{R}^{n+1}\}_{\ell \in \mathbb{N}}$ satisfying (4.A), (4.B) and (4.C). This completes the proof. \square

5. PROOF OF THEOREM 3

Take any f of $A(U_n, \mathbb{R}^{n+1})$ and fix it. By the assertion (1) of Theorem 2, there exists a sequence $\{f_i\}_{i=1,2,\dots} \subset \text{PRF}(U_n, \mathbb{R}^{n+1})$ satisfying $\lim_{i \rightarrow \infty} f_i = f$. Take any point \mathbf{x} of U_n and fix it. Since $f_i \in \text{PRF}(U_n, \mathbb{R}^{n+1})$ for any i ($i = 1, 2, \dots$), by the assertion (1) of Theorem 2, there must exist a sequence of points $\{\mathbf{x}_j\}_{j=1,2,\dots} \subset U_n$ satisfying $\lim_{j \rightarrow \infty} \mathbf{x}_j = \mathbf{x}$, $\{\mathbf{x}_j\}_{j=1,2,\dots} \subset \text{Reg}(f_i)$ for any i ($i = 1, 2, \dots$) and $f_i(\mathbf{x})$ is completely recovered from the sequence of Legendre data $\left\{ \{\nu_i(\mathbf{x}_j), a_i(\mathbf{x}_j)\}_{\mathbf{x}_j \in \text{Reg}(\nu_i)} \right\}_{i=1,2,\dots}$ of f_i . On the other hand, $f(\mathbf{x}) = \lim_{i \rightarrow \infty} f_i(\mathbf{x})$. Therefore, $f(\mathbf{x})$ is completely recovered from the sequence of Legendre data $\left\{ \{\nu_i(\mathbf{x}_j), a_i(\mathbf{x}_j)\}_{\mathbf{x}_j \in \text{Reg}(\nu_i)} \right\}_{i=1,2,\dots}$ of f_i . This completes the proof. \square

6. EXAMPLES

Example 3 (Constant mapping). Let $\mathbf{X}_0 \in \mathbb{R}^{n+1}$ be a fixed point, and consider the constant mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ given by $f(\mathbf{x}) = \mathbf{X}_0$. Then f is a frontal but not a regular frontal.

Now consider the sequence of concentric spheres $f_k : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ given by

$$f_k(\mathbf{x}) = \mathbf{X}_0 + \frac{1}{k} (\cos x_1 \cdots \cos x_n, \cos x_1 \cdots \cos x_{n-1} \sin x_n, \cos x_1 \cdots \cos x_{n-2} \sin x_{n-1} \cdots, \sin x_1),$$

We see that $\text{Reg}(f_k) = \mathbb{R}^n$ and $f_k(\mathbb{R}^n)$ is the sphere of centre \mathbf{X}_0 and radius $1/k$, hence its Gauss map $\nu_k : \mathbb{R}^n \rightarrow S^n$ is given by

$$\nu(\mathbf{x}) = k(f_k(\mathbf{x}) - \mathbf{X}_0).$$

Therefore, $\text{Reg}(\nu_k) = \mathbb{R}^n$ and f_k is a regular frontal for any $k \in \mathbb{N}$. We also have that

$$a_k(\mathbf{x}) = f_k(\mathbf{x}) \cdot \nu(\mathbf{x}) = \mathbf{X}_0 \cdot \nu(\mathbf{x}) + \frac{1}{k}.$$

It is clear by the definition of ν_k that $\theta_j(x) = x_j$ for $j = 1, \dots, n$, meaning that

$$b_j = b_j \frac{\partial \theta_j}{\partial x_j} = b_1 \frac{\partial \theta_1}{\partial x_j} + \cdots + b_n \frac{\partial \theta_n}{\partial x_j} = \frac{\partial a_k}{\partial x_j} = x_0 \cdot \frac{\partial \nu}{\partial x_j} = x_0 \cdot \mu_j,$$

and from here we obtain

$$a_k \nu + b_1 \mu_1 + \cdots + b_n \mu_n = (x_0 \cdot \nu) \nu + (x_0 \cdot \mu_1) \mu_1 + \cdots + (x_0 \cdot \mu_n) \mu_n + \frac{1}{k} \nu = x_0 + \frac{1}{k} [k(f_k - x_0)] = f_k.$$

Example 4 (Cuspidal cross-cap). Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the folded Whitney umbrella,

$$f(x, y) = (x, y^2, xy^3).$$

This is a regular frontal with $\text{Reg}(f) = \text{Reg}(\nu) = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$. The Legendre data associated to f is

$$\nu(x, y) = \frac{(-2y^3, -3xy, 2)}{\sqrt{9x^2y^2 + 4y^6 + 4}}; \quad a(x, y) = -\frac{3xy^3}{\sqrt{9x^2y^2 + 4y^6 + 4}}.$$

Writing $\nu(x, y) = \nu^2(\theta_1(x, y), \theta_2(x, y))$ gives us functions

$$\begin{aligned} \sin \theta_1(x, y) &= \frac{2}{\sqrt{9x^2y^2 + 4y^6 + 4}}; & \cos \theta_1(x, y) &= \sqrt{\frac{9x^2y^2 + 4y^6}{9x^2y^2 + 4y^6 + 4}}; \\ \sin \theta_2(x, y) &= -\frac{3xy}{\sqrt{9x^2y^2 + 4y^6}}; & \cos \theta_2(x, y) &= -\frac{2y^3}{\sqrt{9x^2y^2 + 4y^6}}. \end{aligned}$$

Applying the chain rule, we see that

$$\begin{aligned} \frac{\partial \theta_1}{\partial x} &= \frac{1}{\cos \theta_1} \frac{\partial \sin \theta_1}{\partial x} = -\frac{18xy^2}{\sqrt{9x^2y^2 + 4y^6} (9x^2y^2 + 4y^6 + 4)}; & \frac{\partial \theta_2}{\partial x} &= \frac{1}{\cos \theta_2} \frac{\partial \sin \theta_2}{\partial x} = \frac{6y^2}{9x^2 + 4y^4}; \\ \frac{\partial \theta_1}{\partial y} &= \frac{1}{\cos \theta_1} \frac{\partial \sin \theta_1}{\partial y} = -\frac{6(3x^2y + 4y^5)}{\sqrt{9x^2y^2 + 4y^6} (9x^2y^2 + 4y^6 + 4)}; & \frac{\partial \theta_2}{\partial y} &= \frac{1}{\cos \theta_2} \frac{\partial \sin \theta_2}{\partial y} = -\frac{12xy}{9x^2 + 4y^4}. \end{aligned}$$

Since f is a regular frontal, the system of equations $da = b_1 d\theta_1 + b_2 d\theta_2$ has a unique solution,

$$b_1(x, y) = \frac{xy^2(9x^2y^2 + 4y^6 + 10)}{\sqrt{9x^2 + 4y^4} \sqrt{9x^2y^2 + 4y^6 + 4}}; \quad b_2(x, y) = \frac{3x^2y - 2y^5}{\sqrt{9x^2y^2 + 4y^6 + 4}}$$

The vector fields $\mu_1, \mu_2: \mathbb{R}^2 \rightarrow S^2$ completing the orthonormal basis $\{\nu(x, y), \mu_1(x, y), \mu_2(x, y)\}$ for $T_{f_n(x, y)}\mathbb{R}^3$ are given by

$$\mu_1(x, y) = -\frac{(4y^3, 6xy, 9x^2y^2 + 4y^6)}{\sqrt{9x^2y^2 + 4y^6} \sqrt{9x^2y^2 + 4y^6 + 4}}; \quad \mu_2(x, y) = \left(\frac{3xy}{\sqrt{9x^2y^2 + 4y^6}}, -\frac{2y^3}{\sqrt{9x^2y^2 + 4y^6}}, 0 \right)$$

Then we have that

$$a(x, y)\nu(x, y) + b_1(x, y)\mu_1(x, y) + b_2(x, y)\mu_2(x, y) = (x, y^2, xy^3) = f(x, y)$$

and f can be recovered from the Legendre data (ν, a) .

Example 5 (Cross-cap). Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the Whitney umbrella,

$$f(x, y) = (x, y^2, xy).$$

This is a pseudo regular frontal with $\text{Reg}(f) = \text{Reg}(\nu) = \mathbb{R}^2 \setminus \{(0, 0)\}$, but it is not a frontal. The Legendre data associated to f is

$$\nu(x, y) = \frac{(-2y^2, -x, 2y)}{\sqrt{x^2 + 4(y^4 + y^2)}}; \quad a(x, y) = -\frac{xy^2}{\sqrt{x^2 + 4(y^4 + y^2)}}.$$

Writing $\nu(x, y) = \nu^2(\theta_1(x, y), \theta_2(x, y))$ gives us functions

$$\begin{aligned} \sin \theta_1(x, y) &= \frac{2y}{\sqrt{x^2 + 4(y^4 + y^2)}}; & \cos \theta_1(x, y) &= \sqrt{\frac{x^2 + 4y^4}{x^2 + 4y^4 + 4y^2}}; \\ \sin \theta_2(x, y) &= \frac{x}{\sqrt{x^2 + 4y^4}}; & \cos \theta_2(x, y) &= -\frac{2y^2}{\sqrt{x^2 + 4y^4}}. \end{aligned}$$

Applying the chain rule, we see that

$$\begin{aligned}\frac{\partial \theta_1}{\partial x} &= \frac{1}{\cos \theta_1} \frac{\partial \sin \theta_1}{\partial x} = -\frac{2xy}{\sqrt{x^2 + 4y^4}(x^2 + 4(y^4 + y^2))}; & \frac{\partial \theta_2}{\partial x} &= \frac{1}{\cos \theta_2} \frac{\partial \sin \theta_2}{\partial x} = \frac{2y^2}{\sqrt{x^2 + 4y^4}\sqrt{x^2 + 4(y^4 + y^2)}}; \\ \frac{\partial \theta_1}{\partial y} &= \frac{1}{\cos \theta_1} \frac{\partial \sin \theta_1}{\partial y} = \frac{2(x^2 - 4y^4)}{\sqrt{x^2 + 4y^4}(x^2 + 4(y^4 + y^2))}; & \frac{\partial \theta_2}{\partial y} &= \frac{1}{\cos \theta_2} \frac{\partial \sin \theta_2}{\partial y} = -\frac{4xy(x^2 + 2y^2(2y^2 + 2))}{\sqrt{x^2 + 4y^4}(x^2 + 4(y^4 + y^2))^{3/2}}.\end{aligned}$$

Since f is a pseudo regular frontal, the system of equations $da = b_1 d\theta_1 + b_2 d\theta_2$ has a unique solution,

$$b_1(x, y) = \frac{xy(x^2 + 4y^4 + 6y^2)}{\sqrt{x^2 + 4y^4}\sqrt{x^2 + 4y^4 + 4y^2}}; \quad b_2(x, y) = \frac{x^2 - 2y^4}{\sqrt{x^2 + 4y^4}}.$$

The vector fields $\mu_1, \mu_2: \mathbb{R}^2 \rightarrow S^2$ completing the orthonormal basis $\{\nu(x, y), \mu_1(x, y), \mu_2(x, y)\}$ for $T_{f_n(x, y)}\mathbb{R}^3$ are given by

$$\mu_1(x, y) = -\frac{(4y^3, 2xy, x^2 + 4y^4)}{\sqrt{x^2 + 4y^4}\sqrt{x^2 + 4(y^4 + y^2)}}; \quad \mu_2(x, y) = \left(\frac{x}{\sqrt{x^2 + 4y^4}}, -\frac{2y^2}{\sqrt{x^2 + 4y^4}}, 0 \right)$$

Then we have that

$$a(x, y)\nu(x, y) + b_1(x, y)\mu_1(x, y) + b_2(x, y)\mu_2(x, y) = (x, y^2, xy) = f(x, y)$$

and f can be recovered from the Legendre data (ν, a) .

Example 6 (Cuspidal edge). Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the cuspidal edge, given by $f(x, y) = (x, y^2, y^3)$. The mapping f is a frontal, but it is not a regular frontal; nonetheless, f is the pointwise limit of the sequence of mappings $f_n: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$f_n(x, y) = \left(x, \frac{x^2}{n} + y^2, y^3 \right),$$

which are regular frontals with $\text{Reg}(f_n) = \text{Reg}(\nu_n) = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$. Moreover, we can transform f_n into f for any $n \in \mathbb{N}$ by applying a suitable change of coordinates in the source and target.

Let $n \in \mathbb{N}$: the Legendre data associated to f_n is

$$\nu_n(x, y) = \frac{(6xy, -3ny, 2n)}{\sqrt{9n^2y^2 + 4n^2 + 36x^2y^2}}; \quad a_n(x, y) = \frac{y(3x^2 - ny^2)}{\sqrt{9n^2y^2 + 4n^2 + 36x^2y^2}}.$$

Writing $\nu_n(x, y) = \nu^2(\theta_1(x, y), \theta_2(x, y))$ gives us functions

$$\begin{aligned}\sin \theta_1(x, y) &= \frac{2n}{\sqrt{9n^2y^2 + 4n^2 + 36x^2y^2}}; & \cos \theta_1(x, y) &= 3y\sqrt{\frac{n^2 + 4x^2}{9n^2y^2 + 4n^2 + 36x^2y^2}}; \\ \sin \theta_2(x, y) &= -\frac{n}{\sqrt{n^2 + 4x^2}}; & \cos \theta_2(x, y) &= \frac{2x}{\sqrt{n^2 + 4x^2}}.\end{aligned}$$

Note that when n goes to infinity, $\sin \theta_2(x, y)$ goes to 1 and $\cos \theta_2(x, y)$ goes to 0 for any $(x, y) \in \text{Reg}(f_n)$, meaning that θ_2 must be a constant function and the differential equation

$$da = b_1 d\theta_1 + b_2 d\theta_2$$

admits infinite solutions (b_1, b_2) .

Applying the chain rule, we see that

$$\begin{aligned}\frac{\partial \theta_1}{\partial x} &= \frac{1}{\cos \theta_1} \frac{\partial \sin \theta_1}{\partial x} = -\frac{24nxy}{\sqrt{n^2 + 4x^2}(9n^2y^2 + 4n^2 + 36x^2y^2)}; & \frac{\partial \theta_2}{\partial x} &= \frac{1}{\cos \theta_2} \frac{\partial \sin \theta_2}{\partial x} = \frac{2n}{n^2 + 4x^2}; \\ \frac{\partial \theta_1}{\partial y} &= \frac{1}{\cos \theta_1} \frac{\partial \sin \theta_1}{\partial y} = -\frac{6n\sqrt{n^2 + 4x^2}}{n^2(9y^2 + 4) + 36x^2y^2}; & \frac{\partial \theta_2}{\partial y} &= \frac{1}{\cos \theta_2} \frac{\partial \sin \theta_2}{\partial y} = 0.\end{aligned}$$

Since f_n is a regular frontal, the system of equations $da = b_1 d\theta_1 + b_2 d\theta_2$ has a unique solution,

$$b_1(x, y) = \frac{3n^2y^4 + 2n^2y^2 - 2nx^2 + 12x^2y^4}{\sqrt{n^2 + 4x^2}\sqrt{9n^2y^2 + 4n^2 + 36x^2y^2}}; \quad b_2(x, y) = \frac{3xy(n^2 + 2ny^2 + 2x^2)}{n\sqrt{9n^2y^2 + 4n^2 + 36x^2y^2}}$$

The vector fields $\mu_1, \mu_2: \mathbb{R}^2 \rightarrow S^2$ completing the orthonormal basis $\{\nu_n(x, y), \mu_1(x, y), \mu_2(x, y)\}$ for $T_{f_n(x, y)}\mathbb{R}^3$ are given by

$$\mu_1(x, y) = -\frac{(-4nx, 2n^2, 3y(n^2 + 4x^2))}{\sqrt{n^2 + 4x^2}\sqrt{9n^2y^2 + 4n^2 + 36x^2y^2}}; \quad \mu_2(x, y) = \left(\frac{n}{\sqrt{n^2 + 4x^2}}, \frac{2x}{\sqrt{n^2 + 4x^2}}, 0\right)$$

Then we have that

$$a_n(x, y)\nu_n(x, y) + b_1(x, y)\mu_1(x, y) + b_2(x, y)\mu_2(x, y) = \left(x, \frac{x^2}{n} + y^2, y^3\right) = f_n(x, y)$$

and f can be recovered from the sequence of Legendre data $\{(\nu_n, a_n)\}_{n \in \mathbb{N}}$.

Example 7 (Swallowtail). Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the swallowtail singularity, given by $f(x, y) = (x, 2y^3 + xy, 3y^4 + xy^2)$. The mapping f is a frontal, but it is not a regular frontal; nonetheless, f is the pointwise limit of the sequence of mappings $f_n: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$f_n(x, y) = \left(x, xy + 2y^3, \frac{x^2}{n} + xy^2 + 3y^4\right),$$

which are regular frontals with $\text{Reg}(\nu_n) = \text{Reg}(f_n) = \{(x, y) \in \mathbb{R}^2 : x \neq -6y^2\}$. Moreover, we can transform f_n into f for any $n \in \mathbb{N}$ by applying a suitable change of coordinates in the source and target.

Let $n \in \mathbb{N}$: the Legendre data associated to f_n is

$$\nu_n(x, y) = \frac{(ny^2 - 2x, 2ny, n)}{\sqrt{4n^2y^2 + n^2 + (2x - ny^2)^2}}; \quad a_n(x, y) = -\frac{ny^4 + x^2}{\sqrt{n^2y^4 + 4n^2y^2 + n^2 - 4nxy^2 + 4x^2}}.$$

Writing $\nu_n(x, y) = \nu^2(\theta_1(x, y), \theta_2(x, y))$ gives us functions

$$\begin{aligned} \sin \theta_1(x, y) &= \frac{n}{\sqrt{4n^2y^2 + n^2 + (2x - ny^2)^2}}; & \cos \theta_1(x, y) &= \sqrt{\frac{n^2y^4 + 4n^2y^2 - 4nxy^2 + 4x^2}{n^2y^4 + 4n^2y^2 + n^2 - 4nxy^2 + 4x^2}}; \\ \sin \theta_2(x, y) &= -\frac{2ny}{\sqrt{n^2y^4 + 4n^2y^2 - 4nxy^2 + 4x^2}}; & \cos \theta_2(x, y) &= \frac{ny^2 - 2x}{\sqrt{n^2y^4 + 4n^2y^2 - 4nxy^2 + 4x^2}}. \end{aligned}$$

Applying the chain rule, we see that

$$\begin{aligned} \frac{\partial \theta_1}{\partial x} &= -\frac{2n(2x - ny^2)}{\sqrt{n^2y^4 + 4n^2y^2 - 4nxy^2 + 4x^2}(n^2y^4 + 4n^2y^2 + n^2 - 4nxy^2 + 4x^2)}; \\ \frac{\partial \theta_2}{\partial x} &= -\frac{4ny}{n^2y^4 + 4n^2y^2 - 4nxy^2 + 4x^2}; \\ \frac{\partial \theta_1}{\partial y} &= \frac{2n^2y(-ny^2 - 2n + 2x)}{\sqrt{n^2y^4 + 4n^2y^2 - 4nxy^2 + 4x^2}(n^2y^4 + 4n^2y^2 + n^2 - 4nxy^2 + 4x^2)}; \\ \frac{\partial \theta_2}{\partial y} &= \frac{2n(ny^2 + 2x)}{n^2y^4 + 4n^2y^2 - 4nxy^2 + 4x^2}. \end{aligned}$$

Since f_n is a regular frontal, the system of equations $da = b_1 d\theta_1 + b_2 d\theta_2$ has a unique solution,

$$\begin{aligned} b_1(x, y) &= \frac{n^3y^2(xy^4 + 4xy^2 + x + 3y^6 + 12y^4 + 4y^2) + n^2x(x(-3y^4 + 4y^2 + 2) - 12y^6) + 12nx^2y^4 + 4x^4}{n\sqrt{n^2y^2(y^2 + 4) - 4nxy^2 + 4x^2}\sqrt{n^2(y^4 + 4y^2 + 1) - 4nxy^2 + 4x^2}}; \\ b_2(x, y) &= \frac{y(nxy^2 + 2nx + 2ny^4 - 2x^2 - 4xy^2)}{\sqrt{n^2y^4 + 4n^2y^2 + n^2 - 4nxy^2 + 4x^2}} \end{aligned}$$

The vector fields $\mu_1, \mu_2: \mathbb{R}^2 \rightarrow S^2$ completing the orthonormal basis $\{\nu_n(x, y), \mu_1(x, y), \mu_2(x, y)\}$ for $T_{f_n(x, y)}\mathbb{R}^3$ are given by

$$\begin{aligned} \mu_1(x, y) &= \frac{(n(2x - ny^2), 2n^2y, n^2y^4 + 4n^2y^2 - 4nxy^2 + 4x^2)}{\sqrt{n^2y^4 + 4n^2y^2 - 4nxy^2 + 4x^2}\sqrt{4n^2y^2 + n^2 + (2x - ny^2)^2}}; \\ \mu_2(x, y) &= \frac{(2ny, ny^2 - 2x)}{\sqrt{n^2y^4 + 4n^2y^2 - 4nxy^2 + 4x^2}}. \end{aligned}$$

Then we have that

$$a_n(x, y)\nu_n(x, y) + b_1(x, y)\mu_1(x, y) + b_2(x, y)\mu_2(x, y) = \left(x, xy + 2y^3, \frac{x^2}{n} + xy^2 + 3y^4\right) = f_n(x, y)$$

and f can be recovered from the sequence of Legendre data $\{(\nu_n, a_n)\}_{n \in \mathbb{N}}$.

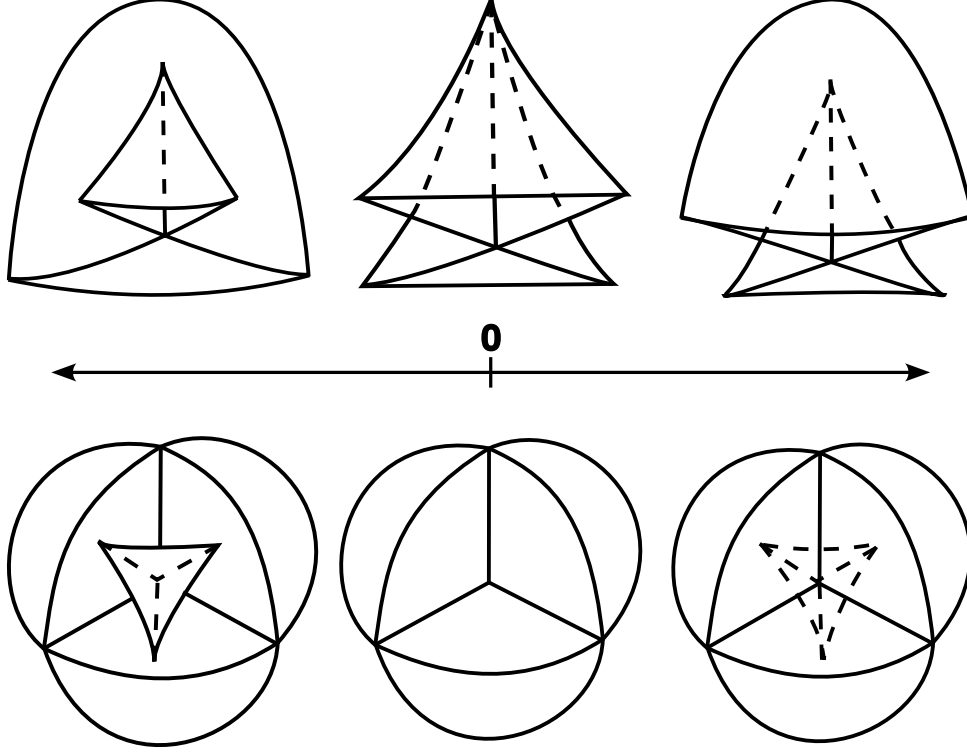


FIGURE 2. Image of the D_4^+ (top) and D_4^- (bottom) singularities, projected into \mathbb{R}^3 via the map $(u, x, y, z) \mapsto (x, y, z)$ for different values of u .

Example 8 (D_4 singularity). Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be the D_4^+ singularity, given by $f(u, v, w) = (u, vw, 2uv + 3v^2 + w^2, uv^2 + 2v^3 + 2vw^2)$ (see Figure 2). The mapping f is a frontal, but it is not a regular frontal; nonetheless, f is the pointwise limit of the sequence of mappings $f_n: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ given by

$$f_n(x, y) = \left(u, vw, 2uv + 3v^2 + w^2, \frac{u^2}{n} + uv^2 + 2v^3 + 2vw^2\right),$$

which are regular frontals with $\text{Reg}(\nu_n) = \text{Reg}(f_n) = \{(u, v, w) \in \mathbb{R}^3 : w^2 \neq uv + 3v^2\}$. Moreover, we can transform f_n into f for any $n \in \mathbb{N}$ by applying a suitable change of coordinates in the source and target.

Let $n \in \mathbb{N}$: the Legendre data associated to f_n is

$$\nu_n(u, v, w) = \frac{(nv^2 - 2u, -2nw, -nv, n)}{\sqrt{n^2v^2 + 4n^2w^2 + n^2 + (nv^2 - 2u)^2}}.$$

Writing $\nu_n(u, v, w) = \nu^3(\theta_1(u, v, w), \theta_2(u, v, w), \theta_3(u, v, w))$ gives us functions

$$\begin{aligned} \sin \theta_1 &= \frac{n}{\sqrt{n^2v^2 + 4n^2w^2 + n^2 + (nv^2 - 2u)^2}}; & \cos \theta_1 &= \sqrt{\frac{n^2v^4 + n^2v^2 + 4n^2w^2 - 4nvw^2 + 4u^2}{n^2v^4 + n^2v^2 + 4n^2w^2 + n^2 - 4nvw^2 + 4u^2}}; \\ \sin \theta_2 &= -\frac{nv}{\sqrt{n^2v^4 + n^2v^2 + 4n^2w^2 - 4nvw^2 + 4u^2}}; & \cos \theta_2 &= \frac{\sqrt{n^2v^4 + 4n^2w^2 - 4nvw^2 + 4u^2}}{\sqrt{n^2v^4 + n^2v^2 + 4n^2w^2 - 4nvw^2 + 4u^2}}; \\ \sin \theta_3 &= -\frac{2nw}{\sqrt{n^2v^4 + 4n^2w^2 - 4nvw^2 + 4u^2}}; & \cos \theta_3 &= \frac{nv^2 - 2u}{\sqrt{n^2v^4 + 4n^2w^2 - 4nvw^2 + 4u^2}}. \end{aligned}$$

Applying the chain rule, we see that

$$\begin{aligned}
\frac{\partial \theta_1}{\partial u} &= -\frac{2n(2u - nv^2)}{\sqrt{n^2v^4 + n^2v^2 + 4n^2w^2 - 4nvw^2 + 4u^2}(n^2v^4 + n^2v^2 + 4n^2w^2 + n^2 - 4nvw^2 + 4u^2)}; \\
\frac{\partial \theta_2}{\partial u} &= -\frac{2nv(nv^2 - 2u)}{\sqrt{n^2v^4 + 4n^2w^2 - 4nvw^2 + 4u^2}(n^2v^4 + n^2v^2 + 4n^2w^2 - 4nvw^2 + 4u^2)}; \\
\frac{\partial \theta_3}{\partial u} &= -\frac{4nw}{n^2(v^4 + 4w^2) - 4nvw^2 + 4u^2}; \\
\frac{\partial \theta_1}{\partial v} &= -\frac{n^2v(2nv^2 + n - 4u)}{\sqrt{n^2v^4 + n^2v^2 + 4n^2w^2 - 4nvw^2 + 4u^2}(n^2v^4 + n^2v^2 + 4n^2w^2 + n^2 - 4nvw^2 + 4u^2)}; \\
\frac{\partial \theta_2}{\partial v} &= -\frac{n(-n^2v^4 + 4n^2w^2 + 4u^2)}{\sqrt{n^2v^4 + 4n^2w^2 - 4nvw^2 + 4u^2}(n^2v^4 + n^2v^2 + 4n^2w^2 - 4nvw^2 + 4u^2)}; \\
\frac{\partial \theta_3}{\partial v} &= \frac{4n^2vw}{n^2(v^4 + 4w^2) - 4nvw^2 + 4u^2}; \\
\frac{\partial \theta_1}{\partial w} &= -\frac{4n^3w}{\sqrt{n^2v^4 + n^2v^2 + 4n^2w^2 - 4nvw^2 + 4u^2}(n^2v^4 + n^2v^2 + 4n^2w^2 + n^2 - 4nvw^2 + 4u^2)}; \\
\frac{\partial \theta_2}{\partial w} &= \frac{4n^3vw}{\sqrt{n^2v^4 + 4n^2w^2 - 4nvw^2 + 4u^2}(n^2v^4 + n^2v^2 + 4n^2w^2 - 4nvw^2 + 4u^2)}; \\
\frac{\partial \theta_3}{\partial w} &= -\frac{2n(nv^2 - 2u)}{n^2(v^4 + 4w^2) - 4nvw^2 + 4u^2}.
\end{aligned}$$

Since f_n is a regular frontal, the system of equations $da = b_1 d\theta_1 + b_2 d\theta_2 + b_3 d\theta_3$ has a unique solution given by

$$\begin{aligned}
b_1(u, v, w) &= \frac{n^3v(u(v^5 + v^3 + 4vw^2 + v) + 2v^6 + 2v^4(w^2 + 1) + v^2(10w^2 + 3) + w^2(8w^2 + 3))}{n\sqrt{n^2(v^4 + v^2 + 4w^2) - 4nvw^2 + 4u^2}\sqrt{n^2(v^4 + v^2 + 4w^2 + 1) - 4nvw^2 + 4u^2}} + \\
&\quad + \frac{n^2u(u(-3v^4 + v^2 + 4w^2 + 2) - 8v^3(v^2 + w^2)) + 8nu^2v(v^2 + w^2) + 4u^4}{n\sqrt{n^2(v^4 + v^2 + 4w^2) - 4nvw^2 + 4u^2}\sqrt{n^2(v^4 + v^2 + 4w^2 + 1) - 4nvw^2 + 4u^2}}; \\
b_2(u, v, w) &= \frac{n^2(u(2v^5 + v^3 + 8vw^2) + 3v^6 + v^4w^2 + 10v^2w^2 + 4w^4)}{\sqrt{n^2(v^4 + 4w^2) - 4nvw^2 + 4u^2}\sqrt{n^2(v^4 + v^2 + 4w^2 + 1) - 4nvw^2 + 4u^2}} + \\
&\quad + \frac{-2nvw(4uv^2 + u + 6v^3 + 2vw^2) + 4u^2(2uv + 3v^2 + w^2)}{\sqrt{n^2(v^4 + 4w^2) - 4nvw^2 + 4u^2}\sqrt{n^2(v^4 + v^2 + 4w^2 + 1) - 4nvw^2 + 4u^2}}; \\
b_3(u, v, w) &= \frac{w(n(2u + v^3) - 2uv)}{\sqrt{n^2(v^4 + v^2 + 4w^2 + 1) - 4nvw^2 + 4u^2}}.
\end{aligned}$$

The vector fields $\mu_1, \mu_2, \mu_3: \mathbb{R}^3 \rightarrow S^3$ completing the orthonormal basis $\{\nu_n, \mu_1, \mu_2, \mu_3\}$ for $T_{f_n(u,v,w)}\mathbb{R}^4$ are given by

$$\begin{aligned}
\mu_1(u, v, w) &= \frac{(n(nv^2 - 2u), 2n^2w, n^2v, n^2v^4 + n^2v^2 + 4n^2w^2 - 4nvw^2 + 4u^2)}{\sqrt{n^2v^4 + n^2v^2 + 4n^2w^2 - 4nvw^2 + 4u^2}\sqrt{n^2v^2 + 4n^2w^2 + n^2 + (nv^2 - 2u)^2}}; \\
\mu_2(u, v, w) &= \frac{(nv(nv^2 - 2u), 2n^2vw, n^2v^4 + 4n^2w^2 - 4nvw^2 + 4u^2, 0)}{\sqrt{n^2v^4 + 4n^2w^2 - 4nvw^2 + 4u^2}\sqrt{n^2v^4 + n^2v^2 + 4n^2w^2 - 4nvw^2 + 4u^2}}; \\
\mu_3(u, v, w) &= \left(\frac{2nw}{\sqrt{n^2v^4 + 4n^2w^2 - 4nvw^2 + 4u^2}}, \frac{nv^2 - 2u}{\sqrt{n^2v^4 + 4n^2w^2 - 4nvw^2 + 4u^2}}, 0, 0 \right).
\end{aligned}$$

Then we have that

$$\begin{aligned}
&a_n(x, y)\nu_n(x, y) + b_1(x, y)\mu_1(x, y) + b_2(x, y)\mu_2(x, y) + b_3(x, y)\mu_3(x, y) \\
&= \left(u, vw, 2uv + 3v^2 + w^2, \frac{u^2}{n} + uv^2 + 2v^3 + 2vw^2 \right) = f_n(x, y)
\end{aligned}$$

and f can be recovered from the sequence of Legendre data $\{(\nu_n, a_n)\}_{n \in \mathbb{N}}$.

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C. MUÑOZ-CABELLO, DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT DE VALÈNCIA, CAMPUS DE BURJASSOT, 46100 BURJASSOT, SPAIN

Email address: Christian.Munoz@uv.es

T. NISHIMURA, RESEARCH INSTITUTE OF ENVIRONMENT AND INFORMATION SCIENCES, YOKOHAMA NATIONAL UNIVERSITY, YOKOHAMA 240-8501, JAPAN

Email address: nishimura-takashi-yx@ynu.ac.jp

R. OSET SINHA, DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT DE VALÈNCIA, CAMPUS DE BURJASSOT, 46100 BURJASSOT, SPAIN

Email address: Raul.Oset@uv.es