Homework 1 Solutions

1. Chapter 2, excercise 2

    a. There are two cases: we draw a blue marble from urn one and we draw a yellow marble from urn one. If we draw a blue marble (with probability 3/6), then the probability of drawing a blue marble from urn 2 is $(5+1)/13$. If we draw a yellow marble from urn 1 (with probability 1/2), then the probability of drawing a blue marble from urn 2 is $5/(13)$. Therefore the total probability is

```
(3/6)*(6/13) + (3/6)*(5/13)
```

```
## [1] 0.4230769
```

    b. The probability the first marble is blue given the second marble is blue is the first of the above probabilities divided by the above.

```
(3/6)*(6/13)/((3/6)*(6/13) + (3/6)*(5/13))
```

```
## [1] 0.5454545
```

2. Chapter 2, excercise 4 The number of ways that two out of five programs can fail is $\binom{5}{2} = 10$ and each occurs with probability $0.2^2 * (1 - 0.2)^3$, giving a total probability of

```
choose(5, 2)*0.2^2*(1 - 0.2)^3
```

```
## [1] 0.2048
```

3. Chapter 2, excercise 6

    a. Note first that $\Pr(\text{at least one prize}) = 1 - \Pr(\text{no prizes})$. The probability of winning no prizes is equal to the probability that none of the five tickets drawn match your three, leaving 57 numbers to choose from. Therefore the number of ways to do this is $\binom{57}{5}$ out of $\binom{60}{5}$ total ways.

```
1 - choose(57, 5)/choose(60, 5)
```

```
## [1] 0.2333431
```

    b. The probability that you win exactly one prize is the probability that exactly one of the five drawn number matches one of your three. The number of ways this can occur is $\binom{3}{1} \cdot \binom{57}{4}$.

```
choose(3, 1)*choose(57, 4)/choose(60, 5)
```

```
## [1] 0.2169784
```

4. Chapter 2, excercise 12 (a and b only)

    a. The probability of winning is $\Pr(N = 3|N \geq 3)$. Note that $3 \leq N \leq 5$). You can get $(N = 3)$ 3 ways: 3 on first roll, 1 on first roll and 2 on second, 2 on first roll and 1 on second, and finally 1 on first 3 rolls. The probability of these are 1/3, 1/9, 1/9, and 1/27, respectively. To get 4, one can roll a 2 first roll and a 2 the second roll, a 1 the first roll and 3 the second roll, or a 1 the first two rolls and a 2 the third roll. The probabilities are 1/9, 1/9, and 1/27. To get a 5 you can roll a 2 the first roll and three the second or a 1 the first two rolls and 3 the second. The probability of these are 1/9 and 1/27. Therefore $\Pr(N = 3|N \geq 3) =$

```
(1/3 + 1/9 + 1/9 + 1/27)/(1/3 + 1/9 + 1/9 + 1/27 + 1/9 + 1/9 + 1/27 + 1/9 + 1/27)
```

```
## [1] 0.5925926
```

b. Using the above calculated outcomes, if we won then the first roll was a 1 if we get a 1 on the first roll and 2 on the second or a 1 on all three rolls. Therefore $\Pr(\text{first roll is } 1|\text{win}) =$

```
(1/9 + 1/27)/(1/3 + 1/9 + 1/9 + 1/27)
```

```
## [1] 0.25
```

5. Chapter 2, excercise 14

a. The probability that the number of turns is one is the probability that either Jack or Jill shows a 1 and the other does not. This equals

```
(1/4)*(2/3) + (3/4)*(1/3)
```

```
## [1] 0.4166667
```

In order for the number of turns to equal 2, first it must not equal 1. Then either must roll a 1 but not both. This occurs with probability

```
(1 - (1/4)*(2/3) + (3/4)*(1/3))*((1/4)*(2/3) + (3/4)*(1/3))
```

```
## [1] 0.4513889
```

b. If $N = 2$, then in the first turn one of two things happened: neither rolled a 1 (with probability $(3/4) * (2/3)$), or both rolled a 1 (with probability $(1/4) * (1/3)$). Therefore $\Pr(\text{first turn was a tie}|N = 2) =$

```
(1/4)*(1/3)/((1/4)*(1/3) + (3/4)*(2/3))
```

```
## [1] 0.1428571
```

6. Chapter 2, excercise 21

There are 3 cases: the two people are in the 3, 4, and 5 group. We will assume they are filled in that order. The total number of ways to fill the groups is $\binom{20}{3}\binom{17}{4}\binom{13}{5}$ In the first case we fix the two people in group 3, then there are 18 people to choose for the remainder of the group, 17 people to choose for the remainder of group 4, and 13 people for group 5. This gives $18 \cdot \binom{17}{4} \cdot \binom{13}{5}$ ways.
In the second case, we fix the two people in group 4. That leaves 18 people to fill group 3, 15 people to fill the remainder of group 4, and 13 people to fill group 5. This gives $\binom{18}{3}\binom{15}{2}\binom{13}{5}$ ways. In the last case, we fix the two people in group 5. That leaves 18 people to fill group 3, 15 people to fill group 4, and 11 people to fill the remainder of group 5. This gives $\binom{18}{3}\binom{15}{4}\binom{11}{5}$ ways.
The total probability is then

```
(18*choose(17, 4)*choose(13, 5) + choose(18, 3)*choose(15, 2)*choose(13, 5)
 + choose(18, 3)*choose(15, 4)*choose(11, 5))/
  (choose(20, 3)*choose(17, 4)*choose(13, 5))
```

```
## [1] 0.1947368
```

7. Suppose that a commitee of four people is randomly selected from a group of 20, consisting of 8 men and 12 women. Assume that each person is equally likely to be chosen. Let X denote the number of women on the committee.

   a. Write out the probability mass function for X. Include only the possible values that X can take.
   X can take 5 values: 0, 1, 2, 3, and 4. The probilities of these are

$$f(x) = \begin{cases} \frac{\binom{8}{4}}{\binom{20}{4}} & x = 0 \\ \frac{\binom{8}{3}\binom{12}{1}}{\binom{20}{4}} & x = 1 \\ \frac{\binom{8}{2}\binom{12}{2}}{\binom{20}{4}} & x = 2 \\ \frac{\binom{8}{1}\binom{12}{3}}{\binom{20}{4}} & x = 3 \\ \frac{\binom{12}{4}}{\binom{20}{4}} & x = 4 \end{cases}$$

   These correspond to

```r
denom = choose(20, 4)
choose(8, 4)/denom
```

```
## [1] 0.01444788
```

```r
choose(8, 3)*choose(12, 1)/denom
```

```
## [1] 0.1386997
```

```r
choose(8, 2)*choose(12, 2)/denom
```

```
## [1] 0.3814241
```

```r
choose(8, 1)*choose(12, 3)/denom
```

```
## [1] 0.3632611
```

```r
choose(12, 4)/denom
```

```
## [1] 0.1021672
```

   b. What is the expectation of X?

The expectation of X is $\sum_{x=0}^{4} x f(x) =$

```
r   0*choose(8, 4)/denom + 1*choose(8, 3)*choose(12, 1)/denom + 2*choose(8, 2)*choose(12,
2)/denom +   3*choose(8, 1)*choose(12, 3)/denom + 4*choose(12, 4)/denom
```

```
## [1] 2.4
```

   c. What is the standard deviation of X?

The standard deviation is the square root of the variance, so we'll calculate the variance first. Since we already have $\mathrm{E}X$, all we need to calculate is $\mathrm{E}X^2 = \sum_{x=0}^{4} x^2 f(x)$

```
r   0^{2}*choose(8, 4)/denom + 1^{2}*choose(8, 3)*choose(12, 1)/denom + 2^{2}*choose(8,
2)*choose(12, 2)/denom +   3^{2}*choose(8, 1)*choose(12, 3)/denom + 4^{2}*choose(12,
4)/denom
```

`## [1] 6.568421` The variance is then $\mathrm{E}X^2 - (\mathrm{E}X)^2$.

r   6.5684214 - 2.4^{2}

[1] 0.8084214 And the standard deviation is

r   sqrt(0.808421)

[1] 0.8991223

8. Suppose that you throw two six-sided dice. Let X be the sum of the two dice.

   a. Write out the probability mass function for X. Include only the possible values that X can take.

$$f(x) = \begin{cases} \frac{1}{36} & x = 2, 12 \\ \frac{2}{36} & x = 3, 11 \\ \frac{3}{36} & x = 4, 10 \\ \frac{4}{36} & x = 5, 9 \\ \frac{5}{36} & x = 6, 8 \\ \frac{6}{36} & x = 7 \end{cases}$$

   b. What is the expectation of X?

```
2*1/36 + 3*2/36 + 4*3/36 + 5*4/36 + 6*5/36 + 7*6/36 +
  8*5/36 + 9*4/36 + 10*3/36 + 11*2/36 + 12*1/36
```

```
## [1] 7
```

   c. What is the standard deviation of X? $\mathrm{E}X^2 =$

```
2^{2}*1/36 + 3^{2}*2/36 + 4^{2}*3/36 + 5^{2}*4/36 + 6^{2}*5/36 + 7^{2}*6/36 +
  8^{2}*5/36 + 9^{2}*4/36 + 10^{2}*3/36 + 11^{2}*2/36 + 12^{2}*1/36
```

```
## [1] 54.83333
```

$\mathrm{Var}(X) =$

```
54.83333 - 7^2
```

```
## [1] 5.83333
```

and the standard deviation equals

```
sqrt(5.83333)
```

```
## [1] 2.415229
```

9. Chapter 2 excercise 16 The only way ar least \$ 0.10 falls out is if both coins that fall out are nickels and dimes. This occurs in $\binom{8}{2}$ ways and there are $\binom{10}{2}$ ways for two coins to fall out. This gives the probability

```
choose(8, 2)/choose(10, 2)
```

```
## [1] 0.6222222
```

The number of ways for the two coins to both be pennies is 1, nickels is $\binom{3}{2}$, and dimes is $\binom{5}{2}$. This gives the probability

```
(1 + choose(3, 2) + choose(5, 2))/choose(10, 2)
```

```
## [1] 0.3111111
```

10. From Chapter 2, excercise 16; what is the expected value of the total amount of money lost? There are 6 possible outcomes: \$ 0.02, \$ 0.06, \$ 0.10, \$ 0.11, \$ 0.15, \$ 0.20. The corresponding number of ways to get these outcomes are 1, $2 \cdot 3$, $\binom{3}{2}$, $2 \cdot 5$, $3 \cdot 5$, and $\binom{5}{2}$. The total number of ways is $\binom{10}{2}$. Therefore the expected value is

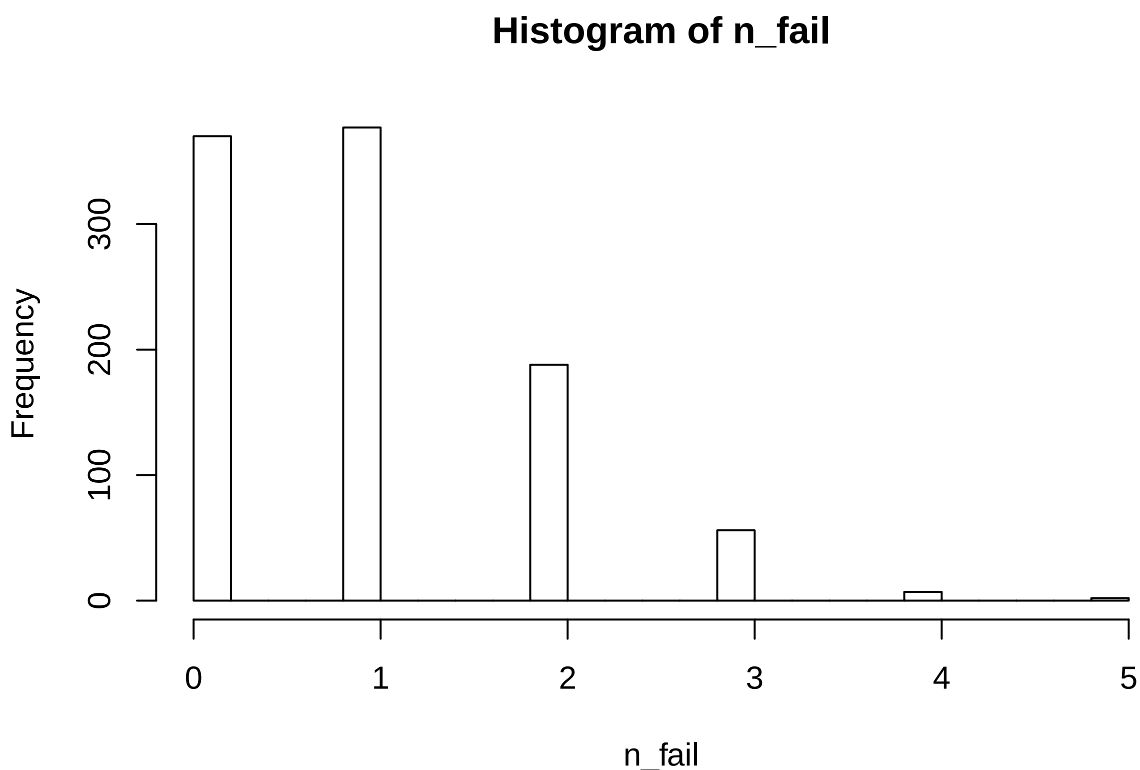
```
denom = choose(10, 2)
0.02*1/denom + 0.06*2*3/denom + 0.1*choose(3, 2)/denom + 0.11*2*5/denom +
  0.15*3*5/denom + 0.2*choose(5, 2)/denom
```

```
## [1] 0.134
```

11. Suppose that a financial services company creats a security instrument that combines 10 individual securities. Each individual security has a 1% chance of defaulting each year. For this excercise, please include your code.

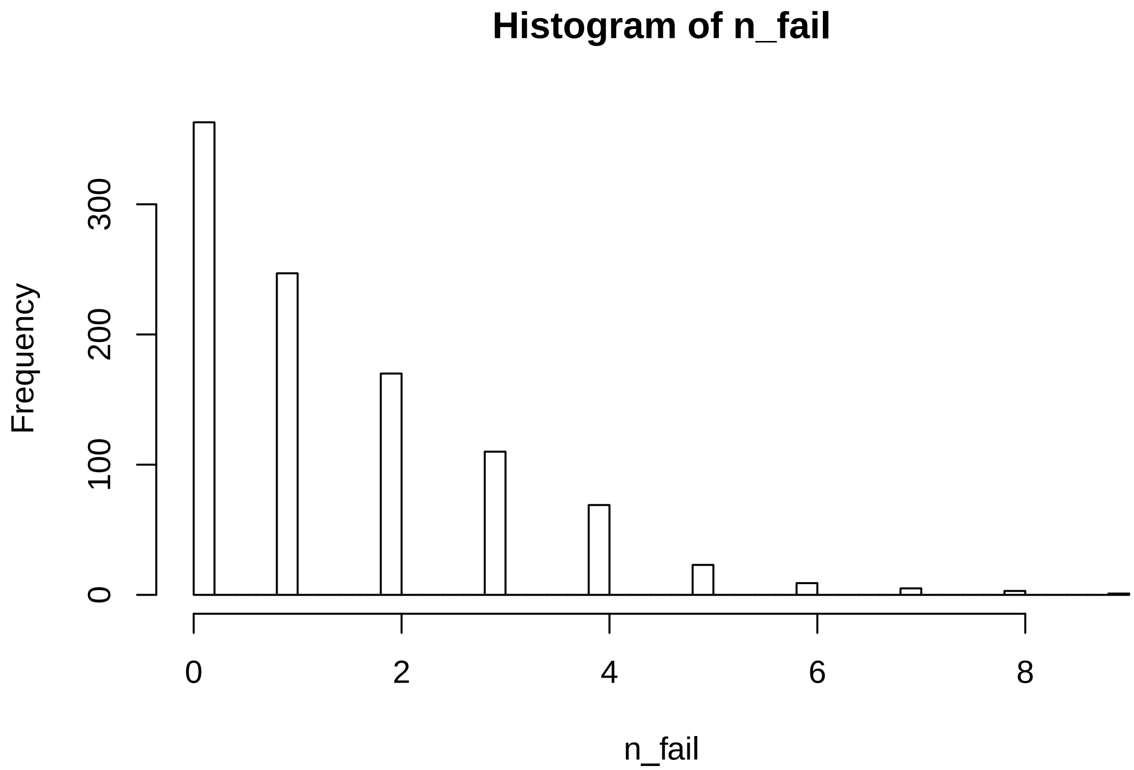
    a. Suppose that the securities are independent. Run 1000 simulations to estimate the expected number of securities that will default in a 10 year period. Plot the 1000 simulated number of defaults in a histogram (the command in R is hist).

```
n_sim = 1000
t = 10
n_fail = rep(0, times = n_sim)
for(i in 1:n_sim){
  n_sec = 10
  for(j in 1:t){
    n_sec = sum(rbinom(n_sec, 1, 0.99))
  }
  n_fail[i] = 10 - n_sec
}
hist(n_fail, breaks = 20)
```



b. Now suppose that a default in one year increases the chance that any of the remaining individual securities defaults by 1% (e.g. if one defaults the first year, the remaining 9 each have a $2 % chance of defaulting the next year). Estimate the expected number of securities that will default in a 10 year period and plot the 1000 simulated number of defaults in a histogram.

```
n_sim = 1000
t = 10
n_fail = rep(0, times = n_sim)
for(i in 1:n_sim){
  n_sec = 10
  for(j in 1:t){
    n_sec = sum(rbinom(n_sec, 1, 0.99 - 0.01*(10 - n_sec)))
  }
  n_fail[i] = 10 - n_sec
}
hist(n_fail, breaks = 40)
```


Histogram of n_fail
Frequency
0
100
200
300
0
2
4
6
8
n_fail