Model selection

Timothy Daley

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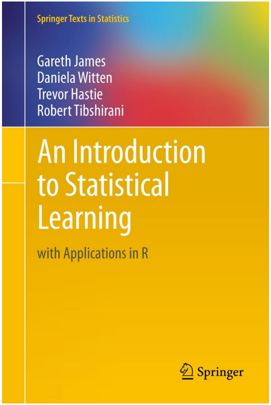
Book change for this week

This week we will be using the book *An Introduction to Statistical Learning with Applications in R*, available for free at [http://www-bcf.usc.edu/~gareth/ISL/]. The material for this week is covered in chapter 6.



Multiple regression: regress everything!

- Suppose we have a bunch of data and we just run a regression on everything.

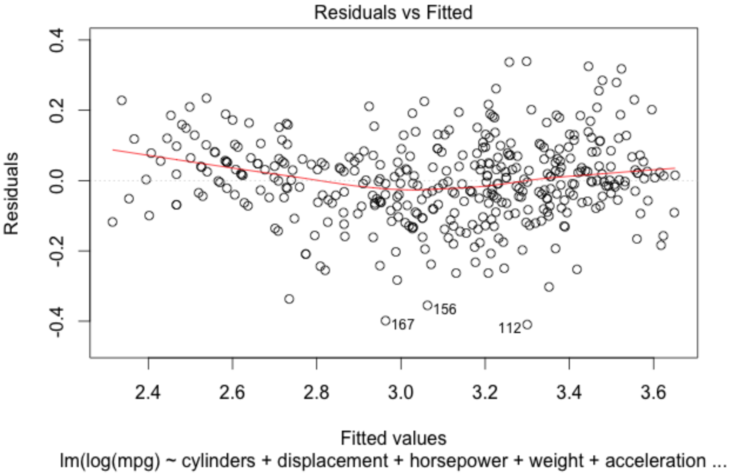
```
log_mpg_vs_all.lm = lm(log(mpg) ~ cylinders + displacement
                       + horsepower + weight
                       + acceleration + year
                       + origin, data=Auto)
```

Multiple regression: regress everything!

Table 1: Fitting linear model: log(mpg) ~ cylinders + displacement + horsepower + weight + acceleration + year + origin

	Estimate	Std. Error	t value	Pr(>|t|)
cylinders	-0.02795	0.01157	-2.415	0.01619
displacement	0.0006362	0.000269	2.365	0.01852
horsepower	-0.001475	0.0004935	-2.989	0.002984
weight	-0.0002551	2.334e-05	-10.93	2.12e-24
acceleration	-0.001348	0.003538	-0.381	0.7034
year	0.02958	0.001824	16.21	2.13e-45
origin	0.04071	0.009955	4.089	5.276e-05
(Intercept)	1.751	0.1662	10.53	5.845e-23
R^2	0.87951			

Multiple regression: regress everything!



How to choose which variables are important?

- Method 1: Successively remove variables that are not important.

<table>
<tr><th>Est</th><th>imate Std</th><th>. Error t v</th><th>alue Pr(</th><th>>|t|)</th></tr>
<tr><td>cylinders</td><td>-0.02795</td><td>0.01157</td><td>-2.415</td><td>0.01619*</td></tr>
<tr><td>displacement</td><td>0.0006362</td><td>0.000269</td><td>2.365</td><td>0.01852*</td></tr>
<tr><td>horsepower</td><td>-0.001475</td><td>0.0004935</td><td>-2.989</td><td>0.002984*</td></tr>
<tr><td>weight -0.0</td><td>002551 2.3</td><td>34e-05 -10</td><td>.93 2.1</td><td>2e-24*</td></tr>
<tr><td>acceleration</td><td>-0.001348</td><td>0.003538</td><td>-0.381</td><td>0.7034</td></tr>
<tr><td>year 0.0</td><td>2958 0.</td><td>001824 16</td><td>.21 2.1</td><td>3e-45*</td></tr>
<tr><td>origin 0.0</td><td>4071 0.</td><td>009955 4.</td><td>089 5.27</td><td>6e-05*</td></tr>
<tr><td>(Intercept)</td><td>1.751</td><td>0.1662</td><td>10.53</td><td>5.845e-23*</td></tr>
<tr><td>R^2 0.8</td><td>7951</td><td></td><td></td><td></td></tr>
</table>

Removing non-significant variables

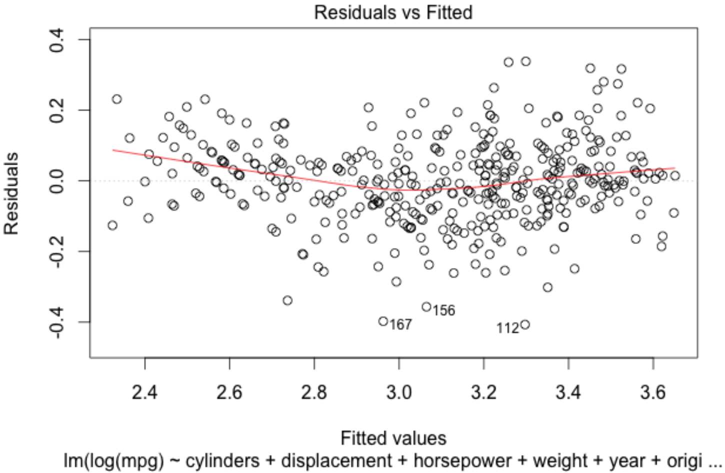
```
log_mpg_vs_sig.lm.1 = lm(log(mpg) ~ cylinders + displacemen
                         + horsepower + weight + year
                         + origin, data = Auto)
```

Removing non-significant variables

Table 3: Fitting linear model: log(mpg) ~ cylinders + displacement + horsepower + weight + year + origin

	Estimate	Std. Error	t value	Pr(>|t|)
cylinders	-0.02772	0.01154	-2.402	0.01679*
displacement	0.0006466	0.0002673	2.419	0.01601*
horsepower	-0.001359	0.0003876	-3.505	0.0005099*
weight	-0.0002594	2.044e-05	-12.69	4.433e-31*
year	0.02963	0.001817	16.31	7.782e-46*
origin	0.04067	0.009944	4.09	5.247e-05*
(Intercept)	1.723	0.1493	11.54	1.174e-26*
R^2	0.87946			

Removing non-significant variables



How to choose which variables are important?

- Method 2:
  - Start with nothing (intercept only)
  - Test adding variables one at a time
  - Add most significant
  - Repeat until no variables are significant

Adding significant variables

- Individual regression estimates:

	Estimate	Std. Error	t value	Pr(>
cylinders	-0.1648	0.005678	-29.03	1.675e-99
displacement	-0.00277	8.569e-05	-32.37	1.478e-112
horsepower	-0.00733	0.0002494	-29.41	5.393e-101
weight	-0.00035	9.79e-06	-35.81	2.392e-125*
acceleration	0.05516	0.005581	9.884	1.045e-20
year	0.05329	0.003817	13.96	3.28e-36
origin	0.2366	0.0177	13.37	8.27e-34

Adding significant variables

- Add weight, now the other added regression coefficients are:

	Estimate	Std. Error	t value	Pr(>
cylinders	-0.04193	2.182e-05	-12.6	8.793e-31
displacement	-0.00092	0.000216	-4.262	2.549e-05
horsepower	-0.00256	0.00041	-6.231	1.205e-09
acceleration	0.01232	0.003261	3.777	0.0001835
year	0.03129	0.00177	17.68	8.933e-52*
origin	0.03096	0.01265	2.448	0.01481

Adding significant variables

- Add weight and year, now the other added regression coefficients are:

	Estimate	Std. Error	t value	Pr(>
cylinders	-0.01872	0.00831	-2.253	0.02483
displacement	-0.00024	0.000169	-1.429	0.1537
horsepower	-0.00087	0.000335	-2.612	0.009355
acceleration	0.0043	0.00251	1.714	0.08735
origin	0.03081	0.009372	3.288	0.001103*

Adding significant variables

- Add weight, year, and origin, now the other added regression coefficients are:

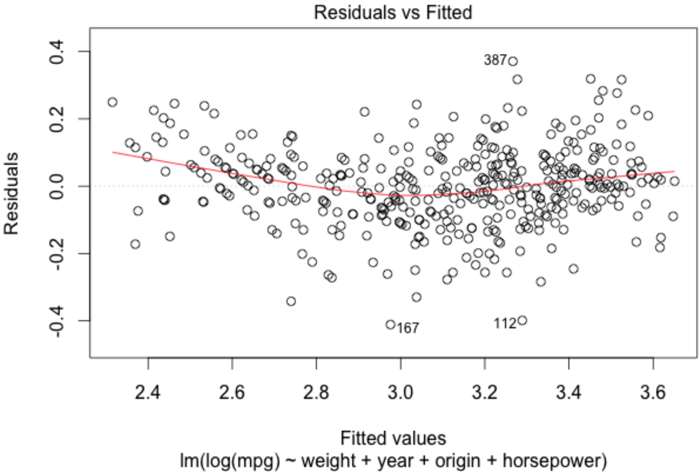
	Estimate	Std. Error	t value	Pr(>
cylinders	-0.01559	0.008287	-1.882	0.06064
displacement	-0.00012	0.000173	-0.6686	0.5042
horsepower	-0.00104	0.000332	-3.116	0.001968*
acceleration	0.00466	0.002479	1.881	0.06077

Adding significant variables

- Final model: weight, year, origin, and horsepower

	Estimate	Std. Error	t value	Pr($>$
weight	-0.00025	1.58e-05	-15.89	3.804e-44
year	0.0295	0.001821	16.2	1.941e-45
origin	0.03463	0.009349	3.704	0.0002434
horsepower	-0.001034	0.0003323	-3.116	0.001968
(Intercept)	1.658	0.148	11.2	1.989e-25

Adding significant variables



Adding vs subtracting variables

- Subtracting variables:
  - Final model:
  - log(mpg) $\sim$ cylinders + displacement + horsepower + weight + year + origin
- Adding variables:
  - Final model:
  - log(mpg) $\sim$ weight + year + origin + horsepower
- Which model is better?

Adjusted R^2

- R^2 always increases with the number of variables
- Penalize R^2 for the number of variables
- adjusted $R^2 = R^2 - (1 - R^2)\frac{p}{n-p-1}$
    - n is the number of observations (sample size)
    - p is the number of variables in regression

Mallow's C_p

- $C_p = \frac{1}{n}(RSS + 2p\hat{\sigma}^2)$
  - RSS = residual sum of squares = $\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$
  - $\hat{\sigma}^2$ is the estimated variance of the residuals
- Smaller C_p is better.

AIC: Akaike Information Criterion

- In general: $\mathrm{AIC} = 2p - 2\log L$
- For linear regression: $\mathrm{AIC} = 2p + n\log RSS$
- With more predictors, RSS gets smaller while p gets larger
- Smaller AIC is preferable
- C_p is a special case of AIC.

BIC: Bayesian Information Criterion

- In general: $\mathrm{BIC} = p \log n - 2 \log L$
- For linear regression: $\mathrm{BIC} = p \log n + n \log(RSS/n)$
- Like AIC but larger penalty on number of parameters
  - Tends to pick smaller models than AIC

Regularization methods

- In linear models the goal is to minimize the residual sum of squares

$$RSS = \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2$$

- What if we add a penalty term for the β_j's?
  - Don't include β_0 since we could just set $\beta_0 = \bar{y}$ and de-mean the y's

Ridge regression

- Ridge regression adds a penalty proportional to β_j^2:

$$\sum_{i=1}^{n}\left(y_i - \beta_0 - \sum_{j=1}^{p}\beta_j x_{ij}\right)^2 + \lambda\sum_{j=1}^{p}\beta_j^2$$

- This "shrinks" the β_j's towards 0
  - But not all the way.

LASSO (least absolute shrinkage and selection operator)

- Lasso adds a penalty proportional to $|\beta_j|$:

$$\sum_{i=1}^{n}\left(y_i - \beta_0 - \sum_{j=1}^{p}\beta_j x_{ij}\right)^2 + \lambda\sum_{j=1}^{p}|\beta_j|$$

- This "shrinks" the β_j's to be 0

How to choose λ?

- $\lambda = 0$ is equivalent to plain regression
- $\lambda = \infty$ will make all $\beta_j = 0$.
- Want $0 < \lambda < \infty$, but what is optimal?