# Probabilistic Reasoning and Decision Making

**INF581 - Lecture Notes**

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These lecture notes are intended to offer partial support to the lecture (not as a replacement). Attend the lecture. See also: the lecture slides, and references given below.

## 1. Agenda and Goals of this Lecture

- About the course (objectives, assessment, deadlines, modalities, teachers, ...)
- Discussion on *what is an autonomous agent*
- Context and main concepts of this course;
  - agents: mapping $f : \mathcal{X} \to \mathcal{A}$, and the pipeline/components within
  - rational agents (vs ...)
  - Bayesian Networks as a representation of knowledge (and an expression of uncertainty)
  - Reasoning (inference): evidence, queries, marginalizing out
  - environments: observations, states, actions, rewards
  - sources of uncertainty
  - decision making under uncertainty

## 2. Reading Material and References

A good reference for the material is David Barber's *Bayesian Reasoning and Machine Learning* [1]; especially, chapters/sections

- 1.1 Probability Refresher (if needed) (Or: Chapter 3 of *The Deep Learning Book* [2][1])
- 1.2 Probabilistic Reasoning
- 3.3 Belief Networks (i.e., Bayesian Networks)
- 7.1 on Expected Utility (i.e., Expected Reward)
- 7.2 Decision Trees (i.e., Probabilistic Trees)
- 7.3 Extending Bayesian Networks to Influence Diagrams

N.B. Due to the focus of the course on autonomous agents, we replace some terminology (in the slides) with that of reinforcement learning, utility = reward, expected utility = value, ....

[1] https://www.deeplearningbook.org/contents/prob.html

## 3. Introduction to the Course: What is an Autonomous Agent?

In this course we follow a thread through the large and active area that involves the study of *Autonomous Agents*. The initial component of *Advanced Machine Learning*, including learning representations and complex inference tasks, provides the necessary framework that we will build on later in the context of reinforcement learning. This first lecture answers the question of *what is as autonomous agent*, and reviews the main components involved in constructing one.

It's easy to be impressed by recent advances in AI as black box magic, but we should be aware out some of its fundamental components in isolation, its open challenges and its limitations.

## 4. Basic Notation for Probabilistic Reasoning

Random variable $X$, domain $\mathcal{X}$. Probability mass function

$$P(X = x)$$

for query $x \in \mathcal{X}$ (shorthand: $P(x)$). Distribution

$$P(X)$$

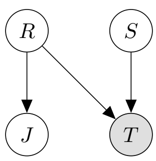
which can be represented by a table (sometimes denoted $\theta$), e.g., fair-coin flip:

| $x$ | $P(X = x)$ |
|---|---|
| heads | 0.5 |
| tails | 0.5 |

where, $\theta_x = P(X = x)$. If function $\ell$ is a function of $x$, then the expected value, $\mathbb{E}_X[\ell]$.

## 5. Inference on Tracey's Grass: Fully Worked Example

First we should recall the Bayesian Network (including probability tables) representing the knowledge required to solve the problem. Then we proceed to inference with a clear question, dividing variables into query, evidence, and latent variables.




| | |
|---|---|
| $P(R = 1) = 0.2$ | (sometimes it $R$ains) |
| $P(S = 1) = 0.1$ | (sometimes $S$prinkler is on) |
| $P(T = 1 \mid R = 1, S = 0) = 1$ | ($R$ain always wets grass) |
| $P(T = 1 \mid R = 0, S = 0) = 0$ | (only $R$ain/$S$prinkler cause wet) |
| $P(T = 1 \mid R = 0, S = 1) = 0.9$ | ($S$prinkler usually wets grass) |
| $P(T = 1 \mid R = 1, S = 1) = 1$ | (...and with rain, always) |

Let $\hat{s} = 1$ be the query (springer was left on overnight?). Let $\tilde{t} = 1$ be the evidence/observation (Tracey's grass is wet).

$$
\begin{aligned}
P(\hat{s}|\tilde{t}) &= \frac{P(\hat{s},\tilde{t})}{P(\tilde{t})} \quad \textcolor{teal}{\triangleright \text{ conditional probability}} \\
&= \frac{\sum_{j\in\{0,1\},r\in\{0,1\}} P(\tilde{t},j,r,\hat{s})}{\sum_{j\in\{0,1\},r\in\{0,1\},s\in\{0,1\}} P(\tilde{t},j,r,s)} \quad \textcolor{teal}{\triangleright \text{ marginalize out nuisance variables}} \\
&= \frac{\sum_{j\in\{0,1\},r\in\{0,1\}} P(j|r)P(\tilde{t}|r,\hat{s})P(r)P(\hat{s})}{\sum_{j\in\{0,1\},r\in\{0,1\},s\in\{0,1\}} P(j|r)P(\tilde{t}|r,s)P(r)P(s)} \quad \textcolor{teal}{\triangleright \text{ factorized according to net}} \\
&= \frac{P(\hat{s})\sum_{r\in\{0,1\}} P(\tilde{t}|r,s)P(r)}{\sum_{s\in\{0,1\}} P(s)\sum_{r\in\{0,1\}} P(\tilde{t}|r,s)P(r)} \quad \textcolor{teal}{\triangleright \sum_j P(j|r)=1\text{; push sums right}} \\
&= \frac{0.1\cdot(0.9\cdot 0.8+1\cdot 0.2)}{0.9\cdot 0.8\cdot 0.1+1\cdot 0.2\cdot 0.1+0\cdot 0.8\cdot 0.9+1\cdot 0.2\cdot 0.9} \\
&= 0.3382
\end{aligned}
$$

Remark: the $P(r)$ *does not* cancel, and nor does the $\sum_r$ disappear because of the $r$ 'trapped' inside the function, specifically inside $P(\tilde{t}|r,s)$. And neither does the $\sum_s P(s)$ for similar reasons. Note that, for example,

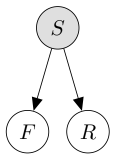
$$\sum_s P(s)\sum_r P(\tilde{t}|r,s)P(r) \neq \sum_s\sum_r P(\tilde{t}|r,s)P(r)$$

even though $\sum_s P(s) = 1$.

The concept of explaining away wrt Tracey's grass: if we learn it's been raining, that *explains away* (to some extent) the wet grass.

See also the 'Earthquake' example in [1].

## 6. Speeding Example




| $s=0$ | $s=1$ |
|---|---|
| 0.9 | 0.1 |

$P(S)$

| | $s=0$ | $s=1$ |
|---|---|---|
| $f=0$ | 1 | 0.1 |
| $f=1$ | 0 | 0.9 |

$P(F|S)$

| | $s=0$ | $s=1$ |
|---|---|---|
| $r=0$ | 0.91 | 0.86 |
| $r=1$ | 0.09 | 0.14 |

$P(R|S)$

The probability of getting a $Fine$ ($\hat{f} \Leftrightarrow F=1$), given that I was $S$peeding ($\tilde{s} \Leftrightarrow S=1$) (and possibly have a *R*ed car; $R \in \{0,1\}$):

$$
\begin{aligned}
P(F=1|S=1) &= \frac{P(\hat{f}|\tilde{s},R)}{P(\tilde{s},F,R)} && \triangleright \text{ conditional probability} \\
&= \frac{\sum_{r\in\{0,1\}} P(\hat{f}|\tilde{s})P(r|\tilde{s})P(\tilde{s})}{\sum_{r\in\{0,1\}}\sum_{f\in\{0,1\}} P(f|\tilde{s})P(r|\tilde{s})P(\tilde{s})} && \triangleright \text{ factorize joint, marginalize out} \\
&= \frac{P(\hat{f}|\tilde{s})P(\tilde{s})\sum_{r\in\{0,1\}} P(r|\tilde{s})}{\sum_{f\in\{0,1\}} P(f|\tilde{s})P(\tilde{s})\sum_{r\in\{0,1\}} P(r|\tilde{s})} && \triangleright \text{ push sum right} \\
&= \frac{P(\hat{f}|\tilde{s})P(\tilde{s})}{\sum_{f\in\{0,1\}} P(f|\tilde{s})P(\tilde{s})} && \triangleright \text{ (remove } \textstyle\sum = 1) \\
&= \frac{P(\hat{f},\tilde{s})}{P(\tilde{s})} \\
&= P(F=1|S=1)
\end{aligned}
$$

i.e., the type/color of the car is *not relevant*. We could have also determined this by visual inspection via $S=\tilde{s}=1$ being common evidence, and thus implying conditional independence.

The probability of getting a *F*ine ($\hat{f} \Leftrightarrow F=1$) and possibly *S*peeding, and possibly have a *R*ed car:

$$
\begin{aligned}
P(F=1) &= P(S,\hat{f},R) \\
&= \sum_{s\in\{0,1\}} P(\hat{f}|s)P(s) \sum_{r\in\{0,1\}} P(r|s) && \triangleright \text{ factorize, marginalize, push right}
\end{aligned}
$$

We can check what the difference is:

$$
\begin{aligned}
P(F=1|R=1) &= P(S,\hat{f},\tilde{r})/P(\tilde{r}) \\
&= \frac{\sum_{s\in\{0,1\}} P(\hat{f}|s)P(s)P(\tilde{r}|s)}{\sum_{s\in\{0,1\}} P(s)\sum_{f\in\{0,1\}} P(f|s)P(\tilde{r}|s)}
\end{aligned}
$$

So, what is the probability of getting a fine ($F=1$), *given that I drive a red car* ($R=1$), but not knowing if I have been speeding or not?

## 7. Umbrella Example

A rational agent takes optimal action (which maximizes value).

### 7.1. Part I: We know it will rain ($s=1$)

Just read straight from the $r$-table, column $s_1$ (Rain)):

$$Q(a_1) = r(s=1,a_1) = 10$$

and $Q(a_2) = r(s=1,a_2) = -40$.

i.e., take the umbrella!

### 7.2. Part II: We are uncertain about the state

But we have some knowledge/beliefs about it, namely $P(S = s_1) = 0.4$.

$$Q(a) = \mathbb{E}_S[r(S,a)] = \sum_s r(s,a)p(s)$$

We evaluate this:

$$Q(a_1) = \mathbb{E}_S[r(S,a_1)] = \sum_{s \in \mathcal{S}} r(s,a_1)P(s) = 10 \cdot 0.4 - 20 \cdot 0.6 = -8$$

$$Q(a_2) = \mathbb{E}_S[r(S,a_2)] = \sum_{s \in \mathcal{S}} r(s,a_2)P(s) = -40 \cdot 0.4 - 60 \cdot 0.6 = 20$$

So, leave the umbrella!

### 7.3. Part III: Observations of the state

We observe $x$ = red sky in the morning.

$$\begin{aligned}
Q(a) = \mathbb{E}_S[r(S,a)] &= \sum_s r(s,a)p(s) \\
&= \sum_s r(s,a)p(s \mid \boldsymbol{x}) \\
&= \sum_s r(a,s)p(s,x)/p(x) && \triangleright \text{ Bayes} \\
&= \sum_s r(a,s)p(x|s)p(s) && \triangleright \text{ from the graph (and } x \text{ constant)} \\
&= r(a,s_1)\underbrace{p(x|s_1)}_{0.6}\underbrace{p(s_1)}_{0.4} + r(a,s_2)\underbrace{p(x|s_2)}_{0.4}\underbrace{p(s_2)}_{0.6}
\end{aligned}$$

where $s_1 = \mathsf{Rain}$.

We observe $x$ = red sky at night.

$$Q(a) = r(a,s_1)\underbrace{p(x|s_1)}_{0.1}\underbrace{p(s_1)}_{0.4} + r(a,s_2)\underbrace{p(x|s_2)}_{0.9}\underbrace{p(s_2)}_{0.6}$$

# A. A Remark on Normalization / Marginalization

Suppose we want

$$P(Y|x)$$

We know that

$$P(Y|x) = p(Y,x)/p(x)$$

It could be that $p(x)$ is really difficult to calculate. But note that

$$P(x) = P(Y,x) = \sum_{y \in \mathcal{Y}} P(y,x) = Z$$

i.e., marginalize out $y$; which involves summing over every *possible* value of $y$ (i.e., where $P(y, \cdot) > 0$). If we are lucky as to be able to have that, then

$$P(Y|x) = \frac{1}{Z} p(Y, x)$$

$Z$ is, in practical terms here, a constant, though it is in fact a random variable in the sense that $Z(X)$ should work for any $X$.

## References


[1] David Barber. *Bayesian Reasoning and Machine Learning.* Cambridge University Press, 2012. http://web4.cs.ucl.ac.uk/staff/D.Barber/textbook/200620.pdf.

[2] Ian Goodfellow, Yoshua Bengio, and Aaron Courville. *Deep Learning.* The MIT Press, 2016. https://www.deeplearningbook.org.