ECO567A: Solution to Lecture 1 Exercise

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Note about this document. This document provides a fully detailed answer to the exercise that we covered in class (cf. the end of Lecture 1 slides). We try to have both a mathematical and a graphical approach. You may read this document and make sure you understand each step. Of course, it is *much* more detailed than the answers expected in the exam.

Exercise. Let $P(Q) = 305 - Q$ be the inverse demand for an industry, where P is the price in dollars and Q is the quantity. The marginal private cost is given by $MPC(Q) = 5+3Q$. Production generates a negative externality in the form of air pollution, the marginal external costs are given by $MEC(Q) = 2Q$.

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1 Question 1

Find the unregulated equilibrium (price and quantity) under perfect competition.

First, the inverse demand function takes a quantity as input and outputs the price at which consumers demand this quantity. It corresponds to the demand curve, which we plot in the (Q, P) graphical space. Second, we saw in the first lecture that the supply curve is confounded with the marginal private cost curve. So, we can plot demand and supply curves. In Figure 1, we show the supply curve in blue and the demand curve in red.

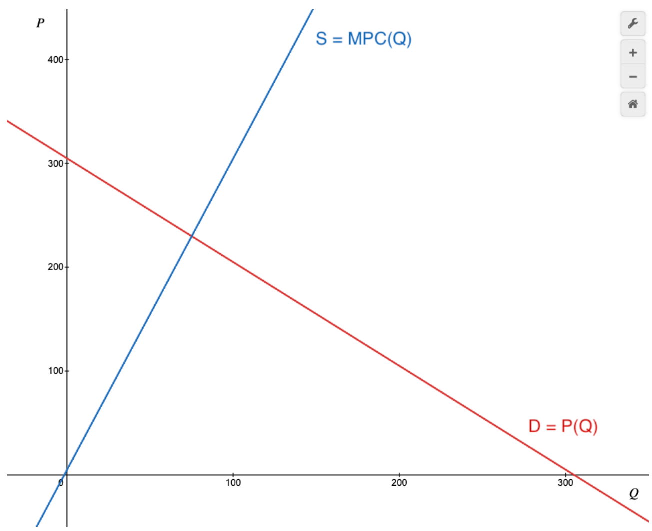


Figure 1: Basic environment

The equilibrium is obtained where the demand and supply curves meet. Graphically, this corresponds to the intersection of the two curves. Figure 2 highlights it.

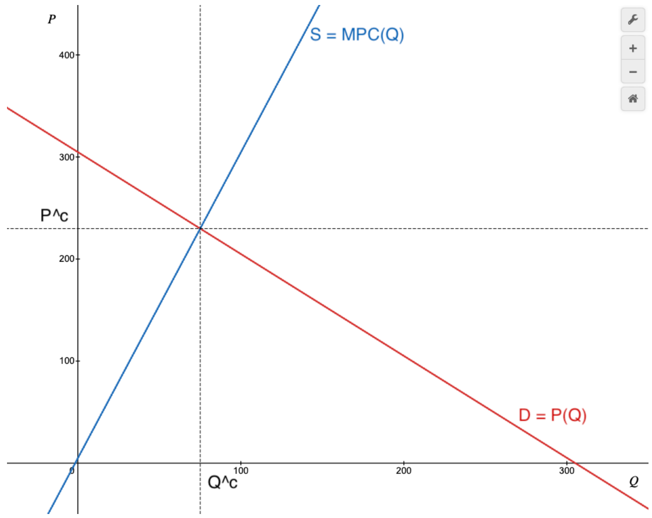


Figure 2: Initial equilibrium

Mathematically, we derive the resulting quantity Q^c by solving the equation:

$$\text{Demand curve} = \text{Supply curve} \iff P\left(Q^c\right) = MPC\left(Q^c\right)$$

Replacing with the expressions defined at the beginning of the exercise:

$$305 - Q^c = 5 + 3Q^c \iff 300 = 4Q^c \iff Q^c = 75$$

Now that we have the equilibrium quantity, we can deduce the equilibrium price by plugging it into the inverse demand function (or, indifferently, into the marginal private cost function):

$$P^c = P\left(Q^c\right) = 305 - Q^c = 305 - 75 = 230$$

We conclude that the initial equilibrium is $(Q^c, P^c) = (75, 230)$.

2 Question 2

Now assume there is only one firm (i.e., a monopolist). Find the monopolist's unregulated equilibrium.

We characterize the problem of the monopolist. It is useful to start from the perfect competition case. In perfect competition, the firm is small relatively to the rest of the economy and thus takes the price as given. From there, its problem consists in choosing the quantity to maximize profits. The firm's problem writes as:

$$\max_{Q} \left[\Pi \left(Q\right)\right] = \max_{Q} \left[PQ - C\left(Q\right)\right]$$

Where, in the eyes of the small firm, price P is a constant.

The monopolist faces a different problem because it can choose both the price and the quantity. But it cannot choose any pair of price and quantity.[1] When the monopolist sets a given price, it cannot sell a larger quantity than that demanded by consumers at this price. This imposes a constraint on the monopolist's problem. This problem writes as:

$$\max_{P,Q} \left[PQ - C\left(Q\right)\right] \text{ subject to } Q = D\left(P\right)$$

Where the function $D\left(.\right)$ corresponds to the demand function, that takes as input a price and outputs the quantity demanded by consumers at this price. We can alternatively write the constraint with the inverse demand function: $Q = D\left(P\right) \iff P\left(Q\right) = P$. We plug this directly into the monopolist's objective and the problem becomes:

$$\max_{Q} \left[P\left(Q\right)Q - C\left(Q\right)\right]$$

Where $R^m\left(Q\right) = P\left(Q\right)Q$ corresponds to the monopolist's revenue as a function of quantity. How can we solve this problem? We said in class that any profit-maximizing firm—be it in perfect competition, in a monopoly, etc.—chooses the quantity that verifies the following equality:

$$\text{Marginal revenue}\left(Q\right) = \text{Marginal cost}\left(Q\right) \tag{1}$$

The monopolist's marginal revenue function is given by the derivative of its revenue function with respect to quantity:

$$MR^m\left(Q\right) = \frac{\partial R^m\left(Q\right)}{\partial Q} = \frac{\partial\left[P\left(Q\right)Q\right]}{\partial Q} = \frac{\partial\left(305Q - Q^2\right)}{\partial Q} = 305 - 2Q$$

So, in the monopolist's case, our optimality condition of Equation 1 becomes:

$$MR^m\left(Q^m\right) = MPC\left(Q^m\right) \iff 305 - 2Q^m = 5 + 3Q^m \iff Q^m = 60$$

[1] To see this, imagine what happens if the monopolist sets an infinite price in the hope of getting infinite profits: Demand would then be null, and the monopolist would not be able to sell any quantity, thus making zero profits.

We deduce the price that the monopolist will impose by plugging the quantity Q^m into the inverse demand function:

$$P^m = P(Q^m) = 305 - Q^m = 245$$

So, under a monopoly, the equilibrium is $(Q^m, P^m) = (60, 245)$.

Graphically, we must add another line. We must plot the marginal revenue function of the monopolist. We add it in orange in Figure 3.

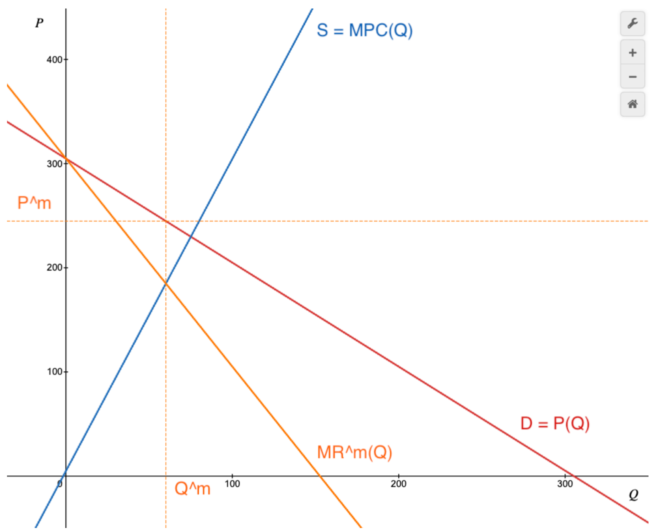


Figure 3: Equilibrium under monopoly

The monopolist chooses the quantity at which its marginal revenue curve intersects with the marginal private cost curve in blue, which is how we draw Q^m on the graph. We read the monopolist's price P^m by evaluating the inverse demand function (in red on the graph) at the monopolist's quantity Q^m.

3 Question 3

Find the socially optimal quantity and price.

Let us come back to the competitive equilibrium. We obtained the equilibrium quantity by solving the following equation:

$$P\left(Q^c\right) = MPC\left(Q^c\right)$$

Which equates the inverse demand function, that we can interpret as the marginal benefit to consumers, and the marginal private cost function. But this equation sort of ignores a part of the costs that society as a whole incurs with production. We only consider the marginal private cost for the firm. Instead, the marginal *social* cost would also include the marginal external cost of pollution:

$$MSC\left(Q\right) = MPC\left(Q\right) + MEC\left(Q\right) = 5 + 3Q + 2Q = 5 + 5Q$$

Visually, we should add the marginal social cost curve to our graph. To keep the graph light, we start from the very first one and add the marginal social cost curve in green (we forget about the monopoly case). This gives Figure 4. We highlight with the purple arrow the fact that the gap between the marginal social and private cost curves corresponds to the marginal external cost.

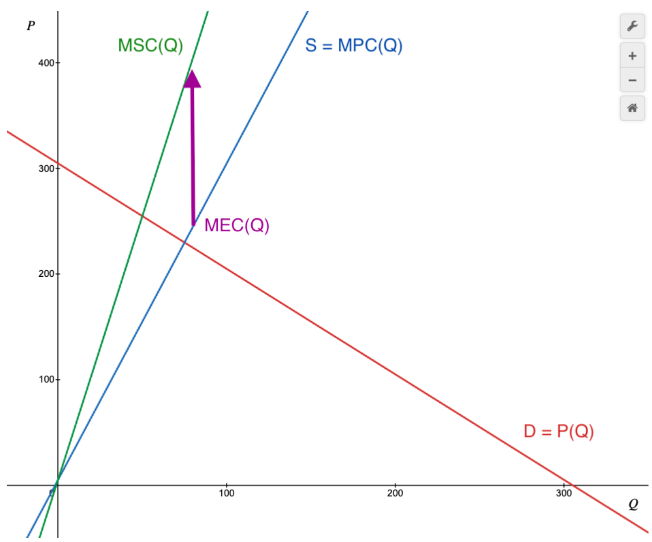


Figure 4: Adding the marginal social cost curve

Then, the socially optimal allocation is obtained as the intersection of the marginal social cost curve with the inverse demand function. This corresponds to the optimal allocation when we account for the cost of pollution. We highlight it in Figure 5 below:

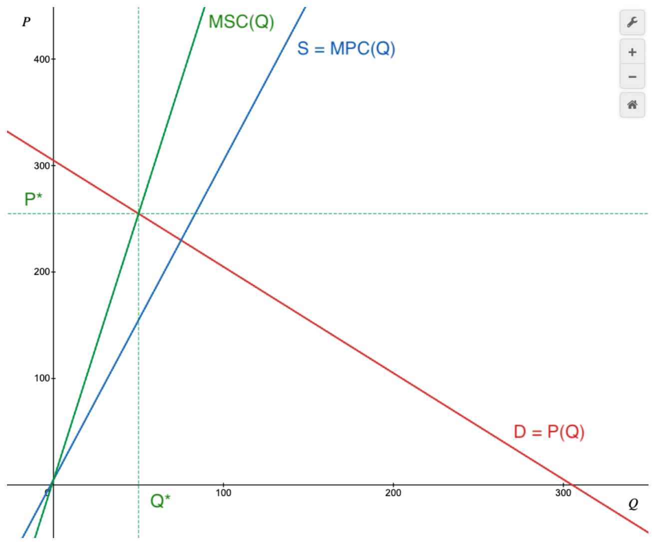


Figure 5: Socially optimal allocation

To determine the socially optimal quantity, we can thus equate the inverse demand function with the marginal social cost function. We solve the equation:

$$P(Q^*) = MSC(Q^*) \iff 305 - Q^* = 5 + 5Q^* \iff 6Q^* = 300 \iff Q^* = 50$$

And, as usual, we move from quantity to price via the inverse demand function. We plug the socially optimal quantity into the inverse demand function:

$$P^* = P(Q^*) = 305 - Q^* = 305 - 50 = 255$$

So, the socially optimal allocation is $(Q^*, P^*) = (50, 255)$.

4 Question 4

Find the deadweight loss under an unregulated monopoly and under perfect competition.

What is the deadweight loss? The socially optimal allocation that we have derived in Question 3 maximizes total surplus. For any other allocation that we consider (the one obtained in perfect competition, the monopoly equilibrium, etc.), the deadweight loss is the difference between the maximum total surplus obtained with the socially optimal allocation and the total surplus obtained with the allocation considered.

In the perfect competition case, we have:

$$DWL^c = TS^* - TS^c$$

In the monopoly case, we have:

$$DWL^m = TS^* - TS^m$$

Where TS^* denotes, in both cases, the maximum total surplus obtained with the socially optimal allocation of Question 3.

Now, what is total surplus? Consider an allocation with price P and quantity Q. Total surplus corresponds to the sum of the consumer surplus and the producer surplus, to which we subtract the cost of the pollution externality for society. This writes as:

$$TS = CS + PS - EC$$

We said in class that the consumer surplus is defined as the area below the demand curve and above the price line. Mathematically, it corresponds to the integral of the difference between the inverse demand function and the price over the interval of quantities traded and consumed:

$$CS = \int_0^Q \left[P(q) - P\right] dq$$

Similarly, we said that the producer surplus (equal to variable profits by the way) is the area above the supply curve and below the price line. Formally, we compute it as the integral of the difference between the price and the marginal private cost function over the interval of quantities produced and sold:

$$PS = \int_0^Q \left[P - MPC(q)\right] dq$$

Eventually, the external cost is obtained by integrating the marginal external cost over the interval of quantities traded:

$$EC = \int_0^Q MEC(q)\, dq$$

So, total surplus is given by:

$$\begin{aligned}TS &= \int_0^Q [P(q) - P]\,dq + \int_0^Q [P - MPC(q)]\,dq - \int_0^Q MEC(q)\,dq \\ &= \int_0^Q [P(q) - MPC(q) - MEC(q)]\,dq \\ &= \int_0^Q [P(q) - MSC(q)]\,dq\end{aligned}$$

To get the maximum total surplus, obtained with the socially optimal allocation, we can replace Q by Q^* in the equation above. Similarly, for the monopoly and the perfect competition case, we can replace Q by respectively Q^m and Q^c. This gives:

$$\begin{cases} TS^* &= \int_0^{Q^*} [P(q) - MSC(q)]\,dq \\ TS^c &= \int_0^{Q^c} [P(q) - MSC(q)]\,dq \\ TS^m &= \int_0^{Q^m} [P(q) - MSC(q)]\,dq \end{cases}$$

We deduce the deadweight loss in the monopoly case:

$$\begin{aligned}DWL^m &= TS^* - TS^m \\ &= \int_0^{Q^*} [P(q) - MSC(q)]\,dq - \int_0^{Q^m} [P(q) - MSC(q)]\,dq \\ &= \int_{Q^m}^{Q^*} [P(q) - MSC(q)]\,dq \\ &= \int_{Q^*}^{Q^m} [MSC(q) - P(q)]\,dq\end{aligned}$$

And, quite similarly, in the perfect competition case:

$$DWL^c = TS^* - TS^c = \int_{Q^*}^{Q^c} [MSC(q) - P(q)]\,dq$$

In both cases, the deadweight loss is computed as the integral of the difference between the marginal social cost function (how costly producing a given unit is to society) and the inverse demand function (how valuable this same unit is for consumers) over the interval of quantities traded *in excess of the socially optimal quantity.* This difference being positive reflects the idea that the production of each of these excess units costs more to society than the benefit that consumers draw from consuming it. We finalize the computations.

In the monopoly case:

$$
\begin{aligned}
DWL^m &= \int_{Q^*}^{Q^m} [MSC(q) - P(q)]\, dq \\
&= \int_{Q^*}^{Q^m} [5 + 5q - 305 + q]\, dq \\
&= -300(Q^m - Q^*) + 3\int_{Q^*}^{Q^m} 2q dq \\
&= -300(60 - 50) + 3\left[q^2\right]_{50}^{60} \\
&= -3000 + 3\left(60^2 - 50^2\right) \\
&= 300
\end{aligned}
$$

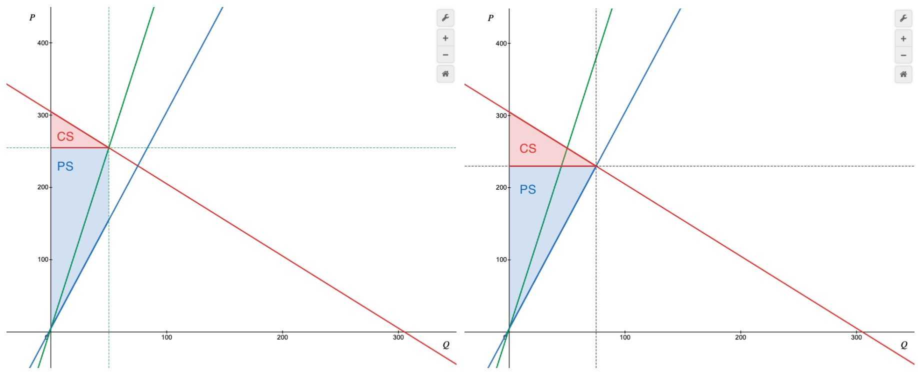
In the perfect competition case:

$$
\begin{aligned}
DWL^c &= \int_{Q^*}^{Q^c} [MSC(q) - P(q)]\, dq \\
&= \int_{Q^*}^{Q^c} [5 + 5q - 305 + q]\, dq \\
&= -300(Q^c - Q^*) + 3\int_{Q^*}^{Q^c} 2q dq \\
&= -300(75 - 50) + 3\left[q^2\right]_{50}^{75} \\
&= -300 \times 25 + 3\left(75^2 - 50^2\right) \\
&= 1875
\end{aligned}
$$

So, the deadweight loss is $DWL^m = 300$ with an unregulated monopoly, and $DWL^c = 1875$ under perfect competition.

Let us visualize the deadweight loss graphically. We focus on the perfect competition case. Do not hesitate to do the monopoly case on your own and to ask me if you have any question.

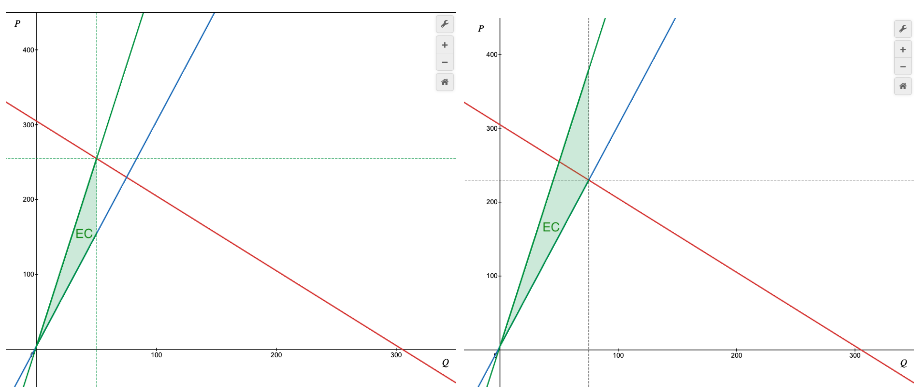
In Figure 6, we compare consumer and producer surplus with the social allocation (Panel 6a) and under perfect competition (Panel 6b). Clearly, the sum of both surplus is larger under perfect competition. With the socially optimal allocation, we “lose” the triangle formed by the demand and supply curves between Q^* and Q^c.



(a) Socially optimal allocation (b) Perfect competition equilibrium

Figure 6: Consumer and producer surplus

But total surplus also accounts for the cost of pollution! In Figure 7, we now compare the external cost with the social allocation (Panel 7a) and under perfect competition (Panel 7b). We see that the social allocation generates a much lower external cost than the perfect competition equilibrium.



(a) Socially optimal allocation (b) Perfect competition equilibrium

Figure 7: External cost

Actually, in the external cost saved by moving from the unregulated equilibrium to the socially optimal allocation, not only do we recover the triangle mentioned above, but we save another portion of external cost. We do end up with a larger total surplus with the socially optimal allocation, and the remaining green area is exactly the deadweight loss under perfect competition. We highlight it in Figure 8.

Figure 8: Deadweight loss

5 Question 5

Which market structure is closer to the social optimum: perfect competition or monopoly?

From all perspectives, the monopoly yields an allocation closer to the socially optimal one than perfect competition. With the monopoly, the quantity traded (60) is lower than under perfect competition (75) and closer the socially optimal one (50). The price (245) is higher than under perfect competition (230) and closer to the socially optimal one (255). Most importantly (as a direct consequence of quantity and price being closer to the social optimum), the deadweight loss is lower under the monopoly (300) than under perfect competition (1875).

We are saying that introducing a market distortion, the monopoly, brings the equilibrium closer to the socially optimal allocation than perfect competition. This may sound like a direct contradiction of the First Fundamental Theorem of Welfare Economics, which states that any allocation obtained as the equilibrium of a perfectly competitive market is socially optimal.[2]

In fact, the First Fundamental Theorem fails in the presence of an externality. Here, the unregulated equilibrium under perfect competition results in *overproduction* of the polluting good. $Q^c = 75$ largely exceeds the socially optimal quantity $Q^* = 50$ because agents neglect (more formally, "do not internalize") the cost of pollution to society. So, our initial equilibrium is not socially optimal.

A monopoly tends to produce a lower quantity, which allows it to sell at a higher price and make larger profits. This is true even beyond this specific exercise. As a consequence, the monopoly mitigates the overproduction issue under perfect competition. Here, it offsets overproduction only partially so that we end up in-between the socially optimal allocation and the perfectly competitive equilibrium.

All in all, the monopoly is the market structure closer to the social optimum.

[2] You may think about this theorem as Adam Smith's "invisible hand".

6 Question 6

Find the optimal tax or subsidy that the government needs to impose to make the monopolist produce on the social optimum.

Denote this tax or subsidy by τ. It is a tax if $\tau > 0$ and a subsidy if $\tau < 0$. For each unit that it produces, the monopolist will pay or receive τ dollars. τ enters the marginal private cost. A tax increases the marginal cost of production, a subsidy reduces it. We have:

$$MPC^{\text{Regulated}}(Q, \tau) = MPC(Q) + \tau$$

Given a tax or subsidy τ, what quantity does the monopolist choose to produce? As before, the monopolist equates the marginal cost of production with the marginal revenue. So, it chooses a quantity $Q^m(\tau)$ that depends on the tax or subsidy and satisfies:

$$MR^m[Q^m(\tau)] = MPC^{\text{Regulated}}[Q^m(\tau), \tau]$$

We solve for $Q^m(\tau)$:

$$\begin{aligned} & MR^m[Q^m(\tau)] = MPC[Q^m(\tau)] + \tau \\ \Longleftrightarrow\ & 305 - 2Q^m(\tau) = 5 + 3Q^m(\tau) + \tau \\ \Longleftrightarrow\ & 5Q^m(\tau) = 300 - \tau \\ \Longleftrightarrow\ & Q^m(\tau) = 60 - \frac{\tau}{5} \end{aligned}$$

So, given a tax or subsidy τ, the monopolist produces $Q^m(\tau) = 60 - \frac{\tau}{5}$. Knowing this, the regulator chooses the tax or subsidy so that the monopolist produces the socially optimal quantity. We denote it by τ^m below:

$$Q^m(\tau^m) = Q^*$$

We solve for τ^m:

$$Q^m(\tau^m) = Q^* \iff 60 - \frac{\tau^m}{5} = 50 \iff \tau^m = 50$$

So, the tax or subsidy that the government needs to impose to make the monopolist produce on the social optimum is $\tau^m = 50$.

Let us postpone the graphical representation to the perfect competition case in the next question, as the graph is a bit more readable. Again, do not hesitate to do the monopoly case on your own and to ask me if you have any question.

7 Question 7

Find the optimal tax or subsidy that the government needs to impose to make the monopolist produce on the social optimum.

Under perfect competition, in Question 1, we used the following condition to derive the equilibrium quantity being traded:

$$P\left(Q^c\right) = MPC\left(Q^c\right)$$

Given a tax or subsidy τ, this condition becomes:

$$P\left[Q^c\left(\tau\right)\right] = MPC^{\text{Regulated}}\left[Q^m\left(\tau\right), \tau\right]$$

So, we can solve for the quantity traded under perfect competition given a tax or subsidy τ, $Q^c\left(\tau\right)$:

$$\begin{aligned} & P\left[Q^c\left(\tau\right)\right] = MPC\left[Q^c\left(\tau\right)\right] + \tau \\ \iff & 305 - Q^c\left(\tau\right) = 5 + 3Q^c\left(\tau\right) + \tau \\ \iff & 4Q^c\left(\tau\right) = 300 - \tau \\ \iff & Q^c\left(\tau\right) = 75 - \frac{\tau}{4} \end{aligned}$$

So, under perfect competition and given a tax or subsidy τ, a quantity $Q^c\left(\tau\right) = 75 - \frac{\tau}{4}$. Knowing this, the regulator chooses the tax or subsidy to match this quantity with the socially optimal one. We denote it by τ^c below:

$$Q^c\left(\tau^c\right) = Q^*$$

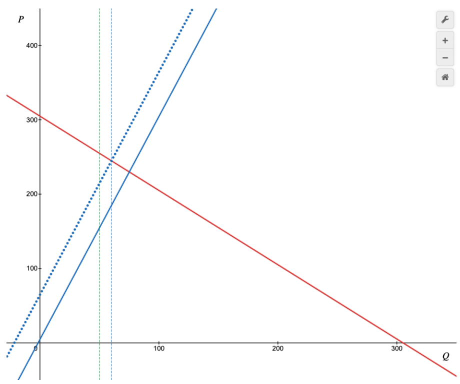
We solve for τ^c:

$$Q^c\left(\tau^c\right) = Q^* \iff 75 - \frac{\tau^c}{4} = 50 \iff \tau^c = 100$$

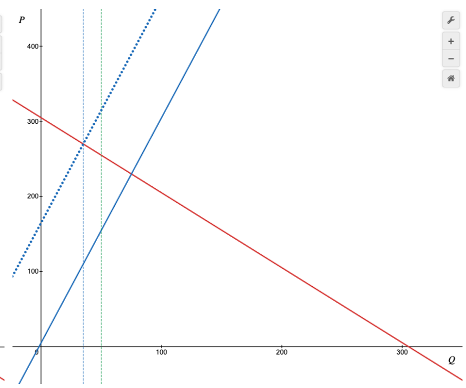
So, the tax or subsidy that the government needs to impose under perfect competition to reach the socially optimal allocation is $\tau^c = 100$. Considering our answer to Question 5, it is not surprising that the tax on the monopolist is lower than the one under perfect competition.

Graphically, the tax or subsidy τ shifts the supply curve (which represents the marginal private cost function) in a parallel way. In the case of a tax, the curve is shifted to the left; in the case of a subsidy, the curve is shifted to the right. Because the equilibrium quantity is determined at the intersection of the demand and supply curves, the goal of the government is to find a tax or subsidy such that this intersection coincides with the socially optimal allocation.

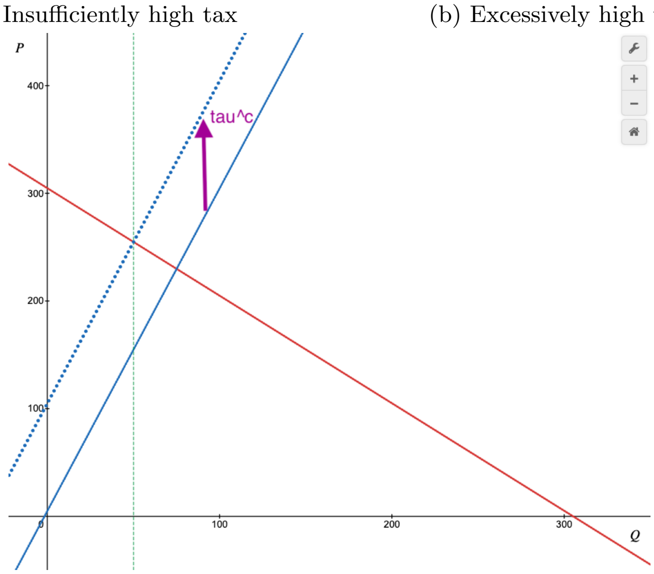
Figure 9 shows different scenarios. In Panel 9a, the tax is insufficient. We shift the supply curve to the left and end up with a lower quantity than in the unregulated equilibrium, but we do not go far enough to the left to reach the socially optimal quantity. In Panel 9b, the tax is excessive on the contrary. We shift the supply curve too far to the left, and production is lower than the social optimum. Eventually, in Panel 9c, the tax is optimal and we match the socially optimal quantity. As highlighted in the graph, we can read the optimal tax τ^c as the gap between the initial and the shifted supply curves.



(a) Insufficiently high tax



(b) Excessively high tax



(c) Optimal tax

Figure 9: Different taxes under perfect competition