ECO567a – PC # 3 – Roback Example

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Contents

1 Assumptions

We add two assumptions with respect to the lecture notes.

On the one hand, we specify firms' production function:

$$F(N_p, L_p; z) = z^{\theta} N_p^{\beta} L_p^{1-\beta}$$

Where N_p stands for the quantity of labor employed by the firm, L_p for the quantity of land rented by the firm, and z for pollution. θ corresponds to the elasticity of the production function with respect to pollution. β eventually denotes the share of labor in the production process.

This production function is known as the "Cobb-Douglas" production function. We make three remarks about this specification. First, the z^{θ} term is called total factor productivity: When it increases, a given increment in the quantity of any of the two factors has a larger effect on total production. Second, the function is homogeneous of degree 1. This means that scaling both inputs by a scalar s multiplies total production by s. In economics, we say that we assume constant returns to scale (doubling all inputs doubles output for any level of production). Third, this functional form implicitly pins down the elasticity of substitution between labor and land to 1. The elasticity of substitution roughly describes how easily the firm can replace land with labor and vice-versa in the production process.

On the other hand, we specify households' utility function and we again assume a Cobb-Douglas form:

$$U(x, L_c; z) = z^{-\gamma} x^{\alpha} L_c^{1-\alpha}$$

Where x stands for the quantity of final good consumed by the household, L_c for the quantity of land rented by the household, and z for pollution. $(-\gamma)$ corresponds to the elasticity of the utility function with respect to pollution. α eventually denotes the share of the final good in utility.

2 Firms

2.1 Stating the Problem

Given a quantity x of the traded good that they must produce, firms choose a quantity of labor to employ (N_p) and a quantity of land to rent (L_p) so as to minimize total costs. Formally, this problem writes as:

$$\min_{N_p, L_p} \{wN_p + rL_p\} \text{ subject to } F(N_p, L_p; z) \geq x$$

2.2 First Approach: Substituting the Constraint

We develop a first approach to solving this problem. We notice that, at the firm's optimum, the production constraint must be binding. Indeed, if for a given choice of labor and land, production is strictly larger than x, then the firm can reduce one of the two inputs to equate back production with x, and this will result in lower costs. Denoting the firm's optimal choice by (N_p^*, L_p^*), we thus have:

$$F(N_p^*, L_p^*; z) = x \iff z^\theta \left(N_p^*\right)^\beta \left(L_p^*\right)^{1-\beta} = x \iff N_p^* = \left[x \left(L_p^*\right)^{\beta-1} z^{-\theta}\right]^{\frac{1}{\beta}}$$

We can then (i) plug this expression for the quantity of labor as a function of the quantity of land into the firm's objective, (ii) solve for the optimal quantity of land without any constraint, and (iii) deduce the optimal quantity of labor. Said otherwise, the firm's cost minimization problem comes down to:

$$\min_{L_p}\{w \left[x \left(L_p\right)^{\beta-1} z^{-\theta}\right]^{\frac{1}{\beta}} + rL_p\}$$

This alternative framing of the version directly incorporates the binding production constraint. We differentiate the objective with respect to the quantity of land rented by the firm:

$$\frac{\partial \left(w \left[x \left(L_p\right)^{\beta-1} z^{-\theta}\right]^{\frac{1}{\beta}} + rL_p\right)}{\partial L_p} = \left(1 - \frac{1}{\beta}\right) w \left(xz^{-\theta}\right)^{\frac{1}{\beta}} L_p^{-\frac{1}{\beta}} + r$$

The derivative is positive if and only if:

$$\left(1 - \frac{1}{\beta}\right) w \left(xz^{-\theta}\right)^{\frac{1}{\beta}} L_p^{-\frac{1}{\beta}} + r > 0 \iff L_p > \left[\frac{r}{\left(\frac{1}{\beta} - 1\right) w \left(xz^{-\theta}\right)^{\frac{1}{\beta}}}\right]^{-\beta}$$

Besides, the derivative is strictly increasing in the quantity of land, which ensures the convexity of the objective function that we want to minimize. Because the objective function is convex, decreasing up to the quantity of land defined at the far right of the expression above, and increasing afterwards, we know that a unique global minimum is reached at that point. We deduce that:

$$L_p^* = \left[\frac{r}{\left(\frac{1}{\beta} - 1\right) w \left(xz^{-\theta}\right)^{\frac{1}{\beta}}}\right]^{-\beta} = xz^{-\theta} \left(\frac{1-\beta}{\beta}\right)^\beta \left(\frac{r}{w}\right)^{-\beta}$$

The optimal quantity of land is therefore increasing in the quantity to produce, x; increasing in the weight of land relative to labor in the production process, $\frac{1-\beta}{\beta}$; decreasing in the price of land relative to labor, $\frac{r}{w}$. We can also compute the optimal quantity of labor with the relation derived previously:

$$\begin{aligned}
N_p^* &= \left[x \left(L_p^*\right)^{\beta-1} z^{-\theta}\right]^{\frac{1}{\beta}} \\
&= \left(x \left[x^{\beta-1} z^{-\theta(\beta-1)} \left(\frac{1-\beta}{\beta}\right)^{\beta(\beta-1)} \left(\frac{r}{w}\right)^{-\beta(\beta-1)}\right] z^{-\theta}\right)^{\frac{1}{\beta}} \\
&= xz^{-\theta} \left(\frac{\beta}{1-\beta}\right)^{1-\beta} \left(\frac{w}{r}\right)^{\beta-1}
\end{aligned}$$

Furthermore, we can compute the firm's total cost function, which maps the quantity of goods x into the cost of the inputs required to produce it:

$$\begin{aligned}
c(x,w,r;z) &= wN_p^* + rL_p^* \\
&= wxz^{-\theta}\left(\frac{\beta}{1-\beta}\right)^{1-\beta}\left(\frac{w}{r}\right)^{\beta-1} + rxz^{-\theta}\left(\frac{1-\beta}{\beta}\right)^{\beta}\left(\frac{r}{w}\right)^{-\beta} \\
&= xz^{-\theta}\left(\frac{1-\beta}{\beta}\right)^{\beta}\left(\frac{r}{w}\right)^{-\beta}\left[w\frac{\beta}{1-\beta}\left(\frac{w}{r}\right)^{-1} + r\right] \\
&= xz^{-\theta}\left(\frac{1-\beta}{\beta}\right)^{\beta}\left(\frac{r}{w}\right)^{-\beta}\left(r\frac{\beta}{1-\beta} + r\right) \\
&= xz^{-\theta}r^{1-\beta}w^{\beta}\underbrace{\beta^{-\beta}\left(1-\beta\right)^{\beta-1}}_{\equiv k_2}
\end{aligned}$$

This defines a constant unit and marginal cost of production:

$$c^1(w,r;z) = c(1,w,r;z) = z^{-\theta}k_2 r^{1-\beta}w^{\beta}$$

Besides, under perfect competition as assumed here, the price and the marginal cost of production are equal at the firm's optimum. Having normalised the price of the traded good at 1, we know that $c^1(w,r;z) = 1$.

Assume that $\theta > 0$. In this case, pollution is "productive" in the sense that a higher pollution allows to produce more with the same inputs. Then, holding everything else constant, higher pollution in a location reduces the unit cost of production there. This incentivises firms to move to this location, raises the demand for labor and land, and generates upward pressures on the wage rate and on the rental rate for land. w and r increase up to the point where the marginal cost of production $c^1(w,r;z)$ is again equal to 1.

2.3 Second Approach: Lagrangian

One can also solve the firm's cost minimization problem with the method of the Lagrangian. Denoting the Lagrange multiplier associated with the production constraint by λ, the Lagrangian writes as:

$$\mathcal{L}(N_p,L_p,\lambda) = wN_p + rL_p - \lambda\left[F(N_p,L_p;z) - x\right] = wN_p + rL_p - \lambda\left[z^{\theta}N_p^{\beta}L_p^{1-\beta} - x\right]$$

The first-order conditions with respect to labor and land write as:

$$\begin{cases} w = \lambda\beta z^{\theta}\left(N_p^*\right)^{\beta-1}\left(L_p^*\right)^{1-\beta} \\ r = \lambda\left(1-\beta\right)z^{\theta}\left(N_p^*\right)^{\beta}\left(L_p^*\right)^{-\beta} \end{cases}$$

And the complementary slackness condition states that either the Lagrange multiplier is null or the production constraint is binding at optimum:

$$\lambda\left[F(N_p^*,L_p^*;z) - x\right] = 0 \iff \lambda\left[z^{\theta}\left(N_p^*\right)^{\beta}\left(L_p^*\right)^{1-\beta} - x\right] = 0$$

In the first-order conditions, all terms but λ are known to be (strictly) positive. So, for the equalities to hold, the Lagrange multiplier must be positive. The complementary slackness condition then implies that the production constraint is binding at optimum.

Additionally, by taking the ratio of both first-order conditions, we have:

$$\frac{w}{r} = \frac{\beta}{1-\beta} \frac{L_p^*}{N_p^*} \iff N_p^* = \frac{r\beta}{w\left(1-\beta\right)} L_p^*$$

Let us plug this into the binding production constraint. We can deduce the optimal quantity of land rented by the firm:

$$\begin{aligned} & z^{\theta} \left(N_p^*\right)^{\beta} \left(L_p^*\right)^{1-\beta} = x \\ \iff & z^{\theta} \left[\frac{r\beta}{w\left(1-\beta\right)} L_p^*\right]^{\beta} \left(L_p^*\right)^{1-\beta} = x \\ \iff & L_p^* = x z^{-\theta} \left(\frac{1-\beta}{\beta}\right)^{\beta} \left(\frac{r}{w}\right)^{-\beta} \end{aligned}$$

We can now deduce the optimal quantity of labor employed by the firm:

$$N_p^* = \frac{r\beta}{w\left(1-\beta\right)} L_p^* = x z^{-\theta} \left(\frac{\beta}{1-\beta}\right)^{1-\beta} \left(\frac{w}{r}\right)^{\beta-1}$$

We retrieve the solution obtained with the previous method, which consisted in substituting the binding production constraint into the firm's objective. The rest of the discussion follows.

3 Workers

3.1 Stating the Problem

The worker' problem consists in choosing the quantity of the traded good that she will consume and the quantity of land that she will rent in order to maximize utility, subject to the budget constraint. Formally, this problem writes as:

$$\max_{x, L_c} \{U\left(x, L_c; z\right)\} = \max_{x, L_c} \{z^{-\gamma} x^{\alpha} L_c^{1-\alpha}\} \text{ subject to } x + rL_c \leq w$$

The budget constraint states that the consumer's expenditures on the consumption good and land cannot exceed her labor income. Expenditures on the consumption good are given by the quantity times the price of the consumption good, which we have normalized to 1; expenditures on land are equal to the quantity of land demanded times the rental rate; labor income is given by the quantity of labor supplied by the worker, which we also assume is fixed to 1, times the wage rate.

3.2 First Approach: Substituting the Constraint

Just like for firms, we develop a first approach to solving this problem. We notice that, at the worker's optimum, the budget constraint must be binding. Indeed, if for a given consumption bundle, some labor income is left, then the worker can buy more of the consumption good to equate back expenditures with w, and this will result in a higher utility. Denoting the worker's optimal choice by (x^*, L_c^*), we thus have:

$$x^* + rL_c^* = w \iff x^* = w - rL_c^*$$

We can then (i) plug this expression for the quantity of consumption good as a function of the quantity of land into the worker's objective, (ii) solve for the optimal quantity of land without any constraint, and (iii) deduce the optimal quantity of consumption good. Said otherwise, the worker's utility maximization problem comes down to:

$$\max_{L_p}\{z^{-\gamma}\left(w - rL_c\right)^{\alpha} L_c^{1-\alpha}\}$$

This alternative framing of the version directly incorporates the binding budget constraint. Besides the term $z^{-\gamma}$ is positive, the following problem will yield the same optimal quantity of land:

$$\max_{L_p}\{\left(w - rL_c\right)^{\alpha} L_c^{1-\alpha}\}$$

We differentiate the objective with respect to the quantity of land demanded by the household:

$$-r\alpha\left(w - rL_c\right)^{\alpha-1} L_c^{1-\alpha} + \left(1-\alpha\right)\left(w - rL_c\right)^{\alpha} L_c^{-\alpha}$$

The derivative is positive, meaning that the objective is increasing in the quantity of land, if and only if:

$$-r\alpha L_c + \left(1-\alpha\right)\left(w - rL_c\right) > 0 \iff -rL_c + \left(1-\alpha\right)w > 0 \iff L_c < \frac{w\left(1-\alpha\right)}{r}$$

We compute the second-order derivative of the objective function:

$$\begin{aligned} &r^2\alpha\left(\alpha-1\right)\left(w - rL_c\right)^{\alpha-2} L_c^{1-\alpha} - r\alpha\left(1-\alpha\right)\left(w - rL_c\right)^{\alpha-1} L_c^{-\alpha} \\ &-r\alpha\left(1-\alpha\right)\left(w - rL_c\right)^{\alpha-1} L_c^{-\alpha} - \alpha\left(1-\alpha\right)\left(w - rL_c\right)^{\alpha} L_c^{-\alpha-1} \end{aligned}$$

This expression is unambiguously negative. This ensures the concavity of the objective function that we want to maximize. Because the objective function is concave, increasing up to the quantity of land defined above, and decreasing afterwards, we know that a unique global maximum is reached at that point. We deduce that:

$$L_c^* = \frac{w\left(1-\alpha\right)}{r}$$

The optimal quantity of land demanded by households is therefore increasing in labor income, w; increasing in the weight of land relative to the final consumption good in utility, $(1-\alpha\ )$; decreasing in the price of land, r. We can also compute the optimal quantity of consumption good with the relation derived previously:

$$x^* = w - rL_c^* \iff x^* = \alpha w$$

We further notice that, with a Cobb-Douglas function, an optimizing household dedicates a constant share of its income to each component of her utility. Indeed, the optimal expenditures on land rL^* are equal to a share $(1-\alpha\ )$ of labor income w, and the expenditures on the consumption good x^* (keep in mind that the price is normalized to 1) account for a share α of income.

3.3 Second Approach: Lagrangian

One can also solve the household's utility maximization problem with the method of the Lagrangian. Denoting the Lagrange multiplier associated with the budget constraint by μ, the Lagrangian writes as:

$$\mathcal{L}(x, L_c, \mu) = U\left(x, L_c; z\right) - \mu\left(x + rL_c - w\right) = z^{-\gamma} x^{\alpha} L_c^{1-\alpha} - \mu\left(x + rL_c - w\right)$$

The first-order conditions with respect to the quantity of consumption good and the quantity of land write as:

$$\begin{cases} \alpha z^{-\gamma}\left(x^*\right)^{\alpha-1}\left(L_c^*\right)^{1-\alpha} = \mu \\ \left(1-\alpha\right) z^{-\gamma}\left(x^*\right)^{\alpha}\left(L_c^*\right)^{-\alpha} = \mu r \end{cases}$$

And the complementary slackness condition states that either the Lagrange multiplier is null or the budget constraint is binding at optimum:

$$\mu\left(x^* + rL_c^* - w\right) = 0$$

In the first-order conditions, all terms but μ are known to be (strictly) positive. So, for the equalities to hold, the Lagrange multiplier must be positive. The complementary slackness condition then implies that the budget constraint is binding at optimum.

Additionally, by taking the ratio of both first-order conditions, we have:

$$\frac{1}{r} = \frac{\alpha}{1-\alpha}\frac{1}{x^*}L_c^* \iff L_c^* = \frac{1-\alpha}{\alpha}\frac{x^*}{r}$$

Let us plug this into the binding budget constraint. We can deduce the optimal quantity of consumption good demanded by the household:

$$\begin{aligned} & x^* + rL_c^* = w \\ \iff & x^* + \frac{1-\alpha}{\alpha}x^* = w \\ \iff & \frac{1}{\alpha}x^* = w \\ \iff & x^* = \alpha w \end{aligned}$$

Then, the optimal demand for land writes as:

$$L_c^* = \frac{(1-\alpha)\,w}{r}$$

We retrieve the solution obtained with the previous method, which consisted in substituting the binding budget constraint into the household's objective. The rest of the discussion follows.

3.4 Indirect Utility Function

Now that we have derived the optimal quantities of final consumption good and land demanded by households, we can compute their indirect utility function. This function, which we denote by V, takes as input the parameters of households' program (i.e., the wage rate and the rental rate for land) and returns the utility derived from optimal choices. Following this definition, we have:

$$V\left(w,r;z\right) = U\left(x^*, L_c^*; z\right) = z^{-\gamma}\left(\alpha w\right)^{\alpha}\left[\frac{(1-\alpha)\,w}{r}\right]^{1-\alpha} = z^{-\gamma} w r^{\alpha-1} \underbrace{\left[\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}\right]}_{\equiv k_1}$$

So, the indirect utility function is defined by:

$$V\left(w,r;z\right) = z^{-\gamma} w r^{\alpha-1} k_1$$

In a spatial equilibrium where migration is frictionless, we know that indirect utility must be equal across locations. Indeed, if two places i and j had different indirect utilities, with $V_i < V_j$, then people would move from i to j. Out-migration from i would create an upward pressure on the wage rate (via reduced labor supply) and a downward pressure on rents (via reduced demand for land), which would raise indirect utility in i up to the point where it is equal to that in location j and migration stops. So, in each location, we can equate indirect utility with a constant, k_0:

$$V\left(w,r;z\right) = k_0$$

What happens when pollution increases in a location? Holding everything else constant, this tends to reduce indirect utility and thus motivates out-migration. Out-migration creates an upward pressure on the wage rate (via reduced labor supply) and a downward pressure on rents (via reduced demand for land). These countervailing effects bring indirect utility back to k_0.

4 Equilibrium

We now characterize the general equilibrium of this economy. We have two equilibrium conditions. On firms' side, the optimal choice of inputs can be summarized by the unit cost function and the marginal cost of production is equal to the price of the final good. On households' side, the optimal quantities of consumption good and land are summarized in the indirect utility function and the spatial equilibrium condition equates the indirect utility function with a constant, k_0, across all locations. We end up with the following system:

$$\begin{cases} z^{-\theta} k_2 r^{1-\beta} w^{\beta} = 1 \\ z^{-\gamma} w r^{\alpha-1} k_1 = k_0 \end{cases}$$

We rewrite the second equilibrium condition as:

$$w = \frac{z^{\gamma} r^{1-\alpha} k_0}{k_1}$$

And we plug this expression into the first equilibrium condition:

$$z^{-\theta} k_2 r^{1-\beta} \left(\frac{z^{\gamma} r^{1-\alpha} k_0}{k_1} \right)^{\beta} = 1$$

We solve for the rental rate r:

$$\begin{aligned} & z^{-\theta+\gamma\beta} k_2 k_0^{\beta} k_1^{-\beta} r^{1-\beta+\beta(1-\alpha)} = 1 \\ \Longleftrightarrow\ & r = z^{\theta-\gamma\beta} k_2^{-1} k_0^{-\beta} k_1^{\beta} \\ \Longleftrightarrow\ & r = z^{\frac{\theta-\gamma\beta}{1-\alpha\beta}} \underbrace{\left(k_2^{-1} k_0^{-\beta} k_1^{\beta} \right)^{\frac{1}{1-\alpha\beta}}}_{\equiv k_3} \end{aligned}$$

We can write the equilibrium rental rate as a function of pollution:

$$r^* = k_3 z^{\frac{\theta-\gamma\beta}{1-\alpha\beta}}$$

And we can derive a similar expression for the equilibrium wage rate:

$$w^* = \frac{z^{\gamma} \left(r^*\right)^{1-\alpha} k_0}{k_1} = z^{\gamma + \frac{(1-\alpha)(\theta-\gamma\beta)}{1-\alpha\beta}} \underbrace{\frac{k_0 k_3}{k_1}}_{\equiv k_4}$$

We simplify the exponent:

$$\begin{aligned} \gamma + \frac{(1-\alpha)(\theta-\gamma\beta)}{1-\alpha\beta} &= \frac{\gamma - \alpha\beta\gamma + \theta - \alpha\theta - \gamma\beta + \alpha\beta\gamma}{1-\alpha\beta} \\ &= \frac{\gamma + \theta - \alpha\theta - \gamma\beta}{1-\alpha\beta} \\ &= \frac{\gamma(1-\beta) + \theta(1-\alpha)}{1-\alpha\beta} \end{aligned}$$

So that we obtain eventually:

$$w^* = k_4 z^{\frac{\gamma(1-\beta)+\theta(1-\alpha)}{1-\alpha\beta}}$$

When pollution goes up in the location, w and r change in equilibrium. w unambiguously increases, while the evolution of r depends on the sign of $\theta-\gamma\beta$. This induces changes in households' disposable income. In terms of utility, the direct negative effect of higher pollution is exactly offset by these indirect, general equilibrium adjustments. The same logic is at play as above, where we discuss why indirect utility remains equal to constant k_0.

Similarly, if the government, say, reduces pollution through abatement investment or regulation, we would not expect an improvement in local workers' utility. General equilibrium adjustments would kick in and maintain indirect utility equal to the constant k_0.

5 Marginal Willingness To Pay (MWTP)

We can back out the marginal willingness to pay (MWTP) for reduced pollution from the observed changes in the wage and rental rates at equilibrium. As seen in class, as it quantifies how much workers value the (dis)amenity, MWTP can then be a precious input for policy-makers.

How can we back this value out? We start from the equality of indirect utility with the constant k_0 and introduce a change in pollution dz:

$$V\left(w,r;z\right)=k_0 \implies \underbrace{\frac{\partial V}{\partial w}\frac{\partial w}{\partial z}dz}_{\text{Impact through } w} + \underbrace{\frac{\partial V}{\partial r}\frac{\partial r}{\partial z}dz}_{\text{Impact through } r} + \underbrace{\frac{\partial V}{\partial z}dz}_{\text{Direct impact}} = 0$$

We can simplify dz and rework slightly the equation. To lighten it, we introduce the following notations for marginal benefits: $\frac{\partial V}{\partial w}=V_w$, $\frac{\partial V}{\partial r}=V_r$, and $\frac{\partial V}{\partial z}=V_z$. We obtain:

$$V_w\frac{\partial w}{\partial z}+V_r\frac{\partial r}{\partial z}=-V_z$$

We divide by the marginal benefit to the wage rate, V_w. This yields:

$$\frac{\partial w}{\partial z}+\frac{V_r}{V_w}\frac{\partial r}{\partial z}=-\frac{V_z}{V_w}$$

Let us compute the $\frac{V_r}{V_w}$ ratio. Using the expression found above for the indirect utility function, we have:

$$\begin{cases} V_r &= \frac{\partial V}{\partial r}=\left(\alpha-1\right)z^{-\gamma}wr^{\alpha-2}k_1 \\ V_w &= \frac{\partial V}{\partial w}=z^{-\gamma}r^{\alpha-1}k_1 \end{cases}$$
$$\implies \frac{V_r}{V_w}=\left(\alpha-1\right)\frac{w}{r}=-\frac{w\left(1-\alpha\right)}{r}$$

And we recognize in the last ratio the consumer's optimal demand for land L_c^*. So:

$$\frac{V_r}{V_w}=-L_c^*$$

In fact, this result is neither random, nor dependent on the specification of the model. This corresponds to a general result in microeconomics, which we call the "Roy identity". We use it to replace $\frac{V_r}{V_w}$ in the equality that we were discussing previously. This gives:

$$\frac{\partial w}{\partial z} - L_c^* \frac{\partial r}{\partial z} = -\frac{V_z}{V_w}$$

This result is nice for two reasons. First, all the terms in the left-hand side of the expression could be observed in the data, possibly with some simplifications. We would need to measure the causal effects of pollution on wages and rents respectively, and then plug these estimates into the expression. L_c^* can be observed from households' residential choices. Second, the right-hand side exactly corresponds to households' MWTP to reduce pollution. We provide some intuition for this right below.

To see this last point, let us come back to the definition of MWTP. It is the dollar amount that households are ready to pay to reduce pollution by one unit. What is the benefit from such a transaction in utility terms? Households enjoy the unit drop in pollution by:

$$dzV_z = (-1)\, V_z = -V_z$$

What is the cost of the transaction in utility terms? Households must pay the dollar amount MWTP, which comes down to a wage cut by MWTP. What is the cost of such a change in the wage?

$$-dwV_w = -\left(-\text{MWTP}\right) V_w = \text{MWTP} \times V_w$$

By definition, the MWTP equates the utility benefit and cost of the transaction. So, we end up with:

$$\text{MWTP} \times V_w = -V_z \iff \text{MWTP} = -\frac{V_z}{V_w}$$

Note that, since V_z is negative, the MWTP is positive as we would expect. We conclude:

$$\text{MWTP} = \underbrace{\frac{\partial w}{\partial z} - L_c^* \frac{\partial r}{\partial z}}_{\text{Estimable / Observable}}$$

Then, in the future, if you need for instance to decide on cleaning up a site or not, you can apply the MWTP to your cost-benefit analysis. Or, if you need to derive the optimal Pigouvian tax on pollution, you can also appeal to MWTP.

6 Comparative Statics

We want to further analyse the effect of pollution on equilibrium outcomes, the wage rate and the rental rate. We distinguish two cases.

6.1 First Case

The first case that we consider assumes $\theta = 0$, meaning that pollution is neutral on the production side. We plug this assumption in the above expressions for the wage and rental as a function of pollution:

$$\begin{cases} w^* & = k_4 z^{\frac{\gamma(1-\beta)}{1-\alpha\beta}} \\ r^* & = k_3 z^{-\frac{\gamma\beta}{1-\alpha\beta}} \end{cases}$$

We differentiate the two expressions with respect to pollution:

$$\begin{cases} \frac{\partial w^*}{\partial z} & = k_4 \frac{\gamma(1-\beta)}{1-\alpha\beta} z^{\frac{\gamma(1-\beta)}{1-\alpha\beta}-1} > 0 \\ \frac{\partial r^*}{\partial z} & = -k_3 \frac{\gamma\beta}{1-\alpha\beta} z^{-\frac{\gamma\beta}{1-\alpha\beta}} < 0 \end{cases}$$

If pollution only affects consumers and weighs down on their utility, then more pollution increases the wage rate and reduces the rental rate of land. The mechanism corresponds to the out-migration story discussed previously.

6.2 Second Case

The second case that we consider assumes $\theta > 0$. In this case, pollution is beneficial for producers as it makes the given quantities of inputs more productive. We come back the more general expressions:

$$\begin{cases} w^* & = k_4 z^{\frac{\gamma(1-\beta)+\theta(1-\alpha)}{1-\alpha\beta}} \\ r^* & = k_3 z^{\frac{\theta-\gamma\beta}{1-\alpha\beta}} \end{cases}$$

We again differentiate the two expressions with respect to pollution:

$$\begin{cases} \frac{\partial w^*}{\partial z} & = k_4 \frac{\gamma(1-\beta)+\theta(1-\alpha)}{1-\alpha\beta} z^{\frac{\gamma(1-\beta)+\theta(1-\alpha)}{1-\alpha\beta}-1} \\ \frac{\partial r^*}{\partial z} & = k_3 \frac{\theta-\gamma\beta}{1-\alpha\beta} z^{\frac{\theta-\gamma\beta}{1-\alpha\beta}} \end{cases}$$

The first expression is unambiguously positive meaning that more pollution translates into a higher wage rate in equilibrium. The second expression is positive if and only if: $\theta > \gamma\beta$. When the production benefit of pollution (governed by the parameter θ) is sufficiently large, the consumption-side cost of pollution (governed by the parameter γ) sufficiently small, or the weight of labor in production (governed by β) sufficiently small, then higher pollution results in a higher rental rate for land.