2 Derivation of Navier-Stokes equation

The goal of this section is to find an equation, which describes the spatio-temporal evolution of the velocity field $\vec{u}(\vec{r}, t)$. This is the Navier-Stokes equation, which is the most fundamental equation in Fluid Dynamics.

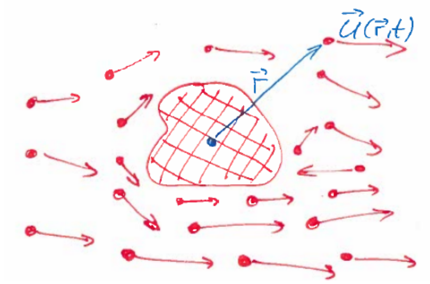

Figure 8: Velocity field around an object.

We want to describe the motion of a fluid particle. We start with Newton's second law of Classical Mechanics

$$\vec{F} = \frac{d}{dt}(m\vec{u}). \tag{2.1}$$

The equation states that the forces acting on the particle equals its change (the time derivative) of momentum (the parenthesis).

> **Remark:** in Fluid Dynamics we look not only at one fluid particle, but at all fluid particles.

$$\frac{d}{dt}(m\vec{u}) = \frac{dm}{dt}\vec{u} + m\frac{d\vec{u}}{dt} = \rho\Delta V\frac{d\vec{u}}{dt}. \tag{2.2}$$

The mass of a fluid particle is constant and does not change over time, hence the time derivative term is zero. We also used the relation

$$m = \rho\Delta V. \tag{2.3}$$

Given the field description $\vec{u} = \vec{u}(\vec{r}, t)$, we have to be a little careful with $\frac{d\vec{u}}{dt}$. The following is wrong:

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}(\vec{r}, t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{u}(\vec{r}, t + \Delta t) - \vec{u}(\vec{r}, t)}{\Delta t} \tag{2.4}$$

See the example in Figure 9.

The correct approach is to follow one fluid particle on its pathline (trajectory) $\vec{r} = \vec{r}(\vec{r}_0, t_0; t)$. See Figure 10.

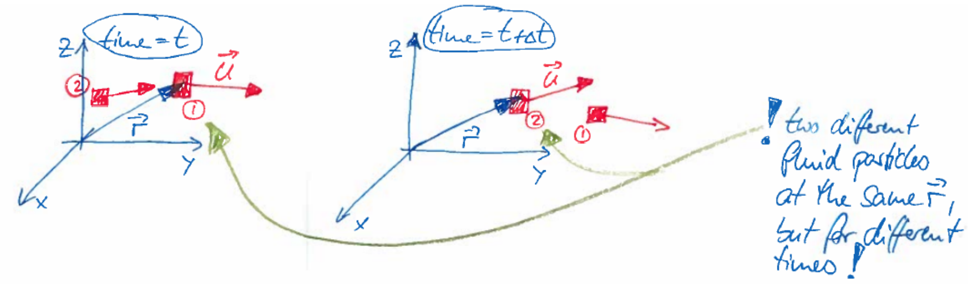

Figure 9: Two fluid particles can have the same coordinate vector $\vec{r}$ but for different times.

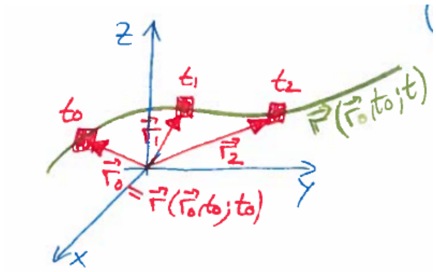

Figure 10: The path and changing coordinates of a single fluid particle.

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}\left(\vec{r}\left(\vec{r}_0, t_0; t\right), t\right)}{dt} \tag{2.5}$$

$$= \frac{d\vec{u}(x(\vec{r}_0, t_0; t), y(\vec{r}_0, t_0; t), z(\vec{r}_0, t_0; t))}{dt} \tag{2.6}$$

$$= \frac{\partial \vec{u}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{u}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{u}}{\partial z}\frac{dz}{dt} + \frac{\partial \vec{u}}{\partial t}\frac{dt}{dt} \tag{2.7}$$

$$= \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right)\vec{u} + \frac{\partial \vec{u}}{\partial t} \tag{2.8}$$

Short notation for partial derivative:

$$\frac{\partial}{\partial x} = \partial_x. \tag{2.9}$$

Here x can be replaced by y, z or t.

Short notation for velocity:

$$\frac{dx}{dt} = u_x \tag{2.10}$$

The terms in parentheses is the dot product between $\vec{u}$ and $\vec{\nabla} = (\partial_x, \partial_y, \partial_z)$, calculated using the chain rule of differentiation:

$$\frac{du(f(t))}{dt} = \frac{\partial u}{\partial f}\frac{df}{dt} \tag{2.11}$$

$$\frac{du(f(t),g(t))}{dt} = \frac{\partial u}{\partial f}\frac{df}{dt} + \frac{\partial u}{\partial g}\frac{dg}{dt} \tag{2.12}$$

$$\frac{du(f(t),g(t),h(t))}{dt} = \frac{\partial u}{\partial f}\frac{df}{dt} + \frac{\partial u}{\partial g}\frac{dg}{dt} + \frac{\partial u}{\partial h}\frac{dh}{dt}. \tag{2.13}$$

This leads to the final expression

$$\frac{d\vec{u}}{dt} = \frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}. \tag{2.14}$$

The material derivative is defined as

$$\frac{d}{dt}\partial_t + \vec{u}\cdot\vec{\nabla} = \frac{D}{Dt} = D_t. \tag{2.15}$$

Whenever we have the time derivative of a field, like $\vec{u}(\vec{r},t)$, then we have to "go with the fluid particle" and use the material derivative.

We now go back to Newton's second equation:

$$\frac{d}{dt}(m\vec{u}) = \rho\Delta V\left(\partial_t + \vec{u}\cdot\vec{\nabla}\right)\vec{u} = \vec{F} = \vec{F}_{\text{external}} + \vec{F}_{\text{surrounding}} \tag{2.16}$$

The surrounding force can be decomposed into

$$\vec{F}_{\text{surrounding}} = \vec{F}_{\text{pressure}} + \vec{F}_{\text{friction}} \tag{2.17}$$

The surrounding fluid particles push the "sandwiched" fluid particle around; they exert pressure. Mutual friction between neighboring fluid particles due to relative and rotational motion, deformation and compression.

Example: The force of gravity is an example of an external force:

$$\vec{F}_{\text{external}} = \vec{F}_{\text{grav}} = \underbrace{\rho\Delta V}_{m}\,\vec{g} = \rho\vec{g}\Delta V, \tag{2.18}$$

with the gravitational constant defined as

$$\vec{g} = -\begin{pmatrix} 0 \\ 0 \\ 9.81\,\frac{\text{m}}{\text{s}} \end{pmatrix} = -g\vec{e}_z. \tag{2.19}$$

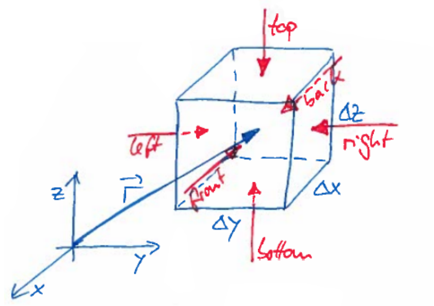

Figure 11: Illustration of the pressure force on each side of a fluid particle.

2.1 Pressure force

$$\left(\vec{F}_{\text{pressure}}\right)_z = \vec{F}^{\text{top}}_{\text{pressure}} + \vec{F}^{\text{bottom}}_{\text{pressure}} \tag{2.20}$$

$$= -p\left(x, y, z + \frac{\Delta z}{2}\right)\Delta x \Delta y + p\left(x, y, z - \frac{\Delta z}{2}\right)\Delta x \Delta y \tag{2.21}$$

$$\begin{aligned} = &- \left(p(x,y,z) + \frac{\partial p(x,y,z)}{\partial z}\frac{\Delta z}{2}\right)\Delta x \Delta y \\ &+ \left(p(x,y,z) + \frac{\partial p(x,y,z)}{\partial z}\left(-\frac{\Delta z}{2}\right)\right)\Delta x \Delta y \end{aligned} \tag{2.22}$$

$$= -\frac{\partial p(x,y,z)}{\partial z}\Delta x \Delta y \Delta z \tag{2.23}$$

Using $\Delta x \Delta y \Delta z = \Delta V$ the pressure force density is

$$\vec{f}_{\text{pressure}} = \frac{\vec{F}_{\text{pressure}}}{\Delta V} = -\vec{\nabla} p(\vec{r}, t) \tag{2.24}$$

2.2 Friction force

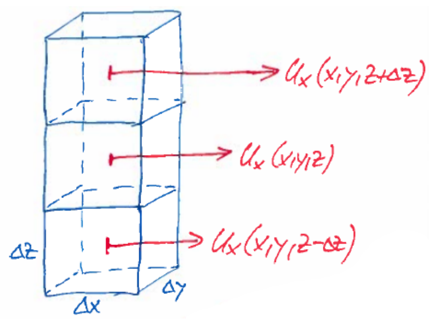

Figure 12: Illustration of the friction force on each side of a fluid particle.

We consider the neighboring fluid particles above and below as sketched in Figure 12.

$$\left(\vec{F}^{\text{top+bottom}}_{\text{friction}}\right)_x = \frac{\mu}{\Delta z}\Delta x \Delta y \left(u_x(x,y,z+\Delta z) - u_x(x,y,z)\right) + \frac{\mu}{\Delta z}\Delta x \Delta y \left(u_x(x,y,z-\Delta z) - u_x(x,y,z)\right) \tag{2.25}$$

If $u_x(x,y,z+\Delta z) > u_x(x,y,z)$, then the fluid particle above pulls the sandwiched fluid particle with it.

Taylor series expansion up to second-order terms:

$$\begin{aligned}\left(\vec{F}^{\text{top+bot}}_{\text{friction}}\right)_x &= \mu \Delta x \Delta y \left\{ \frac{u_x(x,y,z+\Delta z) - u_x(x,y,z)}{\Delta z} + \frac{u_x(x,y,z-\Delta z) - u_x(x,y,z)}{\Delta z} \right\} \\ &= \frac{\mu \Delta x \Delta y}{\Delta z} \left\{ u_x(x,y,z) + \frac{\partial u_x(x,y,z)}{\partial z}\Delta z + \frac{\partial^2 u_x(x,y,z)}{\partial z^2}\frac{\Delta z^2}{2} - u_x(x,y,z) \right. \\ &\quad \left. + u_x(x,y,z) + \frac{\partial u_x(x,y,z)}{\partial z}(-\Delta z) + \frac{\partial^2 u_x(x,y,z)}{\partial z^2}\frac{(-\Delta z)^2}{2} - u_x(x,y,z) \right\} \end{aligned} \tag{2.26}$$

$$= \mu \Delta x \Delta y \Delta z \frac{\partial^2 u_x(x,y,z)}{\partial z^2} \tag{2.27}$$

Front + back:

$$\left(\vec{F}^{\text{front+back}}_{\text{friction}}\right)_x = \mu \Delta V \frac{\partial^2 u_x(x,y,z)}{\partial y^2} \tag{2.28}$$

Most general expression of the friction force:

$$\frac{\vec{F}_{\text{friction}}}{\Delta V} = \vec{f}_{\text{friction}}\left(\frac{\partial^2 u_i}{\partial x_k \partial x_l}\right) = \vec{f}_{\text{friction}}(\vec{\nabla}, \vec{\nabla}, \vec{u}) \tag{2.29}$$

The task is to build a vector $\vec{f}$ from a combination of three vectors $\vec{a} = \vec{\nabla}, \vec{b} = \vec{\nabla}, \vec{c} = \vec{u}$, such that

$$\vec{f} = \alpha \left(\vec{a} \cdot \vec{b}\right) \vec{c} + \beta \left(\vec{a} \cdot \vec{c}\right) \vec{b} + \alpha \left(\vec{b} \cdot \vec{c}\right) \vec{a} \tag{2.30}$$

The solution:

$$\vec{f}_{\text{friction}} = \mu \left(\vec{\nabla} \cdot \vec{\nabla}\right) \vec{u} + \left(\mu_v + \frac{\mu}{3}\right) \vec{\nabla} \left(\vec{\nabla} \cdot \vec{u}\right), \tag{2.31}$$

where μ is the shear (dynamic) viscosity and μ_v is the compression (bulk) viscosity.

Navier-Stokes equation:

$$\rho \left(\partial_t + \left(\vec{u} \cdot \vec{\nabla}\right)\right) \vec{u} = \vec{f}_{\text{ext}} - \vec{\nabla} p + \mu \left(\vec{\nabla} \cdot \vec{\nabla}\right) \vec{u} + \left(\mu_v + \frac{\mu}{3}\right) \vec{\nabla} \left(\vec{\nabla} \cdot \vec{u}\right) \tag{2.32}$$

where

$$\vec{u} = \vec{u}\left(\vec{r}, t\right) \tag{2.33}$$

$$p = p\left(\vec{r}, t\right) \tag{2.34}$$

$$\rho = \rho\left(\vec{r}, t\right) \tag{2.35}$$

$$\vec{f}_{\text{ext}} = \vec{f}_{\text{ext}}\left(\vec{r}, t\right) \tag{2.36}$$

2.3 Navier-Stokes equation in components

$$\rho \begin{pmatrix} \frac{\partial u_x}{\partial t} \\ \frac{\partial u_y}{\partial t} \\ \frac{\partial u_z}{\partial t} \end{pmatrix} + \rho \begin{pmatrix} \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_x \\ \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_y \\ \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_z \end{pmatrix} = \begin{pmatrix} f_x^{\text{ext}} \\ f_y^{\text{ext}} \\ f_z^{\text{ext}} \end{pmatrix} - \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{pmatrix}$$
$$+ \mu \begin{pmatrix} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u_x \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u_y \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u_z \end{pmatrix}$$
$$+ \left(\mu_v + \frac{\mu}{3}\right) \begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\ \frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \end{pmatrix} \tag{2.37}$$

Remark: There are only a few exact analytical solutions; many approximate analytical solutions (guided by intuition). Computational fluid dynamics can give us "exact" numerical solutions for approximations to the Navier-Stokes equation.

We now have three coupled differential equations for five fields: $u_x(\vec{r}, t)$, $u_y(\vec{r}, t)$, $u_z(\vec{r}, t)$, $p(\vec{r}, t)$, and $\rho(\vec{r}, t)$. This means we are missing two equations.

The first missing equation

From thermodynamics we have an equation of state

$$g(p, \rho) = 0. \tag{2.38}$$

For an incompressible flow, the equaiton of state is simply

$$\rho = \text{constant}. \tag{2.39}$$

For a compressible flow, the equation of state can be found with the law of ideal gases:

$$pV = NkT, \tag{2.40}$$

from which we get

$$\rho = \frac{N}{V} = \frac{1}{kT} p \tag{2.41}$$
$$\frac{p}{\rho} = kT = \text{constant}. \tag{2.42}$$

This only holds if the temperature is constant.

The second missing equation

Equation of continuity, local mass conservation.

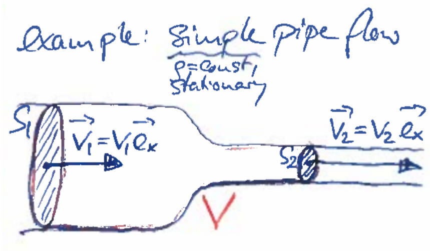

Figure 13: A simple pipe flow to illustrate mass conservation.

Example: Simple pipe flow. See Figure 13.

$$M_{\text{in}} = \rho S_1 v_1 \Delta t \tag{2.43}$$
$$M_{\text{out}} = \rho S_2 v_2 \Delta t \tag{2.44}$$

Mass conservation:

$$M_{\text{in}} = M_{\text{out}} \tag{2.45}$$
$$\Downarrow$$
$$v_1 S_1 = v_2 S_2 \tag{2.46}$$

This is Leonardo's law.

Local mass conservation in a small volume element:

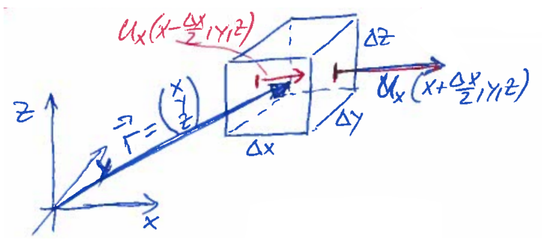

Figure 14: Local mass conservation in a small volume element.

Mass flux through surface of volume $\Delta V = \Delta x \Delta y \Delta z$ in x-direction:

$$
\begin{aligned}
\frac{dM_x^S}{dt} &= \rho\left(x+\frac{\Delta x}{2}, y, z, t\right) u_x\left(x+\frac{\Delta x}{2}, y, z, t\right) \Delta y \Delta z \\
&\quad - \rho\left(x-\frac{\Delta x}{2}, y, z\right) u_x\left(x-\frac{\Delta x}{2}, y, z\right) \Delta y \Delta z \\
&= \left(\rho(x,y,z) + \frac{\partial \rho(x,y,z)}{\partial x}\frac{\Delta x}{2}\right)\left(u_x(x,y,z) + \frac{\partial u_x(x,y,z)}{\partial x}\frac{\Delta x}{2}\right) \Delta y \Delta z \\
&\quad - \left[\rho(x,y,z) + \frac{\partial \rho(x,y,z)}{\partial x}\left(-\frac{\Delta x}{2}\right)\right]\left[u_x(x,y,z) + \frac{\partial u_x(x,y,z)}{\partial x}\left(-\frac{\Delta x}{2}\right)\right] \Delta y \Delta z \\
&= \frac{\partial \rho(x,y,z)}{\partial x} u_x(x,y,z) \Delta x \Delta y \Delta z + \rho(x,y,z)\frac{\partial u_x(x,y,z)}{\partial x} \Delta x \Delta y \Delta z \\
&= \frac{\partial\left(\rho(x,y,z,t) u_x(x,y,z,t)\right)}{\partial x} \Delta V
\end{aligned} \tag{2.47}
$$

Mass flux in y and z-direction:

$$\frac{dM_y^S}{dt} = \frac{\partial(\rho(x,y,z)u_y(x,y,z))}{\partial y}\Delta V \tag{2.48}$$

$$\frac{dM_z^S}{dt} = \frac{\partial(\rho(x,y,z)u_z(x,y,z))}{\partial z}\Delta V \tag{2.49}$$

Sum of mass fluxes through volume in all directions:

$$\frac{dM^S}{dt} = \frac{dM_x^S}{dt} + \frac{dM_y^S}{dt} + \frac{dM_z^S}{dt} \tag{2.50}$$

$$= \frac{\partial(\rho u_x)}{\partial x}\Delta V + \frac{\partial(\rho u_y)}{\partial y}\Delta V + \frac{\partial(\rho u_z)}{\partial z}\Delta V \tag{2.51}$$

$$= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \rho u_x \\ \rho u_y \\ \rho u_z \end{pmatrix} \Delta V \tag{2.52}$$

$$= \vec{\nabla} \cdot \left(\rho(x,y,z)\vec{u}(x,y,z)\right) \Delta V \tag{2.53}$$

Increase of mass within fixed volume ΔV:

$$\frac{\partial M^V}{\partial t} = \frac{\partial(\rho(\vec{r},t)\Delta V)}{\partial t} = \frac{\partial \rho(\vec{r},t)}{\partial t}\Delta V \tag{2.54}$$

Local mass conservation

$$\frac{dM^V}{dt} = -\frac{dM^S}{dt} \tag{2.55}$$

If mass within the volume increases, then less has to flow out of the surface than to flow in

$$\frac{\partial \rho(\vec{r},t)}{\partial t} + \vec{\nabla} \cdot \left(\rho(\vec{r},t)\vec{u}(\vec{r},t)\right) = 0 \tag{2.56}$$

This is the equation of continuity.

2.4 Summary

Navier-Stokes equation:

$$\rho\left(\frac{\partial}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\right)\vec{u}=\vec{f}_{\text{ext}}-\vec{\nabla}p+\mu\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{u}+\left(\mu_v+\frac{\mu}{3}\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}\right) \tag{2.57}$$

$$\vec{u}=\vec{u}(\vec{r},t)=\vec{u}(x,y,z,t) \tag{2.58}$$

$$\rho=\rho(\vec{r},t) \tag{2.59}$$

$$p=p(\vec{r},t) \tag{2.60}$$

Equation of continuity:

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{u})=0 \tag{2.61}$$

Equation of state:

$$g(p,\rho;T)=0 \tag{2.62}$$

Heat equation (if the temperature also becomes a field $T(\vec{r},t)$):

$$\left(\frac{\partial}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\right)T(\vec{r},r)=\kappa\left(\vec{\nabla}\cdot\vec{\nabla}\right)T(\vec{r},t), \tag{2.63}$$

where κ is the thermal diffusion.