# 4 Ideal flow: planar 2-dimensional potential flow around cylinder

For further details see section 4.3, 4.9 and 7.1-6 in the KCD book

For a planar two-dimensional stationary flow the velocity field

$$\vec{u} = \begin{pmatrix} u_x(x,y) \\ u_y(x,y) \\ 0 \end{pmatrix} \tag{4.1}$$

does not depend on $z$ and $t$ and the vector does not have a $z$-component. The flow is defined to be "ideal" once the viscosity $\mu = 0$ is put to zero. The mass density $\rho = \rho_0$ is assumed to be constant. We will determine the two velocity components $u_x(x,y)$ and $u_y(x,y)$ from the two equations $\vec{\nabla} \cdot \vec{u} = 0$ and $\vec{\nabla} \times \vec{u} = 0$ defining incompressible and irrotational flows. The pressure field $p(x,y)$ is then determined via Bernoulli's equation.

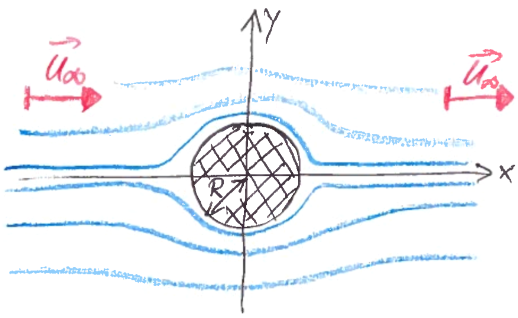


Figure 16: 2-dimensional ideal flow around a cylinder.

Questions to Figure 16:

1. How does the velocity field $\vec{u} = \vec{u}(x,y)$ look like?

2. How do the pathlines (streamlines) look like?

$$\vec{\nabla} \cdot \vec{u} = 0 \Rightarrow \vec{u}(\vec{r}) = \vec{\nabla}\phi(\vec{r}) \tag{4.2}$$

where $\phi(\vec{r})$ is the velocity potential.

$$\vec{\nabla} \times \vec{u} = 0 \tag{4.3}$$

$$\Downarrow$$

$$\vec{\nabla} \cdot \vec{\nabla}\phi(\vec{r}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y) = 0 \tag{4.4}$$

This second order differential equation is the Laplace equation.

Question: How does the cylinder (obstacle) enter in solving the Laplace equation?

There are two boundary conditions.

First boudary condition:

$$\vec{u}(|\vec{r}| \to \infty) = \vec{u}_\infty = u_\infty \vec{e}_x \tag{4.5}$$
$$\Downarrow$$
$$\phi(|\vec{r}| \to \infty) = u_\infty x + \text{constant} \tag{4.6}$$

Second boundary condition:

$$0 = \vec{u}_{\text{surface}} \cdot \vec{n} = \vec{\nabla}\phi \Big|_{\text{surface}} \cdot \vec{n} \tag{4.7}$$

The fluid particle does not flow into/out of the surface; only tangential component.

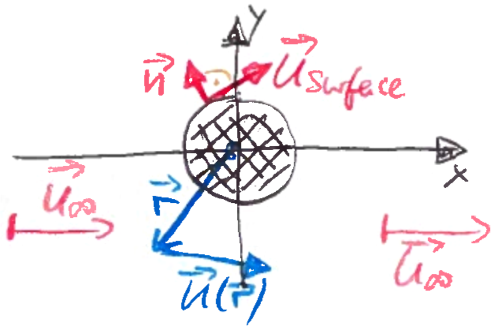


Figure 17: Tangential components of the velocity field.

For the flow around the cylinder the solution of the Laplace equation with the two boundary conditions "falls from the sky" (for the moment):

$$\phi(x, y) = u_\infty x \left(1 + \frac{R^2}{x^2 + y^2}\right). \tag{4.8}$$

It fulfills Laplace's equation and the two boundary conditions.

Velocity field:

$$\vec{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \vec{\nabla}\phi(x, y) = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{pmatrix} \tag{4.9}$$

$$\begin{aligned} u_x &= u_\infty \left(1 + \frac{R^2(y^2 - x^2)}{(x^2 + y^2)^2}\right) \\ u_y &= -u_\infty \frac{2xyR^2}{(x^2 + y^2)^2} \end{aligned} \tag{4.10}$$

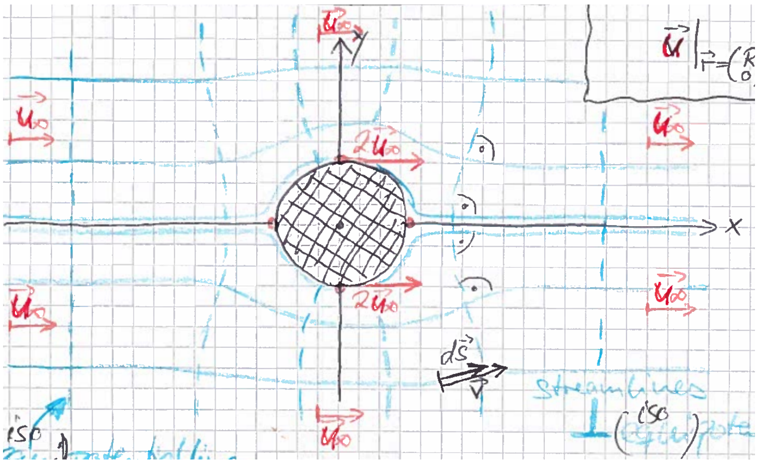

Figure 18: Streamlines around the cylinder.

Examples:

$$\vec{u}\Big|_{x\to\pm\infty} = u_\infty \vec{e}_x = \vec{u}\Big|_{y\to\pm\infty} \tag{4.11}$$

$$\vec{u}\Big|_{\vec{r}=\begin{pmatrix}0\\R\end{pmatrix}} = 2u_\infty \vec{e}_x \tag{4.12}$$

$$\vec{u}\Big|_{\vec{r}=\begin{pmatrix}R\\0\end{pmatrix}} = 0 = \vec{u}\Big|_{\vec{r}=\begin{pmatrix}-R\\0\end{pmatrix}} \tag{4.13}$$

The two points in (4.13) with $\vec{u} = 0$ are called stagnation points.

## 4.1 Pathline around a cylinder

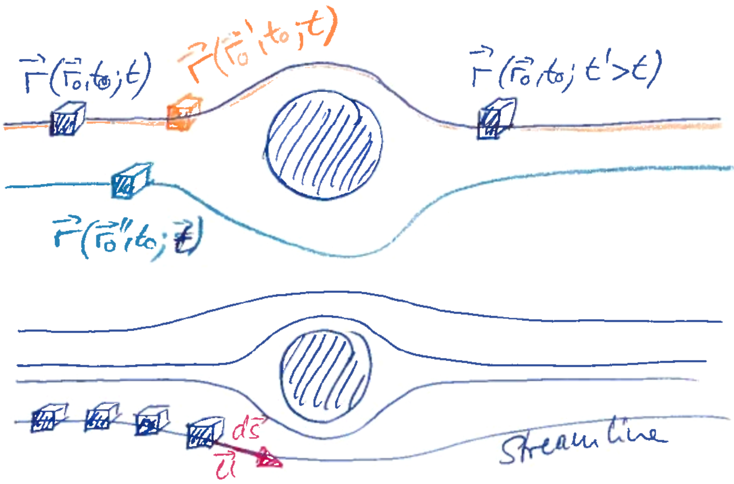

Figure 19: Path of fluid particles around the cylinder.

Top part of Figure 19 shows the Lagrangian picture: a particle is followed through all times. The bottom part shows the Eulerian picture, which is a snapshot of all fluid particles at one particular time.

Connection between Lagrangian and Eulerian picture:

$$\frac{d\vec{r}}{dt} = \vec{u}(\vec{r}, t) \tag{4.14}$$

Given the snapshots $\vec{u}(\vec{r}, t)$, we can calculate the pathlines. Given the pathlines, we can construct the snapshots. For stationary flows

$$\vec{u}(\vec{r}, t) = \vec{u}(\vec{r}) \Rightarrow \text{pathline} = \text{streamline}. \tag{4.15}$$

Question: How to calculate the streamlines?

First approach: Definition of streamline:

$$d\vec{s} \parallel \vec{u}, \tag{4.16}$$

where $\vec{ds}$ is a line element of a streamline.

$$0 = d\vec{s} \times \vec{u} = \begin{vmatrix} 0 & 0 & \vec{e}_z \\ dx & dy & 0 \\ u_x & u_y & 0 \end{vmatrix} = (u_y dx - u_x dy)\vec{e}_z \tag{4.17}$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{u_y}{u_x} = -\frac{2xyR^2}{(x^2+y^2)^2 + R^2(y^2 - x^2)}. \tag{4.18}$$

We will not try to solve this ugly non-linear differential equation.

Second approach: introduce the streamfunction $\psi(x, y)$.

Incompressibility gives us:

$$\vec{\nabla} \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{4.19}$$

which leads to the ansatz

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x} \tag{4.20}$$

$$\begin{aligned} d\vec{s} \times \vec{u} &= (u_y dx - u_x dy)\vec{e}_x & (4.21) \\ &= \left(-\frac{\partial \psi}{\partial x} dx - \frac{\partial \psi}{\partial y} dy\right)\vec{e}_z & (4.22) \\ &= -d\psi \vec{e}_z \stackrel{!}{=} 0 & (4.23) \end{aligned}$$

The streamfunction is constant along a streamline. This represents an *isopotential line* of the streamfunction and describes a streamline.

From the defining functions of the streamfunction in (4.20) and the $u_x$, $u_y$ solution for the ideal flow around a cylinder in (4.10), we can determine $\psi$ by partial integration

$$\psi(x,y) = v_\infty y \left(1 - \frac{R^2}{x^2+y^2}\right) \tag{4.24}$$

$$= v_\infty \sin\phi \left(r - \frac{R^2}{r}\right) \tag{4.25}$$

In the last step we have introduced the cylindrical coordinates $x = r\cos\phi$ and $y = r\sin\phi$. The intermediate steps of the partial integration have been left out.

**Remark:** relationship between velocity potential and streamfunction
$\phi =$ constant, $\psi =$ constant represent an orthogonal set of curves, because

$$\left(\vec{\nabla}\phi\right) \cdot \left(\vec{\nabla}\psi\right) = \begin{pmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial\psi}{\partial x} \\ \frac{\partial\psi}{\partial y} \end{pmatrix} = \begin{pmatrix} u_x \\ u_y \end{pmatrix} \cdot \begin{pmatrix} -u_y \\ u_x \end{pmatrix} = 0 \tag{4.26}$$

**Back to the derivation of (4.8):** two dimensional potential flow around an infinitely long cylinder. Because of cylinder symmetry we can transform from Cartesian to cylindrical coordinates

$$x = r\cos\phi \tag{4.27}$$

$$y = r\sin\phi \tag{4.28}$$

$$\Phi(x,y) \to \Phi(r,\phi) \tag{4.29}$$

$$\Delta\Phi(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi(x,y) \tag{4.30}$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\phi}\right]\Phi(r,\phi) \tag{4.31}$$

$$= 0 \tag{4.32}$$

$$\Downarrow$$

$$r\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = -\frac{\partial^2\Phi}{\partial\phi^2} \tag{4.33}$$

Ansatz: factorization

$$\Phi(r,\phi) = f(r)g(\phi) \tag{4.34}$$

$$\frac{1}{f\cdot g}r\frac{\partial}{\partial r}\left(r\frac{\partial(f(r)g(\phi))}{\partial r}\right) = -\frac{1}{f\cdot g}\frac{\partial^2(f(r)g(\phi))}{\partial\phi^2} \tag{4.35}$$

$$\Downarrow$$

$$\frac{1}{f(r)}r\frac{\partial}{\partial r}\left(r\frac{\partial f(r)}{\partial r}\right) = -\frac{1}{g(\phi)}\frac{\partial^2 g(\phi)}{\partial\phi^2} \stackrel{!}{=} m^2 \tag{4.36}$$

Left part depends only on $r$, and the middle part depends only on $\phi$. As a consequence, both have to be equal to a constant, which does neither depend on $r$ nor $\phi$.

$$\frac{\partial^2 g(\phi)}{\partial \phi^2} = -m^2 g(\phi) \tag{4.37}$$
$$\Downarrow$$
$$g(\phi) = e^{im\phi} = \cos m\phi + i \sin m\phi \tag{4.38}$$

Requirement:

$$g(\phi) = g(\phi + 2\pi) \tag{4.39}$$
$$\Downarrow$$
$$e^{im\phi} = e^{im(\phi+2m)} \tag{4.40}$$
$$\Downarrow$$
$$e^{2\pi i m} = 1. \tag{4.41}$$

This fixes $m$ to integer values:

$$m = \ldots, -2, -1, 0, 1, 2, \ldots \tag{4.42}$$

$$r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} f(r)\right) = m^2 f(r) \tag{4.43}$$

Polynomial ansatz:

$$f(r) = r^\alpha \tag{4.44}$$

$$r\frac{\partial}{\partial r}\left(r\alpha r^{\alpha-1}\right) = \alpha^2 r^\alpha \stackrel{!}{=} m^2 r^\alpha \tag{4.45}$$
$$\Downarrow$$
$$\alpha = \pm m \tag{4.46}$$

$$\Phi(r,\phi) = f(r)g(\phi) = r^{\pm m} e^{im\phi} \tag{4.47}$$

Since the Laplace equation is linear in $\Phi$, the most general solution for $\Phi$ is a linear superposition of all possible solutions:

$$\Phi(r,\phi) = \sum_{m=-\infty}^{\infty} \left(a_m r^m + b_m r^{-m}\right) e^{im\phi} \tag{4.48}$$
$$= \sum_{m=1}^{\infty} \left[\left(a_m r^m + b_m r^{-m}\right) e^{im\phi} + \left(c_m r^m + d_m r^{-m}\right) e^{-im\phi}\right] \tag{4.49}$$

Remark:

$$\Phi_{m=0} = a_0 + b_0 = \text{constant} \tag{4.50}$$
$$\Downarrow$$
$$\vec{u}_{m=0} = \vec{\nabla}\Phi_{m=0} = 0 \tag{4.51}$$

Remark: Another $m = 0$ solution is $\phi_{m=0} = c \ln r$. It fulfills (4.43). However, it is not able to fulfill the boundary condition at $r = R$, and has to be discarded.

Determination of amplitudes $a_m$, $b_m$, $c_m$, and $d_m$ via boundary conditions:

$$\Phi(r \to \infty, \phi) = u_\infty x = u_\infty r \cos\phi \tag{4.52}$$

$$\begin{aligned} \Phi(r \to \infty, \phi) &= \sum_{m=1}^{\infty} \left( a_m r^m e^{im\phi} + c_m r^m e^{-im\phi} \right) && (4.53) \\ &\stackrel{!}{=} u_\infty r \cos\phi && (4.54) \\ &= u_\infty r \frac{e^{i\phi} + e^{-i\phi}}{2} && (4.55) \end{aligned}$$

$$a_2 = a_3 = \cdots = c_2 = c_3 = \cdots = 0 \tag{4.56}$$

$$a_1 = \frac{u_\infty}{2} = c_1 \tag{4.57}$$

$$\Phi(r, \phi) = u_\infty r \frac{e^{i\phi} + e^{-i\phi}}{2} + \sum_{m=1}^{\infty} \left( \frac{b_m}{r^m} e^{im\phi} + \frac{d_m}{r^m} e^{-i\phi} \right) \tag{4.58}$$

$$\vec{u} \cdot \vec{e}_r|_{r=R} = \vec{\nabla}\Phi \cdot \vec{e}_r|_{r=R} = \left.\frac{\partial \Phi}{\partial r}\right|_{r=R} = 0 \tag{4.59}$$

$$\left.\frac{\partial \Phi(r, \phi)}{\partial r}\right|_{r=R} = u_\infty \frac{e^{i\phi} + e^{-i\phi}}{2} + \sum_{m=1}^{\infty} \left.\frac{(-m)}{r^{m+1}}\right|_{r=R} \left( b_m e^{im\phi} + d_m e^{-im\phi} \right) \stackrel{!}{=} 0 \tag{4.60}$$

$$b_2 = b_3 = \cdots = d_2 = d_3 = \cdots = 0 \tag{4.61}$$

$$\frac{u_\infty}{2} - \frac{b_1}{R^2} = 0 = \frac{u_\infty}{2} - \frac{d_1}{R^2} \tag{4.62}$$

$$\Downarrow$$

$$b_1 = d_1 = \frac{u_\infty R^2}{2} \tag{4.63}$$

$$\Phi(r, \phi) = u_\infty r \frac{e^{i\phi} + e^{-i\phi}}{2} + \frac{u_\infty R^2}{2} \frac{e^{i\phi} + e^{-i\phi}}{2} \tag{4.64}$$

where

$$\frac{e^{i\phi} + e^{-i\phi}}{2} = \cos\phi \tag{4.65}$$

$$\Phi(r, \phi) = u_\infty r \cos\phi \left( 1 + \frac{R^2}{r^2} \right) = u_\infty x \left( 1 + \frac{R^2}{x^2 + y^2} \right) = \Phi(x, y) \tag{4.66}$$

# 5 More on ideal potential flows

**Opening remark:** 2-dimensional potential flow solutions will often look like

$$\Phi(x,y) = u_\infty x + f(x,y). \tag{5.1}$$

Any function $f(x,y)$, which fulfills

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \qquad \text{and} \qquad f(|x|,|y| \to \infty) = 0, \tag{5.2}$$

describes a flow around some obstacle. The question is: which obstacle? Let's play around with $f(x,y)$.

**Example 1:**

$$\Phi(x,y) = \frac{m}{2\pi} \ln \sqrt{x^2 + y^2} \tag{5.3}$$

represents the radial flow resulting from a source with strength $m$. See Figure 20.

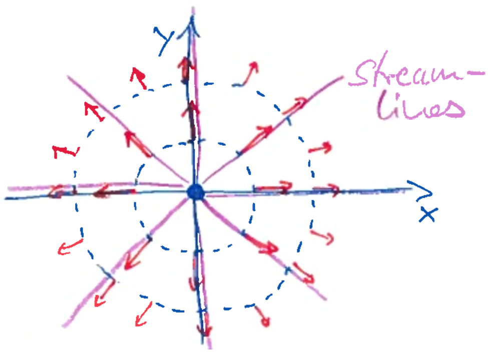


Figure 20: Two-dimensional radial flow resulting from a source line along the z-axis.

$$u_x = \frac{\partial \Phi}{\partial x} = \frac{m}{2\pi} \frac{x}{x^2 + y^2} = \frac{m}{2\pi} \frac{\cos\phi}{r} \tag{5.4}$$

$$u_y = \frac{\partial \Phi}{\partial y} = \frac{m}{2\pi} \frac{y}{x^2 + y^2} = \frac{m}{2\pi} \frac{\sin\phi}{r} \tag{5.5}$$

**Example 2 (method of images):** Source flow in front of a wall. See Figure 21.

Boundary condition: no flow through the wall; only tangential component.

$$\phi(x,y) = \frac{m}{2\pi} \ln \sqrt{(x+a)^2 + y^2} + \frac{m}{2\pi} \ln \sqrt{(x-a)^2 + y^2} \tag{5.6}$$

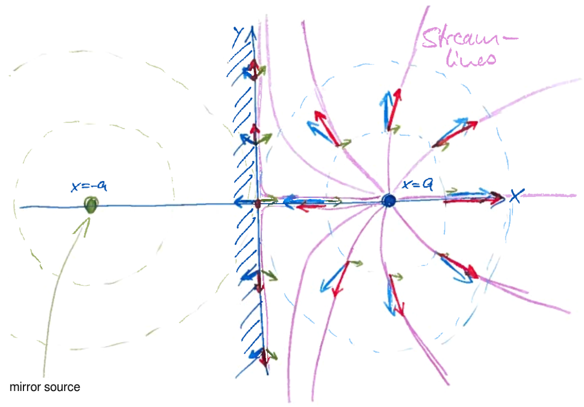


Figure 21: Source flow in front of a wall.

**Example 3:** flow past a 2-dimensional half-body. See Figure 22.

$$\Phi = u_\infty x + \frac{m}{2\pi} \ln \sqrt{x^2 + y^2} \tag{5.7}$$

$$\Downarrow$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{5.8}$$

$$u_x(x, y) = u_\infty + \frac{m}{2\pi} \frac{x}{x^2 + y^2} \tag{5.9}$$

$$u_y(x, y) = u_\infty + \frac{m}{2\pi} \frac{y}{x^2 + y^2} \tag{5.10}$$

Engineering flow interpretations:

1. An example of the beginning of the half-body is the leading edge of an airfoil
2. pedestrian on a bridge looking down: front part of a bridge pier
3. flow over a cliff

**Example 4 ("beauty of mathematics"):** Conformal mappings.

Complex potential

$$w(z) = \phi(x, y) + i\psi(x, y) \tag{5.11}$$

where $z = x + iy$ and $i^2 = -1$.

Velocity:

$$\frac{dw(z)}{dz} = \frac{dw(z)}{dz}\bigg|_{dz=dx} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u_x - iu_y \tag{5.12}$$

$$= \frac{dw(z)}{dz}\bigg|_{dz=idy} = \frac{\partial \phi}{i\partial y} + i\frac{\partial \psi}{i\partial y} = \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y} = u_x - iu_y \tag{5.13}$$

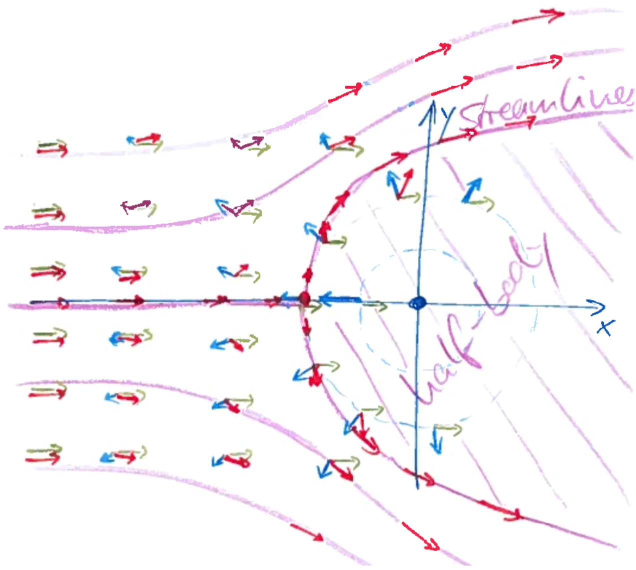


Figure 22: Flow past 2-dimensional half-body.

**Example 5:**

$$w(z) = u_\infty z = u_\infty (x + iy) = u_\infty x + i u_\infty y \tag{5.14}$$

describes the constant flow $\vec{u} = u_\infty \vec{e}_x$.

**Example 6:**

$$\begin{aligned} w(z) &= \frac{m}{2\pi} \ln z = \frac{m}{2\pi} \ln(x + iy) && (5.15)\\ &= \frac{m}{2\pi} \ln \left( r e^{i\theta} \right) && (5.16)\\ &= \frac{m}{2\pi} \ln r + \frac{m}{2\pi} \ln e^{i\theta} && (5.17)\\ &= \frac{m}{2\pi} \ln \sqrt{x^2 + y^2} + i \frac{m}{2\pi} \theta && (5.18) \end{aligned}$$

The two terms in the last line are the velocity potential and the stream function of a radial source flow (see "Example 1").

**Example 7:**

$$\begin{aligned} w(z) &= \frac{A}{2} z^2 = \frac{A}{2} (x + iy)^2 && (5.19)\\ &= \frac{A}{2} (x^2 - y^2) + i A x y && (5.20) \end{aligned}$$

**Example 8:** flow around cylinder with radius $R$

$$w(z) = \phi(x,y) + i\psi(x,y) = u_\infty x\left(1+\frac{R^2}{x^2+y^2}\right) + iu_\infty y\left(1-\frac{R^2}{x^2+y^2}\right) \tag{5.21}$$

$$= u_\infty(x+iy) + u_\infty R^2\frac{x-iy}{x^2+y^2} \tag{5.22}$$

$$= u_\infty(x+iy) + \frac{u_\infty R^2}{x+iy} \tag{5.23}$$

$$= u_\infty\left(z+\frac{R^2}{z}\right) \tag{5.24}$$

Change of variable ($z \to \tilde{z}$):

$$z = z(\tilde{z}) \tag{5.25}$$

Example:

$$\tilde{z} = (z+z_0) + \frac{1}{z+z_0} \tag{5.26}$$

$$\Downarrow$$

$$w_{\text{new obstacle}}(\tilde{z}) = w_{\text{cylinder}}(z(\tilde{z})) \tag{5.27}$$

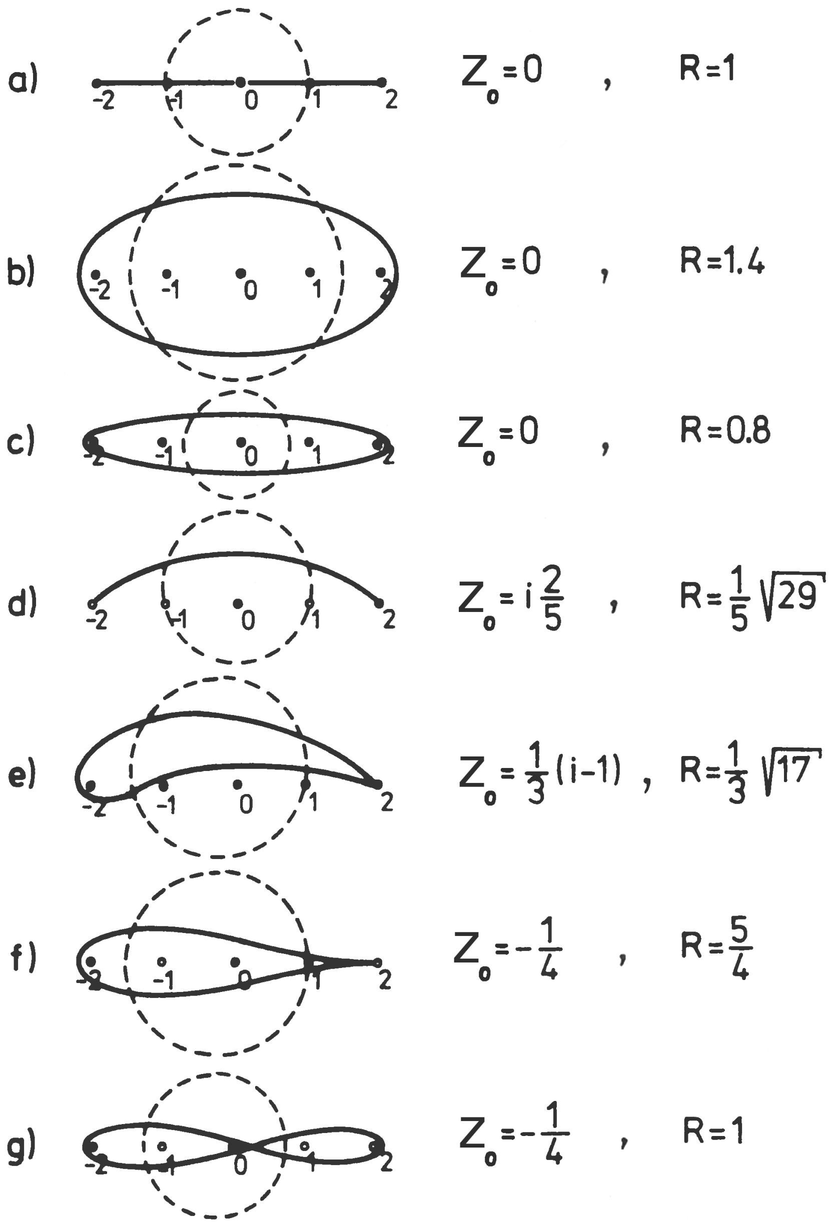


Figure 23: Several examples for different $z_0$ and $R$. e) is known as Joukowski's airfoil.