

Fluid Dynamics + Turbulence (fall 2018)

Homework Problems I + voluntary Exercises

Posted: August 28, 2018

Deadline: September 11 (Tuesday) at 08:30 am (on Blackboard).

Homework problem 1.1: Why does an Airbus A380 fly?

- (a) Use the Navier-Stokes equation in the static limit $\vec{u} = 0$, the gravitational force $\vec{f}_{\text{ext}} = -\rho g \vec{e}_z$, and the equation of state $p/p_0 = \rho/\rho_0$ to derive the barometric height formula

$$\frac{p(z)}{p_0} = \frac{\rho(z)}{\rho_0} = \exp\left(-\frac{g\rho_0 z}{p_0}\right),$$

where z is the height above ground, $p_0 = p(z = 0) = 1.01 \times 10^5 \frac{\text{kg}}{\text{m sec}^2}$ is the air pressure at ground, $\rho_0 = \rho(z = 0) = 1.20 \text{ kg/m}^3$ is the air density at ground, and $g = 9.81 \text{ m/sec}^2$ is the acceleration of gravity. How much is the air density reduced at the height $z = 10 \text{ km}$ above ground?

- (b) An Airbus A380 is typically cruising with the velocity $u = 945 \text{ km/h}$ at a height 10 km above ground. The lift force acting on the air wings is balancing the weight force. Use Bernoulli's equation to make a rough estimate of the velocity difference above and below the air wings. All you need are the results from (a), the wing area $A_{\text{wing}} = 846 \text{ m}^2$ and the weight $m \approx 500 \text{ t}$ of the A380.

Homework problem 1.2: The friction force

The friction force in the Navier-Stokes equation

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} \right) = \vec{f}_{\text{ext}} - \vec{\nabla} p + \mu (\vec{\nabla} \cdot \vec{\nabla}) \vec{u} + \left(\mu_v + \frac{\mu}{3} \right) \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) \quad (1)$$

contains the term

$$\frac{\partial^2 u_y}{\partial x \partial y} \quad (2)$$

in x-direction. Why does this term represent a force pointing in the x-direction?

Homework problem 1.3: A simple velocity field and its path-lines

Consider the following two-dimensional velocity field:

$$u_x = u_0 \quad (3)$$

$$u_y = \kappa x^2 \quad (4)$$

- (a) Sketch the velocity vector at selected positions along the x -axis.
- (b) Calculate the divergence $\vec{\nabla} \cdot \vec{u}$ and curl $\vec{\nabla} \times \vec{u}$ of the velocity field.
- (c) Use the differential equations

$$\frac{dx}{dt} = u_x, \quad \frac{dy}{dt} = u_y \quad (5)$$

to show that if a particle is in position (x_0, y_0) at time t_0 , then at time t its position is given by

$$x(t) = x_0 + u_0(t - t_0) \quad (6)$$

$$y(t) = y_0 + \frac{\kappa}{3u_0} [x(t)^3 - x_0^3]. \quad (7)$$

- (d) Sketch some particle trajectories (path lines) with $t_0 = x_0 = 0$.

Exercise 1.1

An infinitesimal closed line integral can be approximated as

$$\begin{aligned} \oint \vec{v}(x, y) \cdot d\vec{s} &\approx v_x \left(x, y + \frac{\Delta y}{2} \right) \Delta x \\ &+ v_y \left(x + \frac{\Delta x}{2}, \Delta y \right) (-\Delta y) \\ &+ v_x \left(x, y - \frac{\Delta y}{2} \right) (-\Delta x) \\ &+ v_y \left(x - \frac{\Delta x}{2}, y \right) \Delta y. \end{aligned} \quad (8)$$

Use a Taylor series expansion and show that

$$\oint \vec{v} \cdot d\vec{s} = - \left(\vec{\nabla} \times \vec{v} \right) \cdot \vec{e}_z \Delta x \Delta y \quad (9)$$

Given this result, illustrate what the irrotational flow condition $\vec{\nabla} \times \vec{v} = 0$ means.

Exercise 1.2

When we derived the Bernoulli equation

$$\frac{\rho_0}{2} \vec{v}^2 + p = \text{constant}, \quad (10)$$

we stumbled over the relation

$$\vec{\nabla} \left(\frac{\rho_0}{2} \vec{v}^2 + p \right) = \rho_0 \vec{v} \times \left(\vec{\nabla} \times \vec{v} \right), \quad (11)$$

where we simply evoked the irrotational flow condition to put the right-hand side to zero. Now we want to be more careful.

- (a) Show that when we multiply (scalar product) the right-hand side with a segment $d\vec{s}$ of a streamline, that

$$d\vec{s} \cdot \left[\vec{v} \times \left(\vec{\nabla} \times \vec{v} \right) \right] = 0. \quad (12)$$

- (b) Show that the integration of the left-hand side,

$$\int \vec{\nabla} \left(\frac{\rho_0}{2} \vec{v}^2 + p \right) \cdot d\vec{s} = 0, \quad (13)$$

along a streamline, leads to

$$\frac{\rho_0}{2} \vec{v}^2 + p = \text{constant} = c \quad (14)$$

along the streamline.

- (c) What can you say about the constant c for different stream lines?