Fluid Dynamics Q3 2017
Homework + Exercise Problems III

Posted:
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Deadline for submission of homework problem:
Friday February 17 at 10.00 am (on paper at 1531-119).

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1 Homework problem 3: Two-dimensional ideal potential flow

Consider the stream function $\Psi(x, y) = Axy$ with $A = 1\,\mathrm{m/sec}$ for the parts (a)-(e).

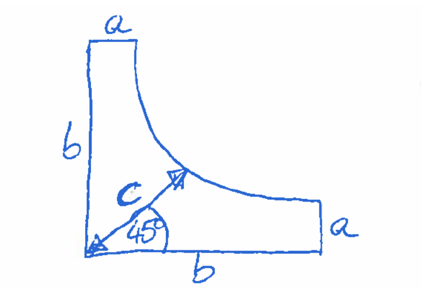
(a) Determine the x- and y-component of the velocity field $\vec{v}(x, y)$. Sketch the velocity field. Also sketch the streamline going through the point $(x, y) = (1\,\mathrm{m}, 1\,\mathrm{m})$.

(b) Determine the mass flux going through the "area" between the two points $(0\,\mathrm{m},\ 0\,\mathrm{m})$ and $(1\,\mathrm{m},\ 1\,\mathrm{m})$.

(c) Determine the mass flux going through the "area" between the two points $(4\,\mathrm{m},\ 0\,\mathrm{m})$ and $(4\,\mathrm{m},\ 0.25\,\mathrm{m})$.

(d) Why are the results of (b) and (c) identical?

(e) We can use the velocity field from (a) to approximately describe the flow in a pipe with corner. Given $a = 10\,\mathrm{cm}$ and $b = 1\,\mathrm{m}$, how large should the length c be?



(f) Now, consider the velocity potential $\Phi(x, y) = Axy$. Determine the x- and y.component of the velocity field $\vec{v}(x, y)$. Sketch the velocity field and the streamlines.

(g) How is the flow in (f) related to the flow in (a)?

Solution

(a)

$$v_x = \frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y} Axy = Ax$$
$$v_y = -\frac{\partial \Psi}{\partial x} = -\frac{\partial}{\partial x} Axy = -Ay$$
$$\vec{v} = \begin{pmatrix} Ax \\ -Ay \end{pmatrix}$$

The velocity field is visualised in figure 1.

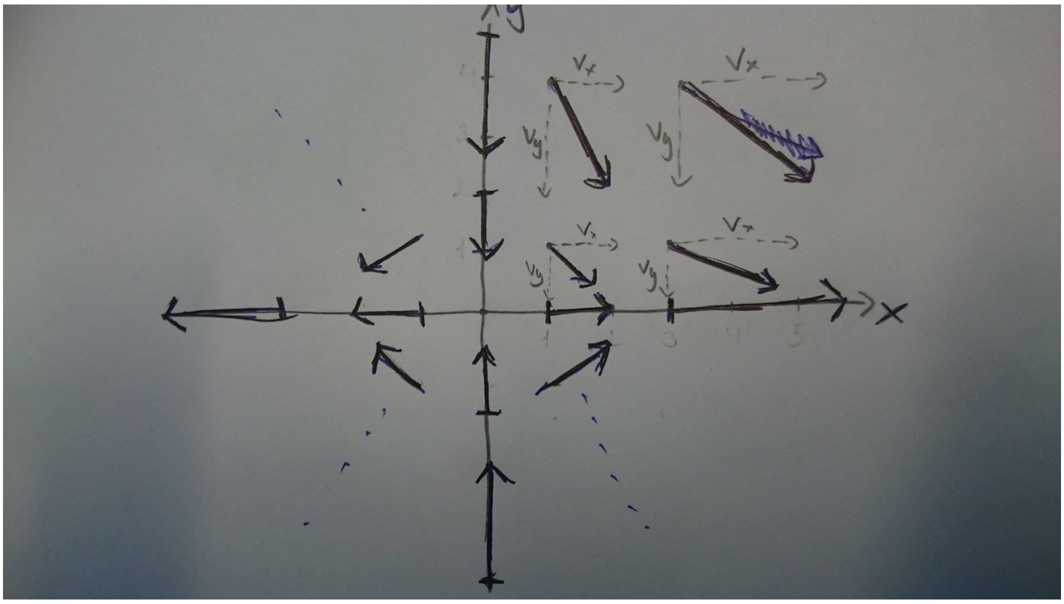


Figure 1: Velocity field $\vec{v}$

Streamlines are found by putting $\Psi(x, y) = $ const. $= c$.

$$\Psi(1,1) = 1 * 1 * 1 = 1 = c$$

$$\Psi(x, y) = c \overset{\text{Result from above}}{\Rightarrow} Axy = 1 \Rightarrow y = \frac{c}{A}\frac{1}{x} \overset{\text{c=1 and A=1}}{\Rightarrow} y = \frac{1}{x}$$

The streamline through $(x, y) = (1\text{m}, 1\text{m})$ is visualised in figure 2. Keep in mind that the stream function is constantly equal 1 on this streamline.

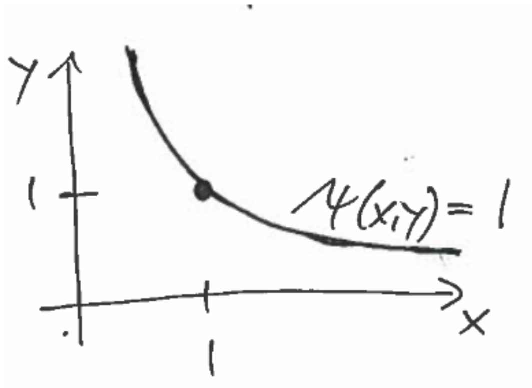


Figure 2: Streamline through the point $(x, y) = (1\text{m}, 1\text{m})$.

(b) The mass flux is found by using the following formula:

$$\frac{dM}{dt} = \rho \Delta z \int (\vec{v} \cdot \vec{n}) \, \mathrm{ds} \tag{1}$$

The unit normal, $\vec{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, is perpendicular to the line between $(x, y) = (0\text{m}, 0\text{m})$ and $(x, y) = (1\text{m}, 1\text{m})$ and parallel to $\vec{v}$.

$\mathrm{ds} = \sqrt{\mathrm{dx}^2 + \mathrm{dy}^2}$ is a small line segment on the line between $(x, y) = (0\text{m}, 0\text{m})$ and $(x, y) = (1\text{m}, 1\text{m})$. See figure 3.

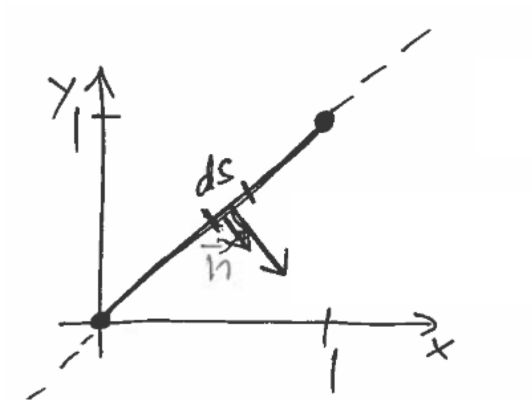


Figure 3: Mass flux through the between $(x, y) = (0\text{m}, 0\text{m})$ and $(x, y) = (1\text{m}, 1\text{m})$.

Use the expressions for $\vec{n}, \vec{v}$ and ds in formula 1

$$
\begin{aligned}
\frac{dM}{dt} &= \rho\Delta z \int (\vec{v}\cdot\vec{n})\,\mathrm{ds} \\
&= \frac{\rho\Delta z}{\sqrt{2}} \int_0^{\sqrt{2}} \left(\begin{pmatrix} Ax \\ -Ay \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \sqrt{\mathrm{dx}^2 + \mathrm{dy}^2} \\
&= \frac{\rho\Delta z}{\sqrt{2}} \int_0^{\sqrt{2}} (Ax + Ay)\sqrt{\mathrm{dx}^2 + \mathrm{dy}^2} \\
&\overset{\text{x=y}}{=} \frac{2A\rho\Delta z}{\sqrt{2}} \int_0^1 \sqrt{2}x\,\mathrm{dx} \\
&= 2A\rho\Delta z \left(\frac{1^2}{2} - \frac{0^2}{2} \right) \\
&= A\rho\Delta z
\end{aligned}
$$

(c) We use equation 1 again. $\vec{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. See figure 4.

$$
\begin{aligned}
\frac{dM}{dt} &= \rho\Delta z \int (\vec{v}\cdot\vec{n})\,\mathrm{ds} \\
&= \rho\Delta z \int_0^{\frac{1}{4}} \left(\begin{pmatrix} Ax \\ -Ay \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \sqrt{\mathrm{dx}^2 + \mathrm{dy}^2} \\
&= \rho\Delta z \int_0^{\frac{1}{4}} Ax\sqrt{\mathrm{dx}^2 + \mathrm{dy}^2} \\
&\overset{\text{dx=0 and x=4}}{=} 4A\rho\Delta z \int_0^{\frac{1}{4}} \mathrm{dy} \\
&= A\rho\Delta z
\end{aligned}
$$

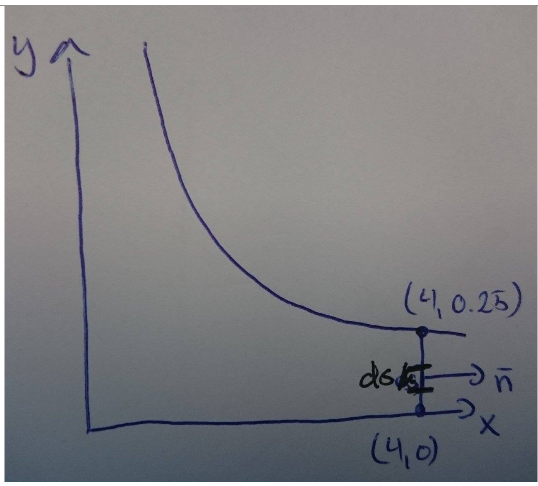


Figure 4: Mass flux through the between $(x, y) = (4\text{m}, 0\text{m})$ and $(x, y) = (4\text{m}, 0.25\text{m})$.

(d) The flow is along the negative y-direction and positive x-direction, as the vector field and streamlines indicate. There is no out flow through the walls. The integral from part (b) and (c) are equal since what flows in must flow out. Equation of continuity, Leonardos law.

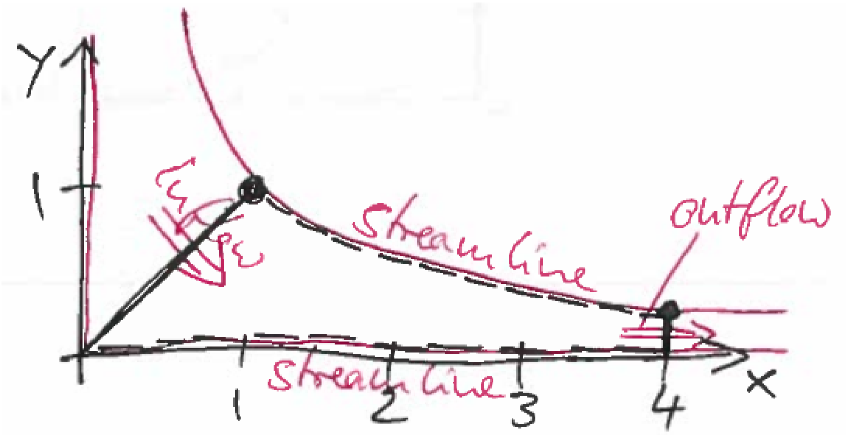


Figure 5: Flow through the object.

(e) Remember that $\Psi(x, y) = c$ along one specific streamline, see figure 6. This fact will be used to solve this part.

$$\Psi\left(x = 0.1, y = 1\right) = 0.1 \cdot 1 \cdot 1 = 0.1 = \frac{1}{10}$$

$$\Psi\left(x = \frac{c}{\sqrt{2}}, y = \frac{c}{\sqrt{2}}\right) = \frac{c}{\sqrt{2}} \cdot \frac{c}{\sqrt{2}} = \frac{c^2}{2}$$

$$\Psi\left(x = 0.1, y = 1\right) \stackrel{!}{=} \Psi\left(x = \frac{c}{\sqrt{2}}, y = \frac{c}{\sqrt{2}}\right)$$

$$\Downarrow$$

$$\frac{1}{10} = \frac{c^2}{2}$$

$$\Downarrow$$

$$c = \frac{1}{\sqrt{5}}$$

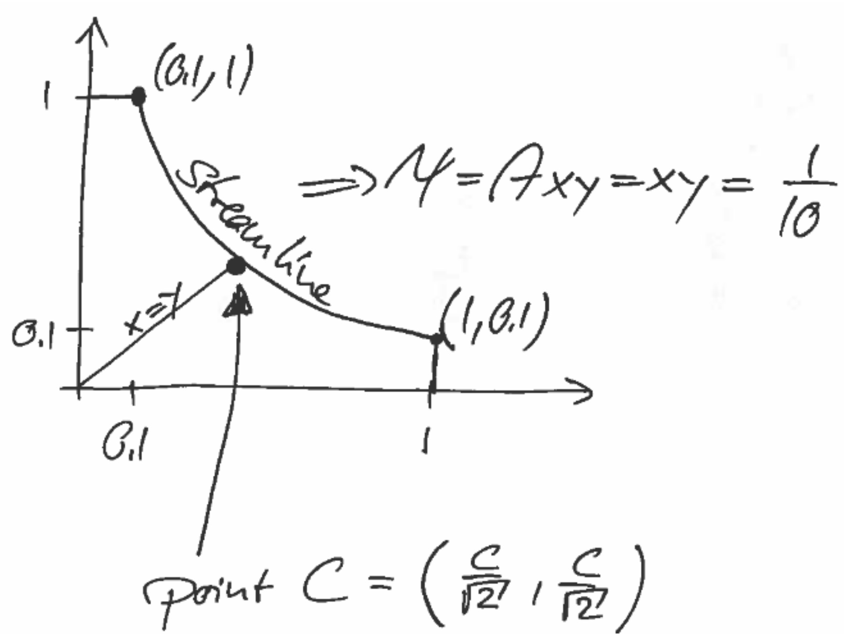


Figure 6: Illustration of the problem in this exercise.

(f)

$$
\begin{aligned}
v_x &= \frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x} Axy = Ay \\
v_y &= \frac{\partial \Phi}{\partial y} = \frac{\partial}{\partial x} Axy = Ax \\
\vec{v} &= \begin{pmatrix} Ay \\ Ax \end{pmatrix}
\end{aligned}
$$

The velocity field and streamlines are seen in figure 7

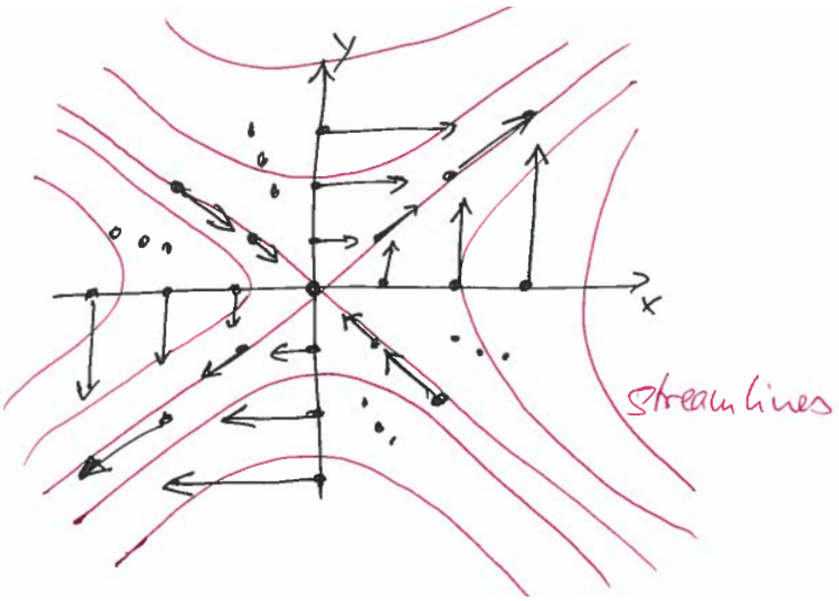


Figure 7: Velocity field and streamlines for $\Psi(x, y)$

(g) The flow in part (a) is rotated by 45 degrees in clockwise direction.

Introduction

The exercises for this week are strongly connected as they all describe a potential flow resulting from an ideal source.

In exercise 3.1 we will show that the stream lines (characterised by a constant stream function $\Phi(x, y) = c_1$) and the isopotential lines (characterised by a constant velocity potential $\Psi(x, y) = c_2$) are orthogonal to each other.

In exercise 3.2 we will treat a potential flow resulting from an ideal source located at the origin.

In exercise 3.3 a wall along the y-axis is introduced.

In exercise 3.4 we will treat a potential flow resulting from an ideal source combined with a free flow.

2 Exercise 3.1

Show that for a potential flow the streamlines and the isopotential lines are orthogonal to each other. The streamlines are characterized by the stream function $\Psi(x, y)$ = constant = c_1, and the isopotential lines are characterized by the velocity potential $\Phi(x, y)$ = constant = c_2.

Solution

When $\Phi = \text{const}_1$, and $\Psi = \text{const}_2$ represent an orthogonal set of curves, because:

$$\left(\vec{\nabla}\Phi\right) \cdot \left(\vec{\nabla}\Psi\right) = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \Psi}{\partial x} \\ \frac{\partial \Psi}{\partial y} \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \cdot \begin{pmatrix} -v_y \\ v_x \end{pmatrix} = 0$$

When $\Phi(x, y)$ = const. we will have the following relation:

$$0 = d\Phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} = \vec{\nabla}\Phi \cdot d\vec{s}$$

3 Exercise 3.2

The velocity potential

$$\Phi(x,y) = \frac{m}{2\pi} \ln \sqrt{x^2 + y^2} = \frac{m}{2\pi} \ln r \tag{2}$$

describes the potential flow resulting from an ideal source located at the origin; see the figure below.

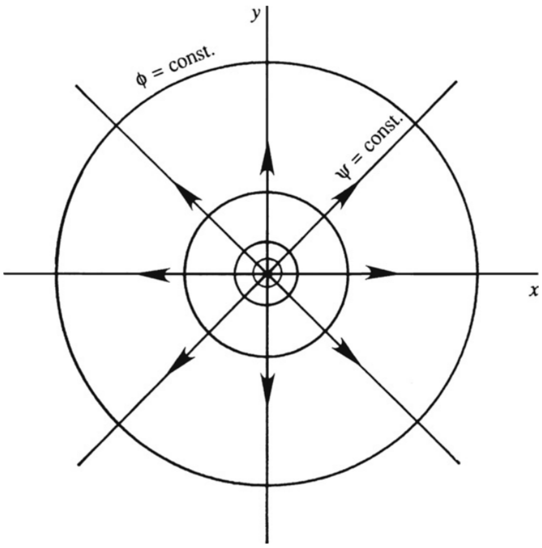


Figure 8: The flow field of an ideal source located at the origin of coordinates in two dimensions. The streamlines are radials and the potential lines are circles. Figure 6.5 from the KCD book

(a) Determine the velocity components v_x and v_y.

(b) Determine the stream function $\Psi(x,y)$.

(c) Visualize the streamlines by plotting the contour lines of the stream function. Compare your results with the figure above.

(d) What is the fluid mass flowing through a circle with radius R? What is the interpretation of the parameter m?

Solution

(a)

$$v_x = \frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{m}{2\pi}\ln\sqrt{x^2+y^2}\right) = \frac{m}{2\pi}\frac{1}{\sqrt{x^2+y^2}}\frac{2x}{2\sqrt{x^2+y^2}}$$
$$= \frac{m}{2\pi}\frac{x}{x^2+y^2} = \frac{m}{2\pi}\frac{r\cos\phi}{r^2} = \frac{m}{2\pi}\frac{\cos\phi}{r}$$

$$v_y = \frac{\partial \Phi}{\partial y} = \frac{\partial}{\partial x}\left(\frac{m}{2\pi}\ln\sqrt{x^2+y^2}\right) = \frac{m}{2\pi}\frac{1}{\sqrt{x^2+y^2}}\frac{2y}{2\sqrt{x^2+y^2}}$$
$$= \frac{m}{2\pi}\frac{y}{x^2+y^2} = \frac{m}{2\pi}\frac{r\sin\phi}{r^2} = \frac{m}{2\pi}\frac{\sin\phi}{r}$$

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{m}{2\pi r}\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \frac{m}{2\pi r}\vec{e_r}$$

(b) $\Psi = \text{const.}$ leads to stream lines parallel to $\vec{v} \parallel \vec{e_r}$. As seen in figure 8 and 9, streamlines are in the radial direction, characterised by angle $\phi = \text{const.}$

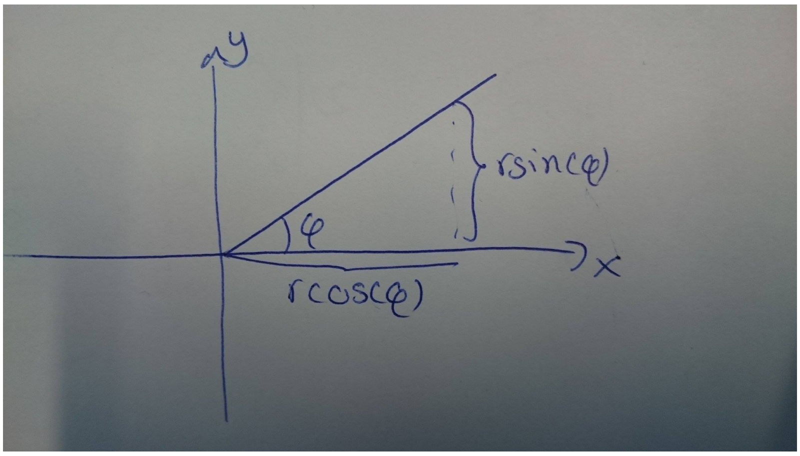


Figure 9: For a constant value of ϕ the stream line propagating radially out will be constant with respect to the angle ϕ.

As a consequence we will state the following relationship:

$$\Psi = a\phi \tag{3}$$

where a is a constant to be determined.

For this we will make use of the following trigonometric relation in both cartesian and polar coordinates:

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{y}{x}$$

The calculation of the constant a consists of fours steps as below:

step 1: (differentiate $\tan\phi$ with respect to ϕ)

$$\begin{aligned}\frac{d\tan\phi}{d\phi} &= \frac{d}{d\phi}\left(\frac{\sin\phi}{\cos\phi}\right) = \frac{\cos\phi\cos\phi + \sin\phi\sin\phi}{\cos^2\phi} \\ &= 1 + \frac{\sin^2\phi}{\cos^2\phi} = 1 + \tan^2\phi\end{aligned}$$

step 2: (differentiate $\tan\phi(x,y)$ with respect to x)

$$\frac{\partial\tan\phi(x,y)}{\partial x} \overset{\text{chain rule}}{=} \frac{\partial\tan\phi}{\partial\phi}\frac{\partial\phi}{\partial x} \overset{\text{step 1}}{=} \left(1+\tan^2\phi\right)\frac{\partial\phi}{\partial x} \tag{4}$$

$$\tan\phi = \frac{y}{x} \Rightarrow \frac{\partial\tan\phi}{\partial x} = \frac{-y}{x^2} \tag{5}$$

step 3: unite equation 4 and 5

$$\begin{aligned}\frac{\partial\tan\phi(x,y)}{\partial x} &= \frac{\partial\tan\phi}{\partial x} \\ &\Downarrow \\ \left(1+\tan^2\phi\right)\frac{\partial\phi}{\partial x} &= \frac{-y}{x^2} \\ &\Downarrow \\ \frac{\partial\phi}{\partial x} &= \frac{1}{1+\tan^2\phi}\frac{-y}{x^2} \\ &= \frac{1}{1+\frac{y^2}{x^2}}\frac{-y}{x^2} \\ &= \frac{-y}{x^2+y^2}\end{aligned} \tag{6}$$

$\partial\phi/\partial x$ is now know. The same calculation has to be performed to determine $\partial\phi/\partial y$.

$$\frac{\partial\phi}{\partial y} = \frac{x}{x^2+y^2} \tag{7}$$

step 4: Unite equation 3, 6 and 7

$$\frac{\partial \Psi}{\partial x} = a\frac{\partial \phi}{\partial x} = -a\frac{y}{x^2+y^2} \stackrel{!}{=} -v_y = -\frac{m}{2\pi}\frac{y}{x^2+y^2} \Rightarrow a = \frac{m}{2\pi} \tag{8}$$

This result will be used in the differentiation with respect to y:

$$\frac{\partial \Psi}{\partial y} = \frac{m}{2\pi}\frac{\partial \phi}{\partial y} = \frac{m}{2\pi}\frac{x}{x^2+y^2} = v_x \text{ as it should be} \tag{9}$$

This result will be used in exercise 3.3 and 3.4 to simplify the calculations when finding the stream function.

To determine $\Psi(x,y)$ we will integrate equation 8 and 9 with respect to x and y, respectively.

$$\begin{gathered}\frac{\partial \Psi(x,y)}{\partial y} = \frac{m}{2\pi}\frac{x}{x^2+y^2} \\ \Downarrow \\ \int \partial \Psi(x,y) = \frac{m}{2\pi}x\int \frac{1}{x^2+y^2}dy \\ \Downarrow \\ \Psi(x,y) = \frac{m}{2\pi}x\int \frac{1}{x^2+y^2}dy\end{gathered}$$

To integrate the right hand side we will make use of the relation:

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

In appendix 1 this relation is showed.

$$\begin{aligned}\Psi(x,y) &= \frac{m}{2\pi}x\int \frac{1}{x^2+y^2}dy \\ &= \frac{m}{2\pi}x\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right) + f(x) \\ &= \frac{m}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) + f(x) \qquad (10)\end{aligned}$$

since $\Psi(x,y)$ is both x and y dependent and we only integrate with respect to y there could be some x-dependency which we add as $f(x)$.

The same integration is performed with respect to x:

$$\begin{aligned}
\frac{\partial\Psi(x,y)}{\partial x} &= -\frac{m}{2\pi}\frac{y}{x^2+y^2} \\
&\Downarrow \\
\int \partial\Psi(x,y) &= -\frac{m}{2\pi}y\int\frac{1}{x^2+y^2}dx \\
&\Downarrow \\
\Psi(x,y) &= -\frac{m}{2\pi}y\int\frac{1}{x^2+y^2}dy \\
&= -\frac{m}{2\pi}y\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)+g(y) \\
&= -\frac{m}{2\pi}\tan^{-1}\left(\frac{x}{y}\right)+g(y) \qquad (11) \\
& \qquad (12)
\end{aligned}$$

Since $\Psi(x,y)$ is both x and y dependent and we only integrate with respect to x there could be some y-dependency which we add as $g(y)$. Realising that

$$\tan\phi = \frac{y}{x} \text{ and } \tan\left(\frac{\pi}{2}-\phi\right) = \frac{x}{y} \qquad (13)$$

we can rewrite equation 10 and 12 as

$$\begin{aligned}
\Psi_1(x,y) &= \frac{m}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)+f(x) = \frac{m}{2\pi}\tan^{-1}\left(\tan\phi\right)+f(x) \\
&= \frac{m}{2\pi}\phi+f(x) \qquad (14)
\end{aligned}$$

$$\begin{aligned}
\Psi_2(x,y) &= -\frac{m}{2\pi}\tan^{-1}\left(\frac{x}{y}\right)+g(y) = -\frac{m}{2\pi}\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\phi\right)\right)+g(y) \\
&= -\frac{m}{2\pi}\left(\frac{\pi}{2}-\phi\right)+g(y) = \frac{m}{2\pi}\phi-\frac{m}{4}+g(y) \qquad (15)
\end{aligned}$$

Documentation for the relations in 13 can be found in appendix 2.

$$\begin{aligned}
\Psi_1(x,y) &\overset{!}{=} \Psi_2(x,y) \\
\frac{m}{2\pi}\phi+f(x) &= \frac{m}{2\pi}\phi-\frac{m}{4}+g(y) \\
f(x) &= -\frac{m}{4}+g(y)
\end{aligned}$$

For the right hand side to equal the left hand side, both expressions must be constant. This is due to the dependency of different variables, respectively. $\vec{v}$ is important as it does not change if wee add a constant to Ψ. This leads to:

$$f(x) = c_1 = 0$$
$$g(y) = c_1 + \frac{m}{4} = \frac{m}{4}$$

Finally, replacing $f(x)$ and $g(x)$ in 14 and 15 by the relation above, we determine $\Psi(x,y)$ as:

$$\Psi(x,y) = \frac{m}{2\pi}\phi$$

Overview:

$$\Phi(x,y) = \frac{m}{2\pi}\ln r$$

$$\Psi(x,y) = \frac{m}{2\pi}\phi$$

These functions are in correspondence with figure 8. Φ is radial dependent and Ψ is angle dependent.

(c) Figure 10 shows the contour lines of the stream function.

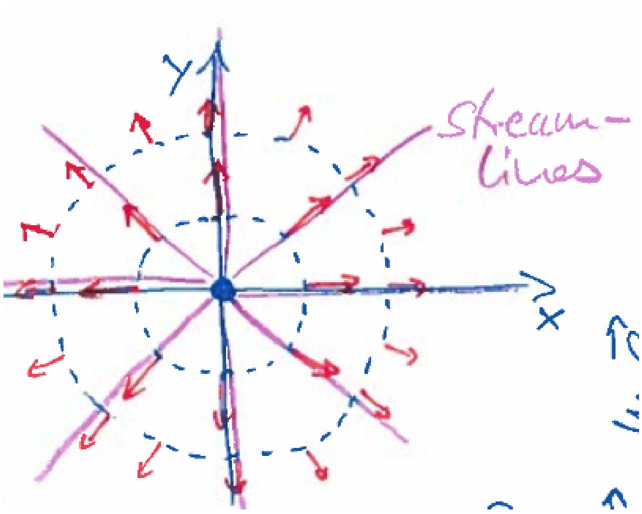


Figure 10: Visualisation of the streamlines by plotting the contour lines of the stream function

(d) The mass flux flowing through an obstacle is given by the following formula:

$$\frac{\Delta M}{\Delta t} = \rho \int (\vec{v} \cdot \vec{n}) ds \tag{16}$$

From part (a) we have that $\vec{v} = \frac{m}{2\pi r}\begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix}$, the normal vector $\vec{n} = \begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix}$ and $ds = rd\phi$. Applying these quantities in equation 16 we end up with

$$\begin{aligned}
\frac{\Delta M}{\Delta t} &= \rho \int (\vec{v} \cdot \vec{n}) ds \\
&= \frac{m}{2\pi}\rho \int_0^{2\pi} \frac{1}{r} \left(\begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix}\right) \cdot \left(\begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix}\right) rd\phi \\
&= \frac{m}{2\pi}\rho \int_0^{2\pi} (\cos^2\phi + \sin^2\phi) d\phi \\
&= \frac{m}{2\pi}\rho \int_0^{2\pi} d\phi \\
&= m\rho \\
&\Downarrow \\
m &= \frac{1}{\rho}\frac{\Delta M}{\Delta t}
\end{aligned}$$

4 Exercise 3.3

What potential flow is described by the following velocity potential?

$$\Phi(x,y) = \frac{m}{2\pi} \ln \sqrt{(x-a)^2 + y^2} + \frac{m}{2\pi} \ln \sqrt{(x+a)^2 + y^2} \tag{17}$$

(a) Determine the velocity components v_x and v_y.

(b) Determine the stream function $\Psi(x,y)$.

(c) Visualize the streamlines by plotting the contour lines of the stream function.

(d) Why does the line $x = 0$ represent a streamline? Determine the velocity components v_x and v_y at $x = 0$.

Solution

The velocity potential in exercise 3.2 is similar to the velocity potential in this exercise and as consequence we can reuse the results from the former. In this exercise the source is not placed at origo, but at some position $x = a$. That is why x is replaced by $x - a$ in this exercise.

Furthermore, in this exercise we introduce a wall on the y-axis, which we take into consideration by imagining a mirror source on the position $x = -a$, therefore the velocity potential in this exercise introduces a second term.

(a) Using the results from exercise 3.1 we can state v_x and v_y without calculating.

$$v_x = \frac{m}{2\pi}\frac{x-a}{(x-a)^2+y^2} + \frac{m}{2\pi}\frac{x+a}{(x+a)^2+y^2}$$

$$v_y = \frac{m}{2\pi}\frac{y}{(x-a)^2+y^2} + \frac{m}{2\pi}\frac{y}{(x+a)^2+y^2}$$

(b) In exercise 3.1 we saw that for an ideal source $\Psi(x,y) = \frac{m}{2\pi}\phi$. In this exercise we have an ideal source but also an imaginary source, both with streamlines propagating radially out. The angle of the streamlines with the x-axis is called ϕ_1 and ϕ_2, see figure 11. By this we state $\Psi(x,y)$ as:

$$\Psi(x,y) = \frac{m}{2\pi}(\phi_1 + \phi_2)$$

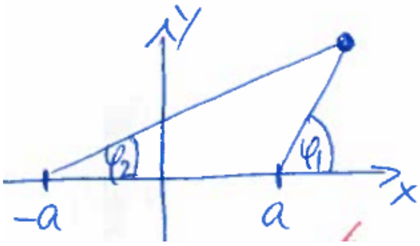


Figure 11: Visualisation of the streamlines with angles ϕ_1 and ϕ_2.

(c) To find the streamlines we put $\Psi = const.$ and as a consequence $\phi_1 + \phi_2 = const.$

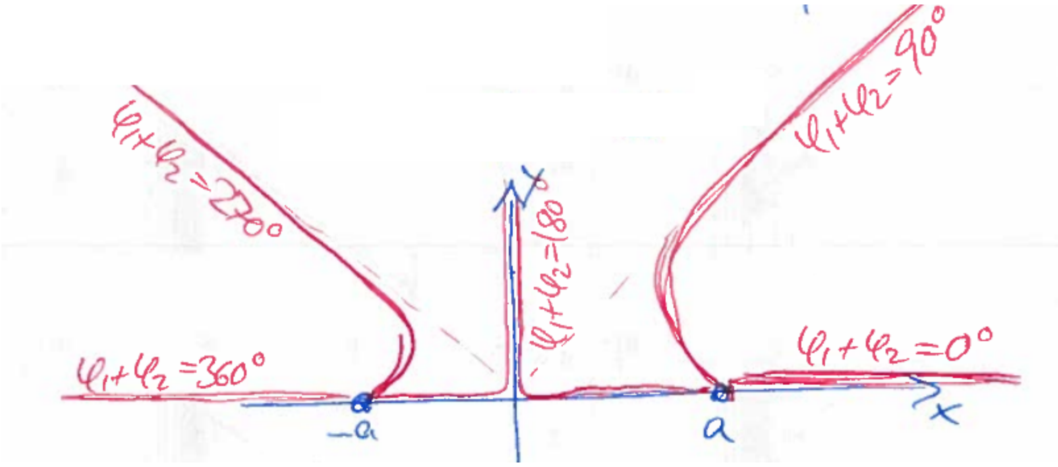


Figure 12: Visualisation of the streamlines for $\Psi(x, y)$

(d) At $x = 0$ there will be no x component of the velocity field as:

$$
\begin{aligned}
v_x &= \frac{m}{2\pi} \frac{x-a}{(x-a)^2 + y^2} + \frac{m}{2\pi} \frac{x+a}{(x+a)^2 + y^2} \\
&\overset{x=0}{=} \frac{m}{2\pi} \frac{-a}{(a^2 + y^2} + \frac{m}{2\pi} \frac{a}{a^2 + y^2} = 0
\end{aligned}
$$

and

$$\vec{v}\,|_{x=0} = v_y \vec{e_y} \| \text{ streamline!}$$

5 Exercise 3.4

The velocity potential

$$\Phi(x, y) = v_\infty x + \frac{m}{2\pi} \ln \sqrt{x^2 + y^2} \tag{18}$$

$$= v_\infty r \cos\theta + \frac{m}{2\pi} \ln r \tag{19}$$

describes a potential flow around a half-body; see the figure below.

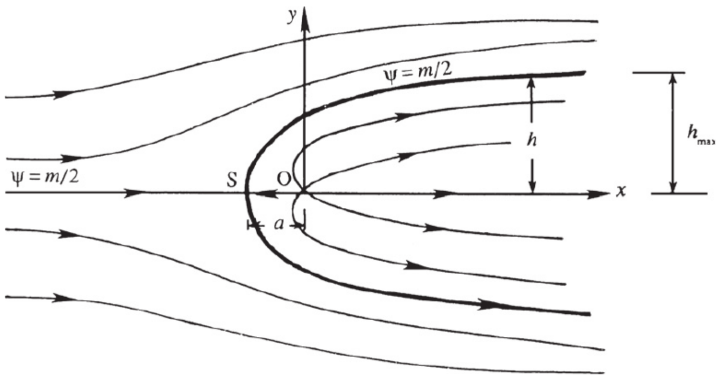


Figure 13: Ideal flow past a two-dimensional half-body formed from a horizontal free stream and a point source at the origin. The boundary streamline, shown as a darker curve, is given by $\Psi = m/2$. Figure 6.7 from the KCD book

(a) Determine the velocity components v_x and v_y.

(b) Determine the stream function $\Psi(x, y)$.

(c) Visualize the streamlines by plotting the contour lines of the stream function. Compare your results with the figure above.

(a) Once again we treat an ideal source with a "free-stream" modification in the x-direction as $v_\infty x$. Using the result from exercise 3.1 we write v_x and v_y as:

$$v_x = v_\infty + \frac{m}{2\pi}\frac{x}{x^2+y^2}$$

$$v_y = \frac{m}{2\pi}\frac{y}{x^2+y^2}$$

(b) As earlier, stating that $\Psi(x,y) = \frac{m}{2\pi}\phi$ we write $\Psi(x,y)$ as:

$$\Psi(x,y) = v_\infty y + \frac{m}{2\pi}\phi$$

$$= v_\infty r \sin\phi + \frac{m}{2\pi}\phi$$

(c) Figure 14 shows the streamlines for $\Psi(x,y)$.

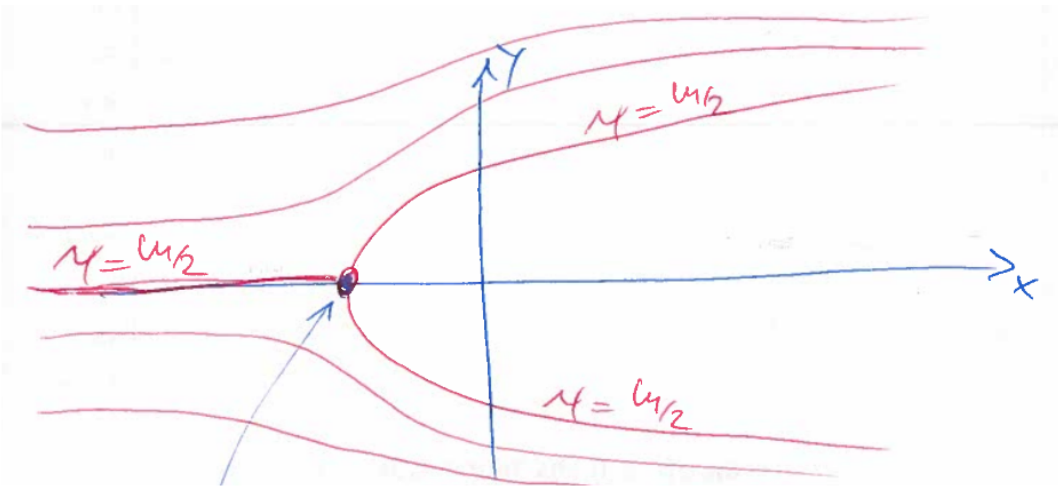


Figure 14: Visualisation of the streamlines for $\Psi(x,y)$. The blue arrow is pointing at the stagnation point.

At the stagnation point $v_x = v_y = 0$. In case of $v_y = 0$ we have:

$$\text{if } v_y = 0 \Rightarrow y = 0$$
$$\text{if } v_x = 0 \Rightarrow v_\infty + \frac{m}{2\pi}\frac{1}{x} = 0 \Rightarrow x = -\frac{m}{2\pi v_\infty}$$

and Ψ evaluated at this set of (x,y) gives:

$$\Psi(x = \frac{m}{2\pi v_\infty}, 0) = \frac{m}{2}$$

As seen in figure 13 and 14.

Appendix 1

Here we show the following relation:

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \tag{20}$$

We will make use of the following substitution: $x = a \tan y$

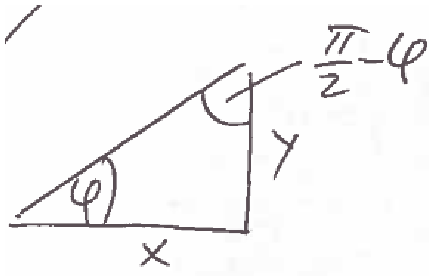
$$\begin{aligned} x &= a \tan y \qquad (21) \\ &\Downarrow \\ \frac{dx}{dy} &= \frac{a}{\cos^2 y} \\ &\Downarrow \\ dx &= a \frac{dy}{\cos^2 y} \qquad (22) \end{aligned}$$

Substitute equation 21 and 22 into 20:
We first deal with the denominator in equation 20

$$\begin{aligned} a^2 + x^2 = a^2 + \left(a^2 + \tan^2 y\right) &= a\left(1 + \tan^2 y\right) \\ = a^2 \frac{\cos^2 y + \sin^2 y}{\cos^2 y} &= \frac{a^2}{\cos^2 y} \end{aligned}$$

$$\begin{aligned} \int \frac{1}{a^2 + x^2} dx = \int \frac{\cos^2 y}{a^2} \frac{a}{\cos^2 y} dy = \frac{1}{a} \int dy = \frac{y}{a} \\ = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \end{aligned}$$

6 Appendix 2



$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{y}{x}$$

$$\cot\phi = \frac{\cos\phi}{\sin\phi} = \frac{x}{y} = \tan\left(\frac{\pi}{2} - \phi\right)$$