# Fluid Dynamics + Turbulence (fall 2017)
# Homework Problems II + voluntary Exercises

**Posted:** August 28, 2018

**Deadline:** September 11 (Tuesday) at 08:30 am (on Blackboard).

## Homework problem 1.1: Why does an Airbus A380 fly?

**(a)** Use the Navier-Stokes equation in the static limit $\vec{u} = 0$, the gravitational force $\vec{f}_{\text{ext}} = -\rho g \vec{e}_z$, and the equation of state $p/p_0 = \rho/\rho_0$ to derive the barometric height formula

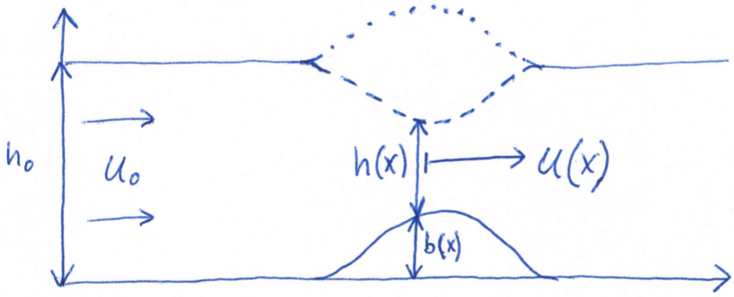
$$\frac{p(z)}{p_0} = \frac{\rho(z)}{\rho_0} = \exp\left(-\frac{g\rho_0 z}{p_0}\right),$$

where $z$ is the height above ground, $p_0 = p(z=0) = 1.01 \times 10^5 \, \frac{\text{kg}}{\text{m}\,\text{sec}^2}$ is the air pressure at ground, $\rho_0 = \rho(z=0) = 1.20\,\text{kg/m}^3$ is the air density at ground, and $g = 9.81\,\text{m/sec}^2$ is the acceleration of gravity. How much is the air density reduced at the height $z = 10\,\text{km}$ above ground?

**(b)** An Airbus A380 is typically cruising with the velocity $u = 945\,\text{km/h}$ at a height 10 km above ground. The lift force acting on the air wings is balancing the weight force. Use Bernoulli's equation to make a rough estimate of the velocity difference above and below the air wings. All you need are the results from (a), the wing area $A_{wing} = 846\,\text{m}^2$ and the weight $m \approx 500\,\text{t}$ of the A380.

## Homework problem 2.2: Surface bump or dip in a shallow river flow?

Consider an ideal two-dimensional shallow river flow over a small bump at the river bed; see sketch.




**(a)** Explain the two conservation laws:

$$u_0 h_0 = u(x)h(x)\,, \tag{1}$$

$$\frac{u_0^2}{2} + g h_0 = \frac{u^2(x)}{2} + g\,(b(x) + h(x))\ . \tag{2}$$

**(b)** Equations (1) and (2) are two equations for the two unknown functions $h(x)$ and $u(x)$. Eliminate $u(x)$ and derive an equation only for $h(x)$. Show that this equation is consistently solved by $h(x) = h_0$ when $b(x) = 0$. In order to solve the equation also for $b(x) > 0$, assume the bump $b(x) \ll h_0$ and the change of river surface height $\Delta h(x) = h(x) - h_0 \ll h_0$ to be small. Derive the solution

$$\Delta h(x) = \frac{b(x)}{\left(\frac{u_0^2}{g h_0} - 1\right)} \tag{3}$$

by keeping first-order terms in small $b(x)/h_0$ and $\Delta h(x)/h_0$ in a consistent way.

**(c)** Use equation (3) in the two limits $u_0^2 \ll g h_0$ and $u_0^2 \gg g h_0$ to explain when a surface bump or dip occurs.

**(d)** What is the catch with equation (3) when $u_0^2 \approx g h_0$?

**(e)** Use the numbers $b_{max} = 0.1\,\mathrm{m}$, $h_0 = 1\,\mathrm{m}$, $g = 9.81\,\mathrm{m/s^2}$ to calculate the extremum of $\Delta h(x)$ for $u_0 = 1\,\mathrm{m/s}$ and $u_0 = 10\,\mathrm{m/s}$.

## Exercise 2.1

The friction force in the Navier-Stokes equation

$$\rho \left( \frac{\partial \vec{u}}{\partial t} + \left( \vec{u} \cdot \vec{\nabla} \right) \vec{u} \right) = \vec{f}_{\text{ext}} - \vec{\nabla} p + \mu \left( \vec{\nabla} \cdot \vec{\nabla} \right) \vec{u} + \left( \mu_v + \frac{\mu}{3} \right) \vec{\nabla} \left( \vec{\nabla} \cdot \vec{u} \right) \quad (4)$$

contains the term

$$\frac{\partial^2 u_y}{\partial x \partial y} \quad (5)$$

in x-direction. Why does this term represent a force pointing in the x-direction?

## Exercise 2.2

An infinitesimal closed line integral can be approximated as

$$\begin{aligned} \oint \vec{v}(x,y) \cdot d\vec{s} \approx v_x & \left( x, y + \frac{\Delta y}{2} \right) \Delta x \\ + v_y & \left( x + \frac{\Delta x}{2}, \Delta y \right) (-\Delta y) \\ + v_x & \left( x, y - \frac{\Delta y}{2} \right) (-\Delta x) \\ + v_y & \left( x - \frac{\Delta x}{2}, y \right) \Delta y. \end{aligned} \quad (6)$$

Use a Taylor series expansion and show that

$$\oint \vec{v} \cdot d\vec{s} = - \left( \vec{\nabla} \times \vec{v} \right) \cdot \vec{e}_z \Delta x \Delta y \quad (7)$$

Given this result, illustrate what the irrotational flow condition $\vec{\nabla} \times \vec{v} = 0$ means.

## Exercise 2.3

Consider the following two-dimensional velocity field:

$$u_x = u_0 \quad (8)$$

$$u_y = \kappa x^2 \quad (9)$$

**(a)** Sketch the velocity vector at selected positions along the $x$-axis.

**(b)** Calculate the divergence $\vec{\nabla} \cdot \vec{u}$ and curl $\vec{\nabla} \times \vec{u}$ of the velocity field.

**(c)** Use the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_x, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = u_y \tag{10}$$

to show that if a particle is in position $(x_0, y_0)$ at time $t_0$, then at time $t$ its position is given by

$$x(t) = x_0 + u_0(t - t_0) \tag{11}$$

$$y(t) = y_0 + \frac{\kappa}{3u_0}\left[x(t)^3 - x_0^3\right]. \tag{12}$$

**(d)** Sketch some particle trajectories (path lines) with $t_0 = x_0 = 0$.

## Exercise 2.4

When we derived the Bernoulli equation

$$\frac{\rho_0}{2}\vec{v}^2 + p = \text{constant}, \tag{13}$$

we stumbled over the relation

$$\vec{\nabla}\left(\frac{\rho_0}{2}\vec{v}^2 + p\right) = \rho_0\vec{v} \times \left(\vec{\nabla} \times \vec{v}\right), \tag{14}$$

where we simply evoked the irrotational flow condition to put the right-hand side to zero. Now we want to be more careful.

**(a)** Show that when we multiply (scalar product) the right-hand side with a segment $d\vec{s}$ of a streamline, that

$$d\vec{s} \cdot \left[\vec{v} \times \left(\vec{\nabla} \times \vec{v}\right)\right] = 0. \tag{15}$$

**(b)** Show that the integration of the left-hand side,

$$\int \vec{\nabla}\left(\frac{\rho_0}{2}\vec{v}^2 + p\right) \cdot d\vec{s} = 0, \tag{16}$$

along a streamline, leads to

$$\frac{\rho_0}{2}\vec{v}^2 + p = \text{constant} = c \tag{17}$$

along the streamline.

**(c)** What can you say about the constant $c$ for different stream lines?