Lecture Notes on Fluid Dynamics

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1 Warm-up: poor man's approach to Fluid Dynamics

This simple approach is capable of quite a few important applications!

1.1 Leonardo's Law: mass conservation

What streams into a volume has to stream out again.

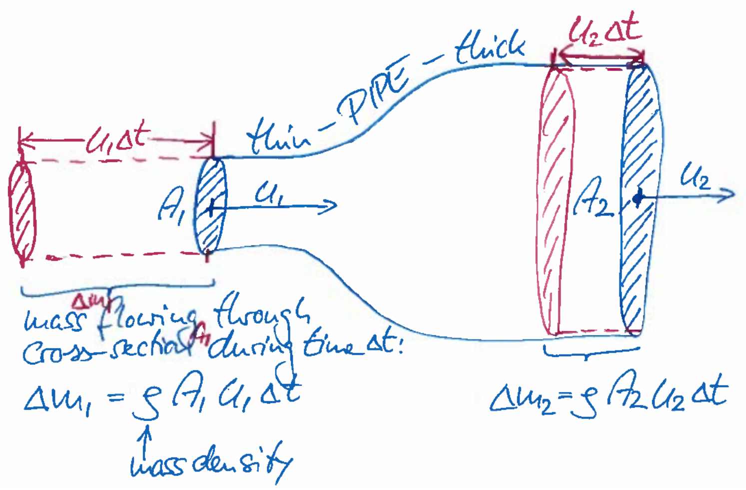

Figure 1: An illustration of mass conservation.

Mass conservation means that the inflow on the left side must equal the outflow on the right side. That is

$$\Delta m_1 = \Delta m_2 \tag{1.1}$$
$$\Downarrow$$
$$\rho A_1 u_1 \Delta t = \rho A_2 u_2 \Delta t \tag{1.2}$$
$$\Downarrow$$
$$A_1 u_1 = A_2 u_2 \tag{1.3}$$

Here (1.3) is known as Leonardo's Law. It has the following properties:

$$A_1 < A_2 \Rightarrow u_1 > u_2 \tag{1.4}$$
$$A_1 > A_2 \Rightarrow u_1 < u_2. \tag{1.5}$$

1.1.1 Example 1: why is it always windy on Aarhus Ø?

In front of the houses:

$$\Delta m_1 = \rho_{\text{air}} A_1 u_1 \Delta t \tag{1.6}$$

Between the two houses:

$$\Delta m_2 = \rho_{\text{air}} A_2 u_2 \Delta t \tag{1.7}$$

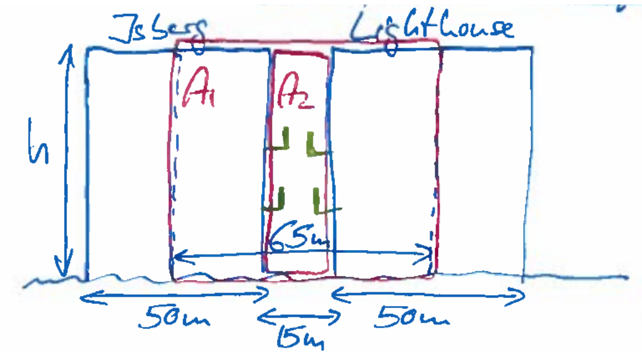

Figure 2: The gap between adjacent apartment buildings seen from the side.

Equating (1.6) and (1.7) gives

$$u_2 = \frac{A_1}{A_2} u_1 \tag{1.8}$$

Plugging in "realistic" numbers:

$$u_2 = \frac{65\,\mathrm{m} \cdot \mathrm{h}}{15\,\mathrm{m} \cdot \mathrm{h}} \cdot 10\,\mathrm{m/s} = 43.3\,\mathrm{m/s} \tag{1.9}$$

Question: Why balconies?

Answer: Architects are not engineers/physicists!

1.1.2 Example 2: falling stream of liquid

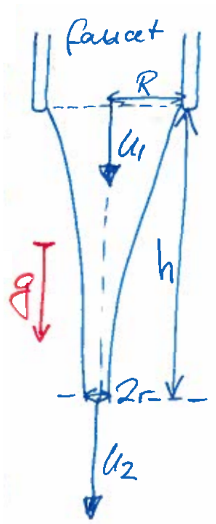

Figure 3: A stream of falling liquid.

We use Leonardo's law:

$$\pi R^2 u_1 = \pi r^2 u_2 \tag{1.10}$$

Energy conservation tells us that the sum of kinetic and potential energy is conserved:

$$\frac{m}{2}u_2^2 = \frac{m}{2}u_1^2 + mgh \tag{1.11}$$

$$\Downarrow$$

$$u_2^2 = u_1^2 + 2gh, \tag{1.12}$$

where $g = 9.82\,\mathrm{m/S^2}$ is the acceleration of gravity.

$$\frac{r}{R} = \left(\frac{\pi r^2}{\pi R^2}\right)^{\frac{1}{2}} = \left(\frac{A_2}{A_1}\right)^{\frac{1}{2}} = \left(\frac{u_1}{u_2}\right)^{\frac{1}{2}} \tag{1.13}$$

$$= \left(\frac{u_1^2}{u_2^2}\right)^{\frac{1}{4}} = \left(\frac{u_1^2}{u_1^2 + 2gh}\right)^{\frac{1}{4}} \tag{1.14}$$

Remark: The narrowing of a falling stream of liquid holds only for the upper part of the stream. At some height h the stream becomes too thin and drop formation sets in.

1.1.3 Example 3: wave energy converter

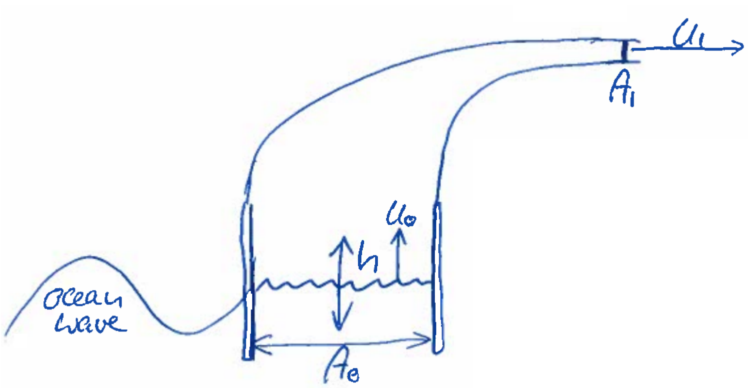

Figure 4: Schematic of a wave energy converter.

The ocean waves induce an oscillating water surface height, which induces an oscillating air stream. A turbine is placed at the nozzle (with cross-section $A_1 \ll A_0$) and extracts power from the moving air stream.

Oscillating height:

$$h(t) = H \sin \omega t, \qquad \omega = 2\pi f = \frac{2\pi}{T}, \tag{1.15}$$

where f is the frequency, T is the oscillating period and ω is the angular frequency.

$$u_0(t) = \frac{dh(t)}{dt} \tag{1.16}$$
$$= H\omega \cos \omega t \tag{1.17}$$

$$A_0 u_0(t) = A_1 u_1(t) \tag{1.18}$$
$$\Downarrow$$
$$u_1(t) = \frac{A_0}{A_1} u_0 \cos \omega t \tag{1.19}$$

$$A_1 \ll A_0 \Rightarrow u_1(t) \gg u_0(t) \tag{1.20}$$

1.1.4 Example 4: wake flow behind a wind turbine and wind farm optimization

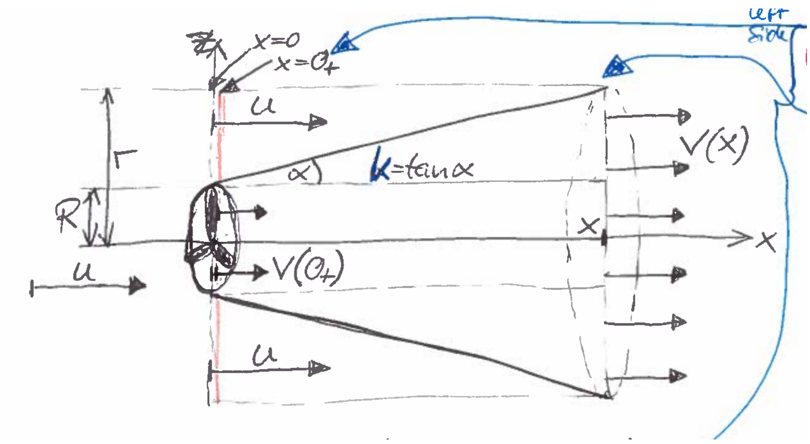

Figure 5: The expanding wake behind a turbine.

Far-field modeling of a wake flow behind a wind turbine. We use the linear wake expansion:

$$r = R + kx. \tag{1.21}$$

We use the equation of continuity (Leonardo's law):

$$\left.\frac{\Delta m}{\Delta t}\right|_{x=0^+} = \rho\pi R^2 v(0^+) + \rho\pi(r^2 - R^2)u = \rho\pi r^2 v(x) = \left.\frac{\Delta m}{\Delta t}\right|_x . \tag{1.22}$$

In words this equation states that the in-flow through the left side of the cylinder is equal to the out-flow through the right side of the cylinder. Rearranging this equation we can get an expression for the wind speed of the wake behind the turbine:

$$v(x) = \frac{R^2}{r^2} v(0_+) + \frac{r^2 - R^2}{r^2} u = u - \frac{R^2}{r^2} (u - v(0_+)) \tag{1.23}$$

$$= u \left\{ 1 - \frac{1 - \frac{v(0_+)}{u}}{\left(1 + \frac{kx}{R}\right)^2} \right\}. \tag{1.24}$$

The ratio

$$q = \frac{v(0_+)}{u} \tag{1.25}$$

is called the axial induction factor.

Consistency checks:

$$\lim_{x \to 0_+} v(x) = v(0_+) \tag{1.26}$$

$$\lim_{x \to \infty} v(x) = u \tag{1.27}$$

Betz theory: power produced by a wind turbine

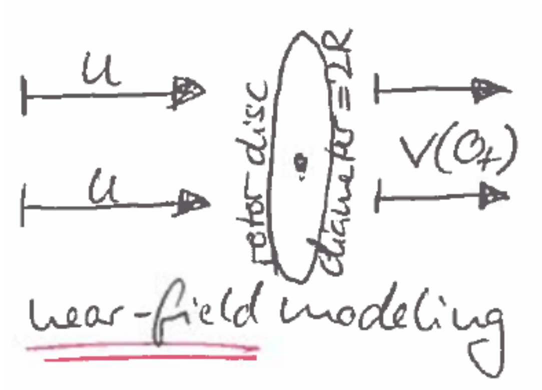

Figure 6: The velocity deficit caused by the rotor disc.

A wind turbine extracts kinetic energy out of the wind flow:

$$E_{\text{extracted}} = \frac{m}{2} \left(u^2 - v^2(0_+)\right) \tag{1.28}$$

$$= \frac{1}{2} \rho \pi R^2 \frac{u + v(0_+)}{2} \Delta t (u^2 - v^2(0_+)) \tag{1.29}$$

$$P_{\text{turbine}} = \frac{dE_{\text{extracted}}}{dt} \tag{1.30}$$

$$= \frac{\rho \pi R^2 u^3}{2} \left\{ \frac{(1+q)}{2} \left(1 - q^2\right) \right\} \tag{1.31}$$

The term in front is the kinetic energy contained in the upstream wind (volume). The term within the curly brackets is the efficiency of the wind turbine also called the power coefficient:

$$C_p = C_p(q) = \frac{(1+q)}{2}\left(1-q^2\right). \tag{1.32}$$

The maximum efficiency of a turbine can be calculated by requiring

$$\frac{dC_p(q)}{dq} = \frac{d}{dq}\left(\frac{1}{2}(1+q)(1-q^2)\right) \stackrel{!}{=} 0 \tag{1.33}$$

This gives the optimal q value

$$q = \frac{1}{3} \tag{1.34}$$

$$\Downarrow$$

$$v(0_+) = \frac{1}{3}u. \tag{1.35}$$

We can then calculate the power coefficient

$$\max_q C_p(q) = C_p(q = \frac{1}{3}) = \frac{16}{27} \approx 0.59. \tag{1.36}$$

This is known as the Betz limit. Real turbines have about $C_p \approx 0.40 - 0.50$.

1.1.5 Power optimization of a two-turbine wind farm

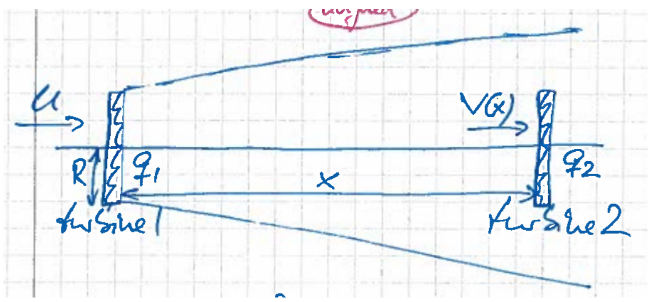

Figure 7: A very simple wind farm consisting of two turbines. The wind is approaching from the left.

We look at a wind farm with two turbines and a wind direction aligned along the connecting line. The total output of the two turbines is

$$P_{1+2} = \frac{\rho\pi R^2}{2} C_{p_1}(q_1)u^3 + \frac{\rho\pi R^2}{2} C_{p_2}(q_2)v^3(x) \tag{1.37}$$

There are no turbines behind turbine 2, so we configure it to extract the maximum amount of energy from the wind

$$q_2 = \frac{1}{3} \Rightarrow C_{p_2}(q_2) = \frac{16}{27} \tag{1.38}$$

Using previous expressions for $C_p(q)$ (1.33) and $v(x)$ (1.24) the total output of the two turbines is

$$P_{1+2} = \frac{\rho \pi R^2}{2} u^3 \left\{ \frac{1}{2}(1+q_1)(1-q_1^2) + \frac{16}{27} \left[1 - \frac{1-q_1}{\left(1+\frac{kx}{R}\right)} \right]^3 \right\} \tag{1.39}$$

Similar to before we find the optimal value of q_1 by the requirement

$$\frac{dP_{1+2}}{dq_1} \stackrel{!}{=} 0. \tag{1.40}$$

We fix the values

$$\begin{aligned} k &= 0.04 \\ \frac{x}{R} &= 8. \end{aligned}$$

The optimal q-value for turbine 1 is then

$$q_1 = 0.58 > \frac{1}{3}, \tag{1.41}$$

so turbine 1 let's through more wind.
Comparing this result with a q-value of $\frac{1}{3}$ gives

$$P_{1+2}(q_1 = 0.58) = 1.07 \cdot P_{1+2}\left(q_1 = \frac{1}{3}\right), \tag{1.42}$$

which is a 7% gain.

2 Derivation of Navier-Stokes equation

The goal of this section is to find an equation, which describes the spatio-temporal evolution of the velocity field $\vec{u}(\vec{r}, t)$. This is the Navier-Stokes equation, which is the most fundamental equation in Fluid Dynamics.

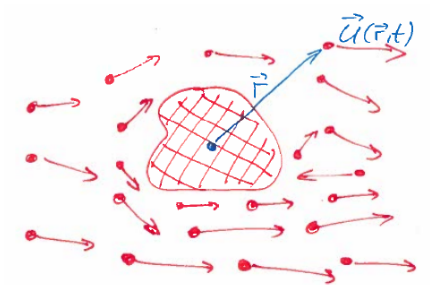

Figure 8: Velocity field around an object.

We want to describe the motion of a fluid particle. We start with Newton's second law of Classical Mechanics

$$\vec{F} = \frac{d}{dt}(m\vec{u}). \tag{2.1}$$

The equation states that the forces acting on the particle equals its change (the time derivative) of momentum (the parenthesis).

> **Remark:** in Fluid Dynamics we look not only at one fluid particle, but at all fluid particles.

$$\frac{d}{dt}(m\vec{u}) = \frac{dm}{dt}\vec{u} + m\frac{d\vec{u}}{dt} = \rho\Delta V\frac{d\vec{u}}{dt}. \tag{2.2}$$

The mass of a fluid particle is constant and does not change over time, hence the time derivative term is zero. We also used the relation

$$m = \rho\Delta V. \tag{2.3}$$

Given the field description $\vec{u} = \vec{u}(\vec{r}, t)$, we have to be a little careful with $\frac{d\vec{u}}{dt}$. The following is wrong:

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}(\vec{r}, t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{u}(\vec{r}, t + \Delta t) - \vec{u}(\vec{r}, t)}{\Delta t} \tag{2.4}$$

See the example in Figure 9.

The correct approach is to follow one fluid particle on its pathline (trajectory) $\vec{r} = \vec{r}(\vec{r}_0, t_0; t)$. See Figure 10.

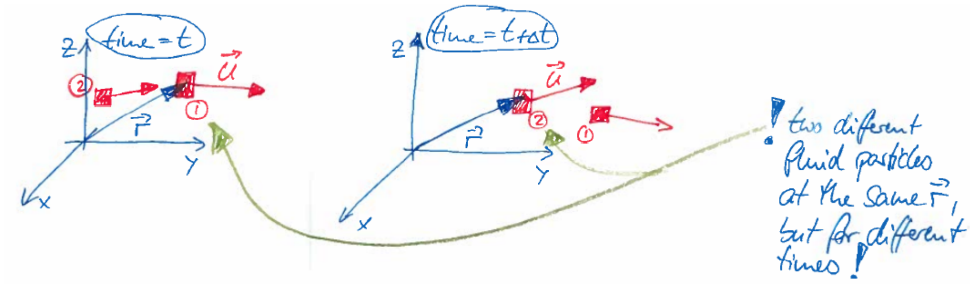

Figure 9: Two fluid particles can have the same coordinate vector $\vec{r}$ but for different times.

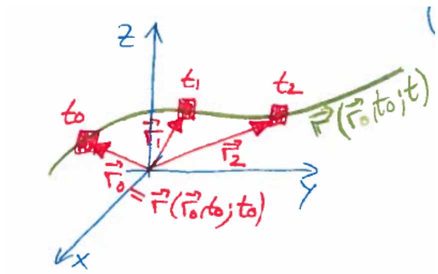

Figure 10: The path and changing coordinates of a single fluid particle.

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}\left(\vec{r}\left(\vec{r}_0, t_0; t\right), t\right)}{dt} \tag{2.5}$$

$$= \frac{d\vec{u}(x(\vec{r}_0, t_0; t), y(\vec{r}_0, t_0; t), z(\vec{r}_0, t_0; t))}{dt} \tag{2.6}$$

$$= \frac{\partial \vec{u}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{u}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{u}}{\partial z}\frac{dz}{dt} + \frac{\partial \vec{u}}{\partial t}\frac{dt}{dt} \tag{2.7}$$

$$= \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right)\vec{u} + \frac{\partial \vec{u}}{\partial t} \tag{2.8}$$

Short notation for partial derivative:

$$\frac{\partial}{\partial x} = \partial_x. \tag{2.9}$$

Here x can be replaced by y, z or t.

Short notation for velocity:

$$\frac{dx}{dt} = u_x \tag{2.10}$$

The terms in parentheses is the dot product between $\vec{u}$ and $\vec{\nabla} = (\partial_x, \partial_y, \partial_z)$, calculated using the chain rule of differentiation:

$$\frac{du(f(t))}{dt} = \frac{\partial u}{\partial f}\frac{df}{dt} \tag{2.11}$$

$$\frac{du(f(t),g(t))}{dt} = \frac{\partial u}{\partial f}\frac{df}{dt} + \frac{\partial u}{\partial g}\frac{dg}{dt} \tag{2.12}$$

$$\frac{du(f(t),g(t),h(t))}{dt} = \frac{\partial u}{\partial f}\frac{df}{dt} + \frac{\partial u}{\partial g}\frac{dg}{dt} + \frac{\partial u}{\partial h}\frac{dh}{dt}. \tag{2.13}$$

This leads to the final expression

$$\frac{d\vec{u}}{dt} = \frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u}. \tag{2.14}$$

The material derivative is defined as

$$\frac{d}{dt}\partial_t + \vec{u} \cdot \vec{\nabla} = \frac{D}{Dt} = D_t. \tag{2.15}$$

Whenever we have the time derivative of a field, like $\vec{u}(\vec{r},t)$, then we have to "go with the fluid particle" and use the material derivative.

We now go back to Newton's second equation:

$$\frac{d}{dt}(m\vec{u}) = \rho \Delta V \left(\partial_t + \vec{u} \cdot \vec{\nabla}\right)\vec{u} = \vec{F} = \vec{F}_{\text{external}} + \vec{F}_{\text{surrounding}} \tag{2.16}$$

The surrounding force can be decomposed into

$$\vec{F}_{\text{surrounding}} = \vec{F}_{\text{pressure}} + \vec{F}_{\text{friction}} \tag{2.17}$$

The surrounding fluid particles push the "sandwiched" fluid particle around; they exert pressure. Mutual friction between neighboring fluid particles due to relative and rotational motion, deformation and compression.

Example: The force of gravity is an example of an external force:

$$\vec{F}_{\text{external}} = \vec{F}_{\text{grav}} = \underbrace{\rho \Delta V}_{m}\,\vec{g} = \rho \vec{g} \Delta V, \tag{2.18}$$

with the gravitational constant defined as

$$\vec{g} = -\begin{pmatrix} 0 \\ 0 \\ 9.81\,\frac{\text{m}}{\text{s}} \end{pmatrix} = -g\vec{e}_z. \tag{2.19}$$

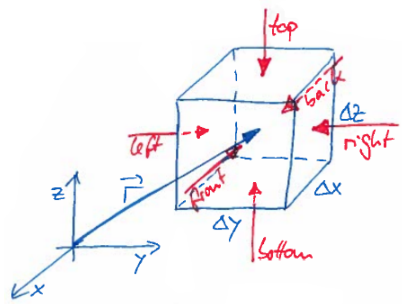

Figure 11: Illustration of the pressure force on each side of a fluid particle.

2.1 Pressure force

$$\left(\vec{F}_{\text{pressure}}\right)_z = \vec{F}^{\text{top}}_{\text{pressure}} + \vec{F}^{\text{bottom}}_{\text{pressure}} \tag{2.20}$$

$$= -p\left(x, y, z + \frac{\Delta z}{2}\right)\Delta x \Delta y + p\left(x, y, z - \frac{\Delta z}{2}\right)\Delta x \Delta y \tag{2.21}$$

$$\begin{aligned} &= -\left(p(x,y,z) + \frac{\partial p(x,y,z)}{\partial z}\frac{\Delta z}{2}\right)\Delta x \Delta y \\ &\quad + \left(p(x,y,z) + \frac{\partial p(x,y,z)}{\partial z}\left(-\frac{\Delta z}{2}\right)\right)\Delta x \Delta y \end{aligned} \tag{2.22}$$

$$= -\frac{\partial p(x,y,z)}{\partial z}\Delta x \Delta y \Delta z \tag{2.23}$$

Using $\Delta x \Delta y \Delta z = \Delta V$ the pressure force density is

$$\vec{f}_{\text{pressure}} = \frac{\vec{F}_{\text{pressure}}}{\Delta V} = -\vec{\nabla} p(\vec{r}, t) \tag{2.24}$$

2.2 Friction force

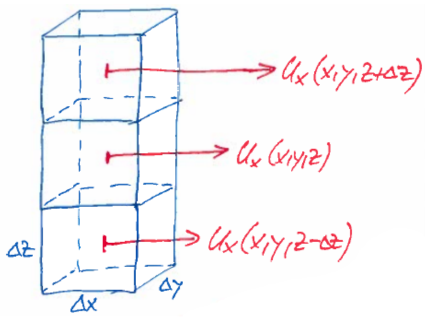

Figure 12: Illustration of the friction force on each side of a fluid particle.

We consider the neighboring fluid particles above and below as sketched in Figure 12.

$$
\begin{aligned}
\left(\vec{F}_{\text{friction}}^{\text{top+bottom}}\right)_x = &\frac{\mu}{\Delta z}\Delta x \Delta y \left(u_x(x,y,z+\Delta z) - u_x(x,y,z)\right) \\
&+ \frac{\mu}{\Delta z}\Delta x \Delta y \left(u_x(x,y,z-\Delta z) - u_x(x,y,z)\right)
\end{aligned}
\tag{2.25}
$$

If $u_x(x,y,z+\Delta z) > u_x(x,y,z)$, then the fluid particle above pulls the sandwiched fluid particle with it.

Taylor series expansion up to second-order terms:

$$
\begin{aligned}
\left(\vec{F}_{\text{friction}}^{\text{top+bot}}\right)_x &= \mu \Delta x \Delta y \left\{ \frac{u_x(x,y,z+\Delta z) - u_x(x,y,z)}{\Delta z} + \frac{u_x(x,y,z-\Delta z) - u_x(x,y,z)}{\Delta z} \right\} \\
&= \frac{\mu \Delta x \Delta y}{\Delta z} \left\{ u_x(x,y,z) + \frac{\partial u_x(x,y,z)}{\partial z}\Delta z + \frac{\partial^2 u_x(x,y,z)}{\partial z^2}\frac{\Delta z^2}{2} - u_x(x,y,z) \right. \\
&\quad \left. + u_x(x,y,z) + \frac{\partial u_x(x,y,z)}{\partial z}(-\Delta z) + \frac{\partial^2 u_x(x,y,z)}{\partial z^2}\frac{(-\Delta z)^2}{2} - u_x(x,y,z) \right\}
\end{aligned}
\tag{2.26}
$$

$$
= \mu \Delta x \Delta y \Delta z \frac{\partial^2 u_x(x,y,z)}{\partial z^2}
\tag{2.27}
$$

Front + back:

$$
\left(\vec{F}_{\text{friction}}^{\text{front+back}}\right)_x = \mu \Delta V \frac{\partial^2 u_x(x,y,z)}{\partial y^2}
\tag{2.28}
$$

Most general expression of the friction force:

$$
\frac{\vec{F}_{\text{friction}}}{\Delta V} = \vec{f}_{\text{friction}}\left(\frac{\partial^2 u_i}{\partial x_k \partial x_l}\right) = \vec{f}_{\text{friction}}(\vec{\nabla}, \vec{\nabla}, \vec{u})
\tag{2.29}
$$

The task is to build a vector $\vec{f}$ from a combination of three vectors $\vec{a} = \vec{\nabla}, \vec{b} = \vec{\nabla}, \vec{c} = \vec{u}$, such that

$$
\vec{f} = \alpha\left(\vec{a}\cdot\vec{b}\right)\vec{c} + \beta\left(\vec{a}\cdot\vec{c}\right)\vec{b} + \alpha\left(\vec{b}\cdot\vec{c}\right)\vec{a}
\tag{2.30}
$$

The solution:

$$
\vec{f}_{\text{friction}} = \mu\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{u} + \left(\mu_v + \frac{\mu}{3}\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}\right),
\tag{2.31}
$$

where μ is the shear (dynamic) viscosity and μ_v is the compression (bulk) viscosity.

Navier-Stokes equation:

$$
\rho\left(\partial_t + \left(\vec{u}\cdot\vec{\nabla}\right)\right)\vec{u} = \vec{f}_{\text{ext}} - \vec{\nabla}p + \mu\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{u} + \left(\mu_v + \frac{\mu}{3}\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}\right)
\tag{2.32}
$$

where

$$
\vec{u} = \vec{u}\left(\vec{r}, t\right) \tag{2.33}
$$

$$
p = p\left(\vec{r}, t\right) \tag{2.34}
$$

$$
\rho = \rho\left(\vec{r}, t\right) \tag{2.35}
$$

$$
\vec{f}_{\text{ext}} = \vec{f}_{\text{ext}}\left(\vec{r}, t\right) \tag{2.36}
$$

2.3 Navier-Stokes equation in components

$$
\rho \begin{pmatrix} \frac{\partial u_x}{\partial t} \\ \frac{\partial u_y}{\partial t} \\ \frac{\partial u_z}{\partial t} \end{pmatrix} + \rho \begin{pmatrix} \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_x \\ \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_y \\ \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_z \end{pmatrix} = \begin{pmatrix} f_x^{\text{ext}} \\ f_y^{\text{ext}} \\ f_z^{\text{ext}} \end{pmatrix} - \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{pmatrix}
$$
$$
+ \mu \begin{pmatrix} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u_x \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u_y \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u_z \end{pmatrix}
$$
$$
+ \left(\mu_v + \frac{\mu}{3}\right) \begin{pmatrix} \frac{\partial}{\partial x} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\ \frac{\partial}{\partial y} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\ \frac{\partial}{\partial z} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \end{pmatrix} \tag{2.37}
$$

Remark: There are only a few exact analytical solutions; many approximate analytical solutions (guided by intuition). Computational fluid dynamics can give us "exact" numerical solutions for approximations to the Navier-Stokes equation.

We now have three coupled differential equations for five fields: $u_x(\vec{r}, t)$, $u_y(\vec{r}, t)$, $u_z(\vec{r}, t)$, $p(\vec{r}, t)$, and $\rho(\vec{r}, t)$. This means we are missing two equations.

The first missing equation

From thermodynamics we have an equation of state

$$g(p, \rho) = 0. \tag{2.38}$$

For an incompressible flow, the equaiton of state is simply

$$\rho = \text{constant}. \tag{2.39}$$

For a compressible flow, the equation of state can be found with the law of ideal gases:

$$pV = NkT, \tag{2.40}$$

from which we get

$$\rho = \frac{N}{V} = \frac{1}{kT} p \tag{2.41}$$
$$\frac{p}{\rho} = kT = \text{constant}. \tag{2.42}$$

This only holds if the temperature is constant.

The second missing equation

Equation of continuity, local mass conservation.

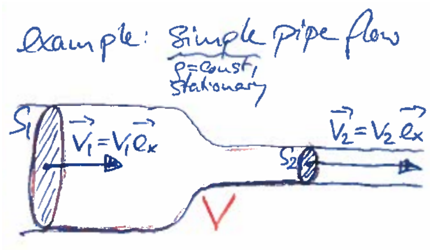

Figure 13: A simple pipe flow to illustrate mass conservation.

Example: Simple pipe flow. See Figure 13.

$$M_{\text{in}} = \rho S_1 v_1 \Delta t \tag{2.43}$$
$$M_{\text{out}} = \rho S_2 v_2 \Delta t \tag{2.44}$$

Mass conservation:

$$M_{\text{in}} = M_{\text{out}} \tag{2.45}$$
$$\Downarrow$$
$$v_1 S_1 = v_2 S_2 \tag{2.46}$$

This is Leonardo's law.

Local mass conservation in a small volume element:

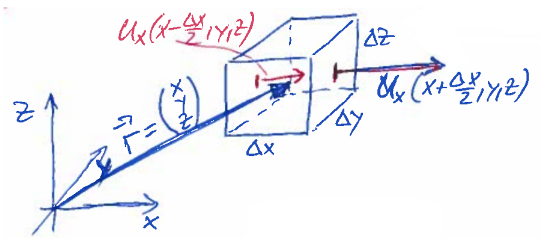

Figure 14: Local mass conservation in a small volume element.

Mass flux through surface of volume $\Delta V = \Delta x \Delta y \Delta z$ in x-direction:

$$\begin{aligned}
\frac{dM_x^S}{dt} &= \rho\left(x+\frac{\Delta x}{2}, y, z, t\right) u_x\left(x+\frac{\Delta x}{2}, y, z, t\right) \Delta y \Delta z \\
&\quad - \rho\left(x-\frac{\Delta x}{2}, y, z\right) u_x\left(x-\frac{\Delta x}{2}, y, z\right) \Delta y \Delta z \\
&= \left(\rho(x,y,z) + \frac{\partial \rho(x,y,z)}{\partial x}\frac{\Delta x}{2}\right)\left(u_x(x,y,z) + \frac{\partial u_x(x,y,z)}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z \\
&\quad - \left[\rho(x,y,z) + \frac{\partial \rho(x,y,z)}{\partial x}\left(-\frac{\Delta x}{2}\right)\right]\left[u_x(x,y,z) + \frac{\partial u_x(x,y,z)}{\partial x}\left(-\frac{\Delta x}{2}\right)\right]\Delta y \Delta z \\
&= \frac{\partial \rho(x,y,z)}{\partial x} u_x(x,y,z)\Delta x \Delta y \Delta z + \rho(x,y,z)\frac{\partial u_x(x,y,z)}{\partial x}\Delta x \Delta y \Delta z \\
&= \frac{\partial\left(\rho(x,y,z,t) u_x(x,y,z,t)\right)}{\partial x}\Delta V
\end{aligned} \tag{2.47}$$

Mass flux in y and z-direction:

$$\frac{dM_y^S}{dt} = \frac{\partial(\rho(x,y,z)u_y(x,y,z))}{\partial y}\Delta V \tag{2.48}$$

$$\frac{dM_z^S}{dt} = \frac{\partial(\rho(x,y,z)u_z(x,y,z))}{\partial z}\Delta V \tag{2.49}$$

Sum of mass fluxes through volume in all directions:

$$\frac{dM^S}{dt} = \frac{dM_x^S}{dt} + \frac{dM_y^S}{dt} + \frac{dM_z^S}{dt} \tag{2.50}$$

$$= \frac{\partial(\rho u_x)}{\partial x}\Delta V + \frac{\partial(\rho u_y)}{\partial y}\Delta V + \frac{\partial(\rho u_z)}{\partial z}\Delta V \tag{2.51}$$

$$= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \rho u_x \\ \rho u_y \\ \rho u_z \end{pmatrix} \Delta V \tag{2.52}$$

$$= \vec{\nabla} \cdot \left(\rho(x,y,z)\vec{u}(x,y,z)\right)\Delta V \tag{2.53}$$

Increase of mass within fixed volume ΔV:

$$\frac{\partial M^V}{\partial t} = \frac{\partial(\rho(\vec{r},t)\Delta V)}{\partial t} = \frac{\partial \rho(\vec{r},t)}{\partial t}\Delta V \tag{2.54}$$

Local mass conservation

$$\frac{dM^V}{dt} = -\frac{dM^S}{dt} \tag{2.55}$$

If mass within the volume increases, then less has to flow out of the surface than to flow in

$$\frac{\partial \rho(\vec{r},t)}{\partial t} + \vec{\nabla} \cdot \left(\rho(\vec{r},t)\vec{u}(\vec{r},t)\right) = 0 \tag{2.56}$$

This is the equation of continuity.

2.4 Summary

Navier-Stokes equation:

$$\rho \left(\frac{\partial}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \right) \vec{u} = \vec{f}_{\text{ext}} - \vec{\nabla} p + \mu \left(\vec{\nabla} \cdot \vec{\nabla} \right) \vec{u} + \left(\mu_v + \frac{\mu}{3} \right) \vec{\nabla} \left(\vec{\nabla} \cdot \vec{u} \right) \tag{2.57}$$

$$\vec{u} = \vec{u}(\vec{r}, t) = \vec{u}(x, y, z, t) \tag{2.58}$$

$$\rho = \rho(\vec{r}, t) \tag{2.59}$$

$$p = p(\vec{r}, t) \tag{2.60}$$

Equation of continuity:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \tag{2.61}$$

Equation of state:

$$g(p, \rho; T) = 0 \tag{2.62}$$

Heat equation (if the temperature also becomes a field $T(\vec{r}, t)$):

$$\left(\frac{\partial}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \right) T(\vec{r}, r) = \kappa \left(\vec{\nabla} \cdot \vec{\nabla} \right) T(\vec{r}, t), \tag{2.63}$$

where κ is the thermal diffusion.

3 Simplification of the Navier-Stokes equation

3.1 Simplification I: incompressible flows

Incompressiblity is a very good approximation for most liquids, including water. In 1000 m depth the density of seawater is only 0.4% larger than at the surface. For gas flows incompressiblity is also a good approximation as long as $|\vec{u}_{\text{gas}}| \ll$ speed of sound. Compressibility becomes important when discussing e.g. sound waves.

Incompressibility means that the density of a fluid particle (moving along its pathline) remains constant.

$$\begin{aligned}
0 = \frac{d\rho(\vec{r},t)}{dt} &= \frac{\partial\rho}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\rho && (3.1)\\
&= -\vec{\nabla}\left(\rho\vec{u}\right) + \left(\vec{u}\cdot\vec{\nabla}\right)\rho && (3.2)\\
&= -\left(\vec{u}\cdot\vec{\nabla}\right)\rho - \rho\left(\vec{\nabla}\cdot\vec{u}\right) + \left(\vec{u}\cdot\vec{\nabla}\right)\rho && (3.3)\\
&= -\rho\left(\vec{\nabla}\cdot\vec{u}\right) && (3.4)\\
&= -\rho\,\mathrm{div}\,\vec{u} && (3.5)
\end{aligned}$$

In the first line we used the material derivative. In the step to the second line we used continuity equation. For incompressibility the divergence must be zero:

$$\vec{\nabla}\cdot\vec{u} = 0 \tag{3.6}$$

Navier-Stokes equation for incompressible flows (3 equations):

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\right)\vec{u} = f_{\text{ext}} - \vec{\nabla}p + \mu\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{u} \tag{3.7}$$

> Remark: It looks simple, but these nonlinear differential equations remain a formidable challenge to engineers, physicists and mathematicians.

Fourth equation:

$$\vec{\nabla}\cdot\vec{u} = 0. \tag{3.8}$$

Fifth equation: equation of state in the simplest form with constant density

$$p = p(\rho) \Rightarrow \rho = \rho_0 = \text{constant}. \tag{3.9}$$

3.2 Simplification II: incompressible, ideal, stationary, irrotational flows

We use the incompressibility result:

$$\vec{\nabla}\cdot\vec{u} = 0 \tag{3.10}$$

Ideal means no friction. To eliminate friction forces we set $\mu = 0$.

Euler equation:

$$\rho_0 \left(\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} \right) = \vec{f}_{\text{ext}} - \vec{\nabla} p \tag{3.11}$$

stationary:

$$\vec{u}(\vec{r}, t) = \vec{u}(\vec{r}) \tag{3.12}$$

$$\Downarrow$$

$$\frac{\partial \vec{u}}{\partial t} = 0 \tag{3.13}$$

$$\rho_0 \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \vec{f}_{\text{ext}} - \vec{\nabla} p \tag{3.14}$$

no external forces: $\vec{f}_{\text{ext}} = 0$

$$\rho_0 \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = -\vec{\nabla} p \tag{3.15}$$

We now look at the convective term on the lefthand side (see the proof below):

$$\left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \frac{1}{2} \vec{\nabla} \underbrace{\left(\vec{u} \cdot \vec{u} \right)}_{\vec{u}^2} - \vec{u} \times \left(\vec{\nabla} \times \vec{u} \right) \tag{3.16}$$

$$\Downarrow$$

$$\vec{\nabla} \left(\frac{\rho_0}{2} \vec{u}^2 + p \right) = \rho_0 \vec{u} \times \left(\vec{\nabla} \times \vec{u} \right) \tag{3.17}$$

Assuming irrotational flow: $\vec{\nabla} \times \vec{u} = 0$.

$$\vec{\nabla} \underbrace{\left(\frac{\rho_0}{2} \vec{u}^2 + p \right)}_{\text{constant}} = 0 \tag{3.18}$$

Bernoulli's equation

$$\frac{\rho_0}{2} \vec{u}^2 + p = \text{constant} \tag{3.19}$$

$$\vec{\nabla} \cdot \vec{u} = 0 \tag{3.20}$$

$$\vec{\nabla} \times \vec{u} = 0 \tag{3.21}$$

Given all the assumptions, this set of equations is equivalent to the Navier-Stokes equation.

Proof of

$$\left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} = \frac{1}{2} \vec{\nabla} \left(\vec{v}^2 \right) - \vec{v} \times \left(\vec{\nabla} \times \vec{v} \right). \tag{3.22}$$

First we look at the x-component of the left-hand side:

$$\left[\left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} \right]_x = \left(v_x \partial_x + v_y \partial_y + v_z \partial_z \right) v_x \tag{3.23}$$

Now we look at the rightmost term on the right-hand side:

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ v_x & v_y & v_z \end{vmatrix} = \left(\partial_y v_z - \partial_z v_y\right) \vec{e}_x + \left(\partial_z v_x - \partial_x v_z\right) \vec{e}_y + \left(\partial_x v_y - \partial_y v_x\right) \vec{e}_z \tag{3.24}$$

Now we can show that the x-component of the right-hand side is equal to the x-component of the left-hand side:

$$\begin{aligned} \left[\frac{1}{2}\vec{\nabla}\left(\vec{v}^2\right) - \vec{v} \times \left(\vec{\nabla} \times \vec{v}\right)\right]_x &= \frac{1}{2}\partial_x \left(v_x^2 + v_y^2 + v_z^2\right) \\ &\quad - \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ v_x & v_y & v_z \\ \partial_y v_z - \partial_z v_y & \partial_z v_x - \partial_x v_z & \partial_x v_y - \partial_y v_x v_x \left(\partial_x v_x\right) \end{vmatrix}_x && (3.25) \\ &= v_x \left(\partial_x v_x\right) + v_y \left(\partial_x v_y\right) + v_z \left(\partial_x v_z\right) \\ &\quad - v_y \left(\partial_x v_y - \partial_y v_x\right) + v_z \left(\partial_z v_x - \partial_x v_z\right) && (3.26) \\ &= v_x \left(\partial_x v_x\right) + v_y \left(\partial_y v_y\right) + v_z \left(\partial_z v_z\right) && (3.27) \\ &= \left[\left(\vec{v} \cdot \vec{\nabla}\right) \vec{v}\right]_x && (3.28) \end{aligned}$$

3.3 Derivation of Bernoulli's equation

The equation

$$\vec{\nabla}\left(\frac{\rho_0}{2}\vec{v}^2 + p\right) = \rho_0 \vec{v} \times \left(\vec{\nabla} \times \vec{v}\right) \tag{3.29}$$

is (scalar) multiplied with $d\vec{s} \parallel \vec{v}$, where $d\vec{s}$ describes an increment of a specific streamline (here pathline since $\vec{v}(\vec{r}, t) = \vec{v}(\vec{r})$).

$$d\vec{s} \cdot \left[\vec{v} \times \left(\vec{\nabla} \times \vec{v}\right)\right] = 0 \tag{3.30}$$

Since $d\vec{s} \parallel \vec{v}$ it must be that $d\vec{s} \perp \vec{v} \times \left(\vec{\nabla} \times \vec{v}\right)$.

Subsequent path-integration along a streamline yields

$$0 \stackrel{!}{=} \int_{\text{streamline}} \vec{\nabla} \underbrace{\left(\frac{\rho_0}{2}\vec{v}^2 + p\right)}_{W} \cdot d\vec{s} \tag{3.31}$$

$$= \int_{\text{streamline}} \begin{pmatrix} \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial y} \\ \frac{\partial W}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{3.32}$$

$$= \int_{\text{streamline}} \left(\frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz\right) \tag{3.33}$$

$$= \int_{\text{streamline}} dW = \int_{\text{streamline}} d\left(\frac{\rho_0}{2}\vec{v}^2 + p\right) \tag{3.34}$$

$$\Downarrow$$

$$\frac{\rho_0}{2}\vec{v}^2 + p = \text{constant} \qquad \text{(along a streamline)} \tag{3.35}$$

For another streamline the constant might in principle be different. For examples like the velocity in the far-field regime is everywhere the same. The same holds true for the pressure. Then the "Bernoulli constant" has to be everywhere (far and nea-field) the same. From here we then also conclude that $\vec{\nabla} \times \vec{v} = 0$ everywhere.

3.4 Example: Why does an airplane fly?

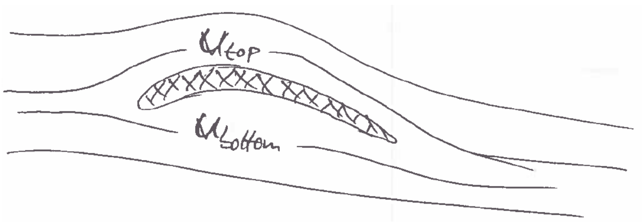

Figure 15: The profile of the wing of an airplane.

The wind speed difference between the top and bottom of the wing creates a pressure difference:

$$u_{\text{top}} > u_{\text{bottom}} \tag{3.36}$$

$$\Downarrow$$

$$p_{\text{top}} < p_{\text{bottom}}. \tag{3.37}$$

This results in a lifting force.