Fluid Dynamics + Turbulence (fall 2017)
Midterm Exam Project II

Posted:
Friday October 06, 2017.

Deadline for submission of part (b): Friday October 20 at 09.15 am (on Blackboard).

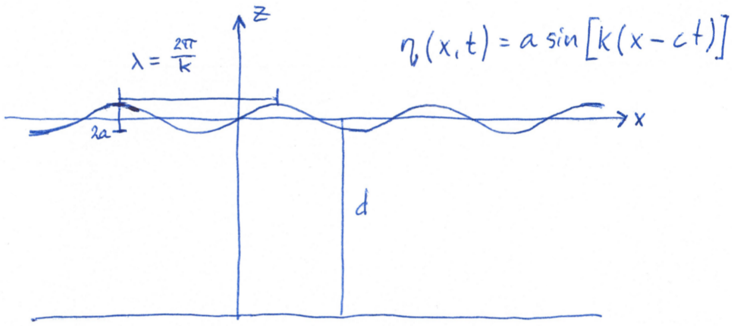
Oral presentation + defence: during week 43.

Remark: the Midterm Exam Project I or II (your choice!) will count 20% to your final course grade.

Midterm exam problem II: surface waves

(a) Read Section 8.2 (old edition: 7.2) of the book KCD: Fluid Mechanics (see the folder 'Miscellaneous Reading' from the course homepage). This will prepare you for the next part (b).

(b) Consider the small amplitude surface waves with amplitude a, speed c, and wave length λ moving in water with finite depth d:



We assume that the wave amplitude is small compared to both the wave length and the water depth,

$$a \ll \lambda \quad \text{and} \quad a \ll d. \tag{1}$$

In the lectures, you saw solutions for $d = \infty$, but here we will keep d finite and only require that the two conditions in (1) are fulfilled.
(b.1) For the present problem, the Navier-Stokes equation is reduced to

$$\frac{\partial \vec{u}}{\partial t} = -\vec{\nabla}\left(gz + \frac{p}{\rho_0}\right) \tag{2}$$

where g is the gravitational acceleration and ρ_0 is the (constant) density of the water. Describe briefly which approximations and assumptions have been made.
(b.2) Equation (2) means that we can write

$$\vec{u}(x,z,t) = \vec{\nabla}\phi(x,z,t) \tag{3}$$

for some potential function ϕ. Show that imcompressibility leads to

$$\vec{\nabla}\cdot\vec{\nabla}\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0. \tag{4}$$

(b.3) Assume that ϕ has the following form:

$$\phi(x,z,t) = \left(A_+ e^{kz} + A_- e^{-kz}\right)\cos[k(x-ct)]. \tag{5}$$

Determine the velocity components $u_x(x,z,t)$ and $u_z(x,z,t)$ from the velocity potential. Show that the condition $\vec{\nabla}\times\vec{u} = 0$ of an irrotational flow is fulfilled. Show that the boundary condition on $\vec{u}$ at the bottom, $z = -d$, leads to

$$A_+ e^{-kd} = A_- e^{kd}. \tag{6}$$

(b.4) Use the kinematic boundary condition

$$\frac{\partial \eta(x,t)}{\partial t} = u_z(x, z=0, t) \tag{7}$$

at the surface to express the amplitude a of the surface wave $\eta(x,t)$ in terms of A_+ and A_-.
(b.5) Another boundary condition at the surface can be written as

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = \left.\frac{1}{g}\frac{\partial^2 \phi}{\partial t^2}\right|_{z=0}. \tag{8}$$

Use it to show that the wave velocity c must be given by

$$c = \frac{\sqrt{gk\tanh(kd)}}{k}, \tag{9}$$

where the hyperbolic tangent is given by

$$\tanh x = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}. \tag{10}$$

(b.6) Consider waves with a wavelength $\lambda = 25\,\text{m}$. At which speed are the wavetops moving if

1. the water depth is $d = 3\,\mathrm{m}$?
2. the water depth is $d = 10\,\mathrm{m}$?
3. the water depth is approximated as infinite?

(b.7) How do the pathlines of the fluid particles look like? Give a quantitative answer based on the results obtained in part (b.3).

(c) Read and understand Sections 8.4+5 (old edition: 7.4+5) of the book KCD: Fluid Mechanics (see the folder 'Miscellaneous Reading' from the course homepage).

(d) Read and understand Section 16.1-4 on Ocean Engines of the book A.V. da Rosa: Fundamentals of Renewable Energy Processes (see the folder 'Miscellaneous Reading' from the course homepage).