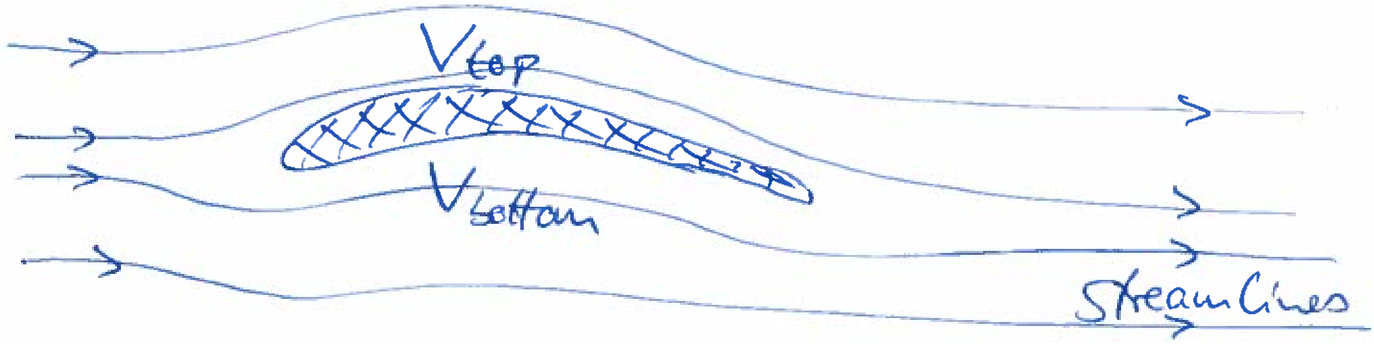
Classical Betz theory II: aerodynamic forces and design of rotor (blades)

(14.09.2016)

opening remark: why does an airplane fly?



$V_{top} > V_{bottom} \Longrightarrow P_{top} < P_{bottom} \Longrightarrow$ lift force!

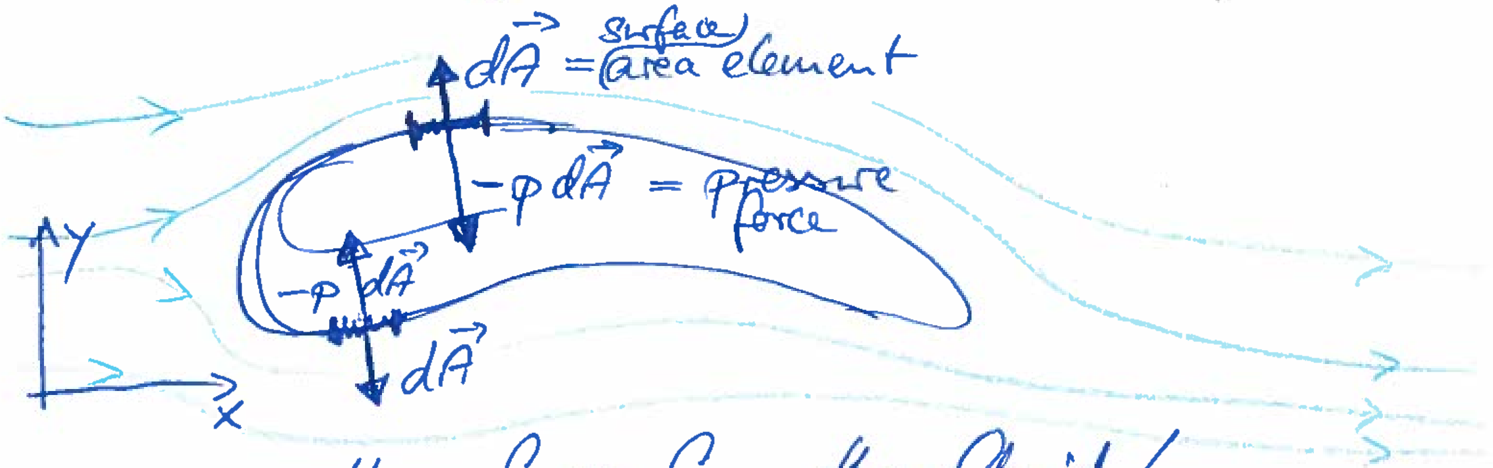
$$\boxed{\text{Bernoulli equation: } \quad \frac{\rho}{2}\vec{v}^{\,2} + p = \underbrace{\text{Const}}_{=p_0}}$$

remember Fluid Dynamics:

Navier Stokes eq. $\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\vec{v} = \vec{f}_{ext} - \vec{\nabla}p + \mu(\vec{\nabla}\cdot\vec{\nabla})\vec{v}$

+ incompressibility $\vec{\nabla}\cdot\vec{v} = 0$
+ ideal flow $\mu = 0$
+ stationary flow $\vec{v}(\vec{r},t) = \vec{v}(\vec{r})$
+ no external forces $\vec{f}_{ext} = 0$
+ irrotational flow $\vec{\nabla}\times\vec{v} = 0$

Lift + drag forces:

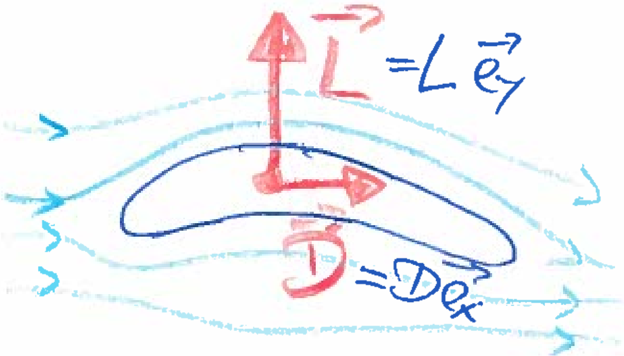


$\Longrightarrow$ resulting force from the fluid/gas streaming around the blade element:

$$\vec{F} = \int_{\text{Surface}} \left(-p(\vec{r})\right) d\vec{A} = -\int_{\text{Surface}} \left(p_0 - \frac{\rho}{2}\vec{v}^{\,2}(\vec{r})\right) d\vec{A}$$

$$= \frac{\rho}{2} \int_{\text{Surface}} \vec{v}^{\,2}(\vec{r})\, d\vec{A}$$

$$= \underbrace{L}_{\text{lift force}}\, \vec{e}_y + \underbrace{D}_{\text{drag force}}\, \vec{e}_x$$

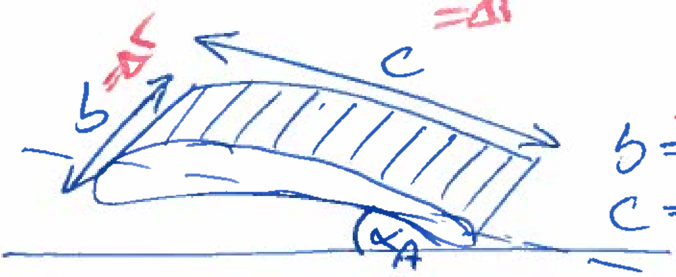


dimensioning:

Lift Coefficient — drag Coefficient

$$L = \frac{\rho}{2} V_\infty^2 \, b \, c \, C_L \, , \quad D = \frac{\rho}{2} V_\infty^2 \, b \, c \, C_D$$

Δr

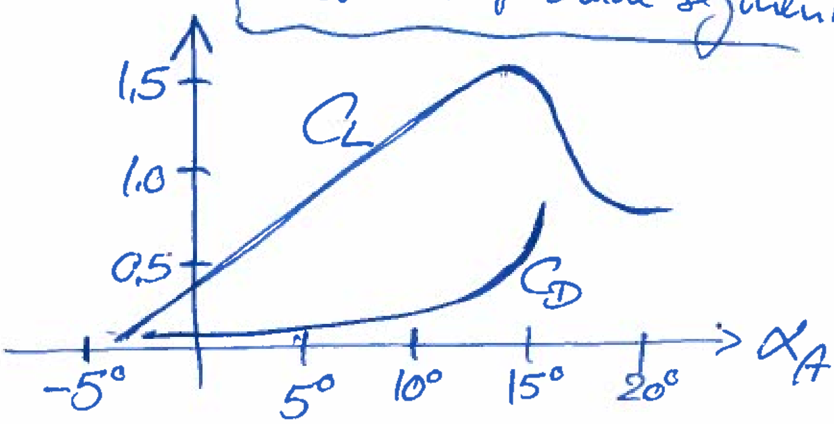


b $= \Delta r$ = width of blade element (= section)

c = chord length = distance between leading and trailing edge of the blade = width of blade segment

α_A = angle of attack

generic dependence of C_L and C_D on the angle of attack α_A:



lift-to-drag ratio:

$$\frac{L}{D} = \frac{C_L(\alpha_A)}{C_D(\alpha_A)}$$

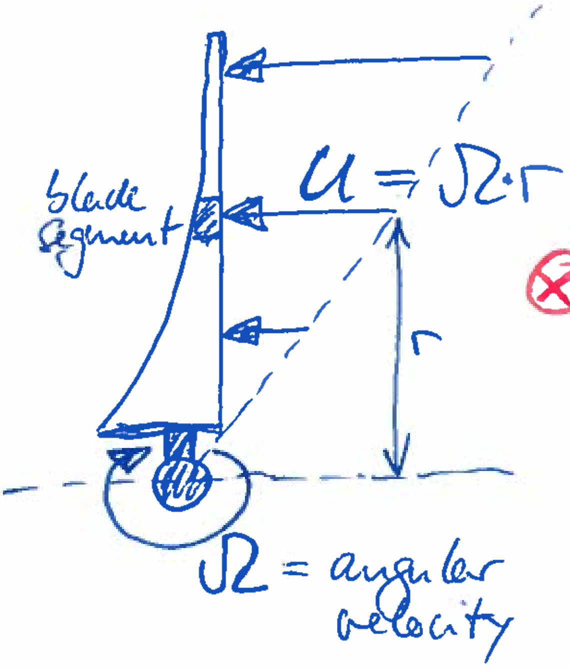
$\Longrightarrow$ measure for the quality of a blade profile

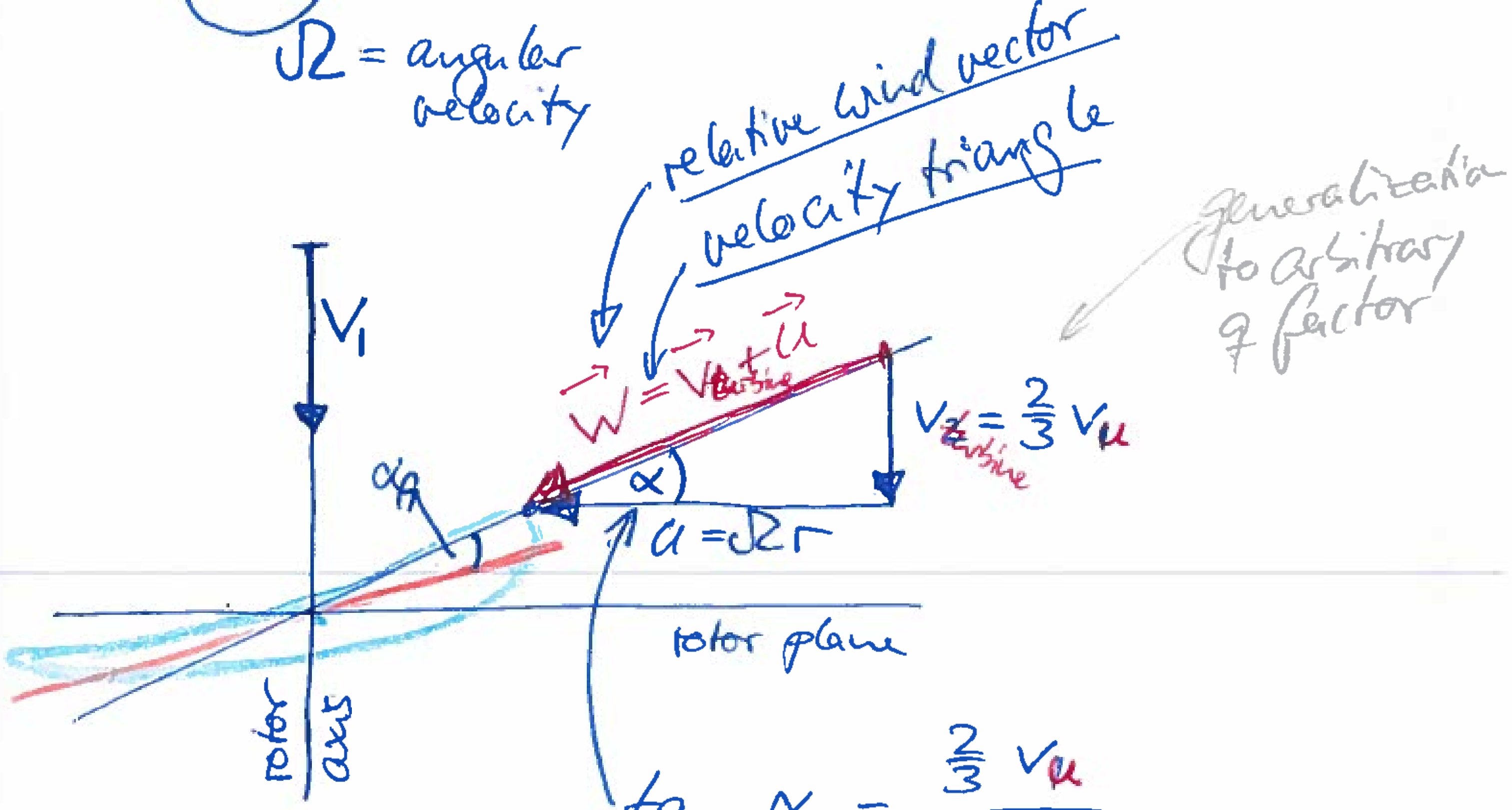
$\Longrightarrow$ high-quality profiles $\boxed{\frac{L}{D} \approx 60}$

at moderate angle of attack $2° \lesssim \alpha_A \lesssim 6°$, when $0.8 \lesssim C_L \lesssim 1.1$.

Supplementary reading:
Kundu, Cohen + Dowling: "Fluid Mechanics", Sections 6.5 + 14.1–7.

What is the wind for a rotating blade segment?





$$\tan \alpha = \frac{\frac{2}{3} V_u}{\Omega r}$$

remark: α decreases with increasing r.

$\Rightarrow$ big α at the shaft, small α at the tip of the blade.

Comparison between $u = \Omega r$ and $v_{z_{\text{turbine}}} = \frac{2}{3} v_u$

$\Longrightarrow$ tip-speed ratio: $\lambda = \dfrac{\Omega R}{v_u}$

example: $\Omega = 2\pi\nu$, $\nu = 15$ rpm

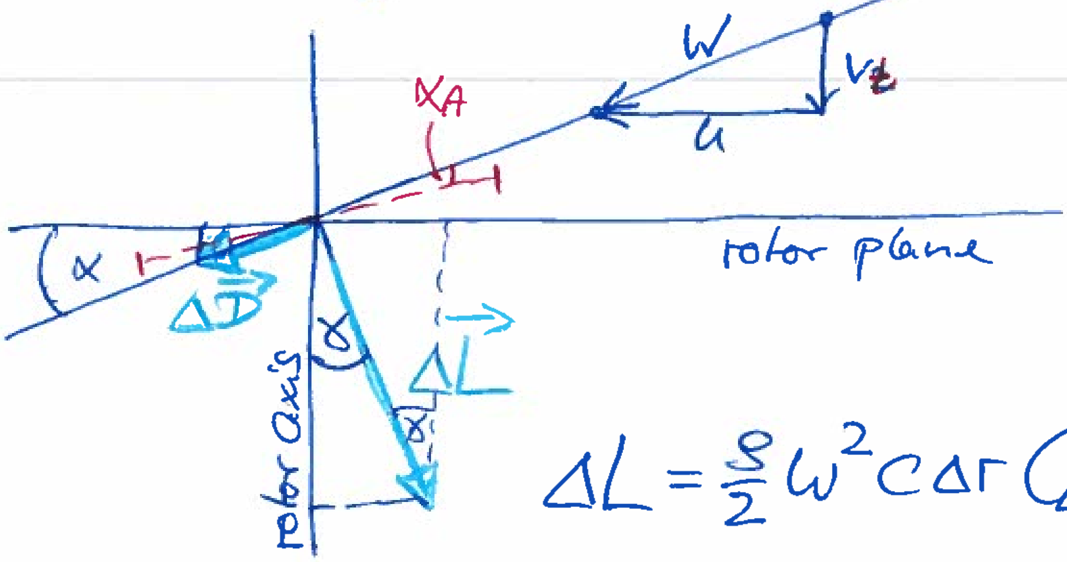
$R = 75\,m$,

$v_u = 12 \frac{m}{sec}$

$$\Longrightarrow \lambda = \frac{2\pi \cdot \frac{0.25}{sec} \cdot 75\,m}{12 \frac{m}{sec}} \approx 10$$

rotor tip moves 10 times faster than the wind!

Lift and drag forces for the blade element: (Δr at r)



$$\Delta L = \frac{\rho}{2} W^2 c \Delta r \, C_L(\alpha_A)$$

$$\Delta D = \frac{\rho}{2} W^2 c \Delta r \, C_D(\alpha_A)$$

$\Rightarrow$ projections onto rotor plane (circumferential direction) and rotor axis (axial direction):

$$\Delta U = \Delta L \sin\alpha - \Delta D \cos\alpha$$
$$= \frac{\rho}{2} w^2 c \Delta r \left[C_L \sin\alpha - C_D \cos\alpha \right]$$

$$\Delta T = \Delta L \cos\alpha + \Delta D \sin\alpha$$
$$= \frac{\rho}{2} w^2 c \Delta r \left[C_L \cos\alpha + C_D \sin\alpha \right]$$

approximations:

$$\tan\alpha = \frac{\frac{2}{3} v_w}{\Omega r} = \frac{2}{3} \frac{R}{r} \frac{1}{\lambda} \underset{r<R}{\geq} \frac{2}{3\lambda} \underset{\lambda \approx 10}{\approx} \frac{1}{15}$$

$$\Rightarrow \alpha \gtrsim 4^\circ$$

$$\Rightarrow \sin 4^\circ \approx 0.07, \quad \cos 4^\circ \approx 0.998$$

$$\Rightarrow \frac{C_L \sin\alpha}{C_D \cos\alpha} \gtrsim 60 \cdot \frac{0.07}{0.998} \approx 4$$

$$\Rightarrow \boxed{C_L \sin\alpha \gg C_D \cos\alpha}$$

$$\Rightarrow \Delta U \approx \frac{\rho}{2} C_L(\alpha_A)\, w^2 c \Delta r \sin\alpha$$

$$\Delta T \approx \frac{\rho}{2} C_L(\alpha_A)\, w^2 c \Delta r \underbrace{\cos\alpha}_{\approx 1}$$

ΔU is small compared to ΔL,
ΔU is the projection of ΔL onto the rotor plane.

$\Rightarrow$ important conclusion: the circumferential force $\Delta U > 0$ is pulling the blade segment forward!

Design of rotor blades

already noticed: $\tan\alpha = \dfrac{\frac{2}{3}v_u}{\Omega r}$

$\Rightarrow$ α decreases with increasing r

$\Rightarrow$ blade twist angle: $\beta(r) = \alpha(r) - \alpha_A$

now: $\boxed{c = c(r)}$ so far, the chord length is the only remaining unknown design variable.

$$\Delta P_{turbine}(r) = \Delta U(r) \cdot \Omega r \cdot N$$

$\Delta U(r)$: circumferential force

Ωr: circumferential velocity

N: number of rotor blades

$$= N \, \Omega r \, \frac{\rho}{2} C_L(\alpha_A) \, w^2 c(r) \, \Delta r \sin\alpha$$

Betz-optimal wind-turbine power:

$$P_{turbine}^{Betz} = C_P^{Betz} \frac{\rho \pi R^2 v_u^3}{2} = \frac{16}{27} \frac{\rho}{2} v_1^3 \int_0^R 2\pi r \, dr$$

$$\Rightarrow \Delta P_{turbine}^{Betz}(r) = \frac{16}{27} \frac{\rho}{2} v_u^3 \, 2\pi r \, \Delta r$$

$$\Delta P_{turbine}(r) = \Delta P^{Betz}_{turbine}(r)$$

$$\Longrightarrow c(r) = \frac{\frac{16}{27}\frac{\varrho}{2} v_u^3 2\pi r \Delta r}{N c \Omega r \frac{\varrho}{2} C_L(\alpha_A) w^2 \Delta r \sin\alpha}$$

$$= \frac{1}{N}\frac{16}{27}\frac{2\pi}{C_L}\frac{v_u^3}{w^2 \Omega \sin\alpha}$$

$$\left\{ \sin\alpha = \frac{v_z}{w} = \frac{\frac{2}{3}v_u}{w} \right.$$

$$= \frac{1}{N}\frac{8}{9}\frac{2\pi}{C_L}\frac{v_u^2}{w\Omega}$$

$$\left\{ \text{tip-speed ratio: } \lambda = \frac{\Omega R}{v_u} \right.$$

$$= 2\pi R \frac{1}{N}\frac{8}{9 C_L}\frac{1}{\lambda}\frac{v_u}{w}$$

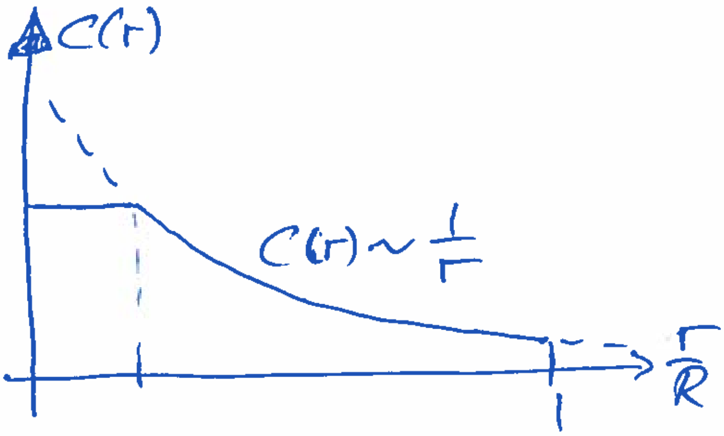
$$\left\{ \begin{aligned} w^2(r) &= (\Omega r)^2 + v_z^2 \\ &= \Omega^2 R^2 \frac{r^2}{R^2} + \frac{4}{9} v_u^2 \\ &= \lambda^2 v_u^2 \frac{r^2}{R^2} + \frac{4}{9} v_u^2 \end{aligned} \right.$$

$$= 2\pi R \frac{1}{N}\frac{8}{9 C_L}\frac{1}{\lambda\sqrt{\lambda^2\left(\frac{r}{R}\right)^2 + \frac{4}{9}}}$$

$$\left\{ \lambda \approx 10, \; r/R \geq 0.15 \;\Longrightarrow\; \frac{\lambda^2 r^2}{R^2} \approx 2.25 \gg 0.44 \right.$$

$$\Longrightarrow \boxed{c(r) \approx \frac{1}{N}\frac{8}{9 C_L}\frac{2\pi R^2}{\lambda^2 r}}$$

practical choice:
$C_L(\alpha_A) \approx 0.6 - 1.2$
$\Longleftrightarrow \alpha_A \approx 2^\circ - 6^\circ$



also: the larger the target tip-speed ratio, the smaller the chord length $c(r)$!

remark:

real wind turbines generate less than the Betz-optimal power. Their power coefficient $c_p \approx 0.40 (-0.50)$ is smaller than $c_p^{Betz} = 0.59$.
This is due to losses.

$\Longrightarrow$ profile losses due to drag forces

$\Longrightarrow$ tip losses

$\Longrightarrow$ losses due to wake rotation

further reading:

R. Gasch + J. Twele: Wind Power plants
$\Longrightarrow$ Sects. 5.2–5.