Homework 2 solution.

(a) start from simplifying the N-S eq

$$\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = -\vec{\nabla}p + \vec{f}_{ext} + \mu(\vec{\nabla}\cdot\vec{\nabla})\vec{u}$$

$$\vec{u}=0 \Rightarrow 0 = -\vec{\nabla}p + \vec{f}_{ext}$$

$$\Rightarrow 0 = -\begin{pmatrix}\frac{\partial p}{\partial x}\\ \frac{\partial p}{\partial y}\\ \frac{\partial p}{\partial z}\end{pmatrix} + \begin{pmatrix}0\\0\\-\rho g\end{pmatrix} \Rightarrow \frac{\partial p}{\partial z} = -\rho g.$$

$$\frac{p}{p_0} = \frac{\rho}{\rho_0} \qquad \frac{\partial p}{\partial z} = -g\cdot\frac{\rho_0}{p_0}\cdot p$$

then solve this differential equation with its initial values.

$$\frac{\partial p}{\partial z} = -\frac{\rho_0 g}{p_0}\cdot p$$

$$\frac{1}{p}dp = -\frac{\rho_0 g}{p_0}dz$$

$$\int_{p_0}^{p}\frac{1}{p^*}dp^* = -\int_0^z \frac{\rho_0 g}{p_0}dz^*$$

$$\left[\ln p^*\right]_{p_0}^{p} = \left[-\frac{\rho_0 g}{p_0}z^*\right]_0^z$$

$$\ln p - \ln p_0 = -\frac{\rho_0 g}{p_0}z$$

$$\ln\frac{p}{p_0} = -\frac{\rho_0 g}{p_0}z$$

$$\frac{p}{p_0} = \exp\left(-\frac{\rho_0 g}{p_0}z\right)$$

from $\frac{p}{p_0} = \frac{\rho}{\rho_0}$

$$\Rightarrow \rho(z) = \rho_0\exp\left(-\frac{\rho_0 g}{p_0}z\right)$$

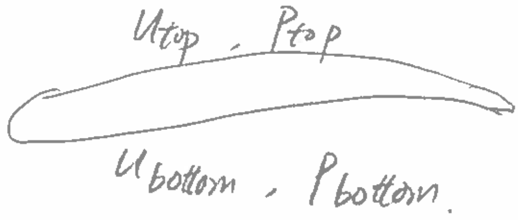
$\rho(z=10km) = 1.20\ kg/m^3$.

$$\exp\left(-\frac{1.20\frac{kg}{m^3}\cdot 9.81\frac{m}{s^2}}{1.01\cdot 10^5\frac{kg}{m\,sec^2}}\cdot 10km\right)$$

after calculation show $\frac{\rho(z=10km)}{\rho(z=0)} = 0.31$

(b)

Bernoulli eq.



$$\frac{\rho u_{top}^2}{2} + P_{top} = \frac{\rho u_{bottom}^2}{2} + P_{bottom}$$

$$P_{bottom} - P_{top} = \frac{\rho}{2}\left(u_{top}^2 - u_{bottom}^2\right)$$

the pressure difference results in a lift force L, which is equal to gravity force. $\Rightarrow$ $L = mg$.

$$L = (P_{bottom} - P_{top}) \cdot A_{wing}$$

$$= \frac{\rho}{2}\left(u_{top}^2 - u_{bottom}^2\right) A_{wing} \overset{!}{=} mg$$

from here we assume $u = \frac{u_{top} + u_{bottom}}{2}$, meaning the plane speed is the average of u_{top} and u_{bottom}. $\Rightarrow$

$$u_{top} + u_{bottom} = 2u$$

and notice $u_{top}^2 - u_{bottom}^2 = (u_{top} + u_{bottom})\underbrace{(u_{top} - u_{bottom})}_{\Delta u}$

$$= 2u \cdot \Delta u$$

$$\Rightarrow \quad \frac{\rho}{2} \cdot 2u \cdot \Delta u \cdot A_{wing} = mg$$

$$\Delta u = \frac{mg}{A_{wing} \cdot \rho \cdot u}$$

$$= \frac{5 \cdot 10^5\,\mathrm{kg} \cdot 9.81\,\frac{N}{kg}}{846\,\mathrm{m}^2 \cdot 1.20\,\frac{kg}{m^3} \cdot 945\,\frac{km}{h}}$$

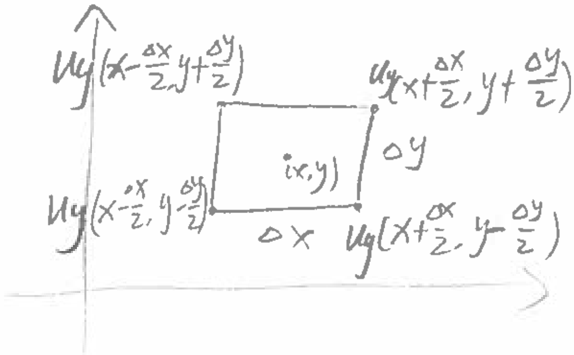
$$\approx 5.9\ \mathrm{m/s}.$$

Remark: other assumptions like $u_{top} = u$ or $u_{bottom} = u$ are also accepted.

Exercise 2.1

$\vec{\nabla}(\vec{\nabla}\cdot\vec{u})$ has a term $\dfrac{\partial^2 u_y}{\partial x \partial y}$, this means u_y could give a force pointing to the x- direction.

Imagine a fluid particle at (x,y) with width Δx, height Δy and ignore z component.

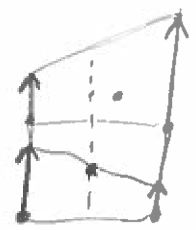


$$\frac{\partial^2 u_y}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial y}\right) \xrightarrow[\text{difference}]{\text{central}}$$

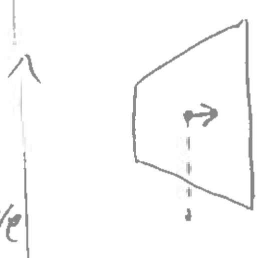
$$= \frac{\partial}{\partial x}\left(\frac{u_y(x, y+\frac{\Delta y}{2}) - u_y(x, y-\frac{\Delta y}{2})}{\Delta y}\right)$$

$$\overset{\text{central}}{\underset{\text{difference}}{=}} \frac{1}{\Delta x}\cdot\frac{1}{\Delta y}\left\{ u_y\left(x+\frac{\Delta x}{2}, y+\frac{\Delta y}{2}\right) - u_y\left(x-\frac{\Delta x}{2}, y+\frac{\Delta y}{2}\right) - u_y\left(x+\frac{\Delta x}{2}, y-\frac{\Delta y}{2}\right) + u_y\left(x-\frac{\Delta x}{2}, y-\frac{\Delta y}{2}\right)\right\}$$

assume $\dfrac{\partial^2 u_y}{\partial x \partial y} > 0$, then the resulting force would be in the positive x direction. the sum in $\{\}$ should be positive, therefore a possiblity would be



$$\begin{cases} u_y(x+\frac{\Delta x}{2}, y+\frac{\Delta y}{2}) > u_y(x-\frac{\Delta x}{2}, y+\frac{\Delta y}{2}) \\ u_y(x-\frac{\Delta x}{2}, y-\frac{\Delta y}{2}) > u_y(x+\frac{\Delta x}{2}, y-\frac{\Delta y}{2}) \end{cases}$$

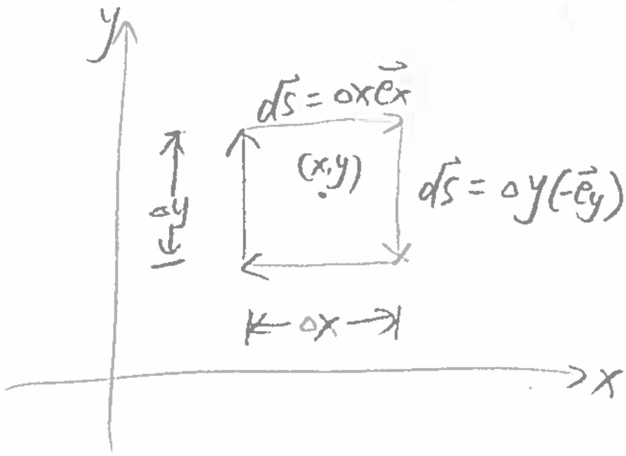


consult the sketches, you can see a positive $\dfrac{\partial^2 u_y}{\partial x \partial y}$ shifts mass center to the right.

Remark: Don't expect a problem like this in the exam.

Exercise 2.2

from problem description.



$$\oint \vec{v}(x,y)\cdot d\vec{s} \approx V_x\left(x, y+\frac{\Delta y}{2}\right)\Delta x + V_y\left(x+\frac{\Delta x}{2}, y\right)(-\Delta y) + V_x\left(x, y-\frac{\Delta y}{2}\right)(-\Delta x) + V_y\left(x-\frac{\Delta x}{2}, y\right)\Delta y$$

Taylor Series for each term

$$\left(V_x(x,y) + \frac{\partial V_x}{\partial y}\cdot\frac{\Delta y}{2}\right)\Delta x - \left(V_y(x,y) + \frac{\partial V_y}{\partial x}\cdot\frac{\Delta x}{2}\right)\Delta y - \left(V_x(x,y) + \frac{\partial V_x}{\partial y}\cdot\left(-\frac{\Delta y}{2}\right)\right)\Delta x + \left(V_y(x,y) + \frac{\partial V_y}{\partial x}\cdot\left(-\frac{\Delta x}{2}\right)\right)\Delta y$$

$$= \underbrace{\left(\frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x}\right)}_{\text{this is the z component in } -(\vec{\nabla}\times\vec{v})}\Delta x\Delta y$$

$$= -(\vec{\nabla}\times\vec{v})\cdot\vec{e}_z\cdot\Delta x\Delta y$$

if $\vec{\nabla}\times\vec{v} = 0$ (irrotational flow), then the line integral of the velocity field in a infiniesmall closed circle is 0.

$\Rightarrow$ no net velocity along a closed loop.

Exercise 2.3

$$\vec{v}(\vec{r},t) = \begin{pmatrix} a \\ bt \\ 0 \end{pmatrix}$$

(a) stream line segments ($d\vec{s}$) are always parallel to $\vec{v}$

$$\Rightarrow \quad d\vec{s} \times \vec{v} = 0$$

$$\begin{cases} d\vec{s} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \\ \vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} a \\ bt \\ 0 \end{pmatrix} \end{cases}$$

$$d\vec{s} \times \vec{v} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ dx & dy & dz \\ a & bt & 0 \end{vmatrix} = \begin{pmatrix} -bt\,dz \\ a\,dz \\ bt\,dx - a\,dy \end{pmatrix} \stackrel{!}{=} 0$$

$$\Rightarrow \quad dz = 0, \quad \underbrace{bt\,dx - a\,dy = 0}_{\text{solved this differential eq.}}$$

$$\left.\begin{aligned} \frac{dy}{dx} = \frac{bt}{a} \quad &\Rightarrow \quad y = \frac{bt}{a}x + C_1 \\ dz = 0 \quad &\Rightarrow \quad z = C_2 \end{aligned}\right\} \Rightarrow \text{stream lines.}$$

(b) path lines are trajectories of individual particles.

$$\vec{v} = \frac{d\vec{r}}{dt} \stackrel{!}{=} \begin{pmatrix} a \\ bt \\ 0 \end{pmatrix} \Rightarrow \vec{r} = \begin{pmatrix} at + C_3 \\ \frac{1}{2}bt^2 + C_4 \\ C_5 \end{pmatrix} \Rightarrow \left.\begin{aligned} x &= at + C_3 \\ y &= \tfrac{1}{2}bt^2 + C_4 \\ z &= C_5 \end{aligned}\right\}$$

(C) ignore the z components.

streamline : $y = \frac{bt}{a}x + C_1$

pathline : $y = \frac{a^2 b}{2}(x - C_3)^2 + C_4$

} not the same. one of them is time-dependent.

Exercise 2.4

(a) show $d\vec{s} \cdot [\vec{v} \times (\underbrace{\vec{\nabla} \times \vec{v}}_{\text{a vector, call it } \vec{w}})] = 0$:

$\vec{v} \times \vec{w}$ is a vector perpendicular to $\vec{v}$ } property of cross products

$\Rightarrow \vec{v} \perp \vec{v} \times \vec{w}$ (†)

streamline segments $d\vec{s}$ is alway parrellel to $\vec{v}$

$\Rightarrow d\vec{s} \parallel \vec{v}$ (*)

combining (†) (*)

$$d\vec{s} \parallel \vec{v} \perp \vec{v} \times (\vec{\nabla} \times \vec{v})$$

$\Rightarrow d\vec{s} \perp \vec{v} \times (\vec{\nabla} \times \vec{v})$ } property of dot products

$\Leftrightarrow d\vec{s} \cdot \vec{v} \times (\vec{\nabla} \times \vec{v}) = 0$

(b) $$0 \overset{!}{=} \int_{\text{A streamline}} \vec{\nabla} (\underbrace{\tfrac{\rho_0}{2}\vec{v}^2 + p}_{H}) \cdot d\vec{s} = \int_{\text{a streamline}} \vec{\nabla} H \cdot d\vec{s}$$

$$= \int_{\text{a streamline}} \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \\ \frac{\partial H}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \int_{\text{a streamline}} \left(\frac{\partial H}{\partial x} \cdot dx + \frac{\partial H}{\partial y} dy + \frac{\partial H}{\partial z} dz \right)$$

$$\underset{\text{differential}}{\overset{\text{total}}{=}} \int_{\text{a streamline}} dH = \int_{\text{a streamline}} d\left(\tfrac{\rho_0}{2}\vec{v}^2 + p\right) \overset{!}{=} 0$$

$\Rightarrow$ along a streamline : $\frac{\rho_0}{2}\vec{v}^2 + p = \text{constant}$

(C) For another streamline, the constant might be different, in principle.

But, in the far upstream, where there's no objects disturbing the ideal flow.

the velocity there must be the same everywhere, the same holds for the pressure there.

Then along certain streamline, the constant does not change.

$\Rightarrow$ the constants are the same everywhere.