Fluid Dynamics Q3 2017
Homework + Exercise Problems I

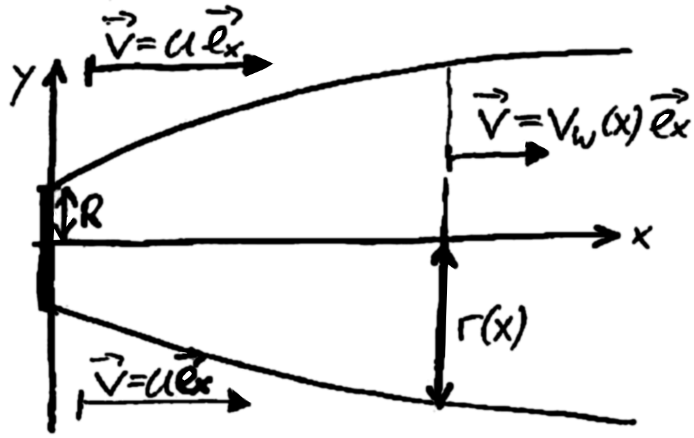
Posted:
Friday January 27, 2017.

Deadline for submission of homework problem:
Friday February 3 at 10.00 am (on paper at 1531-119).

Contents

1 Homework problem 1: Wind turbine wake

Consider the following simple model for the wake behind a wind turbine:



Outside the wake, the unperturbed wind is $v = ue_x$. Inside the wake, the x-component of the wind is $v_w(x)$, i.e. it only depends on the downwind distance x from the turbine. The wake is assumed to have a circular cross section with radius

$$r(x) = R \quad 1 + k\frac{x}{R} \quad , \tag{1}$$

where R is the rotor radius of the turbine. k and are two parameters describing the wake expansion.

(a) Show that for $x > 0$ the x-component of the wind inside the wake is given by

$$v_w(x) = u - \frac{u - v_w(0)}{1 + \frac{kx}{R} \quad {}^{2}} \tag{2}$$

(b) Let $R = 70\,\text{m}$, $v_w(0) = u/3$, $k = 0.35$ and $= 0.5$. Find $x_{90\%}$ and $x_{99\%}$ such that $v_w(x_{90\%}) = 0.9u$ and $v_w(x_{99\%}) = 0.99u$, respectively.

(c) A wind turbine generates the power $P = \frac{1}{2}\ {}_{\text{air}}\ R^2 C_p v^3$. The Power coefficient is assumed to be $C_p = 0.4$, ${}_{\text{air}} = 1.225\,\text{kg/m}^3$, and again $R = 70\,\text{m}$.

How much power does the turbine at $x = 0$ produce for a typical wind velocity $u = 10\,\text{m/sec}$?

How much power does a second turbine produce, which stands in the full wake of the first turbine at $x = 15R$?

Use the same parameters as in part (b), and give your results in the unit $\text{MW} = 1 \times 10^6\,\text{kg}\,\text{m}^2/\text{sec}^3$.

Solution

(a) To solve this part we will make use of Leonardo's law (mass conservation in cylindrical volume of radius $r(x)$ - what flows in must flow out)

$$\frac{m_1}{t}_{\;x=0} = \frac{m_2}{t}_{\;x=0}$$

Left hand side consists of turbine incoming wind -and surrounding wind

$$A_1 v_w(0) + \quad (A_2 - A_1)\, u = \quad A_2 v_w(x)$$

$$R^2 v_w(0) + \qquad r^2(x) - \quad R^2 \quad u = \qquad r^2(x) v_w(x)$$

Divide by and

$$R^2 v_w(0) + \quad r^2(x) - R^2 \quad u = r^2(x) v_w(x)$$

Now solve for $v_w(x)$

$$\begin{aligned} v_w(x) &= u - \frac{R^2}{r^2(x)}\,(u - v_w(0)) \\ &= u - \frac{u - v_w(0)}{1 + k\frac{x}{R}\;^2} \end{aligned}$$

(b) We continue the steps from above and replace x by x_q:

$$v_w(x_q) = u - \frac{u - v_w(0)}{1 + k\frac{x_q}{R}\;^2}$$

Switch around terms

$$\frac{u - v_w(0)}{1 + k\frac{x_q}{R}\;^2} = u - v_w(x_q)$$

Switch around factors

$$\frac{u - v_w(0)}{u - v_w(x_q)} = \quad 1 + k\frac{x_q}{R}\;^2$$

Use $\quad = 0.5$

$$\frac{u - v_w(0)}{u - v_w(x_q)} = 1 + k\frac{x_q}{R}$$

solve for x_q

$$x_q = \frac{R}{k} \quad \frac{u - v_w(0)}{u - v_w(x_q)} - 1$$

Put in the given values and calculate

$$q = 0.90 \qquad x_{90\%} = \frac{70\,\text{m}}{0.35} \quad \frac{u - \frac{u}{3}}{u - 0.9u} - 1 \quad = \frac{70\,\text{m}}{0.35} \quad \frac{\frac{2}{3}}{0.1} - 1$$
$$= \frac{17}{3} \cdot 200\,\text{m} = 1133\,\text{m}$$

$$q = 0.99 \qquad x_{99\%} = \frac{70\,\text{m}}{0.35} \quad \frac{u - \frac{u}{3}}{u - 0.99u} - 1 \quad = \frac{70\,\text{m}}{0.35} \quad \frac{\frac{2}{3}}{0.1} - 1$$
$$= \frac{197}{3} \cdot 200\,\text{m} = 13\,133\,\text{m}$$

(c1)

$$\begin{aligned} P &= \frac{1}{2}\ {}_{\text{air}} \quad R^2 C_p v^3 \\ &= \frac{1}{2} \cdot 1.225\,\text{kg}/\text{m}^3 \cdot \quad \cdot 70^2\text{m}^2 \cdot 0.4 \cdot v^3 \\ &= 3770\,\text{kg}/\text{m} \cdot v^3 \end{aligned}$$

Setting $v = 10\,\text{m}/\text{sec}$ gives:

$$\begin{aligned} P &= 3770\,\text{kg}/\text{m} \cdot 10^3\text{m}^3/\text{sec}^3 \\ &= 3.77 \times 10^6\,\text{kgm}^2/\text{sec}^3 \\ &= 3.77 MW \end{aligned}$$

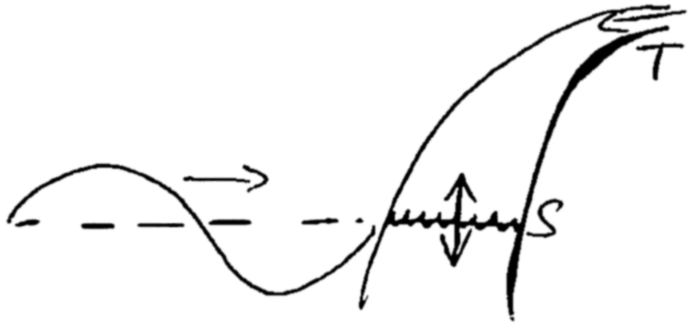
(c2) Put v equal to $v_w(x)$, $u = 10\,\text{m}/\text{s}$ and calculate

$$\begin{aligned} P &= 3770\,\text{kg}/\text{m} \cdot \quad u - \frac{u - v_w(0)}{(1 + \frac{kx}{R})^2} \quad ^3 \\ &= 3770\,\text{kg}/\text{m} \cdot \quad 10\,\text{m}/\text{s} - \frac{10\,\text{m}/\text{s} - \frac{10}{3}\text{m}/\text{s}}{(1 + \frac{0.35 \cdot 15R}{R})^{2 \cdot 0.5}} \quad ^3 \\ &= 2.69 \cdot 10^6\text{kgm}^2/\text{sec}^3 \\ &= 2.69\,\text{MW} \end{aligned}$$

(c3) This part is included in part (c2)

2 Exercise 1.1: The oscillating water column wave energy converter

Consider the sketch below of a device for converting ocean waves to electrical power.



The waves induce an oscillating water surface height in the bottom of the device (labeled S) and this drives air in and out of the top of the device (labeled T). A turbine is placed at the top nozzle to extract power from the moving air stream.

(a) Assume that the water surface height h is oscillating with an amplitude H and an angular frequency , i.e.

$$h(t) = H \sin \quad t. \tag{3}$$

Argue that if we ignore the influence of the turbine, the air velocity at the top nozzle is then given by

$$v(t) = \frac{A_S}{A_T} \quad H \cos \quad t, \tag{4}$$

where A_S and A_T are cross-sectional areas of the device at S, respectively T.

(b) A fluid moving at a speed v carried kinetic energy through a perpendicular plane of area A at the rate $P = \frac{1}{2} \quad Av^3$. Assuming that the turbine can extract 30% of the theoretically available power, calculate its peak output given:

$$\begin{aligned}
A_S &= 50\,\mathrm{m}^2 \\
A_T &= 1\,\mathrm{m}^2 \\
&= 2 \quad \cdot 0.5 \frac{1}{\mathrm{sec}} \\
H &= 0.5\,\mathrm{m} \\
{}_{\mathrm{air}} &= 1.2\,\frac{\mathrm{kg}}{\mathrm{m}^3}
\end{aligned}$$

Solution

(a) Mass conservation gives:

$$A_S v_S = A_T v_T$$

$$v_T = \frac{A_S}{A_T} v_S.$$

The speed at S is given by the water surface:

$$v_S(t) = \frac{d}{dt} h(t) = H \quad \cos(\quad t).$$

So finally:

$$v_T(t) = \frac{A_S}{A_T} \quad H \cos(\quad t)$$

(b) We put in the numbers:

$$\begin{aligned} v_T^{\max} &= \frac{A_S}{A_T} \quad H = \frac{50\,\mathrm{m}^2}{1\,\mathrm{m}^2} 2 \quad \times 0.5\,\mathrm{s}^{-1} 0.5\,\mathrm{m} \\ &= 78.5\,\mathrm{m/s}. \end{aligned}$$

As $V_m(cos) = [-1, 1]$ we exclude this term since we want v_T^{max}

With 30% efficiency we get

$$\begin{aligned} P_{\text{extracted}}^{\max} &= 0.3 \cdot \frac{1}{2} \cdot 1.2\,\mathrm{kg/m^3} \cdot 1\,\mathrm{m}^2 \cdot (78.5\,\mathrm{m/s})^3 \\ &= 8.70 \times 10^4\,\mathrm{W} = 87.0\,\mathrm{kW} \end{aligned}$$

3 Exercise 1.2

Consider the following two-dimensional velocity field:

$$u_x = u_0 \tag{5}$$
$$u_y = \quad x^2 \tag{6}$$

(a) Sketch the velocity vector at selected positions along the x-axis.

(b) Calculate the divergence $\quad \cdot u$ and curl $\quad \times u$ of the velocity field.

(c) Use the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u_x, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = u_y \tag{7}$$

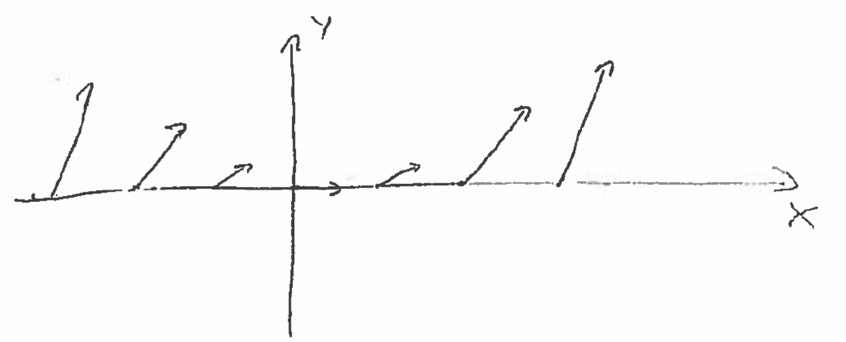
to show that if a particle is in position (x_0, y_0) at time t_0, then at time t its position is given by

$$x(t) = x_0 + u_0(t - t_0) \tag{8}$$
$$y(t) = y_0 + \frac{\quad}{3u_0} \quad x(t)^3 - x_0^3 \quad . \tag{9}$$

(d) Sketch some particle trajectories (path lines) with $t_0 = x_0 = 0$.

Solution

(a) Sketch of the velocity vector at selected positions along the x-axis.



(b)

$$u = \begin{matrix} u_x \\ u_y \\ u_z \end{matrix} = \begin{matrix} u_0 \\ x^2 \\ 0 \end{matrix} \quad \text{and} \quad = \begin{matrix} \frac{}{x} \\ \frac{}{y} \\ \frac{}{z} \end{matrix}$$

$$\cdot u = \frac{}{x}u_x + \frac{}{y}u_y + \frac{}{y}u_z = 0 + 0 + 0 = 0$$

$$\times u = \begin{matrix} \frac{}{y}u_z - \frac{}{z}u_y \\ \frac{}{z}u_x - \frac{}{x}u_z \\ \frac{}{x}u_y - \frac{}{y}u_x \end{matrix} = \begin{matrix} 0 \\ 0 \\ 2\ \ x \end{matrix}$$

Another way of solving this exercise could be as the following:

Since v is in the xy-plane we have

$$\times u \;_{x} = \quad \times u \;_{y} = 0$$

$$\times u \;_{z} = \;_{x}u_y - \;_{y}u_x = 2\ \ x - 0$$

i.e.

$$\times u = 2\ \ x\hat{e}_z$$

(c) The velocity of a particle in a certain position x is given as:

$$v_x = \frac{dx(t)}{dt}$$

To find an expression for a particles position at x at time t we will perform an integration with respect to time t. To avoid confusion put $t = t$

$$
\begin{aligned}
v_x &= \frac{dx(t)}{dt} \\
&\text{integrate over } t \\
\int_{t_0}^{t} v_x dt &= \int_{t_0}^{t} \frac{dx(t)}{dt} dt \\
v_x \int_{t_0}^{t} dt &= \int_{t_0}^{t} dx(t) \\
v_x(t - t_0) &= x(t) - x(t_0) \\
&\text{put } v_x = v_0 \text{ and } x(t_0) = x_0 \\
v_0(t - t_0) &= x(t) - x_0 \\
&\text{rearrange} \\
x(t) &= x_0 + v_0(t - t_0)
\end{aligned}
$$

The velocity of a particle in a certain position x is given as:

$$
v_y = \frac{dy(t)}{dt}
$$

$v_y = x^2$ and putting in the equation for $x(t)$ we get that $v_y(t) = (x_0 + v_0(t - t_0))^2$. Again, we put $t = t$:

$$
v_y = \frac{dy(t)}{dt}
$$

$$
\int_{t_0}^{t} v_y(t) dt = \int_{t_0}^{t} \frac{dy(t)}{dt} dt
$$

$$
\int_{t_0}^{t} (x_0 + v_0(t - t_0))^2 dt = \int_{t_0}^{t} dy(t)
$$

To integrate this expression we will use the method of substitution. Put $u = x_0 + v_0(t - t_0)$ and we have then that

$$
\begin{aligned}
\frac{du}{dt} &= \frac{d}{dt}(x_0 + v_0(t - t_0)) \\
&= \frac{d}{dt} x_0 + v_0 \frac{d}{dt}(t - t_0) \\
&= v_0
\end{aligned}
$$

so

$$\frac{du}{dt} = v_0$$

$$dt = \frac{1}{v_0} du$$

To find the new integration boundaries we put t_0 and t into the substitution. Here the lower range is x_0 and the upper range is $x(t)$. Now we will insert our findings into the integration.

$$\int_{t_0}^{t} (x_0 + v_0(t - t_0))^2 dt = \int_{t_0}^{t} dy(t)$$

$$\frac{}{v_0} \int_{x_0}^{x(t)} u^2 du = y(t) - y(t_0)$$

$$\frac{}{3v_0}(x^3(t) - x_0^3) + y_0 = y(t)$$

(d) If $x_0 = t_0 = 0$ we have

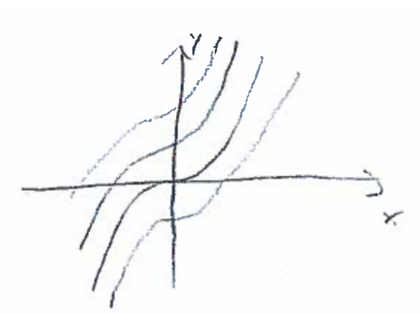
$$x(t) = u_0 t$$

$$y(t) = y_0 + \frac{}{3u_0}[x(t)]^3$$

i.e.

$$y = y_0 + \frac{}{3u_0} x^3$$

Sketch of some particle trajectories (path lines) with $t_0 = x_0 = 0$.



Since v is time independent, then the stream lines are the same as path trajectories.