Lecture Notes on Fluid Dynamics

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1 Warm-up: poor man's approach to Fluid Dynamics

This simple approach is capable of quite a few important applications!

1.1 Leonardo's Law: mass conservation

What streams into a volume has to stream out again.

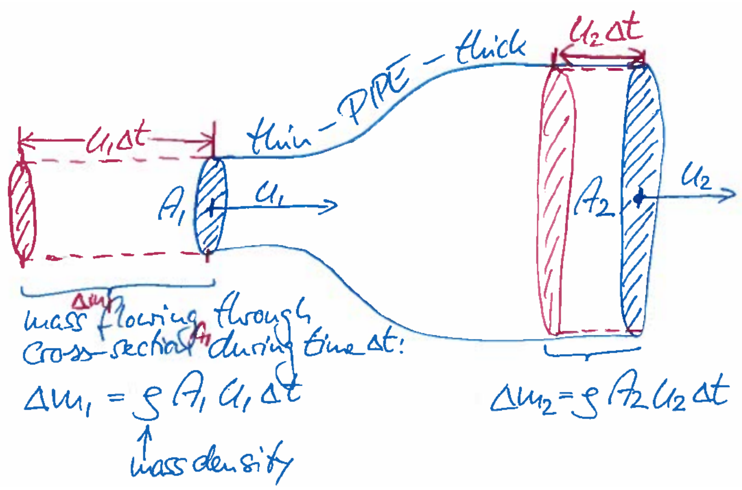


Figure 1: An illustration of mass conservation.

Mass conservation means that the inflow on the left side must equal the outflow on the right side. That is

$$\Delta m_1 = \Delta m_2 \tag{1.1}$$

$$\Downarrow$$

$$\rho A_1 u_1 \Delta t = \rho A_2 u_2 \Delta t \tag{1.2}$$

$$\Downarrow$$

$$A_1 u_1 = A_2 u_2 \tag{1.3}$$

Here (1.3) is known as Leonardo's Law. It has the following properties:

$$A_1 < A_2 \Rightarrow u_1 > u_2 \tag{1.4}$$

$$A_1 > A_2 \Rightarrow u_1 < u_2. \tag{1.5}$$

1.1.1 Example 1: why is it always windy on Aarhus Ø?

In front of the houses:

$$\Delta m_1 = \rho_{\text{air}} A_1 u_1 \Delta t \tag{1.6}$$

Between the two houses:

$$\Delta m_2 = \rho_{\text{air}} A_2 u_2 \Delta t \tag{1.7}$$

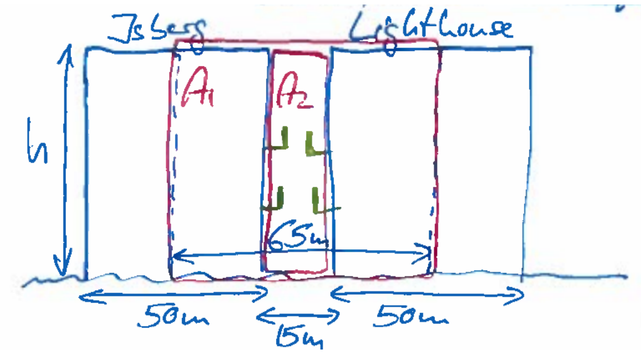


Figure 2: The gap between adjacent apartment buildings seen from the side.

Equating (1.6) and (1.7) gives

$$u_2 = \frac{A_1}{A_2} u_1 \tag{1.8}$$

Plugging in "realistic" numbers:

$$u_2 = \frac{65\,\mathrm{m} \cdot \mathrm{h}}{15\,\mathrm{m} \cdot \mathrm{h}} \cdot 10\,\mathrm{m/s} = 43.3\,\mathrm{m/s} \tag{1.9}$$

Question: Why balconies?

Answer: Architects are not engineers/physicists!

1.1.2 Example 2: falling stream of liquid

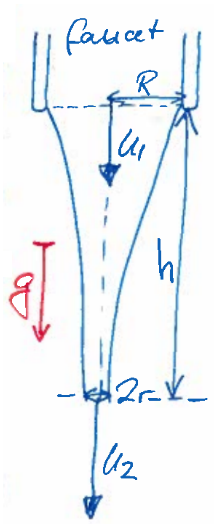


Figure 3: A stream of falling liquid.

We use Leonardo's law:

$$\pi R^2 u_1 = \pi r^2 u_2 \tag{1.10}$$

Energy conservation tells us that the sum of kinetic and potential energy is conserved:

$$\frac{m}{2}u_2^2 = \frac{m}{2}u_1^2 + mgh \tag{1.11}$$

$$\Downarrow$$

$$u_2^2 = u_1^2 + 2gh, \tag{1.12}$$

where $g = 9.82\,\text{m}/\text{S}^2$ is the acceleration of gravity.

$$\frac{r}{R} = \left(\frac{\pi r^2}{\pi R^2}\right)^{\frac{1}{2}} = \left(\frac{A_2}{A_1}\right)^{\frac{1}{2}} = \left(\frac{u_1}{u_2}\right)^{\frac{1}{2}} \tag{1.13}$$

$$= \left(\frac{u_1^2}{u_2^2}\right)^{\frac{1}{4}} = \left(\frac{u_1^2}{u_1^2 + 2gh}\right)^{\frac{1}{4}} \tag{1.14}$$

Remark: The narrowing of a falling stream of liquid holds only for the upper part of the stream. At some height h the stream becomes too thin and drop formation sets in.

1.1.3 Example 3: wave energy converter

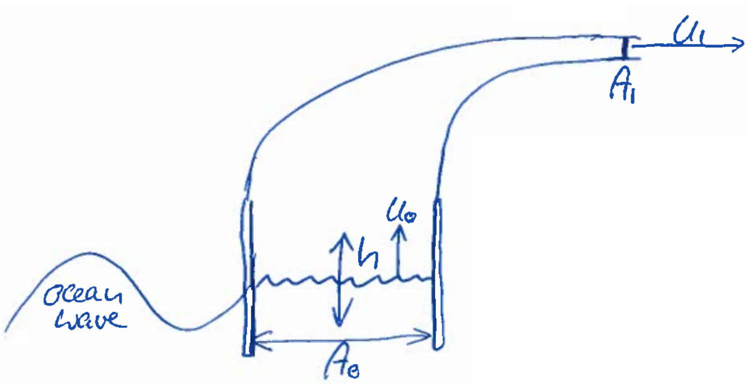


Figure 4: Schematic of a wave energy converter.

The ocean waves induce an oscillating water surface height, which induces an oscillating air stream. A turbine is placed at the nozzle (with cross-section $A_1 \ll A_0$) and extracts power from the moving air stream.

Oscillating height:

$$h(t) = H \sin \omega t, \qquad \omega = 2\pi f = \frac{2\pi}{T}, \tag{1.15}$$

where f is the frequency, T is the oscillating period and ω is the angular frequency.

$$u_0(t) = \frac{dh(t)}{dt} \tag{1.16}$$

$$= H\omega \cos \omega t \tag{1.17}$$

$$A_0 u_0(t) = A_1 u_1(t) \tag{1.18}$$

$$\Downarrow$$

$$u_1(t) = \frac{A_0}{A_1} u_0 \cos \omega t \tag{1.19}$$

$$A_1 \ll A_0 \Rightarrow u_1(t) \gg u_0(t) \tag{1.20}$$

1.1.4 Example 4: wake flow behind a wind turbine and wind farm optimization

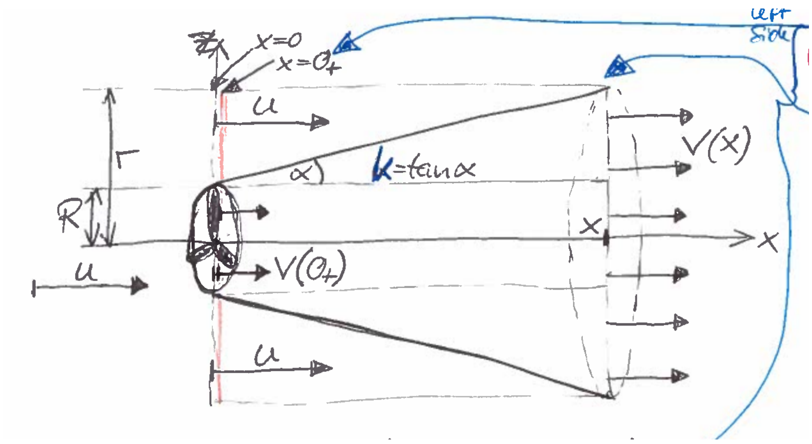


Figure 5: The expanding wake behind a turbine.

Far-field modeling of a wake flow behind a wind turbine. We use the linear wake expansion:

$$r = R + kx. \tag{1.21}$$

We use the equation of continuity (Leonardo's law):

$$\left.\frac{\Delta m}{\Delta t}\right|_{x=0^+} = \rho\pi R^2 v(0^+) + \rho\pi(r^2 - R^2)u = \rho\pi r^2 v(x) = \left.\frac{\Delta m}{\Delta t}\right|_{x}. \tag{1.22}$$

In words this equation states that the in-flow through the left side of the cylinder is equal to the out-flow through the right side of the cylinder. Rearranging this equation we can get an expression for the wind speed of the wake behind the turbine:

$$v(x) = \frac{R^2}{r^2} v(0_+) + \frac{r^2 - R^2}{r^2} u = u - \frac{R^2}{r^2} (u - v(0_+)) \tag{1.23}$$

$$= u \left\{ 1 - \frac{1 - \frac{v(0_+)}{u}}{\left(1 + \frac{kx}{R}\right)^2} \right\}. \tag{1.24}$$

The ratio

$$q = \frac{v(0_+)}{u} \tag{1.25}$$

is called the axial induction factor.

Consistency checks:

$$\lim_{x \to 0_+} v(x) = v(0_+) \tag{1.26}$$

$$\lim_{x \to \infty} v(x) = u \tag{1.27}$$

Betz theory: power produced by a wind turbine

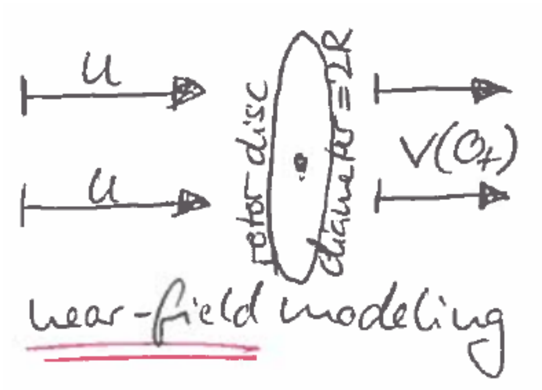


Figure 6: The velocity deficit caused by the rotor disc.

A wind turbine extracts kinetic energy out of the wind flow:

$$E_{\text{extracted}} = \frac{m}{2} \left(u^2 - v^2(0_+)\right) \tag{1.28}$$

$$= \frac{1}{2} \rho \pi R^2 \frac{u + v(0_+)}{2} \Delta t (u^2 - v^2(0_+)) \tag{1.29}$$

$$P_{\text{turbine}} = \frac{dE_{\text{extracted}}}{dt} \tag{1.30}$$

$$= \frac{\rho \pi R^2 u^3}{2} \left\{ \frac{(1+q)}{2} \left(1 - q^2\right) \right\} \tag{1.31}$$

The term in front is the kinetic energy contained in the upstream wind (volume). The term within the curly brackets is the efficiency of the wind turbine also called the power coefficient:

$$C_p = C_p(q) = \frac{(1+q)}{2}\left(1-q^2\right). \tag{1.32}$$

The maximum efficiency of a turbine can be calculated by requiring

$$\frac{dC_p(q)}{dq} = \frac{d}{dq}\left(\frac{1}{2}(1+q)(1-q^2)\right) \stackrel{!}{=} 0 \tag{1.33}$$

This gives the optimal q value

$$q = \frac{1}{3} \tag{1.34}$$

$$\Downarrow$$

$$v(0_+) = \frac{1}{3}u. \tag{1.35}$$

We can then calculate the power coefficient

$$\max_q C_p(q) = C_p(q = \frac{1}{3}) = \frac{16}{27} \approx 0.59. \tag{1.36}$$

This is known as the Betz limit. Real turbines have about $C_p \approx 0.40 - 0.50$.

1.1.5 Power optimization of a two-turbine wind farm

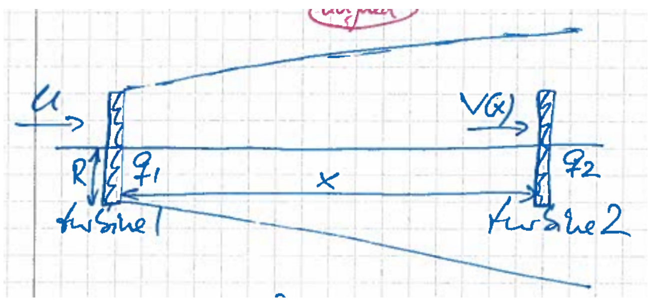


Figure 7: A very simple wind farm consisting of two turbines. The wind is approaching from the left.

We look at a wind farm with two turbines and a wind direction aligned along the connecting line. The total output of the two turbines is

$$P_{1+2} = \frac{\rho\pi R^2}{2}C_{p_1}(q_1)u^3 + \frac{\rho\pi R^2}{2}C_{p_2}(q_2)v^3(x) \tag{1.37}$$

There are no turbines behind turbine 2, so we configure it to extract the maximum amount of energy from the wind

$$q_2 = \frac{1}{3} \Rightarrow C_{p_2}(q_2) = \frac{16}{27} \tag{1.38}$$

Using previous expressions for $C_p(q)$ (1.33) and $v(x)$ (1.24) the total output of the two turbines is

$$P_{1+2} = \frac{\rho\pi R^2}{2} u^3 \left\{ \frac{1}{2}(1+q_1)(1-q_1^2) + \frac{16}{27}\left[1 - \frac{1-q_1}{\left(1+\frac{kx}{R}\right)}\right]^3 \right\} \tag{1.39}$$

Similar to before we find the optimal value of q_1 by the requirement

$$\frac{dP_{1+2}}{dq_1} \stackrel{!}{=} 0. \tag{1.40}$$

We fix the values

$$\begin{aligned} k &= 0.04 \\ \frac{x}{R} &= 8. \end{aligned}$$

The optimal q-value for turbine 1 is then

$$q_1 = 0.58 > \frac{1}{3}, \tag{1.41}$$

so turbine 1 let's through more wind.
Comparing this result with a q-value of $\frac{1}{3}$ gives

$$P_{1+2}(q_1 = 0.58) = 1.07 \cdot P_{1+2}\left(q_1 = \frac{1}{3}\right), \tag{1.42}$$

which is a 7% gain.

2 Derivation of Navier-Stokes equation

The goal of this section is to find an equation, which describes the spatio-temporal evolution of the velocity field $\vec{u}(\vec{r}, t)$. This is the Navier-Stokes equation, which is the most fundamental equation in Fluid Dynamics.

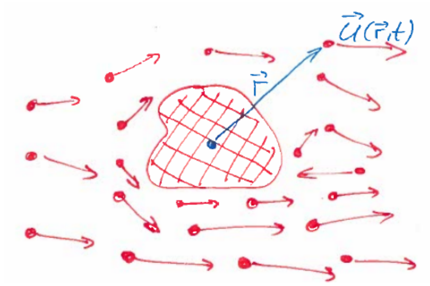


Figure 8: Velocity field around an object.

We want to describe the motion of a fluid particle. We start with Newton's second law of Classical Mechanics

$$\vec{F} = \frac{d}{dt}(m\vec{u}). \tag{2.1}$$

The equation states that the forces acting on the particle equals its change (the time derivative) of momentum (the parenthesis).

> **Remark:** in Fluid Dynamics we look not only at one fluid particle, but at all fluid particles.

$$\frac{d}{dt}(m\vec{u}) = \frac{dm}{dt}\vec{u} + m\frac{d\vec{u}}{dt} = \rho \Delta V \frac{d\vec{u}}{dt}. \tag{2.2}$$

The mass of a fluid particle is constant and does not change over time, hence the term dm/dt is zero. We also used the relation

$$m = \rho \Delta V. \tag{2.3}$$

Given the field description $\vec{u} = \vec{u}(\vec{r}, t)$, we have to be a little careful with $\frac{d\vec{u}}{dt}$. The following is wrong:

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}(\vec{r}, t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{u}(\vec{r}, t + \Delta t) - \vec{u}(\vec{r}, t)}{\Delta t} \tag{2.4}$$

See the example in Figure 9.

The correct approach is to follow one fluid particle on its pathline (trajectory) $\vec{r} = \vec{r}(\vec{r}_0, t_0; t)$. See Figure 10.

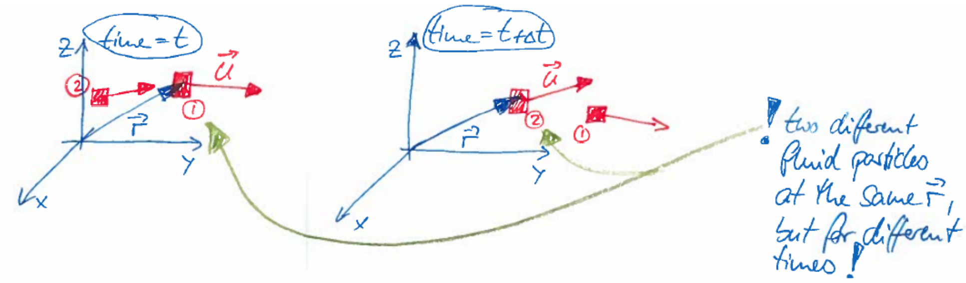


Figure 9: Two fluid particles can have the same coordinate vector $\vec{r}$ but for different times.

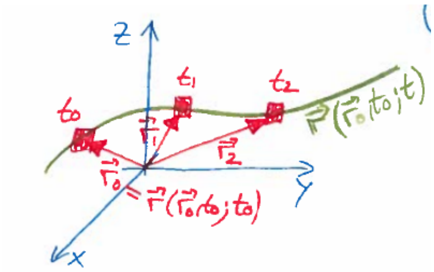


Figure 10: The path and changing coordinates of a single fluid particle.

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}\left(\vec{r}\left(\vec{r}_0, t_0; t\right), t\right)}{dt} \tag{2.5}$$

$$= \frac{d\vec{u}(x(\vec{r}_0, t_0; t), y(\vec{r}_0, t_0; t), z(\vec{r}_0, t_0; t))}{dt} \tag{2.6}$$

$$= \frac{\partial \vec{u}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{u}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{u}}{\partial z}\frac{dz}{dt} + \frac{\partial \vec{u}}{\partial t}\frac{dt}{dt} \tag{2.7}$$

$$= \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right)\vec{u} + \frac{\partial \vec{u}}{\partial t} \tag{2.8}$$

Short notation for partial derivative:

$$\frac{\partial}{\partial x} = \partial_x. \tag{2.9}$$

Here x can be replaced by y, z or t.

Short notation for velocity:

$$\frac{dx}{dt} = u_x \tag{2.10}$$

The terms in parentheses is the dot product between $\vec{u}$ and $\vec{\nabla} = \left(\partial_x,\ \partial_y,\ \partial_z\right)$, calculated using the chain rule of differentiation:

$$\frac{du(f(t))}{dt} = \frac{\partial u}{\partial f}\frac{df}{dt} \tag{2.11}$$

$$\frac{du(f(t),g(t))}{dt} = \frac{\partial u}{\partial f}\frac{df}{dt} + \frac{\partial u}{\partial g}\frac{dg}{dt} \tag{2.12}$$

$$\frac{du(f(t),g(t),h(t))}{dt} = \frac{\partial u}{\partial f}\frac{df}{dt} + \frac{\partial u}{\partial g}\frac{dg}{dt} + \frac{\partial u}{\partial h}\frac{dh}{dt}. \tag{2.13}$$

This leads to the final expression

$$\frac{d\vec{u}}{dt} = \frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}. \tag{2.14}$$

The material derivative is defined as

$$\partial_t + \vec{u}\cdot\vec{\nabla} = \frac{D}{Dt} = D_t. \tag{2.15}$$

Whenever we have the time derivative of a field, like $\vec{u}(\vec{r},t)$, then we have to "go with the fluid particle" and use the material derivative.

We now go back to Newton's second equation:

$$\frac{d}{dt}(m\vec{u}) = \rho\Delta V\left(\partial_t + \vec{u}\cdot\vec{\nabla}\right)\vec{u} = \vec{F} = \vec{F}_{\text{external}} + \vec{F}_{\text{surrounding}} \tag{2.16}$$

The surrounding force can be decomposed into

$$\vec{F}_{\text{surrounding}} = \vec{F}_{\text{pressure}} + \vec{F}_{\text{friction}} \tag{2.17}$$

The surrounding fluid particles push the "sandwiched" fluid particle around; they exert pressure. Mutual friction between neighboring fluid particles due to relative and rotational motion, deformation and compression.

Example: The force of gravity is an example of an external force:

$$\vec{F}_{\text{external}} = \vec{F}_{\text{grav}} = \underbrace{\rho\Delta V}_{m}\,\vec{g} = \rho\vec{g}\Delta V, \tag{2.18}$$

with the gravitational constant defined as

$$\vec{g} = -\begin{pmatrix} 0 \\ 0 \\ 9.81\,\frac{\text{m}}{\text{s}} \end{pmatrix} = -g\vec{e}_z. \tag{2.19}$$

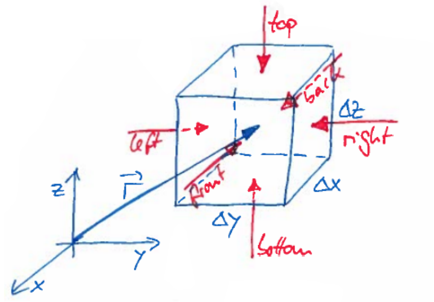


Figure 11: Illustration of the pressure force on each side of a fluid particle.

2.1 Pressure force

$$
\begin{aligned}
\left(\vec{F}_{\text{pressure}}\right)_z &= \vec{F}^{\text{top}}_{\text{pressure}} + \vec{F}^{\text{bottom}}_{\text{pressure}} &(2.20)\\
&= -p\left(x,y,z+\frac{\Delta z}{2}\right)\Delta x\Delta y + p\left(x,y,z-\frac{\Delta z}{2}\right)\Delta x\Delta y &(2.21)\\
&= -\left(p(x,y,z)+\frac{\partial p(x,y,z)}{\partial z}\frac{\Delta z}{2}\right)\Delta x\Delta y \\
&\quad + \left(p(x,y,z)+\frac{\partial p(x,y,z)}{\partial z}\left(-\frac{\Delta z}{2}\right)\right)\Delta x\Delta y &(2.22)\\
&= -\frac{\partial p(x,y,z)}{\partial z}\Delta x\Delta y\Delta z &(2.23)
\end{aligned}
$$

Using $\Delta x\Delta y\Delta z = \Delta V$ the pressure force density is

$$\vec{f}_{\text{pressure}} = \frac{\vec{F}_{\text{pressure}}}{\Delta V} = -\vec{\nabla}p(\vec{r},t) \qquad (2.24)$$

2.2 Friction force

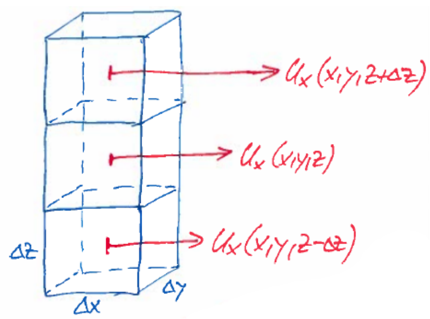


Figure 12: Illustration of the friction force on each side of a fluid particle.

We consider the neighboring fluid particles above and below as sketched in Figure 12.

$$\left(\vec{F}^{\text{top+bottom}}_{\text{friction}}\right)_x = \frac{\mu}{\Delta z}\Delta x \Delta y \left(u_x(x,y,z+\Delta z) - u_x(x,y,z)\right) + \frac{\mu}{\Delta z}\Delta x \Delta y \left(u_x(x,y,z-\Delta z) - u_x(x,y,z)\right) \tag{2.25}$$

If $u_x(x,y,z+\Delta z) > u_x(x,y,z)$, then the fluid particle above pulls the sandwiched fluid particle with it.

Taylor series expansion up to second-order terms:

$$\begin{aligned}\left(\vec{F}^{\text{top+bot}}_{\text{friction}}\right)_x &= \mu \Delta x \Delta y \left\{\frac{u_x(x,y,z+\Delta z) - u_x(x,y,z)}{\Delta z} + \frac{u_x(x,y,z-\Delta z) - u_x(x,y,z)}{\Delta z}\right\} \\ &= \frac{\mu \Delta x \Delta y}{\Delta z}\left\{u_x(x,y,z) + \frac{\partial u_x(x,y,z)}{\partial z}\Delta z + \frac{\partial^2 u_x(x,y,z)}{\partial z^2}\frac{\Delta z^2}{2} - u_x(x,y,z)\right. \\ &\quad \left. + u_x(x,y,z) + \frac{\partial u_x(x,y,z)}{\partial z}(-\Delta z) + \frac{\partial^2 u_x(x,y,z)}{\partial z^2}\frac{(-\Delta z)^2}{2} - u_x(x,y,z)\right\}\end{aligned} \tag{2.26}$$

$$= \mu \Delta x \Delta y \Delta z \frac{\partial^2 u_x(x,y,z)}{\partial z^2} \tag{2.27}$$

Front + back:

$$\left(\vec{F}^{\text{front+back}}_{\text{friction}}\right)_x = \mu \Delta V \frac{\partial^2 u_x(x,y,z)}{\partial y^2} \tag{2.28}$$

Most general expression of the friction force:

$$\frac{\vec{F}_{\text{friction}}}{\Delta V} = \vec{f}_{\text{friction}}\left(\frac{\partial^2 u_i}{\partial x_k \partial x_l}\right) = \vec{f}_{\text{friction}}(\vec{\nabla}, \vec{\nabla}, \vec{u}) \tag{2.29}$$

The task is to build a vector $\vec{f}$ from a combination of three vectors $\vec{a} = \vec{\nabla}, \vec{b} = \vec{\nabla}, \vec{c} = \vec{u}$, such that

$$\vec{f} = \alpha \left(\vec{a} \cdot \vec{b}\right) \vec{c} + \beta \left(\vec{a} \cdot \vec{c}\right) \vec{b} + \gamma \left(\vec{b} \cdot \vec{c}\right) \vec{a} \tag{2.30}$$

The solution:

$$\vec{f}_{\text{friction}} = \mu \left(\vec{\nabla} \cdot \vec{\nabla}\right) \vec{u} + \left(\mu_v + \frac{\mu}{3}\right) \vec{\nabla} \left(\vec{\nabla} \cdot \vec{u}\right), \tag{2.31}$$

where μ is the shear (dynamic) viscosity and μ_v is the compression (bulk) viscosity.

Navier-Stokes equation:

$$\rho \left(\partial_t + \left(\vec{u} \cdot \vec{\nabla}\right)\right) \vec{u} = \vec{f}_{\text{ext}} - \vec{\nabla} p + \mu \left(\vec{\nabla} \cdot \vec{\nabla}\right) \vec{u} + \left(\mu_v + \frac{\mu}{3}\right) \vec{\nabla} \left(\vec{\nabla} \cdot \vec{u}\right) \tag{2.32}$$

where

$$\vec{u} = \vec{u}\left(\vec{r}, t\right) \tag{2.33}$$

$$p = p\left(\vec{r}, t\right) \tag{2.34}$$

$$\rho = \rho\left(\vec{r}, t\right) \tag{2.35}$$

$$\vec{f}_{\text{ext}} = \vec{f}_{\text{ext}}\left(\vec{r}, t\right) \tag{2.36}$$

2.3 Navier-Stokes equation in components

$$
\rho \begin{pmatrix} \frac{\partial u_x}{\partial t} \\ \frac{\partial u_y}{\partial t} \\ \frac{\partial u_z}{\partial t} \end{pmatrix} + \rho \begin{pmatrix} \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_x \\ \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_y \\ \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_z \end{pmatrix} = \begin{pmatrix} f_x^{\text{ext}} \\ f_y^{\text{ext}} \\ f_z^{\text{ext}} \end{pmatrix} - \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{pmatrix}
+ \mu \begin{pmatrix} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u_x \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u_y \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u_z \end{pmatrix}
+ \left(\mu_v + \frac{\mu}{3}\right) \begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\ \frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \end{pmatrix} \tag{2.37}
$$

Remark: There are only a few exact analytical solutions; many approximate analytical solutions (guided by intuition). Computational fluid dynamics can give us "exact" numerical solutions for approximations to the Navier-Stokes equation.

We now have three coupled differential equations for five fields: $u_x(\vec{r}, t)$, $u_y(\vec{r}, t)$, $u_z(\vec{r}, t)$, $p(\vec{r}, t)$, and $\rho(\vec{r}, t)$. This means we are missing two equations.

The first missing equation

From thermodynamics we have an equation of state

$$g(p, \rho) = 0. \tag{2.38}$$

For an incompressible flow, the equaiton of state is simply

$$\rho = \text{constant}. \tag{2.39}$$

For a compressible flow, the equation of state can be found with the law of ideal gases:

$$pV = NkT, \tag{2.40}$$

from which we get

$$\rho = \frac{N}{V} = \frac{1}{kT} p \tag{2.41}$$

$$\frac{p}{\rho} = kT = \text{constant}. \tag{2.42}$$

This only holds if the temperature is constant.

The second missing equation

Equation of continuity, local mass conservation.

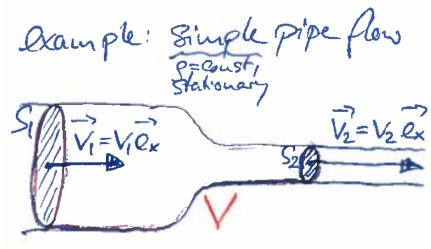


Figure 13: A simple pipe flow to illustrate mass conservation.

Example: Simple pipe flow. See Figure 13.

$$M_{\text{in}} = \rho S_1 v_1 \Delta t \tag{2.43}$$

$$M_{\text{out}} = \rho S_2 v_2 \Delta t \tag{2.44}$$

Mass conservation:

$$M_{\text{in}} = M_{\text{out}} \tag{2.45}$$

$$\Downarrow$$

$$v_1 S_1 = v_2 S_2 \tag{2.46}$$

This is Leonardo's law.

Local mass conservation in a small volume element:

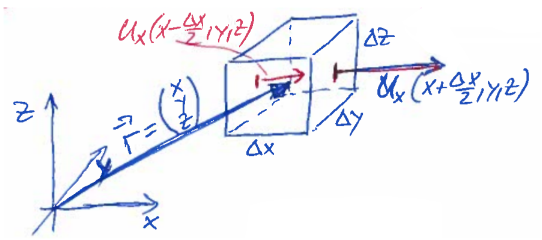


Figure 14: Local mass conservation in a small volume element.

Mass flux through surface of volume $\Delta V = \Delta x \Delta y \Delta z$ in x-direction:

$$\begin{aligned}
\frac{dM_x^S}{dt} &= \rho\left(x + \frac{\Delta x}{2}, y, z, t\right) u_x \left(x + \frac{\Delta x}{2}, y, z, t\right) \Delta y \Delta z \\
&\quad - \rho\left(x - \frac{\Delta x}{2}, y, z\right) u_x \left(x - \frac{\Delta x}{2}, y, z\right) \Delta y \Delta z \\
&= \left(\rho(x,y,z) + \frac{\partial \rho(x,y,z)}{\partial x}\frac{\Delta x}{2}\right)\left(u_x(x,y,z) + \frac{\partial u_x(x,y,z)}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z \\
&\quad - \left[\rho(x,y,z) + \frac{\partial \rho(x,y,z)}{\partial x}\left(-\frac{\Delta x}{2}\right)\right]\left[u_x(x,y,z) + \frac{\partial u_x(x,y,z)}{\partial x}\left(-\frac{\Delta x}{2}\right)\right]\Delta y \Delta z \\
&= \frac{\partial \rho(x,y,z)}{\partial x} u_x(x,y,z)\Delta x \Delta y \Delta z + \rho(x,y,z)\frac{\partial u_x(x,y,z)}{\partial x}\Delta x \Delta y \Delta z \\
&= \frac{\partial\left(\rho(x,y,z,t)u_x(x,y,z,t)\right)}{\partial x}\Delta V
\end{aligned} \tag{2.47}$$

Mass flux in y and z-direction:

$$\frac{dM_y^S}{dt} = \frac{\partial(\rho(x,y,z)u_y(x,y,z))}{\partial y}\Delta V \tag{2.48}$$

$$\frac{dM_z^S}{dt} = \frac{\partial(\rho(x,y,z)u_z(x,y,z))}{\partial z}\Delta V \tag{2.49}$$

Sum of mass fluxes through volume in all directions:

$$\frac{dM^S}{dt} = \frac{dM_x^S}{dt} + \frac{dM_y^S}{dt} + \frac{dM_z^S}{dt} \tag{2.50}$$

$$= \frac{\partial(\rho u_x)}{\partial x}\Delta V + \frac{\partial(\rho u_y)}{\partial y}\Delta V + \frac{\partial(\rho u_z)}{\partial z}\Delta V \tag{2.51}$$

$$= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \rho u_x \\ \rho u_y \\ \rho u_z \end{pmatrix} \Delta V \tag{2.52}$$

$$= \vec{\nabla} \cdot \left(\rho(x,y,z)\vec{u}(x,y,z)\right) \Delta V \tag{2.53}$$

Increase of mass within fixed volume ΔV:

$$\frac{\partial M^V}{\partial t} = \frac{\partial(\rho(\vec{r},t)\Delta V)}{\partial t} = \frac{\partial \rho(\vec{r},t)}{\partial t}\Delta V \tag{2.54}$$

Local mass conservation

$$\frac{dM^V}{dt} = -\frac{dM^S}{dt} \tag{2.55}$$

If mass within the volume increases, then less has to flow out of the surface than to flow in

$$\frac{\partial \rho(\vec{r},t)}{\partial t} + \vec{\nabla} \cdot \left(\rho(\vec{r},t)\vec{u}(\vec{r},t)\right) = 0 \tag{2.56}$$

This is the equation of continuity.

2.4 Summary

Navier-Stokes equation:

$$\rho\left(\frac{\partial}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\right)\vec{u}=\vec{f}_{\text{ext}}-\vec{\nabla}p+\mu\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{u}+\left(\mu_v+\frac{\mu}{3}\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}\right) \tag{2.57}$$

$$\vec{u}=\vec{u}(\vec{r},t)=\vec{u}(x,y,z,t) \tag{2.58}$$

$$\rho=\rho(\vec{r},t) \tag{2.59}$$

$$p=p(\vec{r},t) \tag{2.60}$$

Equation of continuity:

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{u})=0 \tag{2.61}$$

Equation of state:

$$g(p,\rho;T)=0 \tag{2.62}$$

Heat equation (if the temperature also becomes a field $T(\vec{r},t)$):

$$\left(\frac{\partial}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\right)T(\vec{r},r)=\kappa\left(\vec{\nabla}\cdot\vec{\nabla}\right)T(\vec{r},t), \tag{2.63}$$

where κ is the thermal diffusion.

3 Simplification of the Navier-Stokes equation

3.1 Simplification I: incompressible flows

Incompressiblity is a very good approximation for most liquids, including water. In 1000 m depth the density of seawater is only 0.4% larger than at the surface. For gas flows incompressiblity is also a good approximation as long as $|\vec{u}_{\text{gas}}| \ll$ speed of sound. Compressibility becomes important when discussing e.g. sound waves.

Incompressibility means that the density of a fluid particle (moving along its path-line) remains constant.

$$\begin{aligned} 0 = \frac{d\rho(\vec{r},t)}{dt} &= \frac{\partial\rho}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\rho && (3.1) \\ &= -\vec{\nabla}\left(\rho\vec{u}\right) + \left(\vec{u}\cdot\vec{\nabla}\right)\rho && (3.2) \\ &= -\left(\vec{u}\cdot\vec{\nabla}\right)\rho - \rho\left(\vec{\nabla}\cdot\vec{u}\right) + \left(\vec{u}\cdot\vec{\nabla}\right)\rho && (3.3) \\ &= -\rho\left(\vec{\nabla}\cdot\vec{u}\right) && (3.4) \\ &= -\rho\,\mathrm{div}\,\vec{u} && (3.5) \end{aligned}$$

In the first line we used the material derivative. In the step to the second line we used continuity equation. For incompressibility the divergence must be zero:

$$\vec{\nabla}\cdot\vec{u} = 0 \tag{3.6}$$

Navier-Stokes equation for incompressible flows (3 equations):

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\right)\vec{u} = f_{\text{ext}} - \vec{\nabla}p + \mu\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{u} \tag{3.7}$$

Remark: It looks simple, but these nonlinear differential equations remain a formidable challenge to engineers, physicists and mathematicians.

Fourth equation:

$$\vec{\nabla}\cdot\vec{u} = 0. \tag{3.8}$$

Fifth equation: equation of state in the simplest form with constant density

$$p = p(\rho) \Rightarrow \rho = \rho_0 = \text{constant}. \tag{3.9}$$

3.2 Simplification II: incompressible, ideal, stationary, irrotational flows

We use the incompressibility result:

$$\vec{\nabla}\cdot\vec{u} = 0 \tag{3.10}$$

Ideal means no friction. To eliminate friction forces we set $\mu = 0$.
Euler equation:

$$\rho_0 \left(\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} \right) = \vec{f}_{\text{ext}} - \vec{\nabla} p \tag{3.11}$$

stationary:

$$\vec{u}(\vec{r}, t) = \vec{u}(\vec{r}) \tag{3.12}$$

$$\Downarrow$$

$$\frac{\partial \vec{u}}{\partial t} = 0 \tag{3.13}$$

$$\rho_0 \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \vec{f}_{\text{ext}} - \vec{\nabla} p \tag{3.14}$$

no external forces: $\vec{f}_{\text{ext}} = 0$

$$\rho_0 \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = -\vec{\nabla} p \tag{3.15}$$

We now look at the convective term on the lefthand side (see the proof below):

$$\left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \frac{1}{2} \vec{\nabla} \underbrace{(\vec{u} \cdot \vec{u})}_{\vec{u}^2} - \vec{u} \times \left(\vec{\nabla} \times \vec{u} \right) \tag{3.16}$$

$$\Downarrow$$

$$\vec{\nabla} \left(\frac{\rho_0}{2} \vec{u}^2 + p \right) = \rho_0 \vec{u} \times \left(\vec{\nabla} \times \vec{u} \right) \tag{3.17}$$

Assuming irrotational flow: $\vec{\nabla} \times \vec{u} = 0$.

$$\vec{\nabla} \underbrace{\left(\frac{\rho_0}{2} \vec{u}^2 + p \right)}_{\text{constant}} = 0 \tag{3.18}$$

Bernoulli's equation

$$\frac{\rho_0}{2} \vec{u}^2 + p = \text{constant} \tag{3.19}$$

$$\vec{\nabla} \cdot \vec{u} = 0 \tag{3.20}$$

$$\vec{\nabla} \times \vec{u} = 0 \tag{3.21}$$

Given all the assumptions, this set of equations is equivalent to the Navier-Stokes equation.

Proof of

$$\left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \frac{1}{2} \vec{\nabla} \left(\vec{u}^2 \right) - \vec{u} \times \left(\vec{\nabla} \times \vec{u} \right). \tag{3.22}$$

First we look at the x-component of the left-hand side:

$$\left[\left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} \right]_x = \left(u_x \partial_x + u_y \partial_y + u_z \partial_z \right) u_x \tag{3.23}$$

Now we look at the rightmost term on the right-hand side:

$$\vec{\nabla} \times \vec{u} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ u_x & u_y & u_z \end{vmatrix} = \left(\partial_y u_z - \partial_z u_y\right) \vec{e}_x + \left(\partial_z u_x - \partial_x u_z\right) \vec{e}_y + \left(\partial_x u_y - \partial_y u_x\right) \vec{e}_z \tag{3.24}$$

Now we can show that the x-component of the right-hand side is equal to the x-component of the left-hand side:

$$\begin{aligned} \left[\frac{1}{2}\vec{\nabla}\left(\vec{u}^2\right) - \vec{u} \times \left(\vec{\nabla} \times \vec{u}\right)\right]_x &= \frac{1}{2}\partial_x \left(u_x^2 + u_y^2 + u_z^2\right) \\ &\quad - \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ u_x & u_y & u_z \\ \partial_y u_z - \partial_z u_y & \partial_z u_x - \partial_x u_z & \partial_x u_y - \partial_y u_x \end{vmatrix}_x && (3.25) \\ &= u_x\left(\partial_x u_x\right) + u_y\left(\partial_x u_y\right) + u_z\left(\partial_x u_z\right) \\ &\quad - u_y\left(\partial_x u_y - \partial_y u_x\right) + u_z\left(\partial_z u_x - \partial_x u_z\right) && (3.26) \\ &= u_x\left(\partial_x u_x\right) + u_y\left(\partial_y u_y\right) + u_z\left(\partial_z u_z\right) && (3.27) \\ &= \left[\left(\vec{u} \cdot \vec{\nabla}\right)\vec{u}\right]_x && (3.28) \end{aligned}$$

3.3 Derivation of Bernoulli's equation

The equation

$$\vec{\nabla}\left(\frac{\rho_0}{2}\vec{u}^2 + p\right) = \rho_0 \vec{u} \times \left(\vec{\nabla} \times \vec{u}\right) \tag{3.29}$$

is (scalar) multiplied with $d\vec{s} \parallel \vec{u}$, where $d\vec{s}$ describes an increment of a specific streamline (here pathline since $\vec{u}(\vec{r}, t) = \vec{u}(\vec{r})$).

$$d\vec{s} \cdot \left[\vec{u} \times \left(\vec{\nabla} \times \vec{u}\right)\right] = 0 \tag{3.30}$$

Since $d\vec{s} \parallel \vec{u}$ it must be that $d\vec{s} \perp \vec{u} \times \left(\vec{\nabla} \times \vec{u}\right)$.

Subsequent path-integration along a streamline yields

$$0 \stackrel{!}{=} \int_{\text{streamline}} \vec{\nabla} \underbrace{\left(\frac{\rho_0}{2}\vec{u}^2 + p\right)}_{W} \cdot d\vec{s} \tag{3.31}$$

$$= \int_{\text{streamline}} \begin{pmatrix} \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial y} \\ \frac{\partial W}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{3.32}$$

$$= \int_{\text{streamline}} \left(\frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz\right) \tag{3.33}$$

$$= \int_{\text{streamline}} dW = \int_{\text{streamline}} d\left(\frac{\rho_0}{2}\vec{u}^2 + p\right) \tag{3.34}$$

$$\Downarrow$$

$$\frac{\rho_0}{2}\vec{u}^2 + p = \text{constant} \qquad \text{(along a streamline)} \tag{3.35}$$

For another streamline the constant might in principle be different. Often the velocity in the far-field regime (away from the obstacle) is everywhere the same. The same holds true for the pressure. Then the "Bernoulli constant" has to be everywhere (far and near-field) the same. From here we then also conclude that $\vec{\nabla} \times \vec{u} = 0$ everywhere.

3.4 Example: Why does an airplane fly?

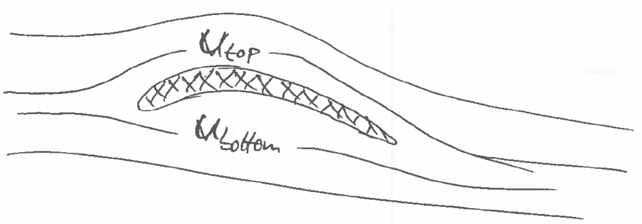


Figure 15: The profile of the wing of an airplane.

According to the Bernoulli's equation, the wind speed difference between the top and bottom of the wing creates a pressure difference:

$$u_{\text{top}} > u_{\text{bottom}} \tag{3.36}$$

$$\Downarrow$$

$$p_{\text{top}} < p_{\text{bottom}}. \tag{3.37}$$

This results in a lifting force.

4 Ideal flow: planar 2-dimensional potential flow around cylinder

For further details see sections 4.3, 4.9 and 7.1-6 in the KCD book

For a planar two-dimensional stationary flow the velocity field

$$\vec{u} = \begin{pmatrix} u_x(x,y) \\ u_y(x,y) \\ 0 \end{pmatrix} \tag{4.1}$$

does not depend on z and t and the vector does not have a z-component. The flow is defined to be "ideal" once the viscosity $\mu = 0$ is put to zero. The mass density $\rho = \rho_0$ is assumed to be constant. We will determine the two velocity components $u_x(x,y)$ and $u_y(x,y)$ from the two equations $\vec{\nabla} \cdot \vec{u} = 0$ and $\vec{\nabla} \times \vec{u} = 0$ defining incompressible and irrotational flows. The pressure field $p(x,y)$ is then determined via Bernoulli's equation.

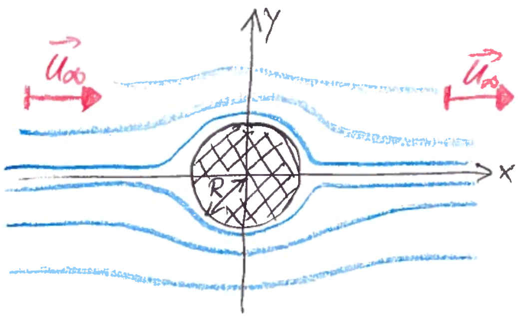


Figure 16: 2-dimensional ideal flow around a cylinder.

Questions to Figure 16:

1. How does the velocity field $\vec{u} = \vec{u}(x,y)$ look like?
2. How do the pathlines (streamlines) look like?

$$\vec{\nabla} \times \vec{u} = 0 \Rightarrow \vec{u}(\vec{r}) = \vec{\nabla}\phi(\vec{r}) \tag{4.2}$$

where $\phi(\vec{r})$ is the velocity potential.

$$\vec{\nabla} \cdot \vec{u} = 0 \tag{4.3}$$

$$\Downarrow$$

$$\vec{\nabla} \cdot \vec{\nabla}\phi(\vec{r}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y) = 0 \tag{4.4}$$

This second order differential equation is the Laplace equation.

Question: How does the cylinder (obstacle) enter in solving the Laplace equation?

There are two boundary conditions.

First boudary condition:

$$\vec{u}(|\vec{r}| \to \infty) = \vec{u}_\infty = u_\infty \vec{e}_x \tag{4.5}$$
$$\Downarrow$$
$$\phi(|\vec{r}| \to \infty) = u_\infty x + \text{constant.} \tag{4.6}$$

Second boundary condition:

$$0 = \vec{u}_{\text{surface}} \cdot \vec{n} = \vec{\nabla}\phi \Big|_{\text{surface}} \cdot \vec{n} \tag{4.7}$$

The fluid particle does not flow into/out of the surface; only tangential component.

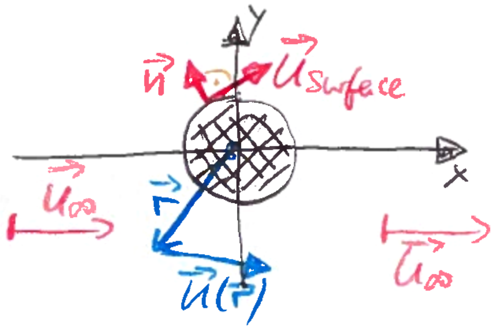


Figure 17: Tangential components of the velocity field.

For the flow around the cylinder the solution of the Laplace equation with the two boundary conditions "falls from the sky" (for the moment):

$$\phi(x,y) = u_\infty x \left(1 + \frac{R^2}{x^2+y^2}\right). \tag{4.8}$$

It fulfills Laplace's equation and the two boundary conditions.

Velocity field:

$$\vec{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \vec{\nabla}\phi(x,y) = \begin{pmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{pmatrix} \tag{4.9}$$

$$\begin{aligned} u_x &= u_\infty \left(1 + \frac{R^2(y^2-x^2)}{(x^2+y^2)^2}\right) \\ u_y &= -u_\infty \frac{2xyR^2}{(x^2+y^2)^2} \end{aligned} \tag{4.10}$$

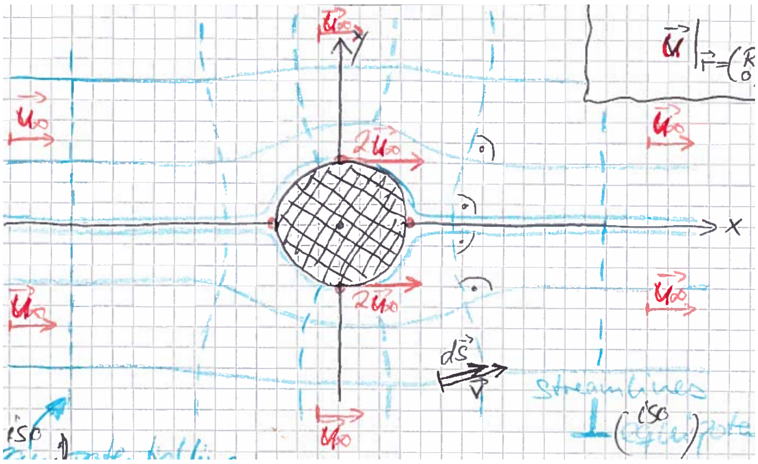


Figure 18: Streamlines around the cylinder.

Examples:

$$\vec{u}\Big|_{x\to\pm\infty} = u_\infty \vec{e}_x = \vec{u}\Big|_{y\to\pm\infty} \tag{4.11}$$

$$\vec{u}\Big|_{\vec{r}=\begin{pmatrix}0\\R\end{pmatrix}} = 2u_\infty \vec{e}_x \tag{4.12}$$

$$\vec{u}\Big|_{\vec{r}=\begin{pmatrix}R\\0\end{pmatrix}} = 0 = \vec{u}\Big|_{\vec{r}=\begin{pmatrix}-R\\0\end{pmatrix}} \tag{4.13}$$

The two points in (4.13) with $\vec{u} = 0$ are called stagnation points.

4.1 Stream line around a cylinder

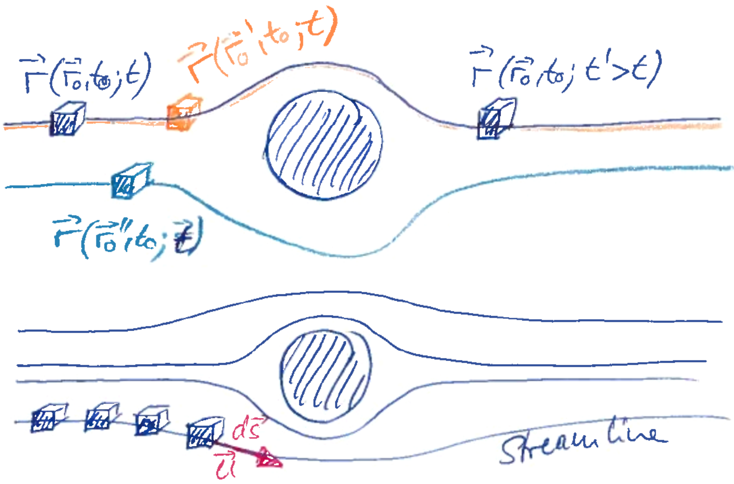


Figure 19: Path of fluid particles around the cylinder.

Top part of Figure 19 shows the Lagrangian picture: a particle is followed through all times. The bottom part shows the Eulerian picture, which is a snapshot of all fluid particles at one particular time.

Connection between Lagrangian and Eulerian picture:

$$\frac{d\vec{r}}{dt} = \vec{u}(\vec{r}, t) \tag{4.14}$$

Given the snapshots $\vec{u}(\vec{r}, t)$, we can calculate the pathlines. Given the pathlines, we can construct the snapshots. For stationary flows

$$\vec{u}(\vec{r}, t) = \vec{u}(\vec{r}) \Rightarrow \text{pathline} = \text{streamline}. \tag{4.15}$$

Question: How to calculate the streamlines?

First approach: Definition of streamline:

$$d\vec{s} \parallel \vec{u}, \tag{4.16}$$

where $\vec{ds}$ is a line element of a streamline.

$$0 = d\vec{s} \times \vec{u} = \begin{vmatrix} 0 & 0 & \vec{e}_z \\ dx & dy & 0 \\ u_x & u_y & 0 \end{vmatrix} = (u_y dx - u_x dy)\vec{e}_z \tag{4.17}$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{u_y}{u_x} = -\frac{2xyR^2}{(x^2+y^2)^2 + R^2(y^2 - x^2)}. \tag{4.18}$$

We will not try to solve this ugly non-linear differential equation.

Second approach: introduce the streamfunction $\psi(x, y)$.

Incompressibility gives us:

$$\vec{\nabla} \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{4.19}$$

which leads to the ansatz

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x} \tag{4.20}$$

$$d\vec{s} \times \vec{u} = (u_y dx - u_x dy)\vec{e}_x \tag{4.21}$$

$$= \left(-\frac{\partial \psi}{\partial x}dx - \frac{\partial \psi}{\partial y}dy\right)\vec{e}_z \tag{4.22}$$

$$= -d\psi \vec{e}_z \stackrel{!}{=} 0 \tag{4.23}$$

The streamfunction is constant along a streamline. This represents an *isopotential line* of the streamfunction and describes a streamline.

From the defining functions of the streamfunction in (4.20) and the u_x, u_y solution for the ideal flow around a cylinder in (4.10), we can determine ψ by partial integration

$$\psi(x,y) = v_\infty y \left(1 - \frac{R^2}{x^2+y^2}\right) \tag{4.24}$$

$$= v_\infty \sin\phi \left(r - \frac{R^2}{r}\right) \tag{4.25}$$

In the last step we have introduced the cylindrical coordinates $x = r\cos\phi$ and $y = r\sin\phi$. The intermediate steps of the partial integration have been left out.

Remark: relationship between velocity potential and streamfunction
$\phi =$ constant, $\psi =$ constant represent an orthogonal set of curves, because

$$\left(\vec{\nabla}\phi\right)\cdot\left(\vec{\nabla}\psi\right) = \begin{pmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial\psi}{\partial x} \\ \frac{\partial\psi}{\partial y} \end{pmatrix} = \begin{pmatrix} u_x \\ u_y \end{pmatrix} \cdot \begin{pmatrix} -u_y \\ u_x \end{pmatrix} = 0 \tag{4.26}$$

Back to the derivation of (4.8): two dimensional potential flow around an infinitely long cylinder. Because of cylinder symmetry we can transform from Cartesian to cylindrical coordinates

$$x = r\cos\phi \tag{4.27}$$

$$y = r\sin\phi \tag{4.28}$$

$$\Phi(x,y) \to \Phi(r,\phi) \tag{4.29}$$

$$\Delta\Phi(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi(x,y) \tag{4.30}$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\phi}\right]\Phi(r,\phi) \tag{4.31}$$

$$= 0 \tag{4.32}$$

$$\Downarrow$$

$$r\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = -\frac{\partial^2\Phi}{\partial\phi^2} \tag{4.33}$$

Ansatz: factorization

$$\Phi(r,\phi) = f(r)g(\phi) \tag{4.34}$$

$$\frac{1}{f\cdot g}r\frac{\partial}{\partial r}\left(r\frac{\partial(f(r)g(\phi))}{\partial r}\right) = -\frac{1}{f\cdot g}\frac{\partial^2(f(r)g(\phi))}{\partial\phi^2} \tag{4.35}$$

$$\Downarrow$$

$$\frac{1}{f(r)}r\frac{\partial}{\partial r}\left(r\frac{\partial f(r)}{\partial r}\right) = -\frac{1}{g(\phi)}\frac{\partial^2 g(\phi)}{\partial\phi^2} \stackrel{!}{=} m^2 \tag{4.36}$$

Left part depends only on r, and the middle part depends only on ϕ. As a consequence, both have to be equal to a constant, which does neither depend on r nor ϕ.

$$\frac{\partial^2 g(\phi)}{\partial \phi^2} = -m^2 g(\phi) \tag{4.37}$$
$$\Downarrow$$
$$g(\phi) = e^{im\phi} = \cos m\phi + i \sin m\phi \tag{4.38}$$

Requirement:

$$g(\phi) = g(\phi + 2\pi) \tag{4.39}$$
$$\Downarrow$$
$$e^{im\phi} = e^{im(\phi+2m)} \tag{4.40}$$
$$\Downarrow$$
$$e^{2\pi i m} = 1. \tag{4.41}$$

This fixes m to integer values:

$$m = \ldots, -2, -1, 0, 1, 2, \ldots \tag{4.42}$$

$$r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} f(r)\right) = m^2 f(r) \tag{4.43}$$

Polynomial ansatz:

$$f(r) = r^\alpha \tag{4.44}$$

$$r\frac{\partial}{\partial r}\left(r\alpha r^{\alpha-1}\right) = \alpha^2 r^\alpha \stackrel{!}{=} m^2 r^\alpha \tag{4.45}$$
$$\Downarrow$$
$$\alpha = \pm m \tag{4.46}$$

$$\Phi(r,\phi) = f(r)g(\phi) = r^{\pm m} e^{im\phi} \tag{4.47}$$

Since the Laplace equation is linear in Φ, the most general solution for Φ is a linear superposition of all possible solutions:

$$\Phi(r,\phi) = \sum_{m=-\infty}^{\infty} \left(a_m r^m + b_m r^{-m}\right) e^{im\phi} \tag{4.48}$$
$$= \sum_{m=1}^{\infty} \left[\left(a_m r^m + b_m r^{-m}\right) e^{im\phi} + \left(c_m r^m + d_m r^{-m}\right) e^{-im\phi}\right] \tag{4.49}$$

Remark:

$$\Phi_{m=0} = a_0 + b_0 = \text{constant} \tag{4.50}$$
$$\Downarrow$$
$$\vec{u}_{m=0} = \vec{\nabla}\Phi_{m=0} = 0 \tag{4.51}$$

Remark: Another $m = 0$ solution is $\phi_{m=0} = c \ln r$. It fulfills (4.43). However, it is not able to fulfill the boundary condition at $r = R$, and has to be discarded.

Determination of amplitudes a_m, b_m, c_m, and d_m via boundary conditions:

$$\Phi(r \to \infty, \phi) = u_\infty x = u_\infty r \cos\phi \tag{4.52}$$

$$\begin{aligned} \Phi(r \to \infty, \phi) &= \sum_{m=1}^{\infty} \left(a_m r^m e^{im\phi} + c_m r^m e^{-im\phi} \right) && (4.53) \\ &\stackrel{!}{=} u_\infty r \cos\phi && (4.54) \\ &= u_\infty r \frac{e^{i\phi} + e^{-i\phi}}{2} && (4.55) \end{aligned}$$

$$a_2 = a_3 = \cdots = c_2 = c_3 = \cdots = 0 \tag{4.56}$$

$$a_1 = \frac{u_\infty}{2} = c_1 \tag{4.57}$$

$$\Phi(r, \phi) = u_\infty r \frac{e^{i\phi} + e^{-i\phi}}{2} + \sum_{m=1}^{\infty} \left(\frac{b_m}{r^m} e^{im\phi} + \frac{d_m}{r^m} e^{-i\phi} \right) \tag{4.58}$$

$$\vec{u} \cdot \vec{e}_r|_{r=R} = \vec{\nabla}\Phi \cdot \vec{e}_r|_{r=R} = \frac{\partial \Phi}{\partial r}\bigg|_{r=R} = 0 \tag{4.59}$$

$$\frac{\partial \Phi(r, \phi)}{\partial r}\bigg|_{r=R} = u_\infty \frac{e^{i\phi} + e^{-i\phi}}{2} + \sum_{m=1}^{\infty} \frac{(-m)}{r^{m+1}}\bigg|_{r=R} \left(b_m e^{im\phi} + d_m e^{-im\phi} \right) \stackrel{!}{=} 0 \tag{4.60}$$

$$b_2 = b_3 = \cdots = d_2 = d_3 = \cdots = 0 \tag{4.61}$$

$$\frac{u_\infty}{2} - \frac{b_1}{R^2} = 0 = \frac{u_\infty}{2} - \frac{d_1}{R^2} \tag{4.62}$$

$$\Downarrow$$

$$b_1 = d_1 = \frac{u_\infty R^2}{2} \tag{4.63}$$

$$\Phi(r, \phi) = u_\infty r \frac{e^{i\phi} + e^{-i\phi}}{2} + \frac{u_\infty R^2}{2} \frac{e^{i\phi} + e^{-i\phi}}{2} \tag{4.64}$$

where

$$\frac{e^{i\phi} + e^{-i\phi}}{2} = \cos\phi \tag{4.65}$$

$$\Phi(r, \phi) = u_\infty r \cos\phi \left(1 + \frac{R^2}{r^2} \right) = u_\infty x \left(1 + \frac{R^2}{x^2 + y^2} \right) = \Phi(x, y) \tag{4.66}$$

5 More on ideal potential flows

Opening remark: 2-dimensional potential flow solutions will often look like

$$\Phi(x,y) = u_\infty x + f(x,y). \tag{5.1}$$

Any function $f(x,y)$, which fulfills

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \qquad \text{and} \qquad f(|x|,|y| \to \infty) = 0, \tag{5.2}$$

describes a flow around some obstacle. The question is: which obstacle? Let's play around with $f(x,y)$.

Example 1:

$$\Phi(x,y) = \frac{m}{2\pi} \ln \sqrt{x^2 + y^2} \tag{5.3}$$

represents the radial flow resulting from a source with strength m. See Figure 20.

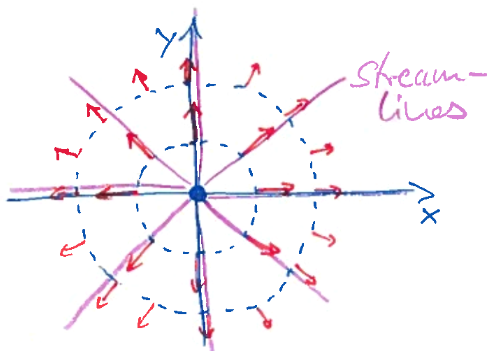


Figure 20: Two-dimensional radial flow resulting from a source line along the z-axis.

$$u_x = \frac{\partial \Phi}{\partial x} = \frac{m}{2\pi} \frac{x}{x^2 + y^2} = \frac{m}{2\pi} \frac{\cos\phi}{r} \tag{5.4}$$

$$u_y = \frac{\partial \Phi}{\partial y} = \frac{m}{2\pi} \frac{y}{x^2 + y^2} = \frac{m}{2\pi} \frac{\sin\phi}{r} \tag{5.5}$$

Example 2 (method of images): Source flow in front of a wall. See Figure 21.

Boundary condition: no flow through the wall; only tangential component.

$$\phi(x,y) = \frac{m}{2\pi} \ln \sqrt{(x+a)^2 + y^2} + \frac{m}{2\pi} \ln \sqrt{(x-a)^2 + y^2} \tag{5.6}$$

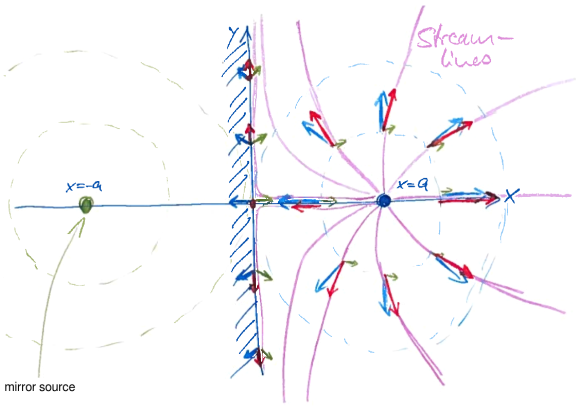


Figure 21: Source flow in front of a wall.

Example 3: flow past a 2-dimensional half-body. See Figure 22.

$$\Phi = u_\infty x + \frac{m}{2\pi} \ln \sqrt{x^2 + y^2} \tag{5.7}$$

$$\Downarrow$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{5.8}$$

$$u_x(x, y) = u_\infty + \frac{m}{2\pi} \frac{x}{x^2 + y^2} \tag{5.9}$$

$$u_y(x, y) = u_\infty + \frac{m}{2\pi} \frac{y}{x^2 + y^2} \tag{5.10}$$

Engineering flow interpretations:

1. An example of the beginning of the half-body is the leading edge of an airfoil
2. pedestrian on a bridge looking down: front part of a bridge pier
3. flow over a cliff

Example 4 ("beauty of mathematics"): Conformal mappings.

Complex potential

$$w(z) = \phi(x, y) + i\psi(x, y) \tag{5.11}$$

where $z = x + iy$ and $i^2 = -1$.

Velocity:

$$\frac{dw(z)}{dz} = \left.\frac{dw(z)}{dz}\right|_{dz=dx} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u_x - iu_y \tag{5.12}$$

$$= \left.\frac{dw(z)}{dz}\right|_{dz=idy} = \frac{\partial \phi}{i\partial y} + i\frac{\partial \psi}{i\partial y} = \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y} = u_x - iu_y \tag{5.13}$$

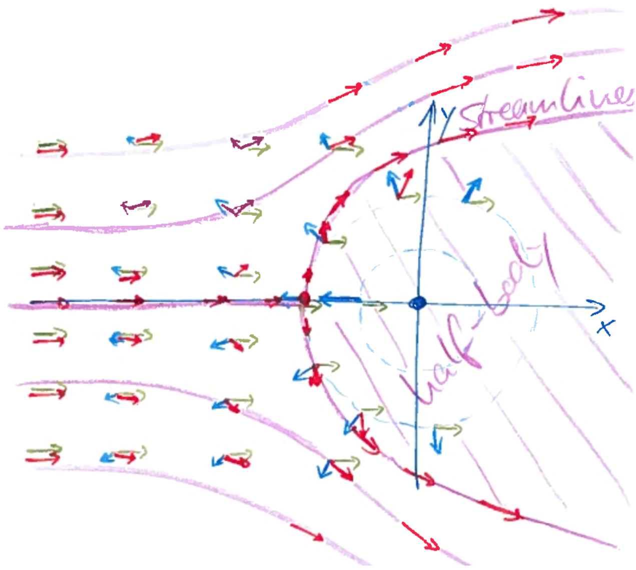


Figure 22: Flow past 2-dimensional half-body.

Example 4.1:

$$w(z) = u_\infty z = u_\infty(x+iy) = u_\infty x + iu_\infty y \tag{5.14}$$

describes the constant flow $\vec{u} = u_\infty \vec{e}_x$.

Example 4.2:

$$w(z) = \frac{m}{2\pi}\ln z = \frac{m}{2\pi}\ln(x+iy) \tag{5.15}$$

$$= \frac{m}{2\pi}\ln\left(re^{i\theta}\right) \tag{5.16}$$

$$= \frac{m}{2\pi}\ln r + \frac{m}{2\pi}\ln e^{i\theta} \tag{5.17}$$

$$= \frac{m}{2\pi}\ln\sqrt{x^2+y^2} + i\frac{m}{2\pi}\theta \tag{5.18}$$

The two terms in the last line are the velocity potential and the stream function of a radial source flow (see "Example 1").

Example 4.3:

$$w(z) = \frac{A}{2}z^2 = \frac{A}{2}(x+iy)^2 \tag{5.19}$$

$$= \frac{A}{2}(x^2-y^2) + iAxy \tag{5.20}$$

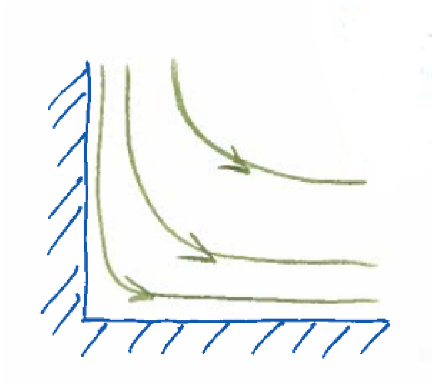


Figure 23: Stream line around a corner

Example 4.4: flow around cylinder with radius R

$$w(z) = \phi(x,y) + i\psi(x,y) = u_\infty x \left(1 + \frac{R^2}{x^2+y^2}\right) + i u_\infty y \left(1 - \frac{R^2}{x^2+y^2}\right) \tag{5.21}$$

$$= u_\infty (x+iy) + u_\infty R^2 \frac{x-iy}{x^2+y^2} \tag{5.22}$$

$$= u_\infty (x+iy) + \frac{u_\infty R^2}{x+iy} \tag{5.23}$$

$$= u_\infty \left(z + \frac{R^2}{z}\right) \tag{5.24}$$

Example 4.5:

Change of variable $(z \to \tilde{z})$:

$$z = z(\tilde{z}) \tag{5.25}$$

$$\tilde{z} = (z + z_0) + \frac{1}{z + z_0} \tag{5.26}$$

$$\Downarrow$$

$$w_{\text{new obstacle}}(\tilde{z}) = w_{\text{cylinder}}(z(\tilde{z})) \tag{5.27}$$

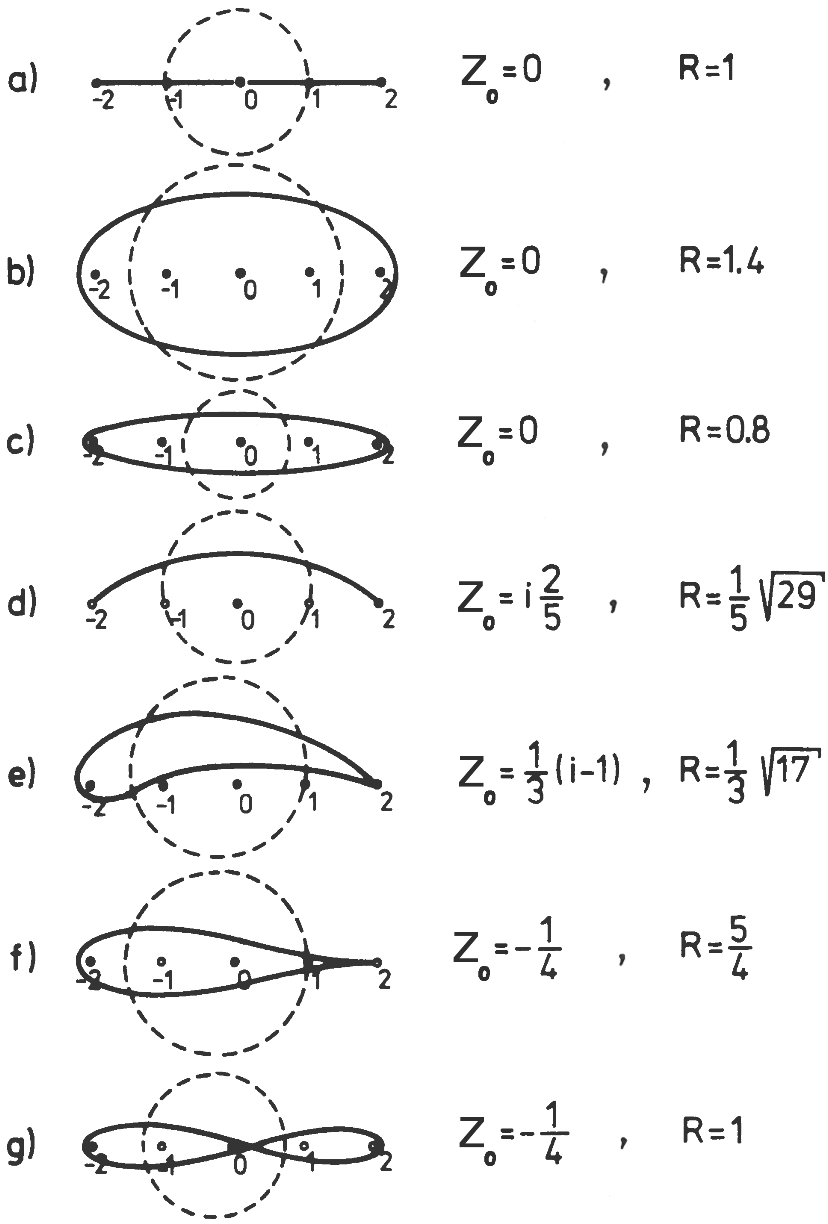


Figure 24: Several examples for different z_0 and R. e) is known as Joukowski's airfoil.

6 Forces on a 2-dimensional body

With the Bernoulli equation,

$$\vec{\nabla}\left(\frac{\rho}{2}\vec{u}^2 + p\right) = 0. \tag{6.1}$$

we can calculate the force on the "obstacle" (Figure 25) from the surrounding flow:

$$\vec{F} = \int_S (-p(\vec{r}))d\vec{A} = -\int_S \left(p_0 - \frac{\rho}{2}\vec{u}^2(\vec{r})\right) d\vec{A} \tag{6.2}$$

$$= \frac{\rho}{2}\int_S \vec{u}^2(\vec{r})d\vec{A} \tag{6.3}$$

$$= L\vec{e}_y + \underbrace{D\vec{e}_x}_{=0}. \tag{6.4}$$

$\vec{L}$ is the lift force and $\vec{D}$ is the drag force. The last term equals zero because there is no friction in ideal flows.

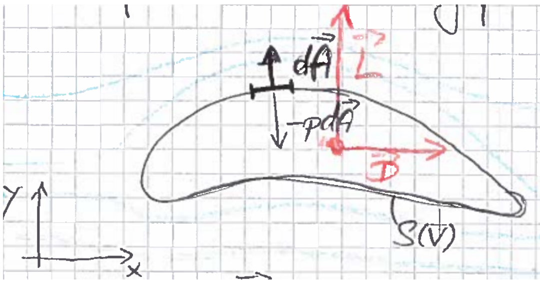


Figure 25: Lift and drag forces.

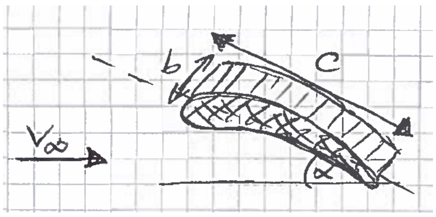


Figure 26: Angle of attack.

Lift coefficient

$$C_L = \frac{L}{\frac{\rho}{2}v_\infty^2 bc} \tag{6.5}$$

Drag coefficient (for the case that friction is non-zero):

$$C_D = \frac{D}{\frac{\rho}{2}v_\infty^2 bc} \tag{6.6}$$

6.1 Turbine blade

The lift force pulls the rotor blade of a wind turbine forward. See Figure 28.

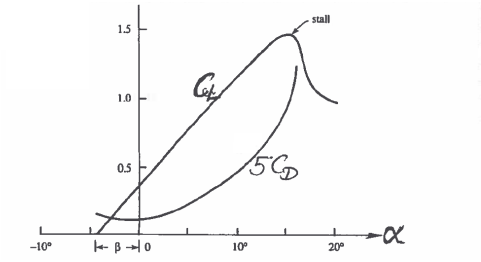


Figure 27: Generic lift and drag coefficients vs. angle of attack.

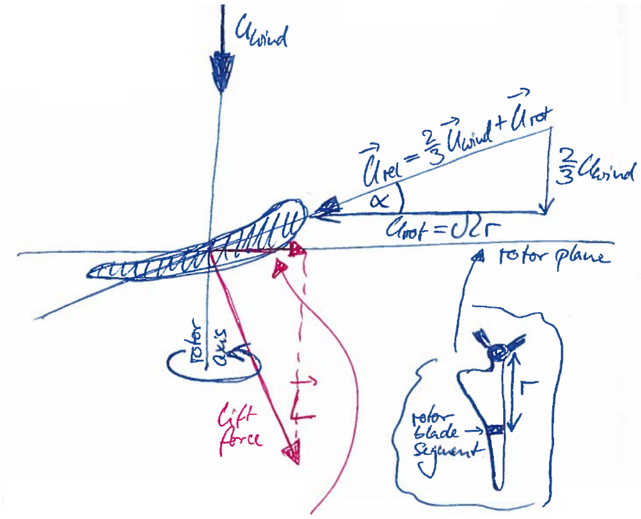


Figure 28: Lift force on the rotor blade of a wind turbine.

6.2 Sailing against the wind (KCD 14.9)

People have sailed without the aid of an engine for thousands of years and have known how to reach an upwind destination. Actually, it is not possible to sail exactly against the wind, but it is possible to sail at ≈ 40-45° to the wind. Figure 29 shows how this is made possible by the aerodynamic lift on the sail, which is a piece of stretched and stiffened cloth. The wind speed is U, and the sailing speed is V, so that the apparent wind speed relative to the boat is U_r. If the sail is properly oriented, this gives rise to a lift force perpendicular to U_r and a drag force parallel to U_r. The resultant force F can be resolved into a driving component (thrust) along the motion of the boat and a lateral component. The driving component performs work in moving the boat; most of this work goes into overcoming the frictional drag and in generating the gravity waves that radiate outward from the hull. The lateral component does not cause much sideways drift because of the shape of the hull. It is clear that the thrust decreases as the angle θ decreases and normally vanishes when θ is ≈ 40-45° . The energy for sailing comes from the wind field, which loses kinetic energy after passing the sail.

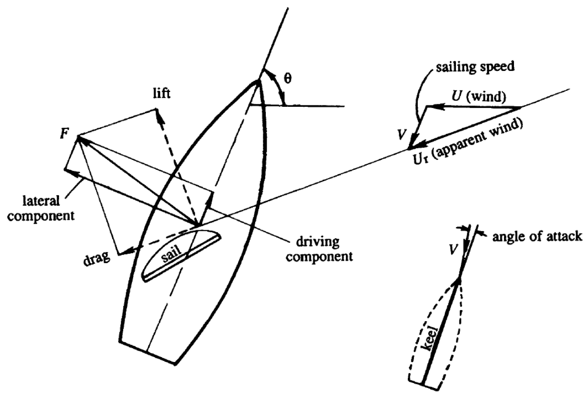


Figure 29: Principle of sailing against the wind. A small component of the sails lift pushes the boat forward at an angle $\theta < 90°$ to the wind. Thus by traversing a zig-zag course at angles $\pm\theta$, a sailboat can reach an upwind destination. A sailboats keel may make a contribution to its upwind progress too.

7 Reynolds number

Fluid around a cylinder can create several real flow patterns. See Figure 30.

Questions:

1. Why so many different flows?
2. What causes and characterizes them?

Certainly friction, i.e. viscosity, will have something to do with it.

$$0 = \rho_0 \left[\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} \right] + \vec{\nabla} p - \mu \left(\vec{\nabla} \cdot \vec{\nabla} \right) \vec{u} \tag{7.1}$$

$$= \rho_0 \left[\frac{U}{T} \frac{\partial \vec{u}'}{\partial t'} + \frac{U^2}{L} \left(\vec{u}' \cdot \vec{\nabla}' \right) \vec{u}' \right] + \frac{\rho_0 U^2}{L} \vec{\nabla}' p' - \mu \frac{U}{L^2} \left(\vec{\nabla} \cdot \vec{\nabla} \right)' \vec{u}' \tag{7.2}$$

$$= \rho_0 \frac{U^2}{L} \left\{ \frac{\partial \vec{u}'}{\partial t'} + \left(\vec{u}' \cdot \vec{\nabla}' \right) \vec{u}' + \vec{\nabla}' p' - \frac{\mu}{\rho_0 L V} \left(\vec{\nabla} \cdot \vec{\nabla} \right)' \vec{u}' \right\} \tag{7.3}$$

where L is the characteristic length, U is the characteristic velocity, $T = L/U$ is the characteristic time, and $P = \rho_0 U^2$ the characteristic pressure.

Reynolds number:

$$Re = \frac{\rho_0 L U}{\mu} \tag{7.4}$$

> **Remark:** law of similarity
>
> If two flows have the same geometry (object) and the same Reynolds number, but a different absolute scale it means that the two flows are similar (identical, except for a scale transformation). Applications of this is wind tunnel experiments of air wings, wind turbine blades, cars, etc.

The Reynolds number

$$Re = \frac{\rho_0 L U}{\mu} = \frac{\rho_0 U^2 / L}{\mu U / L}, \tag{7.5}$$

is the inertia force density divided by the friction force density.

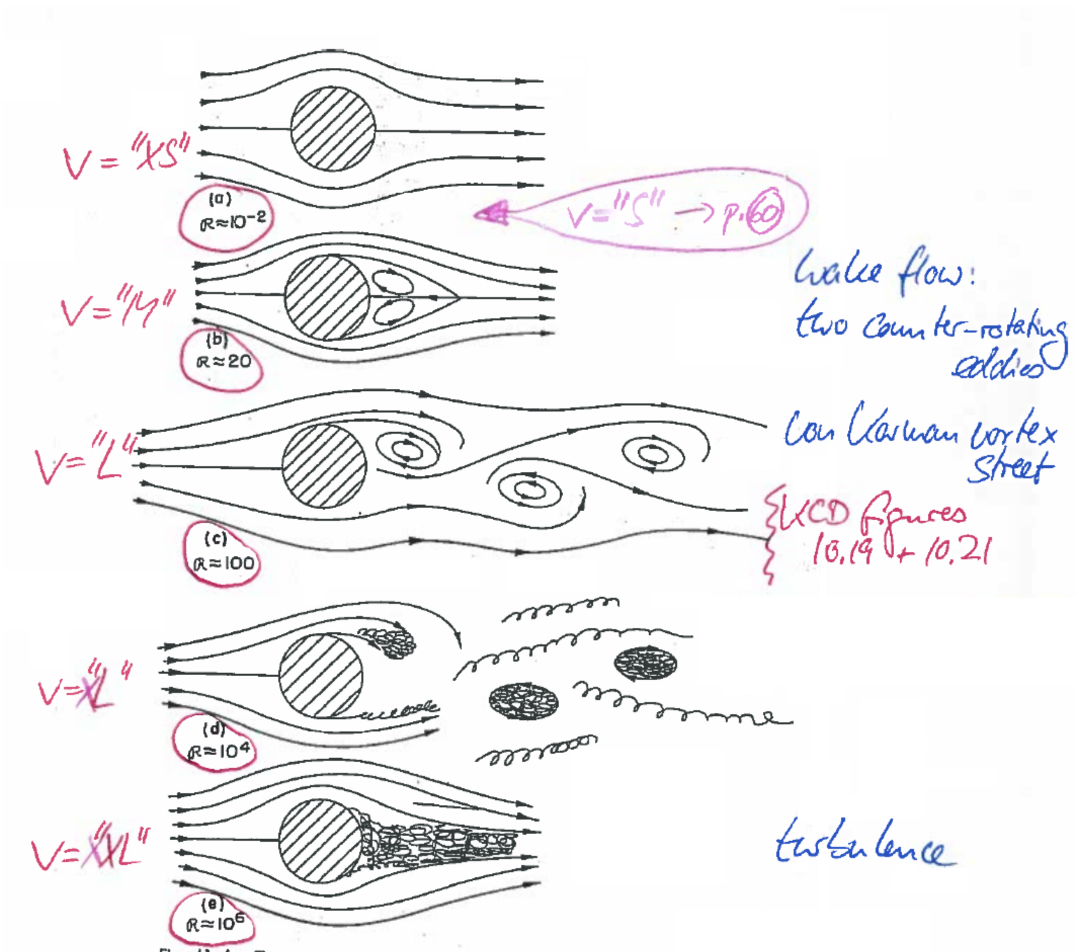


Fig. 41–6. Flow past a cylinder for various Reynolds numbers.

Figure 30: Flow around a cylinder. From laminar to turbulent with increasing velocity.

$Re \ll 1$

inertia $\ll$ friction force density

$\Rightarrow$ due to large friction (coupling), neighbouring fluid particles move in an ordered way.

$\Rightarrow$ first (top) flow pattern on p. 57

NS eq.: $\Delta = \vec{\nabla}\cdot\vec{\nabla}$

$$\cancel{\rho_0\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right]} = -\vec{\nabla}p + \boxed{\mu \Delta \vec{v}}$$

$=0$

$\Rightarrow$ creeping flows (example: marble falling down in honey)

$\Rightarrow$ the resulting flow pattern looks like potential flow: same streamlines, but different velocities close to the cylinder surface due to ~~same~~ no-slip boundary condition

$$\boxed{\vec{v}\big|_{\text{boundary}} = 0} \quad \text{for viscous flows}$$

! no-slip condition !

$Re \gg 1$

inertia $\gg$ friction force density

$\Rightarrow$ due to weak friction (coupling) neighbouring fluid particles move in a disordered way.

$\Rightarrow$ last (bottom) flow pattern on p. 57

NS eq.:

$$\boxed{\rho_0\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla}p} + \cancel{\mu \Delta \vec{v}}$$

Euler eq.

??? $\Rightarrow$ but then this looks like a potential flow, and $Re \gg 1$ should describe the first (top) flow pattern of p. 57 ???

mathematical problem: by neglecting the friction forces, we turn the 2nd order NS eq. into a first order differential equation.

$\Rightarrow$ boundary condition

$$\boxed{\vec{v}\big|_{\text{boundary}} = 0} \quad \text{for viscous flows}$$

can no longer be fulfilled

stability analysis:

$$\vec{v}(\vec{r},t) = \underbrace{\vec{V}(\vec{r})}_{\text{ideal flow}} + \underbrace{\delta\vec{v}(\vec{r},t)}_{\text{perturbation}}$$

$$\Rightarrow \delta\vec{v}(\vec{r},t) \sim e^{\lambda t}\,\delta\vec{v}(\vec{r},0), \quad \lambda > 0 \quad \Rightarrow \text{unstable flow}$$

Figure 31: Extreme examples of Reynolds number.

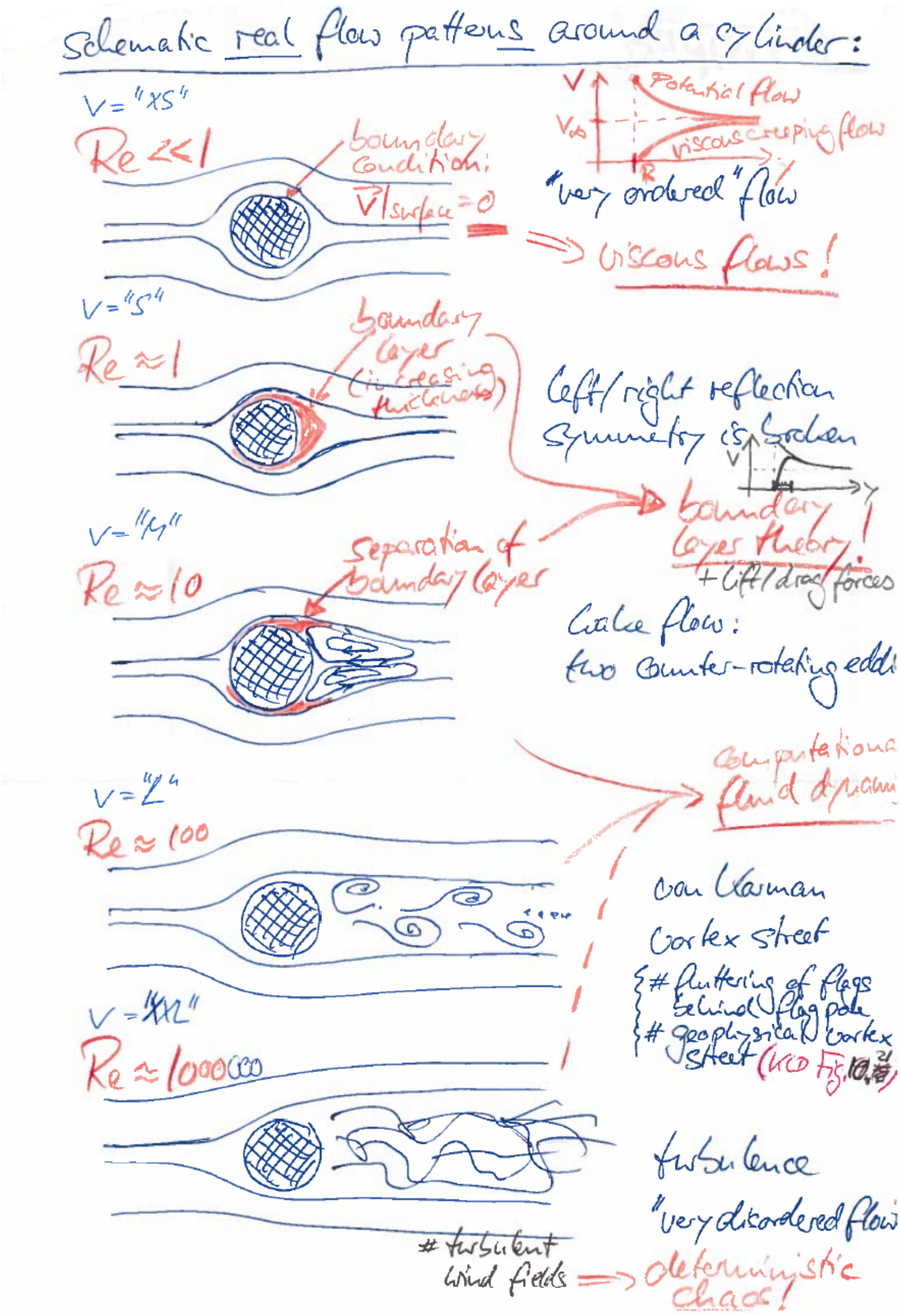


Figure 32: Schematic real flow patterns around a cylinder with different Reynolds numbers.

8 Viscous pipe flow

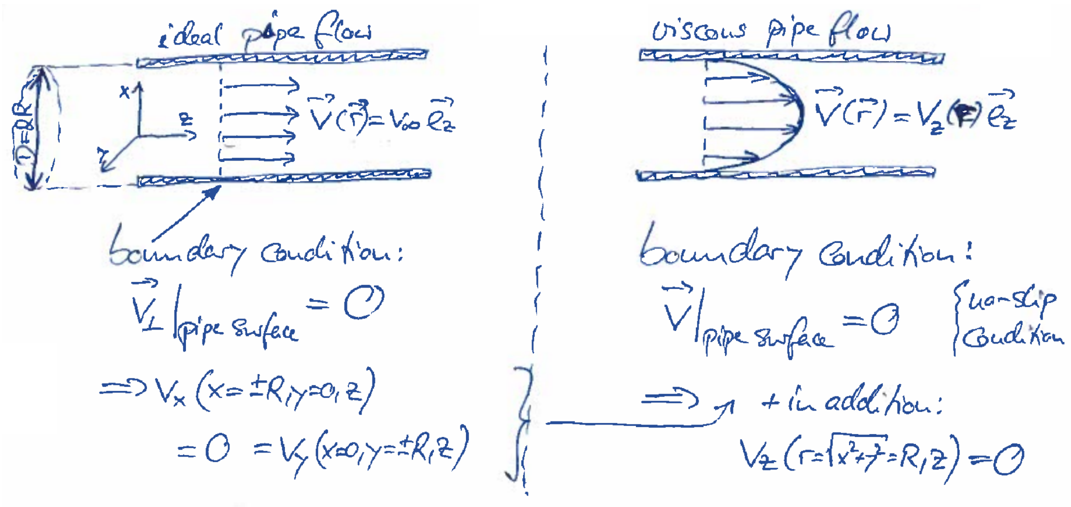


Figure 33: Ideal and viscous pipe flow.

Task: calculate velocity profile $u_z = u_z(r)$ for the viscous pipe flow.

Navier-Stokes equation

$$\rho_0 \underbrace{\frac{\partial \vec{u}}{\partial t}}_{=0} + \rho_0 \underbrace{\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}}_{=0} = \underbrace{\vec{f}_{\text{ext}}}_{=0} - \vec{\nabla} p + \mu \left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{u} \tag{8.1}$$

The second term on the left-hand side vanishes because of the incompressibility condition

$$0 = \vec{\nabla}\cdot\vec{u} = \partial_x u_x + \partial_y u_y + \partial_z u_z = \partial_z u_z \tag{8.2}$$

with $u_x = u_y = 0$ and

$$\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} = u_z \partial_z \begin{pmatrix} 0 \\ 0 \\ u_z \end{pmatrix} = 0. \tag{8.3}$$

$$\vec{\nabla} p = \mu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u_z \vec{e}_z \tag{8.4}$$

$$\Downarrow$$

$$\left(\vec{\nabla} p\right)_x = \left(\vec{\nabla} p\right)_y = 0 \tag{8.5}$$

$$\Downarrow$$

$$p = p(z) \tag{8.6}$$

$$\mu\left(\partial_x^2 + \partial_y^2\right) u_z(r) = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z(r)}{\partial r}\right) \tag{8.7}$$

$$= \frac{\partial p(z)}{\partial z} \stackrel{!}{=} \text{constant} \tag{8.8}$$

$$\frac{\partial p(z)}{\partial z} = c \tag{8.9}$$

$$\Downarrow$$

$$\begin{aligned} p(z) &= cz + d \qquad (8.10)\\ &= \frac{p(z=L) - p(z=0)}{L} z + p(z=0) \qquad (8.11)\\ &= -\frac{\Delta p}{L} z + p(z=0) \qquad (8.12) \end{aligned}$$

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z(r)}{\partial r}\right) = c = -\frac{\Delta p}{L} \tag{8.13}$$

$$\Downarrow$$

$$r\frac{\partial u_z(r)}{\partial r} = -\frac{\Delta p}{\mu L}\frac{r^2}{2} + D_1 \tag{8.14}$$

$$\Downarrow$$

$$u_z(r) = -\frac{\Delta p}{4\mu L} r^2 + D_1 \ln r + D_2 \tag{8.15}$$

$$\tag{8.16}$$

D_1 and D_2 are determined from the boudnary conditions $u_z(r = R) = 0$ (no slip condition) and $|u_z(r = 0)| < \infty$ (finiteness).

$$u_z(r) = \frac{\Delta p}{4\mu L}\left(R^2 - r^2\right) \tag{8.17}$$

Fluid mass per time passing through pipe cross-section:

$$\frac{dM}{dt} = \int_0^R \rho_0 u_z(r) 2\pi r dr = \frac{\pi \rho_0 R^4 \Delta p}{8\mu L} \tag{8.18}$$

This is the Hagen-Poiseuille law.

Remark: this law allows to determine the viscosity:

$$\left\{ \underbrace{\rho_0, R, L}_{\text{known}} \quad , \quad \underbrace{\Delta p, \frac{dM}{dt}}_{\text{measured}} \right\} \Rightarrow \mu \tag{8.19}$$

Remark: "Ohm's Law"

$$\Delta p = \Delta U\,, \quad \frac{dM}{dt} = I \tag{8.20}$$

$$\Downarrow$$

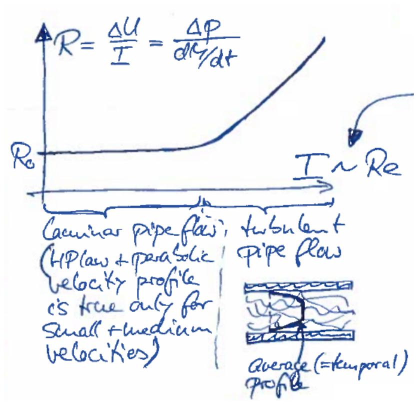
$$I = \frac{\pi \rho_0 R^4}{8 L \mu} \Delta U = \frac{\Delta U}{R_0} \tag{8.21}$$

$$\text{where } R_0 = \frac{8 L \mu}{\pi \rho_0 R^4} = \text{pipe resistance} \tag{8.22}$$

$$\text{Reynolds number in Equation 7.4} \tag{8.23}$$

$$R_e = \frac{\rho_0 2 R \bar{u}_z}{\mu} = \frac{\rho_0 2 R}{\mu} \frac{dM/dt}{\rho_0 \pi R^2} = \frac{2}{\pi \mu R} I \tag{8.24}$$

$$\tag{8.25}$$



$$R = R_0 \cdot f(Re) \tag{8.26}$$

with

$$f(Re \to 0) = 1. \tag{8.27}$$

Turbulence occurs when the velocity becomes large; it increases the pipe resistance. This is important for the operation of oil and gas pipelines.

9 Boundary layers

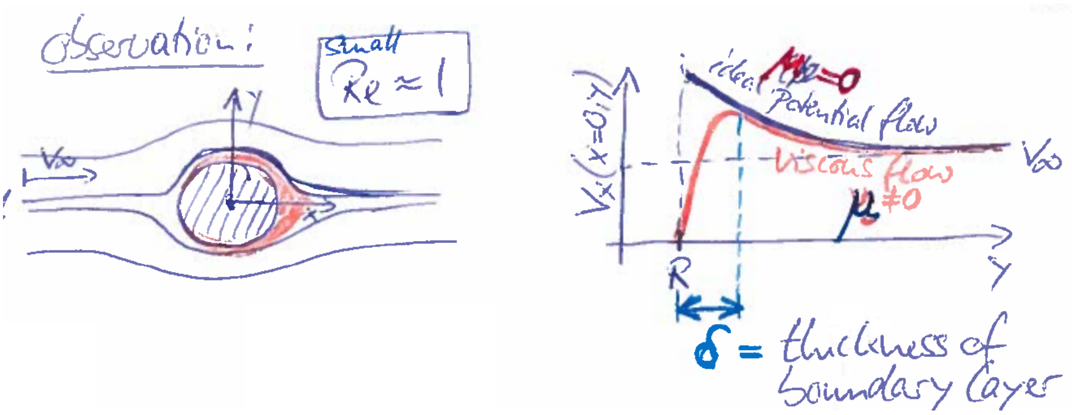


Figure 34: Example of boundary layer for flow around a cylinder.

Idea of boundary layer theory:

1. within the boundary layer the velocity increases from zero to the ideal flow velocity
2. inside the boundary layer we use the Navier-Stokes equation (with friction)
3. outside the boundary layer we use the Euler equation without friction i.e. ideal potential flow
4. at the boundary surface we match the inside solution with the outside solution

Derivation of the (laminar) boundary layer equations. Approximation to the Navier-Stokes equation inside the boundary layer. See Figure 35.

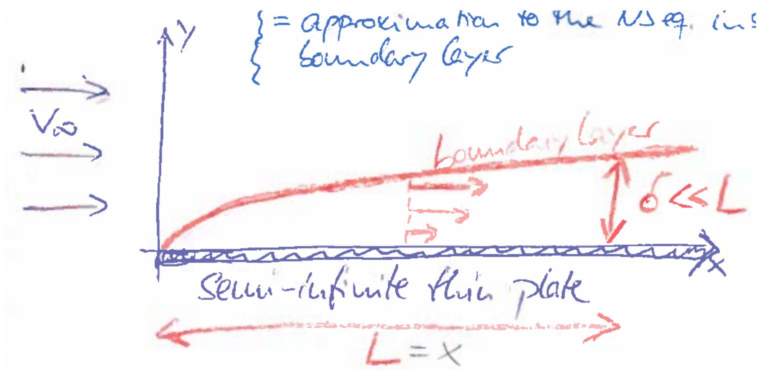


Figure 35: Laminar boundary layer.

In the following we determine $\delta = \delta(x)$ (Figure 35) without solving the Navier-Stokes equation.

Incompressible flow:

$$\vec{\nabla} \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{9.1}$$

$$\Downarrow$$

$$\mathcal{O}\left(\frac{u_\infty}{L}\right) + \mathcal{O}\left(\frac{u_y}{\delta}\right) = 0 \tag{9.2}$$

$$\Downarrow$$

$$\mathcal{O}(u_y) = \delta\frac{u_\infty}{L} = \frac{\delta}{L}u_\infty \tag{9.3}$$

Navier-Stokes equation (x-component):

$$u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\mu}{\rho_0}\frac{\partial^2 u_x}{\partial x^2} + \frac{\mu}{\rho_0}\frac{\partial^2 u_x}{\partial y^2} \tag{9.4}$$

$$\mathcal{O}\left(\frac{u_\infty^2}{L}\right) \stackrel{!}{=} \mathcal{O}\left(\frac{\mu}{\rho_0}\frac{u_\infty}{\delta^2}\right) \tag{9.5}$$

$$\left(\frac{\delta}{L}\right)^2 \sim \frac{\mu}{\rho_0 L u_\infty} = \frac{1}{Re} \tag{9.6}$$

The larger the Reynolds number, the thinner the boundary layer. This holds for $Re \leq 1 \times 10^5 - 1 \times 10^6$, above that the boundary layer becomes turbulent, and is no longer laminar.

$$\delta(x) \sim \sqrt{\frac{\mu x}{\rho_0 u_\infty}} \tag{9.7}$$

Navier-Stokes equation (y-component):

$$u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\mu}{\rho_0}\frac{\partial^2 u_y}{\partial x^2} + \frac{\mu}{\rho_0}\frac{\partial^2 u_y}{\partial y^2} \tag{9.8}$$

Ther first term on the right-hand side is the only large term. All other terms are neglected, which leads to

$$\frac{\partial p}{\partial y} = 0 \Rightarrow p = p(x) \tag{9.9}$$

Prandtl equations: laminar boundary layer equations

$$u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p(x)}{\partial x} + \frac{\mu}{\rho_0}\frac{\partial^2 u_x}{\partial y^2} \tag{9.10}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{9.11}$$

Example: Solution of Prandtl equations for laminar boundary flow around semi-infinite plate

$$p = p(x) \Rightarrow p(x)|_{\text{inside}} = p(x)|_{\text{outside}} \tag{9.12}$$

Outside boundary layer:

$$u_x|_{\text{outside}} = u_x = u_\infty = \text{constant} \tag{9.13}$$
$$u_y|_{\text{outside}} = 0 \tag{9.14}$$

Bernoulli equation:

$$p(x) + \frac{\rho_0}{2} u_x^2 = \text{constant} \tag{9.15}$$
$$\Downarrow$$
$$p(x) = \text{constant} \tag{9.16}$$
$$\Downarrow$$
$$\frac{\partial p(x)}{\partial x} = 0 \tag{9.17}$$

Prandtl equations inside boundary layer:

$$u_x \partial_x u_x + u_y \partial_y u_y = \frac{\mu}{\rho_0} \partial_y^2 u_x \tag{9.18}$$
$$\partial_x u_x + \partial_y u_y = 0 \tag{9.19}$$

Solution: similarity ansatz.

$$u_x(x, y) = u_\infty g\left(\frac{y}{\delta(x)}\right) \tag{9.20}$$

Except for a rescaling with $\delta(x)$ the velocity $u_x(x, y)$ looks like the same for all x.

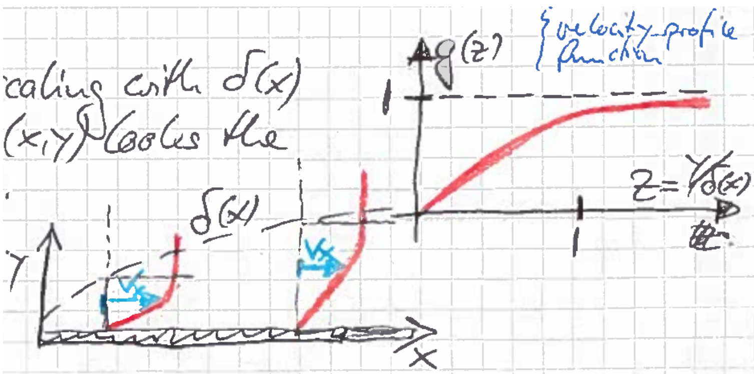


Figure 36: Similarity ansatz for the laminar boundary layer.

Question: does the similarity ansatz work?

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{9.21}$$

$$\Downarrow$$

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x} \tag{9.22}$$

$$\psi(x, y) = u_\infty \delta(x) f\left(\frac{y}{\delta(x)}\right) \tag{9.23}$$

$$\begin{aligned} u_x = \frac{\partial \psi}{\partial y} &= u_\infty \delta(x) \frac{df(z)}{dz} \frac{dz}{dy} && (9.24) \\ &= u_\infty \delta(x) \frac{df(z)}{dz} \frac{1}{\delta(x)} && (9.25) \\ &= u_\infty \frac{df(z)}{dz} && (9.26) \\ &\overset{!}{=} u_\infty g(z) && (9.27) \end{aligned}$$

$$f' = \frac{df(z)}{dz} g(z) \tag{9.28}$$

$$\begin{aligned} u_y &= -\frac{\partial \psi}{\partial x} && (9.29) \\ &= -u_\infty \frac{d\delta(x)}{dx} f(z) - u_\infty \delta(x) \frac{df(z)}{dz} \frac{dz}{dx} && (9.30) \\ &= u_\infty \left[-f + \frac{yf'}{\delta}\right] \frac{d\delta(x)}{dx} && (9.31) \end{aligned}$$

$$\begin{aligned} \frac{\partial u_x}{\partial x} &= \left(u_\infty f''\right) \left(\frac{-y}{\delta^2}\right) \frac{d\delta(x)}{dx} = -\frac{u_\infty y f''}{\delta^2} \frac{d\delta(x)}{dx} && (9.32) \\ \frac{\partial u_x}{\partial y} &= \left(u_\infty f''\right) \frac{1}{\delta} = \frac{u_\infty f''}{\delta} && (9.33) \\ \frac{\partial u_x}{\partial y} &= \frac{u_\infty f''}{\delta^2} && (9.34) \end{aligned}$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} - \frac{\mu}{\rho_0} \frac{\partial^2 u_x}{\partial y^2} = -(u_\infty f') \left(\frac{u_\infty y f''}{\delta^2} \frac{d\delta(x)}{dx} \right)$$

$$+ \left(u_\infty \left[-f + \frac{y f'}{\delta} \right] \frac{d\delta(x)}{dx} \right) \left(\frac{u_\infty f''}{\delta} \right) \tag{9.35}$$

$$- \frac{\mu}{\rho_0} \left(\frac{u_\infty f''}{\delta^2} \right)$$

$$= -\frac{u_\infty^2}{\delta} \frac{d\delta}{dx} f f'' - \frac{\mu}{\rho_0} \frac{u_\infty}{\delta^2} f''' \tag{9.36}$$

$$= 0 \tag{9.37}$$

$$\frac{\rho_0 u_\infty}{\mu} \delta(x) \frac{d\delta(x)}{dx} = -\frac{f'''(z)}{f(z) f''(z)} \stackrel{!}{=} c_1^2 \tag{9.38}$$

$$\delta \frac{d\delta}{dx} = \frac{1}{2} \frac{d\delta^2}{dx} = c_1^2 \frac{\mu}{\rho_0 u_\infty} \tag{9.39}$$

$$\delta^2 = 2 c_1^2 \frac{\mu}{\rho_0 u_\infty} x + c_2 \tag{9.40}$$

$c_2 = 0$ since $\delta(x = 0) = 0$.

$$\delta(x) = c_1 \sqrt{\frac{2\mu}{\rho_0 u_\infty} x} \tag{9.41}$$

$$\delta(x) = \sqrt{\frac{\mu}{\rho_0 u_\infty} x} \tag{9.42}$$

Freedom of choice $c_1 = \frac{1}{\sqrt{2}}$ because of arbitrary definition of δ; for example, $u_x(y = \delta) = 0.99 u_\infty$ or $u_x(x = \delta) = 0.95 u_\infty$.

This is the same result as the order of magnitude calculation when we calculated the x-component of the Navier-stokes equation earlier in this section.

$$f'''(z) + \frac{1}{2} f(z) f''(z) = 0 \tag{9.43}$$

$$f(z) \frac{d^2 f(z)}{dz^2} + 2 \frac{d^3 f(z)}{dz^3} = 0 \tag{9.44}$$

This is Blasius' equation. It is a special case of the more general Falker-Skan equation. The Blasius equation can only be solved numerically. The boundary conditions for the solutions are:

$$u_x(y = 0) = 0 \Rightarrow f'(0) = 0, \tag{9.45}$$

$$u_y(y = 0) = 0 \Rightarrow f(0) = 0, \tag{9.46}$$

$$u_x(y = \infty) = u_\infty \Rightarrow f'(\infty) = 1. \tag{9.47}$$

The solution is sketched in Figure 37.

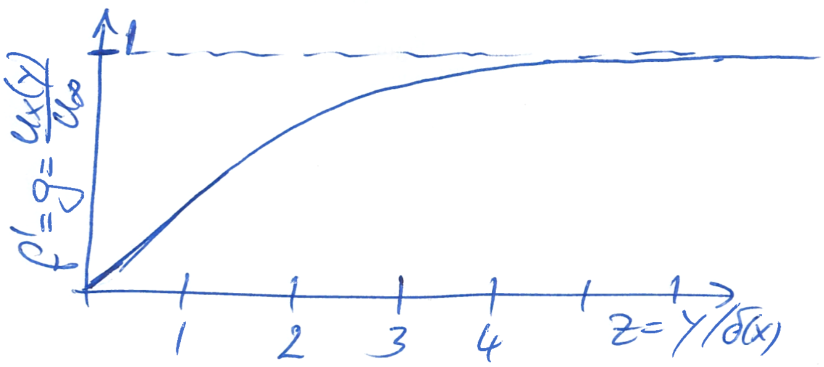


Figure 37: Solution to the Blasius equation.

9.1 Separation of boundary layers

Boundary condition at the wall:

$$u_x(x, y = 0) = u_y(x, y = 0) = 0 \tag{9.48}$$

Prandtl equation (with pressure)

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \frac{\mu}{\rho_0} \frac{\partial^2 u_x}{\partial y^2} \tag{9.49}$$

If we are very close to the wall, the two terms on the left side equal zero. We then have

$$\frac{\partial p(x, y = 0)}{\partial x} = \mu \frac{\partial^2 u_x(x, y = 0)}{\partial y^2} \tag{9.50}$$

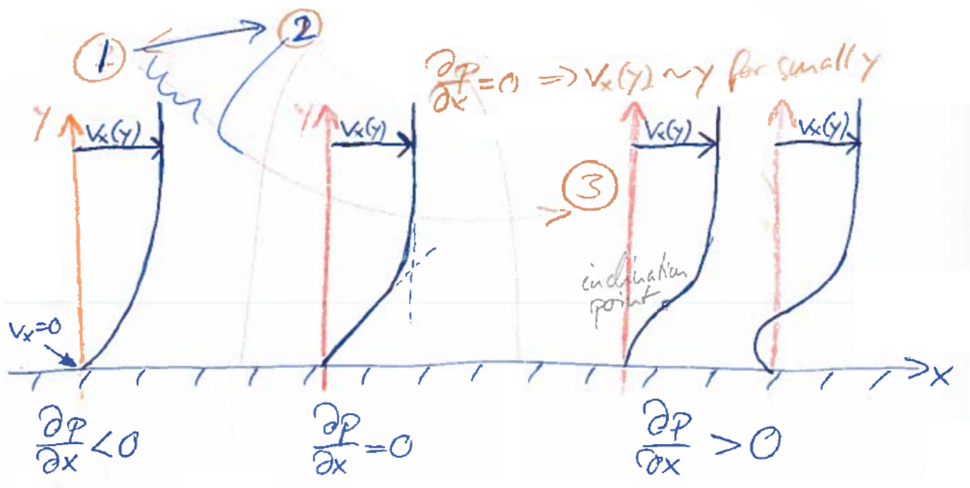


Figure 38: Shape of the boundary layer for different pressure profiles.

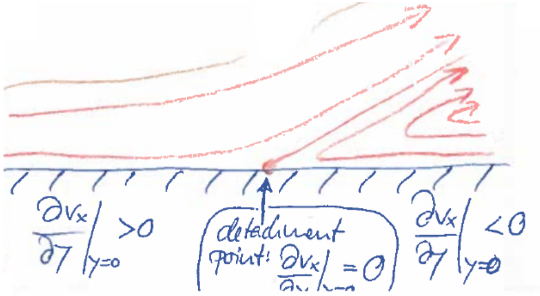


Figure 39: Detachment point of the boundary layer.

Example: flow around cylinder

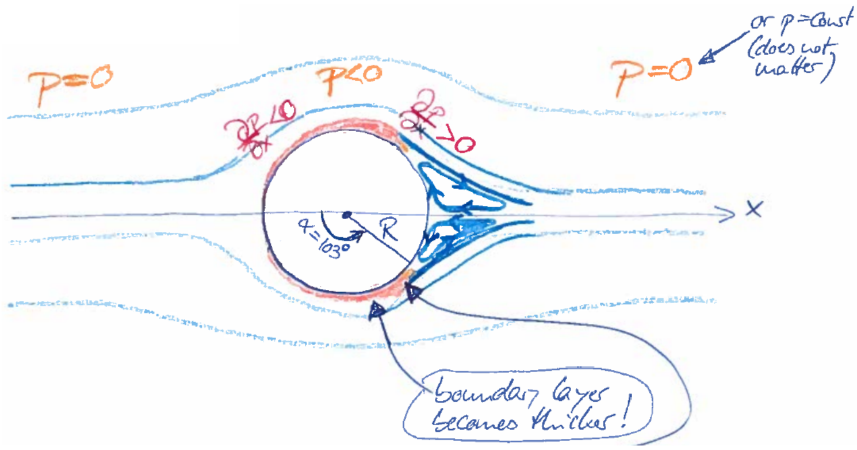
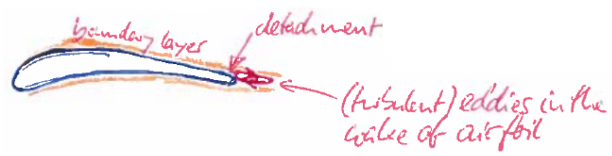


Figure 40: Separation of boundary layer and detachment point for flow around a cylinder.

The detachment point is at the separation of the boundary layer. When $v \approx 0$ there is no kinetic energy to run against the pressure gradient.

Remark: separation of boundary layers is a big issue in mechanical engineering; for example: design of airfoils, wind-turbine blades etc.



In case of separation the lift decreases, which can lead to airplane crash. It would also lead to a substantial rise in the overall drag (more friction). This would require more engine power and therefore more fuel for an airplane. For a wind turbine it would means less power generation.

Engineer's dream: construct airfoils without turbulent wake, with e.g. shark skin or fish scales.

9.2 Solution of Prandtl equations for free boundary layers

Figure 41 shows a 2-dimensional laminar jet flow generated from a flow through a long slit streaming into a resting fluid.

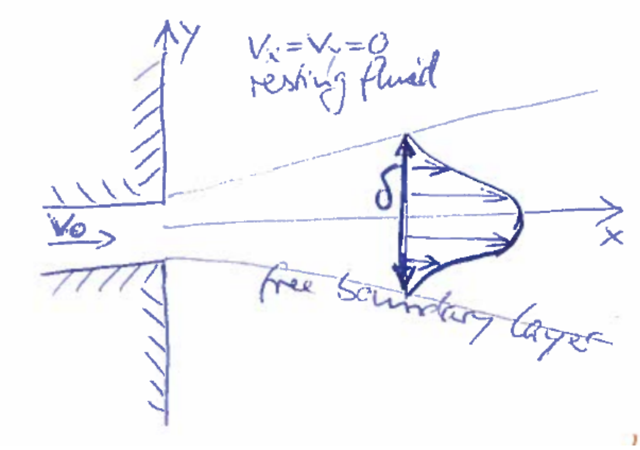


Figure 41: 2-dimensional laminar jet flow.

We use the Prandtl equations with the similarity ansatz:

$$u_x(x,y) = u_{\max}(x) g\left(\frac{y}{\delta(x)}\right) \tag{9.51}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \Rightarrow u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x} \tag{9.52}$$

$$\psi = u_{\max}(x)\delta(x) f\left(\frac{y}{\delta(x)}\right) \tag{9.53}$$

$$\frac{\partial \psi}{\partial y} = u_{\max}\delta f' \frac{1}{\delta} = u_{\max} f' = u_{\max} g = u_x \tag{9.54}$$

$$u_y = -\frac{\partial \psi}{\partial x} = -u'_{\max}\delta f - u_{\max}\delta' f - u_{\max}\delta f' \frac{(-y)\delta'}{\delta^2} \tag{9.55}$$

$$\partial_x u_x = u'_{\max} f' + u_{\max} f'' \frac{(-y)\delta'}{\delta^2} \tag{9.56}$$

$$\partial_y u_x = u_{\max} f'' \frac{1}{\delta} \tag{9.57}$$

$$\partial_y^2 u_x = \frac{u_{\max}}{\delta^2} f''' \tag{9.58}$$

Insertion into Prandtl's equation with $p(x) = \text{constant}$:

$$u_x \partial_x u_x + u_y \partial_y u_x - \frac{\mu}{\rho_0} \partial_y^2 u_x = u_{\max} f' \left\{ u'_{\max} f' - u_{\max} \frac{y\delta'}{\delta^2} f'' \right\}$$
$$- \left\{ u'_{\max} \delta f + u_{\max} \delta' f - u_{\max} \frac{y\delta'}{\delta} f' \right\} u_{\max} \frac{1}{\delta} f'' \quad (9.59)$$
$$- \frac{\mu}{\rho_0^2} \frac{u_{\max}}{\delta^2} f'''$$

$$= u_{\max} u'_{\max} f'^2 - u_{\max} u'_{\max} f f'' - u_{\max}^2 \frac{\delta'}{\delta} f f'' - \frac{\mu}{\rho_0} \frac{u_{\max}}{\delta^2} f''' \quad (9.60)$$

$$\stackrel{!}{=} 0 \quad (9.61)$$

All four terms in (9.60) have the form $\alpha_i(x)\beta_i(x)$ for $(i = 1, ..., 4)$. The sum of these four terms has to be zero. This means that $\alpha_1(x) \sim \alpha_2(x) \sim \alpha_3(x) \sim \alpha_4(x)$.

Ansatz:

$$u_{\max}(x) = c_1 x^m \quad (9.62)$$
$$\delta(x) = c_2 x^n \quad (9.63)$$

Remark: we expect $m < 0$ (decreasing velocity with penetration depth) and $n > 0$ (increasing thickness of jet with penetration depth).

Sum of the 4 terms:

$$c_1^2 m x^{2m-1} (f'^2 - f f'') - c_1^2 x^{2m} \frac{n}{x} f f'' - \frac{\mu}{\rho_0} \frac{c_1}{c_2^2} \frac{x^m}{x^{2n}} f''' = 0 \quad (9.64)$$

$$2m - 1 = m - 2n \quad (9.65)$$

$$m(f'^2 - f f'') - n f f'' - \frac{\mu}{\rho_0} \frac{1}{c_1 c_2^2} f''' = 0 \quad (9.66)$$

This differential equation determines the velocity profile

$$g\left(\frac{y}{\delta(x)}\right) = f'\left(\frac{y}{\delta(x)}\right) \quad (9.67)$$

of the jet. We are not going to solve this, but we want to know m and n, because they determine $u_{\max}(x)$ and $\delta(x)$. We need a second equation for m and n.

Second equation: conservation of momentum flux. Momentum flux through the red plane in Figure 42 is identical to the momentum flux through the blue plane. This means that the integrated momentum flux does not depend on x.

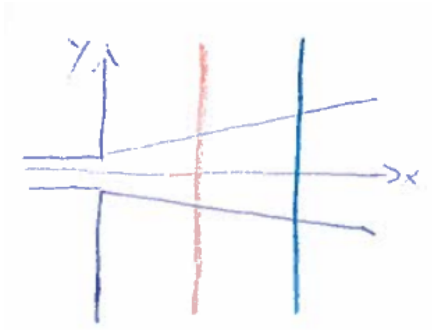


Figure 42: Momentum flux for 2-dimensional laminar jet flow.

$$\begin{aligned} \text{momentum} &= \rho_0 \Delta V \cdot u_x && (9.68) \\ &= \rho_0 \Delta A u_x \Delta t u_x && (9.69) \end{aligned}$$

Momentum flux:

$$\frac{\text{momentum}}{\Delta A \Delta t} = \rho_0 u_x^2 \tag{9.70}$$

Proof of conservation of momentum flux

If

$$\int_{-\infty}^{\infty} \rho_0 u_x^2(x) dy = \text{constant} \tag{9.71}$$

then

$$\frac{d}{dx} \int_{-\infty}^{\infty} \rho_0 u_x^2(x) dy = 0 \tag{9.72}$$

$$\begin{aligned} \frac{d}{dx} \int_{-\infty}^{\infty} \rho_0 u_x^2(x) dy &= 2\rho_0 \int_{-\infty}^{\infty} \left(u_x \frac{\partial u_x}{\partial x} \right) dy && (9.73) \\ &= 2\mu \frac{\partial u_x}{\partial y} \bigg|_{-\infty}^{\infty} - 2\rho_0 \int_{-\infty}^{\infty} u_y \frac{\partial u_x}{\partial y} dy && (9.74) \\ &= -2\rho_0 u_x u_y \bigg|_{-\infty}^{\infty} + 2\rho_0 \int_{-\infty}^{\infty} \frac{\partial u_y}{\partial y} dy && (9.75) \\ &= -2\rho_0 \int_{-\infty}^{\infty} u_x \frac{\partial u_x}{\partial x} dy && (9.76) \\ &= 0 && (9.77) \end{aligned}$$

Prandtl's equation (9.61) has been inserted into (9.73) to obtain (9.74). The first term in (9.74) is zero because $u_x(y) = \text{constant}$ for $y \to \pm\infty$. Partial integration has been used to arrive at (9.75). The incompressibility condition $\partial_x u_x + \partial_y u_y$ has been used to get (9.76). Since (9.76) is equal to (9.73) the integral has to be zero.

$$\int_{-\infty}^{\infty} \rho_0 u_x^2 dy = \rho_0 \int_{-\infty}^{\infty} u_{\text{max}}^2(x) g^2 \left(\frac{y}{\delta(x)} \right) dy \tag{9.78}$$

$$= \rho_0 u_{\text{max}}^2(x) \delta(x) \int_{-\infty}^{\infty} g^2(z) dz \tag{9.79}$$

$$\stackrel{!}{=} \text{constant} \tag{9.80}$$

$$u_{\text{max}}^2(x)\delta(x) = \text{constant} \tag{9.81}$$

$$c_1^2 x^{2m} c_2 x^n = \text{constant} \tag{9.82}$$

$$2m + n = 0 \tag{9.83}$$

$$m + 2n = 1 \,, \qquad 2m + n = 0 \tag{9.84}$$

$$\Downarrow$$

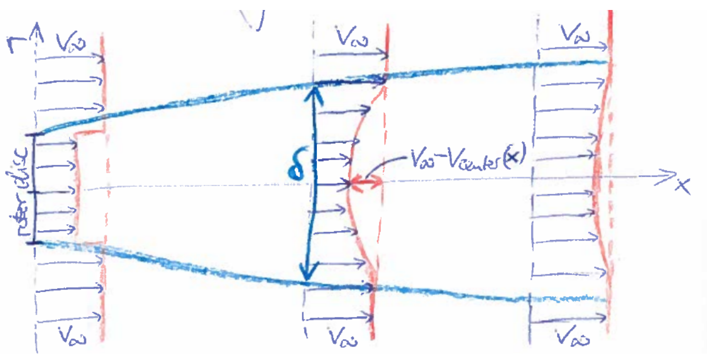
$$m = -\frac{1}{3} \,, \qquad n = \frac{2}{3} \tag{9.85}$$

$$u_{\text{max}}(x) \sim \frac{1}{x^{1/3}} \tag{9.86}$$

$$\delta(x) \sim x^{2/3} \tag{9.87}$$

Remark: negative jet flow

Wake behind a wind turbine can be modeled as a negative jet.



$$u_x(x,r) = u_\infty - u_{\text{center}}(x) g \left(\frac{r}{\delta(x)} \right) \tag{9.88}$$

10 Small-amplitude surface waves

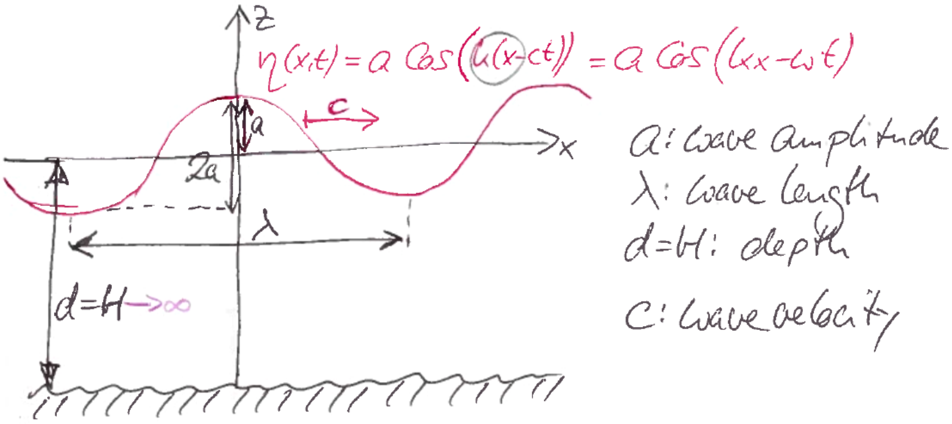


Figure 43:

Figure 43 shows a surface wave. Here k is the wave number, τ the oscillation period, and ω the circular frequency.

$$k = \frac{2\pi}{\lambda} \tag{10.1}$$

$$\omega = \frac{2\pi}{\tau} = 2\pi f \tag{10.2}$$

$$c = \frac{\lambda}{\tau} = \lambda f = \frac{\omega}{k} \tag{10.3}$$

Questions:

1. how does c depend on λ, d, a, ...
2. how does the fluid particles (below the surface) move? Pathlines? (expectation: more fluid motion close to the surface than in great depth.)

Assumptions: Incompressibility:

$$\rho = \rho_0 \,, \quad \vec{\nabla} \cdot \vec{u} = 0 \tag{10.4}$$

No friciton. Euler equation:

$$\rho_0 \left(\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} \right) = \vec{f}_{\text{ext}} - \vec{\nabla} p \tag{10.5}$$

Gravitational force density:

$$\vec{f}_{\text{ext}} = -\rho_0 g \vec{e}_z = -\vec{\nabla}(\rho_0 g z) \tag{10.6}$$

small amplitude waves:

$$a \ll \lambda, d \tag{10.7}$$

deep water waves:

$$\lambda \ll d \tag{10.8}$$

Hierarchy:

$$a \ll \lambda \ll d \tag{10.9}$$

$$\left| \frac{\partial \vec{u}}{\partial t} \right| \approx \frac{\Delta u}{\Delta t} \approx \frac{u - 0}{\tau/4} \approx \frac{u}{\tau} \approx \frac{a}{\tau^2} \tag{10.10}$$

$$|(\vec{u} \cdot \vec{\nabla})\vec{u}| \approx u \frac{\Delta u}{\Delta x} \tag{10.11}$$

$$\approx u \frac{u}{\lambda} = \frac{u^2}{\lambda} = \frac{1}{\lambda} \frac{a^2}{\tau^2} \tag{10.12}$$

$$\frac{|(\vec{u} \cdot \vec{\nabla})\vec{u}|}{|\partial \vec{u}/\partial t|} \approx \frac{\frac{1}{\lambda}\frac{a^2}{\tau^2}}{\frac{a}{\tau^2}} = \frac{a}{\lambda} \ll 1 \tag{10.13}$$

"Surviving" part of the Euler equation:

$$\frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} \left(gz + \frac{p}{\rho_0} \right) \tag{10.14}$$

We do the curl

$$\vec{\nabla} \times \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} \times \vec{\nabla} \left(gz + \frac{p}{\rho_0} \right) = 0 \tag{10.15}$$

$$\vec{\nabla} \times \vec{u} = \text{constant} \overset{!}{=} 0 \tag{10.16}$$

This constant has to be the same everywhere:

$$\text{constant}(z = 0) = \text{constant}(z = -\infty) = 0 \tag{10.17}$$

$$\vec{u} = \vec{\nabla}\Phi \tag{10.18}$$

with incompressibility

$$0 = \vec{\nabla}\vec{u} = \vec{\nabla} \cdot \vec{\nabla}\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \Phi(x, z, t) \tag{10.19}$$

Question: how shall we solve this linear differential equation?

We solve it via factorization:

$$\Phi(x, z, t) = X(x)Z(z)T(t) \tag{10.20}$$

$$\frac{1}{X(x)Z(z)T(t)}\left[\frac{\partial^2}{\partial x^2}(X(x)Z(z)T(t))+\frac{\partial^2}{\partial z^2}(X(x)Z(z)T(t))\right]$$
$$=\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} \tag{10.21}$$
$$=0 \tag{10.22}$$

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}=-k^2=\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} \tag{10.23}$$

$$\frac{\partial^2 X(x)}{\partial x^2}+k^2 X(x)=0 \tag{10.24}$$
$$\frac{\partial^2 Z(z)}{\partial z^2}-k^2 Z(z)=0 \tag{10.25}$$

$$X(x)=e^{\pm ikx} \tag{10.26}$$
$$Z(z)=e^{\pm kz} \tag{10.27}$$

$$\Phi(x,z,t)=Ae^{\pm ikx}e^{\pm kz}T(t) \tag{10.28}$$

10.1 First boundary condition: $(z=-d=-\infty)$

$$\vec{u}|_{z=-\infty}=0 \tag{10.29}$$
$$\Downarrow$$
$$\Phi(z=-\infty)=\text{constant}=0. \tag{10.30}$$

The second part of our general solution

$$\Phi(x,z,t)=f_+(x,t)e^{kz}+f_-(x,t)e^{-kz} \tag{10.31}$$

does not fulfill this boundary condition, and has to be set to zero, i.e. $f_-(x,t)=0$.

$$\Phi(x,c,t)=Ae^{\pm ikx}e^{kz}T(t) \tag{10.32}$$

10.2 Second boundary condition: (surface)

$$p(x,z,t)|_{z=\eta(x,t)}=p_0 \tag{10.33}$$

$$\frac{\partial \vec{u}}{\partial t} = \frac{\partial}{\partial t}\left(\vec{\nabla}\Phi\right) = -\vec{\nabla}\left(gz + \frac{p}{\rho_0}\right) \tag{10.34}$$

$$\Downarrow$$

$$\vec{\nabla}\left(\frac{\partial \Phi}{\partial t} + gz + \frac{p}{\rho_0}\right) = 0 \tag{10.35}$$

$$\Downarrow$$

$$\left(\frac{\partial \Phi}{\partial t} + gz + \frac{p}{\rho_0}\right)\Bigg|_{z=\eta(x,t)} = \left(\frac{\partial \Phi}{\partial t} + gz\right)\Bigg|_{z=\eta(x,t)} + \frac{p_0}{\rho_0} = \text{constant} \tag{10.36}$$

Gauge transformation:

$$\Phi \to \Phi + \left(\text{constant} - \frac{p_0}{\rho_0}\right) t \tag{10.37}$$

We are allowed to do this because the velocity field $\vec{u} = \vec{\nabla}\Phi$ does not change with this transformation.

$$\left(\frac{\partial \Phi}{\partial t} + gz\right)\Bigg|_{z=\eta(x,t)} = 0 \tag{10.38}$$

$$\frac{\partial \Phi}{\partial t}\Bigg|_{z=\eta(x,t)} = -g\eta(x,t) \tag{10.39}$$

If we want to determine Φ i.e. $T(t)$ we need another equation relating Φ and η, so that we get rid of η. This is going to be the kinematic boundary condition.

10.3 Kinematic boundary condition

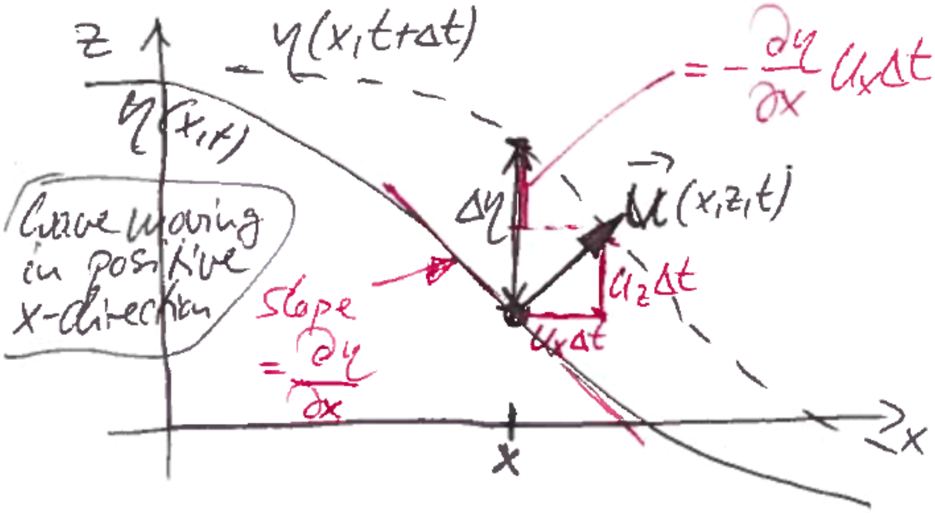


Figure 44:

$$\Delta\eta = u_z \Delta t - \frac{\partial \eta}{\partial x} u_x \Delta t \tag{10.40}$$

$$\Downarrow$$

$$\frac{\partial \eta}{\partial t} \approx \frac{\Delta \eta}{\Delta t} \tag{10.41}$$

$$= u_z - \frac{\partial \eta}{\partial x} u_x \tag{10.42}$$

$$\approx u_z. \tag{10.43}$$

In the last step we have used

$$\frac{\partial \eta}{\partial x} \sim \frac{a}{\lambda} \ll 1. \tag{10.44}$$

$$\left.\frac{\partial \Phi}{\partial x}\right|_{z=\eta} = \left. u_z \right|_{z=\eta(x,t)} = \frac{\partial \eta(x,t)}{\partial t} \tag{10.45}$$

$$= \frac{\partial}{\partial t}\left(\frac{(-1)}{g} \left.\frac{\partial \Phi}{\partial t}\right|_{z=\eta} \right) \tag{10.46}$$

$$= -\frac{1}{g} \left.\frac{\partial^2 \Phi}{\partial t^2}\right|_{z=\eta}. \tag{10.47}$$

Again $a \ll \lambda$

$$\left.\frac{\partial \Phi}{\partial z}\right|_{z\approx 0} + \frac{1}{g} \left.\frac{\partial^2 \Phi}{\partial t^2}\right|_{z\approx 0} = 0 \tag{10.48}$$

Now we can determine $T(t)$ by inserting the expression for Φ in (10.32).

$$\left.\frac{\partial \Phi}{\partial z}\right|_{z=0} = \left. A e^{\pm ikx} k e^{kz} T(t) \right|_{z=0} \tag{10.49}$$

$$= A k e^{\pm ikx} T(t) \tag{10.50}$$

$$\stackrel{!}{=} -\frac{1}{g} \left.\frac{\partial^2 \Phi}{\partial t^2}\right|_{z=0} = -\frac{1}{g} A e^{\pm ikx} \frac{\partial^2 T(t)}{\partial t^2} \tag{10.51}$$

$$\frac{\partial^2 T(t)}{\partial t^2} + gkT(t) = 0 \tag{10.52}$$

$$T(t) = e^{\pm i\sqrt{gk}t} = e^{\pm i\omega t} \tag{10.53}$$

$$\Phi(x, z, t) = A e^{kz} e^{\pm ikx} e^{\pm i\omega t} \tag{10.54}$$

Using Euler relations one of the four combinations can be rewritten as

$$\Phi(x, z, t) = A e^{kz} \sin(kx - \omega t). \tag{10.55}$$

From this expression for the velocity potential we can calculate the surface function $\eta(x, t)$ via the relation (10.39):

$$\eta(x, t) = \frac{(-1)}{g} \left.\frac{\partial \Phi}{\partial t}\right|_{z=\eta(x,t)} \tag{10.56}$$

$$= \frac{(-1)}{g} A(-\omega) \cos(kx - \omega t) \left. e^{kz} \right|_{z=\eta} \tag{10.57}$$

$$= \frac{\omega}{g} A \cos(kx - \omega t) e^{k\eta(x,t)} \tag{10.58}$$

$$= \frac{\omega}{g} A \cos(kx - \omega t) \tag{10.59}$$

This is the expression as in Figure 43. The wave velocity is given by

$$c = \frac{\omega}{k} = \frac{\sqrt{gk}}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} \tag{10.60}$$

Surface waves with a large wave length propagate faster than those with a short wave length.

Question: how does the motion of the fluid particles look like?

$$u_x = \frac{\partial \Phi}{\partial x} = A k e^{kz} \cos(kx - \omega t) \tag{10.61}$$

$$u_z = \frac{\partial \Phi}{\partial z} = A k e^{kz} \sin(kx - \omega t) \tag{10.62}$$

Order of magnitude estimate for the velocity amplitude ($a = \frac{\omega}{g}A$):

$$\mathcal{O}(u_x) = \mathcal{O}(u_z) = Ak = \frac{g}{\omega}ak = a\frac{gk^2}{\omega^2}\frac{\omega}{k} \tag{10.63}$$

$$= a\frac{gk^2}{\sqrt{gk}^2}c = akc \tag{10.64}$$

$$= a\frac{2\pi}{\lambda}c = 2\pi\frac{a}{\lambda}c \tag{10.65}$$

$$\mathcal{O}(u_x) = \mathcal{O}(u_x) \ll c \tag{10.66}$$

Fluid particle is not moving with the wave velocity. Its velocity is much smaller. The fluid particle is more or less at a fixed position. It is oscillating around this fixed position with a small amplitude:

$$u_x \approx Ake^{kz_0}\cos(kx_0 - \omega t) \tag{10.67}$$

$$u_z \approx Ake^{kz_0}\sin(kx_0 - \omega t) \tag{10.68}$$

Pathline of a fluid particle

$$u_x = \frac{dx}{dt} \tag{10.69}$$

$$u_z = \frac{dz}{dt} \tag{10.70}$$

$$x - x_0 = -\frac{Ak}{\omega}e^{kz_0}\sin(kx_0 - \omega t) \tag{10.71}$$

$$z - z_0 = \frac{Ak}{\omega}e^{kz_0}\cos(kx_0 - \omega t) \tag{10.72}$$

Fluid particle moves in a circle (see Figure 45):

$$(x - x_0)^2 + (z - z_0^2) = \left(\frac{Ak}{\omega}e^{kz_0}\right)^2 \tag{10.73}$$

with radius

$$R = \frac{Ak}{\omega}e^{kz_0} = \frac{g}{\omega}\frac{k}{\omega}ae^{kz_0} \tag{10.74}$$

$$= ae^{kz_0}. \tag{10.75}$$

So far we have discussed a pure wave, which is characterized by the wave number k and which travels with $c = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$. The most general solution describing the spatio-temporal dynamics of the water surface is obtained by a superposition of different pure waves:

$$\begin{aligned}\Phi(x, z, t) = \int_0^\infty dk\, \Big[&A(k)e^{i(kx+\omega t)} + B(k)e^{-i(kx+\omega t)} \\ &+ C(k)e^{i(kx-\omega t)} + D(k)e^{-i(kx-\omega t)}\Big]\end{aligned} \tag{10.76}$$

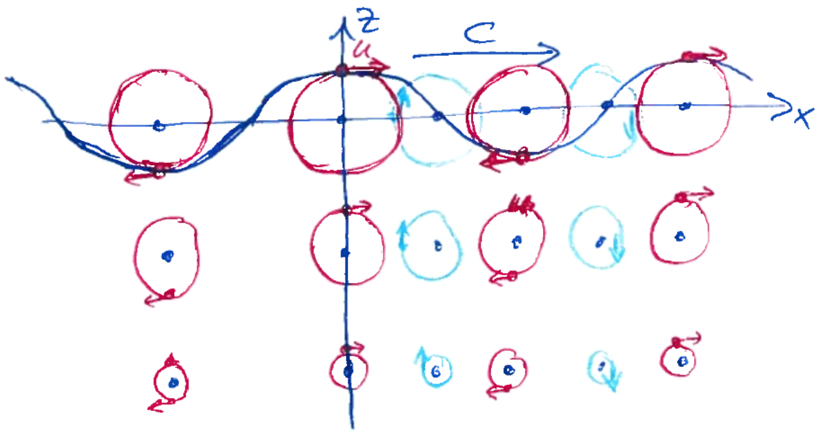


Figure 45:

Remark: perturbation of a flat surface

A wave generated by this disturbance contains several different wave numbers k. Initial wave form is not stable. It decays since different k components propagate with different phase velocities.

Remark: small-amplitude surface waves at a finite depth ($d < \infty$)

$$c = \sqrt{\frac{g}{k}\tanh(kd)} \tag{10.77}$$

$$= \sqrt{\frac{g\lambda}{2\pi}\tanh\left(2\pi\frac{d}{\lambda}\right)} \tag{10.78}$$

$$d \gg \lambda : \tanh(x \gg 1) = 0 \Rightarrow c = \sqrt{\frac{g\lambda}{2\pi}} \tag{10.79}$$

$$d \ll \lambda : \tanh(x \ll 1) = 0 \Rightarrow c = \sqrt{gd} \tag{10.80}$$

More surface waves:

- non-linear waves ($\lambda \gg d$): solitons
- monster waves
- wind-wave interaction
- wave energy

11 Sound waves

Sound waves are pressure (density) waves in a compressible medium.

$$p = p_0 + \tilde{p}, \quad \tilde{p} \ll p_0 \tag{11.1}$$

$$\rho = \rho_0 + \tilde{\rho}, \quad \tilde{\rho} \ll \rho_0 \tag{11.2}$$

Equation of state:

$$\rho = \rho_0 (1 + \kappa (p - p_0)) \tag{11.3}$$

The parameter κ is called compressibility.

$$\tilde{\rho} = \rho_0 \kappa \tilde{p} \tag{11.4}$$

Assumption: small velocities.

$$\vec{u} = 0 + \tilde{\vec{u}}. \tag{11.5}$$

Euler equation without friction:

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} \right) = -\vec{\nabla} p \tag{11.6}$$

$$(\rho_0 + \tilde{\rho}) \frac{\partial \tilde{\vec{u}}}{\partial t} + (\rho_0 + \tilde{\rho}) \left(\tilde{\vec{u}} \cdot \vec{\nabla} \right) \tilde{\vec{u}} = -\vec{\nabla} \tilde{p} \tag{11.7}$$

$$\frac{\partial \tilde{\vec{u}}}{\partial t} = -\frac{1}{\rho_0} \vec{\nabla} \tilde{p} \tag{11.8}$$

Equation of continuity:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} (\rho \vec{u}) = 0 \tag{11.9}$$

$$\Downarrow$$

$$\frac{\partial \tilde{\rho}}{\partial t} + \vec{\nabla} \left((\rho_0 + \tilde{\rho}) \tilde{\vec{u}} \right) = 0 \tag{11.10}$$

$$\Downarrow$$

$$\frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \vec{\nabla} \tilde{\vec{u}} = 0 \tag{11.11}$$

Divergence of (11.8):

$$\vec{\nabla} \frac{\partial \tilde{\vec{u}}}{\partial t} = -\frac{1}{\rho_0} \vec{\nabla} \cdot \vec{\nabla} \tilde{p} \tag{11.12}$$

Time derivative of (11.11):

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} + \rho_0 \frac{\partial}{\partial t} \vec{\nabla} \tilde{\vec{u}} = 0 \tag{11.13}$$

$$\Delta\tilde{p} = -\rho_0 \vec{\nabla}\frac{\partial \vec{u}}{\partial t} \tag{11.14}$$
$$= \frac{\partial^2 \tilde{\rho}}{\partial t^2} \tag{11.15}$$

Wave equation:

$$\Delta\tilde{p} = \rho_0 \kappa \frac{\partial^2 \tilde{p}}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 \tilde{p}}{\partial t^2} \tag{11.16}$$

In 1+1 dimensions:

$$\frac{\partial^2 \tilde{p}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \tilde{p}}{\partial t^2} \tag{11.17}$$

Plane-wave solution:

$$\tilde{p} = A_p \cos(x - ct) \tag{11.18}$$

Here $c = \frac{1}{\sqrt{\rho_o \kappa}}$ is the speed of sound. Examples:

$$c_{\text{air}} = 340\,\text{m/s} \tag{11.19}$$
$$c_{\text{water}} = 1500\,\text{m/s} \tag{11.20}$$

"Density wave":

$$\frac{\partial^2 \tilde{\rho}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \tilde{\rho}}{\partial t^2} \Rightarrow \tilde{\rho} = A_\rho \cos(x - ct) \tag{11.21}$$

Longitudinal "velocity wave":

$$\frac{\partial \tilde{u}_x}{\partial t} = -\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x} = \frac{A_p}{\rho_0}\sin(x - ct) \tag{11.22}$$
$$\Downarrow$$
$$\tilde{u}_x = \frac{A_p}{\rho_0 c}\cos(x - ct) = A_u \cos(x - ct) \tag{11.23}$$

Validity of approximation, which has neglected small quadratic terms in the Euler equation:

$$\frac{|\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}|}{|\frac{\partial \vec{u}}{\partial t}|} = \frac{|v_x \frac{\partial v_x}{\partial x}|}{|\frac{\partial v_x}{\partial t}|} \approx \frac{A_u^2}{A_u c} = \frac{A_u}{c} \approx \frac{|\vec{u}|}{c} \tag{11.24}$$

Typical velocity oscillations are much smaller than the speed of sound:

$$\frac{|\vec{u}|}{c} \ll 1. \tag{11.25}$$

Example: loudspeaker

$$f = 100 - 2000\,\text{Hz} \Rightarrow f_{\text{typical}} = 1000\,\text{Hz} \tag{11.26}$$

Amplitude of membrane displacement $\Delta x \approx 1\,\text{mm}$

$$\Delta u \approx \frac{\Delta x}{\Delta t/2} \approx \frac{2 \times 10^{-3}\,\text{m}}{10^{-3}\,\text{s}} \tag{11.27}$$

$$= 2\,\text{m/s} \ll c_{\text{air}} \tag{11.28}$$

11.1 Outlook: shock waves

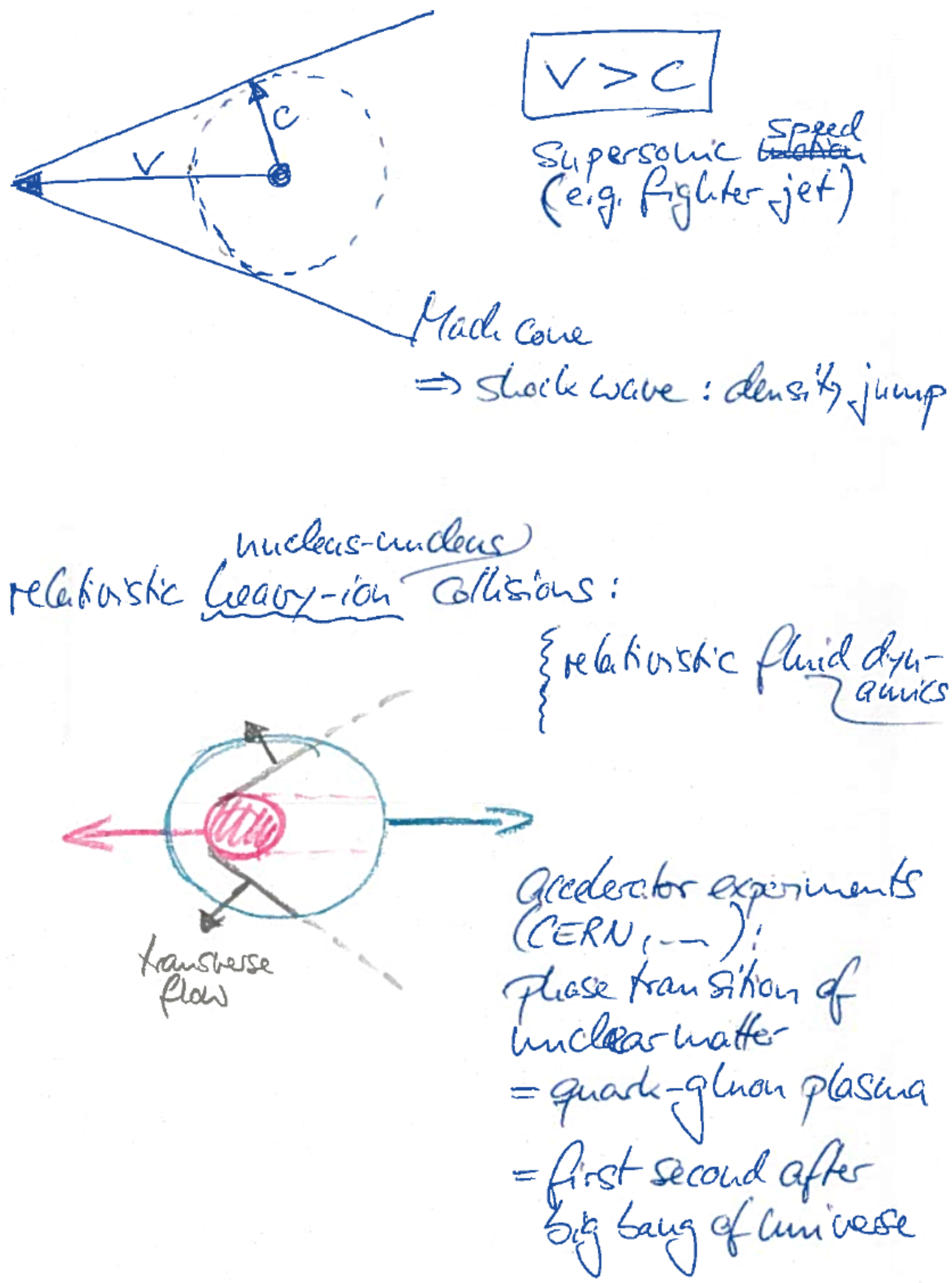


Figure 46: Sketches of shock waves.

12 Instabilities (not corrected)

Motivation: Navier-Stokes equation in non-dimensional units with no external forces:

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\vec{\nabla} p + \frac{1}{Re} \vec{\nabla}^2 \vec{v} \tag{12.1}$$

Reynolds number:

$$Re = \frac{\rho L V}{\mu} = \frac{\rho V^2 / L}{\mu V / L^2} = \frac{\text{inertia force density}}{\text{friction force density}} \tag{12.2}$$

As the Reynolds number increases, a stable flow becomes unstable, and a new stable flow emerges.

Big question: when does a stable flow become unstable?

Answer: linear stability analysis

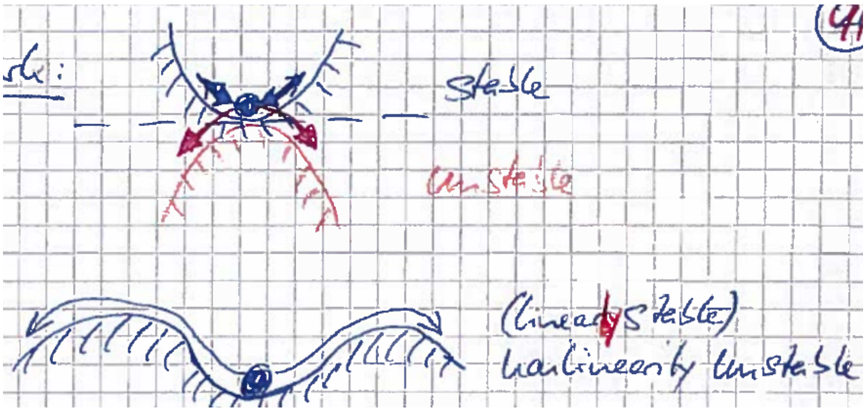


Figure 47:

Sketch of linear stability analysis

Simplifying assumptions: stable flows are steady (stationary)

$$\vec{v}(\vec{r}, t) = \vec{U}(\vec{r}) \;, \quad p(\vec{r}, t) = P(\vec{r}) \tag{12.3}$$

Introduce perturbations

$$\vec{v}(\vec{r}, t) = \vec{U}(\vec{r}) + \vec{u}(\vec{r}, t) \tag{12.4}$$

$$p(\vec{r}, t) = P(\vec{r}) + \tilde{p}(\vec{r}, t) \tag{12.5}$$

$$\frac{\partial(\vec{U} + \vec{u})}{\partial t} + \left[(\vec{U} + \vec{u}) \cdot \vec{\nabla}\right] (\vec{U} + \vec{u}) = -\vec{\nabla}(P + \tilde{p}) + \frac{1}{Re} \left(\vec{\nabla} \cdot \vec{\nabla}\right) (\vec{U} + \vec{u}) \tag{12.6}$$

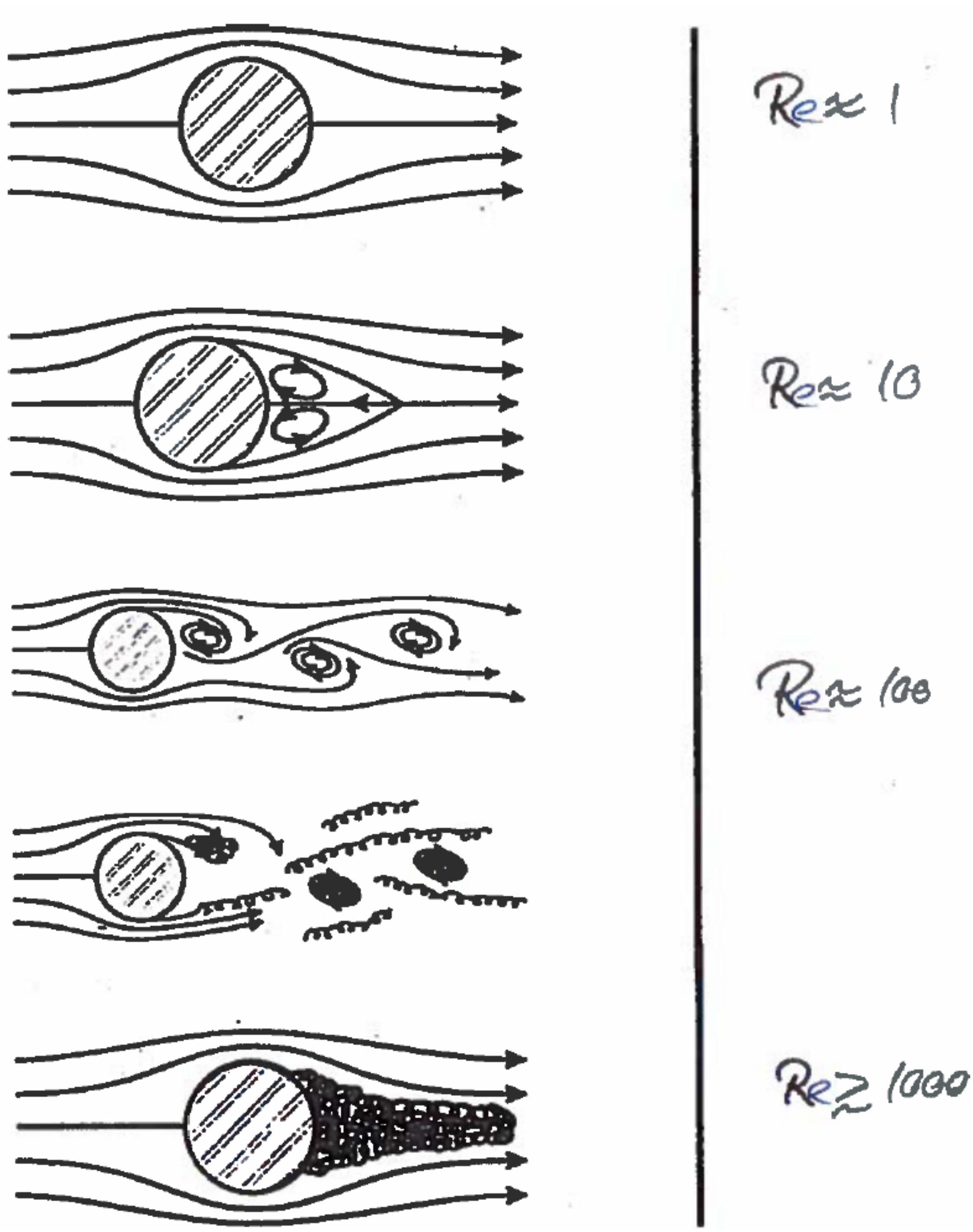


Figure 48:

$$\frac{\partial \vec{U}}{\partial t} + (\vec{U} + \vec{\nabla})\vec{U} + \frac{\partial \vec{u}}{\partial t} + (\vec{U} + \vec{\nabla})\vec{u}$$
$$+(\vec{u} + \vec{\nabla})\vec{U} + (\vec{u} + \vec{\nabla})\vec{u} = -\vec{\nabla} P \frac{1}{Re}\left(\vec{\nabla} \cdot \vec{\nabla}\right)\vec{U} - \vec{\nabla}\tilde{p} + \frac{1}{Re}\left(\vec{\nabla} \cdot \vec{\nabla}\right)\vec{u} \tag{12.7}$$

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{U} \cdot \vec{\nabla}\right)\vec{u} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{U} = -\vec{\nabla}\tilde{p} + \frac{1}{Re}\left(\vec{\nabla} \cdot \vec{\nabla}\right)\vec{u} \tag{12.8}$$

$$\nabla \vec{\cdot}\, \vec{u} = 0 \tag{12.9}$$

$$\vec{u}(\vec{r}, t) = e^{\lambda t}\vec{u}(\vec{r, 0}) \tag{12.10}$$

$$\lambda\vec{u}(\vec{r}, t) = \frac{1}{Re}\left(\vec{\nabla} \cdot \vec{\nabla}\right)\vec{u}(\vec{r}, t) - \vec{\nabla}\tilde{p} - \left(\vec{U} \cdot \vec{\nabla}\right)\vec{u}(\vec{r}, t) - \left(\vec{u} \cdot \vec{\nabla}\right)\vec{U}(\vec{r}, t) \tag{12.11}$$

$$\vec{\nabla} \cdot \vec{u}(\vec{r}, t) = 0 \tag{12.12}$$

This is an eigenvalue equation.

$$\lambda = \lambda(Re, \vec{U}) \tag{12.13}$$

depends on Re and $\vec{U}(\vec{r})$ and $\vec{u}(\vec{r}, 0)$.

If $\lambda < 0$: $\vec{U}(\vec{r})$ is stable and the perturbations damps out. If $\lambda > 0$: $\vec{U}(\vec{r})$ is unstable and the perturbation grows.

Linear stability analysis of the poor man's Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t} = -\left[\left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} + \vec{\nabla} p\right] + \frac{1}{Re}\vec{\nabla}^2\vec{v} + \vec{f} \tag{12.14}$$

"analogy"

$$v_{t+1} - v_t = -2v_t^2 - v_t + 1 \tag{12.15}$$

Oversimplification: discrete time steps $\Delta t = 1$, no spatial structure, no vector.

$$v_{t+1} = 1 - 2v_t^2\,, \qquad -1 \le v_t \le 1 \tag{12.16}$$

substitution

$$v_t = 2x_t - 1 \tag{12.17}$$
$$\Downarrow$$
$$x_{t+1} = 4x_t(1 - x_t)\,, \quad 0 \le x_t \le 1 \tag{12.18}$$

generalization

$$x_{t+1} = rx_t(1 - x_t) \ , \quad 0 \leq r \leq 4 \tag{12.19}$$

This is the famous logistic (quadratic) map of deterministic chaos.

The order parameter r is analogous to Re: different dynamic (temporal) patterns for different r.

As r increases an "old pattern" becomes unstable, and a "new pattern" emerges.

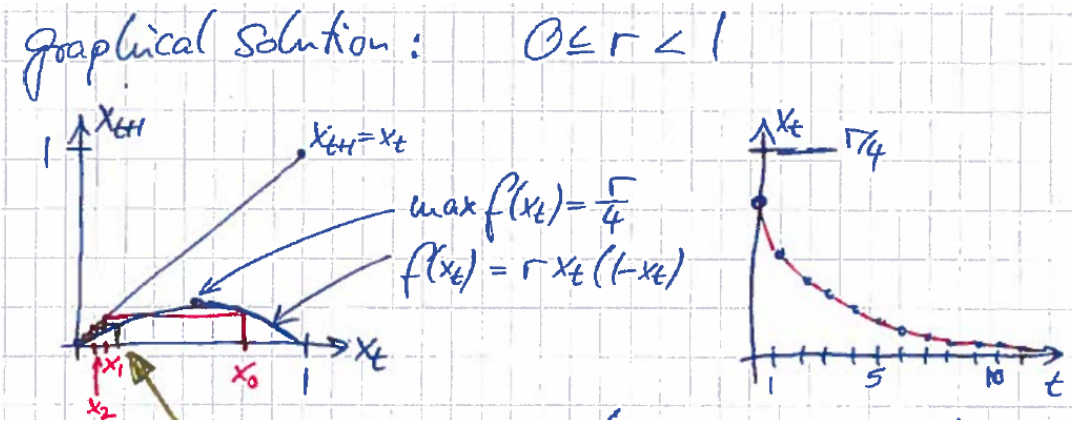


Figure 49:

Linear stability analysis

$$x_{t+1} = x^* + \delta x_{t+1} = f(x_t) = f(x^* + \delta x_t) \tag{12.20}$$

$$\approx f(x^*) + f'(x^*)\delta x_t 0 + \cdots \tag{12.21}$$

Perturbations are only kept up to first order (linearization). Quadratic and higher order terms are neglected.

$$\left| \frac{\delta x_{t+1}}{\delta x_t} \right| = |f'(x^*)| \tag{12.22}$$

$$|f'(x^*)| < 1 : \quad |\delta x_t| = |f'(x^*)|^t |\delta x_0| = e^{\lambda t} |\delta x_0 \tag{12.23}$$

$$|f'(x^*)| > 1 : \quad |\delta x_t| = e^{\lambda t} |\delta(x_0)| \tag{12.24}$$

$$f(x) = rx(1-z) \tag{12.25}$$
$$\Downarrow$$
$$f'(x) = r[(1-x)-x] = r(1-2x) \tag{12.26}$$
$$\Downarrow$$
$$|f'(x^* = 0)| = r \tag{12.27}$$
$$\Downarrow$$
$$x^* = 0 \tag{12.28}$$

Stable fixed point for $r < 1$. Unstable fixed point for $r > 1$.

Question: what happens for $r > 1$?

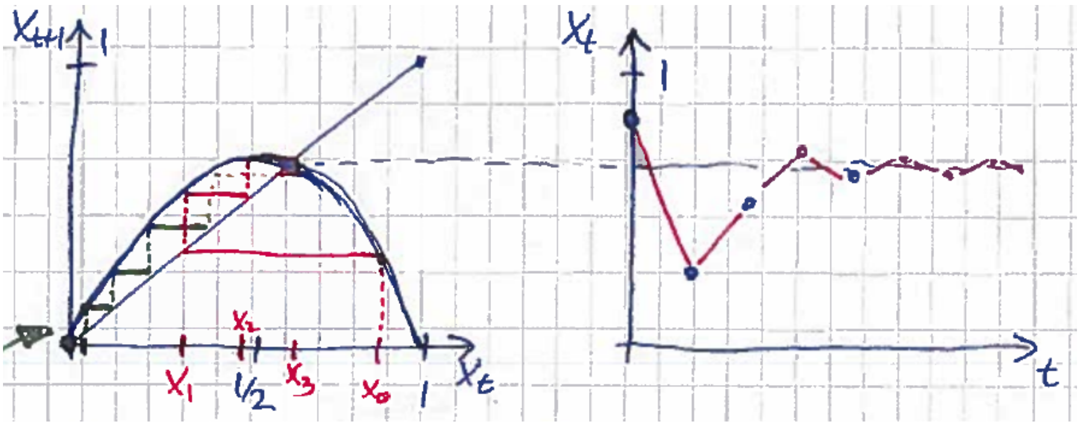


Figure 50:

Fixed points:

$$x^* = f(x^*) = rx^*(1-x^*) \tag{12.29}$$
$$\Downarrow$$
$$x_0^* = 0\,, \quad x_1^* = \frac{r-1}{r} \tag{12.30}$$

Stability:

$$f'(x_0^*) = r > 1 \tag{12.31}$$

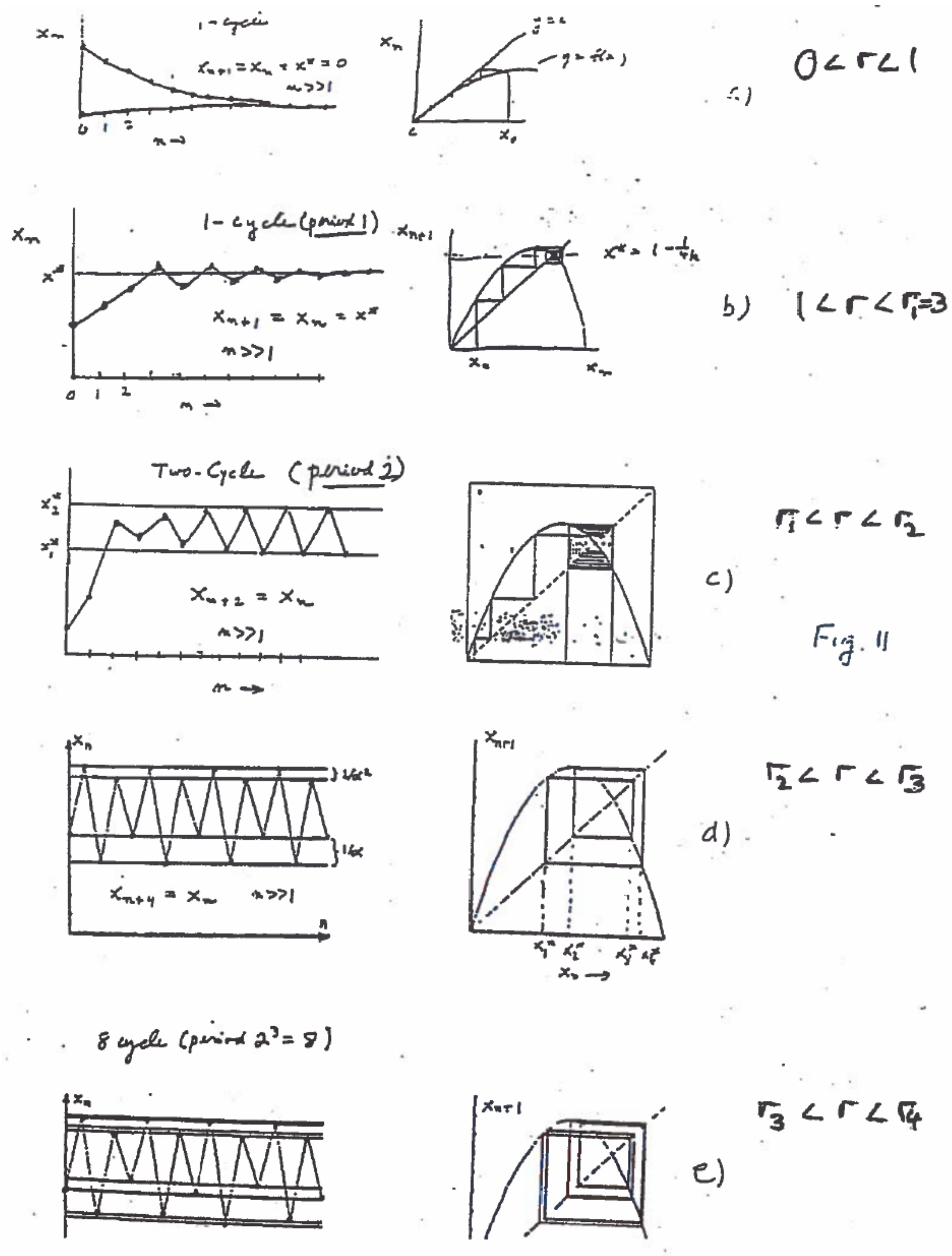
which means that x_0^* is an unstable fixed point.

$$|f'(x_1^*)| = \left| r\left(1 - 2\frac{r-1}{r}\right) \right| = |2-r| \tag{12.32}$$

$$|f'(x_1^*)| < 1 \Rightarrow 1 < r < 4 \tag{12.33}$$

which means that x_1^* is a stable fixed point.

Question: what happens for $r > 3$?



1-cycle
$x_{n+1} = x_n = x^* = 0$
$n >> 1$
$0 < r < 1$
a)
1-cycle (period 1)
$x^* = 1 - \frac{1}{4r}$
$x_{n+1} = x_n = x^*$
$n >> 1$
b)
$1 < r < r_1 = 3$
Two-Cycle (period 2)
$x_{n+2} = x_n$
$n >> 1$
$r_1 < r < r_2$
c)
Fig. 11
$x_{n+4} = x_n$
$n >> 1$
$r_2 < r < r_3$
d)
8 cycle (period $2^3 = 8$)
$r_3 < r < r_4$
e)

Figure 51:

two cycle ($r_1 = 3 < r < r_2 =?$): jumps between two values x_1^* and x_2^*.

$$x_2^* = f(x_1^*)\ , \quad x_1^* = f(x_2^*) \tag{12.34}$$
$$\Downarrow$$
$$x_2^* = f(f(x_2^*))\ , \quad x_1^* = f(f(x_1^*)) \tag{12.35}$$

$$f^{(2)}(x) = f(f(x)) = rf(x)(1 - f(x)) \tag{12.36}$$
$$= r^2 x(x-1)[1 - rx(1-x)] \tag{12.37}$$

$$\left| \frac{df^{(2)}(x)}{dx} \right| < 1 \Rightarrow r_1 = 3 < r < r_2 =? \tag{12.38}$$

stable two-cycle

$r_2 < r < r_3$ is a stable four-cycle

$$f^{(4)}(x) = f(f(f(f(x)))) \tag{12.39}$$
$$\left| \frac{df^{(4)}(x)}{dx} \right| < 1 \tag{12.40}$$

$$3 < r \leq r_\infty = 3.57... \tag{12.41}$$

2-cycle, 4-cycle, 8-cycle, ...

$$3.57... \leq r \leq 4 \tag{12.42}$$

Mostly chaotic.

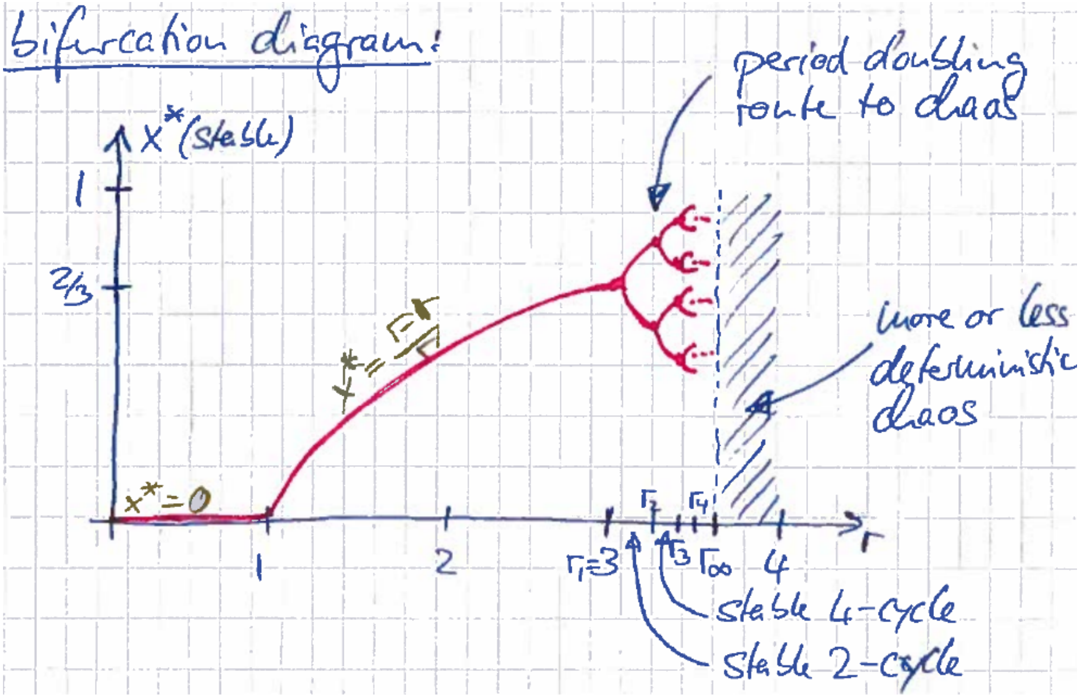


Figure 52:

Question: how can we characterize deterministic chaos?

Liapunov exponent:

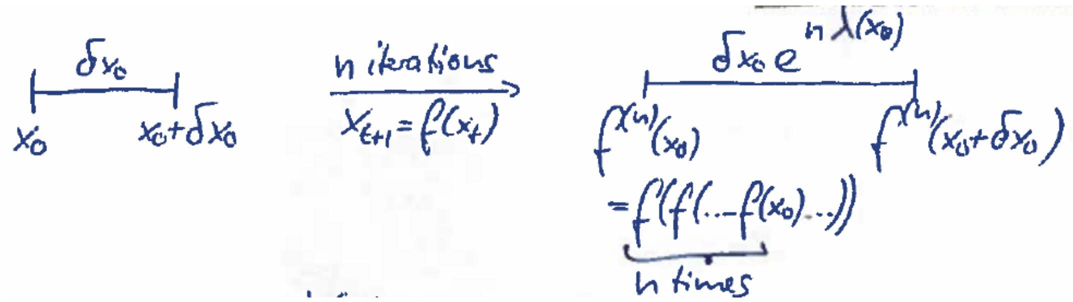


Figure 53:

$$\delta x_0 e^{n\lambda(x_0)} = |f^{(n)}(x_0 + \delta x_0) - f^{(n)}(x_0)| \tag{12.43}$$

$$\lambda(x_0) = \lim_{n\to\infty} \lim_{\delta x_0 \to 0} \frac{1}{n} \ln \left| \frac{f^{(n)}(x_0 + \delta x_0 - f^{(n)}(x_0)}{\delta x_0} \right. \tag{12.44}$$

$$= \lim_{n\to\infty} \frac{1}{n} \ln |(f^{(n)}(x_0))'| \tag{12.45}$$

$$= \lim_{n\to\infty} \frac{1}{n} \ln |f'(x_{n-1}) \cdot f'(x_{n-2}) \cdot \ldots \cdot f'(x_1) \cdot f'(x_0)| \tag{12.46}$$

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \tag{12.47}$$

Attractive stable limit cycle (with period $T = m$)

$$\lambda(x_0) = \frac{1}{m} \sum_{i=0}^{m-1} \ln |f'(x_i^*)| < 0 \tag{12.48}$$

Neighboring phase space trajectories with nearly identical initial conditions converge towards each other.

Deterministic chaos:

$$\lambda(x_0) > 0 \tag{12.49}$$

neighboring trajectories diverge

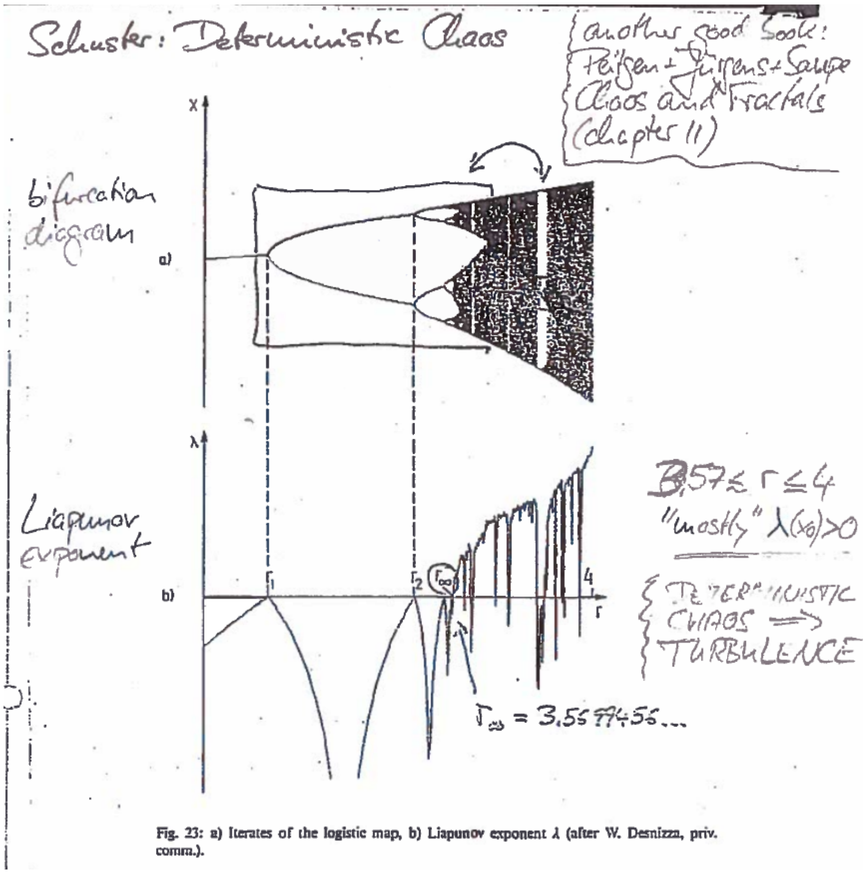


Fig. 23: a) Iterates of the logistic map, b) Liapunov exponent λ (after W. Desnizza, priv. comm.).

Figure 54:

Special case (of the logistic map): $r = 4$

Observation: $\lambda(x_0) > 0$ leads to divergence of neighboring trajectories. All values $0 \le x_t \le 1$ occur (as t runs its course).

Question: what is the probability density for a specific x-value?

$$x_{t+1} = 4x_t(1 - x_t) \tag{12.50}$$

Substitution:

$$x_t = \frac{1}{2}[1 - \cos(2\pi y_t)] \tag{12.51}$$

$$\begin{aligned}
t_{t+1} &= \frac{1}{2}[1 - \cos(2\pi y_{t+1})] = 4x_t(1 - x_t) &(12.52)\\
&= \frac{4}{2}[1 - \cos(2\pi y_t)]\left[1 - \frac{1}{2} + \frac{1}{2}\cos(2\pi y_t)\right] &(12.53)\\
&= [1 - \cos(2\pi y_t)][1 + \cos(2\pi y_t)] &(12.54)\\
&= 1 - \cos^2(2\pi y_t &(12.55)\\
&= \frac{1}{2} + \frac{1}{2}[\sin^2(2\pi y_t) + \cos^2(2\pi y_t)] - \cos^2(2\pi y_t) &(12.56)\\
&= \frac{1}{2} + \frac{1}{2}[\sin^2(2\pi y_t) - \cos^2(2\pi y_t)] &(12.57)\\
&= \frac{1}{2}[1 - \cos(4\pi y_t)] &(12.58)
\end{aligned}$$

Solution:

$$y_{t+1} = 2y_t\,, \quad 0 \leq y_t \leq 1 \tag{12.59}$$

$$y_t = 2^t y_0 \tag{12.60}$$

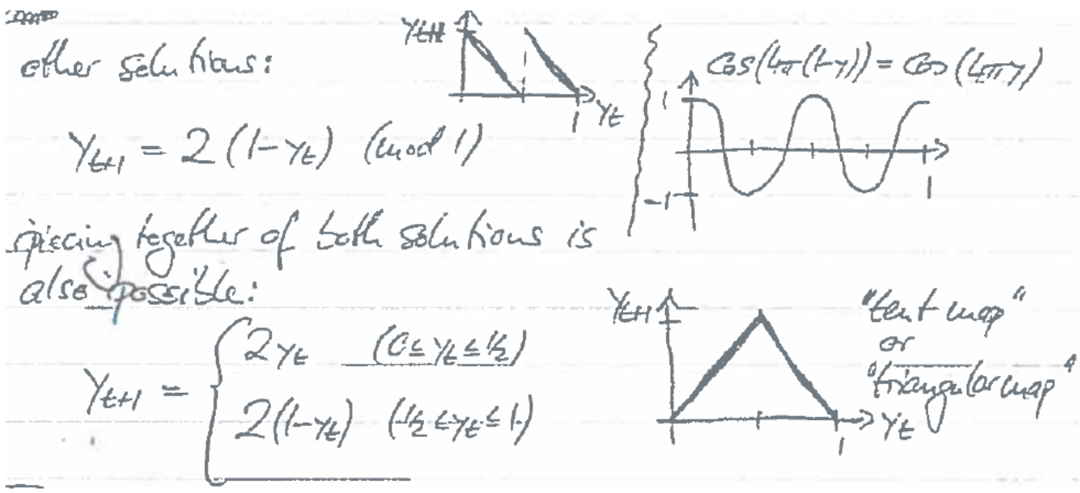


Figure 55:

Once again: sensitive dependence on initial conditions

$$y_0 = \alpha_1 2^{-1} + \alpha_2 2^{-2} + \alpha_3 2^{-3} + \cdots = \sum_{i=1}^{\infty} \alpha_i 2^{-i} \tag{12.61}$$

$$\begin{aligned} y_1 &= 2y_0 && (12.62)\\ &= \alpha_2 2^{-2} + \alpha_3 2^{-3} + \alpha_4 2^{-4} + \ldots && (12.63)\\ &= \sum_{i=1+1}^{\infty} \alpha_{i+1} 2^{-i} && (12.64) \end{aligned}$$

$$\begin{aligned} y_2 &= \cdots = \sum_{i=1+1}^{\infty} \alpha_{i+2} 2^{-i} && (12.65)\\ &\vdots && (12.66)\\ y_t &= \sum_{i=1+1}^{\infty} \alpha_{i+t} 2^{-i} && (12.67) \end{aligned}$$

$$\begin{aligned} y_0 &= 0.\alpha_1\alpha_2 \ldots \alpha_n \alpha_{n+1} \ldots && (12.68)\\ y_0' &= 0.\alpha_1\alpha_2 \ldots \alpha_n \alpha_{n+1}' \ldots && (12.69) \end{aligned}$$

$$|y_0 - y_0'| \leq \underbrace{0.00\ldots01}_{(n-1)\text{ times}} = \frac{1}{2^n} \tag{12.70}$$

During the first n time steps the two trajectories $y_i = f^{(i)}(y_0)$ and $y_i' = f^{(i)}(y_0')$ stay close together; thereafter they separate completely. Sensitive dependence on initial condition.

$$y_0 = 0.\alpha_1\alpha_2\alpha_3 \cdots = \sum_{i=1}^{\infty} \alpha_i 2^{-i} \tag{12.71}$$

with random numbers $\alpha_i = \{0, 1\}$.
$\{y_t = f^{(t)}(y_0)\}$ uniformly distributed on $[0, 1]$

$$\rho(y) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-0} \delta(y - y_t) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{else} \end{cases} \tag{12.72}$$

Back to our original question:

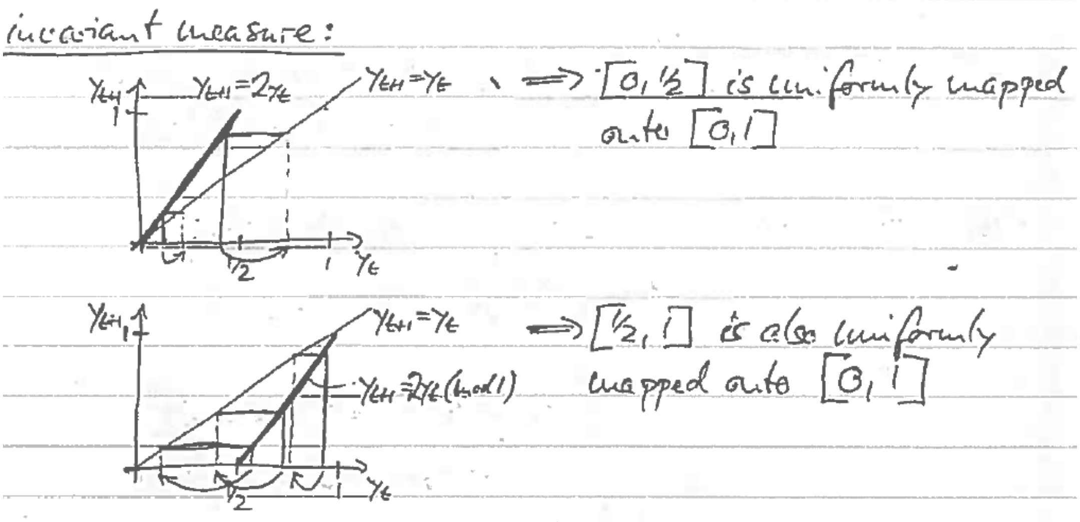


Figure 56:

$$1 = \int_0^1 \rho(y)dy = \int_0^1 dy = 2\int_0^{\frac{1}{2}} dy \tag{12.73}$$

$$= 2\int_0^1 \frac{dx}{\pi \sin(2\pi y)} = \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{1-\cos^2(2\pi y)}} \tag{12.74}$$

$$= \frac{2}{\pi}\int_0^1 \frac{dx}{[1-(1-2x)^2]^{1/2}} = \frac{2}{\pi}\int_0^1 \frac{dx}{[4x-4x^2]^{1/2}} = \int_0^1 \frac{1}{\pi[x(1-x)]^{1/2}}dx \tag{12.75}$$

$$\rho(x) = \frac{1}{\pi[x(1-x)]^{1/2}} \tag{12.76}$$

invariant measure for the map $x_{t+1} = 4x_t(1-x_t)$.

Transparency:

$$\rho(x)\ , \quad x_{t+1} = 4x_t(1-x_t) \tag{12.77}$$

$$\rho(v)\ , \quad v_{t+1} = 1 - 2v_t^2 \tag{12.78}$$

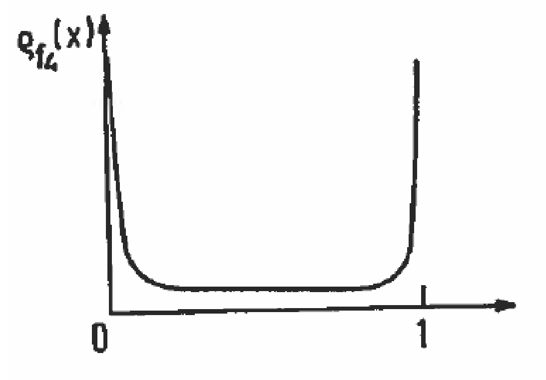


Figure 57:

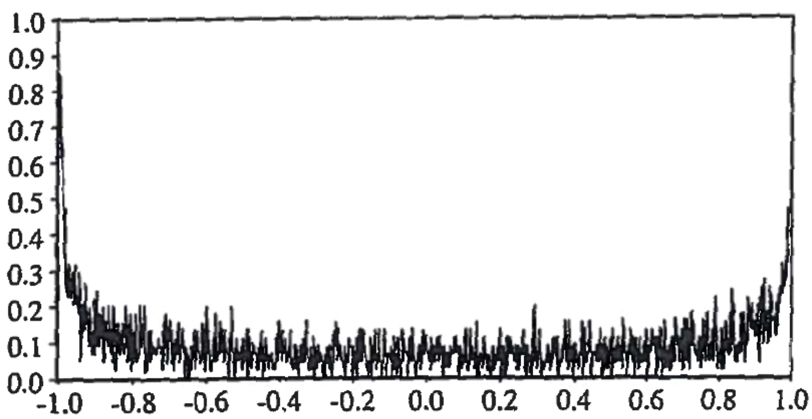


Figure 58: