3 Simplification of the Navier-Stokes equation

3.1 Simplification I: incompressible flows

Incompressiblity is a very good approximation for most liquids, including water. In 1000 m depth the density of seawater is only 0.4% larger than at the surface. For gas flows incompressiblity is also a good approximation as long as $|\vec{u}_{\text{gas}}| \ll$ speed of sound. Compressibility becomes important when discussing e.g. sound waves.

Incompressibility means that the density of a fluid particle (moving along its pathline) remains constant.

$$\begin{aligned} 0 = \frac{d\rho(\vec{r},t)}{dt} &= \frac{\partial \rho}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\rho && (3.1)\\ &= -\vec{\nabla}\left(\rho\vec{u}\right) + \left(\vec{u}\cdot\vec{\nabla}\right)\rho && (3.2)\\ &= -\left(\vec{u}\cdot\vec{\nabla}\right)\rho - \rho\left(\vec{\nabla}\cdot\vec{u}\right) + \left(\vec{u}\cdot\vec{\nabla}\right)\rho && (3.3)\\ &= -\rho\left(\vec{\nabla}\cdot\vec{u}\right) && (3.4)\\ &= -\rho\,\mathrm{div}\,\vec{u} && (3.5)\end{aligned}$$

In the first line we used the material derivative. In the step to the second line we used continuity equation. For incompressibility the divergence must be zero:

$$\vec{\nabla}\cdot\vec{u} = 0 \tag{3.6}$$

Navier-Stokes equation for incompressible flows (3 equations):

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\right)\vec{u} = f_{\text{ext}} - \vec{\nabla}p + \mu\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{u} \tag{3.7}$$

> Remark: It looks simple, but these nonlinear differential equations remain a formidable challenge to engineers, physicists and mathematicians.

Fourth equation:

$$\vec{\nabla}\cdot\vec{u} = 0. \tag{3.8}$$

Fifth equation: equation of state in the simplest form with constant density

$$p = p(\rho) \Rightarrow \rho = \rho_0 = \text{constant}. \tag{3.9}$$

3.2 Simplification II: incompressible, ideal, stationary, irrotational flows

We use the incompressibility result:

$$\vec{\nabla}\cdot\vec{u} = 0 \tag{3.10}$$

Ideal means no friction. To eliminate friction forces we set $\mu = 0$.

Euler equation:

$$\rho_0 \left(\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} \right) = \vec{f}_{\text{ext}} - \vec{\nabla} p \tag{3.11}$$

stationary:

$$\vec{u}(\vec{r}, t) = \vec{u}(\vec{r}) \tag{3.12}$$

$$\Downarrow$$

$$\frac{\partial \vec{u}}{\partial t} = 0 \tag{3.13}$$

$$\rho_0 \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \vec{f}_{\text{ext}} - \vec{\nabla} p \tag{3.14}$$

no external forces: $\vec{f}_{\text{ext}} = 0$

$$\rho_0 \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = -\vec{\nabla} p \tag{3.15}$$

We now look at the convective term on the lefthand side (see the proof below):

$$\left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \frac{1}{2} \vec{\nabla} \underbrace{(\vec{u} \cdot \vec{u})}_{\vec{u}^2} - \vec{u} \times \left(\vec{\nabla} \times \vec{u} \right) \tag{3.16}$$

$$\Downarrow$$

$$\vec{\nabla} \left(\frac{\rho_0}{2} \vec{u}^2 + p \right) = \rho_0 \vec{u} \times \left(\vec{\nabla} \times \vec{u} \right) \tag{3.17}$$

Assuming irrotational flow: $\vec{\nabla} \times \vec{u} = 0$.

$$\vec{\nabla} \underbrace{\left(\frac{\rho_0}{2} \vec{u}^2 + p \right)}_{\text{constant}} = 0 \tag{3.18}$$

Bernoulli's equation

$$\frac{\rho_0}{2} \vec{u}^2 + p = \text{constant} \tag{3.19}$$

$$\vec{\nabla} \cdot \vec{u} = 0 \tag{3.20}$$

$$\vec{\nabla} \times \vec{u} = 0 \tag{3.21}$$

Given all the assumptions, this set of equations is equivalent to the Navier-Stokes equation.

Proof of

$$\left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \frac{1}{2} \vec{\nabla} \left(\vec{u}^2 \right) - \vec{u} \times \left(\vec{\nabla} \times \vec{u} \right). \tag{3.22}$$

First we look at the x-component of the left-hand side:

$$\left[\left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} \right]_x = \left(u_x \partial_x + u_y \partial_y + u_z \partial_z \right) u_x \tag{3.23}$$

Now we look at the rightmost term on the right-hand side:

$$\vec{\nabla} \times \vec{u} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ u_x & u_y & u_z \end{vmatrix} = \left(\partial_y u_z - \partial_z u_y\right) \vec{e}_x + \left(\partial_z u_x - \partial_x u_z\right) \vec{e}_y + \left(\partial_x u_y - \partial_y u_x\right) \vec{e}_z \tag{3.24}$$

Now we can show that the x-component of the right-hand side is equal to the x-component of the left-hand side:

$$\left[\frac{1}{2}\vec{\nabla}\left(\vec{u}^2\right) - \vec{u} \times \left(\vec{\nabla} \times \vec{u}\right)\right]_x = \frac{1}{2}\partial_x \left(u_x^2 + u_y^2 + u_z^2\right) - \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ u_x & u_y & u_z \\ \partial_y u_z - \partial_z u_y & \partial_z u_x - \partial_x u_z & \partial_x u_y - \partial_y u_x \end{vmatrix}_x \tag{3.25}$$

$$= u_x \left(\partial_x u_x\right) + u_y \left(\partial_x u_y\right) + u_z \left(\partial_x u_z\right) - u_y \left(\partial_x u_y - \partial_y u_x\right) + u_z \left(\partial_z u_x - \partial_x u_z\right) \tag{3.26}$$

$$= u_x \left(\partial_x u_x\right) + u_y \left(\partial_y u_y\right) + u_z \left(\partial_z u_z\right) \tag{3.27}$$

$$= \left[\left(\vec{u} \cdot \vec{\nabla}\right) \vec{u}\right]_x \tag{3.28}$$

3.3 Derivation of Bernoulli's equation

The equation

$$\vec{\nabla}\left(\frac{\rho_0}{2}\vec{u}^2 + p\right) = \rho_0 \vec{u} \times \left(\vec{\nabla} \times \vec{u}\right) \tag{3.29}$$

is (scalar) multiplied with $d\vec{s} \parallel \vec{u}$, where $d\vec{s}$ describes an increment of a specific streamline (here pathline since $\vec{u}(\vec{r}, t) = \vec{u}(\vec{r})$).

$$d\vec{s} \cdot \left[\vec{u} \times \left(\vec{\nabla} \times \vec{u}\right)\right] = 0 \tag{3.30}$$

Since $d\vec{s} \parallel \vec{u}$ it must be that $d\vec{s} \perp \vec{u} \times \left(\vec{\nabla} \times \vec{u}\right)$.

Subsequent path-integration along a streamline yields

$$0 \stackrel{!}{=} \int_{\text{streamline}} \vec{\nabla} \underbrace{\left(\frac{\rho_0}{2}\vec{u}^2 + p\right)}_{W} \cdot d\vec{s} \tag{3.31}$$

$$= \int_{\text{streamline}} \begin{pmatrix} \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial y} \\ \frac{\partial W}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{3.32}$$

$$= \int_{\text{streamline}} \left(\frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz\right) \tag{3.33}$$

$$= \int_{\text{streamline}} dW = \int_{\text{streamline}} d\left(\frac{\rho_0}{2}\vec{u}^2 + p\right) \tag{3.34}$$

$$\Downarrow$$

$$\frac{\rho_0}{2}\vec{u}^2 + p = \text{constant} \qquad \text{(along a streamline)} \tag{3.35}$$

For another streamline the constant might in principle be different. Often the velocity in the far-field regime (away from the obstacle) is everywhere the same. The same holds true for the pressure. Then the "Bernoulli constant" has to be everywhere (far and near-field) the same. From here we then also conclude that $\vec{\nabla} \times \vec{u} = 0$ everywhere.

3.4 Example: Why does an airplane fly?

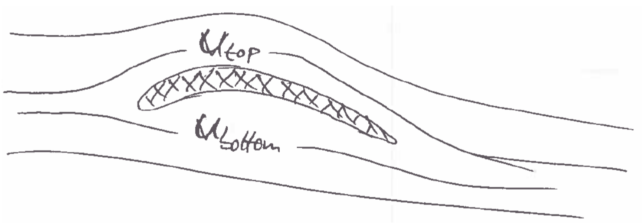

Figure 15: The profile of the wing of an airplane.

According to the Bernoulli's equation, the wind speed difference between the top and bottom of the wing creates a pressure difference:

$$u_{\text{top}} > u_{\text{bottom}} \tag{3.36}$$

$$\Downarrow$$

$$p_{\text{top}} < p_{\text{bottom}}. \tag{3.37}$$

This results in a lifting force.