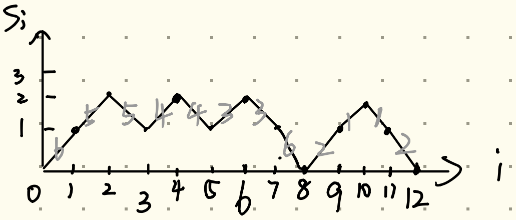
Exercises

1. Given the sequence $(++-+-+--++--)$ of ±1s with nonnegative partial sums, find or draw the corresponding
a) well-formed sequence of six pairs of parentheses
b) nonnegative path from the origin to $(12, 0)$
c) stack permutation of 123456.

1、a) (( ) ( ) ( )) (( ))

b)



c) 2 1 6 3 4 5

2. Given the sequence $(++-+--++--)$ of ±1s with nonnegative partial sums, find or draw the corresponding
a) well-parenthesized product of six variables
b) full binary tree with six leaves whose vertices are labeled with well-parenthesized products
c) triangulation of a convex septagon.

△ 规则：(是 1 ，变是 -1.

2. a)

$q = ((x_1(x_2x_3((x_4x_5$

$q' = ((x_1(x_2x_3((x_4x_5x_6$

$= ((x_1(x_2x_3(y_1x_6$ $\qquad y_1 = (x_4x_5)$

$= ((x_1(x_2x_3y_2$ $\qquad y_2 = (y_1x_6)$

$= ((x_1y_3y_2$ $\qquad y_3 = (x_2x_3)$

$= (y_4y_2$ $\qquad y_4 = (x_1y_3)$

$= y_5$ $\qquad y_5 = (y_4y_2)$

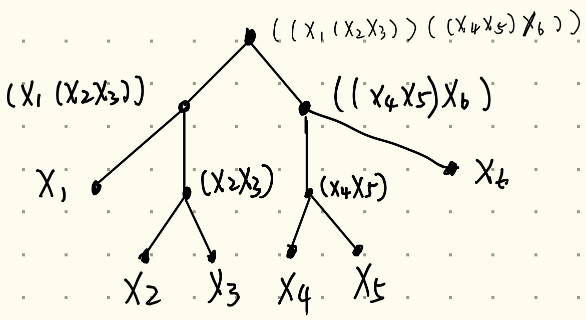
设 $p = y_5$

$= (y_4y_2)$

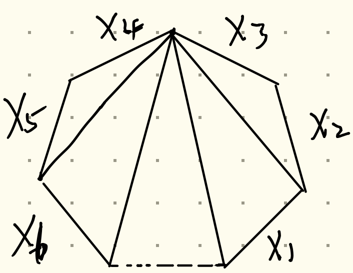
$= ((x_1y_3)(y_1x_6))$

$= ((x_1(x_2x_3))((x_4x_5)x_6))$

b)

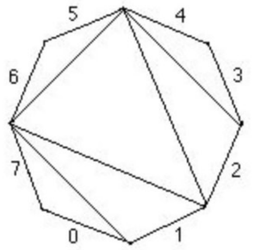


c)



3. Given the following triangulation of a convex octagon, find or draw the corresponding

a) well-parenthesized product of seven variables
b) sequence of twelve ±1s with nonnegative partial sums
c) nonnegative path from the origin to $(12, 0)$
d) stack permutation of 123456.

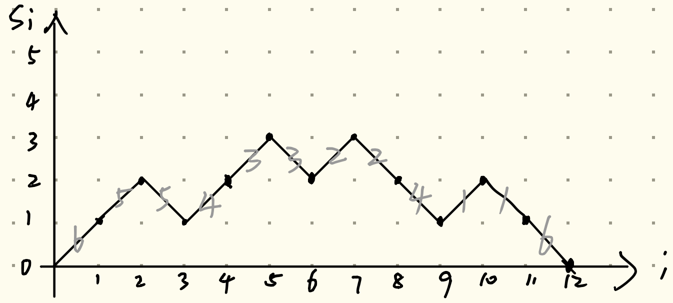


(: 1
变量 : −1.

a) $((x_1((x_2(x_3x_4))(x_5x_6)))x_7)$

b) $(+1, +1, -1, +1, +1, -1, +1, -1, -1, +1, -1, -1)$

c)



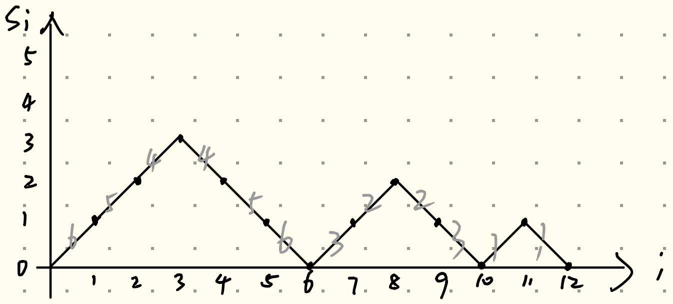
d) 6 1 4 2 3 5

4. Given the well-parenthesized product $(((12)3)((45)(67)))$ of seven variables (with x_i denoted by i), find or draw the corresponding

a) sequence of twelve ±1's with nonnegative partial sums
b) stack permutation of 123456
c) triangulation of a convex octagon.

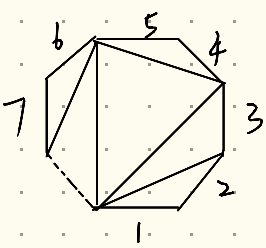
4. a) $(+1, +1, +1, -1, -1, -1, +1, +1, -1, -1, +1, -1)$

b)



stack permutation : 1 3 2 6 5 4

c)

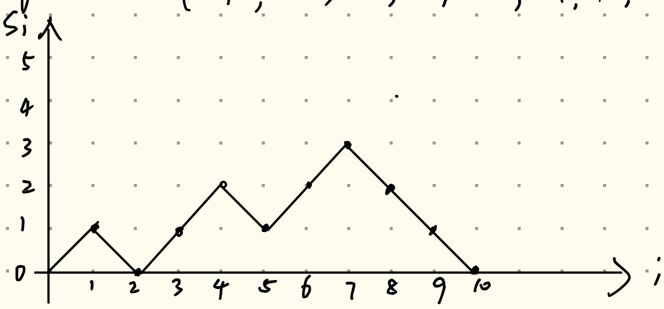


5. Find the sequence of ten pushes (+) and pops (−) that produces the stack permutation 42135, and draw the corresponding nonnegative path.

5.

α [6] β	operation	Sequence.
[ ] 42135		
[4] 2135	4 poped	−
[24] 135	2 poped	−
[124] 35	1 poped	−
1 [24] 35	1 pushed	+
12 [4] 35	2 pushed	+
12 [34] 5	3 poped	−
123 [4] 5	3 pushed	+.
1234 [ ] 5	4 pushed	+
1234 [5]	5 poped	−
12345 [ ]	5 pushed	+

Sequence: (+, −, +, +, −, +, +, −, −, −)



6. Prove that the Catalan numbers c_n satisfy the recurrence relation (11).

6. 定义 a_n：n个变量的 well-parenthsized product 的数量.

a_{n+1} 是 Catalan 数 C_n .

由 (AB)，A是前 i个变量的，B是后 n-i 个变量的.

$\therefore a_n = a_1 a_{n-1} + a_2 a_{n-2} + \cdots + a_{n-1} a_1 = \sum_{i=1}^{n-1} a_i a_{n-i}$

代入 C_j 并调整 i 的范围

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$$

∴ 第 n 个 Catalan 数 = 前面 n-1 个 Catalan 数的乘积之和.

$$\therefore C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!}$$

$$\therefore \frac{C_n}{C_{n-1}} = \frac{(2n)!}{(n+1)!\,\cancel{n!}} \cdot \frac{\cancel{n!}\,(n-1)!}{(2n-2)!} = \frac{(2n-1)\cdot 2n}{n\cdot(n+1)} = \frac{4n-2}{n+1}$$

7. Use the Stirling approximation (12) for $n!$ to prove that the Catalan number c_n satisfies (13).

In Exercises 8–10, suppose that T is a triangulation of a convex $(n+2)$-gon P with diagonal set D. Let $v_0v_1\cdots v_{n+1}$ be the vertices of P in order, and let s_i be the side $v_{i-1}v_i$ for $1 \le i \le n+1$, $s_0 = v_0v_{n+1}$. Denote by $\mathbf{p}$ the corresponding well-parenthesized product $s_1s_2\cdots s_{n+1}$ of its sides other than s_0.

$$S_n = \sqrt{2\pi n}\, e^{-n} n^n \qquad (12)$$

$$C_n = O(n^{-\frac{3}{2}} 4^n) \qquad (13)$$

7. Stirling 公式 对大n时 $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n = \sqrt{2\pi n}\, e^{-n} n^n$

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\frac{(2n)!}{(n!)^2}$$

$$\sim \frac{1}{n+1}\frac{\sqrt{2\pi\cdot 2n}\; e^{-2n}\,(2n)^{2n}}{2\pi n\cdot e^{-2n}\; n^{2n}}$$

$$= \frac{4^n}{(n+1)\sqrt{\pi n}}$$

$$\sim \frac{4^n}{n\cdot\sqrt{\pi n}} = \frac{4^n}{\sqrt{\pi}\cdot n^{\frac{3}{2}}} = O(n^{-\frac{3}{2}} 4^n)$$

8. Prove that the diagonal v_iv_j is in the diagonal set D if and only if the product $s_{i+1}s_{i+2}\cdots s_j$ is well-parenthesized in $\mathbf{p}$.

8. ① if $v_iv_j \in D$，则 $s_{i+1}s_{i+2}\cdots s_j$ 是 well-parenthesized.

∵ $v_iv_j \in D$，

∴ triangulation 中至少有一个 △ 是由边 s_i, s_{i+1} 和 v_iv_j 组成的.

通过将 diagonal v_iv_j 加入 triangulation，

即创建一个封闭括号对 $(s_i\ s_{i+1})$

逐步向内推进，①可得证.

② if $s_{i+1}s_{i+2}\cdots s_j$ 是 well-parenthesized.，那么 $v_iv_j \in D$.

在 triangulation 中，某个 △ 由相邻边 s_i，s_{i+1} 和 diagonal v_iv_j 组

这个 △ 外边与子串的括号匹配.

∴ 对角线 v_iv_j 完成了一个外部 △

∴ $v_iv_j \in D$

9. Let $1 \le k \le n-1$. Prove that the nonnegative path corresponding to T meets the x-axis at the point $(2k,0)$ if and only if D contains the diagonal $v_k v_{n+1}$.

9. ① 非负 path 在 $(2k,0)$ 与 x 轴相交 $\longrightarrow$ diagonal $V_k V_{n+1} \in D$.

即 path 走了 $2k$ 步（经过 $2k$ 条对角线），并且在这个位置返回到 $y=0$.

需要适当的对角线确保每一步与 triangulation 匹配

$\therefore$ 一定有 diagonal $V_n V_{k+1} \in D$.

② diagonal $V_k V_{n+1} \in D$. $\longrightarrow$ 非负 path 在 $(2k,0)$ 与 x 轴相交

$V_k V_{n+1}$ 会将多边形分为两部分，

让 path 在 $x=2k$ 时回到 $y=0$，即在 $(2k,0)$ 与 x 轴相交.

10. Prove that the nonnegative path corresponding to T is positive if and only if D contains the diagonal $v_0 v_n$.

① 若对角线 $V_0 V_n$ 存在 $\longrightarrow$ 非负 path 是正的

$V_0 V_n$ 将 n-gon 分成 2 个部分，确保逆时针遍历时

不会回到 x 轴

即 path 是 positive 的.

② 非负 path 是正的 $\longrightarrow$ 对角线 $V_0 V_n$ 存在

path 每一步都不会回到 x 轴.

若 $V_0 V_n$ 不存在，则路径可能多次经过其它对角线分割.

纵坐标可能会下降至 x 轴或以下

与条件矛盾

故假设不成立

$\therefore$ 对角线 $V_0 V_n$ 存在