Available online at www.sciencedirect.com
ScienceDirect
Int. J. Production Economics 106 (2007) 204–216
international journal of production economics
www.elsevier.com/locate/ijpe

Using vector autoregressive residuals to monitor multivariate processes in the presence of serial correlation

Xia Pan[a,*], Jeffrey Jarrett[b]

[a]*Faculty of Management and Administration, Macao University of Science and Technology, Macao*
[b]*College of Business Administration, University of Rhode Island, Kingston, RI 02881, USA*

Received 28 November 2005; accepted 28 June 2006
Available online 12 October 2006

Abstract

Traditional literature on statistical quality control discusses separately multivariate control charts for independent processes and univariate control charts for autocorrelated processes. We extend univariate residual monitoring to the multivariate environment, and propose using vector autoregressive residuals (VAR) to monitor multivariate processes in the presence of serial correlation. We mathematically examine the effects of shifts in process parameters on the VAR residual chart, and give examples using data from a plastic production process. The results indicate the feasibility of VAR residual chart to achieve quality control and improvement.

Keywords: Quality control; SPC; Multivariate and serially correlated processes; Vector autoregression (VAR)

1. Introduction

Technology dramatically changed the characteristics of modern production processes and its control and improvement. With the development of information technology, large data sets with short sample intervals available at low cost became available to examine for analytical and decision-making purposes (see Tagaras (1998) and Shiau (2003) for a survey of recent monitoring and inspection developments up to 2003). Hence, new quality control procedures to more efficiently supply reliable assurance on product quality and business operations are developing. Assessing the performance of high dimensional, cross and serially correlated processes is now an issue.

Statistical process control (SPC) as a method of quality improvement and control has a long history, and its development has been reinforced by the Six Sigma concept popularly adopted by blue chip organizations such as allied signal and GE (Dale et al., 2001). Previous investigations on multivariate quality control (process monitoring) for independent observations include Hotelling (1947), Alt (1984), Jackson (1985), Lowry et al. (1992), Wierda (1994), Mason et al. (1995), Lowry and Montgomery (1995), Aparisi and Haro (2003), Yeh et al. (2004), Villalobos et al. (2005), and Yang and Rahim (2005). Studies of SPC for monitoring the univariate autocorrelated process include Ermer

*Corresponding author. Tel.: +8538972193; fax: +853880022.
E-mail addresses: Panpapers@yahoo.com (X. Pan), jejarrett@mail.uri.edu (J. Jarrett).

0925-5273/$ - see front matter
doi:10.1016/j.ijpe.2006.07.002

et al. (1979), Alwan and Roberts (1988), Harris and Ross (1991), Montgomery and Mastrangelo (1991), English et al. (1991), Alwan (1992), Wardell et al. (1992), Lu and Reynolds (1999), Jiang et al. (2000), West et al. (2002), Testik (2005), Holmes and Mergen (2005), and Pan (2006).

Alwan and Roberts's (1988) monitoring approach is to separate the variation in the time series into two parts and construct two charts: common cause chart and special cause chart. The former essentially accounts for the process's systematic variation that is represented by the autoregressive integrated moving average (ARIMA) model, and the latter is for detecting assignable causes assigned in the residuals of the ARIMA model. They designed the special cause chart as a Shewhart-type chart to monitor the residuals filtered and whitened from the autocorrelated process. Previously, there is little literature on SPC charts for production processes having both multivariate and autocorrelation characteristics appear. Among the exceptions are Kramer and Schmid (1997) using EWMA scheme on a two-dimension AR(1) process, and Pan and Jarrett (2004) using state space model for multivariate time series. In this paper, we extend Alwan and Roberts's special cause approach to multivariate cases, using a vector autoregressive (VAR) model.

The traditional multivariate charts do not account for autocorrelation and assume the processes are serially independent. In Phase I of the multivariate chart, when the n-variable processes are in-control, we collect m samples of size r, calculate each sample's mean $\bar{\vec{x}}_i$ and variance–covariance matrix S_i, and estimate the process mean vector $\vec{\mu}$ and process covariance matrix Σ, respectively, as the grand mean $\bar{\bar{\vec{x}}} = \sum_{i=1}^{m} \bar{\vec{x}}_i$ and the pooled sample variance–covariance matrix $\bar{S} = \sum_{i=1}^{m} S_i/m$. Under the assumption of multivariate normality, the chart statistic is Hotelling's T^2 (Hotelling, 1947). In Phase II, the T^2 is constructed sequentially using $\bar{S}$ and a sample of size r each time, and the chart statistic follows an F distribution. If the Phase I m is large enough ($m > 100$), the estimates of the process parameters are good enough and the chart statistic approximates a $\chi^2(n)$ distribution.

Alwan and Roberts's approach is to set up a Shewhart chart monitoring the residuals of the ARIMA model of the process. For an underlying process from an integrated order of d, autoregressive (AR) order of p and moving average (MA) order of q model, we difference the ARIMA(p, d, q) to be a stationary ARMA(p, q) model,

$$x_t = c + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q}. \quad (1)$$

For processes with the presence of MA terms, ordinary least square (OLS) estimation is not available. Only when the model has just AR terms, can OLS be used. Maximum likelihood estimation (MLE) can be used to estimate the coefficients in (1) to arrive at the one-step-ahead forecast:

$$\hat{x}_{t|t-1} = \hat{c} + \hat{\phi}_1 x_{t-1} + \cdots + \hat{\phi}_p x_{t-p} + \hat{\theta}_1 \hat{e}_{t-1} + \cdots + \hat{\theta}_q \hat{e}_{t-q}, \quad (2)$$

where the residual series $\hat{e}_t = x_t - \hat{x}_{t|t-1}$ is acquired recursively from the historical data. Process (1) contains systematic patterns, while in the residuals the systematic patterns are filtered out. A control chart on the residuals can reflect the assignable causes if the process is out-of-control.

Usually, MLE is not as easy as OLS for estimation. In practice, the true underlying process is never known. Researchers seek parsimonious models to approximately express the underlying process. As estimating an AR model is much easier than estimating a model with MA terms (MA or ARMA), one may hope the parsimonious model is an AR. Unfortunately, this is often not available for univariate processes, and one must deal with the MA terms.

2. VAR residual chart

In a multivariate process system with presence of serial correlation, we use VAR models to approximate the system, and estimate and monitor the VAR residuals as a serially independent multivariate series. Using a VAR to approximate a linear time-series system is appropriate due to the physical principles of the process dynamics. It is conceivable that a serially correlated real process system can have moving average terms, which are composed of disturbances. For univariate models, one regards the disturbances as a collection of the effects of other influential factors. These factors are often process variables and/or environmental variables. Some of them are not measurable, others are measurable but not measured in the simple univariate model. If these factors are autocorrelated, then it is reasonable to model the disturbances as MA. A second source of disturbances is measurement errors. This disturbance is not from the process system and its environment, but from the measurement instruments. The systematic cause should not include measurement errors, because the measurement disturbance would not be induced if the quality inspection were not applied. Therefore, if the true disturbance factors are not autocorrelated, the systematic cause is not moving average.

In multivariate processes, more measurable variables are included in the model. These measurable variables would have had effects in the disturbance term if they were not included in the model explicitly. Hence, with explicit inclusion of more variables in the multivariate model, the effects of the disturbance factor should be less than that in the univariate case. In other words, we shift some potentially autocorrelated components of the disturbance to the AR component. Therefore, the systematic MA is less important in the multivariate case than in the univariate case. Estimating the multivariate AR(p) model instead of a multivariate ARMA(p) model is more reasonable than estimating a univariate AR instead of a univariate ARMA. Therefore, a multivariate AR(p) is more likely to fit the true underlying process. In an ideal example, all of the systematic variables may be included in the VAR(p), and only the white noise and measurement error are in the disturbance. By this reasoning, we use VAR models to filer the systematic cause from the process. The biggest advantage of VAR models is that they can be estimated with OLS method instead of the more complex MLE method.

Denote the n variables of a VAR process in a $(n \times 1)$ vector $\vec{y}_t = (y_{1t}, y_{2t}, \ldots, y_{nt})'$. The VAR model of order p is

$$\vec{y}_t = \vec{c} + (\Phi_1 L + \Phi_2 L^2 + \cdots + \Phi_p L^p)\vec{y}_t + \vec{e}_t, \tag{3}$$

where L is the backshift operator, $\vec{c} = (c_1, c_2, \ldots, c_n)'$ is the constant vector, and $\vec{e}_t = (e_{1t}, e_{2t}, \ldots, e_{nt})'$ is the error term vector. Each Φ_j is a $(n \times n)$ coefficient matrix for the jth lag. The error term vector $\vec{e}_t$ is time independent but correlated cross-sectionally, i.e., $E(\vec{e}_t \vec{e}_t') = \Omega$ is invariant about time but may not be a diagonal $(n \times n)$ matrix. Strictly we assume $\vec{e}_t \sim$ i.i.d. $\mathrm{N}(\vec{0}, \Omega)$. For an in-control process, the systematic cause is invariant about time. To estimate the in control process parameters in Phase I, we denote the coefficient matrix $\Pi' = (\vec{c}\,\Phi)$ with $\Phi = \left(\Phi_1\, \Phi_2\, \cdots\, \Phi_p\right)$ and the vector $\vec{x}_t' = (1, \vec{y}_{t-1}', \vec{y}_{t-2}', \ldots, \vec{y}_{t-p}')'$. Then (3) becomes $\vec{y}_t = \Pi' \vec{x}_t + \vec{e}_t$. For Gaussian VAR($p$) process ($\vec{e}_t \sim$ i.i.d. $\mathrm{N}(\vec{0}, \Omega)$), the OLS estimates are the same as the maximum likelihood estimates:

$$\hat{\Pi}' = \left[\sum_{t=1}^{T} \vec{y}_t \vec{x}_t'\right] \left[\sum_{t=1}^{T} \vec{x}_t \vec{x}_t'\right]^{-1}, \tag{4}$$

where T is the number of observations in Phase I. The estimated systematic model and the error term are

$$\hat{\vec{y}}_t = \hat{\Pi}' \vec{x}_t \text{ and } \hat{\vec{e}}_t = \vec{y}_t - \hat{\vec{y}}_t, \tag{5}$$

respectively. One constructs the special-cause chart on $\hat{\vec{e}}_t$. The variance–covariance matrix of $\vec{e}$ is estimated as

$$\hat{\Omega} = (1/T) \sum_{t=1}^{T} \hat{\vec{e}}_t \hat{\vec{e}}_t'. \tag{6}$$

If the process is in-control and (3) is adequate and well estimated, the residuals $\hat{\vec{e}}_t$ should be also asymptotically i.i.d. normal with zero means. One applies the traditional Hotelling T^2 chart to $\hat{\vec{e}}_t$. For a Phase II observation $\vec{y}_i$, the chart statistic is

$$T_i^2 = \hat{\vec{e}}_i' \hat{\Omega}^{-1} \hat{\vec{e}}_i \sim \chi^2(n). \tag{7}$$

In practice, determining the autoregressive order p may be the issue before chart setting. To find an appropriate VAR order p, we can use the Akaike information criterion (AIC) and the related criteria. For example, p can be determined by minimizing AIC (Akaike, 1976):

$$\mathrm{AIC}_p = \ln\left(\left|\hat{\Omega}_p\right|\right) + 2pn^2/(T - p), \tag{8a}$$

or HQ (Hannan and Quinn, 1979):

$$\mathrm{HQ}_p = \ln\left(\left|\hat{\Omega}_p\right|\right) + 2pn^2 \ln(\ln T)/T. \tag{8b}$$

In principle, we should determine the upper control limit (UCL) by considering the ARL for in-control and out-of-control processes. The simple Shewhart chart with known control limits of "3-*sigma*" setting corresponds to a Type I error of 0.0027 or an in-control ARL of 370. If the VAR processes is exactly known, the VAR residual chart will have ARL of 370 when the Type I error is set at 0.0027. However, the VAR residuals are only asymptotically i.i.d. Therefore, to use the convenient Type I error way of setting up control limits for the VAR chart, we need to have the Phase I data size large enough in order to guarantee the serial independence of the residuals.

3. The effects of parameter shift

There are three classes of parameters in VAR model: the process mean$\vec{\mu}$, the covariance matrix of error term Ω, and the autoregressive coefficients of the model Φ. Although the shift on the mean $\vec{\mu}$ of the variables of the process is the most important shift of interest, any one or all of the three types of shift change the distributions of the chart statistic and induce out of control signals. The effect of a shift depends on the magnitude of the shift, the coefficient estimates, and the covariance matrix estimate $\hat{\Omega}$. The out-of-control ARL is then determined. The results of the examination in shift effects are as follows and one may observe the proofs of the lemmas in the appendices.

Lemma 1. *For a* VAR(p) *process, a shift* $\vec{\eta}$ *in the process mean vectors makes the Hotelling* T^2 *on VAR residuals a non-central* χ^2*distribution*,

$$_{\text{shift}}T_t^2 = {}_{\text{noshift}}T_t^2 + 2\vec{\eta}'\hat{\Omega}^{-1}\hat{\vec{e}}_t + \vec{\eta}'\hat{\Omega}^{-1}\vec{\eta}. \tag{9}$$

This lemma considers the situation that the shift is on the process mean instead of on the VAR constant vector. Jarrett and Pan (2006) discussed the latter case in which the shift occurs on the constant vector. In the steady-state form, the first term of the chart statistic after shift in mean is the in-control statistic $_{\text{noshift}t}T_t^2 \sim \chi^2(n)$ with mean of n and variance of $2n$. Since asymptotically $\hat{\vec{e}}_t \sim \mathrm{N}(\vec{0}, \Omega)$, the second term $-2\vec{\eta}'\hat{\Omega}^{-1}\hat{\vec{e}}_t$ is a linear combination of normal distributions so that it is normally distributed with mean of zeros and variance of $4\vec{\eta}'\hat{\Omega}^{-1}\hat{\Omega}\hat{\Omega}^{-1}\vec{\eta} = 4\vec{\eta}'\hat{\Omega}^{-1}\vec{\eta}$. We view the effect of this second term on the first term as it extends the variation but without moving the mean. The third term of (9b) is a shift from the mean of the first term, and this shift is positive because it is a positive scalar. Therefore, we measure the total effect of the process shift $\vec{\eta}$ on T^2 by $\Delta = \vec{\eta}'\hat{\Omega}^{-1}\vec{\eta}$, which measures both the second and the third terms. The proof of this lemma is given in Appendix A.

Lemma 2. *If the process covariance* Ω *shifts to* Ω_1, *then the effect on the VAR chart is* $_{\text{shift}}T_t^2 = \hat{\vec{e}}_t' D'\hat{\Omega}^{-1}D\hat{\vec{e}}_t$.

That is, the out-of-control statistic is a Hotelling T^2 constructed from the amplified residuals $\hat{\vec{\varepsilon}}_t = D\hat{\vec{e}}_t$ and the original covariance estimate $\hat{\Omega}$. Consequently, if the shift is only in the diagonal elements of the variance–covariance matrix, an increase in the variance will result to an increase in Hotelling T^2. This result tells us that the out-of-control statistic encounters a step shift itself immediately after the occurrence of the shift in the process variance. The sequence of the Hotelling T^2 is independent without time = series patterns except the step shift. The proof of this lemma is given in Appendix B.

Lemma 3. *For a* VAR(p) *process, if the coefficients shift from* Φ *to* $\Phi+\Delta\Phi$ *the effect on the Hotelling* T^2 *for the residuals is serially correlated and is affected by the process status in the past p periods.*

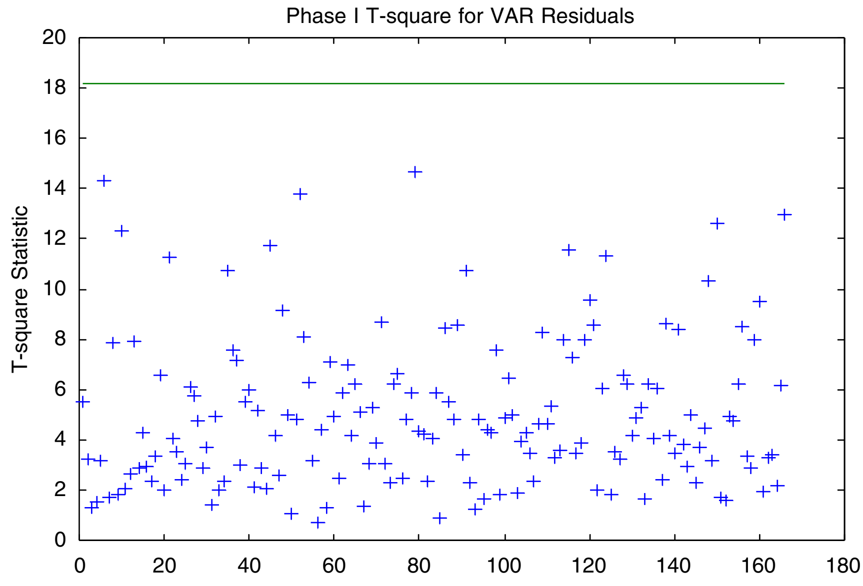
The proof of this lemma is given in Appendix C.

4. VAR residual chart—an example

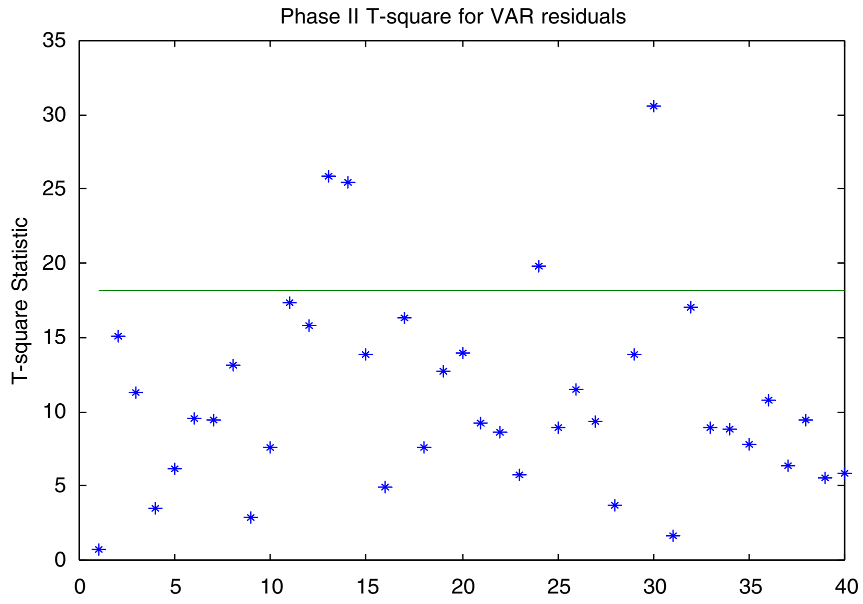
As presented in the above lemmas, the Hotelling T^2 chart on VAR residuals can detect signals when a shift occurs in anyone or more of the three types of the process parameters. In the following example, we wish to show the effectiveness of using VAR residual chart to detect special causes in multivariate time series. This example is for a five-variable real-life production process.

Temperatures at five different detecting points of a plastic mold are recorded. The temperature and its diffusion are important in controlling the quality of the plastic mold during the molding. As the temperature at the point that is near the injection inlet is higher and diffuses to the point that is far from the inlet, the five variables are correlated with time lags. As the heat diffusion is mainly passive (not adjusted by dynamically heating and cooling during the molding), SPC instead of automatic process control (APC) is the appropriate way of monitoring the process. To comply with company confidentiality requirements, the data were transformed and truncated from its original form.

Fig. 1 shows the VAR residual charts. The raw data are found having serial correlation and seasonality of five-period cycle. Note here after deseasonalizing the data we still unable to construct a traditional multivariate control chart, because without knowing the structure of the time series we cannot actually obtain the appropriate control limit. VAR residuals are asymptotically serially independent, so the Hotelling T^2 chart can be applied. In Phase I, we used 165 observations to estimate the VAR parameters using OLS method, and chose a VAR(5) model to approximate the process. The true process is unknown and may have MA terms, but we



(a)

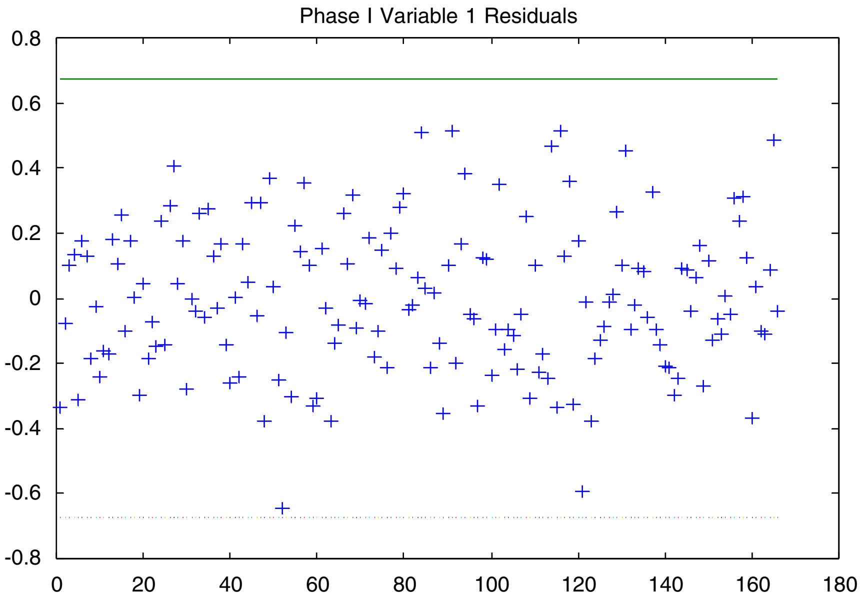


(b)

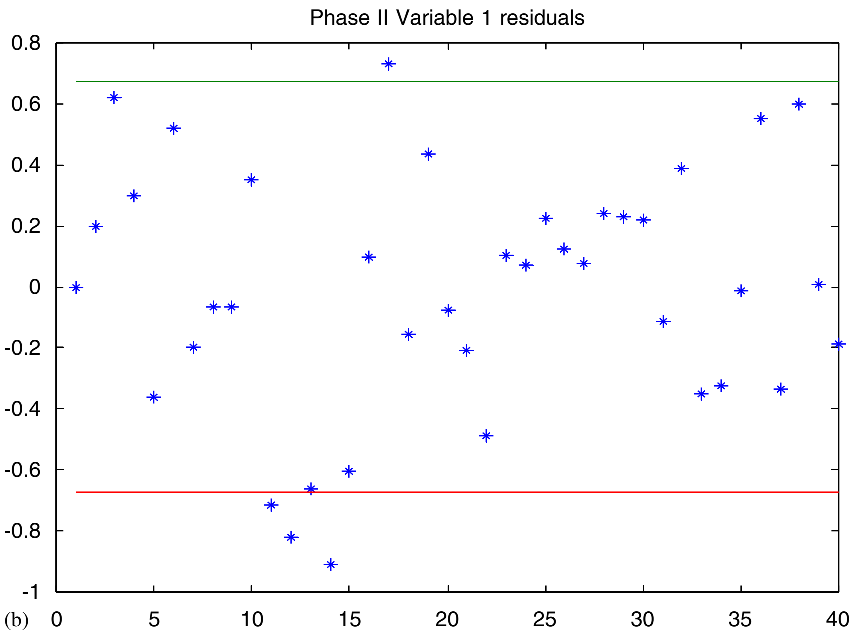
Fig. 1. (a) Phase I T^2 chart for VAR residuals; (b) Phase II T^2 chart for VAR residuals.

chose a VAR(5) as the parsimonious model. Presetting UCL at Type I error of 0.0027 on the VAR residual chart, we found no out-of-limit observations in Phase I (Fig. 1(a)). In Phase II (Fig. 1(b)) of 40 observations, we found signals at the 13th, 14th, 24th, and 30th observations. This indicates that there is something abnormal in the process.

To identify the variable(s) that mainly induce the signals on the T^2 chart, we draw univariate Shewhart charts for each of the five residual series, using the five residual series estimated from the VAR(5) model. These Shewhart charts are shown in Figs. 2–6. It can be seen that Phase I charts for these residuals are all in-control, indicating our VAR(5) estimates the system well. From Phase II Shewhart charts, we can see that the signals of #13 and #14 on Phase II Hotelling T^2 chart are due to Variable 1, and the signal of #30 on Phase II Hotelling T^2 chart



(a)

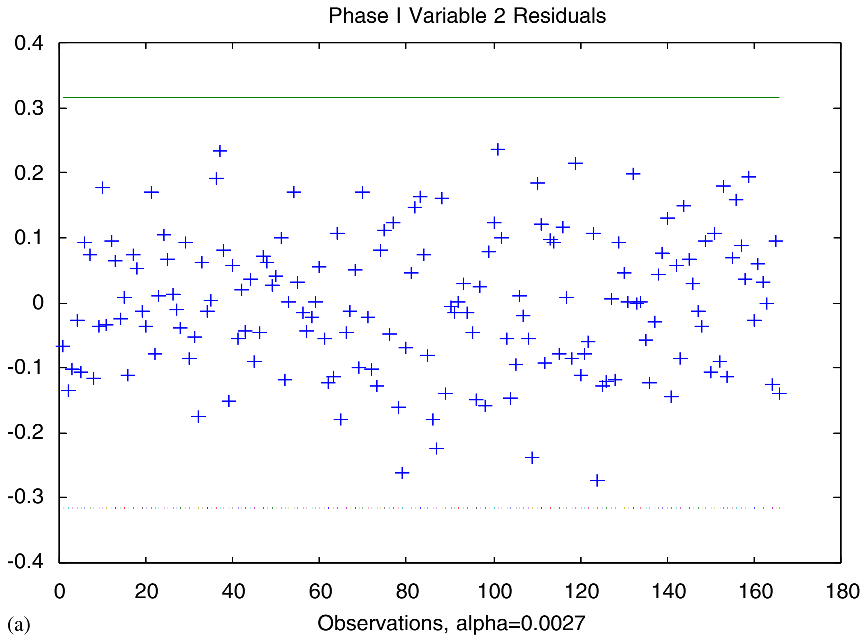


(b)

Fig. 2. (a) Phase I Shewhart chart for Variable 1 residuals; (b) Phase II Shewhart chart for Variable 1 residuals.

is due to Variable 3. However, the signal of #24 on Phase II Hotelling T^2 chart cannot be traced to a single variable. All the five variables jointly contributed to this signal, although none of the five Shewhart charts gave any signal. On the other hand, Variable 1 also signaled on other observations that did not signal on the Hotelling T^2 chart. This is due to the influence of the other variables burying the signals.

The above figures show that modeling the processes with VAR(5) provided an acceptable and appropriate approximation for the purpose of filtering out part of the systematic patterns. One of the special values of the procedure shown in above is the simplicity of the OLS estimation for the VAR model. Without the VAR modeling, the univariate variables would have had to be estimated with the much more complex maximum likelihood



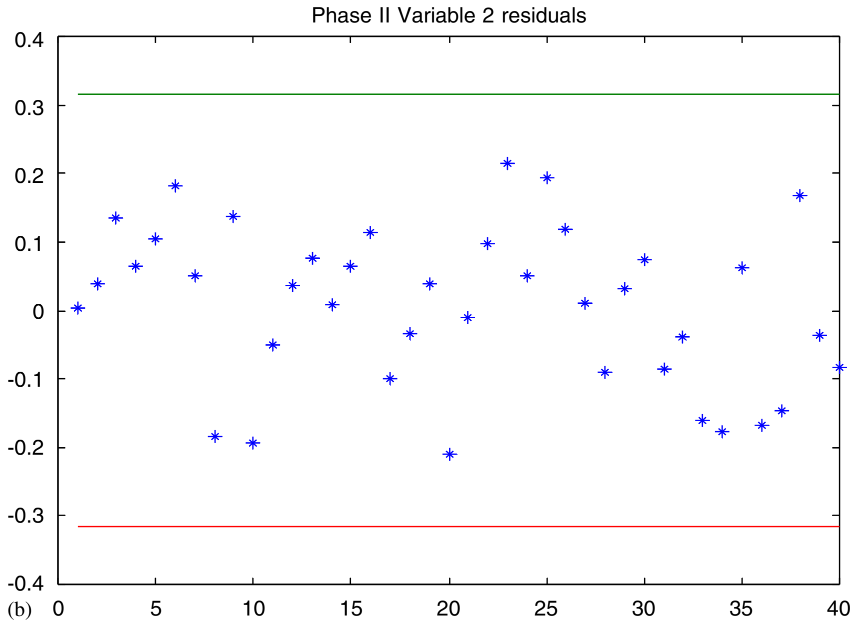


Fig. 3. (a) Phase I Shewhart chart for Variable 2 residuals; (b) Phase II Shewhart chart for Variable 2 residuals.

method—as mentioned in the earlier section, a univariate time series unlikely has a parsimonious expression with only AR terms. Also, as the VAR residuals have been already obtained, constructing the follow-up univariate Shewhart charts do not require re-estimating each of the univariate variables. Hence, our proposed procedure is practical and useful in the sense that it can provide efficient and reliable control of quality of the production. Further techniques and the detailed knowledge on the production mechanism are needed to identify what the assignable causes are.

5. Summary

A disadvantage of single variable monitoring for a single production process is that one may need to monitor and control many variables. Multivariate

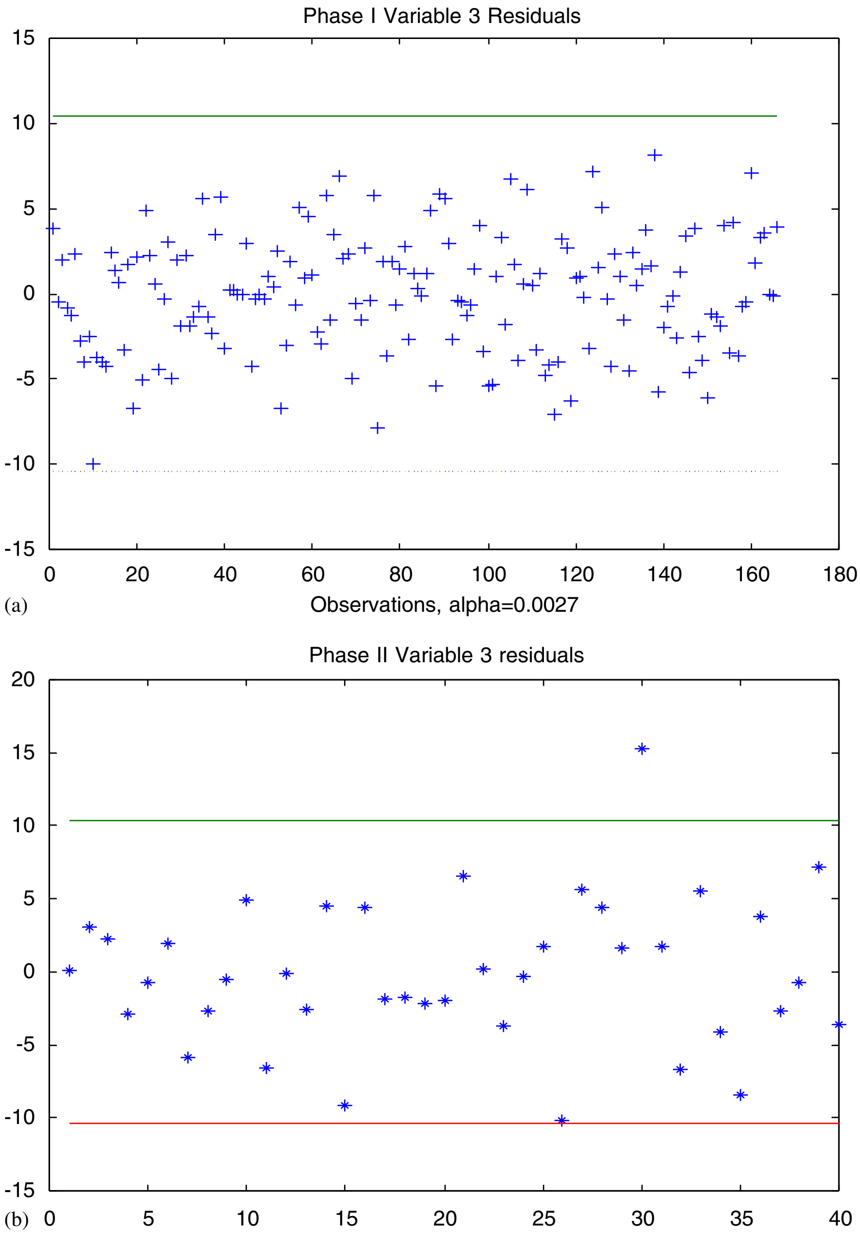


Fig. 4. (a) Phase I Shewhart chart for Variable 3 residuals; (b) Phase II Shewhart chart for Variable 3 residuals.

quality control methods overcome this disadvantage by allowing manufacturers the option to monitor several variables at the same time. However, when the multivariate processes are serially and cross-sectionally correlated, it is hard to set an appropriate control limit. VAR residual charts can overcome this problem. Knowledge of multivariate time series is necessary to set up the charts and do the appropriate estimation, computation, and chart analysis; however, the interpretation of the control charts is not much more difficult than interpreting univariate Shewhart control charts.

We observe in this study that VAR residual charts are useful in monitoring multivariate processes in the presence of serial correlation. Changes in one single parameter have effects on the whole system. The VAR residual chart is effective in that the small changes in parameters of the multivariate processes make a jointly large effect on the chart. We noted in the lemmas on the shift effects

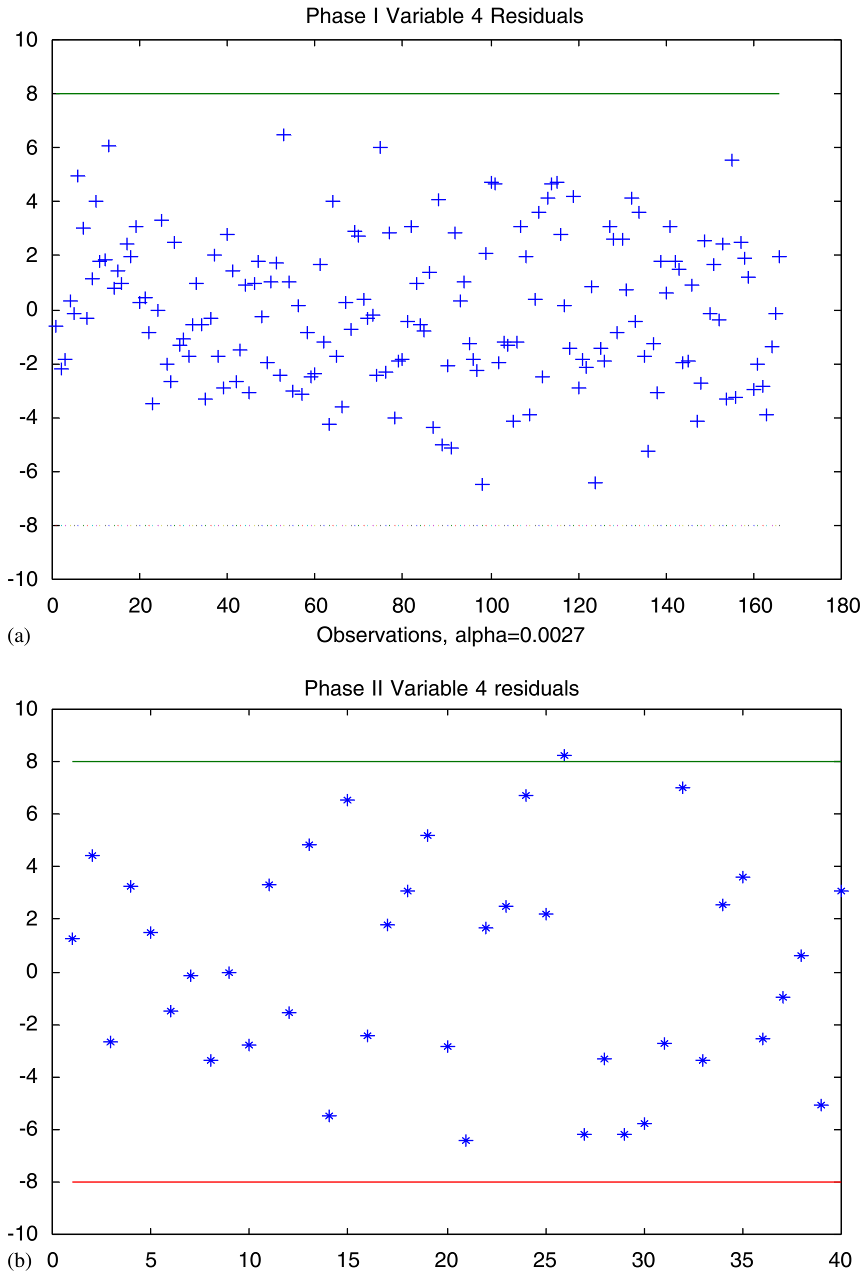


Fig. 5. (a) Phase I Shewhart chart for Variable 4 residuals; (b) Phase II Shewhart chart for Variable 4 residuals.

that changes in anyone of the three parameters can cause the VAR residual chart to signal. When the VAR residual chart detects a signal, in turn, we use univariate charts to determine at which variable the change is. As this may provide misleading information (due to the interaction between variables), we can jointly use approaches similar to Mason et al. (1995) to decompose the reason for the changes. One of the future research areas is how to distinguish the mean shift, the variability shift, and the coefficient shift, and whether we have single or joint causes for the changes in several parameters. The likelihood ratio-based method proposed in Yeh et al. (2004) can be used in monitoring the changes in multivariate variability. The CuScore scheme described in Pan (2006) can be used to detect and monitor the changes in ARMA coefficients.

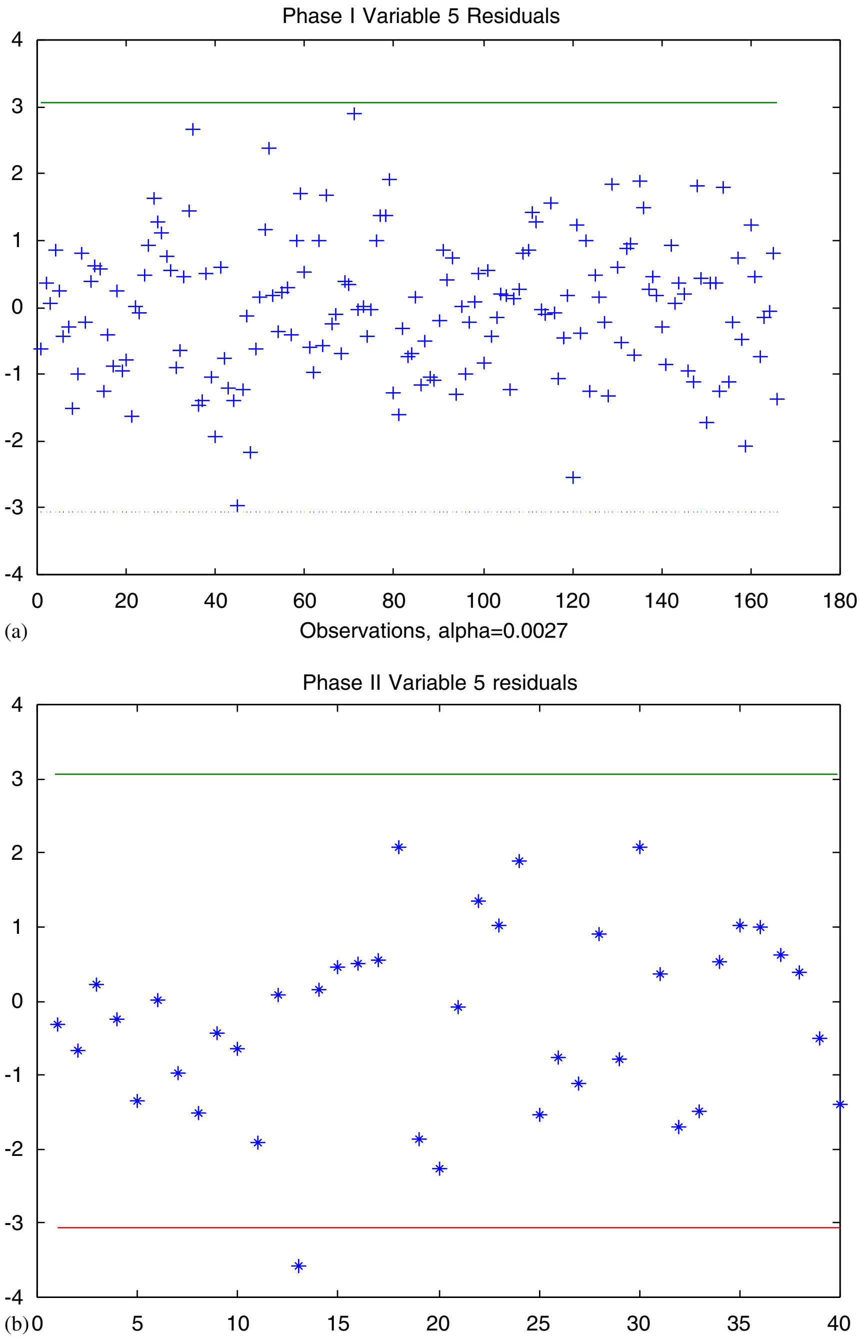


Fig. 6. (a) Phase I Shewhart chart for Variable 5 residuals; (b) Phase II Shewhart chart for Variable 5 residuals.

One of the biggest advantages of VAR residual chart is that the VAR model can be easily estimated with OLS procedures that are available in popular computer software and some spreadsheet software such as MS Excel. As VAR models more likely fit in the unknown real process than the univariate models, practitioners can avoid the complexity of using MLE. Of course, we should note that although OLS is easy to implement, we still require sufficiently large sample sizes to model the processes well and effectively. Computer-aided calculation is needed. Another promising development in chart construction is to combine the VAR modeling with the adjustments of APC suggested in Box and Kramer (1992). Nevertheless, the VAR residual chart showed its potential usefulness.

Appendix A. Proof of Lemma 1

Lemma 1. *For a* VAR(p) *process, a shift* $\vec{\eta}$ *in the process mean vectors makes the Hotelling* T^2 *on VAR residuals a non-central* χ^2*distribution,*

$$_{\text{shift}}T_t^2 = {}_{\text{noshift}\,t}T_t^2 + 2\vec{\eta}'\hat{\Omega}^{-1}\hat{\vec{e}}_t + \vec{\eta}'\hat{\Omega}^{-1}\vec{\eta}.$$

Proof. Before the shift occurrence, the process in (3) can be expressed alternatively as

$$(I_n - \Phi_1 L - \cdots - \Phi_p L^p)(\vec{y}_t - \vec{\mu}) = \vec{e}_t, \tag{A.1}$$

where I_n is an $n \times n$ matrix and $\vec{\mu} = (I_n - \Phi_1 - \cdots - \Phi_p)^{-1}\vec{c}$ is the process mean vector. After a shift of $\vec{\eta}$ in the process mean occurs at time t_0, the process mean becomes$\vec{\mu} + \vec{\eta}$, hence, the process status becomes $\vec{z}_t$, and

$$(I_n - \Phi_1 L - \cdots - \Phi_p L^p)(\vec{z}_t - (\vec{\mu} + \vec{\eta})) = \vec{e}_t. \tag{A.2}$$

Here $\vec{z}_t$ is just the denotation of the observations and it is $\vec{y}_t$ before the shift. The one-step-ahead forecasting error without the shift occurrence is $\hat{\vec{e}}_t = \vec{y}_t - \hat{\vec{y}}_t$, which can be also expressed as

$$\begin{aligned}\hat{\vec{e}}_t &= \vec{y}_t - \hat{\vec{c}} - (\hat{\Phi}_1 L + \cdots + \hat{\Phi} L^p)\vec{y}_t \\ &= (I_n - \hat{\Phi}_1 L - \cdots - \hat{\Phi}_p L^p)(\vec{y}_t - \hat{\vec{\mu}}),\end{aligned} \tag{A.3}$$

where $\hat{\vec{\mu}} = (I_n - \hat{\Phi}_1 - \cdots - \hat{\Phi}_p)^{-1}\hat{\vec{c}}$. After the shift, from (A.2) this same error term can be also expressed as

$$\hat{\vec{e}}_t = (I_n - \hat{\Phi}_1 L - \cdots - \hat{\Phi}_p L^p)(\vec{z}_t - (\hat{\vec{\mu}} + \vec{\eta})). \tag{A.4}$$

However, one still bases the forecasting on the estimated coefficients $\hat{\Pi}' = (\hat{\vec{c}}\,\hat{\Phi})$, since the shift is unknown before the detection. Thus, similarly

$$\hat{\vec{z}}_t = \hat{\vec{c}} + (\hat{\Phi}_1 L + \cdots + \hat{\Phi}_p L^p)\vec{z}_t \ \text{ when } t \geqslant t_0 + p. \tag{A.4$'$}$$

Then the one-step-ahead forecasting error after the shift is

$$\begin{aligned}\hat{\vec{\varepsilon}}_t &= \vec{z}_t - \hat{\vec{z}}_t = \vec{z}_t - \hat{\vec{c}} - (\hat{\Phi}_1 L + \cdots + \hat{\Phi} L^p)\vec{z}_t \\ &= (I_n - \hat{\Phi}_1 L - \cdots - \hat{\Phi}_p L^p)(\vec{z}_t - \hat{\vec{\mu}}) \\ &= \hat{\vec{e}}_t + \vec{\eta}.\end{aligned} \tag{A.5}$$

With the shifted $\hat{\vec{\varepsilon}}_t = \hat{\vec{e}}_t + \hat{\vec{\eta}}$, the Hotelling T^2 statistic is a non-central χ^2distribution:

$$\begin{aligned}{}_{\text{shift}}T_t^2 &= \hat{\vec{\varepsilon}}_t'\hat{\Omega}^{-1}\hat{\vec{\varepsilon}}_t = (\hat{\vec{e}}_t + \vec{\eta})'\hat{\Omega}^{-1}(\hat{\vec{e}}_t + \vec{\eta}) \\ &= \hat{\vec{e}}_t'\hat{\Omega}^{-1}\hat{\vec{e}}_t + 2\vec{\eta}'\hat{\Omega}^{-1}\hat{\vec{e}}_t + \vec{\eta}'\hat{\Omega}^{-1}\vec{\eta} \\ &= {}_{\text{noshift}}T_t^2 + 2\vec{\eta}'\hat{\Omega}^{-1}\hat{\vec{e}}_t + \vec{\eta}'\hat{\Omega}^{-1}\vec{\eta}.\end{aligned} \tag{A.6}$$

This is a positive shift on the mean of T^2 and on the variance of T^2. □

Appendix B. Proof of Lemma 2

Lemma 2. *If the process covariance* Ω *shifts to* Ω_1, *then the effect on the VAR chart is* ${}_{\text{shift}}T_t^2 = \hat{\vec{e}}_t' D'^{-1}\hat{\Omega}^{-1}D^{-1}\hat{\vec{e}}_t$.

Proof. If the process covariance Ω shifts to Ω_1, then since the covariance matrices are always positive definite symmetric matrices, there must exist a non-singular matrix D so that $\Omega_1 = D\Omega D'$. This can be easily seen from the decomposition $\Omega = P\Lambda P'$ and $\Omega_1 = P_1\Lambda_1 P'_1$, where $\Lambda_1 = C^{1/2}\Lambda C^{1/2}$, $\Lambda^{1/2} = C^{-1/2}\Lambda_1^{1/2}$, $C^{1/2} = \Lambda_1^{1/2}\Lambda^{-1/2}$ are all positive diagonal matrices, and $I = PP' = P_1 P_1{}'$. We have, $D = P_1 C^{1/2} P'$ and $D' = PC^{1/2}P_1{}'$.

From Cholesky factorization $\Omega = (P\Lambda^{1/2})(\Lambda^{1/2}P')$, $\Omega_1 = (P_1\Lambda_1^{1/2})(\Lambda_1^{1/2}P_1{}')$, with estimated in-control covariance matrix $\hat{\Omega} = (\hat{P}\hat{\Lambda}^{1/2})(\hat{\Lambda}^{1/2}\hat{P}')$, we have the residuals before the covariance shift, $\hat{\vec{e}}_t = (P\Lambda^{1/2})\vec{u}_t$. Moreover, after the covariance shifts, we express the residuals as $\hat{\vec{\varepsilon}}_t = (P_1\Lambda_1^{1/2})\vec{u}_t$, where $\vec{u}_t \sim \mathrm{N}(\vec{0}, I)$ is a normalized sequence.

Without knowing the covariance shift, we base the Hotelling T^2 statistic on the in-control variance–covariance matrix estimate and the updated one-step-ahead forecasting errors. the effect of the error term variance–covariance shift on VAR chart is (note $I = \Lambda^{-1/2}P'P\Lambda^{1/2}$),

$$\begin{aligned}{}_{\text{shift}}T_t^2 &= \hat{\vec{\varepsilon}}_t'\hat{\Omega}^{-1}\hat{\vec{\varepsilon}}_t = \vec{u}_t'\Lambda_1^{1/2}P_1\Omega^{-1}P_1\Lambda_1^{1/2}\vec{u}_t \\ &= \vec{u}_t'\Lambda^{1/2}P'P\Lambda^{-1/2}\Lambda_1^{1/2}P_1\Omega^{-1}P_1\Lambda_1^{1/2}\Lambda^{-1/2}P'P\Lambda^{1/2}\vec{u}_t \\ &= \hat{\vec{e}}_t'D'\hat{\Omega}^{-1}D\hat{\vec{e}}_t,\end{aligned} \tag{B.1}$$

where $D = P_1\Lambda_1^{1/2}\Lambda^{-1/2}P' = P_1 C^{1/2}P'$ and $\hat{\vec{\varepsilon}}_t = D\hat{\vec{e}}_t$. Hence, the out-of-control statistic is Hotelling-style constructed from the amplified residuals and the original variance–covariance matrix. □

Appendix C. Proof of Lemma 3

Lemma 3. *For a* VAR(p) *process, if the coefficients shift from* Φ *to* $\Phi+\Delta\Phi$ *the effect on the Hotelling* T^2 *for the residuals is*

$$ {}_{\text{shift}}T_t^2 = {}_{\text{noshift}}T_t^2 + 2\sum_{i=1}^{p}(\Delta\Phi_i(\vec{y}_t - \vec{\mu}))'\Omega^{-1}\vec{e}_t + \sum_{i=1}^{p}\left(\sum_{j=1}^{p}(\Delta\Phi_i(\vec{y}_t - \vec{\mu}))'\Omega^{-1}\Delta\Phi_j(\vec{y}_t - \vec{\mu})\right). $$

Proof. From (3), the one-step-ahead forecasting based on non-shifted coefficients is

$$ \hat{\vec{y}}_t = \vec{c} + (\Phi_1 L + \Phi_2 L^2 + \cdots + \Phi_p L^p)\vec{y}_t. \tag{C.1} $$

Since the means of the processes do not shift, and noticing $\vec{c} = (I_n - \Phi_1 - \cdots - \Phi_p)\vec{\mu}$, then the processes after the coefficient shift is

$$ \vec{y}_t = (1 - \Phi - \Delta\Phi)\vec{\mu} + (\Phi_1 L + \Delta\Phi_1 L + \Phi_2 L^2 + \Delta\Phi_2 L^2 \cdots + \Phi_p L^p + \Delta\Phi_p L^p)\vec{y}_t + \vec{e}_t \tag{C.2} $$

or

$$ (I - (\Phi_1 + \Delta\Phi_1)L - \cdots - (\Phi_p + \Delta\Phi_p)L^p)(\vec{y}_t - \vec{\mu}) = \vec{e}_t. $$

Hence, the residuals after the coefficient shift are

$$ \begin{aligned} \hat{\vec{e}}_t &= \vec{y}_t - \hat{\vec{y}}_t = \vec{e}_t + (\Delta\Phi_1 L + \cdots + \Delta\Phi_p L^p)\vec{y}_t - \Delta\Phi\vec{\mu} \\ &= \vec{e}_t + (\Delta\Phi_1 L + \cdots + \Delta\Phi_p L^p)(\vec{y}_t - \vec{\mu}). \end{aligned} \tag{C.3} $$

Therefore, the Hotelling T^2 statistic will be

$$ \begin{aligned} {}_{\text{shift}}T_t^2 &= \hat{\vec{e}}_t{}'\Omega^{-1}\hat{\vec{e}}_t \\ &= {}_{\text{noshift}}T_t^2 + 2\sum_{i=1}^{p}(\Delta\Phi_i L^i(\vec{y}_t - \vec{\mu}))'\Omega^{-1}\vec{e}_t \\ &\quad + \sum_{i=1}^{p}\left(\sum_{j=1}^{p}(\Delta\Phi_i L^i(\vec{y}_t - \vec{\mu}))'\Omega^{-1}\Delta\Phi_j L^i(\vec{y}_t - \vec{\mu})\right) \\ &= {}_{\text{noshift}}T_t^2 + 2\sum_{j}^{p}\left(\sum_{i=1}^{p}(\Delta\Phi_i L^i(\vec{y}_t - \vec{\mu}))'\Omega^{-1} \right. \\ &\quad \left. \times (I - (\Phi_j + \Delta\Phi_j)L^j)(\vec{y}_t - \vec{\mu})\right) \\ &\quad + \sum_{i=1}^{p}\left(\sum_{j=1}^{p}(\Delta\Phi_i L^i(\vec{y}_t - \vec{\mu}))'\Omega^{-1}\Delta\Phi_j L^j(\vec{y}_t - \vec{\mu})\right). \end{aligned} \tag{C.4} $$

Both the second and the third items include p^2 terms each. All these terms are composed of the lagged periods. Therefore, the ${}_{\text{shift}}T_t^2$ is serially correlated and affected by the process status of the past p periods. □

References

Akaike, H., 1976. Canonical correlations analysis of time series and the use of an information criterion. In: Mehra, R., Lainiotis, D.G. (Eds.), System Identification: Advances and Case Studies. Academic Press, New York.

Alt, F.B., 1984. Multivariate quality control. The Encyclopedia of Statistical Sciences, 110–122.

Alwan, L.C., 1992. Effects of autocorrelation on control charts. Communications in Satistics—Theory and Methods 21, 1025–1049.

Alwan, L.C., Roberts, H.V., 1988. Time-series modeling for detecting level shifts of autocorrelated processes. Journal of Business and Economics Statistics 6, 87–96.

Aparisi, F., Haro, C.L., 2003. A Comparison of T^2 control charts with variable sampling schemes as opposed to MEWMA chart. International Journal of Production Economics 41, 2169–2182.

Box, G.E.P., Kramer, T., 1992. Statistical process monitoring and feedback adjustment—A discussion. Technometrics 34, 251–285.

Dale, B.G., Elkjaer, M.B.F., van der Wiele, A., Williams, A.R.T., 2001. Fad, fashion and fit: An examination of quality circles, business process re-engineering and statistical process control. International Journal of Production Economics 73, 137–152.

English, J.R., Krishnamurthi, M., Sastri, T., 1991. Quality monitoring of continuous flow processes. Computers and Industrial Engineering 20, 251–260.

Ermer, D.S., Chow, M.C., Wu, S.M., 1979. A time series control chart for a nuclear reactor. In: in: Proceedings of the 1979 Annual Reliability and Maintainability Symposium. Institute of Electrical and Electronic Engineers, New York.

Hannan, E.J., Quinn, B., 1979. Determination of the order of an autoregression. Journal of The Royal Statistical Society Series B-Methodological 41, 190–191.

Harris, T.J., Ross, W.H., 1991. Statistical process control procedures for correlated observations. Canadian Journal of Chemical Engineering 69, 48–57.

Holmes, D.S., Mergen, A.E., 2005. Effect of multivariate process on principal component analysis: A case study. The AAPS Journal 7, 106–117.

Hotelling, H., 1947. Multivariate quality control. In: Eisenhart, C., Hastay, M.W., Wallis, W.A. (Eds.), Techniques of Statistical Analysis. McGraw-Hill, New York.

Jackson, J.E., 1985. Multivariate quality control. Communications in Statistics—Theory and Methods 14, 2657–2688.

Jarrett, J.E., Pan, X., 2006. The quality control chart for monitoring multivariate autocorrelated processes. Computational Statistics and Data Analysis, in press, Corrected Proof, Available online 30 March 2006.

Jiang, W., Tsui, K.L., Woodall, W.H., 2000. A new SPC monitoring method: The ARMA chart. Technometrics 42, 399–410.

Kramer, H.G., Schmid, W., 1997. EWMA charts for multivariate time series. Sequential Analysis 16, 131–154.

Lowry, C.A., Woodall, W., Champ, C.W., Rigdon, S.E., 1992. A multivariate exponentially weighted moving average control chart. Technometrics 34, 46–53.

Lowry, C.A., Montgomery, D.C., 1995. A review of multivariate control charts. IIE Transactions 27, 800–810.

Lu, C.W., Reynolds, M.R., 1999. Control charts for monitoring the mean and variance of autocorrelated processes. Journal of Quality Technology 31, 259–274.

Mason, R.L., Tracy, N.D., Young, J.C., 1995. Decomposition of T^2 for multivariate control charts interpretation. Journal of Quality Technology 27, 99–108.

Montgomery, D.C., Mastrangelo, C.M., 1991. Some statistical process control methods for autocorrelated data. Journal of Quality Technology 23, 179–193.

Pan, X., 2006. A simple method using cuscore to monitor and detect coefficient changes in ARMA residuals. Communications in Statistics—Simulation and Computation 35 (2).

Pan, X., Jarrett, J., 2004. Applying state space to SPC: Monitoring multivariate time series. Journal of Applied Statistics 31, 397–418.

Shiau, Y.R., 2003. Inspection allocation planning for a multiple quality characteristic advanced manufacturing system. Advanced Manufacturing Technology 21, 494–500.

Tagaras, G., 1998. A Survey of recent developments in the design of adaptive control charts. Journal of Quality Technology 30, 212–231.

Testik, M.C., 2005. Model inadequacy and residuals control charts for autocorrelated processes. Quality and Reliability Engineering International 21, 115–130.

Villalobos, R.J., Muñoz, L., Gutierrez, M.A., 2005. Using fixed and adaptive multivariate SPC charts for online SMD assembly monitoring. International Journal of Production Economics 95, 109–121.

Wardell, D., Moskowitz, H., Plante, R., 1992. Control charts in presence of data correlation. Management Science 38, 1084–1105.

West, D., Delana, S., Jarrett, J., 2002. Transfer function modeling of processes with dynamic inputs. Journal of Quality Technology 34, 315–321.

Wierda, S.J., 1994. Multivariate statistical process control: recent results and directions for future researches. Stastica Neerlandica 48, 147–168.

Yang, S.F., Rahim, M.A., 2005. Economic statistical process control for multivariate quality characteristics under Weibull shock model. International Journal of Production Economics 98, 215–226.

Yeh, A.B., Hwang, L., Wu, Y., 2004. A likelihood-ratio-based EWMA chart for monitoring variability of multivariate normal processes. IIE Transactions 36, 865–879.