Geometric complexes and topological persistence

A panoramic view

Ulrich Bauer

Technical University of Munich (TUM)

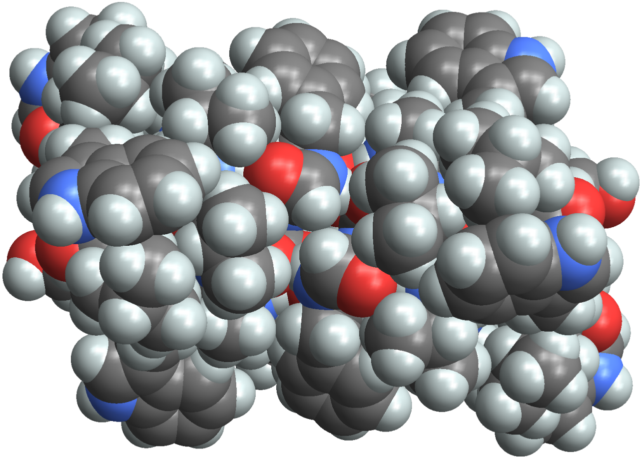
Aug 1, 2025

TSVP Thematic Program
TDA PARTI: Topological Data Analysis, Persistence And Representation Theory Intertwined

OIS] OIST, Okinawa

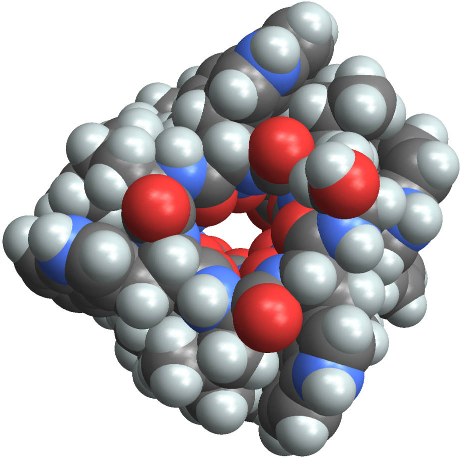
Funded by
Deutsche Forschungsgemeinschaft
German Research Foundation
SFB TRR 109 Discretization in Geometry and Dynamics
Technical University of Munich
Munich Center for Machine Learning

Biogeometry

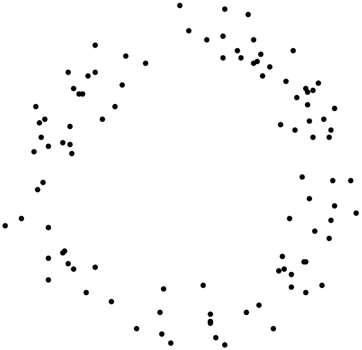


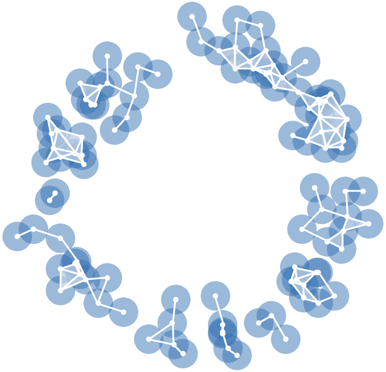
Gramicidin (an antibiotic functioning as an ion channel)

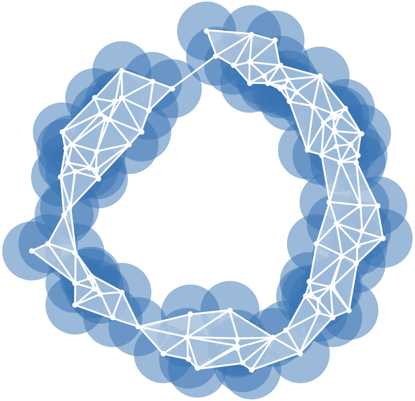
Biogeometry

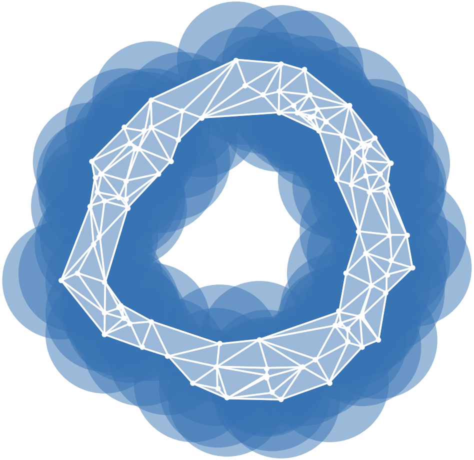


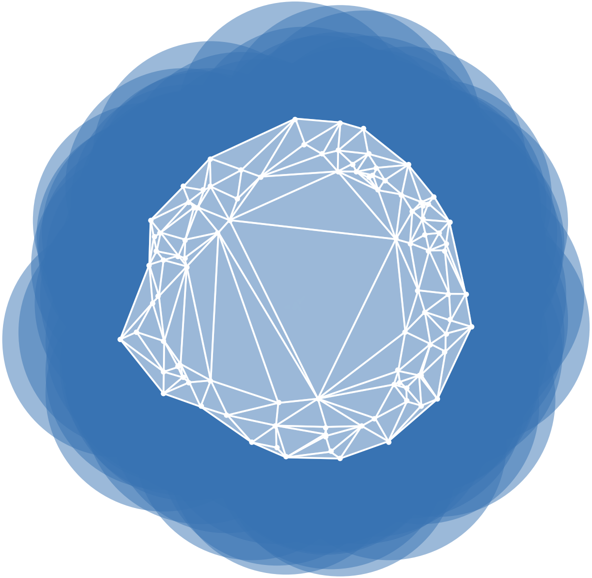
Gramicidin (an antibiotic functioning as an ion channel)





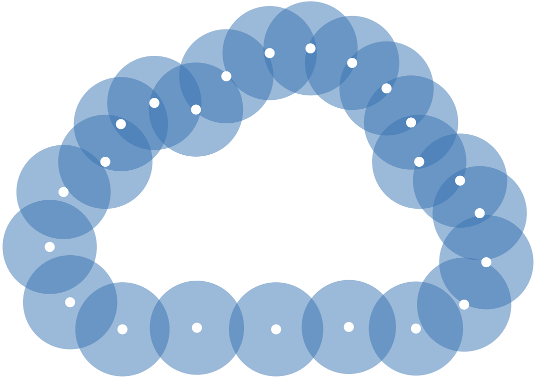




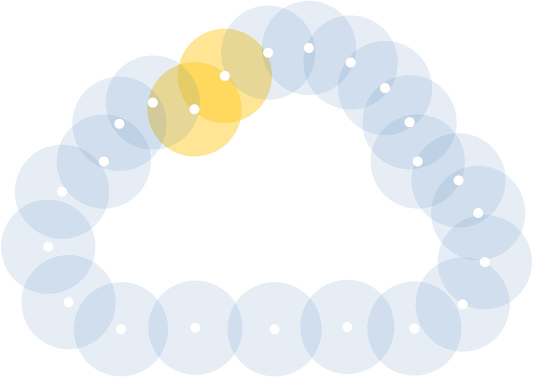


Geometric complexes

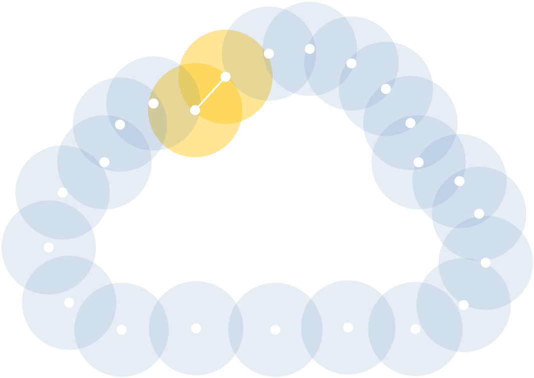
Čech complexes



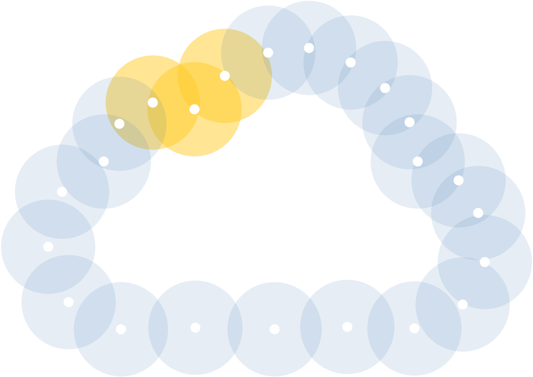
Čech complexes



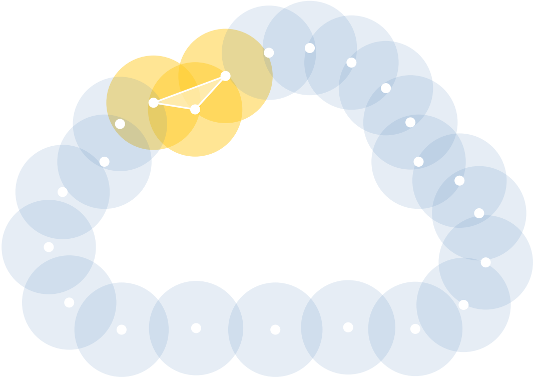
Čech complexes



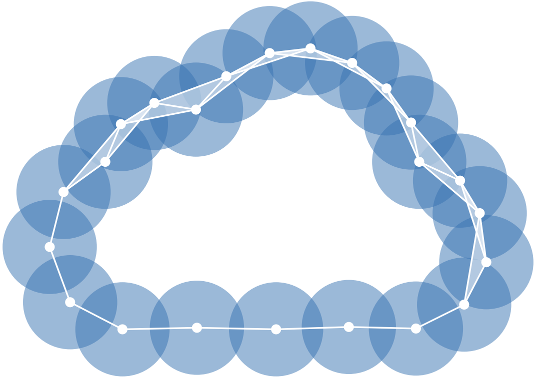
Čech complexes



Čech complexes



Čech complexes



Nerves

Definition (Alexandrov 1928)

Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\operatorname{Nrv}(\mathcal{U}) = \{J \subseteq I \mid 0 < |J| < \infty \text{ and } \bigcap_{j \in J} U_j \neq \varnothing\}.$$

Nerves

Definition (Alexandrov 1928)

Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid 0 < |J| < \infty \text{ and } \bigcap_{j \in J} U_j \neq \varnothing\}.$$

Theorem (Borsuk 1948, and many more)

Let $\mathcal{U}$ *be a* good cover *of* X. *Then* $\mathrm{Nrv}(\mathcal{U})$ *is homotopy equivalent to* X.

Nerves

Definition (Alexandrov 1928)

Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid 0 < |J| < \infty \text{ and } \bigcap_{j \in J} U_j \neq \varnothing\}.$$

Theorem (Borsuk 1948, and many more)

Let $\mathcal{U}$ be a good cover *of X. Then $\mathrm{Nrv}(\mathcal{U})$ is homotopy equivalent to X.*

Here *good* can mean different things:

- open, numerable cover, contractible intersections
- closed, convex finite cover

Nerves

Definition (Alexandrov 1928)

Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid 0 < |J| < \infty \text{ and } \bigcap_{j \in J} U_j \neq \varnothing\}.$$

Theorem (Borsuk 1948, and many more)

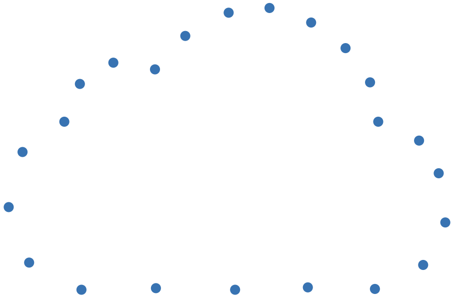
Let $\mathcal{U}$ *be a* good cover *of* X. *Then* $\mathrm{Nrv}(\mathcal{U})$ *is homotopy equivalent to* X.

Here *good* can mean different things:

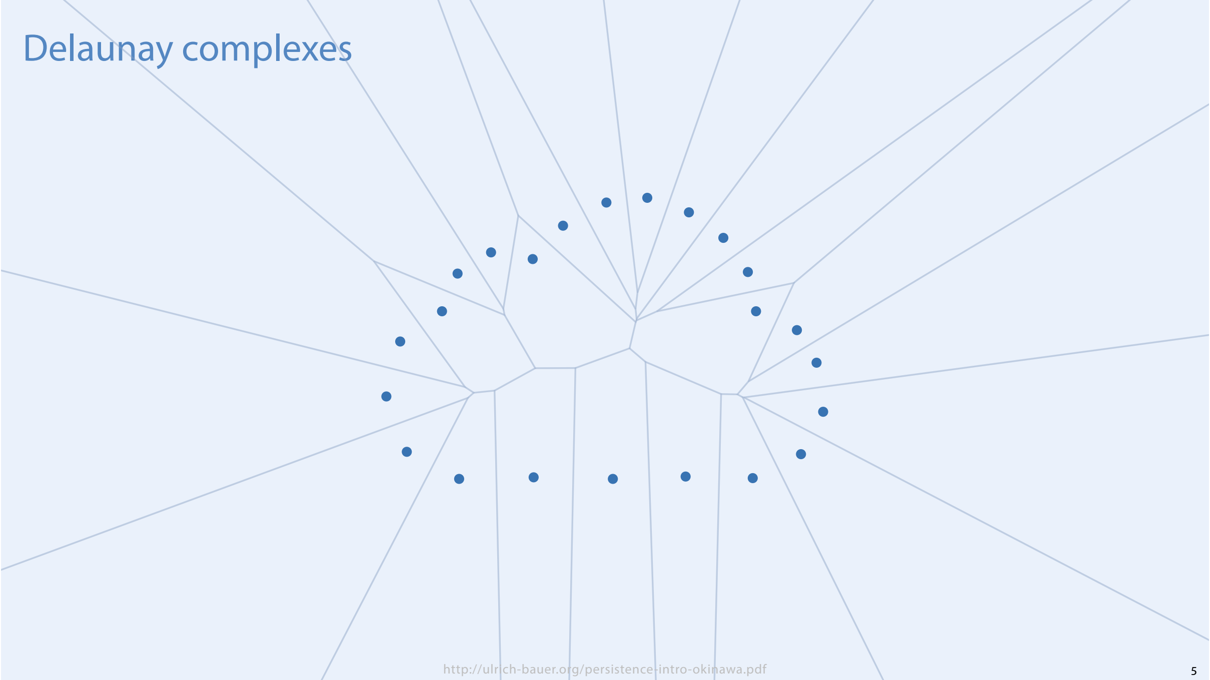
- open, numerable cover, contractible intersections
- closed, convex finite cover

U. Bauer, M. Kerber, F. Roll, and A. Rolle
A Unified View on the Functorial Nerve Theorem and its Variations
Expositiones Mathematicae, 2023. doi:10.1016/j.exmath.2023.04.005

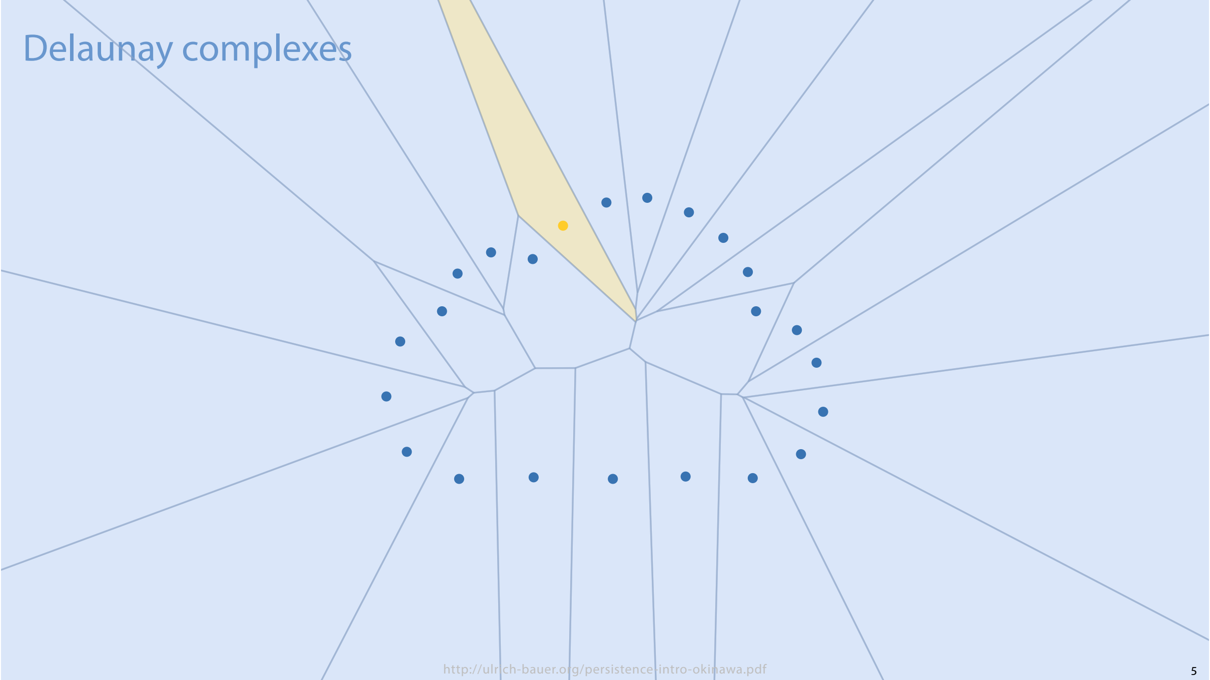
Delaunay complexes



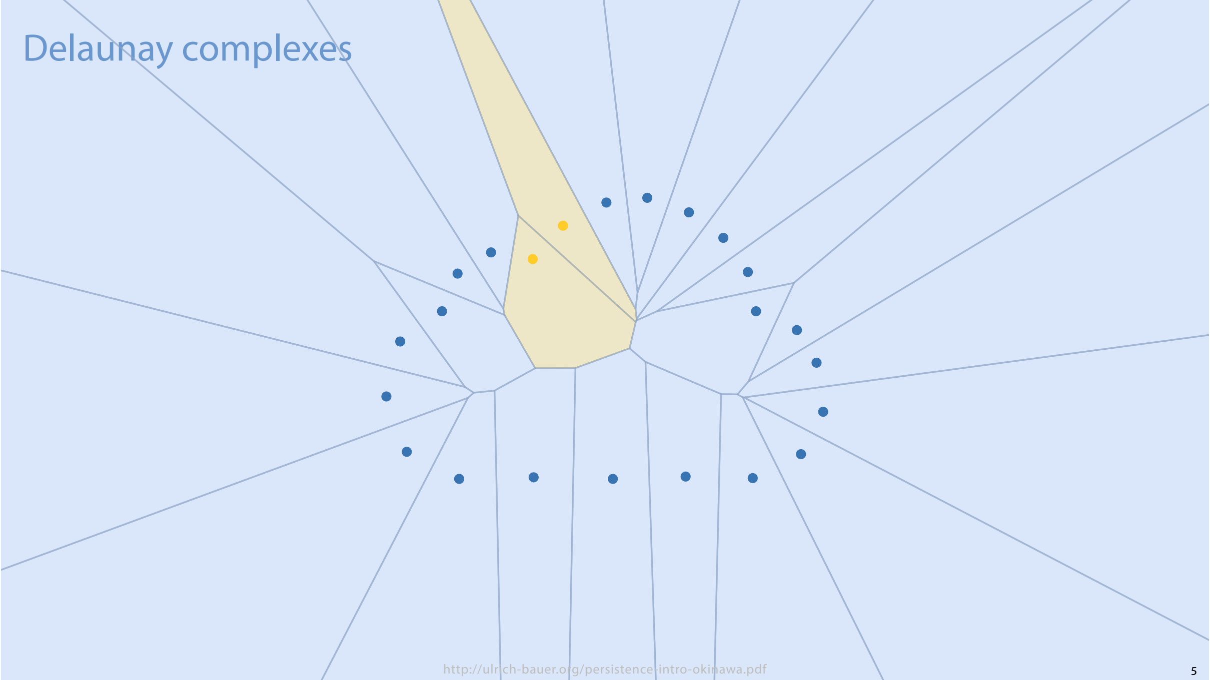
Delaunay complexes



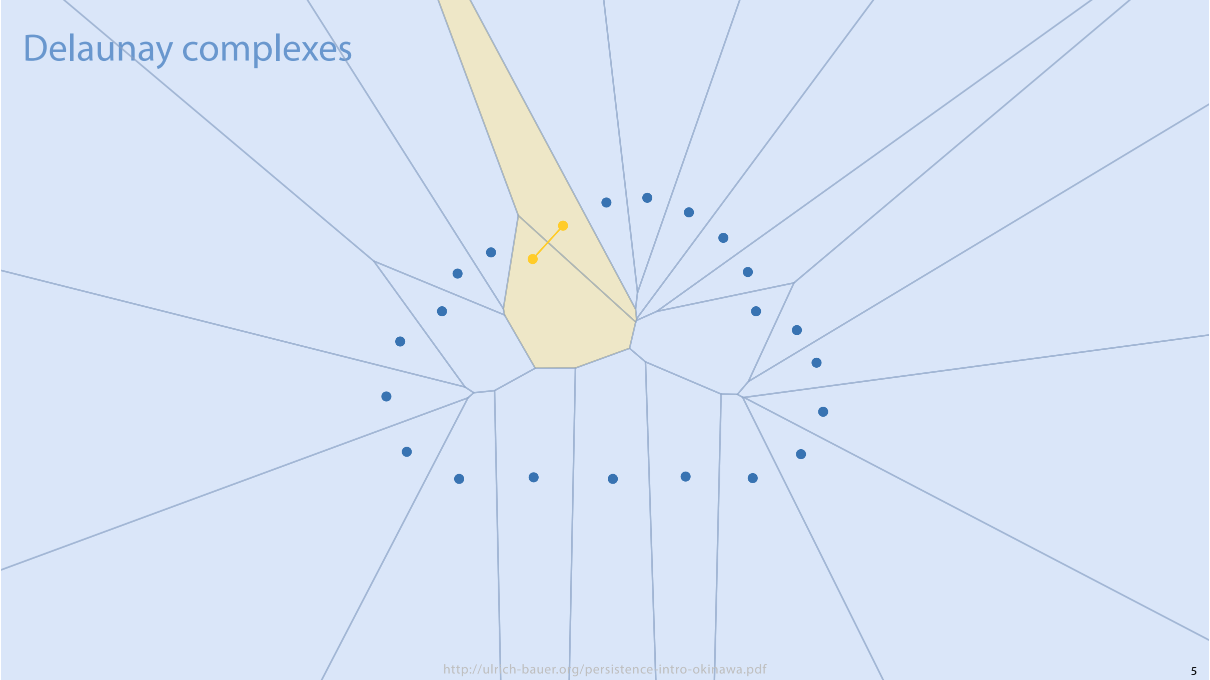
Delaunay complexes



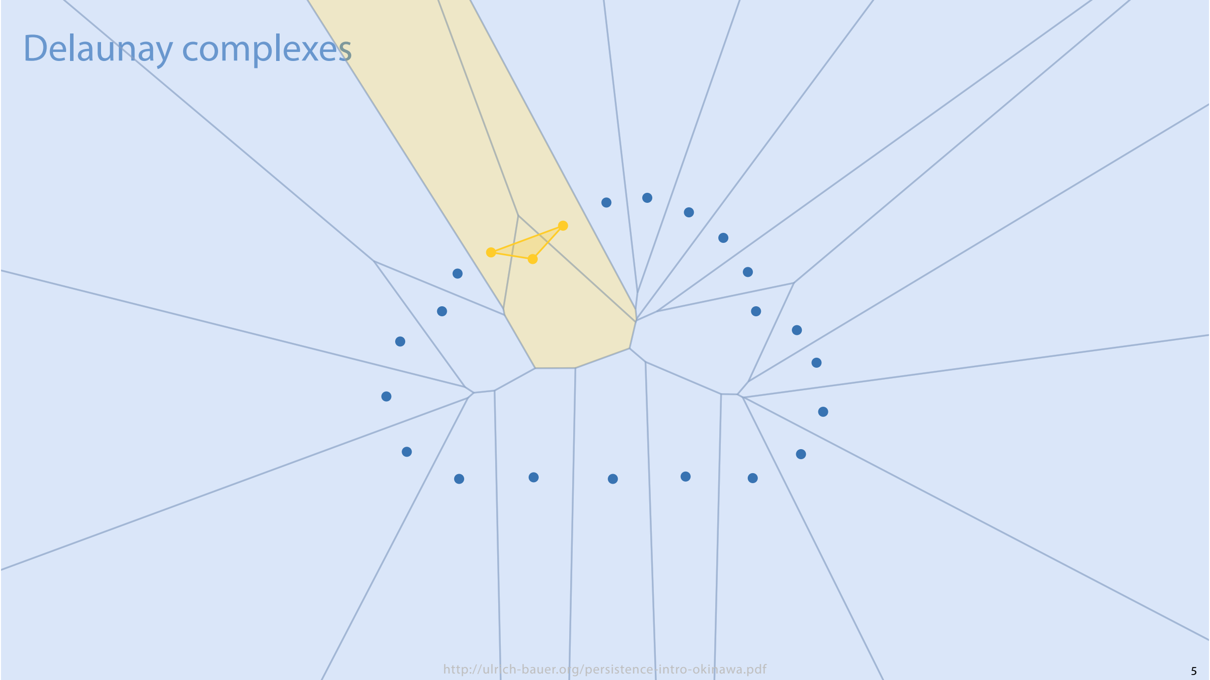
Delaunay complexes



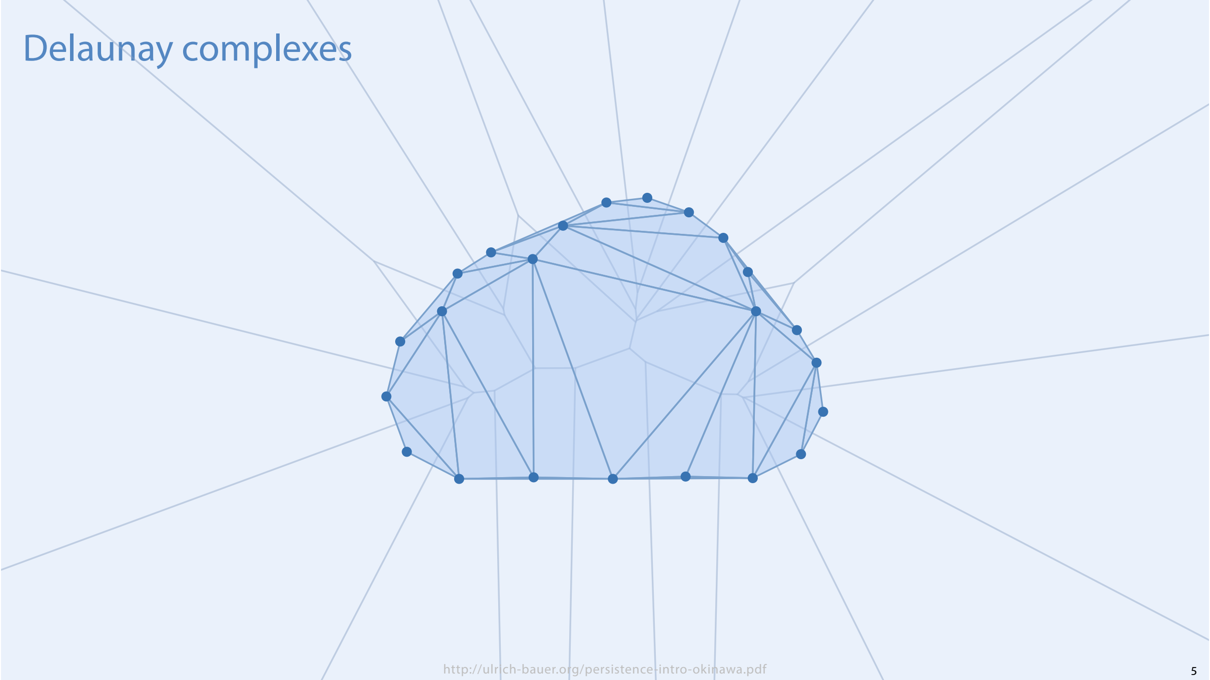
Delaunay complexes



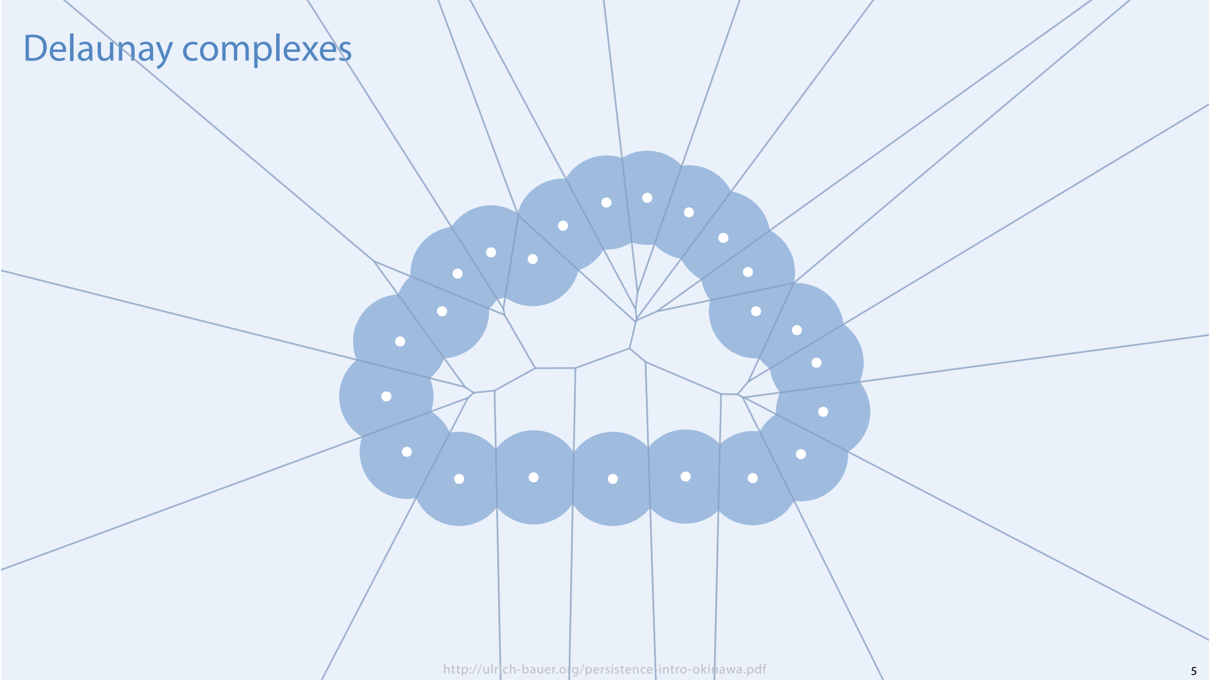
Delaunay complexes



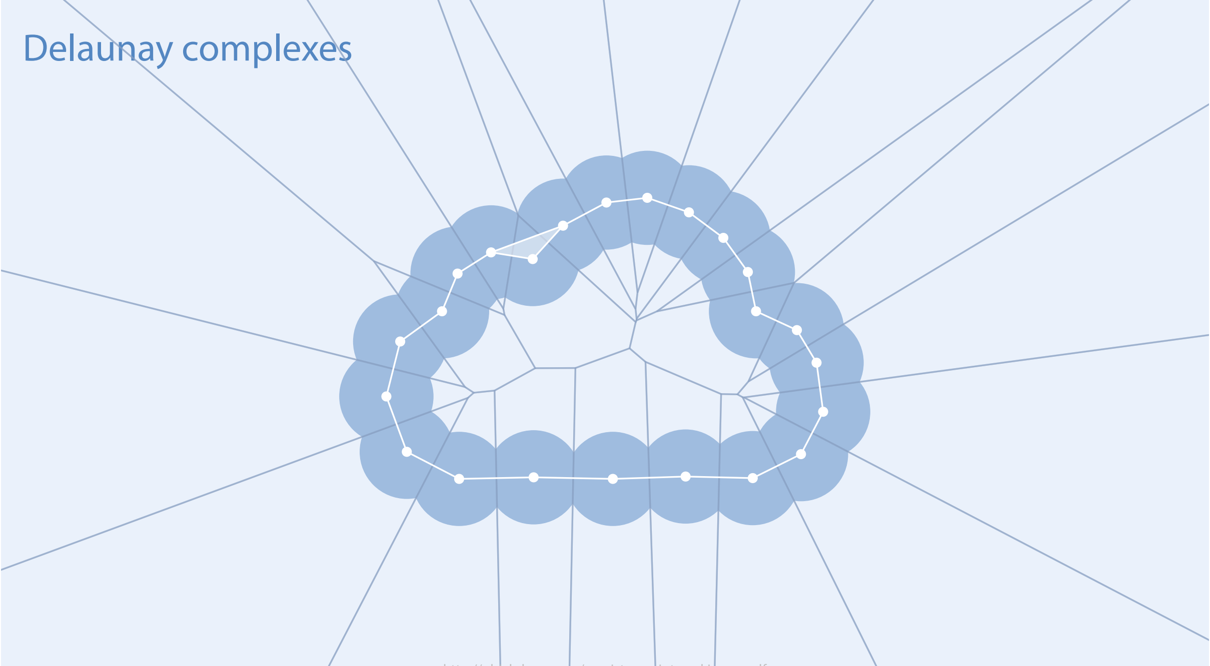
Delaunay complexes



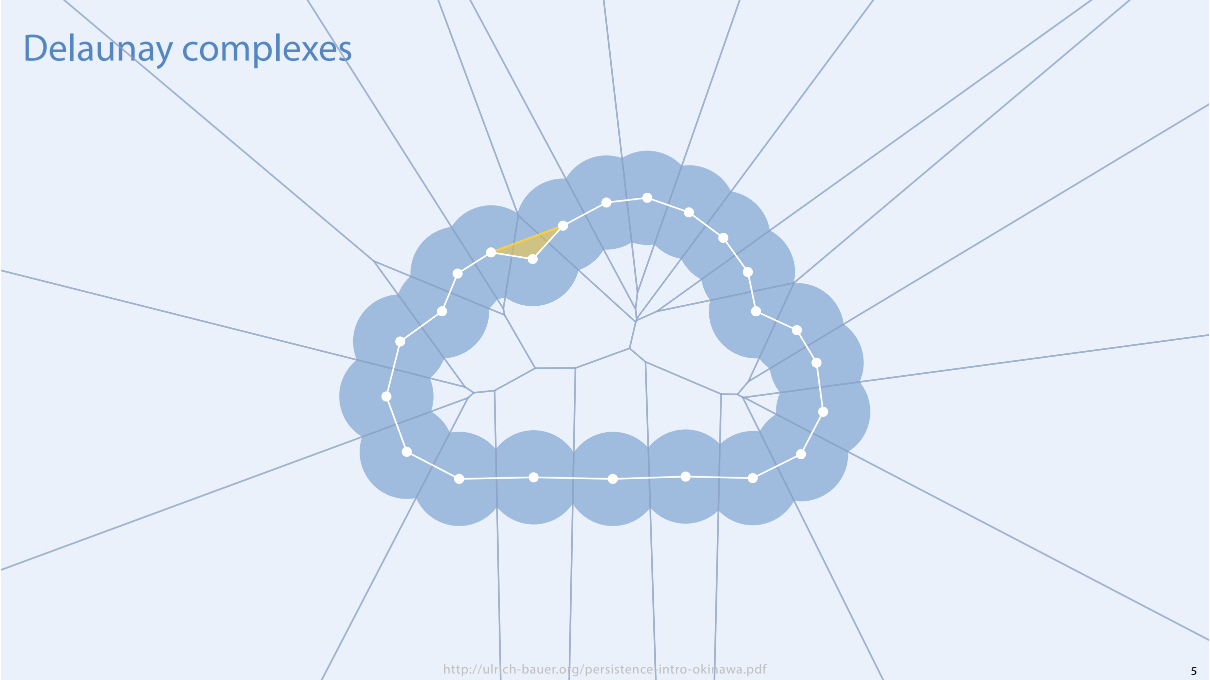
Delaunay complexes



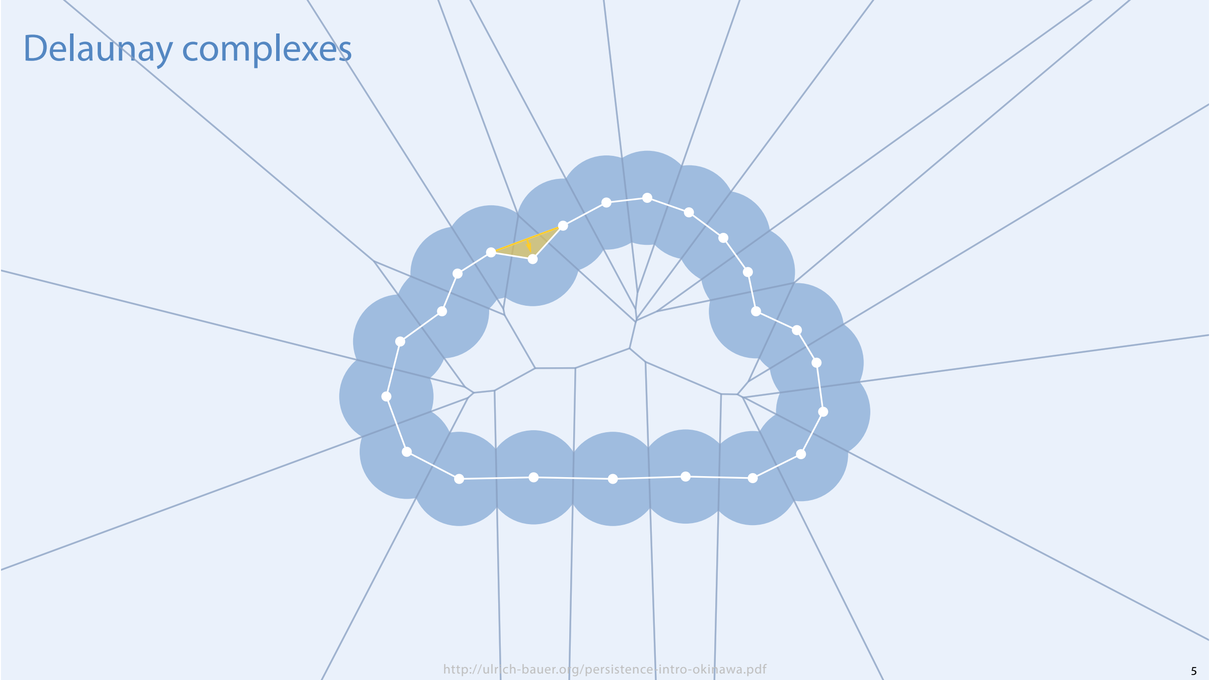
Delaunay complexes



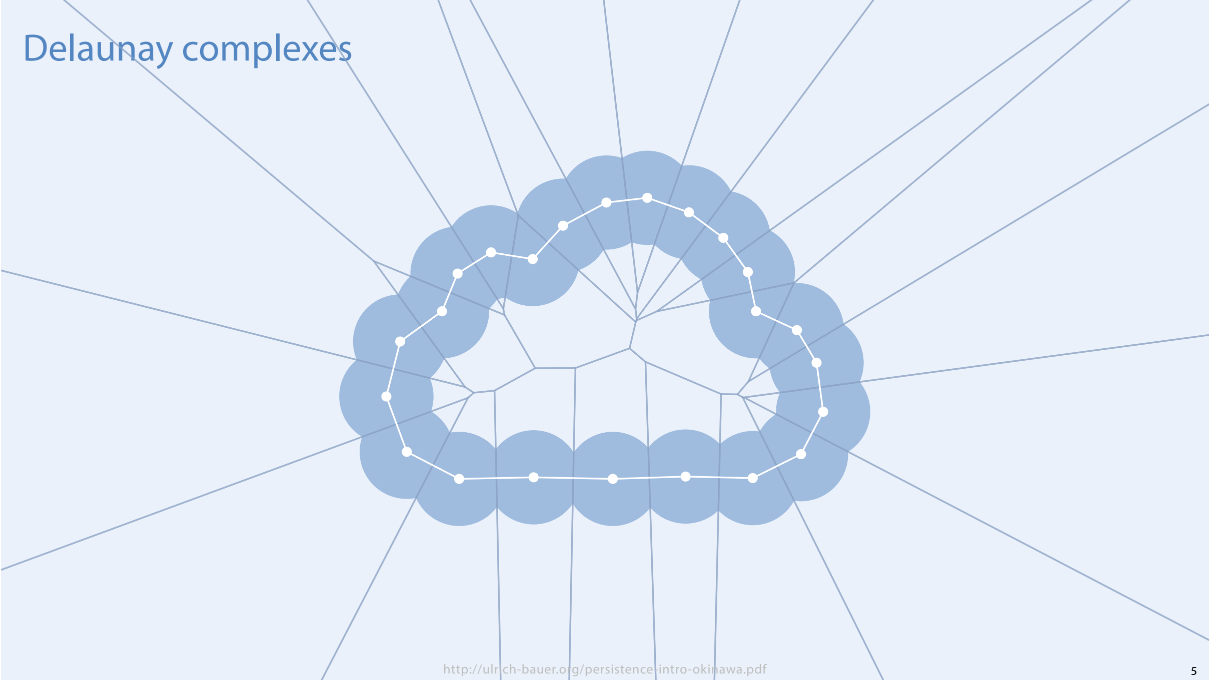
Delaunay complexes



Delaunay complexes



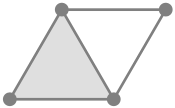
Delaunay complexes



Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

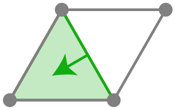


Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

- A *free face* of K is a simplex that has a unique proper coface.

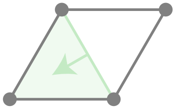


Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

- A *free face* of K is a simplex that has a unique proper coface.
- We can *collapse* K to a subcomplex L:

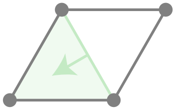


Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

- A *free face* of K is a simplex that has a unique proper coface.
- We can *collapse* K to a subcomplex L:
  - remove a free face and its proper coface.

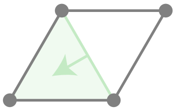


Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

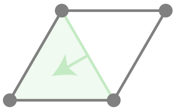
- A *free face* of K is a simplex that has a unique proper coface.
- We can *collapse* K to a subcomplex L:
  - remove a free face and its proper coface.
- L is homotopy equivalent to K.



Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.



- A *free face* of K is a simplex that has a unique proper coface.
- We can *collapse* K to a subcomplex L:
  - remove a free face and its proper coface.
- L is homotopy equivalent to K.

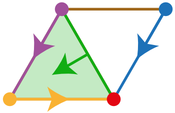
If there is a sequence of such elementary collapses from K to M, we say that K *collapses* to M (written as $K \searrow M$).

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.

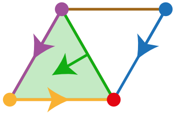


Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

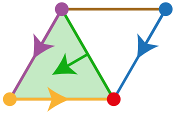
- the sublevel sets $f^{-1}(-\infty, t]$ are subcomplexes, and

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

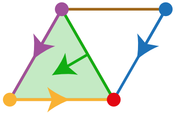
- the sublevel sets $f^{-1}(-\infty, t]$ are subcomplexes, and

Discrete Morse theory

Definition (Forman 1998)

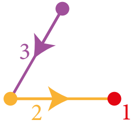
A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

- the sublevel sets $f^{-1}(-\infty, t]$ are subcomplexes, and

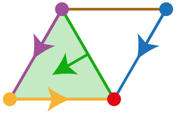


Discrete Morse theory

Definition (Forman 1998)

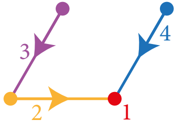
A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

- the sublevel sets $f^{-1}(-\infty, t]$ are subcomplexes, and

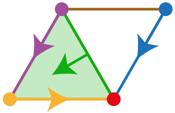


Discrete Morse theory

Definition (Forman 1998)

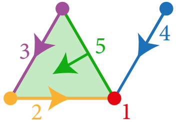
A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

- the sublevel sets $f^{-1}(-\infty, t]$ are subcomplexes, and

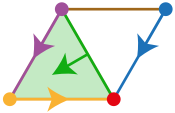


Discrete Morse theory

Definition (Forman 1998)

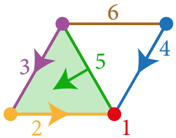
A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

- the sublevel sets $f^{-1}(-\infty, t]$ are subcomplexes, and

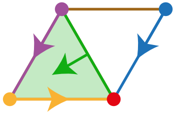


Discrete Morse theory

Definition (Forman 1998)

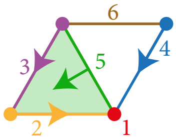
A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.

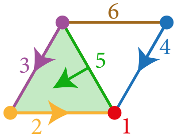


A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

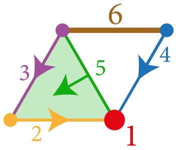
- the sublevel sets $f^{-1}(-\infty, t]$ are subcomplexes, and
- the level sets $f^{-1}(t)$ form a discrete vector field.



Fundamental theorem of discrete Morse theory



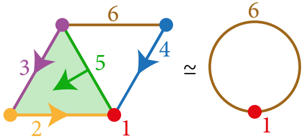
Fundamental theorem of discrete Morse theory



Fundamental theorem of discrete Morse theory

Theorem (Forman 1998)

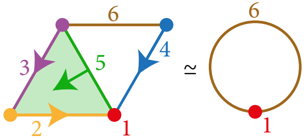
A simplicial complex with a discrete Morse function f is homotopy equivalent to a space (a CW complex*) built from the critical simplices of f.*



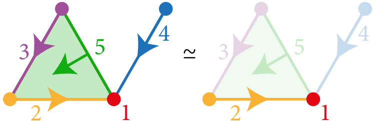
Fundamental theorem of discrete Morse theory

Theorem (Forman 1998)

A simplicial complex with a discrete Morse function f is homotopy equivalent to a space (a CW complex*) built from the critical simplices of f.*



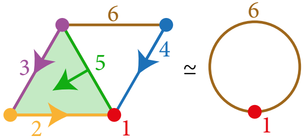
Discrete Morse functions – and their gradients – encode *collapses* of sublevel sets:



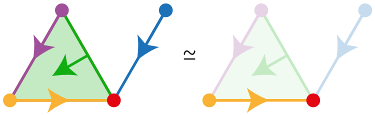
Fundamental theorem of discrete Morse theory

Theorem (Forman 1998)

A simplicial complex with a discrete Morse function f is homotopy equivalent to a space (a CW complex*) built from the critical simplices of f.*



Discrete Morse functions – and their gradients – encode *collapses* of sublevel sets:

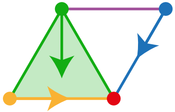


Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex K is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}$$



- indicated by an arrow from L to U

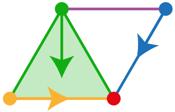
Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex K is a partition of the simplices into intervals of the face poset:

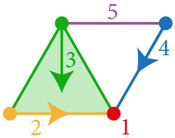
$$[L, U] = \{Q : L \subseteq Q \subseteq U\}$$

- indicated by an arrow from L to U



A *generalized discrete Morse function* $f : K \to \mathbb{R}$ satisfies:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes (for all $t \in \mathbb{R}$)
- the level sets $f^{-1}(t)$ form a generalized vector field (the *discrete gradient* of f)



Morse theory for Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

The Čech complexes and the Delaunay complexes (alpha shapes) are sublevel sets of (generalized) discrete Morse functions.

Morse theory for Čech and Delaunay complexes

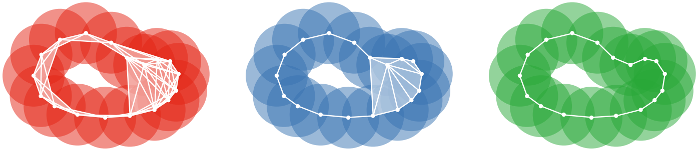
Proposition (B, Edelsbrunner 2014)

The Čech complexes and the Delaunay complexes (alpha shapes) are sublevel sets of (generalized) discrete Morse functions.

Theorem (B, Edelsbrunner 2017)

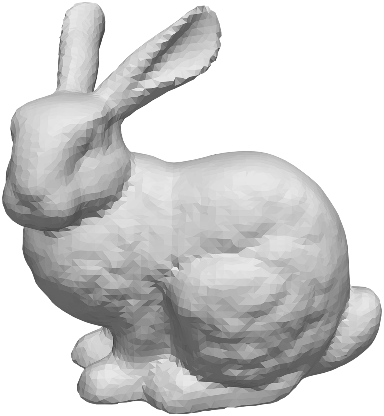
Čech, Delaunay, and Wrap complexes (at any scale r) of a point set $X \subset \mathbb{R}^d$ in general position are related by collapses encoded by a single discrete gradient field:

$$\mathrm{Cech}_r X \searrow \mathrm{Del}_r X \searrow \mathrm{Wrap}_r X.$$



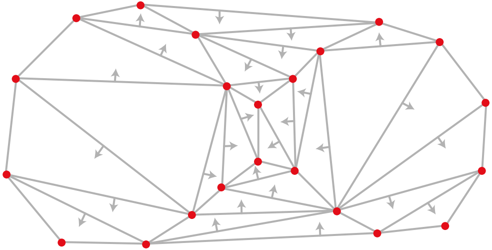
Wrap complexes

Foundation of the surface reconstruction software *Wrap* (Edelsbrunner 1995, Geomagic)



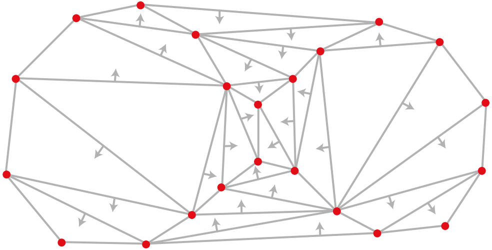
Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



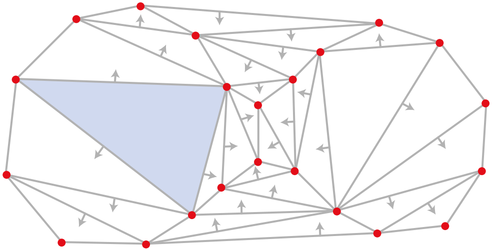
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



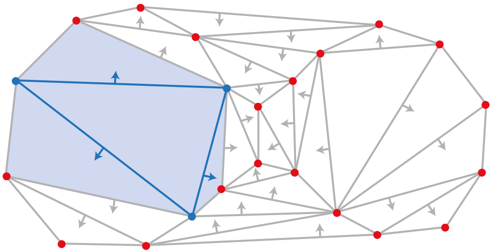
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



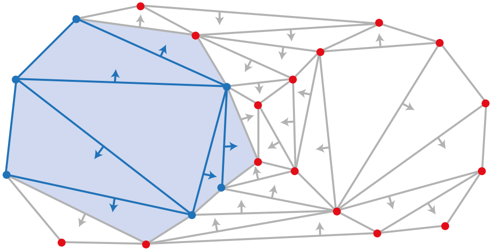
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



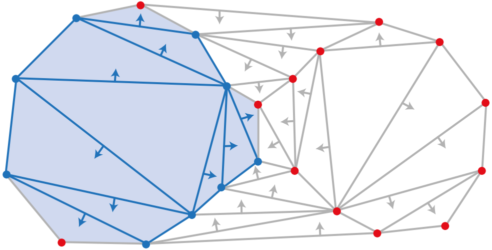
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



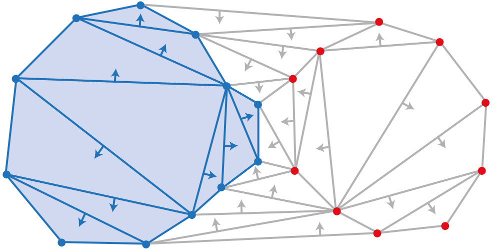
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



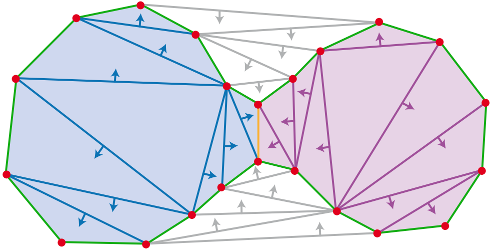
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r\, X$:

- the smallest subcomplex of $\mathrm{Del}_r\, X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



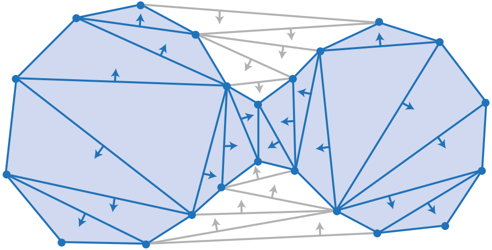
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r\, X$:

- the smallest subcomplex of $\mathrm{Del}_r\, X$ that
    - contains all critical simplices
    - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

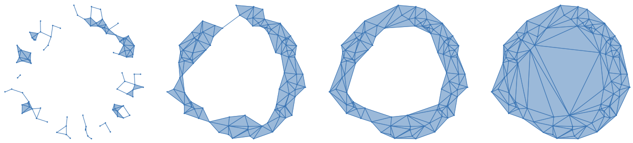
- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Delaunay and Wrap complexes

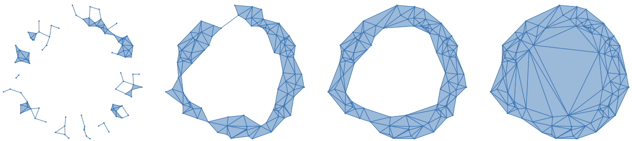
Delaunay and Wrap complexes

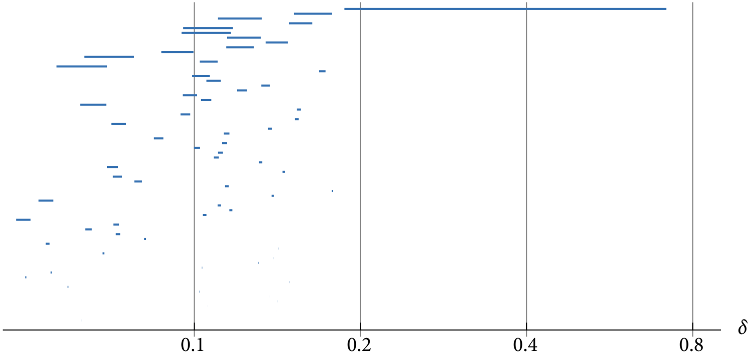
Persistent homology

Persistence barcodes

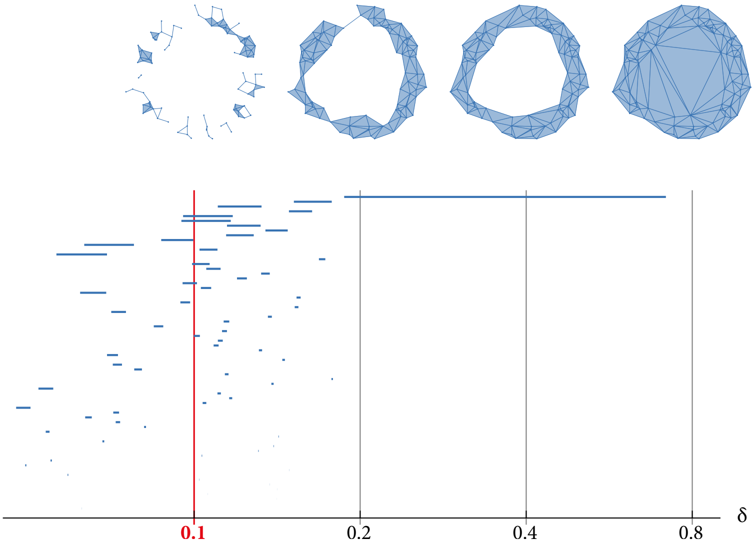


Persistence barcodes



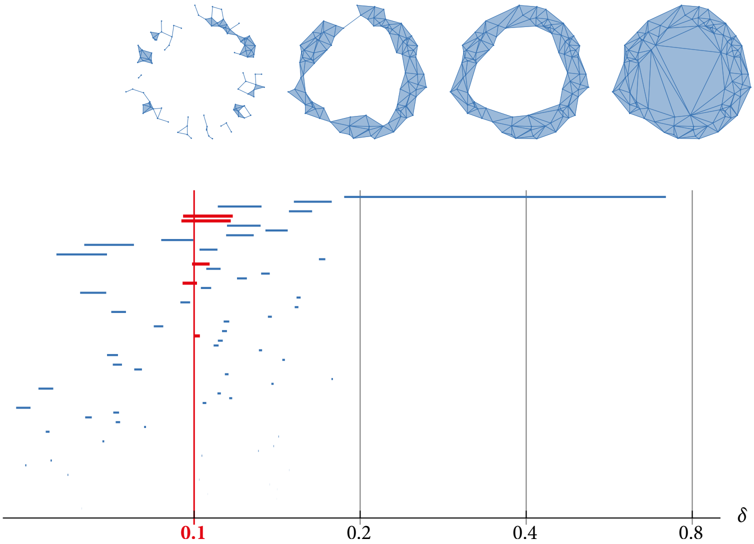


Persistence barcodes

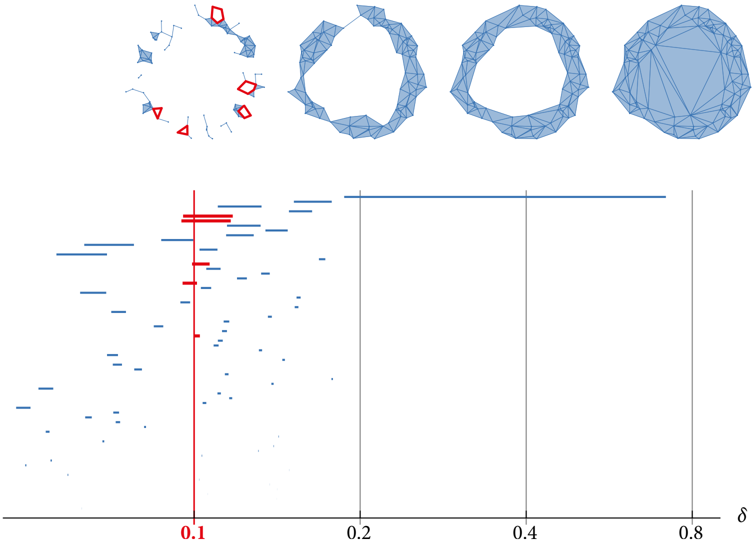


http://ulrich-bauer.org/persistence-intro-okinawa.pdf

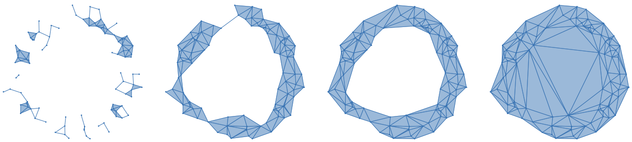
Persistence barcodes

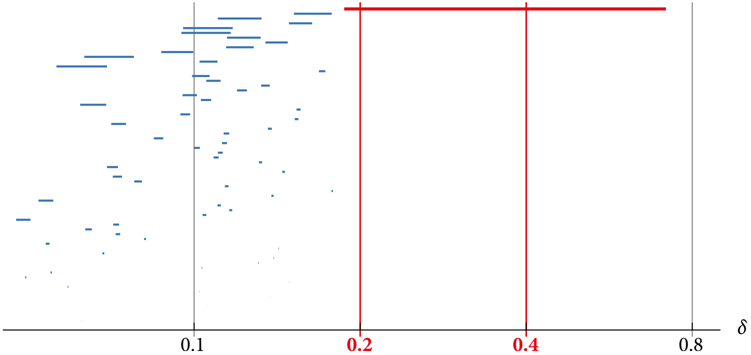


Persistence barcodes

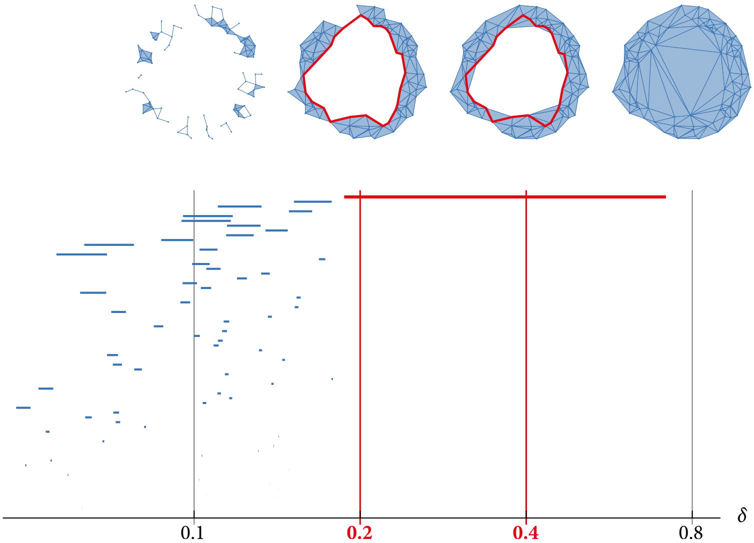


Persistence barcodes





Persistence barcodes



Computing persistent homology via matrix reduction

Algorithm (matrix reduction; a variant of Gauss elimination)

Require: D: $m \times n$ matrix

Ensure: V is full rank upper triangular, $R = D \cdot V$ has unique column pivots

function Reduce(D)

  $R = D$

  $V = I(n)$

  while there exist $i < j$ such that $\operatorname{pivot} R_i = \operatorname{pivot} R_j$ **do**

    add column R_i to column R_j $\triangleright$ eliminate the nonzero entry in row $\operatorname{pivot} R_i$

    add column V_i to column V_j

  return R, V

Computing persistent homology via matrix reduction

Algorithm (exhaustive matrix reduction; a variant of Gauss–Jordan elimination)

Require: D: $m \times n$ matrix

Ensure: V is full rank upper triangular, $R = D \cdot V$ has unique column pivots

function Reduce(D)

$R = D$

$V = I(n)$

while there exist $i < j$ such that R_j has a nonzero entry in row $\mathrm{pivot}\, R_i$ **do**

add column R_i to column R_j ▷ eliminate the nonzero entry in row $\mathrm{pivot}\, R_i$

add column V_i to column V_j

return R, V

Computing persistent homology via matrix reduction

Algorithm (exhaustive matrix reduction; a variant of Gauss–Jordan elimination)

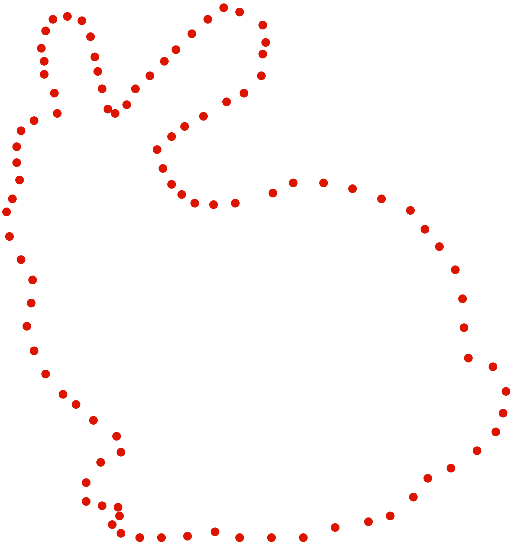
Require: D: $m \times n$ matrix
Ensure: V is full rank upper triangular, $R = D \cdot V$ has unique column pivots

function Reduce(D)
  $R = D$
  $V = I(n)$
  while there exist $i < j$ such that R_j has a nonzero entry in row $\operatorname{pivot} R_i$ **do**
    add column R_i to column R_j ▷ eliminate the nonzero entry in row $\operatorname{pivot} R_i$
    add column V_i to column V_j
  return R, V

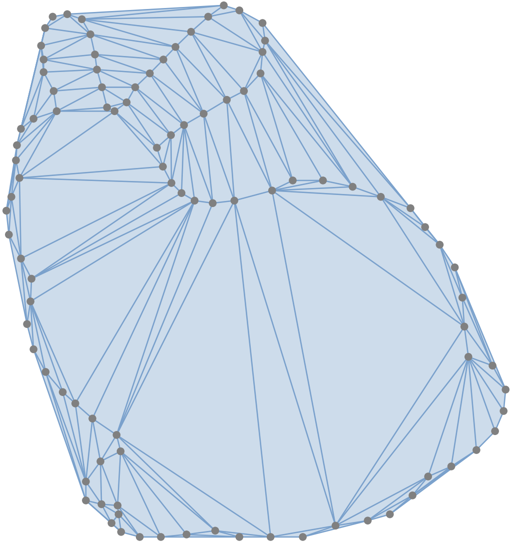
Proposition

The resulting columns R_j are minimal (in a lexicographic order) within their homology class (in K_{j-1}).

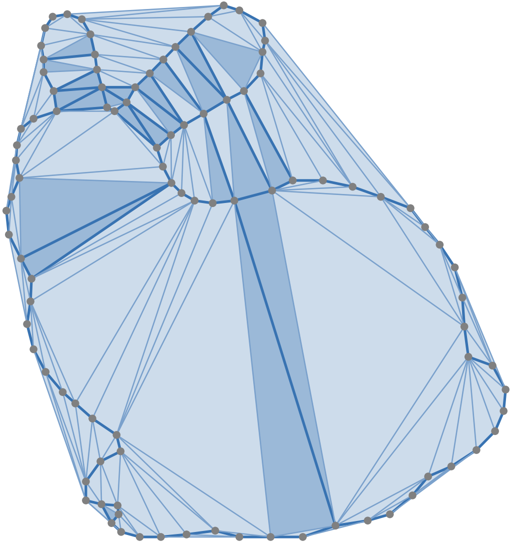
Wrap complexes and lexicographically minimal cycles



Wrap complexes and lexicographically minimal cycles

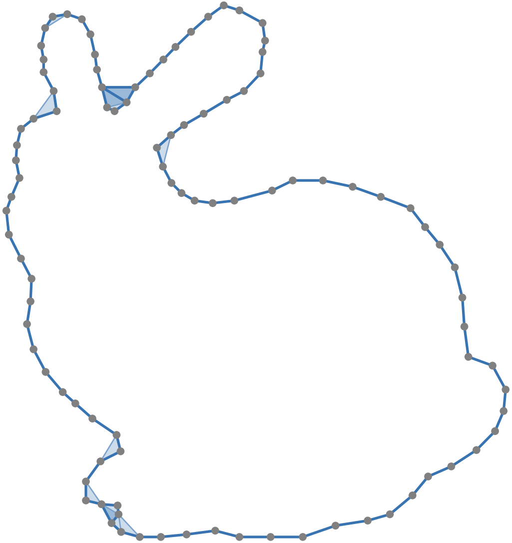


Wrap complexes and lexicographically minimal cycles

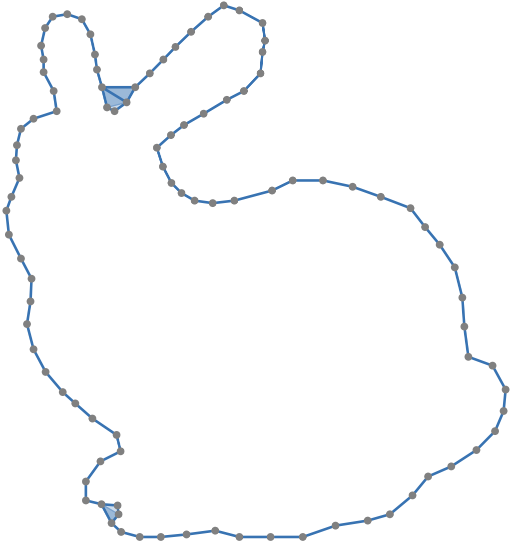


http://ulrich-bauer.org/persistence-intro-okinawa.pdf

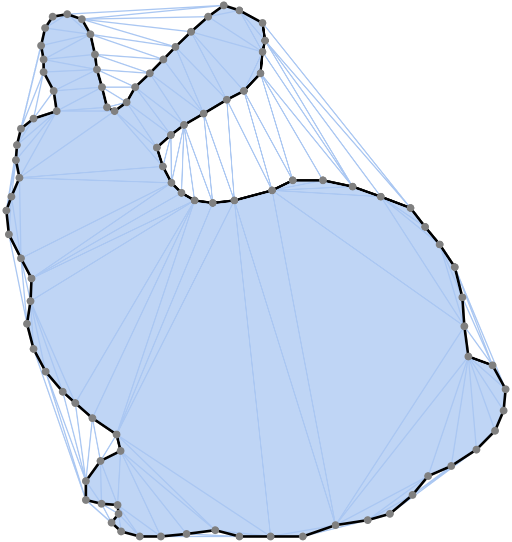
Wrap complexes and lexicographically minimal cycles



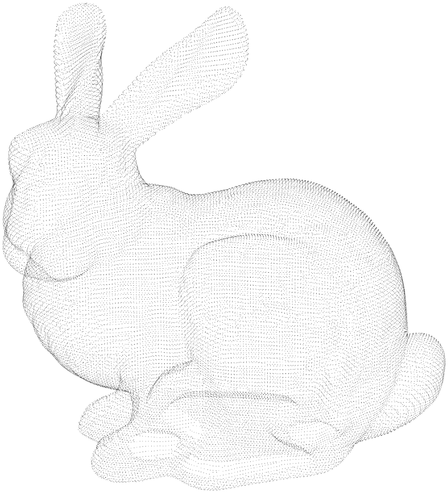
Wrap complexes and lexicographically minimal cycles



Wrap complexes and lexicographically minimal cycles

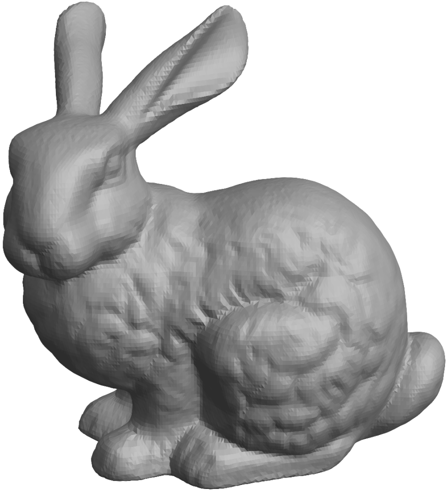


Wrap complexes and lexicographically minimal cycles



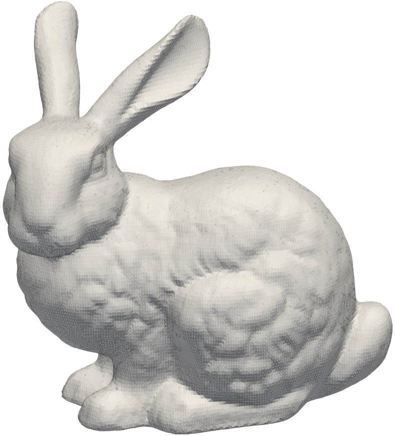
http://ulrich-bauer.org/persistence-intro-okinawa.pdf

Wrap complexes and lexicographically minimal cycles

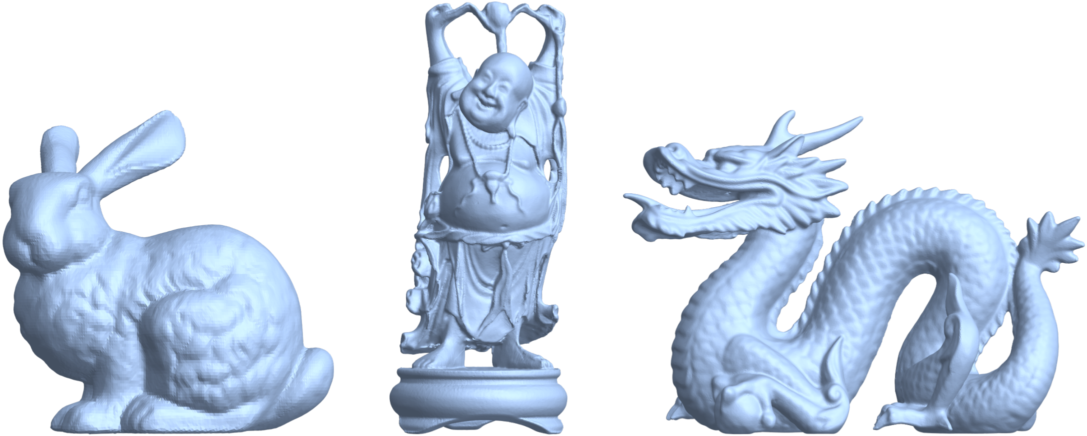


Standard reduction and exhaustive reduction

Standard reduction and exhaustive reduction



Point cloud reconstruction with minimal cycles

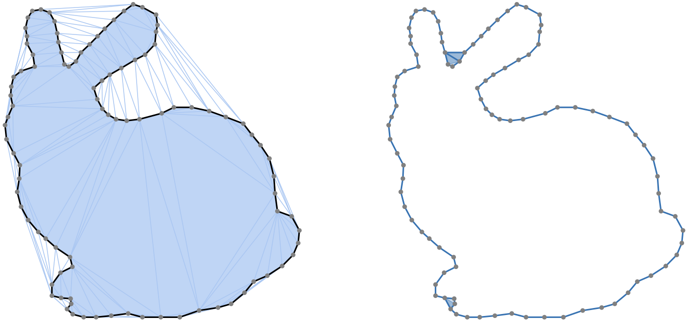


Wrap complexes support minimal cycles

Theorem (B, Roll 2024)

Let $X \subset \mathbb{R}$ be a finite subset in general position and let $r \in \mathbb{R}$.

- *Exhaustive matrix reduction computes the minimal cycles homologous to a simplex boundary.*
- *Any lexicographically minimal cycle of $\mathrm{Del}_r(X)$ is supported on $\mathrm{Wrap}_r(X)$.*



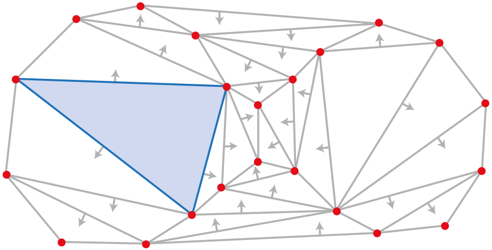
Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

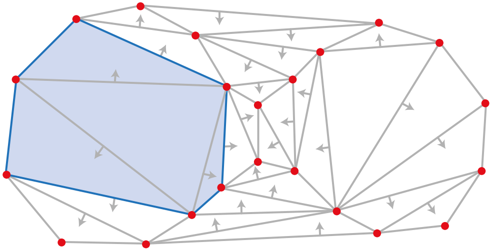
- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

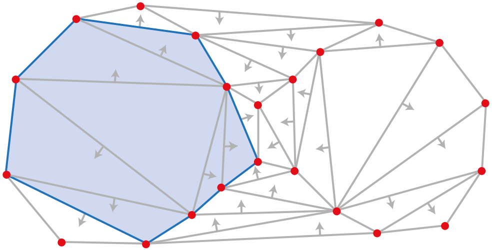
- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

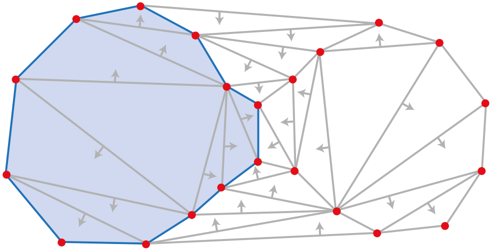
- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

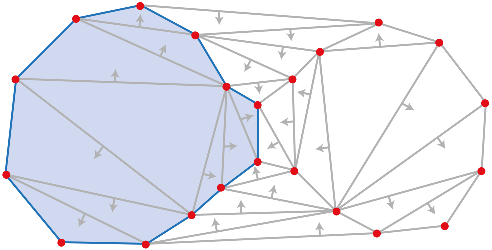
- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)

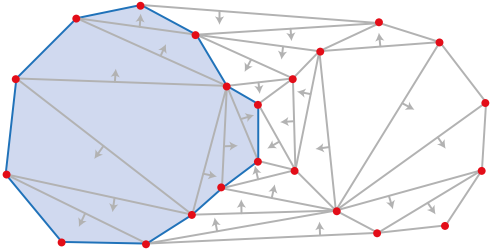


- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)

Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



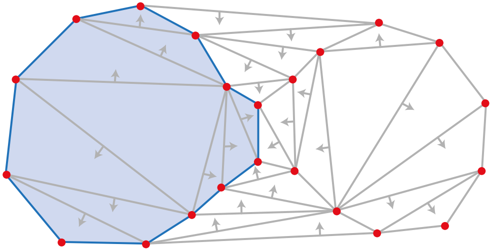
- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)

We interpret persistent homology in terms of algebraic Morse theory:

Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)

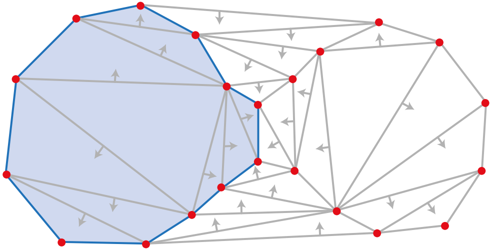
We interpret persistent homology in terms of algebraic Morse theory:

- Persistence pairs correspond to algebraic gradient pairs

Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)

We interpret persistent homology in terms of algebraic Morse theory:

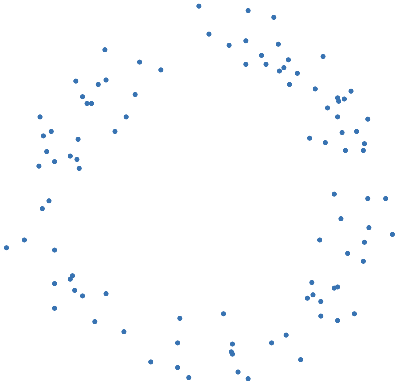
- Persistence pairs correspond to algebraic gradient pairs
- Matrix reduction (exhaustive) corresponds to gradient flow

Vietoris–Rips persistence

Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

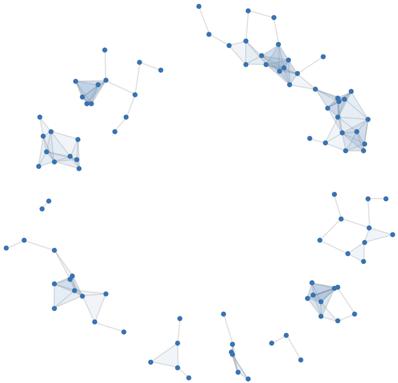
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

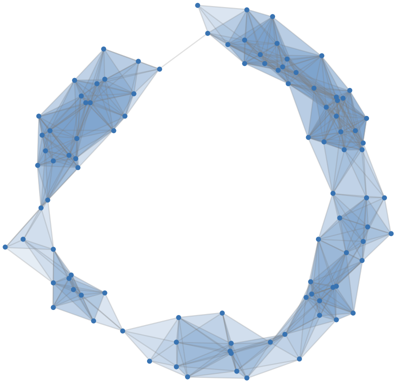
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

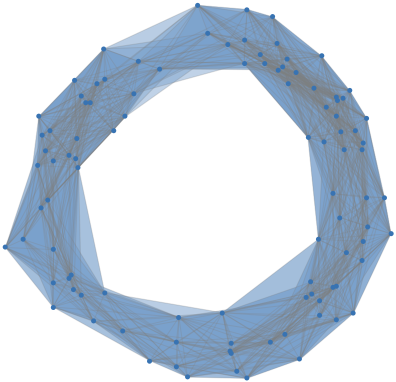
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

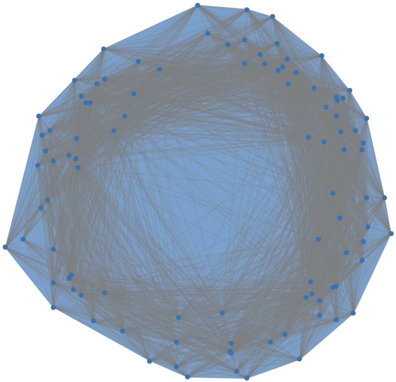
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Vietoris–Rips complexes

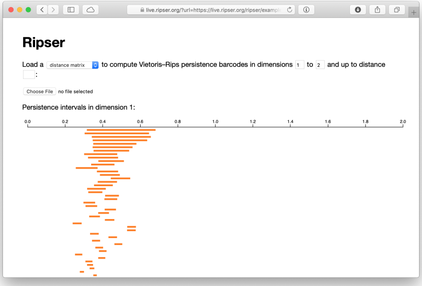
For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

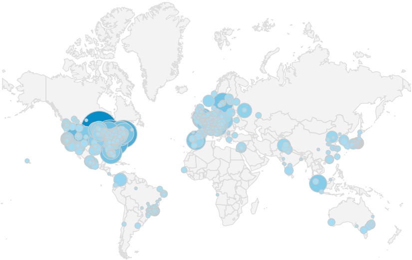
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Ripser: software for computing Vietoris–Rips persistence barcodes

Open source software (`ripser.org`)

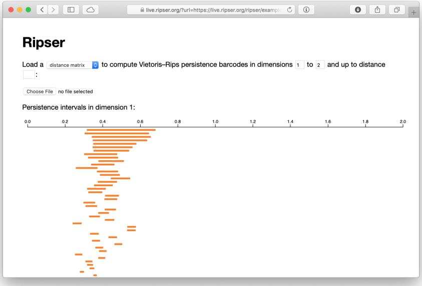


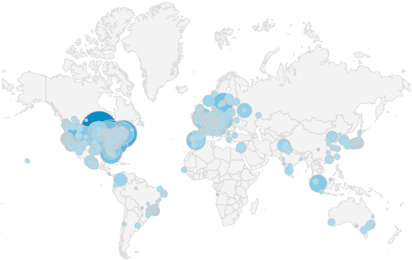


Ripser users across the globe

Ripser: software for computing Vietoris–Rips persistence barcodes

Open source software (`ripser.org`)





Ripser users across the globe

Computational improvements based on

- *implicit matrix representations*
- *apparent pairs*, connecting persistence to discrete Morse theory

Apparent pairs

Ripser uses the following pairing of simplices (breaking ties in the filtration lexicographically):

Definition (B 2016, 2021)

In a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, two simplices (σ_i, σ_j) form an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

Apparent pairs

Ripser uses the following pairing of simplices (breaking ties in the filtration lexicographically):

Definition (B 2016, 2021)

In a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, two simplices (σ_i, σ_j) form an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

Special cases and equivalent definitions have been (re)discovered independently multiple times

- Kahle 2011, Robbins 2015, Henselmann-Petrusek 2017, Lampret 2020, Feichtner-Kozlov 2020, …

Apparent pairs

Ripser uses the following pairing of simplices (breaking ties in the filtration lexicographically):

Definition (B 2016, 2021)

In a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, two simplices (σ_i, σ_j) form an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

Special cases and equivalent definitions have been (re)discovered independently multiple times

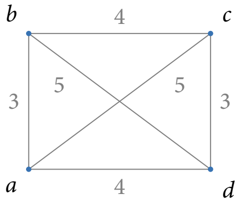
- Kahle 2011, Robbins 2015, Henselmann-Petrusek 2017, Lampret 2020, Feichtner-Kozlov 2020, …

Proposition (B 2021)

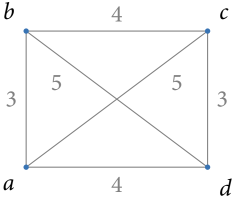
The apparent pairs are both

- *persistence pairs (creating/destroying a feature in homology) and*
- *gradient pairs (in the sense of discrete Morse theory).*

Apparent pairs of the diameter-lexicographic filtration

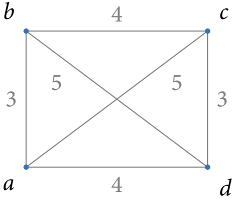


Apparent pairs of the diameter-lexicographic filtration



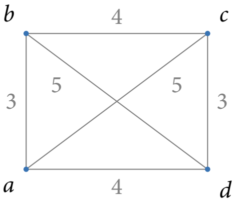
$$\partial_1 = \begin{array}{c} \begin{array}{cccccc} (a,b){:}3 & (c,d){:}3 & (a,d){:}4 & (b,c){:}4 & (a,c){:}5 & (b,d){:}5 \end{array} \\ \begin{pmatrix} 1 & & 1 & & 1 & \\ \mathbf{1} & & & 1 & & 1 \\ & 1 & & 1 & 1 & \\ & \mathbf{1} & 1 & & & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ (a) \\ (b) \\ (c) \\ (d) \end{array}$$

Apparent pairs of the diameter-lexicographic filtration



$$
\partial_1 = \begin{array}{cl}
\begin{array}{cccccc} (a,b){:}3 & (c,d){:}3 & (a,d){:}4 & (b,c){:}4 & (a,c){:}5 & (b,d){:}5 \end{array} & \\
\begin{pmatrix} 1 & & 1 & & 1 & \\ \mathbf{1} & & & 1 & & 1 \\ & 1 & & 1 & 1 & \\ & \mathbf{1} & 1 & & & 1 \end{pmatrix} & \begin{array}{l} (a) \\ (b) \\ (c) \\ (d) \end{array}
\end{array}
\qquad
\partial_2 = \begin{array}{cl}
\begin{array}{cccc} (a,b,c){:}5 & (a,b,d){:}5 & (a,c,d){:}5 & (b,c,d){:}5 \end{array} & \\
\begin{pmatrix} 1 & 1 & & \\ & & 1 & 1 \\ & 1 & 1 & \\ 1 & & & 1 \\ \mathbf{1} & & 1 & \\ & \mathbf{1} & & 1 \end{pmatrix} & \begin{array}{l} (a,b){:}3 \\ (c,d){:}3 \\ (a,d){:}4 \\ (b,c){:}4 \\ (a,c){:}5 \\ (b,d){:}5 \end{array}
\end{array}
$$

Apparent pairs of the diameter-lexicographic filtration



$$
\partial_1 = \begin{array}{c}
\begin{array}{cccccc} (a,b){:}3 & (c,d){:}3 & (a,d){:}4 & (b,c){:}4 & (a,c){:}5 & (b,d){:}5 \end{array} \\
\begin{pmatrix}
1 & & 1 & & 1 & \\
\mathbf{1} & & & 1 & & 1 \\
 & 1 & & 1 & 1 & \\
 & \mathbf{1} & 1 & & & 1
\end{pmatrix}
\end{array}
\begin{array}{l} \\ (a) \\ (b) \\ (c) \\ (d) \end{array}
$$

$$
\partial_2 = \begin{array}{c}
\begin{array}{cccc} (a,b,c){:}5 & (a,b,d){:}5 & (a,c,d){:}5 & (b,c,d){:}5 \end{array} \\
\begin{pmatrix}
1 & 1 & & \\
 & & 1 & 1 \\
 & 1 & 1 & \\
1 & & & 1 \\
\mathbf{1} & & 1 & \\
 & \mathbf{1} & & 1
\end{pmatrix}
\end{array}
\begin{array}{l} \\ (a,b){:}3 \\ (c,d){:}3 \\ (a,d){:}4 \\ (b,c){:}4 \\ (a,c){:}5 \\ (b,d){:}5 \end{array}
$$

$$
\partial_3 = \begin{array}{c}
(a,b,c,d) \\
\begin{pmatrix}
1 \\
1 \\
1 \\
\mathbf{1}
\end{pmatrix}
\end{array}
\begin{array}{l} \\ (a,b,c) \\ (a,b,d) \\ (a,c,d) \\ (b,c,d) \end{array}
$$

A shortcut for finding pivots

$$\begin{array}{cl} (a,b,c) & \\ \left(\begin{array}{c} 1 \\ 1 \\ \\ 1 \\ \\ \\ \end{array}\right) & \begin{array}{l} (a,b) \\ (a,c) \\ (a,d) \\ (b,c) \\ (b,d) \\ (c,d) \end{array} \end{array} \qquad \begin{array}{cl} (a,b,c){:}5 & \\ \left(\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right) & \begin{array}{l} (a,b){:}3 \\ (c,d){:}3 \\ (a,d){:}4 \\ (b,c){:}4 \\ (a,c){:}5 \\ (b,d){:}5 \end{array} \end{array}$$

Shortcut for finding the pivot (latest) facet of a simplex τ:

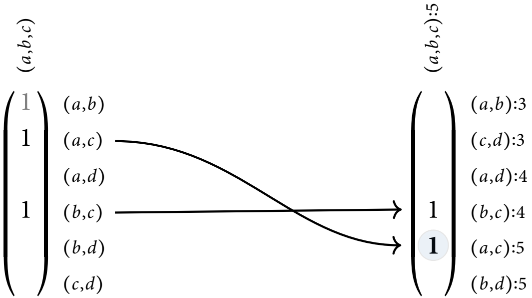
A shortcut for finding pivots

$$
\begin{array}{cc}
(a,b,c) & \\
\begin{pmatrix} 1 \\ 1 \\ \\ 1 \\ \\ \\ \end{pmatrix} & \begin{array}{l} (a,b) \\ (a,c) \\ (a,d) \\ (b,c) \\ (b,d) \\ (c,d) \end{array}
\end{array}
\longrightarrow
\begin{array}{cc}
(a,b,c){:}5 & \\
\begin{pmatrix} \\ \\ \\ 1 \\ \\ \\ \end{pmatrix} & \begin{array}{l} (a,b){:}3 \\ (c,d){:}3 \\ (a,d){:}4 \\ (b,c){:}4 \\ (a,c){:}5 \\ (b,d){:}5 \end{array}
\end{array}
$$

Shortcut for finding the pivot (latest) facet of a simplex τ:

- Enumerate facets σ of τ in (reverse) lexicographic order

A shortcut for finding pivots



Shortcut for finding the pivot (latest) facet of a simplex τ:

- Enumerate facets σ of τ in (reverse) lexicographic order
- The first facet σ with $\operatorname{diam} \sigma = \operatorname{diam} \tau$ is the pivot of τ

A shortcut for finding pivots

$$
\begin{array}{cl}
(a,b,c) & \\
\left(\begin{array}{c} 1 \\ 1 \\ \\ 1 \\ \\ \end{array}\right) & \begin{array}{l} (a,b) \\ (a,c) \\ (a,d) \\ (b,c) \\ (b,d) \\ (c,d) \end{array}
\end{array}
\qquad\qquad
\begin{array}{cl}
(a,b,c){:}5 & \\
\left(\begin{array}{c} 1 \\ \\ \\ 1 \\ \mathbf{1} \\ \end{array}\right) & \begin{array}{l} (a,b){:}3 \\ (c,d){:}3 \\ (a,d){:}4 \\ (b,c){:}4 \\ (a,c){:}5 \\ (b,d){:}5 \end{array}
\end{array}
$$

Shortcut for finding the pivot (latest) facet of a simplex τ:

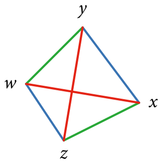
- Enumerate facets σ of τ in (reverse) lexicographic order
- The first facet σ with $\operatorname{diam}\sigma = \operatorname{diam}\tau$ is the pivot of τ

Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ-*hyperbolic* (for $\delta \geq 0$) if for all $w, x, y, z \in X$ we have

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$

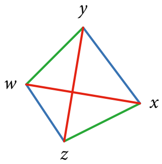


Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ-*hyperbolic* (for $\delta \geq 0$) if for all $w, x, y, z \in X$ we have

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$



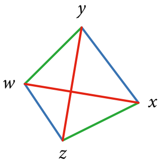
- The hyperbolic plane is $(\ln 2)$-hyperbolic.

Gromov-hyperbolicity

Definition (Gromov 1988)

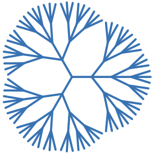
A metric space X is δ-*hyperbolic* (for $\delta \geq 0$) if for all $w, x, y, z \in X$ we have

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$

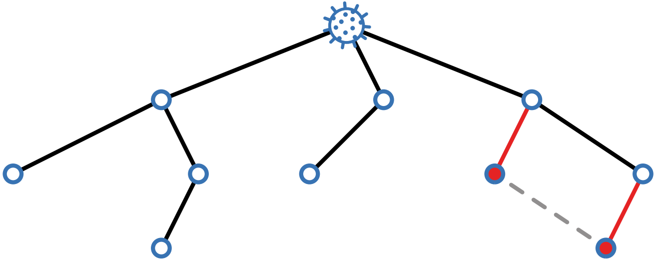


- The hyperbolic plane is $(\ln 2)$-hyperbolic.

- Metric trees and their subspaces are precisely the 0-hyperbolic spaces.

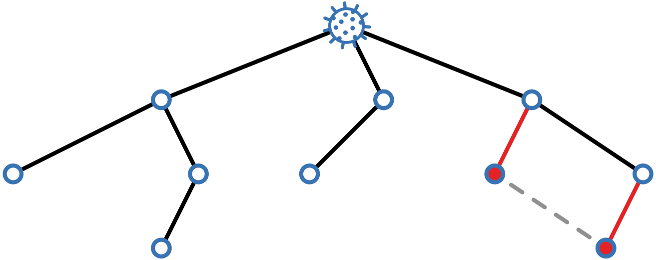


Topology of viral evolution



Joint work with: A. Ott, M. Bleher, L. Hahn (Heidelberg), R. Rabadan, J. Patiño-Galindo (Columbia), M. Carrière (INRIA)

Topology of viral evolution



Joint work with: A. Ott, M. Bleher, L. Hahn (Heidelberg), R. Rabadan, J. Patiño-Galindo (Columbia), M. Carrière (INRIA)

Observation: Ripser runs unusually fast on genetic distance data

- SARS-CoV2 RNA sequences (spike protein)
- 25556 data points (2.8×10^{12} simplices in 2-skeleton)
- $120\,s$ computation time (with data points ordered appropriately)

Rips complexes of hyperbolic spaces

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

Rips complexes of hyperbolic spaces

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

- Generalization to non-geodesic spaces? In particular, finite metric spaces?

Rips complexes of hyperbolic spaces

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

- Generalization to non-geodesic spaces? In particular, finite metric spaces?
- Collapsiblility? By a discrete gradient compatible with the filtration?

Rips complexes of hyperbolic spaces

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

- Generalization to non-geodesic spaces? In particular, finite metric spaces?
- Collapsiblility? By a discrete gradient compatible with the filtration?
- Connection to computation of persistent homology?

Rips complexes of hyperbolic spaces

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

- Generalization to non-geodesic spaces? In particular, finite metric spaces?
- Collapsiblility? By a discrete gradient compatible with the filtration?
- Connection to computation of persistent homology?

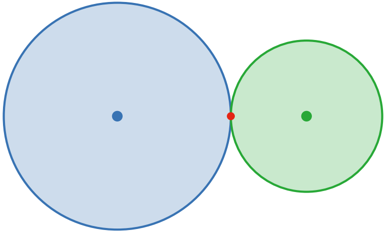
Theorem (B, Roll 2022)

Let X be a finite δ-hyperbolic space. Then there is a single discrete gradient encoding the collapses

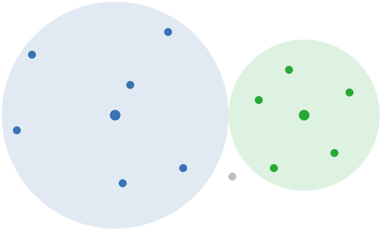
$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect *of X.*

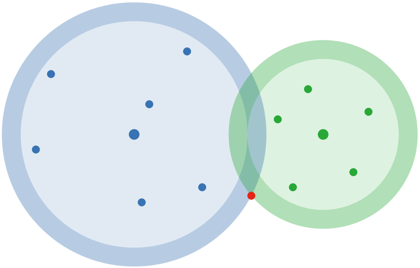
Geodesic defect



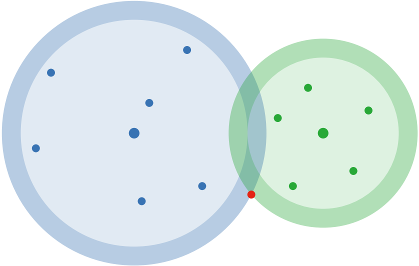
Geodesic defect



Geodesic defect



Geodesic defect



Definition (Bonk, Schramm 2000)

A metric space X is *ν-geodesic* if for all points $x, y \in X$ and all $r, s \geq 0$ with $r + s = d(x, y)$ we have

$$B_{r+\nu}(x) \cap B_{s+\nu}(y) \neq \varnothing.$$

The infimum of all such ν is the *geodesic defect* of X.

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$.

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$*.*
- *The apparent pairs (lexicographic refinement, based on arbitrary vertex order) refine this gradient.*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances).* *Then the diameter function* $\operatorname{diam}: \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$.
- *The apparent pairs (lexicographic refinement, based on arbitrary vertex order) refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$.
- *The apparent pairs (lexicographic refinement, based on arbitrary vertex order) refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

- $\operatorname{Rips}_t(X) \searrow T_t$ *for all* $t \in \mathbb{R}$,

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$*.*
- *The apparent pairs (lexicographic refinement, based on arbitrary vertex order) refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

- $\operatorname{Rips}_t(X) \searrow T_t$ *for all* $t \in \mathbb{R}$*,*
- $\operatorname{Rips}_t(X) \searrow T \searrow \{*\}$ *for* $t \geq \max d(E)$*, and*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}: \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$*.*
- *The apparent pairs (lexicographic refinement, based on arbitrary vertex order) refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

- $\operatorname{Rips}_t(X) \searrow T_t$ *for all* $t \in \mathbb{R}$,
- $\operatorname{Rips}_t(X) \searrow T \searrow \{*\}$ *for* $t \geq \max d(E)$*, and*
- $\operatorname{Rips}_u(X) \searrow \operatorname{Rips}_t(X)$ *whenever no pairwise distance lies in the interval* $(t, u]$.

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$.
- *The apparent pairs (lexicographic refinement, based on arbitrary vertex order) refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

- $\operatorname{Rips}_t(X) \searrow T_t$ *for all* $t \in \mathbb{R}$,
- $\operatorname{Rips}_t(X) \searrow T \searrow \{*\}$ *for* $t \geq \max d(E)$, *and*
- $\operatorname{Rips}_u(X) \searrow \operatorname{Rips}_t(X)$ *whenever no pairwise distance lies in the interval* $(t, u]$.

In particular, the persistent homology is trivial in degrees > 0.

Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.

Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

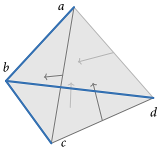
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



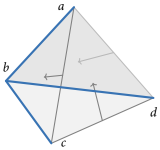
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



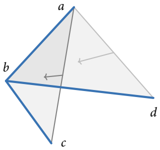
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



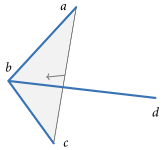
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



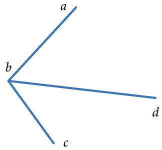
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



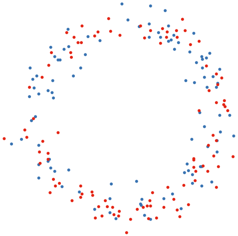
Stability

Stability of persistence barcodes

If two point clouds are close, then their barcodes are also close:

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $X, Y \subset \mathbb{R}^d$ be finite point sets with Hausdorff distance $d_H(X, Y) = \delta$.

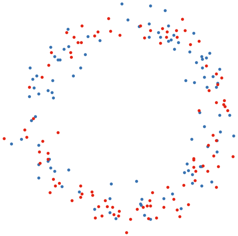


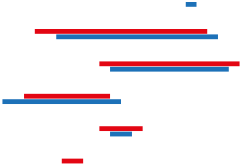
Stability of persistence barcodes

If two point clouds are close, then their barcodes are also close:

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $X, Y \subset \mathbb{R}^d$ *be finite point sets with Hausdorff distance* $d_H(X, Y) = \delta$.
Then there exists a δ-matching *between the intervals in the (Čech or Delaunay) persistence barcodes for* X *and* Y*:*





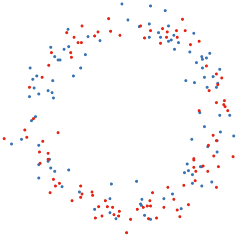
Stability of persistence barcodes

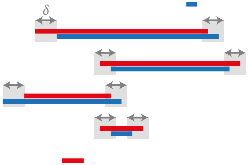
If two point clouds are close, then their barcodes are also close:

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $X, Y \subset \mathbb{R}^d$ be finite point sets with Hausdorff distance $d_H(X, Y) = \delta$.
*Then there exists a δ-*matching *between the intervals in the (Čech or Delaunay) persistence barcodes for X and Y:*

- *matched intervals have endpoints within distance $\leq \delta$, and*





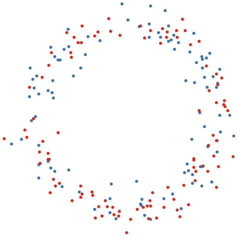
Stability of persistence barcodes

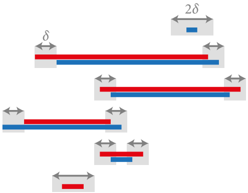
If two point clouds are close, then their barcodes are also close:

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $X, Y \subset \mathbb{R}^d$ be finite point sets with Hausdorff distance $d_H(X, Y) = \delta$.
*Then there exists a δ-*matching *between the intervals in the (Čech or Delaunay) persistence barcodes for X and Y:*

- *matched intervals have endpoints within distance $\leq \delta$, and*
- *unmatched intervals have length $\leq 2\delta$.*





Persistence and stability: the big picture

Data	point cloud	$P \subset \mathbb{R}^d$

Persistence and stability: the big picture

$$\begin{tikzcd} \text{Data} \arrow[d] & \text{point cloud} \arrow[d, "\text{distance}"] & P \subset \mathbb{R}^d \\ \text{Geometry} & \text{function} & f : \mathbb{R}^d \to \mathbb{R} \end{tikzcd}$$

Persistence and stability: the big picture

$$\begin{tikzcd}
\text{Data} \arrow[d] & \text{point cloud} \arrow[d, "\text{distance}"] & P \subset \mathbb{R}^d \\
\text{Geometry} \arrow[d] & \text{function} \arrow[d, "\text{sublevel sets}"] & f : \mathbb{R}^d \to \mathbb{R} \\
\text{Topology} & \text{topological spaces (filtration)} & K : \mathbf{R} \to \mathbf{Top}
\end{tikzcd}$$

Persistence and stability: the big picture

$$
\begin{tikzcd}
\text{Data} \arrow[d] & \text{point cloud} \arrow[d, "\text{distance}"] & P \subset \mathbb{R}^d \\
\text{Geometry} \arrow[d] & \text{function} \arrow[d, "\text{sublevel sets}"] & f : \mathbb{R}^d \to \mathbb{R} \\
\text{Topology} \arrow[d] & \text{topological spaces (filtration)} \arrow[d, "\text{homology}"] & K : \mathbf{R} \to \mathbf{Top} \\
\text{Algebra} & \text{vector spaces (persistence module)} & M : \mathbf{R} \to \mathbf{Vect}
\end{tikzcd}
$$

Persistence and stability: the big picture

$$\begin{tikzcd}
\text{Data} \arrow[d] & \text{point cloud} \arrow[d, "\text{distance}"] & P \subset \mathbb{R}^d \\
\text{Geometry} \arrow[d] & \text{function} \arrow[d, "\text{sublevel sets}"] & f : \mathbb{R}^d \to \mathbb{R} \\
\text{Topology} \arrow[d] & \text{topological spaces (filtration)} \arrow[d, "\text{homology}"] & K : \mathbf{R} \to \mathbf{Top} \\
\text{Algebra} \arrow[d] & \text{vector spaces (persistence module)} \arrow[d, "\text{structure theorem}"] & M : \mathbf{R} \to \mathbf{Vect} \\
\text{Combinatorics} & \text{intervals (persistence barcode)} & B : \mathbf{R} \to \mathbf{Mch}
\end{tikzcd}$$

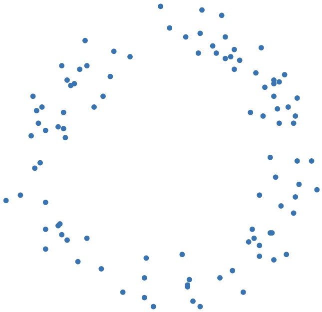
Persistence and stability: the big picture

$$
\begin{tikzcd}[row sep=large]
\text{Data} \arrow[d] & \text{point cloud} \arrow[d, "\text{distance}"] & P \subset \mathbb{R}^d \\
\text{Geometry} \arrow[d] & \text{function} \arrow[d, "\text{sublevel sets}"] \arrow[dddl, bend right, start anchor=west, end anchor=north west, red, shift left=0ex, to path={}] \arrow[ddd, bend right=60, red] & f : \mathbb{R}^d \to \mathbb{R} \\
\text{Topology} \arrow[d] & \text{topological spaces (filtration)} \arrow[d, "\text{homology}"] & K : \mathbf{R} \to \mathbf{Top} \\
\text{Algebra} \arrow[d] & \text{vector spaces (persistence module)} \arrow[d, "\text{structure theorem}"] & M : \mathbf{R} \to \mathbf{Vect} \\
\text{Combinatorics} & \text{intervals (persistence barcode)} & B : \mathbf{R} \to \mathbf{Mch}
\end{tikzcd}
$$

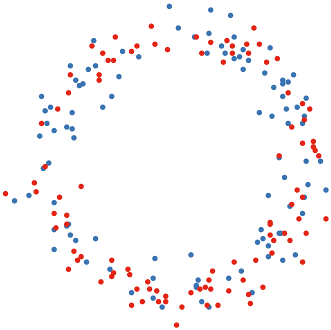
Persistence and stability: the big picture

Data	point cloud	$P \subset \mathbb{R}^d$
$\downarrow$	$\downarrow$ distance	
Geometry	function	$f : \mathbb{R}^d \to \mathbb{R}$
$\downarrow$	$\downarrow$ sublevel sets	
Topology	topological spaces (filtration)	$K : \mathbf{R} \to \mathbf{Top}$
$\downarrow$	$\downarrow$ homology	
Algebra	vector spaces (persistence module)	$M : \mathbf{R} \to \mathbf{Vect}$
$\downarrow$	$\downarrow$ structure theorem	
Combinatorics	intervals (persistence barcode)	$B : \mathbf{R} \to \mathbf{Mch}$

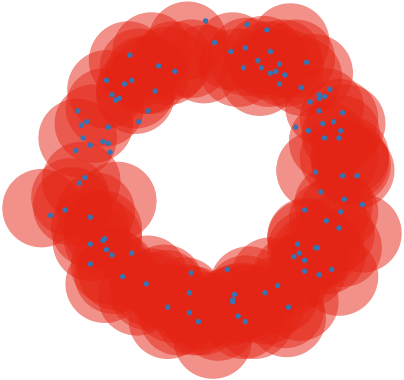
Geometric interleavings



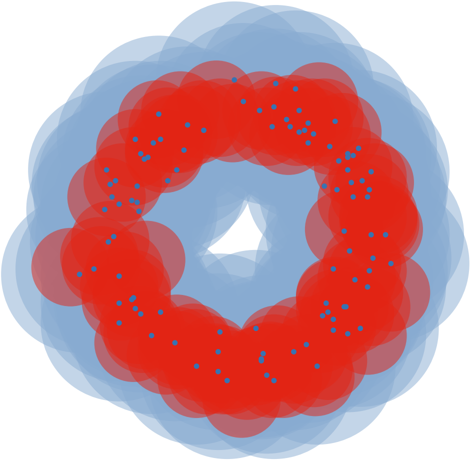
Geometric interleavings



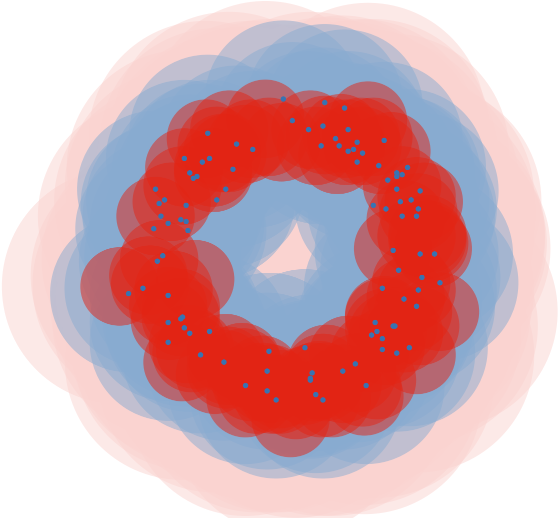
Geometric interleavings



Geometric interleavings



Geometric interleavings



Interleavings

Let $\delta = \|f - g\|_\infty$. Write $F_t = f^{-1}(-\infty, t]$ for the t-sublevel set of f, and similarly $G_t = g^{-1}(-\infty, t]$.

Interleavings

Let $\delta = \|f - g\|_\infty$. Write $F_t = f^{-1}(-\infty, t]$ for the t-sublevel set of f, and similarly $G_t = g^{-1}(-\infty, t]$.

Then the sublevel set filtrations $F, G : \mathbf{R} \to \mathbf{Top}$ are δ-*interleaved:*

$$\begin{tikzcd}
{} \arrow[r, dotted] \arrow[dr, dotted] & F_{t-\delta} \arrow[r] \arrow[dr] & F_t \arrow[r] \arrow[dr] & F_{t+\delta} \arrow[r, dotted] \arrow[dr, dotted] & {} \\
{} \arrow[r, dotted] \arrow[ur, dotted] & G_{t-\delta} \arrow[r] \arrow[ur] & G_t \arrow[r] \arrow[ur] & G_{t+\delta} \arrow[r, dotted] \arrow[ur, dotted] & {}
\end{tikzcd} \qquad \forall t \in \mathbb{R}.$$

Interleavings

Let $\delta = \|f - g\|_\infty$. Write $F_t = f^{-1}(-\infty, t]$ for the t-sublevel set of f, and similarly $G_t = g^{-1}(-\infty, t]$.

Then the sublevel set filtrations $F, G : \mathbf{R} \to \mathbf{Top}$ are δ-*interleaved:*

$$\begin{tikzcd}
{} \arrow[r, dotted] \arrow[dr, dotted] & F_{t-\delta} \arrow[r] \arrow[dr] & F_t \arrow[r] \arrow[dr] & F_{t+\delta} \arrow[r, dotted] \arrow[dr, dotted] & {} \\
{} \arrow[r, dotted] \arrow[ur, dotted] & G_{t-\delta} \arrow[r] \arrow[ur] & G_t \arrow[r] \arrow[ur] & G_{t+\delta} \arrow[r, dotted] \arrow[ur, dotted] & {}
\end{tikzcd} \qquad \forall t \in \mathbb{R}.$$

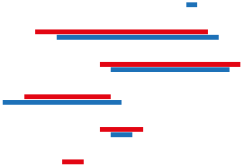
Applying homology, the persistence modules $H_*(F), H_*(G) : \mathbf{R} \to \mathbf{Vect}$ are δ-*interleaved:*

$$\begin{tikzcd}
{} \arrow[r, dotted] \arrow[dr, dotted] & H_*(F_{t-\delta}) \arrow[r] \arrow[dr] & H_*(F_t) \arrow[r] \arrow[dr] & H_*(F_{t+\delta}) \arrow[r, dotted] \arrow[dr, dotted] & {} \\
{} \arrow[r, dotted] \arrow[ur, dotted] & H_*(G_{t-\delta}) \arrow[r] \arrow[ur] & H_*(G_t) \arrow[r] \arrow[ur] & H_*(G_{t+\delta}) \arrow[r, dotted] \arrow[ur, dotted] & {}
\end{tikzcd} \qquad \forall t \in \mathbb{R}.$$

Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2016; B, Lesnick 2015)

If two persistence modules are δ-interleaved,
then their barcodes admit a δ-matching:

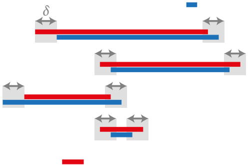


Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2016; B, Lesnick 2015)

If two persistence modules are δ-interleaved,
then their barcodes admit a δ-matching:

- *matched intervals have endpoints within distance $\leq \delta$, and*

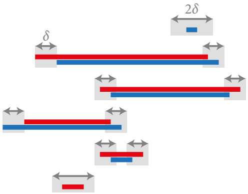


Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2016; B, Lesnick 2015)

If two persistence modules are δ-interleaved,
then their barcodes admit a δ-matching:

- *matched intervals have endpoints within distance $\leq \delta$, and*
- *unmatched intervals have length $\leq 2\delta$.*



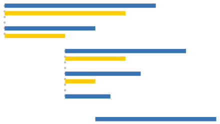
Structure of persistence sub-/quotient modules

Proposition (B, Lesnick 2015)

Let $M \twoheadrightarrow N$ be an epimorphism of persistence modules.
Then there is an injection of barcodes $B(N) \hookrightarrow B(M)$ such that
if J is mapped to I, then

- *I and J are aligned below, and*
- *I bounds J above.*

This construction is functorial (compatible with composition).



Structure of persistence sub-/quotient modules

Proposition (B, Lesnick 2015)

Let $M \twoheadrightarrow N$ be an epimorphism of persistence modules.
Then there is an injection of barcodes $B(N) \hookrightarrow B(M)$ such that
if J is mapped to I, then

- *I and J are aligned below, and*
- *I bounds J above.*

This construction is functorial (compatible with composition).

Dually, for a monomorphism $N \hookrightarrow O$ there is an injection $B(N) \hookrightarrow B(O)$.

Induced matchings

Any morphism of pfd persistence modules $f : M \to N$ factors through its image as an epimorphism followed by a monomorphism:

$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$$

This gives an *induced matching* between their barcodes:

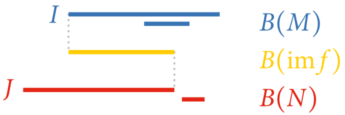
Induced matchings

Any morphism of pfd persistence modules $f : M \to N$ factors through its image as an epimorphism followed by a monomorphism:

$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$$

This gives an *induced matching* between their barcodes:

- compose the injections $B(M) \hookleftarrow B(\operatorname{im} f) \hookrightarrow B(N)$ from before to a matching $B(M) \nrightarrow B(N)$.



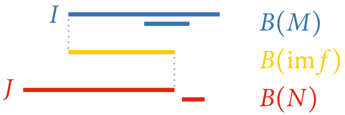
Induced matchings

Any morphism of pfd persistence modules $f : M \to N$ factors through its image as an epimorphism followed by a monomorphism:

$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$$

This gives an *induced matching* between their barcodes:

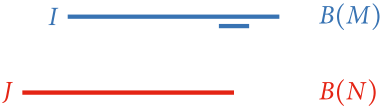
- compose the injections $B(M) \hookleftarrow B(\operatorname{im} f) \hookrightarrow B(N)$ from before to a matching $B(M) \nrightarrow B(N)$.



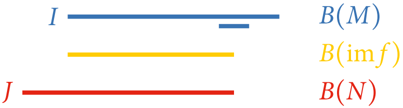
Warning:

- The corresponding morphism of persistence modules may not be isomorphic to f

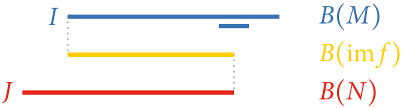
The induced matching theorem



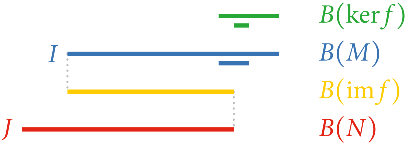
The induced matching theorem



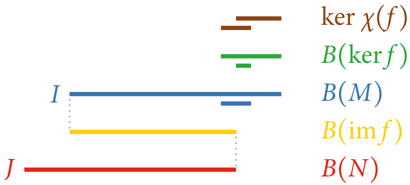
The induced matching theorem



The induced matching theorem



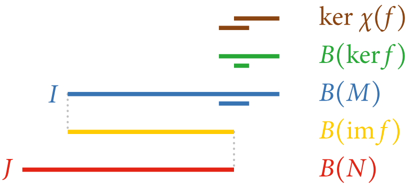
The induced matching theorem



The induced matching theorem

Theorem (B, Lesnick, 2015, 2020)

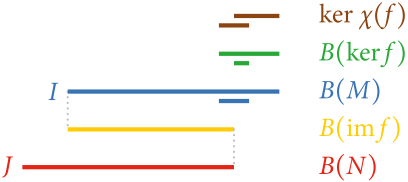
If $f : M \to N$ has δ-small (co)kernel, then so does the induced matching $\chi(f)\colon B(M) \nrightarrow B(N)$.



The induced matching theorem

Theorem (B, Lesnick, 2015, 2020)

If $f : M \to N$ has δ-small (co)kernel, then so does the induced matching $\chi(f)\colon B(M) \nrightarrow B(N)$.



Corollary

If $f\colon M \to N(\delta)$ is a δ-interleaving morphism, then this induced matching is a δ-matching.

Algebraic stability via induced matchings

Consider an interleaving $f_t : M_t \to N_{t+\delta},\ g_t : N_t \to M_{t+\delta}\ (t \in \mathbb{R})$:

$$\begin{tikzcd}
 & M_t \arrow[dr, "f_t"] & \\
N_{t-\delta} \arrow[ur, "g_{t-\delta}"] \arrow[rr, "N_{t-\delta,t+\delta}"] & & N_{t+\delta}
\end{tikzcd}$$

Algebraic stability via induced matchings

Consider an interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(t \in \mathbb{R})$:

$$\begin{tikzcd}
 & & M_t \arrow[dr, two heads] & & \\
 & & & \operatorname{im} f_t \arrow[dr, hook] & \\
N_{t-\delta} \arrow[uurr, "g_{t-\delta}"] \arrow[rr, two heads] & & \operatorname{im} N_{t-\delta,t+\delta} \arrow[ur, hook] \arrow[rr, hook] & & N_{t+\delta}
\end{tikzcd}$$

Construct an induced δ-matching of barcodes:

Algebraic stability via induced matchings

Consider an interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(t \in \mathbb{R})$:

$$
\begin{tikzcd}
 & & M_t \arrow[dr, two heads] & & \\
 & & & \operatorname{im} f_t \arrow[dr, hook] & \\
N_{t-\delta} \arrow[uurr, "g_{t-\delta}"] \arrow[r, two heads] & \operatorname{im} N_{t-\delta,t+\delta} \arrow[urr, hook] \arrow[rrr, hook] & & & N_{t+\delta}
\end{tikzcd}
$$

Construct an induced δ-matching of barcodes:

- Match every interval in $B(N)$ of length $> 2\delta$ to a similar interval in $B(M)$

$B(N)$

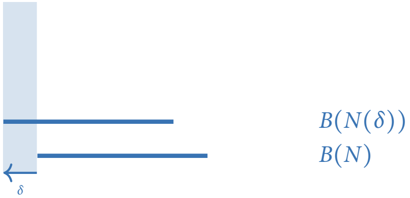
Algebraic stability via induced matchings

Consider an interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(t \in \mathbb{R})$:

$$\begin{tikzcd} & & M_t \arrow[dr, two heads] & & \\ & & & \operatorname{im} f_t \arrow[dr, hook] & \\ N_{t-\delta} \arrow[uurr, "g_{t-\delta}"] \arrow[rr, two heads] & & \operatorname{im} N_{t-\delta,t+\delta} \arrow[ur, hook] \arrow[rr, hook] & & N_{t+\delta} \end{tikzcd}$$

Construct an induced δ-matching of barcodes:

- Match every interval in $B(N)$ of length $> 2\delta$ to a similar interval in $B(M)$



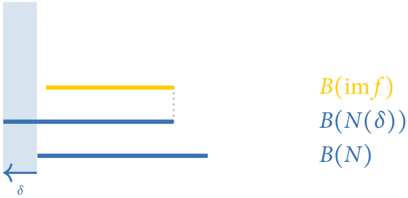
Algebraic stability via induced matchings

Consider an interleaving $f_t : M_t \to N_{t+\delta},\ g_t : N_t \to M_{t+\delta}$ $(t \in \mathbb{R})$:

$$\begin{tikzcd}
 & M_t \arrow[dr, two heads] & & \\
 & & \operatorname{im} f_t \arrow[dr, hook] & \\
N_{t-\delta} \arrow[uur, "g_{t-\delta}"] \arrow[r, two heads] & \operatorname{im} N_{t-\delta,t+\delta} \arrow[ur, hook] \arrow[rr, hook] & & N_{t+\delta}
\end{tikzcd}$$

Construct an induced δ-matching of barcodes:

- Match every interval in $B(N)$ of length $> 2\delta$ to a similar interval in $B(M)$



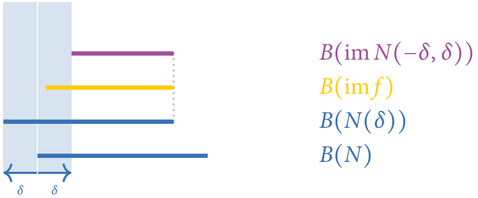
Algebraic stability via induced matchings

Consider an interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(t \in \mathbb{R})$:

$$\begin{tikzcd}
 & & M_t \arrow[dr, two heads] & & \\
 & & & \operatorname{im} f_t \arrow[dr, hook] & \\
N_{t-\delta} \arrow[uurr, "g_{t-\delta}"] \arrow[r, two heads] & \operatorname{im} N_{t-\delta,t+\delta} \arrow[urr, hook] \arrow[rrr, hook] & & & N_{t+\delta}
\end{tikzcd}$$

Construct an induced δ-matching of barcodes:

- Match every interval in $B(N)$ of length $> 2\delta$ to a similar interval in $B(M)$



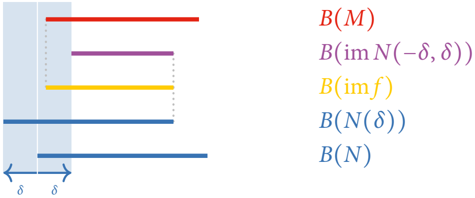
Algebraic stability via induced matchings

Consider an interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(t \in \mathbb{R})$:

$$
\begin{tikzcd}
 & & M_t \arrow[dr, two heads] & & \\
 & & & \operatorname{im} f_t \arrow[dr, hook] & \\
N_{t-\delta} \arrow[uurr, "g_{t-\delta}"] \arrow[rr, two heads] & & \operatorname{im} N_{t-\delta,t+\delta} \arrow[ur, hook] \arrow[rr, hook] & & N_{t+\delta}
\end{tikzcd}
$$

Construct an induced δ-matching of barcodes:

- Match every interval in $B(N)$ of length $> 2\delta$ to a similar interval in $B(M)$



Instabillity

Indecomposables abound in multi-parameter persistence

Theorem (B, Scoccola 2022)

For $n > 1$, almost every finitely presentable n-parameter persistence module is indecomposable.

Indecomposables abound in multi-parameter persistence

Theorem (B, Scoccola 2022)

For $n > 1$, almost every finitely presentable n-parameter persistence module is indecomposable.

- *Given a persistence module M, any neighborhood contains an indecomposable:*

Indecomposables abound in multi-parameter persistence

Theorem (B, Scoccola 2022)

For $n > 1$, almost every finitely presentable n-parameter persistence module is indecomposable.

- *Given a persistence module M, any neighborhood contains an indecomposable:*
  - *For every $\epsilon > 0$, there is an indecomposable N with $d_I(M, N) < \epsilon$.*

Indecomposables abound in multi-parameter persistence

Theorem (B, Scoccola 2022)

For $n > 1$, almost every finitely presentable n-parameter persistence module is indecomposable.

- *Given a persistence module M, any neighborhood contains an indecomposable:*
  - *For every $\epsilon > 0$, there is an indecomposable N with $d_I(M, N) < \epsilon$.*
- *Any indecomposable N has a neighborhood of* almost indecomposables*:*

Indecomposables abound in multi-parameter persistence

Theorem (B, Scoccola 2022)

For $n > 1$, almost every finitely presentable n-parameter persistence module is indecomposable.

- *Given a persistence module M, any neighborhood contains an indecomposable:*
  - *For every $\epsilon > 0$, there is an indecomposable N with $d_I(M, N) < \epsilon$.*
- *Any indecomposable N has a neighborhood of* almost indecomposables*:*
  - *For every $\epsilon > 0$, there is $\delta > 0$ such that any persistence module M within distance δ ($d_I(M, N) < \epsilon$) decomposes into*

Indecomposables abound in multi-parameter persistence

Theorem (B, Scoccola 2022)

For $n > 1$, almost every finitely presentable n-parameter persistence module is indecomposable.

- *Given a persistence module M, any neighborhood contains an indecomposable:*
  - *For every $\epsilon > 0$, there is an indecomposable N with $d_I(M, N) < \epsilon$.*
- *Any indecomposable N has a neighborhood of* almost indecomposables*:*
  - *For every $\epsilon > 0$, there is $\delta > 0$ such that any persistence module M within distance δ ($d_I(M, N) < \epsilon$) decomposes into*
    - *an indecomposable summand and*

Indecomposables abound in multi-parameter persistence

Theorem (B, Scoccola 2022)

For $n > 1$, almost every finitely presentable n-parameter persistence module is indecomposable.

- *Given a persistence module M, any neighborhood contains an indecomposable:*
  - *For every $\epsilon > 0$, there is an indecomposable N with $d_I(M, N) < \epsilon$.*
- *Any indecomposable N has a neighborhood of* almost indecomposables*:*
  - *For every $\epsilon > 0$, there is $\delta > 0$ such that any persistence module M within distance δ ($d_I(M, N) < \epsilon$) decomposes into*
    - *an indecomposable summand and*
    - *an ϵ-small summand (distance $< \epsilon$ to zero module).*

Indecomposables abound in multi-parameter persistence

Theorem (B, Scoccola 2022)

For $n > 1$, almost every finitely presentable n-parameter persistence module is indecomposable.

- *Given a persistence module M, any neighborhood contains an indecomposable:*
  - *For every $\epsilon > 0$, there is an indecomposable N with $d_I(M, N) < \epsilon$.*
- *Any indecomposable N has a neighborhood of* almost indecomposables*:*
  - *For every $\epsilon > 0$, there is $\delta > 0$ such that any persistence module M within distance δ ($d_I(M, N) < \epsilon$) decomposes into*
    - *an indecomposable summand and*
    - *an ϵ-small summand (distance $< \epsilon$ to zero module).*

In other words:

- Indecomposables form a dense subset.

Indecomposables abound in multi-parameter persistence

Theorem (B, Scoccola 2022)

For $n > 1$, almost every finitely presentable n-parameter persistence module is indecomposable.

- *Given a persistence module M, any neighborhood contains an indecomposable:*
  - *For every $\epsilon > 0$, there is an indecomposable N with $d_I(M, N) < \epsilon$.*
- *Any indecomposable N has a neighborhood of* almost indecomposables*:*
  - *For every $\epsilon > 0$, there is $\delta > 0$ such that any persistence module M within distance δ ($d_I(M, N) < \epsilon$) decomposes into*
    - *an indecomposable summand and*
    - *an ϵ-small summand (distance $< \epsilon$ to zero module).*

In other words:

- Indecomposables form a dense subset.
- ϵ-almost indecomposables form an open and dense subset.

Indecomposables abound in multi-parameter persistence

Theorem (B, Scoccola 2022)

For $n > 1$, almost every finitely presentable n-parameter persistence module is indecomposable.

- *Given a persistence module M, any neighborhood contains an indecomposable:*
  - *For every $\epsilon > 0$, there is an indecomposable N with $d_I(M, N) < \epsilon$.*
- *Any indecomposable N has a neighborhood of* almost indecomposables*:*
  - *For every $\epsilon > 0$, there is $\delta > 0$ such that any persistence module M within distance δ ($d_I(M, N) < \epsilon$) decomposes into*
    - *an indecomposable summand and*
    - *an ϵ-small summand (distance $< \epsilon$ to zero module).*

In other words:

- Indecomposables form a dense subset.
- ϵ-almost indecomposables form an open and dense subset.
- Indecomposability is generic (in the closure of the finitely presented persistence modules).

Bottleneck distance and structural stability

The definition of bottleneck (or matching) distance extends to multi-parameter persistence.

Definition

Let $\delta \geq 0$, and let $M, N \colon \mathbf{R}^n \to \mathbf{vect}$.

Bottleneck distance and structural stability

The definition of bottleneck (or matching) distance extends to multi-parameter persistence.

Definition

Let $\delta \geq 0$, and let $M, N: \mathbf{R}^n \to \mathbf{vect}$.

- An δ-*matching* between M and N consists of decompositions $M \cong \bigoplus_{j \in J} M_j$ and $N \cong \bigoplus_{j \in J} N_j$

Bottleneck distance and structural stability

The definition of bottleneck (or matching) distance extends to multi-parameter persistence.

Definition

Let $\delta \geq 0$, and let $M, N\colon \mathbf{R}^n \to \mathbf{vect}$.

- An δ-*matching* between M and N consists of decompositions $M \cong \bigoplus_{j \in J} M_j$ and $N \cong \bigoplus_{j \in J} N_j$
  - indexed by the same set J,

Bottleneck distance and structural stability

The definition of bottleneck (or matching) distance extends to multi-parameter persistence.

Definition

Let $\delta \geq 0$, and let $M, N \colon \mathbf{R}^n \to \mathbf{vect}$.

- An δ-*matching* between M and N consists of decompositions $M \cong \bigoplus_{j \in J} M_j$ and $N \cong \bigoplus_{j \in J} N_j$
  - indexed by the same set J,
  - where each summand is either indecomposable or zero, and

Bottleneck distance and structural stability

The definition of bottleneck (or matching) distance extends to multi-parameter persistence.

Definition

Let $\delta \geq 0$, and let $M, N \colon \mathbf{R}^n \to \mathbf{vect}$.

- An δ-*matching* between M and N consists of decompositions $M \cong \bigoplus_{j \in J} M_j$ and $N \cong \bigoplus_{j \in J} N_j$
  - indexed by the same set J,
  - where each summand is either indecomposable or zero, and
  - such that M_j and N_j are δ-interleaved for every $j \in J$.

Bottleneck distance and structural stability

The definition of bottleneck (or matching) distance extends to multi-parameter persistence.

Definition

Let $\delta \geq 0$, and let $M, N\colon \mathbf{R}^n \to \mathbf{vect}$.

- An δ-*matching* between M and N consists of decompositions $M \cong \bigoplus_{j \in J} M_j$ and $N \cong \bigoplus_{j \in J} N_j$
  - indexed by the same set J,
  - where each summand is either indecomposable or zero, and
  - such that M_j and N_j are δ-interleaved for every $j \in J$.
- The *bottleneck distance* between M and N is

$$d_B(M, N) = \inf\{\delta \geq 0 \mid \text{there exists a } \delta\text{-matching between } M \text{ and } N\}.$$

Bottleneck distance and structural stability

The definition of bottleneck (or matching) distance extends to multi-parameter persistence.

Definition

Let $\delta \geq 0$, and let $M, N\colon \mathbf{R}^n \to \mathbf{vect}$.

- An δ-*matching* between M and N consists of decompositions $M \cong \bigoplus_{j \in J} M_j$ and $N \cong \bigoplus_{j \in J} N_j$
  - indexed by the same set J,
  - where each summand is either indecomposable or zero, and
  - such that M_j and N_j are δ-interleaved for every $j \in J$.
- The *bottleneck distance* between M and N is

$$d_B(M, N) = \inf\{\delta \geq 0 \mid \text{there exists a } \delta\text{-matching between } M \text{ and } N\}.$$

- M is *structurally stable* if passing from interleaving to bottleneck distance is continuous at M.

Bottleneck distance and structural stability

The definition of bottleneck (or matching) distance extends to multi-parameter persistence.

Definition

Let $\delta \geq 0$, and let $M, N \colon \mathbf{R}^n \to \mathbf{vect}$.

- An δ-*matching* between M and N consists of decompositions $M \cong \bigoplus_{j \in J} M_j$ and $N \cong \bigoplus_{j \in J} N_j$
  - indexed by the same set J,
  - where each summand is either indecomposable or zero, and
  - such that M_j and N_j are δ-interleaved for every $j \in J$.
- The *bottleneck distance* between M and N is

$$d_B(M, N) = \inf\{\delta \geq 0 \mid \text{there exists a } \delta\text{-matching between } M \text{ and } N\}.$$

- M is *structurally stable* if passing from interleaving to bottleneck distance is continuous at M.
  - Nearby persistence modules (in the interleaving distance) have a similar structure.

Bottleneck distance and structural stability

We can reinterpret stability of barcodes and density of indecomposables:

Corollary

1-*parameter and multi-parameter persistence modules differ in their different structural stability:*

Bottleneck distance and structural stability

We can reinterpret stability of barcodes and density of indecomposables:

Corollary

1-parameter and multi-parameter persistence modules differ in their different structural stability:

- *Every 1-parameter pfd persistence module is structurally stable.*

Bottleneck distance and structural stability

We can reinterpret stability of barcodes and density of indecomposables:

Corollary

1*-parameter and multi-parameter persistence modules differ in their different structural stability:*

- *Every* 1*-parameter pfd persistence module is structurally stable.*
- *A finitely presentable* n*-parameter persistence module (with* $n > 1$*) is structurally stable if and only if it is indecomposable.*

Origins

When was persistent homology discovered first?

When was persistent homology discovered first?

ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses is satisfied. The critical limits cluster at most at $+\infty$; the critical points are

When was persistent homology discovered first?

Web Images More... Sign in

Google

Scholar 9 results (**0.02** sec) My Citations

- All citations
- Articles
- Case law
- My library

- Any time
- Since 2016
- Since 2015
- Since 2012
- Custom range...

- Sort by relevance
- Sort by date

- include citations

- Create alert

Rank and span in functional topology

Search within citing articles

Exact homomorphism sequences in homology theory ed.ac.uk [PDF]
JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR
The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] ...
Cited by 46 Related articles All 3 versions Cite Save More

Marston Morse and his mathematical works ams.org [PDF]
R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org
American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to ...
Cited by 24 Related articles All 4 versions Cite Save More

Unstable minimal surfaces of higher topological structure
M Morse, CB Tompkins - Duke Math. J, 1941 - projecteuclid.org
1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. Theproblem of the existenceof minimal surfaces of unstable type contains many of the typical difficulties, especially those ofa topological nature. Having studied this ...
Cited by 19 Related articles All 2 versions Cite Save

[PDF] **Persistence in discrete Morse theory** psu.edu [PDF]
U Bauer - 2011 - Citeseer

http://ulrich-bauer.org/persistence-intro-okinawa.pdf

When was persistent homology discovered first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

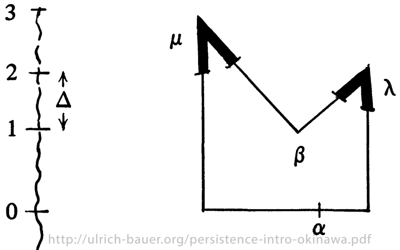
In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was

When was persistent homology discovered first?

inequalities pertain between the dimensions of the A_i and those of $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels [**D**] in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers [**36**] and [**44**] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

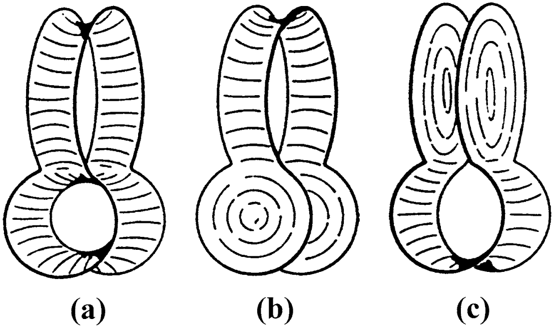
In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.



Motivation and application: minimal surfaces

Problem (Plateau's problem)

Find an immersed disk of least area spanned by a given closed Jordan curve.

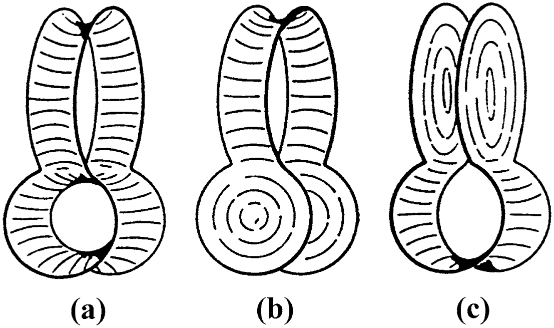


(from Dierkes et al.: *Minimal Surfaces*, 2010)

Motivation and application: minimal surfaces

Problem (Plateau's problem)

Find an immersed disk of least area spanned by a given closed Jordan curve.



(from Dierkes et al.: *Minimal Surfaces*, 2010)

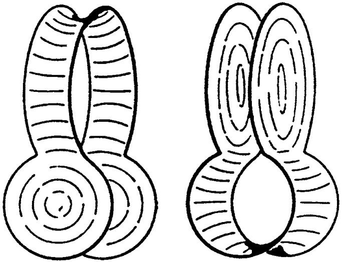
Solution by Douglas (1930; first Fields Medal 1936):

- identifies minimal surfaces with critical points of the *Douglas functional*

Existence of unstable minimal surfaces

Theorem (Morse, Tompkins 1939; Shiffman 1939)

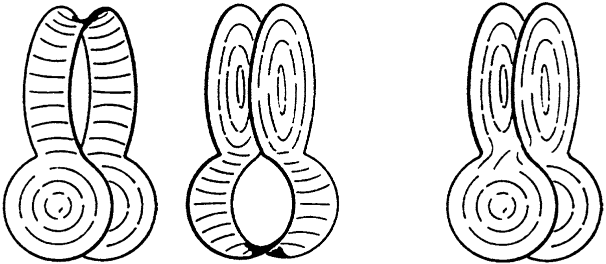
Assume that a given curve bounds two separate stable minimal surfaces.



Existence of unstable minimal surfaces

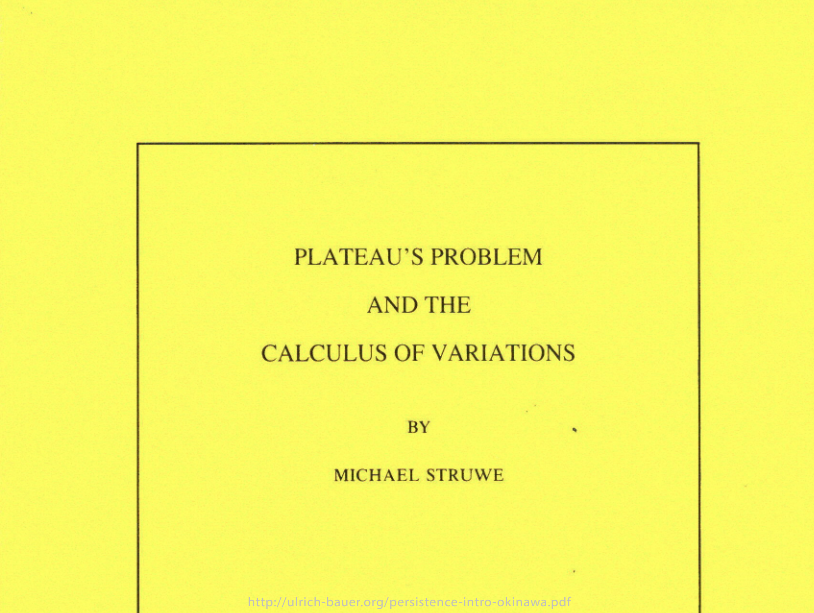
Theorem (Morse, Tompkins 1939; Shiffman 1939)

Assume that a given curve bounds two separate stable minimal surfaces.
Then that curve also bounds an unstable *minimal surface (a critical point that is not a local minimum).*



Whatever happened to functional topology?

Whatever happened to functional topology?



PLATEAU'S PROBLEM

AND THE

CALCULUS OF VARIATIONS

BY

MICHAEL STRUWE

Whatever happened to functional topology?

82 A. The classical Plateau Problem for disc - type minimal surfaces.

The technical complexity and the use of a sophisticated topological machinery (which is not shadowed in our presentation) moreover tend to make Morse-Tompkins' original paper unreadable and inaccessible for the non-specialist, cf. Hildebrandt [4, p. 324].

Confronting Morse-Tompkins' and Shiffman's approach with that given in Chapter 4 we see how much can be gained in simplicity and strength by merely replacing the C^o–topology by the $H^{1/2,2}$–topology and verifying the Palais - Smale - type condition stated in Lemma 2.10.

However, in 1964/65 when Palais and Smale introduced this condition in the calculus of variations it was not clear that it could be meaningful for analyzing the geometry of surfaces, cf. Hildebrandt [4, p. 323 f.].

Q-tame persistence modules

Definition (Chazal et al. 2009)

A persistence module $M : \mathbf{R} \to \mathbf{vect}$ is *q-tame* if all structure maps $M_s \to M_t$ $(s < t)$ have finite rank.

Q-tame persistence modules

Definition (Chazal et al. 2009)

A persistence module $M : \mathbf{R} \to \mathbf{vect}$ is *q-tame* if all structure maps $M_s \to M_t$ $(s < t)$ have finite rank.

- A q-tame persistence module still admits a barcode (disregarding the endpoints).

Q-tame persistence modules

Definition (Chazal et al. 2009)

A persistence module $M : \mathbf{R} \to \mathbf{vect}$ is *q-tame* if all structure maps $M_s \to M_t$ $(s < t)$ have finite rank.

- A q-tame persistence module still admits a barcode (disregarding the endpoints).

Morse's main technical result, in modern language:

- The minimal surface functional has q-tame persistent homology.

Structure of q-tame persistence modules

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

The radical *of a q-tame persistence module* M, $(\operatorname{rad} M)_t = \sum_{s<t} \operatorname{im} M_{s,t}$,

- *admits a barcode decomposition, and*

Structure of q-tame persistence modules

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

The radical *of a q-tame persistence module* M, $(\operatorname{rad} M)_t = \sum_{s<t} \operatorname{im} M_{s,t}$,

- *admits a barcode decomposition, and*
- *the canonical inclusion* $\operatorname{rad} M \hookrightarrow M$ *has an* ephemeral *cokernel:*

Structure of q-tame persistence modules

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

The radical *of a q-tame persistence module* M, $(\operatorname{rad} M)_t = \sum_{s<t} \operatorname{im} M_{s,t}$,

- *admits a barcode decomposition, and*
- *the canonical inclusion* $\operatorname{rad} M \hookrightarrow M$ *has an* ephemeral *cokernel:*
    - *In the kernel, all maps between different indices* $s < t$ *are zero.*

Structure of q-tame persistence modules

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

The radical *of a q-tame persistence module* M*,* $(\operatorname{rad} M)_t = \sum_{s<t} \operatorname{im} M_{s,t}$*,*

- *admits a barcode decomposition, and*
- *the canonical inclusion* $\operatorname{rad} M \hookrightarrow M$ *has an* ephemeral *cokernel:*
  - *In the kernel, all maps between different indices* $s < t$ *are zero.*

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

Every q-tame persistence module has a unique persistence diagram that completely describes its isomorphism type in the observable category.

Structure of q-tame persistence modules

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

The radical *of a q-tame persistence module* M, $(\operatorname{rad} M)_t = \sum_{s<t} \operatorname{im} M_{s,t}$,

- *admits a barcode decomposition, and*
- *the canonical inclusion* $\operatorname{rad} M \hookrightarrow M$ *has an* ephemeral *cokernel:*
  - *In the kernel, all maps between different indices* $s < t$ *are zero.*

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

Every q-tame persistence module has a unique persistence diagram that completely describes its isomorphism type in the observable category.

- *A* persistence diagram *describes the intervals in a barcode, modulo the endpoints.*

Structure of q-tame persistence modules

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

The radical *of a q-tame persistence module* M, $(\operatorname{rad} M)_t = \sum_{s<t} \operatorname{im} M_{s,t}$,

- *admits a barcode decomposition, and*
- *the canonical inclusion* $\operatorname{rad} M \hookrightarrow M$ *has an* ephemeral *cokernel:*
  - *In the kernel, all maps between different indices* $s < t$ *are zero.*

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

Every q-tame persistence module has a unique persistence diagram that completely describes its isomorphism type in the observable category.

- *A* persistence diagram *describes the intervals in a barcode, modulo the endpoints.*
- *The* observable category *is the category of persistence modules, modulo ephemeral modules.*

Generalized Morse inequalities

Assume that the sublevel sets of a bounded function $f : X \to \mathbb{R}$ have q-tame persistent homology.

Generalized Morse inequalities

Assume that the sublevel sets of a bounded function $f : X \to \mathbb{R}$ have q-tame persistent homology.

- The persistence diagram of f has finitely many bars of length $> \epsilon$, for every $\epsilon > 0$.

Generalized Morse inequalities

Assume that the sublevel sets of a bounded function $f : X \to \mathbb{R}$ have q-tame persistent homology.

- The persistence diagram of f has finitely many bars of length $> \epsilon$, for every $\epsilon > 0$.
- This gives generalized Morse inequalities

$$\sum_{i=0}^{d} (-1)^{d-i} (m_i^\epsilon - \beta_i) \geq 0.$$

  - m_i^ϵ counts endpoints in intervals with length $> \epsilon$

Generalized Morse inequalities

Assume that the sublevel sets of a bounded function $f : X \to \mathbb{R}$ have q-tame persistent homology.

- The persistence diagram of f has finitely many bars of length $> \epsilon$, for every $\epsilon > 0$.
- This gives generalized Morse inequalities

$$\sum_{i=0}^{d} (-1)^{d-i} (m_i^\epsilon - \beta_i) \geq 0.$$

  - m_i^ϵ counts endpoints in intervals with length $> \epsilon$
- Morse and Tompkins used this idea to show the existence of an unstable minimal surface.

Vietoris vs singular homology

Vietoris vs singular homology

- A compact sublevel set filtration $(f_{\leq t})_{t \in \mathbb{R}}$ is continuous from above:

$$f_{\leq t} = \bigcap_{u>t} f_{\leq u} = \lim_{u>t} f_{\leq u}$$

Vietoris vs singular homology

- A compact sublevel set filtration $(f_{\leq t})_{t\in\mathbb{R}}$ is continuous from above:

$$f_{\leq t} = \bigcap_{u>t} f_{\leq u} = \lim_{u>t} f_{\leq u}$$

- Vietoris (equivalently, Čech) homology (with field coefficients) preserves limits:

$$\lim \circ \check{H}_* = \check{H}_* \circ \lim$$

Vietoris vs singular homology

- A compact sublevel set filtration $(f_{\leq t})_{t \in \mathbb{R}}$ is continuous from above:

$$f_{\leq t} = \bigcap_{u>t} f_{\leq u} = \lim_{u>t} f_{\leq u}$$

- Vietoris (equivalently, Čech) homology (with field coefficients) preserves limits:

$$\lim \circ \check{H}_* = \check{H}_* \circ \lim$$

- Therefore, persistent Vietoris/Čech homology is also continuous from above:

$$\check{H}_*(f_{\leq t}) = \lim_{u>t} \check{H}_*(f_{\leq u})$$

Weakly LC filtrations

Definition (Morse 1937; paraphrased)

The sublevel set filtration of a function $f: X \to \mathbb{R}$ is said to be *weakly locally connected* if for

Weakly LC filtrations

Definition (Morse 1937; paraphrased)

The sublevel set filtration of a function $f\colon X \to \mathbb{R}$ is said to be *weakly locally connected* if for

- any point $x \in X$,

Weakly LC filtrations

Definition (Morse 1937; paraphrased)

The sublevel set filtration of a function $f : X \to \mathbb{R}$ is said to be *weakly locally connected* if for

- any point $x \in X$,
- any value $t > f(x)$, and

Weakly LC filtrations

Definition (Morse 1937; paraphrased)

The sublevel set filtration of a function $f : X \to \mathbb{R}$ is said to be *weakly locally connected* if for

- any point $x \in X$,
- any value $t > f(x)$, and
- any neighborhood V of x in the sublevel set $f^{-1}(-\infty, t]$

there is

Weakly LC filtrations

Definition (Morse 1937; paraphrased)

The sublevel set filtration of a function $f : X \to \mathbb{R}$ is said to be *weakly locally connected* if for

- any point $x \in X$,
- any value $t > f(x)$, and
- any neighborhood V of x in the sublevel set $f^{-1}(-\infty, t]$

there is

- a value s with $f(x) < s < t$ and

Weakly LC filtrations

Definition (Morse 1937; paraphrased)

The sublevel set filtration of a function $f: X \to \mathbb{R}$ is said to be *weakly locally connected* if for

- any point $x \in X$,
- any value $t > f(x)$, and
- any neighborhood V of x in the sublevel set $f^{-1}(-\infty, t]$

there is

- a value s with $f(x) < s < t$ and
- a neighborhood $U \subseteq V$ of x in the sublevel set $f^{-1}(-\infty, s]$

such that the inclusion $U \to V$ is homotopic to a constant map.

Q-tameness from local connectivity

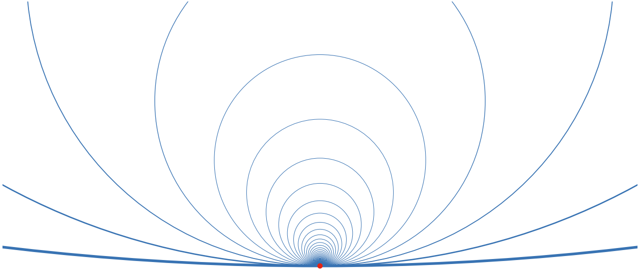
Theorem (Morse, 1937)

If the sublevel set filtration of $f: X \to \mathbb{R}$ is compact and weakly locally connected, *then it has q-tame persistent Vietoris homology.*

Q-tameness from local connectivity

Theorem (Morse, 1937; incorrect)

If the sublevel set filtration of $f : X \to \mathbb{R}$ is compact and weakly locally connected, *then it has q-tame persistent Vietoris homology.*



Homologically locally small filtrations

Definition

The sublevel set filtration of a function $f\colon X \to \mathbb{R}$ is *homologically locally connected (HLC)* if

- for any point $x \in X$,
- any values $f(x) < s < t$, and
- any neighborhood V of x in the sublevel set $f^{-1}(-\infty, t]$,

there is

- a neighborhood $U \subseteq V$ of x in the sublevel set $f^{-1}(-\infty, s]$

such that the inclusion $U \hookrightarrow V$ induces a zero map on homology.

Homologically locally small filtrations

Definition

The sublevel set filtration of a function $f\colon X \to \mathbb{R}$ is *homologically locally connected (HLC)* if

- for any point $x \in X$,
- any values $f(x) < s < t$, and
- any neighborhood V of x in the sublevel set $f^{-1}(-\infty, t]$,

there is

- a neighborhood $U \subseteq V$ of x in the sublevel set $f^{-1}(-\infty, s]$

such that the inclusion $U \hookrightarrow V$ induces a zero map on homology.

Theorem (B, Medina-Mardones, Schmahl 2021)

If the sublevel set fitration of $f\colon X \to \mathbb{R}$ is compact and homologically locally connected, *then their persistent homology is q-tame.*

- f is not required to be continuous

Homologically locally small filtrations

Definition

The sublevel set filtration of a function $f: X \to \mathbb{R}$ is *homologically locally connected (HLC)* if

- for any point $x \in X$,
- any values $f(x) < s < t$, and
- any neighborhood V of x in the sublevel set $f^{-1}(-\infty, t]$,

there is

- a neighborhood $U \subseteq V$ of x in the sublevel set $f^{-1}(-\infty, s]$

such that the inclusion $U \hookrightarrow V$ induces a zero map on homology.

Theorem (B, Medina-Mardones, Schmahl 2021)

If the sublevel set fitration of $f: X \to \mathbb{R}$ is compact and homologically locally connected, *then their persistent homology is q-tame.*

- f is not required to be continuous
- Conditions are satisfied by the Douglas functional

Homologically locally small filtrations

Definition

The sublevel set filtration of a function $f\colon X \to \mathbb{R}$ is *homologically locally connected (HLC)* if

- for any point $x \in X$,
- any values $f(x) < s < t$, and
- any neighborhood V of x in the sublevel set $f^{-1}(-\infty, t]$,

there is

- a neighborhood $U \subseteq V$ of x in the sublevel set $f^{-1}(-\infty, s]$

such that the inclusion $U \hookrightarrow V$ induces a zero map on homology.

Theorem (B, Medina-Mardones, Schmahl 2021)

If the sublevel set fitration of $f\colon X \to \mathbb{R}$ is compact and homologically locally connected, *then their persistent homology is q-tame.*

- f is not required to be continuous
- Conditions are satisfied by the Douglas functional
- Fixes the gap in Morse/Tompkins' proof

Simplification

Topological simplification of functions

Consider the following problem:

Problem (Topological simplification)

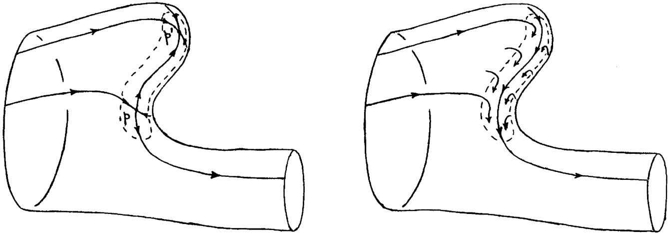
Given a function $f\colon X \to \mathbb{R}$ *and a real number* $\delta \geq 0$*, find a function* f_δ

- *with the minimal number of critical points*
- *subject to* $\|f_\delta - f\|_\infty \leq \delta$.

Persistence and Morse theory

Morse theory (smooth or discrete):

- Relates critical points to homology of sublevel sets
- Provides a method for *canceling* pairs of critical points

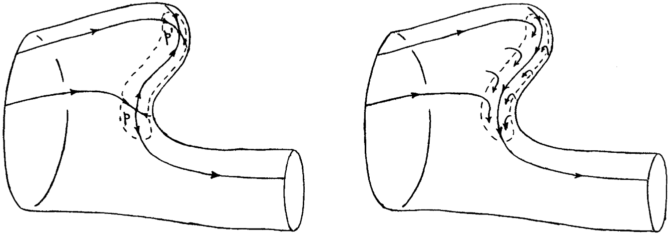


(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

Persistence and Morse theory

Morse theory (smooth or discrete):

- Relates critical points to homology of sublevel sets
- Provides a method for *canceling* pairs of critical points



(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

Persistent homology of a Morse function:

- critical points correspond to endpoints of barcode intervals

Canceling persistence pairs

By stability of persistence barcodes:

Proposition

The critical points of f with persistence $> 2\delta$ provide a lower bound on the number of critical points of any function g with $\|g - f\|_\infty \leq \delta$.

Canceling persistence pairs

By stability of persistence barcodes:

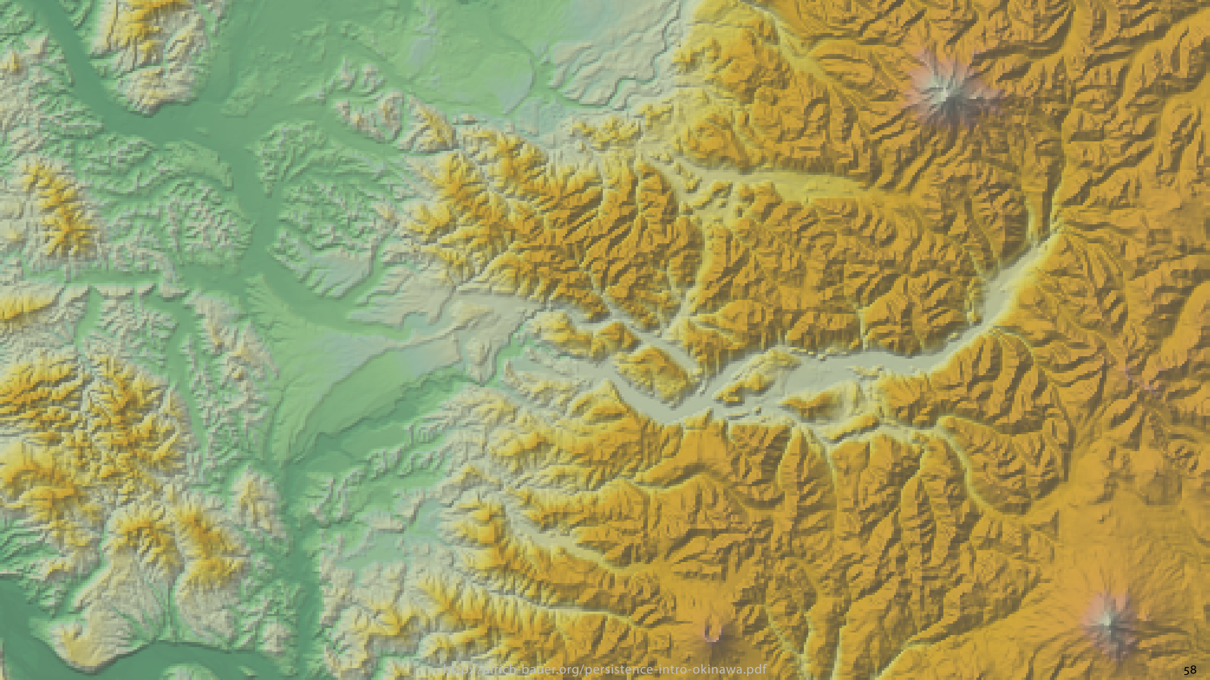
Proposition

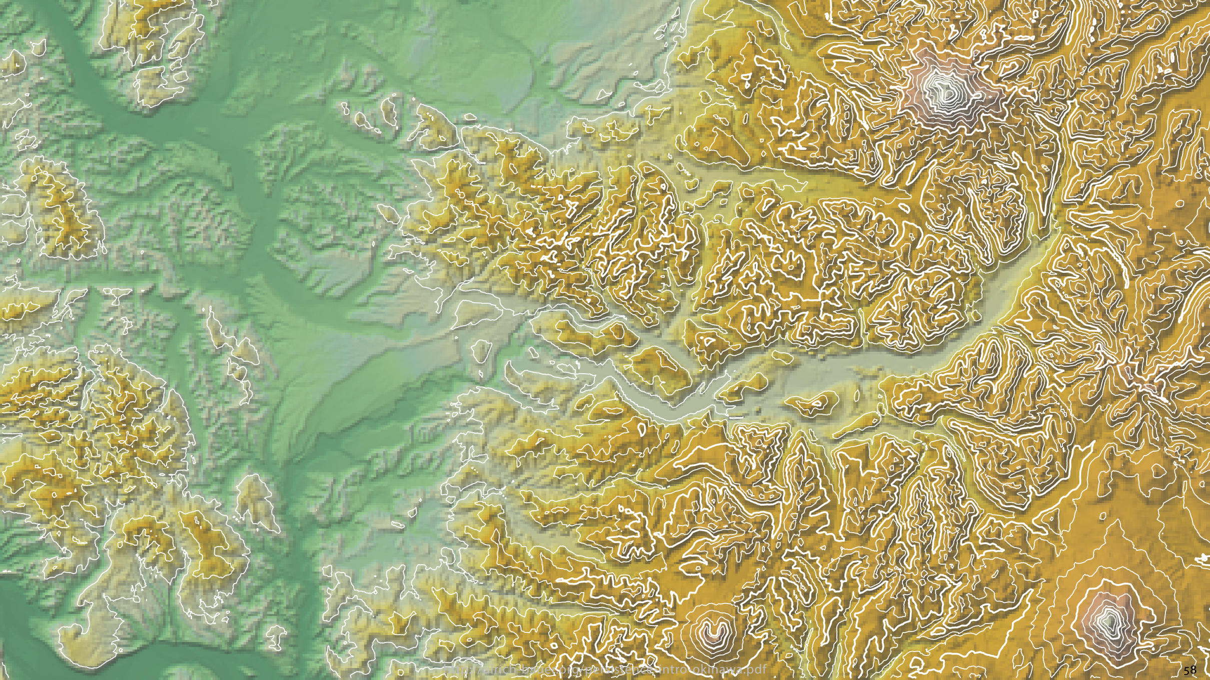
The critical points of f with persistence $> 2\delta$ provide a lower bound on the number of critical points of any function g with $\|g - f\|_\infty \leq \delta$.

Theorem (B, Lange, Wardetzky, 2011)

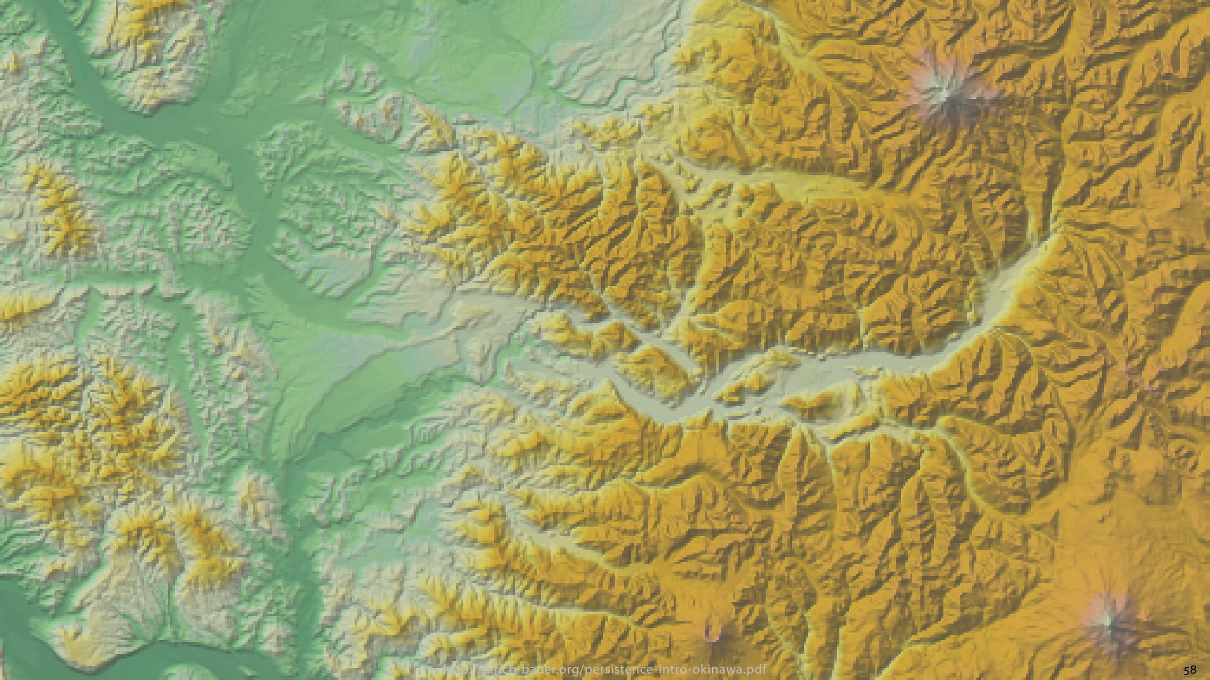
This lower bound can be achieved for a function on a surface

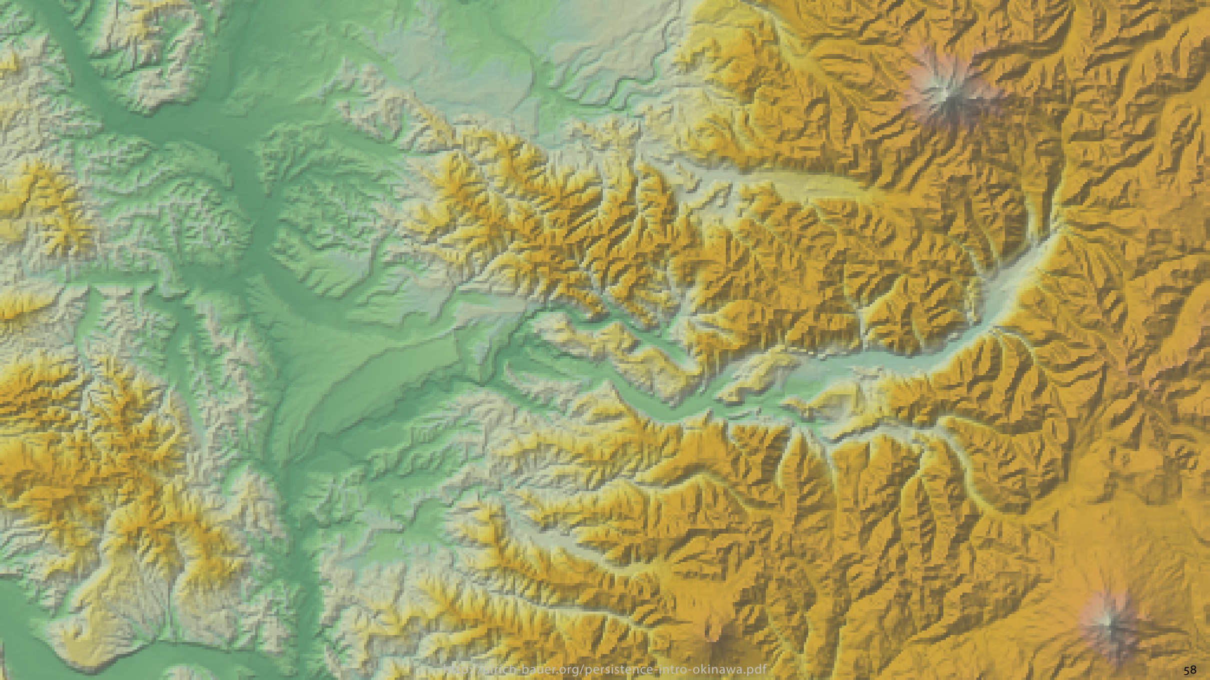
- *by canceling all critical points with persistence $\leq 2\delta$.*



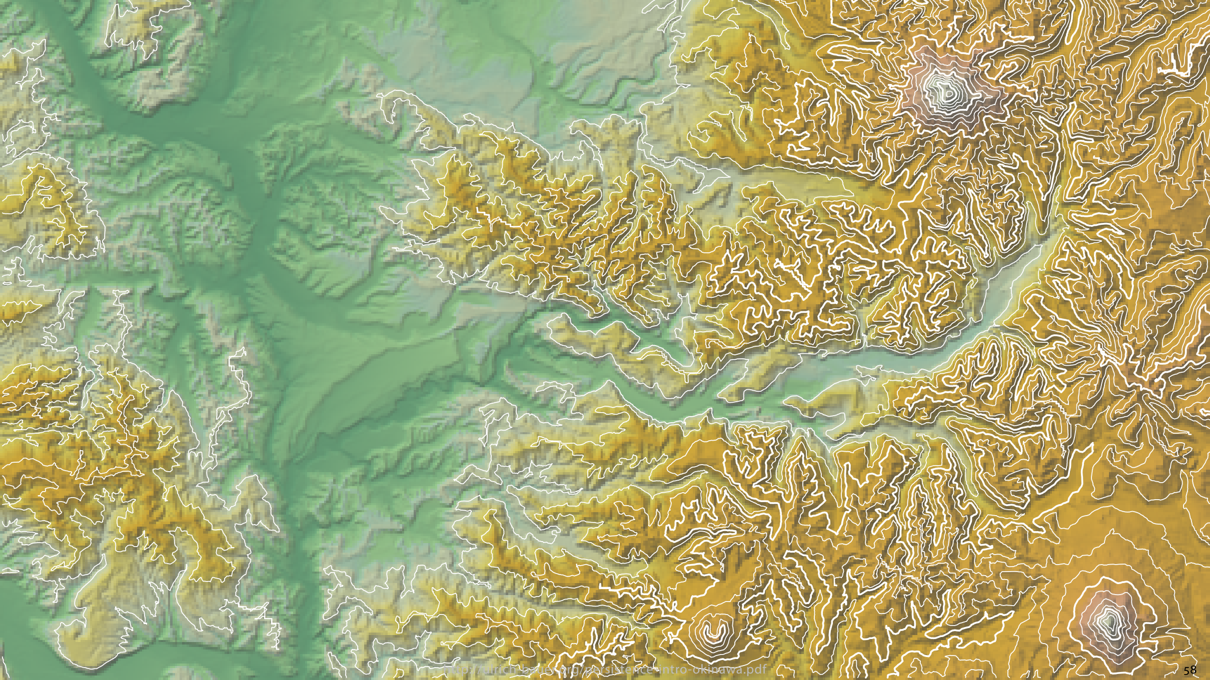


http://ulrich-bauer.org/persistence-intro-okinawa.pdf

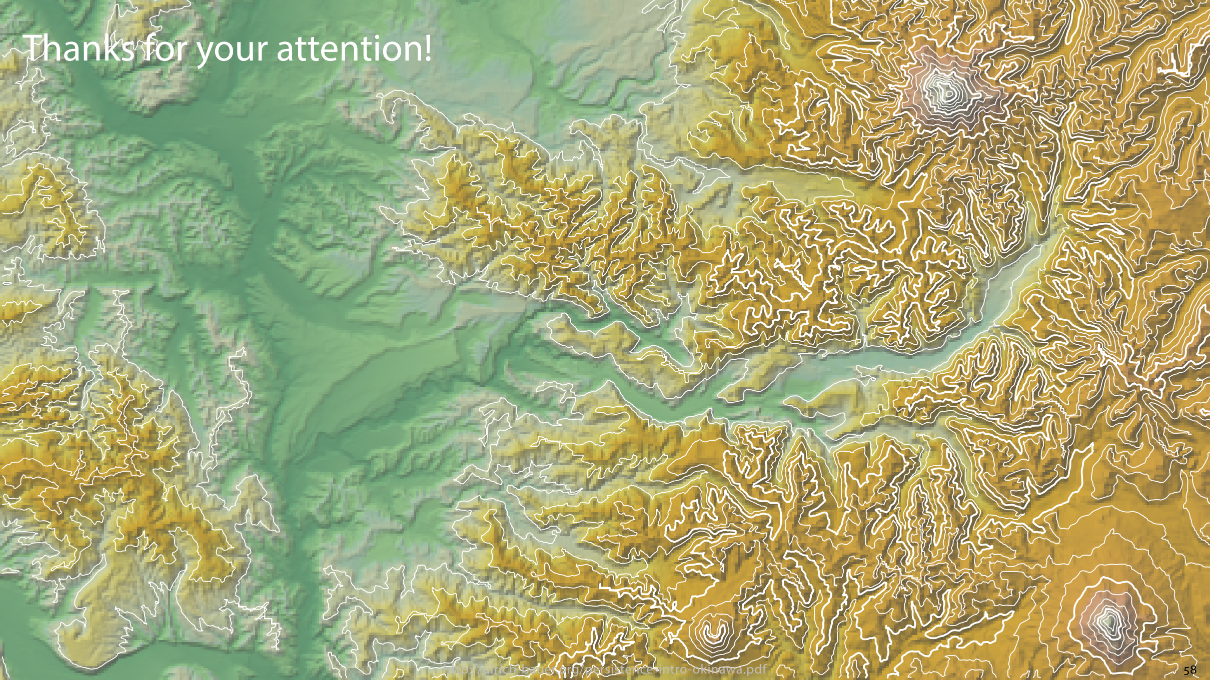




http://ulrich-bauer.org/persistence-intro-okinawa.pdf



Thanks for your attention!



http://ulrich-bauer.org/persistence-intro-okinawa.pdf