An introduction to persistent homology

Part 2: Barcodes

Ulrich Bauer

TUM

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Summer School on Persistent Homology and Barcodes
Schloss Rauischholzhausen

Stability: from point clouds to barcodes

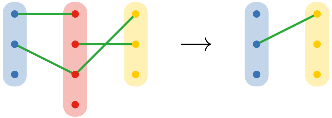
point cloud	$P \subset \mathbb{R}^d$	Hausdorff distance
$\downarrow$ distance		
function	$f : \mathbb{R}^d \to \mathbb{R}$	L_∞ distance
$\downarrow$ sublevel sets		
topological spaces (filtration)	$K : \mathbf{R} \to \mathbf{Top}$	Interleaving distance
$\downarrow$ homology		
vector spaces (persistence module)	$M : \mathbf{R} \to \mathbf{Vect}$	Interleaving distance
$\downarrow$ decomposition		
intervals (persistence barcode)	$B : \mathbf{R} \to \mathbf{Mch}$	Bottleneck/interleaving distance

The category of matchings

Consider the category **Mch** (a subcategory of the category **Rel** of sets and relations) with

- objects: sets,
- morphisms: matchings (partial bijections).

Composition:



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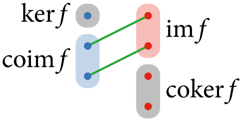

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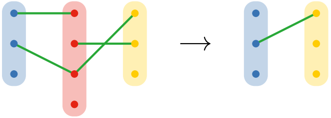

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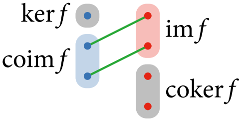

$\mathbf{Mch}$ is *Puppe-exact (p-exact)*:

- it has a zero object ($\varnothing$)
- it has all (co)kernels
- every mono (epi) is (co)kernel
- every morphism $f : A \to B$ has an epi-mono factorization $A \twoheadrightarrow \operatorname{im} f \hookrightarrow B$

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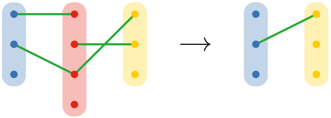
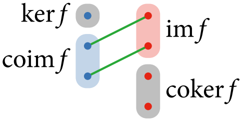
Composition: 

(Co)kernel/(co)image: 

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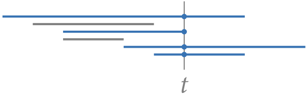
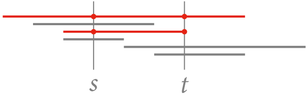
but not additive:

- it does not have all (co)products

From barcodes to matching diagrams (and back)

- A barcode (collection of intervals) can be read as a diagram $\mathbf{R} \to \mathbf{Mch}$:





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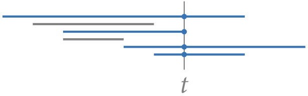

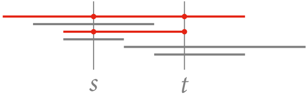

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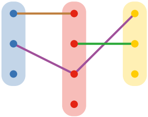

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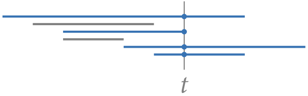

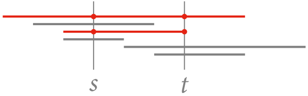

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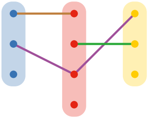

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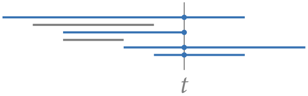

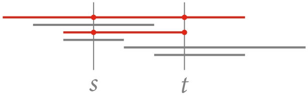

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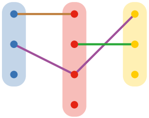

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Turn this into an equivalence of categories $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbf{R}}$

A category of barcodes

Proposition

The functor category $\mathbf{Mch}^{\mathbf{R}}$ *is equivalent to* $\mathbf{Barc}$*, the category with*

- *objects: barcodes (as a disjoint union of intervals),*
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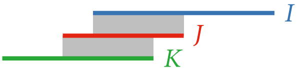
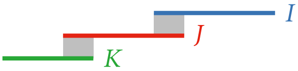
- *composition of overlap matchings:* $\tau \bullet \sigma = \{(I, K) \in \tau \circ \sigma \mid I \text{ overlaps } K \text{ above}\}$
  (where $\tau \circ \sigma$ *is the standard composition of matchings)*



$(I, K) \in \tau \bullet \sigma$ *(overlap)*



$(I, K) \notin \tau \bullet \sigma$ *(no overlap)*

Bottleneck distance as an interleaving distance

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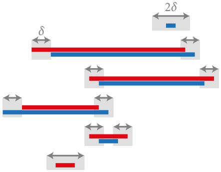

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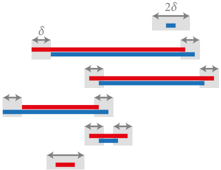

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- δ-*interleaving* between diagrams $X, Y : \mathbf{R} \to \mathcal{C}$ (in any category $\mathcal{C}$):

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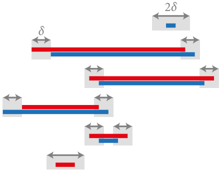

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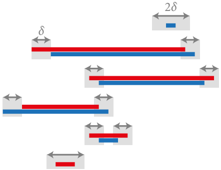

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Proposition

$d_I = d_B$ *(using the equivalence* $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbf{R}}$).

Structure of persistence sub-/quotient modules

Proposition

Let $M \twoheadrightarrow N$ *be an epimorphism.*
Then there is an injection of barcodes $B(N) \hookrightarrow B(M)$ *such that if* J *is mapped to* I*, then*

- I *and* J *are aligned below, and*
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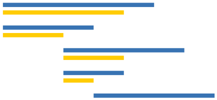

This construction is functorial.

Dually, there is an injection $B(M) \hookrightarrow B(N)$ *for monomorphisms* $M \hookrightarrow N$*.*

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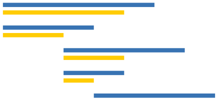
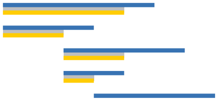


Dually, there is an injection $B(M) \hookrightarrow B(N)$ for monomorphisms $M \hookrightarrow N$.

Rephrased for $\mathbf{Mch}^{\mathbf{R}}$:

Proposition

There is a functor from epimorphisms of persistence modules to epimorphisms of matching diagrams.



Dually, there is a functor from monomorphisms to monomorphisms.

Induced matchings

For $f : M \to N$ a morphism of pfd persistence modules, the epi-mono factorization

$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$$

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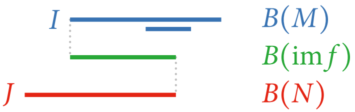

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Theorem

Assume that $\ker f$ *is* δ*-trivial. If* $\chi(f)$ *matches* I *to* J*, then*

1. I *overlaps* J*, and* J *overlaps* $I(\delta)$.
2. *Any unmatched interval of* $B(M)$ *is* δ*-trivial.*

There is a dual statement for $\operatorname{coker} f$ δ*-trivial.*

The categorified induced matching theorem

Induced matching theorem, rephrased in $\mathbf{Mch}^{\mathbf{R}}$:

Theorem

If $f : M \to N$ has δ-trivial (co)kernel, then so does the induced matching $\chi(f)\colon B(M) \nrightarrow B(N)$.

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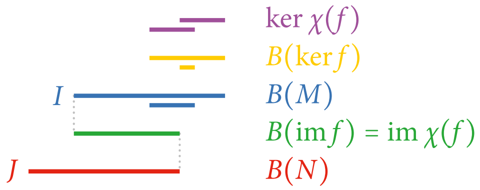

Note:

- We always have $B(\operatorname{im} f) = \operatorname{im} \chi(f)$ by construction.
- But $\ker \chi(f)$ may differ from $B(\ker f)$.
- The induced matching may strictly decrease the triviality of the kernel.

A general criterion for δ-trivial (co)kernels

Lemma

Consider a morphism $f : M \to N$ between diagrams $M, N : \mathbf{R} \to \mathcal{A}$ in a Puppe-exact category $\mathcal{A}$, and let $s : N(-\delta) \to N$ be the internal shift morphism. The following are equivalent:

1. $\operatorname{coker} f$ *is δ-trivial;*
2. *the image monomorphism $\operatorname{im} s \hookrightarrow N$ factors through the image monomorphism $\operatorname{im} f \hookrightarrow N$ as*

$$
\begin{tikzcd}
 & M \arrow[dr, two heads] \arrow[drr, bend left, "f"] & & \\
 & & \operatorname{im} f \arrow[dr, hook] & \\
N(-\delta) \arrow[r, two heads] \arrow[rrr, bend right, "s"'] & \operatorname{im} s \arrow[ur, hook] \arrow[rr, hook] & & N
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A dual statement holds for $\ker f$.

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$\square$

The algebraic stability theorem (categorically)

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Two pfd persistence modules M *and* N *are* δ*-interleaved if and only if their barcodes* $B(M)$ *and* $B(N)$ *are* δ*-interleaved. In particular,* $d_I(M,N) = d_I(B(M), B(N))$.

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Forward direction:

- A δ-interleaving morphism $f : M \to N(\delta)$ has 2δ-trivial kernel and cokernel.

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The algebraic stability theorem (categorically)

Theorem

Two pfd persistence modules M and N are δ-interleaved if and only if their barcodes $B(M)$ and $B(N)$ are δ-interleaved. In particular, $d_I(M,N) = d_I(B(M), B(N))$.

Proof outline.

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Converse direction:

- Apply the canonical functor $\mathbf{Mch} \to \mathbf{Vect}$. $\square$

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- Goal: construct barcode/matching diagram of persistence module *without* decomposition
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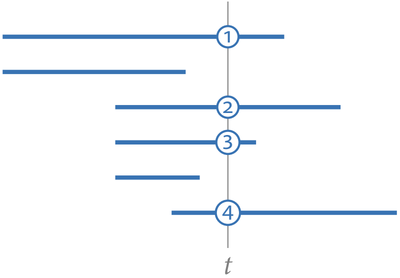

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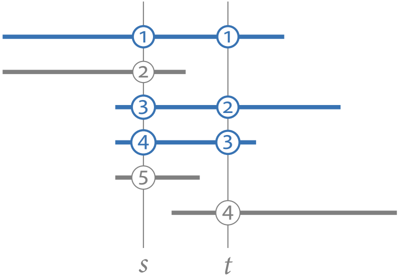

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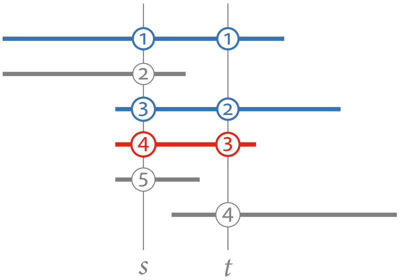

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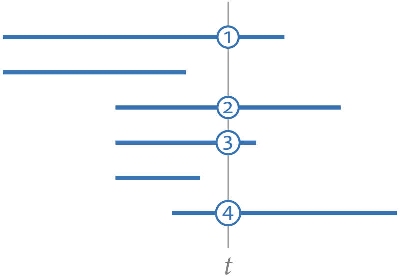

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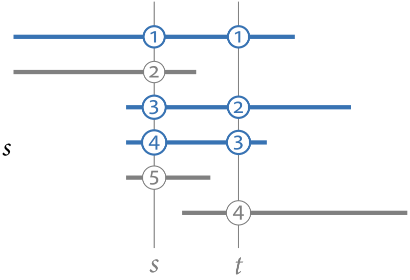

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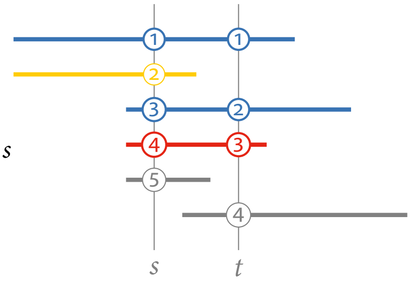

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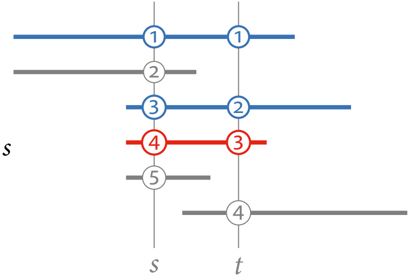

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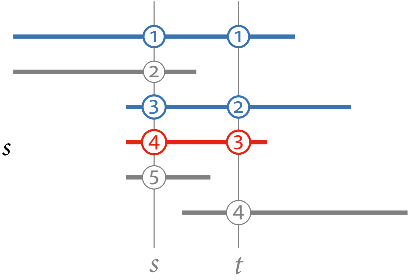

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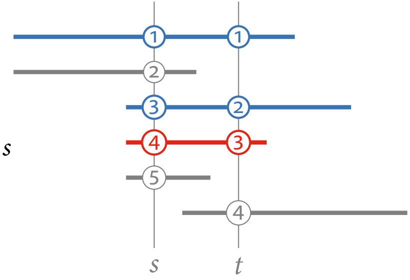

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This specifies the barcode of M (as a matching diagram) based on ranks only.

Functoriality

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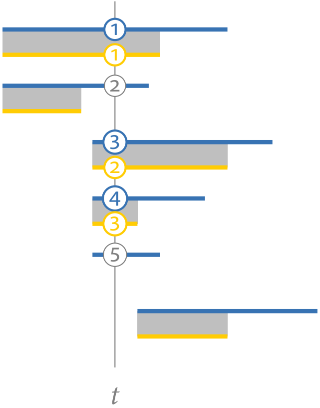

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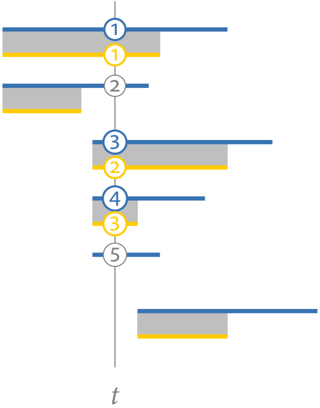

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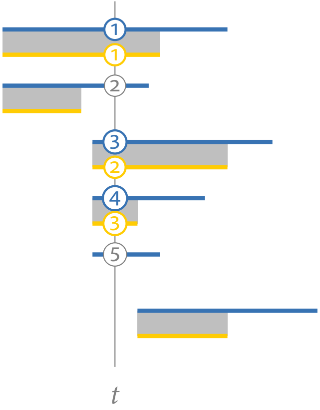

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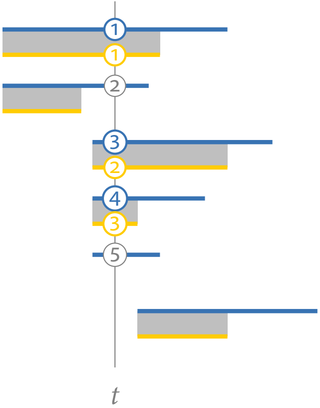

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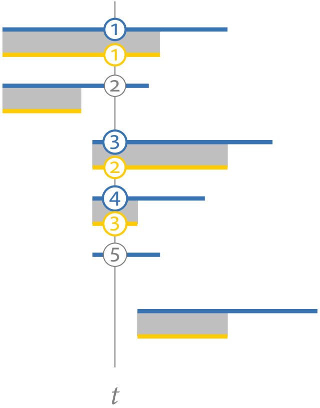

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Obtain induced matching and algebraic stability theorems without invoking the interval decomposition theorem