Induced matchings and the algebraic stability of persistence barcodes

Ulrich Bauer

TUM

July 12, 2015

Geometry Workshop Seggau

Joint work with Michael Lesnick (IMA)

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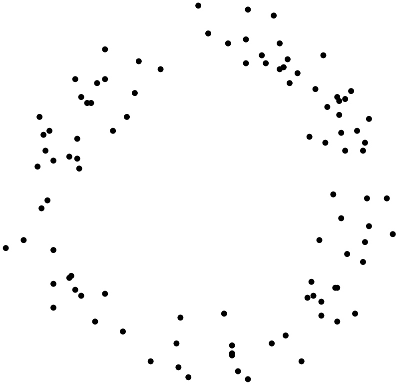

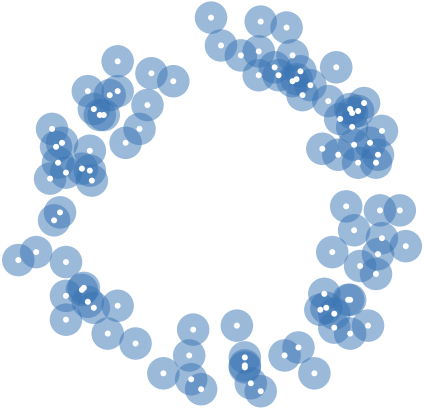

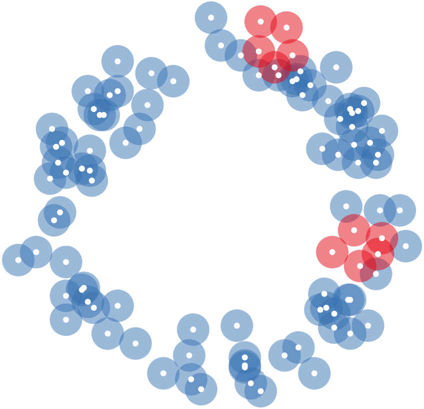

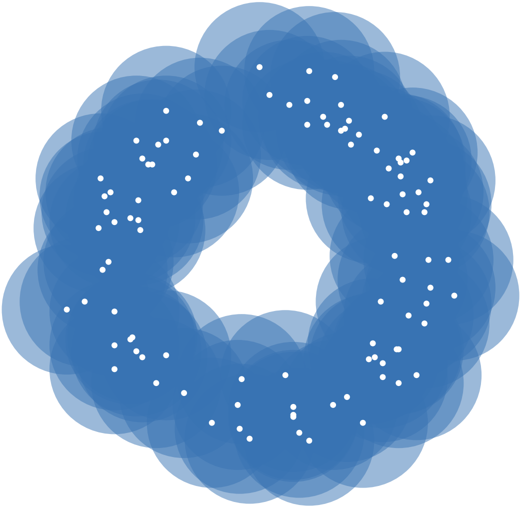

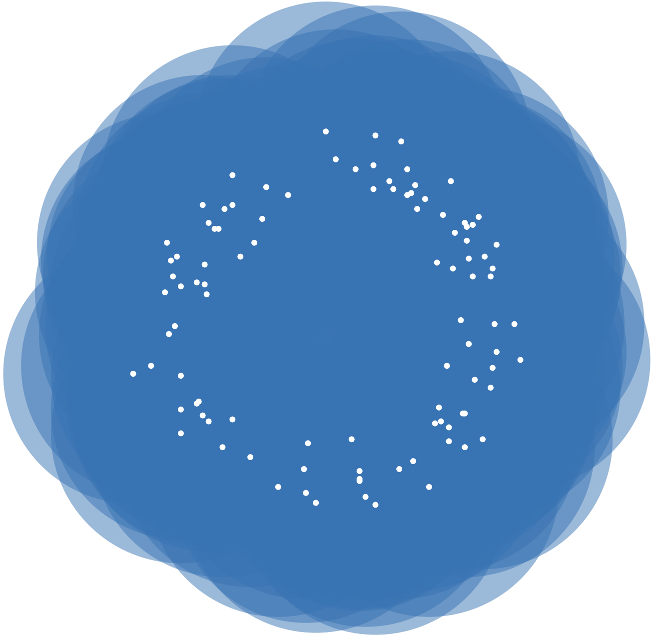

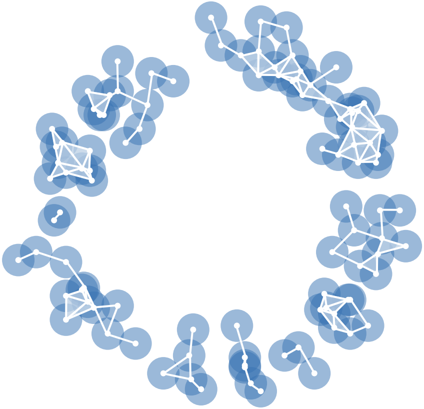

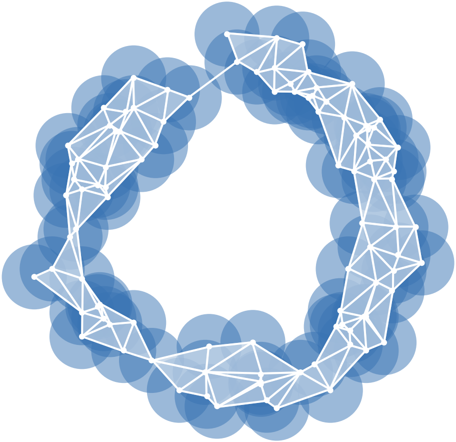

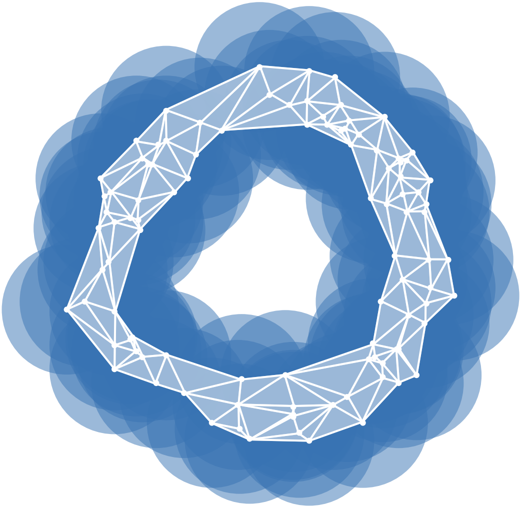

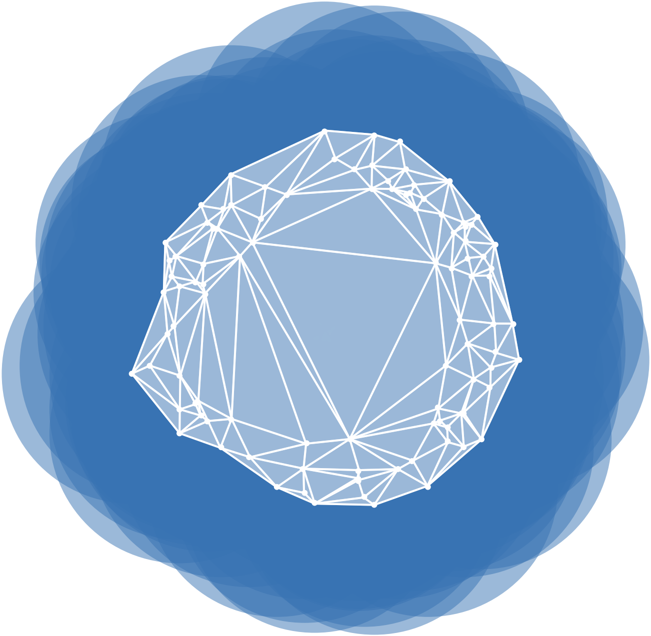

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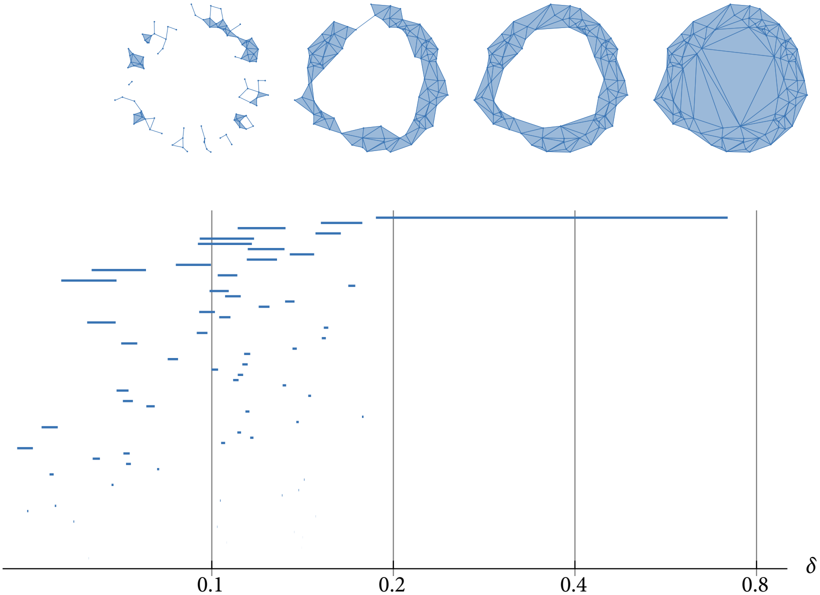

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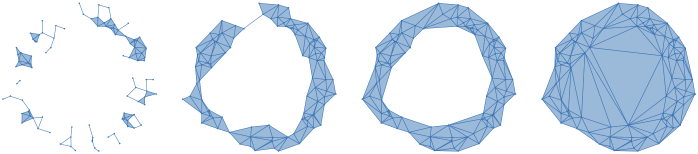

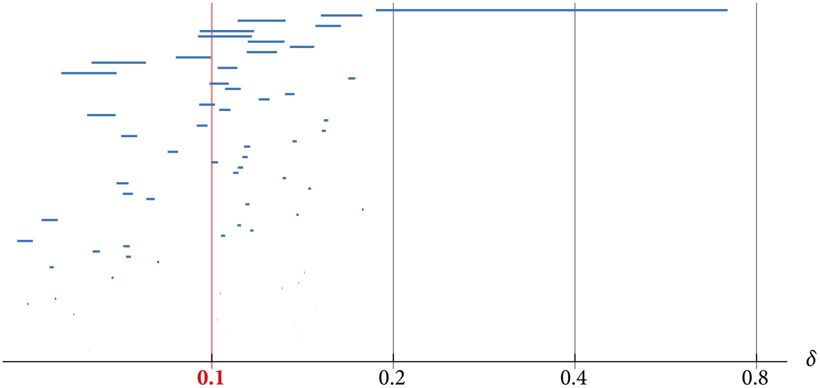

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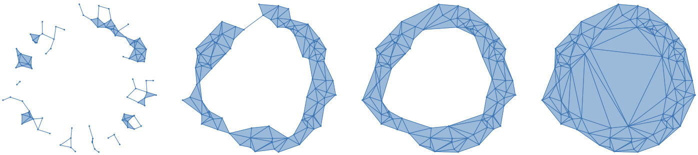

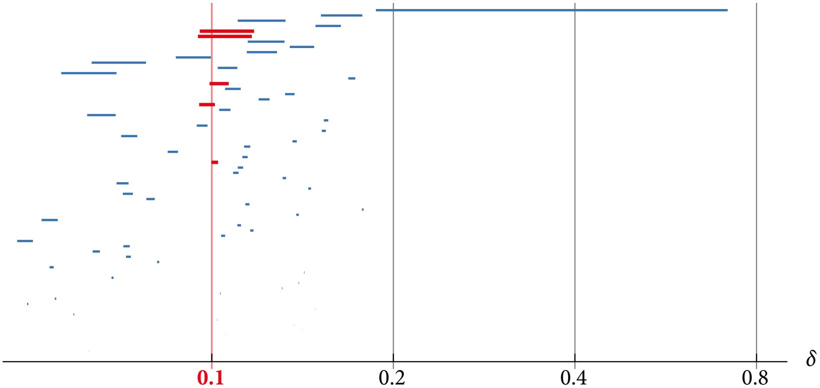

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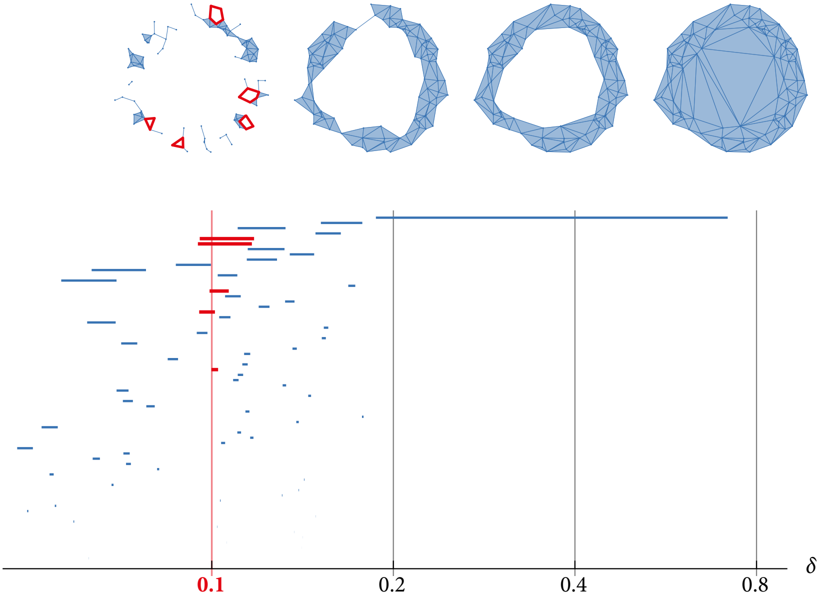

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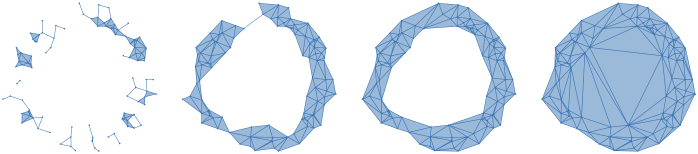

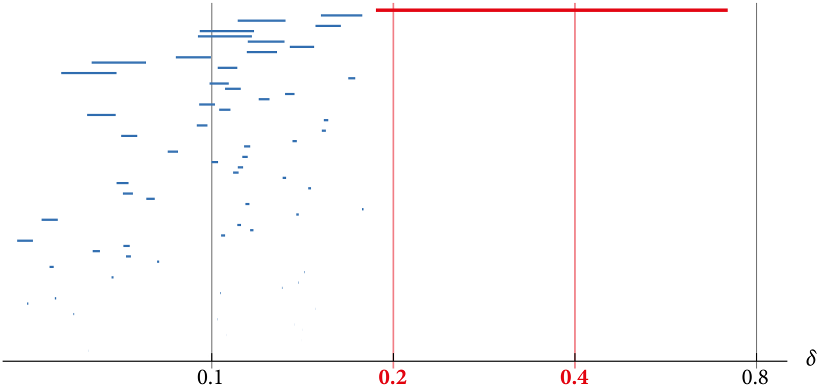

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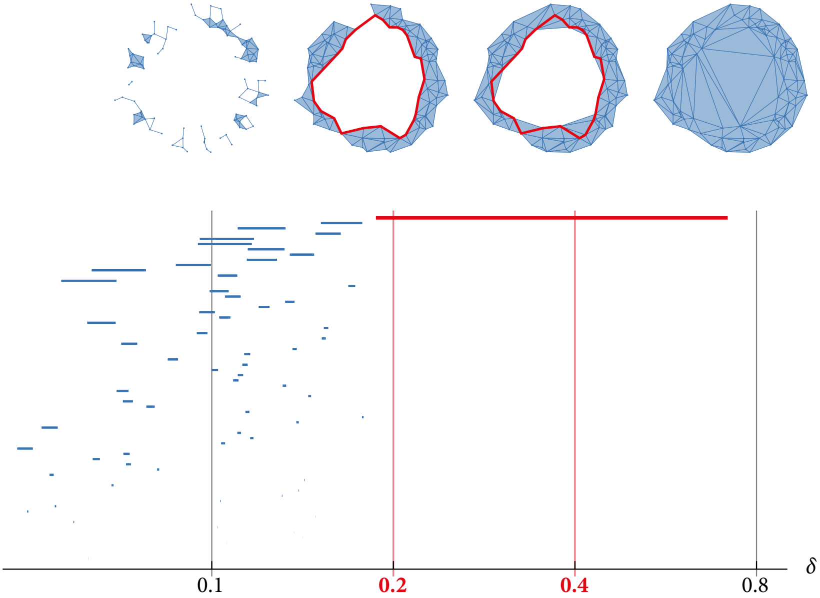

What is persistent homology?

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- A filtration is a certain diagram $K : \mathbf{R} \to \mathbf{Top}$.
  - A topological space K_t for each $t \in \mathbb{R}$
  - An inclusion map $K_s \hookrightarrow K_t$ for each $s \leq t \in \mathbb{R}$

Inference

Homology inference

Problem (Homology inference)

Determine the homology $H_*(\Omega)$ of a shape $\Omega \subset \mathbb{R}^d$ from a finite approximation P close to Ω.

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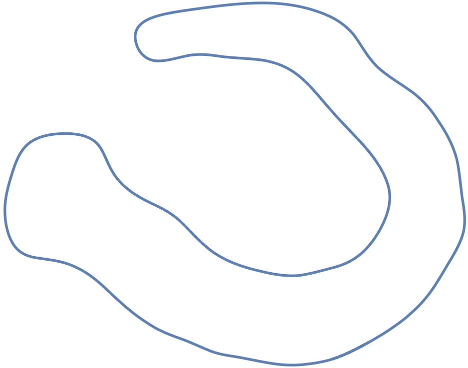
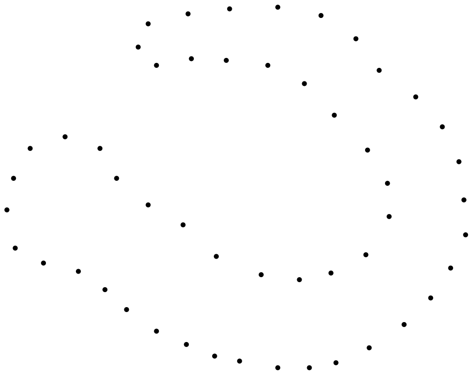
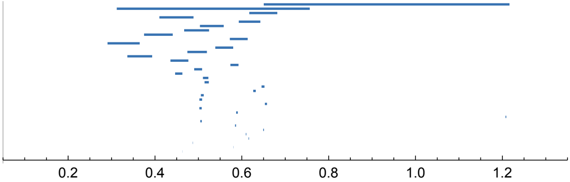
Idea:

- approximate the shape by a thickening $B_\delta(P)$ covering Ω







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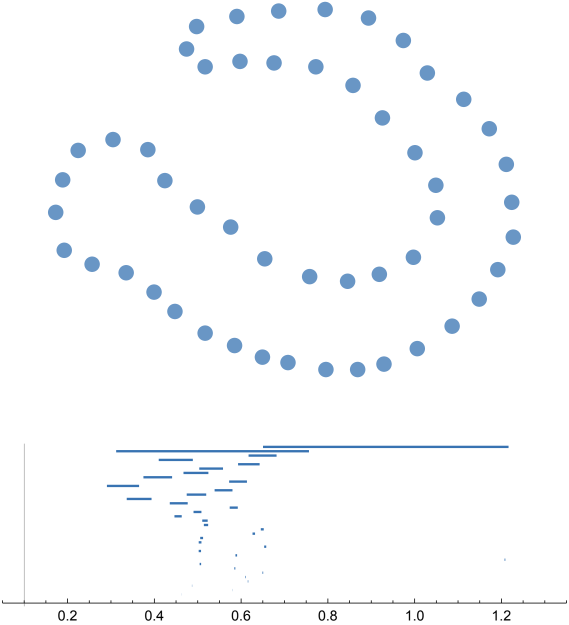
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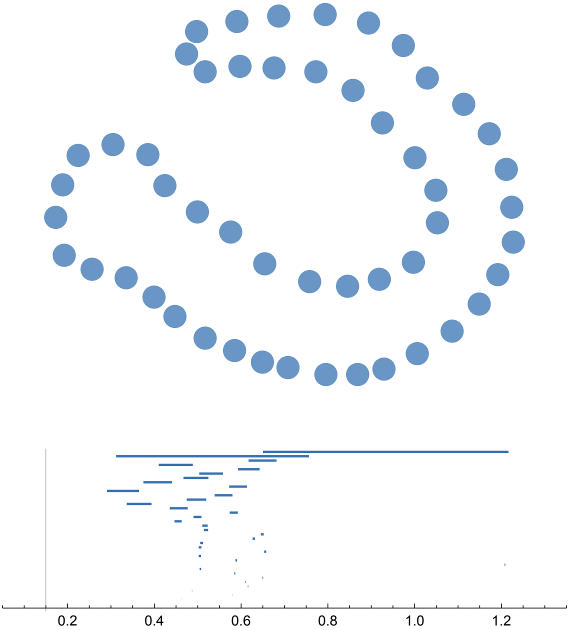
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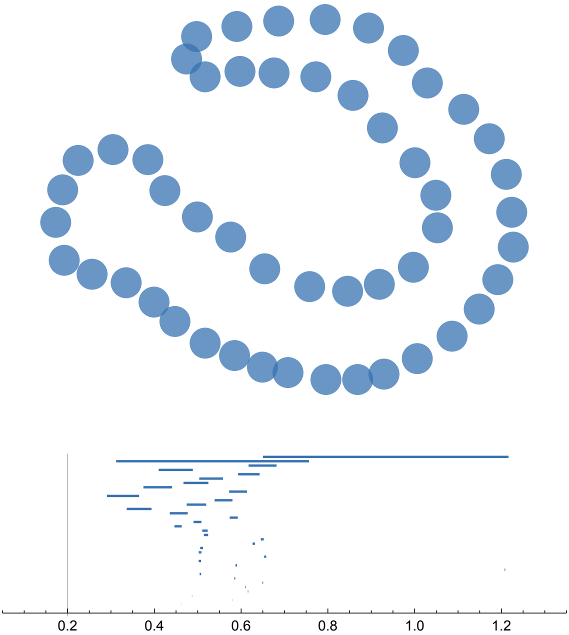
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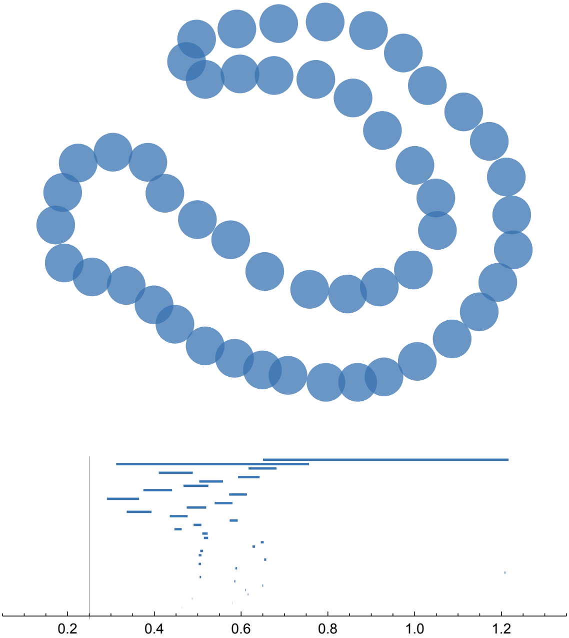
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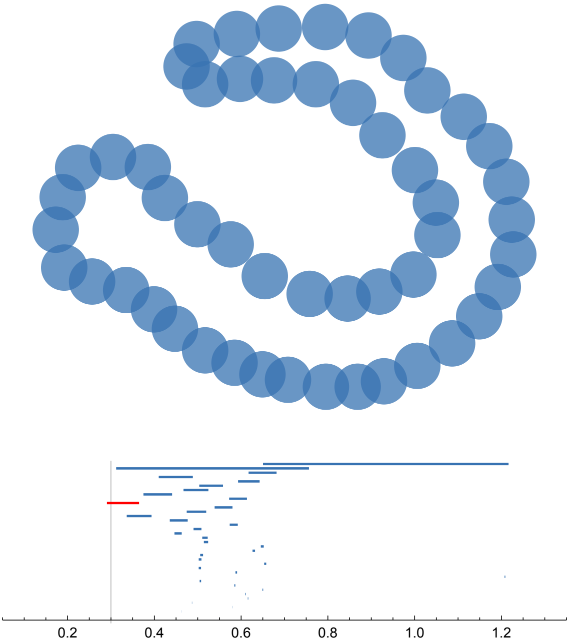

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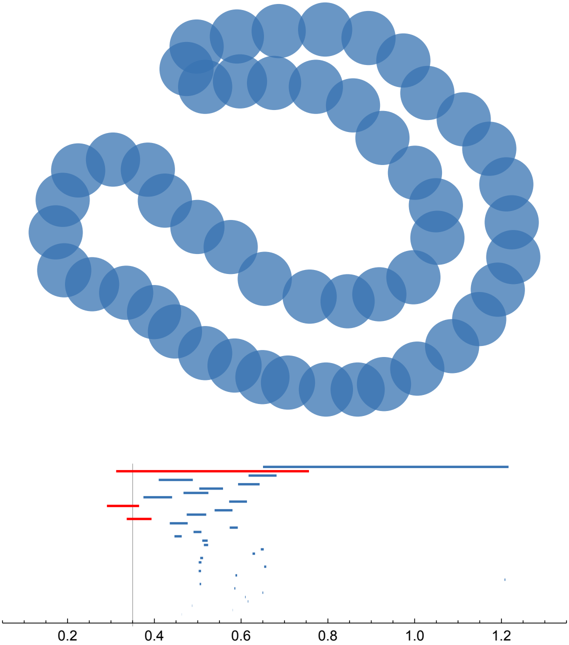
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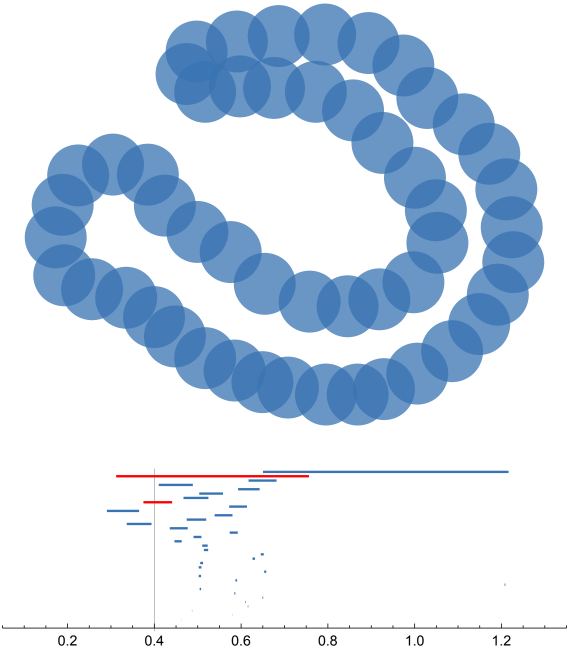
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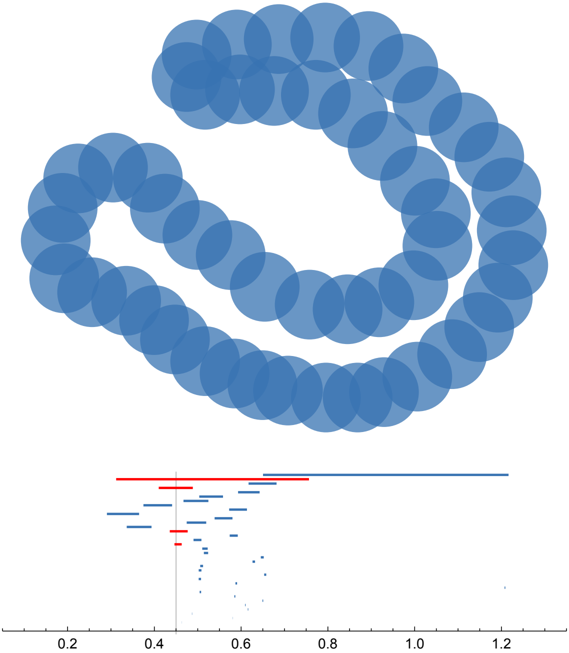
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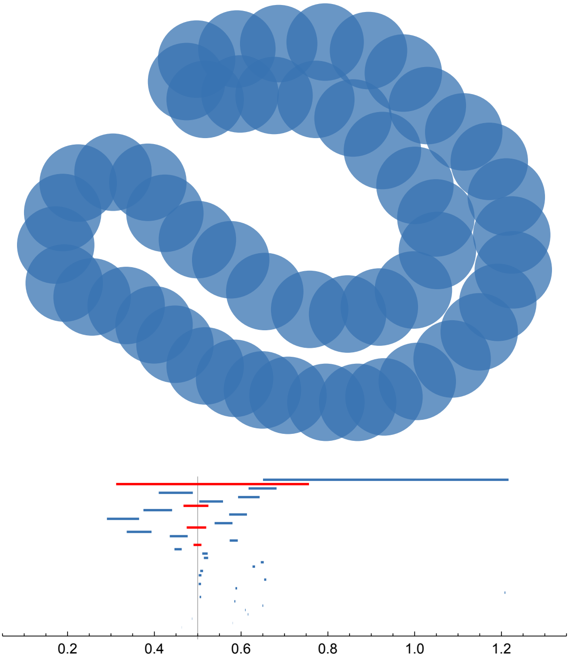
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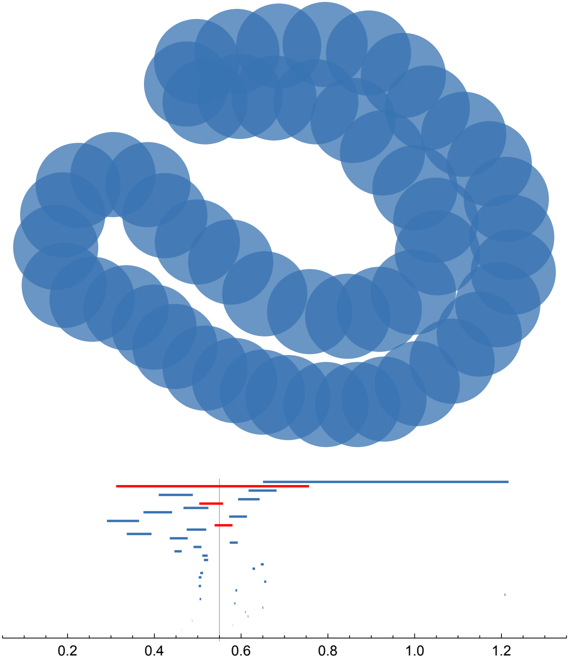
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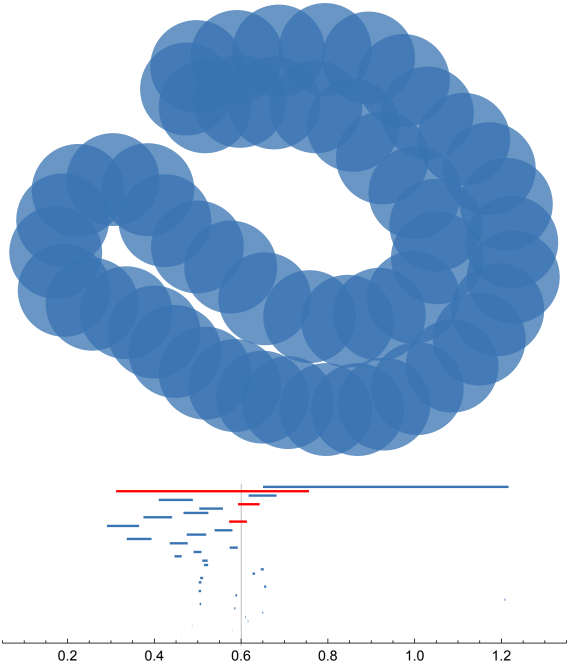

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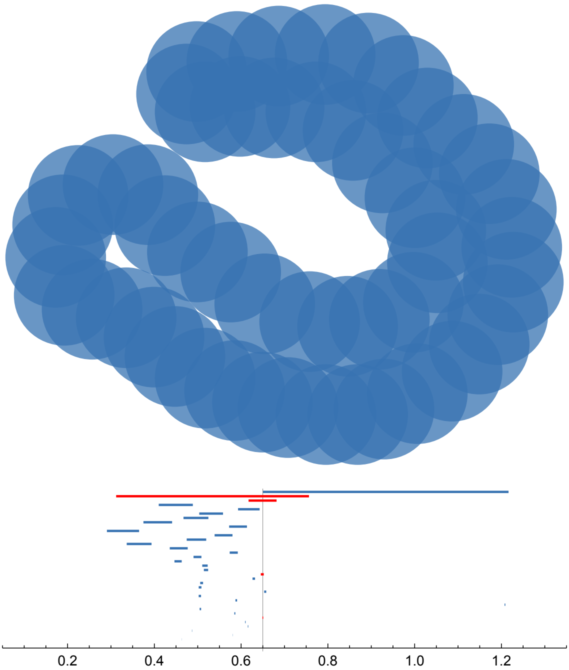
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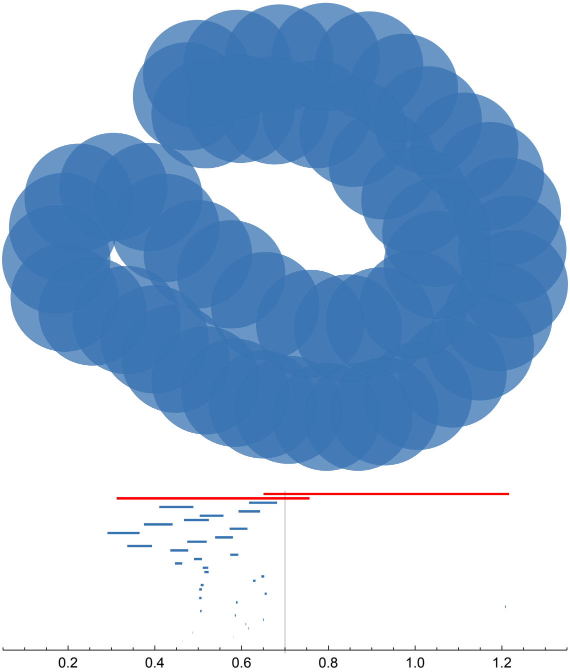
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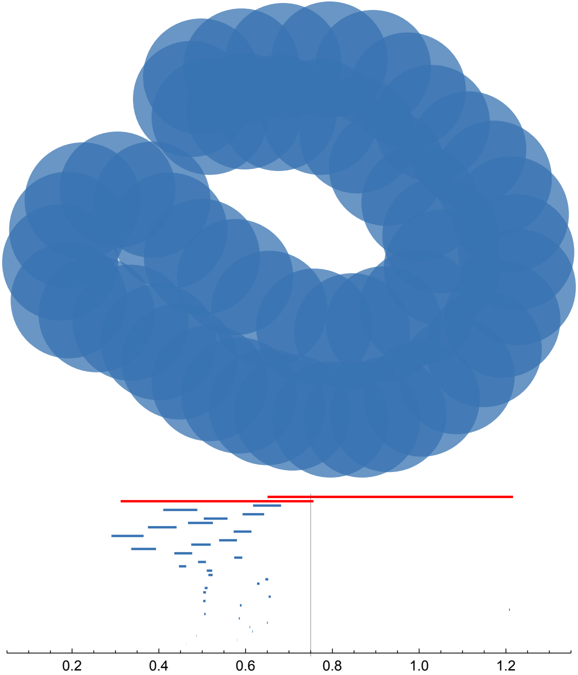
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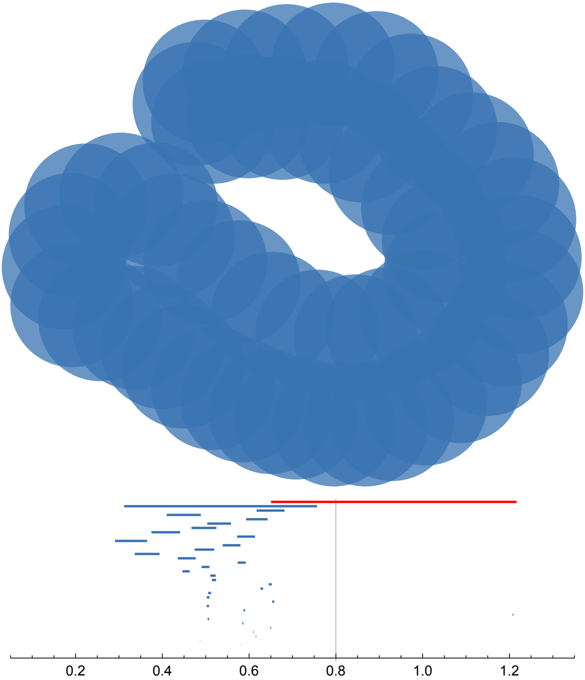
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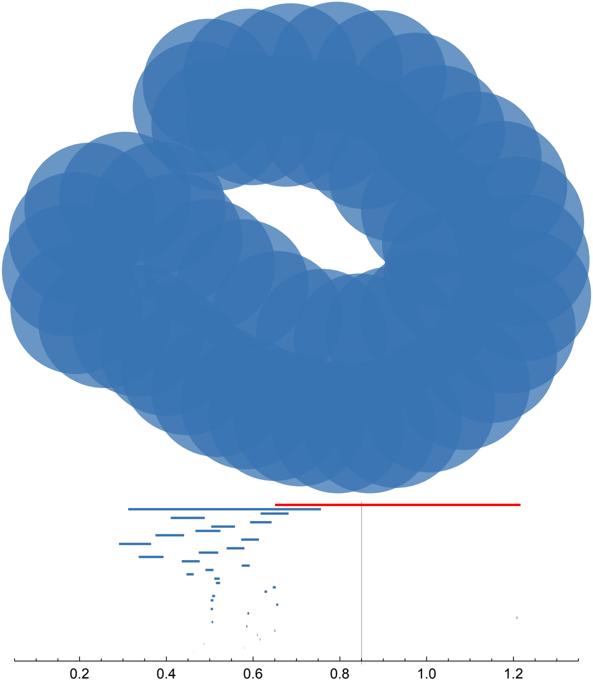
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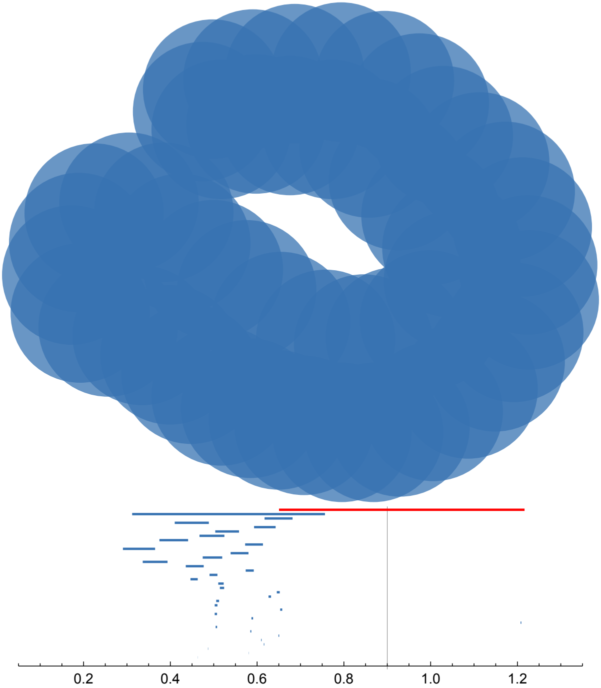
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Homology inference using persistent homology

$P_\delta = B_\delta(P)$: δ-neighborhood (union of balls) around P

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\Omega \subset \mathbb{R}^d$. Let $P \subset \Omega$ be such that

- $\Omega \subseteq P_\delta$ *for some* $\delta > 0$,
- *both* $H_*(\Omega \hookrightarrow \Omega_\delta)$ *and* $H_*(\Omega_\delta \hookrightarrow \Omega_{2\delta})$ *are isomorphisms.*

Then

$$H_*(\Omega) \cong \operatorname{im} H_*(P_\delta \hookrightarrow P_{2\delta}).$$

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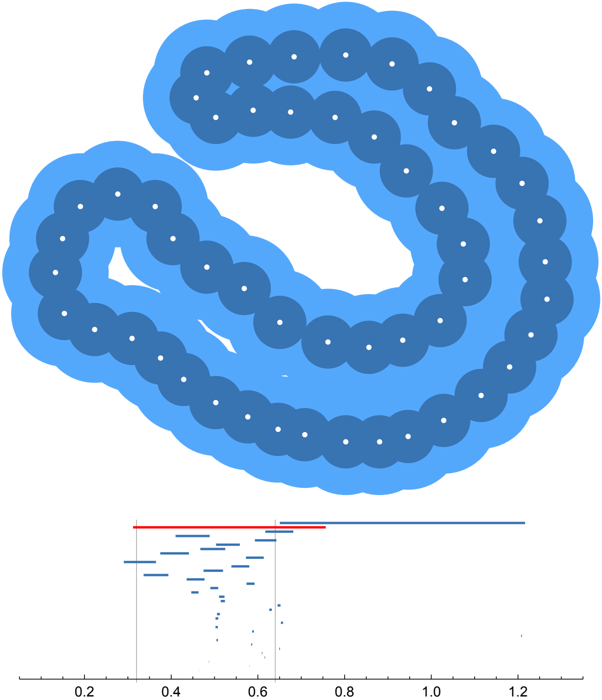

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Proof.

$$\begin{tikzcd}
\Omega \arrow[dr, hook] & & \Omega_\delta \arrow[dr, hook] & & \Omega_{2\delta} \\
& P_\delta \arrow[ur, hook] & & P_{2\delta} \arrow[ur, hook] &
\end{tikzcd}$$

Proof.

$$
\begin{tikzcd}
H_*(\Omega) \arrow[dr] & & H_*(\Omega_\delta) \arrow[dr] & & H_*(\Omega_{2\delta}) \\
& H_*(P_\delta) \arrow[ur] & & H_*(P_{2\delta}) \arrow[ur] &
\end{tikzcd}
$$

Proof.

$$
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H_*(\Omega) \arrow[rr, hook, two heads] \arrow[dr] & & H_*(\Omega_\delta) \arrow[rr, hook] \arrow[dr] & & H_*(\Omega_{2\delta}) \\
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\end{tikzcd}$$

Proof.

$$
\begin{tikzcd}
H_*(\Omega) \arrow[rr, hook, two heads] \arrow[dr] & & H_*(\Omega_\delta) \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, dashed, leftrightarrow, "\cong" description] & & H_*(\Omega_{2\delta}) \\
& H_*(P_\delta) \arrow[ur, two heads] \arrow[dr, two heads] & & H_*(P_{2\delta}) \arrow[ur] & \\
& & \operatorname{im} H_*(P_\delta \hookrightarrow P_{2\delta}) \arrow[ur, hook] & &
\end{tikzcd}
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$$\begin{tikzcd}
H_*(\Omega) \arrow[rr, hook, two heads, "\cong"] \arrow[dr] & & H_*(\Omega_\delta) \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, dashed, leftrightarrow, "\cong"] & & H_*(\Omega_{2\delta}) \\
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\end{tikzcd}$$

$\square$

Stability

The pipeline of topological data analysis

$$
\begin{tikzcd}
\text{point cloud} \arrow[d, "\text{distance}"] \\
\text{function} \arrow[d, "\text{sublevel sets}"] \\
\text{topological spaces} \arrow[d, "\text{homology}"] \\
\text{vector spaces} \arrow[d, "\text{barcode}"] \\
\text{intervals}
\end{tikzcd}
$$

The pipeline of topological data analysis

$$P \subset \mathbb{R}^d$$

$$\begin{tikzcd}
\text{point cloud} \arrow[d, "\text{distance}"] \\
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\end{tikzcd}$$

The pipeline of topological data analysis

$$\begin{tikzcd}
\text{point cloud} \arrow[d, "\text{distance}"] & P \subset \mathbb{R}^d \arrow[d, "x \mapsto d(x,P)"] \\
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\text{vector spaces} \arrow[d, "\text{barcode}"] & \\
\text{intervals} &
\end{tikzcd}
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The pipeline of topological data analysis

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\big\downarrow \text{ sublevel sets} \\
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\end{array}
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The pipeline of topological data analysis

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\text{topological spaces} \arrow[d, "\text{homology}"] & {} \\
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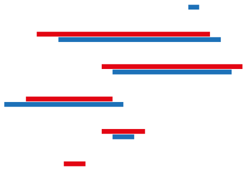
Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

If two functions $f, g : K \to \mathbb{R}$ have distance $\|f - g\|_\infty \leq \delta$, then there exists a δ-matching of their barcodes.



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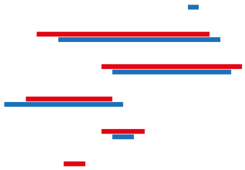

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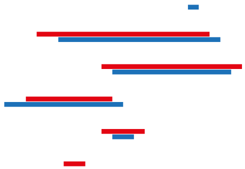

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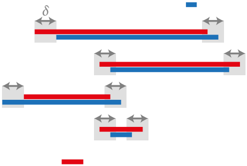

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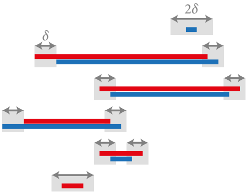

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Homology is a *functor:* homology groups are interleaved too.

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Interval Persistence Modules

Let $\mathbb{K}$ be a field. For an arbitrary interval $I \subseteq \mathbb{R}$, define the *interval persistence module* $C(I)$ by

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- Motivates use of homology with field coefficients
- The decomposition itself is not unique

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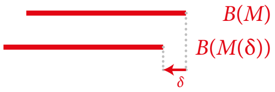
- define $M(\delta)$ by $M(\delta)_t = M_{t+\delta}$ (shift barcode to the left by δ)



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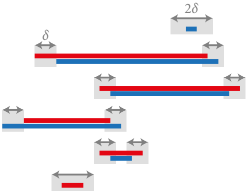

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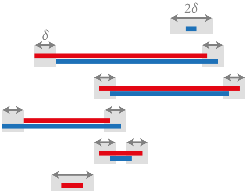

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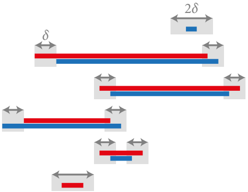

- converse statement also holds (isometry theorem)
- indirect proof, following interpolation argument of original stability theorem

Induced matchings

Structure of submodules and quotient modules

Proposition (B, Lesnick 2014)

For a persistence submodule $K \subseteq M$:

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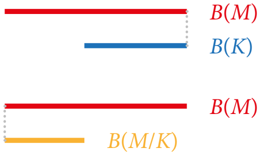

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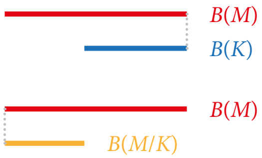

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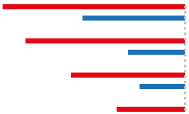
- If multiple bars have same endpoint: match in order of decreasing length



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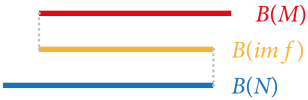

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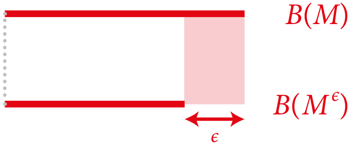
The induced matching theorem

Define M^ϵ by shrinking bars of $B(M)$ from the right by ϵ.

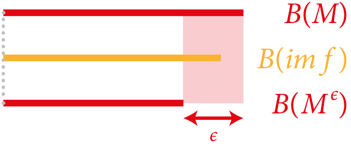


The induced matching theorem

Define M^ϵ by shrinking bars of $B(M)$ from the right by ϵ.

Lemma

Let $f : M \to N$ *be such that* $\ker f$ *is* ϵ-trivial: $(\ker f)^\epsilon = 0$.
Then M^ϵ *is a quotient module of* $\operatorname{im} f$.

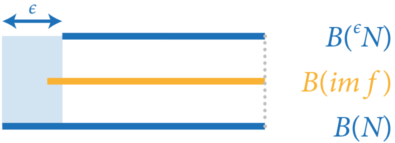


The induced matching theorem

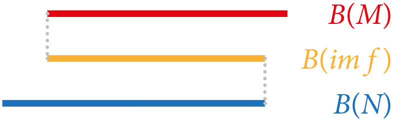
Define ${}^{\epsilon}N$ by shrinking bars of $B(N)$ from the left by ϵ.

Lemma

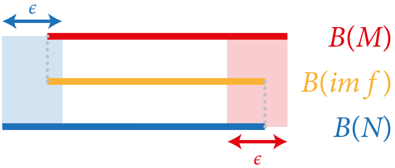
Let $f : M \to N$ *be such that* $\operatorname{coker} f$ *is* ϵ-trivial: $(\operatorname{coker} f)^{\epsilon} = 0$.
Then ${}^{\epsilon}N$ *is a submodule of* $\operatorname{im} f$.



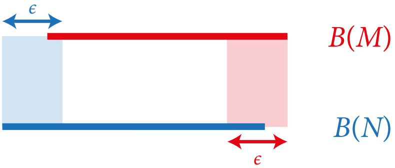
The induced matching theorem



The induced matching theorem



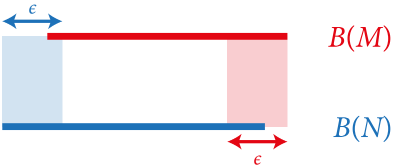
The induced matching theorem



The induced matching theorem

Theorem (B, Lesnick 2014)

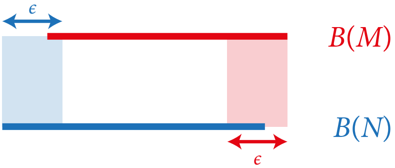
Let $f : M \to N$ be a morphism with $\ker f$ and $\operatorname{coker} f$ ϵ-trivial.



The induced matching theorem

Theorem (B, Lesnick 2014)

Let $f : M \to N$ be a morphism with $\ker f$ and $\operatorname{coker} f$ ϵ-trivial.
Then each interval of length $\geq \epsilon$ is matched by $B(f)$.

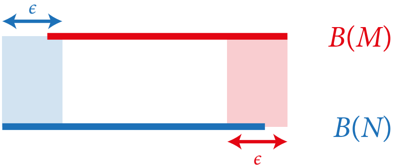


The induced matching theorem

Theorem (B, Lesnick 2014)

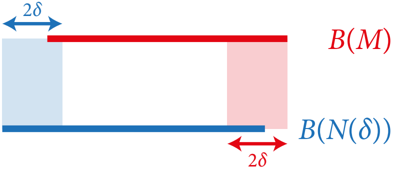
Let $f : M \to N$ *be a morphism with* $\ker f$ *and* $\operatorname{coker} f$ ϵ*-trivial.*
Then each interval of length $\geq \epsilon$ *is matched by* $B(f)$*.*
If $B(f)$ *matches* $[b, d) \in B(M)$ *to* $[b', d') \in B(N)$*, then*

$$b' \leq b \leq b' + \epsilon \text{ and } d - \epsilon \leq d' \leq d.$$



The induced matching theorem

Let $f : M \to N(\delta)$ be an interleaving morphism.
Then $\ker f$ and $\operatorname{coker} f$ are 2δ-trivial.

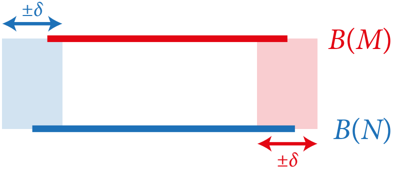


The induced matching theorem

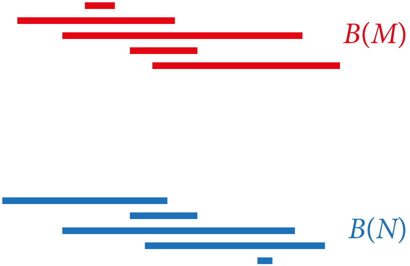
Let $f : M \to N(\delta)$ be an interleaving morphism.
Then $\ker f$ and $\operatorname{coker} f$ are 2δ-trivial.

Corollary (Algebraic stability via induced matchings)

A δ-interleaving between persistence modules induces a δ-matching of their persistence barcodes.



Stability via induced matchings



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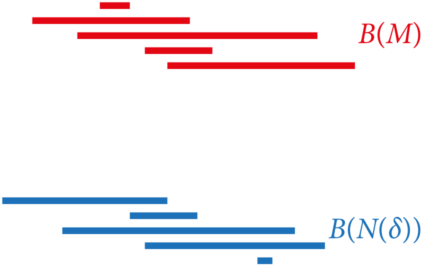
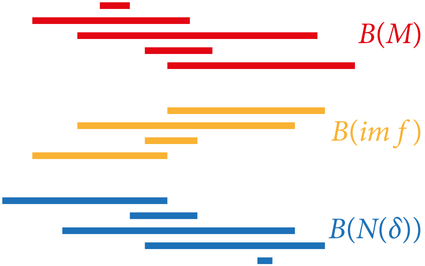
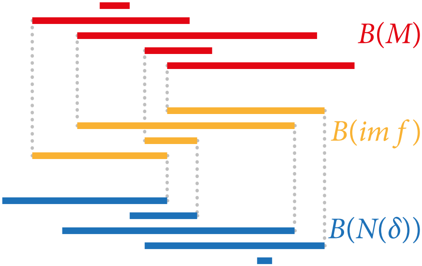


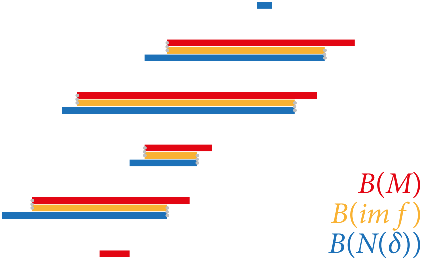
Stability via induced matchings



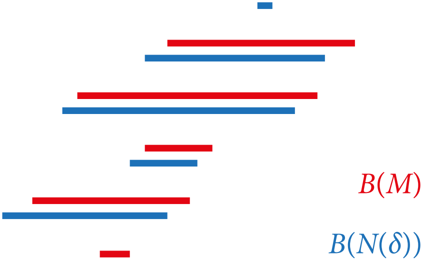
Stability via induced matchings



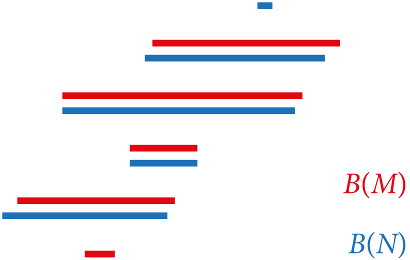
Stability via induced matchings



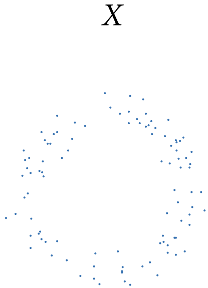
Stability via induced matchings

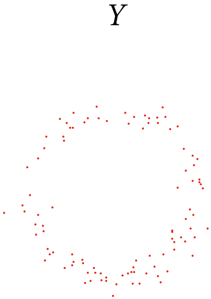


Stability via induced matchings



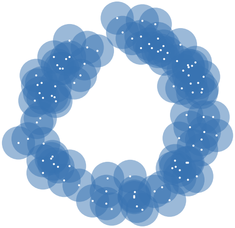
Application: connecting different point clouds



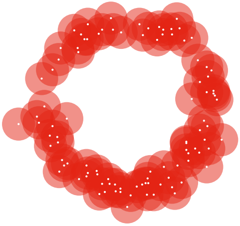


Application: connecting different point clouds

$B_r(X)$

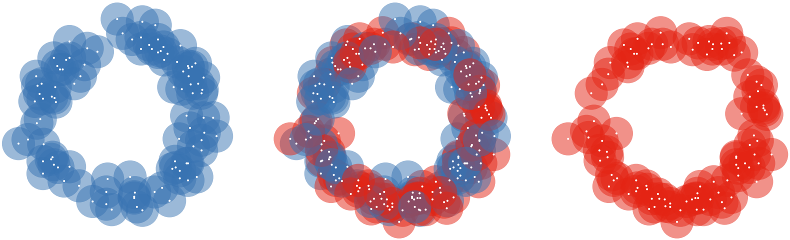


$B_r(Y)$



Application: connecting different point clouds

$$B_r(X) \hookrightarrow B_r(X \cup Y) \hookleftarrow B_r(Y)$$



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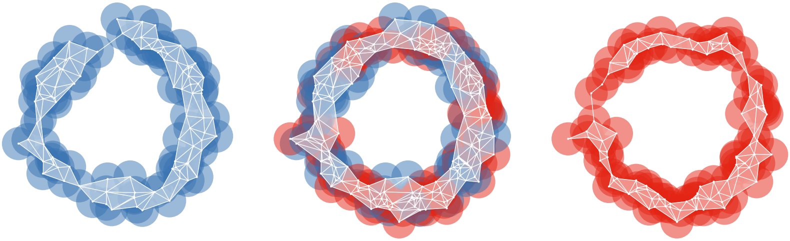

$\mathrm{Del}_r(X)$ $\qquad$ $\mathrm{Del}_r(X \cup Y)$ $\qquad$ $\mathrm{Del}_r(Y)$

Application: connecting different point clouds

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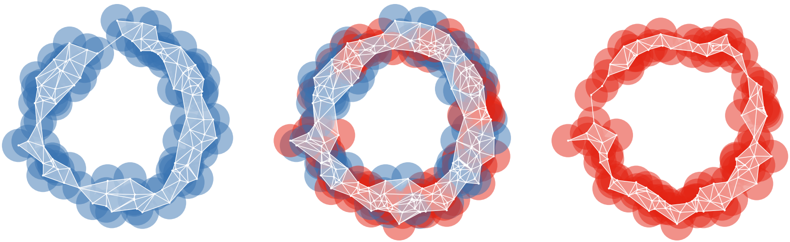

$$\begin{tikzcd}[column sep=small]
\mathrm{Del}_r(X) \arrow[dr, hook] & & \mathrm{Del}_r(X \cup Y) \arrow[dl, hook'] \arrow[dr, hook] & & \mathrm{Del}_r(Y) \arrow[dl, hook'] \\
& \mathrm{Del}_r(X \cup Y, X) & & \mathrm{Del}_r(X \cup Y, Y) &
\end{tikzcd}$$

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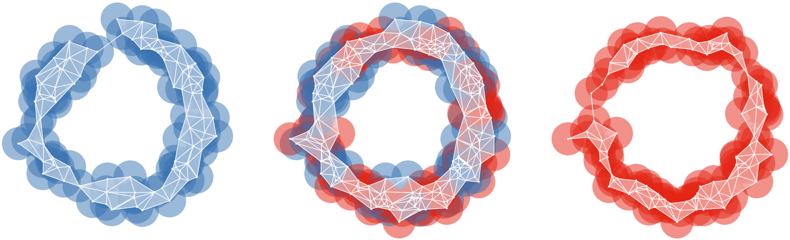

$$\begin{tikzcd}
\mathrm{Del}_r(X) \arrow[dr, hook] & & \mathrm{Del}_r(X \cup Y) \arrow[dl, "\simeq"'] \arrow[dr, hook, "\simeq"] & & \mathrm{Del}_r(Y) \arrow[dl, hook'] \\
& \mathrm{Del}_r(X \cup Y, X) & & \mathrm{Del}_r(X \cup Y, Y) &
\end{tikzcd}$$