Topological simplification problems

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TUM

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Mathematical Signal Processing and Data Analysis – University of Bremen

Discrete Morse theory

Definition (Forman 1998)

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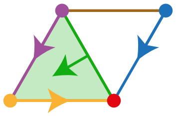
- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



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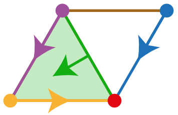

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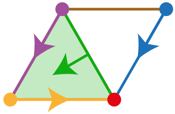

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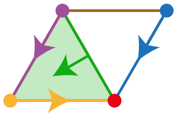

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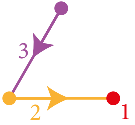

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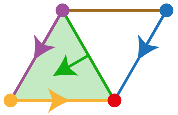

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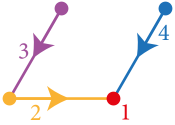

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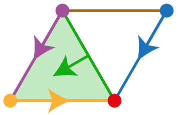

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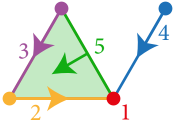

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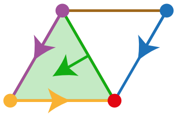

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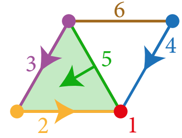

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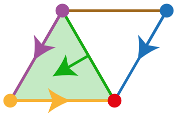

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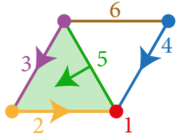

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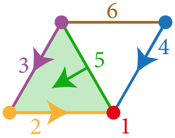

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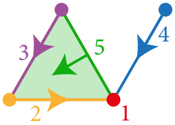

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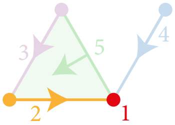

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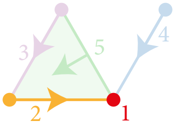

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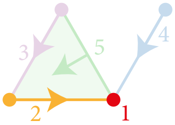

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This homotopy equivalence is compatible with the filtration.

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K and M have isomorphic persistent homology (with regard to f).

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Requires strong assumptions:

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Let M be a submanifold of $\mathbb{R}^d$. Let $P \subset M$, $\delta > 0$ be such that

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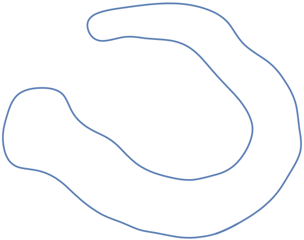

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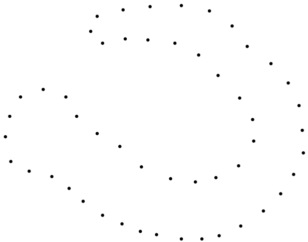

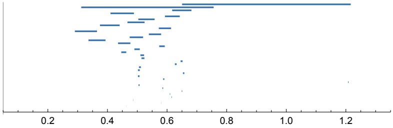

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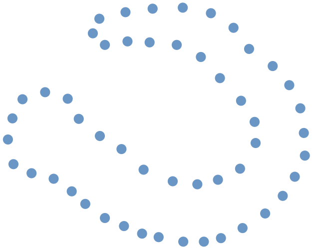

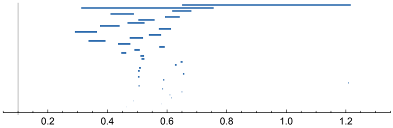

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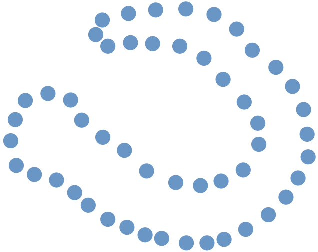

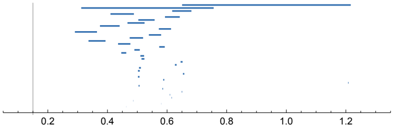

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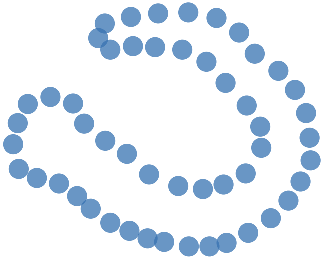

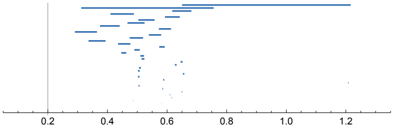

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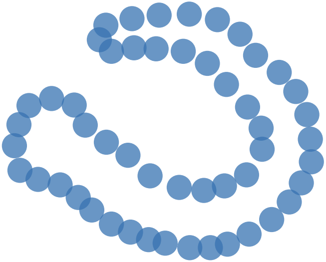

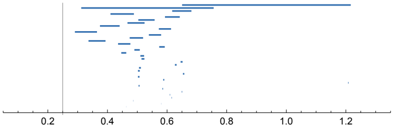

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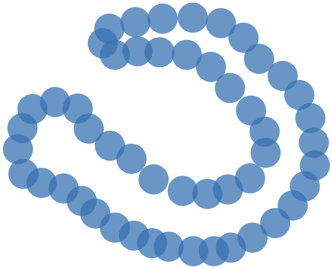

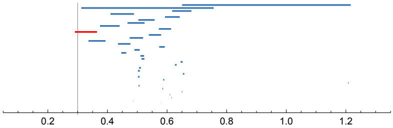

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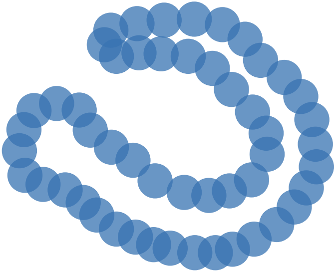

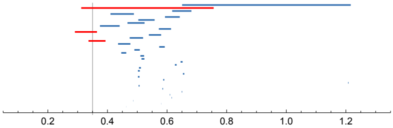

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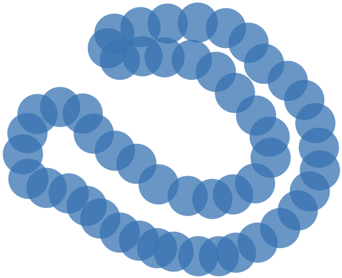

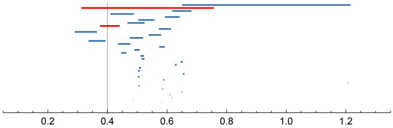

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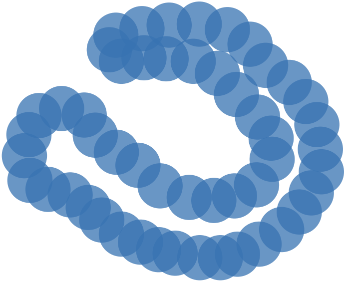

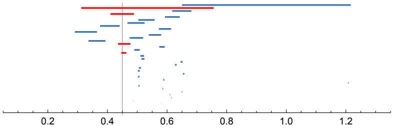

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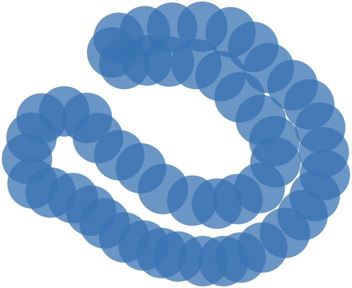

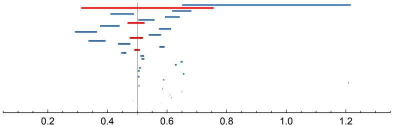

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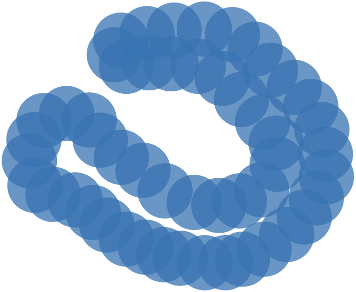

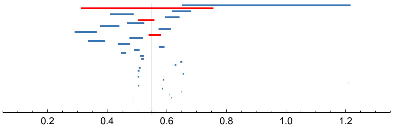

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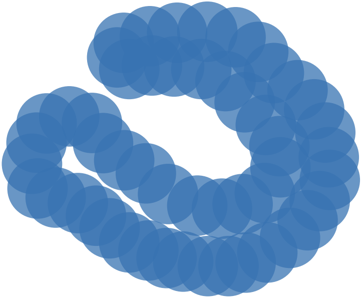

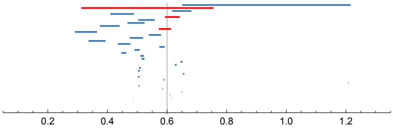

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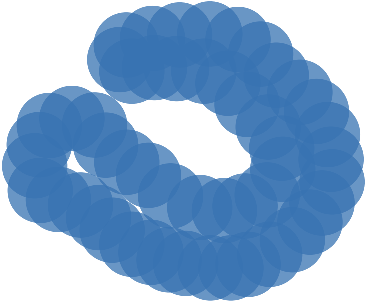

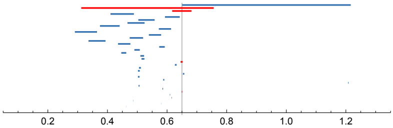

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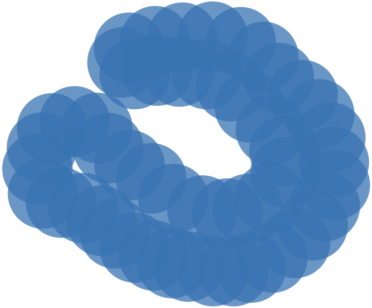

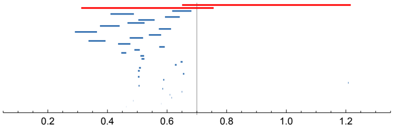

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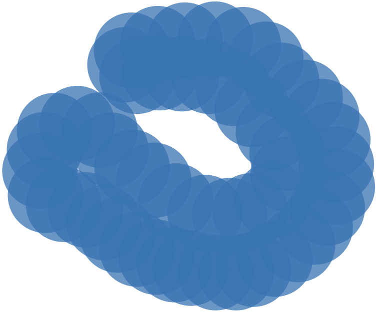

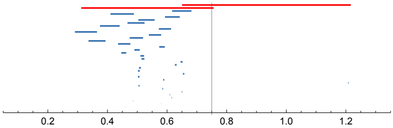

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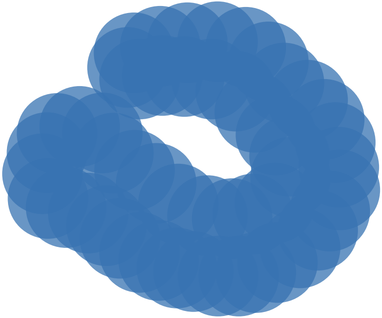

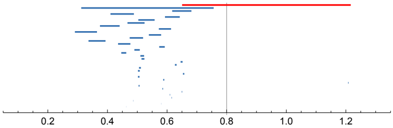

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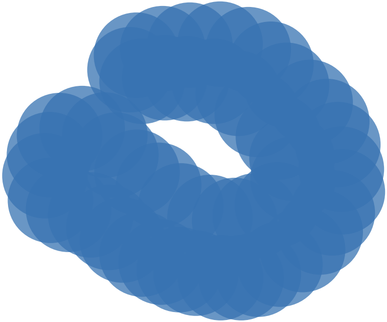

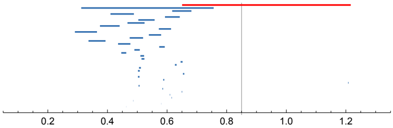

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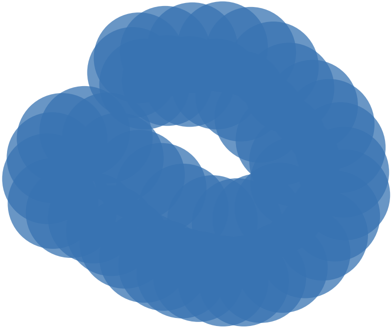
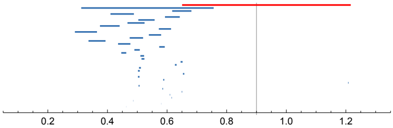




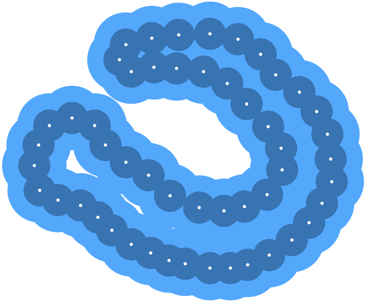
Homology inference using persistence

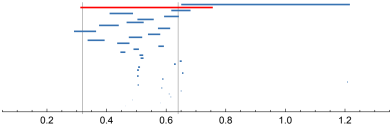
Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\Omega \subset \mathbb{R}^d$. Let $P \subset \Omega$, $\delta > 0$ be such that

- *$B_\delta(P)$ covers Ω, and*
- *the inclusions $\Omega \hookrightarrow B_\delta(\Omega) \hookrightarrow B_{2\delta}(\Omega)$ preserve homology.*

Then $H_*(\Omega) \cong \operatorname{im} H_*(B_\delta(P) \hookrightarrow B_{2\delta}(P))$.





Homological realization

This motivates the *homological realization problem*:

Problem

Given a simplicial pair $L \subseteq K$*, find* X *with* $L \subseteq X \subseteq K$ *such that*

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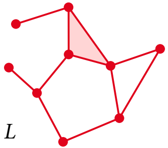

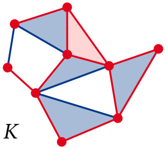

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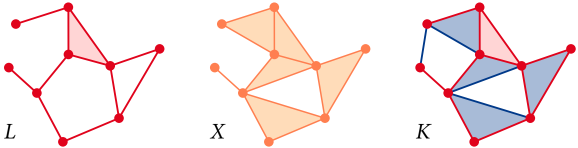

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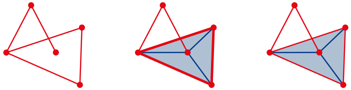

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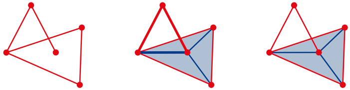

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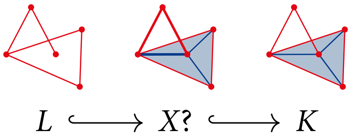

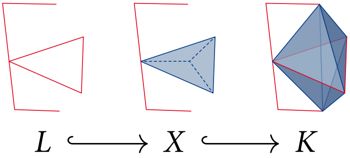

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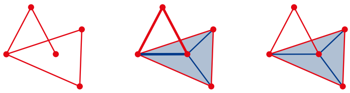

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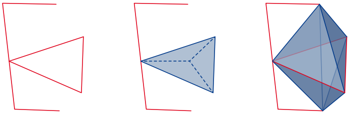

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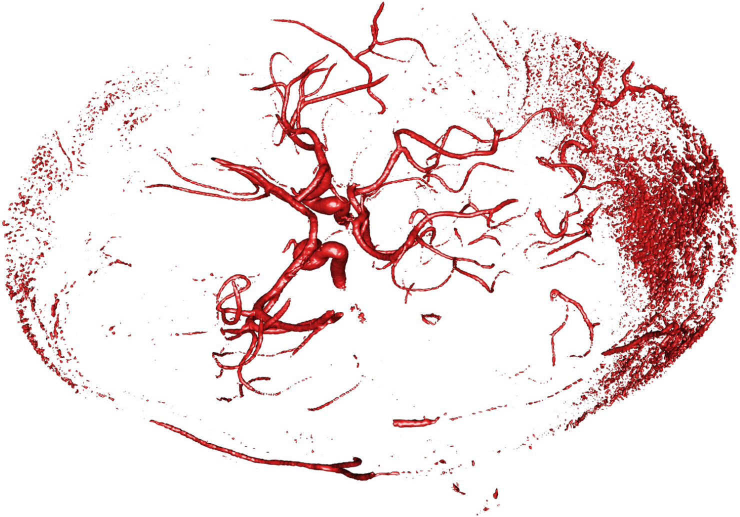
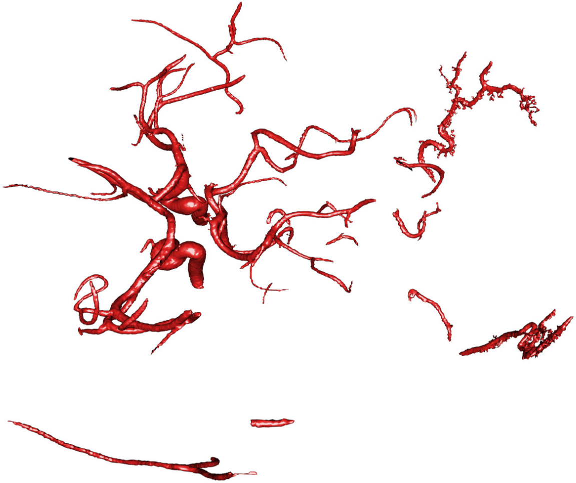
Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

The homological realization problem is NP-hard, even in $\mathbb{R}^3$.

Simplification





Sublevel set simplification

Let $F_{\leq t} = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem (Sublevel set simplification)

Given a function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ *and* $t \in \mathbb{R}$, $\delta > 0$,
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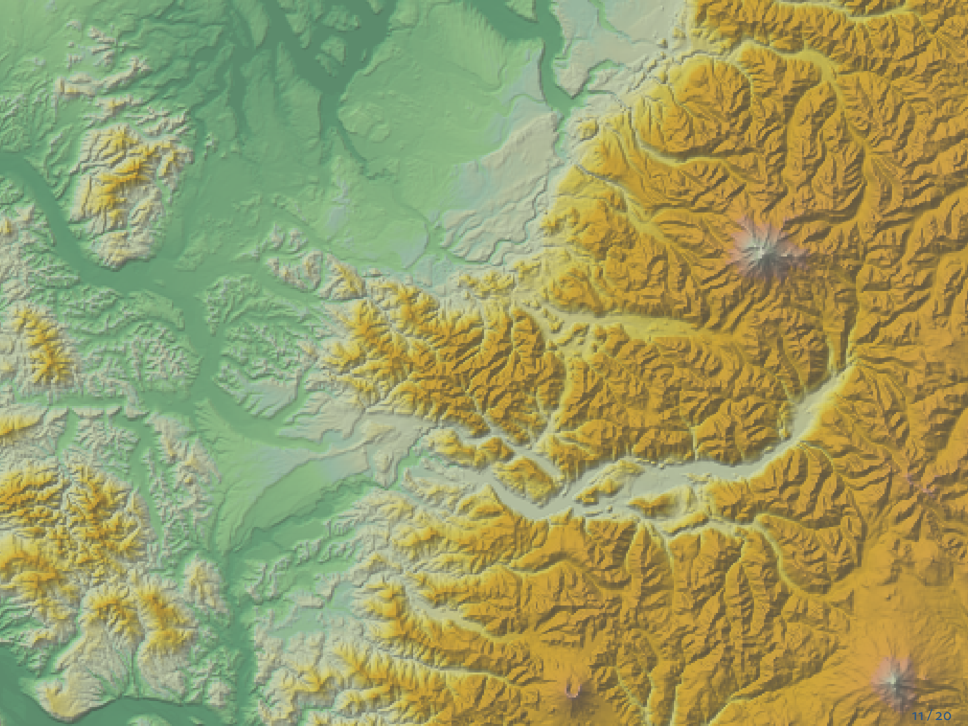
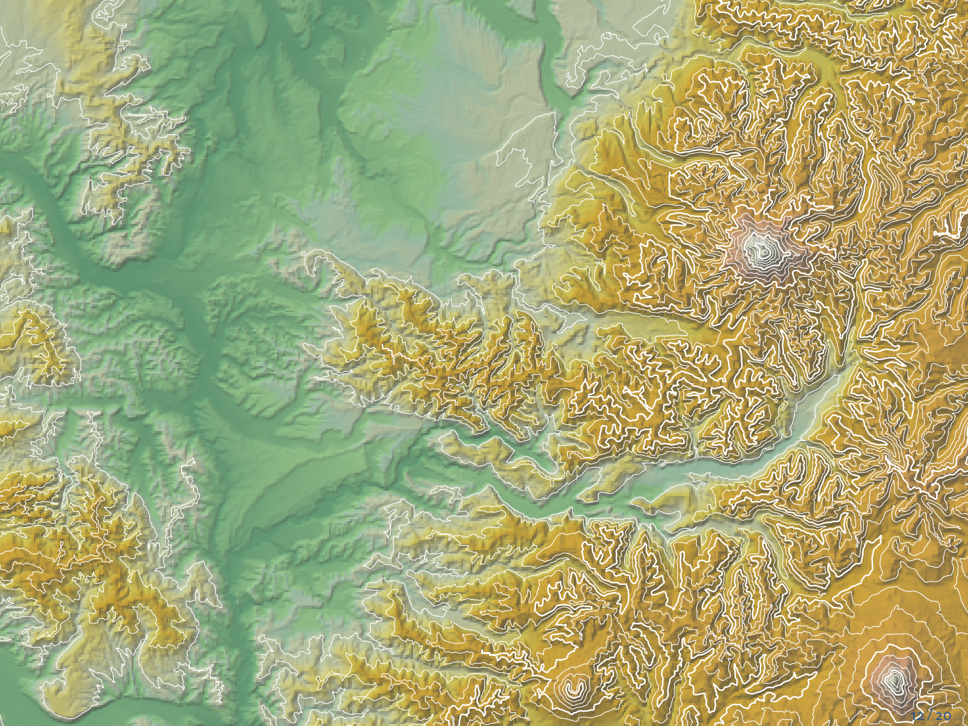
Sublevel set simplification in $\mathbb{R}^3$ is NP-hard.





Topological simplification of functions

Consider the following problem:

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Given a function f and a real number $\delta \geq 0$, find a function f_δ with the minimal number of critical points subject to $\|f_\delta - f\|_\infty \leq \delta$.

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(Discrete) Morse theory:

- Relates critical points to homology of sublevel sets
- Provides method for canceling pairs of critical points

Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\|f - g\|_\infty = \delta$. Then there exists a matching between the intervals of the persistence barcodes of f and g such that

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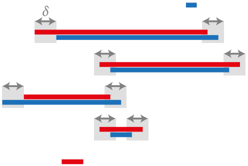

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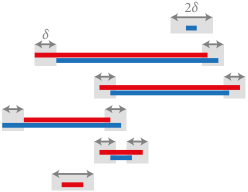

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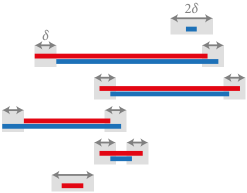
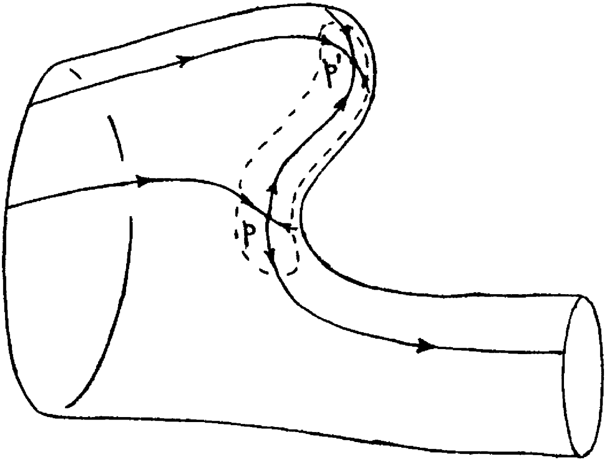


Corollary

Let f be a discrete Morse function and let $\delta > 0$.
Then for every function f_δ with $\|f_\delta - f\|_\infty \leq \delta$ we have:

$$\begin{aligned} &\#\text{ critical points of } f_\delta \\ \geq\ &\#\text{ critical points of } f \text{ with persistence } > 2\delta. \end{aligned}$$

Canceling critical points of a gradient field



from *Lectures on the h-cobordism theorem* (Milnor, 1965)

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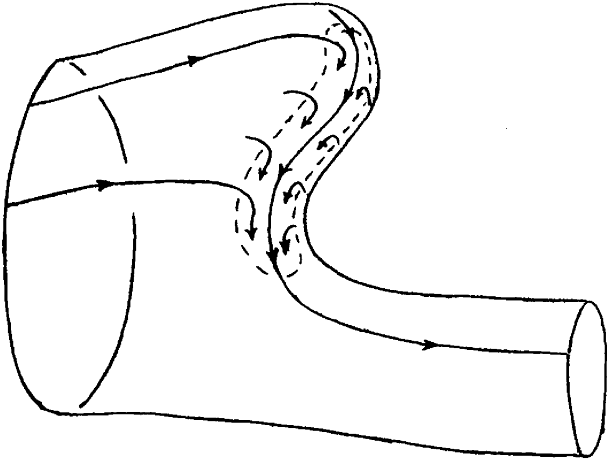

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Side-effects of elimination

Idea for simplifying critical points [Edelsbrunner et al. 2006, Attali et al. 2009]:

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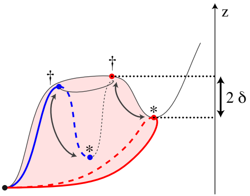

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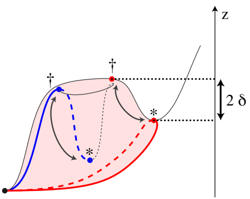

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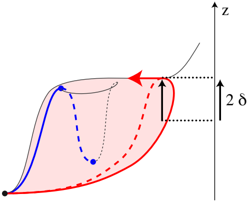

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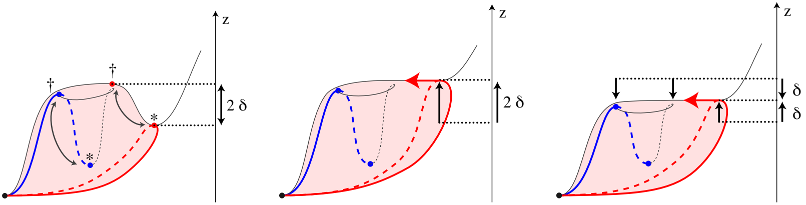

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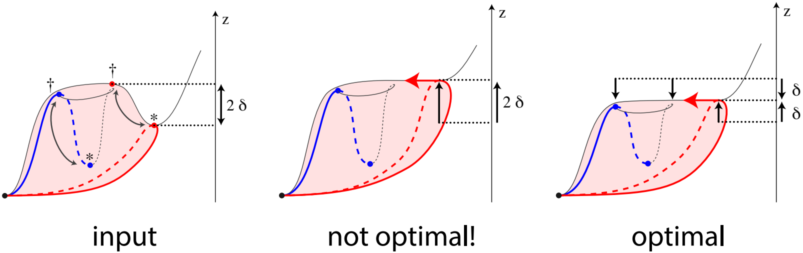
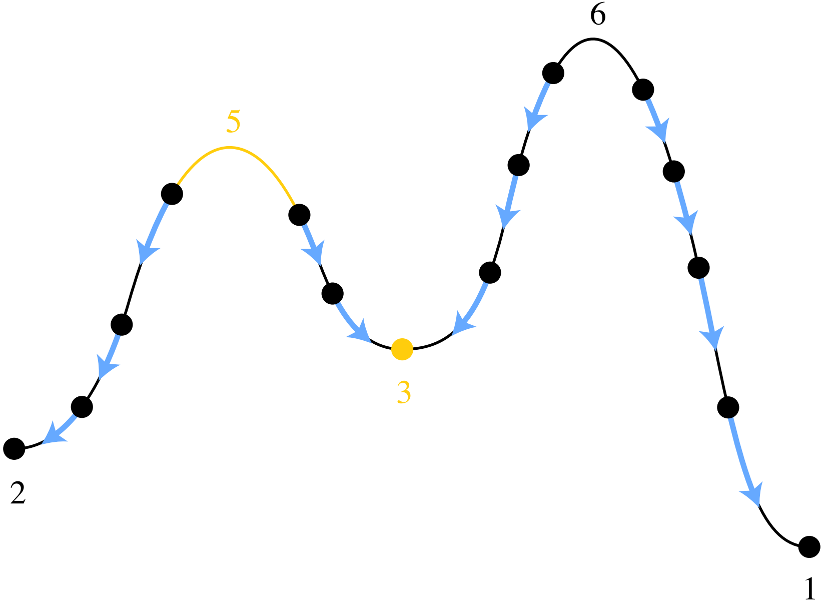


For an optimal solution, we must allow the critical values to change!

Canceling critical points of a function



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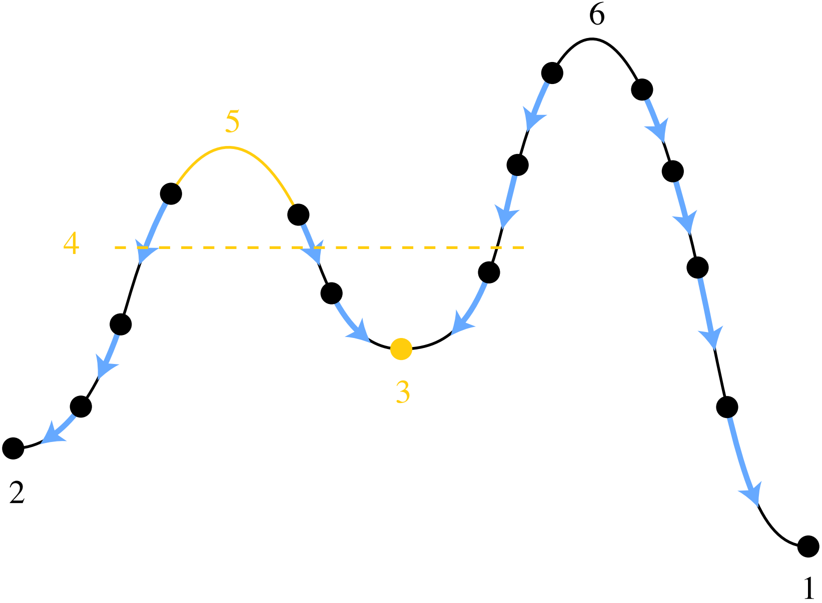

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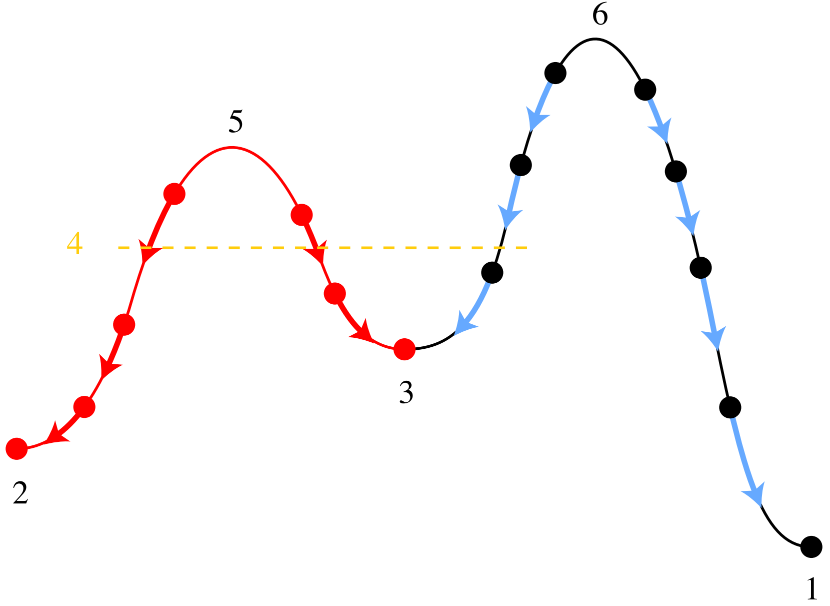

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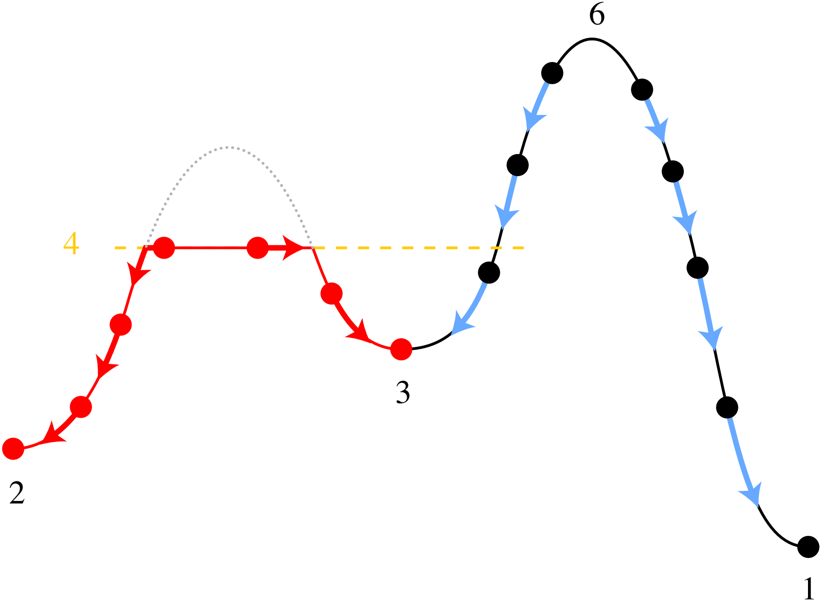

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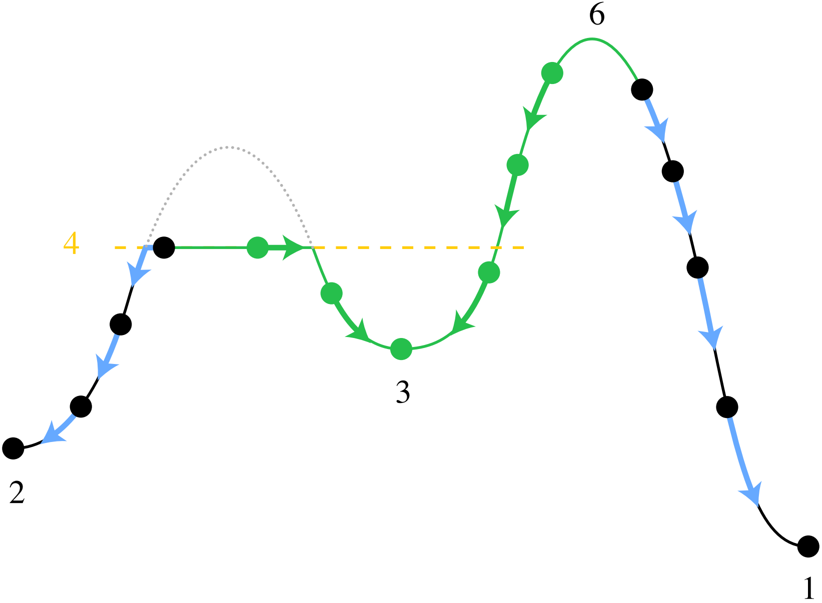

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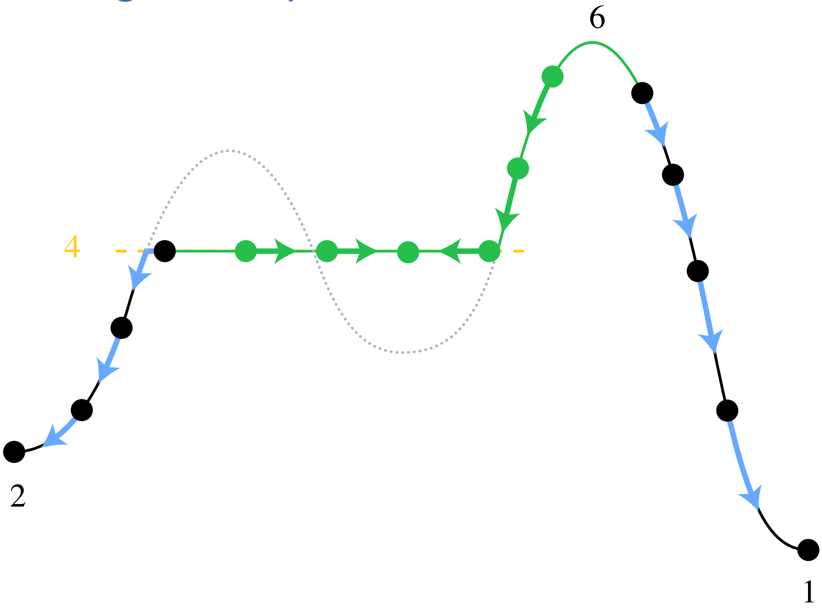

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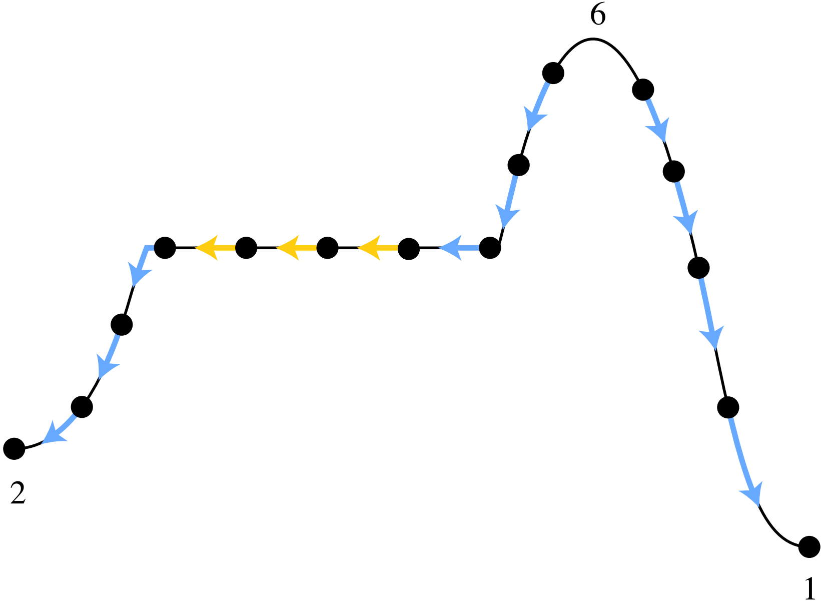

Combining persistence and Morse theory

Theorem (B., Lange, Wardetzky, 2011)

Let f be a discrete Morse function on a surface (with distinct critical values) with gradient field V. Any persistence pair (σ, τ) can be canceled in V after all persistence pairs $(\tilde{\sigma}, \tilde{\tau})$ with

$$f(\sigma) < f(\tilde{\sigma}) < f(\tilde{\tau}) < f(\tau)$$

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Corollary

The function f_δ constructed from f by canceling all persistence pairs with persistence $\leq 2\delta$ achieves the lower bound on the number of critical points subject to $\|f_\delta - f\|_\infty \leq \delta$.

Combination with energy methods

Recall: simplified vector field V_δ imposes inequalities on simplified function consistent with V_δ

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- defines convex polytope of solutions:
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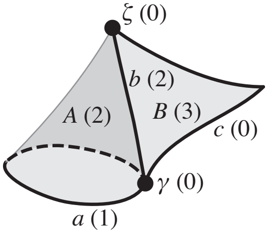
- find the "best" solution using your favorite energy functional

Cancelation conflicts

The result does not generalize beyond surfaces!



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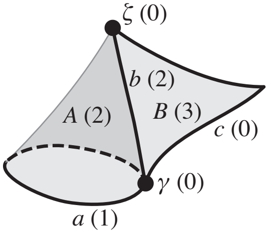

Room for discussions...