# THE SPACE OF REEB GRAPHS


Ulrich Bauer TUM

joint work with:

Yusu Wang, Liz Munch

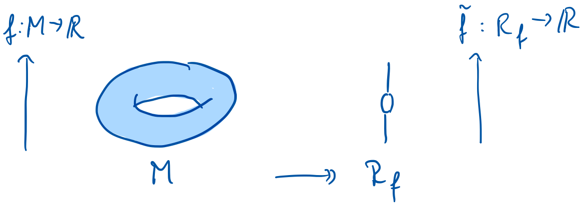
Claudia Landi, Facundo Mémoli

Håvard Bjerkevik, Ben Fluhr

"Here are two things that are reasonably close to each other, and I want to compare them."

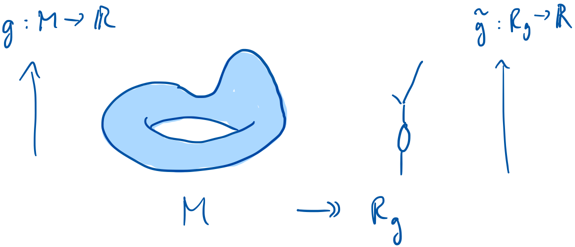
S. WEINBERGER

# REEB GRAPHS




identify components of level sets $f^{-1}(t)$: $R_f = M/\sim_f$

($x \sim_f y \iff x, y$ in same component of some $f^{-1}(t)$, $t \in \mathbb{R}$.)



# FORMAL SETTING

**Reeb domains**: compact triangulable spaces

**Reeb quotient maps**: quotient maps with connected fibers

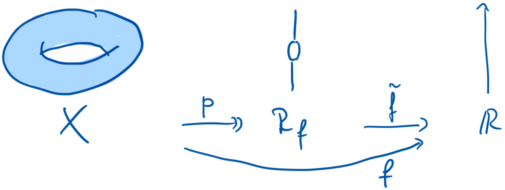
**Reeb graph**:

- a Reeb domain $R_f$, together with
- a function $\tilde{f} : R_f \to \mathbb{R}$ with discrete fibers (Reeb function)

$R_f$ is the Reeb graph of $f : X \to \mathbb{R}$ if

- $f = \tilde{f} \circ p$ for some Reeb quotient map $p : X \twoheadrightarrow R_f$.

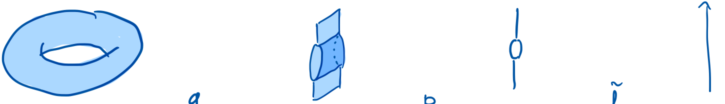
In this case, $R_f \cong X/\sim_f$.



# MULTIPLE SPACES, ONE REEB GRAPH

Let $q : Y \twoheadrightarrow X$ be a Reeb quotient map.
Then $R_f$ is also the Reeb graph of $g = f \circ q$.

- Reeb quotient maps preserve Reeb graphs.



$$\begin{tikzcd} Y \arrow[r, two heads, "q"] \arrow[rrr, bend right, "g"'] & X \arrow[r, two heads, "p"] \arrow[rr, bend right, "f"'] & R_f \arrow[r, "\tilde{f}"] & \mathbb{R} \end{tikzcd}$$

# GOALS

How to compare two Reeb graphs $R_f$, $R_g$? ($f, g : M \to \mathbb{R}$ are unknown)

Assign distance $d(R_f, R_g)$ (extended pseudo-metric).

Desirable properties:

Stability: For any space $X$ and $f, g : X \to \mathbb{R}$ with Reeb graphs $R_f$, $R_g$,

$d(R_f, R_g) \leq \|f - g\|_\infty$. [Cohen-Steiner & al 2005]

Universality: For any other stable distance $d_s$,

$d_s(R_f, R_g) \leq d(R_f, R_g)$. [Di Fabio, Landi 2012] [Lesnick 2015]

# A Canonical Universal Distance

Given Reeb graphs $R_f$, $R_g$ with functions $\tilde{f}$, $\tilde{g}$,

$$d_u(R_f, R_g) := \inf \|f-g\|_\infty$$

$$\begin{tikzcd}
 & & X \arrow[dll, "p_f"'] \arrow[drr, "p_g"] \arrow[ddl, "f"] \arrow[ddr, "g"'] & & \\
R_f \arrow[d, "\tilde{f}"'] & & & & R_g \arrow[d, "\tilde{g}"] \\
\mathbb{R} & & & & \mathbb{R}
\end{tikzcd}$$

over all Reeb domains $X$ and Reeb quotient maps $p_f, p_g$.

- stability: infimum is lower bound on all $\|f-g\|_\infty$
- universality: infimum is greatest lower bound

# TRIANGLE INEQUALITY FOR THE UNIVERSAL DISTANCE

For all

$$\begin{tikzcd}[column sep=small]
 & X \arrow[dl, two heads] \arrow[dr, two heads] & & Y \arrow[dl, two heads] \arrow[dr, two heads] & \\
R_f & & R_g & & R_h
\end{tikzcd}$$

consider pullback

$$\begin{tikzcd}[column sep=small]
 & & P \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dd, phantom, "\lrcorner" very near start] & & \\
 & X \arrow[dl, two heads] \arrow[dr, two heads] & & Y \arrow[dl, two heads] \arrow[dr, two heads] & \\
R_f \arrow[d, "\tilde{f}"'] & & R_g \arrow[d, "\tilde{g}"'] & & R_h \arrow[d, "\tilde{h}"'] \\
\mathbb{R} & & \mathbb{R} & & \mathbb{R}
\end{tikzcd}$$

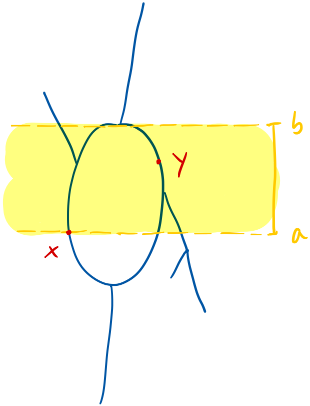
Then $\quad \|f_P - h_P\|_\infty \leq \|f_P - g_P\|_\infty + \|g_P - h_P\|_\infty$

$$= \|f_X - g_X\|_\infty + \|g_Y - h_Y\|_\infty .$$

# A REEB GRAPH AS A METRIC SPACE

- On a Reeb graph $R_f$ with $\tilde{f}: R_f \to \mathbb{R}$, consider the metric

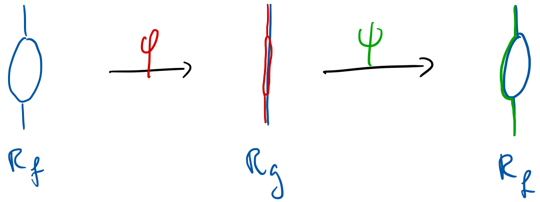
$$d_f : (x,y) \mapsto \inf\{ b-a \mid x, y \text{ connected in } \tilde{f}^{-1}[a,b] \}.$$



# FUNCTIONAL DISTORTION DISTANCE [B., Ge, Wang 2014]

- Given maps $\phi : R_f \to R_g$, $\psi : R_g \to R_f$, consider
  $$G(\phi, \psi) = \{(x, \phi(x)) \mid x \in R_f\} \cup \{(\psi(y), y) \mid y \in R_g\} \subseteq R_f \times R_g.$$

- **Distortion** of $(\phi, \psi)$:
  $$D(\phi, \psi) = \sup_{(x,y),(\tilde{x},\tilde{y}) \in G(\phi,\psi)} \frac{1}{2} \left| d_f(x,\tilde{x}) - d_g(y,\tilde{y}) \right|.$$

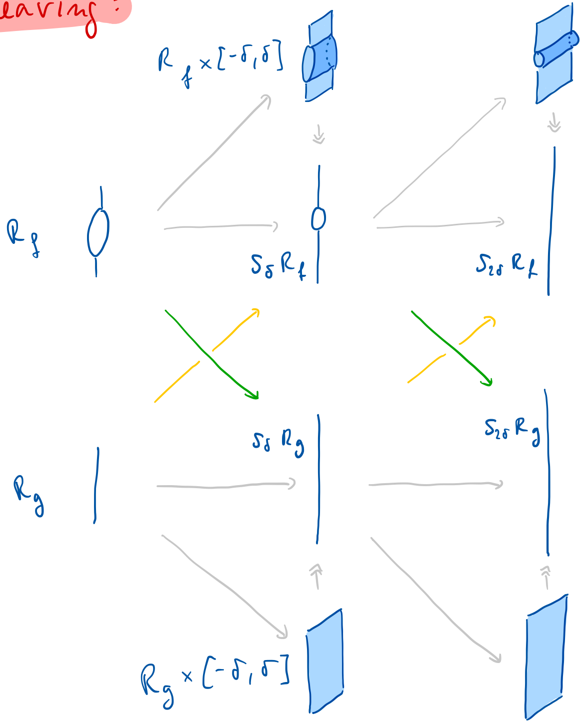



$$D(\varphi, \psi) = \frac{1}{2}$$

- **Functional distortion distance**:
  $$d_{FD}(R_f, R_g) = \inf_{\phi, \psi} \left( \max \{ D(\phi, \psi), \|f - g \circ \phi\|_\infty, \|g - f \circ \psi\|_\infty \} \right).$$

# INTERLEAVING DISTANCE [Bubenik & al. 2015; de Silva & al. 2016]

## δ-Interleaving:




## Interleaving distance:

$\inf \{ \delta \mid \exists\ \delta\text{-interleaving between } R_f, R_g \}$

# LEVEL SET PERSISTENT HOMOLOGY

Thm [Carlsson, de Silva, Morozov 2009]

Given $f : X \to \mathbb{R}$:

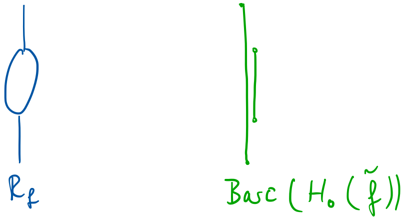
Homology of level sets $H_*(f^{-1}(t); \mathbb{F})$

is encoded (up to isomorphism) by a unique

collection of intervals (level set persistence barcode).

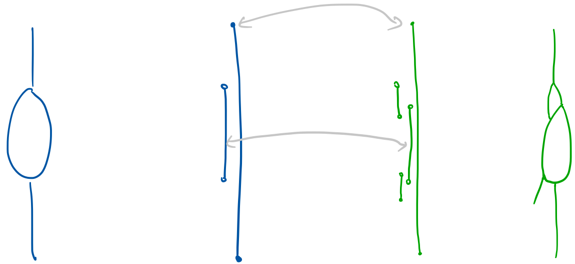
Example for a Reeb graph

($f^{-1}(t)$ is discrete):




(also encodes $f^{-1}(I) \hookrightarrow f^{-1}(J)$ for intervals $I \subseteq J$)

# THE BOTTLENECK DISTANCE BETWEEN PERSISTENCE BARCODES



A **δ-matching** between two barcodes $\mathrm{Barc}(f)$, $\mathrm{Barc}(g)$ satisfies:

- matched intervals $(I, J)$ have distance $d_H(I, J) \leq \delta$
- unmatched intervals have length $\leq 2\delta$

The **bottleneck distance** $d_B(f, g)$ is

$$\inf \delta \ : \ \exists \ \delta\text{-maching between } \mathrm{Barc}(f), \mathrm{Barc}(g)$$

# A ZOO OF DISTANCES AND INEQUALITIES

[Carlsson, de Silva, Morozov 2009]

$$d_B(R_f, R_g) \leq \|f-g\|_\infty$$

tight

[B., Ye, Wang 2014]

$$\frac{1}{3} d_B(R_f, R_g) \leq d_{FD}(R_f, R_g) \leq \|f-g\|_\infty$$

[B., Munch, Wang 2015]

$$\frac{1}{3} d_{FD}(R_f, R_g) \leq d_I(R_f, R_g) \leq d_{FD}(R_f, R_g)$$

[Botnan, Lesnick 2016]

$$\frac{1}{5} d_B(R_f, R_g) \leq d_I(R_f, R_g)$$

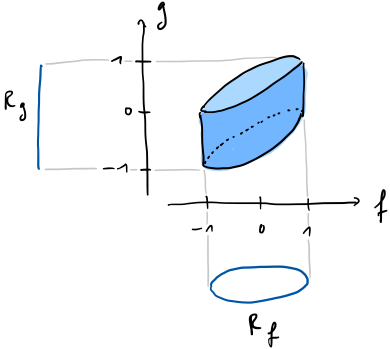
[Bjerkevik 2016]

$$\frac{1}{2} d_B(R_f, R_g) \leq d_I(R_f, R_g)$$

- $d_B$ : bottleneck distance (of level set $H_0$ barcode)
- $d_{FD}$ : functional distortion distance
- $d_I$ : interleaving distance

# Functional Distortion & Interleaving Distances are NOT Universal

Consider a cylinder with two functions $f, g$:




- $d_u(R_f, R_g) = \|f-g\|_\infty = 1$

- $d_I(R_f, R_g) \leq d_{FD}(R_f, R_g) \leq \frac{1}{2} < d_u(R_f, R_g) = 1$

# Metric Equivalence of Reeb Graph Distances

[B., Bjerkevik 2019]

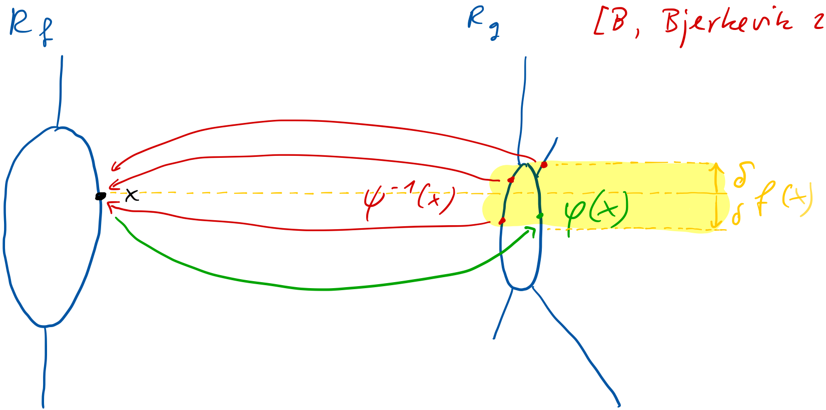
$$\begin{tikzcd}
d_{FC} \arrow[d, "{[1,3]}"'] \arrow[dr, "{[1,3]}" near start] & d_U \arrow[l, "{[1,3]}"'] \arrow[dl, "{[1,3]}"' near start] \arrow[d, "{[1,5]}"] \\
d_{FD} \arrow[r, "{[1,3]}"'] & d_I
\end{tikzcd}$$

$$\frac{d_{FD}}{d_I} \in [1,3]$$

all bounds are sharp!

# The Functional Contortion Distance

[B, Bjerkevik 2019]




$d_{FC}(R_f, R_g) = \inf \{ \delta \mid \exists \ \varphi : R_f \to R_g, \ \psi : R_g \to R_f :$

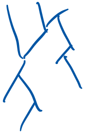
$\forall x \in R_f : \quad \varphi(x), \psi^{-1}(x)$ connected in $g^{-1}[f(x)-\delta, f(x)+\delta]$,

$\forall y \in R_g : \quad [\dots] \}$

# Metric Equivalences for Contour & Merge Trees

**Contour tree**:

a Reeb graph that is a tree

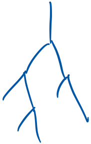


[B., Bjerkevik 2019]

$$
\begin{tikzcd}
 & d_{FC} = d_u \arrow[dl, "{[1,3]}"'] \arrow[dr, "{[1,3]}"] & \\
d_{FD} \arrow[rr, "{[1,3]}"'] & & d_I
\end{tikzcd}
$$

**Merge tree**:

a contour tree without up-forks



[Fluhr 2019]

$$d_I = d_{FD} = d_{FC} = d_u$$

# Topological Edit Zig-Zags

- Consider zig-zag diagrams $Z$ of Reeb quotient maps

$$
\begin{tikzcd}
\mathbb{R} & & \mathbb{R} & & \mathbb{R} & & & \mathbb{R} \\
R_f = R_1 \arrow[u, "\tilde{f}_1"] & & R_2 \arrow[u, "\tilde{f}_2"] & & R_3 \arrow[u, "\tilde{f}_3"] & \cdots & & R_n = R_g \arrow[u, "\tilde{f}_n"] \\
& X_1 \arrow[ul, two heads] \arrow[ur, two heads] & & X_2 \arrow[ul, two heads] \arrow[ur, two heads] & & & X_{n-1} \arrow[ur, two heads] &
\end{tikzcd}
$$

and take the limit $L_Z$

$$
\begin{tikzcd}
\mathbb{R} & & \mathbb{R} & & \mathbb{R} & & & \mathbb{R} \\
R_f = R_1 \arrow[u, "\tilde{f}_1"] & & R_2 \arrow[u, "\tilde{f}_2"] & & R_3 \arrow[u, "\tilde{f}_3"] & \cdots & & R_n = R_g \arrow[u, "\tilde{f}_n"] \\
& X_1 \arrow[ul, two heads] \arrow[ur, two heads] & & X_2 \arrow[ul, two heads] \arrow[ur, two heads] & & & X_{n-1} \arrow[ur, two heads] & \\
& & & L_Z \arrow[ull, two heads] \arrow[u, two heads] \arrow[urrr, two heads] & \cdots & & &
\end{tikzcd}
$$

(note: all maps are Reeb quotient maps).

# The Topological Edit Distance

- Consider zig-zag **limit** $L_z$:

$$
\begin{tikzcd}
\mathbb{R} & & \mathbb{R} & & \mathbb{R} & & & \mathbb{R} \\
R_f = R_1 \arrow[u, "\tilde{f}_1"] & & R_2 \arrow[u, "\tilde{f}_2"] & & R_3 \arrow[u, "\tilde{f}_3"] & \cdots & & R_n = R_g \arrow[u, "\tilde{f}_n"] \\
& X_1 \arrow[ul, two heads] \arrow[ur, two heads] & & X_2 \arrow[ul, two heads] \arrow[ur, two heads] & & & X_{n-1} \arrow[ur, two heads] & \\
& & & L_z \arrow[ull, two heads] \arrow[u, two heads] \arrow[urrr, two heads] & \cdots & & &
\end{tikzcd}
$$

- **Spread** of functions $f_i : L_z \twoheadrightarrow R_i \longrightarrow \mathbb{R}$ :

$$S_z : L_z \to \mathbb{R}, \qquad x \mapsto \max_i f_i(x) - \min_j f_j(x).$$

  - $\|f-g\|_\infty \leq \|S_z\|_\infty$, with equality for $n = 2$

- **Edit distance**:

$$d_e(R_f, R_g) = \inf_z \|S_z\|_\infty .$$

**Prop.** $d_e$ is stable and universal.

# The Reeb Graph Edit Distance

- Consider zig-zag diagrams $Z$ of Reeb quotient maps

$$\begin{tikzcd}[column sep=small]
R_f = R_1 & & R_2 & & R_3 & \cdots & R_n = R_g \\
& G_1 \arrow[ul, two heads] \arrow[ur, two heads] & & G_2 \arrow[ul, two heads] \arrow[ur, two heads] & \cdots & G_{n-1} \arrow[ur, two heads] &
\end{tikzcd}$$

where the $G_j$ are finite graphs.

Interpretation:

$$\begin{tikzcd}[column sep=small]
& \mathbb{R} & \\
& R_{i+1} \arrow[u, "\tilde{f}_i"] & \\
G_i \arrow[ur, two heads] & & G_{i+1} \arrow[ul, two heads]
\end{tikzcd}$$

edit

modify $G_i$ to $G_{i+1}$, maintaining the Reeb graph $R_{i+1}$

$$\begin{tikzcd}[column sep=small]
\mathbb{R} & & \mathbb{R} \\
R_i \arrow[u, "\tilde{f}_i"] & & R_{i+1} \arrow[u, "\tilde{f}_{i+1}"'] \\
& G_i \arrow[ul, two heads] \arrow[ur, two heads] &
\end{tikzcd}$$

relabel

modify $f_i$ to $f_{i+1} : G_i \to \mathbb{R}$, maintaining the domain $G_i$

- Define the Reeb graph edit distance analogously as

$$d_{eGraph}(R_f, R_g) = \inf_Z \|s_Z\|_\infty .$$

# Universality of the Reeb Graph Edit Distance

**Thm** [B., Landi, Mémoli 2018] The Reeb graph edit distance is universal.

- Hard part: stability
  - Given: $f, g : X \to \mathbb{R}$
  - Construct zigzag

$$\begin{tikzcd}[column sep=small]
R_f = R_0 & & R_i & & R_{i+1} & & R_n = R_g \\
& \cdots \arrow[ul] \arrow[ur, dotted] & & G_i \arrow[ul] \arrow[ur] & & \cdots \arrow[ul, dotted] \arrow[ur] &
\end{tikzcd}$$

with spread $\leq \|f-g\|_\infty$

  - Main difficulty: construct the Reeb quotient maps

# CONCLUSION

- All previous distances for Reeb graphs are equivalent
- Thus, there is one (topological) space of Reeb graphs
- There is an "optimal" distance (universal)
- This distance can be understood as an edit distance

## Challenges:

- Computation / Approximation
- Relationship between PL / Top. setting
- Higher - dimensional generalizations