Apparent pairs in geometry & topology

Ulrich Bauer

Technical University of Munich (TUM)

May 27, 2025

Workshop on Combinatorial Topological Dynamics

Jagiellonian University

Funded by

Deutsche Forschungsgemeinschaft
German Research Foundation

SFB TRR 109 Discretization in Geometry and Dynamics

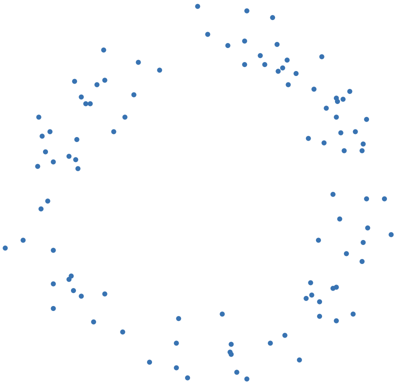
Technical University of Munich

Munich Center for Machine Learning

Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

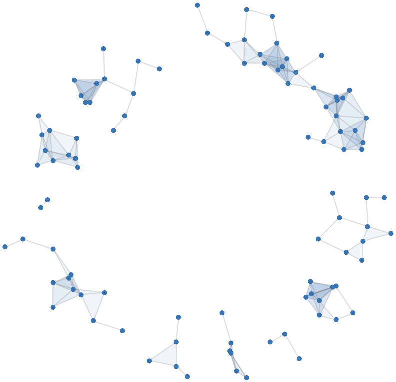
$$\operatorname{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

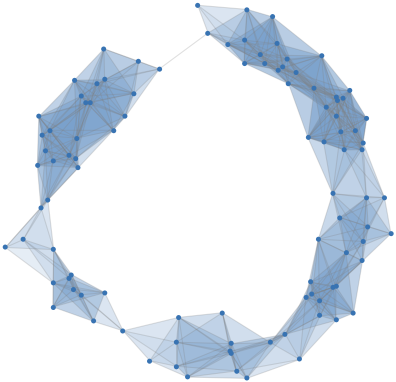
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

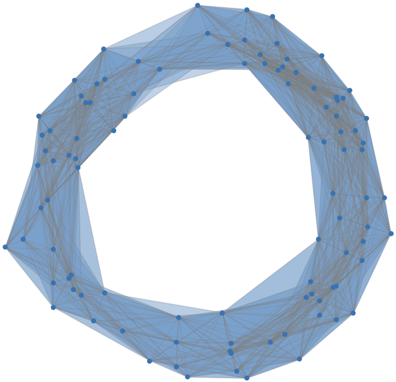
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

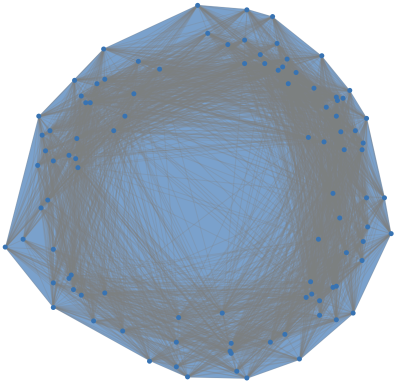
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Vietoris–Rips complexes

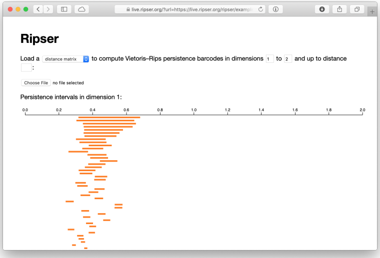
For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

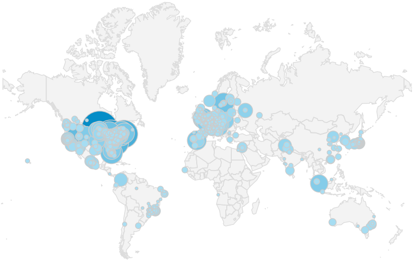
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



Ripser: software for computing Vietoris–Rips persistence barcodes

Open source software (`ripser.org`)

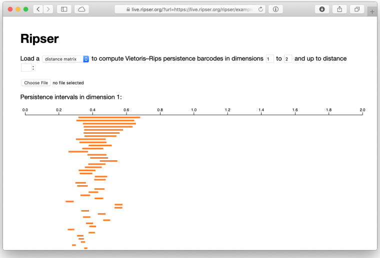


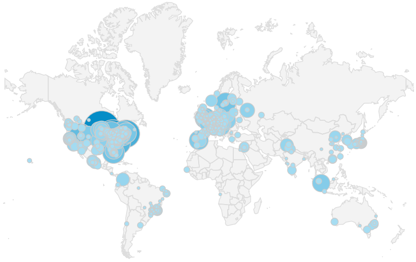


Ripser users worldwide

Ripser: software for computing Vietoris–Rips persistence barcodes

Open source software (`ripser.org`)





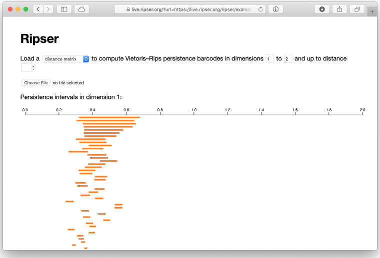
Ripser users worldwide

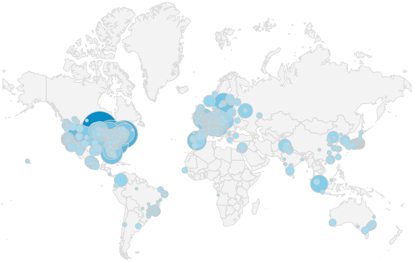
Efficient matrix algorithm based on

- *clearing*: avoiding unnecessary column operations
- computing persistent *cohomology*

Ripser: software for computing Vietoris–Rips persistence barcodes

Open source software (`ripser.org`)





Ripser users worldwide

Efficient matrix algorithm based on

- *clearing*: avoiding unnecessary column operations
- computing persistent *cohomology*

Computational improvements based on

- *implicit matrix representations*
- *apparent pairs*, connecting persistence to discrete Morse theory

Apparent pairs

Ripser uses the following pairing of simplices (breaking ties in the filtration lexicographically):

Definition (B 2016, 2021)

In a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, two simplices (σ_i, σ_j) form an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

Apparent pairs

Ripser uses the following pairing of simplices (breaking ties in the filtration lexicographically):

Definition (B 2016, 2021)

In a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, two simplices (σ_i, σ_j) form an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

Special cases and equivalent definitions have been (re)discovered independently multiple times

- Kahle 2011, Robbins 2015, Henselmann-Petrusek 2017, Lampret 2020, Feichtner-Kozlov 2020, …

Apparent pairs

Ripser uses the following pairing of simplices (breaking ties in the filtration lexicographically):

Definition (B 2016, 2021)

In a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, two simplices (σ_i, σ_j) form an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

Special cases and equivalent definitions have been (re)discovered independently multiple times

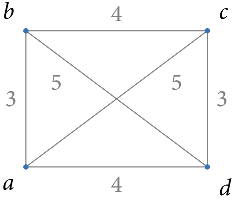
- Kahle 2011, Robbins 2015, Henselmann-Petrusek 2017, Lampret 2020, Feichtner-Kozlov 2020, …

Proposition (B 2021)

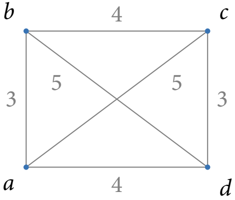
The apparent pairs are both

- *persistence pairs (creating/destroying a feature in homology) and*
- *gradient pairs (in the sense of discrete Morse theory).*

Apparent pairs of the diameter-lexicographic filtration

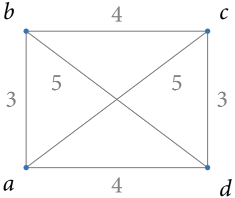


Apparent pairs of the diameter-lexicographic filtration



$$
\partial_1 = \begin{array}{c@{}c}
\begin{array}{cccccc} \rotatebox{90}{$(a,b){:}3$} & \rotatebox{90}{$(c,d){:}3$} & \rotatebox{90}{$(a,d){:}4$} & \rotatebox{90}{$(b,c){:}4$} & \rotatebox{90}{$(a,c){:}5$} & \rotatebox{90}{$(b,d){:}5$} \end{array} & \\
\begin{pmatrix} 1 & & 1 & & 1 & \\ \mathbf{1} & & & 1 & & 1 \\ & 1 & & 1 & 1 & \\ & \mathbf{1} & 1 & & & 1 \end{pmatrix} & \begin{array}{c} (a) \\ (b) \\ (c) \\ (d) \end{array}
\end{array}
$$

Apparent pairs of the diameter-lexicographic filtration



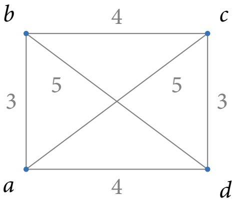
$$
\partial_1 = \begin{array}{c} \\ \end{array}
\begin{array}{cccccc}
(a,b){:}3 & (c,d){:}3 & (a,d){:}4 & (b,c){:}4 & (a,c){:}5 & (b,d){:}5
\end{array}
$$

$$
\partial_1 = \left(\begin{array}{cccccc}
1 & & 1 & & 1 & \\
\mathbf{1} & & & 1 & & 1 \\
 & 1 & & 1 & 1 & \\
 & \mathbf{1} & 1 & & & 1
\end{array}\right)
\begin{array}{l}
(a) \\ (b) \\ (c) \\ (d)
\end{array}
$$

$$
\begin{array}{cccc}
(a,b,c){:}5 & (a,b,d){:}5 & (a,c,d){:}5 & (b,c,d){:}5
\end{array}
$$

$$
\partial_2 = \left(\begin{array}{cccc}
1 & 1 & & \\
 & & 1 & 1 \\
 & 1 & 1 & \\
1 & & & 1 \\
\mathbf{1} & & 1 & \\
 & \mathbf{1} & & 1
\end{array}\right)
\begin{array}{l}
(a,b){:}3 \\ (c,d){:}3 \\ (a,d){:}4 \\ (b,c){:}4 \\ (a,c){:}5 \\ (b,d){:}5
\end{array}
$$

Apparent pairs of the diameter-lexicographic filtration



$$
\partial_1 = \begin{array}{c}
\begin{array}{cccccc} (a,b){:}3 & (c,d){:}3 & (a,d){:}4 & (b,c){:}4 & (a,c){:}5 & (b,d){:}5 \end{array} \\
\begin{pmatrix}
1 & & 1 & & 1 & \\
\mathbf{1} & & & 1 & & 1 \\
 & 1 & & 1 & 1 & \\
 & \mathbf{1} & 1 & & & 1
\end{pmatrix}
\end{array}
\begin{array}{l} (a) \\ (b) \\ (c) \\ (d) \end{array}
\qquad
\partial_2 = \begin{array}{c}
\begin{array}{cccc} (a,b,c){:}5 & (a,b,d){:}5 & (a,c,d){:}5 & (b,c,d){:}5 \end{array} \\
\begin{pmatrix}
1 & 1 & & \\
 & & 1 & 1 \\
 & 1 & 1 & \\
1 & & & 1 \\
\mathbf{1} & & 1 & \\
 & \mathbf{1} & & 1
\end{pmatrix}
\end{array}
\begin{array}{l} (a,b){:}3 \\ (c,d){:}3 \\ (a,d){:}4 \\ (b,c){:}4 \\ (a,c){:}5 \\ (b,d){:}5 \end{array}
\qquad
\partial_3 = \begin{array}{c}
(a,b,c,d) \\
\begin{pmatrix} 1 \\ 1 \\ 1 \\ \mathbf{1} \end{pmatrix}
\end{array}
\begin{array}{l} (a,b,c) \\ (a,b,d) \\ (a,c,d) \\ (b,c,d) \end{array}
$$

A shortcut for finding pivots

$$\begin{array}{c} (a,b,c) \\ \left(\begin{array}{c} 1 \\ 1 \\ \\ 1 \\ \\ \\ \end{array}\right) \end{array} \begin{array}{l} \\ (a,b) \\ (a,c) \\ (a,d) \\ (b,c) \\ (b,d) \\ (c,d) \end{array} \qquad \begin{array}{c} (a,b,c){:}5 \\ \left(\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right) \end{array} \begin{array}{l} \\ (a,b){:}3 \\ (c,d){:}3 \\ (a,d){:}4 \\ (b,c){:}4 \\ (a,c){:}5 \\ (b,d){:}5 \end{array}$$

Shortcut for finding the pivot (latest) facet of a simplex τ:

A shortcut for finding pivots

$$
\begin{array}{cl}
(a,b,c) & \\
\left(\begin{array}{c} 1 \\ 1 \\ \\ 1 \\ \\ \end{array}\right) &
\begin{array}{l} (a,b) \\ (a,c) \\ (a,d) \\ (b,c) \\ (b,d) \\ (c,d) \end{array}
\end{array}
\longrightarrow
\begin{array}{cl}
(a,b,c){:}5 & \\
\left(\begin{array}{c} \\ \\ \\ 1 \\ \\ \end{array}\right) &
\begin{array}{l} (a,b){:}3 \\ (c,d){:}3 \\ (a,d){:}4 \\ (b,c){:}4 \\ (a,c){:}5 \\ (b,d){:}5 \end{array}
\end{array}
$$

Shortcut for finding the pivot (latest) facet of a simplex τ:

- Enumerate facets σ of τ in (reverse) lexicographic order

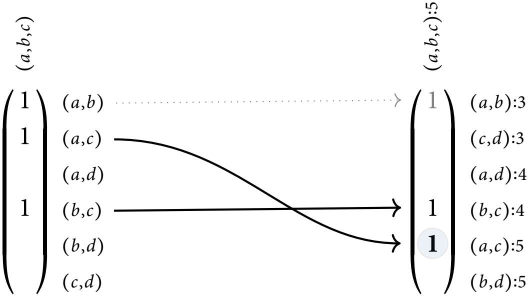
A shortcut for finding pivots

$$
\begin{array}{cc}
(a,b,c) & \\
\left(\begin{array}{c} 1 \\ 1 \\ \\ 1 \\ \\ \end{array}\right) & \begin{array}{l} (a,b) \\ (a,c) \\ (a,d) \\ (b,c) \\ (b,d) \\ (c,d) \end{array}
\end{array}
\qquad\qquad
\begin{array}{cc}
(a,b,c){:}5 & \\
\left(\begin{array}{c} \\ \\ \\ 1 \\ \mathbf{1} \\ \end{array}\right) & \begin{array}{l} (a,b){:}3 \\ (c,d){:}3 \\ (a,d){:}4 \\ (b,c){:}4 \\ (a,c){:}5 \\ (b,d){:}5 \end{array}
\end{array}
$$

Shortcut for finding the pivot (latest) facet of a simplex τ:

- Enumerate facets σ of τ in (reverse) lexicographic order
- The first facet σ with $\operatorname{diam} \sigma = \operatorname{diam} \tau$ is the pivot of τ

A shortcut for finding pivots



Shortcut for finding the pivot (latest) facet of a simplex τ:

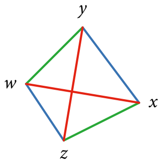
- Enumerate facets σ of τ in (reverse) lexicographic order
- The first facet σ with $\operatorname{diam}\sigma = \operatorname{diam}\tau$ is the pivot of τ

Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ-*hyperbolic* (for $\delta \geq 0$) if for all $w, x, y, z \in X$ we have

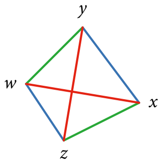
$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$



Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ-*hyperbolic* (for $\delta \geq 0$) if for all $w, x, y, z \in X$ we have



$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$

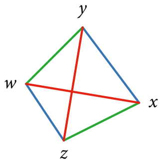
- The hyperbolic plane is $(\ln 2)$-hyperbolic.

Gromov-hyperbolicity

Definition (Gromov 1988)

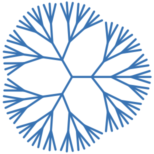
A metric space X is δ-*hyperbolic* (for $\delta \geq 0$) if for all $w, x, y, z \in X$ we have

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$

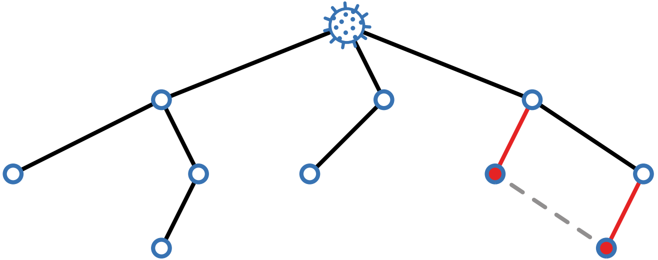


- The hyperbolic plane is $(\ln 2)$-hyperbolic.

- Metric trees and their subspaces are precisely the 0-hyperbolic spaces.

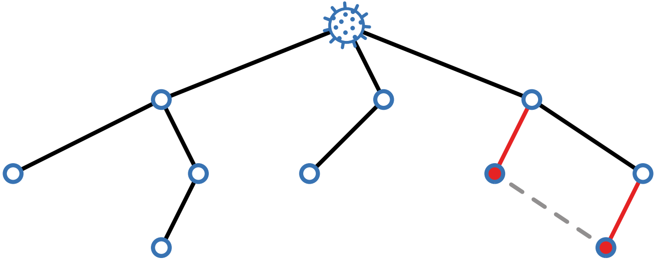


Topology of viral evolution



Joint work with: A. Ott, M. Bleher, L. Hahn (Heidelberg), R. Rabadan, J. Patiño-Galindo (Columbia), M. Carrière (INRIA)

Topology of viral evolution



Joint work with: A. Ott, M. Bleher, L. Hahn (Heidelberg), R. Rabadan, J. Patiño-Galindo (Columbia), M. Carrière (INRIA)

Observation: Ripser runs unusually fast on genetic distance data

- SARS-CoV2 RNA sequences (spike protein)
- 25556 data points (2.8×10^{12} simplices in 2-skeleton)
- $120\,s$ computation time (with data points ordered appropriately)

Rips complexes of hyperbolic spaces

Theorem (Rips; Gromov 1988)

Let X *be a* δ*-hyperbolic geodesic metric space. Then* $\mathrm{Rips}_t(X)$ *is contractible for all* $t \geq 4\delta$*.*

Rips complexes of hyperbolic spaces

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

- Generalization to non-geodesic spaces? In particular, finite metric spaces?

Rips complexes of hyperbolic spaces

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

- Generalization to non-geodesic spaces? In particular, finite metric spaces?
- Collapsiblility? By a discrete gradient compatible with the filtration?

Rips complexes of hyperbolic spaces

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

- Generalization to non-geodesic spaces? In particular, finite metric spaces?
- Collapsiblility? By a discrete gradient compatible with the filtration?
- Connection to computation of persistent homology?

Rips complexes of hyperbolic spaces

Theorem (Rips; Gromov 1988)

Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

- Generalization to non-geodesic spaces? In particular, finite metric spaces?
- Collapsiblility? By a discrete gradient compatible with the filtration?
- Connection to computation of persistent homology?

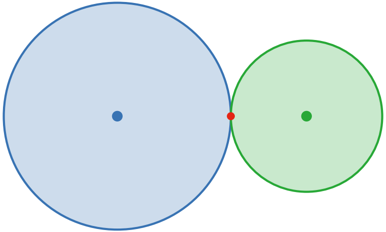
Theorem (B, Roll 2022)

Let X be a finite δ-hyperbolic space. Then there is a single discrete gradient encoding the collapses

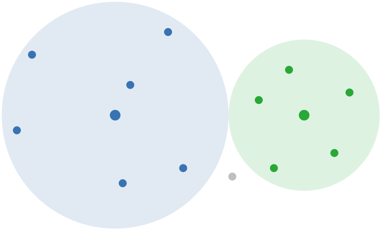
$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect *of X.*

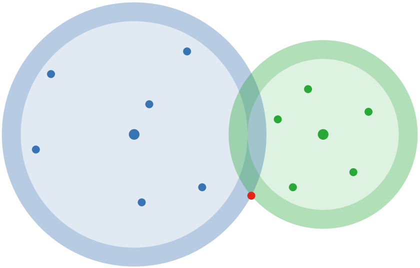
Geodesic defect



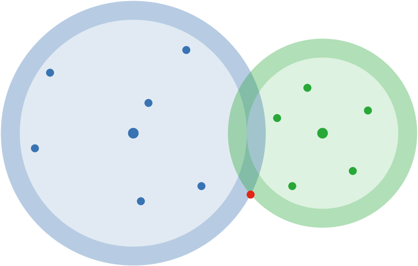
Geodesic defect



Geodesic defect



Geodesic defect



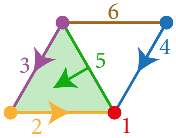
Definition (Bonk, Schramm 2000)

A metric space X is *ν-geodesic* if for all points $x, y \in X$ and all $r, s \geq 0$ with $r + s = d(x, y)$ we have

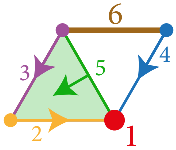
$$B_{r+\nu}(x) \cap B_{s+\nu}(y) \neq \varnothing.$$

The infimum of all such ν is the *geodesic defect* of X.

Discrete Morse theory



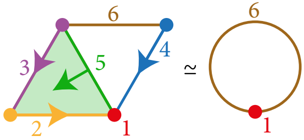
Discrete Morse theory



Discrete Morse theory

Theorem (Forman 1998)

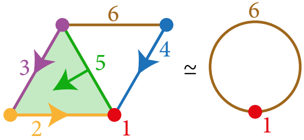
A simplicial complex with a discrete Morse function f is homotopy equivalent to a space (a CW complex*) built from the critical simplices of f.*



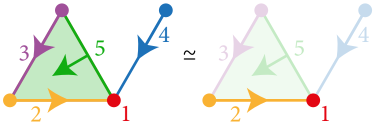
Discrete Morse theory

Theorem (Forman 1998)

A simplicial complex with a discrete Morse function f is homotopy equivalent to a space (a CW complex*) built from the critical simplices of f.*



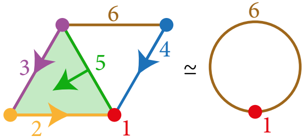
Discrete Morse functions – and their gradients – encode *collapses* of sublevel sets:



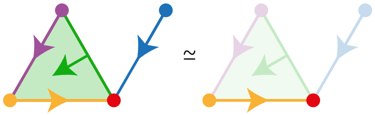
Discrete Morse theory

Theorem (Forman 1998)

A simplicial complex with a discrete Morse function f is homotopy equivalent to a space (a CW complex*) built from the critical simplices of* f.

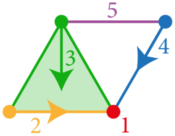


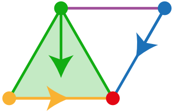
Discrete Morse functions – and their gradients – encode *collapses* of sublevel sets:



Generalizing discrete Morse theory

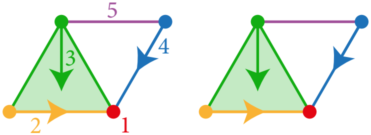
Generalized gradients partition the face poset into intervals (instead of just facet pairs):



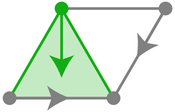


Generalizing discrete Morse theory

Generalized gradients partition the face poset into intervals (instead of just facet pairs):

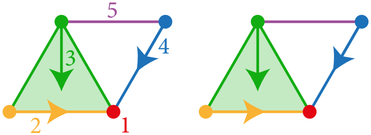


- A generalized vector field V can always be refined to a vector field.

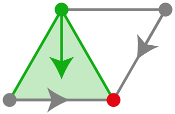


Generalizing discrete Morse theory

Generalized gradients partition the face poset into intervals (instead of just facet pairs):

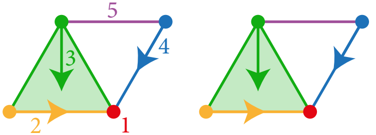


- A generalized vector field V can always be refined to a vector field.

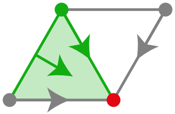


Generalizing discrete Morse theory

Generalized gradients partition the face poset into intervals (instead of just facet pairs):

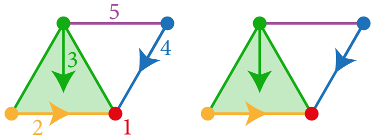


- A generalized vector field V can always be refined to a vector field.

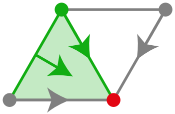


Generalizing discrete Morse theory

Generalized gradients partition the face poset into intervals (instead of just facet pairs):



- A generalized vector field V can always be refined to a vector field.



Proposition (B, Roll 2022)

Let f be a generalized discrete Morse function, breaking ties lexicographically.
Then the apparent pairs of zero persistence form a gradient that

- *refines the gradient of f and*
- *has the same critical simplices.*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$.

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$.
- *The apparent pairs (lexicographic refinement, based on arbitrary vertex order) refine this gradient.*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V,E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$*.*
- *The apparent pairs (lexicographic refinement, based on arbitrary vertex order) refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}: \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$.
- *The apparent pairs (lexicographic refinement, based on arbitrary vertex order) refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

- $\operatorname{Rips}_t(X) \searrow T_t$ *for all* $t \in \mathbb{R}$,

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$*.*
- *The apparent pairs (lexicographic refinement, based on arbitrary vertex order) refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

- $\operatorname{Rips}_t(X) \searrow T_t$ *for all* $t \in \mathbb{R}$,
- $\operatorname{Rips}_t(X) \searrow T \searrow \{*\}$ *for* $t \geq \max d(E)$, *and*

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) *with a* generic *path length metric (distinct pairwise distances). Then the diameter function* $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ *is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices,* $V \cup E$.
- *The apparent pairs (lexicographic refinement, based on arbitrary vertex order) refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X *induces the collapses*

- $\operatorname{Rips}_t(X) \searrow T_t$ *for all* $t \in \mathbb{R}$,
- $\operatorname{Rips}_t(X) \searrow T \searrow \{*\}$ *for* $t \geq \max d(E)$, *and*
- $\operatorname{Rips}_u(X) \searrow \operatorname{Rips}_t(X)$ *whenever no pairwise distance lies in the interval* $(t, u]$.

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) with a generic *path length metric (distinct pairwise distances). Then the diameter function $\operatorname{diam}\colon \Delta(V) \to \mathbb{R}$ is a generalized discrete Morse function.*

- *The critical simplices are the tree simplices, $V \cup E$.*
- *The apparent pairs (lexicographic refinement, based on arbitrary vertex order) refine this gradient.*

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X induces the collapses

- $\operatorname{Rips}_t(X) \searrow T_t$ *for all $t \in \mathbb{R}$,*
- $\operatorname{Rips}_t(X) \searrow T \searrow \{*\}$ *for $t \geq \max d(E)$, and*
- $\operatorname{Rips}_u(X) \searrow \operatorname{Rips}_t(X)$ *whenever no pairwise distance lies in the interval $(t, u]$.*

In particular, the persistent homology is trivial in degrees > 0.

Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.

Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

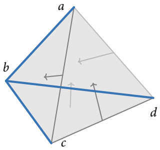
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



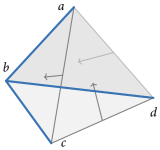
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



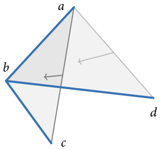
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



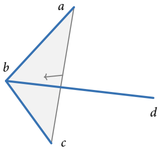
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



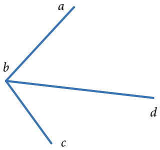
Tree metrics beyond the generic case

Why is Ripser particularly fast on genetic distances (tree-like, non-generic metric)?

- Consider a weighted finite tree $T = (V, E)$, viewed as a metric space X.
- Choose an arbitrary root and extend the rooted tree partial order to a total order.

Theorem (B, Roll 2022)

The apparent pairs gradient for this total order induces the same collapses as in the generic case.



Morse theory for Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

The Čech complexes and the Delaunay complexes (alpha shapes) are sublevel sets of (generalized) discrete Morse functions. Both functions have the same critical simplices/values.

Morse theory for Čech and Delaunay complexes

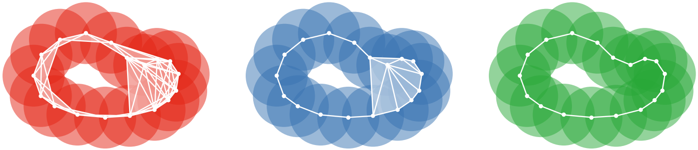
Proposition (B, Edelsbrunner 2014)

The Čech complexes and the Delaunay complexes (alpha shapes) are sublevel sets of (generalized) discrete Morse functions. Both functions have the same critical simplices/values.

Theorem (B, Edelsbrunner 2017)

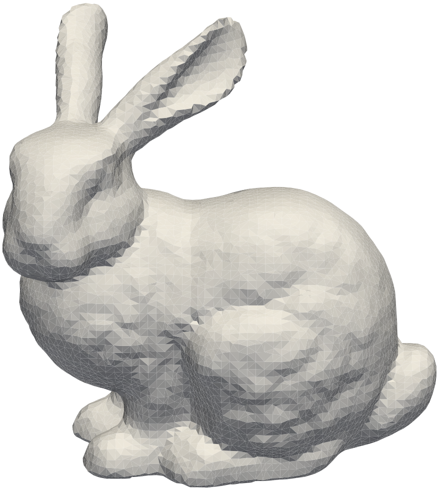
Čech, Delaunay, and Wrap complexes (at any scale r) of a point set $X \subset \mathbb{R}^d$ in general position are related by collapses encoded by a single discrete gradient field:

$$\mathrm{Cech}_r X \searrow \mathrm{Del}_r X \searrow \mathrm{Wrap}_r X.$$

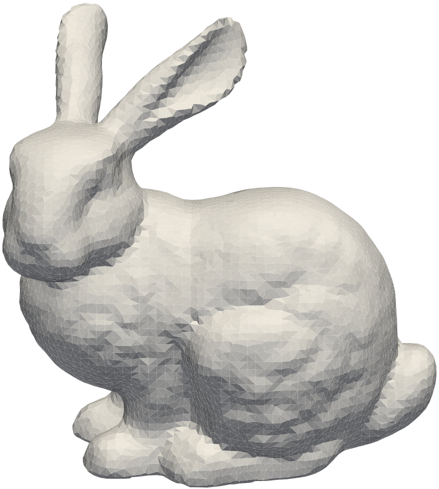


From Delaunay to Wrap complexes

From Delaunay to Wrap complexes



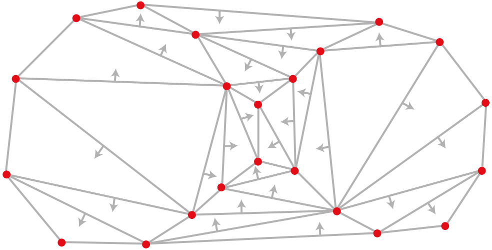
From Delaunay to Wrap complexes



Foundation of the surface reconstruction software *Wrap* (Edelsbrunner 1995, Geomagic)

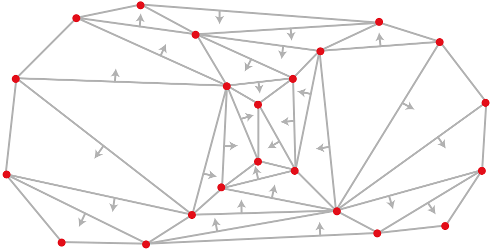
Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



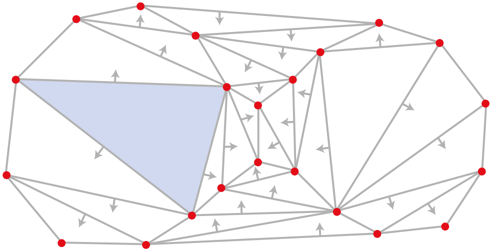
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
    - contains all critical simplices
    - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



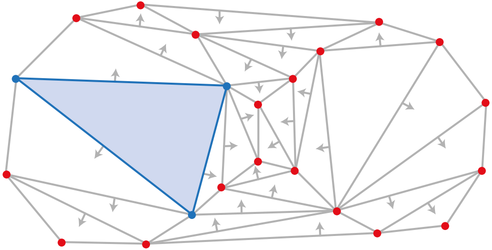
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r\, X$:

- the smallest subcomplex of $\mathrm{Del}_r\, X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



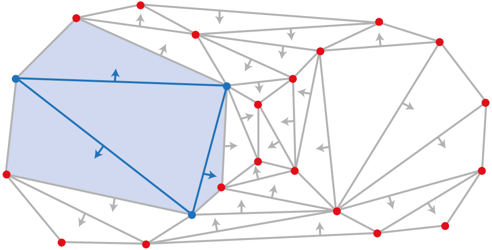
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



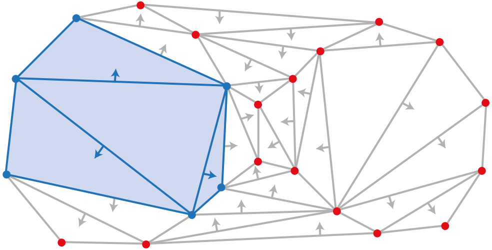
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



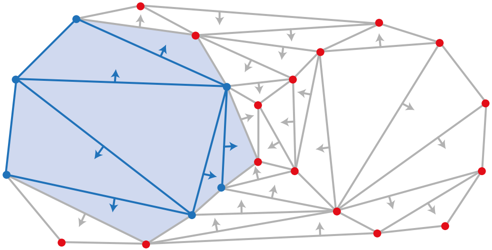
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r\, X$:

- the smallest subcomplex of $\mathrm{Del}_r\, X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



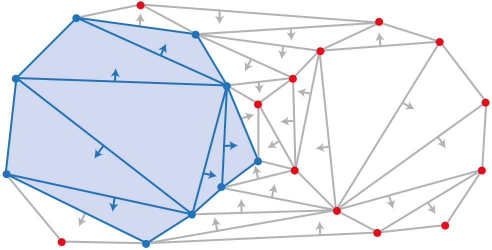
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



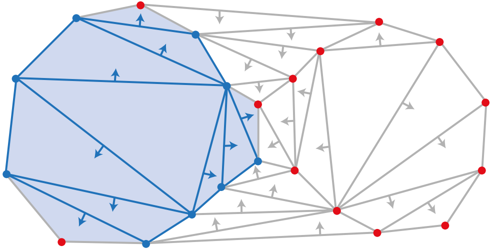
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



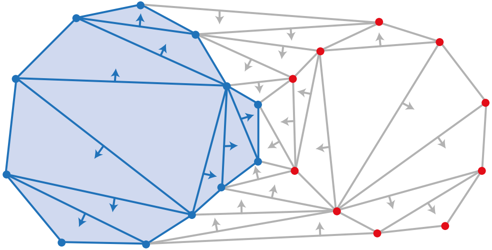
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



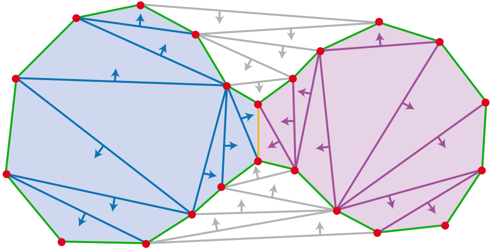
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
  - contains all critical simplices
  - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



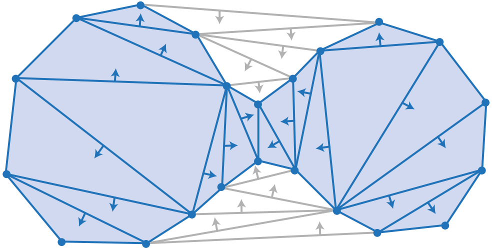
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r\, X$:

- the smallest subcomplex of $\mathrm{Del}_r\, X$ that
    - contains all critical simplices
    - is compatible with V (a union of intervals of V).

Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r\, X$:

- the smallest subcomplex of $\mathrm{Del}_r\, X$ that
    - contains all critical simplices
    - is compatible with V (a union of intervals of V).

Computing persistent homology via matrix reduction

Algorithm (matrix reduction; a variant of Gauss elimination)

Require: D: $m \times n$ matrix

Ensure: V is full rank upper triangular, $R = D \cdot V$ has unique column pivots

function Reduce(D)

$R = D$

$V = I(n)$

while there exist $i < j$ such that $\operatorname{pivot} R_i = \operatorname{pivot} R_j$ **do**

add column R_i to column R_j ▷ eliminate the nonzero entry in row $\operatorname{pivot} R_i$

add column V_i to column V_j

return R, V

Computing persistent homology via matrix reduction

Algorithm (exhaustive matrix reduction; a variant of Gauss–Jordan elimination)

Require: D: $m \times n$ matrix

Ensure: V is full rank upper triangular, $R = D \cdot V$ has unique column pivots

function Reduce(D)

  $R = D$

  $V = I(n)$

  while there exist $i < j$ such that R_j has a nonzero entry in row $\operatorname{pivot} R_i$ **do**

    add column R_i to column R_j ▷ eliminate the nonzero entry in row $\operatorname{pivot} R_i$

    add column V_i to column V_j

  return R, V

Computing persistent homology via matrix reduction

Algorithm (exhaustive matrix reduction; a variant of Gauss–Jordan elimination)

Require: D: $m \times n$ matrix
Ensure: V is full rank upper triangular, $R = D \cdot V$ has unique column pivots

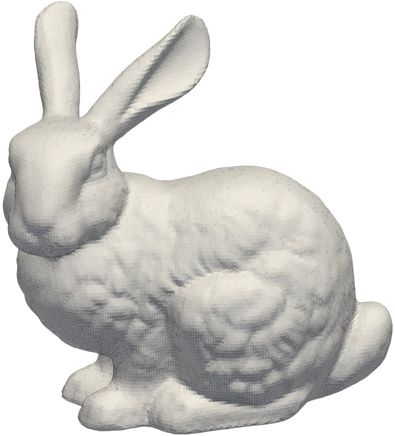
function Reduce(D)
  $R = D$
  $V = I(n)$
  while there exist $i < j$ such that R_j has a nonzero entry in row $\mathrm{pivot}\, R_i$ **do**
    add column R_i to column R_j ▷ eliminate the nonzero entry in row $\mathrm{pivot}\, R_i$
    add column V_i to column V_j
  return R, V

Proposition

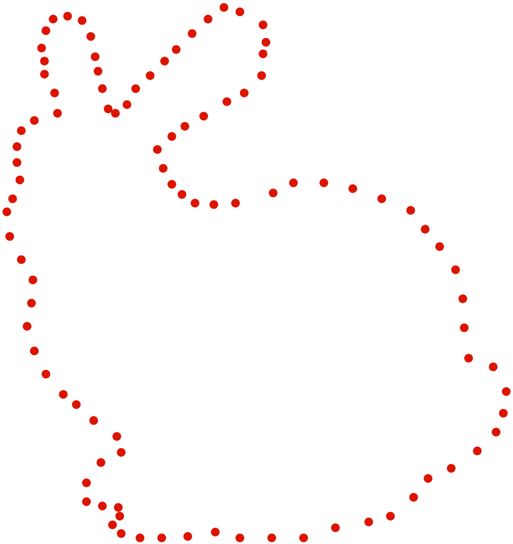
The resulting columns R_j are minimal (in a lexicographic order) within their homology class (in K_{j-1}).

Standard reduction and exhaustive reduction

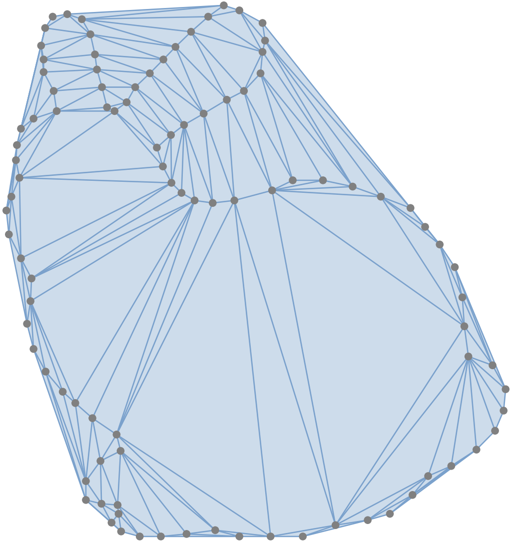
Standard reduction and exhaustive reduction



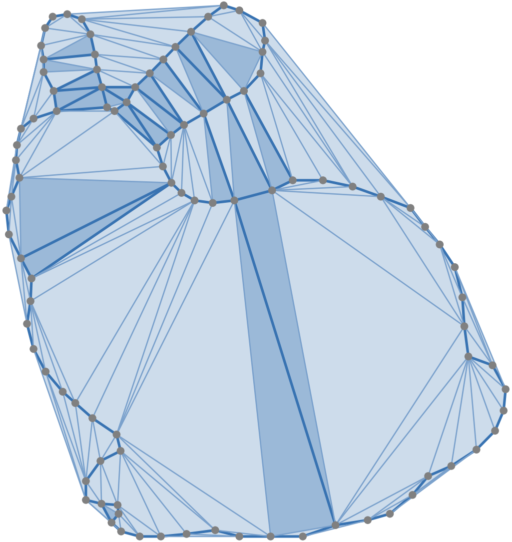
Wrap complexes and lexicographically minimal cycles



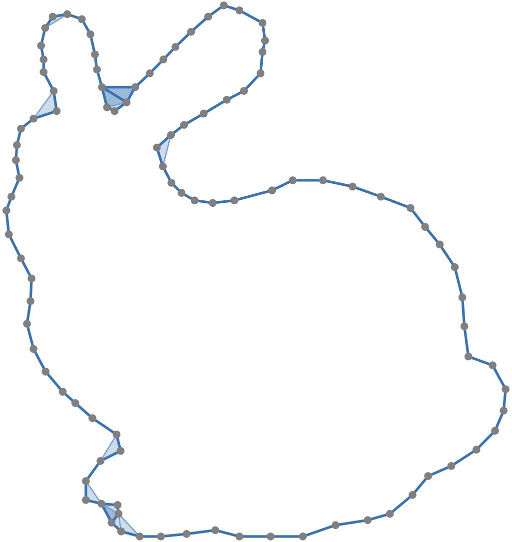
Wrap complexes and lexicographically minimal cycles



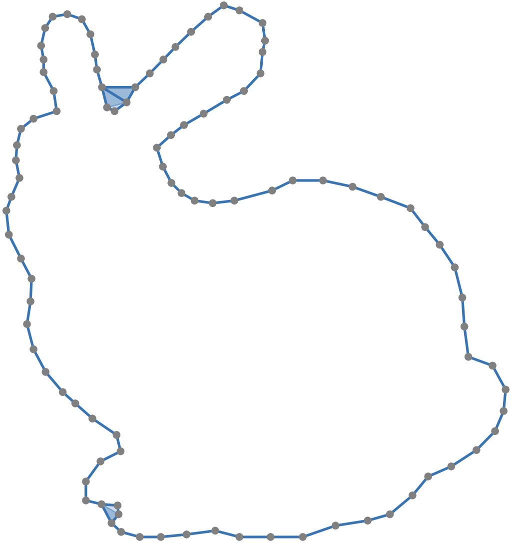
Wrap complexes and lexicographically minimal cycles



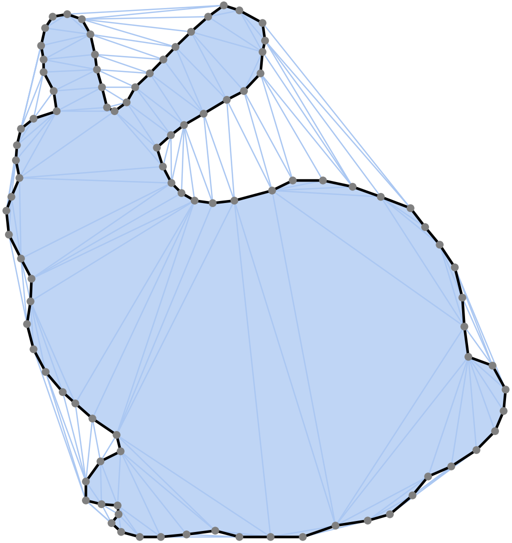
Wrap complexes and lexicographically minimal cycles



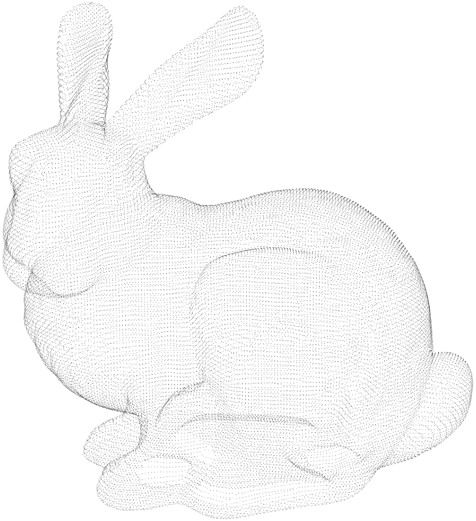
Wrap complexes and lexicographically minimal cycles



Wrap complexes and lexicographically minimal cycles

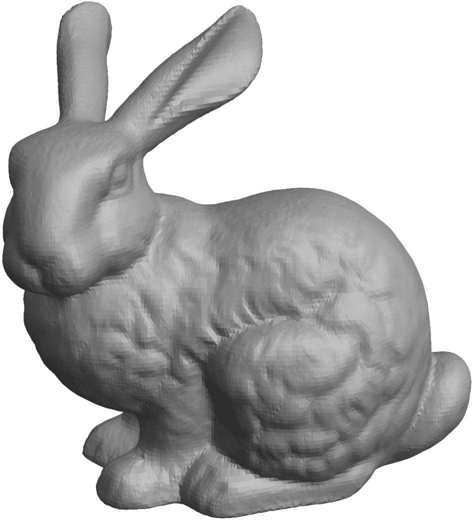


Wrap complexes and lexicographically minimal cycles

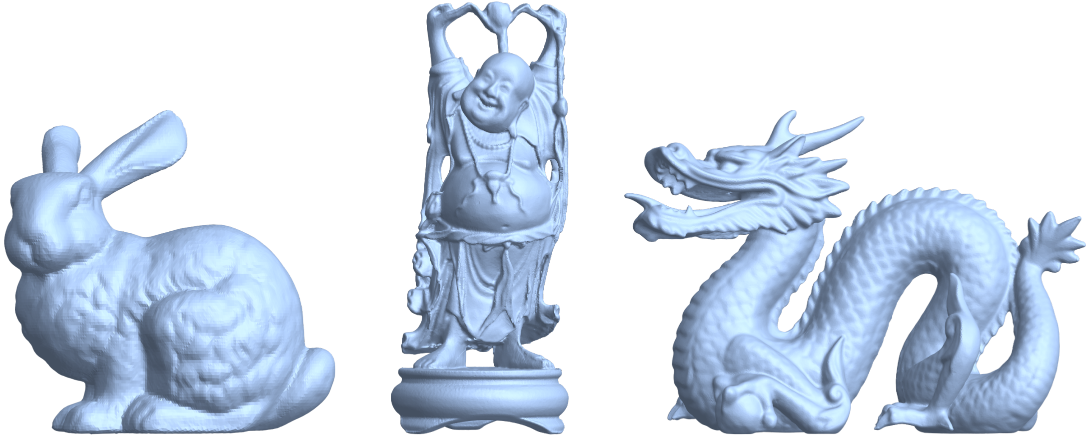


http://ulrich-bauer.org/apparent-pairs-talk-jagiellonian.pdf

Wrap complexes and lexicographically minimal cycles



Point cloud reconstruction with minimal cycles

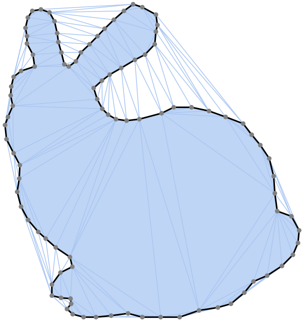


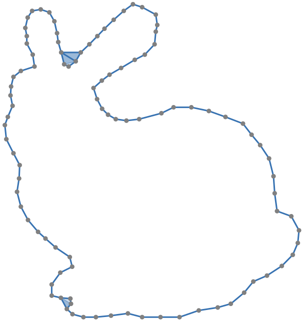
Wrap complexes support minimal cycles

Theorem (B, Roll 2024)

Let $X \subset \mathbb{R}$ be a finite subset in general position and let $r \in \mathbb{R}$.

- *Exhaustive matrix reduction computes the minimal cycles homologous to a simplex boundary.*
- *Any lexicographically minimal cycle of $\mathrm{Del}_r(X)$ is supported on $\mathrm{Wrap}_r(X)$.*





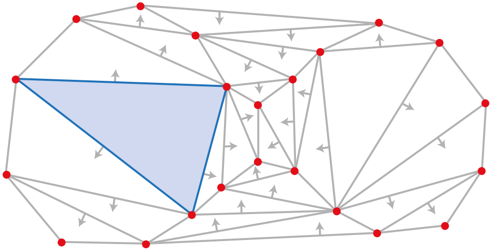
Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

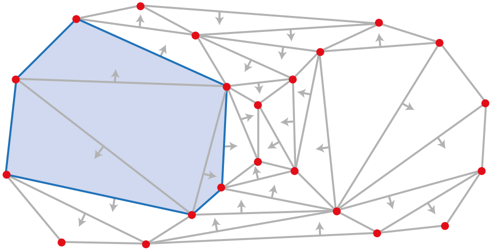
- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

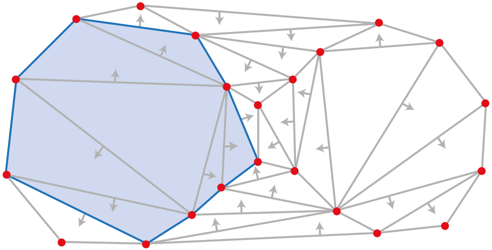
- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

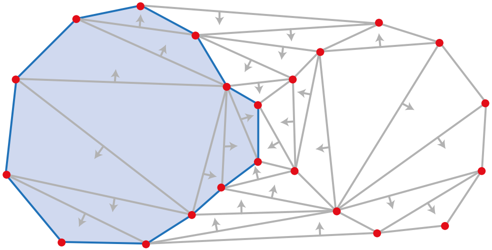
- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

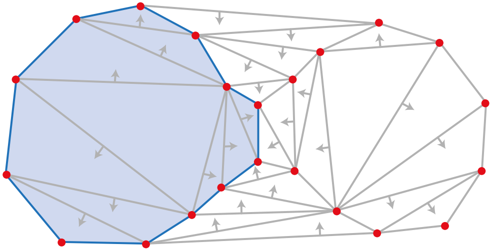
- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)

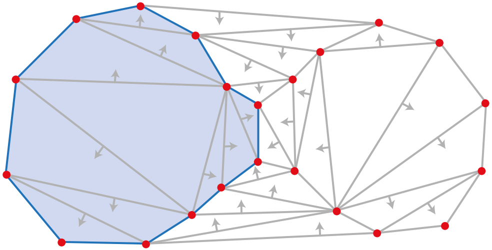


- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)

Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



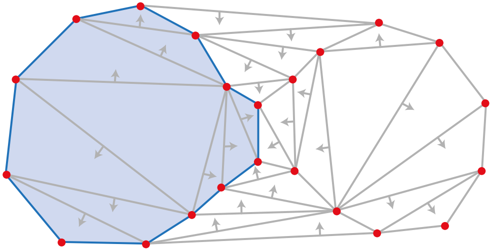
- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)

We interpret persistent homology in terms of algebraic Morse theory:

Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)

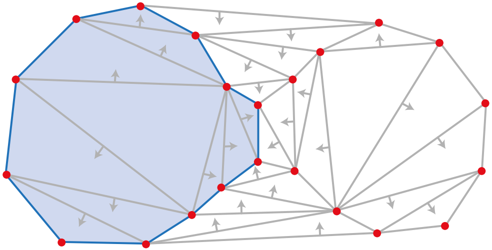
We interpret persistent homology in terms of algebraic Morse theory:

- Persistence pairs correspond to algebraic gradient pairs

Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a gradient (discrete Morse theory)



- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to based chain complexes (*algebraic Morse theory*)

We interpret persistent homology in terms of algebraic Morse theory:

- Persistence pairs correspond to algebraic gradient pairs
- Matrix reduction (exhaustive) corresponds to gradient flow

Making discrete Morse theory algebraic

Algebraic Morse theory expresses discrete Morse theory in purely algebraic terms.

Making discrete Morse theory algebraic

Algebraic Morse theory expresses discrete Morse theory in purely algebraic terms.

Definition

Let (C_*, Σ_*) be a free chain complex with a basis (Lefschetz complex).

Making discrete Morse theory algebraic

Algebraic Morse theory expresses discrete Morse theory in purely algebraic terms.

Definition

Let (C_*, Σ_*) be a free chain complex with a basis (Lefschetz complex).

- A basis element $\sigma \in \Sigma_d$ is a *facet* of another $\tau \in \Sigma_{d+1}$ if $\langle \sigma, \partial\tau \rangle \neq 0$.

Making discrete Morse theory algebraic

Algebraic Morse theory expresses discrete Morse theory in purely algebraic terms.

Definition

Let (C_*, Σ_*) be a free chain complex with a basis (Lefschetz complex).

- A basis element $\sigma \in \Sigma_d$ is a *facet* of another $\tau \in \Sigma_{d+1}$ if $\langle \sigma, \partial\tau \rangle \neq 0$.
- The transitive closure is the *face* poset.

Making discrete Morse theory algebraic

Algebraic Morse theory expresses discrete Morse theory in purely algebraic terms.

Definition

Let (C_*, Σ_*) be a free chain complex with a basis (Lefschetz complex).

- A basis element $\sigma \in \Sigma_d$ is a *facet* of another $\tau \in \Sigma_{d+1}$ if $\langle \sigma, \partial\tau \rangle \neq 0$.
- The transitive closure is the *face* poset.

This generalized Forman's theory for cell complexes:

Making discrete Morse theory algebraic

Algebraic Morse theory expresses discrete Morse theory in purely algebraic terms.

Definition

Let (C_*, Σ_*) be a free chain complex with a basis (Lefschetz complex).

- A basis element $\sigma \in \Sigma_d$ is a *facet* of another $\tau \in \Sigma_{d+1}$ if $\langle \sigma, \partial\tau \rangle \neq 0$.
- The transitive closure is the *face* poset.

This generalized Forman's theory for cell complexes:

- The basis elements Σ_* take the role of the simplices in discrete Morse theory

Making discrete Morse theory algebraic

Algebraic Morse theory expresses discrete Morse theory in purely algebraic terms.

Definition

Let (C_*, Σ_*) be a free chain complex with a basis (Lefschetz complex).

- A basis element $\sigma \in \Sigma_d$ is a *facet* of another $\tau \in \Sigma_{d+1}$ if $\langle \sigma, \partial\tau \rangle \neq 0$.
- The transitive closure is the *face* poset.

This generalized Forman's theory for cell complexes:

- The basis elements Σ_* take the role of the simplices in discrete Morse theory
- All further definitions of discrete Morse theory apply verbatim

Algebraic gradients from persistent homology

Goals for a bridge between persistence pairs and gradient pairs:

Algebraic gradients from persistent homology

Goals for a bridge between persistence pairs and gradient pairs:

- Gradient pairs should correspond to all persistence pairs

Algebraic gradients from persistent homology

Goals for a bridge between persistence pairs and gradient pairs:

- Gradient pairs should correspond to all persistence pairs
- Original gradient pairs (apparent zero persistence pairs) should be preserved

Algebraic gradients from persistent homology

Goals for a bridge between persistence pairs and gradient pairs:

- Gradient pairs should correspond to all persistence pairs
- Original gradient pairs (apparent zero persistence pairs) should be preserved

Let $R = D \cdot V$ be the result of standard reduction of the boundary matrix D (Delaunay filtration, breaking ties lexicographically).

Algebraic gradients from persistent homology

Goals for a bridge between persistence pairs and gradient pairs:

- Gradient pairs should correspond to all persistence pairs
- Original gradient pairs (apparent zero persistence pairs) should be preserved

Let $R = D \cdot V$ be the result of standard reduction of the boundary matrix D (Delaunay filtration, breaking ties lexicographically).

- The pairs of chains $(\mathrm{pivot}\, R_\tau, V_\tau)$ form an algebraic gradient (*reduction gradient*).

Algebraic gradients from persistent homology

Goals for a bridge between persistence pairs and gradient pairs:

- Gradient pairs should correspond to all persistence pairs
- Original gradient pairs (apparent zero persistence pairs) should be preserved

Let $R = D \cdot V$ be the result of standard reduction of the boundary matrix D (Delaunay filtration, breaking ties lexicographically).

- The pairs of chains $(\operatorname{pivot} R_\tau, V_\tau)$ form an algebraic gradient (*reduction gradient*).
- If (σ, τ) is an apparent pair, then $(\operatorname{pivot} R_\tau, V_\tau) = (\sigma, \tau)$.

Algebraic gradients from persistent homology

Goals for a bridge between persistence pairs and gradient pairs:

- Gradient pairs should correspond to all persistence pairs
- Original gradient pairs (apparent zero persistence pairs) should be preserved

Let $R = D \cdot V$ be the result of standard reduction of the boundary matrix D (Delaunay filtration, breaking ties lexicographically).

- The pairs of chains $(\operatorname{pivot} R_\tau, V_\tau)$ form an algebraic gradient (*reduction gradient*).
- If (σ, τ) is an apparent pair, then $(\operatorname{pivot} R_\tau, V_\tau) = (\sigma, \tau)$.
- Thus the apparent pairs gradient is a subset of the reduction gradient.

Different angles on wrapping cycles

For a cycle $z \in Z_*(K_i)$ in a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, the following are equivalent:

Different angles on wrapping cycles

For a cycle $z \in Z_*(K_i)$ in a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, the following are equivalent:

- z is exhaustively reduced.

Different angles on wrapping cycles

For a cycle $z \in Z_*(K_i)$ in a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, the following are equivalent:

- z is exhaustively reduced.
- z is supported only on death and essential simplices.

Different angles on wrapping cycles

For a cycle $z \in Z_*(K_i)$ in a simplexwise filtration $(K_i = \{\sigma_1, \ldots, \sigma_i\})_i$, the following are equivalent:

- z is exhaustively reduced.
- z is supported only on death and essential simplices.
- z is lexicographically minimal in its homology class $[z] \in H_*(K_k)$ for k before the death of z.

Different angles on wrapping cycles

For a cycle $z \in Z_*(K_i)$ in a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, the following are equivalent:

- z is exhaustively reduced.
- z is supported only on death and essential simplices.
- z is lexicographically minimal in its homology class $[z] \in H_*(K_k)$ for k before the death of z.
- z is invariant under the flow induced by the reduction gradient.

Key insights leading to the theorem

Steps from flow invariant cycles to the Wrap complex:

Key insights leading to the theorem

Steps from flow invariant cycles to the Wrap complex:

- Fewer gradient pairs means more flow invariant cycles.

Key insights leading to the theorem

Steps from flow invariant cycles to the Wrap complex:

- Fewer gradient pairs means more flow invariant cycles.
  - Thus, a flow invariant cycle for the reduction gradient is also invariant for the apparent gradient.

Key insights leading to the theorem

Steps from flow invariant cycles to the Wrap complex:

- Fewer gradient pairs means more flow invariant cycles.
  - Thus, a flow invariant cycle for the reduction gradient is also invariant for the apparent gradient.
- The flow depends only on the gradient pairs, independent of the critical basis elements.

Key insights leading to the theorem

Steps from flow invariant cycles to the Wrap complex:

- Fewer gradient pairs means more flow invariant cycles.
    - Thus, a flow invariant cycle for the reduction gradient is also invariant for the apparent gradient.
- The flow depends only on the gradient pairs, independent of the critical basis elements.
    - Thus, the apparent pairs induce the same flow on the standard basis and the reduction basis.

Key insights leading to the theorem

Steps from flow invariant cycles to the Wrap complex:

- Fewer gradient pairs means more flow invariant cycles.
    - Thus, a flow invariant cycle for the reduction gradient is also invariant for the apparent gradient.
- The flow depends only on the gradient pairs, independent of the critical basis elements.
    - Thus, the apparent pairs induce the same flow on the standard basis and the reduction basis.
- All flow-invariant cycles are supported on the descending complex.

Key insights leading to the theorem

Steps from flow invariant cycles to the Wrap complex:

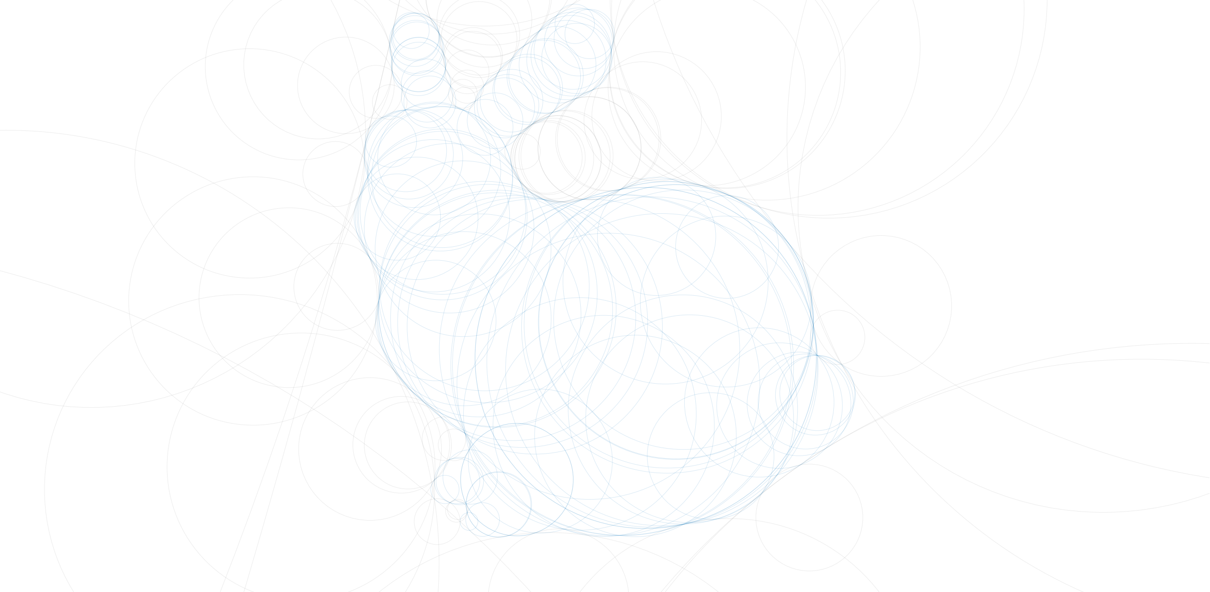
- Fewer gradient pairs means more flow invariant cycles.
  - Thus, a flow invariant cycle for the reduction gradient is also invariant for the apparent gradient.
- The flow depends only on the gradient pairs, independent of the critical basis elements.
  - Thus, the apparent pairs induce the same flow on the standard basis and the reduction basis.
- All flow-invariant cycles are supported on the descending complex.
- Coarser gradients have larger descending complexes.

Key insights leading to the theorem

Steps from flow invariant cycles to the Wrap complex:

- Fewer gradient pairs means more flow invariant cycles.
  - Thus, a flow invariant cycle for the reduction gradient is also invariant for the apparent gradient.
- The flow depends only on the gradient pairs, independent of the critical basis elements.
  - Thus, the apparent pairs induce the same flow on the standard basis and the reduction basis.
- All flow-invariant cycles are supported on the descending complex.
- Coarser gradients have larger descending complexes.
  - Thus, a flow invariant cycle is also supported on the Wrap complex.

Thanks for your attention!



http://ulrich-bauer.org/apparent-pairs-talk-jagiellonian.pdf

Thanks for your attention!

U. Bauer, F. Roll

Gromov hyperbolicity, geodesic defect, and apparent pairs in Vietoris-Rips filtrations

Symposium on Computational Geometry, 2022. doi:10.4230/LIPIcs.SoCG.2022.15

U. Bauer, F. Roll

Wrapping Cycles in Delaunay Complexes: Bridging Persistent Homology and Discrete Morse Theory

Symposium on Computational Geometry, 2024. arXiv:2212.02345

U. Bauer, H. Edelsbrunner

The Morse Theory of Čech and Delaunay Complexes

Transactions of the AMS, 2017. doi:10.1090/tran/6991

U. Bauer

Ripser: efficient computation of Vietoris–Rips persistence barcodes

Journal of Applied and Computational Topology , 2021. doi:10.1007/s41468-021-00071-5