Persistent homology for functionals

The Morse theory of Plateau's minimal surface problem

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June 20, 2022

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Joint work with Anibal Medina-Mardones (MPI Bonn) and Maximilian Schmahl (Heidelberg)

Morse functions

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- A critical point of f is *non-degenerate* if the Hessian at that point is non-singular.
- In this case, the *index* of the critical point is the total multiplicity of negative eigenvalues of the Hessian.
- If f has only non-degenerate critical points, it is a *Morse function*.

Morse inequalities

Theorem (Morse 1925)

Consider a Morse function $f : M \to \mathbb{R}$. The numbers m_i of index i critical points of f are related to the Betti numbers β_i of M by the following inequalities:

$$
\begin{gathered}
m_0 \geq \beta_0 \\
m_1 - m_0 \geq \beta_1 - \beta_0 \\
\vdots \\
m_d - m_{d-1} + \cdots \pm m_0 \geq \beta_d - \beta_{d-1} + \cdots \pm \beta_0
\end{gathered}
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Corollary ("Mountain pass lemma")

If M is connected ($\beta_0 = 1$) and f has two minima ($m_0 \geq 2$), then it has a critical point of index 1:

$$m_1 \geq \beta_1 - \beta_0 + m_0 \geq \beta_1 + 1 \geq 1.$$

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- A filtration is a certain diagram $F : \mathbf{R} \to \mathbf{Top}$ of topological spaces, indexed by $\mathbf{R} := (\mathbb{R}, \leq)$

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- Apply homology $H_* : \mathbf{Top} \to \mathbf{Vect}$
- Persistent homology is a diagram $M = H_* \circ K : \mathbf{R} \to \mathbf{Vect}$ *(persistence module)*:

$$\cdots\cdots\rightarrow M_s \longrightarrow M_t \cdots\cdots\rightarrow$$

Barcodes: the structure of persistence modules

Theorem (Krull–Schmidt–Remak; Crawley-Boewey 2015)

Any persistence module $M : \mathbf{R} \to \mathbf{vect}$ *(of finite dimensional vector spaces over some field* $\mathbb{F}$*) decomposes (in an essentially unique way) as a direct sum of* interval modules

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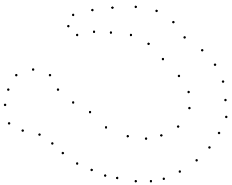

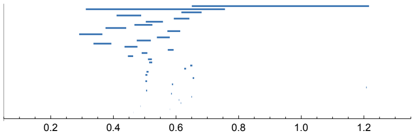

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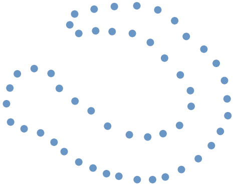

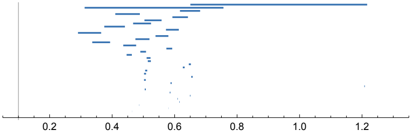

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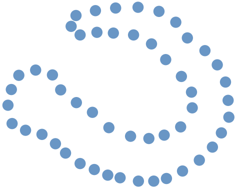

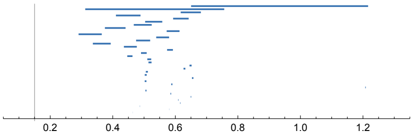

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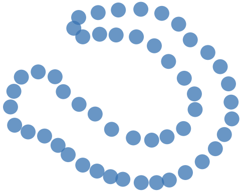

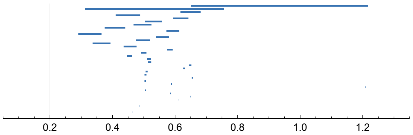

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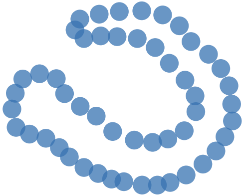

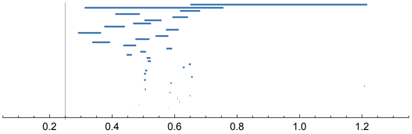

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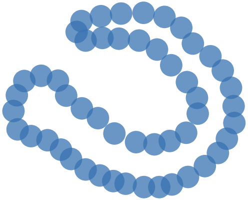

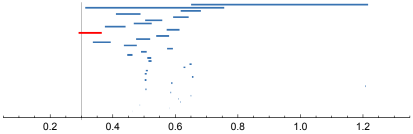

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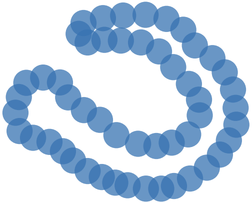

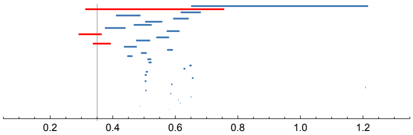

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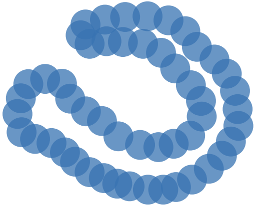

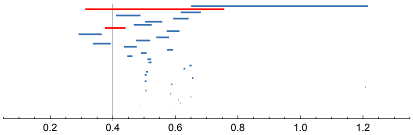

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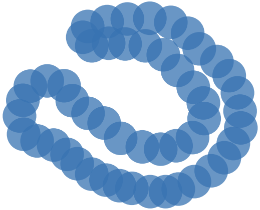

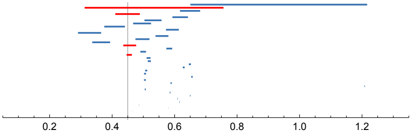

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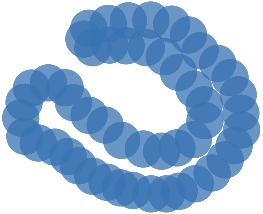

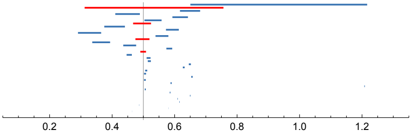

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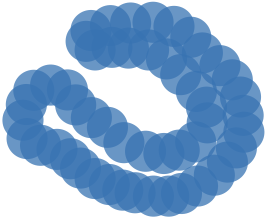

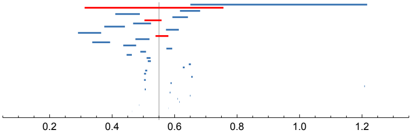

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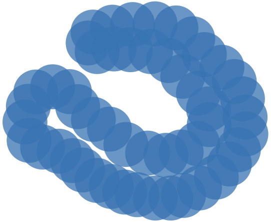

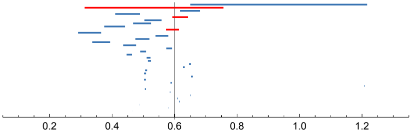

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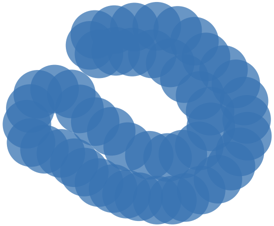

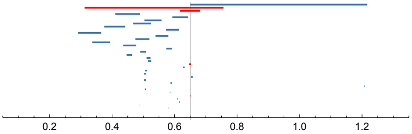

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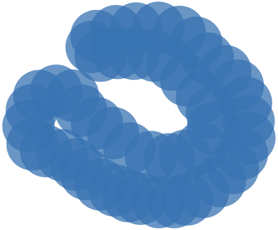

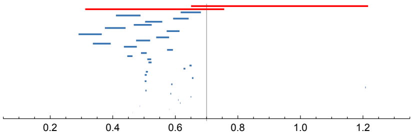

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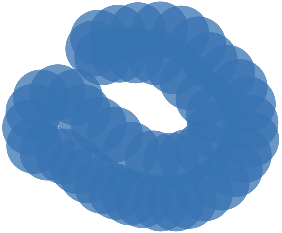

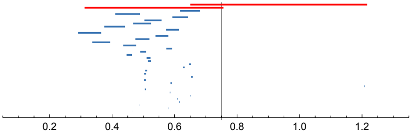

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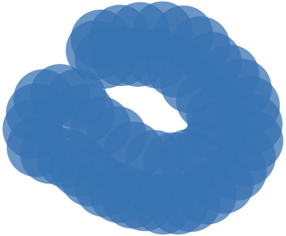

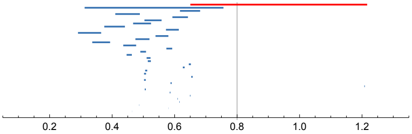

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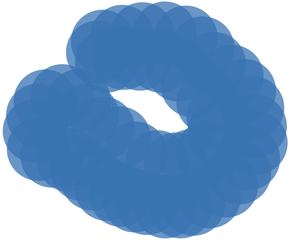

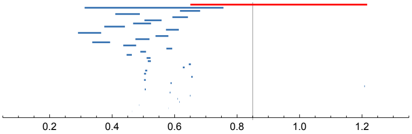

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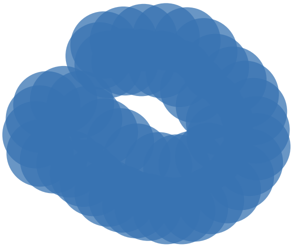

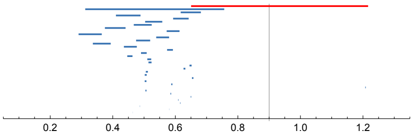

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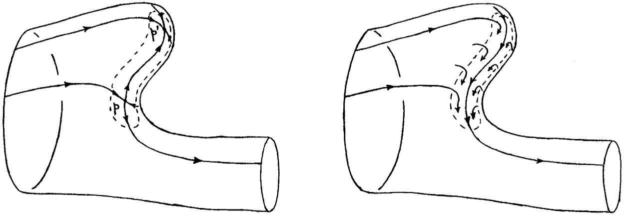
- Relates critical points to homology of sublevel sets
- Provides a method for *canceling* pairs of critical points



(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

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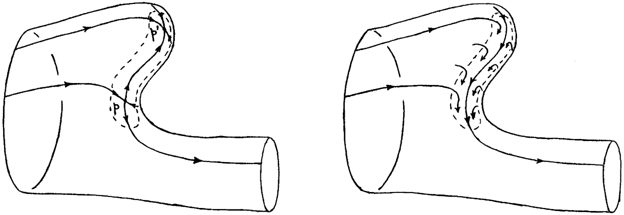

(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

Persistent homology:

- Relates homology of different sublevel set
- Identifies pairs of critical points (birth and death of homology)
- Quantifies their *persistence*

Morse inequalities through the lens of persistence

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- The m_i critical points of index i correspond to α_i births in H_i and ω_{i-1} deaths in H_{i-1}:

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When was persistent homology discovered?

- H. Edelsbrunner, D. Letscher, and A. Zomorodian
  Topological persistence and simplification
  Foundations of Computer Science, 2000
- V. Robbins
  Computational Topology at Multiple Resolutions.
  PhD thesis, University of Colorado Boulder, 2000
- P. Frosini
  A distance for similarity classes of submanifolds of a Euclidean space
  Bulletin of the Australian Mathematical Society, 1990.
- S. A. Barannikov
  The framed Morse complex and its invariants.
  In *Singularities and bifurcations, Adv. Soviet Math. (vol. 21)*, 1994.

When was persistent homology discovered first?

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ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses

When was persistent homology discovered first?

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Rank and span in functional topology

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Exact homomorphism sequences in homology theory — ed.ac.uk [PDF]
JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR
The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] ...
Cited by 46 Related articles All 3 versions Cite Save More

Marston Morse and his mathematical works — ams.org [PDF]
R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org
American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to ...
Cited by 24 Related articles All 4 versions Cite Save More

Unstable minimal surfaces of higher topological structure
M Morse, CB Tompkins - Duke Math. J, 1941 - projecteuclid.org
1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. Theproblem of the existenceof minimal surfaces of unstable type contains many of the typical difficulties, especially those ofa topological nature. Having studied this ...
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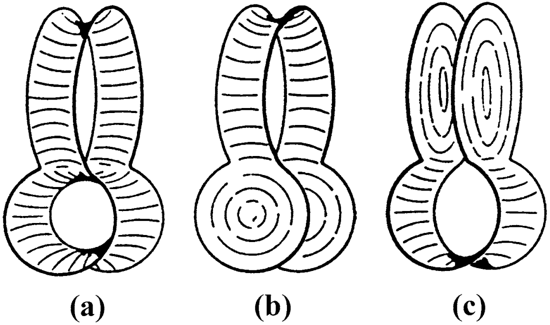
[PDF] **Persistence in discrete Morse theory** — psu.edu [PDF]
U Bauer - 2011 - Citeseer

http://ulrich-bauer.org/persistence-functionals-atmcs.pdf

Motivation and application: minimal surfaces

Problem (Plateau's problem)

Find a surface of least area spanned by a given closed Jordan curve.



(from Dierkes et al.: *Minimal Surfaces*, 2010)

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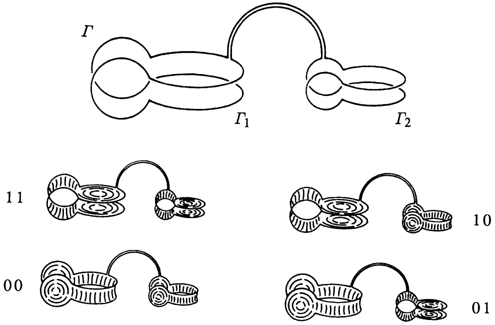

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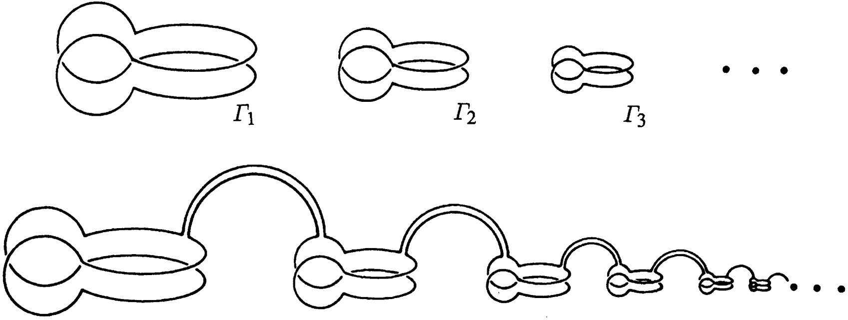

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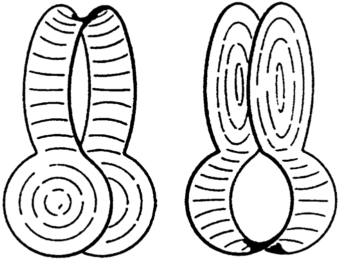
- Any bounded continuous function with compact sublevel sets must have a locally compact domain.
- For Banach spaces, this is equivalent to having finite dimension.

Existence of unstable minimal surfaces

Theorem (Morse, Tompkins 1939; Shiffman 1939)

Assume that a given curve bounds two separate stable minimal surfaces.



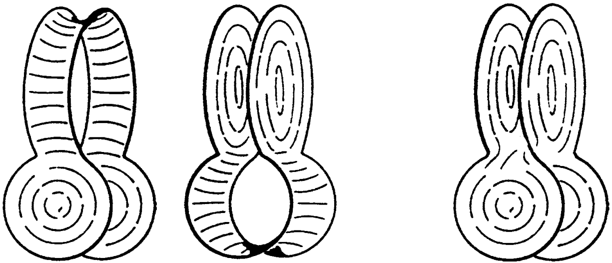
Existence of unstable minimal surfaces

Theorem (Morse, Tompkins 1939; Shiffman 1939)

Assume that a given curve bounds two separate stable minimal surfaces.

Then there also exists an unstable *minimal surface bounding that curve:*

- *a critical point that is not a local minimum.*



Whatever happened to functional topology?

Whatever happened to functional topology?

PLATEAU'S PROBLEM

AND THE

CALCULUS OF VARIATIONS

BY

MICHAEL STRUWE

Whatever happened to functional topology?

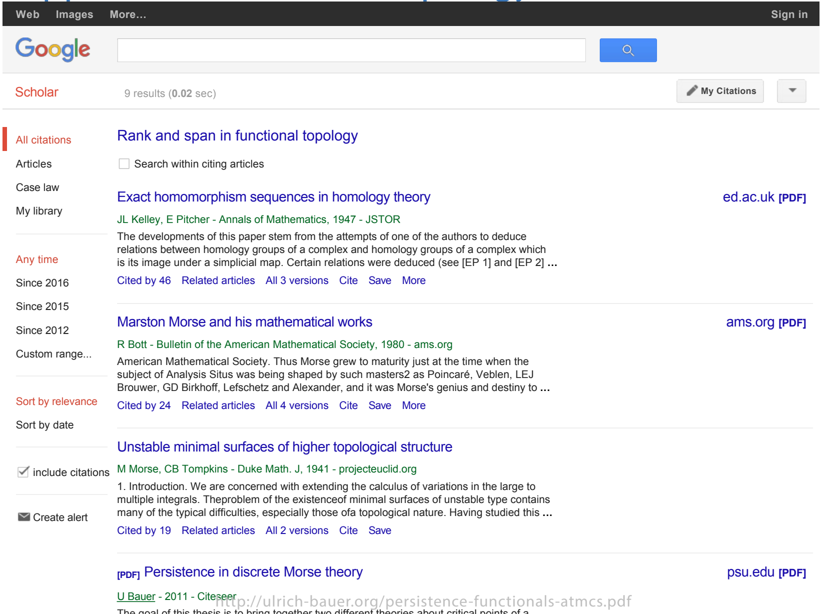
The technical complexity and the use of a sophisticated topological machinery (which is not shadowed in our presentation) moreover tend to make Morse-Tompkins' original paper unreadable and inaccessible for the non-specialist, cf. Hildebrandt [4, p. 324].

Confronting Morse-Tompkins' and Shiffman's approach with that given in Chapter 4 we see how much can be gained in simplicity and strength by merely replacing the C^o–topology by the $H^{1/2,2}$–topology and verifying the Palais - Smale - type condition stated in Lemma 2.10.

However, in 1964/65 when Palais and Smale introduced this condition in the calculus of variations it was not clear that it could be meaningful for analyzing the geometry of surfaces, cf. Hildebrandt [4, p. 323 f.].

Instead, a completely new approach was taken by Böhme and Tromba [1] to tackle the problem of understanding the global structure of the set of minimal surfaces spanning a wire.

Whatever happened to functional topology?



http://ulrich-bauer.org/persistence-functionals-atmcs.pdf

Whatever happened to functional topology?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was in the 1920's to go from the mini-max principle of Birkhoff to the Morse inequalities, let alone extend these inequalities to function spaces, so that by

Whatever happened to functional topology?

inequalities pertain between the dimensions of the A_i and those of $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels [D] in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers [36] and [44] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.

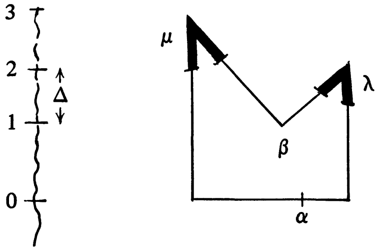


FIGURE 8

Whatever happened to functional topology?

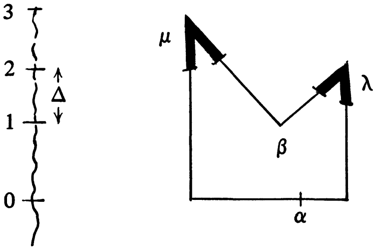


FIGURE 8

The values a where $H(a, a^-) \neq 0$ are indicated on the left, and corresponding to each of these *critical values* a generator of $H(a, a^-)$ is drawn on M, using the singular theory for simplicity. Morse calls such generators "caps". Thus α and β are two "0-caps" and μ and λ two "1-caps". Notice that every cap u defines a definite boundary element ∂u in

$$H(a^-) = \lim_{\varepsilon \to 0^+} H(F < a - \varepsilon);$$

Morse calls a cap u linkable iff $\partial u = 0$. Otherwise it is called *nonlinkable*.

In our example, α, β and μ are linkable while λ is *not*.

Next Morse defines the *span* of a cap u associated to the critical level a in the following manner.

http://ulrich-bauer.org/persistence-functionals-atmcs.pdf

43, he also delivered the Colloquium Lectures of the Mathematical Society and wrote his monumental book on the Calculus of Variations in the Large; it eventually earned him practically every honor of the mathematical community, over twenty honorary degrees, the National Science Medal, the Legion of Honor of France, ...

Nevertheless, when I first met Marston in 1949 he was in a sense a solitary figure, battling the *algebraic topology*, into which his beloved Analysis Situs had grown. For Marston always saw topology from the side of Analysis, Mechanics, and Differential geometry. The unsolved problems he proposed had to do with dynamics–the three body problem, the billiard ball problem, and so on. The development of the algebraic tools of topology, or the project of bringing order into the vast number of homology theories which had sprung up in the thirties–and which was eventually accomplished by the Eilenberg-Steenrod axioms–these had little interest for him. "*The battle between algebra and geometry has been waged from antiquity to the present*" he wrote in his address *Mathematics and the Arts* at Kenyon College in 1949, and

Received by the editors April 15, 1980.

[1]This work was supported in part through funds provided by the National Science Foundation under the grant 33-966-7566-2.

[2]Poincaré was born in 1854, the others all in the 1880's.

Q-tame persistence modules

Definition (Chazal et al. 2009)

A persistence module $M : \mathbf{R} \to \mathbf{Vect}$ is *q-tame* if for every $s < t$ the structure map $M_s \to M_t$ has finite rank.

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  - not necessarily pointwise finite-dimensional [Droz 2012],
  - but always q-tame [Chazal et al. 2016].
- A q-tame persistence module does not necessarily have a barcode decomposition.

Structure of q-tame persistence modules

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

The radical *of a q-tame persistence module M,* $(\operatorname{rad} M)_t = \sum_{s<t} \operatorname{im} M_{s,t}$,

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- *The* observable category *is the quotient of abelian categories $\mathbf{Obs} = \mathbf{Pers}/\mathbf{Eph}$, consisting of persistence modules modulo the ephemeral persistence modules.*

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Morse and Tompkins used this idea to show the existence of an unstable minimal surface.

Generalized Morse inequalities

Assume that $f : X \to \mathbb{R}$ is bounded and its persistent sublevel set homology is q-tame.

- The persistence diagram of f has finitely many bars of length $> \epsilon$, for every $\epsilon > 0$.
- This gives *generalized* Morse inequalities

$$\sum_{i=0}^{d} (-1)^{d-i} (m_i^\epsilon - \beta_i) \geq 0.$$

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- in particular, conditions satisfied by the Douglas functional.

Weakly locally connected filtrations

Definition (Morse 1937; paraphrased in topological terms)

The sublevel set filtration of a function $f : X \to \mathbb{R}$ is *weakly locally connected* if for

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such that the inclusion $U \to V$ induces trivial maps on all homotopy groups.

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If a function $f: X \to \mathbb{R}$ on a metric space X is bounded below and the sublevel set filtration is compact and weakly locally connected, *then it has q-tame persistent Vietoris homology.*

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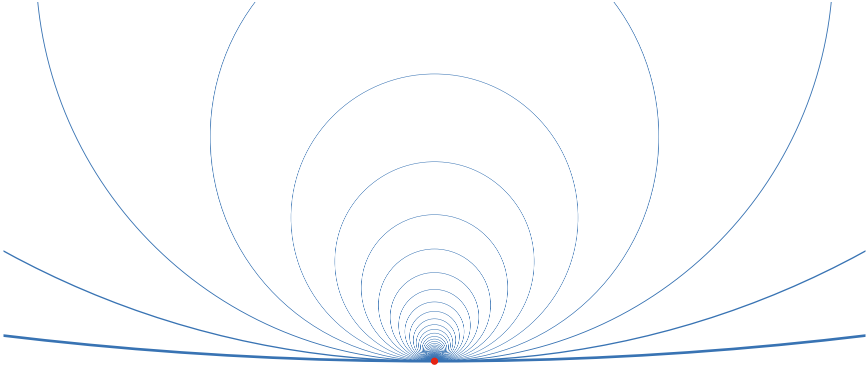

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It is *weakly HLS* if for any $t > f(x)$ there merely *exists* an s satisfying the above conditions.

A sufficient condition for q-tame persistence

Theorem (B, Medina-Mardones, Schmahl 2022)

If the sublevel set filtration of a function $f\colon X \to \mathbb{R}$ is compact and HLS, then it has q-tame persistent homology and hence a persistence diagram.

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- If f is continuous, then weakly HLS is already sufficient
- Conditions are satisfied by Douglas' minimal surface functional
- Fixes the flaw in the proof by Morse–Tompkins
- Applies not only to Vietoris/Čech but also to singular homology

A glimpse at the proof

The crucial ingredient is a version of *Wilder's Finiteness Theorem*.

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Then for any $s < t < u$ the inclusion $C \cap f_{\leq s} \hookrightarrow L \cap f_{\leq t}$ induces a map of finite rank in homology.

surfaces with distinct areas or else the conditions of the theorem are fulfilled.

We shall now state a theorem from which all trace of conditions on the finiteness of the number of critical values is removed.

THEOREM 7.4. *Suppose F and M satisfy conditions* I *to* IV. *Let z be a class of relative k-cycles u with a modulus $F \leqq a$. Suppose the cycles u are mutually homologous but non-bounding mod $F \leqq a$, on $F \cdot \leqq b$ for some value $b > a$. There is then a least number $c > a$ such that $F \leqq a$ contains a cycle of z, and at the level c there is at least one homotopic critical point q such that each critical set which contains q has a positive k^{th} cap type number.*

In this theorem the number a may be less than the absolute minimum of F, in which case u is an ordinary cycle non-bounding on $F \leqq b$. If in addition b is infinite, u is non-bounding on M. In particular u may be a 0-cycle. If $R_0 = 1$ the number c in the theorem is then the absolute minimum of F. In the case of the minimal surface problem this implies the existence of a minimal surface of absolute minimum type. This result is stated merely to give an insight into the way the theorem works.

A critical set of points σ will be termed *minimizing* if there exists a neighborhood N of σ such that whenever p is a point of N not on σ, $F(p)$ exceeds the value of F on σ. With this understood we have the following corollary of the theorem.

COROLLARY 7.1. *Suppose that F and M satisfy conditions* I *to* IV *and that $R_0 = 1$. Corresponding to any two disjoint minimizing critical sets there exists at least one homotopic critical point p such that each critical set which contains p has a positive first type number m_1.*

Both the theorem and the corollary are false if Vietoris cycles are replaced by singular cycles. The theorem and corollary apply at once to the minimal surface problem inasmuch as conditions I to IV are fulfilled by $A(\varphi)$ and Ω, and $R_0 = 1$ on Ω. The application of the theorem is not limited by the existence of infinitely many critical sets or critical values. But in such a case the conclusion that there exists another critical set would add nothing were it not stated that the

Čech/Vietoris vs. singular persistent homology

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- The above statements are not true for singular instead of Čech homology.

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Theorem (Mountain Pass Theorem; Morse–Tompkins 1939, modernized version)

Let M be a connected metric space. Assume that

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- So if t is a death, there must be a critical set of non-minimum type at level t. □

Unstable minimal surfaces via persistence

Theorem (Unstable minimal surface theorem; Morse–Tompkins 1939)

Let $\gamma: S^1 \to \mathbb{R}^3$ be a curve with Lipschitz derivative. Assume that γ bounds two minimal surfaces contained in distinct minimum type critical sets of the Douglas functional A_γ.

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- Any point in the critical set corresponds to a minimal surface spanned by γ. $\square$

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Thanks for your attention!