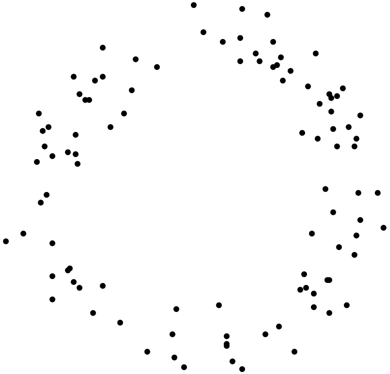
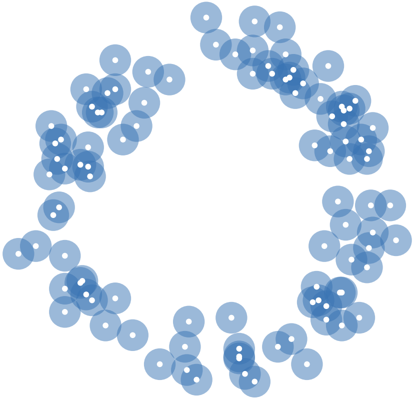
Induced matchings and the algebraic stability of persistence barcodes

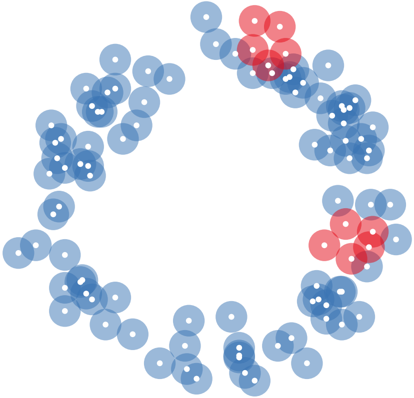
Ulrich Bauer (IST Austria), Michael Lesnick (IMA)

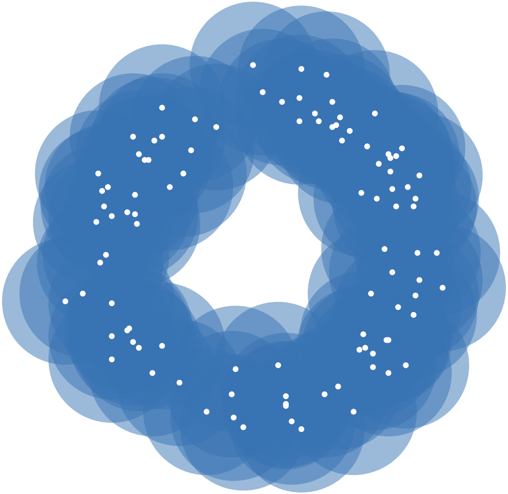
IST Austria

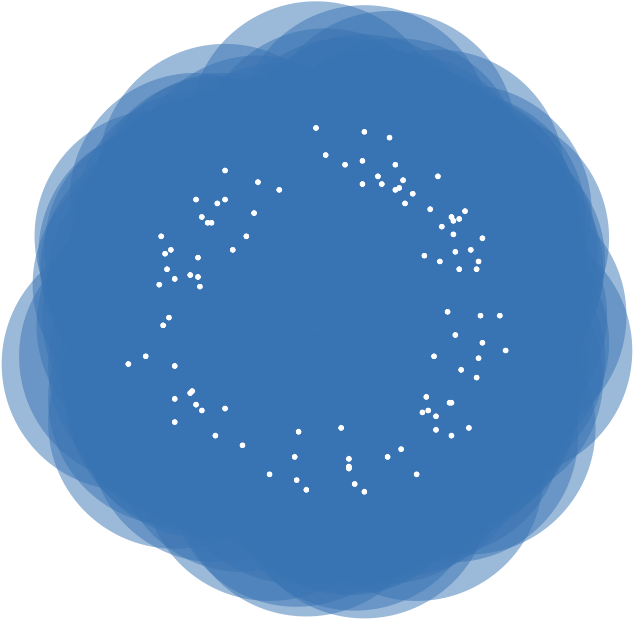
Jul 24, 2014

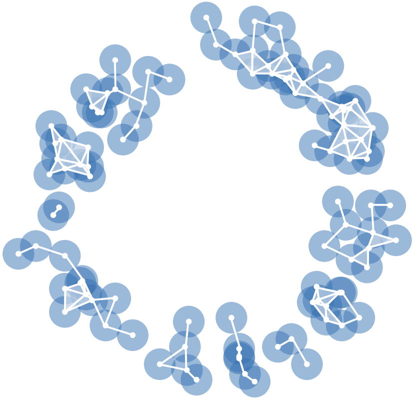


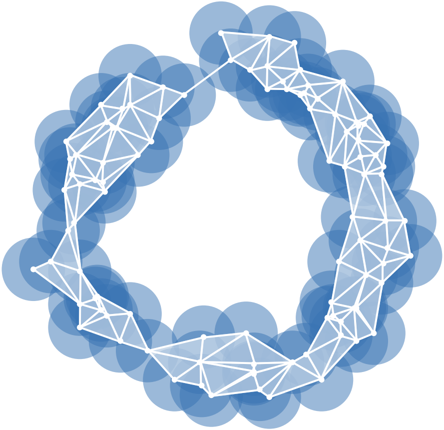


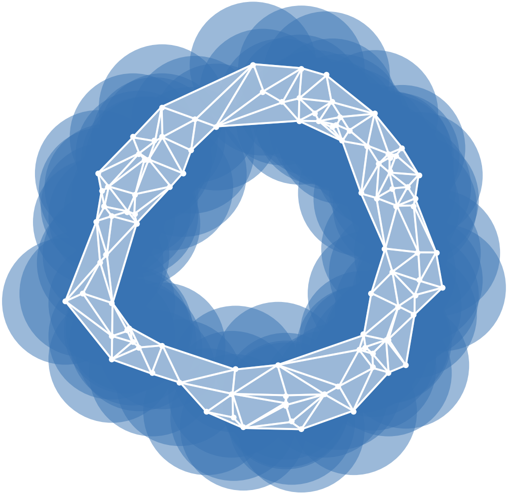


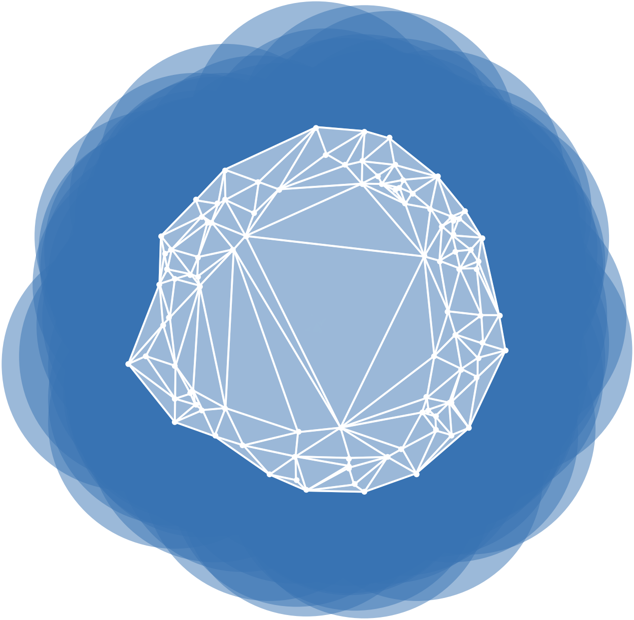












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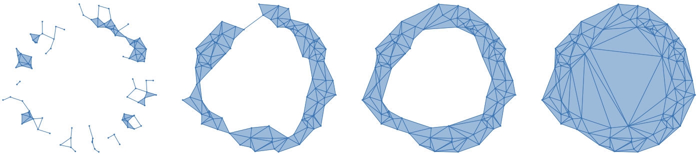

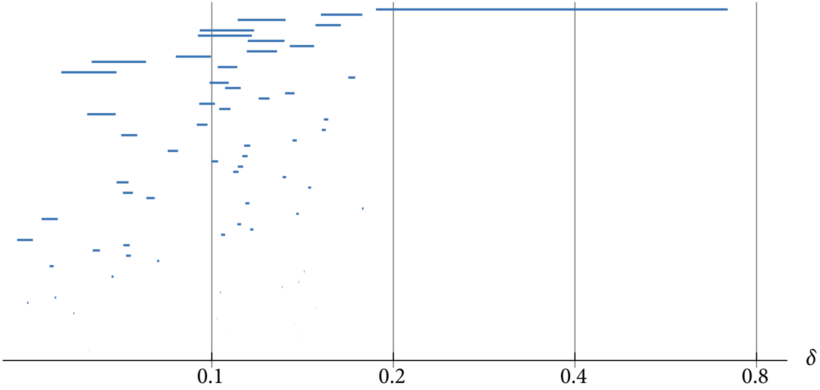

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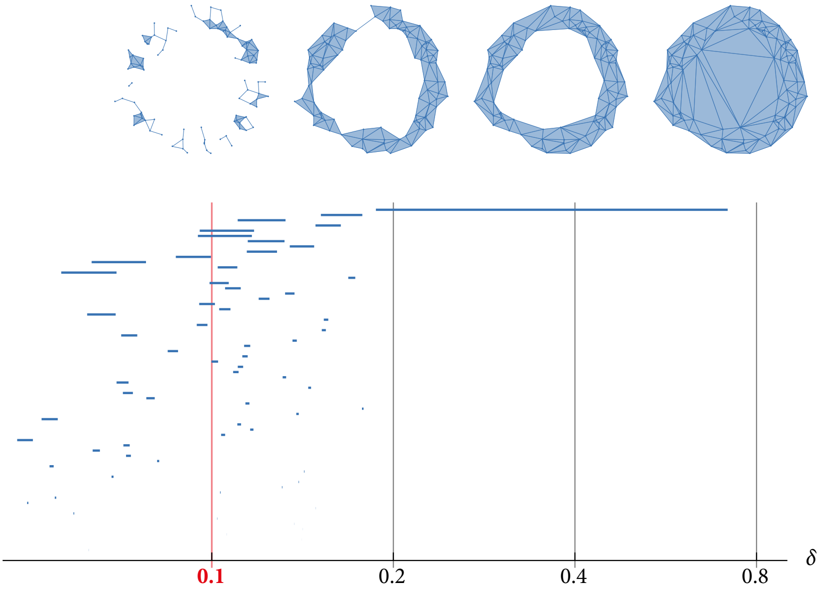

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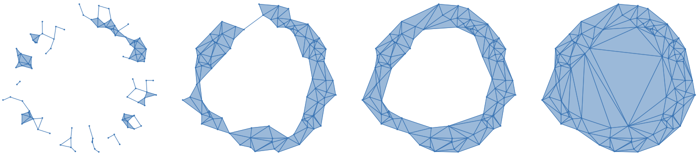

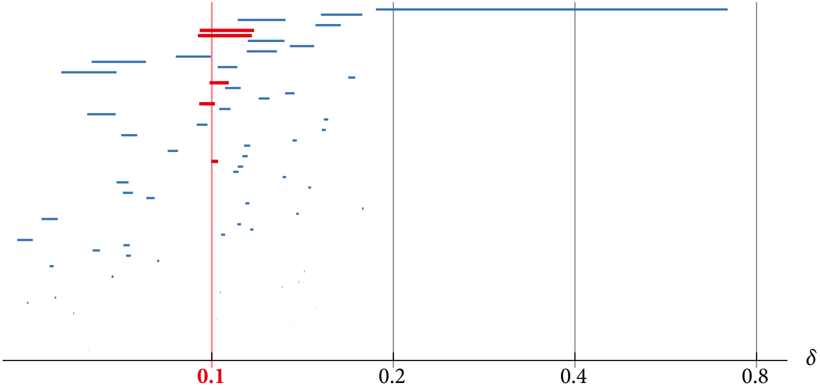

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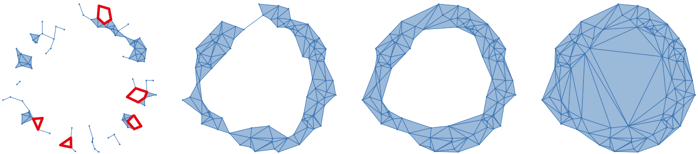

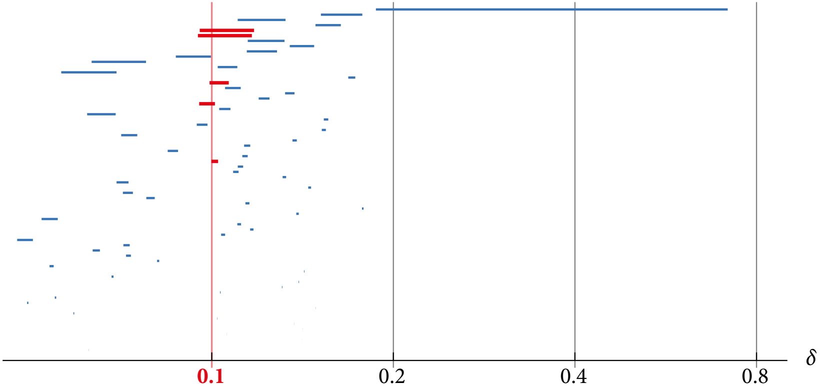

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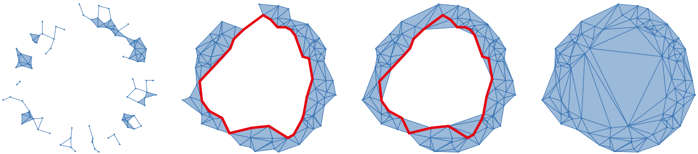

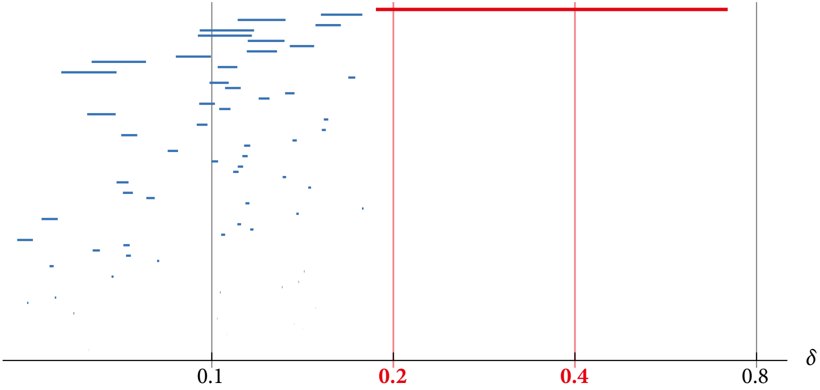

The pipeline of topological data analysis

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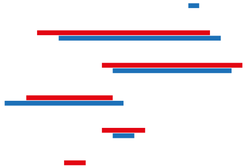
Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

If two functions $f, g : K \to \mathbb{R}$ have distance $\|f - g\|_\infty \leq \delta$ then there exists a δ-matching of their barcodes.



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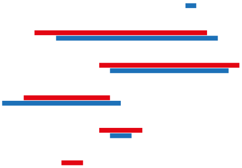

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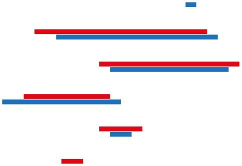

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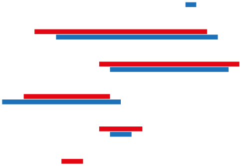

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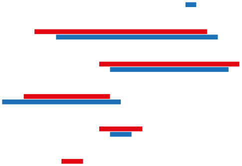

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Stability for functions in the big picture

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So the sublevel sets are *interleaved:*

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Therefore their homology groups are interleaved as well.

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- morphism and transition maps commute:

$$\begin{tikzcd} M_s \arrow[r] \arrow[d, "f_s"'] & M_t \arrow[d, "f_t"] \\ N_s \arrow[r] & N_t \end{tikzcd}$$

Interval Persistence Modules

Let $\mathcal{I}_{\mathbb{R}}$ denote the set of all intervals in $\mathbb{R}$.
For $I \in \mathcal{I}_{\mathbb{R}}$, define the *interval persistence module* $C(I)$ by

$$C(I)_t = \begin{cases} \mathbb{K} & \text{if } t \in I, \\ 0 & \text{otherwise.} \end{cases}$$

$$C(I)_s \to C(I)_t = \begin{cases} \mathrm{id}_{\mathbb{K}} & \text{if } s, t \in I, \\ 0 & \text{otherwise.} \end{cases}$$

The structure of persistence modules

Theorem (Crawley-Boewey 2012)

Let M be a pointwise finite dimensional persistence module:
M_t is finite dimensional for all t.
Then M is interval-decomposable:
there exists a collection of intervals $B(M)$ such that

$$M \cong \bigoplus_{I \in B(M)} C(I).$$

Theorem (Azumaya 1950, de Silva et al. 2012)

If M is interval-decomposable, then $B(M)$ is unique.

$B(M)$ is called the *barcode* of M.

- Motivates use of homology with field coefficients

Interleavings of persistence modules

Definition

Two persistence modules M and N are δ-*interleaved* if there are morphisms

$$f : M \to N(\delta), \qquad g : N \to M(\delta)$$

such that this diagrams commutes for all t:

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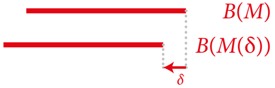

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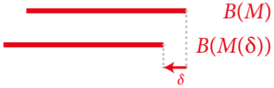

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Theorem (Chazal et al. 2009, 2012)

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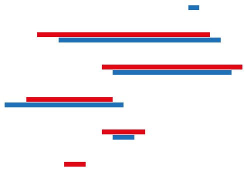

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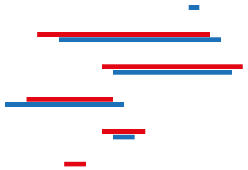

- converse statement also holds (isometry theorem)
- indirect proof, 80 page paper (Chazal et al. 2012)

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- revision and expansion of previous proof

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- stability follows from properties of a *single morphism*, not just from a pair of morphisms
- relies on *partial functoriality* of the induced matching

The matching category

A *matching* $\sigma : S \nleftrightarrow T$ is a bijection $S' \to T'$, where $S' \subseteq S$, $T' \subseteq T$. Formally:

$$\sigma = \{(s,t) \in S' \times T' \mid \sigma(s) = t\} \subseteq S \times T.$$

Composition of matchings $\sigma : S \nleftrightarrow T$ and $\tau : T \nleftrightarrow U$:

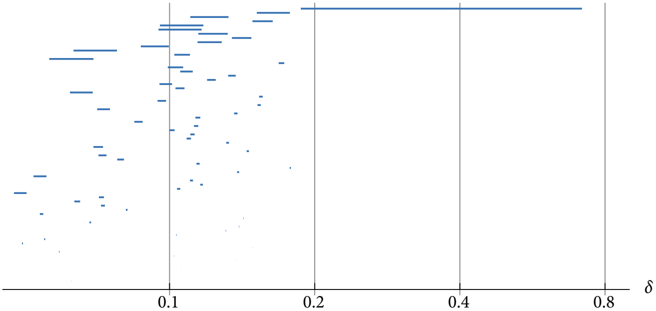
$$\tau \circ \sigma = \{(s,u) \mid (s,t) \in \sigma, (t,u) \in \tau \text{ for some } t \in T\}.$$

Matchings form a category $\mathbf{Mch}$

- objects: sets
- morphisms: matchings

Barcodes as matching diagrams

We can regard a barcode $\mathcal{D}$ as a functor $\mathbf{R} \to \mathbf{Mch}$:



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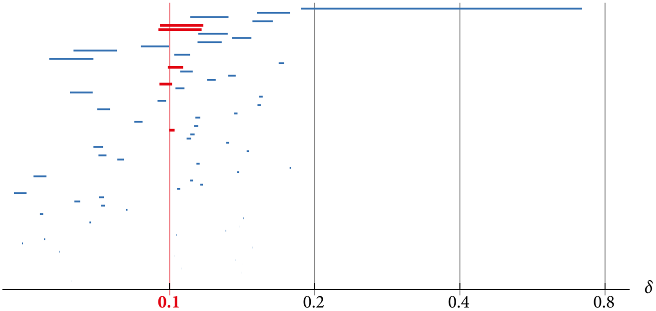
We can regard a barcode $\mathcal{D}$ as a functor $\mathbf{R} \to \mathbf{Mch}$:

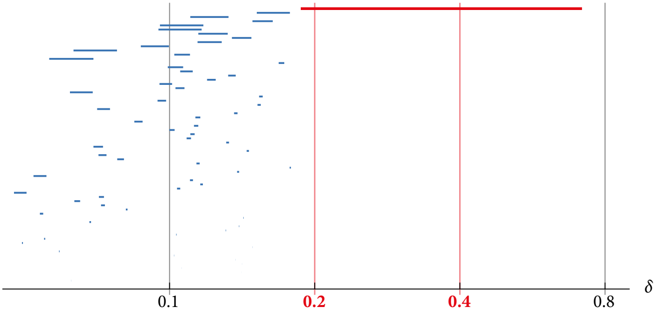
- For each real number t, let $\mathcal{D}_t$ be the subset of $\mathcal{D}$ consisting of intervals which contain t, and



Barcodes as matching diagrams

We can regard a barcode $\mathcal{D}$ as a functor $\mathbf{R} \to \mathbf{Mch}$:

- For each real number t, let $\mathcal{D}_t$ be the subset of $\mathcal{D}$ consisting of intervals which contain t, and
- for each $s \leq t$, define the matching $\mathcal{D}_{s,t} : \mathcal{D}_s \nleftrightarrow \mathcal{D}_t$ to be the identity on $\mathcal{D}_s \cap \mathcal{D}_t$.



Barcode matchings as interleavings

We can regard matchings of barcodes $\sigma : \mathcal{C} \nleftrightarrow \mathcal{D}$ as morphisms.

- Morphisms $\mathcal{C} \to \mathcal{D}$ are natural transformations;
  the matchings $\sigma_t : \mathcal{C}_t \nleftrightarrow \mathcal{D}_t$ are restrictions of σ to $\mathcal{C}_t \times \mathcal{D}_t$:

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This way, we can regard a δ-matching of barcodes $\mathcal{C} \nleftrightarrow \mathcal{D}$ as a δ-interleaving:

$$\begin{tikzcd}
\mathcal{C}_t \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \mathcal{C}_{t+2\delta} \arrow[dr, leftrightarrow] & \\
& \mathcal{D}_{t+\delta} \arrow[ur, leftrightarrow] \arrow[rr, leftrightarrow] & & \mathcal{D}_{t+3\delta}
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Non-functoriality of the persistence barcode

Proposition

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Lemma

There exists no functor $\mathbf{Vect} \to \mathbf{Mch}$ *sending each vector space of dimension* d *to a set of cardinality* d*.*

Structure of submodules and quotient modules

Proposition (B, Lesnick 2013)

For a persistence submodule $K \subseteq M$:

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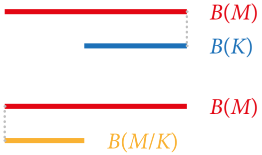

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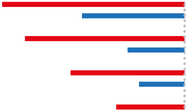
- If multiple bars have same endpoint: match in order of decreasing length



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Similarly for surjections.

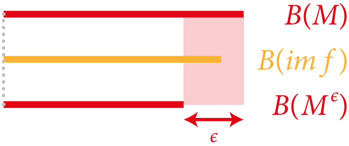
The induced matching theorem

Say K is ϵ-trivial if $K_t \to K_{t+\epsilon}$ is the zero map for all t.

Define M^ϵ by $M^\epsilon_t = \operatorname{coim}(M_t \to M_{t+\epsilon})$.

Lemma

Let $f : M \to N$ be a morphism such that $\ker f$ is ϵ-trivial.
Then M^ϵ is a quotient module of $\operatorname{im} f$.



$$\begin{tikzcd} & M \arrow[dl, two heads] \arrow[dr, two heads] & \\ M^\epsilon & & \operatorname{im} f \arrow[ll, two heads] \end{tikzcd}$$

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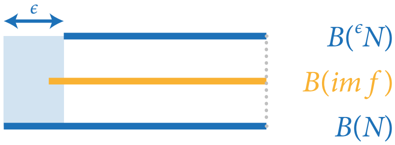
Define ${}^{\epsilon}N$ by ${}^{\epsilon}N_t = \operatorname{im}(N_{t-\epsilon} \to N_t)$.

Lemma

Let $f : M \to N$ be a morphism such that $\operatorname{coker} f$ is ϵ-trivial. Then ${}^{\epsilon}N$ is a submodule of $\operatorname{im} f$.



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 & N & \\
\operatorname{im} f \arrow[ur, hook] & & {}^{\epsilon}N \arrow[ul, hook'] \arrow[ll, hook']
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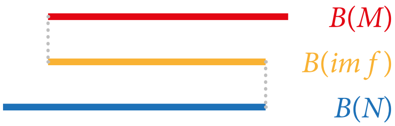

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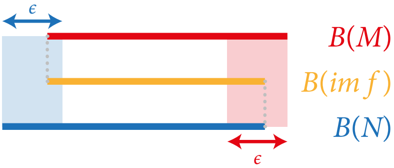

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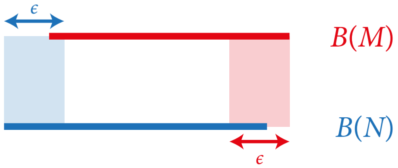
The induced matching theorem

Theorem (B, Lesnick 2013)

Let $f : M \to N$ *be a morphism with* $\ker f$ *and* $\operatorname{coker} f$ ϵ*-trivial.*
Then each interval of length $\geq \epsilon$ *is matched by* $B(f)$*.*
If $B(f)$ *matches* $[b, d) \in B(M)$ *to* $[b', d') \in B(N)$*, then*
$b' \leq b \leq b' + \epsilon$ *and* $d - \epsilon \leq d' \leq d$*.*



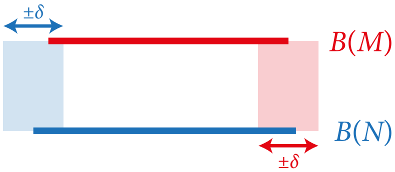
The induced matching theorem

Let $f : M \to N(\delta)$ be an interleaving morphism.

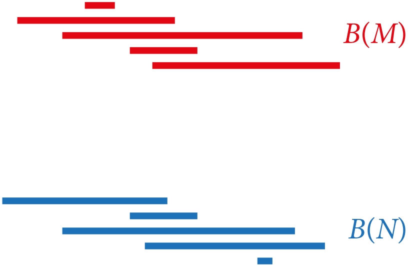
Then $\ker f$ and $\operatorname{coker} f$ are 2δ-trivial.

Corollary

A δ-interleaving between persistence modules induces a δ-matching of their persistence barcodes.



Stability via induced matchings



Thanks for your attention!

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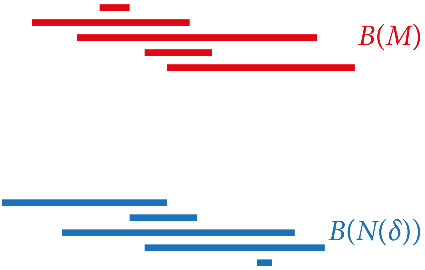

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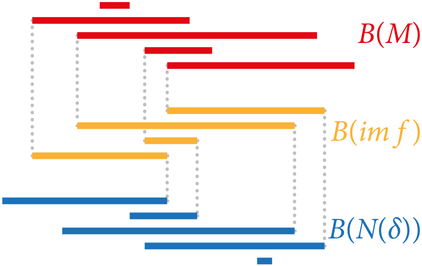

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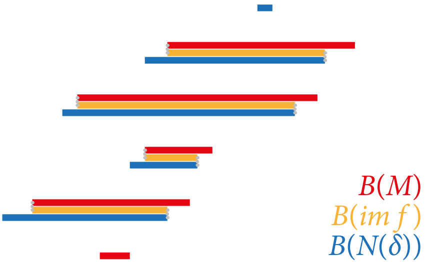

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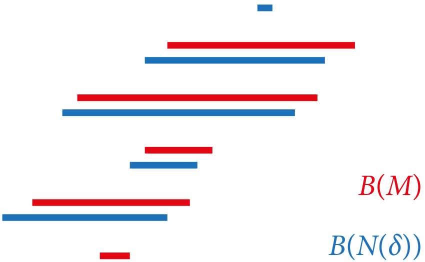

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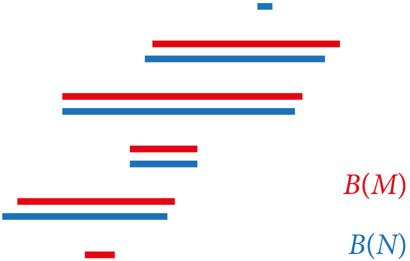

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