DUALITIES AND CLEARING FOR IMAGE PERSISTENCE

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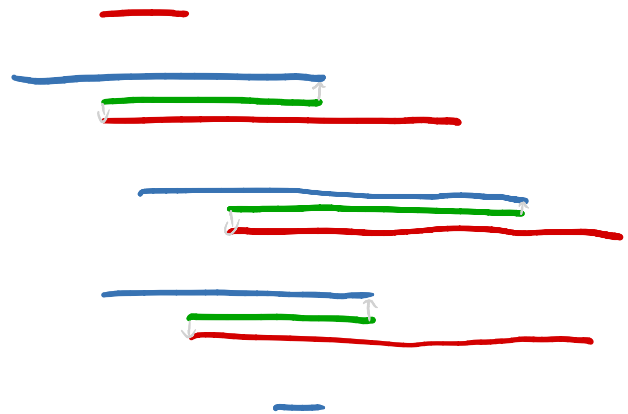

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FH VORARLBERG, DORNBIRN

PERSISTENT HOMOLOGY

Filtration $K_\bullet$: $\cdots \to K_s \hookrightarrow \cdots \hookrightarrow K_t \hookrightarrow \cdots$

(a diagram of complexes, indexed over $\mathbb{R}$, connected by inclusion maps)

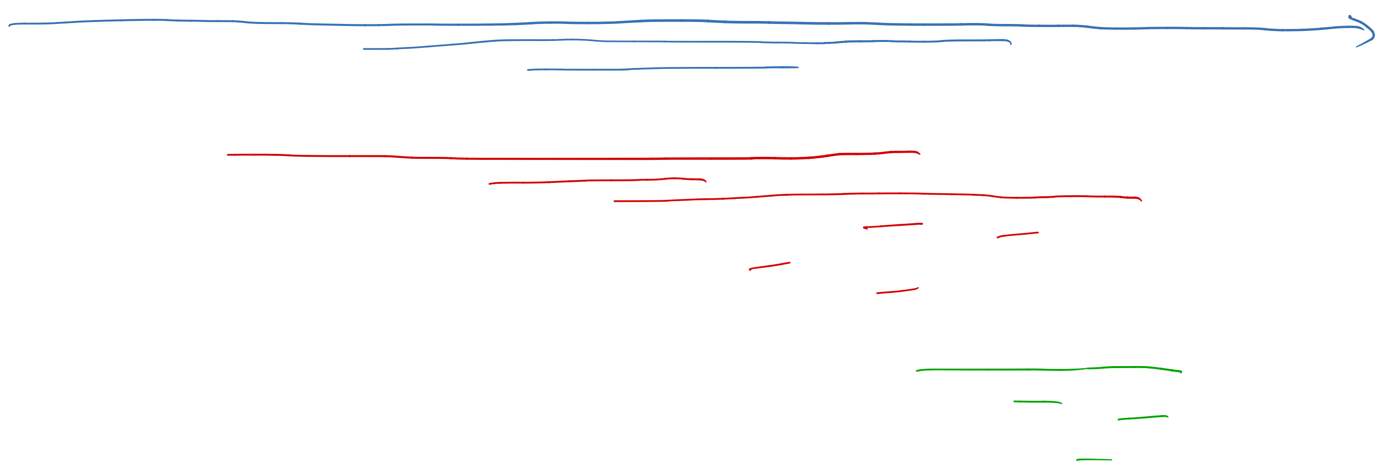
$\rightsquigarrow$ Homology $\rightsquigarrow$

Persistence module $H_*(K_\bullet)$: $\cdots \to H_*(K_s) \to \cdots \to H_*(K_t) \to \cdots$

(a diagram of vector spaces, indexed over $\mathbb{R}$, connected by linear maps)

$\rightsquigarrow$

Barcode $B(H_*(K_\bullet))$



INDUCED MATCHINGS

Two filtrations:

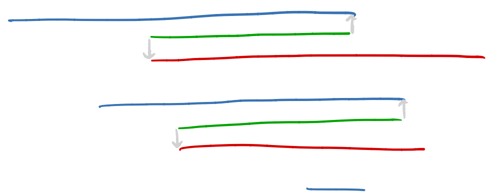
$$
\begin{tikzcd}
\cdots \arrow[rr, hook] & & K_t \arrow[rr, hook] \arrow[dr] & & K_{t+2\delta} \arrow[r, hook] & {} \\
{} \arrow[r, hook] & L_{t-\delta} \arrow[ur] \arrow[rr, hook] & & L_{t+\delta} \arrow[ur] \arrow[r, hook] & \cdots
\end{tikzcd}
$$

Interleaving of persistent homology:

$$
\begin{tikzcd}
\cdots \arrow[rr] & & H_*(K_t) \arrow[rr] \arrow[dr] & & H_*(K_{t+2\delta}) \arrow[r] & {} \\
{} \arrow[r] & H_*(L_{t-\delta}) \arrow[ur] \arrow[rr] & & H_*(L_{t+\delta}) \arrow[ur] \arrow[r] & \cdots
\end{tikzcd}
$$

Maps $H_*(L_t) \xrightarrow{\varphi_t} H_*(K_{t+\delta})$ induce matching of barcodes:

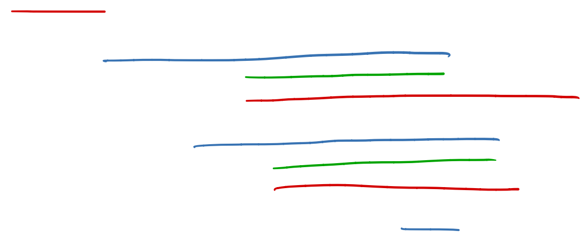
$$B(H_*(L_\bullet)) \hookleftarrow B(\operatorname{im}\varphi_\bullet) \hookrightarrow B(H_*(K_\bullet))$$



Algebraic Stability of Barcodes

<u>Thm</u> [Chazal et al 2009; B, Lesnick 2013]

A δ-interleaving of persistence modules induces a δ-matching of barcodes.



Applications:

extend classical stability

- filtrations of two different domains
- Gromov–Hausdorff stability

COMPUTING IMAGE BARCODES

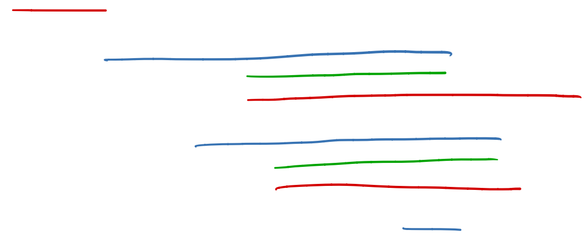
Persistence computation

- Matrix reduction
- Clearing
- Cohomology

Persistence barcodes of images (of morphisms $\varphi_* : H_*(L_\bullet \hookrightarrow K_\bullet)$)

$$\begin{tikzcd} \cdots \arrow[r, hook] & K_s \arrow[r, hook] & \cdots \arrow[r, hook] & K_t \arrow[r, hook] & \cdots \\ \cdots \arrow[r] & L_s \arrow[u, hook] \arrow[r] & \cdots \arrow[r] & L_t \arrow[u, hook] \arrow[r] & \cdots \end{tikzcd}$$

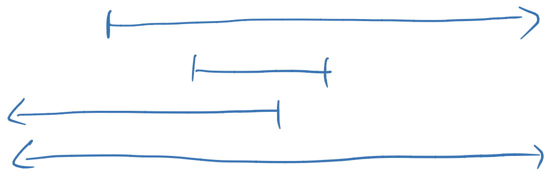
$\rightsquigarrow$



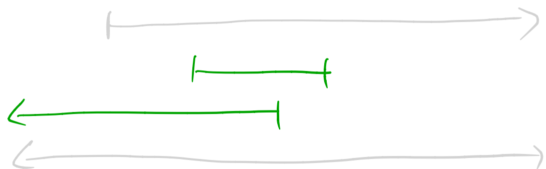
- previous work [Edels., Harer, Morozov 2009]:
  filtrations $K_\bullet$, $L_\bullet$ with $L_t = K_t \cap L$ ("one function")
  no clearing
- present work
  $K_\bullet, L_\bullet$ arbitrary filtrations of A ("two functions")
  clearing & cohomology

BARCODES & SUBBARCODES

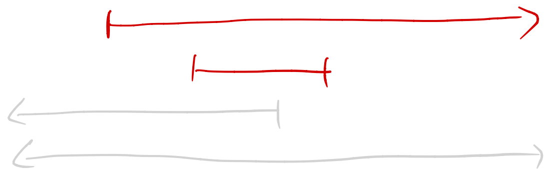
M : persistence module , $\mathcal{B}(M)$: barcode



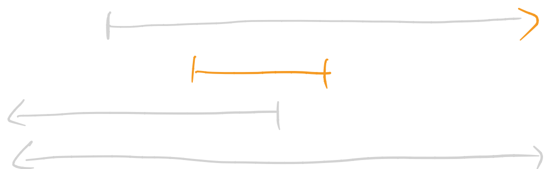
$\mathcal{B}(M)$ barcode



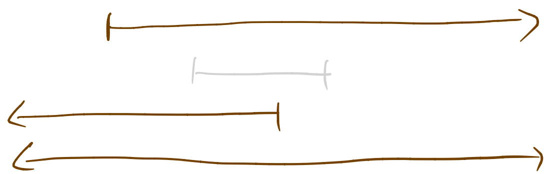
$\mathcal{B}^\flat(M)$ lower part



$\mathcal{B}^\sharp(M)$ upper part



$\mathcal{B}^\circ(M)$ finite part

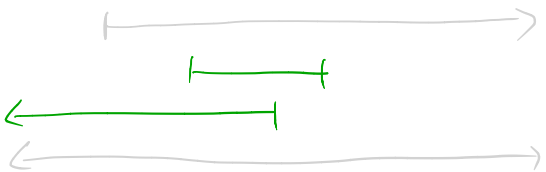


$\mathcal{B}^\infty(M)$ infinite part

LOWER SUBMODULES, UPPER QUOTIENT MODULES

Persistence module M has colimit, $\operatorname{colim} M$, maps $\eta_{M_t} : M_t \to \operatorname{colim} M$
(η : unit of adjunction $\operatorname{colim} \dashv \Delta$)

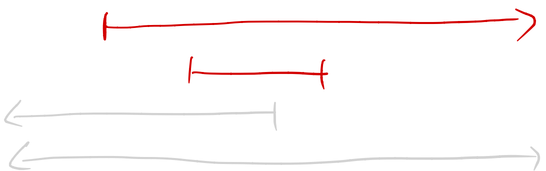
Define submodule $M^\flat$: $\quad M^\flat_t = \ker \eta_{M_t}$



$B^\flat(M) = B(M^\flat)$ lower part

Dually : M has limit, $\lim M$, maps $\varepsilon_{M_t} : \lim M \to M_t$

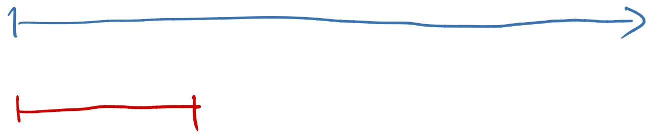
Define quotient module $M^\sharp$: $\quad M^\sharp_t = \operatorname{coker}\left(\varepsilon_{M_t} : \lim M \to M_t\right)$



$B^\sharp(M) = B(M^\sharp)$ upper part

$\flat$, $\sharp$: <u>endofunctors</u> of persistence modules

PROPERTIES OF LOWER / UPPER FUNCTORS



$\mathcal{B}(M)$

$\varphi \downarrow$ $\qquad \mathcal{B}(N) = \mathcal{B}((\operatorname{im}\varphi)^\flat)$

$$\operatorname{im}(\varphi^\flat : M^\flat \to N^\flat) \neq (\operatorname{im}\varphi)^\flat$$

- $\flat$, $\sharp$ are not exact
  - $\flat$ is left-exact
  - $\sharp$ is right-exact (if M satisfies Mittag-Leffler condition; e.g., M_t fin. dim. $\forall\, t$)

- $\flat$, $\sharp$ do not preserve images
  - But if φ is eventually mono, then $\operatorname{im}(\varphi^\flat) = \operatorname{im}(\varphi)^\flat$,
    if φ is initially epi, then $\operatorname{im}(\varphi^\sharp) = \operatorname{im}(\varphi)^\sharp$.

DUALITY & RELATIVITY

- homology and cohomology, absolute & relative:

$$\mathcal{B}(H_*(K_\bullet)) = \mathcal{B}(H^*(K_\bullet)) \qquad \text{[de Silva \& al 2011]}$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$\mathcal{B}(H_*(A, K_\bullet)) = \mathcal{B}(H^*(A, K_\bullet))$$

- generalize to images ($L_\bullet \hookrightarrow K_\bullet$ filtrations of A):

$$\varphi_d : H_d(L_\bullet) \to H_d(K_\bullet) \qquad \psi_d : H_d(A, L_\bullet) \to H_d(A, K_\bullet)$$

$$\varphi^d : H^d(L_\bullet) \leftarrow H^d(K_\bullet) \qquad \psi^d : H^d(A, L_\bullet) \leftarrow H^d(A, K_\bullet)$$

Prop.

$$\mathcal{B}(\operatorname{im} \varphi_d) = \mathcal{B}(\operatorname{im} \varphi^d)$$

$$\mathcal{B}(\operatorname{im} \psi_d) = \mathcal{B}(\operatorname{im} \psi^d)$$

FROM RELATIVE TO ABSOLUTE

Theorem [B, Schmahl 2019]

$$B^0(\operatorname{im} \varphi_d) = B^0(\operatorname{im} \psi_{d+1}) = B^0(\operatorname{im} \psi^{d+1})$$

$$B^\infty(\operatorname{im} \varphi_d) = B^\infty(H_d(L_\bullet)) \leftrightarrow B^\infty(H^d(A, L_\bullet))$$

absolute homology

relative cohomology

BEHIND THE SCENES : ABSOLUTE & RELATIVE

Short exact sequence of persistence modules (with $A_\bullet$: constant filtration of A)

$$\begin{tikzcd}
C_*(K_\bullet) \arrow[r, hook] & C_*(A_\bullet) \arrow[r, two heads] & C_*(A, K_\bullet) \\
C_*(L_\bullet) \arrow[r, hook] \arrow[u] & C_*(A_\bullet) \arrow[r, two heads] \arrow[u, equal] & C_*(A, L_\bullet) \arrow[u]
\end{tikzcd}$$

Long exact sequence in homology

$$\begin{tikzcd}
\cdots \arrow[r] & H_d(K_\bullet) \arrow[r] & H_d(A_\bullet) \arrow[r] & H_d(A, K_\bullet) \arrow[r, "\partial^K"] & H_{d-1}(K_\bullet) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_d(L_\bullet) \arrow[r] \arrow[u] & H_d(A_\bullet) \arrow[r] \arrow[u, equal] & H_d(A, L_\bullet) \arrow[r, "\partial^L"'] \arrow[u] & H_{d-1}(L_\bullet) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

splits at $H_d(K_\bullet)$, $H_d(A, K_\bullet)$, decomposing into

(finite part)

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \operatorname{im} \partial^K \arrow[r, equal] & \operatorname{im} \partial^K \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & \operatorname{im} \partial^L \arrow[r, equal] \arrow[u] & \operatorname{im} \partial^L \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

$\oplus$

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker} \partial^K \arrow[r, hook] & H_d(A_\bullet) \arrow[r, two heads] & \ker \partial^K \arrow[r] & 0 \arrow[r] & \cdots \\
0 \arrow[r] & \operatorname{coker} \partial^L \arrow[r, hook] \arrow[u] & H_d(A_\bullet) \arrow[r, two heads] \arrow[u, equal] & \ker \partial^L \arrow[r] \arrow[u] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

COMPUTATION

Reduce boundary matrix D to $R = D \cdot V$ (V: invertible, upper tri.)

Yields compatible basis for persistence:

- cycles that become boundaries (columns $R_j \neq 0$)
- chains bounding those cycles (corresponding columns V_j)
- cycles that remain non-bounding (columns V_i with $R_i = 0$, $i \notin$ pivots R)

Relative homology: same computation gives compatible basis

Cohomology: reduce D to $S = W \cdot D$ (W: invertible, upper tri),
rows give compatible basis

Clearing: $i =$ pivot R_j (max. row index of nonzero entry in R_j)
$\implies$ $R_i = 0$ (boundaries are cycles!)

COMPUTING IMAGE PERSISTENCE

D^{im} : boundary matrix, col order K, row order L.

$$\begin{array}{cc} & \xrightarrow{L} \\ L\big\downarrow & D^L \end{array} \qquad \begin{array}{cc} & \xrightarrow{K} \\ L\big\downarrow & D^{im} \end{array} \qquad \begin{array}{cc} & \xrightarrow{K} \\ K\big\downarrow & D^K \end{array}$$

Reduce to $R^{im} = D^{im} V^{im}$

<u>Lemma</u> R^{im} and R^L have same col span and pivots.

<u>Theorem</u> [B, Schmahl 2019]

$$B(\operatorname{im}\varphi) = \{[b_j, d_j) \mid b_j, d_j : \text{filtration values for } i, j : i = \text{pivot } R^{im}_j\}$$

$$\cup \{[e_i, \infty) \mid e_i : \text{filtration value for } i : R^L_i = 0,\ i \notin \text{pivots } R\}$$

Note :

- requires both R^{im} and R^L
- clearing additionally requires R^K

CLEARING FOR IMAGE PERSISTENCE

homology:

${}_{L}\!\downarrow \overrightarrow{R^{L}}^{L} = D^{L} V^{L}$	${}_{L}\!\downarrow \overrightarrow{R^{im}}^{K} = D^{im} V^{im}$	${}_{K}\!\downarrow \overrightarrow{R^{K}}^{\varphi} = D^{K} V^{K}$
$B^{\infty}(\operatorname{im}\varphi)$	$B^{0}(\operatorname{im}\varphi)$	$(B^{\infty}(\operatorname{im}\varphi))$
infinite part	finite part	(infinite part rel. hom) clearing

cohomology:

${}_{L}\!\downarrow \overrightarrow{S}^{L} = W D$	${}_{L}\!\downarrow \overrightarrow{S'}^{K} = W' D'$	${}_{K}\!\downarrow \overrightarrow{S''}^{\varphi} = W'' D''$
$B^{\infty}(\operatorname{im}\varphi)$	$B^{0}(\operatorname{im}\varphi)$	$B^{\infty}(\operatorname{im}\varphi)$
(infinite part rel. cohom.)	finite part	infinite part clearing

CONCLUSION

- Image persistence implemented in Ripser
  - two distance matrices for same underlying set
  - cohomology & clearing
  - performance competitive with standard barcode computation

- computation of image barcodes and induced matchings feasible for large data sets

- further possible application:
  geometric matching of barcodes for two point sets in $\mathbb{R}^d$ using Delaunay complexes, collapse $\check{C}ech_r(X) \searrow Del_r(X)$