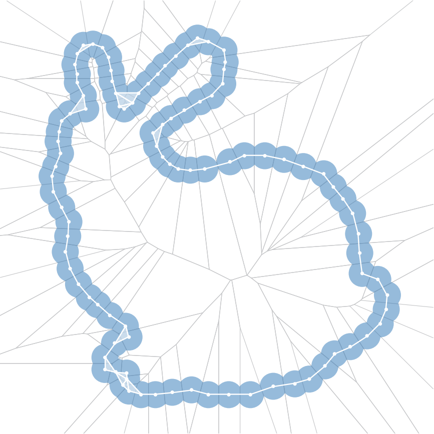


The Morse theory of Čech and Delaunay complexes

Workshop on Random and Statistical Topology
Tohoku University

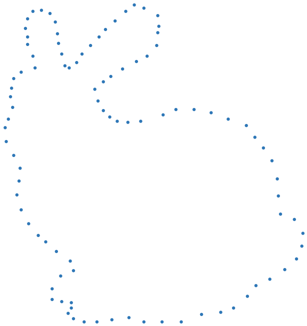
Ulrich Bauer

TUM

February 17, 2016

Joint work with Herbert Edelsbrunner (IST Austria)

Connect the dots: topology from geometry



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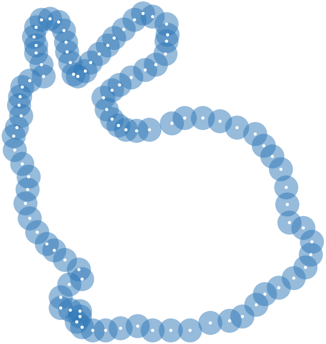

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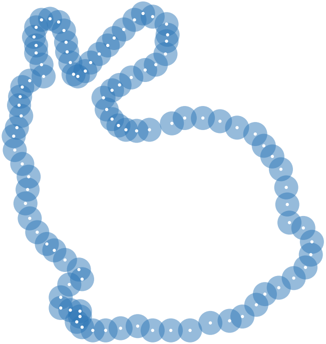

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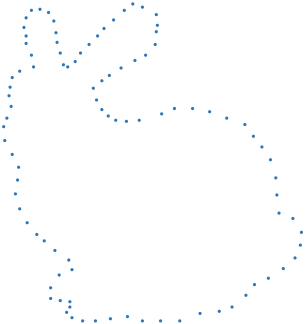

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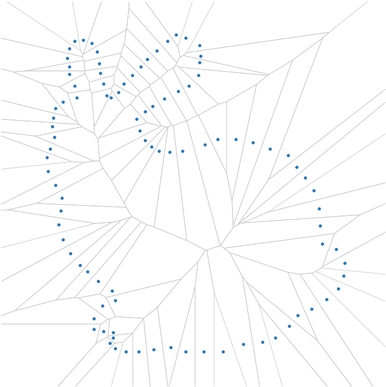

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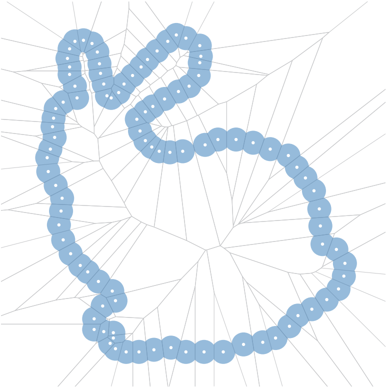

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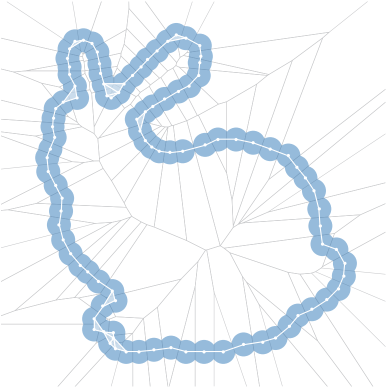

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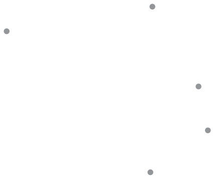
Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)



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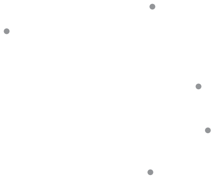

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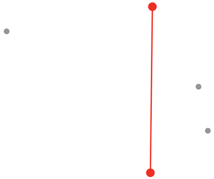

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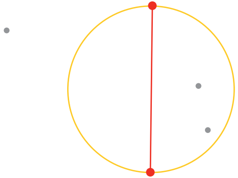

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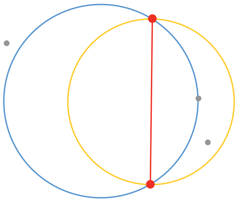

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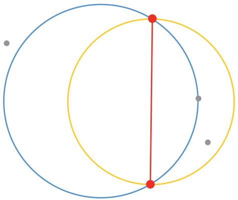

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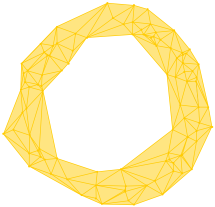

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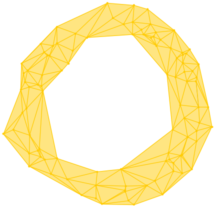

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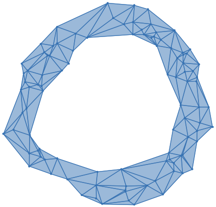

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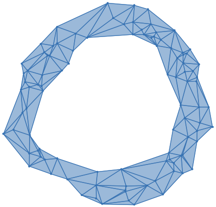

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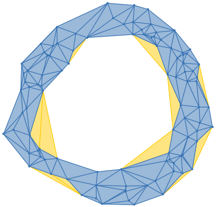

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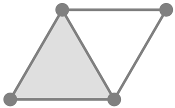
- Are they related by a sequence of simplicial collapses?

Discrete Morse theory

Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.



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Definition (Whitehead 1938)

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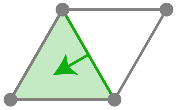
- A *free face* of K is a simplex that has a unique proper coface.



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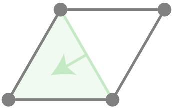

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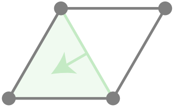

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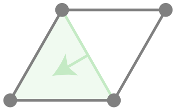

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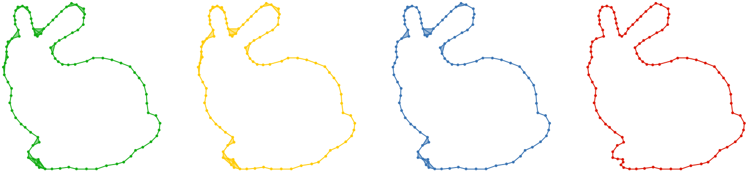
If there is a sequence of such elementary collapses from K to M, we say that K *collapses* to M (written as $K \searrow M$).

Main result: a sequence of collapses

Theorem (B, Edelsbrunner 2014)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent. In particular,

$$\mathrm{Cech}_r \searrow \mathrm{DelCech}_r \searrow \mathrm{Del}_r \searrow \mathrm{Wrap}_r .$$



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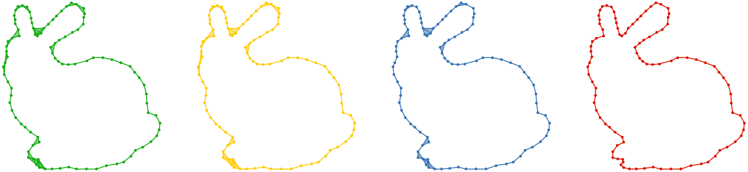

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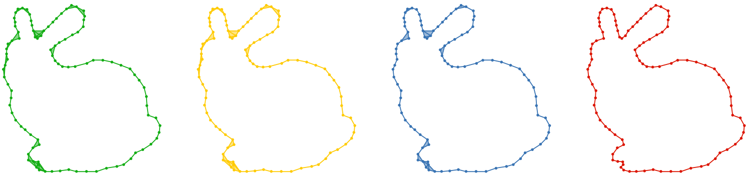
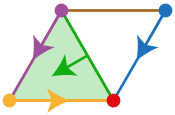


- All collapses are induced by a single *discrete gradient field*
- Also works for weighted point sets

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a simplicial complex
is a partition of the simplices
into singletons and pairs $\{L, U\}$,
where L is a face of U with codimension 1.



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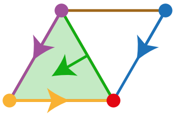

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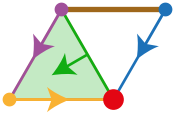

The singletons are called *critical simplices.*

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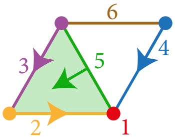

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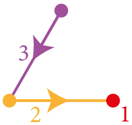

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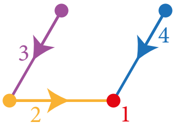

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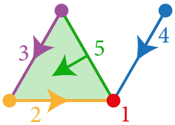

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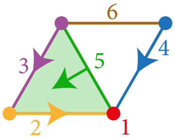

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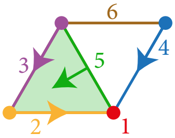

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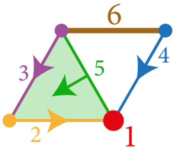
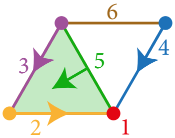


If $f^{-1}(t) = \{Q\}$ then t is a *critical value*.

Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.



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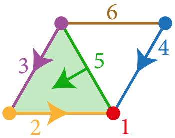

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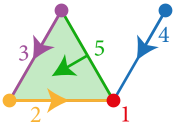

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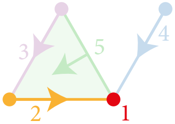

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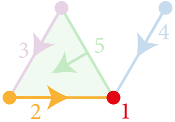
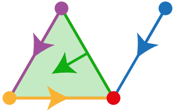


Let V be a discrete gradient field on a simplicial complex K, and let L be a subcomplex of K.

Corollary

If $K \setminus L$ is the union of some pairs of V, then $K \searrow L$.



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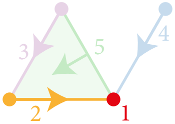

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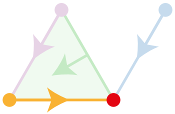

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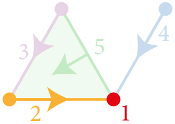

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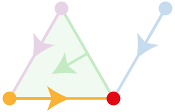

We say that V *induces the collapse* $K \searrow L$.

Unfortunately…

Neither the Čech nor the Delaunay functions
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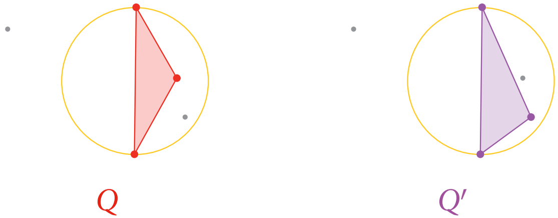
- Example: two simplices Q, Q' with $f_C(Q) = f_C(Q')$ such that neither is a face of the other:

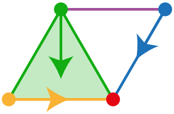


Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}$$



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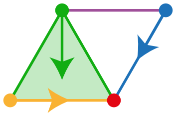

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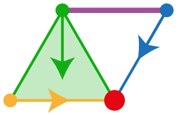

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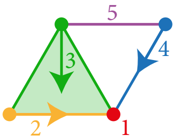
The singletons are called *critical simplices*.

Generalized discrete Morse theory

Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes



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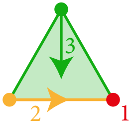

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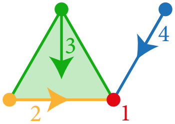

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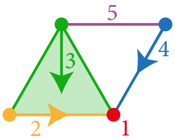

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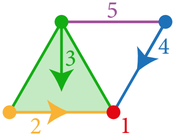
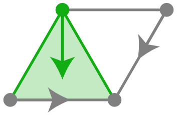


Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:



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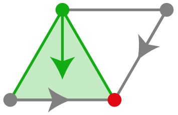

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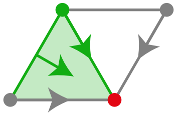

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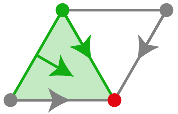

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Therefore the collapsing theorems also hold for generalized discrete Morse functions.

Čech and Delaunay functions

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Proposition (B, Edelsbrunner 2014)

The Čech function and the Delaunay function are generalized discrete Morse functions.

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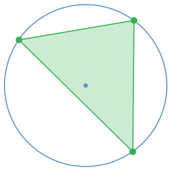

Čech intervals

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Let $Q \subseteq X$ be a simplex with smallest enclosing sphere S.
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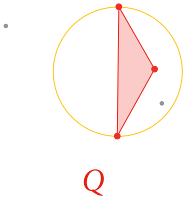

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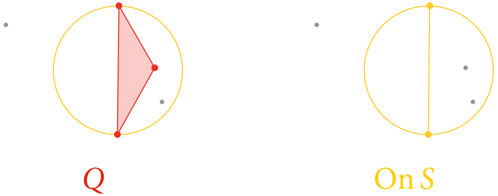

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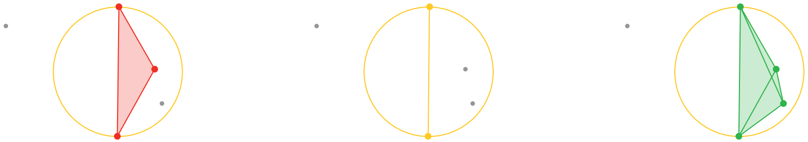

Q | $\operatorname{On} S$ | $\operatorname{Encl} S$

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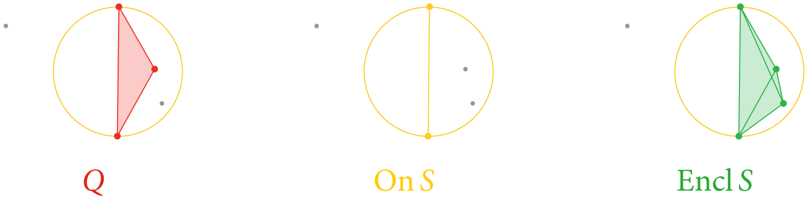

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$$\mathrm{Front}\, S = \{x \in \mathrm{On}\, S \mid \mu_x > 0\},$$

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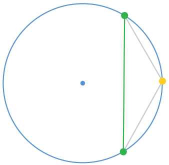

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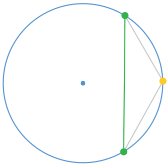

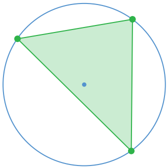

Delaunay intervals

Lemma

Let $Q \subseteq X$ be a simplex with smallest empty circumsphere S. Then $Q' \subseteq X$ has the same smallest empty circumsphere S iff

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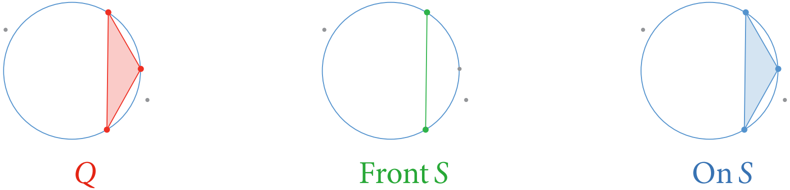

Sphere optimization problems

Both Čech and Delaunay functions are defined using the smallest sphere S satisfying certain constraints:

$$\begin{aligned} \underset{r,z}{\text{minimize}} \quad & r \\ \text{subject to} \quad & \|z - q\| \leq r, && q \in Q, \\ & \|z - e\| \geq r, && e \in E. \end{aligned}$$

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&\underset{a,z}{\text{minimize}} && \|z\|^2 - a && (r^2 = \|z\|^2 - a) \\
&\text{subject to} && \|z - q\|^2 \leq \|z\|^2 - a, && q \in Q, \\
& && \|z - e\|^2 \geq \|z\|^2 - a, && e \in E.
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The Karush–Kuhn–Tucker optimality conditions

Consider an optimization problem of the form

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\begin{aligned}
\underset{x}{\text{minimize}} \quad & f(x) & \\
\text{subject to} \quad & g_i(x) \le 0, & i \in I, \\
& h_j(x) = 0, & j \in J,
\end{aligned}
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where the function f is convex and g_i, h_j are affine.

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where the function f is convex and g_i, h_j are affine.

Theorem (Karush 1939, Kuhn–Tucker 1951)

A feasible point x to the above problem is an optimal solution iff there exist Lagrange multipliers $(\nu_i)_{i \in I}$ *and* $(\lambda_j)_{j \in J}$ *such that*

$$\begin{aligned} \nabla f(x) + \sum_{i \in I} \nu_i \nabla g_i(x) + \sum_{j \in J} \lambda_j \nabla h_j(x) &= 0, && \textit{(stationarity)} \\ \nu_i \, g_i(x) &= 0, \quad i \in I, && \textit{(complementary slackness)} \\ \nu_i &\ge 0, \quad i \in I. && \textit{(dual feasibility)} \end{aligned}$$

KKT conditions for the smallest sphere problem

The KKT conditions for our sphere optimization problem are:

Proposition

A sphere S enclosing Q and excluding E is minimal iff its center z can be written as an affine combination

$$z = \sum_{x \in Q \cup E} \mu_x x, \qquad 1 = \sum_{x \in Q \cup E} \mu_x$$

such that

- $\mu_x = 0$ *for* $x \notin \mathrm{On}\, S$,
- $\mu_x \geq 0$ *for* $x \notin E$, *and*
- $\mu_x \leq 0$ *for* $x \notin Q$.

Čech and Delaunay intervals from KKT

Proposition

A sphere S is the smallest sphere enclosing Q and excluding E iff

S *is the smallest circumsphere of some simplex* $P \subseteq X$,

$Q \in [\operatorname{Front} S, \operatorname{Encl} S]$, *and*

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The Delaunay intervals are of the form $[\operatorname{Front} S, \operatorname{On} S]$.

Selective Delaunay complexes

Define for finite point sets $X, E \subset \mathbb{R}^d$:

- *E-Delaunay function* $f_E(Q)$:
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- *E-Delaunay complex* $\mathrm{Del}_r(X, E) = f_D^{-1}(-\infty, r]$

Selective Delaunay complexes

Define for finite point sets $X, E \subset \mathbb{R}^d$:

- *E-Delaunay function* $f_E(Q)$:
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Theorem (B, Edelsbrunner 2014)

Let $E \subseteq E' \subseteq X$. Then

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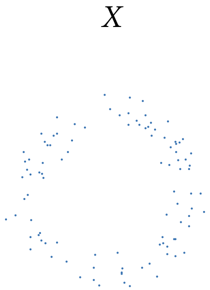
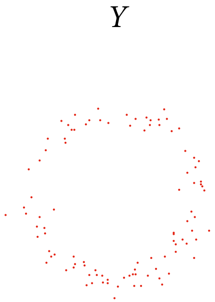
Note: choosing $E = \varnothing$ and $E' = X$ yields

$$\mathrm{Cech}_r(X) \searrow \mathrm{DelCech}_r(X) \searrow \mathrm{Del}_r(X).$$

Connecting different Delaunay complexes





Connecting different Delaunay complexes

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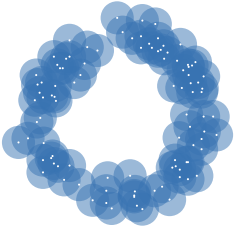

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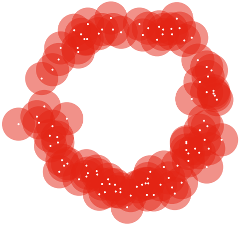
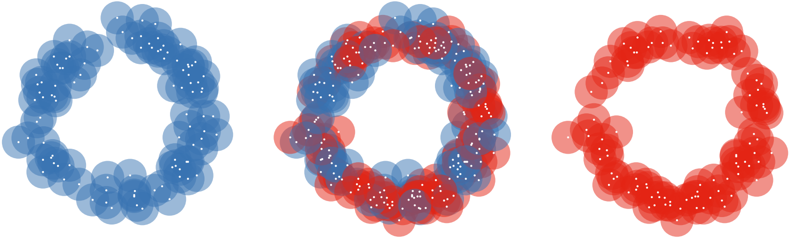


Connecting different Delaunay complexes

$$B_r(X) \hookrightarrow B_r(X \cup Y) \hookleftarrow B_r(Y)$$



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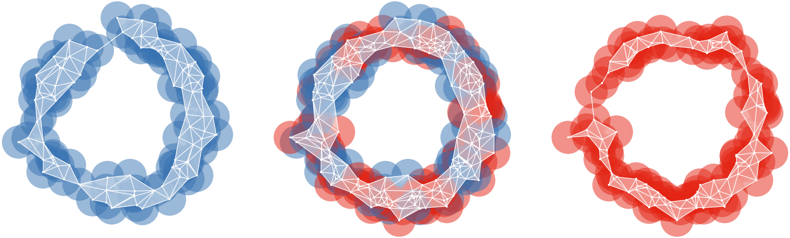
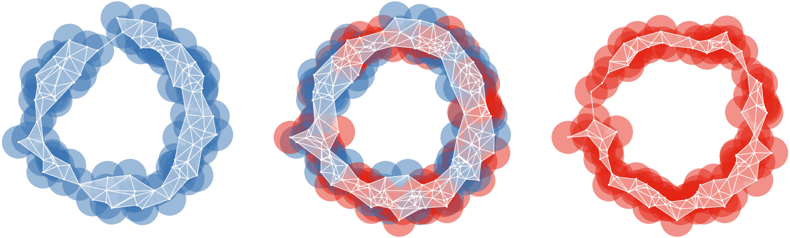


$\mathrm{Del}_r(X)$ $\qquad$ $\mathrm{Del}_r(X \cup Y)$ $\qquad$ $\mathrm{Del}_r(Y)$

Connecting different Delaunay complexes

$$
\begin{tikzcd}
B_r(X) \arrow[r, hook] & B_r(X \cup Y) & B_r(Y) \arrow[l, hook']
\end{tikzcd}
$$



$$
\begin{tikzcd}[column sep=small]
\mathrm{Del}_r(X) \arrow[dr, hook] & & \mathrm{Del}_r(X \cup Y) \arrow[dl, hook'] \arrow[dr, hook] & & \mathrm{Del}_r(Y) \arrow[dl, hook'] \\
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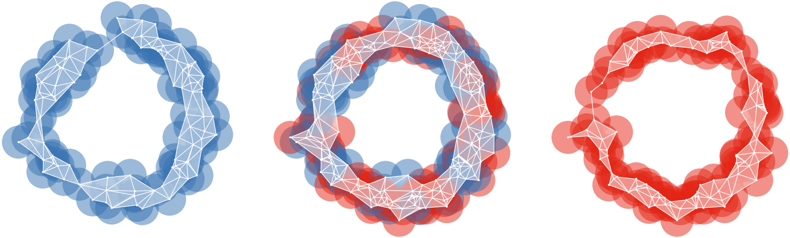

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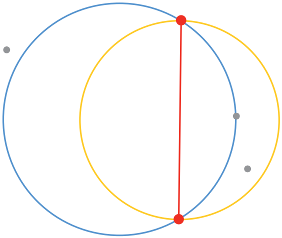
Collapsing from Čech to Delaunay

Collapsing the Delaunay–Čech complex

To construct the collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$:

- Consider a non-critical Delaunay simplex Q



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To construct the collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$:

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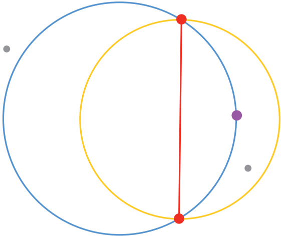

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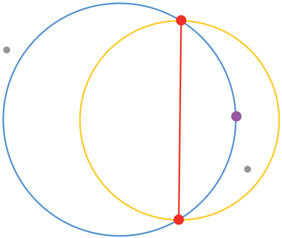

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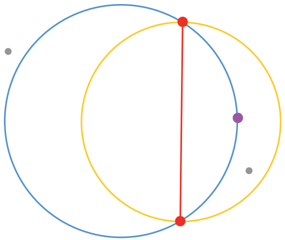

Lemma

The pairs (Q', Q'') yield a discrete gradient.

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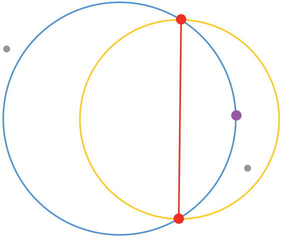

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Lemma

The pairs (Q', Q'') yield a discrete gradient.
This gradient induces a collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$, for any radius r.

Collapsing non-Delaunay simplices

To construct the collapse $\mathrm{Cech}_r \searrow \mathrm{DelCech}_r$:

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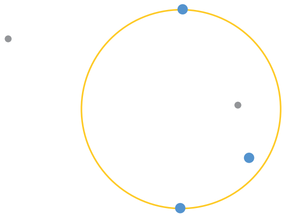

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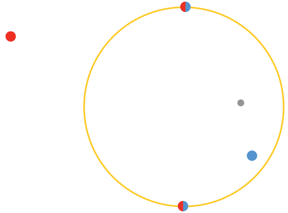

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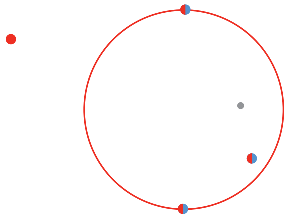

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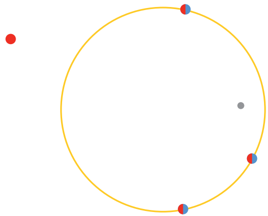

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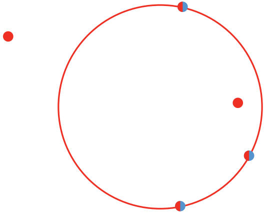

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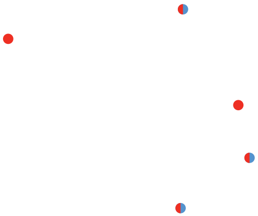

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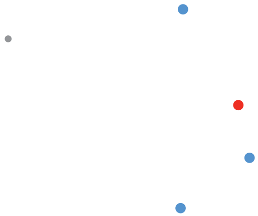

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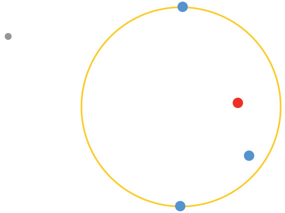

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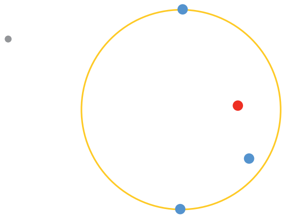

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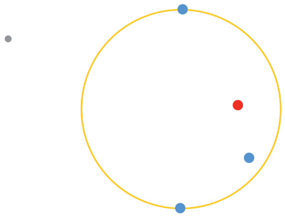

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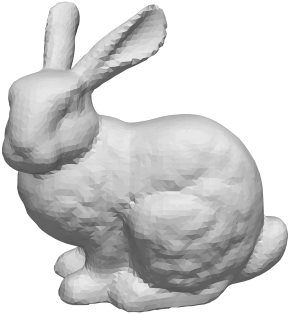
Wrap complexes

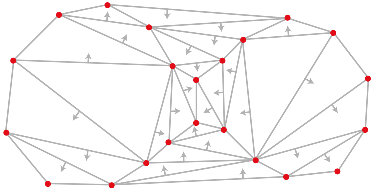
Wrap complexes

Generalizes and greatly simplifies the surface reconstruction algorithm *Wrap* (Edelsbrunner 1995, Geomagic)



Wrap complexes

Consider the Delaunay function f_D of X.



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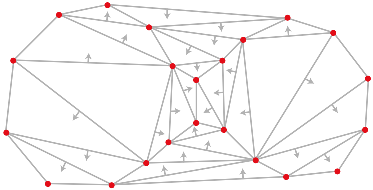

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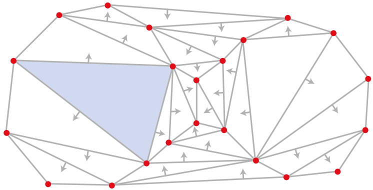

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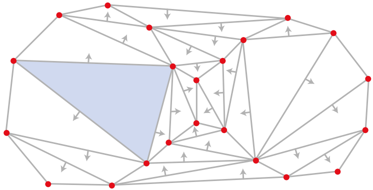

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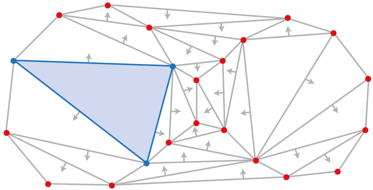

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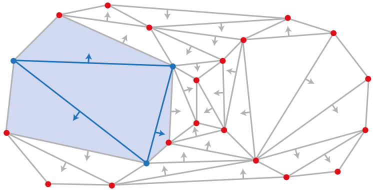

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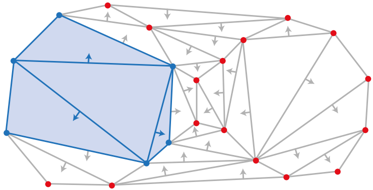

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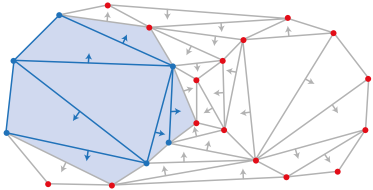

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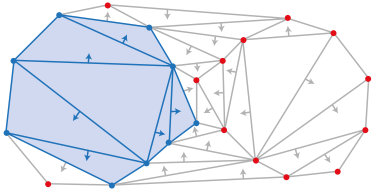

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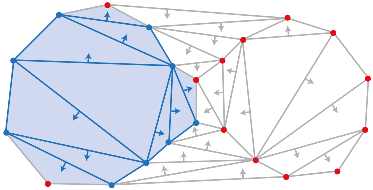

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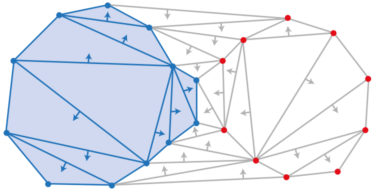

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Define

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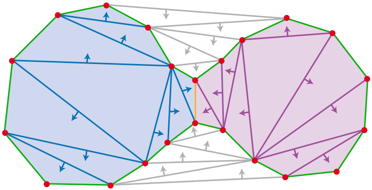

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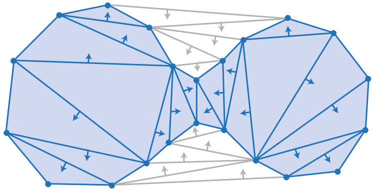

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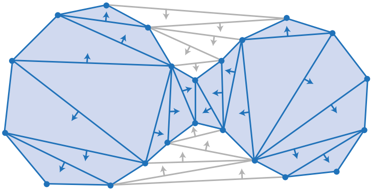

The Delaunay intervals induce a collapse $\mathrm{Del}_r \searrow \mathrm{Wrap}_r$.

Wrapping up

- Čech and Delaunay complexes from Morse functions
- Explicit homotopy equivalence by simplicial collapses
- Simple definition and generalization of *Wrap* complexes