# Persistent Homology and the Stability Theorem

Ulrich Bauer

TUM

February 26, 2018

Workshop *Persistence, Representations, and Computation*, Raitenhaslach

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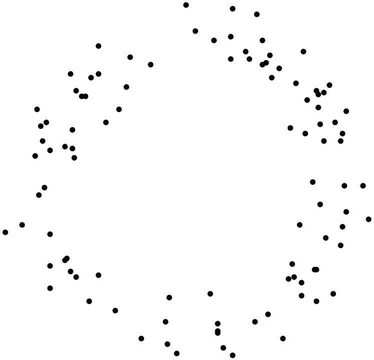

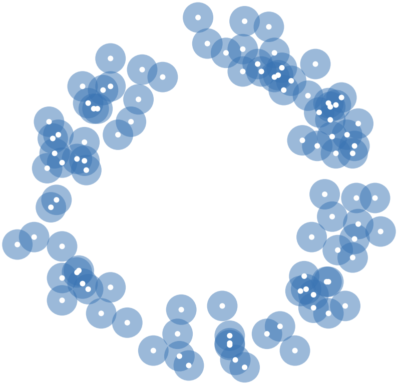

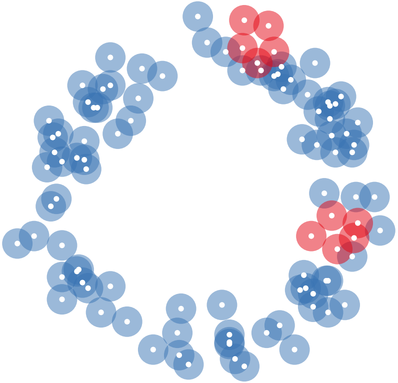

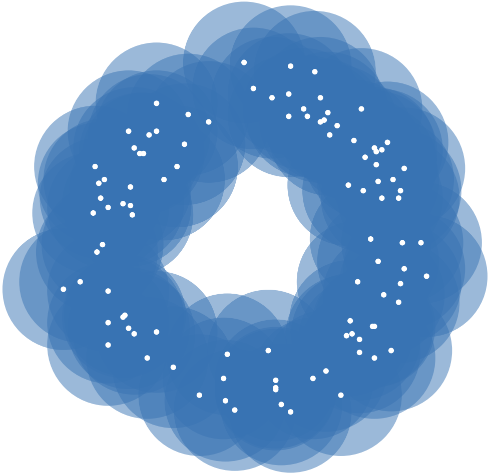

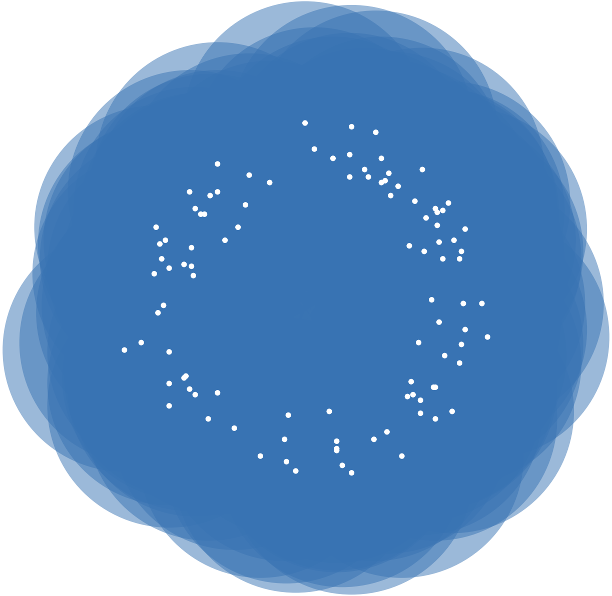

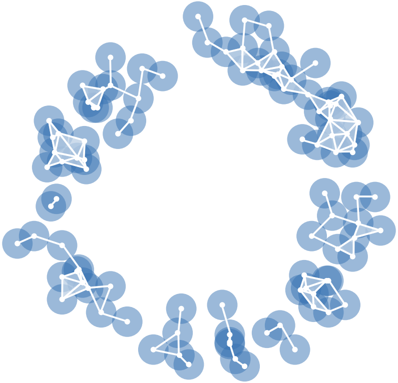

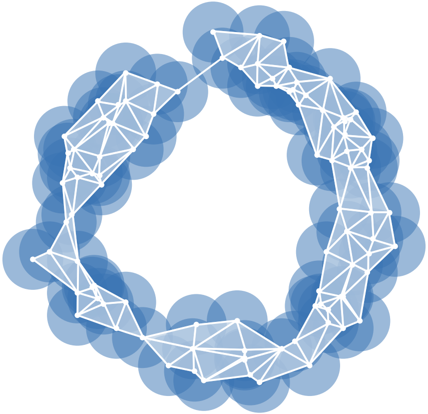

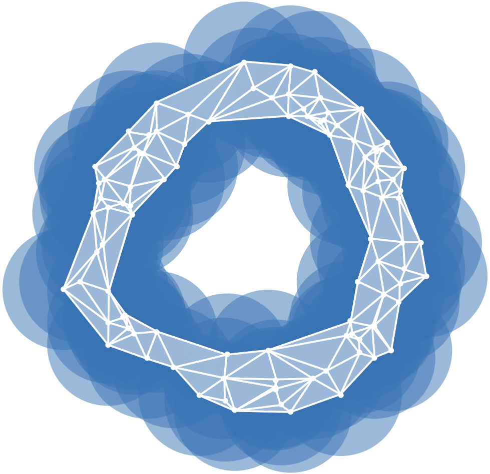

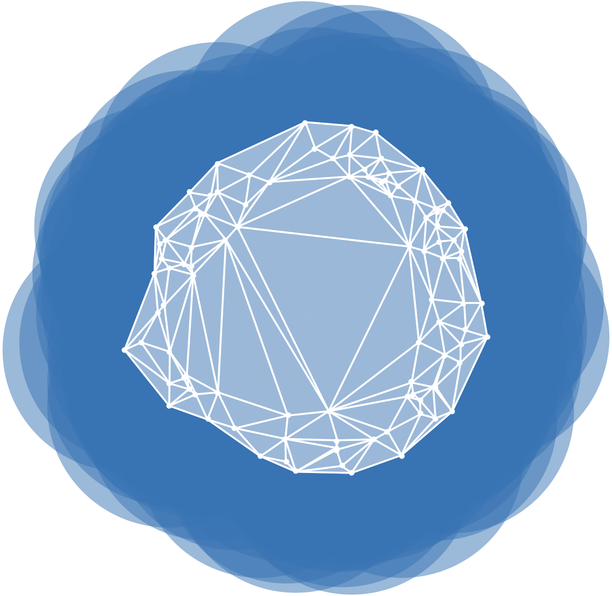

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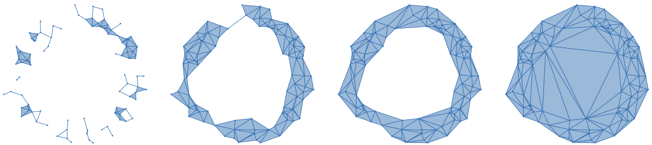

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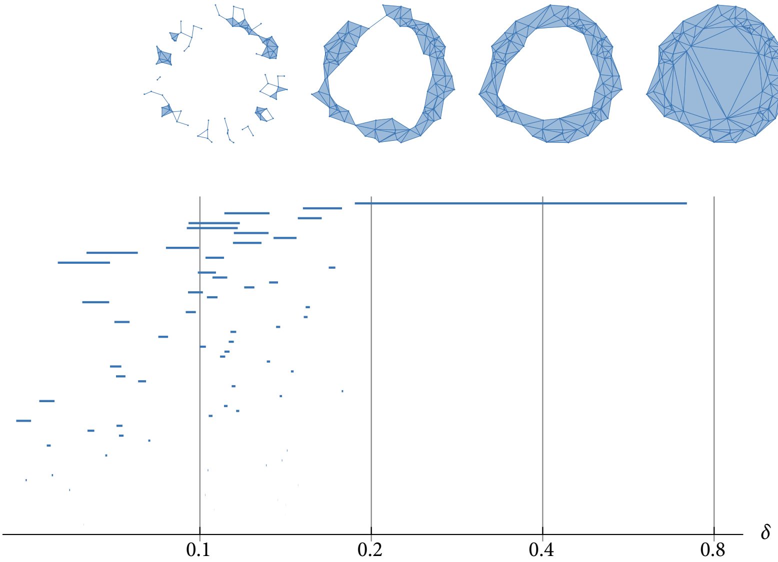

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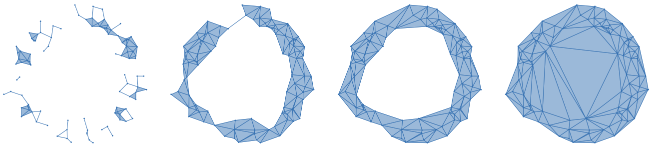

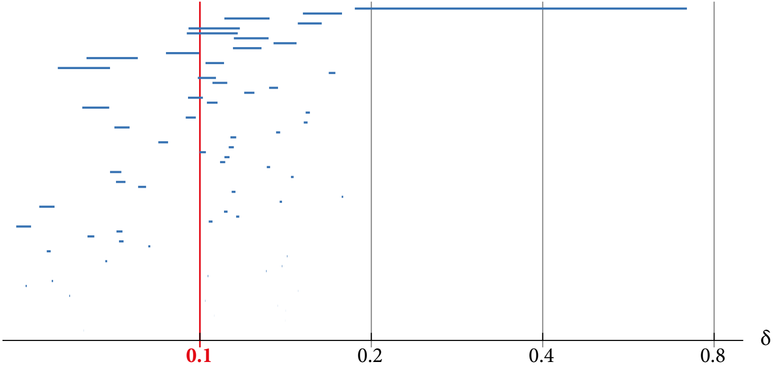

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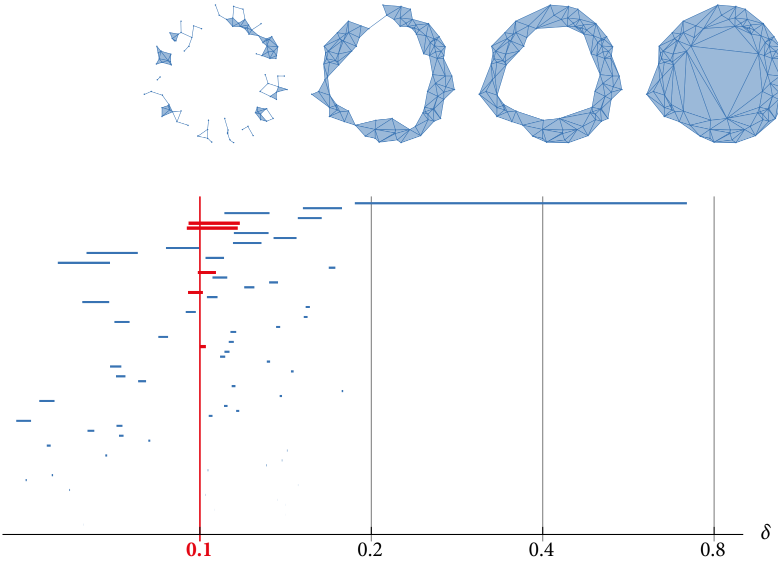

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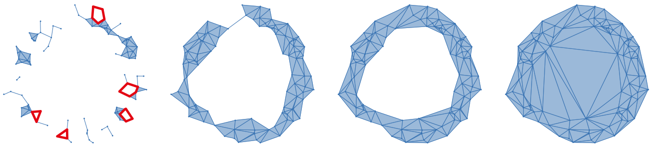

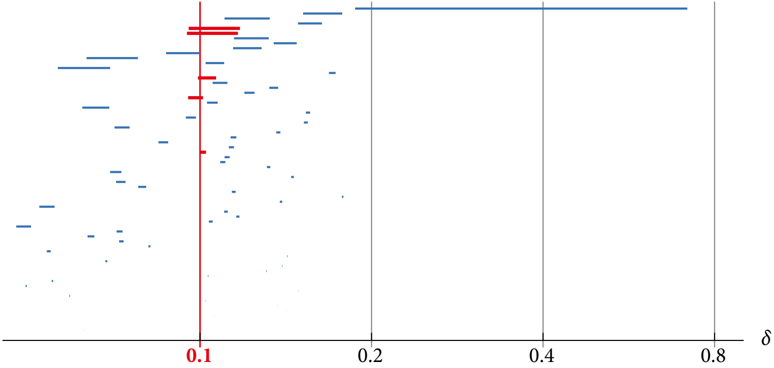

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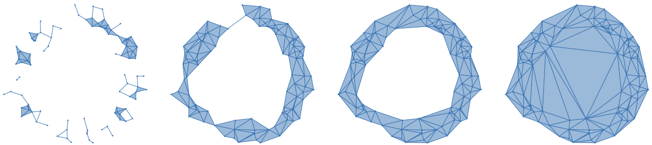

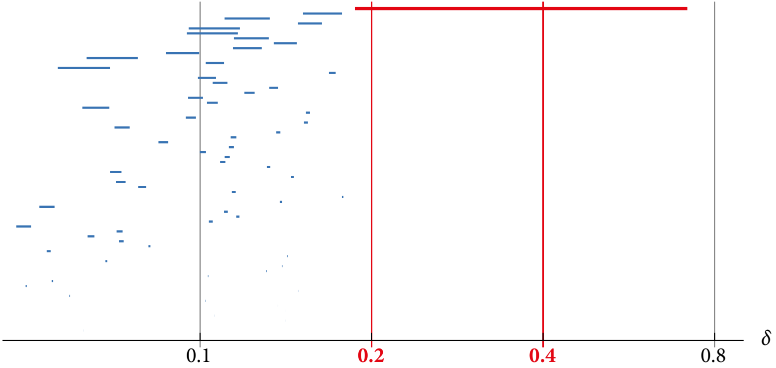

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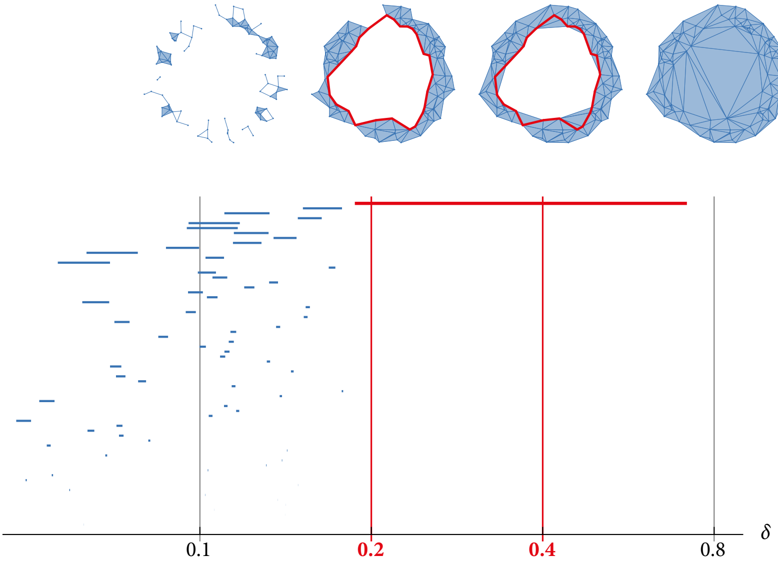

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  - $\mathbf{R}$ is the poset category of $(\mathbb{R}, \leq)$
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- Consider homology with coefficients in a field (often $\mathbb{Z}_2$) $H_* : \mathbf{Top} \to \mathbf{Vect}$
- Persistent homology is a diagram $M : \mathbf{R} \to \mathbf{Vect}$ *(persistence module)*

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Requires strong assumptions:

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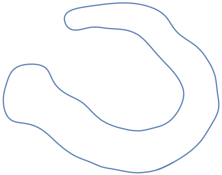

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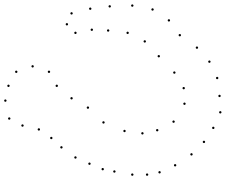

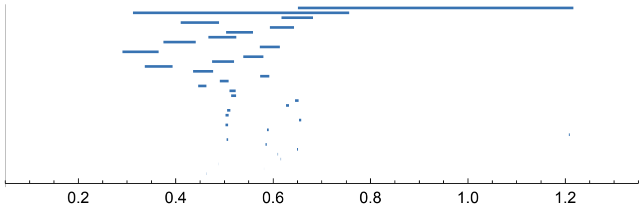

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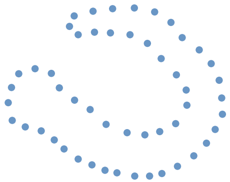

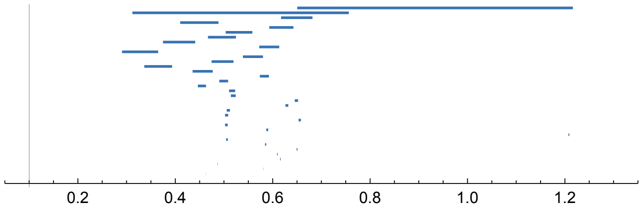

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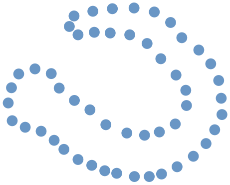

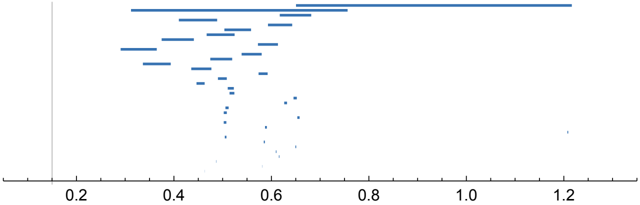

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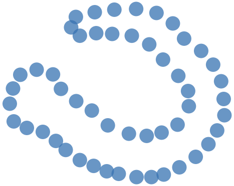

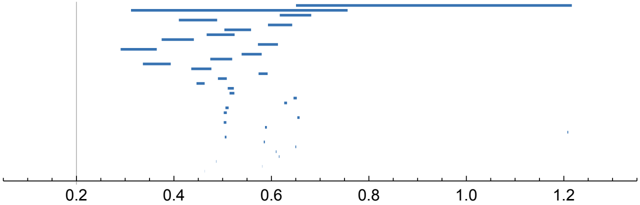

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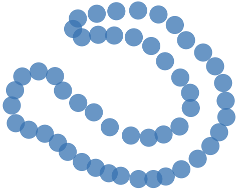

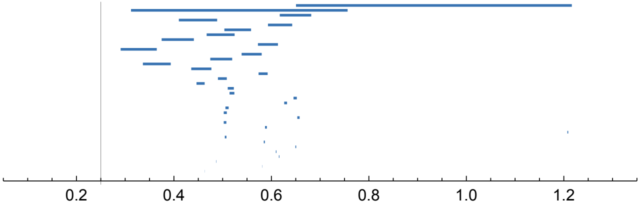

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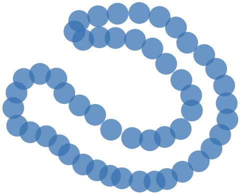

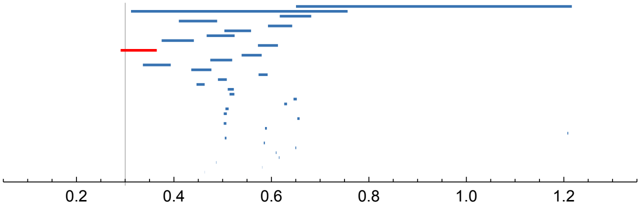

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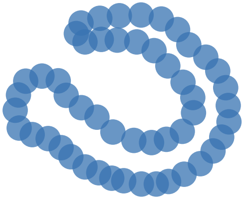

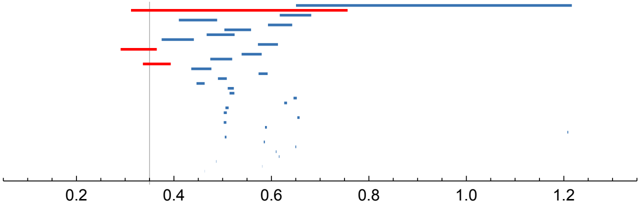

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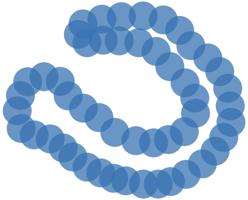

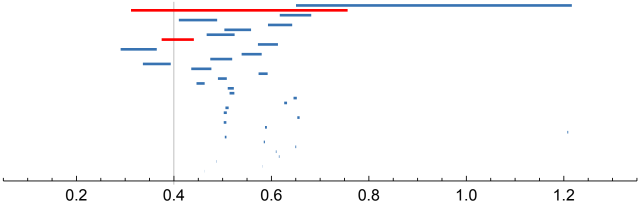

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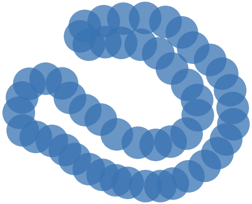

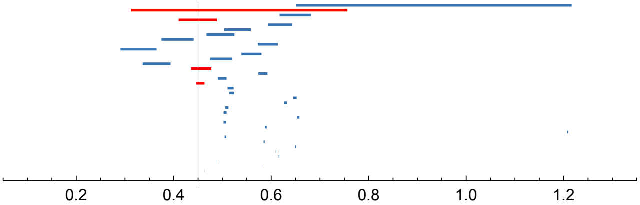

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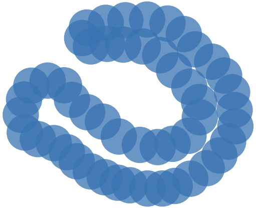

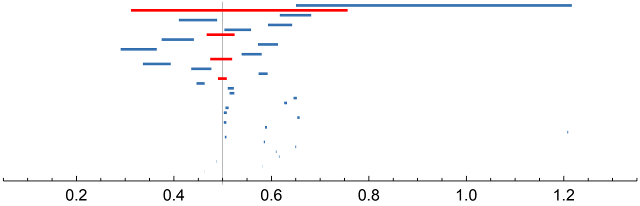

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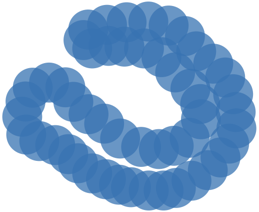

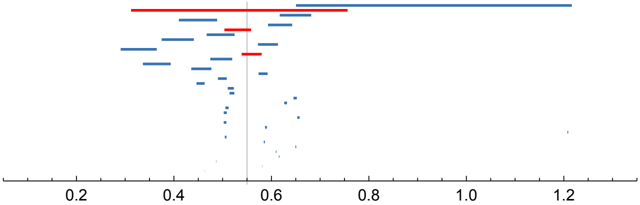

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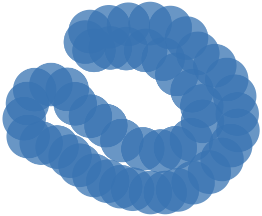

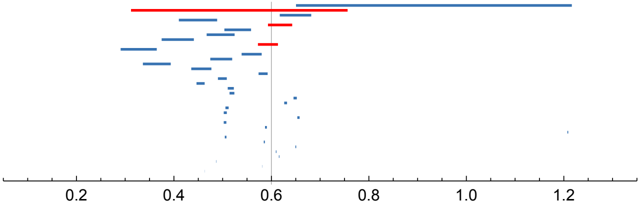

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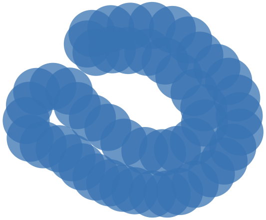

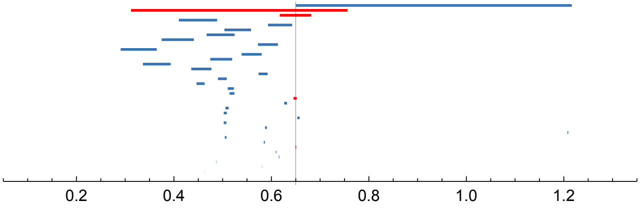

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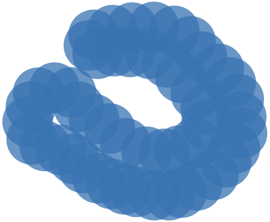

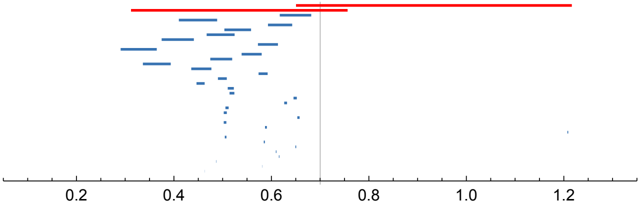

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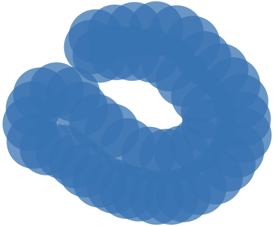

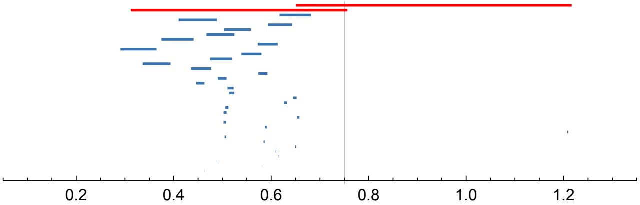

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*Let $M$ be a submanifold of $\mathbb{R}^d$. Let $P \subset M$, $\delta > 0$ be such that*

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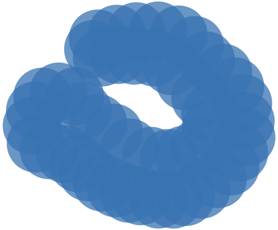

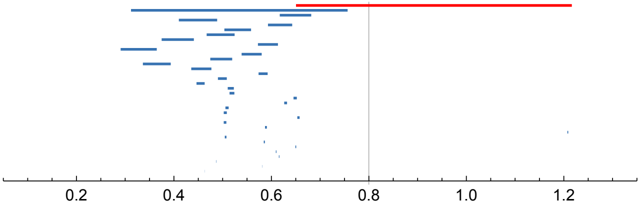

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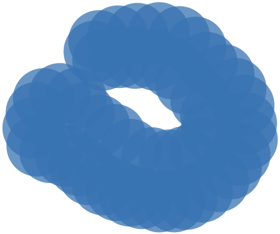

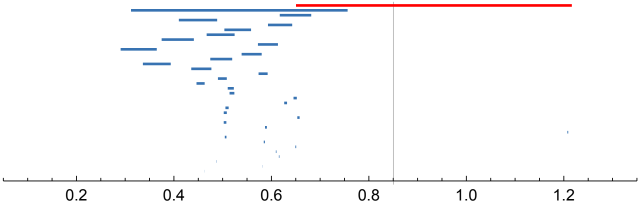

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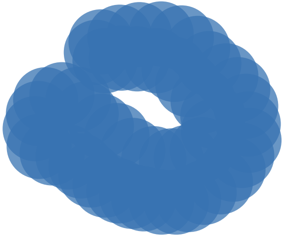
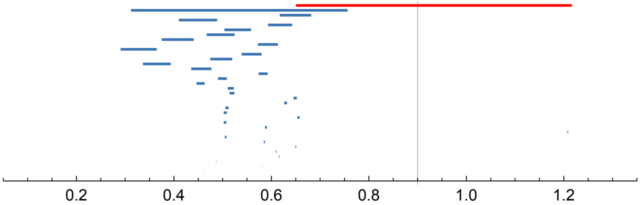




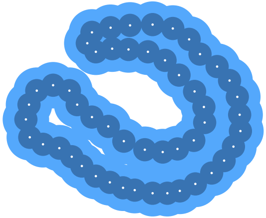
# Homology inference using persistence

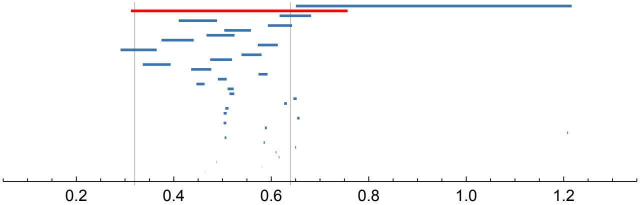
## Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

*Let* $\Omega \subset \mathbb{R}^d$. *Let* $P \subset \Omega$, $\delta > 0$ *be such that*

- $B_\delta(P)$ *covers* $\Omega$, *and*
- *the inclusions* $\Omega \hookrightarrow B_\delta(\Omega) \hookrightarrow B_{2\delta}(\Omega)$ *preserve homology.*

*Then* $H_*(\Omega) \cong \operatorname{im} H_*(B_\delta(P) \hookrightarrow B_{2\delta}(P))$.





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This motivates the *homological realization problem*:

## Problem

*Given a simplicial pair $L \subseteq K$, find $X$ with $L \subseteq X \subseteq K$ such that*

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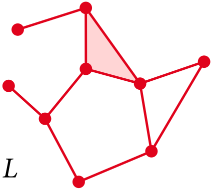

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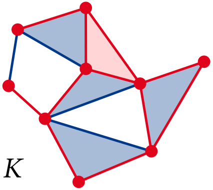

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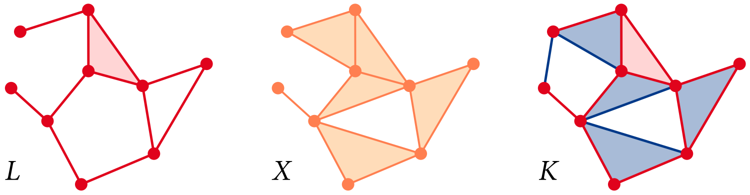

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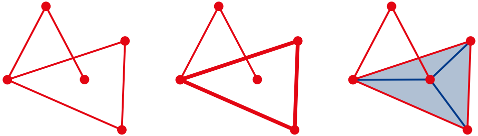

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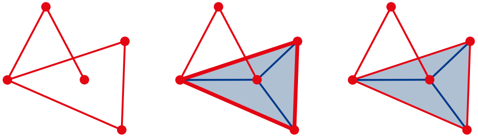

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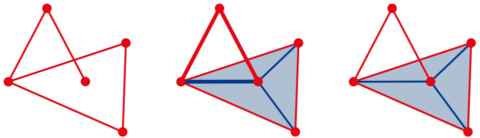

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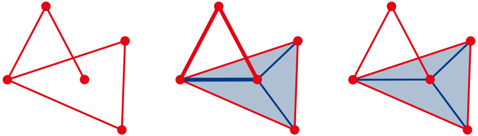

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## Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

*The homological realization problem is NP-hard, even in $\mathbb{R}^3$.*

# Stability

# Persistence and stability: the big picture

| | | |
|---|---|---|
| Data | point cloud | $P \subset \mathbb{R}^d$ |

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| Topology | topological spaces (filtration) | $K : \mathbf{R} \to \mathbf{Top}$ |
| $\downarrow$ | $\downarrow$ homology | |
| Algebra | vector spaces (persistence module) | $M : \mathbf{R} \to \mathbf{Vect}$ |

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# Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

*Let* $\|f - g\|_\infty = \delta$*. Then there exists a matching between the intervals of the persistence barcodes of* $f$ *and* $g$ *such that*

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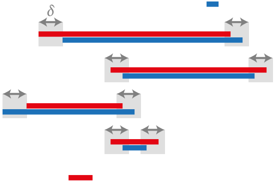

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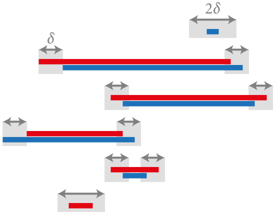

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Let $\delta = \|f - g\|_\infty$. Write $F_t = f^{-1}(-\infty, t]$ and $G_t = g^{-1}(-\infty, t]$.

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Applying homology (functor) preserves commutativity

- persistent homology of $f, g$ yields
  $\delta$-interleaved persistence modules $\mathbf{R} \to \mathbf{Vect}$

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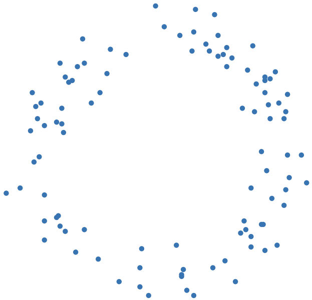

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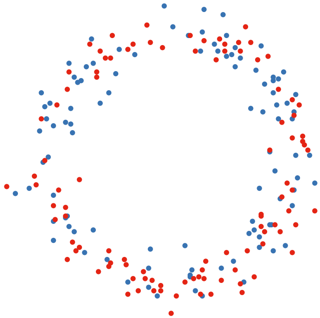

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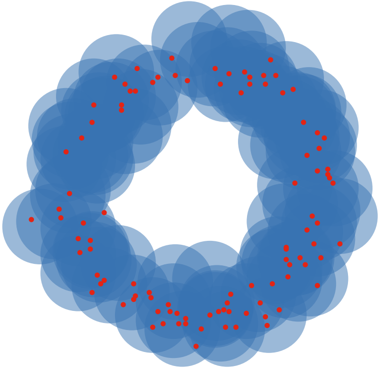

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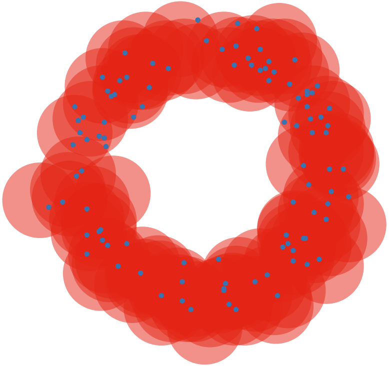

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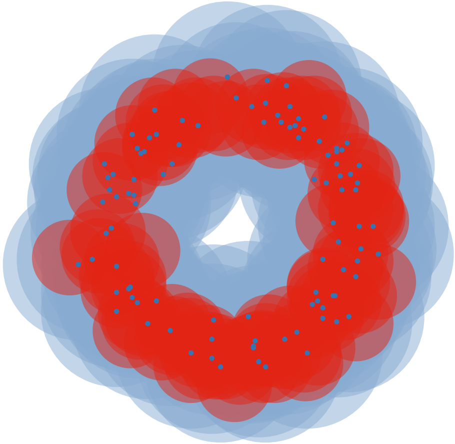

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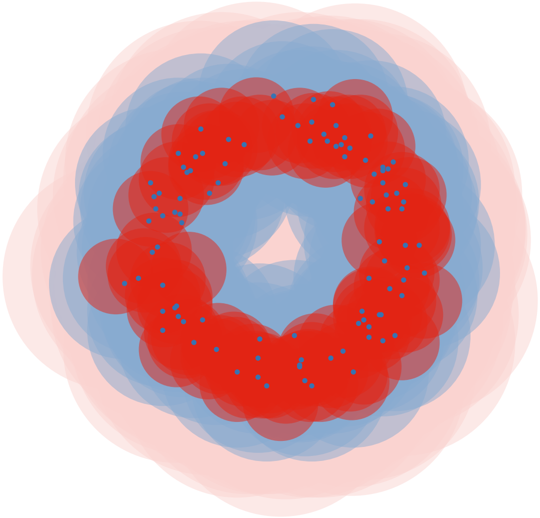

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- morphism and transition maps commute:

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# Interval Persistence Modules

Let $\mathbb{K}$ be a field. For an arbitrary interval $I \subseteq \mathbb{R}$, define the *interval persistence module* $\mathbb{K}(I)$ by

$$\mathbb{K}(I)_t = \begin{cases} \mathbb{K} & \text{if } t \in I, \\ 0 & \text{otherwise,} \end{cases}$$

with transition maps of maximal rank.

Schematic example:

$$\cdots\dashrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{K} \longrightarrow \mathbb{K} \dashrightarrow \mathbb{K} \longrightarrow 0 \longrightarrow 0 \dashrightarrow$$

# Barcodes: the structure of persistence modules

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- The decomposition itself is not unique.
- This is why we use homology with coefficients in a field.

# Algebraic stability of persistence barcodes

## Theorem (Chazal et al. 2009, 2012; B, Lesnick 2015)

*If two persistence modules are $\delta$-interleaved,*
*then there exists a $\delta$-matching of their barcodes:*

- matched intervals have endpoints within distance $\leq \delta$,
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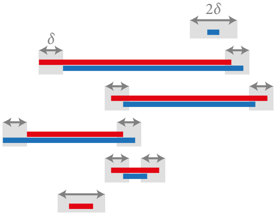
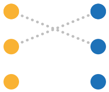


# Barcodes as diagrams

# The matching category

A *matching* $\sigma : S \nrightarrow T$ is a bijection $S' \to T'$, where $S' \subseteq S$, $T' \subseteq T$.



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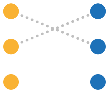
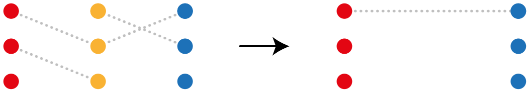


Composition of matchings $\sigma : S \nrightarrow T$ and $\tau : T \nrightarrow U$:



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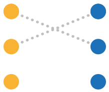

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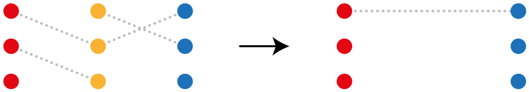
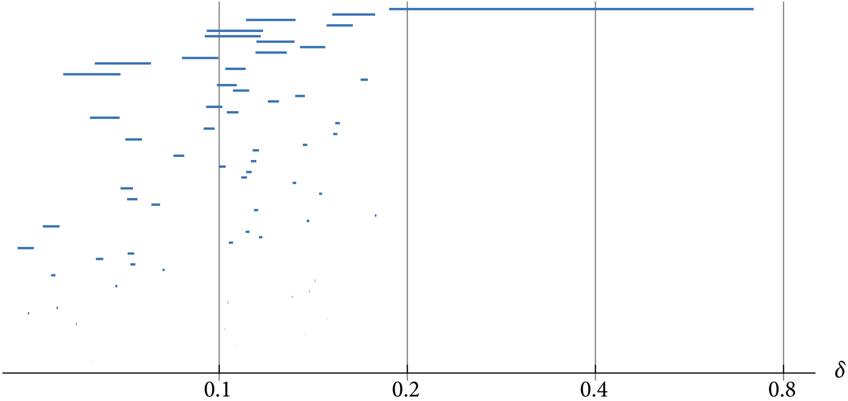


Matchings form a category $\mathbf{Mch}$

- objects: sets
- morphisms: matchings

# Barcodes as matching diagrams

We can regard a barcode $B$ as a functor $\mathbf{R} \to \mathbf{Mch}$:



# Barcodes as matching diagrams

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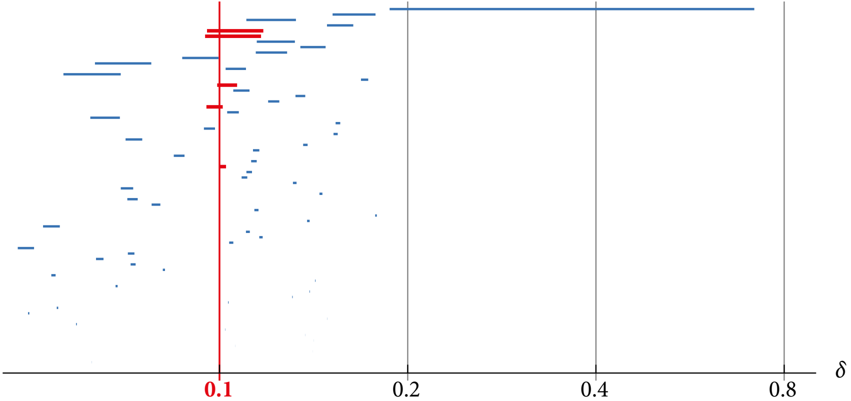
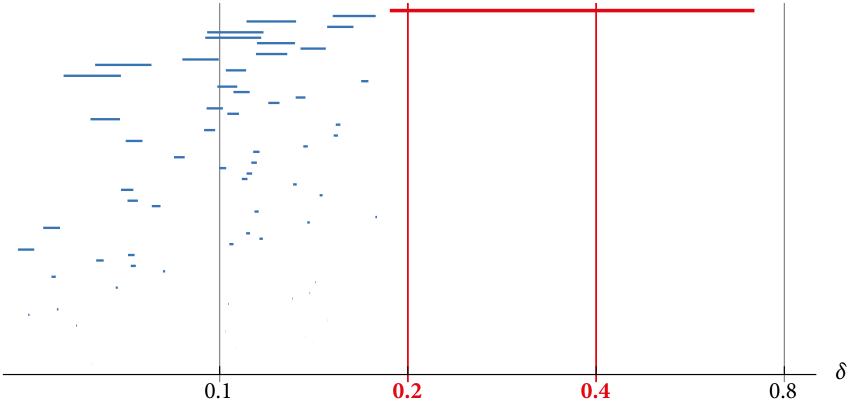
- For each real number $t$, let $B_t$ be the set of intervals of $B$ that contain $t$, and



# Barcodes as matching diagrams

We can regard a barcode $B$ as a functor $\mathbf{R} \to \mathbf{Mch}$:

- For each real number $t$, let $B_t$ be the set of intervals of $B$ that contain $t$, and
- for each $s \leq t$, define the matching $B_s \nrightarrow B_t$
  to be the identity on $B_s \cap B_t$.



# Stability via functoriality?

$$\begin{tikzcd} F_t \arrow[rr, hook] \arrow[dr, hook] & & F_{t+2\delta} \arrow[dr, hook] & \\ & G_{t+\delta} \arrow[ur, hook] \arrow[rr, hook] & & G_{t+3\delta} \end{tikzcd}$$

# Stability via functoriality?

$$
\begin{tikzcd}
H_*(F_t) \arrow[rr] \arrow[dr] & & H_*(F_{t+2\delta}) \arrow[dr] & \\
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\end{tikzcd}
$$

# Stability via functoriality?

$$\begin{tikzcd}
B(H_*(F_t)) \arrow[r] \arrow[dr] & B(H_*(F_{t+2\delta})) \arrow[dr] & \\
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# Non-functoriality of the persistence barcode

### Theorem (B, Lesnick 2015)

*There exists no functor* $\mathbf{Vect}^{\mathbf{R}} \to \mathbf{Mch}^{\mathbf{R}}$ *sending each persistence module to its barcode.*

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### Proposition

*There exists no functor* $\mathbf{Vect} \to \mathbf{Mch}$ *sending each vector space of dimension* $d$ *to a set of cardinality* $d$*.*

# Induced barcode matchings

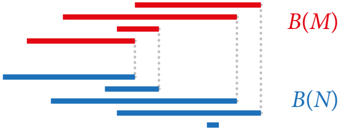
# Structure of persistence submodules / quotients

## Proposition

*Let $f : M \to N$ be a monomorphism of persistence modules: each $f_t : M_t \to N_t$ is injective.*

*Then $f$ induces an injective map $B(M) \to B(N)$ mapping each $I \in B(M)$ to some $J \in B(N)$ with larger or same left and same right endpoint.*

$$\begin{tikzcd} {} \arrow[r, dotted] & M_s \arrow[r] \arrow[d, hook] & M_t \arrow[r, dotted] \arrow[d, hook] & {} \\ {} \arrow[r, dotted] & N_s \arrow[r] & N_t \arrow[r, dotted] & {} \end{tikzcd}$$




Dually for epimorphisms (left and right exchanged).

# Induced matchings

For a general morphism $f : M \to N$ of persistence modules: consider *epi-mono factorization*

$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N.$$

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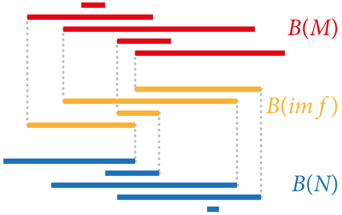
For a general morphism $f : M \to N$ of persistence modules: consider *epi-mono factorization*

$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N.$$

- $\operatorname{im} f \hookrightarrow N$ induces injection $B(\operatorname{im} f) \hookrightarrow B(N)$
- $M \twoheadrightarrow \operatorname{im} f$ induces injection $B(\operatorname{im} f) \hookrightarrow B(M)$
- compose to a matching $B(M) \nrightarrow B(N)$:



# Stability from interleavings

Consider interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(\forall t \in \mathbb{R})$:

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$B(N)$

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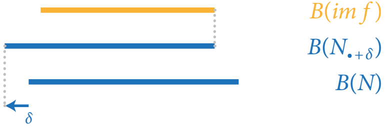

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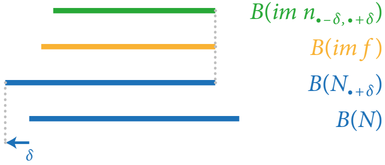

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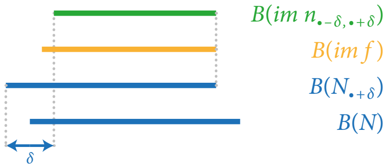

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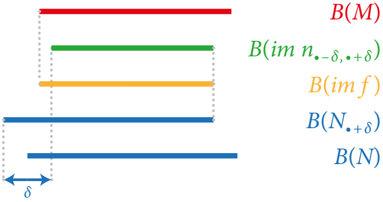

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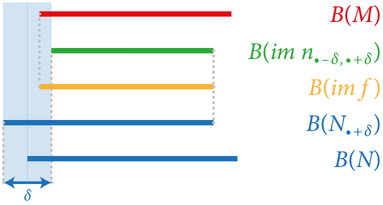

# Computation

# Demo: Ripser

Example data set:

- 192 points on $\mathbb{S}^2$
- persistent homology barcodes up to dimension $2$
- over 56 mio. simplices in 3-skeleton

Some previous software:

- javaplex (Stanford): 3200 seconds, 12 GB
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Ripser: 1.2 seconds, 160 MB

# Ripser

A software for computing Vietoris–Rips persistence barcodes

- around 1000 lines of C++ code, no external dependencies
- support for
  - coefficients in a prime field $\mathbb{Z}/p\mathbb{Z}$
  - sparse distance matrices (for distance threshold)
- open source (`http://ripser.org`)
  - released in July 2016
- online version (`http://live.ripser.org`)
  - launched in August 2016
- (co-)winner of 2016 ATMCS Best New Software Award

# Computing homology

Computing homology $H_* = Z_*/B_*$ (recall: $B_* \subseteq Z_* \subseteq C_*$):

- compute basis for boundaries $B_* = \operatorname{im} \partial_*$
- extend to basis for cycles $Z_* = \ker \partial_*$
- new (non-boundary) basis cycles generate quotient $Z_*/B_*$

# Homology by matrix reduction

Notation:

- $D$: boundary matrix (with $\mathbb{Z}_2$ coefficients)
- $R_i$: $i$th column of $R$

Matrix reduction algorithm (variant of Gaussian elimination):

- $R = D, V = I$
- while $\exists i < j$ with $\operatorname{pivot} R_i = \operatorname{pivot} R_j$
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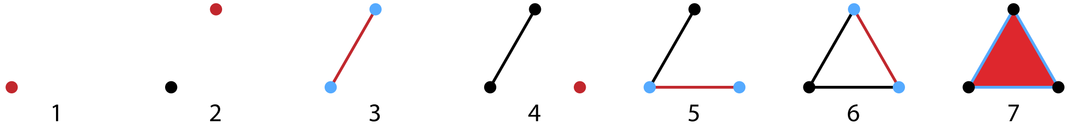
Result:

- $R = D \cdot V$ is reduced (each column has a unique pivot)
- $V$ is full rank upper triangular

# Matrix reduction




| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | | | 1 | | 1 | | |
| 2 | | | 1 | | | 1 | |
| 3 | | | | | | | 1 |
| 4 | | | | | 1 | 1 | |
| 5 | | | | | | | 1 |
| 6 | | | | | | | 1 |
| 7 | | | | | | | |

$R$

$= D \cdot$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | |
| 2 | | 1 | | | | | |
| 3 | | | 1 | | | | |
| 4 | | | | 1 | | | |
| 5 | | | | | 1 | | |
| 6 | | | | | | 1 | |
| 7 | | | | | | | 1 |

$V$

Algorithm:

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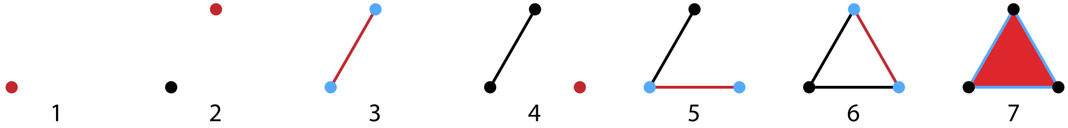
# Matrix reduction




$R$:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | | | 1 | | 1 | | |
| 2 | | | 1 | | | 1 | |
| 3 | | | | | | | 1 |
| 4 | | | | | 1 | 1 | |
| 5 | | | | | | | 1 |
| 6 | | | | | | | 1 |
| 7 | | | | | | | |

$= D \cdot$

$V$:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | |
| 2 | | 1 | | | | | |
| 3 | | | 1 | | | | |
| 4 | | | | 1 | | | |
| 5 | | | | | 1 | | |
| 6 | | | | | | 1 | |
| 7 | | | | | | | 1 |

Algorithm:

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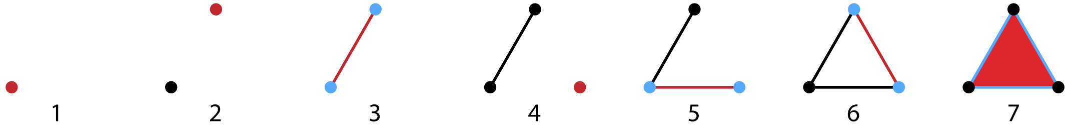
# Matrix reduction




$$\underbrace{\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & & & 1 & & 1 & & \\ 2 & & & 1 & & & 1 & \\ 3 & & & & & & & 1 \\ 4 & & & & & 1 & 1 & \\ 5 & & & & & & & 1 \\ 6 & & & & & & & 1 \\ 7 & & & & & & & \end{array}}_{R} = D \cdot \underbrace{\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & & & & & & \\ 2 & & 1 & & & & & \\ 3 & & & 1 & & & & \\ 4 & & & & 1 & & & \\ 5 & & & & & 1 & & \\ 6 & & & & & & 1 & \\ 7 & & & & & & & 1 \end{array}}_{V}$$

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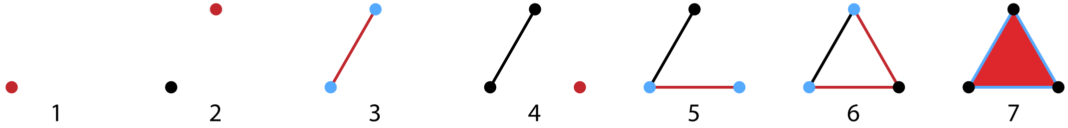


| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | | | 1 | | 1 | 1 | |
| 2 | | | 1 | | | 1 | |
| 3 | | | | | | | 1 |
| 4 | | | | | 1 | 0 | |
| 5 | | | | | | | 1 |
| 6 | | | | | | | 1 |
| 7 | | | | | | | |

$R$

$= D \cdot$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | |
| 2 | | 1 | | | | | |
| 3 | | | 1 | | | | |
| 4 | | | | 1 | | | |
| 5 | | | | | 1 | 1 | |
| 6 | | | | | | 1 | |
| 7 | | | | | | | 1 |

$V$

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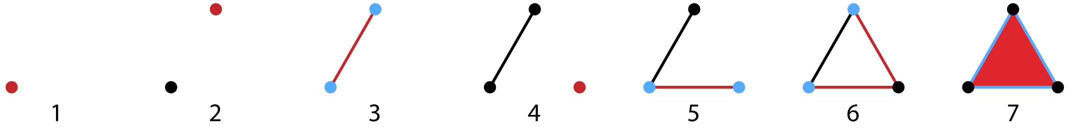
# Matrix reduction

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| 2 | | | 1 | | | 1 | |
| 3 | | | | | | | 1 |
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| 5 | | | | | | | 1 |
| 6 | | | | | | | 1 |
| 7 | | | | | | | |

$R$

$= D \cdot$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | |
| 2 | | 1 | | | | | |
| 3 | | | 1 | | | | |
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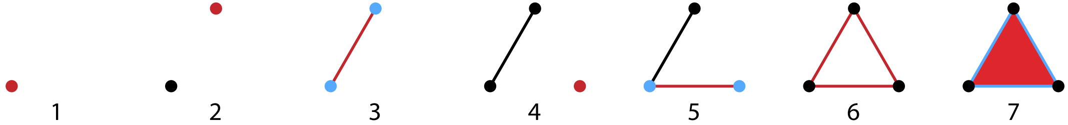


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| 2 | | | 1 | | | 0 | |
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$R$

$= D \cdot$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
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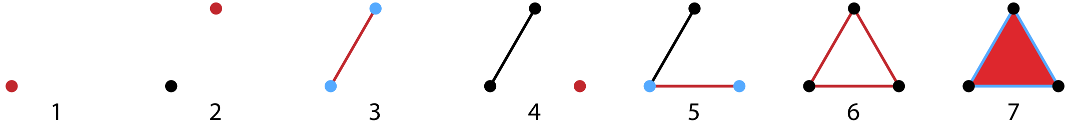


$R$:

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$= D \cdot$

$V$:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | |
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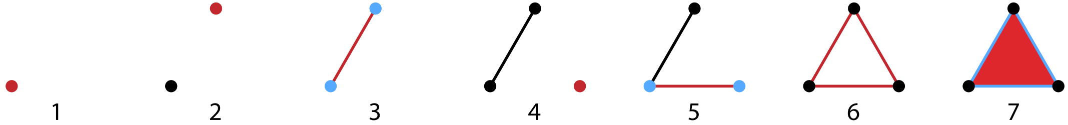


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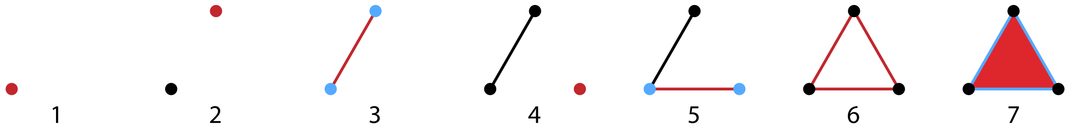
| 5 | | | | | 1 | 1 | |
| 6 | | | | | | 1 | |
| 7 | | | | | | | 1 |

$V$

Algorithm:

- while $\exists i < j$ with $\text{pivot}\, R_i = \text{pivot}\, R_j$
  - add $R_i$ to $R_j$, add $V_i$ to $V_j$

# Matrix reduction




$$\underbrace{\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & \color{red}{5} & 6 & 7 \\ \hline 1 & & & 1 & & 1 & & \\ 2 & & & 1 & & & & \\ 3 & & & & & & & 1 \\ 4 & & & & & \color{red}{1} & & \\ 5 & & & & & & & 1 \\ 6 & & & & & & & 1 \\ 7 & & & & & & & \end{array}}_{R} = D \cdot \underbrace{\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & & & & & & \\ 2 & & 1 & & & & & \\ 3 & & & 1 & & & 1 & \\ 4 & & & & 1 & & & \\ 5 & & & & & 1 & 1 & \\ 6 & & & & & & 1 & \\ 7 & & & & & & & 1 \end{array}}_{V}$$

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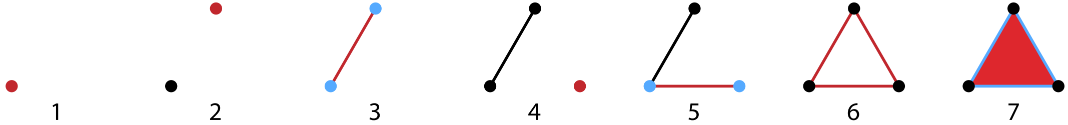


$$\underbrace{\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & \color{red}{5} & 6 & 7 \\ \hline 1 & & & 1 & & 1 & & \\ 2 & & & 1 & & & & \\ 3 & & & & & & & 1 \\ \color{red}{4} & & & & & \color{red}{1} & & \\ 5 & & & & & & & 1 \\ 6 & & & & & & & 1 \\ 7 & & & & & & & \end{array}}_{R} = D \cdot \underbrace{\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & & & & & & \\ 2 & & 1 & & & & & \\ 3 & & & 1 & & & 1 & \\ 4 & & & & 1 & & & \\ 5 & & & & & 1 & 1 & \\ 6 & & & & & & 1 & \\ 7 & & & & & & & 1 \end{array}}_{V}$$

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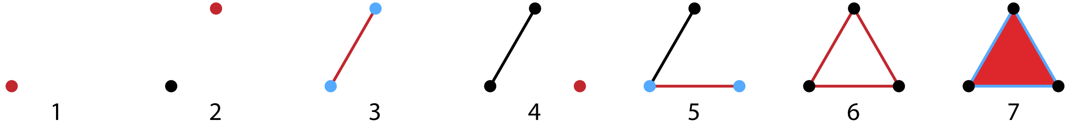


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# The four special ingredients of Ripser

The improved performance of Ripser is based on 4 insights:

- Compute cohomology [de Silva et al. 2011]
  - reduce transposed matrix
- Skip inessential columns [Chen, Kerber 2011]
  - many columns are redundant for homology
- Implicit boundary matrix
  - don't store the matrix $R = D \cdot V$ in memory
- Apparent pairs
  - find column pivot without constructing entire column

# Sending persistence into Hilbert space

# Extending the TDA pipeline

Mapping barcodes into a Hilbert space?

- desirable for (kernel-based) machine learning methods and statistics
- stability (Lipschitz continuity): important for reliable predictions
- inverse stability (bi-Lipschitz): avoid loss of information

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Can we hope for something better?

# No bi-Lipschitz feature maps for persistence

### Theorem (B, Carrière 2018)

*There is no bi-Lipschitz map from the persistence diagrams*
*(with the interleaving or any* $p$*–Wasserstein distance)*
*into any finite-dimensional Hilbert space,*
*even when restricting to bounded range or number of bars.*

# No bi-Lipschitz feature maps for persistence

## Theorem (B, Carrière 2018)

*There is no bi-Lipschitz map from the persistence diagrams (with the interleaving or any* $p$*–Wasserstein distance) into any finite-dimensional Hilbert space, even when restricting to bounded range or number of bars.*

## Theorem (B, Carrière 2018)

*If there was such a bi-Lipschitz map into some Hilbert space, the ratio of the Lipschitz constants would have to go to* $\infty$ *together with the bounds on number or range of bars.*

# History

# When was persistent homology invented?

- [Edelsbrunner/Letscher/Zomorodian 2000]

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- [Leray 1946]?

# When was persistent homology invented first?

When was persistent homology invented first?

ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

# RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

## 1. Introduction.

The analysis of functions $F$ on metric spaces $M$ of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative $k$-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of $k$-limits suitably counted is called the $k^{\text{th}}$ type number $m_k$ of $F$. The theory seeks to establish relations between the numbers $m_k$ and the connectivities $p_k$ of $M$. The numbers $p_k$ are finite in the most important applications. It is otherwise with the

# When was persistent homology invented first?

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## Rank and span in functional topology

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### Exact homomorphism sequences in homology theory
ed.ac.uk **[PDF]**

JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR

The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] **...**

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### Marston Morse and his mathematical works
ams.org **[PDF]**

R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org

American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to **...**

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### Unstable minimal surfaces of higher topological structure

# When was persistent homology invented first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

## MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

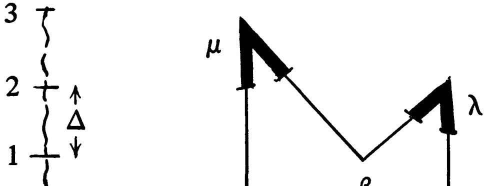
**1. Introduction.** Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

# When was persistent homology invented first?

*inequalities pertain between the dimensions of the $A_i$ and those of $H(A_i)$.* Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels [**D**] in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers [**36**] and [**44**] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on $M$.

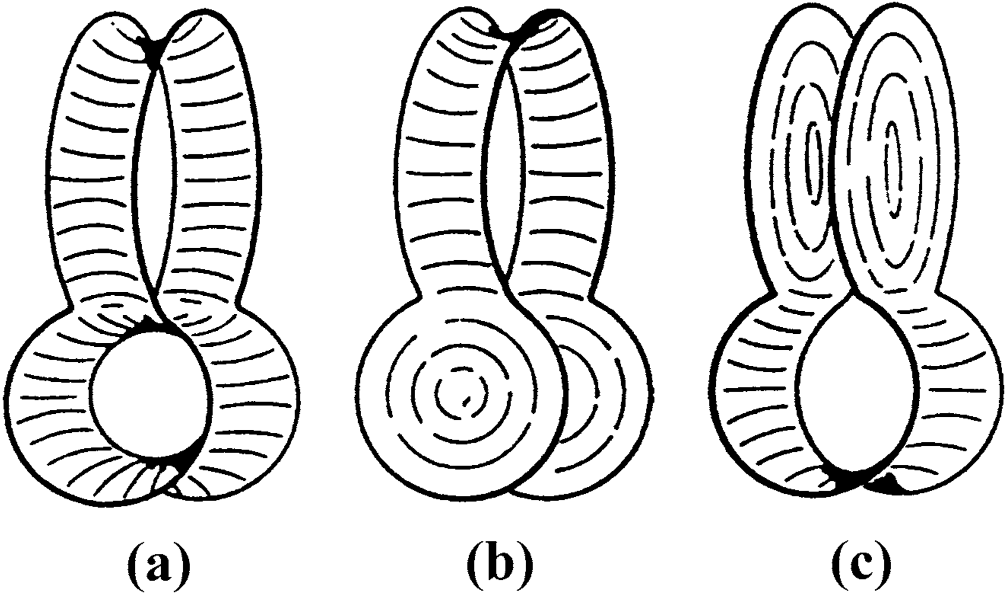


# Morse's *functional topology*

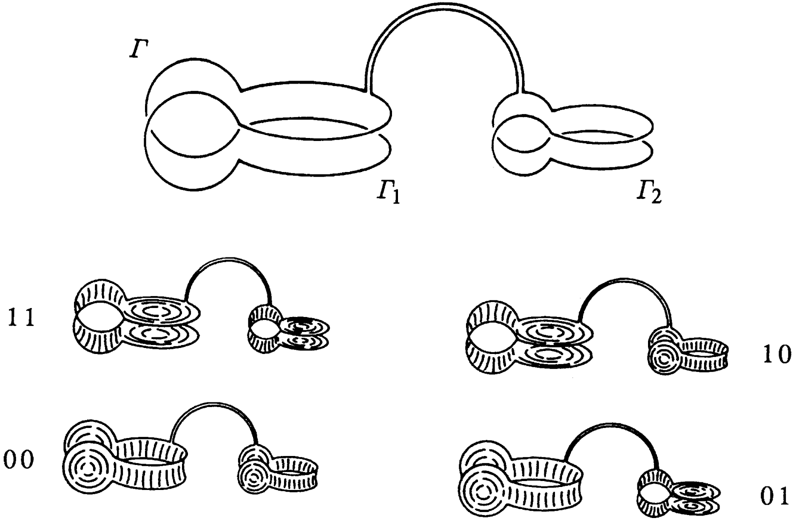
Key aspects:

- early precursor of persistence and spectral sequences
- uses Vietoris homology with field coefficients
- applies to a broad class of functions on metric spaces (not necessarily continuous)
- inclusions of sublevel sets have finite rank homology (*q-tame* persistent homology)
- focus on controlled behavior in pathological cases

# Motivation and application: minimal surfaces

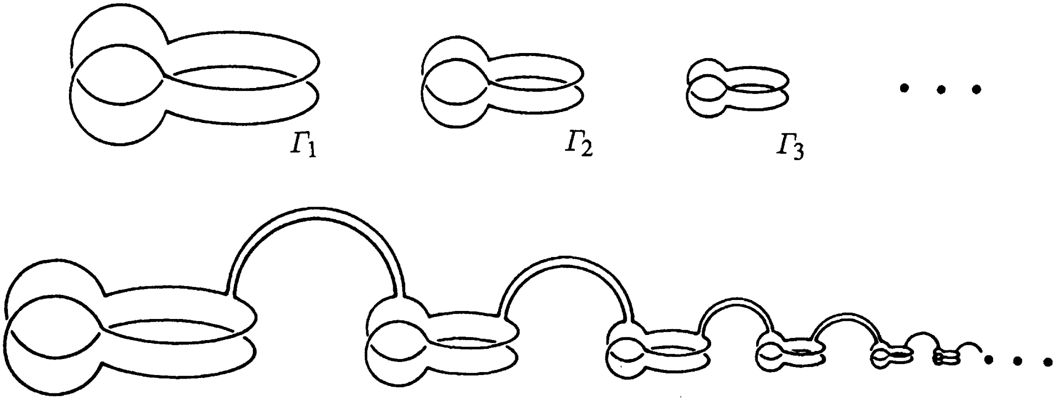


# Motivation and application: minimal surfaces




(from Dierkes et al.: Minimal Surfaces, Springer 2010)

# Motivation and application: minimal surfaces




(from Dierkes et al.: Minimal Surfaces, Springer 2010)

# Existence of unstable minimal surfaces

Using persistent homology:

- Number of $\epsilon$-persistent critical points (minimal surfaces) is finite for any $\epsilon > 0$
- Morse inequalities for $\epsilon$-persistent critical points

## Theorem (Morse, Tompkins 1939)

*There is a $C_1$ curve bounding an unstable minimal surface (an index $1$ critical point of the area functional).*

# Thanks for your attention!

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