

The Morse theory of Čech and Delaunay complexes

Ulrich Bauer

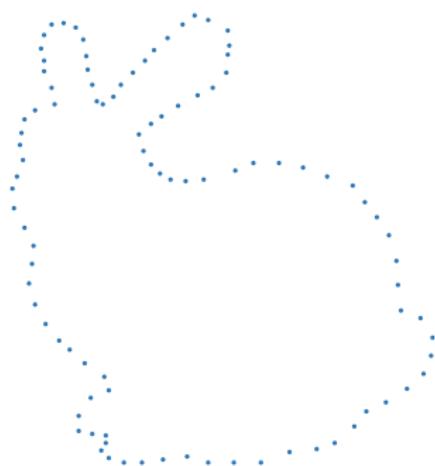
TUM

April 27, 2017

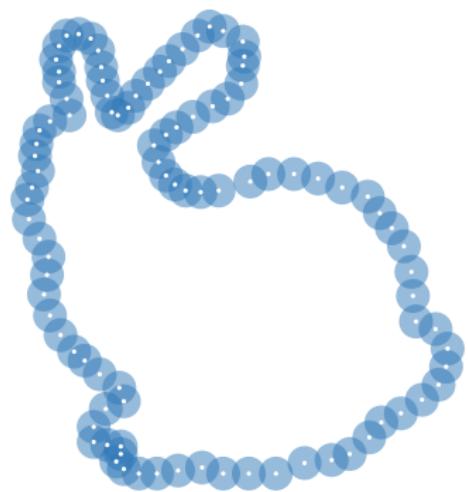
Oberwolfach workshop
Computational geometry

Joint work with Herbert Edelsbrunner (IST Austria)

Connect the dots: topology from geometry



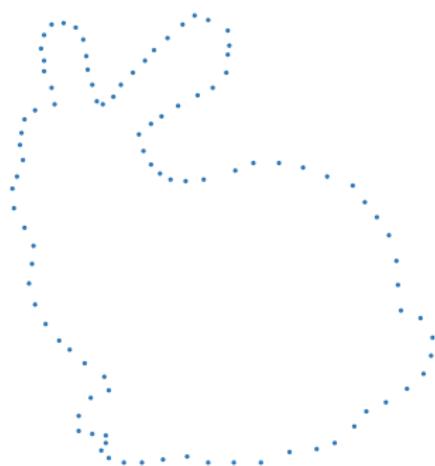
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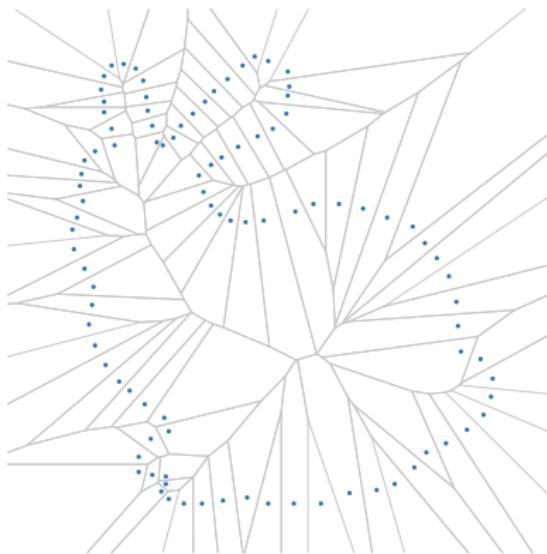
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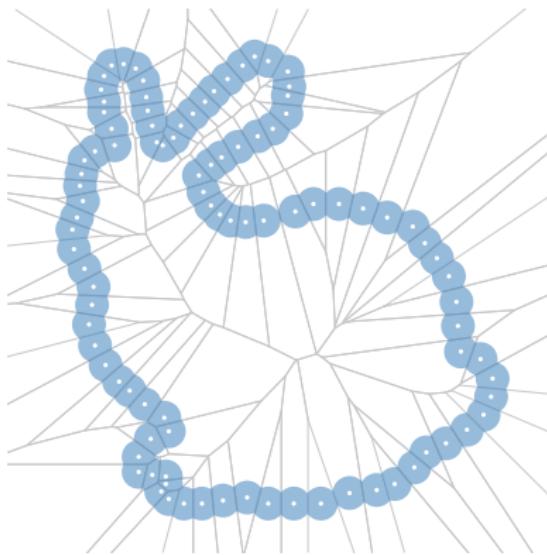
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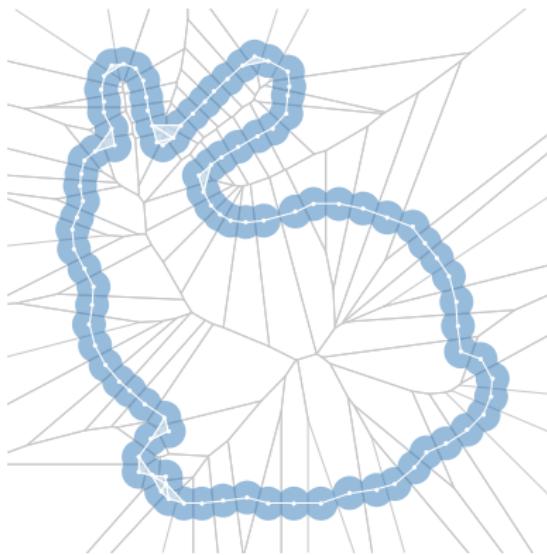
Connect the dots: topology from geometry



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Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)



Čech and Delaunay functions

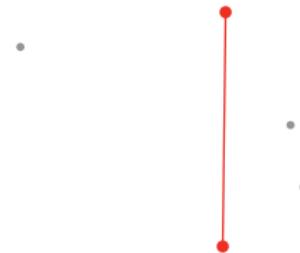
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(abstract simplices)

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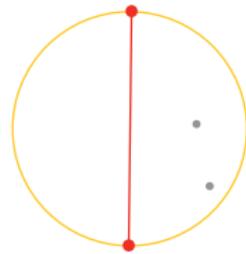
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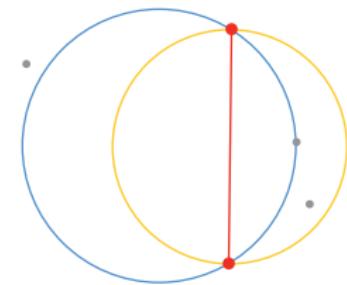
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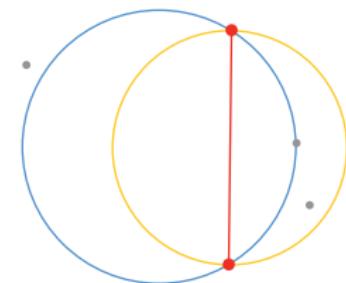
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Čech and Delaunay complexes from functions

Define for any radius r :

Čech and Delaunay complexes from functions

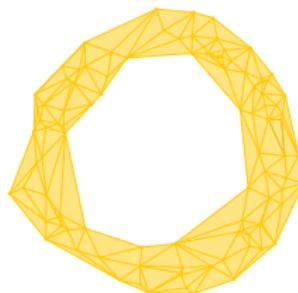
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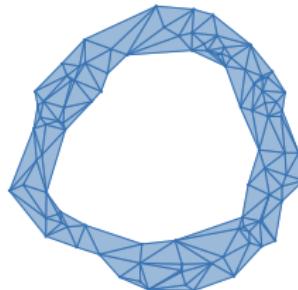
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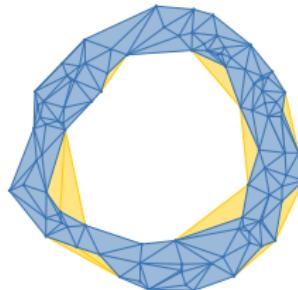
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Properties of Čech and Delaunay complexes

By the *Nerve theorem* (Borsuk 1947):

$$\text{Del}_r(X) \simeq \text{Cech}_r(X) \simeq B_r(X).$$

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- Are all three complexes homotopy equivalent?
- Ca?

Main result: a sequence of collapses

Theorem (B, Edelsbrunner 2014/17, SoCG/Trans. AMS)

$\check{\text{C}}\text{ech}$, Delaunay– $\check{\text{C}}\text{ech}$, Delaunay, and Wrap complexes are homotopy equivalent through a sequence of collapses

$$\text{Cech}_r X \searrow \text{Cech}_r X \cap \text{Del} X \searrow \text{Del}_r X \searrow \text{Wrap}_r X.$$



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- Explicit chain maps inducing isomorphisms in homology
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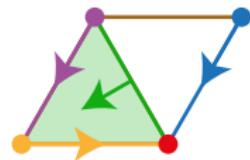
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Discrete Morse theory

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a simplicial complex
is a partition of the simplices
into singletons and pairs $\{L, U\}$,
where L is a face of U with codimension 1.

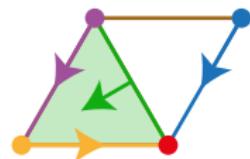


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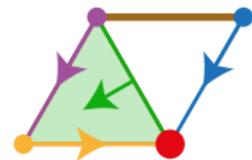
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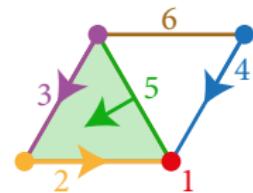
The singletons are called *critical simplices*.

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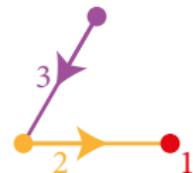


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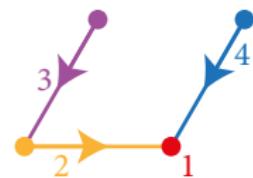


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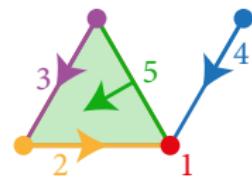


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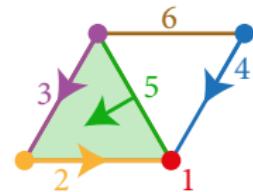


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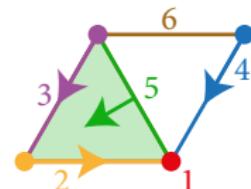


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- the level sets $f^{-1}(t)$ form a discrete vector field (the *discrete gradient* of f)

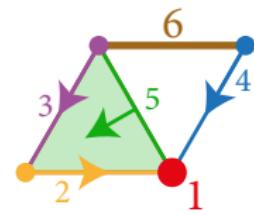


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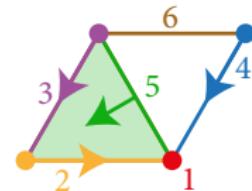
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If $f^{-1}(t) = \{Q\}$ then t is a *critical value*.

Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K .

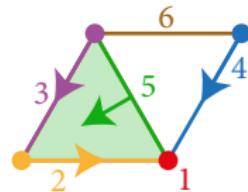


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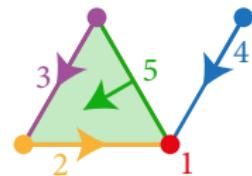


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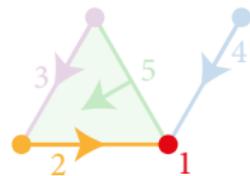


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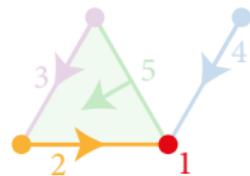


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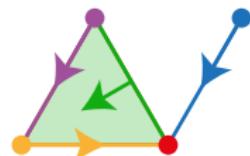
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Let V be a discrete gradient field on a simplicial complex K ,
and let L be a subcomplex of K .

Corollary

*If $K \setminus L$ is the union of some pairs of V ,
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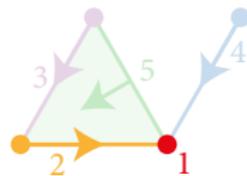


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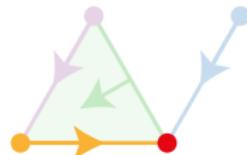
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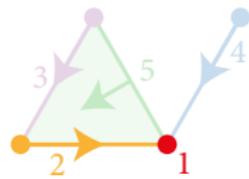


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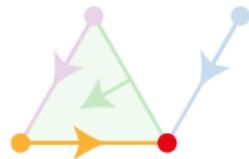


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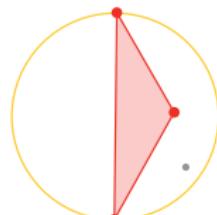
We say that V induces the collapse $K \searrow L$.



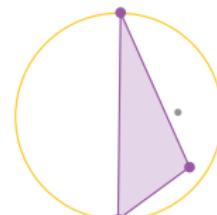
Unfortunately...

Neither the Čech nor the Delaunay functions
are discrete Morse functions!

- Example: two simplices Q, Q' with $f_C(Q) = f_C(Q')$
such that neither is a face of the other:



Q



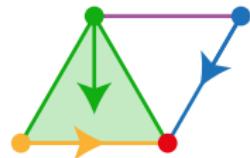
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Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}$$

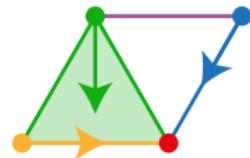


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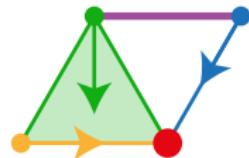
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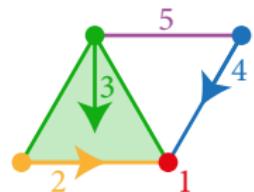
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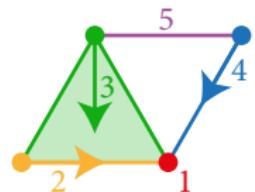


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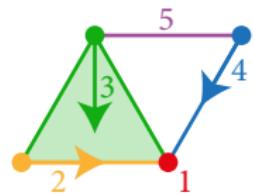


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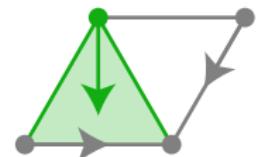
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Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:

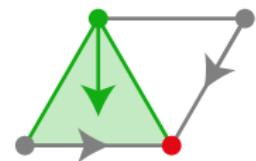


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For each non-critical interval $[L, U] \in V$:

- choose an arbitrary vertex $x \in U \setminus L$

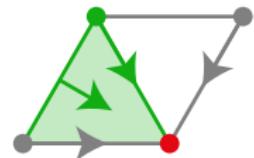


Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:

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- partition $[L, U]$ into pairs $(Q \setminus \{x\}, Q \cup \{x\})$ for all $Q \in [L, U]$.



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Therefore the collapsing theorems also hold for generalized discrete Morse functions.

Čech and Delaunay intervals

Morse theory of Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

*The Čech function and the Delaunay function
are generalized discrete Morse functions.*

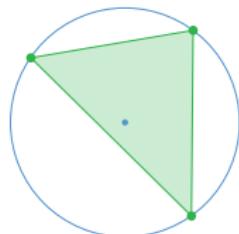
Morse theory of Čech and Delaunay complexes

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The following are equivalent for each simplex $Q \subseteq X$:

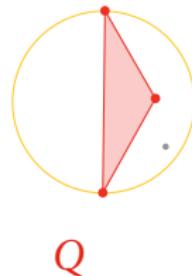
- *Q is a critical simplex of f_C*
- *Q is a critical simplex of f_D*
- $f_D(Q) = f_C(Q)$
- *Q is a centered Delaunay simplex
(containing the circumcenter in its interior)*



Čech intervals

Lemma

Let $Q \subseteq X$ be a simplex with smallest enclosing sphere S .
Then $Q' \subseteq X$ has the same smallest enclosing sphere S iff

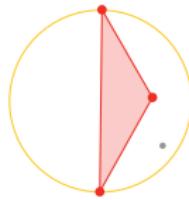


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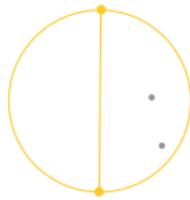
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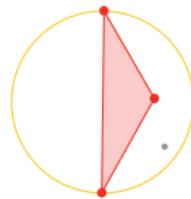
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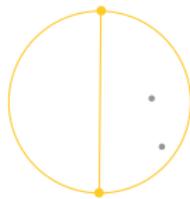
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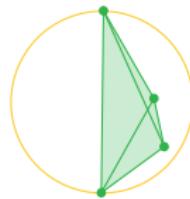
$$\text{On } S \subseteq Q' \subseteq \text{Encl } S.$$



Q



$\text{On } S$



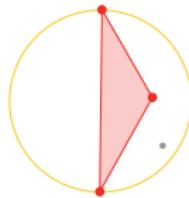
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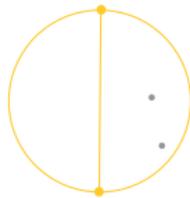
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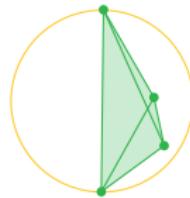
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On S



Encl S

The front and back faces of a simplex

Let S be the smallest circumsphere of a simplex On $S \subseteq X$.

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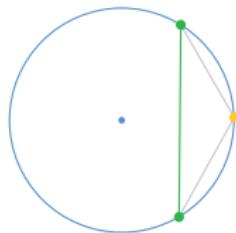
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$$\text{Front } S = \{x \in \text{On } S \mid \mu_x > 0\},$$

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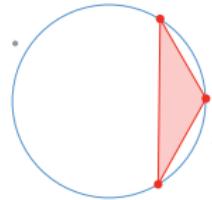


Delaunay intervals

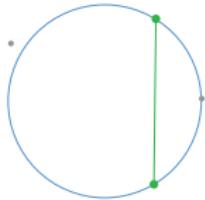
Lemma

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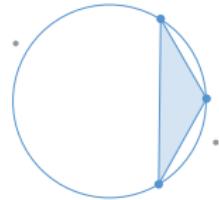
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Q



$\text{Front } S$



$\text{On } S$

Selective Delaunay complexes

Sphere optimization problems

Both Čech and Delaunay functions are defined using the smallest sphere S satisfying certain constraints:

minimize
 r,z

r

subject to

$$\begin{aligned}\|z - q\| &\leq r, & q \in Q, \\ \|z - e\| &\geq r, & e \in E.\end{aligned}$$

Here r is the radius of the sphere S , and z is the center of S .

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- Čech function: choose $E = \emptyset$
- Delaunay function: choose $E = X$

Čech and Delaunay intervals from KKT

The *Karush–Kuhn–Tucker* optimality conditions yield:

Proposition

A sphere S enclosing Q and excluding E is minimal iff

- S is the smallest circumsphere of Q ,
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The $\check{\text{C}}\text{ech}$ intervals are of the form $[\text{On } S, \text{Encl } S]$.

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- *Selective Delaunay function* $f_E(Q)$:
radius of smallest sphere enclosing Q and excluding E

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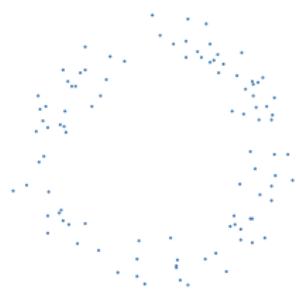
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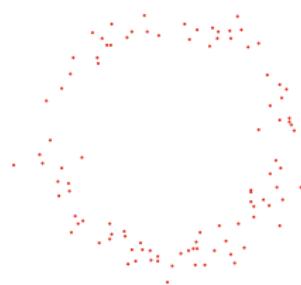
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Connecting different Delaunay complexes

X

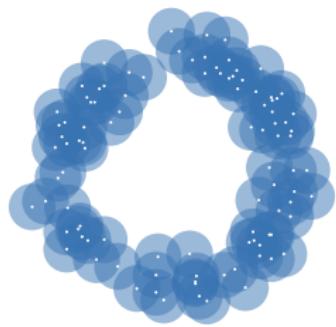


Y



Connecting different Delaunay complexes

$B_r(X)$

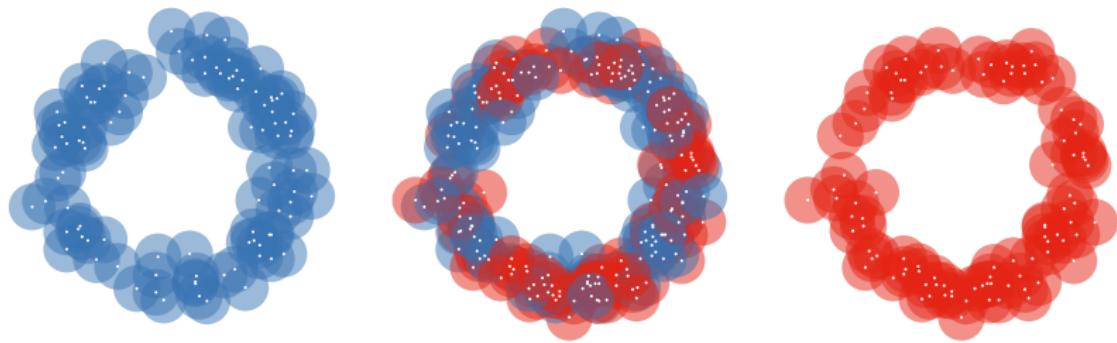


$B_r(Y)$



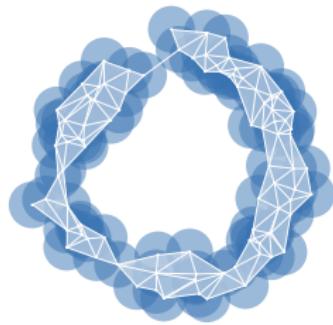
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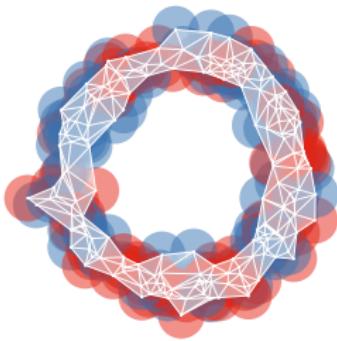


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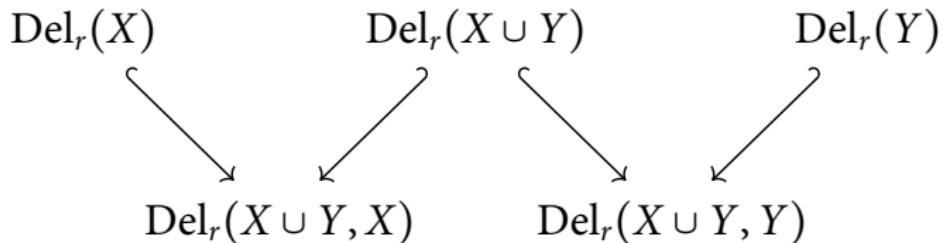
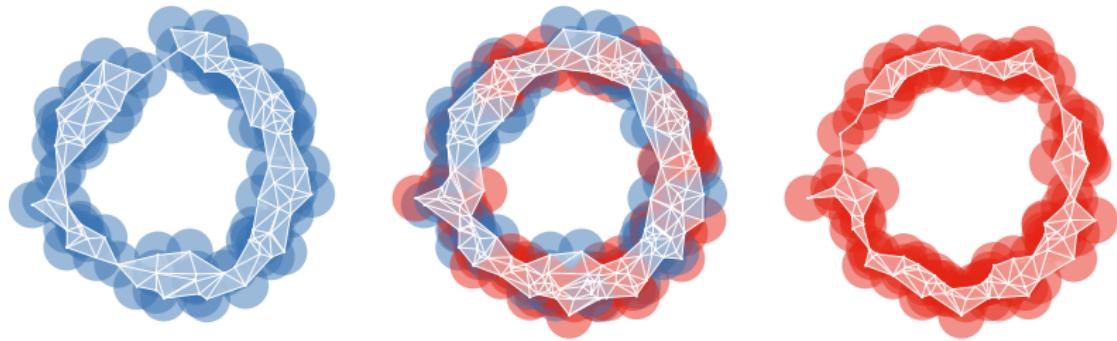
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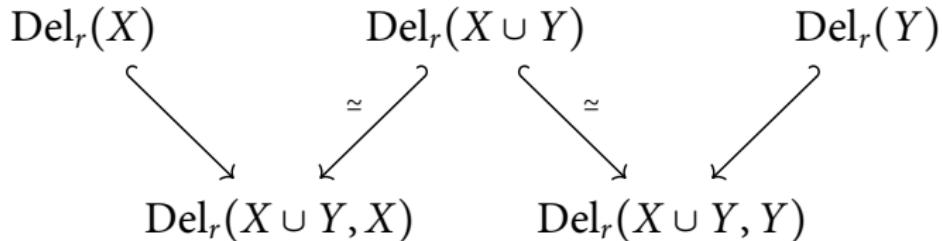
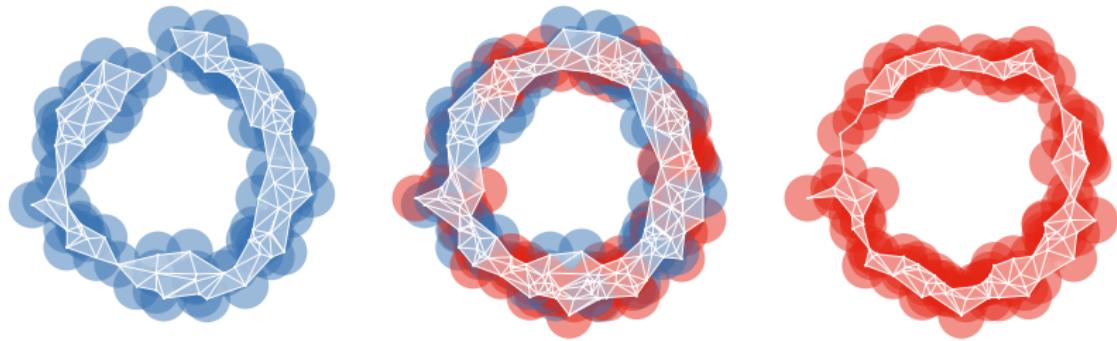
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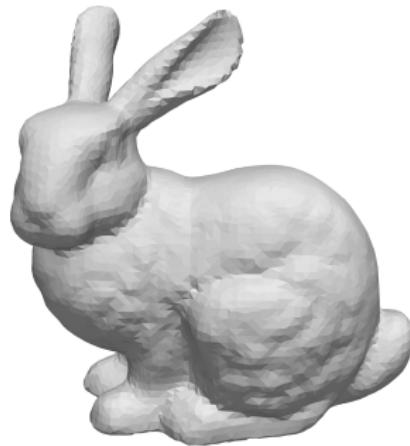
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Wrap complexes

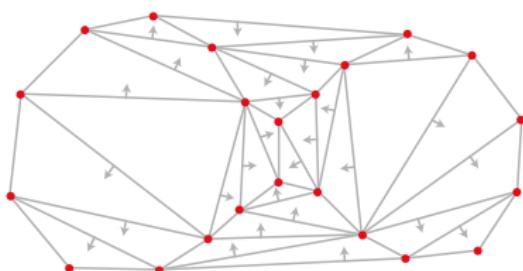
Wrap complexes

Generalizes and greatly simplifies the surface reconstruction algorithm *Wrap* (Edelsbrunner 1995, Geomagic)



Wrap complexes

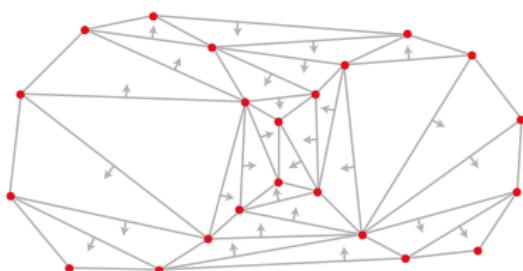
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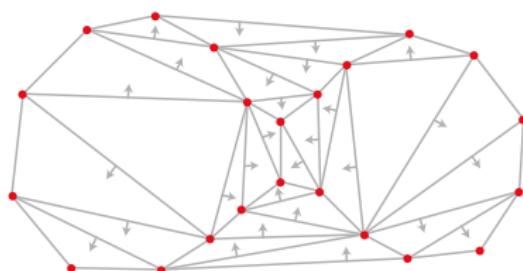
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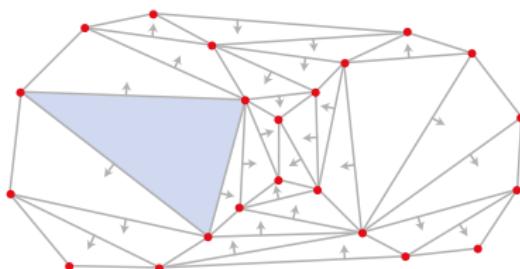
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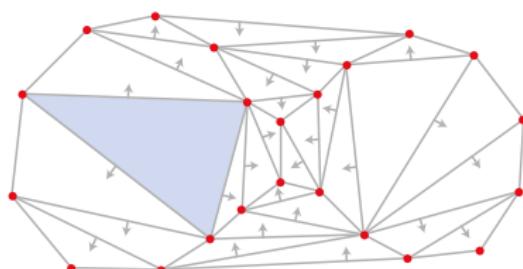
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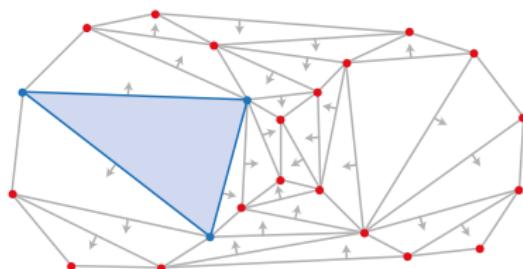
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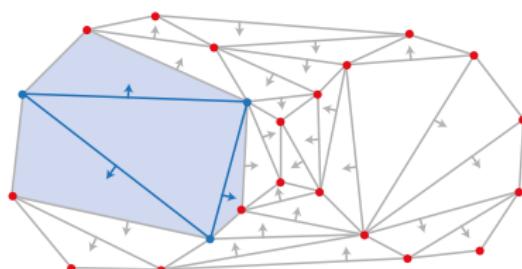
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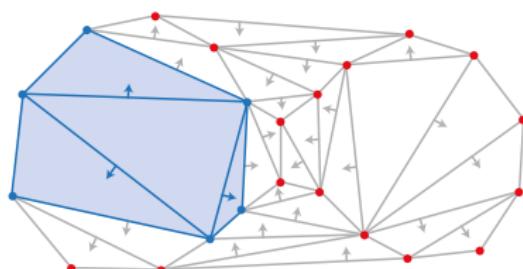
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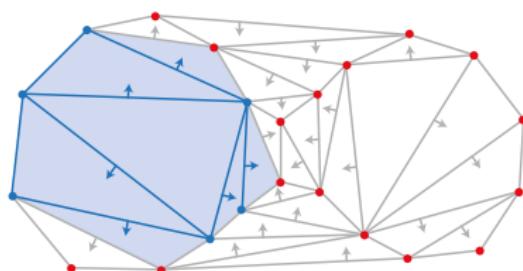
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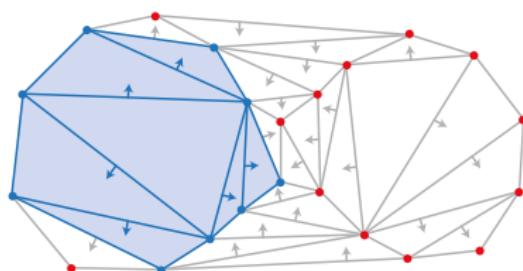
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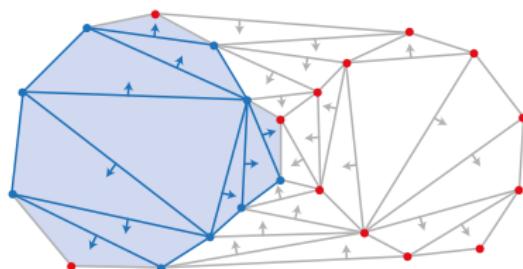
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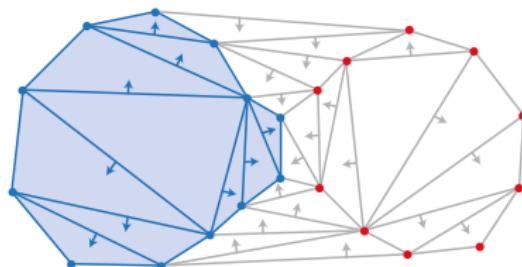
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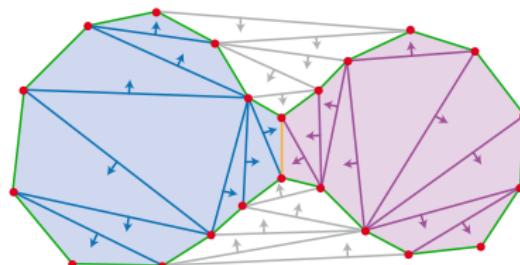
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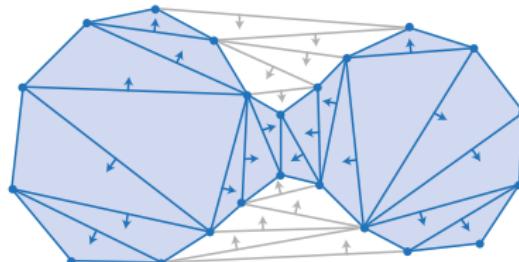
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The Delaunay intervals induce a collapse $\text{Del}_r \searrow \text{Wrap}_r$.

Wrapping up

- Čech and Delaunay complexes from Morse functions
- Explicit homotopy equivalence by simplicial collapses
- Simple definition and generalization of *Wrap* complexes