Persistence diagrams as diagrams

A Categorification of the Stability Theorem

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TUM

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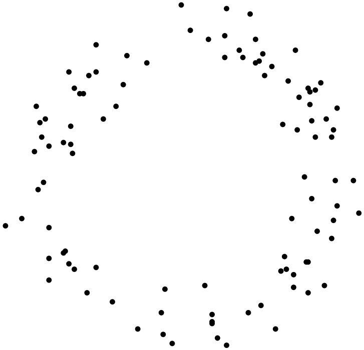
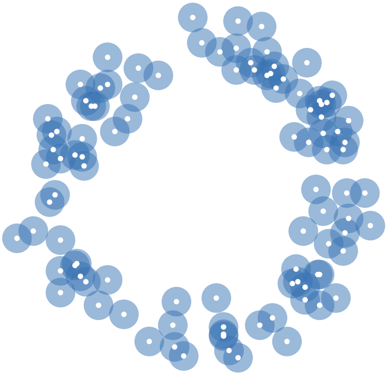
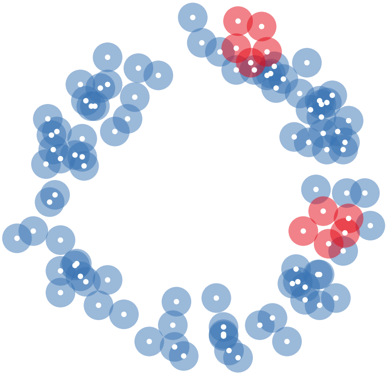
Workshop on Geometry, Topology, and Computation

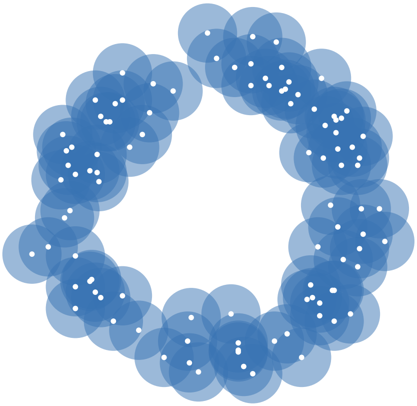
Mathematikon, Heidelberg University

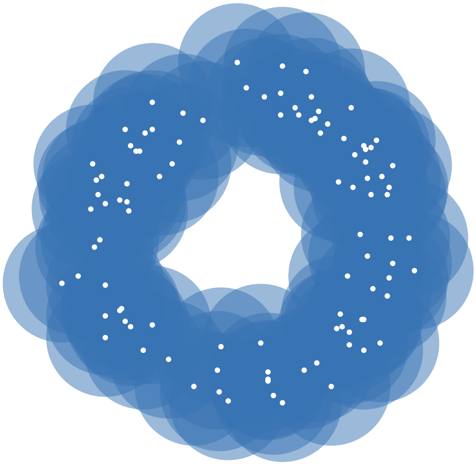
Joint work with Michael Lesnick (Albany)

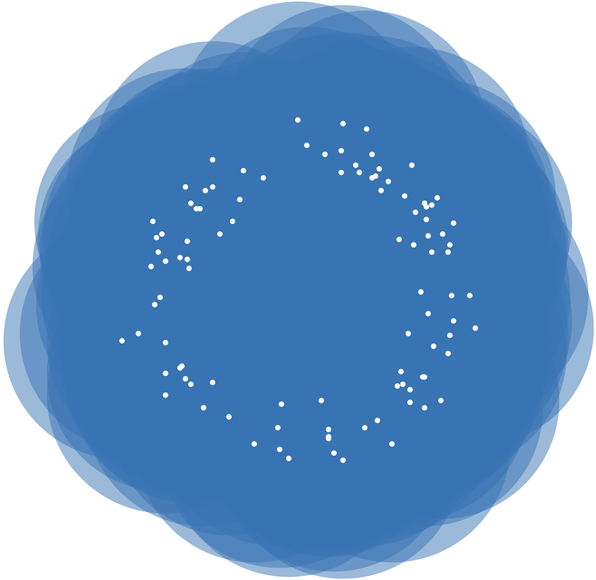




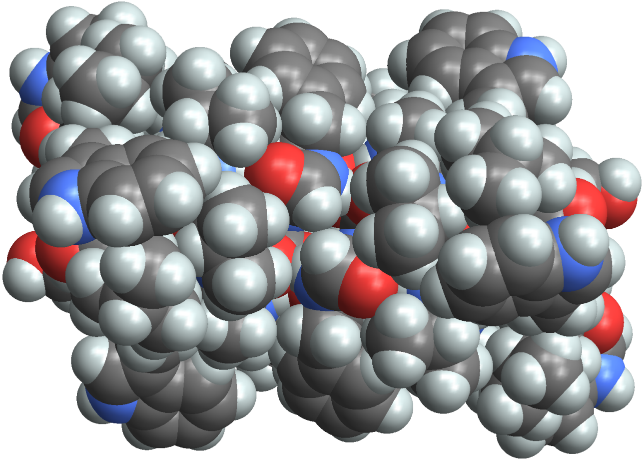








Gramicidin: an antibiotic functioning as ion channel



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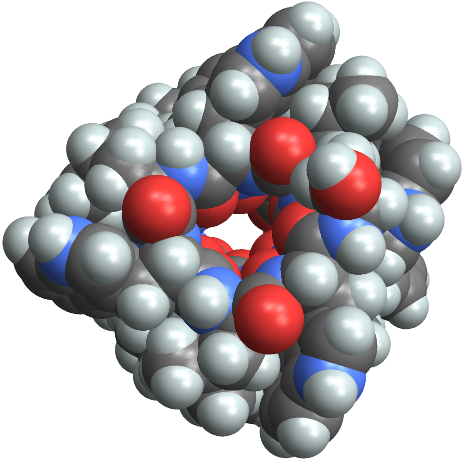

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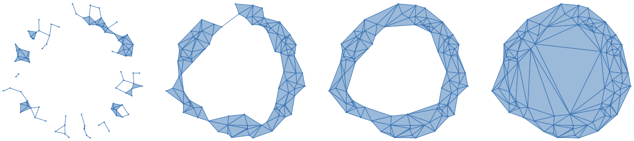

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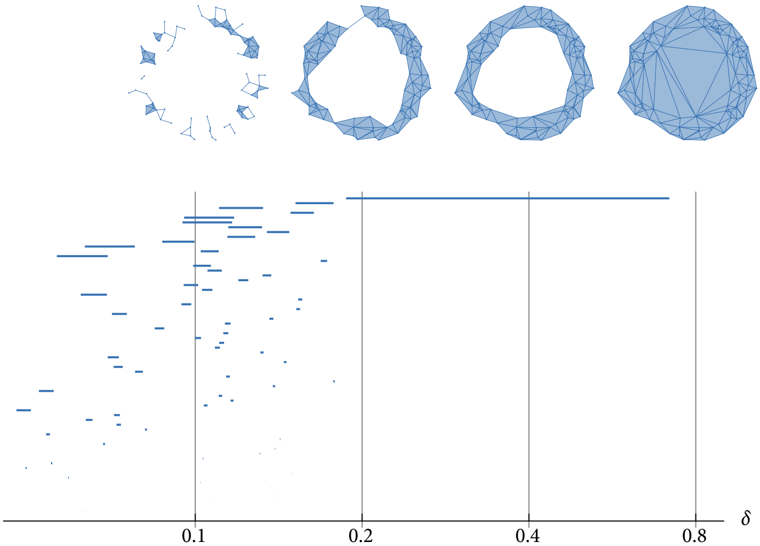

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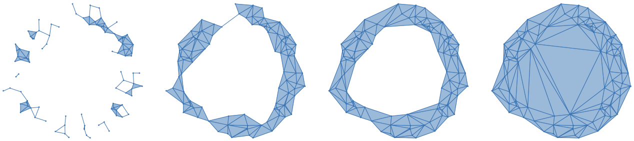

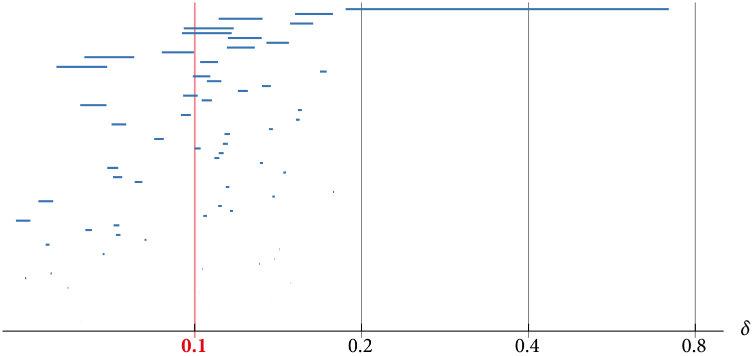

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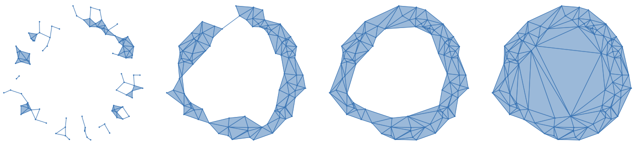

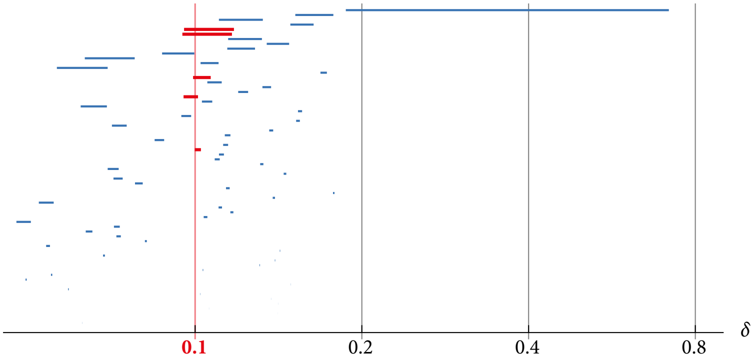

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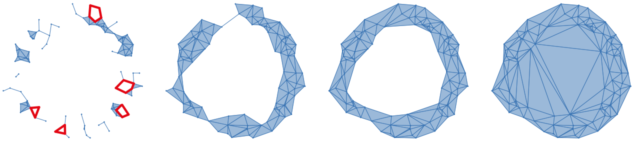

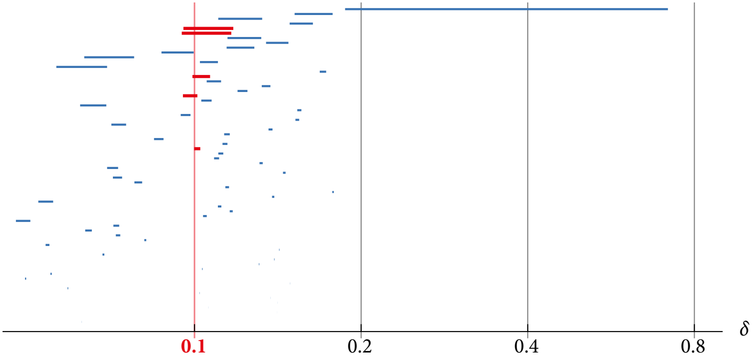

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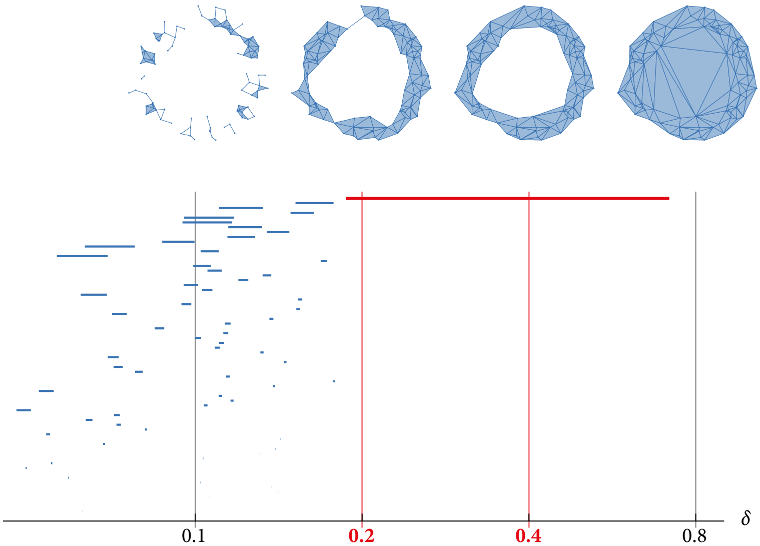

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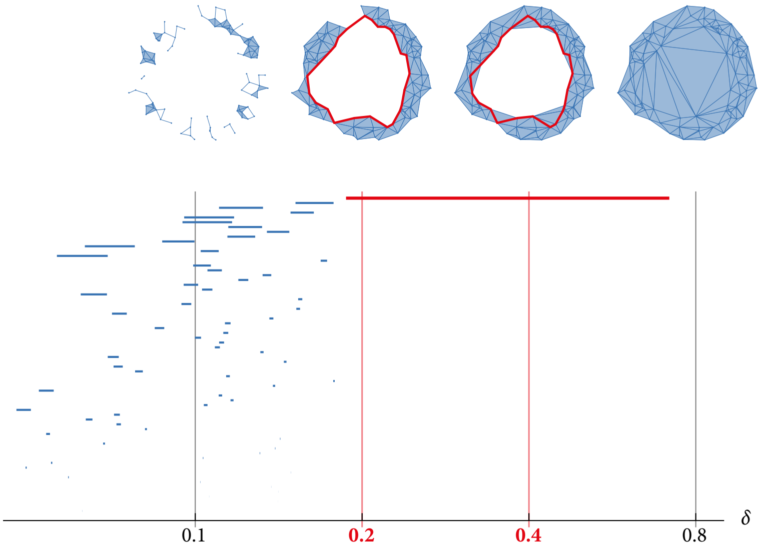

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In this talk, all vector spaces will be finite dimensional.

Barcodes: the structure of persistence modules

Theorem (Crawley-Boevey 2015)

Any persistence module $M : \mathbf{R} \to \mathbf{vect}$ *(of finite dim. vector spaces over some field* $\mathbb{F}$*) decomposes as a direct sum of* interval modules

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- The points in the *persistence diagram* are the endpoints of the intervals in the barcode.
- This is not a diagram in the sense of category theory (functor)!

The many faces of persistence

- Persistence diagrams: multiset of points $(b, d) \in \overline{\mathbb{R}}^2 : b \leq d$
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- Matching diagrams: sequence of partial bijections
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Persistence and stability: the big picture

point cloud	$P \subset \mathbb{R}^d$	Hausdorff distance

Persistence and stability: the big picture

$$\begin{array}{ccc} \text{point cloud} & P \subset \mathbb{R}^d & \text{Hausdorff distance} \\ \big\downarrow \text{distance} & & \\ \text{function} & f : \mathbb{R}^d \to \mathbb{R} & L_\infty \text{ distance} \end{array}$$

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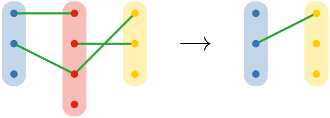
$\downarrow$ decomposition		
intervals (persistence barcode)	$B : \mathbf{R} \to \mathbf{Mch}$	Bottleneck/interleaving distance

The category of matchings

Consider the category **Mch** (a subcategory of the category **Rel** of sets and relations) with

- objects: sets,
- morphisms: matchings (partial bijections).

Composition:



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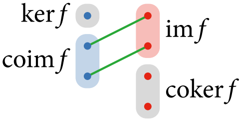

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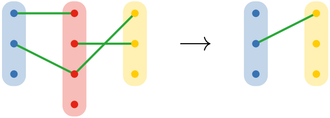

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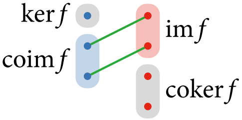

$\mathbf{Mch}$ is *Puppe-exact (p-exact)*:

- it has a zero object ($\varnothing$)
- it has all (co)kernels
- every mono (epi) is (co)kernel
- every morphism $f : A \to B$ has an epi-mono factorization $A \twoheadrightarrow \operatorname{im} f \hookrightarrow B$

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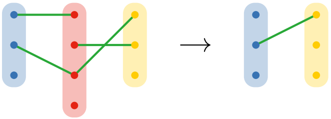

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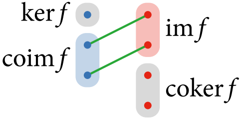

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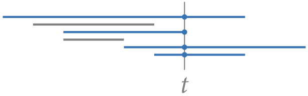
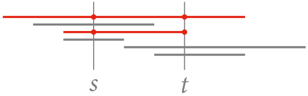
but not additive:

- it does not have all (co)products

From barcodes to matching diagrams (and back)

- A barcode (collection of intervals) can be read as a diagram $\mathbf{R} \to \mathbf{Mch}$:





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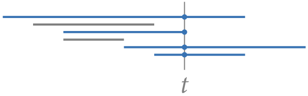

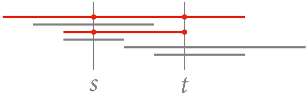

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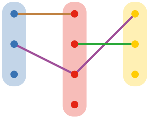

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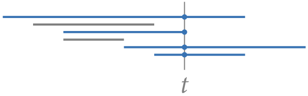

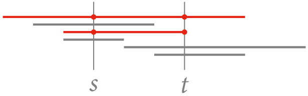

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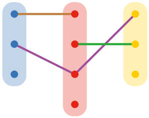

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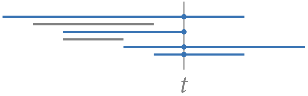

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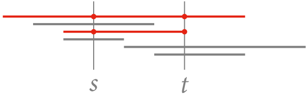

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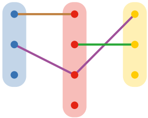

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Turn this into an equivalence of categories $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbf{R}}$

A category of barcodes

Proposition

The functor category $\mathbf{Mch}^{\mathbf{R}}$ *is equivalent to* $\mathbf{Barc}$, *the category with*

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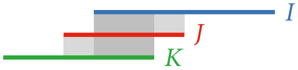
- *composition of overlap matchings:* $\tau \bullet \sigma = \{(I, K) \in \tau \circ \sigma \mid I \text{ overlaps } K \text{ above}\}$
  (where $\tau \circ \sigma$ *is the standard composition of matchings)*



$(I, K) \in \tau \bullet \sigma$ *(overlap)*

$(I, K) \notin \tau \bullet \sigma$ *(no overlap)*

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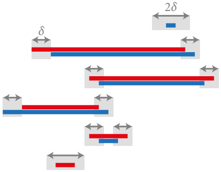
Bottleneck distance: $d_B(U, V) = \inf\{\delta \mid \exists \delta\text{-matching } U \nrightarrow V\}$



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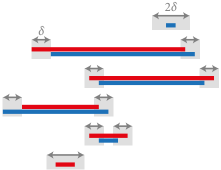

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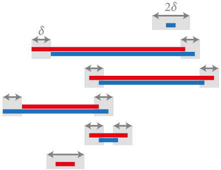

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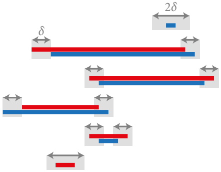

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Proposition

$d_I = d_B$ *(using the equivalence* $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbf{R}}$).

Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2012; B, Lesnick 2015)

If two persistence modules are δ-interleaved,
then there exists a δ-matching of their barcodes:

- matched intervals have endpoints within distance $\leq \delta$,
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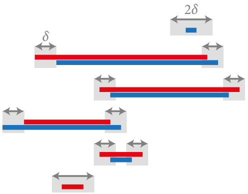

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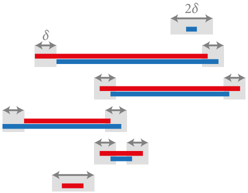

- matched intervals have endpoints within distance $\leq \delta$,
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Equivalently: there exists a δ-interleaving of their barcodes (as diagrams $\mathbf{R} \to \mathbf{Mch}$).

Non-functoriality of persistence barcodes

Can a persistence module M be mapped to its barcode $B(M)$ by a functor $B : \mathbf{vect} \to \mathbf{Mch}$?

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There is no functor $\mathbf{vect} \to \mathbf{Mch}$ *sending every vector space* V *to a set of cardinality* $\dim V$ *(equivalently: sending a linear map* f *to a matching of cardinality* $\operatorname{rank} f$*).*

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But: there is a barcode functor for subcategories of monos/epis of persistence modules $\mathbf{vect}^{\mathbf{R}}$:

Structure of persistence sub-/quotient modules

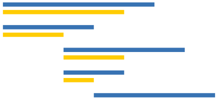
Proposition

Let $M \twoheadrightarrow N$ be an epimorphism.

Then there is an injection of barcodes $B(N) \hookrightarrow B(M)$ such that if J is mapped to I, then

- *I and J are aligned below, and*
- *I bounds J above.*

This construction is functorial.



Dually, there is an injection $B(M) \hookrightarrow B(N)$ for monomorphisms $M \hookrightarrow N$.

Persistence sub-/quotient modules and their matching diagrams

Structure of persistence sub-/quotient modules, rephrased for $\mathbf{Mch}^{\mathbf{R}}$:

Proposition

There is a functor from epimorphisms of persistence modules to epimorphisms of matching diagrams.

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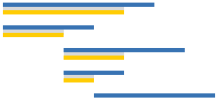

Algebraic stability via induced matchings

Consider an interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(\forall t \in \mathbf{R})$:

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Induced δ-matching of barcodes:

$B(N)$

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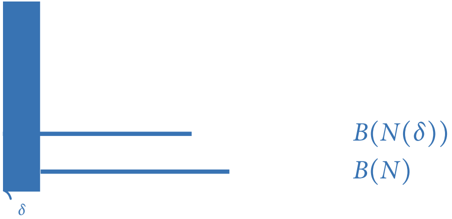

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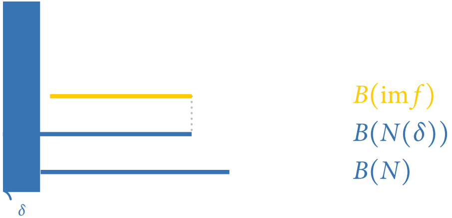

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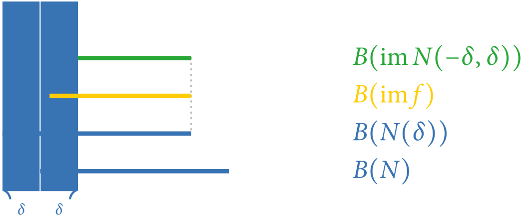

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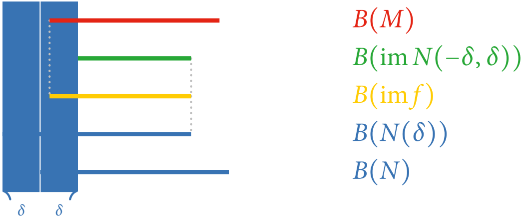

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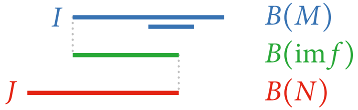

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Theorem

Assume that $\ker f$ *is* δ*-trivial. If* $\chi(f)$ *matches* I *to* J*, then*

(i) *I overlaps J, and J overlaps $I(\delta)$.*

(ii) *Any unmatched interval of $B(M)$ is δ-trivial.*

There is a dual statement for $\operatorname{coker} f$ *δ-trivial.*

The categorified induced matching theorem

Induced matching theorem, rephrased in $\mathbf{Mch}^{\mathbf{R}}$:

Theorem

If $f : M \to N$ has δ-trivial (co)kernel, then so does the induced matching $\chi(f)\colon B(M) \nrightarrow B(N)$.

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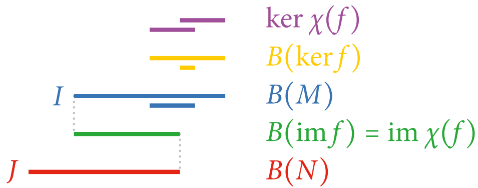

Note:

- We always have $B(\operatorname{im} f) = \operatorname{im} \chi(f)$ by construction.
- But $\ker \chi(f)$ may differ from $B(\ker f)$.
- The induced matching may strictly decrease the triviality of the kernel.

A general criterion for δ-trivial (co)kernels

Lemma

Consider a morphism $f : M \to N$ between diagrams $M, N : \mathbf{R} \to \mathcal{A}$ in a Puppe-exact category $\mathcal{A}$, and let $s : N(-\delta) \to N$ be the internal shift morphism. The following are equivalent:

(i) $\operatorname{coker} f$ *is δ-trivial;*

(ii) *the image monomorphism $\operatorname{im} s \hookrightarrow N$ factors through the image monomorphism $\operatorname{im} f \hookrightarrow N$ as*

$$\begin{tikzcd}
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A dual statement holds for $\ker f$.

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Theorem

Two pfd persistence modules M *and* N *are* δ*-interleaved if and only if their barcodes* $B(M)$ *and* $B(N)$ *are* δ*-interleaved. In particular,* $d_I(M,N) = d_I(B(M),B(N))$.

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Converse direction:

- Apply the free functor $\mathbf{Mch} \to \mathbf{Vect}$. $\square$

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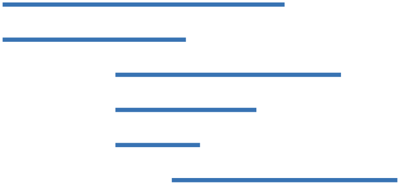

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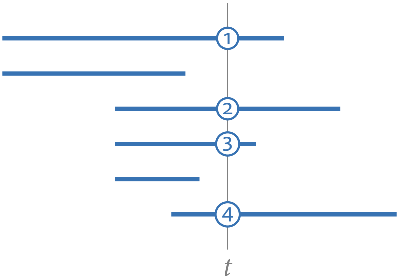

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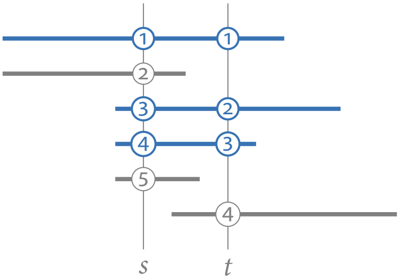

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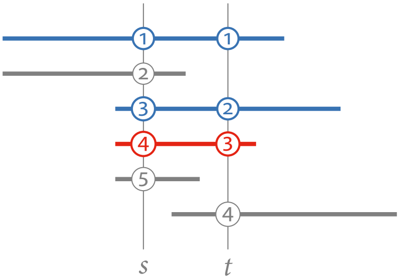

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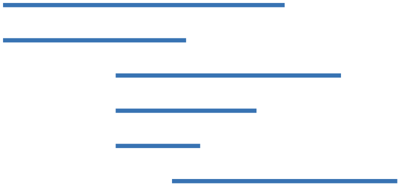
A rank formula for barcodes

M: persistence module, D: corresponding barcode (ordered by birth, then death).



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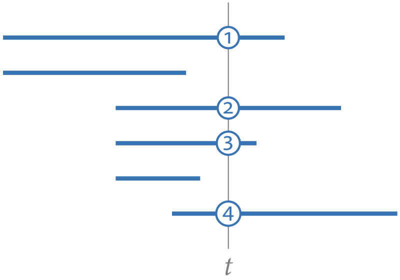
- Which j are in D_t? This is $\{j \mid j \leq \dim M_t\}$.



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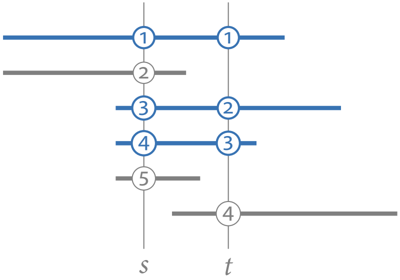

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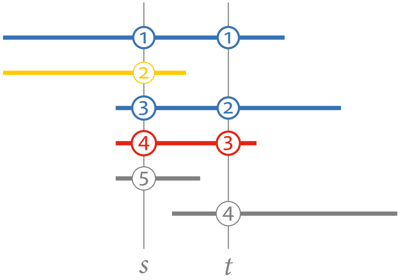

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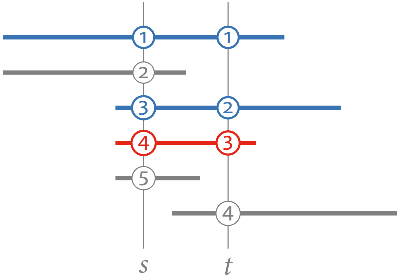

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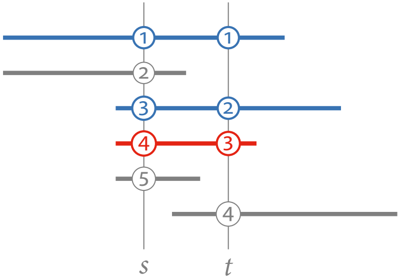

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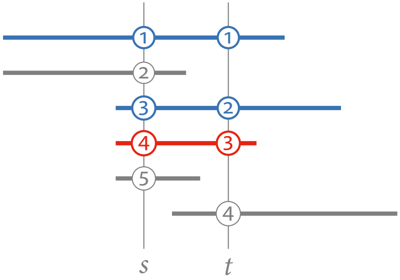
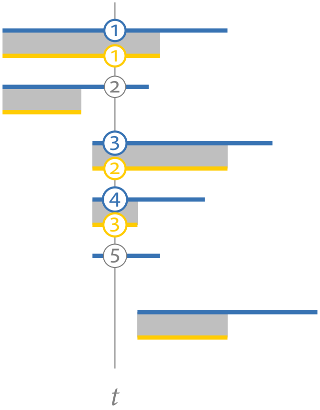


This specifies the barcode of M (as a matching diagram) based on ranks only.

Functoriality

The previous construction extents to a functor of epimorphisms $M \twoheadrightarrow N$ from persistence modules to matching diagrams.



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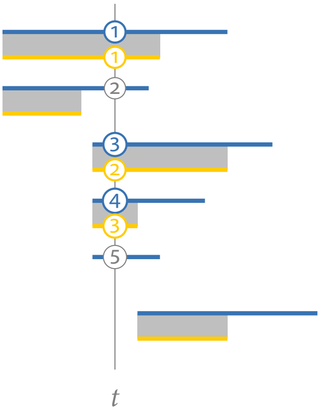

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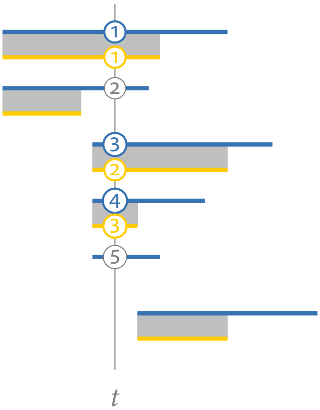

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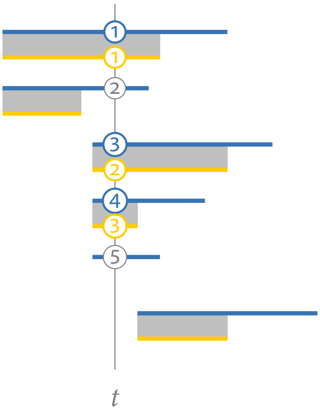

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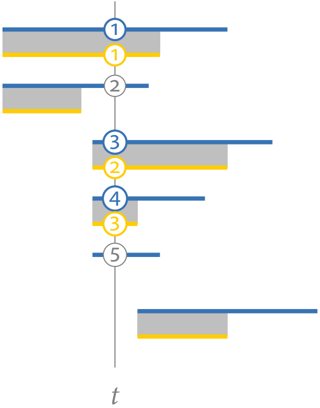

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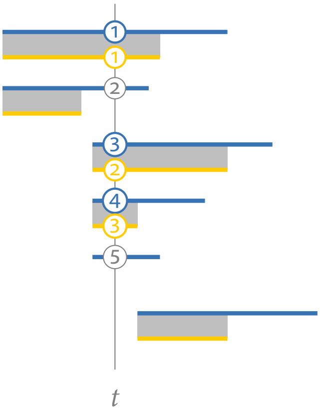
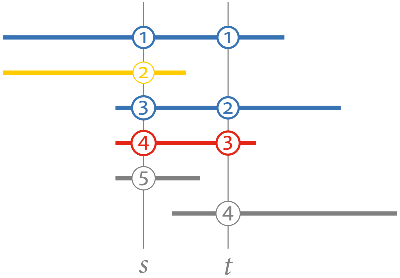


Obtain induced matching and algebraic stability theorems without Crawley-Boewey's interval decomposition

Conclusion

Barcodes as matching diagrams:

- A natural perspective on persistence



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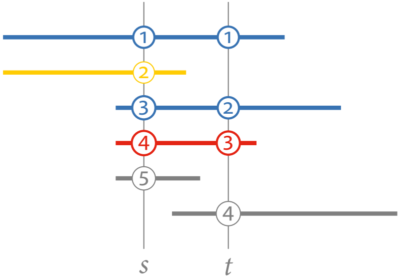

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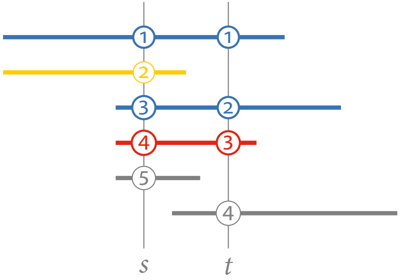

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