INDECOMPOSABLES IN MULTI-PARAMETER PERSISTENCE

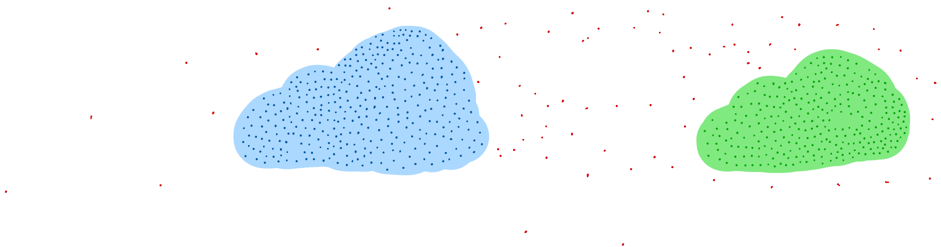
Ulrich Bauer (TUM)

Persistence, Sheaves, and Homotopy Theory Online Seminar

Feb 14, 2023

joint work with:

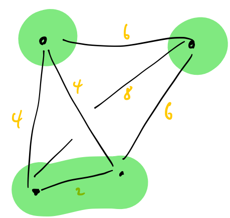
Magnus Botnan / Steffen Oppermann / Johan Steen / Luis Scoccola / Ben Fluhr



CLUSTERING FUNCTIONS

X : finite set

Clustering function φ :
maps a metric $d : X \times X \to \mathbb{R}$ (distance matrix)
to a partition of X.

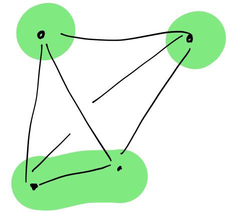


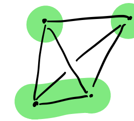
KLEINBERG'S AXIOMS

Desirable properties

- scale invariance :

$$\varphi(d) = \varphi(\lambda \cdot d).$$





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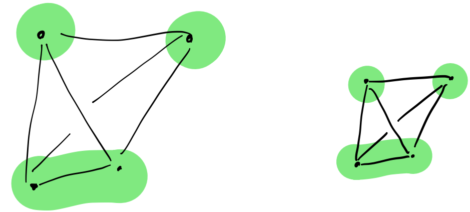

- richness :

every partition is obtained from some d.

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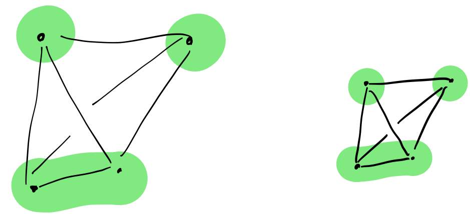
Desirable properties

- **scale invariance:**

  $$\varphi(d) = \varphi(\lambda \cdot d).$$

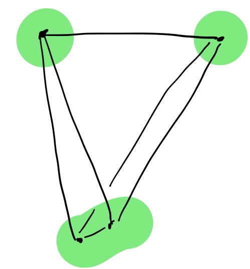


- **richness:**

  every partition is obtained from some d.

- **consistency:**

  decreasing d within clusters / increasing d across clusters does not change the result.



KLEINBERG'S IMPOSSIBILITY THEOREM

Thm [Kleinberg 2002] No clustering function satisfies

- scale-invariant,
- richness, and
- consistency.

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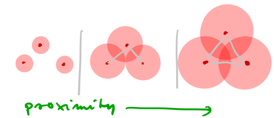
Motivates the use of a scale parameter

⇝ hierarchical clustering

CLUSTERING FROM CONNECTED COMPONENTS

proximity graph

- filter edges by proximity



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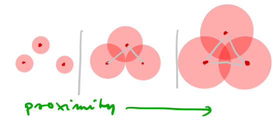
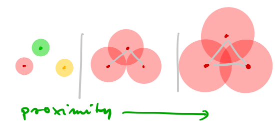
π_0 (connected components)





single-linkage clustering

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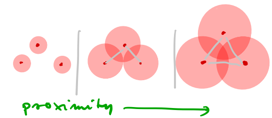

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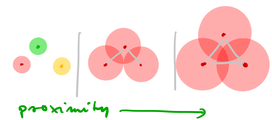


single-linkage clustering

H_0 (homology in deg. 0 with coeffs in K)

$$K^3 \to K \to K$$

proximity

persistent homology

$H_0 = F \circ \pi_0$

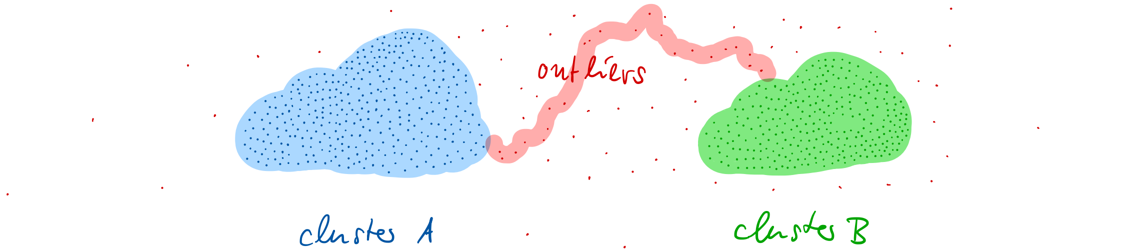
HIERARCHICAL CLUSTERING: EXISTENCE & UNIQUENESS

Thm [Carlsson, Mémoli 2010] Single-linkage clustering is the unique hierarchical clustering method satisfying [... certain axioms similar to Kleinberg's].

But ...

CHAINING EFFECT

Single-linkage clustering is sensitive to outliers

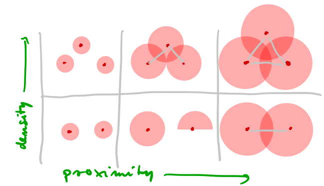


→ not used much in practice!

2-PARAMETER CLUSTERING

density-proximity graph

- filter points by density
- filter edges by proximity



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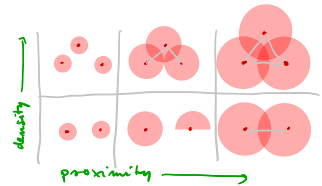

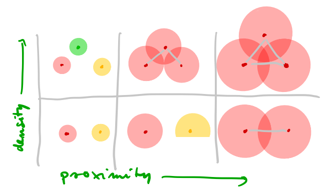

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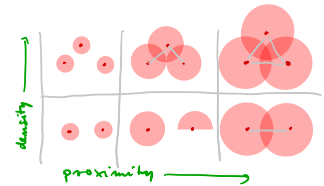

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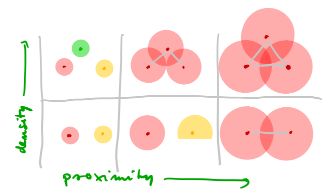

H_0 (homology in deg. 0
with coeffs in K)

$$\begin{tikzcd} K^3 \arrow[r] & K \arrow[r] & K \\ K^2 \arrow[r] \arrow[u] & K^2 \arrow[r] \arrow[u] & K \arrow[u] \end{tikzcd}$$

density

proximity

TRICHOTOMY OF REPRESENTATION TYPES

Given a finite indexing poset P. ($\mathbb{k}$: algebr. closed)

What are the indecomposable diagrams with shape P?

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3 cases (representation types): [Drozd 1977]

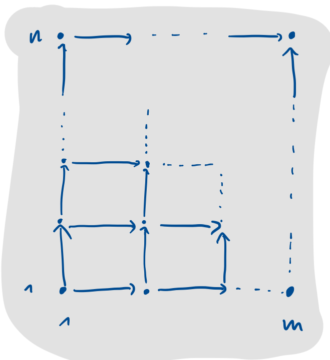
a) A finite list. finite type

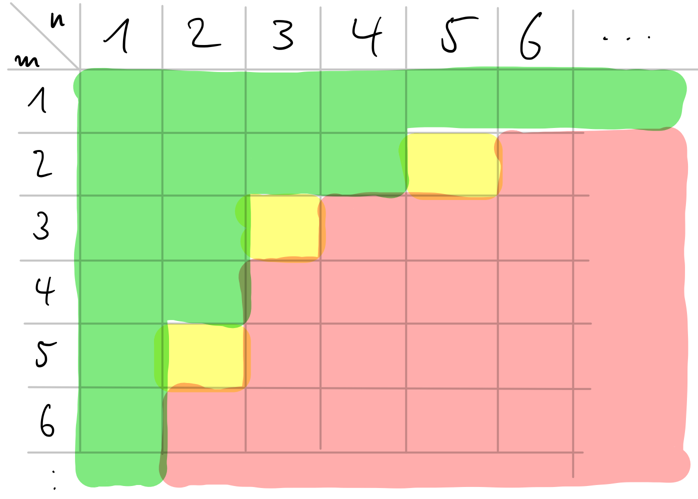
b) 1-param. families. tame

c) It's complicated. wild

(as complicated as modules over any finite-dim. algebra; including undecidable problems)

REPRESENTATION TYPES OF COMMUTATIVE GRIDS





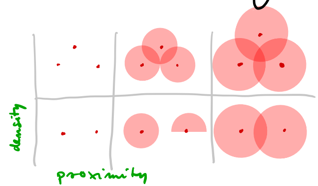
- finite type
- tame
- wild

for $(m-1)(n-1)$ $\left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\}$ 4.

[Leszczyński '94, Skowroński '2000]

GRID DIAGRAMS FROM CLUSTERING

Consider again 2-parameter clustering (proximity / density)



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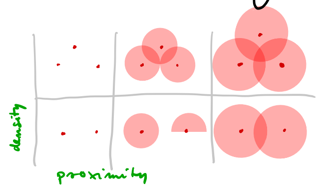

This yields diagrams of the form

$$\begin{tikzcd}
\vdots & \vdots & \\
{} \arrow[u] \arrow[r, two heads] & {} \arrow[u, dash, dotted] & \cdots \\
{} \arrow[u] \arrow[r, two heads] & {} \arrow[u] \arrow[r, two heads] & {} \arrow[r, two heads] & {}
\end{tikzcd}$$

Horizontal maps are surjective!

GRID DIAGRAMS FROM CLUSTERING

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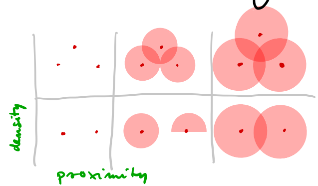

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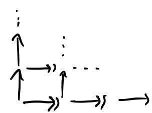
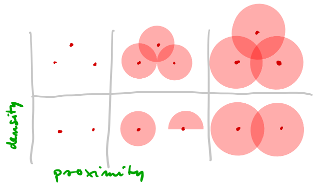


Horizontal maps are surjective!

Does this simplify the picture?

EPIMORPHISMS

Lemma $\mathrm{Rep}^{\twoheadrightarrow}(m,2)$ is finite type.



$$H_0 \rightsquigarrow \begin{tikzcd} K^3 \arrow[r, two heads, "(1\ 1\ 1)"] & K \arrow[r, two heads] & K \\ K^2 \arrow[u, "\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}"] \arrow[r, two heads] & K^2 \arrow[u, "(1\ 1)"'] \arrow[r, two heads, "(1\ 1)"'] & K \arrow[u] \end{tikzcd}$$

$$\cong \begin{tikzcd} K \arrow[r, two heads] & K \arrow[r, two heads] & K \\ K \arrow[u] \arrow[r, two heads] & K \arrow[u] \arrow[r, two heads] & K \arrow[u] \end{tikzcd} \oplus \begin{tikzcd} K \arrow[r, two heads] & 0 \arrow[r, two heads] & 0 \\ K \arrow[u] \arrow[r, two heads] & K \arrow[u] \arrow[r, two heads] & 0 \arrow[u] \end{tikzcd} \oplus \begin{tikzcd} K \arrow[r, two heads] & 0 \arrow[r, two heads] & 0 \\ 0 \arrow[u] \arrow[r, two heads] & 0 \arrow[u] \arrow[r, two heads] & 0 \arrow[u] \end{tikzcd}$$

- $\mathrm{Rep}(m,n)$: all commutative dgms over $m \times n$ grid
- $\mathrm{Rep}^{\twoheadrightarrow}(m,n)$: epis in horizontal direction
- $\mathrm{Rep}^{\twoheaduparrow\twoheadrightarrow}(m,n)$: epis in both directions.

EPIC GRIDS & WILD THINGS

Thm [B, Botnan, Oppermann, Steen 20]

$$
\begin{tikzcd}
\mathrm{Rep}^{\uparrow\twoheadrightarrow}(m,n) \arrow[r, phantom, "\sim"] \arrow[d, phantom, "\wr"] & \mathrm{Rep}^{\uparrow}(m,n-1) \arrow[d, phantom, "\wr"] \\
\mathrm{Rep}^{\twoheadrightarrow}(m-1,n) \arrow[r, phantom, "\sim"] & \mathrm{Rep}(m-1,n-1)
\end{tikzcd}
$$

same representation type

Corollary $\mathrm{Rep}^{\twoheadrightarrow}(m,n)$ is

- finite type
- tame
- wild

for $(m-1)(n-2) \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} 4.$

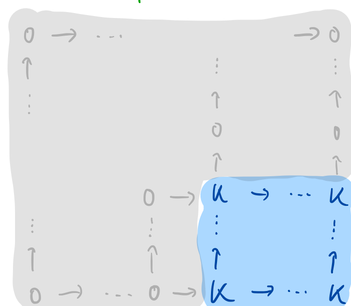
BEHIND THE SCENES

equivalence of categories

$$\frac{\mathrm{Rep}^{\twoheadrightarrow}(m,n)}{\mathrm{Rep}^{\hat{\imath}\twoheadrightarrow}(m,n)} \simeq \mathrm{Rep}(m,n-1)$$

additive quotient $\frac{A}{B}$:
identify morphisms in A whose difference factors through B

indecomposables are of the form



$\Rightarrow$ finite type

THE INSTABILITY OF DECOMPOSITIONS

How useful are indecomposables for TDA?

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Thm [B, Scoccola 22] Generically, 2-parameter persistence modules (finitely presented) are of the form

$$A \oplus B$$

(ε-indecomposable)

indecomposable

$d_I(B, 0) < \varepsilon$

(for any $\varepsilon > 0$).

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indecomposable $\quad d_I(B, 0) < \varepsilon$

(for any $\varepsilon > 0$).

- $\forall \varepsilon \; \exists$ open, dense subset of ε-indecomposables)

THE INSTABILITY OF DECOMPOSITIONS

Proof.

(a) $M : \mathbb{R}^n \to \mathrm{vect}$ f.p., indecomposable

$\forall \varepsilon \; \exists \delta$:

$d_I(M, N) < \delta \implies N$ is ε-indecomposable.

The Instability of Decompositions

Proof.

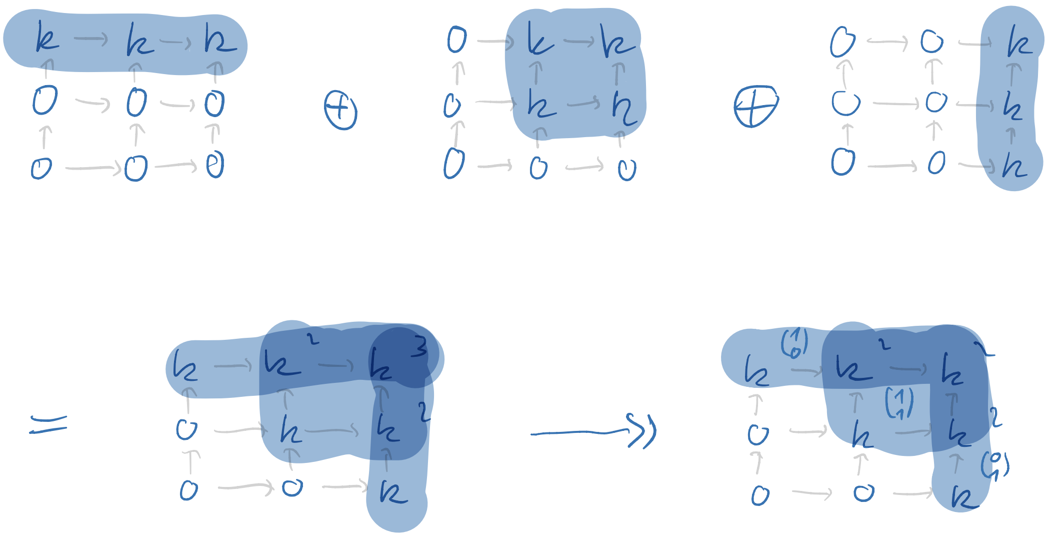
(a) $M : \mathbb{R}^n \to \mathrm{vect}$ f.p., indecomposable

$\forall \varepsilon \; \exists \delta$:

$d_I(M, N) < \delta \Rightarrow N$ is ε-indecomposable.

(b) Indecomposables are dense in interleaving distance.

The Idea of Tacking Indecomposables



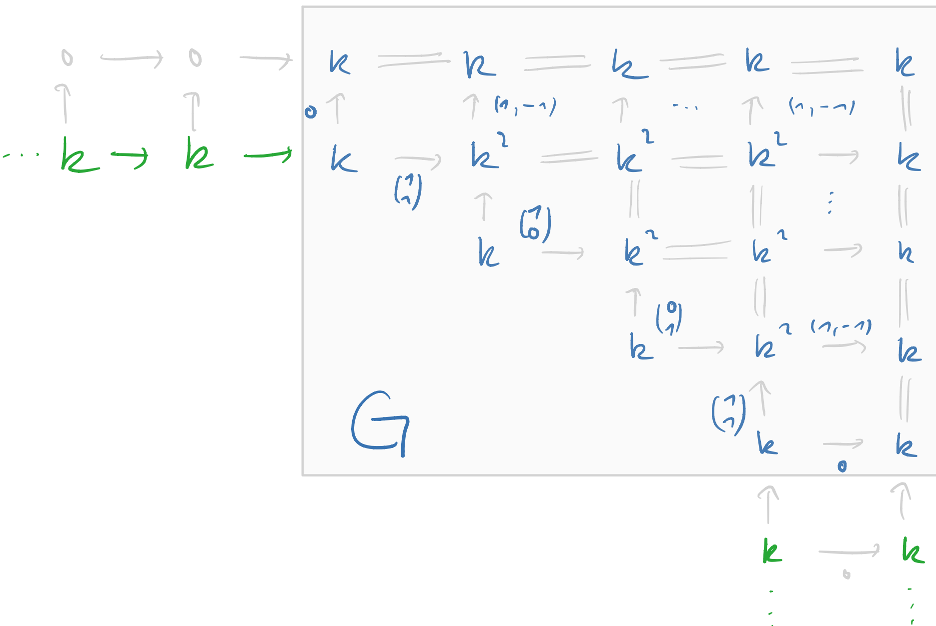
The Tacking Gadget

$$
\begin{tikzcd}
k \arrow[r, equal] & k \arrow[r, equal] & k \arrow[r, equal] & k \arrow[r, equal] & k \\
k \arrow[u, "0"] \arrow[r, "\binom{1}{1}"'] & k^2 \arrow[u, "{(1,-1)}"'] \arrow[r, equal] & k^2 \arrow[u, "\cdots"'] \arrow[r, equal] & k^2 \arrow[u, "{(1,-1)}"'] \arrow[r] & k \arrow[u, equal] \\
& k \arrow[u] \arrow[r, "\binom{1}{0}"] & k^2 \arrow[u, equal] \arrow[r, equal] & k^2 \arrow[u, equal] \arrow[r] & k \arrow[u, equal] \\
& & k \arrow[u] \arrow[r, "\binom{0}{1}"] & k^2 \arrow[u, equal] \arrow[r, "{(1,-1)}"] & k \arrow[u, equal] \\
& & & k \arrow[u, "\binom{1}{1}"] \arrow[r, "0"'] & k \arrow[u, equal]
\end{tikzcd}
$$

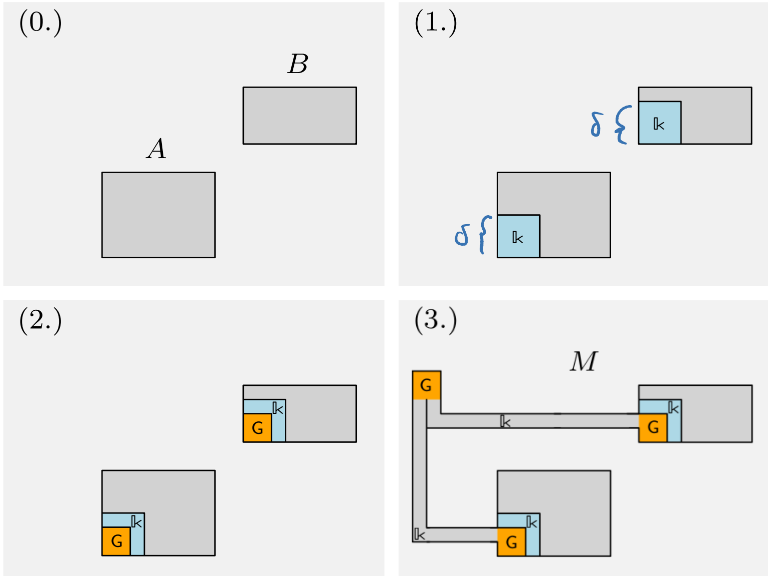
$\vdots$

G

THE TACKING GADGET



TACKING INDECOMPOSABLES

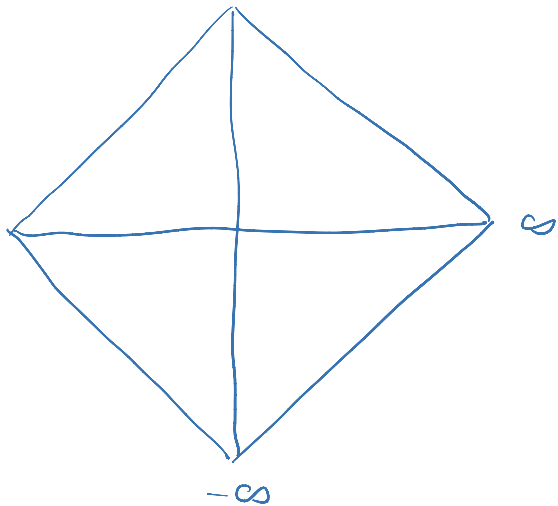


$d_I(A \oplus B, M) < \delta$ for $\delta > 0$ arbitrary

THE MAYER-VIETORIS PYRAMID

[Carlsson et al. 2009]

$f : X \to \mathbb{R}$



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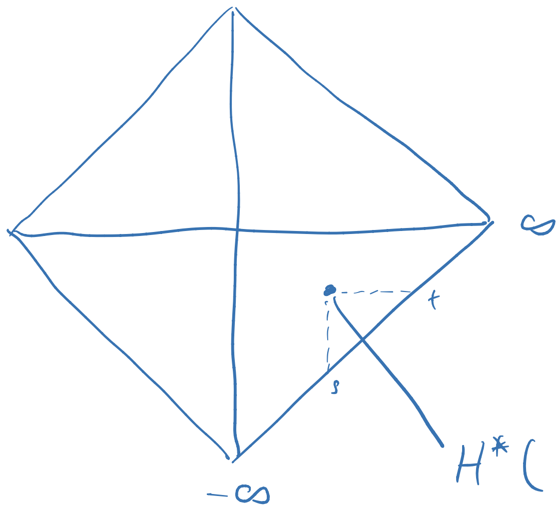

$H^*(f^{-1}(s,t))$

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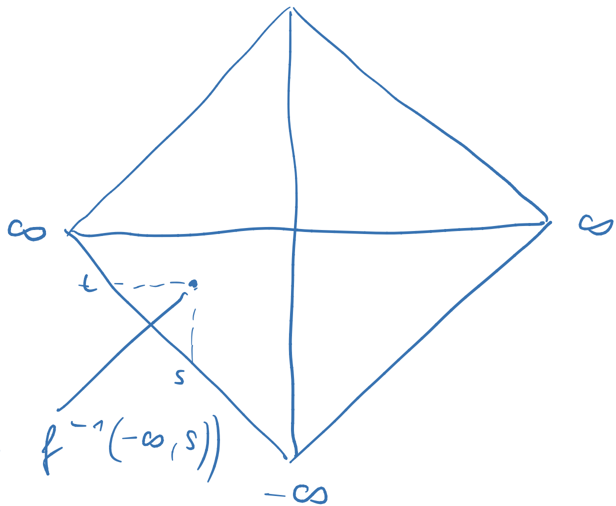

$H^*(f^{-1}(-\infty, t), f^{-1}(-\infty, s))$

THE MAYER-VIETORIS PYRAMID

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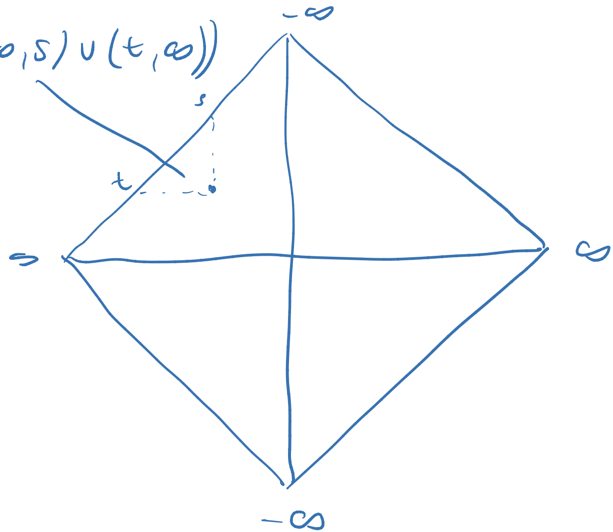
$f : X \to \mathbb{R}$

$H^*(X, f^{-1}((-\infty, s) \cup (t, \infty)))$



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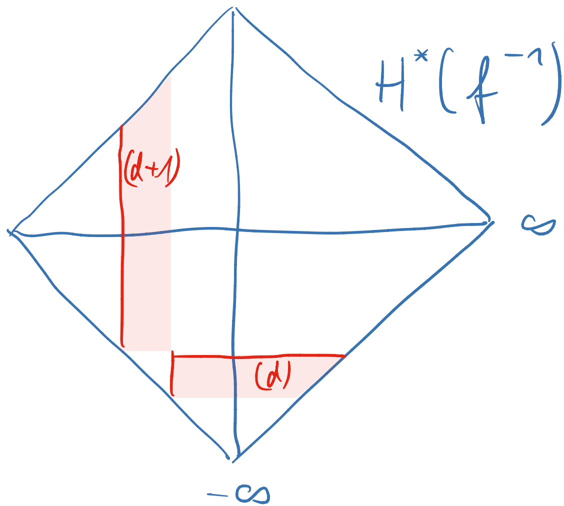

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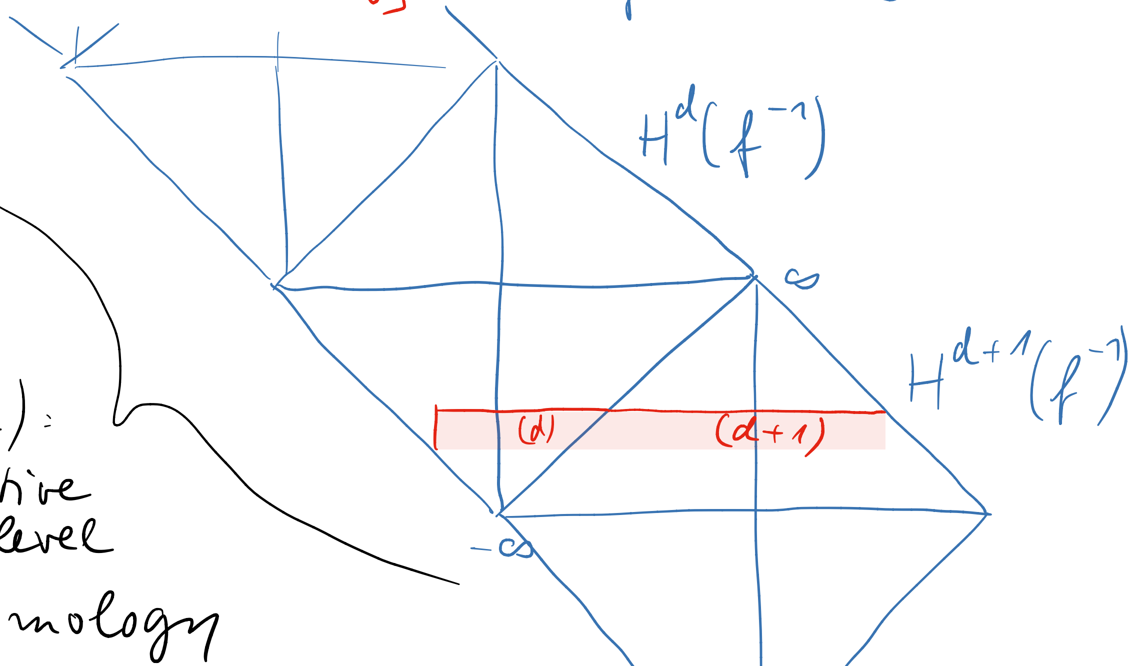

$h(f)$:

Relative
Interlevel
Set
Cohomology

RISC OF A FUNCTION

[B, Botnan, Fluhr 2021] Assume $h(f)$ p.f.d.

Prop. $h(f)$ is a 2-param. pers. module supported on a strip $\mathbb{M} \subseteq \mathbb{R}^2$

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Prop. $h(f)$ is cohomological (equivalently: middle-exact;

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \rightsquigarrow A \to B \oplus C \to D$$

exact)

RISC OF A FUNCTION

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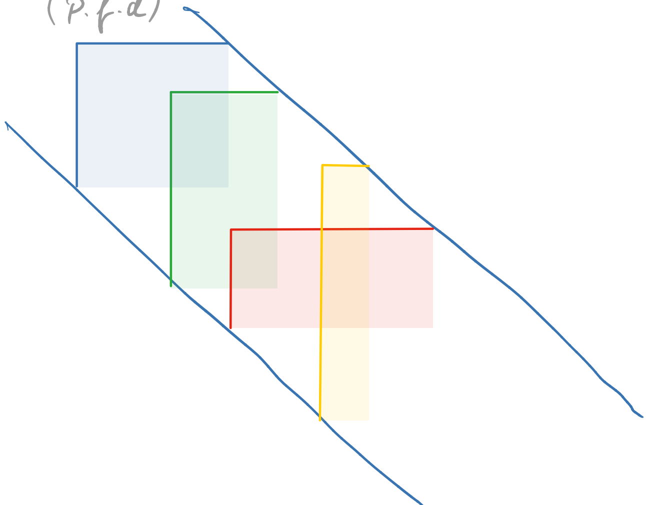
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \rightsquigarrow \underset{\text{exact}}{A \to B \oplus C \to D})$$

Prop. $h(f)$ is sequentially continuous (from below).

DECOMPOSITION OF COHOMOLOGICAL FUNCTORS

Thm [BBF 21'] Any seq. cont. cohomological functor $M \to \mathrm{vect}$ (p.f.d) decomposes into block summands.



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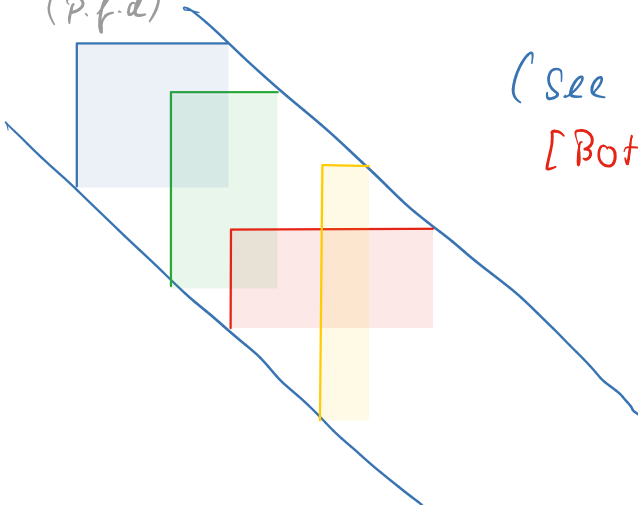

(See also: [Botnan, Lebovici, Oudot '20])

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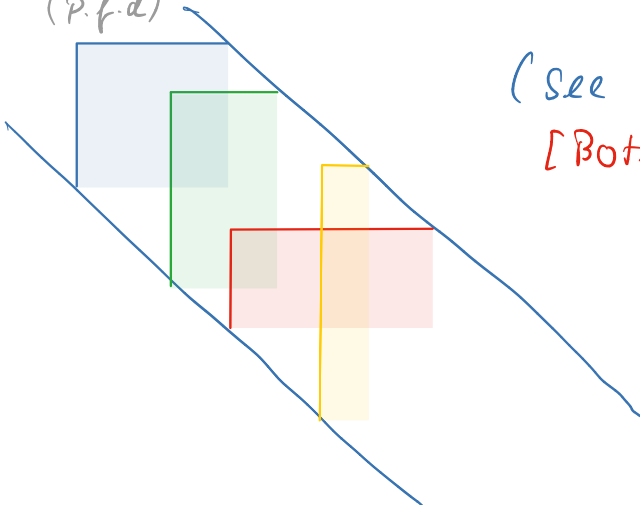

(See also : [Botnan, Lebovici, Oudot '20])

Prop. q-tame & seq-continuous $\Rightarrow$ p.f.d.

INDUCED MORPHISMS & INTERLEAVINGS

- A map φ :
$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & \mathbb{R} & \end{tikzcd}$$
induces

a morphism in RISC $h(\varphi) : h(f) \to h(g)$.

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a morphism in RISC $h(\varphi) : h(f) \to h(g)$.

- Two functions $f, g : X \to \mathbb{R}$, $\delta = \| f - g \|$ induce a δ-interleaving between $h(f)$, $h(g)$.

INDUCED MORPHISMS & INTERLEAVINGS

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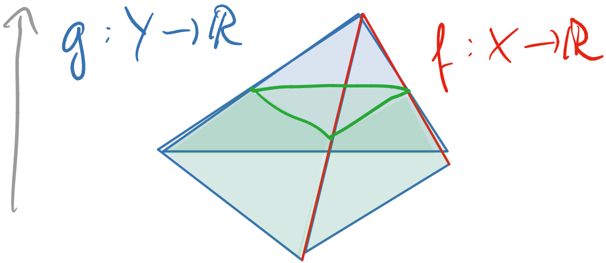
- These are richer than in standard persistence!

WHAT STANDARD PERSISTENCE CAN'T SEE



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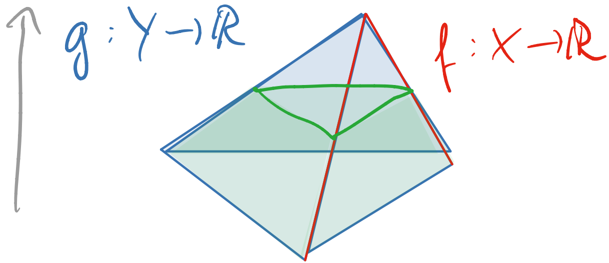
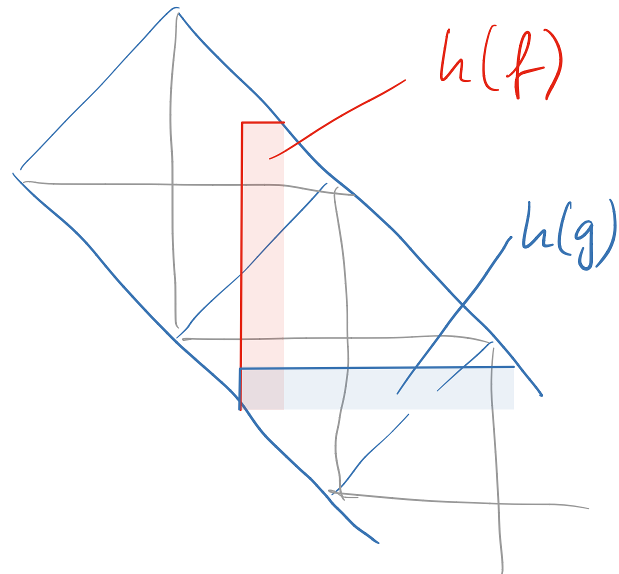


(reduced cohomology)



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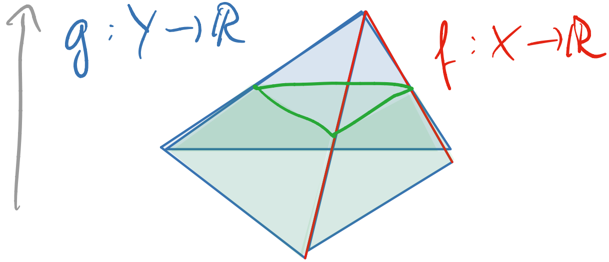
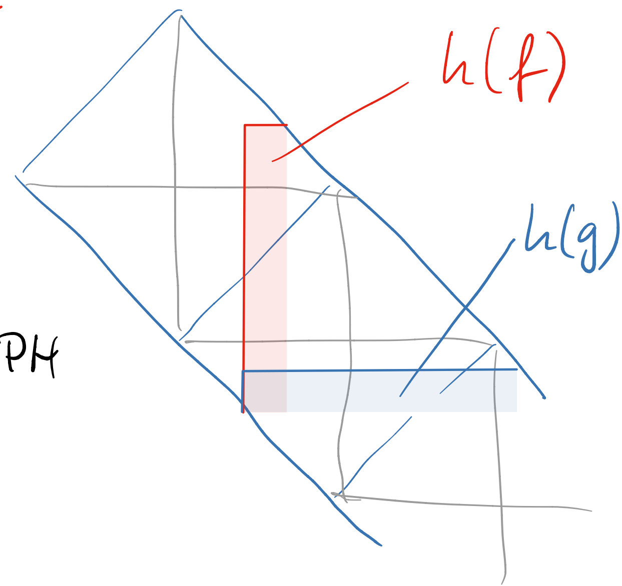


$\varphi : X \hookrightarrow Y$

(reduced cohomology)



The induced map in PH is zero.

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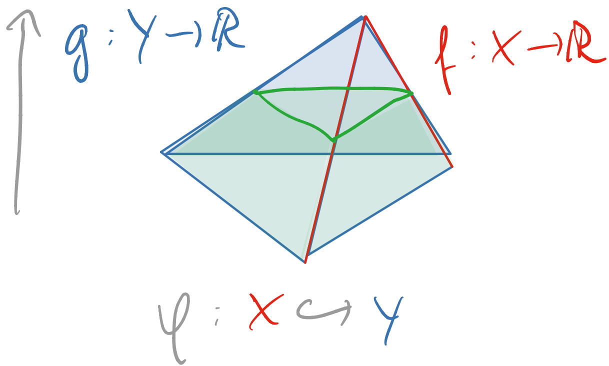

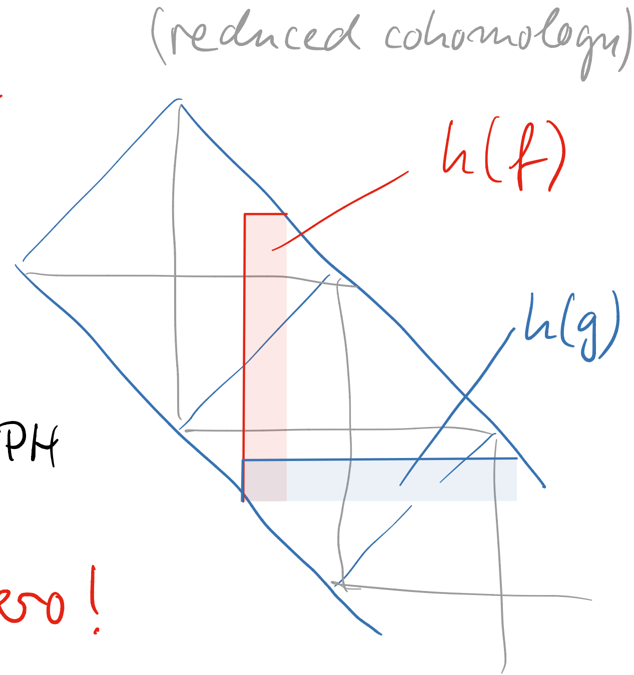

The induced map in PH is zero.

But $h(\varphi)$ is nonzero!

EXTENDED AND LEVEL SET PERSISTENCE

Extended persistence :

$$\cdots \; H_*(f^{-1}(-\infty, s]) \to H_*(f^{-1}(-\infty, t)) \to \cdots \; H^*(X)$$

$$\to H^*(X, [v, \infty)) \longrightarrow H^*(X, [u, \infty)) \to \cdots$$

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Thm [Carlsson et al 09] The persistence diags of extended persistence and of (inter) level set persistence correspond bijectively.

FUNTORIAL EQUIVALENCE

Can we make the extended-level set correspondence functorial?

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Not in the standard way

(graded persistence modules)

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Yes using RISC

(using a derived category)

RISC ≃ DERIVED LEVEL SET SHEAF PERSISTENCE

$$f : X \to \mathbb{R}$$

$$\Big\downarrow \text{RISC}$$

$h(f)$:

$\mathbb{M} \to \mathrm{vect}$ (cohom.)

RISC $\cong$ DERIVED LEVEL SET SHEAF PERSISTENCE

$$
\begin{tikzcd}
 & f : X \to \mathbb{R} \arrow[dl, "\text{DLSP}"'] \arrow[dr, "\text{RISC}"] & \\
Rf_*(k_X) : D^+(\mathbb{R}) & & h(f) : \mathbb{M} \to \text{vect (cohom.)}
\end{tikzcd}
$$

RISC ≅ DERIVED LEVEL SET SHEAF PERSISTENCE

$$
\begin{tikzcd}
 & f : X \to \mathbb{R} \arrow[dl, "\mathrm{DLSP}"'] \arrow[dr, "\mathrm{RISC}"] & \\
D^+(\mathbb{R}) \arrow[rr, "\cong"] & & \mathbb{M} \to \mathrm{vect}\ (\text{cohom.})
\end{tikzcd}
$$

$Rf_*(k_X)$:

$h(f)$:

(tame)

(p.f.d.)

RISC ≅ DERIVED LEVEL SET SHEAF PERSISTENCE

$$
\begin{tikzcd}
 & f\colon X \to \mathbb{R} \arrow[dl, "\mathrm{DLSP}"'] \arrow[dr, "\mathrm{RISC}"] \arrow[d, phantom, "\Swarrow\cong"] & \\
Rf_*(k_X) : \; D^+(\mathbb{R}) \arrow[rr, "\cong"] & & h(f) : \; \mathbb{M} \to \mathrm{vect} \ (\text{cohom.})
\end{tikzcd}
$$

(tame) (p.f.d.)

Thm [B, Fluhr '22] DLSP $\cong$ RISC.

RISC ≅ DERIVED LEVEL SET SHEAF PERSISTENCE

$$
\begin{tikzcd}
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Rf_*(k_X)\colon\ D^+(\mathbb{R}) \ \text{(tame)} \arrow[rr, "\cong"] & & h(f)\colon\ \mathbb{M} \to \mathrm{vect} \ \text{(cohom.)} \ \text{(p.f.d.)}
\end{tikzcd}
$$

Thm [B, Fluhr '22] DLSP ≅ RISC.

Relies crucially on maps across degrees!

SUMMARY

Indecomposables in multi-d persistence ...

- can be very complicated, even for H_0.

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- can be very complicated, even for H_0.
- are dense in the interleaving distance
- clarify structure of interlevel set persistence
- reveal interaction across degrees
- highlight the role of derived categories in persistent homology.