

Bridging Morse theory and persistent homology of geometric complexes

Ulrich Bauer

Technical University of Munich (TUM)

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Combinatorics seminar

Charles University, Prague

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Forschungsgemeinschaft
German Research Foundation



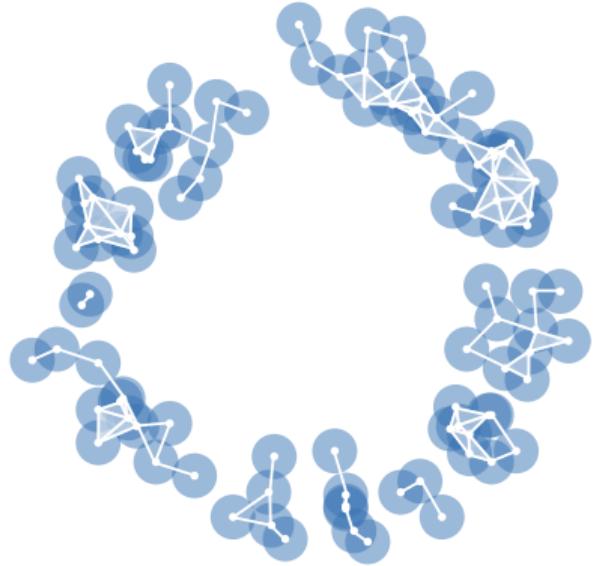
SFB
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Discretization
In Geometry
and Dynamics

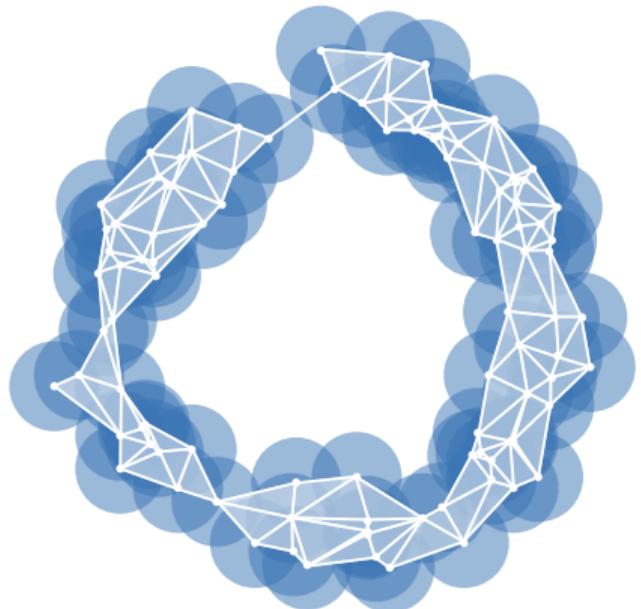
Technical
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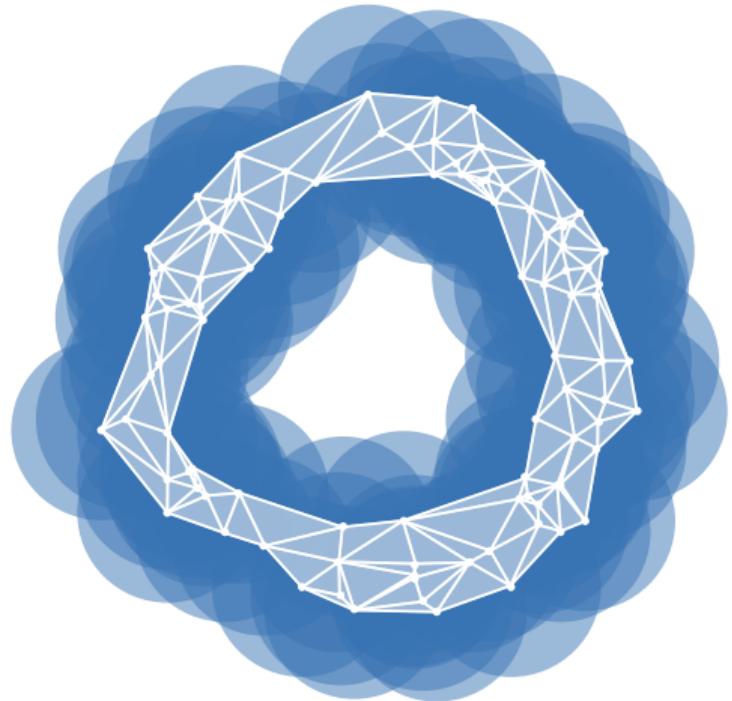


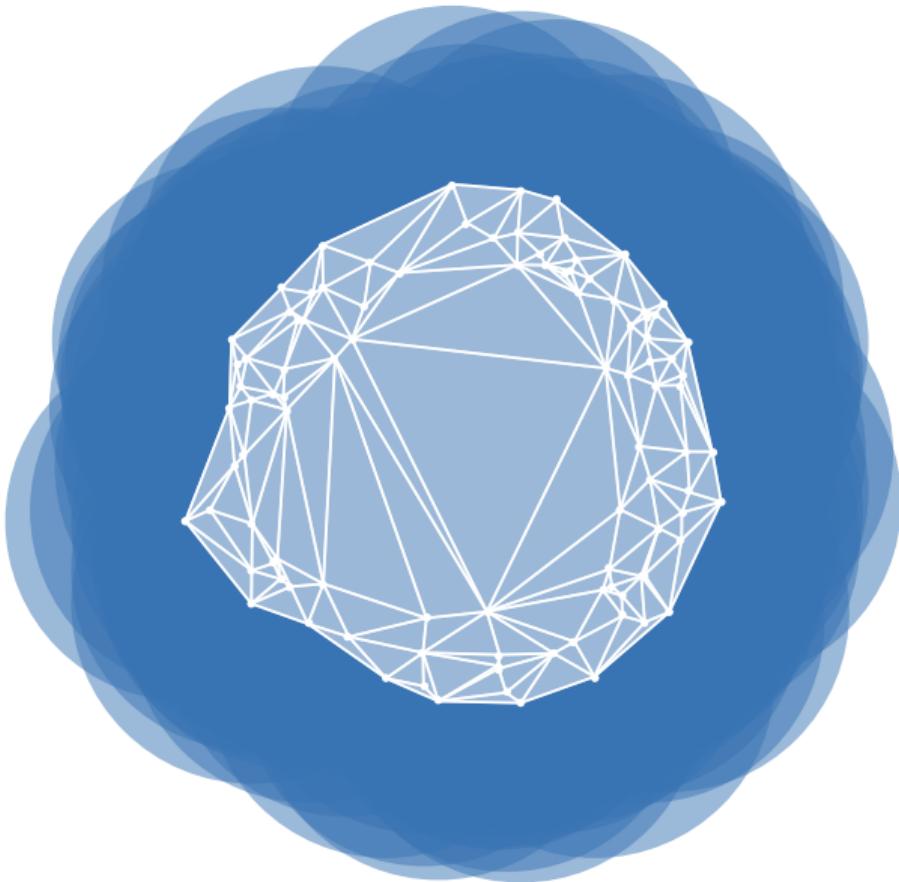
MCSL
Munich Center for Machine Learning





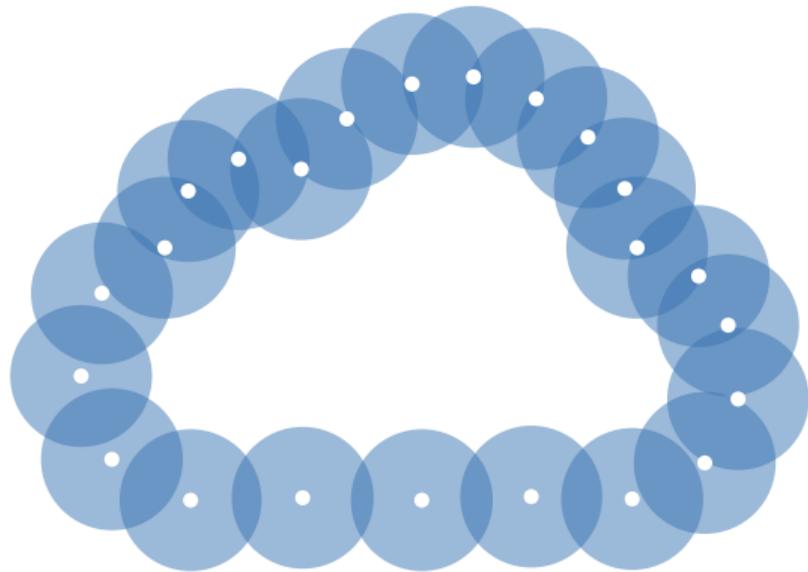




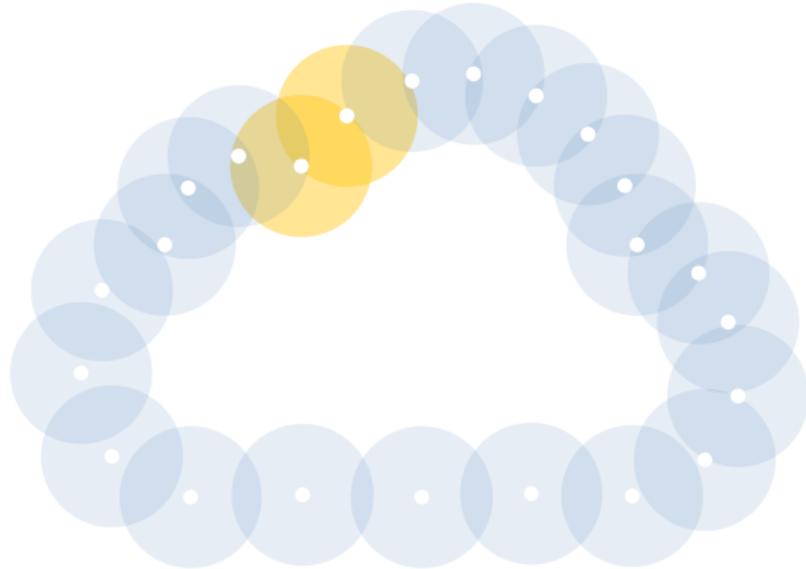


Geometric complexes

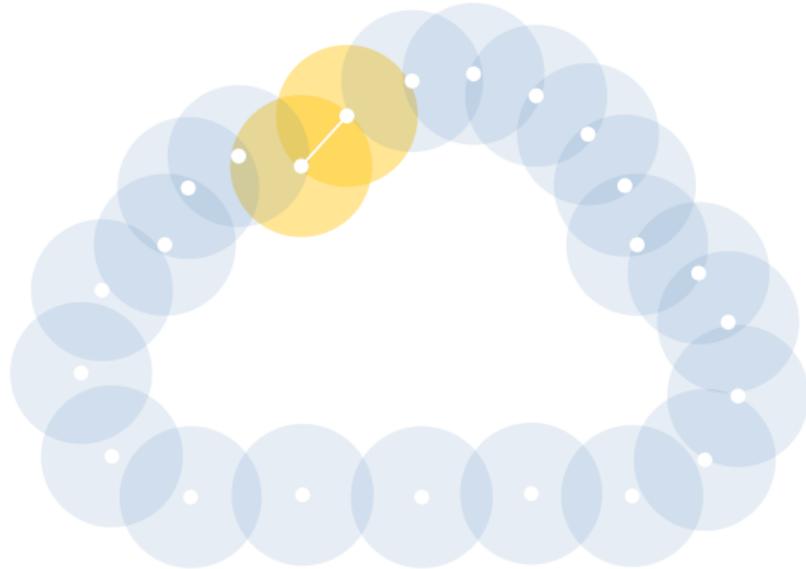
Čech complexes



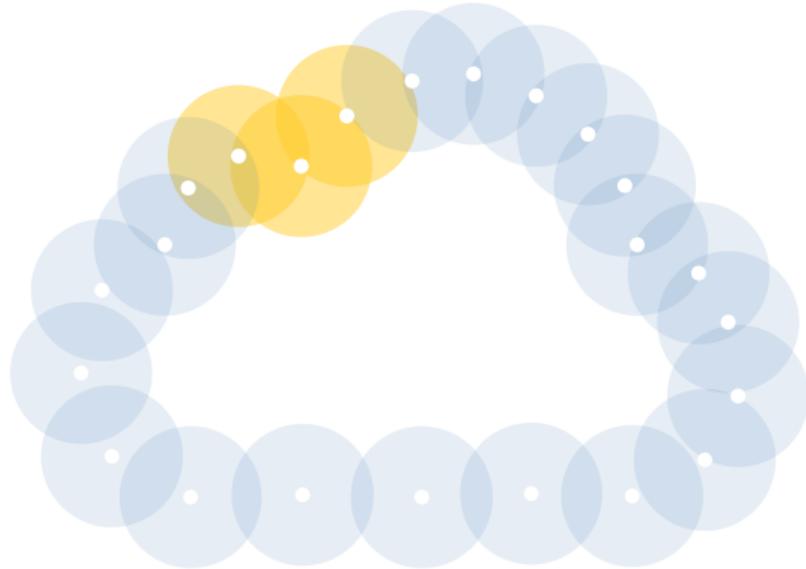
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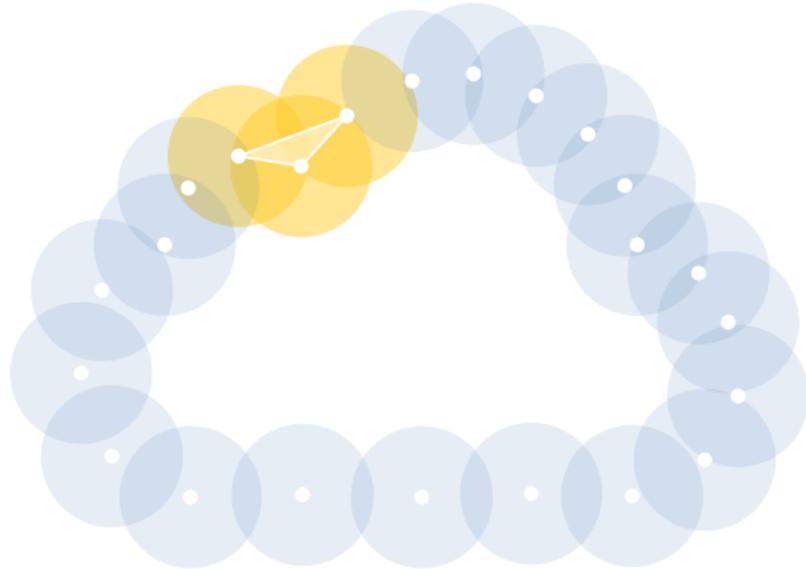
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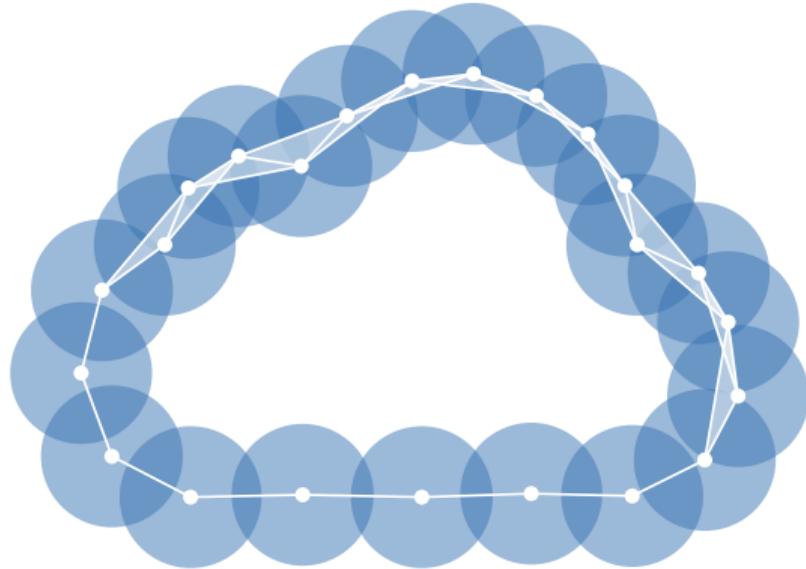
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Nerves

Definition (Alexandrov 1928)

Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space X . The *nerve* of \mathcal{U} is the simplicial complex

$$\text{Nrv}(\mathcal{U}) = \{J \subseteq I \mid 0 < |J| < \infty \text{ and } \bigcap_{j \in J} U_j \neq \emptyset\}$$

recording the nonempty intersections of cover elements.

- Any subset of \mathcal{U} with nonempty intersection corresponds to a simplex in $\text{Nrv}(\mathcal{U})$

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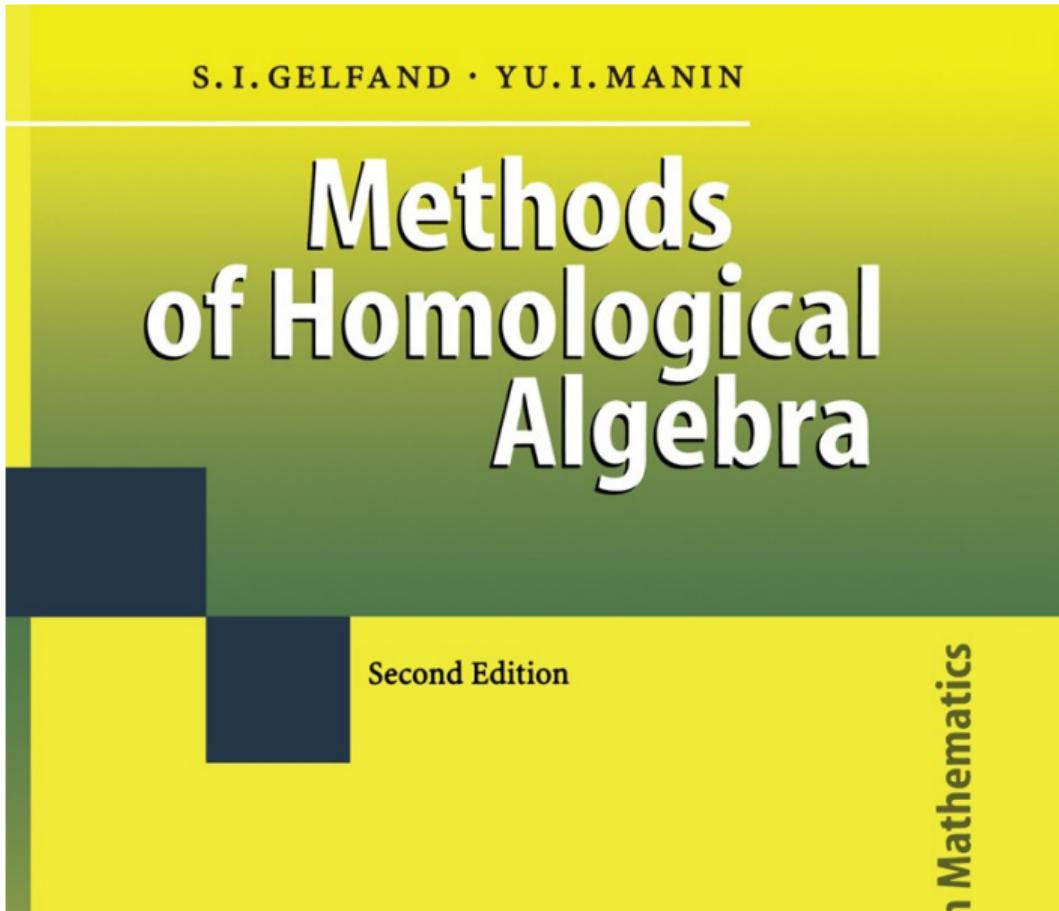


U. Bauer, M. Kerber, F. Roll, and A. Rolle

A Unified View on the Functorial Nerve Theorem and its Variations

Nerve theorems, almost true

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3. Nerve of a Covering

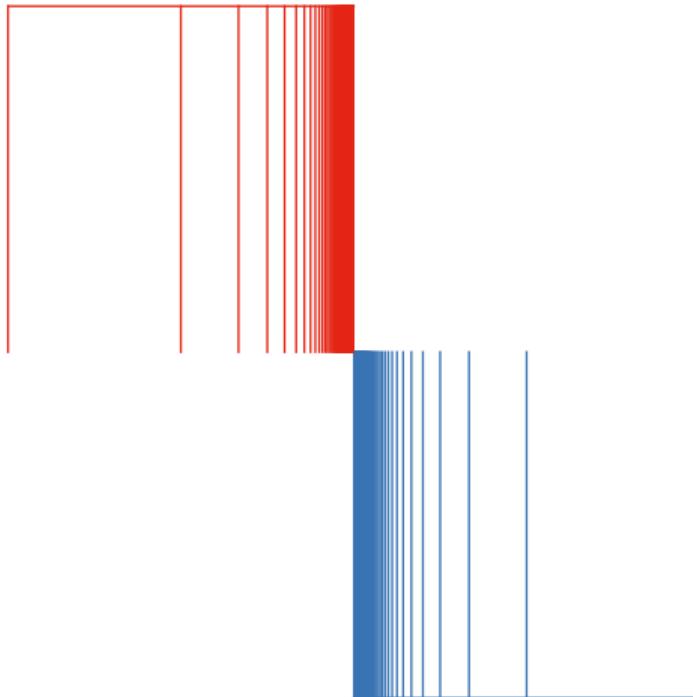
Let Y be a topological space, $U = (U_\alpha)$ be its covering (either by open or by closed sets), α running over some set A . Let

$$X_n = \{(\alpha_0, \dots, \alpha_n) \mid U_{\alpha_0} \cap \dots \cap U_{\alpha_n} \neq \emptyset\},$$

$$X(f)(\alpha_0, \dots, \alpha_n) = (\alpha_{f(0)}, \dots, \alpha_{f(m)}) \quad \text{for } f : [m] \rightarrow [n].$$

This simplicial set reflects the combinatorial structure of a covering. One can show that if the covering U is locally finite and all nonempty finite intersections $U_{\alpha_0} \cap \dots \cap U_{\alpha_n}$ are contractible, then the geometric realization $|X|$ of X is homotopically equivalent to Y , so that the topology can be efficiently encoded into combinatorical data.

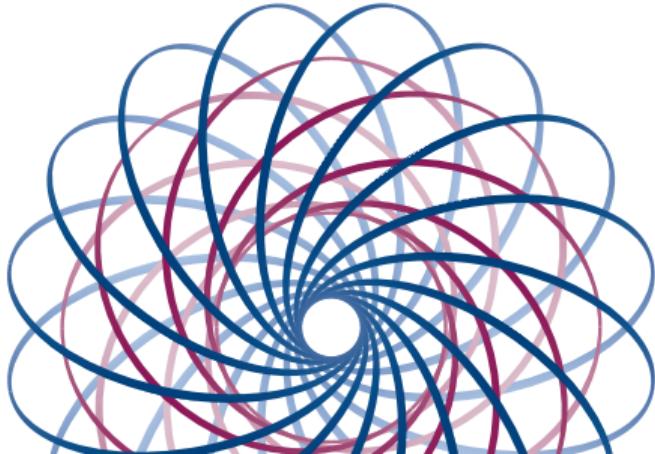
Nerve theorems, almost true



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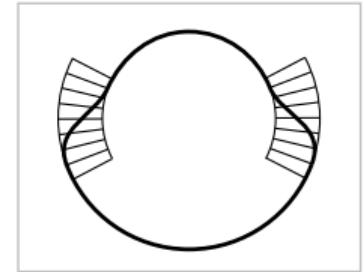
Algebraic Topology

Allen Hatcher



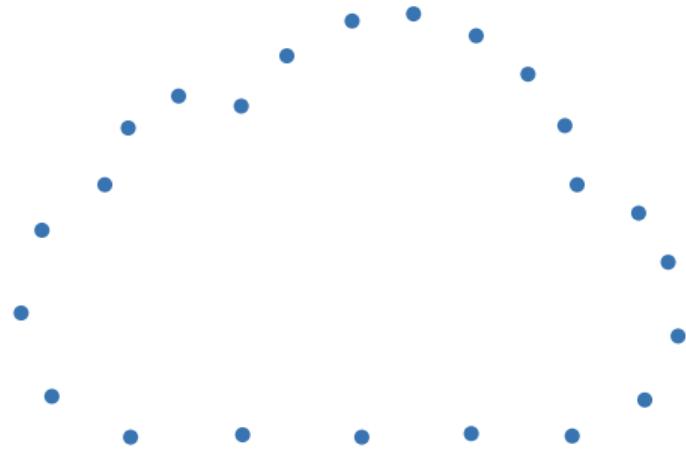
Nerve theorems, almost true

Since X is paracompact there is a partition of unity subordinate to the cover \mathcal{U} . This is a family of maps $\varphi_\alpha : X \rightarrow [0, 1]$ satisfying three conditions: The support of each φ_α is contained in some $X_{i(\alpha)}$, only finitely many φ_α 's are nonzero near each point of X , and $\sum_\alpha \varphi_\alpha = 1$. Define a section $s : X \rightarrow \Delta X_{\mathcal{U}}$ of p by setting $s(x) = \sum_\alpha \varphi_\alpha(x) x_{i(\alpha)}$. The figure shows the case $X = S^1$ with a cover by two arcs, the heavy line indicating the image of s . In the general case the section s embeds X as a retract of $\Delta X_{\mathcal{U}}$, and it is a deformation retract since points in fibers $p^{-1}(x)$ can move linearly along line segments to $s(x)$. \square

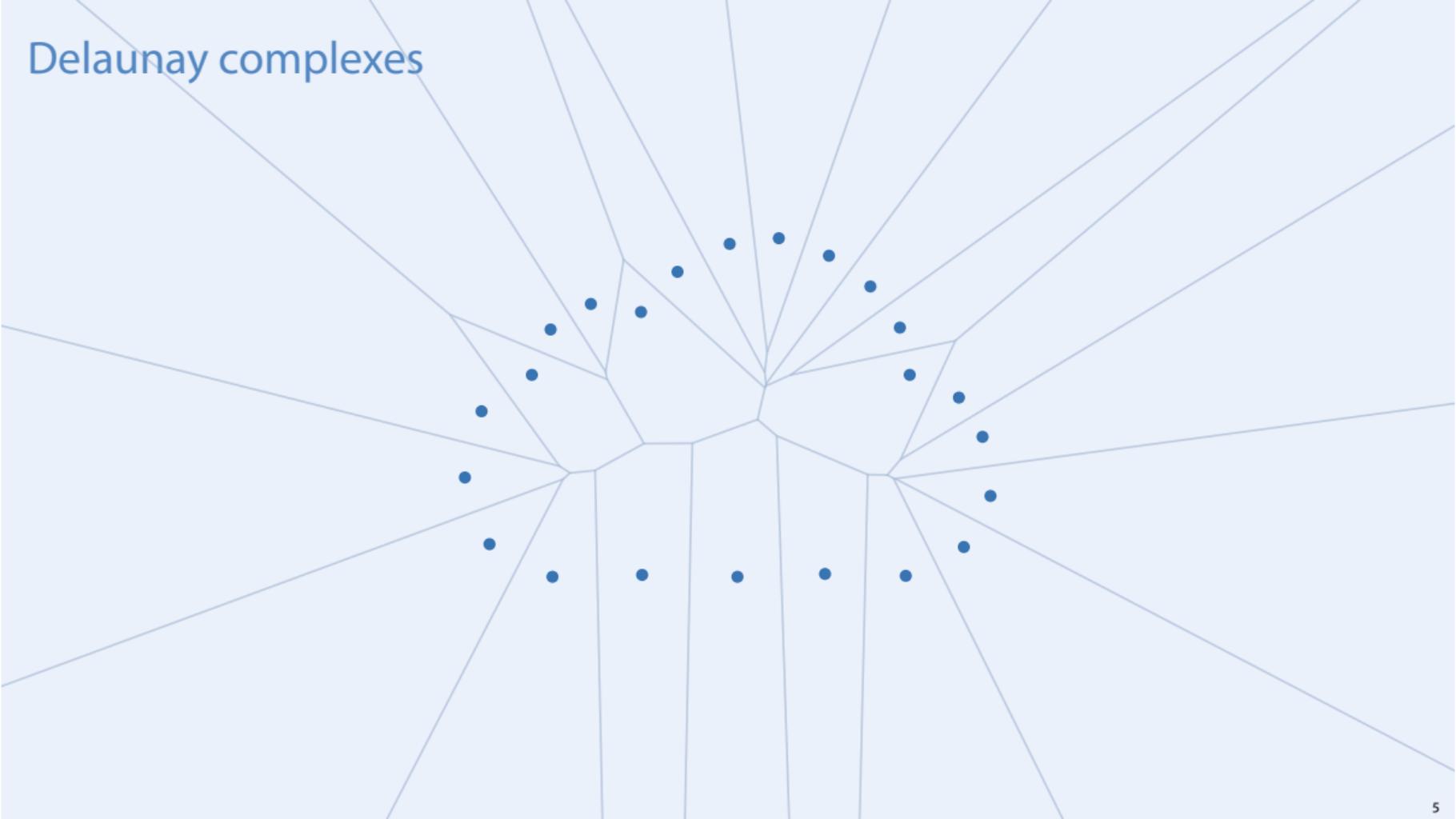


Corollary 4G.3. *If \mathcal{U} is an open cover of a paracompact space X such that every nonempty intersection of finitely many sets in \mathcal{U} is contractible, then X is homotopy equivalent to the nerve $N\mathcal{U}$.*

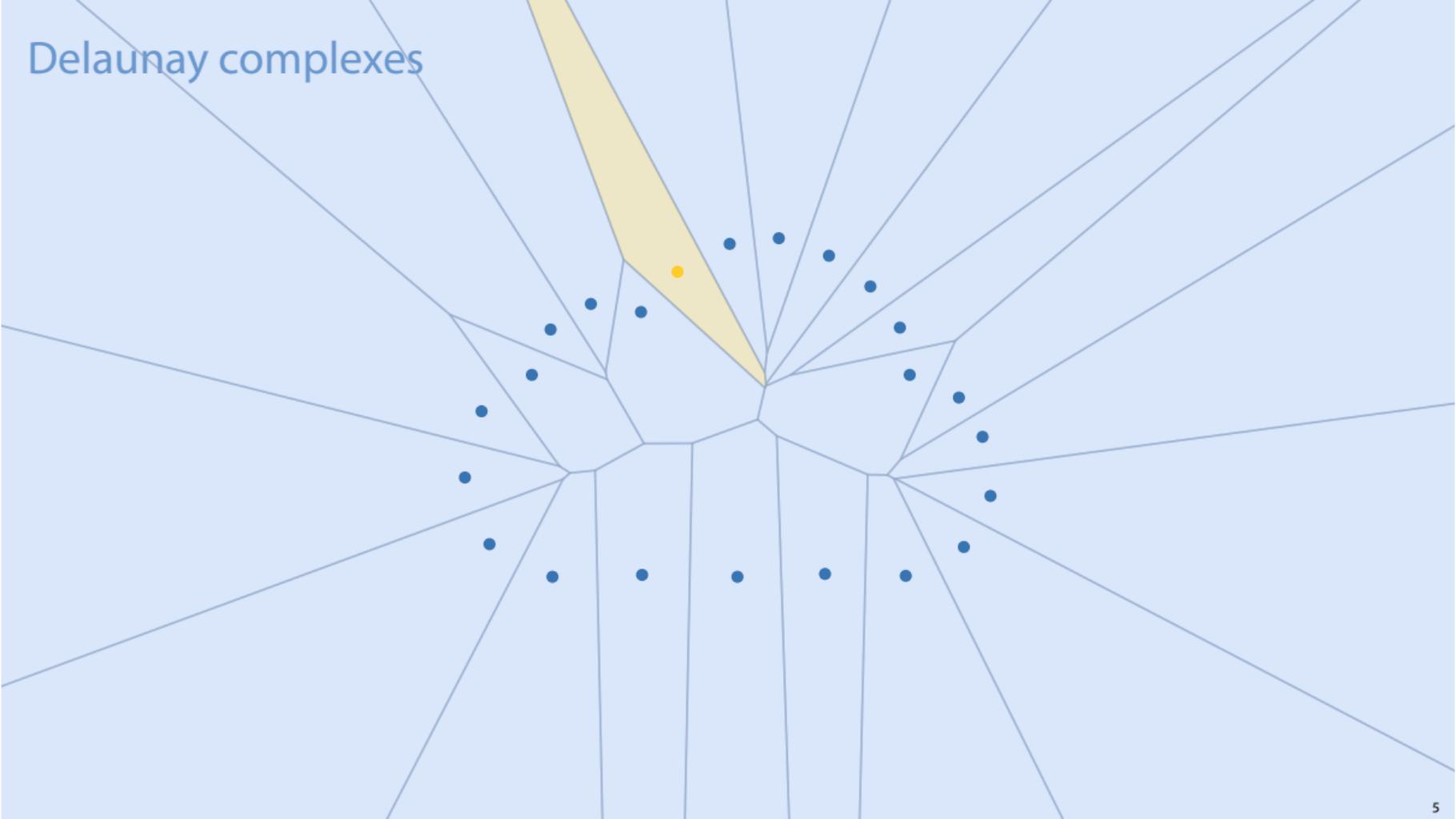
Delaunay complexes



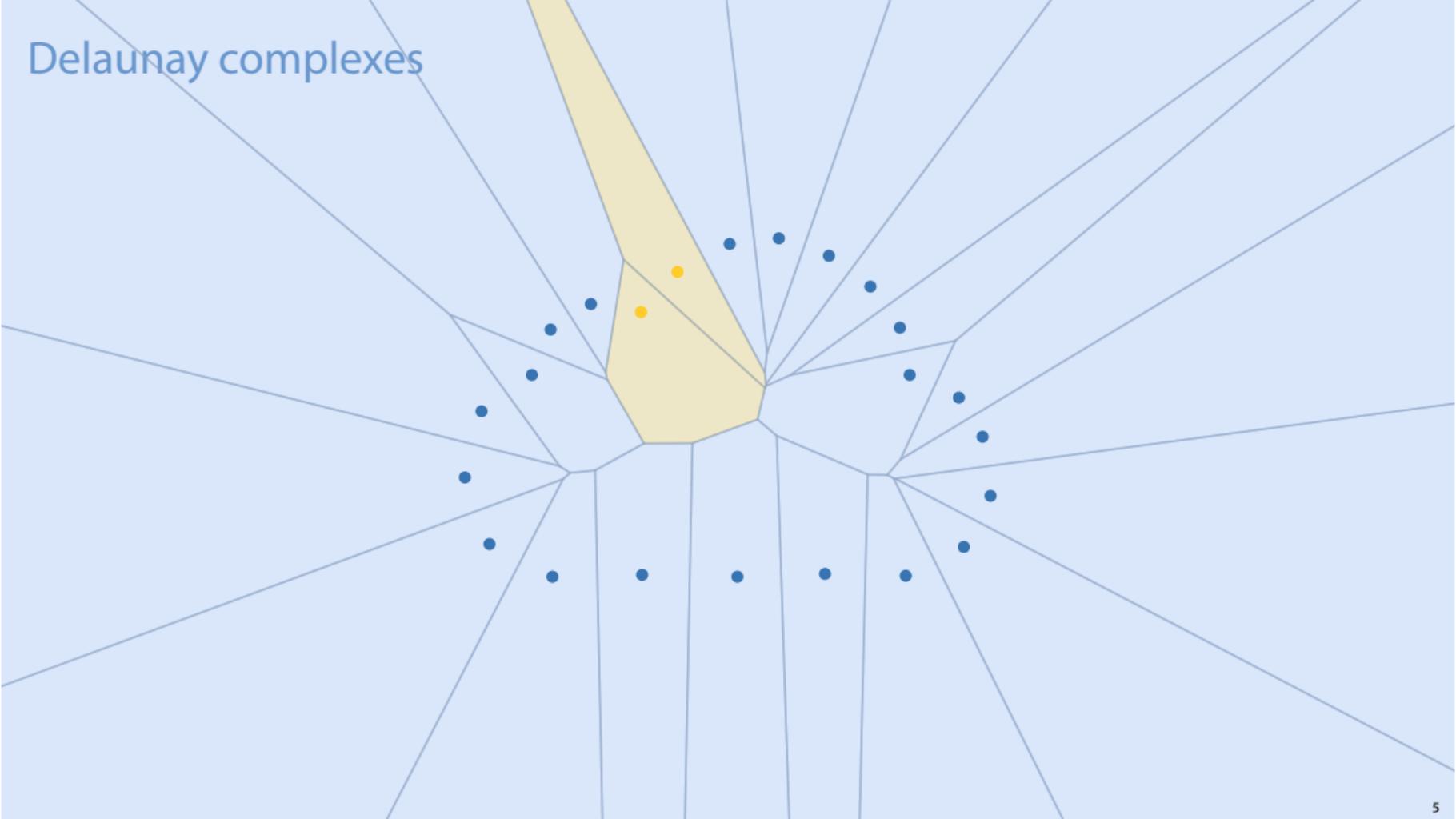
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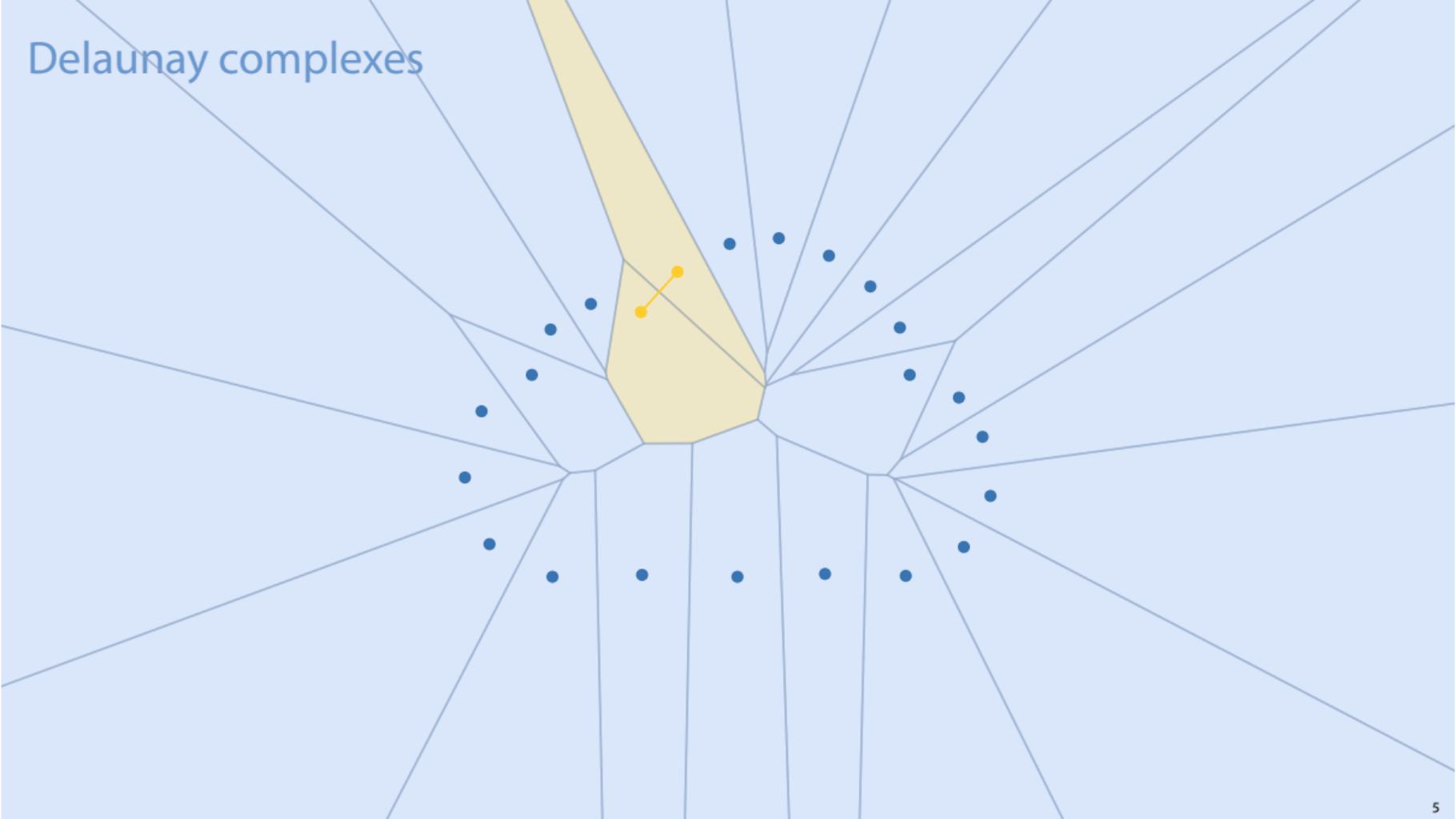
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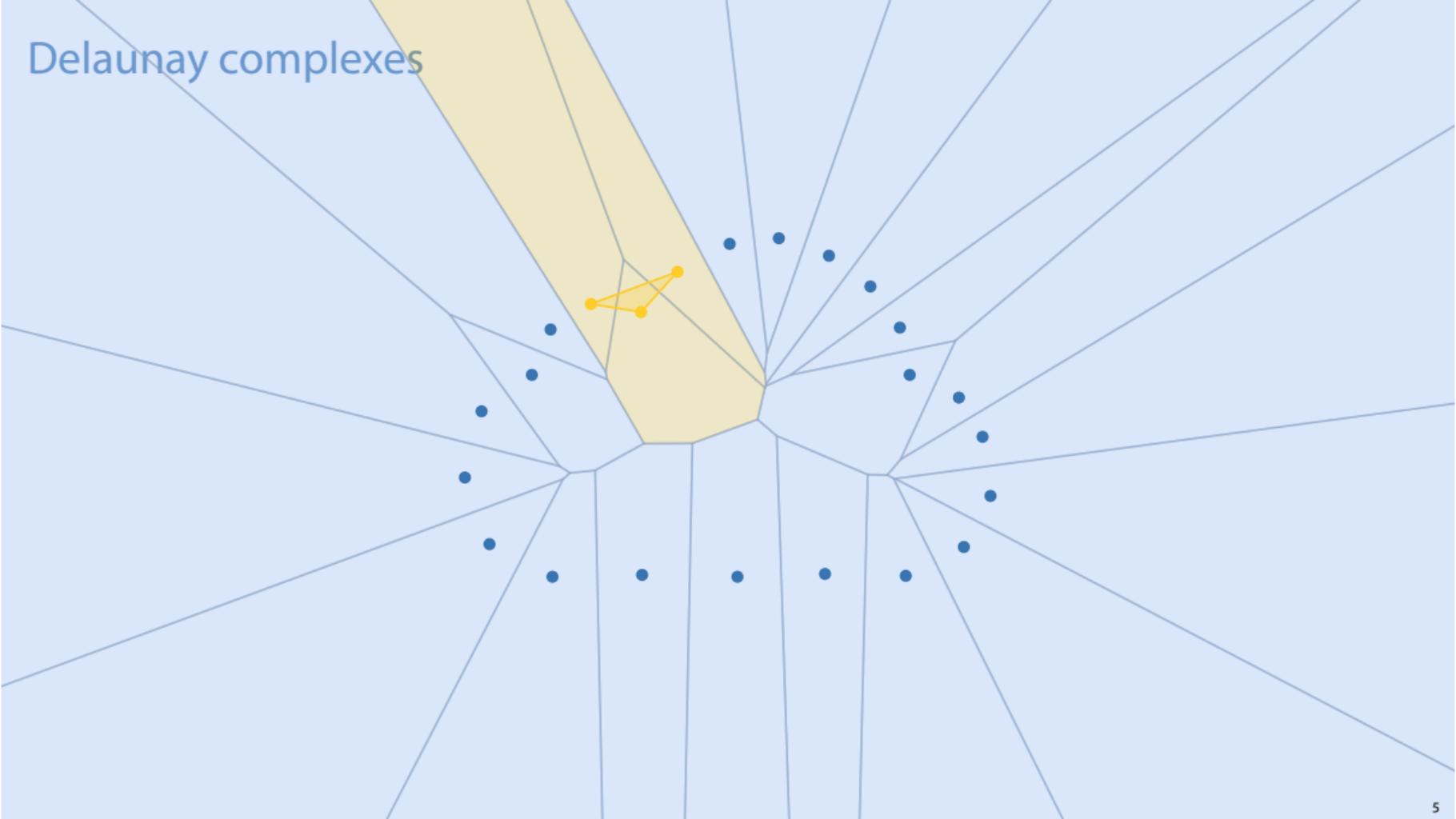
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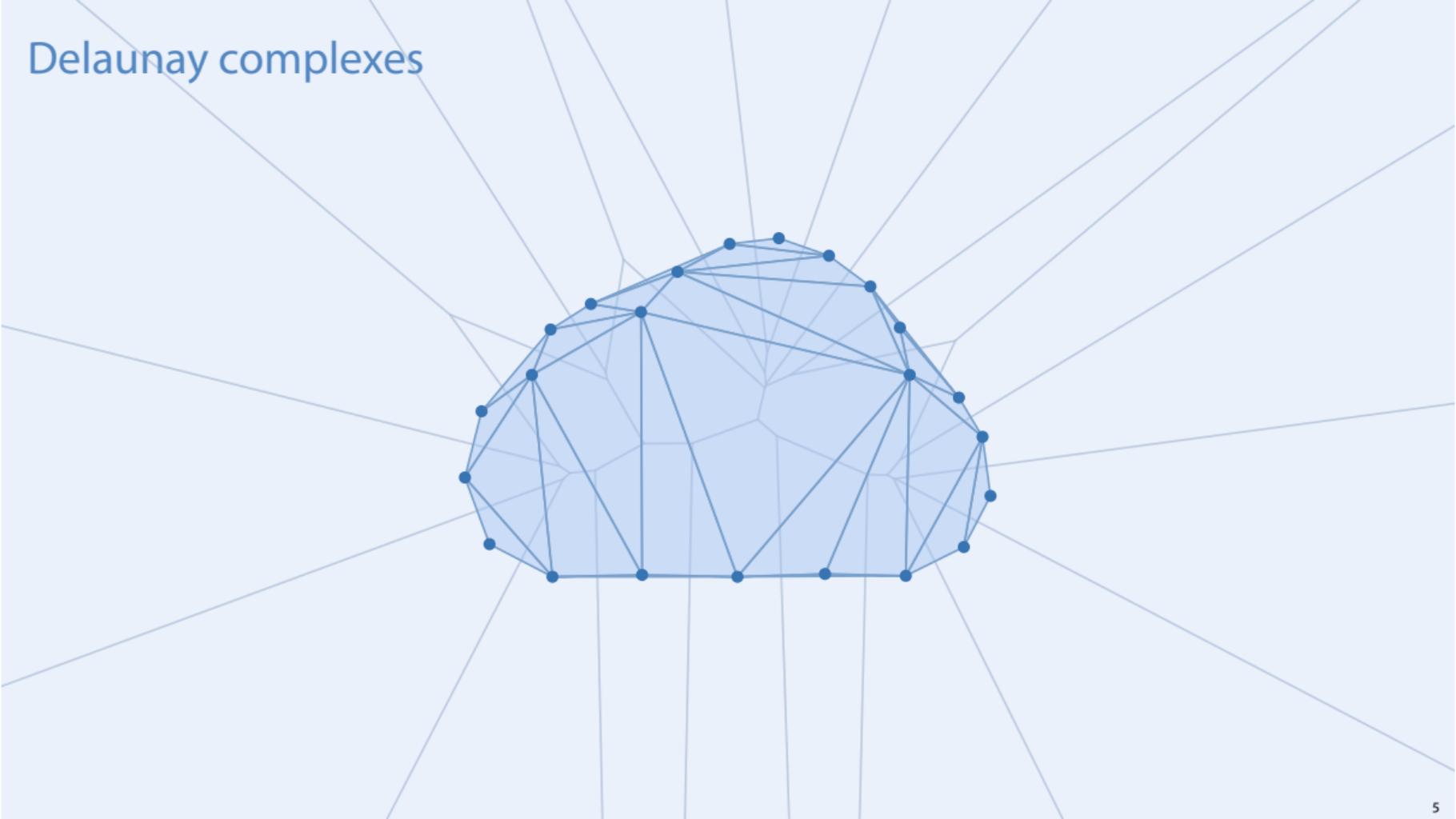
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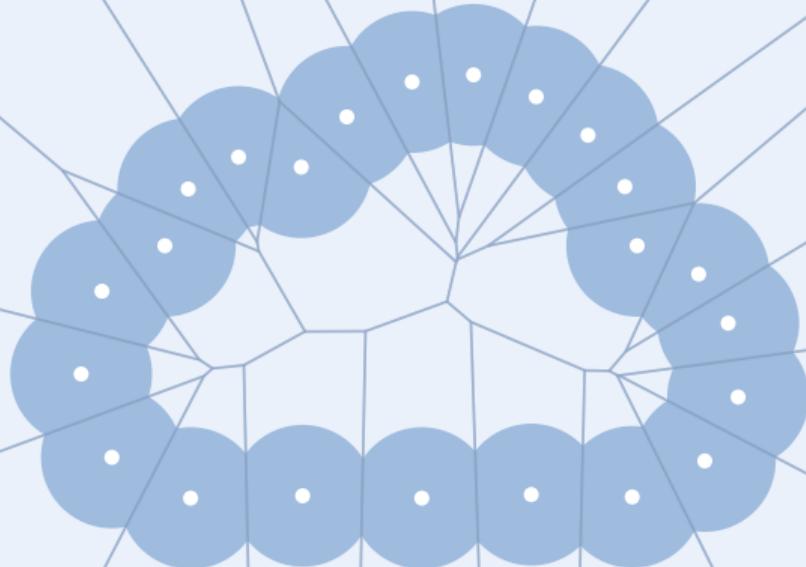
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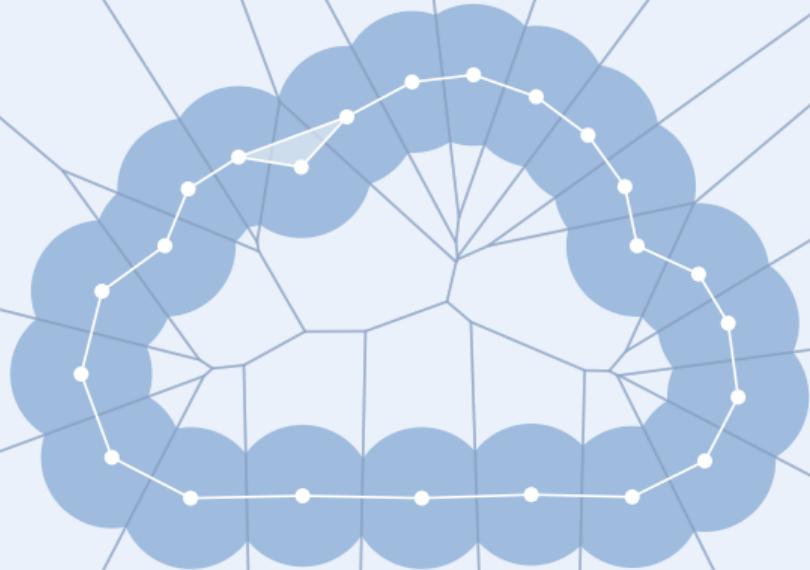
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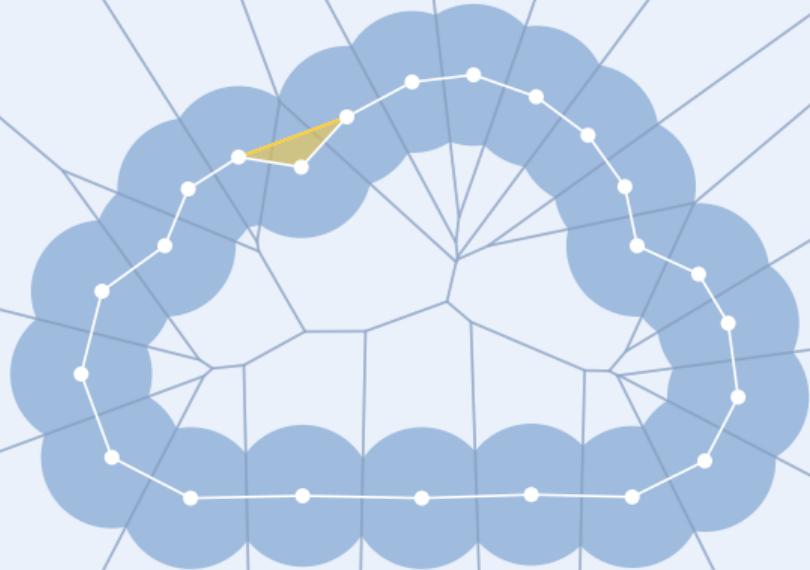
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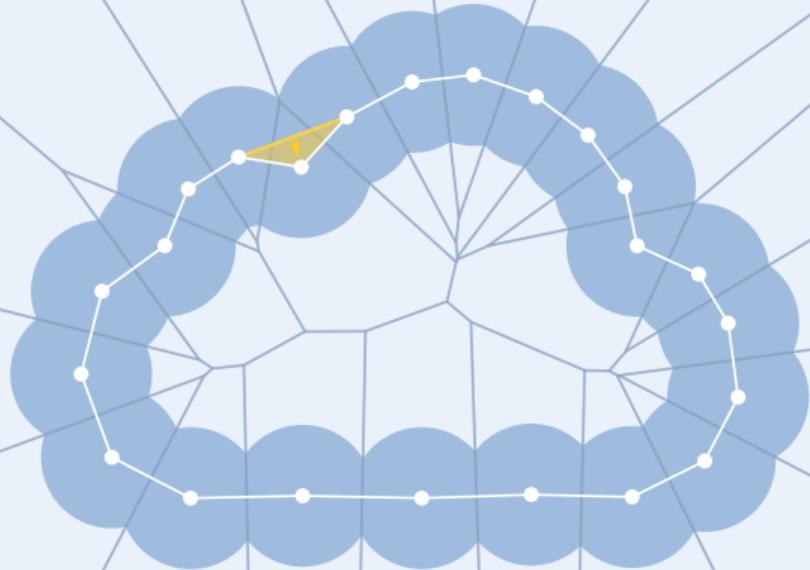
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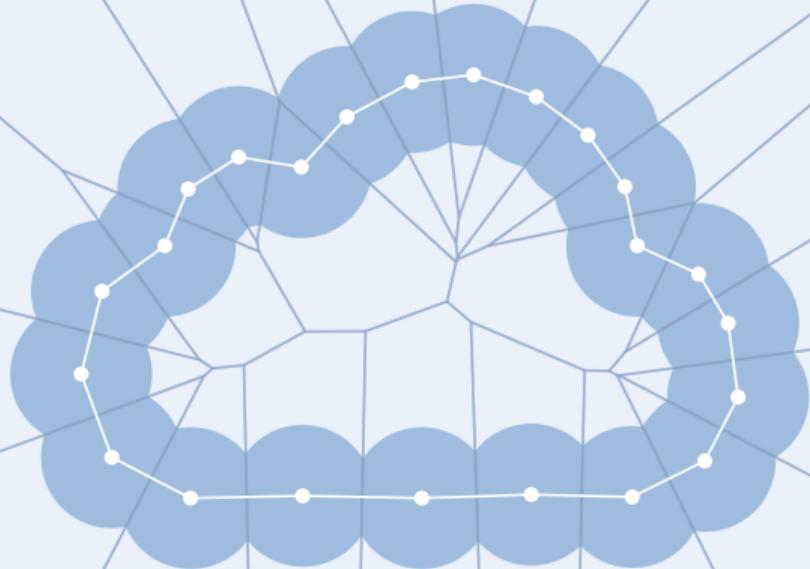
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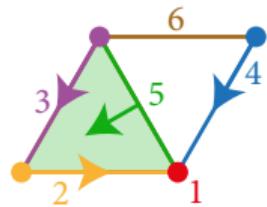
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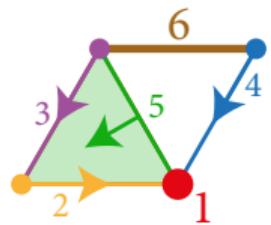
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Discrete Morse theory



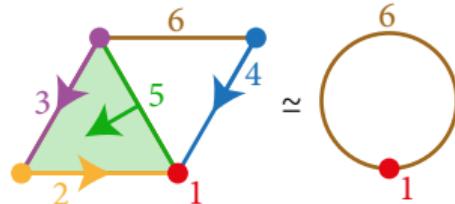
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Theorem (Forman 1998)

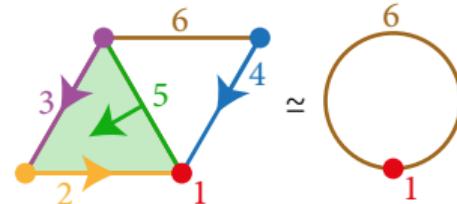
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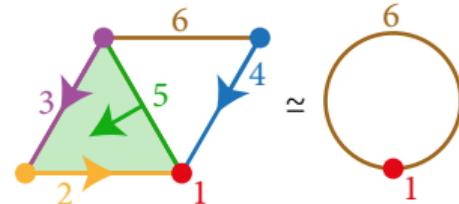
Discrete Morse functions – and their gradients – encode *collapses* of sublevel sets:



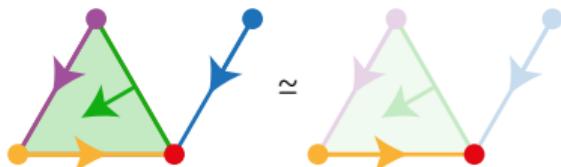
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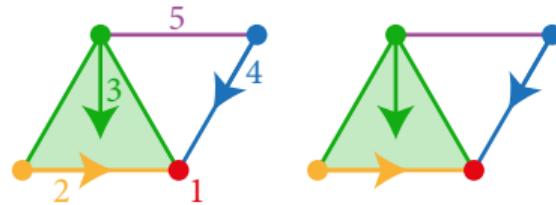


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Generalizing discrete Morse theory

Generalized gradients are a partition into intervals in the face poset (instead of just facet pairs):

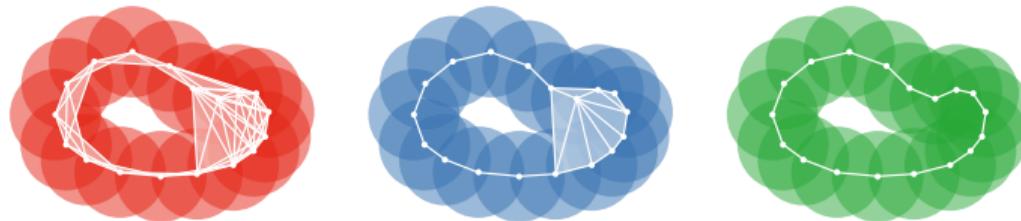


Morse theory for Čech and Delaunay complexes

Theorem (B, Edelsbrunner 2017)

Čech, Delaunay, and Wrap complexes (at any scale r) of a point set $X \subset \mathbb{R}^d$ in general position are related by collapses encoded by a single discrete gradient field:

$$\text{Cech}_r X \searrow \text{Del}_r X \searrow \text{Wrap}_r X.$$



Morse theory of Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

The Čech function and the Delaunay function of a point set $X \subset \mathbb{R}^d$ in general position are generalized discrete Morse functions.

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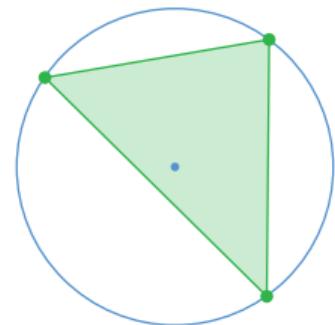
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- $f_D(Q) = f_C(Q)$
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- Q is a critical simplex off f_D
- Q is a centered Delaunay simplex
(containing the circumcenter in the interior)

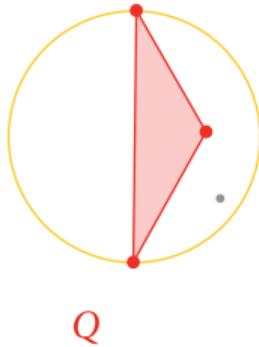


Čech intervals

Lemma

Let $Q \subseteq X$ be a simplex with smallest enclosing sphere S .

Then $Q' \subseteq X$ has the same smallest enclosing sphere S iff



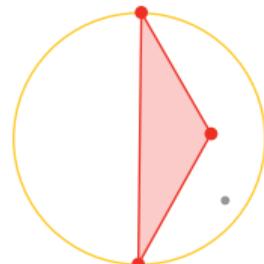
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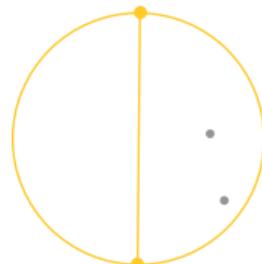
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On $S \subseteq Q'$



Q



On S

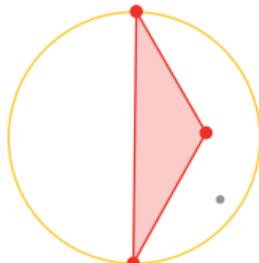
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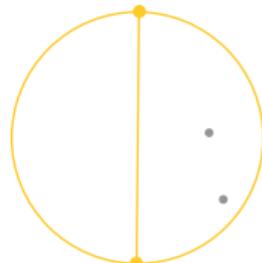
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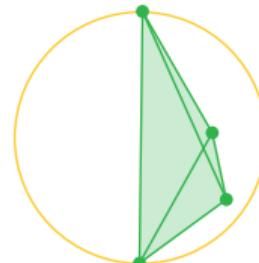
$$\text{On } S \subseteq Q' \subseteq \text{Encl } S.$$



Q



$\text{On } S$



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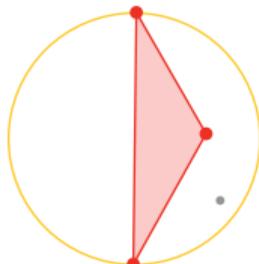
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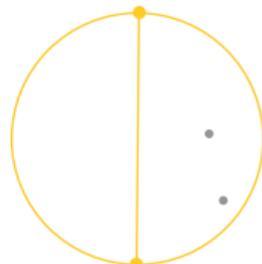
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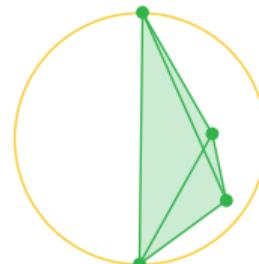
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The front and back faces of a simplex

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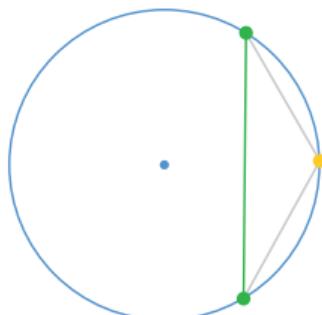
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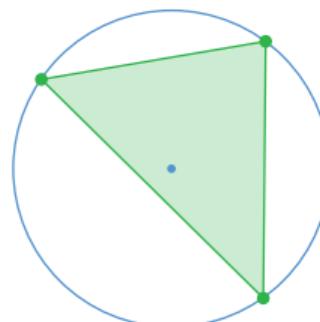
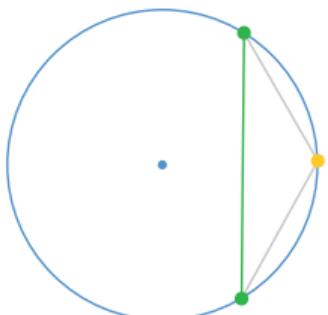
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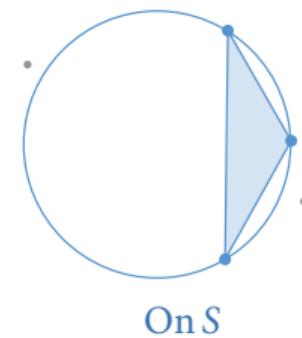
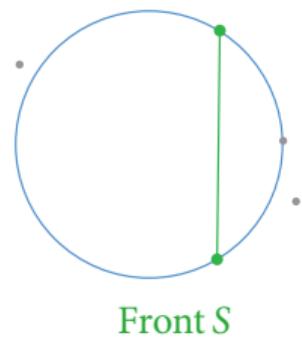
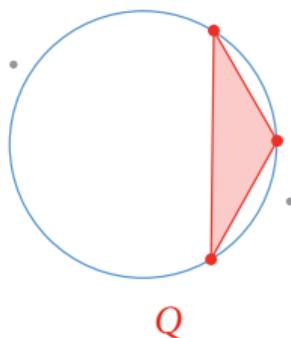


Delaunay intervals

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$\check{\text{C}}$ ech and Delaunay intervals from Karush–Kuhn–Tucker conditions

Proposition

A sphere S is the smallest sphere enclosing Q and excluding E iff

- S is the smallest circumsphere of some simplex $P \subseteq X$,
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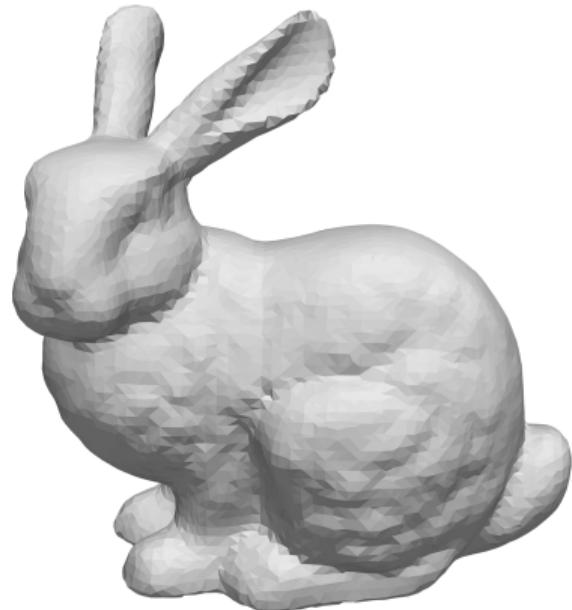
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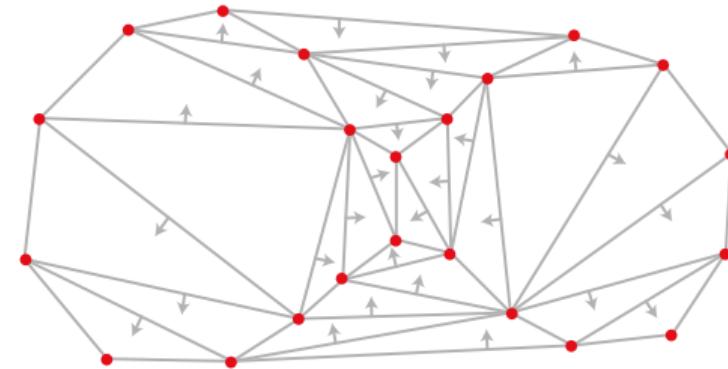
Wrap complexes

Foundation of the surface reconstruction software *Wrap* (Edelsbrunner 1995, Geomagic)



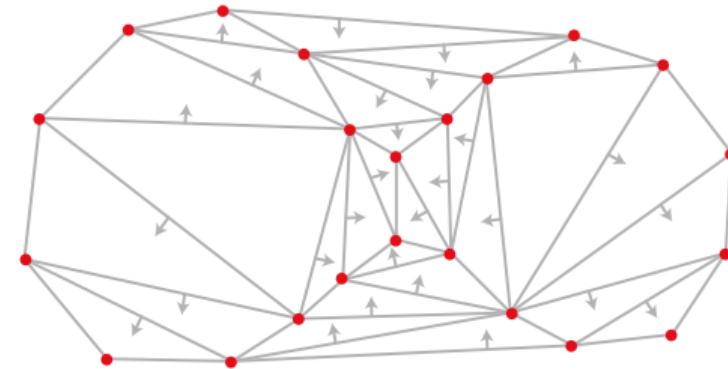
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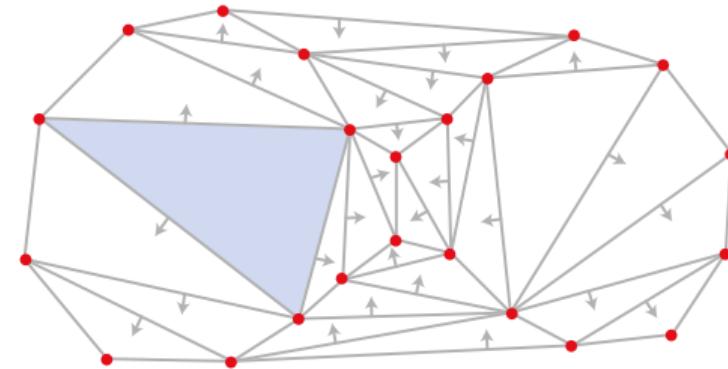
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$\text{Wrap}_r(X)$ is the *descending complex* of V on $\text{Del}_r X$:

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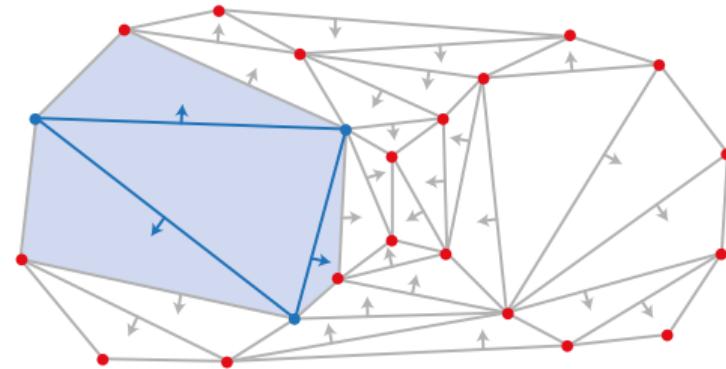
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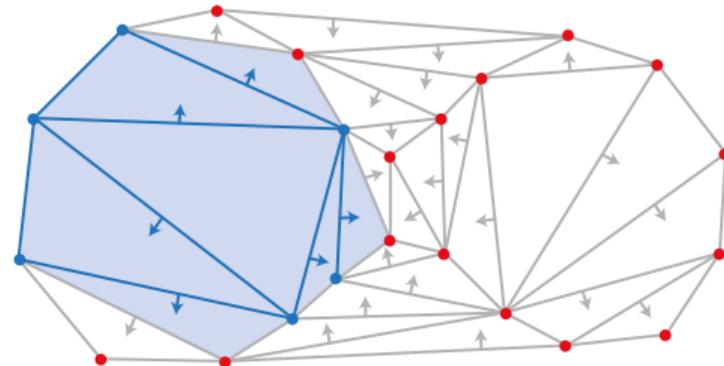
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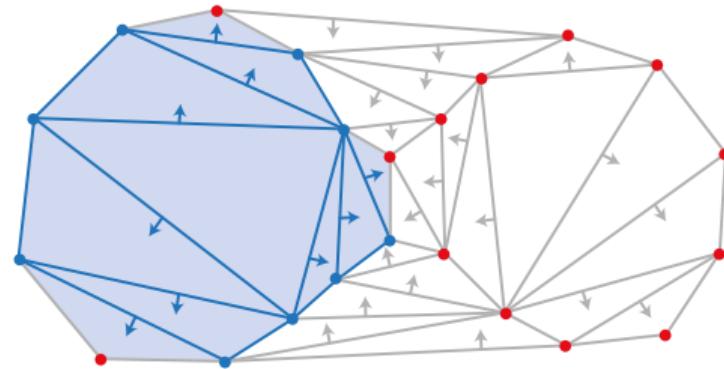
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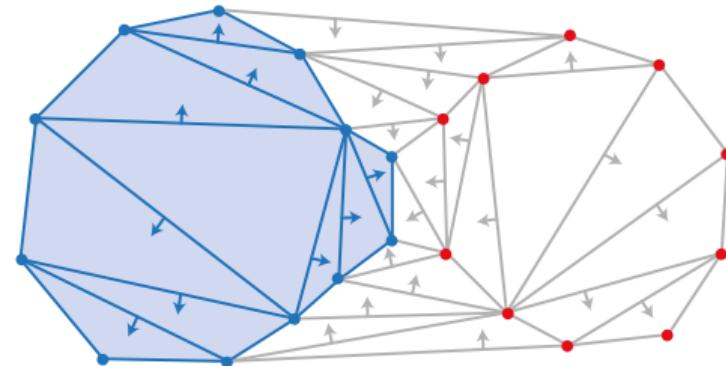
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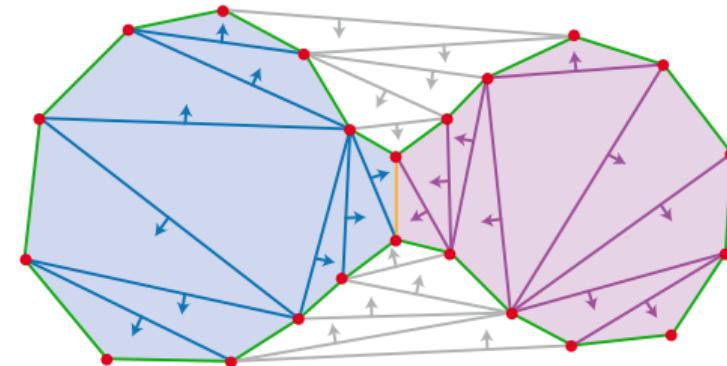
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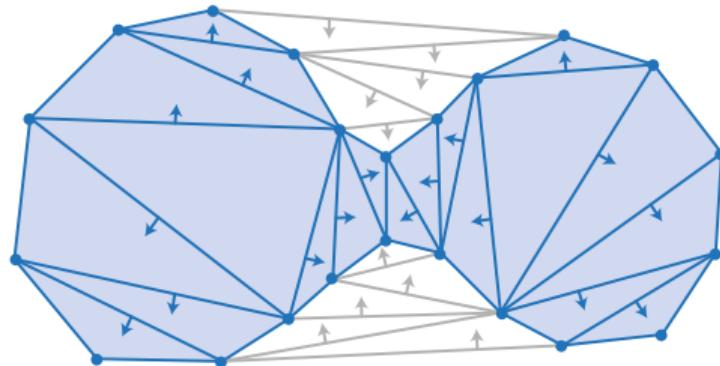
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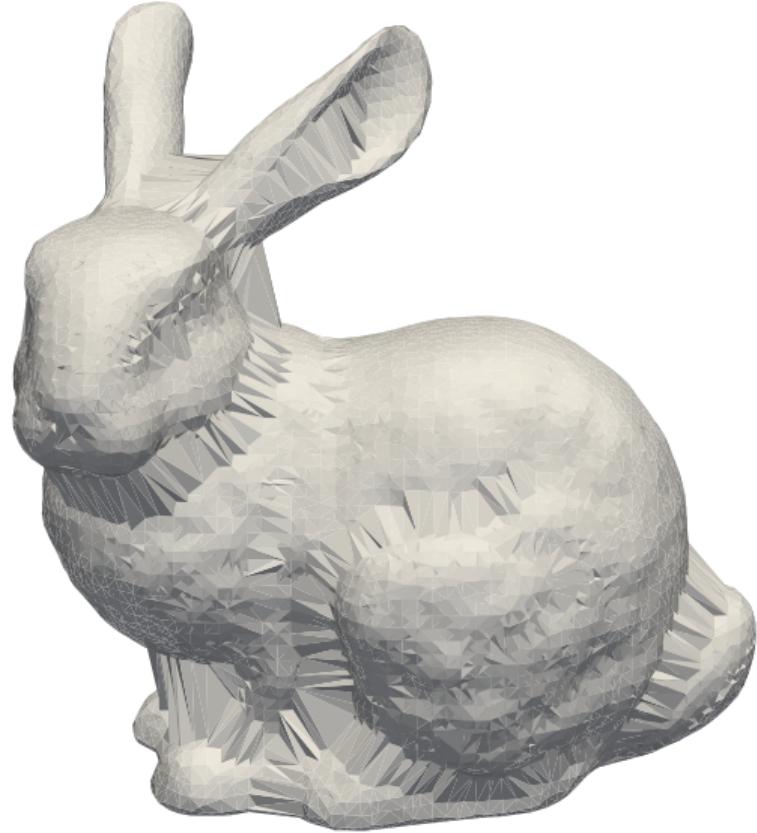


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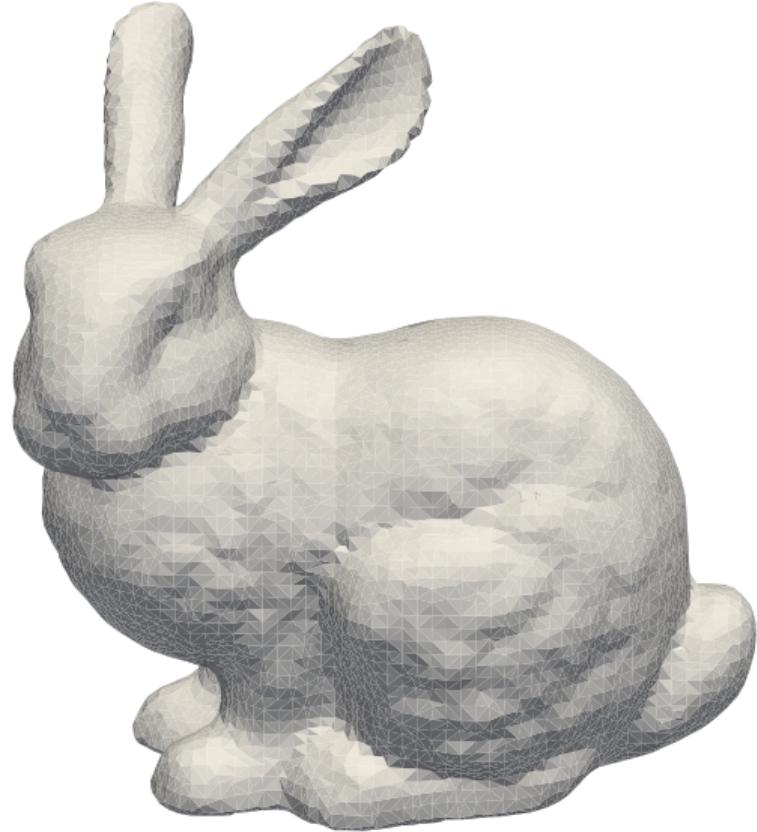
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Delaunay and Wrap complexes



Delaunay and Wrap complexes



Persistence of Delaunay filtrations

Algorithm (matrix reduction)

Require: $D: m \times n$ matrix

Ensure: $R = D \cdot V$ is reduced, V is full rank upper triangular

function Reduce(D)

$R = D, V = I(n)$

while there exist $i < j$ such that pivot $R_i =$ pivot R_j **do**

 add column R_i to column R_j

 add column V_i to column V_j

return R, V

Algorithm (exhaustive matrix reduction)

Require: $D: m \times n$ matrix

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$R = D, V = I(n)$

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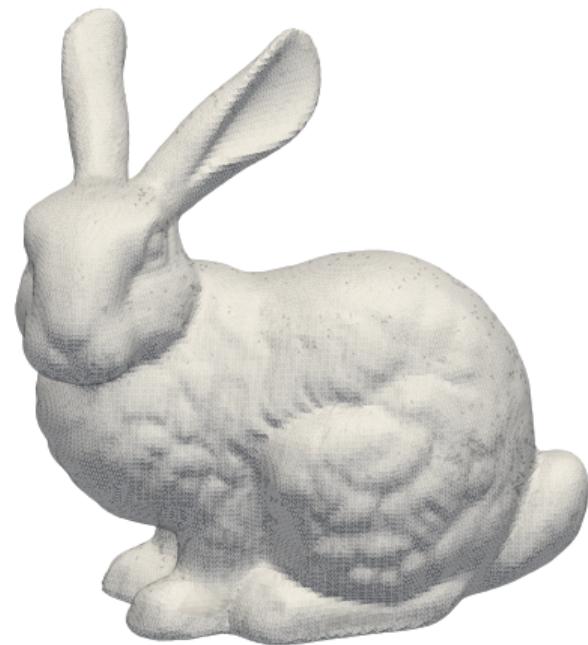
Proposition

The resulting columns R_j are minimal (in a lexicographic order) within their homology class (in K_{j-1}).

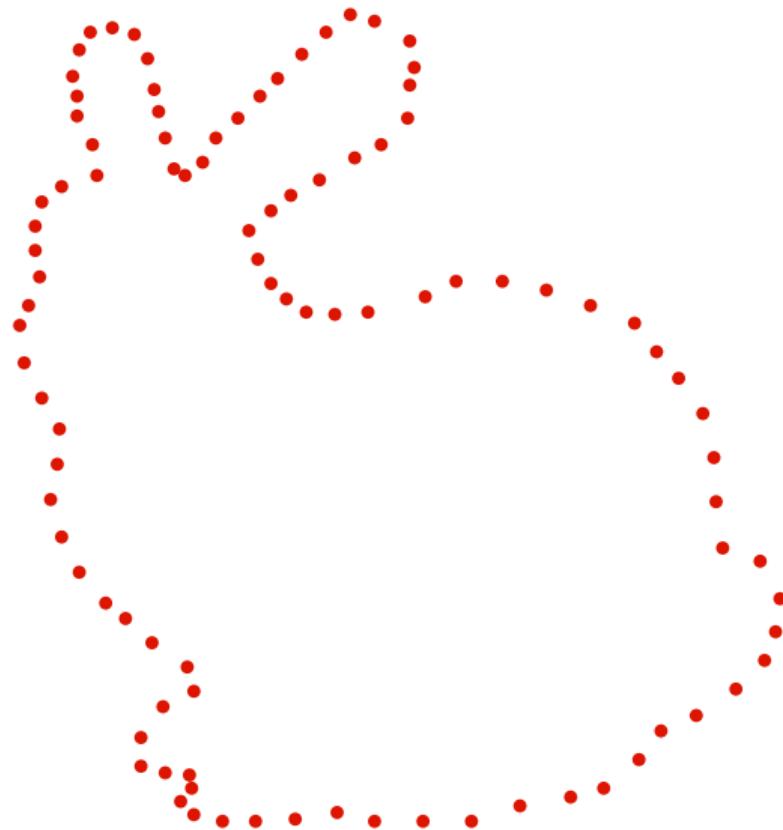
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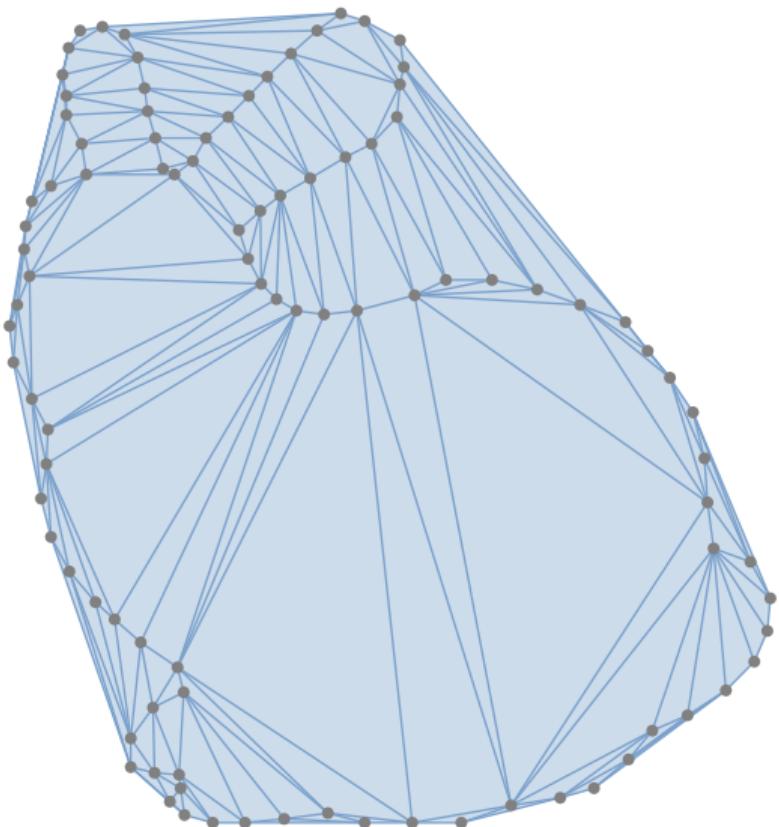
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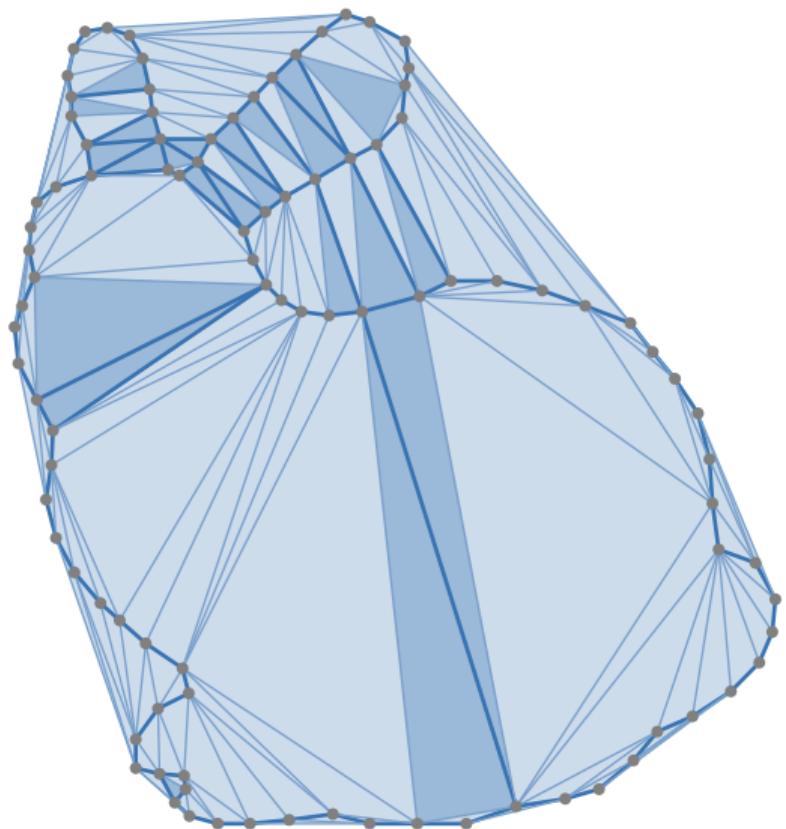
Wrap complexes and lexicographically minimal cycles



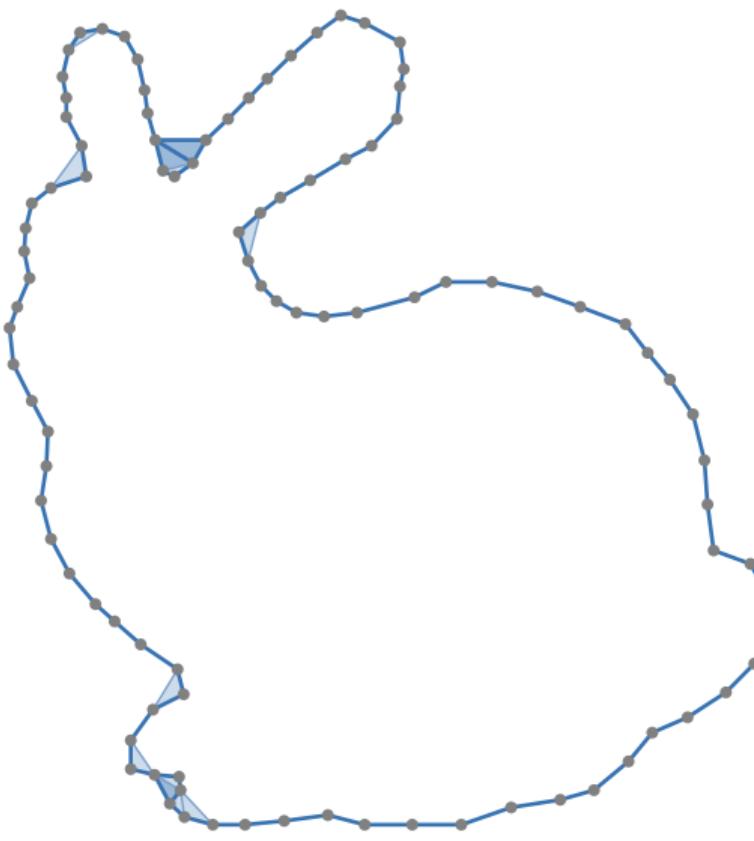
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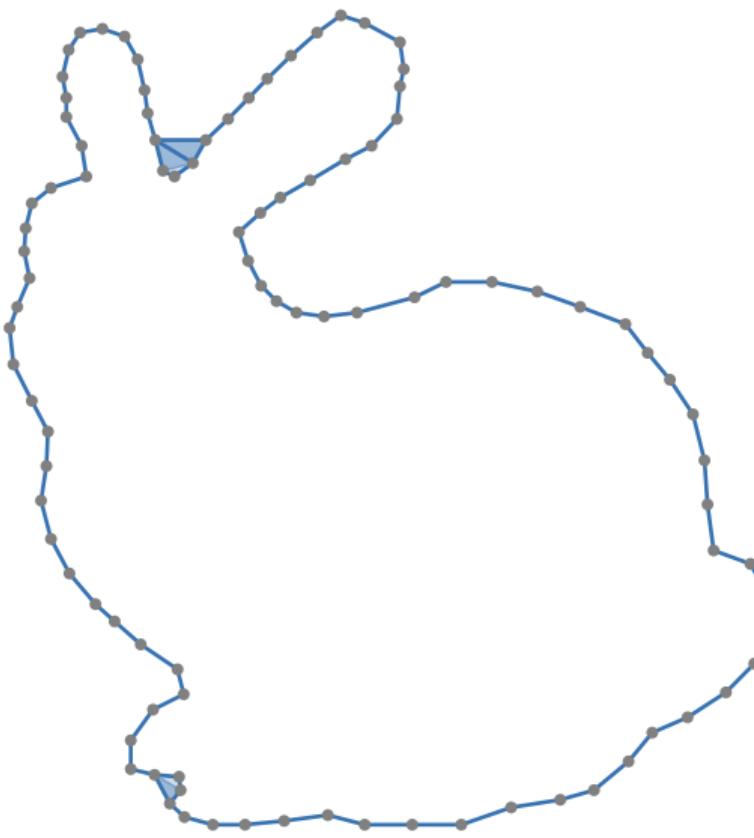
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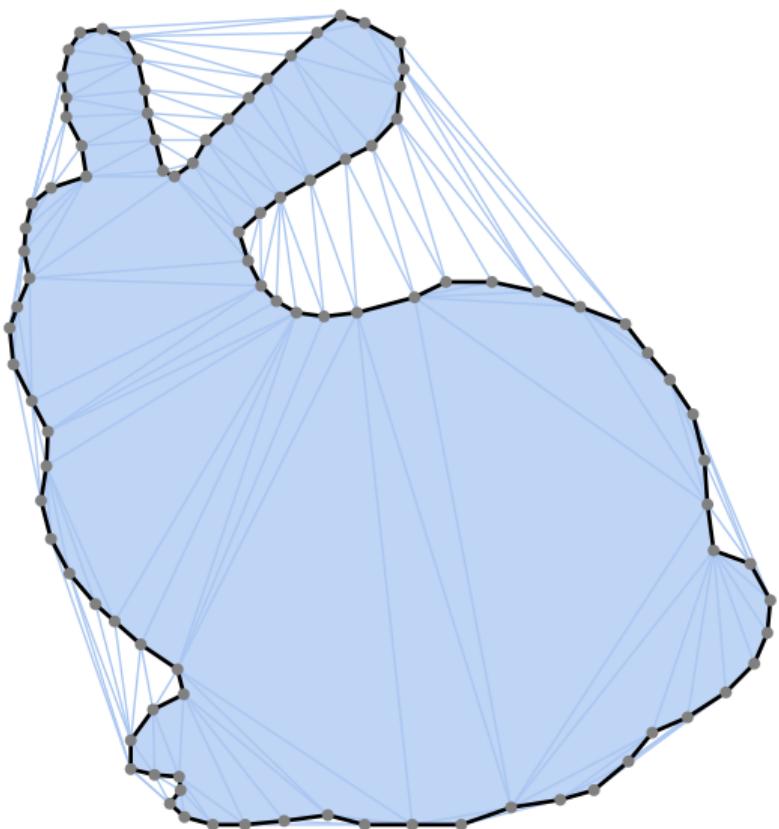
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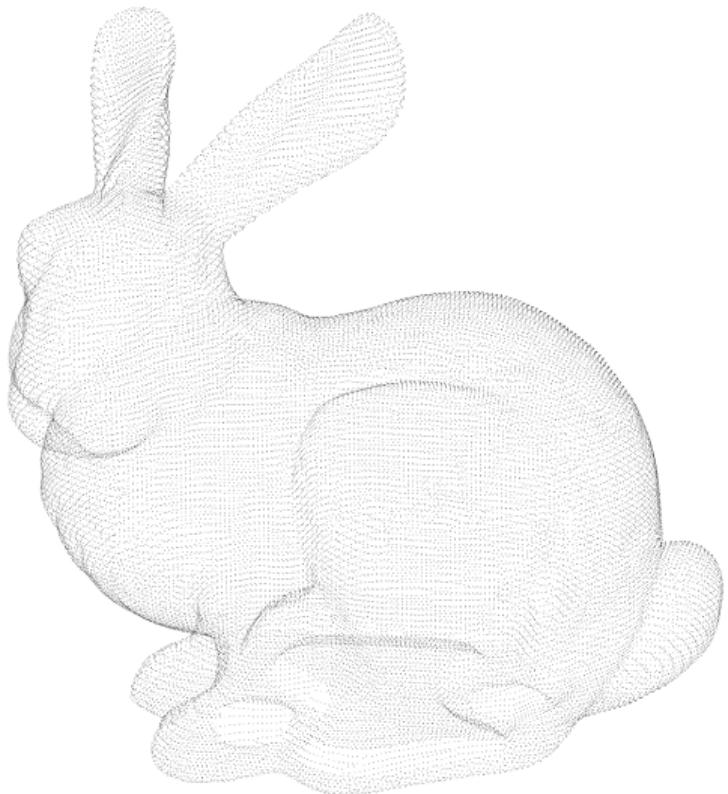
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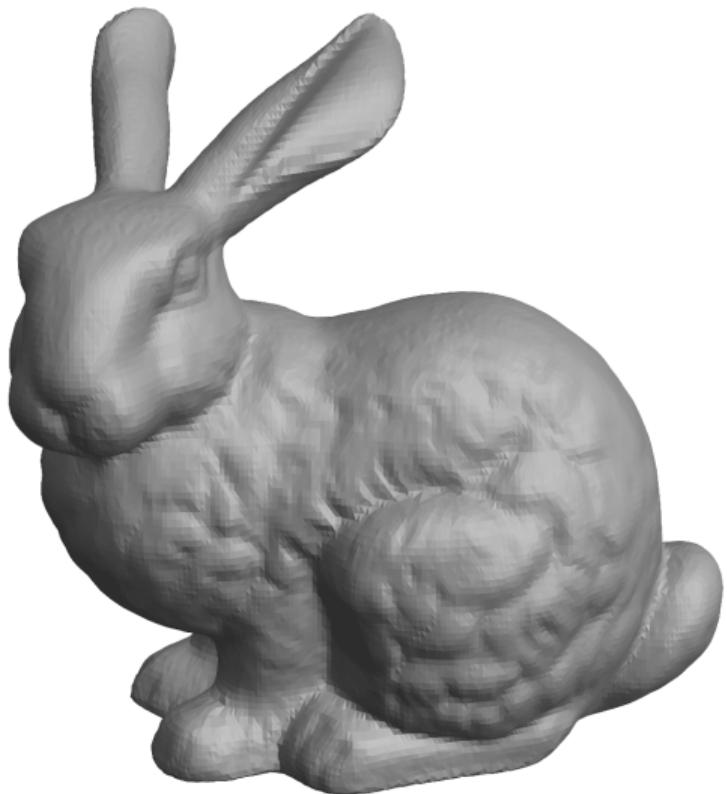
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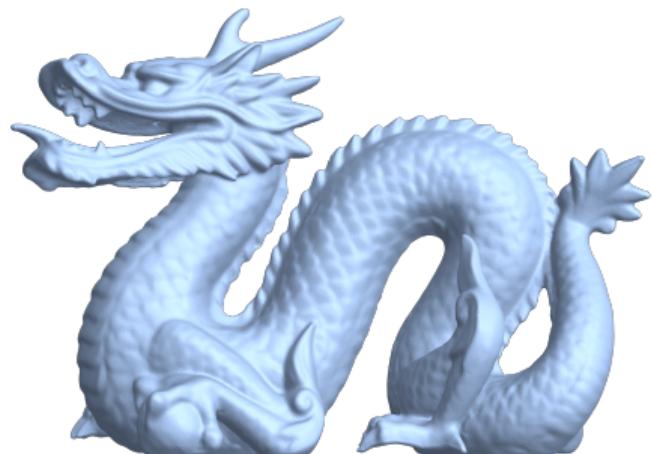
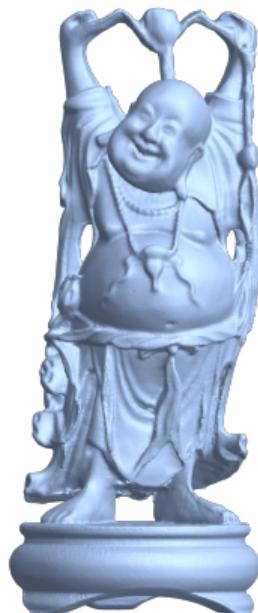
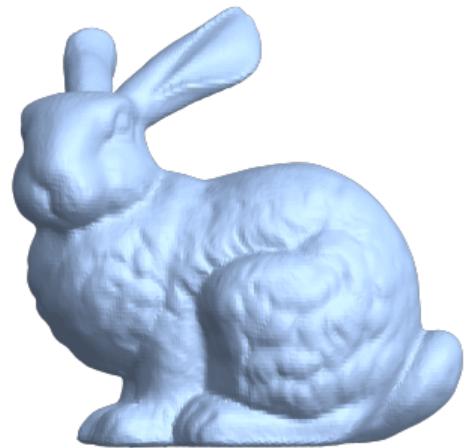
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Point cloud reconstruction with minimal cycles

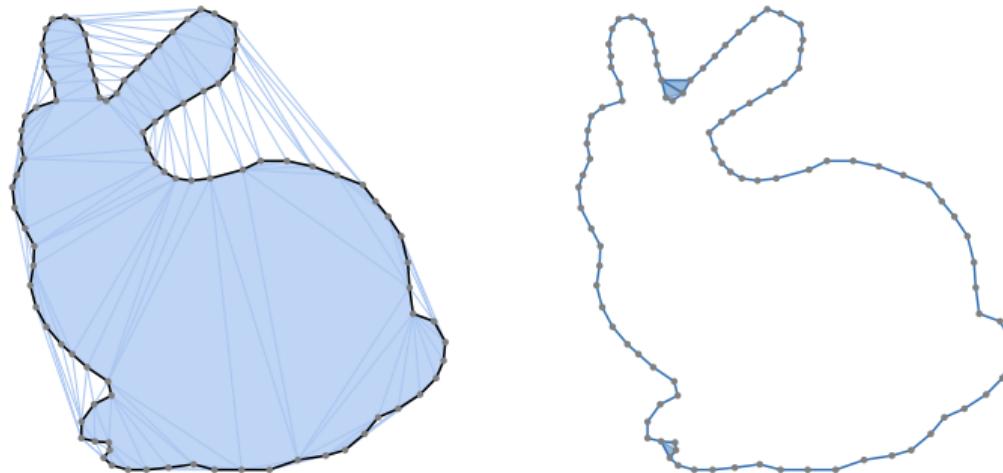


Wrap complexes support minimal cycles

Theorem (B, Roll 2022)

Let $X \subset \mathbb{R}$ be a finite subset in general position and let $r \in \mathbb{R}$.

- Exhaustive matrix reduction computes the minimal cycles homologous to a simplex boundary.
- Any lexicographically minimal cycle of $\text{Del}_r(X)$ is supported on $\text{Wrap}_r(X)$.

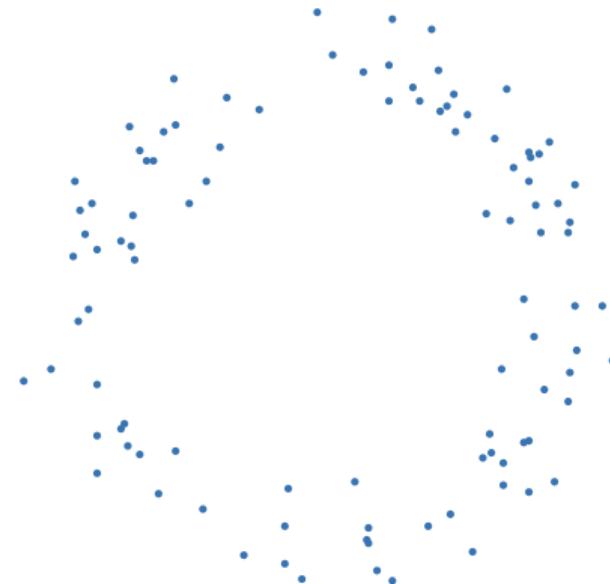


Vietoris–Rips persistence

Vietoris–Rips complexes

For a metric space X , the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

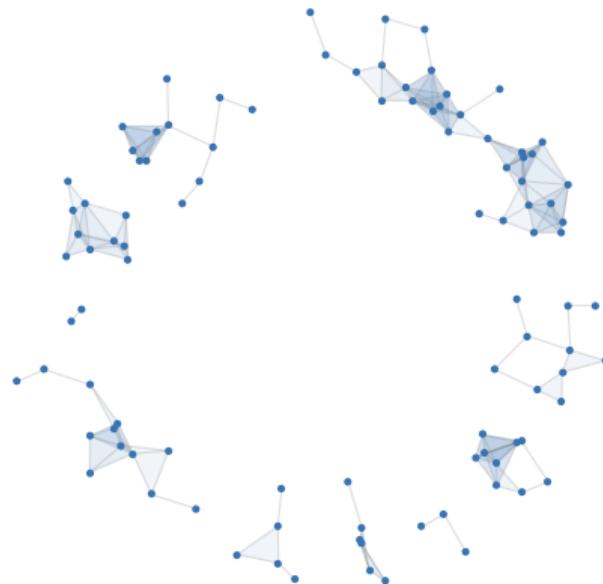
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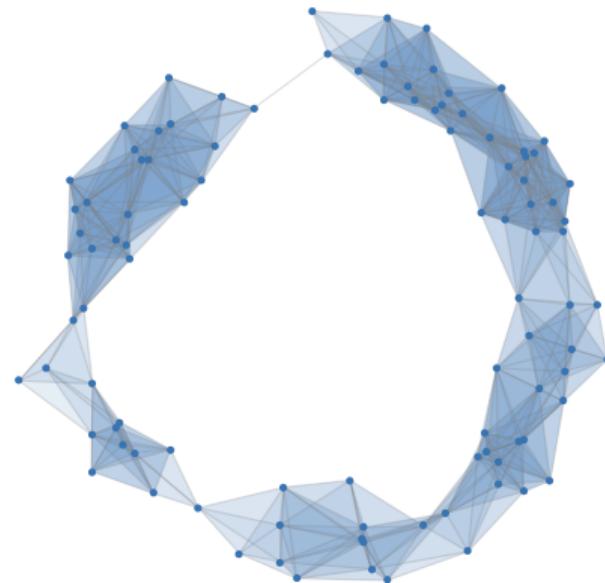
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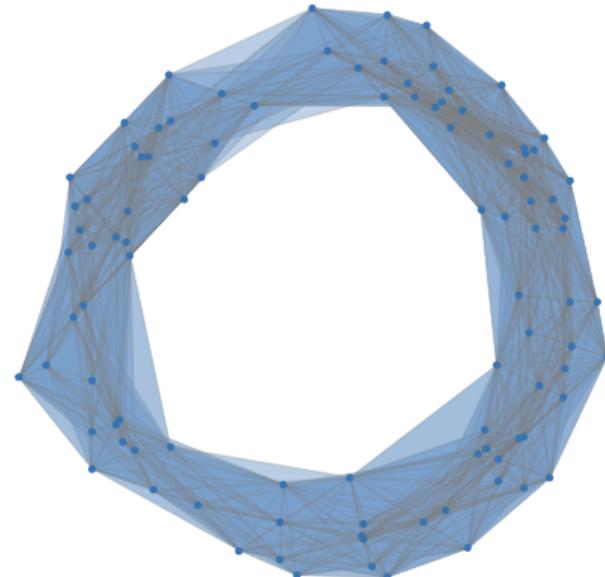
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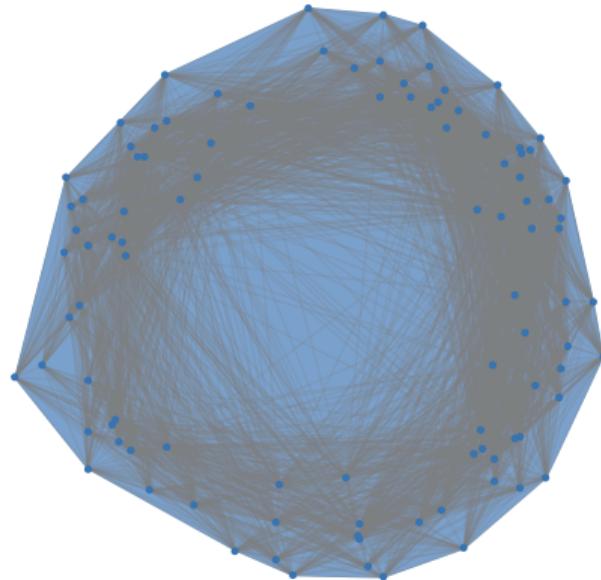
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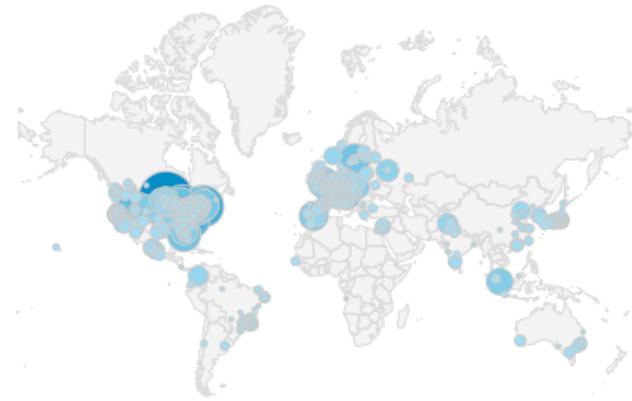
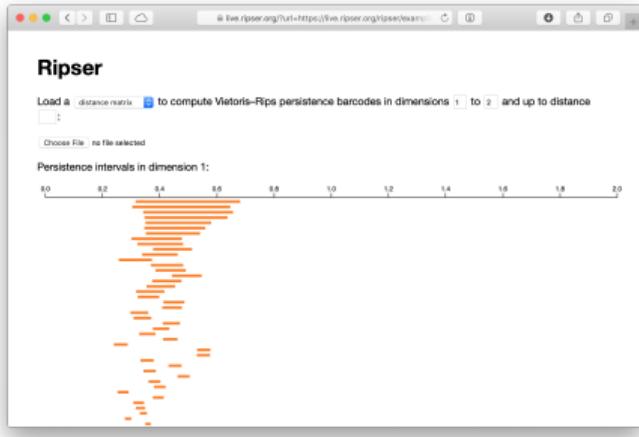
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Ripser: software for computing Vietoris–Rips persistence barcodes

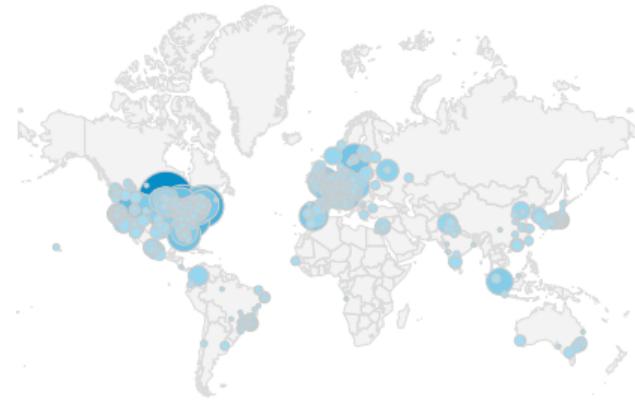
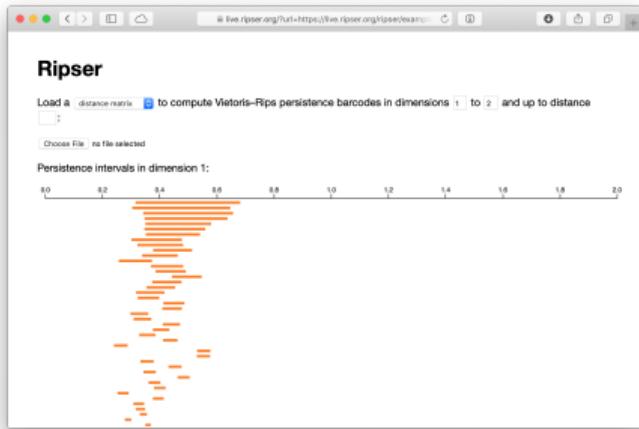
Open source software (ripser.org)



Ripser users across the globe

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Computational improvements based on

- *implicit matrix representations*
- *apparent pairs*, connecting persistence to discrete Morse theory

Apparent pairs

Ripser uses the following pairing of simplices (breaking ties in the filtration lexicographically):

Definition (B 2016, 2021)

In a simplexwise filtration ($K_i = \{\sigma_1, \dots, \sigma_i\}$)_i, two simplices (σ_i, σ_j) form an *apparent pair* if

- σ_i is the latest proper face of σ_j , and
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The apparent pairs are both

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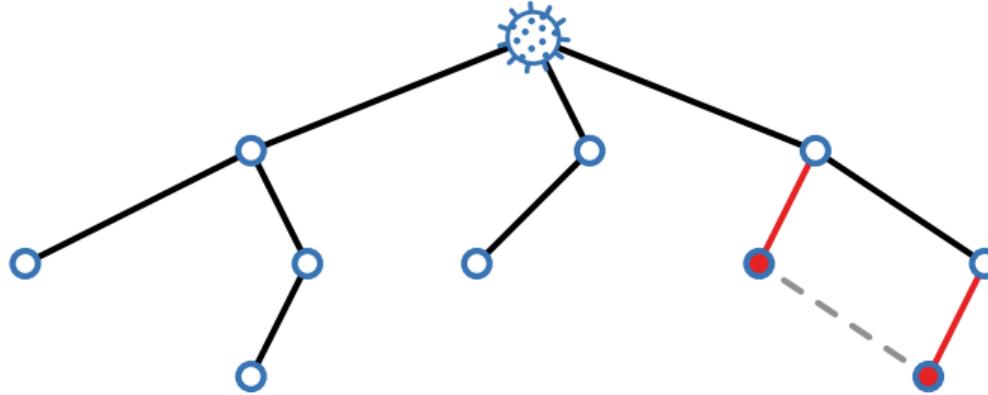
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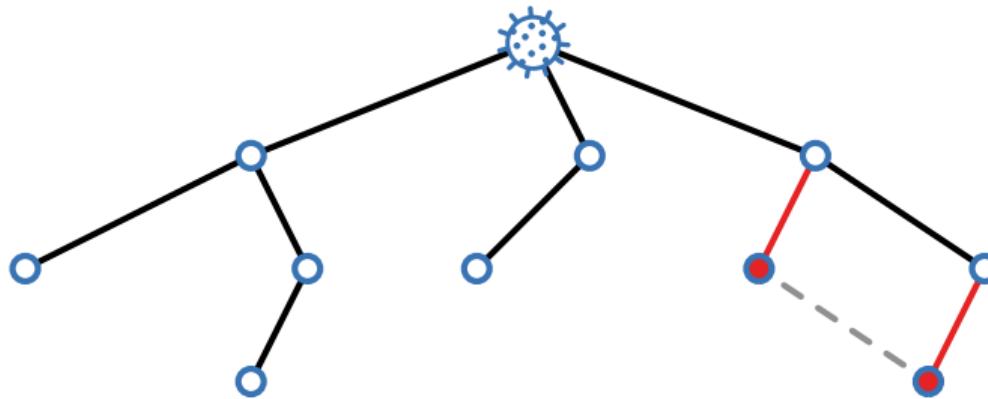
Any generalized discrete Morse function is refined by apparent pairs.

Topology of viral evolution



Joint work with: A. Ott, M. Bleher, L. Hahn (Heidelberg), R. Rabadan, J. Patiño-Galindo (Columbia), M. Carrière (INRIA)

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Observation: Ripser runs unusually fast on genetic distance data

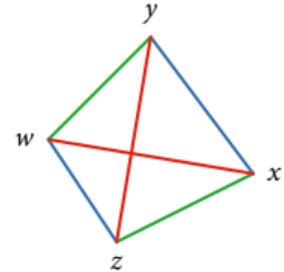
- SARS-CoV2 RNA sequences (spike protein)
- 25556 data points (2.8×10^{12} simplices in 2-skeleton)
- 120 s computation time (with data points ordered appropriately)

Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ -hyperbolic (for $\delta \geq 0$) if for all $w, x, y, z \in X$ we have

$$d(w, x) + d(y, z) \leq \max\{d(w, y) + d(x, z), d(w, z) + d(x, y)\} + 2\delta.$$

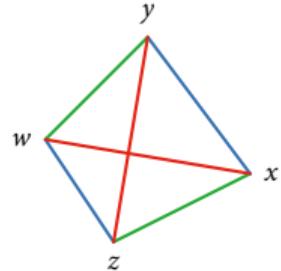


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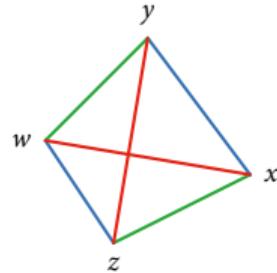


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- The 0-hyperbolic spaces are precisely the metric trees and their subspaces.



Rips Contractibility

Theorem (Rips; Gromov 1988)

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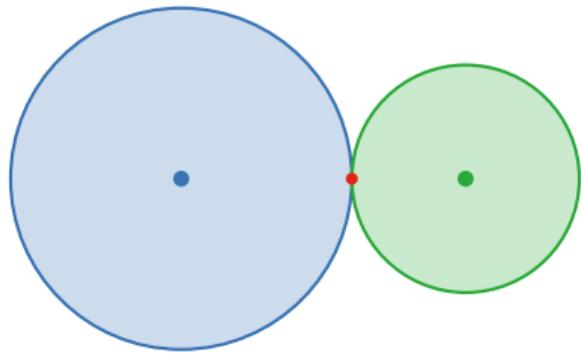
Theorem (B, Roll 2022)

Let X be a finite δ -hyperbolic space. Then there is a single discrete gradient encoding the collapses

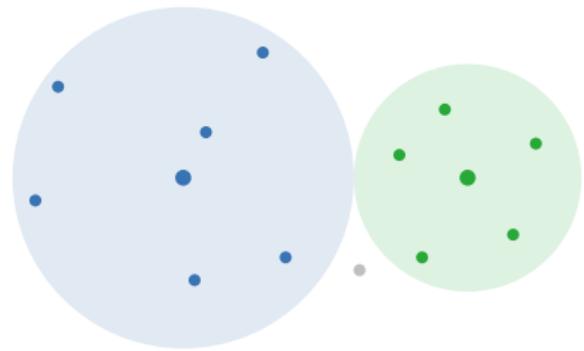
$$\text{Rips}_u(X) \searrow \text{Rips}_t(X) \searrow \{\ast\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect of X .

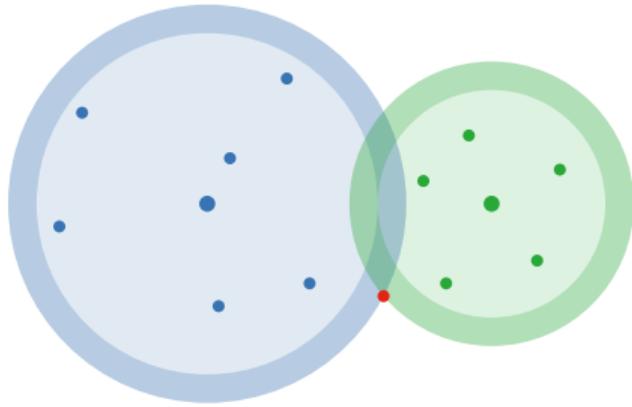
Geodesic defect



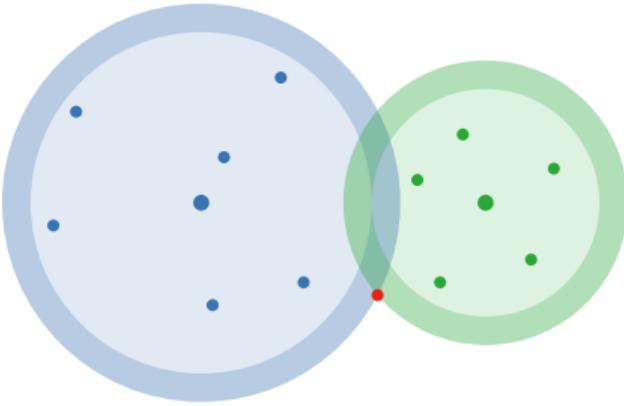
Geodesic defect



Geodesic defect



Geodesic defect



Definition (Bonk, Schramm 2000)

A metric space X is ν -geodesic if for all points $x, y \in X$ and all $r, s \geq 0$ with $r + s = d(x, y)$ we have

$$B_{r+\nu}(x) \cap B_{s+\nu}(y) \neq \emptyset.$$

The infimum of all such ν is the *geodesic defect* of X .

The diameter function of generic trees

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Consider a finite weighted tree (V, E) with a generic path length metric (distinct pairwise distances). Then the diameter function $\text{diam}: \Delta(V) \rightarrow \mathbb{R}$ is a generalized discrete Morse function.

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Consider a finite weighted tree (V, E) with a generic path length metric (distinct pairwise distances). Then the diameter function $\text{diam}: \Delta(V) \rightarrow \mathbb{R}$ is a generalized discrete Morse function.

- The apparent pairs refine this gradient.

Theorem (B, Roll 2022)

The apparent pairs of the diameter function for a generic tree metric space X induces the collapses

- $\text{Rips}_t(X) \searrow T_t$ for all $t \in \mathbb{R}$,
- $\text{Rips}_t(X) \searrow T \searrow \{\ast\}$ for $t \geq \max d(E)$, and
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In particular, the persistent homology is trivial in degrees > 0 .

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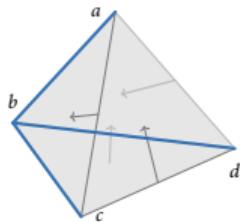
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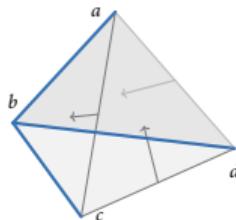
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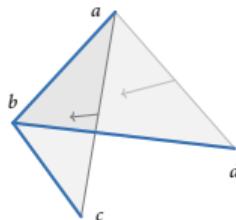
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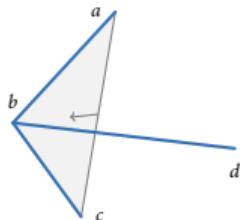
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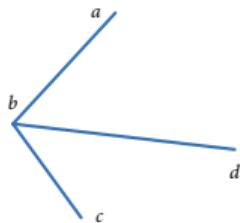
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