Criticality and simplification

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What is a critical point/value?

Classical definition:

For a smooth function $f : M \to \mathbb{R}$:

- x is a *critical point* of f: differential $D_x f \equiv 0$
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For a general smooth map $f : M \to N$:

- x is a *critical point* of f: differential $D_x f$ is not surjective

Homological critical values

$M_t = f^{-1}(-\infty, t]$: *sublevel set* of $f : M \to \mathbb{R}$

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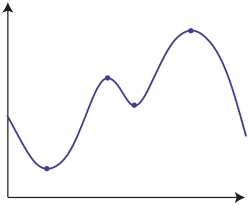
- also applicable to non-smooth functions
- homological critical values are stable
- a homological regular value might be critical in the classical sense!

Topological denoising of a function



Noise (even small) can create lots of critical points

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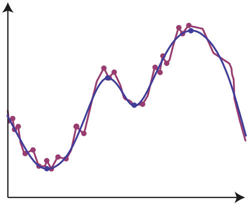

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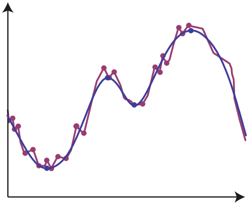

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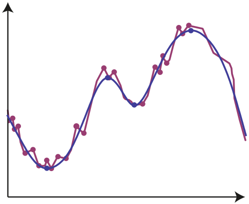

Noise (even small) can create lots of critical points

Topological denoising by simplification of critical set:

- removing critical points caused by noise
- given a function f on a surface and $\delta > 0$,
  find a function f_δ that:
  - minimizes number of critical points
  - stays close to input function: $\|f_\delta - f\|_\infty < \delta$

Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for sublevel sets

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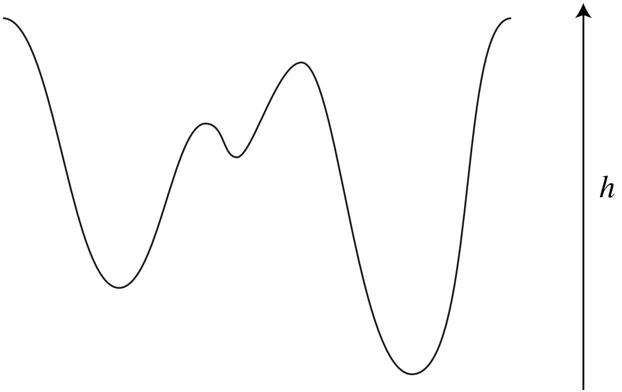
Example: connected components in 1D



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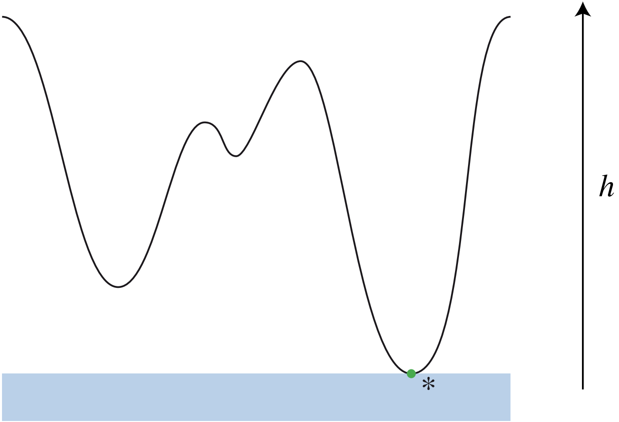

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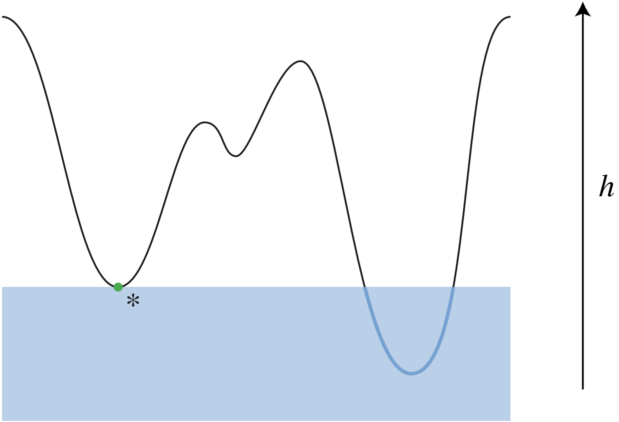

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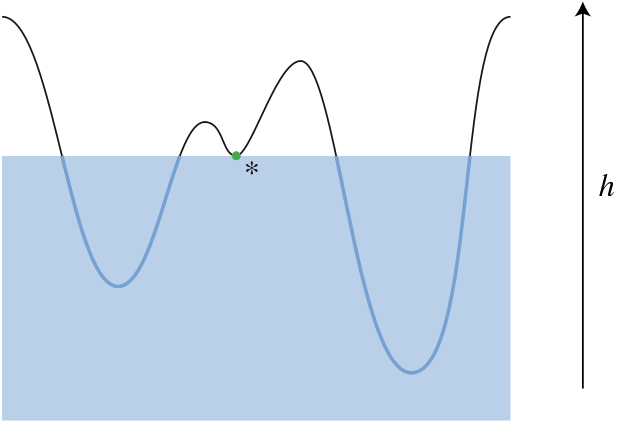

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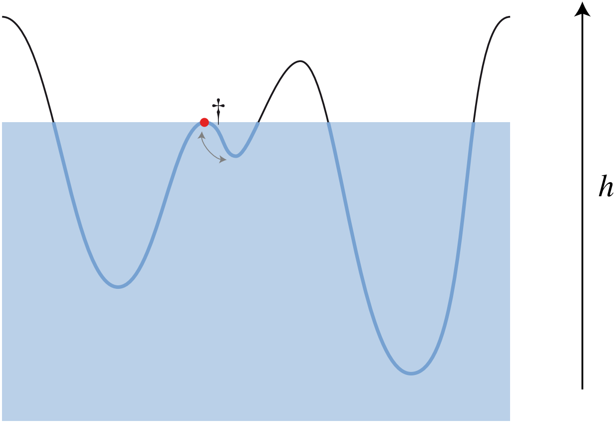

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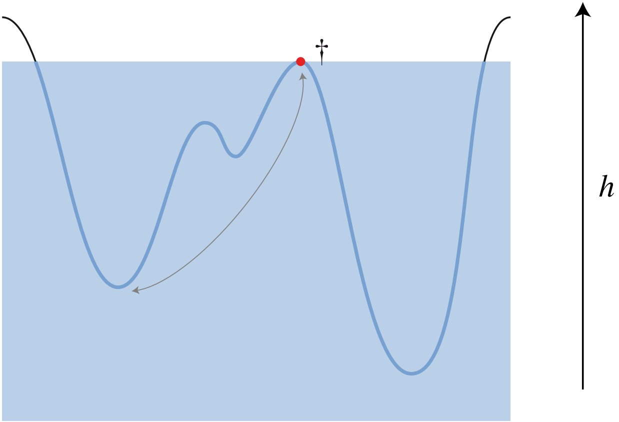

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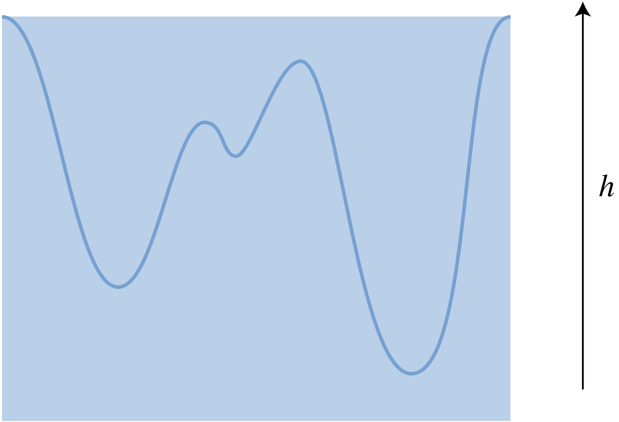
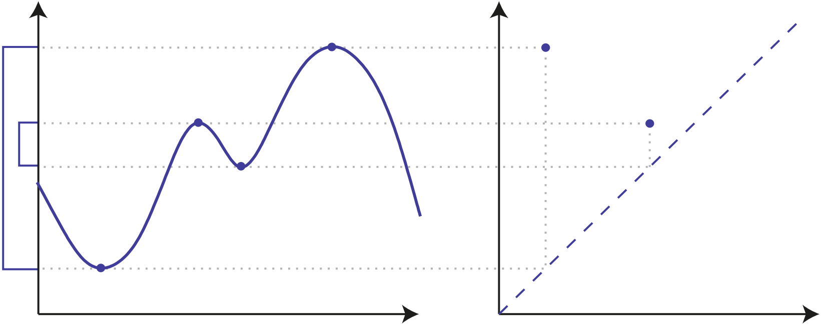


Persistence diagrams [Cohen-Steiner et al., 2005]



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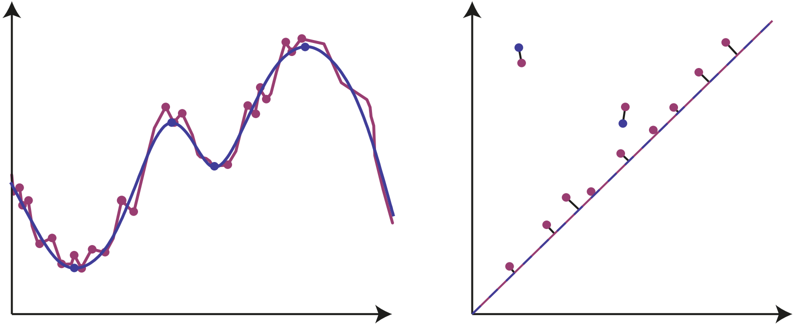
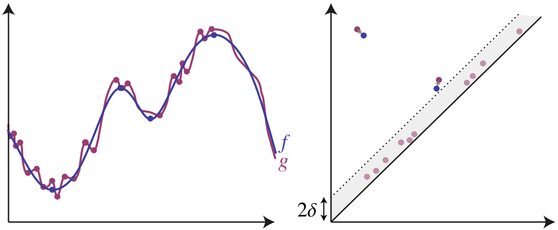


Stability of persistence diagrams

Theorem (Cohen-Steiner et al., 2005)

Let $\|f - g\|_\infty \leq \delta$.

The persistence pairs of f *that have persistence* $> 2\delta$ *can be mapped injectively to the persistence pairs of* g.



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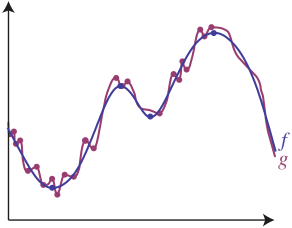
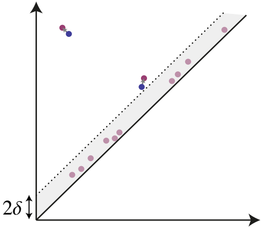
Corresponding points p_f, p_g in the persistence diagrams have distance $\|p_f - p_g\|_\infty \leq \delta$.





A bound on number of critical points

Corollary

Let f be a discrete Morse function on a surface and let $\delta > 0$.

Then for every function f_δ with $\|f_\delta - f\|_\infty < \delta$ we have:

$$\begin{aligned} &\#\text{ critical points of } f_\delta \\ \geq\ &\#\text{ critical points of } f \text{ with persistence} \geq 2\delta. \end{aligned}$$

Persistence-guided simplification

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Idea for simplifying critical points [Edelsbrunner et al. 2006, Attali et al. 2009]:

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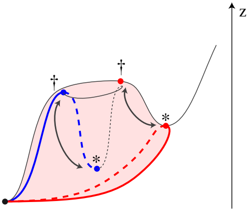
Idea for simplifying critical points [Edelsbrunner et al. 2006, Attali et al. 2009]:

- remove all persistence pairs of f with persistence $< 2\delta$
- leave all other persistence pairs unmodified

Side-effects of elimination



input

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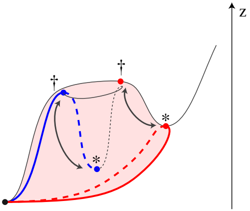

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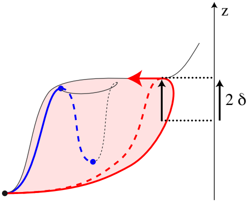

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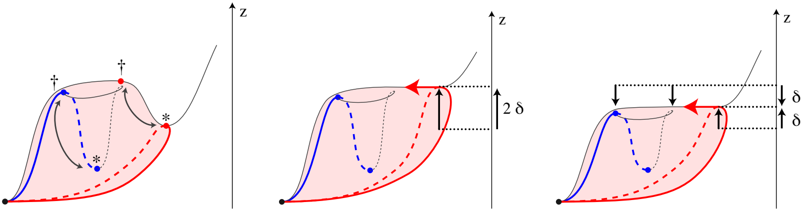
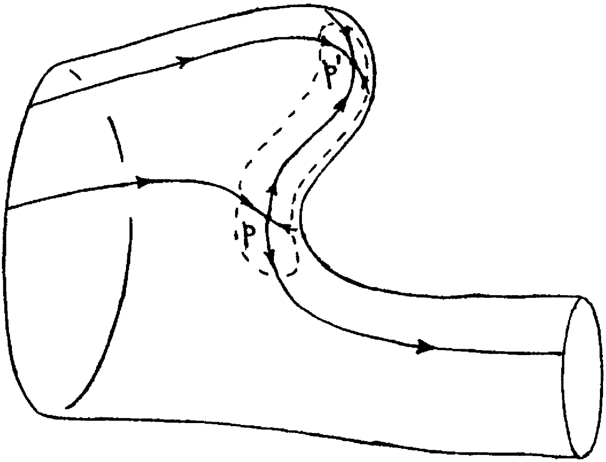


input not optimal! optimal

For an optimal solution, we must allow the critical values to change!

Canceling critical points of a gradient field



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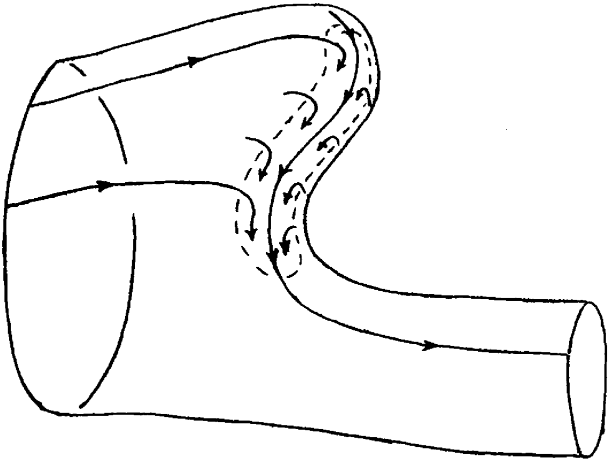

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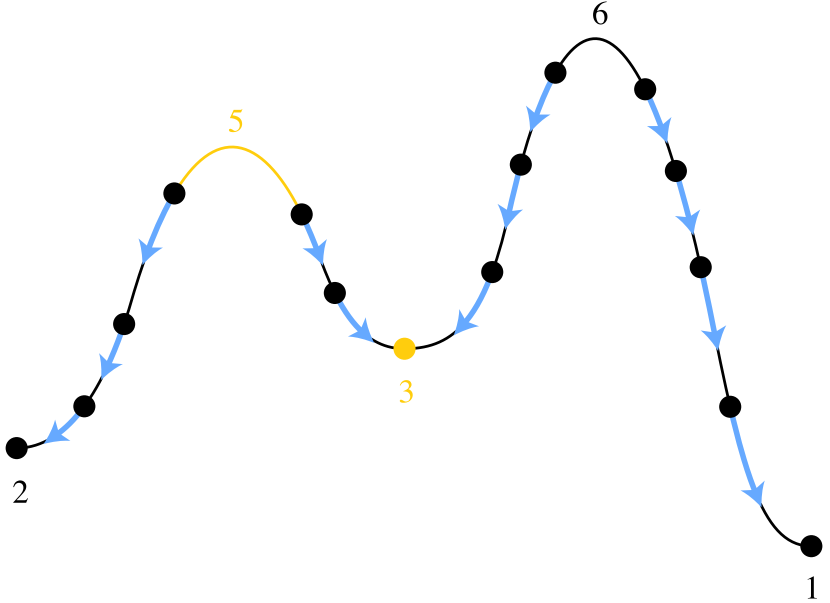

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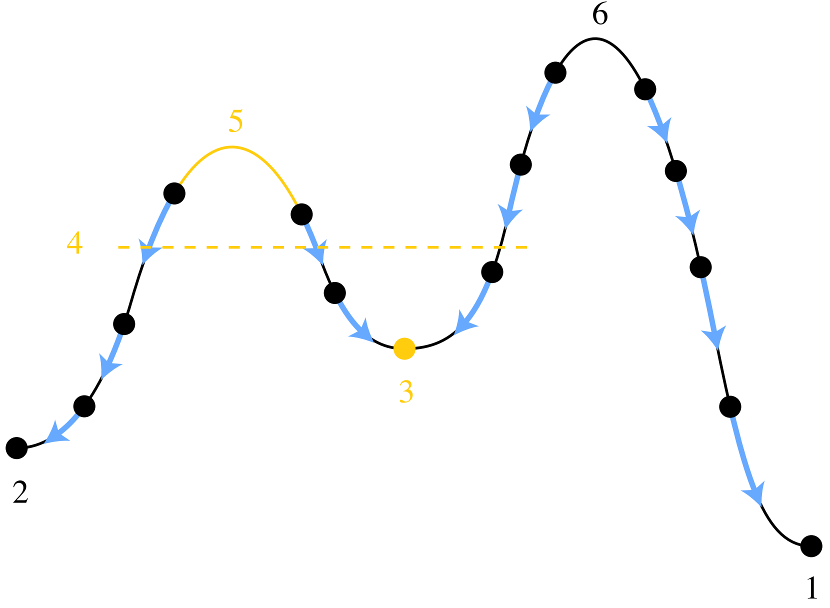

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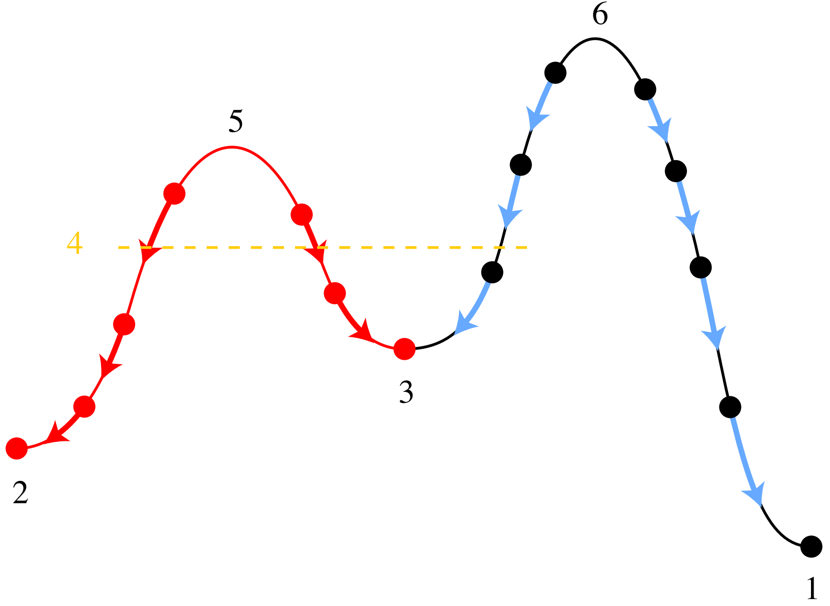

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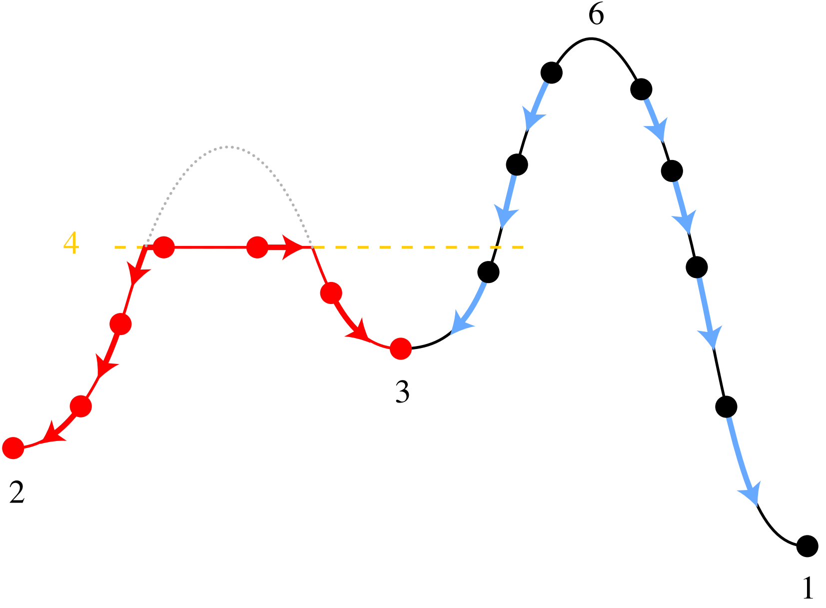

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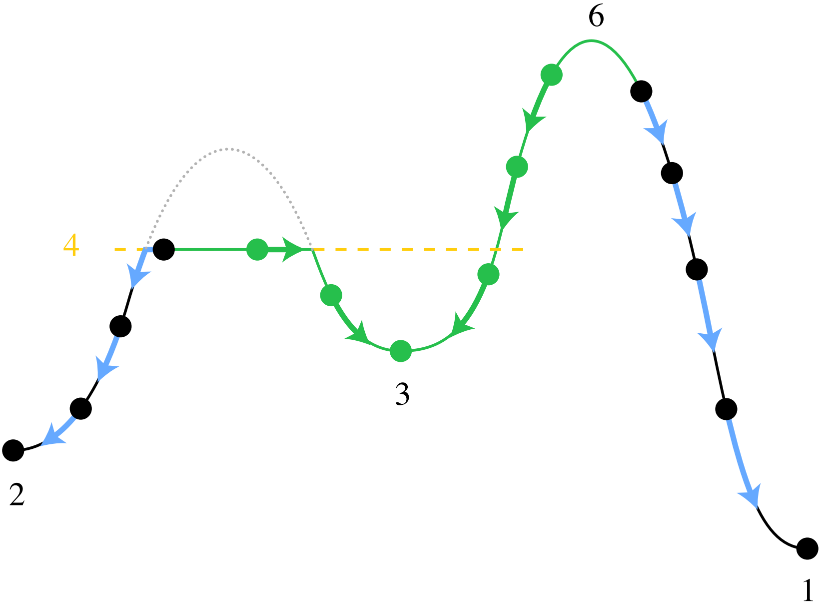

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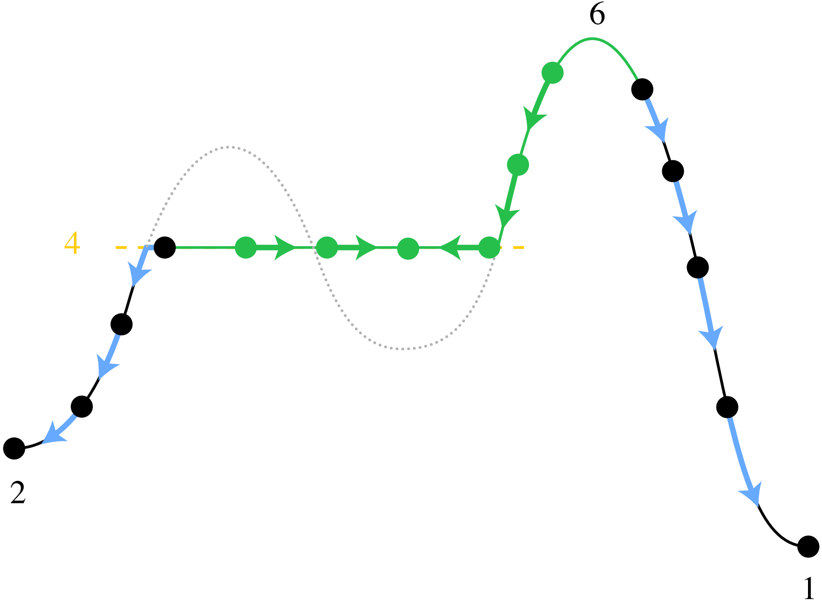

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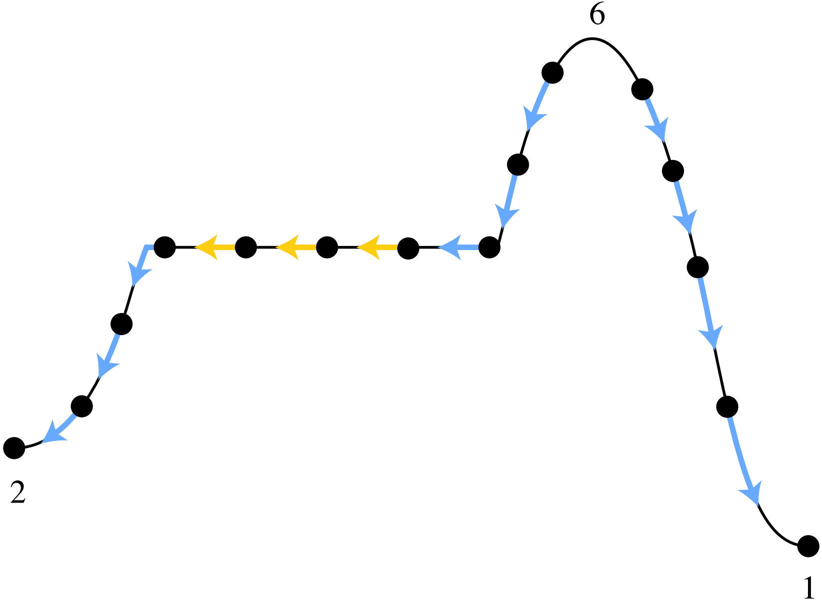

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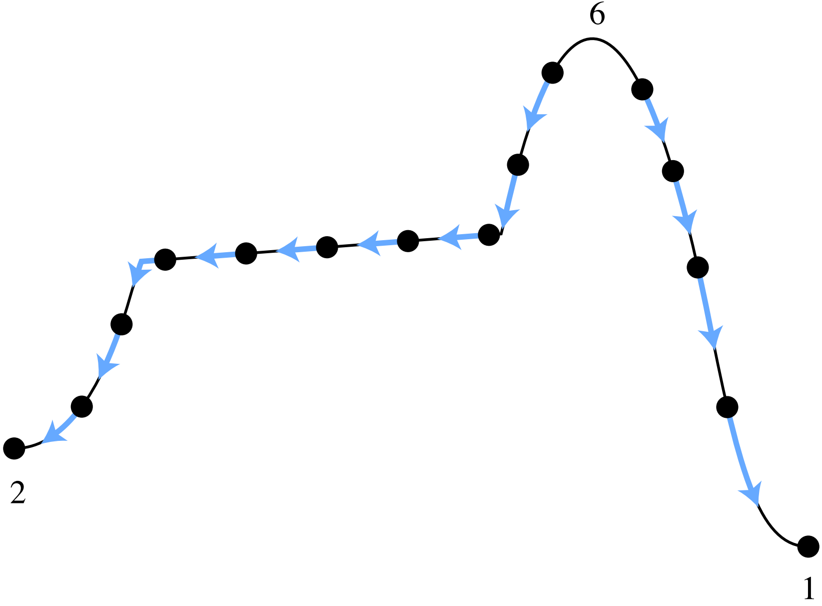

Results on topological simplification

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Does not hold anymore in 3D. Moreover:

- Topological function simplification is NP-hard [Egecioglu, Gonzalez 1995]
- Even simplification of a single isosurface is NP-hard (ongoing joint work with Amenta, Attali, Devillers, Glisse, Lieutier)


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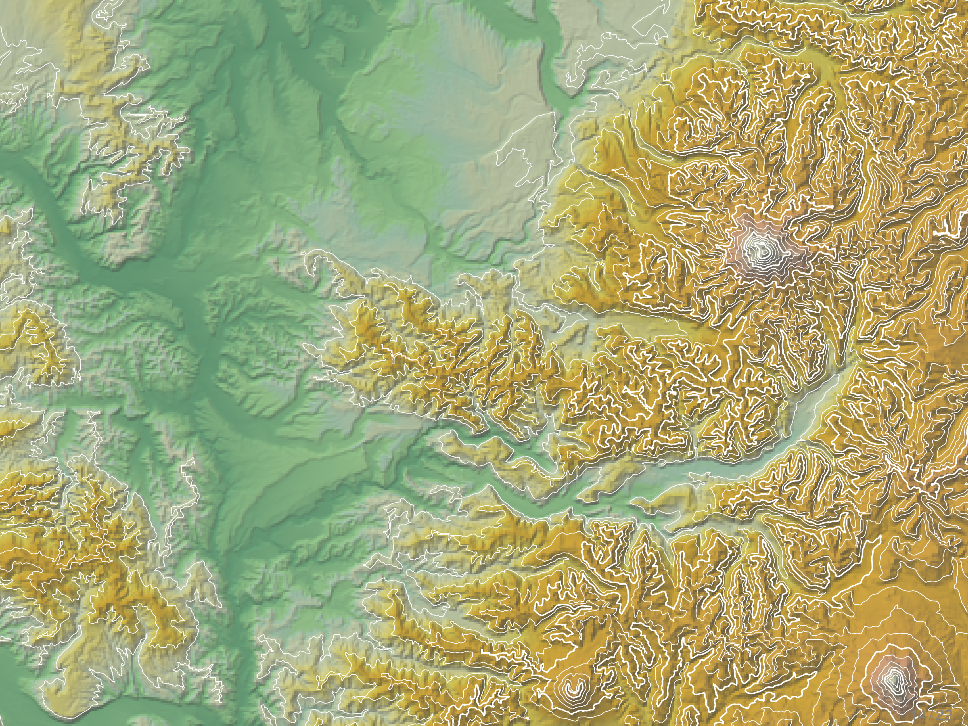

Robustness

What about other tolerance norms than $\|\cdot\|_\infty$?

- Persistence is not robust to outliers!

(ongoing joint work with Schönlieb, Munk, Wardetzky)

A robust denoising method

A well-known denoising method (Rudin, Osher, Fatemi, 1992):
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Recall the definition of TV *(total variation)*:

- For differentiable u: $\mathrm{TV}(u) = \int_a^b |u'(t)| \, \mathrm{d}t$
- Can be continuously extended to L^1

Equivalence with the taut string problem

Theorem (Grasmair, 2006)

Let u_α be a minimizer of the ROF functional. Then $u_\alpha = U'_\alpha$, where U_α minimizes

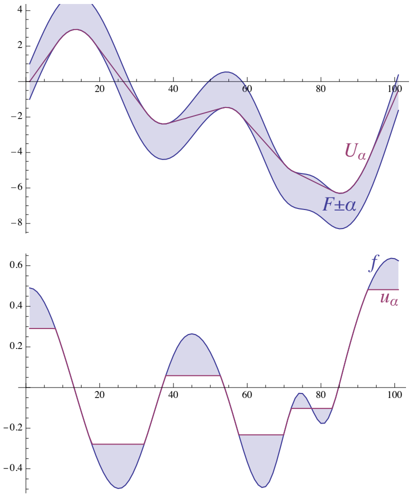
$$\int_a^b \sqrt{1 + U'(t)^2}\,\mathrm{d}t$$

subject to

$$\|U - F\|_\infty \leq \alpha, \quad U(a) = F(a), \quad U(b) = F(b),$$

where $F(t) = \int_a^t f(\tau)\,\mathrm{d}\tau$.

An example



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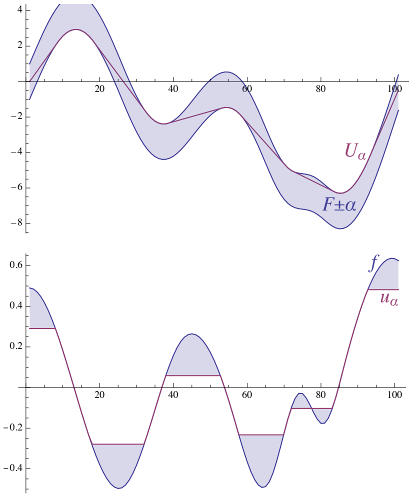

Properties:

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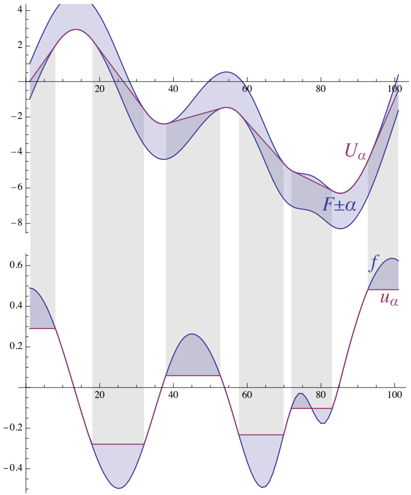

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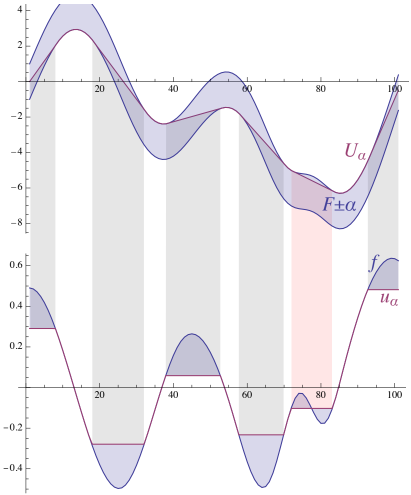

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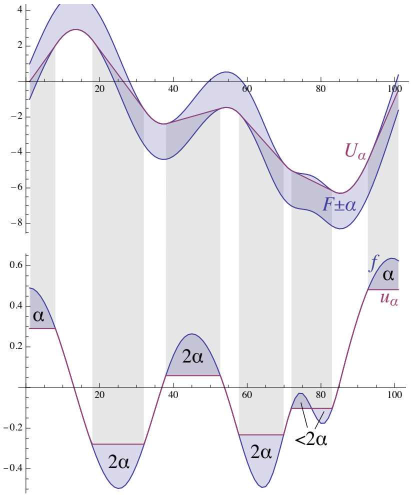

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- u_α coincides with f apart from some intervals, on which it is constant
- Some but not all constant intervals cancel critical points

A stable descriptor for critical points

Consider the values α at which the number of critical points of u_α changes

- Decreasing sequence s_i of "stability" values
- $s_i(f) := \inf_G \|G - F\|_\infty$: $g = G'$ has $\leq i$ critical points

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- Decreasing sequence s_i of "stability" values
- $s_i(f) := \inf_G \|G - F\|_\infty$: $g = G'$ has $\leq i$ critical points

This sequence is stable under perturbations of F:

$$|s_i(f) - s_i(\tilde{f})| \leq \|F - \widetilde{F}\|_\infty$$

Reason: s_i is a distance function
(to the set of functions with $\leq i$ critical points)

Estimating the number of critical points

Consider:

- 1D signal, n measurements with Gaussian noise $\sigma \sim \frac{1}{n}$
- Estimate number of critical points by thresholding stability values

Which one performs better? Persistence or TV/taut string?

- Experimental evidence in favor of TV/taut string
- Behavior for fixed n and different functions? Asymptotic behavior $n \to \infty$?

Simplifying plane maps

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Fold curves can also be eliminated in pairs (bounding an annulus; ongoing joint work with Zorin)