Persistence simplification of discrete Morse functions on surfaces

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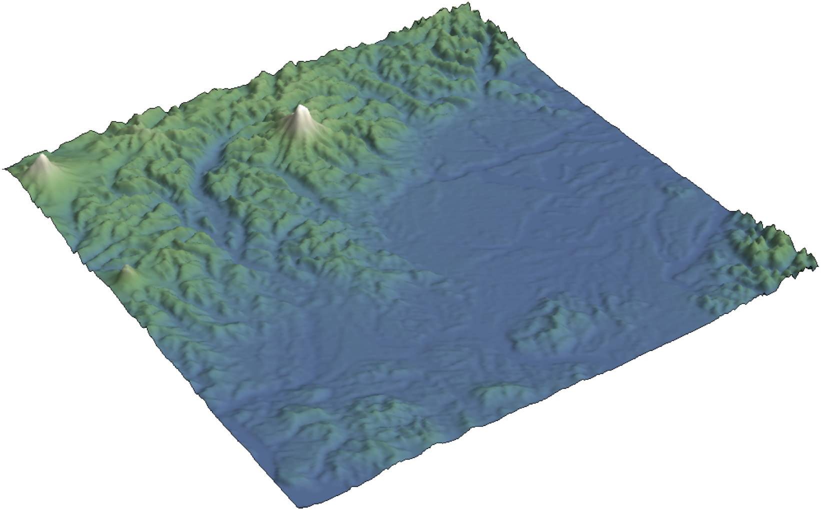

Goal

Topological denoising of functions on surfaces

- minimize number of critical points
- stay close to original function

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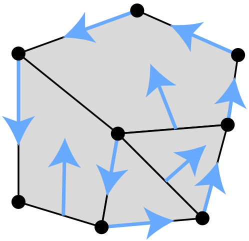
Using:

- Discrete Morse theory [Forman 1998]
  - provides notion of critical point in the discrete setting
- Homological persistence [Edelsbrunner et al. 2002]
  - quantifies homological features

Discrete Morse theory [Forman, 1998]

Consider finite CW complex $\mathcal{K}$.

- Discrete vector field:
  - a set of *pairs* of cells (σ, τ), where σ is a regular facet of τ (arrow from σ to τ)
  - each cell in at most one pair



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- Discrete vector field:
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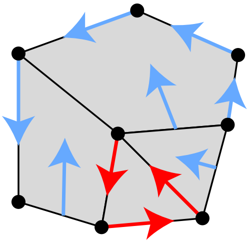
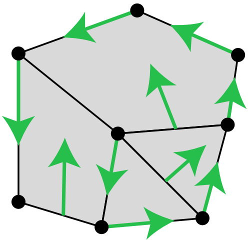
- Discrete gradient vector field:
  - no closed paths





Discrete Morse theory [Forman, 1998]

- Critical cell:
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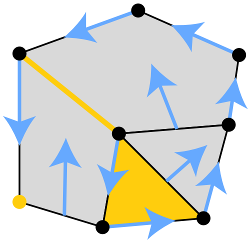

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- Critical cell:
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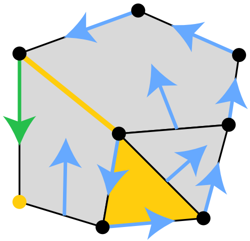
- Cancellation of critical cells:
  - Prerequisite: path between two critical cells



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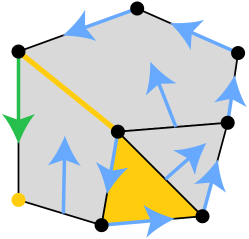

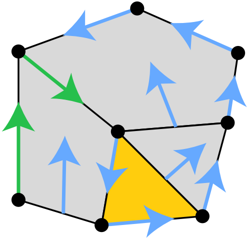

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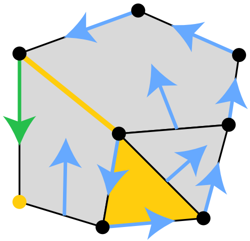

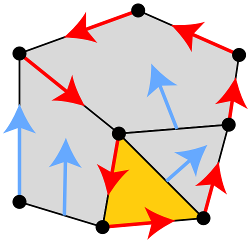

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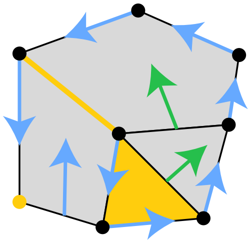

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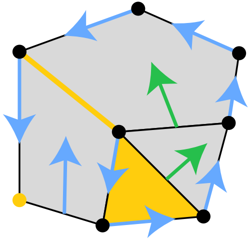
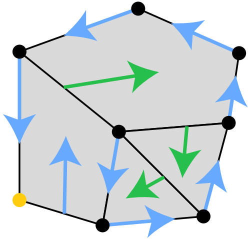


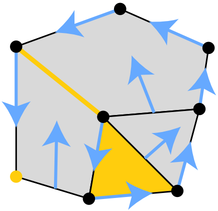


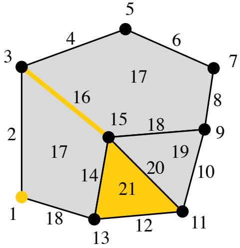
Discrete Morse function [Forman, 1998]

A function $f : \{\text{cells of } \mathcal{K}\} \to \mathbb{R}$ and a gradient vector field V_f with:

- For all σ facet of τ:
  - If there is an arrow $\sigma \to \tau$: $f(\sigma) \geq f(\tau)$
  - Otherwise: $f(\sigma) < f(\tau)$

f is *consistent* with V_f.





Discrete Morse function (Pseudo-Morse function)

A function f : {cells of $\mathcal{K}$} $\to \mathbb{R}$ and a gradient vector field V_f with:

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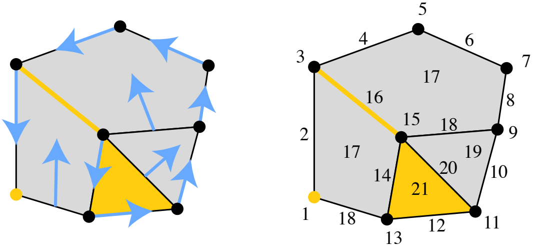
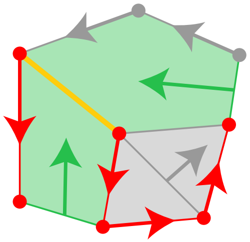


Attracting and repelling sets

A gradient vector field V enforces inequalities on cells

- Attracting set of a (critical) cell σ:
  all cells ρ with $g(\rho) \geq g(\sigma)$ for *any* g consistent with V
- Repelling set: analogously for $g(\rho) \leq g(\sigma)$



Back to our problem

Aim: Cancel critical points from pseudo-Morse function (g, V)

To do: Cancelation requires two steps:

- Reverse gradient vector field (which pairs?)
- Make function consistent to new vector field (how?)

Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for growing spaces

Given:

- CW complex $\mathcal{K}$
- An injective (Pseudo-)Morse function (f, V)
- Critical cells $\{\rho_1, \ldots, \rho_N\}$ of V such that $f(\rho_i) < f(\rho_{i+1})$

Persistent homology [Edelsbrunner et al., 2002]

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Morse theory: homology depends on critical cells

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- Level Subcomplex $\mathcal{K}(\rho)$:
  all cells ϕ with $f(\phi) \leq f(\rho)$ and their faces

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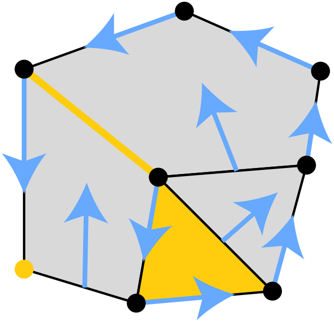
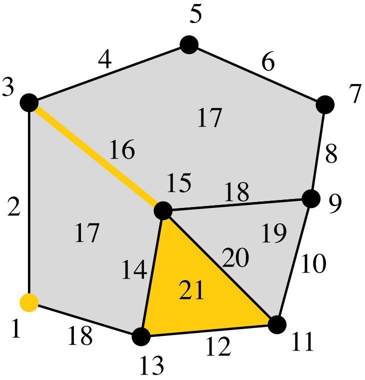
Morse theory: homology depends on critical cells

- Level Subcomplex $\mathcal{K}(\rho)$:
  all cells ϕ with $f(\phi) \leq f(\rho)$ and their faces
- Investigate change of homology of $\mathcal{K}(\rho_i)$ as i increases

Example: level subcomplexes





Example: Persistent homology

$\mathcal{K}(\rho_1)$

$H_0 \cong \mathbb{K}$

$H_1 \cong 0$

$H_2 \cong 0$

ρ_1 *positive* cell

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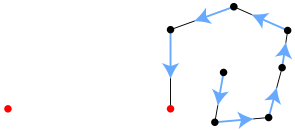

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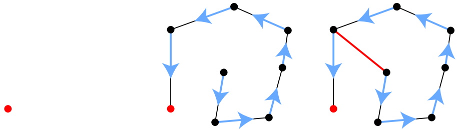

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ρ_1 *positive* cell

$\mathcal{K}(\rho_2)$

$H_0 \cong \mathbb{K}$

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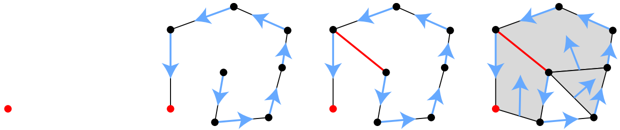
ρ_2 *positive* cell

Example: Persistent homology



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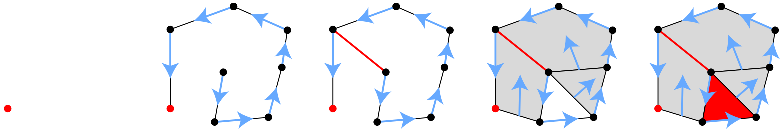


$\mathcal{K}(\rho_1)$

$H_0 \cong \mathbb{K}$

$H_1 \cong 0$

$H_2 \cong 0$

ρ_1 *positive* cell

$\mathcal{K}(\rho_2)$

$H_0 \cong \mathbb{K}$

$H_1 \cong \mathbb{K}$

$H_2 \cong 0$

ρ_2 *positive* cell

$\mathcal{K}(\rho_3)$

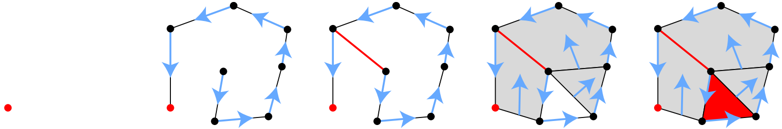
$H_0 \cong \mathbb{K}$

$H_1 \cong 0$

$H_2 \cong 0$

ρ_3 *negative* cell

Example: Persistent homology



$\mathcal{K}(\rho_1)$

$H_0 \cong \mathbb{K}$

$H_1 \cong 0$

$H_2 \cong 0$

ρ_1 *positive* cell

$\mathcal{K}(\rho_2)$

$H_0 \cong \mathbb{K}$

$H_1 \cong \mathbb{K}$

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ρ_2 *positive* cell

$\mathcal{K}(\rho_3)$

$H_0 \cong \mathbb{K}$

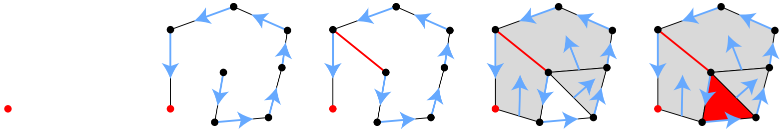
$H_1 \cong 0$

$H_2 \cong 0$

ρ_3 *negative* cell

- (ρ_2, ρ_3) is a *persistence pair*:
  ρ_3 kills homology created at ρ_2

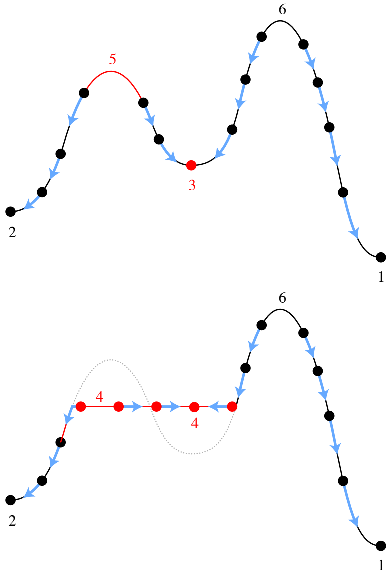
Example: Persistent homology



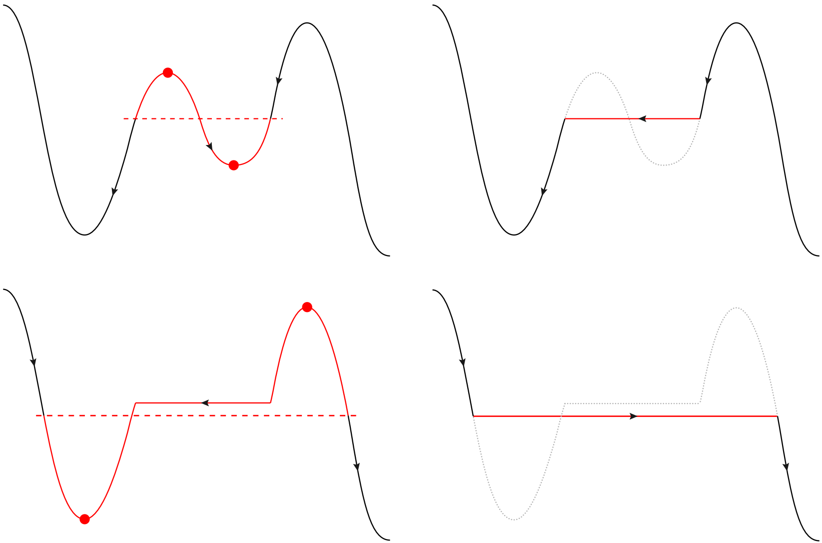
$\mathcal{K}(\rho_1)$	$\mathcal{K}(\rho_2)$	$\mathcal{K}(\rho_3)$
$H_0 \cong \mathbb{K}$	$H_0 \cong \mathbb{K}$	$H_0 \cong \mathbb{K}$
$H_1 \cong 0$	$H_1 \cong \mathbb{K}$	$H_1 \cong 0$
$H_2 \cong 0$	$H_2 \cong 0$	$H_2 \cong 0$
ρ_1 *positive* cell	ρ_2 *positive* cell	ρ_3 *negative* cell

- (ρ_2, ρ_3) is a *persistence pair*:
  ρ_3 kills homology created at ρ_2
- $f(\rho_3) - f(\rho_2)$ is the *persistence* of (ρ_2, ρ_3)

Canceling persistence pairs



Canceling persistence pairs



Natural questions

- Can we cancel all persistence pairs?
  - Are the assumptions for canceling satisfied?
- Can we cancel all persistence pairs with small persistence?
- If yes, how close can we stay to the original function?

Main result

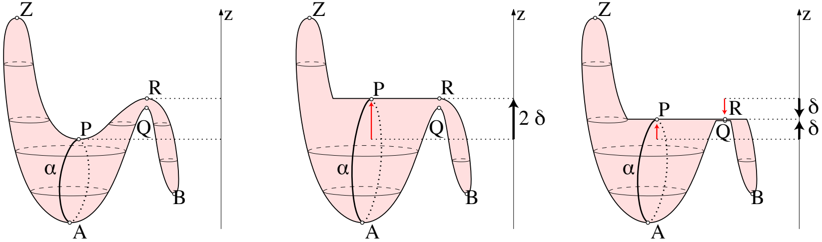
Let (g, V) be a discrete pseudo-Morse function on a combinatorial surface and $\delta > 0$.

Then there exists a pseudo-Morse function (g_δ, V_δ) with:

- $||g - g_\delta||_\infty < \delta$
- All persistence pairs of (g, V) with persistence $< 2\delta$ are canceled

This function achieves the minimal number of critical points.

Comparison with other methods

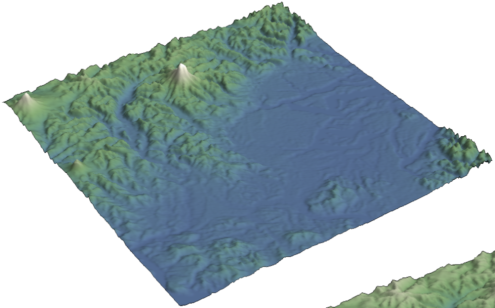


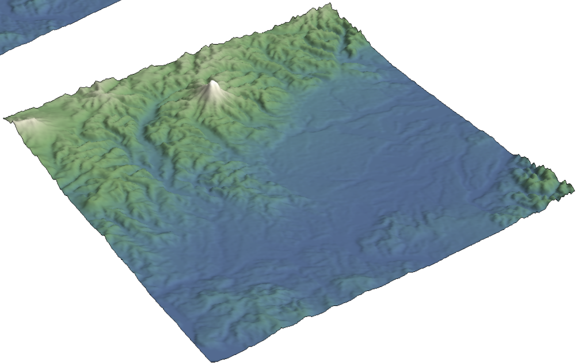
Original function

after [Edelsbrunner et al. 2006, Attali et al. 2009]

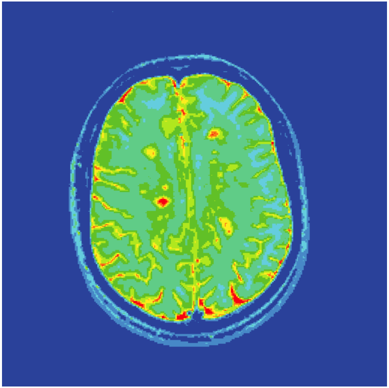
our method

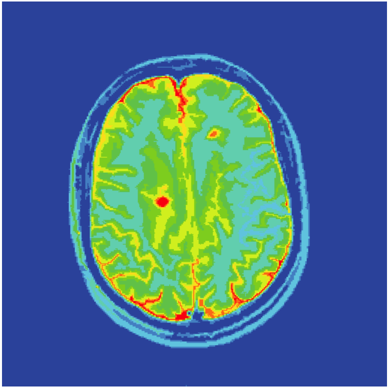
Example: Simplification of terrain





Example: Medical images





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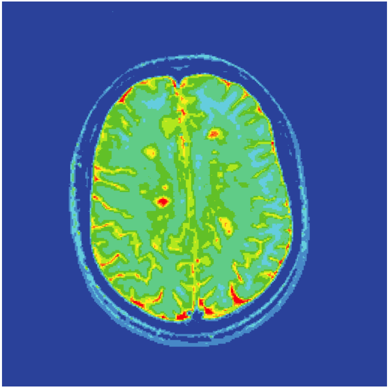

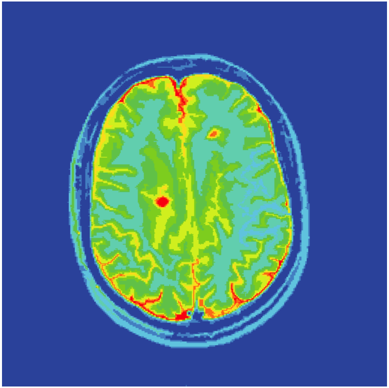

…thanks for your attention!