Persistent Homology

From Theory to Computation

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TUM

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Seminario del Dipartimento di Informatica – Università di Verona

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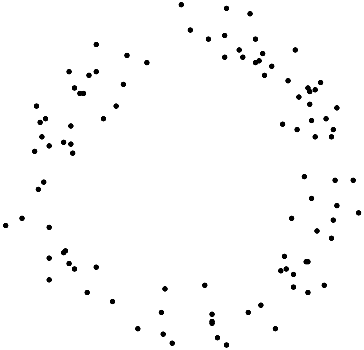
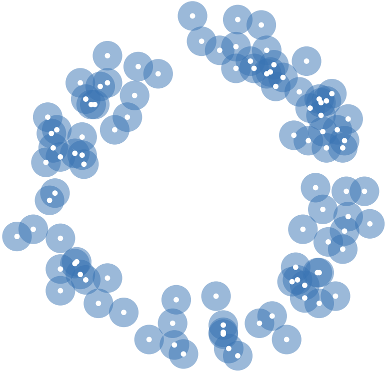
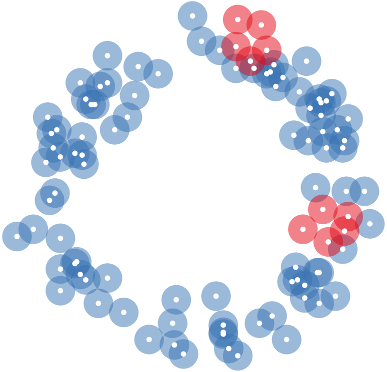
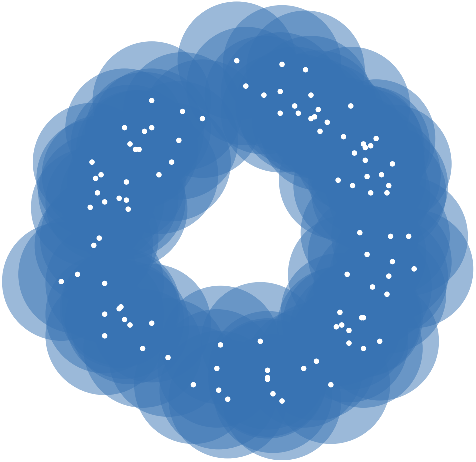
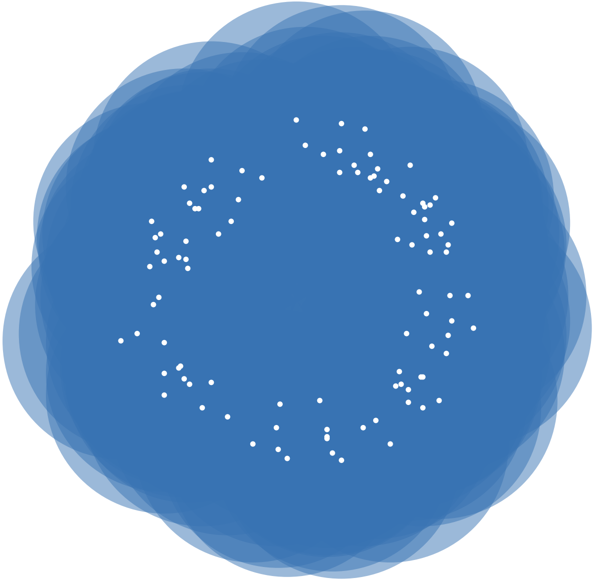
Holes in data

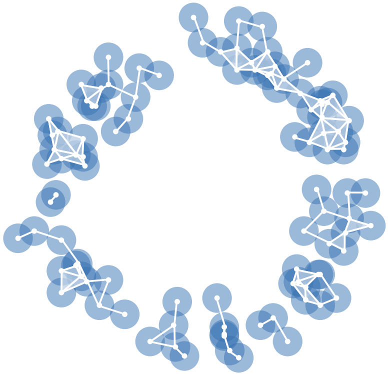


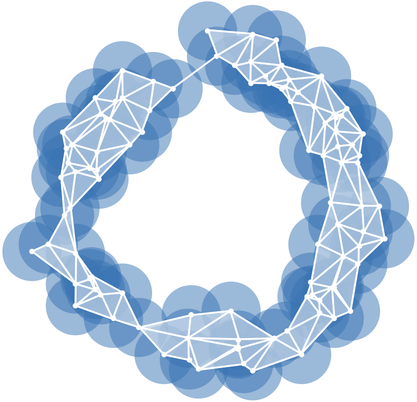


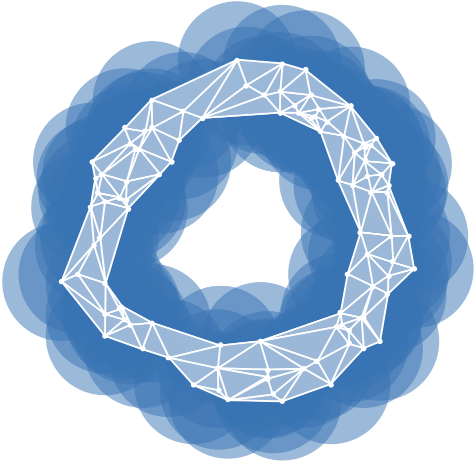


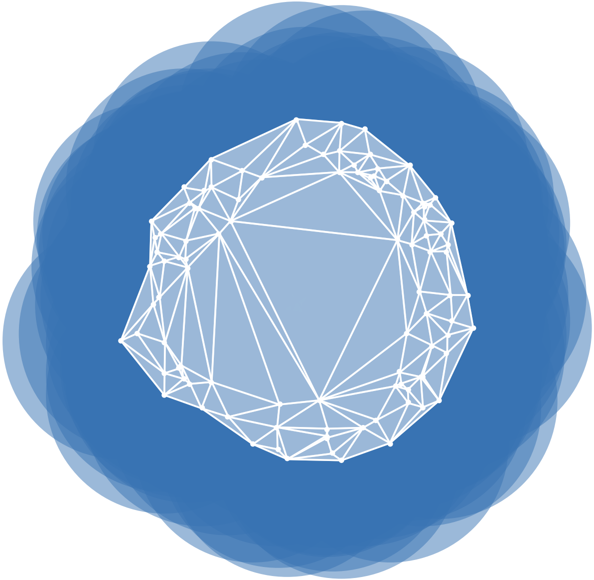




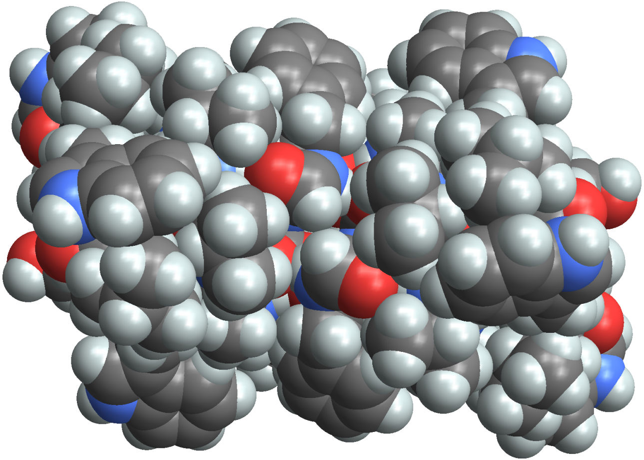








Gramicidin: an antibiotic functioning as ion channel



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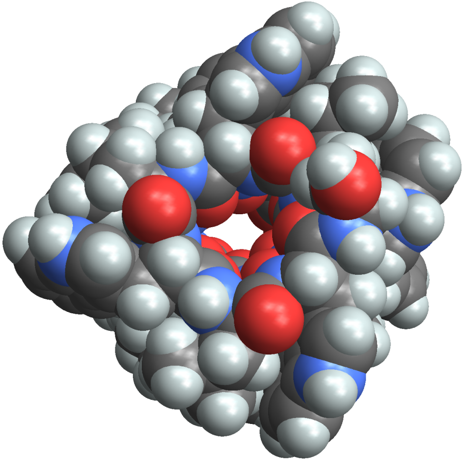

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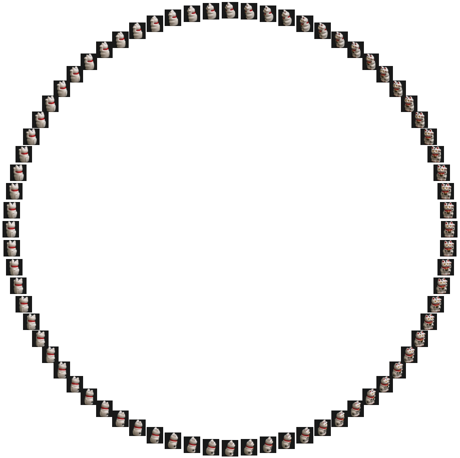

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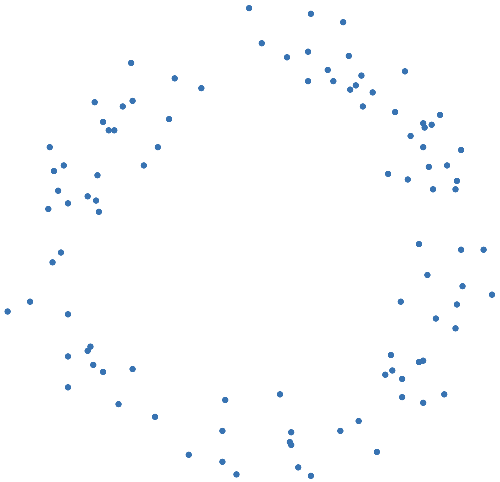
Geometric Complexes

Vietoris–Rips complexes



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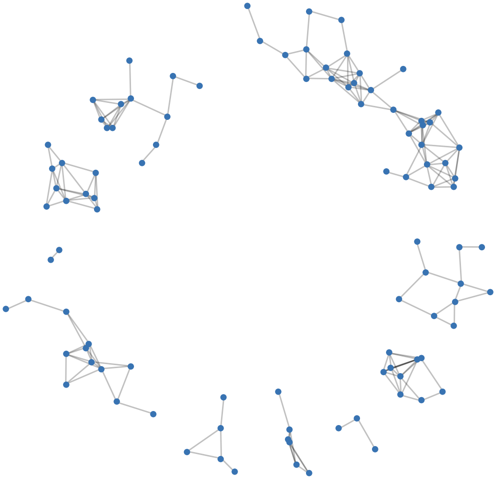

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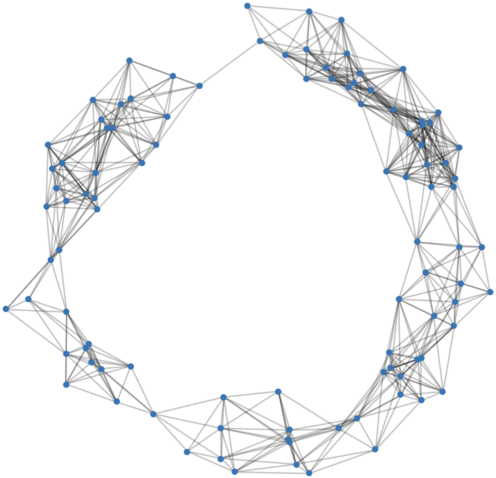

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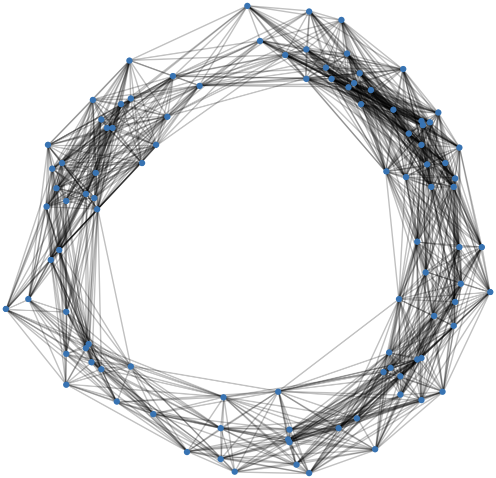

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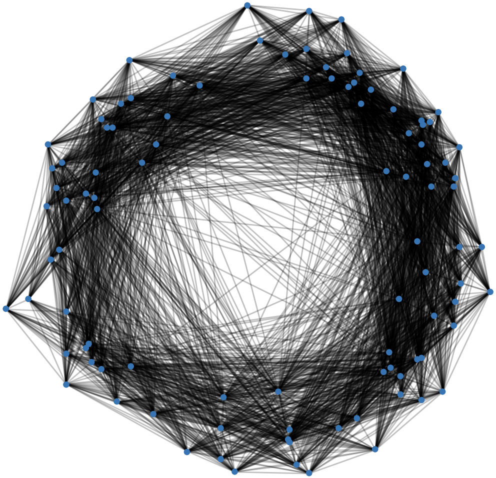

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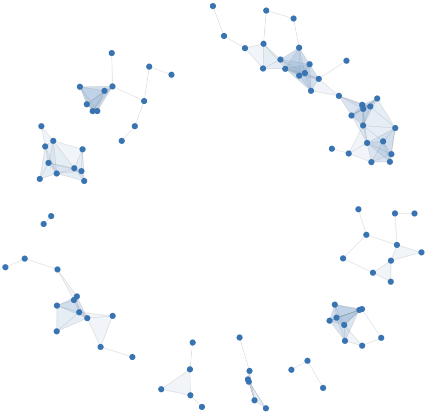

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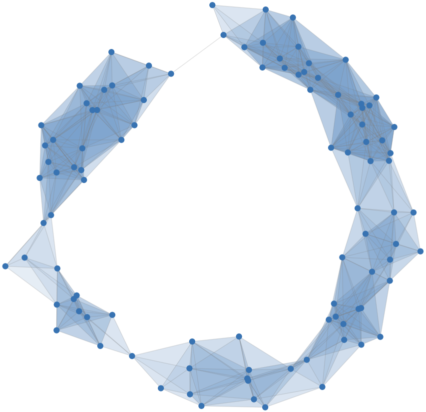

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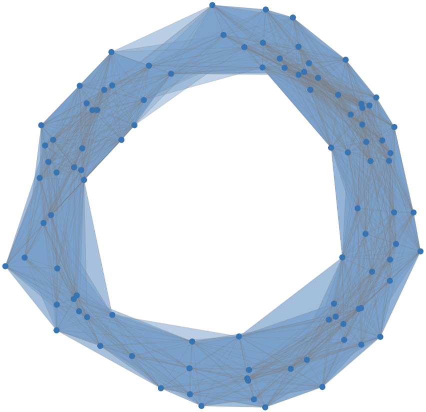

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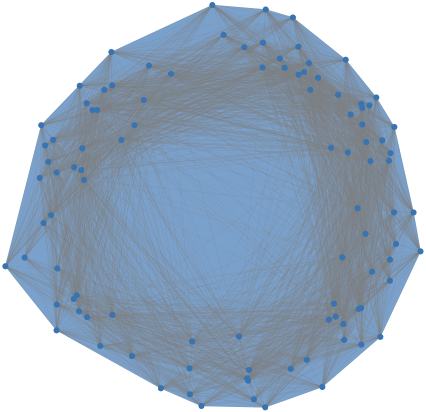
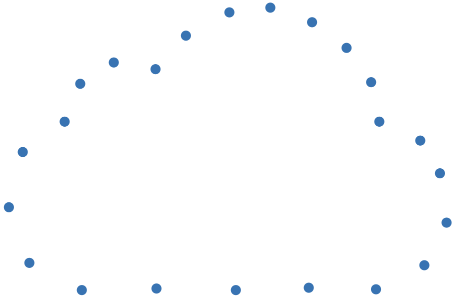


Čech complexes



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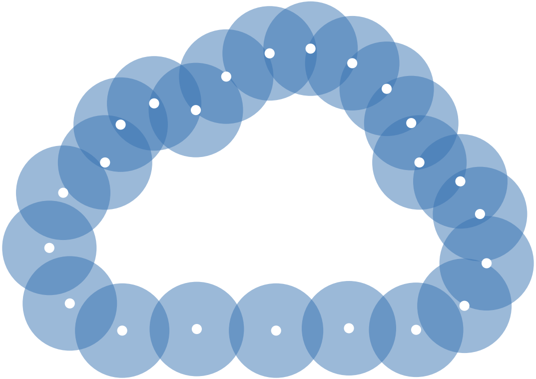

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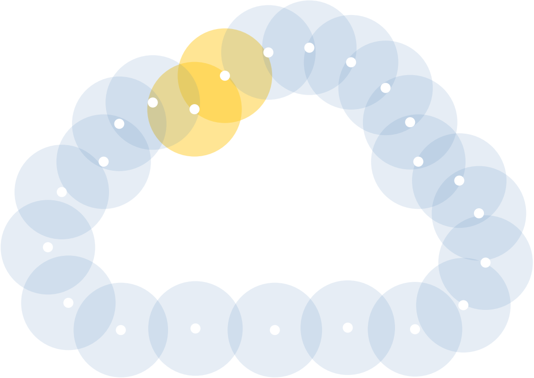

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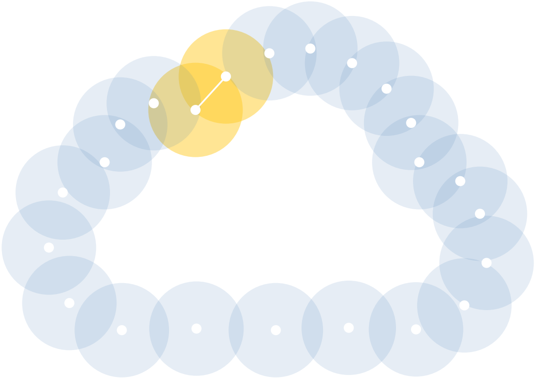

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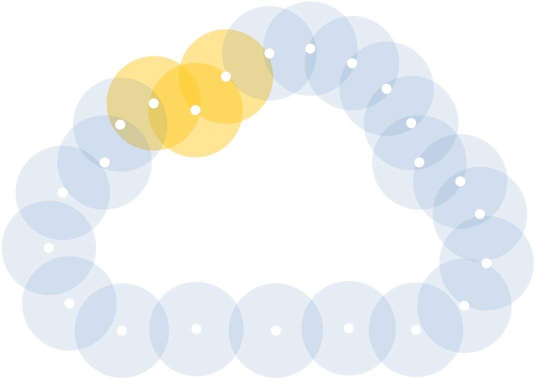

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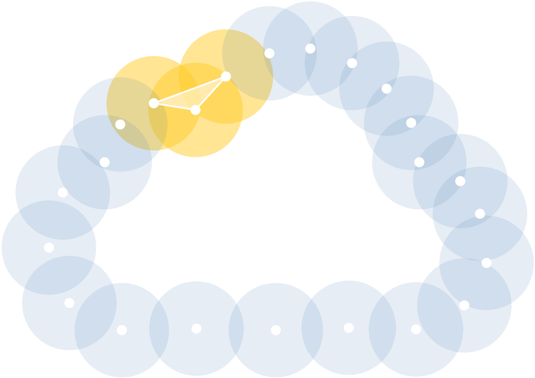

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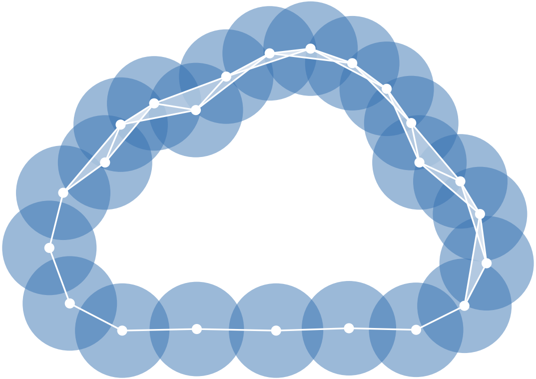

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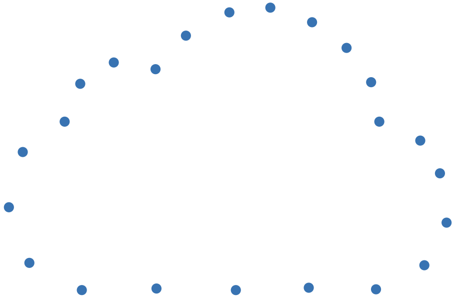

Delaunay complexes

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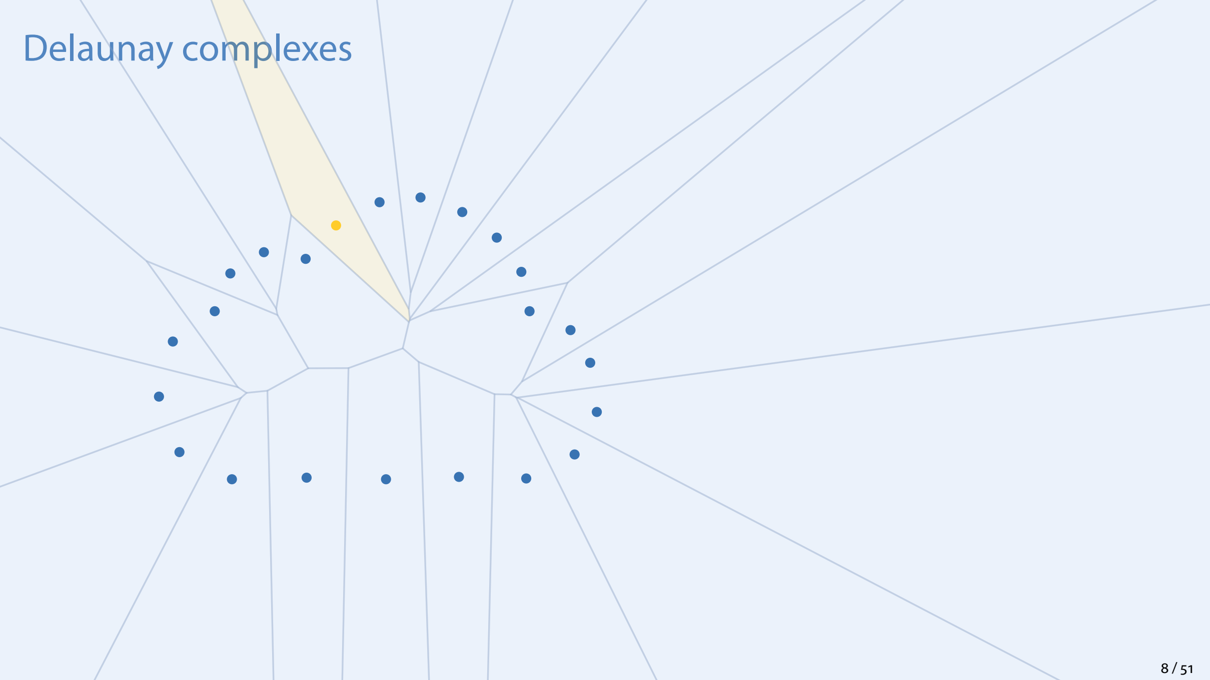

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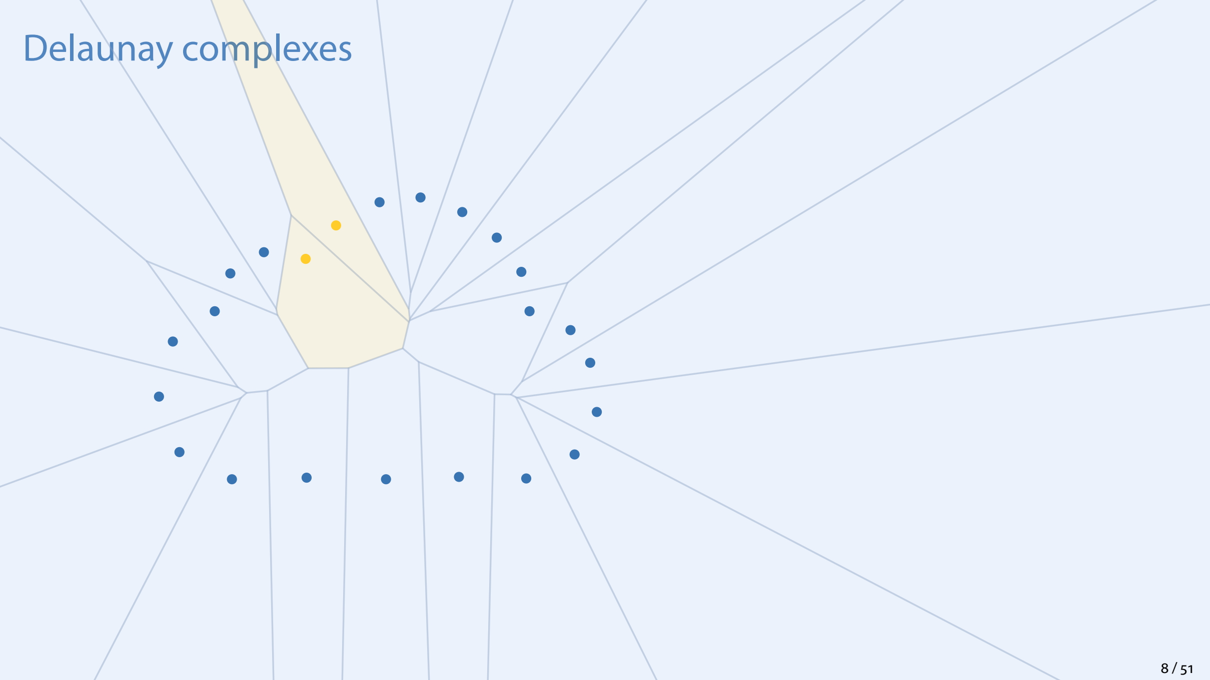

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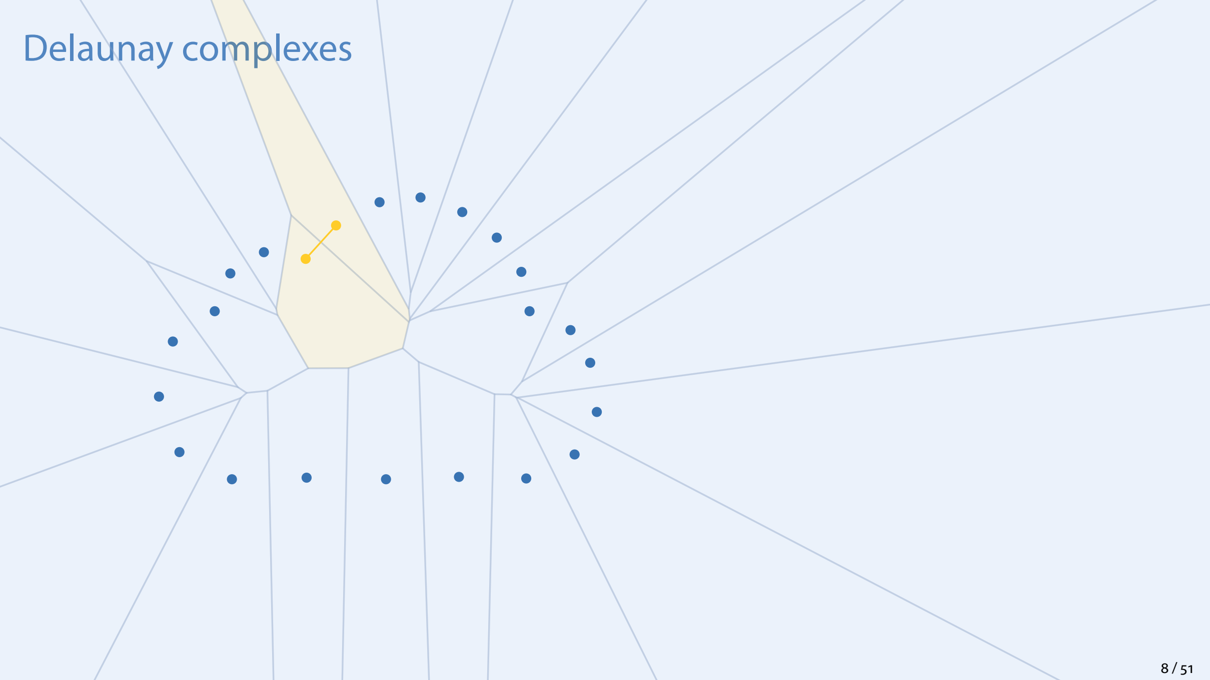

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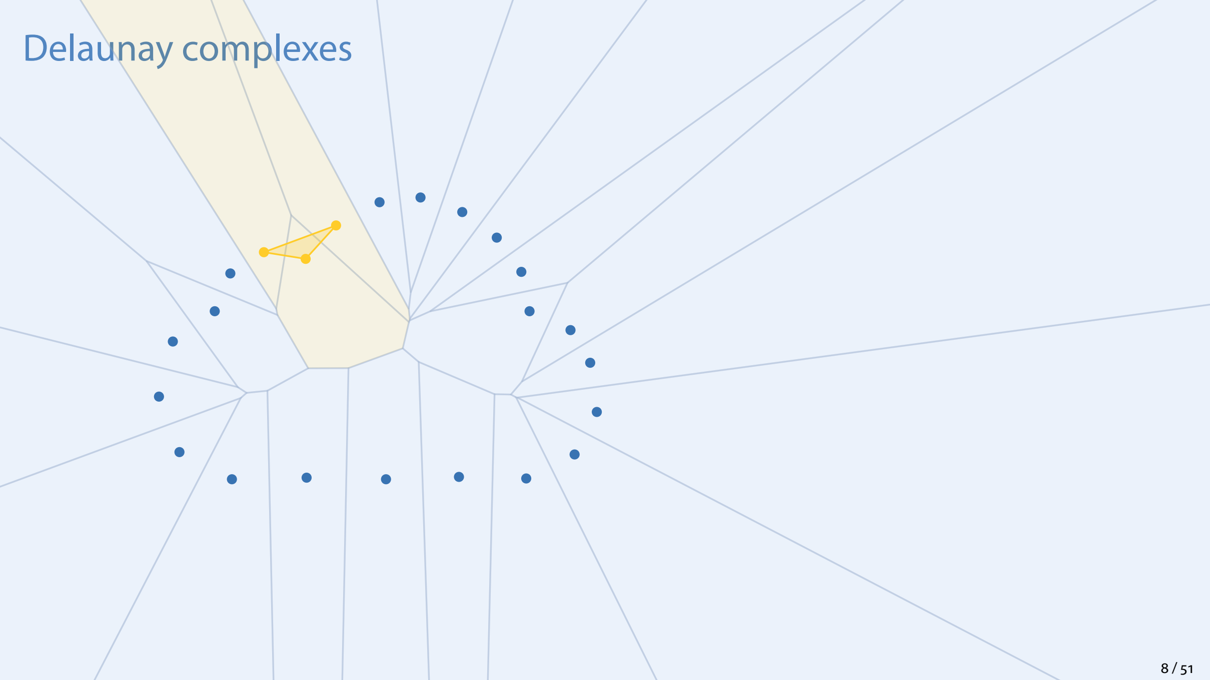

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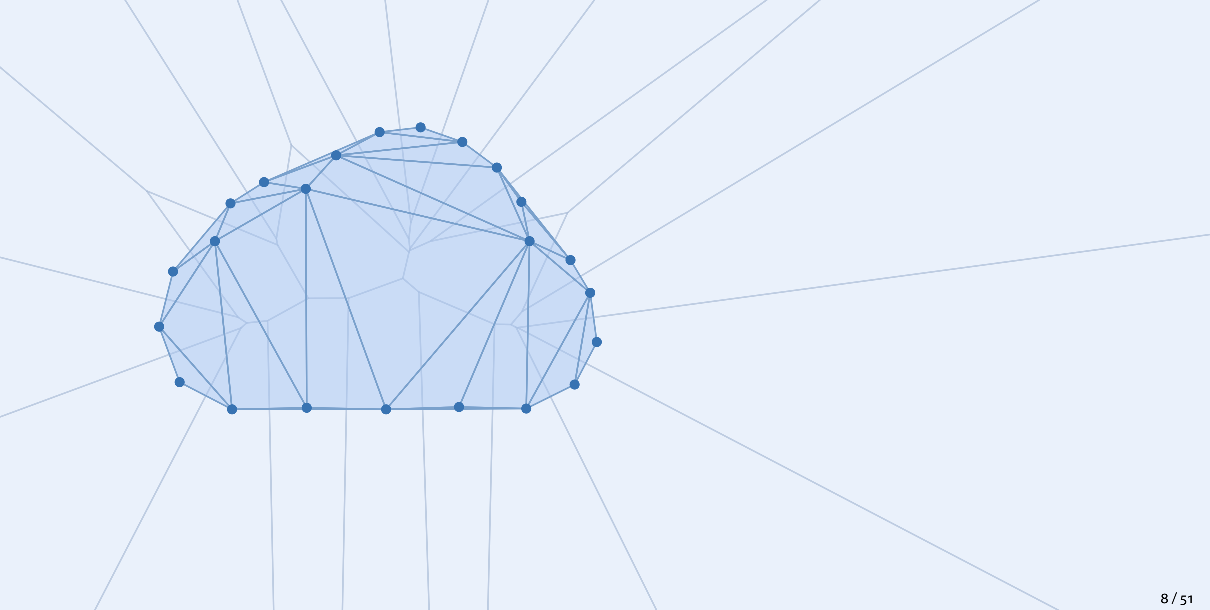

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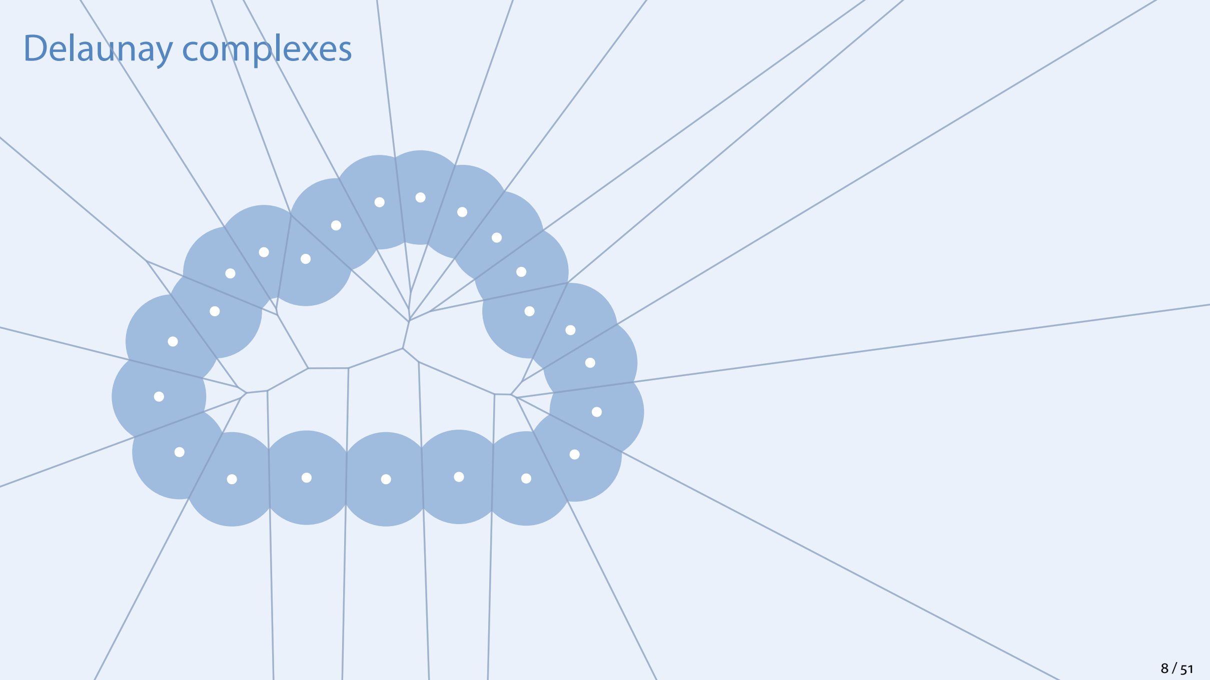

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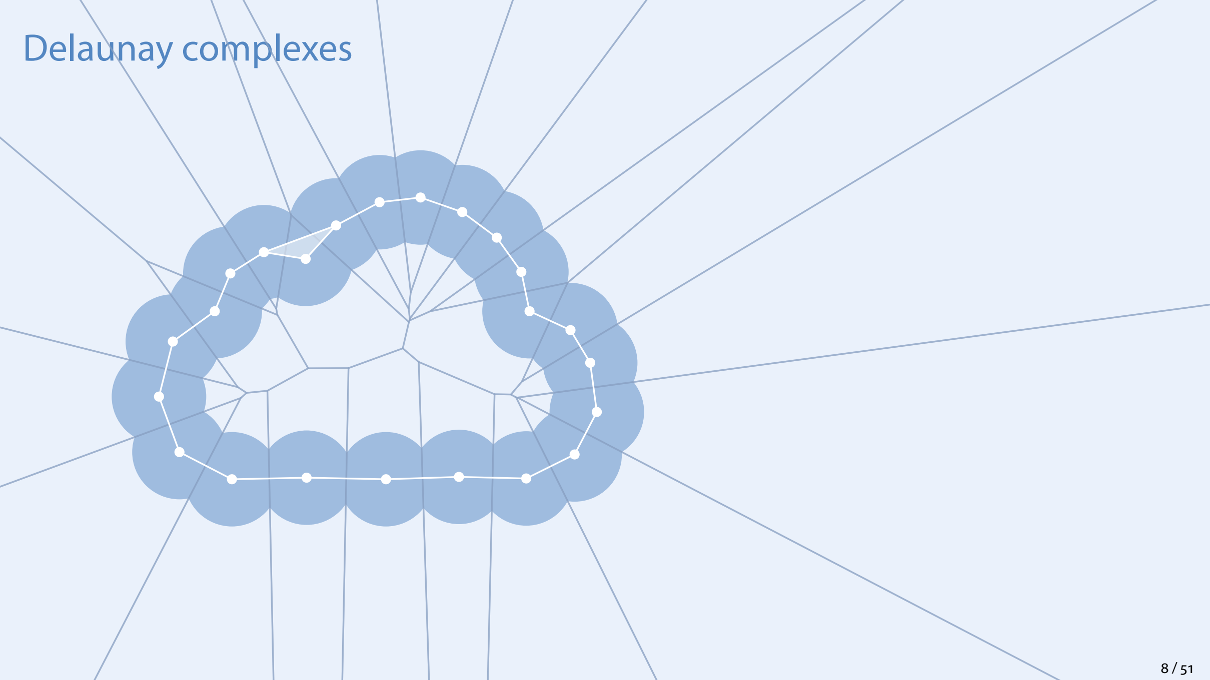

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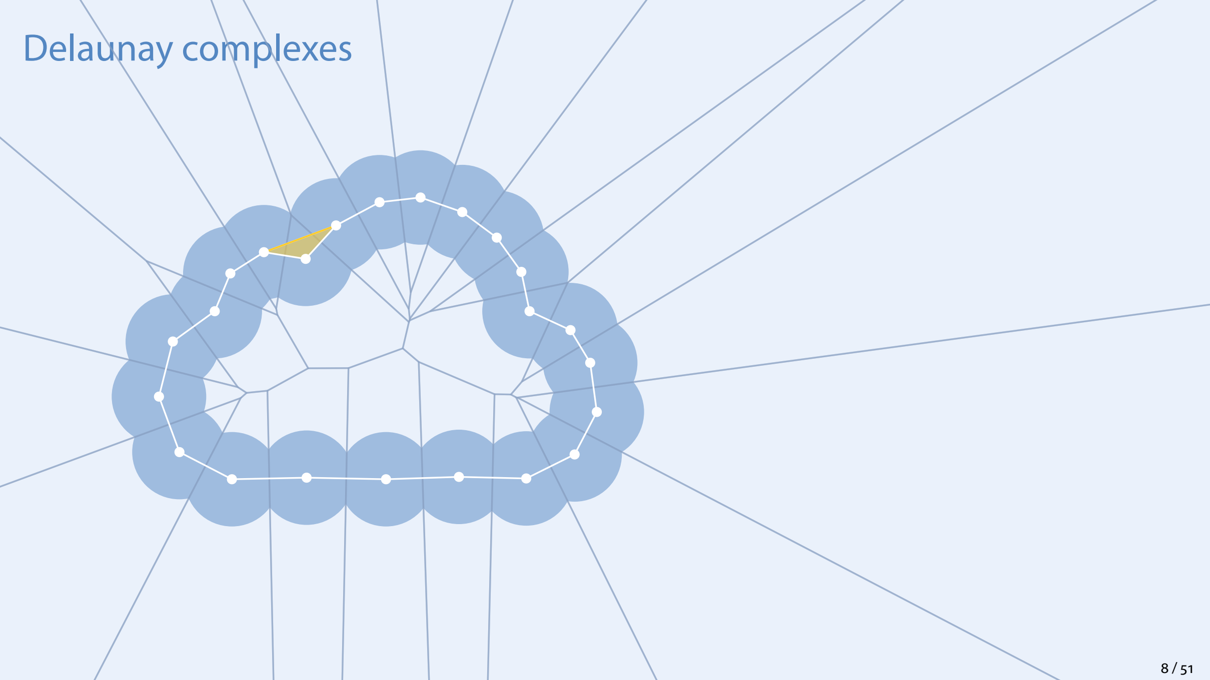

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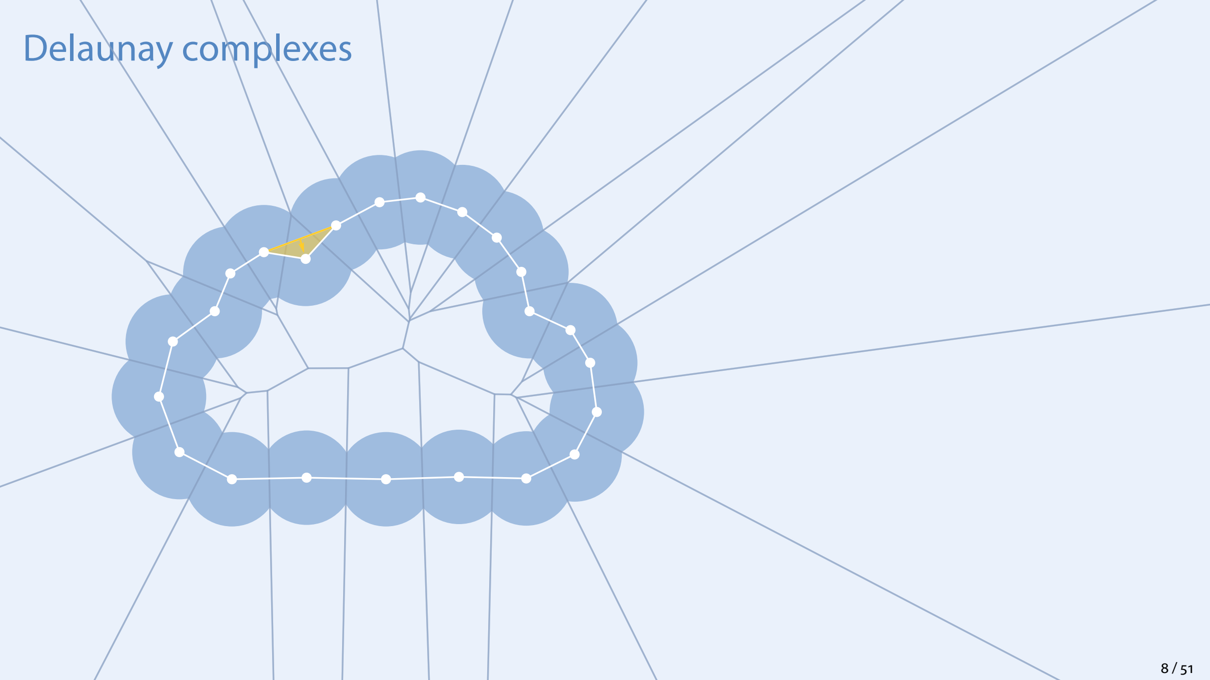

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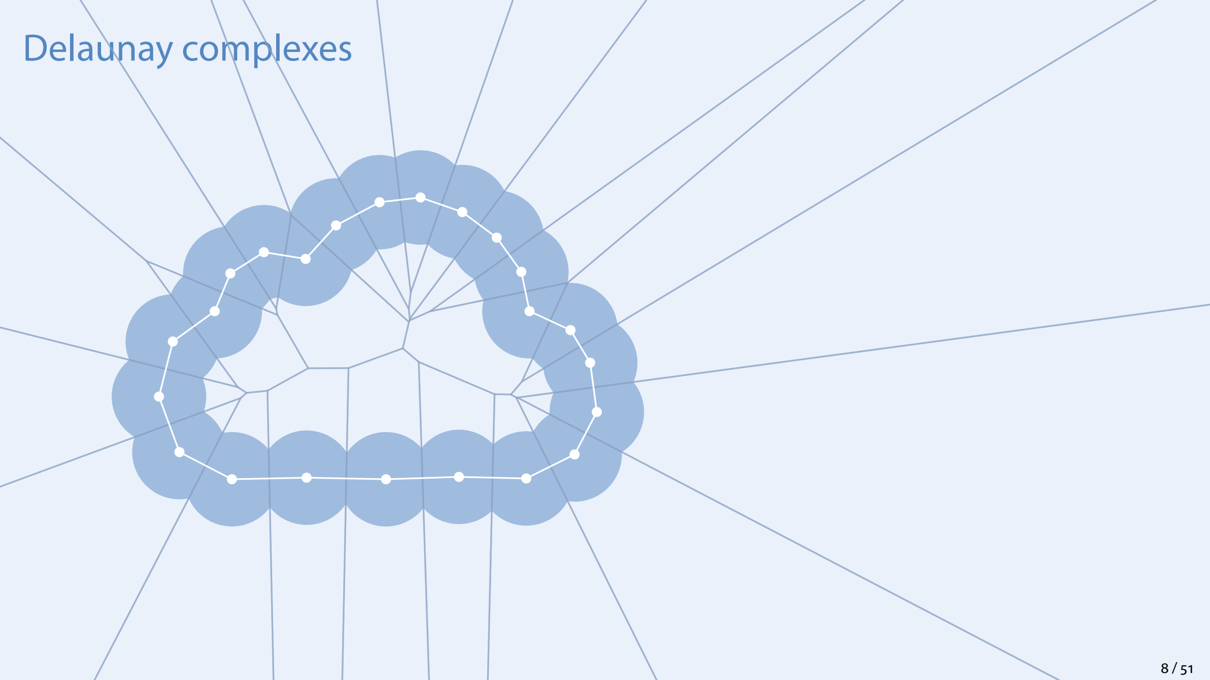
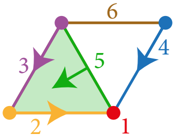


Discrete Morse theory



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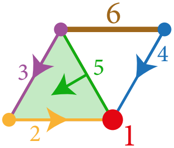
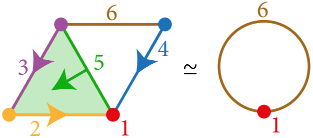


Discrete Morse theory

Theorem (Forman 1998)

A simplicial complex with a discrete Morse function f is homotopy equivalent to a CW complex build from the critical simplices of f.



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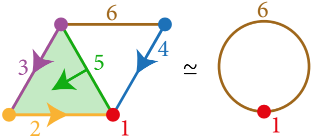
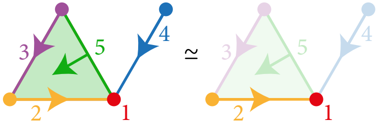


Discrete Morse functions encode collapses of sublevel sets:



Morse theory for Čech and Delaunay complexes

Proposition

The Čech complexes and the Delaunay complexes (alpha shapes) are sublevel sets of (generalized) discrete Morse functions.

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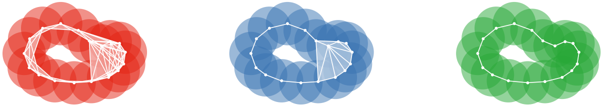
Theorem (B., Edelsbrunner 2017)

Čech, Delaunay, and Wrap complexes are homotopy equivalent through a sequence of collapses

$$\mathrm{Cech}_r X \searrow \mathrm{Cech}_r X \cap \mathrm{Del}\, X \searrow \mathrm{Del}_r X \searrow \mathrm{Wrap}_r X,$$

induced by a single discrete gradient field.



Delaunay and Wrap complexes

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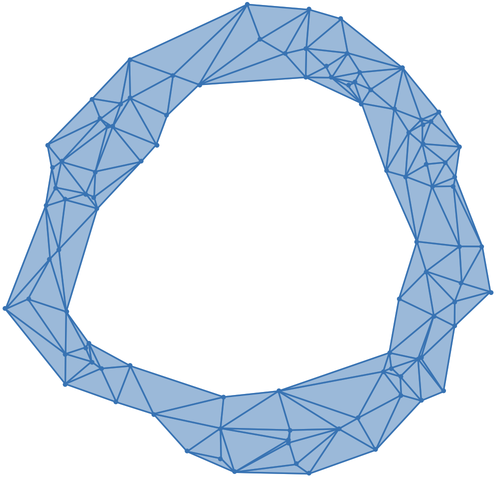
Persistent homology

What is homology?



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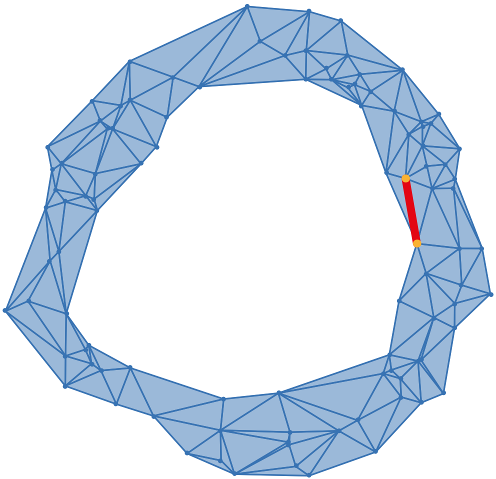

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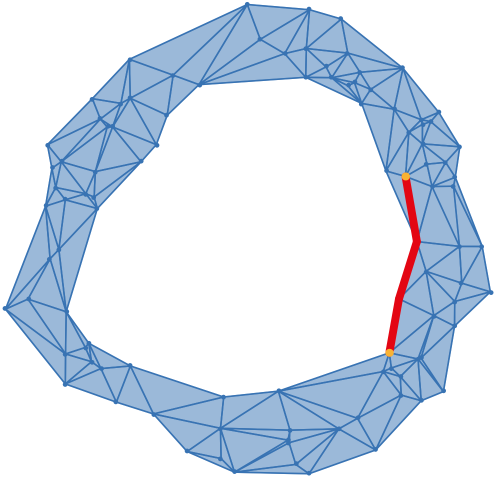

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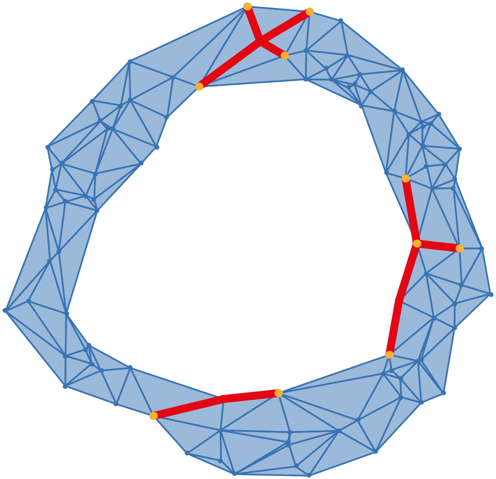

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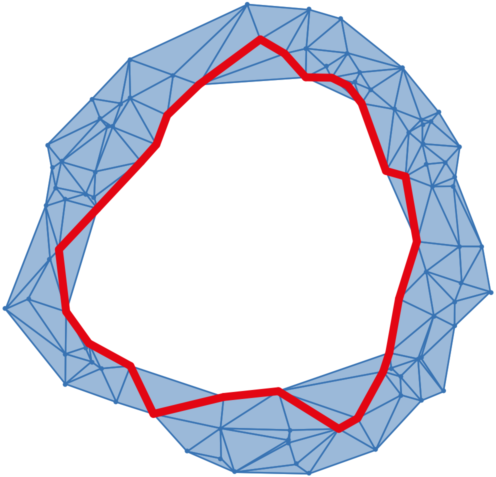

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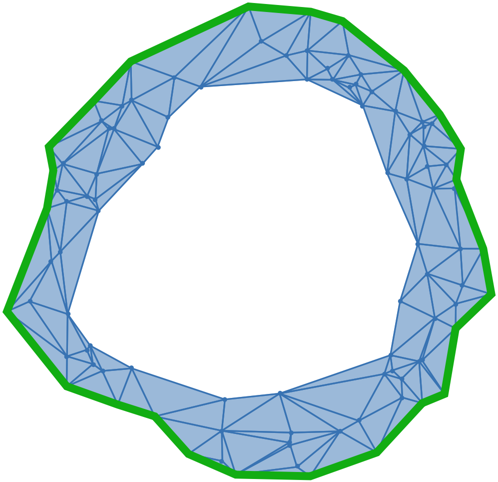

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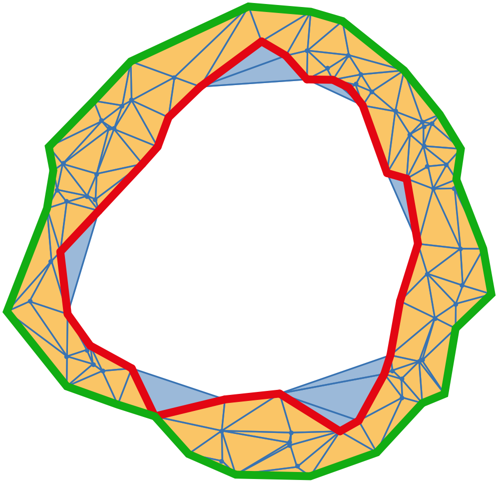

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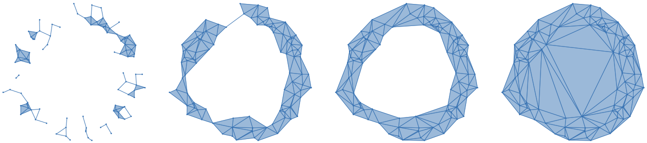

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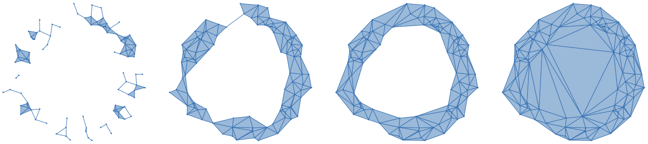

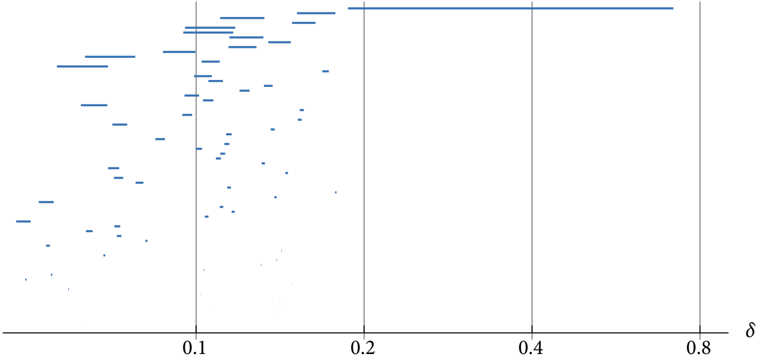

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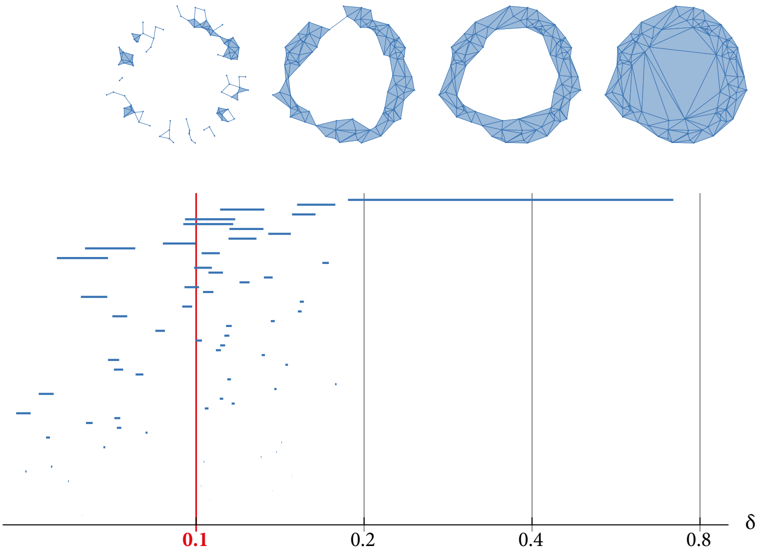

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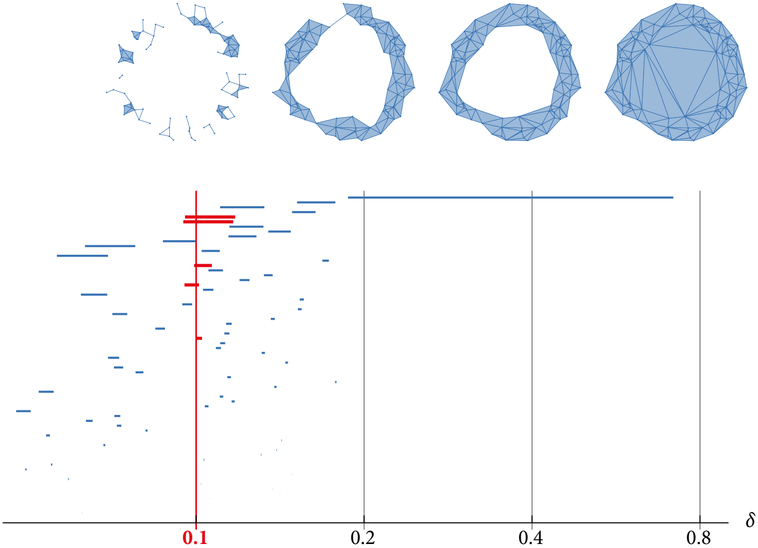

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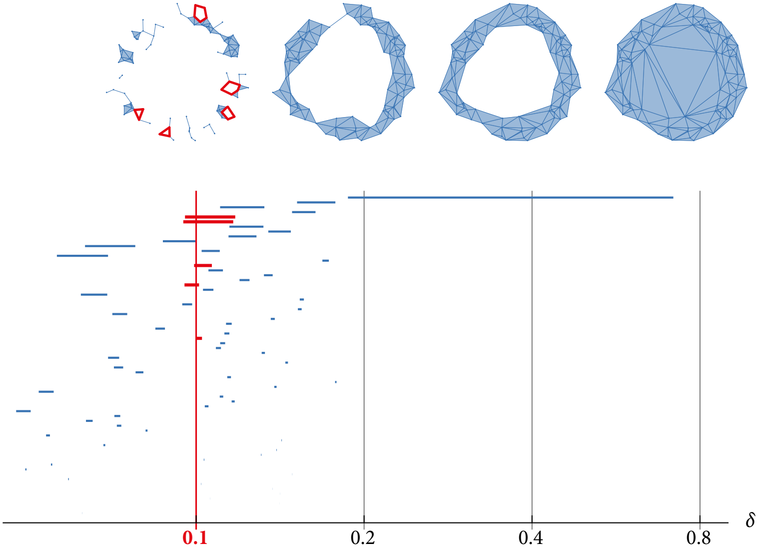

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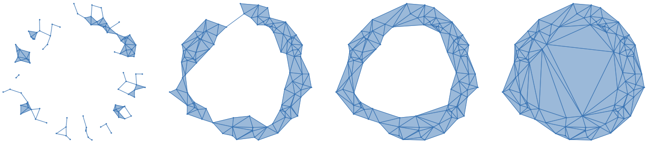

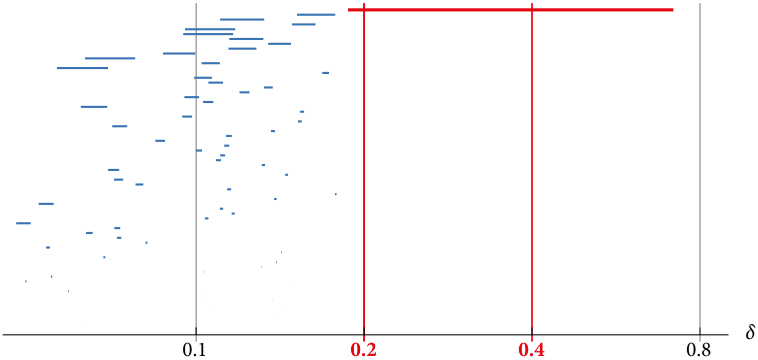

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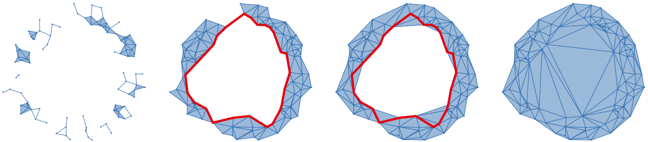

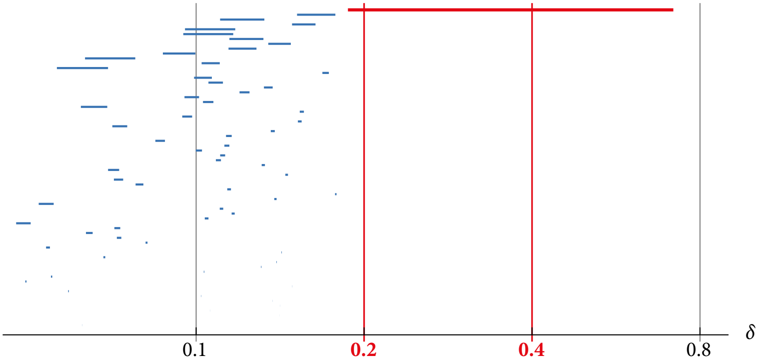

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- A filtration is a certain diagram $K : \mathbf{R} \to \mathbf{Top}$
  - $\mathbf{R}$ is the poset $(\mathbb{R}, \leq)$
  - A topological space K_t for each $t \in \mathbb{R}$
  - An inclusion map $K_s \hookrightarrow K_t$ for each $s \leq t \in \mathbb{R}$

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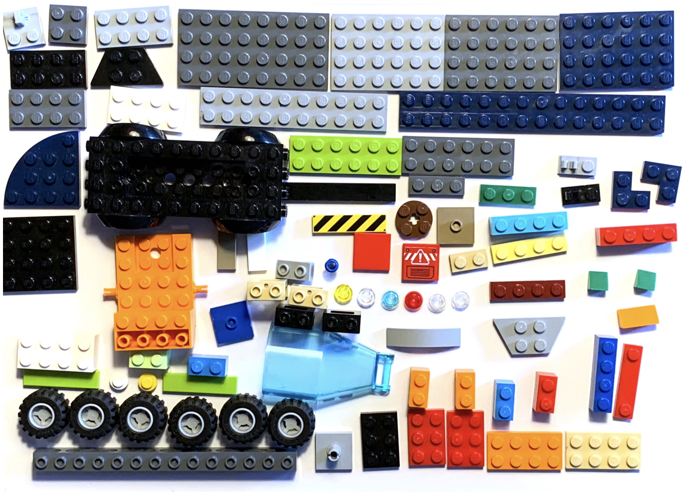
- Persistent homology is a diagram $M : \mathbf{R} \to \mathbf{Vect}$
  (persistence module)



Barcodes: the structure of persistence modules

Theorem (Krull–Schmidt–Remak–Azumaya; Crawley-Boevey 2015)

Any persistence module M of vector spaces over the field $\mathbb{F}$ is a direct sum of interval modules

$$0 \to \cdots \to 0 \to \mathbb{F} \to \cdots \to \mathbb{F} \to 0 \to \cdots \to 0$$

(in an essentially unique way).

- The supporting intervals form the *persistence barcode*.
- We rarely have such a simple structure for other diagrams, like $\mathbf{R}^2 \to \mathbf{vect}$ (2-parameter persistence modules)

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Two-parameter persistence

Consider grid-shaped commutative diagrams of vector spaces:

$$
\begin{tikzcd}
V_{0,0} \arrow[r] \arrow[d] & V_{1,0} \arrow[r] \arrow[d] & V_{2,0} \arrow[r, dotted] \arrow[d] & V_{m,0} \arrow[d] \\
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Two-parameter persistence

Consider grid-shaped commutative diagrams of vector spaces:

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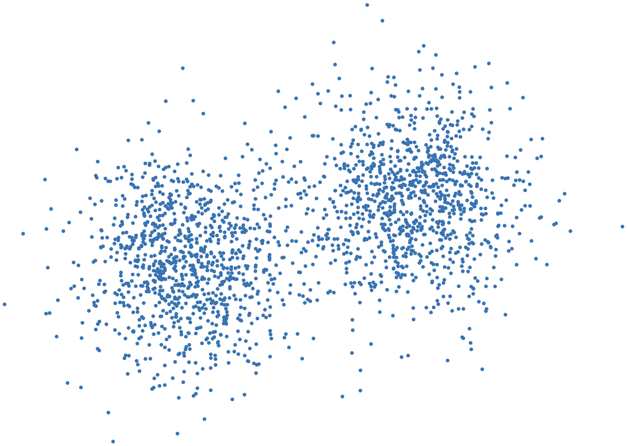
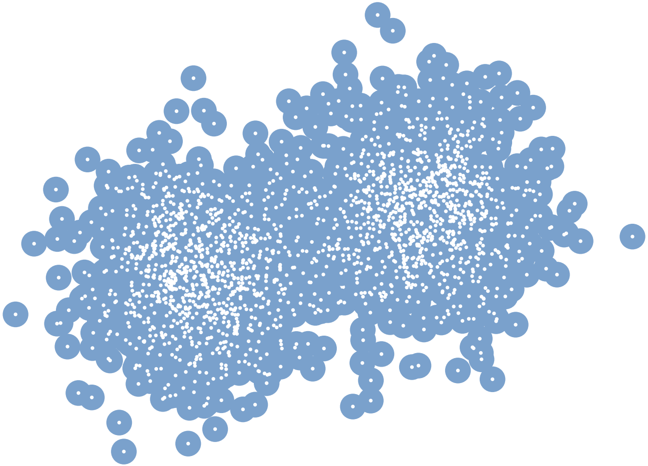
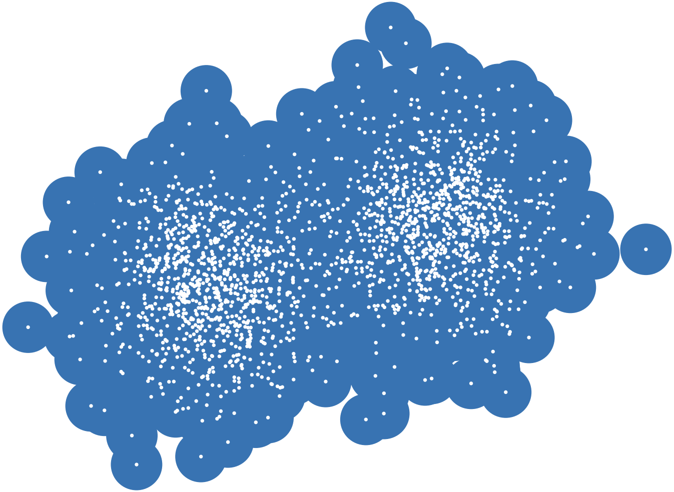
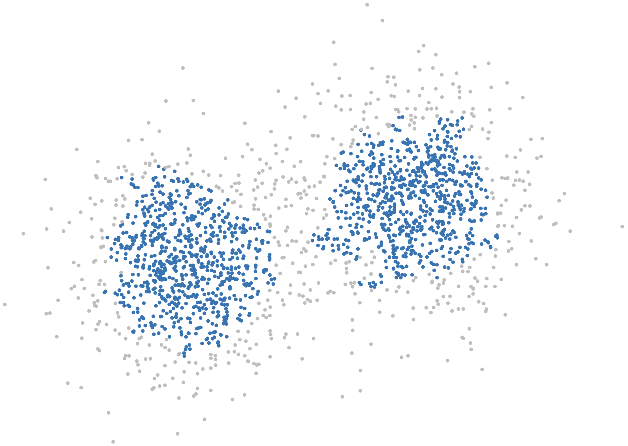
Classification of indecomposables [Drozd 77; Leszczyński 94]:

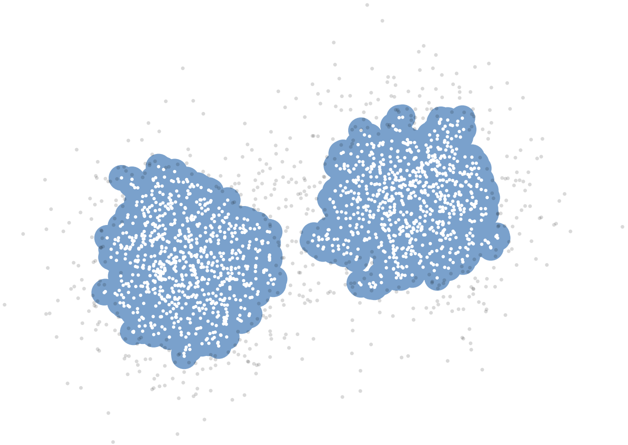
$$m \cdot n \in \begin{cases} \{0,1,2,3\} & \textit{finite type} \text{ (finite classification)} \\ \{4\} & \textit{tame type} \text{ (1-parameter families)} \\ \{5,6,\dots\} & \textit{wild type} \text{ (undecidable theory)} \end{cases}$$

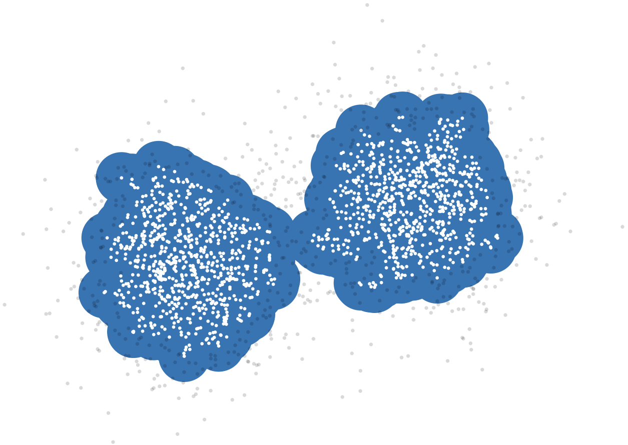


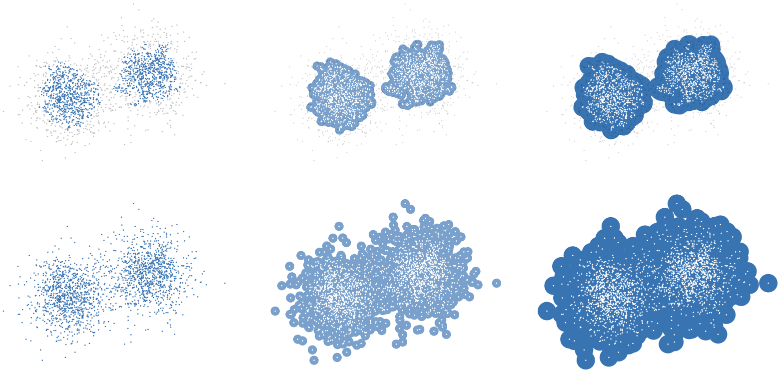












Two-parameter persistence with surjections

Common setup for 2-parameter persistence in degree 0:

- Merging components yields *surjective* horizontal maps

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Does the special structure simplify the picture?

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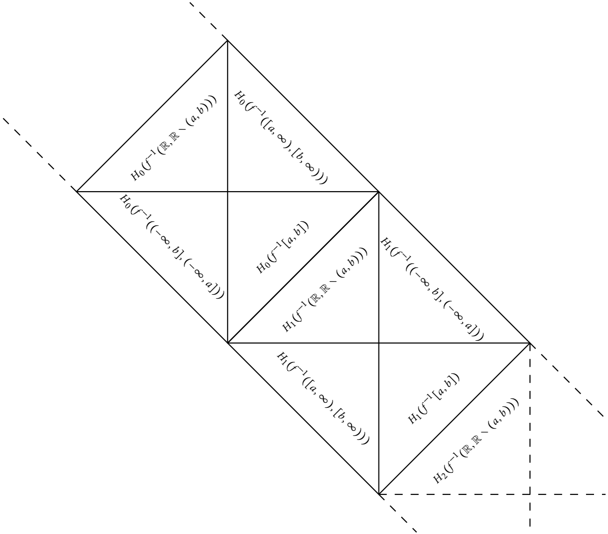
Theorem (B., Botnan, Oppermann, Steen 2020)

The representation type of $m \times n$ grids in which all horizontal maps are surjective is the same as that of general $m \times (n-1)$ grids.

Level and interlevel set persistence



Stability

Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $f, g : X \to \mathbb{R}$ (sufficiently regular) with $\|f - g\|_\infty = \delta$.

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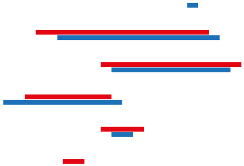

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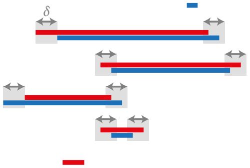

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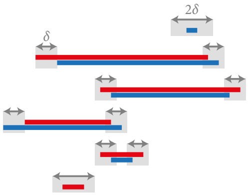
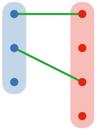


The category of matchings

Consider the category $\mathbf{Mch}$ with

- objects: sets,
- morphisms: matchings (bijections between subsets).



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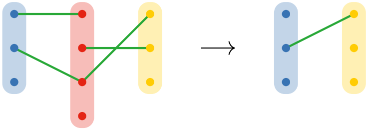
Composition: 

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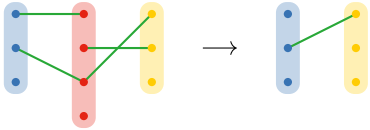
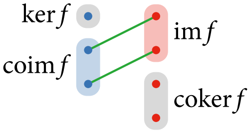


(Co)kernel/(co)image:



From barcodes to matching diagrams (and back)

- A barcode (collection of intervals) defines a diagram
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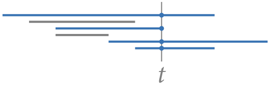

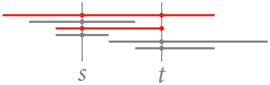

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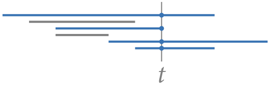

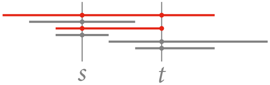

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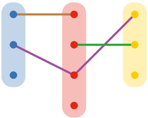

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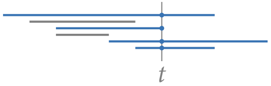

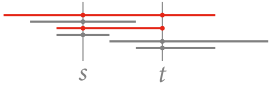

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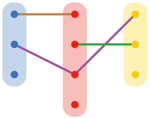

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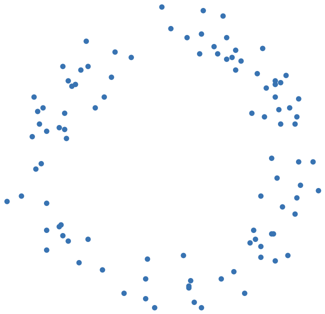
We can turn barcodes into a category $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbf{R}}$

Geometric interleavings



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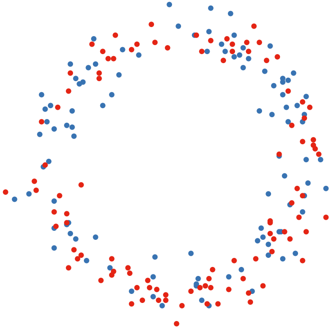

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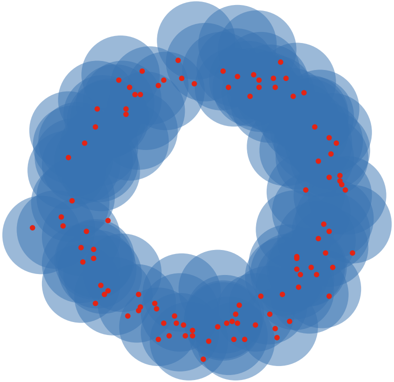

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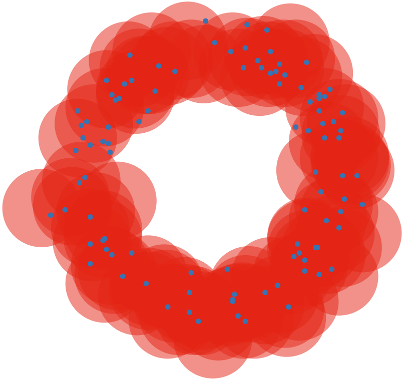

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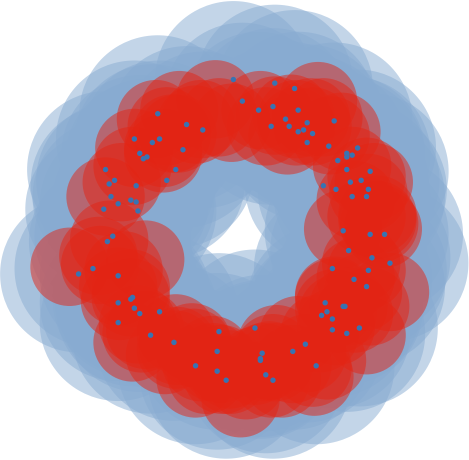

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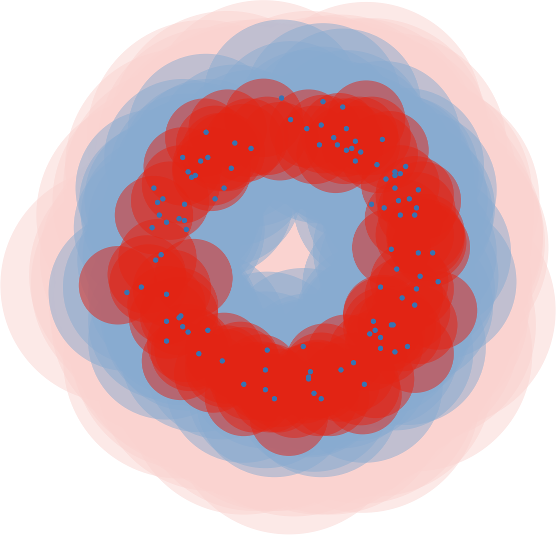

Interleavings

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$$\begin{tikzcd}
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Applying homology (a functor) preserves commutativity

- persistent homology of f, g yields δ-interleaved persistence modules $\mathbf{R} \to \mathbf{Vect}$

Persistence and stability: the big picture

Data	point cloud	$P \subset \mathbb{R}^d$

Persistence and stability: the big picture

$$
\begin{array}{ccc}
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Theorem (Chazal et al. 2009, 2012; B, Lesnick 2015)

If two persistence modules are δ-interleaved, then there exists a δ-matching of their barcodes:

- matched intervals have endpoints within distance $\leq \delta$,
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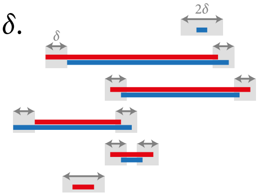

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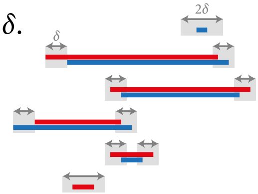

Equivalently: there exists a δ-interleaving of their barcodes
(as diagrams $\mathbf{R} \to \mathbf{Mch}$).

Non-functoriality of persistence barcodes

Can a persistence module M be mapped to its barcode $B(M)$ by a functor $B : \mathbf{vect} \to \mathbf{Mch}$?

- This would preserve δ-interleavings, and thus yield stability of persistence barcodes.

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Proposition

There is no functor $\mathbf{vect} \to \mathbf{Mch}$ *sending a* d*-dimensional vector space to a set with* d *elements.*

Structure of persistence sub-/quotient modules

Proposition

Let $M \twoheadrightarrow N$ be an epimorphism.
Then there is an injection of barcodes $B(N) \hookrightarrow B(M)$ such that
if J is mapped to I, then

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This construction is functorial.

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- In both $B(M)$ and $B(N)$, sort intervals with same left end
- Match up sorted intervals in order

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Let $M \twoheadrightarrow N$ be an epimorphism.
Then there is an injection of barcodes $B(N) \hookrightarrow B(M)$ such that if J is mapped to I, then

- *I and J are aligned below, and*
- *I bounds J above.*

This construction is functorial.

- In both $B(M)$ and $B(N)$, sort intervals with same left end
- Match up sorted intervals in order

Dually, there is an injection $B(M) \hookrightarrow B(N)$ for monos $M \hookrightarrow N$.

Induced matchings

For $f : M \to N$ an arbitrary morphism of pfd persistence modules, the epi-mono factorization

$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$$

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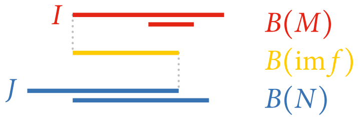
$$\chi(f) : B(M) \nrightarrow B(N).$$



Algebraic stability via induced matchings

Consider an interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(t \in \mathbb{R})$:

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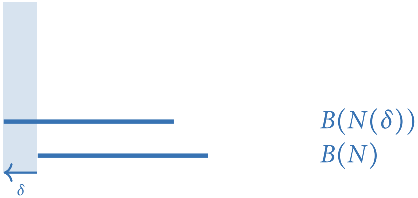

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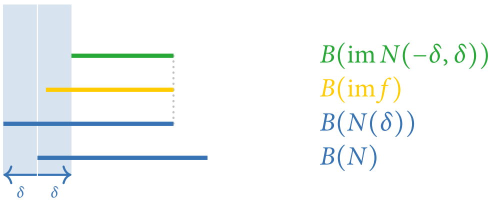

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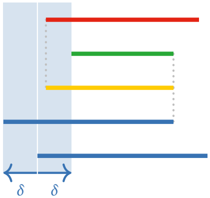

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Requires strong assumptions:

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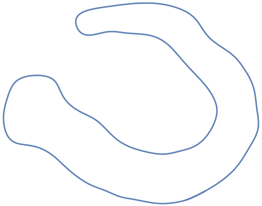

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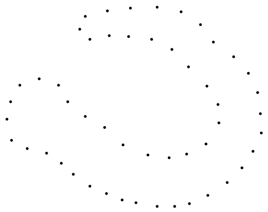

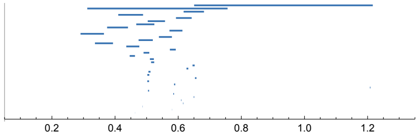

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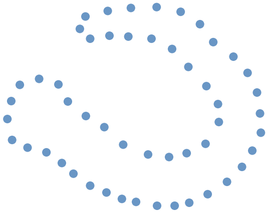

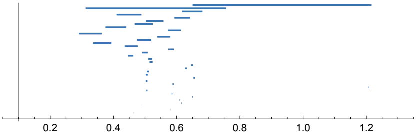

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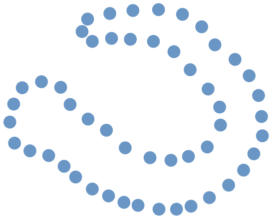

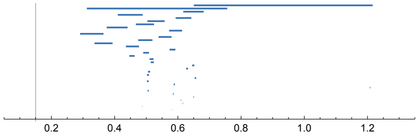

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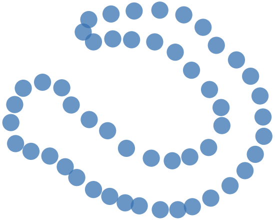

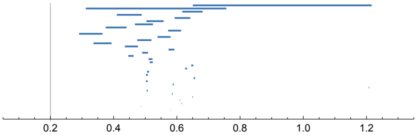

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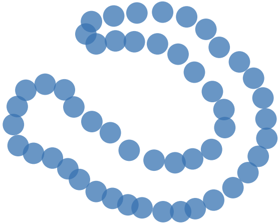

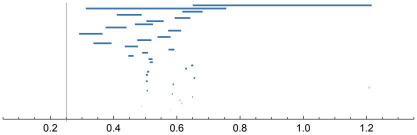

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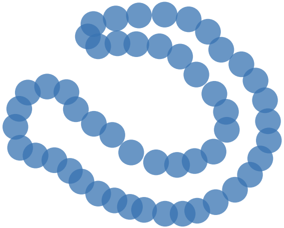

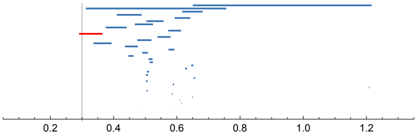

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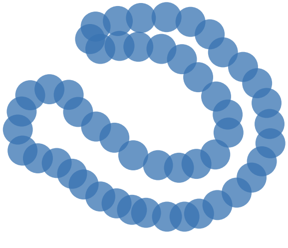

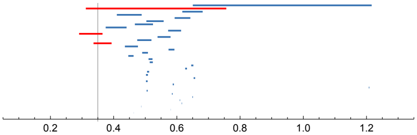

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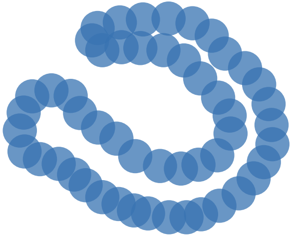

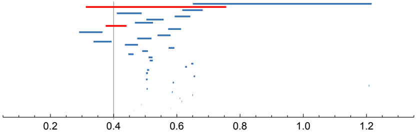

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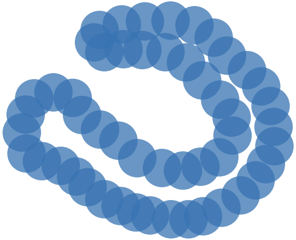

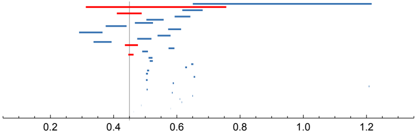

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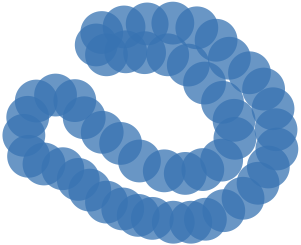

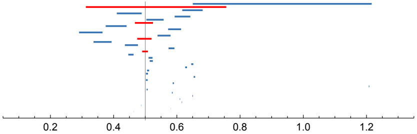

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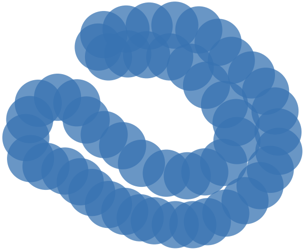

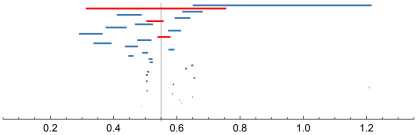

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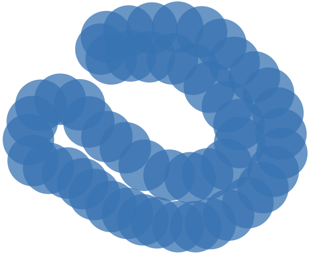

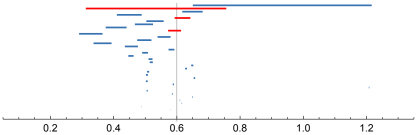

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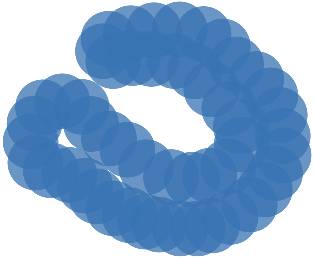

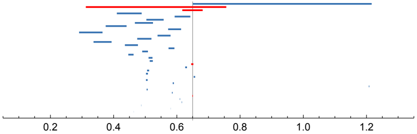

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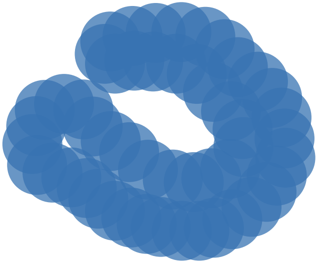

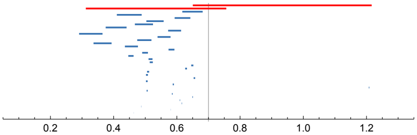

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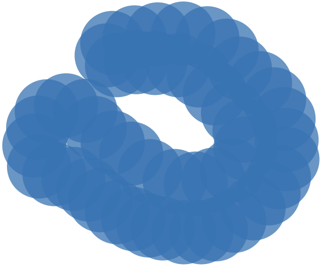

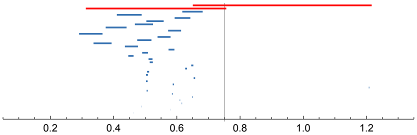

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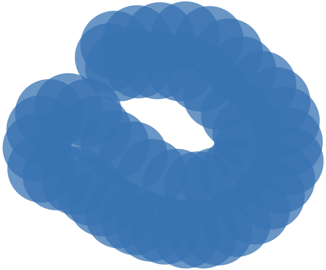

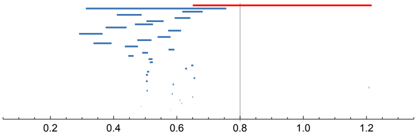

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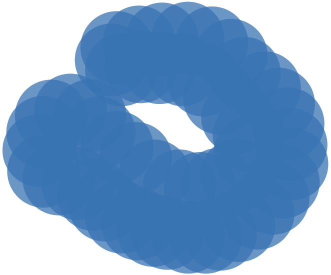
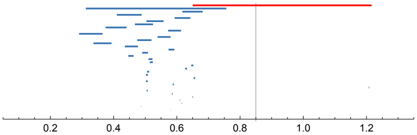




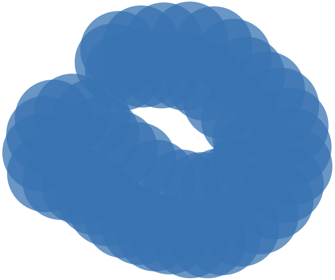
Homology reconstruction by thickening

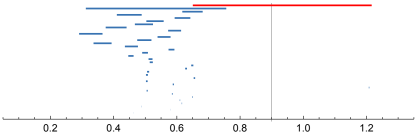
Theorem (Niyogi, Smale, Weinberger 2006)

Let X be a submanifold of $\mathbb{R}^d$. Let $P \subset X$ and $\delta > 0$ be such that

- *P_δ covers X, and*
- *$\delta < \sqrt{3/20}\,\mathrm{reach}(X)$.*

Then $H_*(X) \cong H_*(P_{2\delta})$.





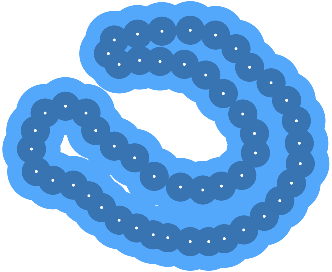
Homology inference using persistence

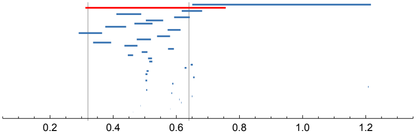
Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $X \subset \mathbb{R}^d$. Let $P \subset X$ and $\delta > 0$ be such that

- *P_δ covers X,*
- *the induced map $H_*(X \hookrightarrow X_\delta)$ is an isomorphism, and*
- *the induced map $H_*(X_\delta \hookrightarrow X_{2\delta})$ is a monomorphism.*

Then $H_*(X) \cong \operatorname{im} H_*(P_\delta \hookrightarrow P_{2\delta})$.





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Proof.

$$\begin{tikzcd}
X \arrow[dr, hook] & & X_\delta \arrow[dr, hook] & & X_{2\delta} \\
& P_\delta \arrow[ur, hook] & & P_{2\delta} \arrow[ur, hook] &
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$\square$

Homological realization

This motivates the *homological realization problem*:

Problem

Given a simplicial pair $L \subseteq K$*, find* X *with* $L \subseteq X \subseteq K$ *such that*

$$H_*(L) \twoheadrightarrow H_*(X) \hookrightarrow H_*(K);$$

equivalently, $\dim H_*(X) = \operatorname{rank} H_*(L \hookrightarrow K)$.

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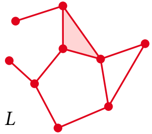

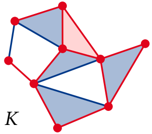

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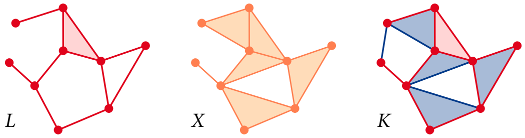

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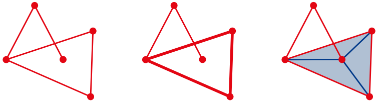

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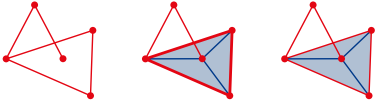

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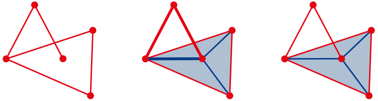

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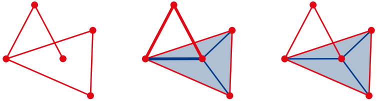
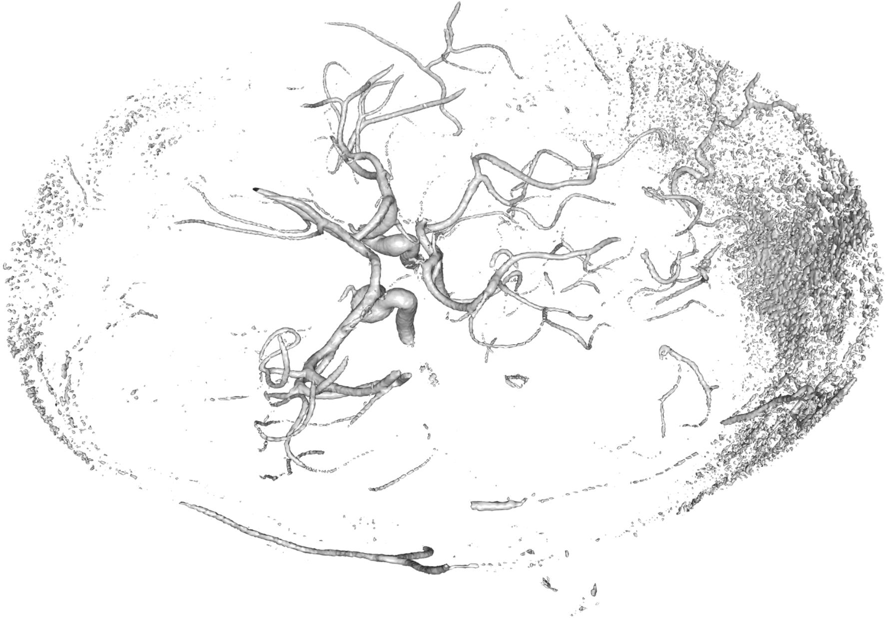


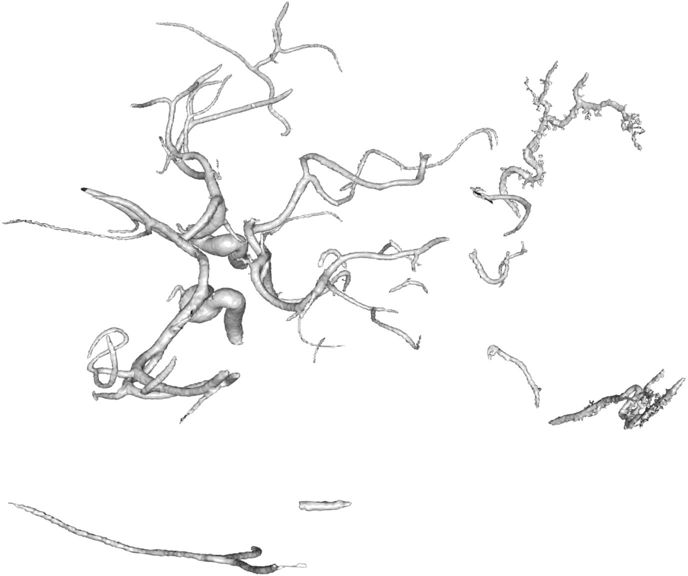
$$L \hookrightarrow X? \hookrightarrow K$$

Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

The homological realization problem is NP-hard, even in $\mathbb{R}^3$.

Simplification





Sublevel set simplification

Let $F_t = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem (Sublevel set simplification)

Given a function $f : X \subset \mathbb{R}^3 \to \mathbb{R}$ *and* $t \in \mathbb{R}$, $\delta > 0$,
find a function g *with* $\|g - f\|_\infty \le \delta$ *minimizing* $\dim H_*(G_t)$.

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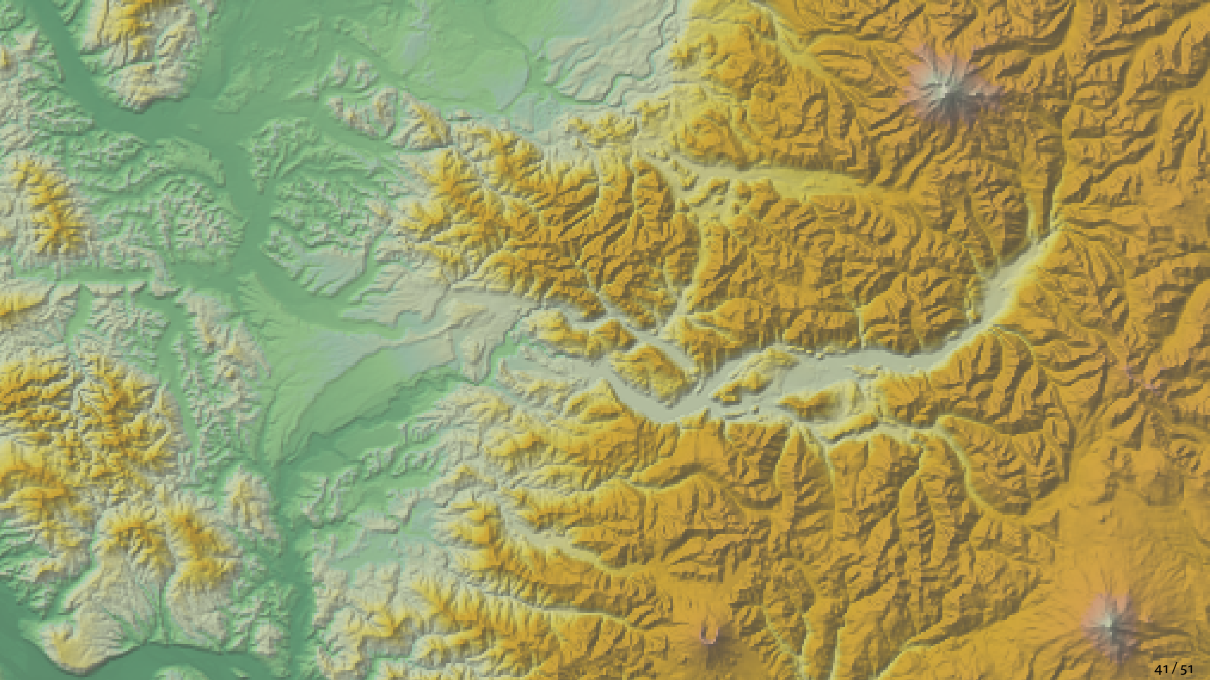
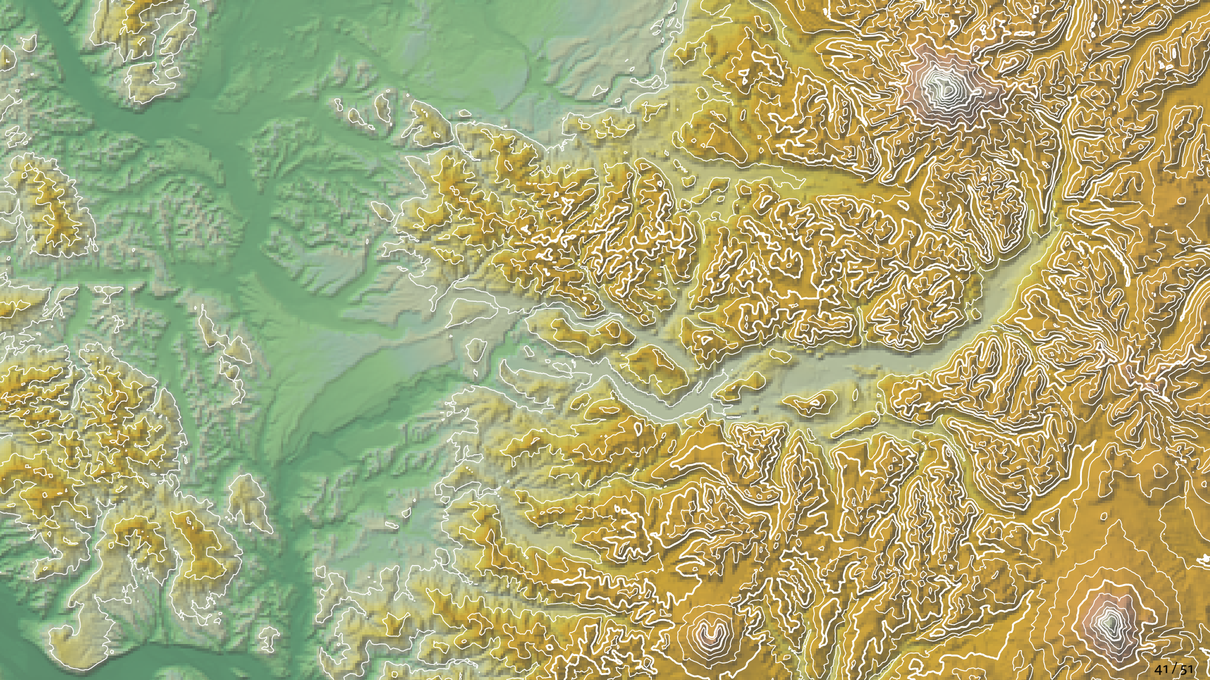
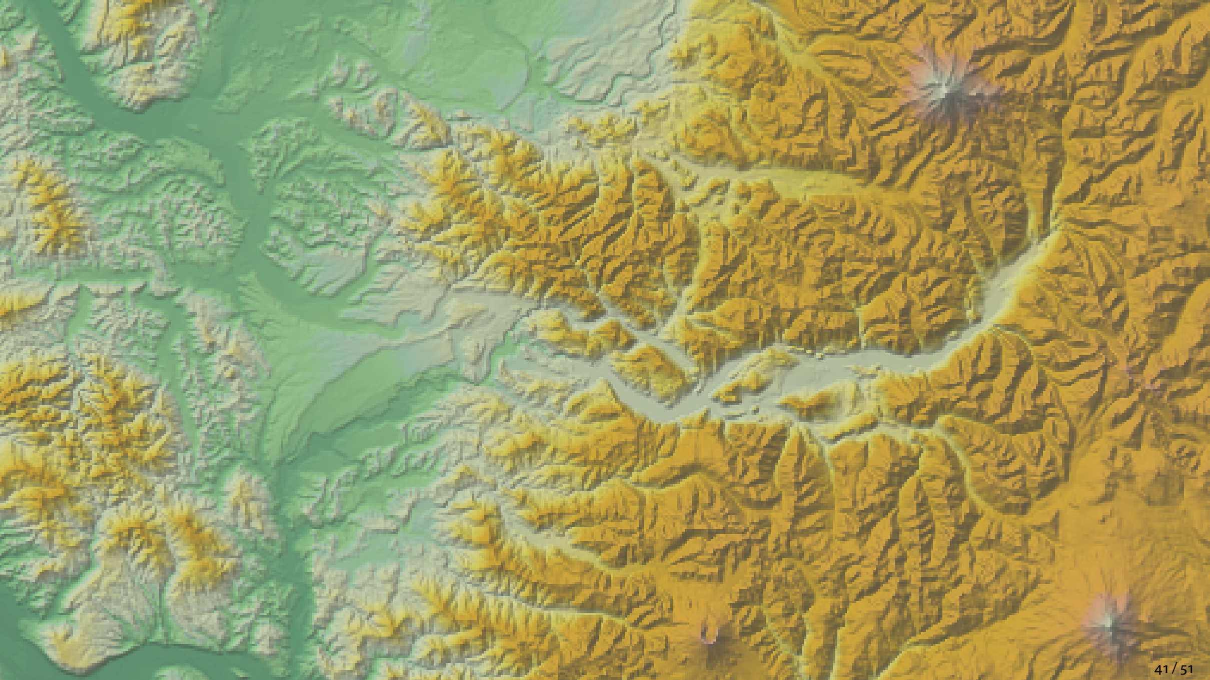
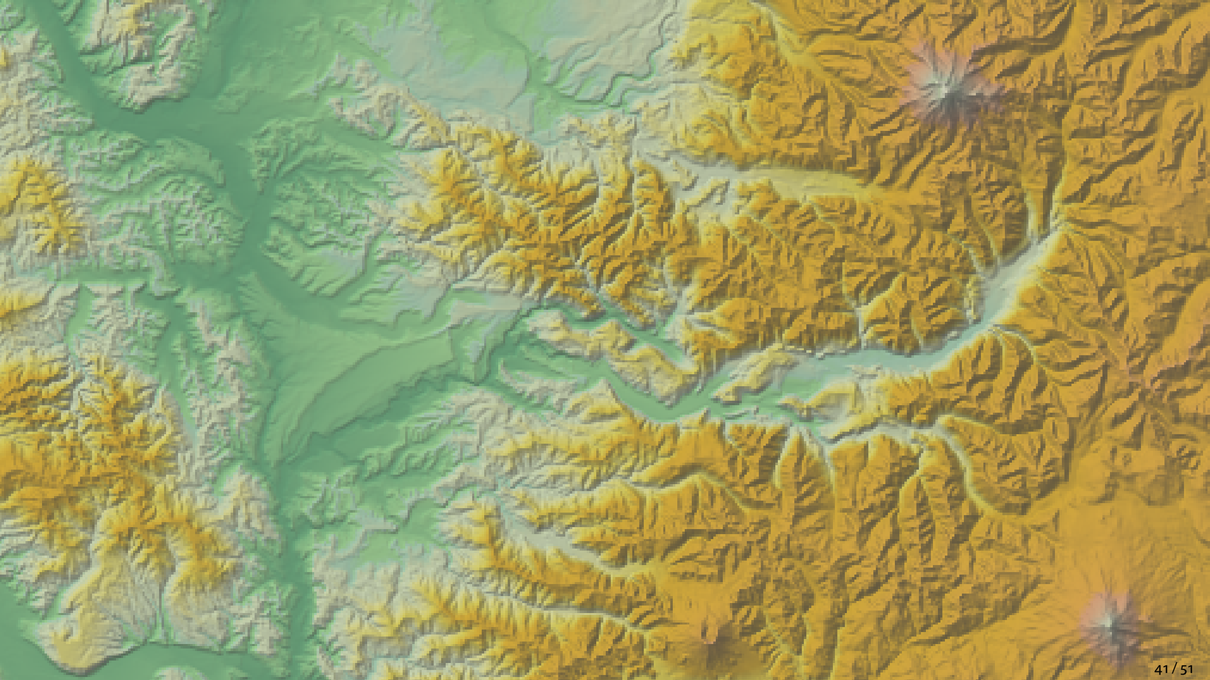
Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

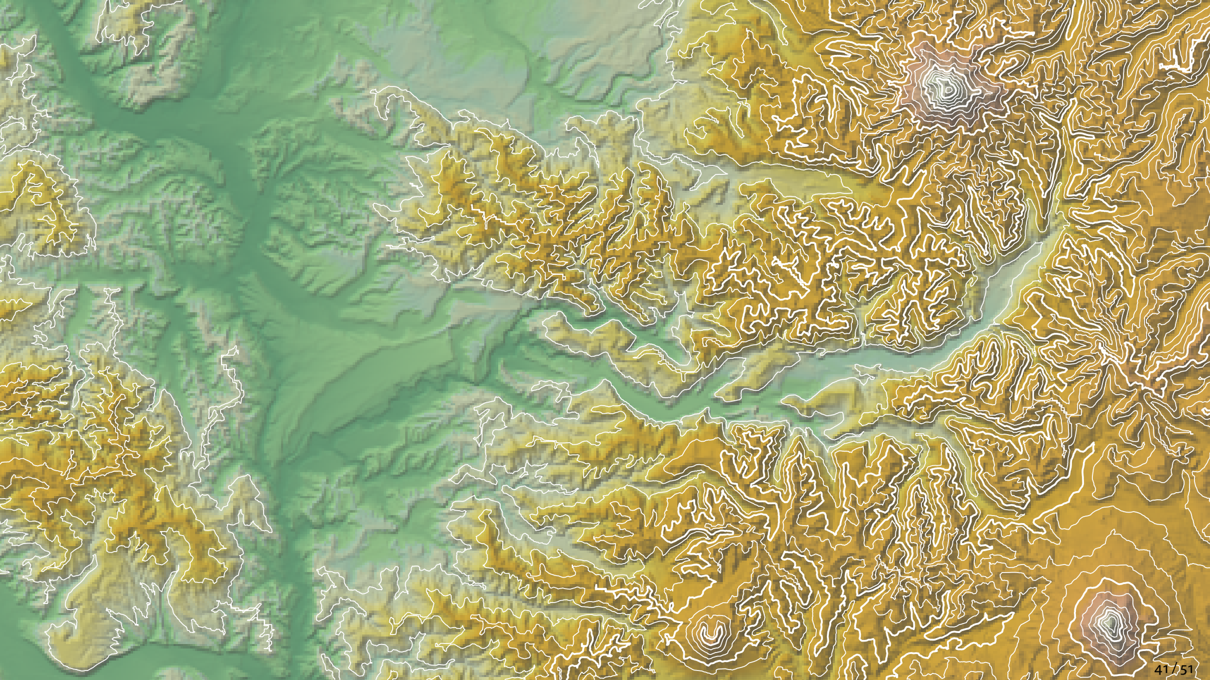
Sublevel set simplification in $\mathbb{R}^3$ is NP-hard.











Topological simplification of functions

Consider the following problem:

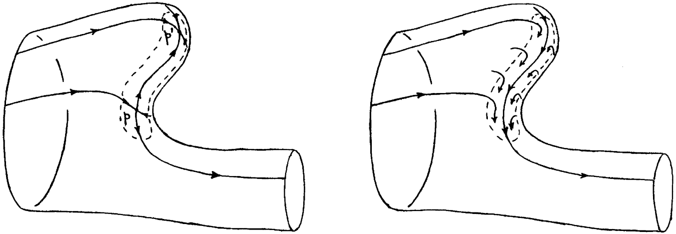
Problem (Topological simplification)

Given a function f and a real number $\delta \geq 0$,
find a function f_δ subject to $\|f_\delta - f\|_\infty \leq \delta$
with the minimal number of critical points.

Persistence and Morse theory

Morse theory (smooth or discrete):

- Relates critical points to homology of sublevel sets
- Provides a method for *canceling* pairs of critical points



(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

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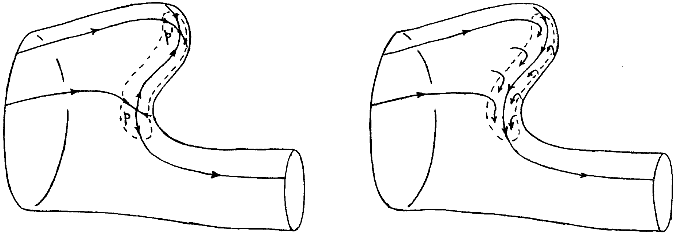

(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

Persistent homology:

- Relates homology of different sublevel set
- Identifies pairs of critical points (birth and death of homology) and quantifies their *persistence*

Persistence and discrete Morse theory

For a Morse fuction:

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By stability of persistence barcodes:

Proposition

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Theorem (B, Lange, Wardetzky, 2011)

Let f be a function on a surface and let $\delta > 0$.
Canceling all pairs with persistence $\leq 2\delta$ yields a function f_δ

- *satisfying $\|f_\delta - f\|_\infty \leq \delta$ and*
- *achieving the lower bound on the number of critical points.*

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Does not generalize to higher-dimensional manifolds!

History

When was persistent homology invented?

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Topological persistence and simplification
Foundations of Computer Science, 2000

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- P. Frosini
  A distance for similarity classes of submanifolds of a Euclidean space
  Bulletin of the Australian Mathematical Society, 1990.

When was persistent homology invented first?

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ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

When was persistent homology invented first?

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Rank and span in functional topology

Search within citing articles

Exact homomorphism sequences in homology theory — ed.ac.uk [PDF]
JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR
The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] ...
Cited by 46 Related articles All 3 versions Cite Save More

Marston Morse and his mathematical works — ams.org [PDF]
R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org
American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to ...
Cited by 24 Related articles All 4 versions Cite Save More

Unstable minimal surfaces of higher topological structure
M Morse, CB Tompkins - Duke Math. J, 1941 - projecteuclid.org
1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. Theproblem of the existenceof minimal surfaces of unstable type contains many of the typical difficulties, especially those ofa topological nature. Having studied this ...
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When was persistent homology invented first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

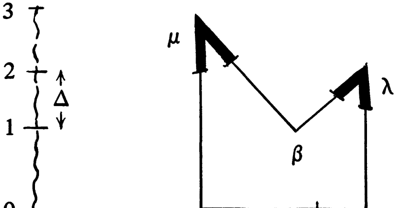
In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However

When was persistent homology invented first?

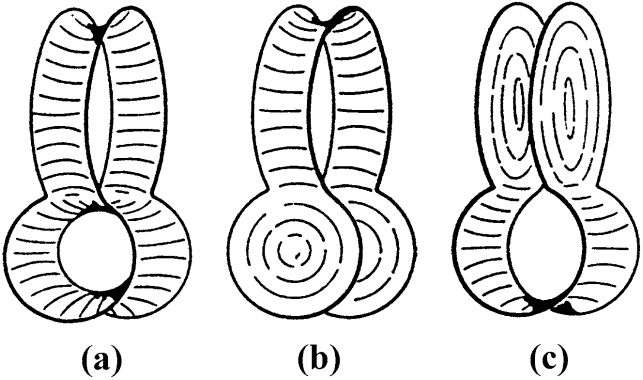
inequalities pertain between the dimensions of the A_i and those of $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels **[D]** in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers **[36]** and **[44]** constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.

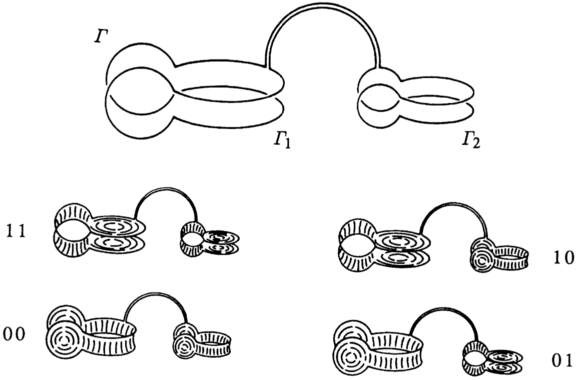


Motivation and application: minimal surfaces



(from Dierkes et al.: *Minimal Surfaces*, 2010)

Motivation and application: minimal surfaces



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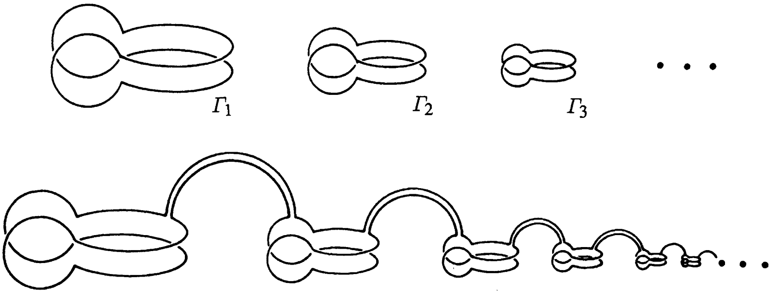
Motivation and application: minimal surfaces



(from Dierkes et al.: *Minimal Surfaces*, 2010)

Existence of unstable minimal surfaces

Using persistent homology:

- Number of ϵ-persistent critical points (minimal surfaces) is finite for any $\epsilon > 0$
- Morse inequalities for ϵ-persistent critical points

Theorem (Morse, Tompkins 1939; Shiffman 1939)

There is a C_1 curve bounding an unstable minimal surface (an index 1 critical point of the area functional).

Computation

An example computation

Example data set:

- 192 points on $\mathbb{S}^2$
- Vietoris–Rips complexes
- persistent homology barcodes up to dimension 2
- over 56 mio. simplices in 3-skeleton

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Some previous software:

- javaplex (Stanford): 3200 seconds, 12 GB
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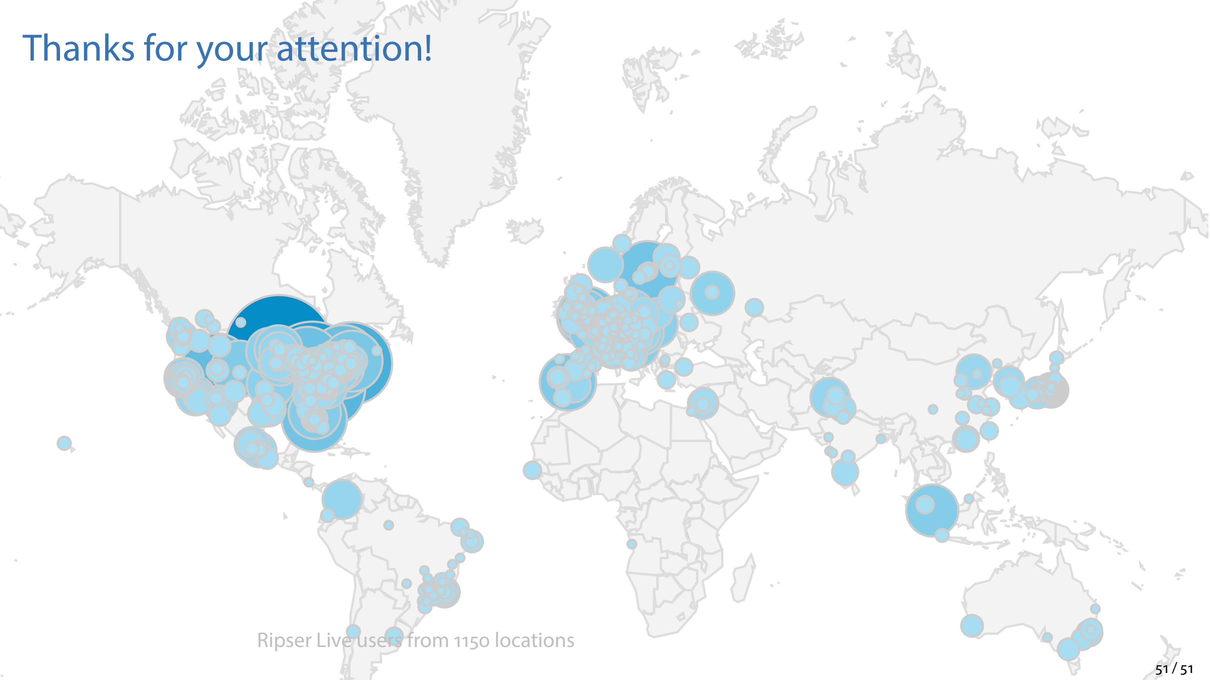
- Ripser: 1.2 seconds, 160 MB

Ripser

A software for computing Vietoris–Rips persistence barcodes

- around 1000 lines of C++ code, no external dependencies
- support for
  - coefficients in a prime field $\mathbb{Z}/p\mathbb{Z}$
  - sparse distance matrices (for distance threshold)
- open source (`http://ripser.org`)
- online version (`http://live.ripser.org`)
- 2016 ATMCS Best New Software Award (joint with RIVET by M. Lesnick and M. Wright)

Thanks for your attention!



Ripser Live users from 1150 locations