Homological reconstruction and simplification in $\mathbb{R}^3$

Ulrich Bauer[1] Dominique Attali[2] Olivier Devillers[3]
Marc Glisse[3] André Lieutier[4]

[1]IST Austria

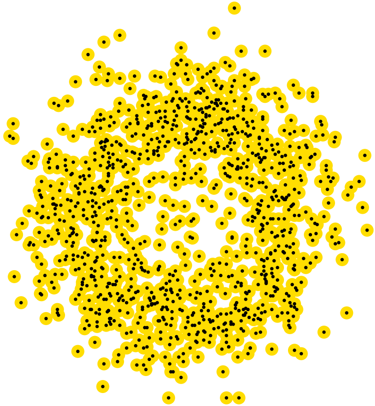
[2]GIPSA-lab

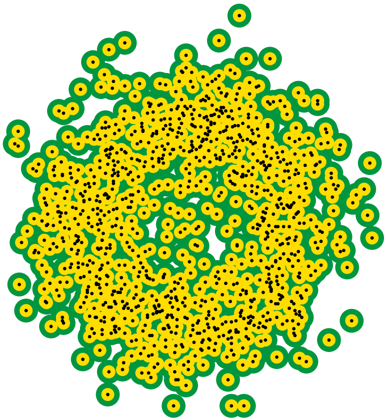
[3]GEOMETRICA (INRIA)

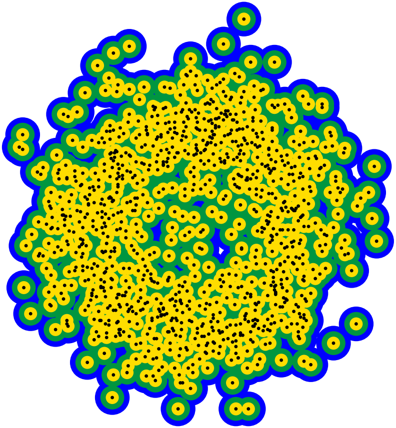
[4]Dassault Systèmes

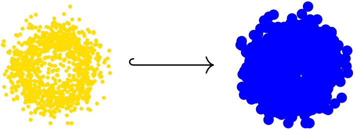
June 18, 2013

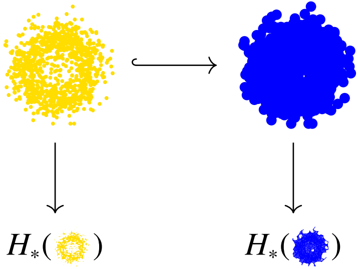
Symposium on Computational Geometry 2013



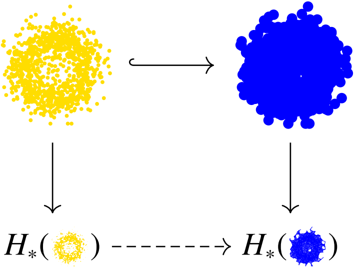








- Homology transforms topological spaces to groups



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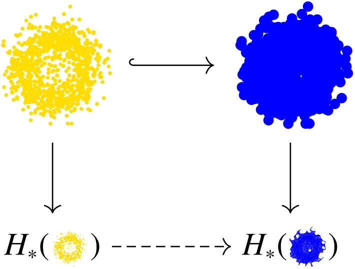

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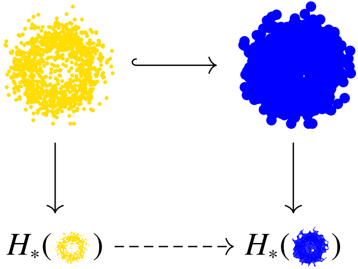

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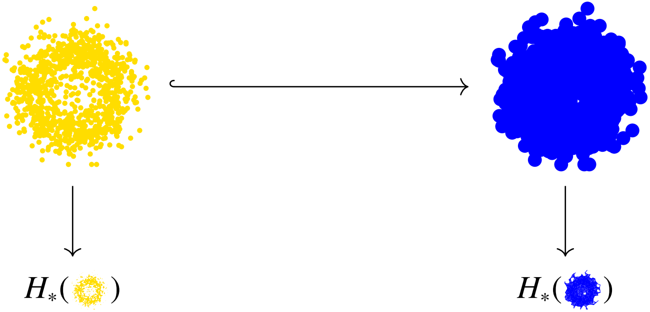
- *Persistent homology group*: $\operatorname{im} H_*($ $\hookrightarrow$  $)$
- Under mild sampling conditions on  :

$\operatorname{im} H_*($ $\hookrightarrow$  $) \sim H_*($  $)$

[Edelsbrunner et al. 2005, Chazal et al. 2005]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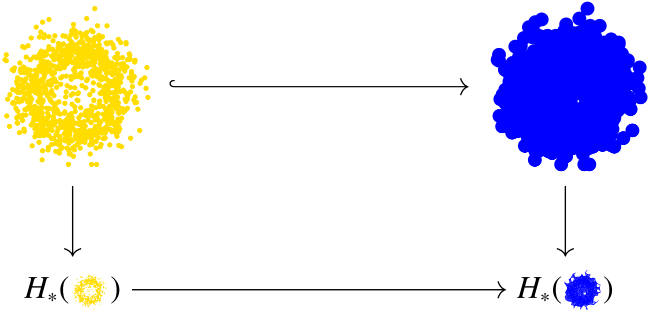
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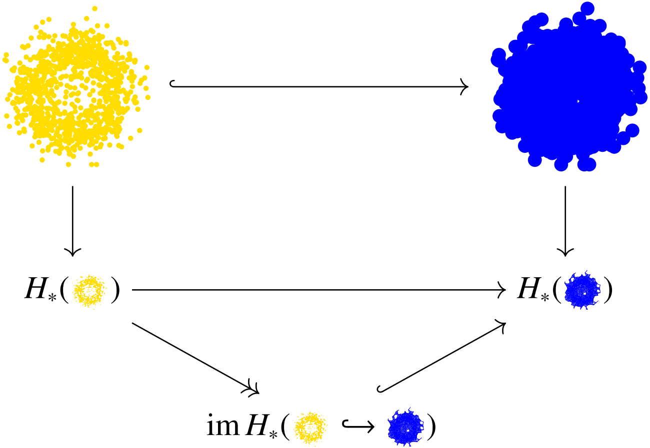
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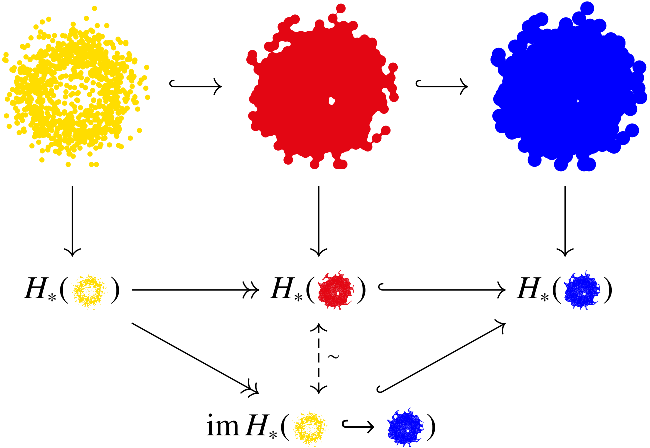
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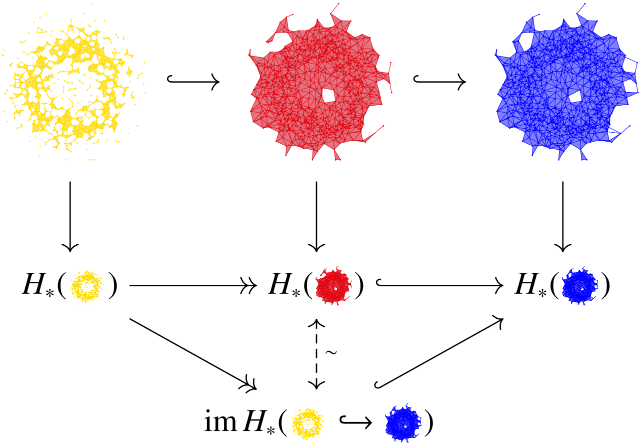

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(Throughout this talk: homology with field coefficients)

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- If $L \subset Y \subset K$ then $\beta(Y) \geq \operatorname{rank} H_*(L \hookrightarrow K)$
- Equality iff Y is a homological reconstruction

Do homological reconstructions always exist?

- Homological reconstructions do not always exists

$$L \hookrightarrow \; ? \; \hookrightarrow K$$

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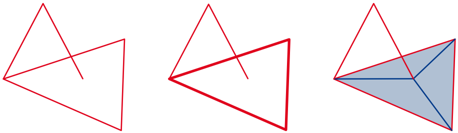

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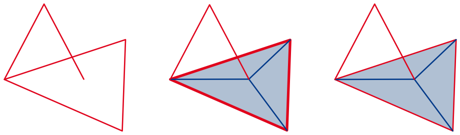

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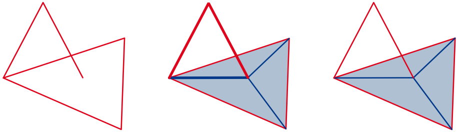

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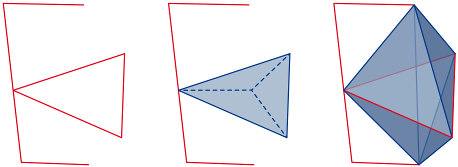

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Homological reconstruction in $\mathbb{R}^3$ is NP-hard

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  - (a) Φ has a satisfying assignment
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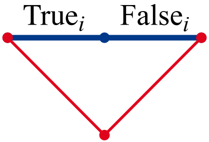
  - (c) (K, L) has a reconstruction as a subcomplex of K

Reduction from 3-SAT: the variable gadget



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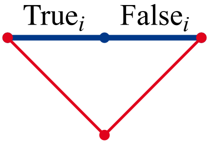

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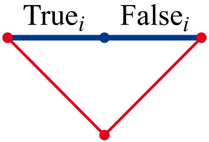

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- edges $\mathrm{True}_i, \mathrm{False}_i$ correspond to possible values of variable x_i

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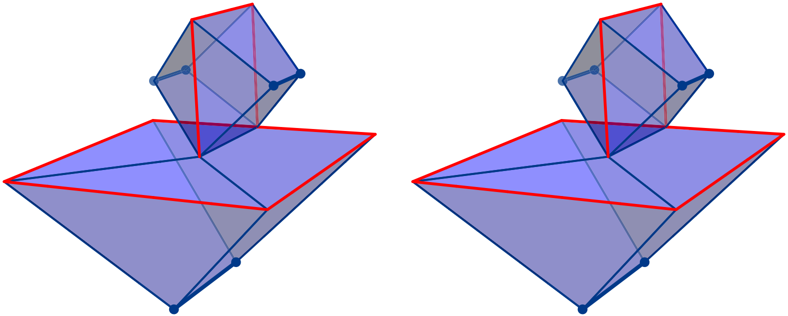

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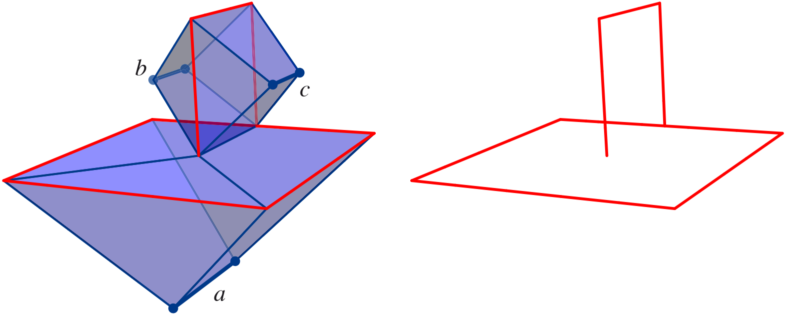

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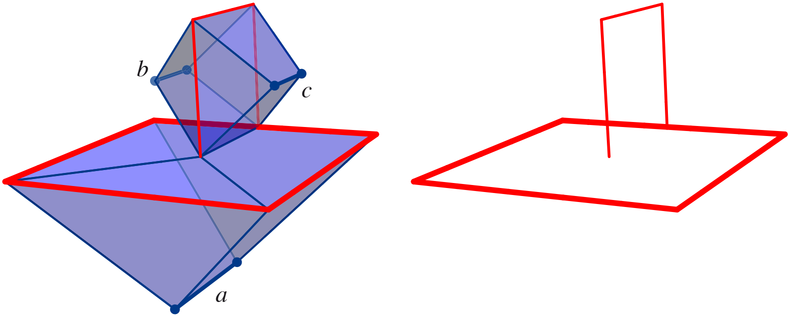

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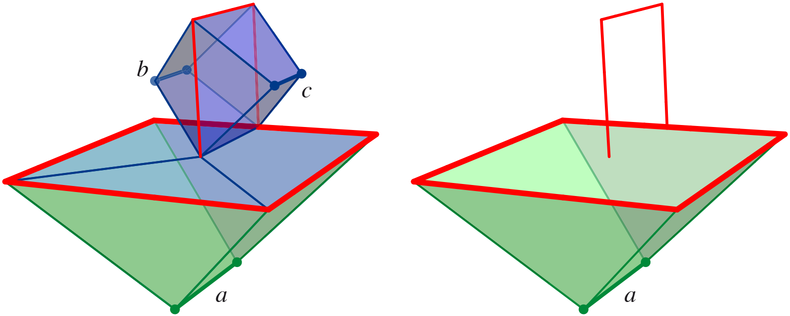

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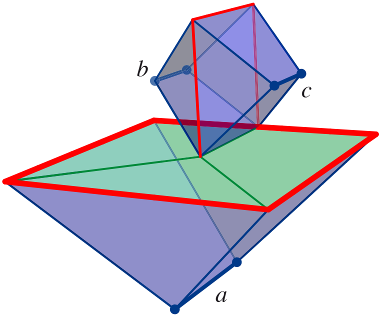

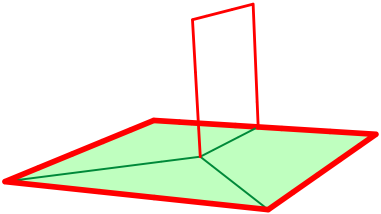

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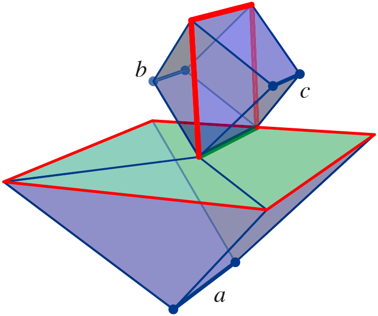

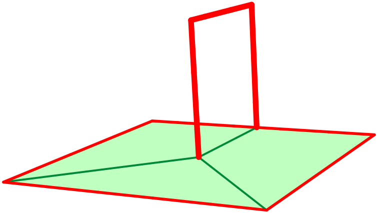

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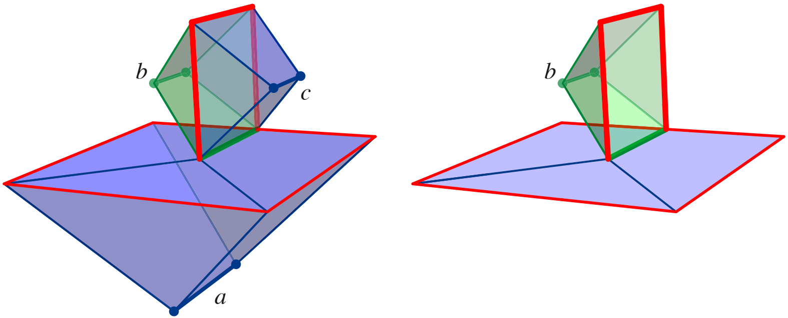

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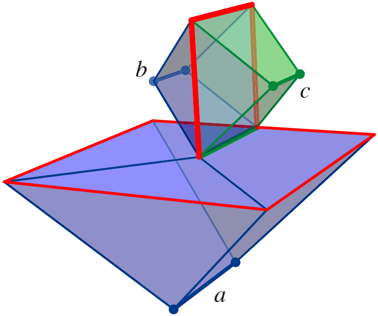

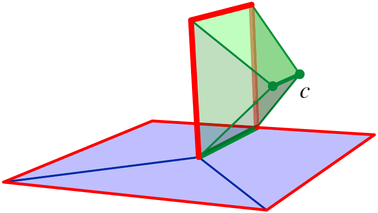

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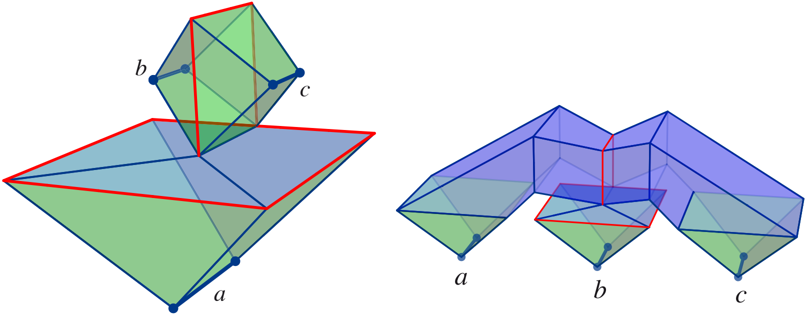

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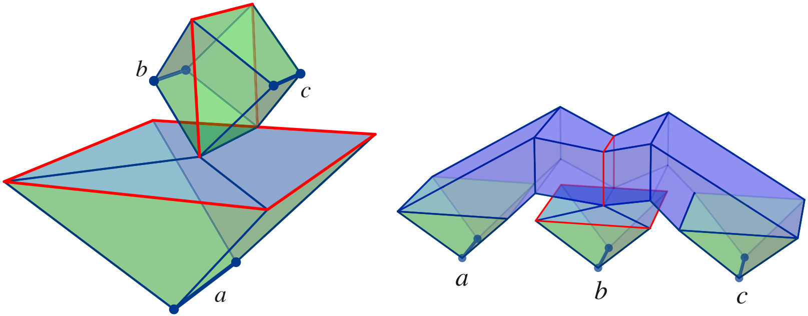

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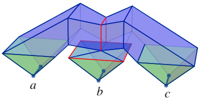
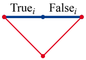
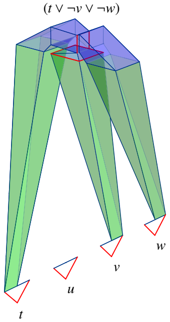
- edges a, b, c correspond to literals in a clause

Reduction from 3-SAT: an example







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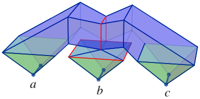

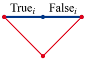

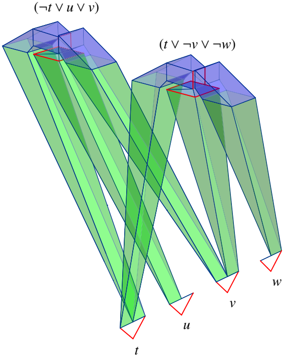

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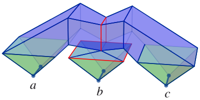

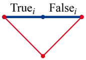

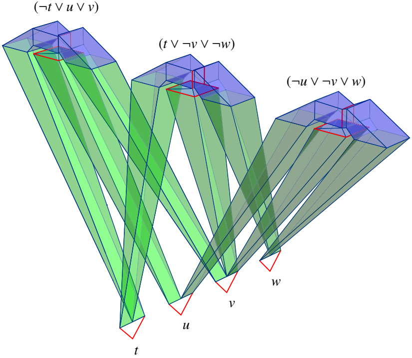

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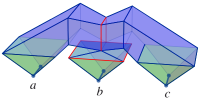

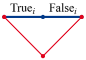

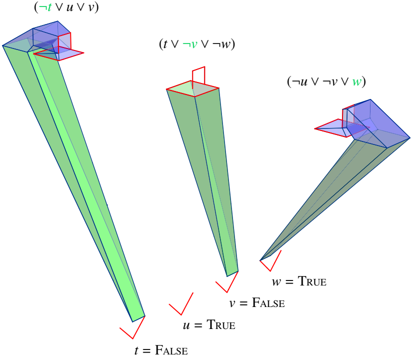

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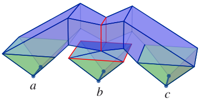

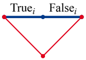

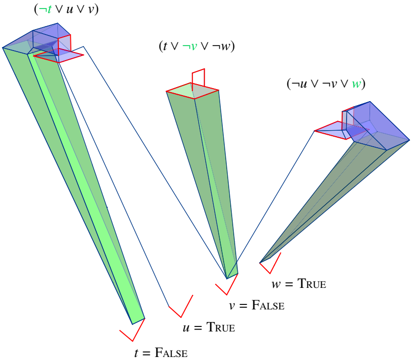

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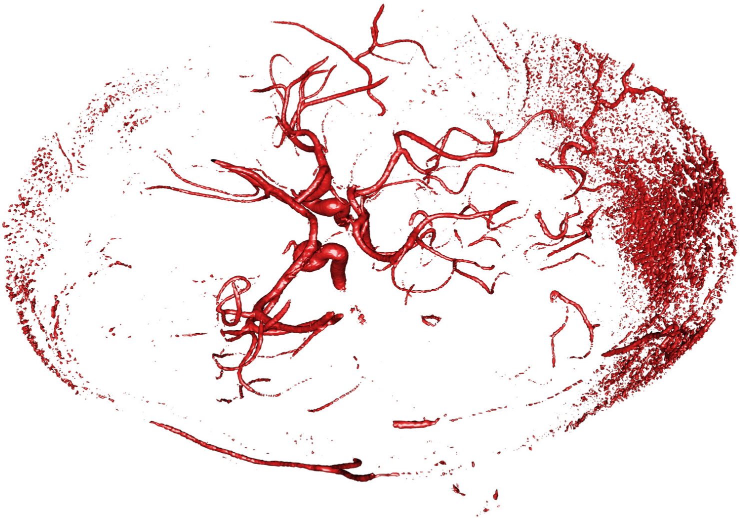

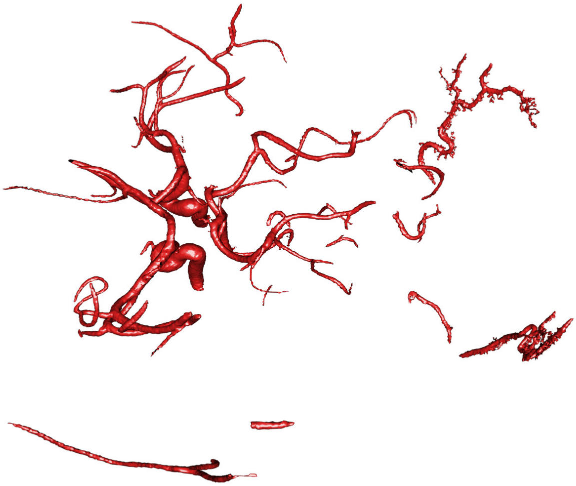

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Let $F_{\leq t} = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem (Sublevel set simplification)

Given a PL function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ *and* $t \in \mathbb{R}$, $\delta > 0$,
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Thus we are looking for X with $L \subset X \subset K$ minimizing $\beta(X)$.

- Lower bound $\beta(G_{\leq t}) \geq \operatorname{rank} H_*(L \hookrightarrow K)$
- *Sublevel set reconstruction*: a function g achieving the lower bound

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Let f be a PL function and $t \in \mathbb{R}$, $\delta \geq 0$. Let $F_{[a,b]} = f^{-1}[a, b]$.

Definition (Well group)

The (t, δ)-*well group* of f is

$$W_*(f, t, \delta) = \bigcap_{g:\|g-f\|_\infty \leq \delta} \operatorname{im} H_*(G_t \hookrightarrow F_{[t-\delta,t+\delta]})$$

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- $\beta(G_t) \geq \operatorname{rank} H_*(G_t \hookrightarrow F_{[t-\delta,t+\delta]}) \geq \operatorname{rank} W_*(f,t,\delta)$
- *Realization* of well group: a function g with $\|g-f\|_\infty \leq \delta$ achieving the lower bound $\beta(G_t) = \operatorname{rank} W_*(f,t,\delta)$.

Well group realization is level set reconstruction

Theorem

Let f be a PL function on $\mathbb{S}^n$ with $t \pm \delta \in \operatorname{int}(\operatorname{im} f)$.
A PL function g realizes the well group $W_*(f, t, \delta)$ if and only if it is a (t, δ)-level set reconstruction of f.

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Corollary

Realization of a well group on $\mathbb{S}^3$ is NP-hard.

Thanks for your attention!