

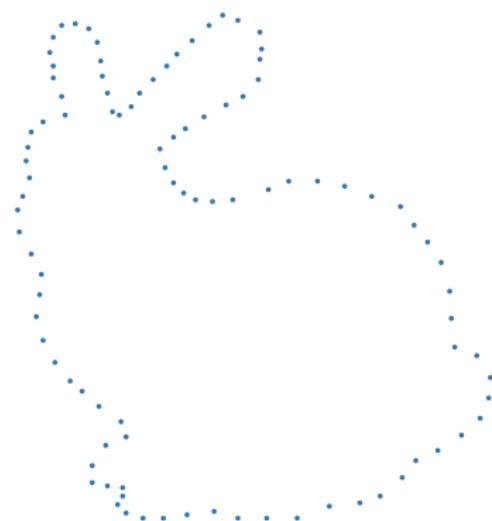
The Morse theory of Čech and Delaunay filtrations

Ulrich Bauer Herbert Edelsbrunner

IST Austria

TOPOSYS meeting
September 9, 2014

Connect the dots: topology from geometry



Connect the dots: topology from geometry

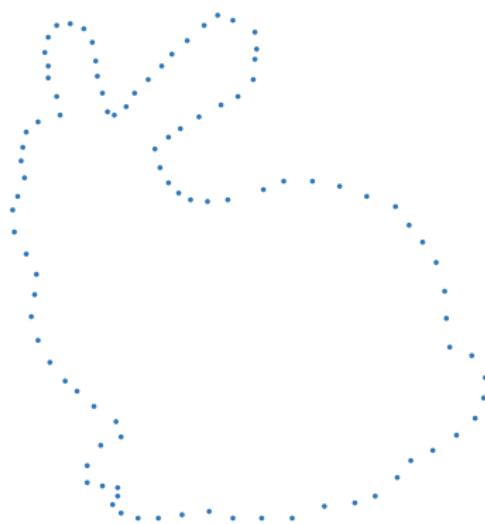


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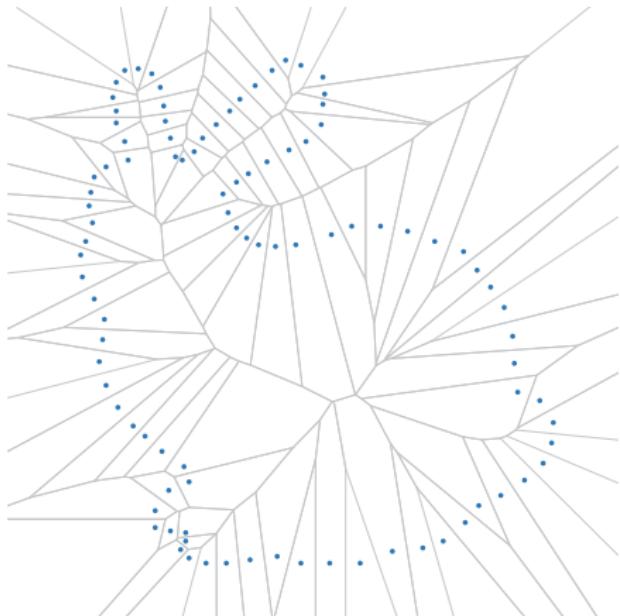
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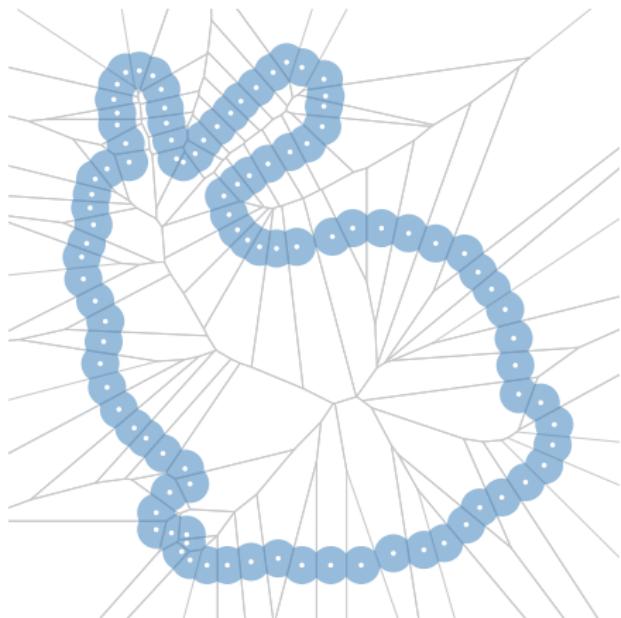
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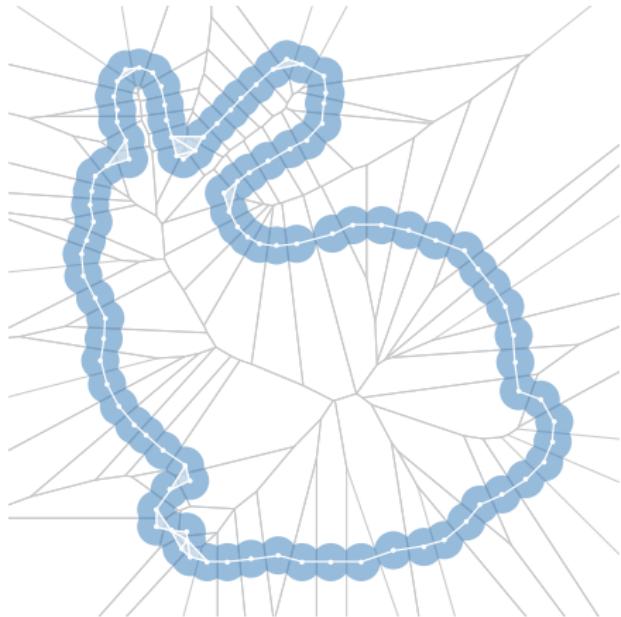
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Čech and Delaunay functions

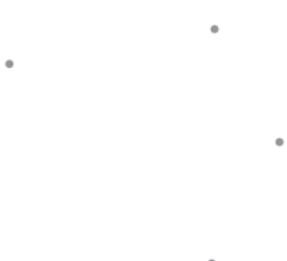
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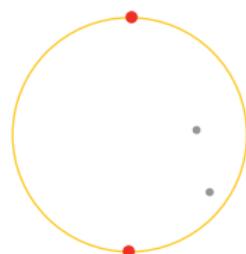
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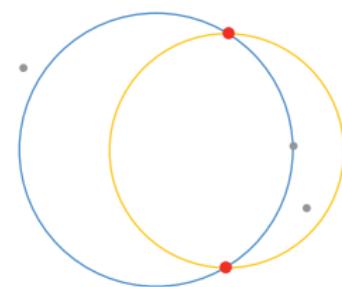
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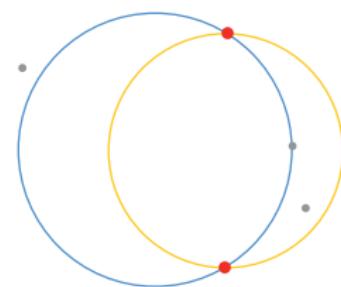
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- defined only if Q has an empty circumsphere: $Q \in \text{Del}(X)$

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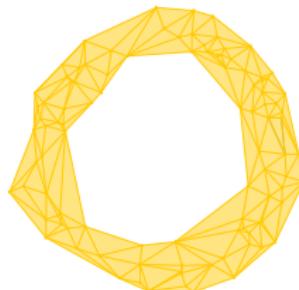
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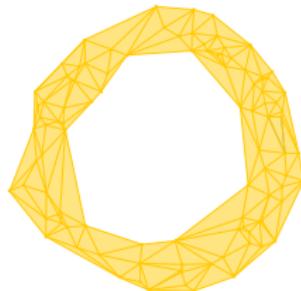
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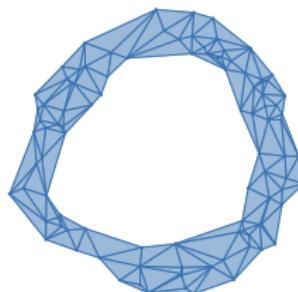
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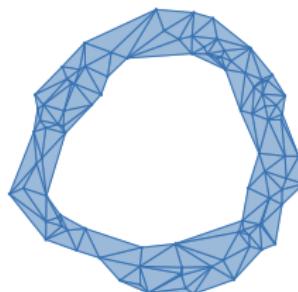
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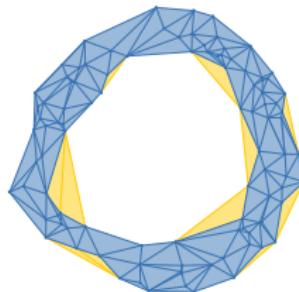
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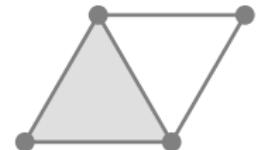
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- Are all three complexes homotopy equivalent?
- Are they related by a sequence of simplicial collapses?

Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

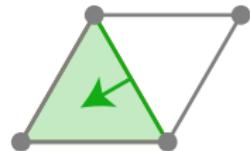


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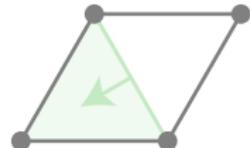


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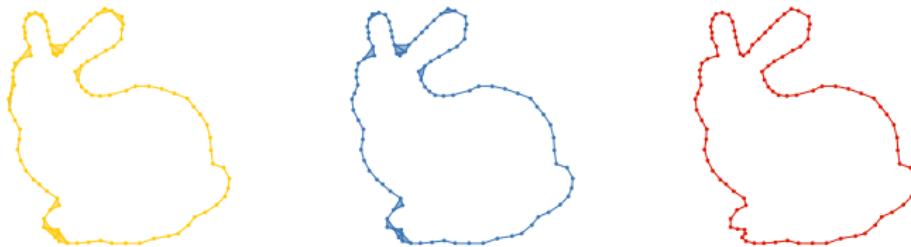
If there is a sequence of such elementary collapses from K to M , we say that K *collapses* to M (written as $K \searrow M$).

Main result: a sequence of collapses

Theorem (B, Edelsbrunner)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent. In particular,

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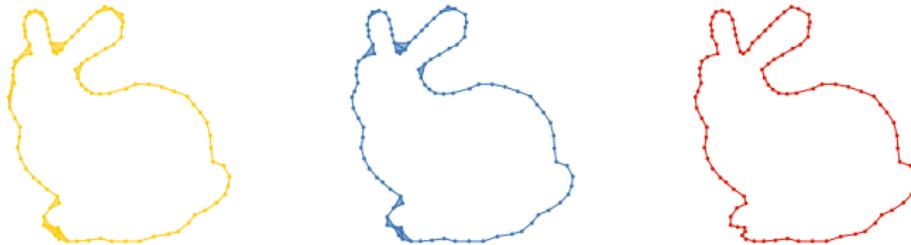


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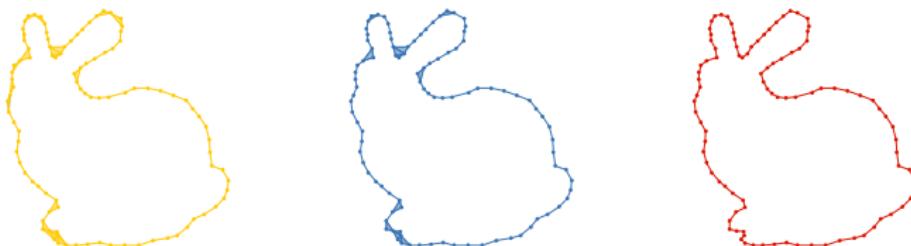
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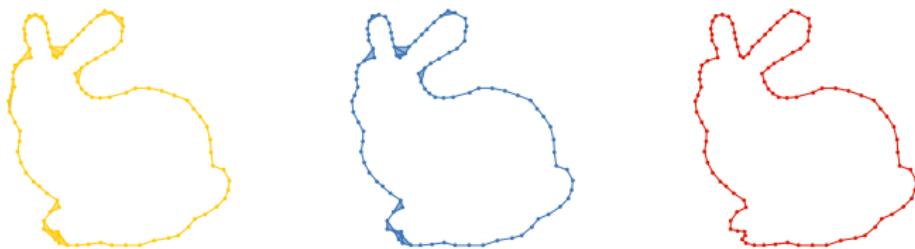
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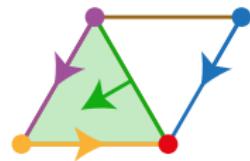


- All collapses are induced by a single *discrete gradient field*
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- Also works for weighted point sets

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a simplicial complex is a partition of the simplices into singletons and *facet pairs* (Q, R) (Q is a face of R with codimension 1).

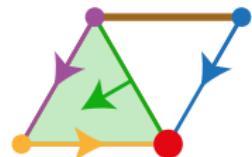


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The singletons are called *critical simplices*.

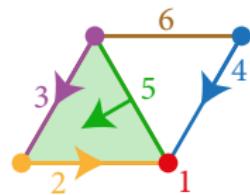


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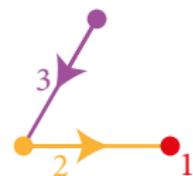


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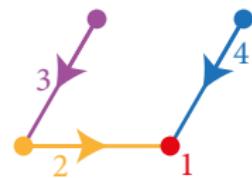


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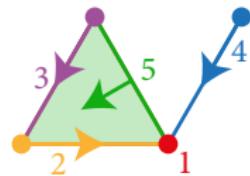


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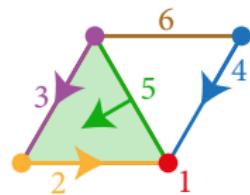


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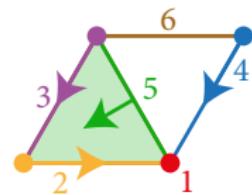


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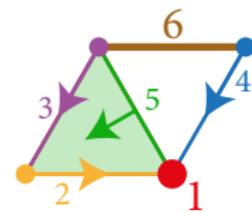


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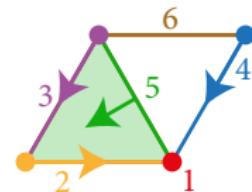
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If $f^{-1}(t) = \{Q\}$ then t is a *critical value*.

Collapses from Morse functions and gradients

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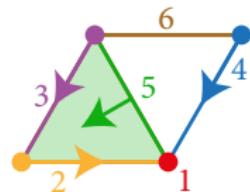


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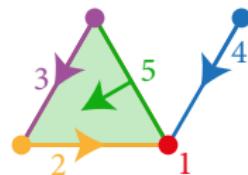


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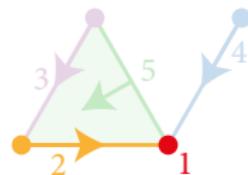


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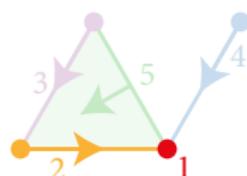


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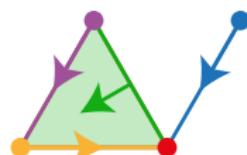
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Corollary

*If $K \setminus L$ is the union of facet pairs of V ,
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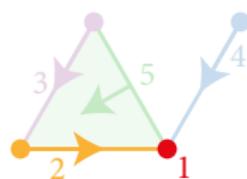


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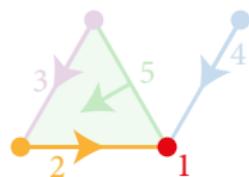


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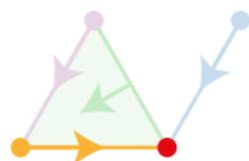
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We say that V induces the collapse $K \searrow L$.

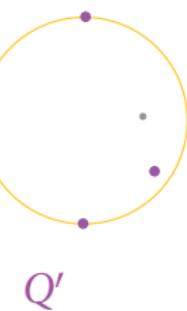
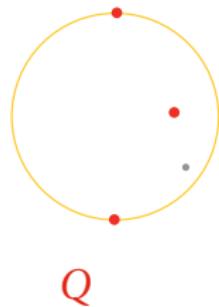
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- Example: two simplices Q, Q' with $f_C(Q) = f_C(Q')$ that do not form a facet pair:

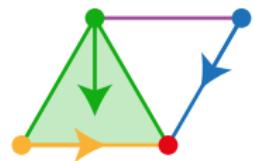


Generalized discrete Morse theory

Definition (Freij 2009)

A *generalized discrete vector field* on a simplicial complex is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}.$$

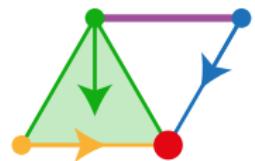


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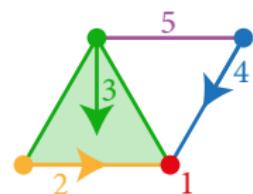
The singletons are called *critical simplices*.

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A function $f : K \rightarrow \mathbb{R}$ on a simplicial complex is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

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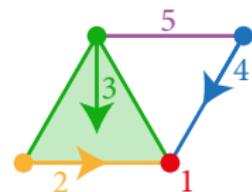


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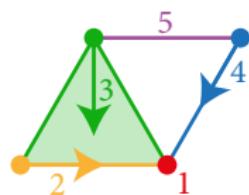


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- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes
- the level sets $f^{-1}(t)$ form a generalized vector field (the *discrete gradient* of f)



Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:

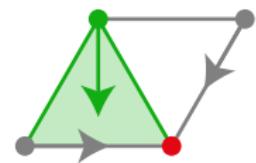


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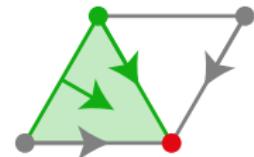


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Therefore the collapsing theorems also hold for generalized discrete Morse functions.

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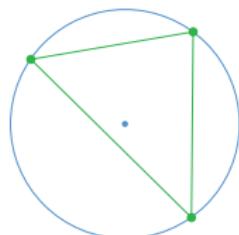
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- Q is a critical simplex of f_D
- Q is a centered Delaunay simplex
(containing the circumcenter in the interior)



Čech intervals

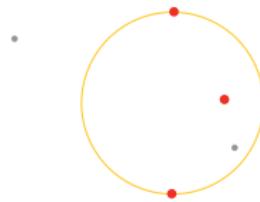
Lemma

Let $Q \subseteq X$ be a simplex with smallest enclosing sphere S .
Then $Q' \subseteq X$ has the same smallest enclosing sphere S iff

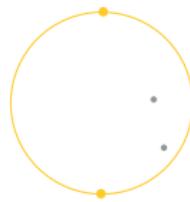
$$Q' \in [L, U],$$

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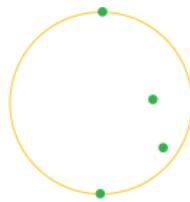
- L : points of X on S ,
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Q



L



U

The front face of a simplex

Let $T \subseteq X$ be a simplex with smallest circumsphere S .

Write the center z of S as an affine combination

$$z = \sum_{x \in T} \mu_x x, \quad 1 = \sum_{x \in T} \mu_x.$$

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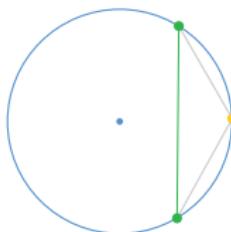
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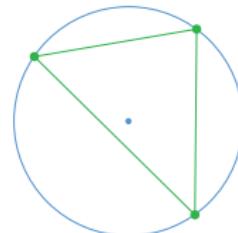
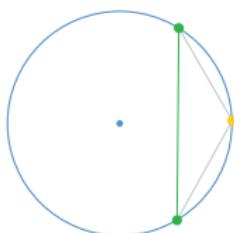
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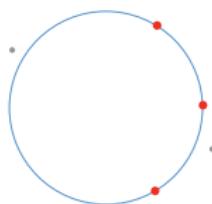
Lemma

Let $Q \subseteq X$ be a simplex with smallest empty circumsphere S .
Then $Q' \subseteq X$ has the same smallest empty circumsphere S iff

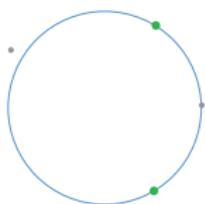
$$Q' \in [F, T],$$

where

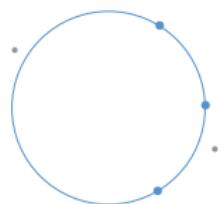
- T : points of X on S ,
- F : front face of T .



Q



F



T

Sphere optimization problems

Both Čech and Delaunay function are defined using the smallest sphere S satisfying certain constraints:

$$\underset{r,z}{\text{minimize}} \quad r$$

$$\begin{aligned} \text{subject to} \quad & \|z - q\| \leq r, \quad q \in Q, \\ & \|z - e\| \geq r, \quad e \in E. \end{aligned}$$

Here r is the radius and z is the center of the sphere S .

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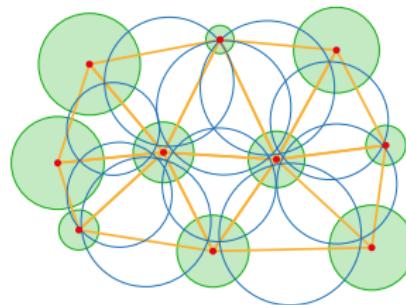
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- Čech function: choose $E = \emptyset$
- Delaunay function: choose $E = X$

Weighted point sets

All constructions can be generalized to weighted point sets

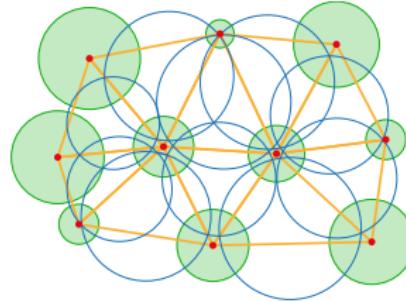
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The optimization problem is:

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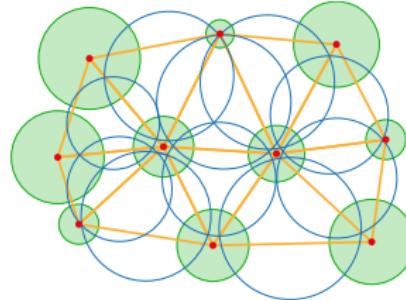
$$\text{subject to} \quad \|z - q\|^2 \leq r^2 + w_q, \quad q \in Q,$$

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The optimization problem (substituting $a = \|z\|^2 - r^2$) is:

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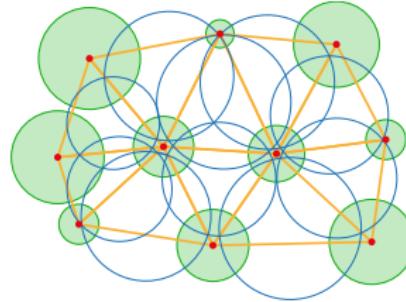
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The Karush–Kuhn–Tucker optimality conditions

Consider an optimization problem

$$\begin{aligned} & \underset{x}{\text{minimize}} && f(x) \\ & \text{subject to} && g_i(x) \leq 0, \quad i \in I, \\ & && h_j(x) = 0, \quad j \in J. \end{aligned}$$

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Assume that the functions g_i, h_j are affine and f is convex.
Then a feasible point x is an optimal solution iff there exist
Lagrange multipliers $(v_i)_{i \in I}$ and $(\lambda_j)_{j \in J}$ such that

$$\nabla f(x) + \sum_{i \in I} v_i \nabla g_i(x) + \sum_{j \in J} \lambda_j \nabla h_j(x) = 0,$$

$$v_i g_i(x) = 0, \quad i \in I,$$

$$\lambda_j \geq 0, \quad j \in J.$$

KKT conditions for smallest spheres

The KKT conditions for our sphere optimization problem are:

Proposition

A sphere S enclosing Q and empty of E is minimal iff its center z can be written as an affine combination

$$z = \sum_{x \in Q \cup E} \mu_x x, \quad 1 = \sum_{x \in Q \cup E} \mu_x$$

such that

- $\mu_x \geq 0$ for $x \notin E$,
- $\mu_x \leq 0$ for $x \notin Q$, and
- $\mu_x = 0$ for $x \notin S$.

Čech and Delaunay intervals from KKT

Corollary

A sphere S enclosing Q and empty of E is minimal iff

$$z \in \text{aff } T,$$

$$Q \in [F, I], \text{ and}$$

$$E \in [B, O],$$

where

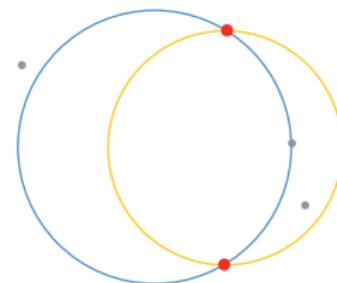
- z : center of S ,
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Intersections of Čech and Delaunay intervals

Lemma (Excluded Singularity)

The intersection of non-singular Čech and Delaunay intervals is a non-singular interval.

- Consider a non-critical Delaunay simplex Q

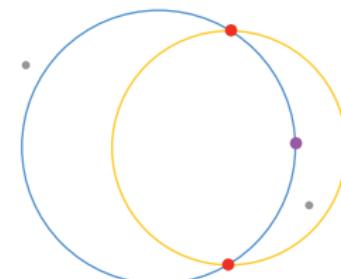


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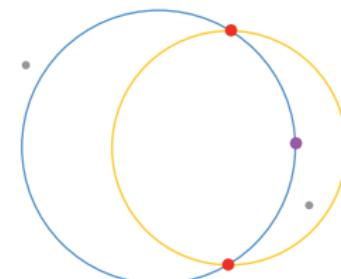


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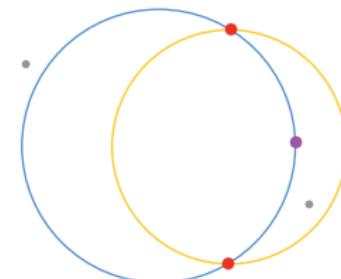


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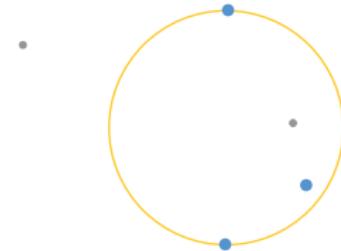


Corollary

The pairs (Q', Q'') yield a vector field that induces a collapse $\text{DelCech}_r \searrow \text{Del}_r$, for any radius r .

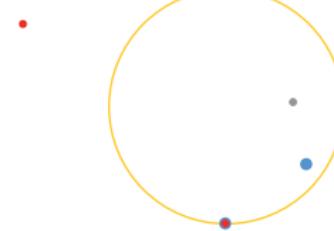
Collapsing non-Delaunay simplices

- Consider a non-Delaunay simplex Q together with its Čech sphere S



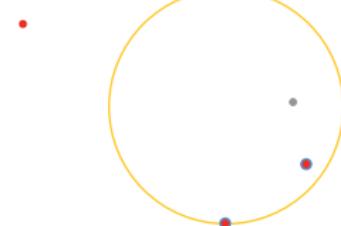
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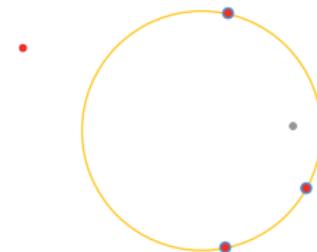
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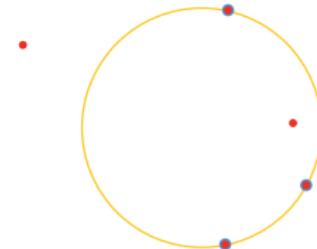
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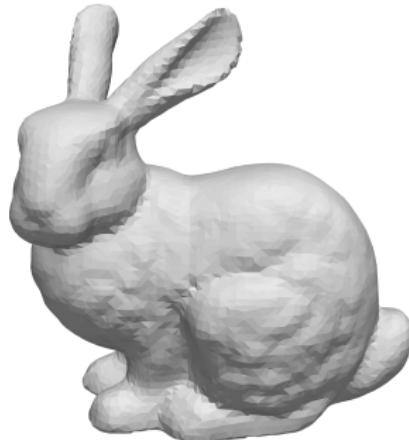
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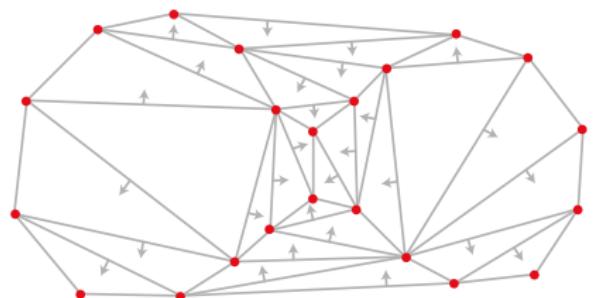
Wrap complexes

Generalizes and greatly simplifies the surface reconstruction algorithm *Wrap* (Edelsbrunner 1995, Geomagic)



Wrap complexes

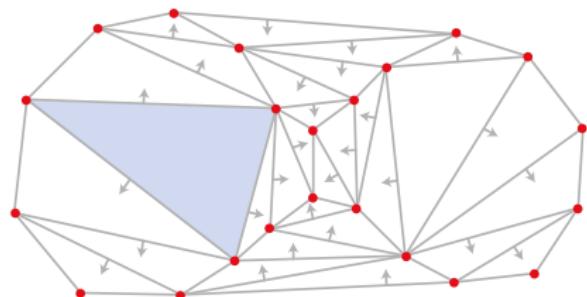
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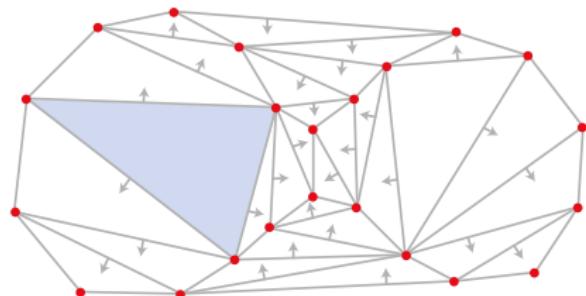
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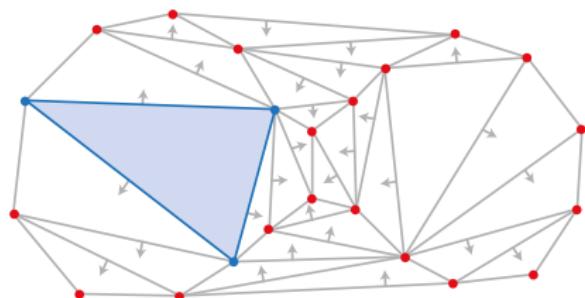
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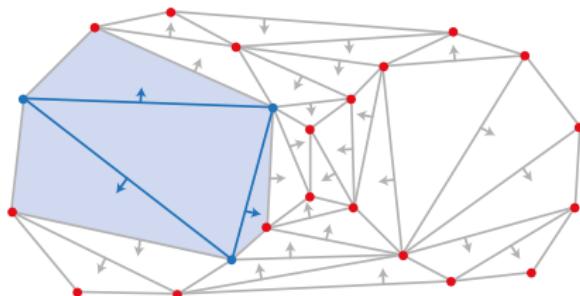
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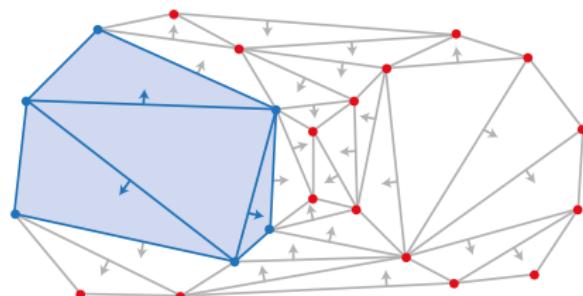
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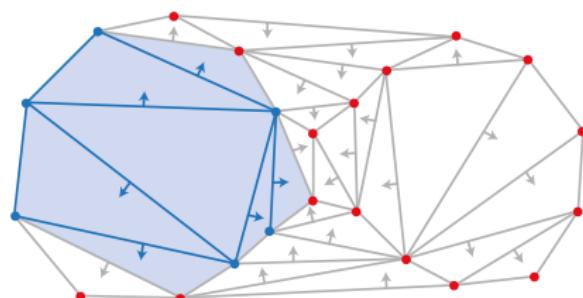
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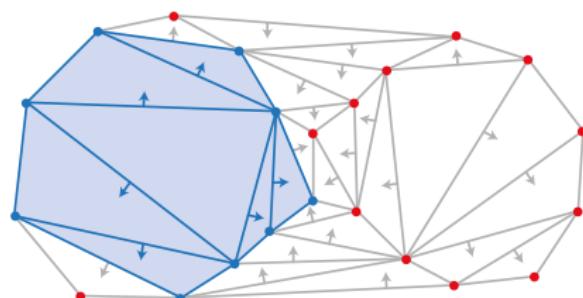
- Let $C \in \text{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$
- Let $\downarrow C$ denote the descending set of Delaunay intervals (with the partial order induced by the face relation)



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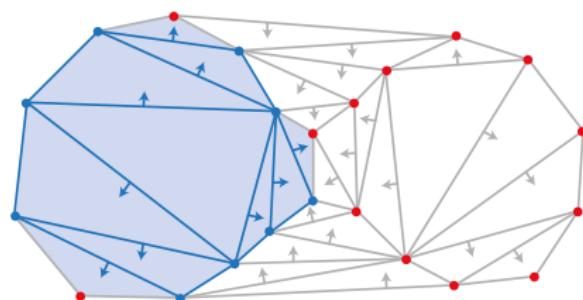
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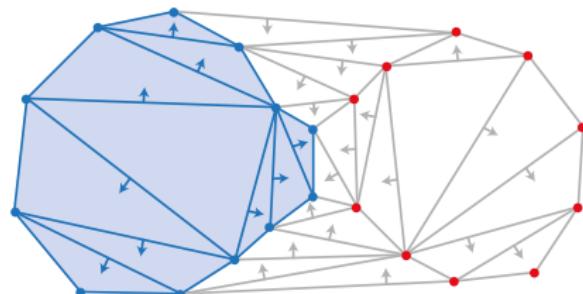
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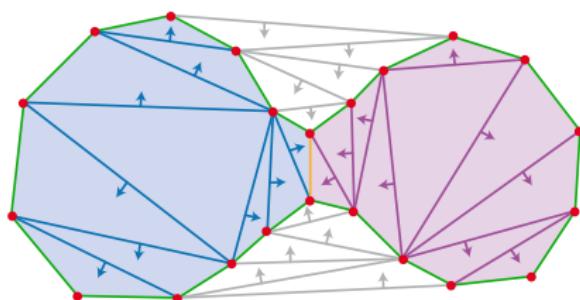
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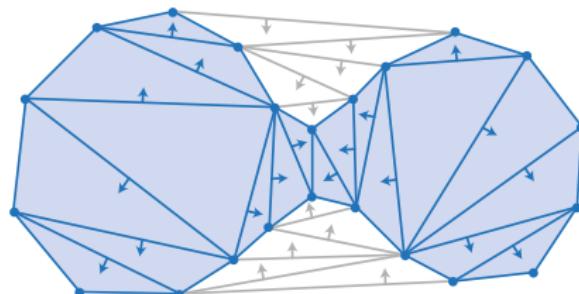
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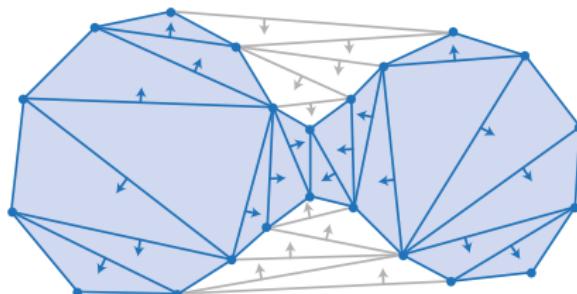
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By construction: $\text{Del}_r \searrow \text{Wrap}_r$.

Wrapping up

- Čech and Delaunay complexes from Morse functions
- Explicit construction of simplicial collapses
- Simple definition and generalization of *Wrap* complexes