Persistent homology

From theory to computation

Ulrich Bauer

TUM

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GAMM workshop
Computational and Mathematical Methods in Data Science
Zuse Instutite Berlin

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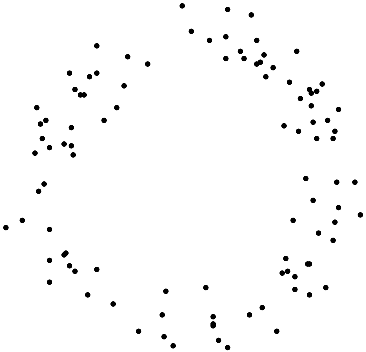
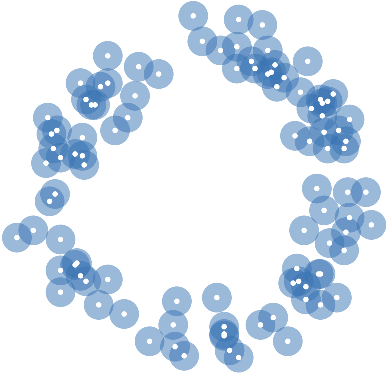
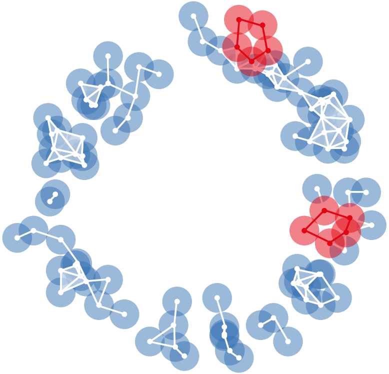
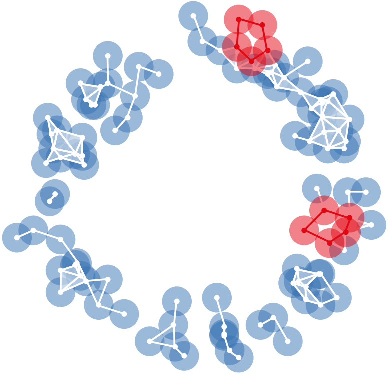
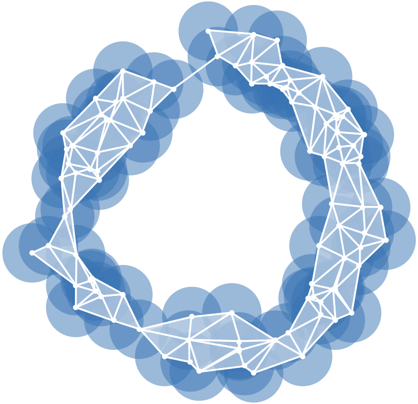
Persistent homology

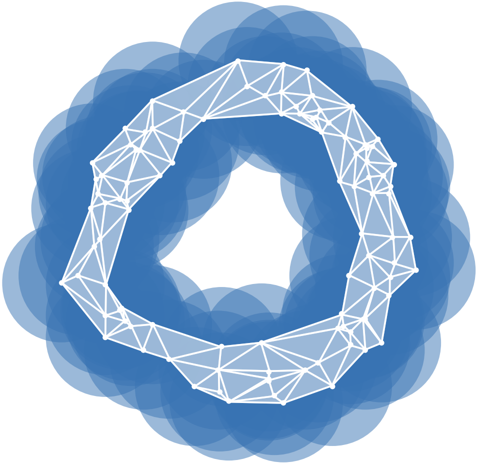


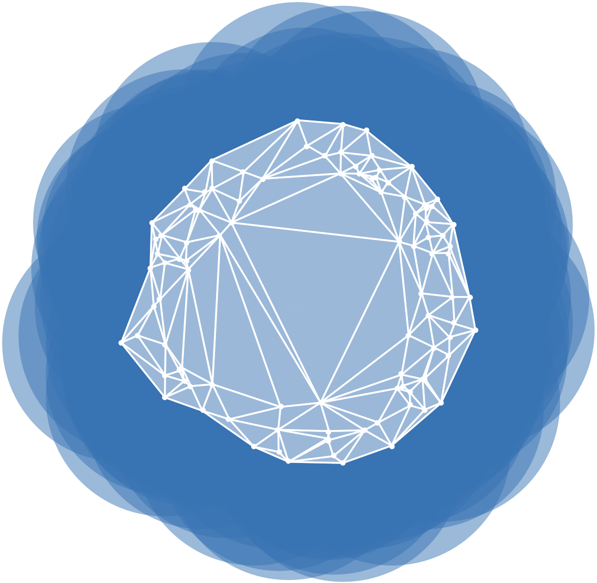




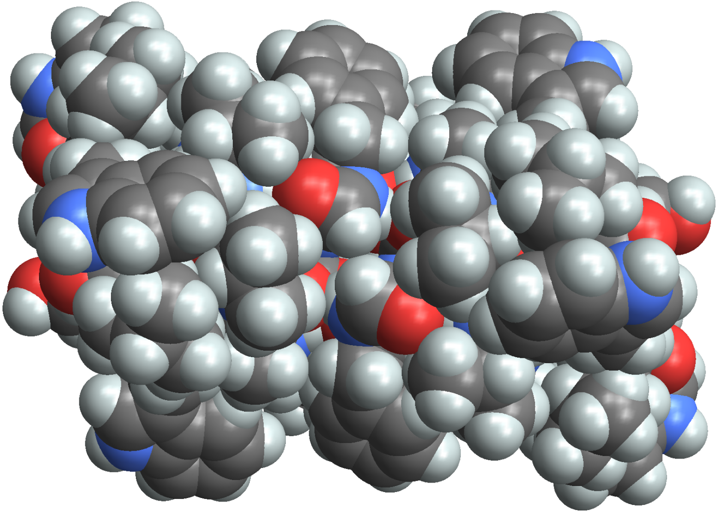








Gramicidin: an antibiotic functioning as ion channel



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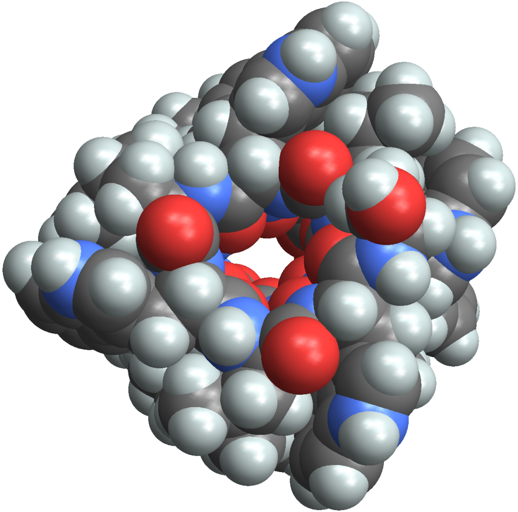

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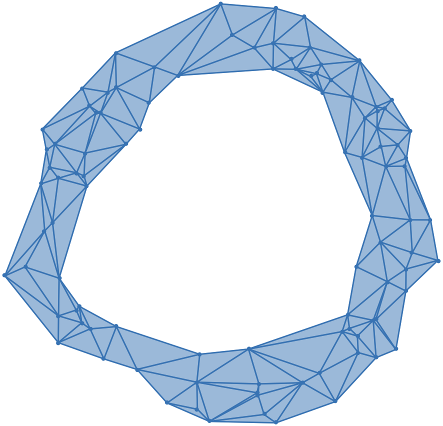

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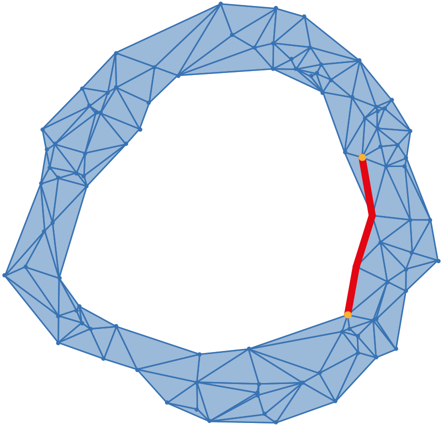

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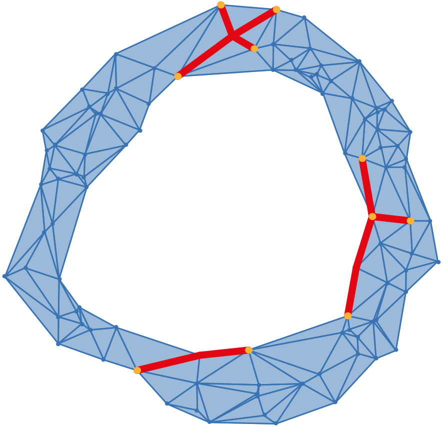

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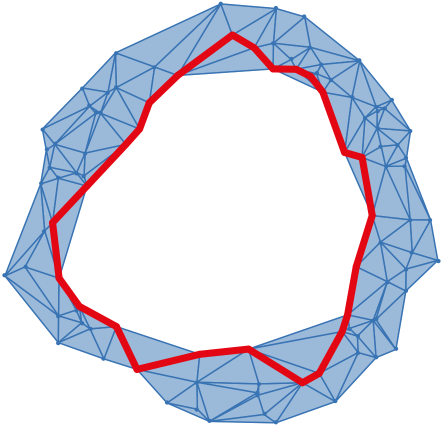

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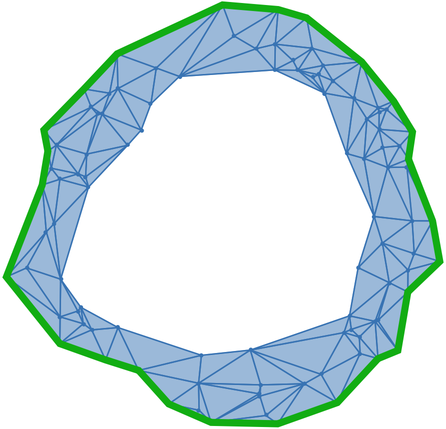

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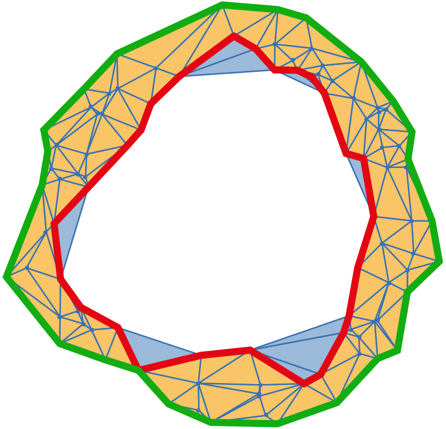

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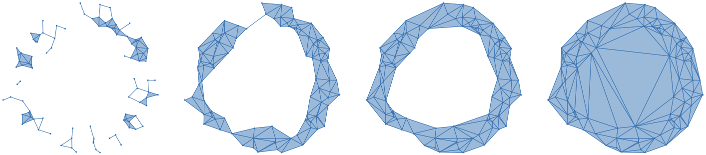

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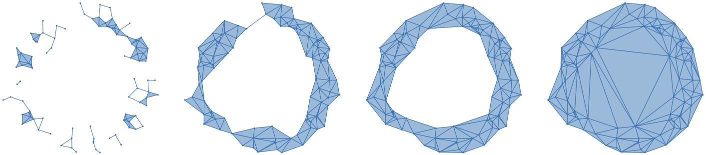

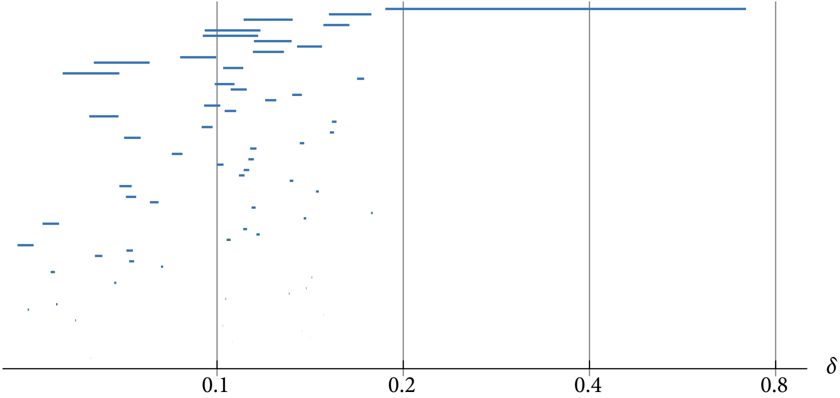

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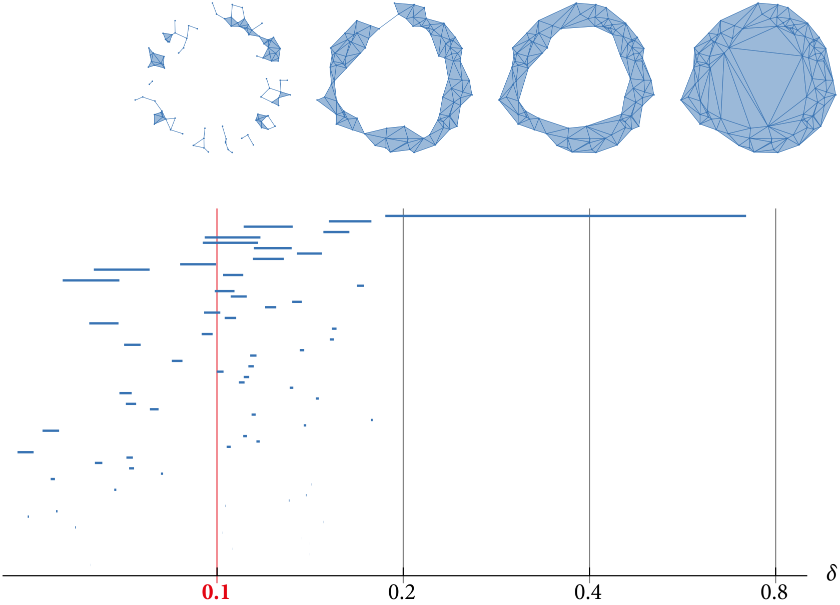

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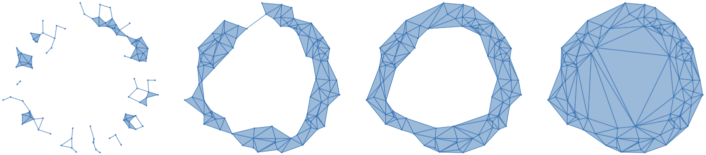

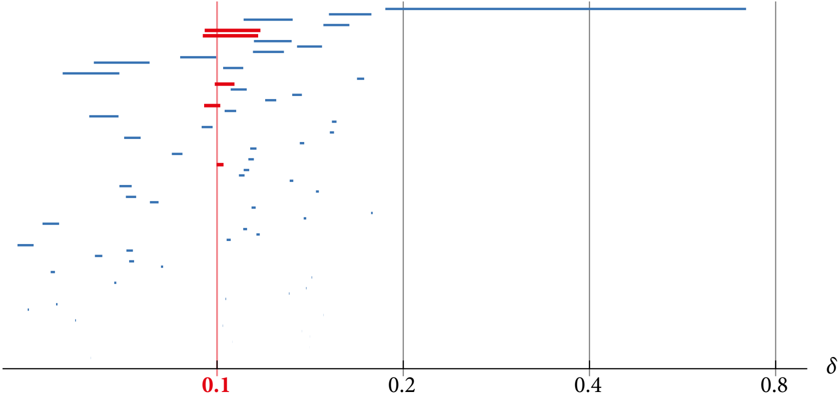

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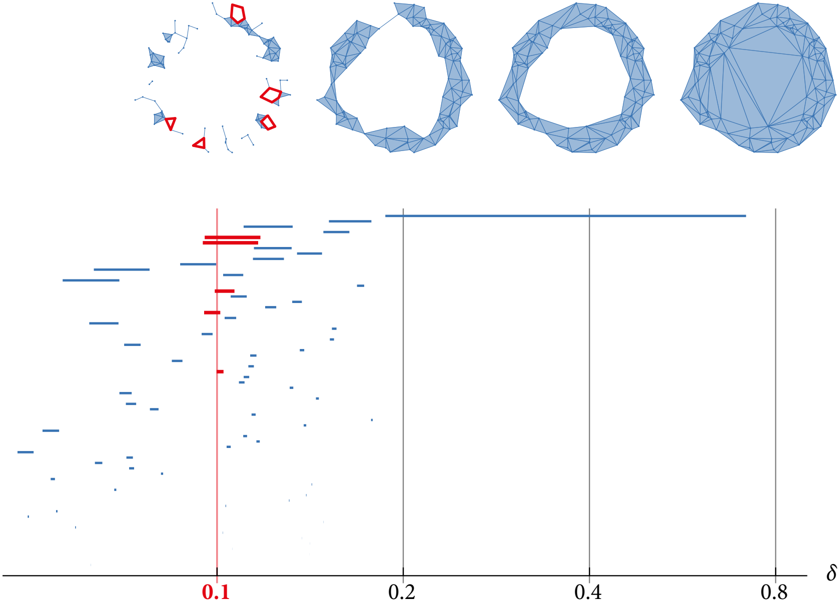

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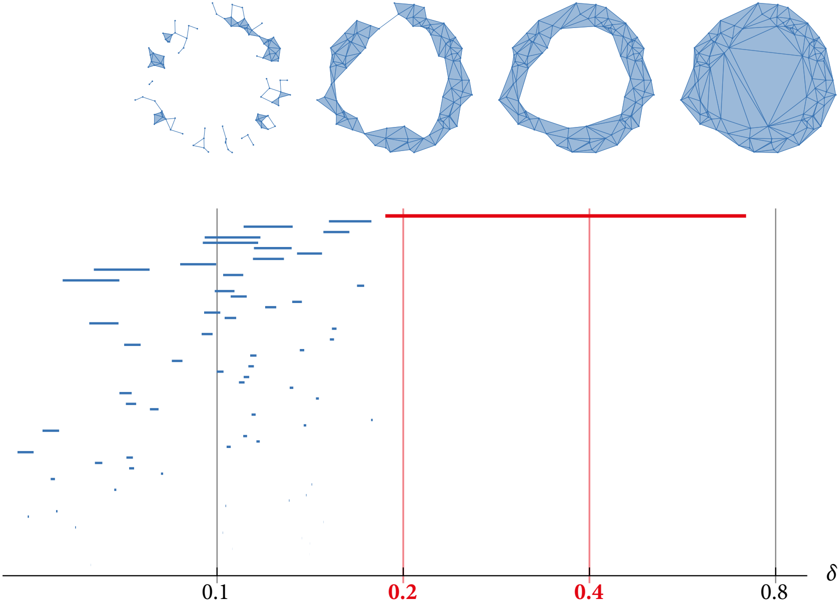

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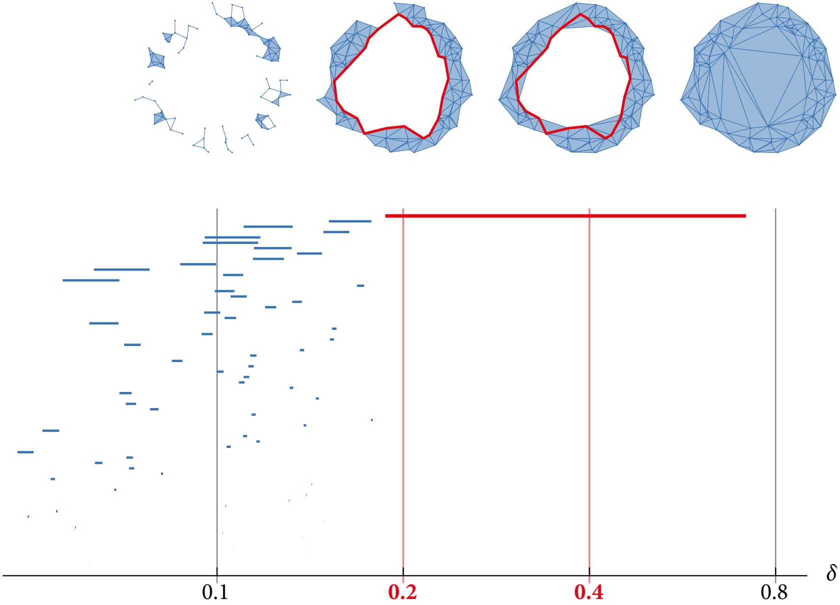

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- A filtration is a certain commutative diagram $K : \mathbf{R} \to \mathbf{Top}$:
  - $\mathbf{R}$ is the partially ordered set $(\mathbb{R}, \leq)$
  - A topological space K_t for each $t \in \mathbb{R}$
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In this talk, all vector spaces will be finite dimensional.

Barcodes: the structure of persistence modules

Theorem (Crawley-Boevey 2015)

Any persistence module $M : \mathbf{R} \to \mathbf{vect}$ *(of finite dim. vector spaces over some field* $\mathbb{F}$*) decomposes as a direct sum of* interval modules

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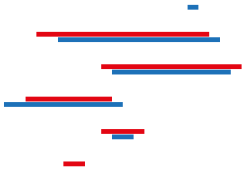
- The barcode completely describes the persistence module (up to isomorphism).

Stability

Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $f, g : X \to \mathbb{R}$ *with* $\|f - g\|_\infty = \delta$ *(and some regularity assumptions).*
Consider the persistence barcodes of (sublevel set filtrations of) f *and* g*.*
Then there exists a matching between their intervals such that



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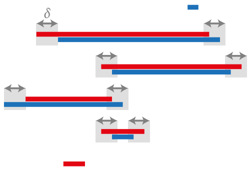

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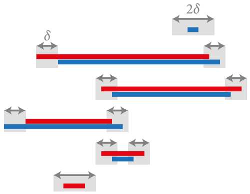

Persistence and stability: the big picture

Data	point cloud	$P \subset \mathbb{R}^d$

Persistence and stability: the big picture

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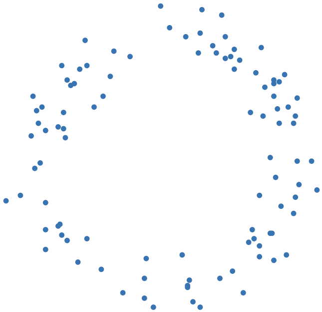

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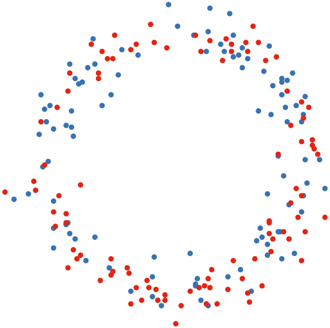

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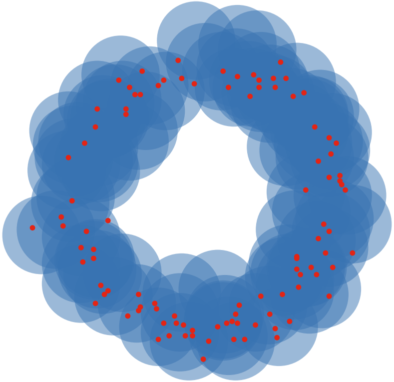

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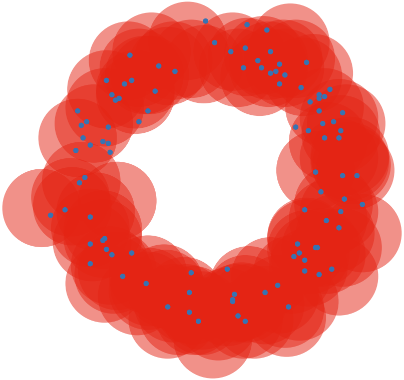

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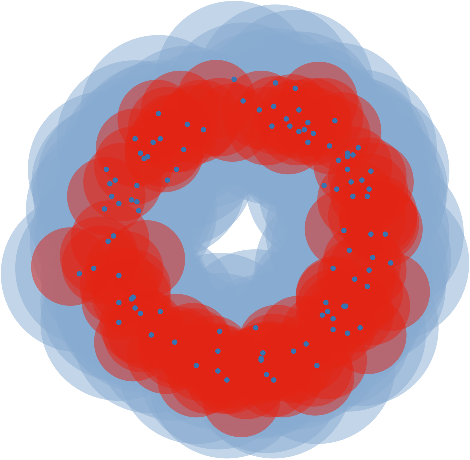

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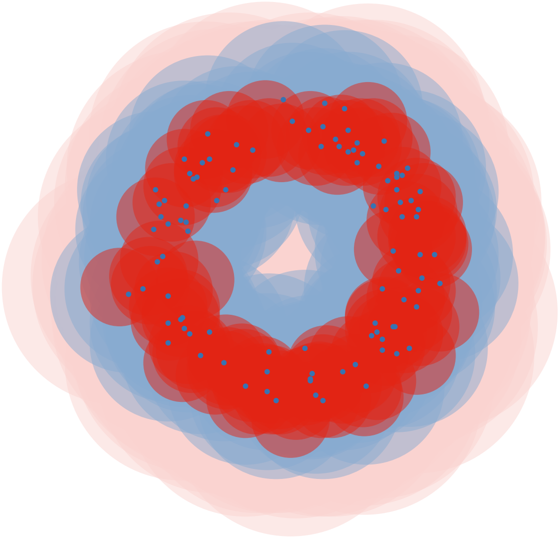

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Applying homology (a functor) preserves commutativity

- persistent homology of f, g yields δ-interleaved persistence modules $\mathbf{R} \to \mathbf{Vect}$

Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2012; B, Lesnick 2015)

If two persistence modules are δ-interleaved, then there exists a δ-matching of their barcodes:

- matched intervals have endpoints within distance $\leq \delta$,
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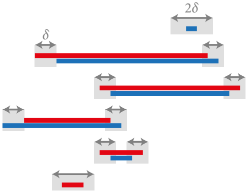

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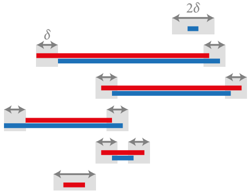

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Equivalently: there exists a δ-interleaving of their barcodes (as diagrams $\mathbf{R} \to \mathbf{Mch}$).

Structure of persistence submodules and quotients

Proposition

Let $M \twoheadrightarrow N$ be an epimorphism.

Then there is an injection of barcodes $B(N) \hookrightarrow B(M)$ such that if J is mapped to I, then

- *I and J are aligned below, and*
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This construction is functorial (compatible with composition and identity).

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Dually, there is an injection $B(N) \hookrightarrow B(O)$ for monomorphisms $N \hookrightarrow O$.

Induced matchings

For $f : M \to N$ a morphism of pfd persistence modules, the epi-mono factorization

$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$$

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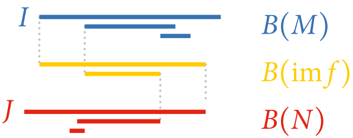

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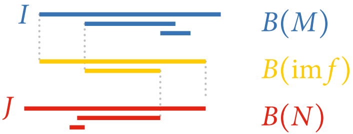

When applied to a δ-interleaving, this yields a δ-matching of barcodes.

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Let X be a submanifold of $\mathbb{R}^d$. Let $P \subset X$ and $\delta > 0$ be such that

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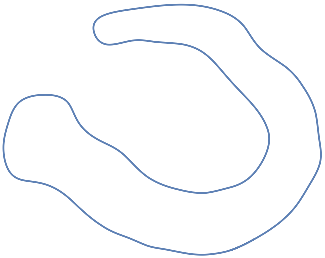

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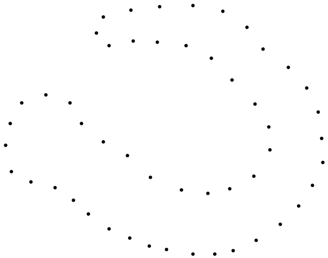

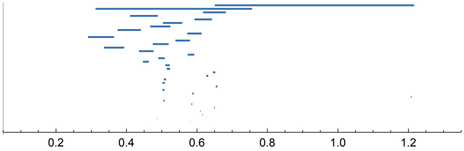

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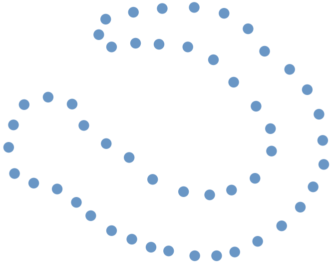

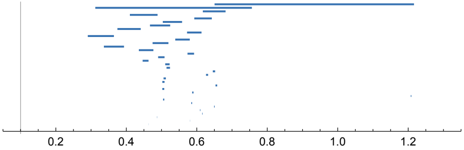

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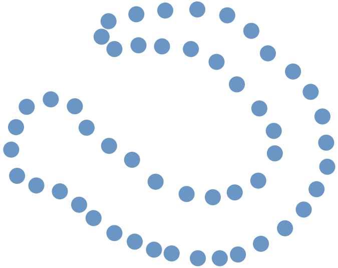

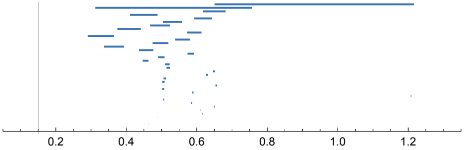

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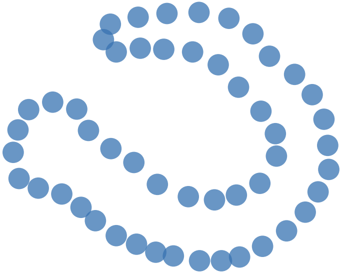

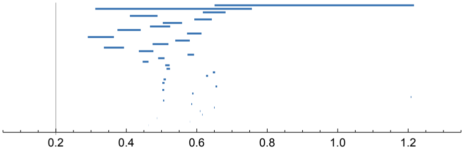

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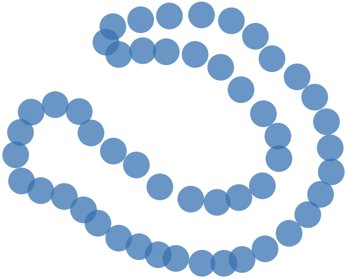

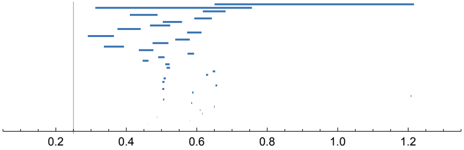

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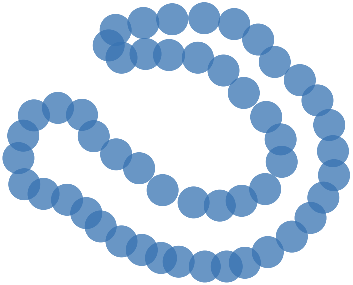

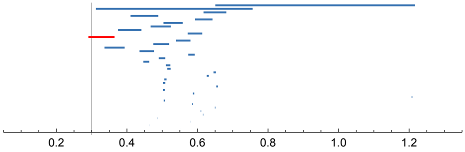

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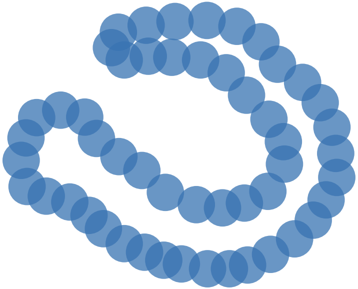

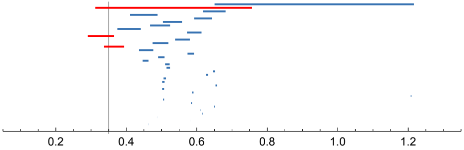

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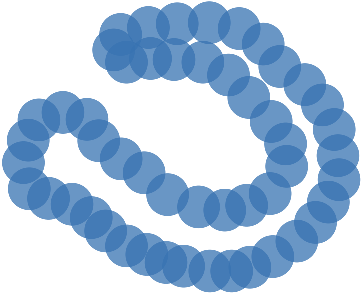

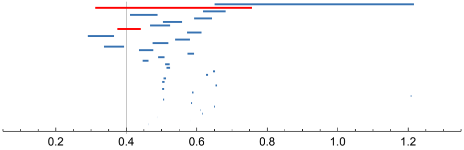

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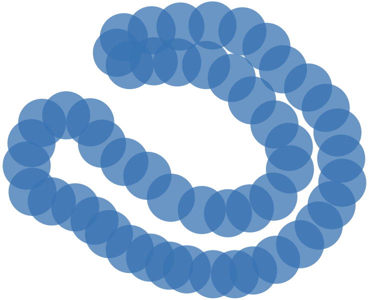

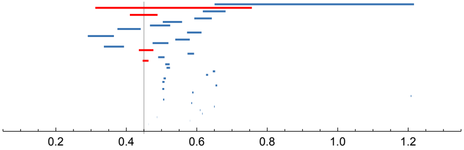

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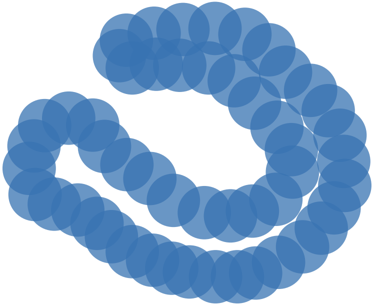

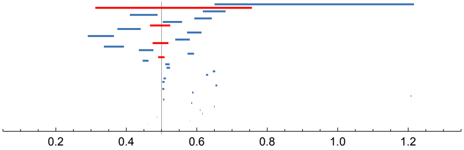

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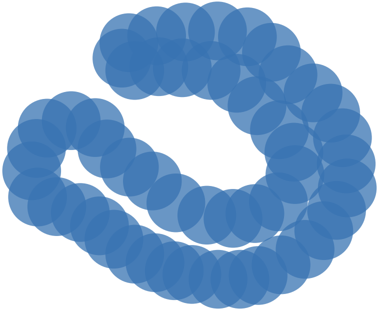

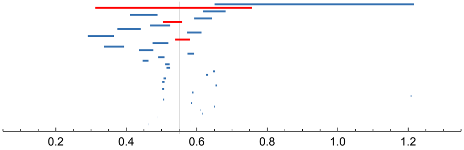

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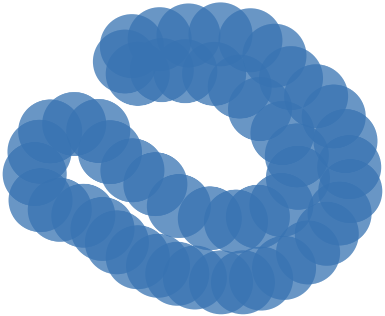

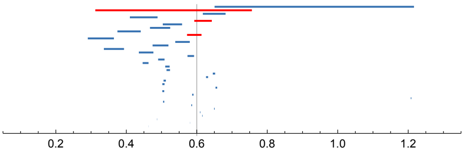

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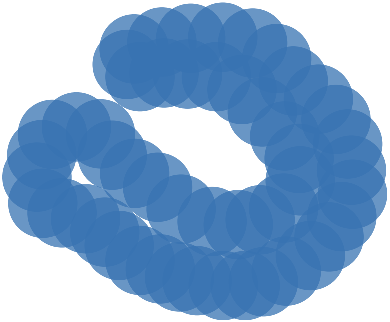

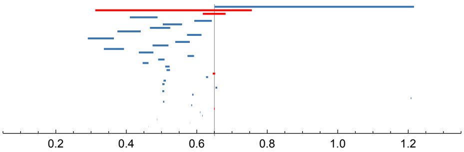

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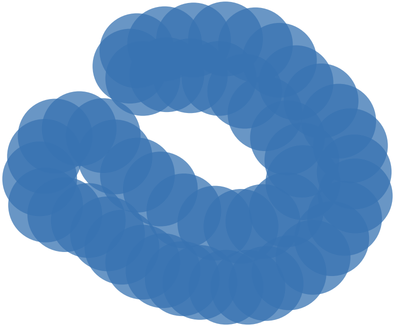

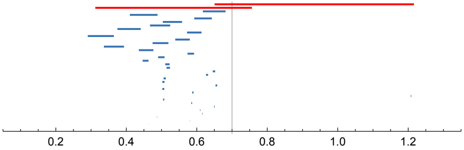

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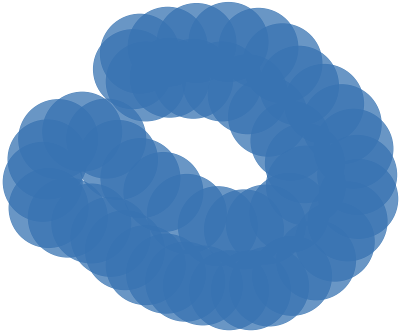

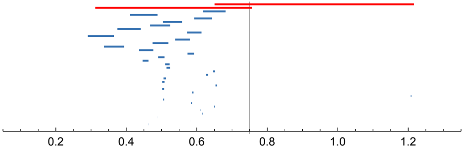

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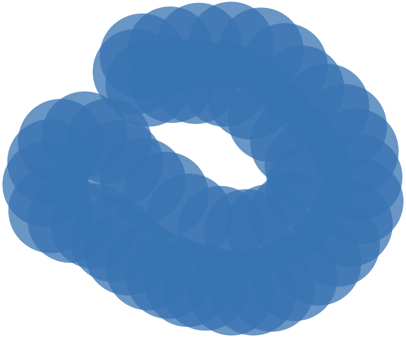

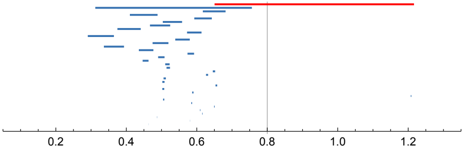

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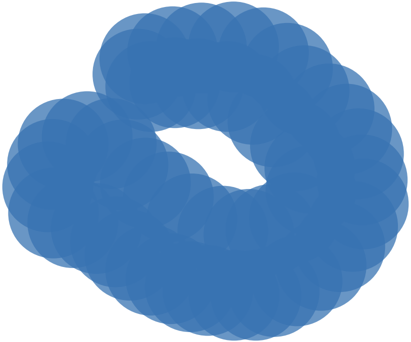

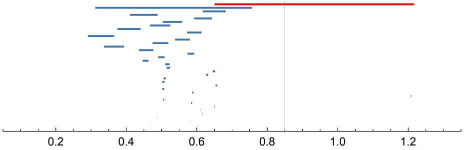

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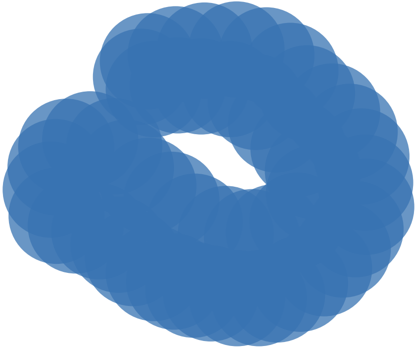
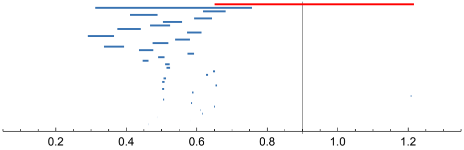




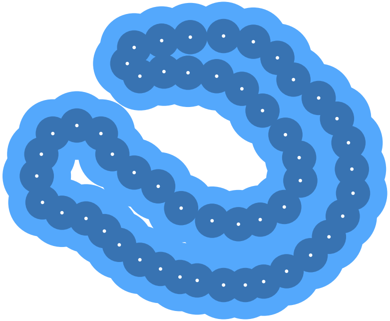
Homology inference using persistence

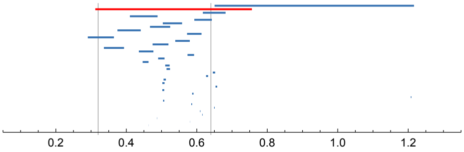
Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $X \subset \mathbb{R}^d$. Let $P \subset X$ and $\delta > 0$ be such that

- *P_δ covers X, and*
- *the induced maps $H_*(X \hookrightarrow X_\delta)$ and $H_*(X_\delta \hookrightarrow X_{2\delta})$ are isomorphisms.*

Then $H_*(X) \cong \operatorname{im} H_*(P_\delta \hookrightarrow P_{2\delta})$.





Homological realization

This motivates the *homological realization problem*:

Problem

Given a pair $L \subseteq K$ of simplicial complexes, find a third complex X with $L \subseteq X \subseteq K$ such that $H_*(X) \cong \operatorname{im} H_*(L \hookrightarrow K)$.

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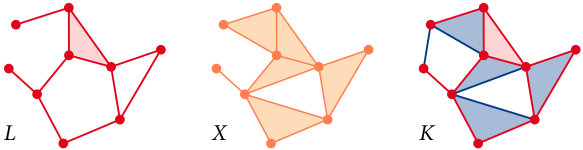

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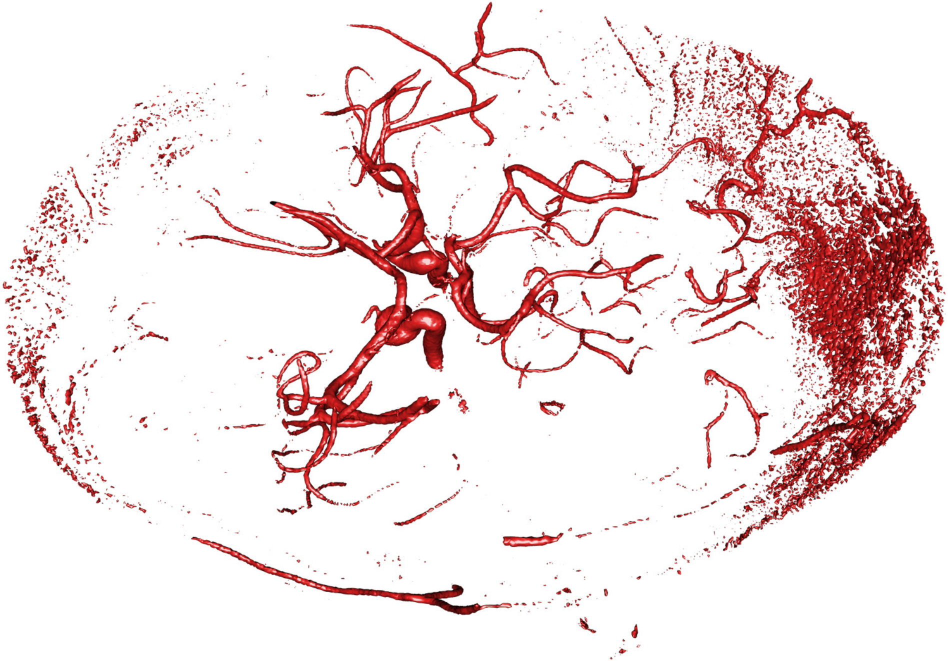

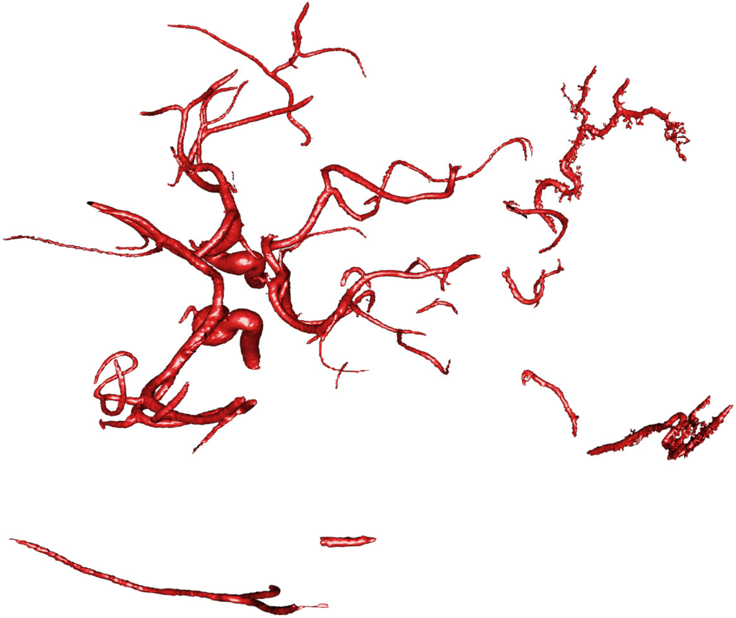

Sublevel set simplification

Let $F_t = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem (Sublevel set simplification)

Given a function $f : X \subset \mathbb{R}^3 \to \mathbb{R}$ and $t \in \mathbb{R}$, $\delta > 0$,
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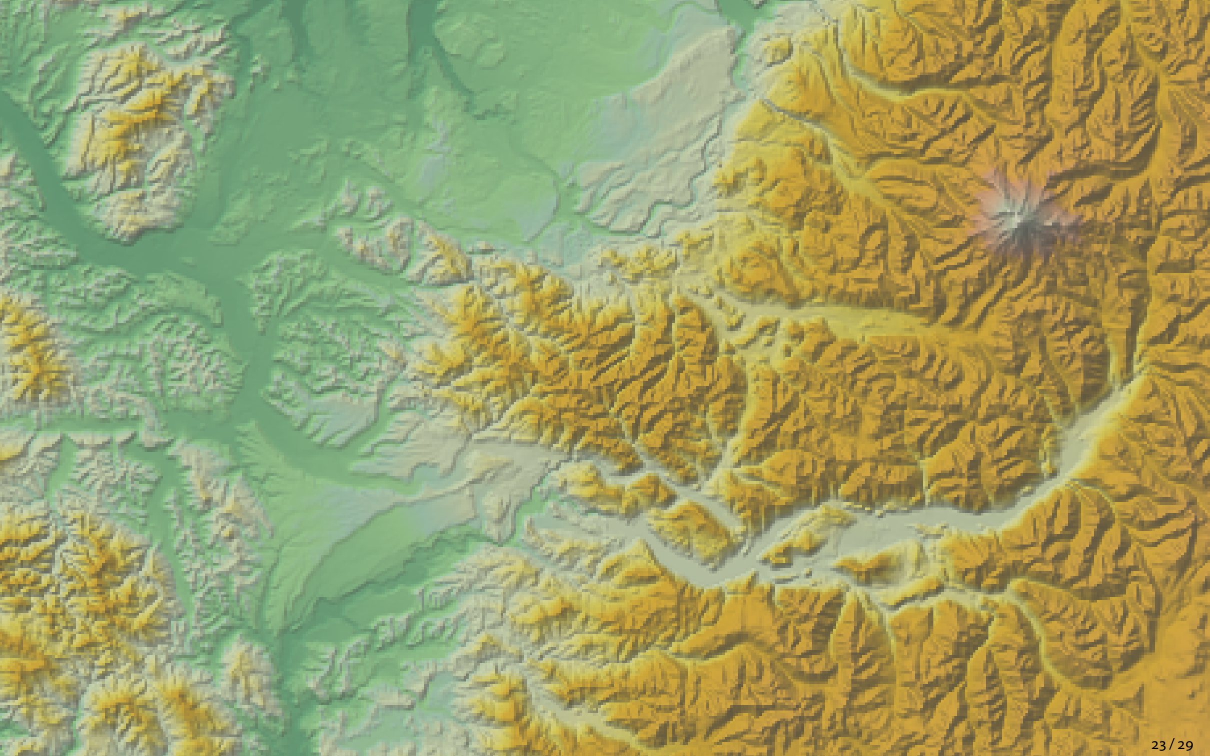
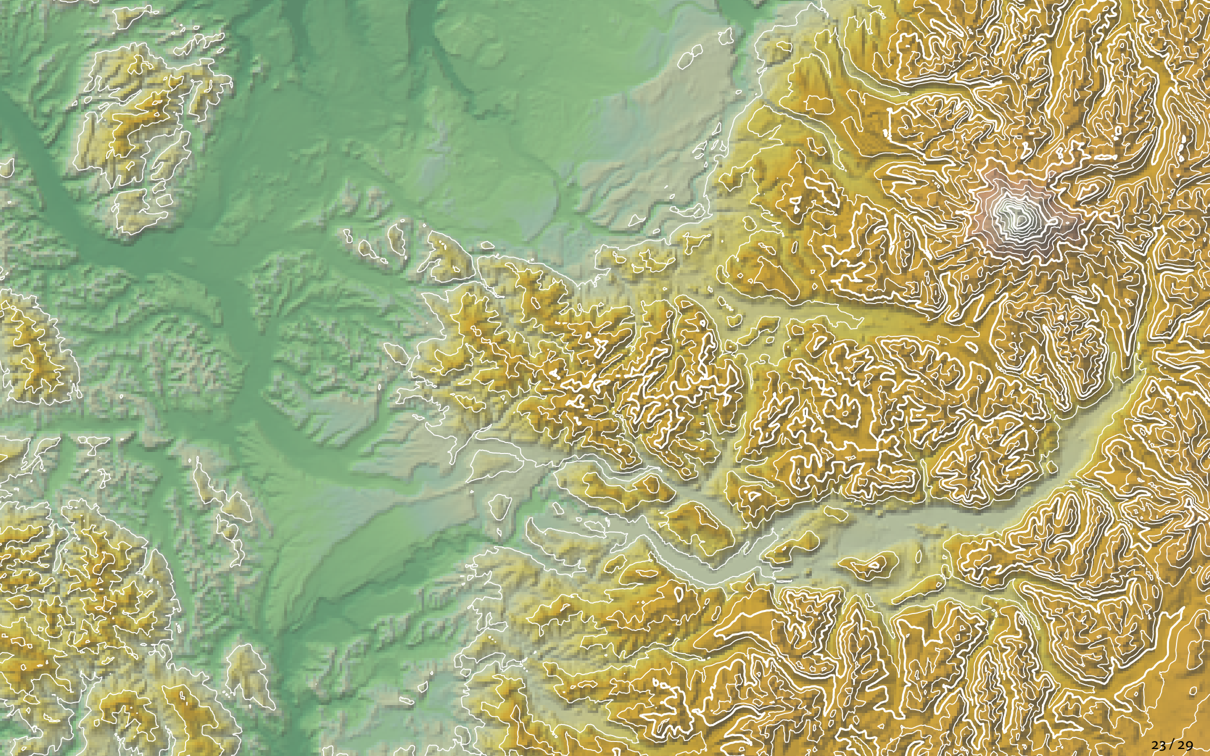
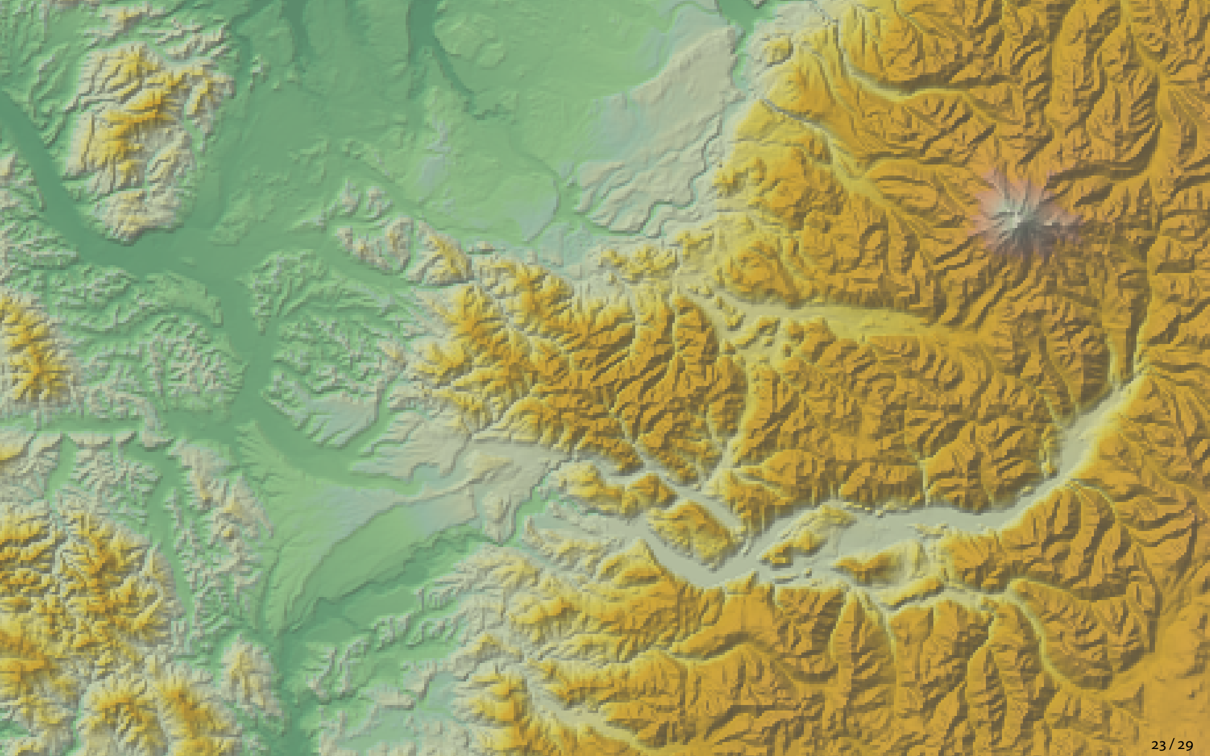
Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

Sublevel set simplification in $\mathbb{R}^3$ is NP-hard.







Topological simplification of functions

Consider the following problem:

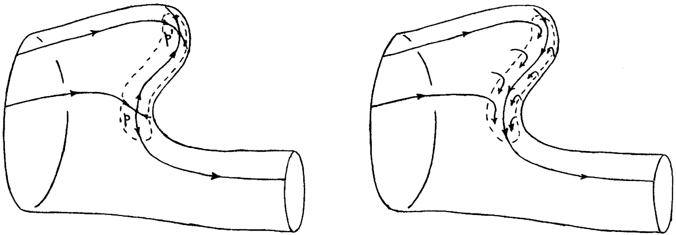
Problem (Topological simplification)

Given a function f and a real number $\delta \geq 0$, find a function f_δ subject to $\|f_\delta - f\|_\infty \leq \delta$ with the minimal number of critical points.

Persistence vs Morse theory

Morse theory (smooth or discrete):

- Relates critical points to homology of sublevel sets
- Provides a method for *canceling* pairs of critical points



(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

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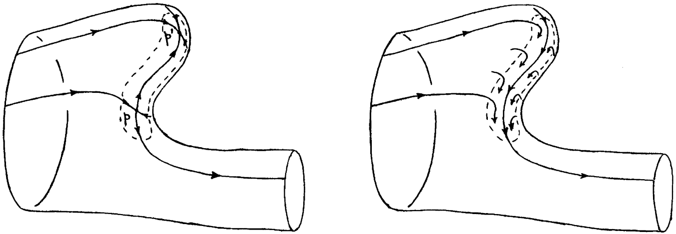

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Persistent homology:

- Relates homology of different sublevel set
- Identifies pairs of critical points (birth and death of homological feature) and quantifies their *persistence*

Combining persistence and discrete Morse theory

By stability of persistence barcodes:

Proposition

The intervals in the barcode of f with persistence $> 2\delta$ provide a lower bound on the number of critical points of any function g with $\|g - f\|_\infty \leq \delta$.

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Theorem (B, Lange, Wardetzky, 2011)

Let f be a function on a surface and let $\delta > 0$.
Then canceling all persistence pairs with persistence $\leq 2\delta$ yields a function f_δ with

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that achieves the lower bound on the number of critical points.

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- Does not generalize to higher-dimensional manifolds!

Computation

Vietoris–Rips complexes

Consider a finite metric space (X, d).
The *Vietoris–Rips complex* is the simplicial complex

$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid \operatorname{diam} S \leq t\}$$

- all edges with pairwise distance $\leq t$
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For large t, $\mathrm{Rips}_t(X)$ is the full simplex with $n = X$ vertices

- Number of d-simplices is $\binom{n}{d+1}$
- Computation is one of the most important challenges in applied topology!

An example computation

Example data set:

- 192 points on $\mathbb{S}^2$
- persistent homology barcodes up to dimension 2
- over 56 mio. simplices in 3-skeleton

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Some previous software:

- javaplex (Stanford): 3200 seconds, 12 GB
- Dionysus (Duke): 596 seconds, 3.4 GB
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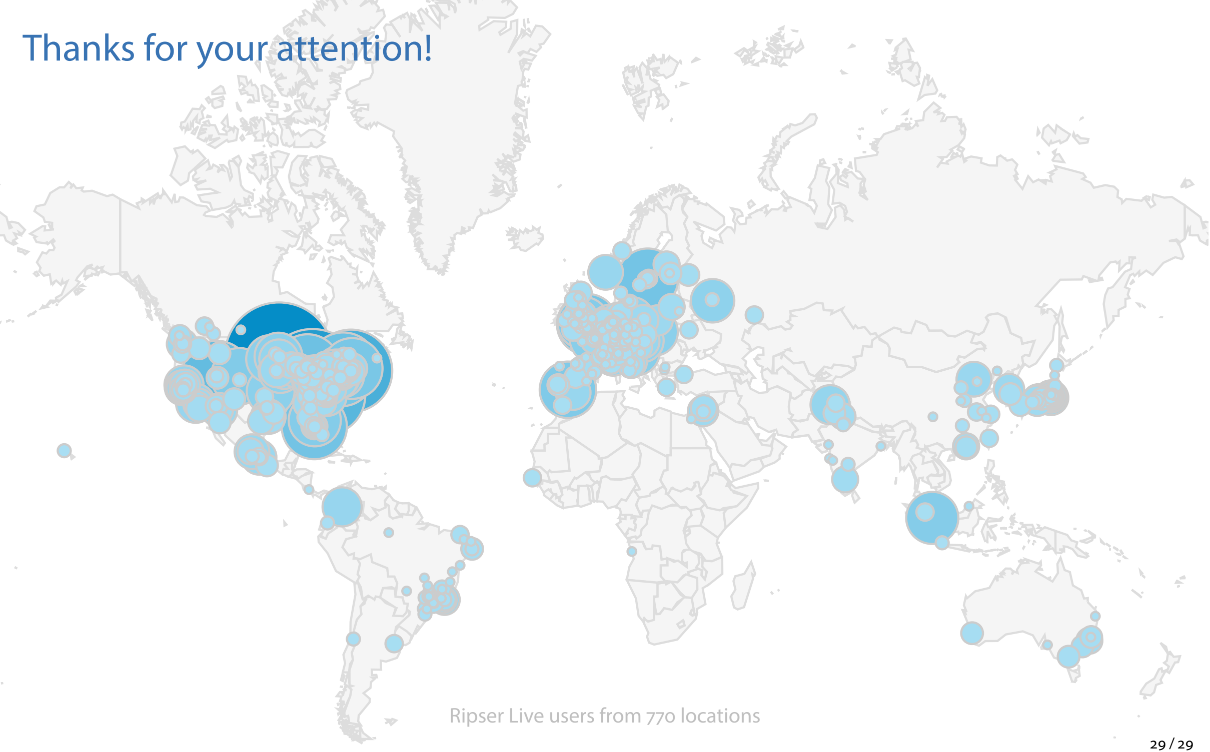
- Ripser: 1.1 seconds, 196 MB

Thanks for your attention!



Ripser Live users from 770 locations