Uniform convergence of discrete curvatures from nets of curvature lines

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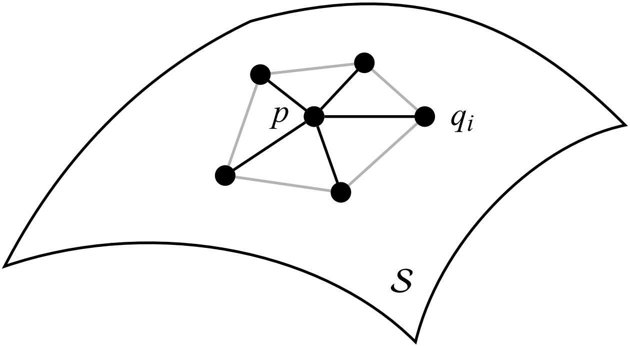
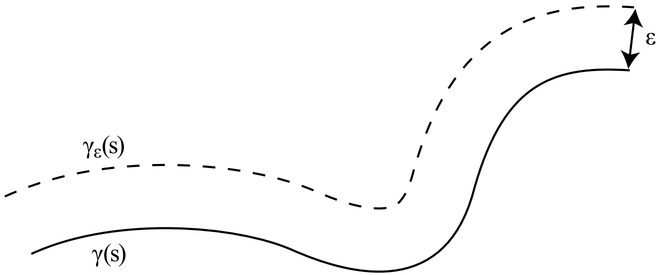
Discrete curvatures



Vertices p, q_i sampled on smooth surface
Discrete curvature: computed using positions of p, q_i only
(*1-local* definition)

Total curvature and the Steiner formula

(Steiner, 1840)
Consider curve $\gamma(s)$ and offset curve $\gamma_\epsilon(s)$

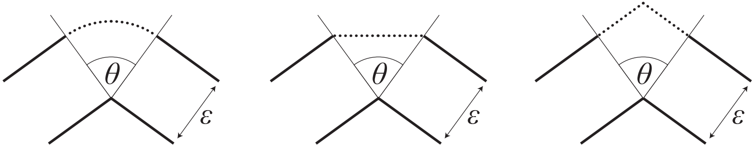


$$\mathit{length}(\gamma_\epsilon) = \mathit{length}(\gamma) + \epsilon \int_\gamma \kappa(s)\, ds$$

$\int_\gamma \kappa(s)\, ds$: *total curvature*
Can be generalized to non-smooth curves

Discrete offset curves

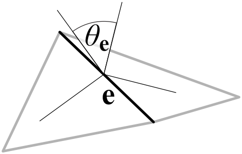
Different ways to construct offset curves for polygons:

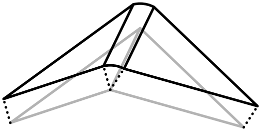


$$\kappa_v \in \{\theta,\ 2\sin\frac{\theta}{2},\ 2\tan\frac{\theta}{2}\}$$

Equivalent in the planar limit up to 2nd order

Discrete curvatures on polyhedral surfaces

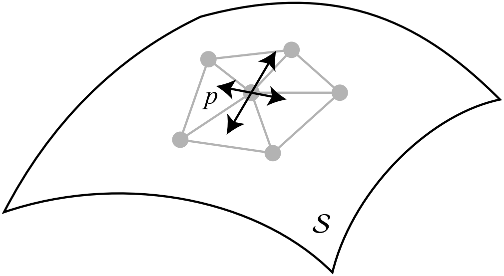




$$\kappa_{\mathbf{e}} \in \left\{ \theta \|\mathbf{e}\|,\ 2\sin\frac{\theta}{2}\|\mathbf{e}\|,\ 2\tan\frac{\theta}{2}\|\mathbf{e}\| \right\}$$

Discrete and smooth curvature

What does discrete curvature tell us about the curvature of $\mathcal{S}$?



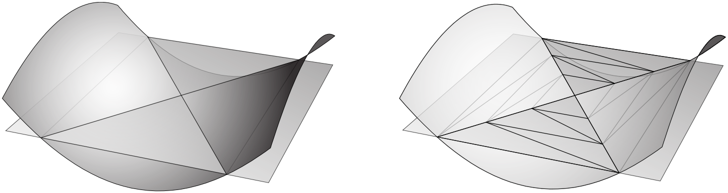
Desiderata:

- *Pointwise* approximation of smooth curvature with an error $\mathcal{O}(\epsilon)$ (ϵ: inrinsic edge length)

Known results

No convergence in general

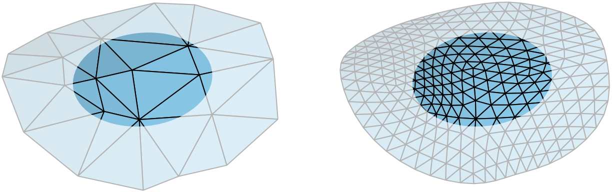
- Planar triangulation of a paraboloid $f(x, y) = 2x^2 - y^2$
- Usual assumptions satisfied
- No convergence for k-local curvature



(Xu, 2005)

Convergence of discrete total curvatures

Consider sequence converging to a *fixed* surface patch

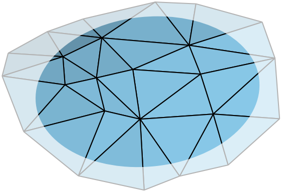


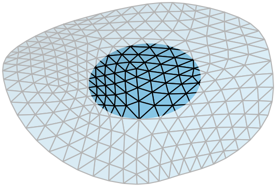
Then total discrete curvature converges to total smooth curvature of limit surface (cf. Cheeger et al. 1984, Fu 1993, Cohen-Steiner & Morvan 2003, W. 2005)

From integrated to pointwise curvature

To obtain pointwise convergence: two limit processes at the same time

- Shrink surface patch
- Refine triangulation of surface patch





Problems:

- Slow convergence
- Not *k*-local!

Synopsis

Discrete curvature

- local curvatures
- generalization of smooth total curvature
- weak convergence

Pointwise convergence

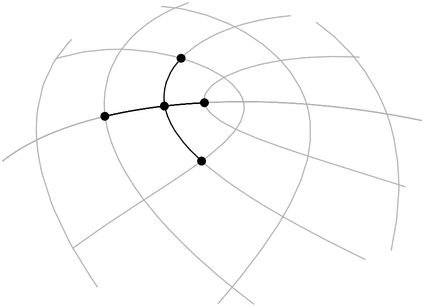
- either special cases
- or give up locality

In general: you can't have both locality and pointwise convergence

Our result

1-local principal curvatures from *nets of curvature lines*

- Approximation of *pointwise* curvatures
- Uniform error bounds
- Even across umbilics!
- Applicable to arbitrary smooth surfaces

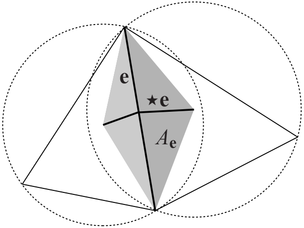


From integrated to pointwise curvature

Goal: avoid refinement inside patch

- Only one edge per patch

Consider *circumcentric area* (cf. Desbrun et al. 2005)
$A_{\mathbf{e}} = \frac{1}{2}\|\mathbf{e}\|\|\star\mathbf{e}\|$



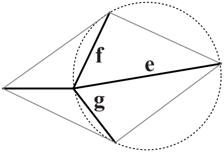
Use

$$\frac{\kappa_{\mathbf{e}}}{2A_{\mathbf{e}}}$$

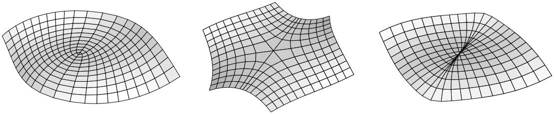
as approximation to pointwise principal curvature *orthogonal* to **e**

Main difficulties

- $A_{\mathbf{e}}$ might be zero

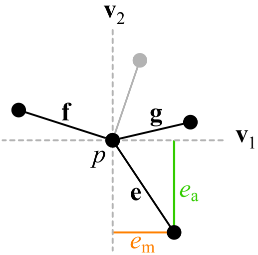


- Unbounded curvature of curvature lines near umbilics



Misalignment to principal direction

e_m is the tangential component of $\mathbf{e}$ *not* aligned to its principal direction



In general: this quantity causes failure of convergence!

Comparing discrete and smooth curvature

Theorem: we have the error estimate

$$\left|\kappa_{\mathbf{e}} - \kappa_1 2A_{\mathbf{e}}\right| \leq C\left|(\delta_\kappa e_m + \delta_\kappa f_m + \delta_\kappa g_m)\epsilon + \epsilon^3\right|$$

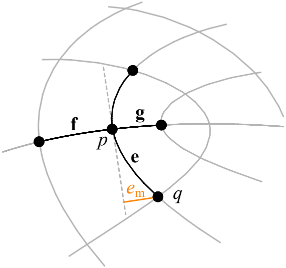
- **f**, **g** adjacent to **e** at vertex p
- e_m: misalignment of edge **e** to principal direction
- $\delta_\kappa = \kappa_1 - \kappa_2$: difference of principal curvatures
- Constant C depends only on $\mathcal{K}$ and ρ

To show ("δ_κ-Lemma"): for each edge **e**,

$$\delta_\kappa e_m \leq C\epsilon^2$$

Misalignment to principal direction

The component e_m is bounded by the distance of q to the tangent line at p



Therefore

$$e_m \leq \frac{\mathcal{K}_{\mathbf{e}}}{2}\epsilon^2$$

where $\mathcal{K}_{\mathbf{e}}$: maximal curvature of curvature line along $\mathbf{e}$ (on curvature line between p and q)

Geodesic curvature of curvature lines

The geodesic curvature of a curvature line $\gamma_{\mathbf{u}}$ along a principal direction $\mathbf{u}$ is

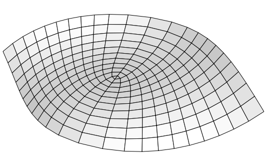
$$\kappa_{\gamma_{\mathbf{u}}} = \frac{(\nabla_{\mathbf{u}} W)\mathbf{u} \cdot \mathbf{v}}{\delta_\kappa}$$

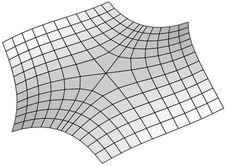
- W: Weingarten map
- $\delta_\kappa = \kappa_1 - \kappa_2$: difference of principal curvatures
- Note: curvature blows up at umbilical points

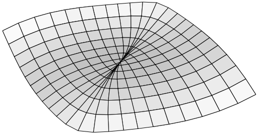
But we multiply by δ_κ:

$$\delta_\kappa e_m \leq \delta_\kappa \left(\frac{C}{\delta_\kappa} \epsilon^2 \right) \leq C\epsilon^2$$

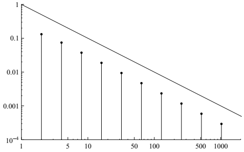
Numerical evaluation

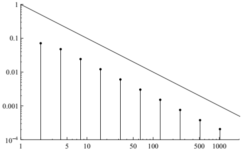


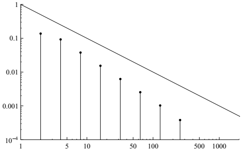




x-axis: $\frac{1}{\epsilon}$; y-axis: error







lemon

$$z = 0.3x^3 + 0.3xy^2$$

star

$$z = 0.4xy^2$$

monstar

$$z = 0.3x^3 + 0.6xy^2$$

Outlook

- Can we show convergence also for circular or conical meshes?
- How to construct curvature line nets?