PERSISTENT MATCHMAKING

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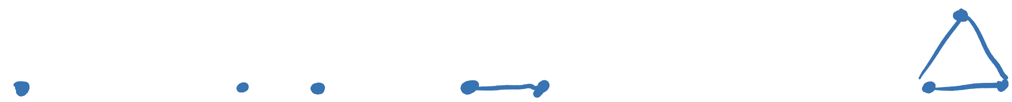
Ripser: Computing Vietoris-Rips Persistence

new version released (v 1.2)

- roughly 2× faster | less memory
- handles H^1 of a data set (COVID genomes) with 93k points (136 trillion simplices)

Persistent Homology



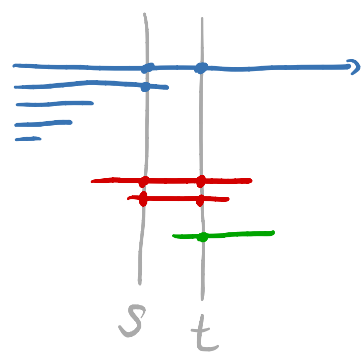
Filtration $K_\bullet$: $K_1 \hookrightarrow K_2 \hookrightarrow K_3 \hookrightarrow \cdots \hookrightarrow K_s \hookrightarrow \cdots$

Homology H_*

Persistence module $H_*(K_\bullet)$: $H_*(K_1) \to \cdots \quad \cdots \to H_*(K_s) \to \cdots$

Barcode B

Matching diagram



INDUCED MATCHINGS

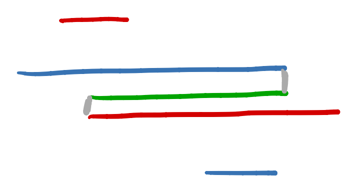
Two filtrations:

$$
\begin{tikzcd}
\cdots \arrow[rr] & & K_t \arrow[rr] \arrow[dr] & & K_{t+2\delta} \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[r] & L_{t-\delta} \arrow[ur] \arrow[rr] & & L_{t+\delta} \arrow[ur] \arrow[rr] & & \cdots
\end{tikzcd}
$$

Interleaving of persistent homology:

$$
\begin{tikzcd}
\cdots \arrow[rr] & & H_*(K_t) \arrow[rr] \arrow[dr] & & H_*(K_{t+2\delta}) \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[r] & H_*(L_{t-\delta}) \arrow[ur] \arrow[rr] & & H_*(L_{t+\delta}) \arrow[ur] \arrow[rr] & & \cdots
\end{tikzcd}
$$

$H_*(L_\bullet) \xrightarrow{\varphi} H_*(K_{\bullet+\delta})$ induces matching of barcodes



$$\mathcal{B}(H_*(L_\bullet)) \hookleftarrow \mathcal{B}(\operatorname{im} \varphi) \hookrightarrow \mathcal{B}(H_*(K_\bullet))$$

ALGEBRAIC STABILITY OF BARCODES

Theorem [Chazal et al. '09] [B, Lesnick '14]
A δ-interleaving of persistence modules ~~admits~~ induces a δ-matching of their barcodes.

Applications: (extending classical stability)

- weaker tameness assumptions
- filtrations of two different domains
- Gromov–Hausdorff stability (Vietoris–Rips barcode)

Persistence Computation: The Basics

(X, d) : metric space

$\mathrm{Rips}_t(X) = \{\emptyset \neq \sigma \subseteq X \mid \operatorname{diam} \sigma \leq t\}$

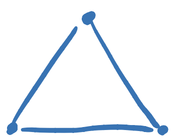
$\rightsquigarrow$ Rips filtration

$\rightsquigarrow$ refine (lexicographically) to simplexwise

D : p-boundary matrix (wrt. ordered simplices)

Matrix reduction: compute $R = D \cdot V$

- R reduced: unique pivots
- V upper triangular, full rank



$$\underset{R}{\begin{pmatrix} 1 & & \\ 1 & 1 & \\ & 1 & \end{pmatrix}} = \underset{D}{\begin{pmatrix} 1 & & 1 \\ 1 & 1 & \\ & 1 & 1 \end{pmatrix}} \cdot \underset{V}{\begin{pmatrix} 1 & & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}} \qquad (\text{in } \mathbb{Z}/(2))$$

Columns of R and some of V form a compatible basis for persistent homology:

R_j generates homology (at index $i = \text{pivot } R_j$)

V_j kills homology (at index j)

If $R_i = 0$:

- V_i generates homology (at index i)
- if $i =$ pivot R_j : we don't need V_i
- else : V_i is essential cycle

Clearing (Chen, Kerber 2011):

- avoid unneeded computations
- requires reducing boundary matrices in decreasing dimensions
- unavailable in top dimension

Persistent cohomology

- vector space dual to homology
- same barcode
- computed by reducing coboundary matrices (transpose boundary matrix & reverse order) in increasing dimension
- unavailable in dimension 0 (where Kruskal's MST algorithm is used)

Filtration $K_1 \hookrightarrow K_2 \hookrightarrow \dots \hookrightarrow K_m = A$

Cochains $C^*(K_1) \twoheadleftarrow C^*(K_2) \twoheadleftarrow \dots \twoheadleftarrow C^*(K_m)$

Relative filt. $C^*(A, K_1) \hookleftarrow C^*(A, K_2) \hookleftarrow \dots \hookleftarrow C^*(A, K_m)$

Reducing the coboundary matrix actually computes persistent relative cohomology!

[dSMV11]: absolute / relative barcodes determine each other.

Consider short exact sequence

$$C_*(K_\bullet) \hookrightarrow C_*(A) \twoheadrightarrow C_*(A, K_\bullet)$$

of filtered chain complexes.

$\rightsquigarrow$ long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_d(K_\bullet) \arrow[r] & H_d(A) \arrow[r] & H_d(A, K_\bullet) \arrow[dll, "\partial"] \\
& H_{d-1}(K_\bullet) \arrow[r, "\eta"'] & H_{d-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

- splits at $H_{d-1}(K_\bullet) = \underbrace{H_{d-1}(K_\bullet)^\infty}_{\operatorname{im}\eta} \oplus \underbrace{H_{d-1}(K_\bullet)^\dagger}_{\operatorname{im}\partial}$

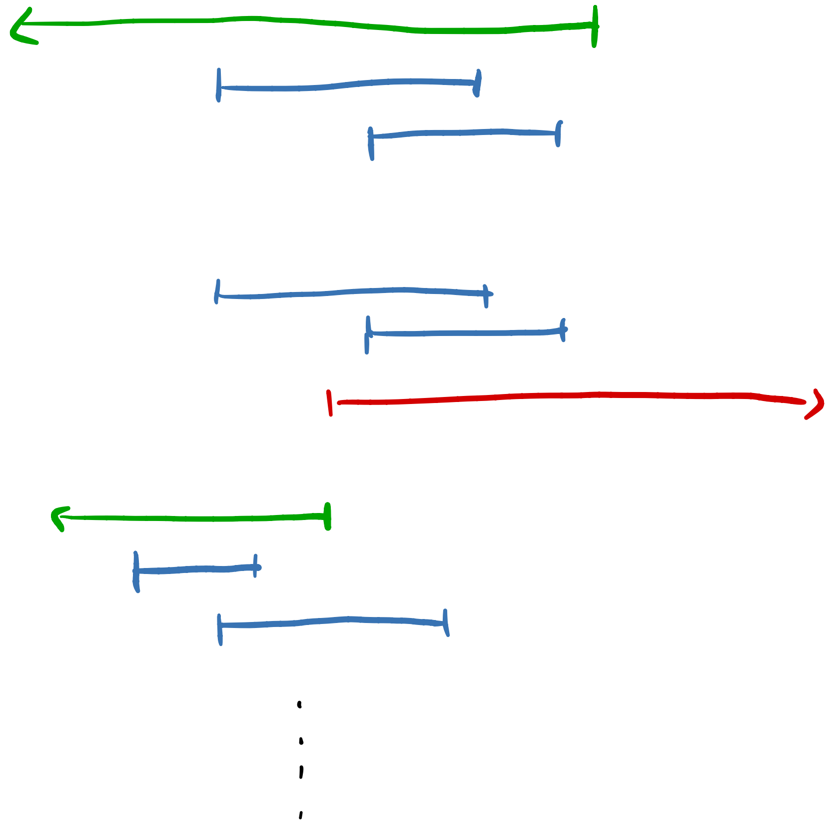
and at $H_d(A, K_\bullet) = \overbrace{H_d(A, K_\bullet)^* \oplus H_d(A, K_\bullet)^{-\infty}}^{\operatorname{im}\partial}$

Example

$H_d(A, K_\bullet)$

$H_{d-1}(K_\bullet)$

$H_{d-1}(A, K_\bullet)$



COMPUTING IMAGE BARCODES

Now consider a pair of filtrations:

$$\begin{array}{ccccccccc}
K_1 & \hookrightarrow & K_2 & \hookrightarrow & \cdots & \hookrightarrow & K_m & = & A \\
\uparrow & & \uparrow & & & & & & \\
L_1 & \hookrightarrow & L_2 & \hookrightarrow & \cdots & \hookrightarrow & L_m & = & A
\end{array}$$

$f : L_\bullet \hookrightarrow K_\bullet$

Compute $\operatorname{im} H_*(f)$?

- Long exact sequence is functorial
- But image may not be an exact sequence

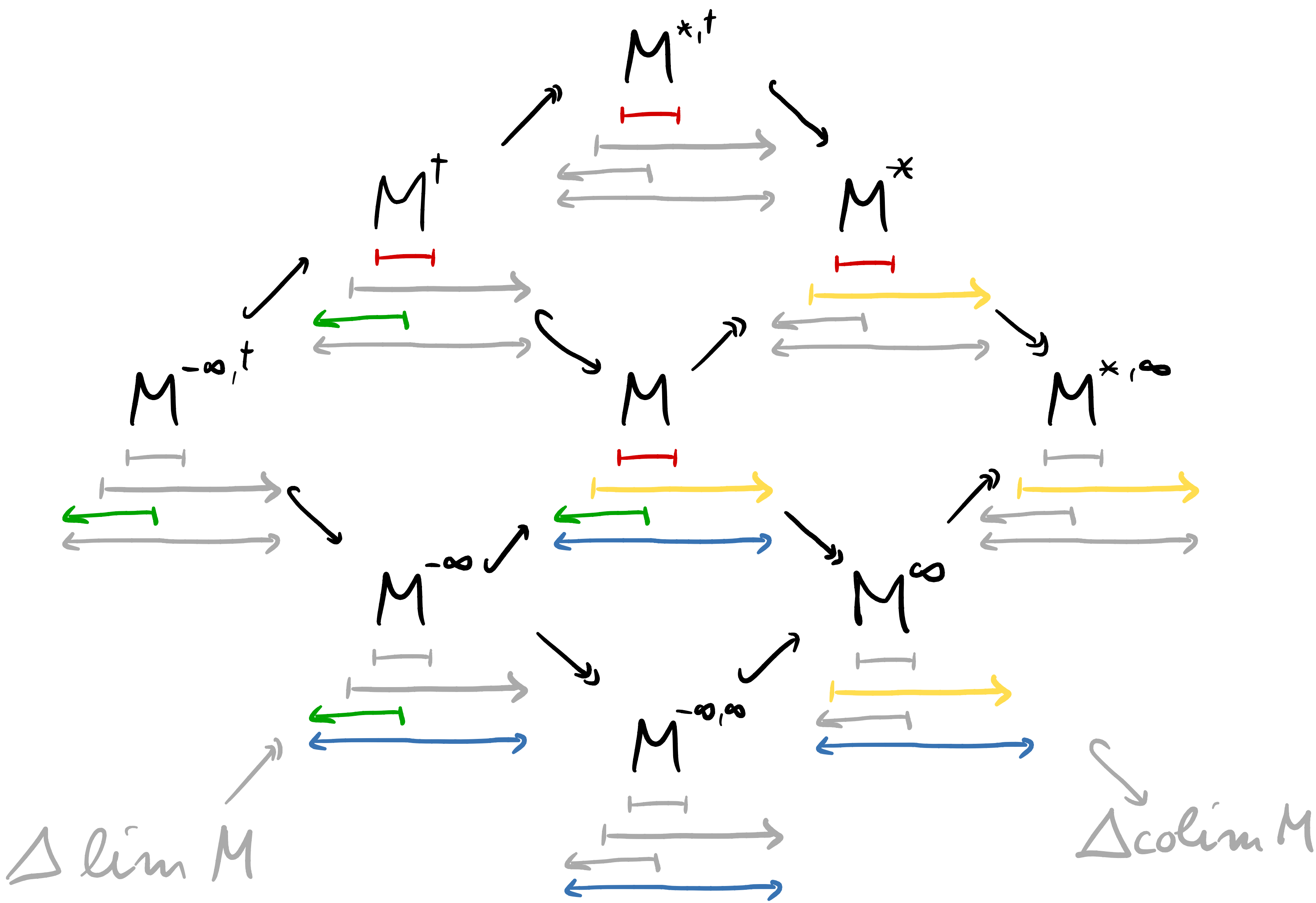
Previous work:

- [Edels., Harer, Morozov '09]:
  - $L_t = K_t \cap L$ (one function)
  - no clearing
- [Skraba, Vej.-Joh '13]
  - for presentations of persistence modules

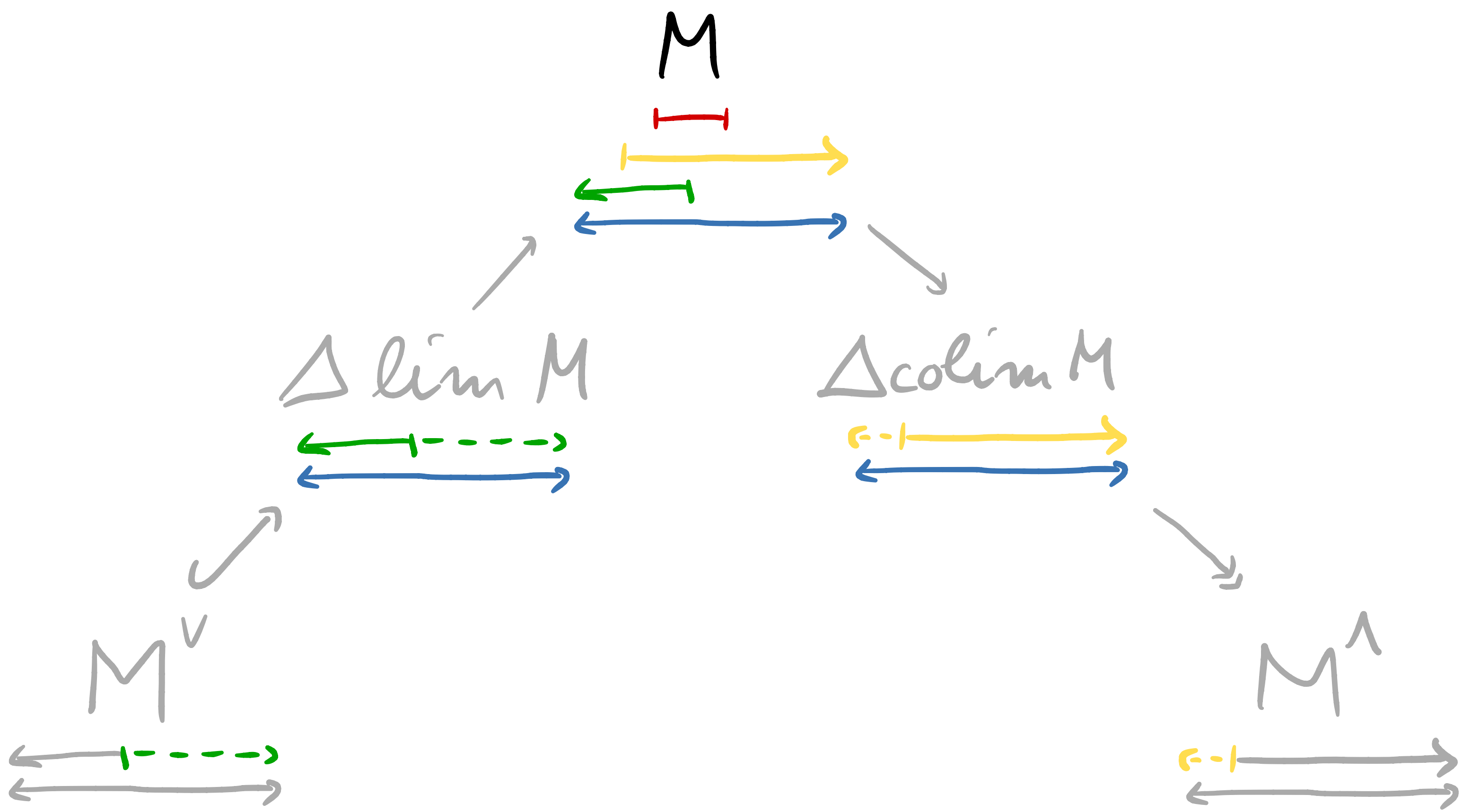
This talk:

- [B, Schmahl '20]
  - $K_\bullet$, $L_\bullet$ arbitrary filtrations of Δ
  - Clearing & cohomology

LIFESPAN FUNCTORS



Complements



interval decomposition not natural

$$
\begin{tikzcd}
H_d(K_\bullet) \cong & H_d(K_\bullet)^{\dagger} \arrow[d, "\cong"'] \arrow[r, phantom, "\oplus"] & H_d(K_\bullet)^{\infty} \arrow[d, "\cong"] \\
& H_{d+1}(A, K_\bullet)^{*} \arrow[dd, leftrightarrow] & H_d(A, K_\bullet)^{\vee} \arrow[d, leftrightarrow] \\
& & H_d(A, K_\bullet)^{-\infty} \arrow[d, leftrightarrow] \\
& H^{d+1}(A, K_\bullet)_{*} & H^{d}(A, K_\bullet)_{-\infty}
\end{tikzcd}
$$

relative l.e.s.

complement

vector space duality

$$f : L_\bullet \hookrightarrow K_\bullet \qquad \phi : (A, L_\bullet) \hookrightarrow (A, K_\bullet)$$

$$\operatorname{im} H_d(f) \cong \left(\operatorname{im} H_d(f)\right)^{\dagger} \oplus \left(\operatorname{im} H_d(f)\right)^{\infty}$$

$$\begin{array}{rcc}
\text{requires colim } H_d(f) \text{ mono} \text{ ——} & \cong & \cong \\
 & \operatorname{im}\left(H_d(f)^{\dagger}\right) & H_d(L_\bullet)^{\infty} \\
\text{relative l.e.s.} \text{ ——} & \cong & \cong \\
 & \operatorname{im}\left(H_{d+1}(\phi)^{*}\right) & H_d(A, L)^{\vee} \\
\text{requires lim } H_{d+1}(\phi) \text{ epi} \text{ ——} & \cong & \updownarrow \\
 & \left(\operatorname{im} H_{d+1}(\phi)\right)^{*} & H_d(A, L_\bullet)^{-\infty}
\end{array}$$

COMPUTING IMAGE PERSISTENCE

$$\begin{array}{cc} & \longrightarrow L \\ L \downarrow & D^L \end{array} \qquad\qquad \begin{array}{cc} & \longrightarrow K \\ L \downarrow & D^{im} \end{array}$$

Reduce to $R^L = D^L V^L$ $\qquad R^{im} = D^{im} V^{im}$

Theorem [B, Schmahl]

$$B(\operatorname{im} H_*(L_\bullet \hookrightarrow K_\bullet)) =$$

$$\{[i,j) \mid i = \operatorname{pivot} R^{im}_j\}$$

$$\cup \{[i,\infty) \mid R^L_i = 0,\ i \in \operatorname{pivots} R^L\}$$

CLEARING FOR IMAGE PERSISTENCE

Homology:

$$
\begin{array}{ccc}
R^L = D^L V^L & R^{im} = D^{im} V^{im} & R^K = D^K V^K \\
H_*(f)^\infty & H_*(f)^+ & H_*(\phi)^\infty \\
 & & \text{\& clearing } R^{im}
\end{array}
$$

Cohomology:

$$
\begin{array}{ccc}
S^L = W^L D^L & S^{im} = W^{im} D^{im} & S^K = W^K D^K \\
H^*(\phi)_\infty & H^*(f)_+ & H^*(f)_\infty \\
 & & \text{\& clearing } S^{im}
\end{array}
$$

IMPLICIT REDUCTION

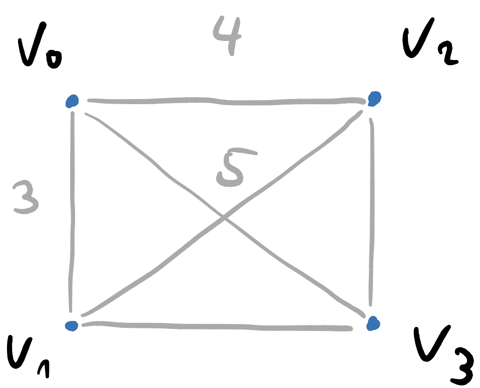
$$R = D \cdot V$$

store

recompute

store pivots (unless apparent)
recompute other entries

Apparent Pairs
(how simplices find the perfect partner)



$$
\begin{array}{c}
\begin{array}{r}
(3)\\(2)\\(1)\\(0)\\(3,2)\\(1,0)\\(3,1)\\(2,0)\\(3,0)\\(2,1)\\(3,2,1)\\(3,2,0)\\(3,1,0)\\(2,1,0)\\(3,2,1,0)
\end{array}
\end{array}
\overset{\begin{array}{ccccccccccccccc}(3)&(2)&(1)&(0)&(3,2)&(1,0)&(3,1)&(2,0)&(3,0)&(2,1)&(3,2,1)&(3,2,0)&(3,1,0)&(2,1,0)&(3,2,1,0)\end{array}}{
\left(\begin{array}{ccccccccccccccc}
 & & & & 1 & & 1 & & 1 & & & & & & \\
 & & & & 1 & & & 1 & & 1 & & & & & \\
 & & & & & 1 & 1 & & & 1 & & & & & \\
 & & & & & 1 & & 1 & 1 & & & & & & \\
 & & & & & & & & & & 1 & 1 & & & \\
 & & & & & & & & & & & & 1 & 1 & \\
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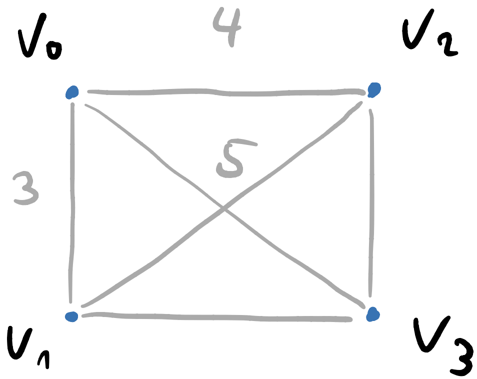
 & & & & & & & & & & & & & & 1\\
 & & & & & & & & & & & & & & 1\\
 & & & & & & & & & & & & & &
\end{array}\right)} = D
$$

APPARENT ZERO PAIRS



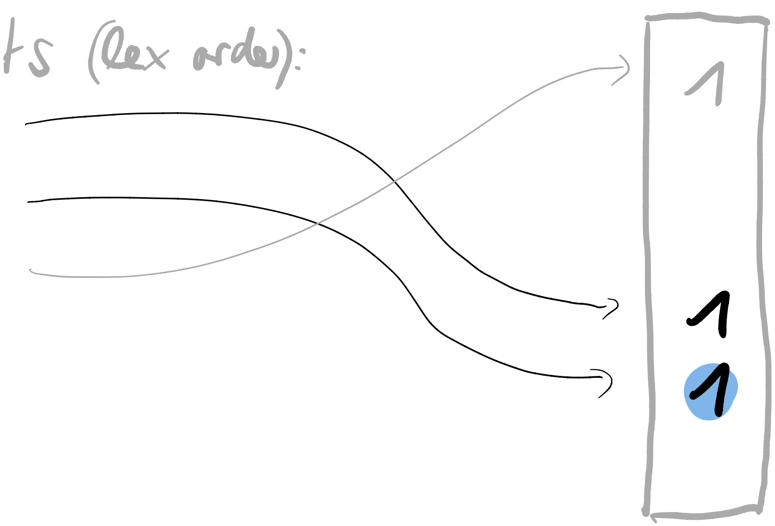
(3, 2, 0): diam = 5

facets (lex order):

(2,0)

(3,0)

(3,2)



(3, 2) : 3

(1, 0) : 3

(3, 1) : 4

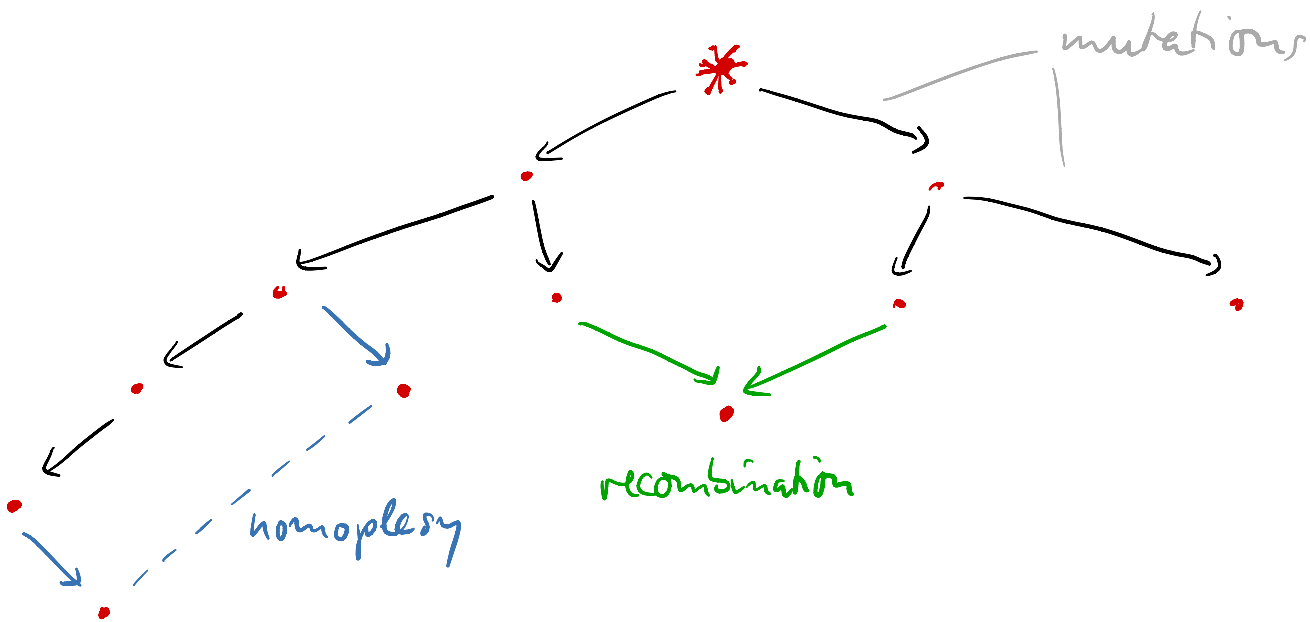
(2, 0) : 4

(3, 0) : 5

(2, 1) : 5

TOPOLOGY OF VIRAL EVOLUTION

[Chen et al. 2013]



Example (Mutation E484K):

- prevalence still low (15.1.21: < 1%)
- but appears in several persistent cycles

ACKNOWLEDGEMENTS

Induced matchings

joint with M. Lesnick (U Albany)

doi: 10.20382/jocg.v6i2a9

doi: 10.1007/978-3-030-43408-3_3

Image persistence

joint with M. Schmahl (U Heidelberg)

arxiv: 2012.12881

Ripser

ripser.org

con'd

joint with: M. Bleher, L. Hahn, A. Ott (U Heidelberg)
J. Patiño-Galindo, R. Rabadan (Columbia U)
M. Carrière (INRIA Sophia-Antipolis)

Funding: DFG (SFB-TR 109)

An Excursion into the Wilderness

Given a finite indexing poset P.

Indecomposable diagrams of K-vector spaces with shape P? (K algebr. closed)

3 cases (representation types):

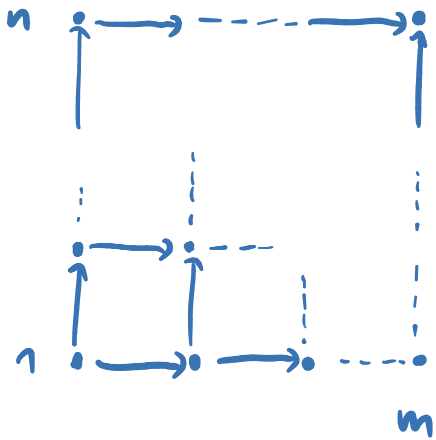
ⓐ A finite list — finite type

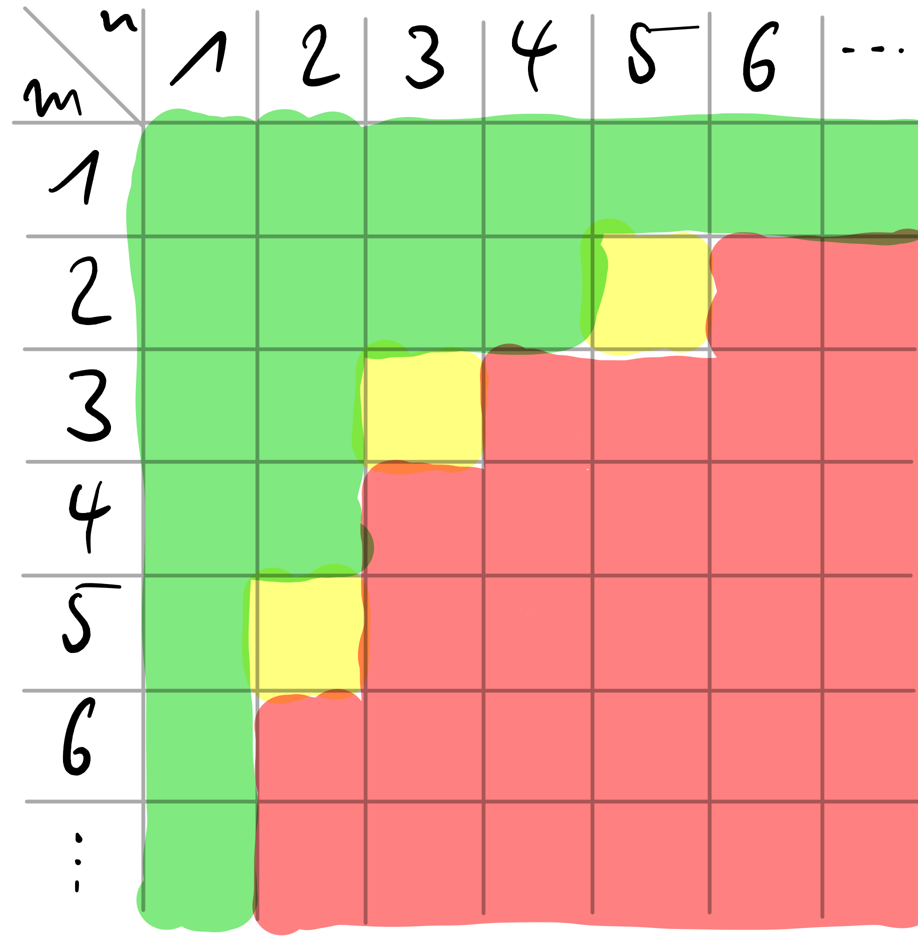
ⓑ A finite list (of 1-param. families) — tame

ⓒ It's complicated. — wild

(as complicated as modules over any finite-dim. algebra; undecidable theory)

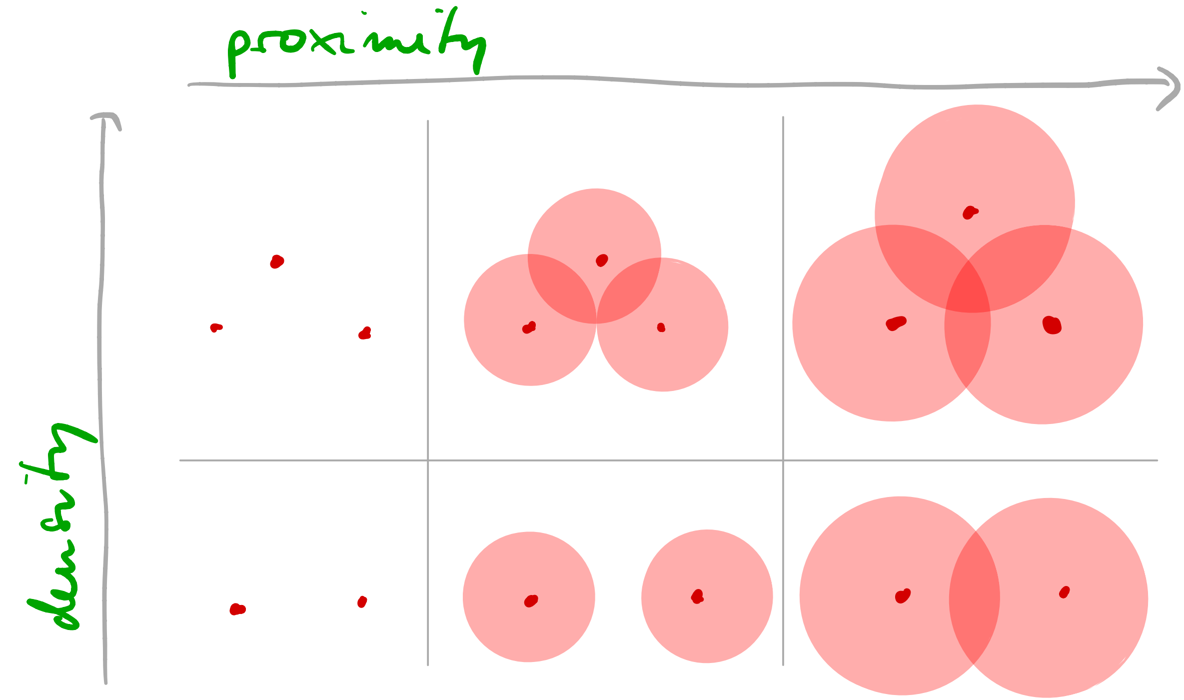
Representation Types of Commutative Grids



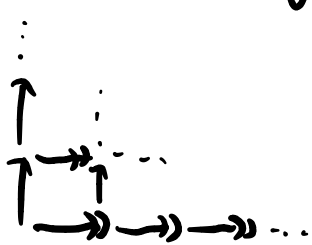


$$\left.\begin{array}{l}\text{finite}\\ \text{tame}\\ \text{wild}\end{array}\right\} : (m-1)(n-1) \left\{\begin{array}{c}<\\ =\\ >\end{array}\right\} 4$$

GRID DIAGRAMS FROM CLUSTERING



Apply $H_0 \rightsquigarrow$ diagram of the form



(horizontal : surjective)

$$
\begin{tikzcd}
K^3 \arrow[r, two heads, "(1\ 1\ 1)"] & K \arrow[r, two heads] & K \\
K^2 \arrow[u, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}"] \arrow[r, two heads] & K^2 \arrow[u, "(1\ 1)"'] \arrow[r, two heads, "(1\ 1)"] & K \arrow[u]
\end{tikzcd}
$$

$$
\cong
\begin{tikzcd}
K \arrow[r, two heads] & K \arrow[r, two heads] & K \\
K \arrow[u] \arrow[r, two heads] & K \arrow[u] \arrow[r, two heads] & K \arrow[u]
\end{tikzcd}
\oplus
\begin{tikzcd}
K \arrow[r, two heads] & 0 \arrow[r, two heads] & 0 \\
K \arrow[u] \arrow[r, two heads] & K \arrow[u] \arrow[r, two heads] & 0 \arrow[u]
\end{tikzcd}
\oplus
\begin{tikzcd}
K \arrow[r, two heads] & 0 \arrow[r, two heads] & 0 \\
0 \arrow[u] \arrow[r, two heads] & 0 \arrow[u] \arrow[r, two heads] & 0 \arrow[u]
\end{tikzcd}
$$

Lemma $\mathrm{Rep}^{\twoheadrightarrow}(m, 2)$ is finite type.

EPIC GRIDS & WILD THINGS

Theorem [B, Botnan, Oppermann, Steen '19]

$$\mathrm{Rep}^{\hat{\uparrow}\twoheadrightarrow}(m,n) \sim \mathrm{Rep}^{\twoheadrightarrow}(m-1,n)$$
$$\wr \qquad\qquad\qquad\qquad \wr$$
$$\mathrm{Rep}^{\twoheadrightarrow}(m-1,n) \sim \mathrm{Rep}(m-1,n-1)$$

Corollary $\mathrm{Rep}^{\twoheadrightarrow}(m,n)$ is

$$\left.\begin{array}{l} \text{finite type} \\ \text{tame} \\ \text{wild} \end{array}\right\} \text{ for } (m-1)(n-2) \left\{\begin{array}{c} < \\ = \\ > \end{array}\right\} 4.$$