

Gromov-hyperbolicity, geodesic defect  
of apparent pairs in Rips filtrations

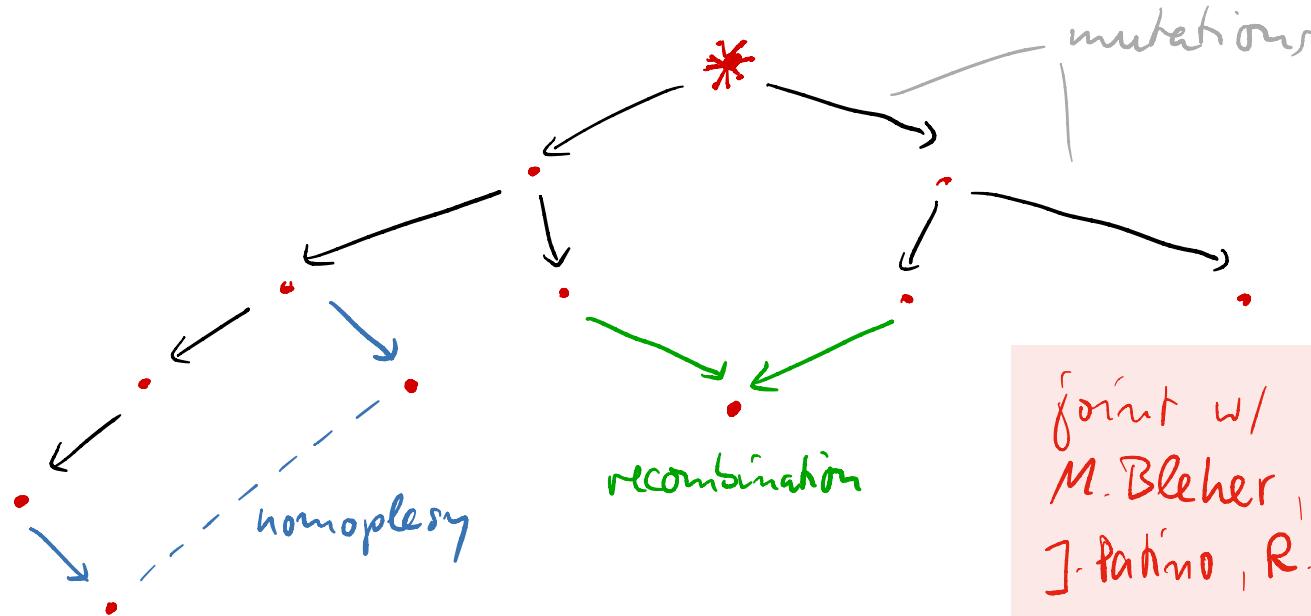
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# Motivation ①: Vietoris - Rips persistence of COVID-19 genetic evolution data



Joint w/  
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- Early detection of adaptive mutations  
(occurring independently in multiple lines)
- Computation faster than usual

New insights  $\rightsquigarrow$  huge speedup; without changes to code!  
( $\sim 1$  day  $\rightsquigarrow$  2 min)

## Motivation ②: The Rips Lemma

Lemma [Rips 1982; Gromov 1988]

Let  $X$  be a  $\delta$ -hyperbolic geodesic space. Then

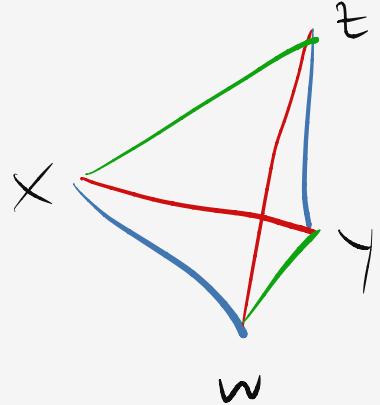
$$\text{Rips}_t(X) \simeq \{*\} \quad \text{for } t \geq 4\delta.$$

- What about non-geodesic spaces?
- In particular, finite metric spaces?
- Connection to Rips? Collapse by apparent pairs?

# Gromov - hyperbolicity

Definition A metric space  $(X, d)$  is  $\delta$ -hyperbolic if  $\forall w, x, y, z \in X$ ,

$$d(x, w) + d(y, z) \leq \max(d(x, y) + d(z, w), d(x, z) + d(y, w)) + 2\delta$$



- Other notions exist for geodesic spaces.
- A metric tree is  $0$ -hyperbolic.

# A Rips Lemma for non-geodesic spaces

Theorem (B, Roll)

Let  $X$  be a  $\delta$ -hyperbolic metric space. Then

$$\text{Rips}_t(X) \simeq \{*\} \quad \text{for } t \geq 4\delta + 2\nu,$$

where  $\nu$  : geodesic defect of  $X$ .

# Filtered Rips Lemma for finite $X$

Theorem (B, Roll)

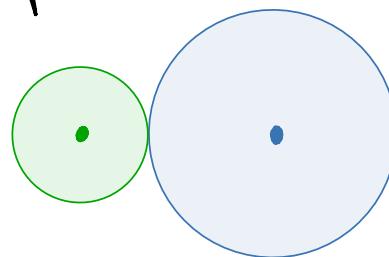
Let  $X$  be a  $\delta$ -hyperbolic finite metric space with geod. defect  $\sigma$ . Then

$$\text{Rips}_n(X) \downarrow \text{Rips}_t(X) \downarrow \{*\}$$

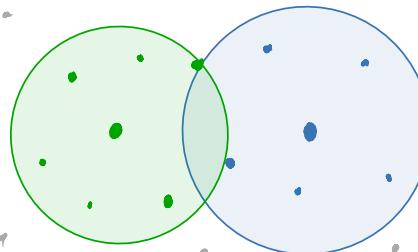
by a single discrete gradient  
for all  $n \geq t \geq 4\delta + 2\sigma$ .

# Disjoint balls

- In a geodesic space, closed balls  $B_r(x)$  and  $B_s(y)$  intersect iff  $r + s \geq d(x, y)$ .



- In general, larger radii may be needed.



# Geodesic defect

Definition      The geodesic defect

of a metric space  $(X, d)$  is

$$\inf \left\{ \nu \geq 0 \mid B_{r+\nu}(x) \cap B_{s+\nu}(y) \neq \emptyset \right. \\ \left. \forall x, y \in X, r + s = d(x, y) \right\}.$$

See also :

- M. Bonk, O. Schramm, 2000 (almost geodesic space)
- U. Lang, 2013 (idea used implicitly)
- P. Johannnad, J. Jost, 2020 (multiplicative local version)

# Bounds on geodesic defect

$v$ : geodesic defect of finite  $(X, d)$

• lower bound:  $v \geq \frac{1}{2} \min_{x \neq y \in X} d(x, y)$

• upper bound:  $v \leq d_H(X, Z)$

for  $Z \subseteq \mathbb{Z}$  a geodesic ambient space

Possible ambient spaces for  $X$ :

•  $\ell^\infty(X)$  (bounded functions  $X \rightarrow \mathbb{R}$ )  $x \mapsto d_x$   
(Kuratowski embedding  $e: X \rightarrow \ell^\infty(X)$ )

• tight span / injective envelope: subspace of  $\ell^\infty(X)$

# Tight span

Given a finite metric space  $(X, d)$ ,  
the tight span is

$$E(X) := \{ f \in \ell^\infty(X) \mid \begin{array}{l} f(x) + f(y) \geq d(x, y) \quad \forall x, y \in X \\ \exists x \exists y : f(x) + f(y) = d(x, y) \end{array}\}$$

- geodesic
- hyperconvex (metric balls have the Helly property)  
↓ [Arveson - Panitchpakdi 1956]
- injective (in the category of metric spaces)
- contains the Kuratowski embedding  $e(X)$
- minimal space with above properties
- contractible [Isbell 1962]

# Density of Kuratowski embedding

Theorem (U. Lang 2013 + ε)

$X$   $\delta$ -hyperbolic, with geodesic defect  $v$ ,

$e : X \hookrightarrow E$  Kuratowski embedding into tight span.

Then

$$d_H(e(X), E) \leq 2\delta + v.$$

## Remark

- $\text{Rips}_{2r}(X) = \check{\text{Cech}}_r(e(X), E)$  (hyperconvexity)
- For  $r \geq 2\delta + v$  :  $\check{\text{Cech}}_r(e(X), E) \simeq E$  (nre lemma)
- $E \simeq \{\infty\}$

# Rips Lemmas for non-geodesic spaces

Corollary  $\text{Rips}_t(X) \simeq \{\ast\}$  for  $t \geq 4\delta + 2\nu$   
not given by an explicit collapse

Extending the argument of Gromov-Rips:

Theorem (B, Roll) A single discrete gradient on  $\Delta(X)$  induces the collapses

$\text{Rips}_n(X) \rightarrow \text{Rips}_t(X) \rightarrow \{\ast\}$   
for all  $n \geq t \geq 4\delta + 2\nu$ .

# VR complexes of trees & discrete Morse

Consider a generic finite metric tree

$T = (V, E)$  (distinct pairwise distances).

- $\text{diam} : \Delta(V) \rightarrow \mathbb{R}$  is a generalized discrete Morse function
- discrete gradient has non-critical intervals  $[e, \Delta_e]$

where

- $e \in \binom{V}{2} \setminus E$  : a non-tree edge
- $\Delta_e$  : unique max coface w/  $\text{diam} = \text{length}(e)$
- Only vertices  $V$  and tree edges  $E$  are critical

# Consequences

- The discrete gradient induces collapses

$\text{Rips}_t(X) \searrow T_t \xleftarrow{\text{filtered tree}} \forall t \in \mathbb{R}$   
and

$\text{Rips}_n(X) \searrow \text{Rips}_t(X) \quad \text{for } [t, n] \cap d(E) = \emptyset$

- For  $\epsilon \geq \max \{ \text{length}(e) \mid e \in T \} :$

$\text{Rips}_t(X) \searrow T \searrow \{*\}$ .

- Prs. homology trivial in  $\deg > 0$ .

# General tree metrics

Example : phylogenetic trees  $(V, E)$

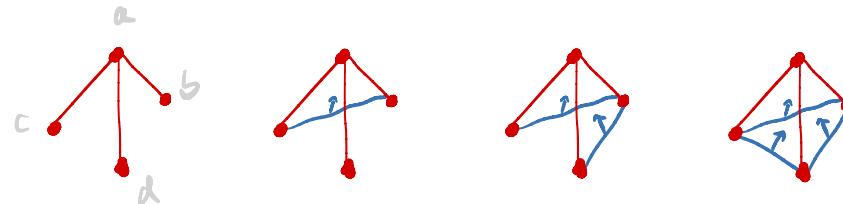
- non-generic tree metric
- diam is not a discrete Morse function

There is still a compatible gradient :

- indep. of choices (canonical gradient)
- for generic trees : equals diam. gradient
- only  $V, E$  critical
- induces the same collapses

# Symbolic perturbation

- Choose total order on vertices ✓
- Order edges lexicographically
- Perturb metric symbolically wrt edge order
  - ~ gradient structure like a generic tree metric
- The canonical gradient (from before) refines any perturbed gradient



# Apparent pairs

- $K$ : simplexwise filtration (total order on  $K$ )

- $(\sigma, \tau)$  apparent pair:
  - $\sigma$  is latest face of  $\tau$
  - $\tau$  is earliest coface of  $\sigma$

~ persistence pairs & discrete gradient

- Rips - lexicographic filtration:

order simplices by

- diameter , then
- lexicographically

# Apparent pairs Rips Lemma for trees

Theorem (B, Roll) Let

- $T = (V, E)$  be a finite metric tree,
- $V$  totally ordered away from root.

Then the apparent pairs induces collapses

$$\text{Rips}_t(V) \rightarrow T_t \quad \forall t \in \mathbb{R}$$

and

$$\text{Rips}_n(V) \hookrightarrow \text{Rips}_t(V) \quad \text{for } [t, n] \cap d(E) = \emptyset$$

- Apparent pairs refine perturbed gradient

# Summary

Rips Lemmas beyond geodesic spaces

- geodesic defect
- filtration compatible
- collapses
- for trees : via apparent pairs

Explains Rips's outstanding performance on evolution data  
(almost tree-like)