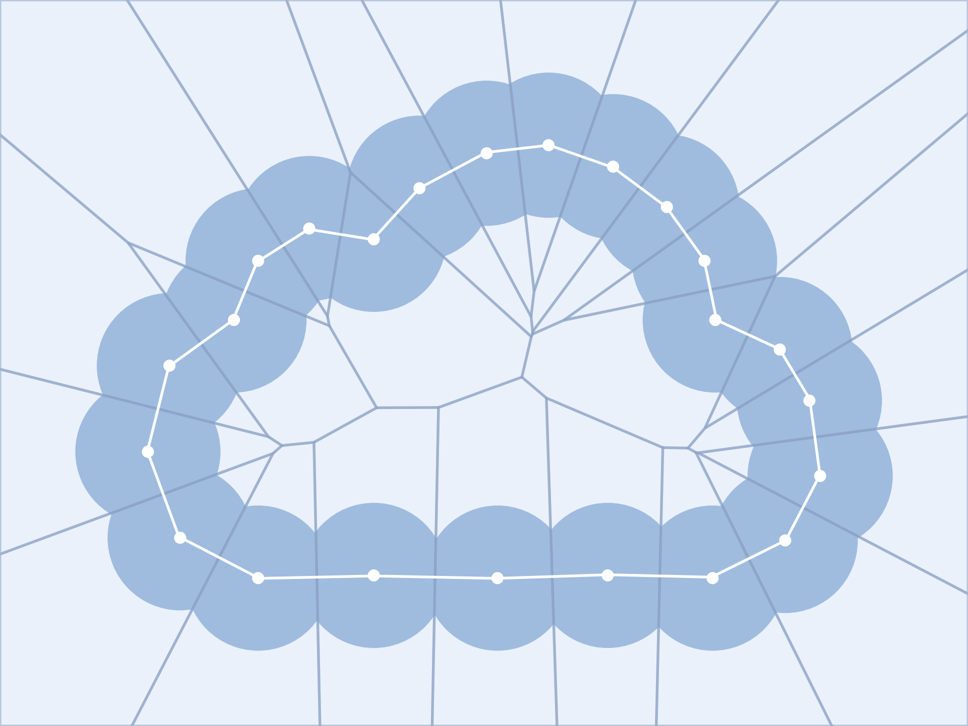

Wrapping clouds

Ulrich Bauer

TUM

Mathematisches Kolloquium, U Osnabrück

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Joint work with Herbert Edelsbrunner (IST Austria)

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The Morse theory of Čech and Delaunay complexes

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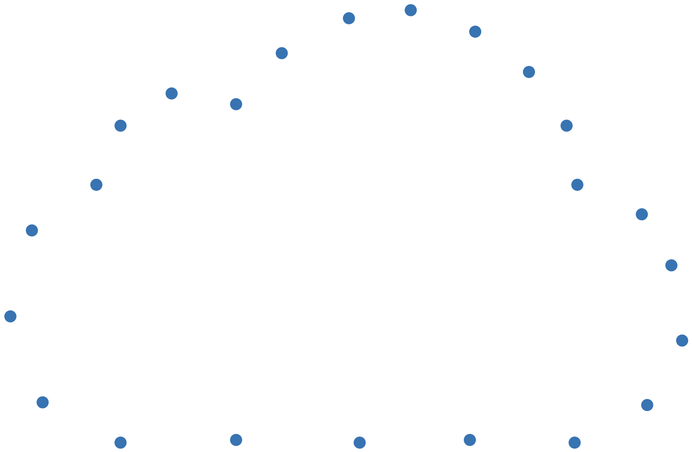

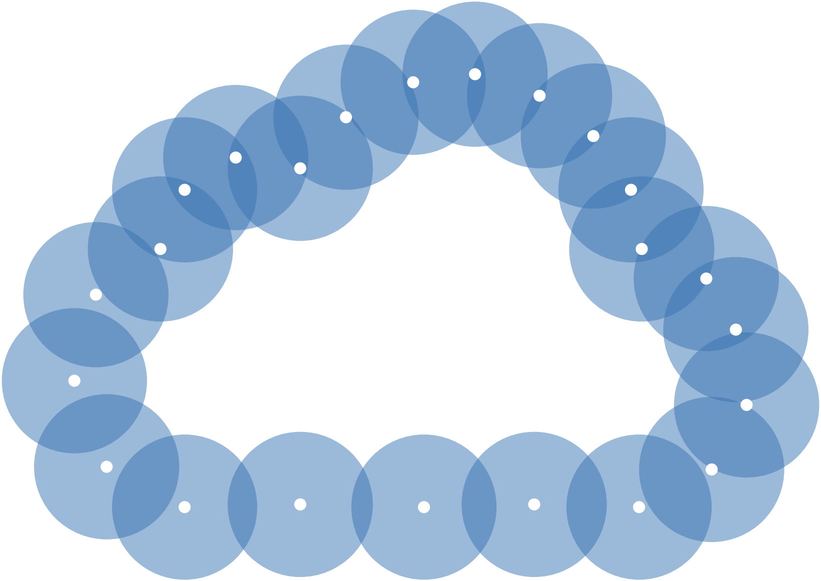

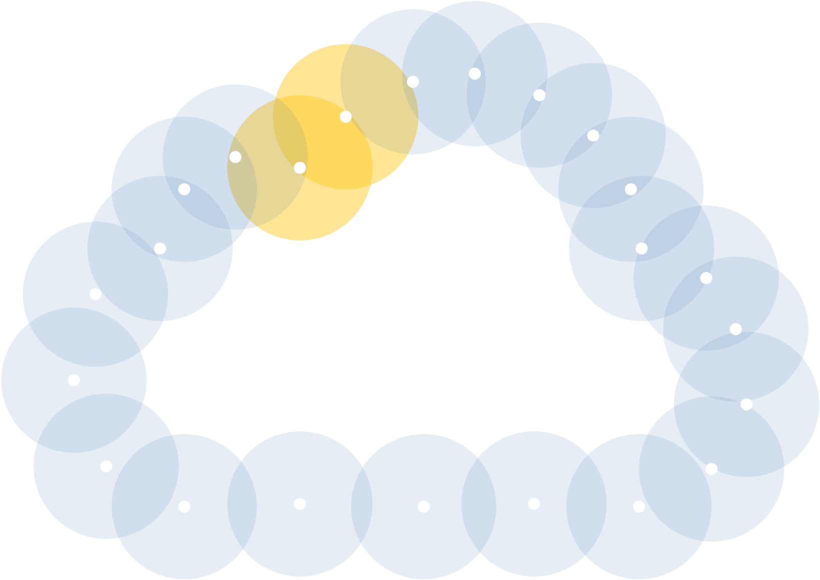

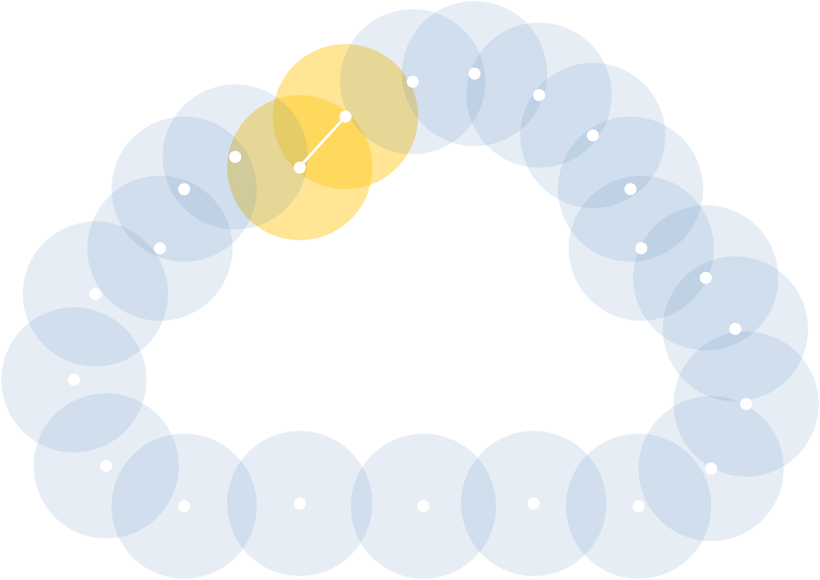

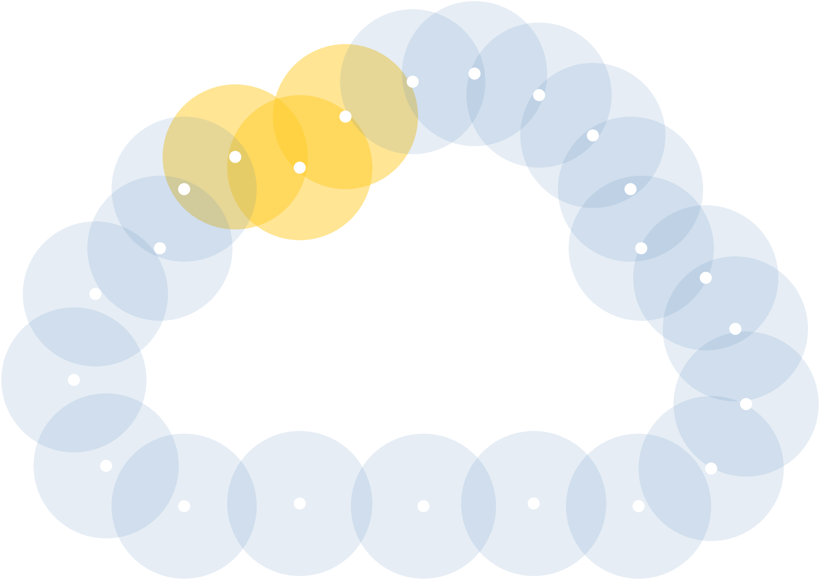

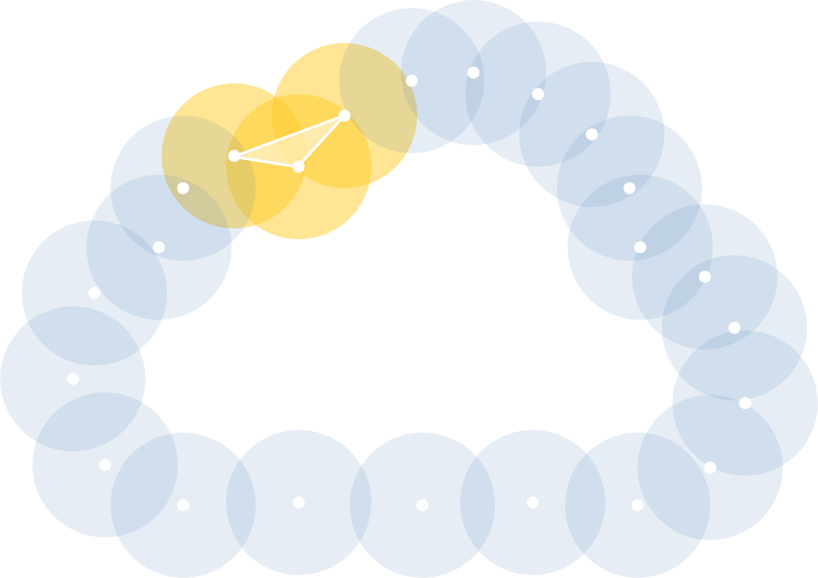

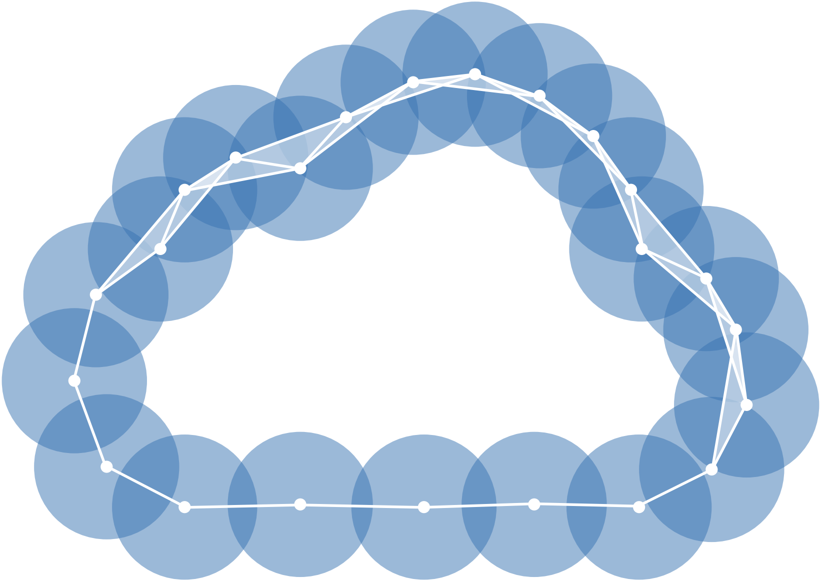

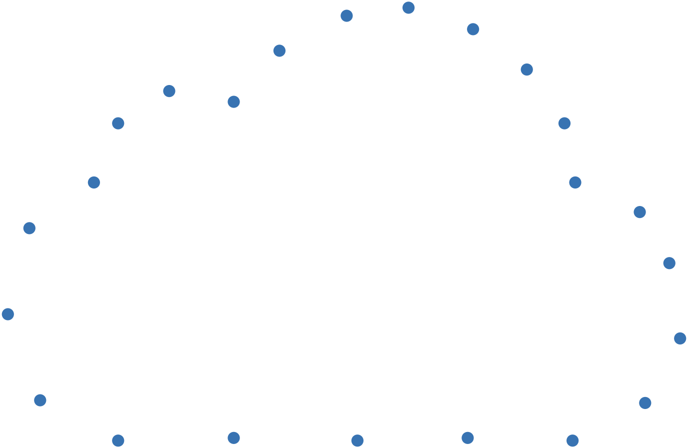

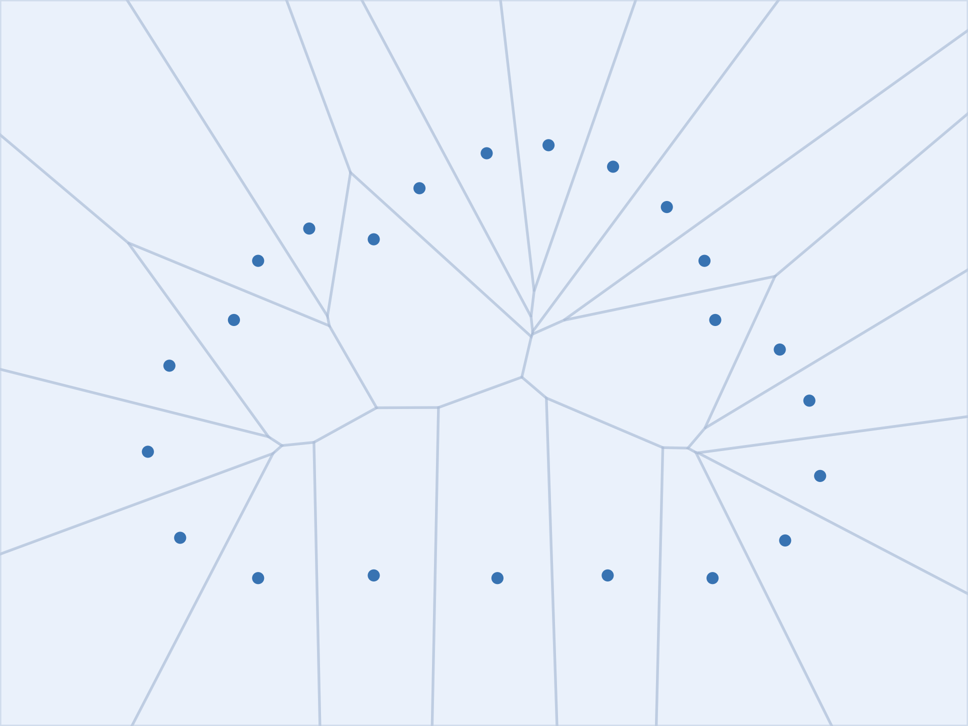

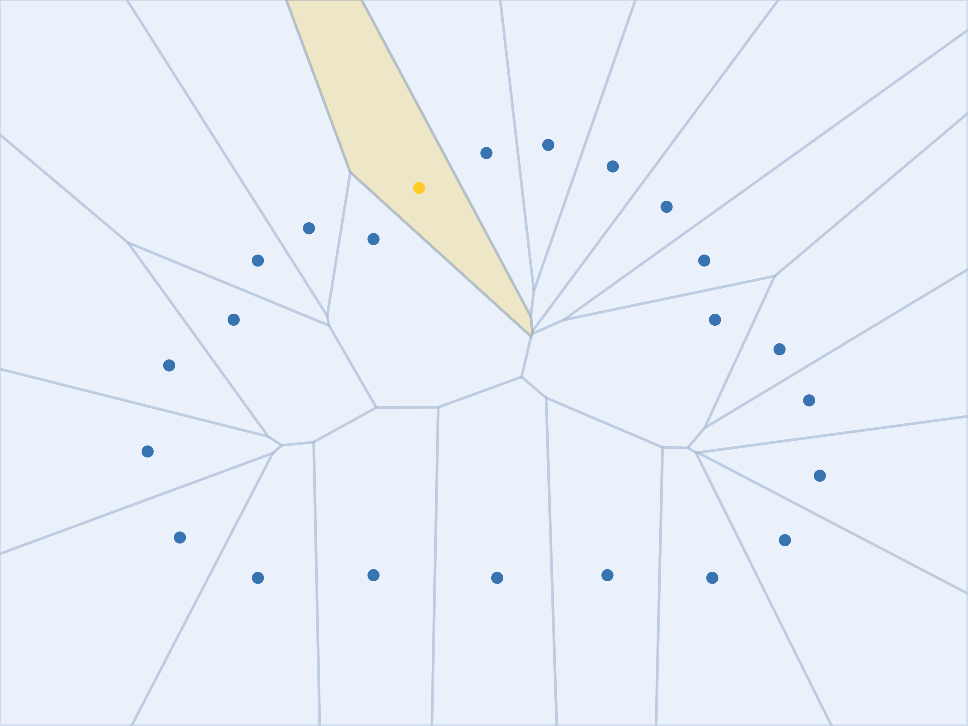

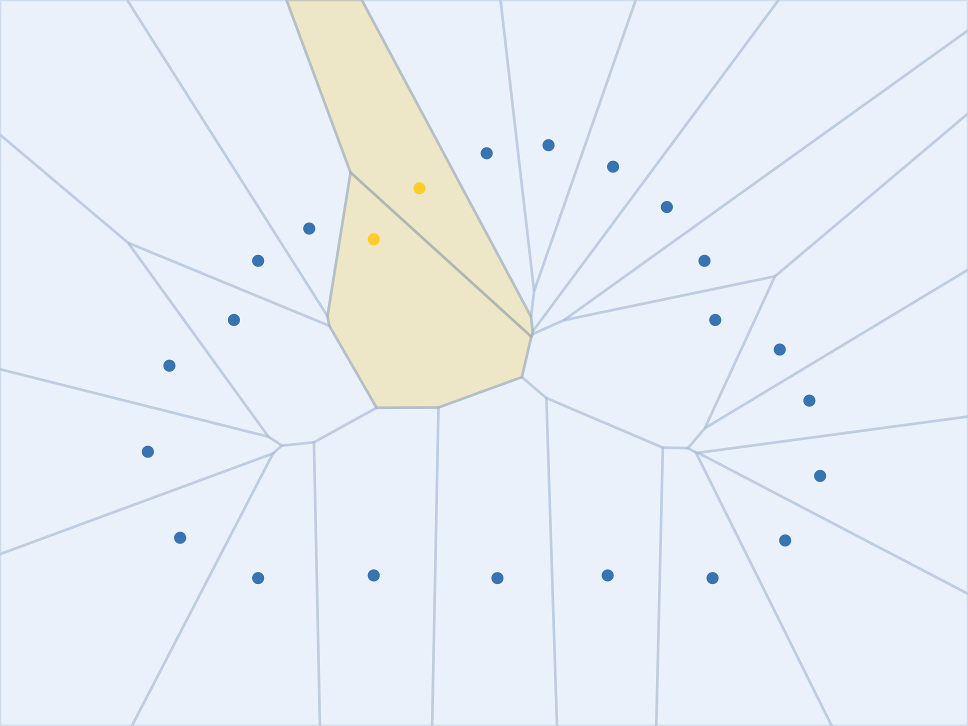

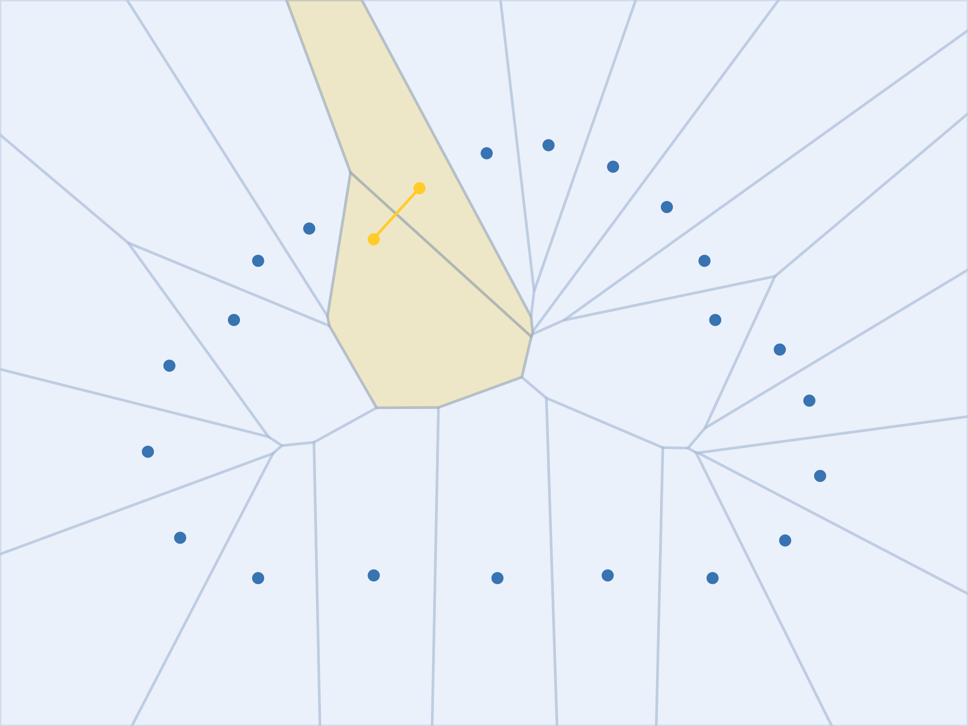

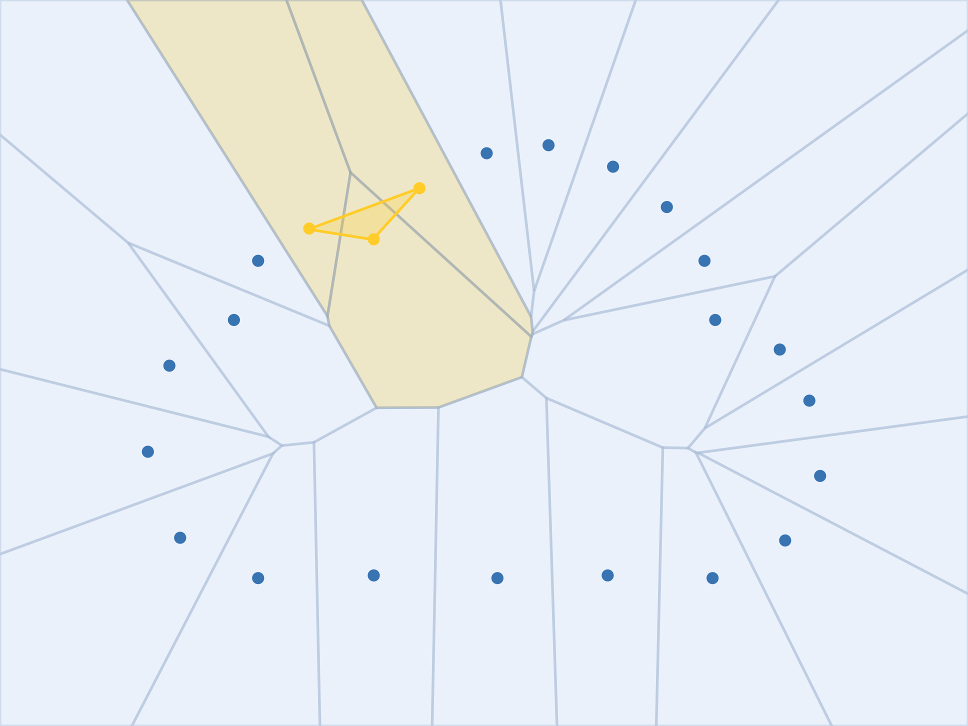

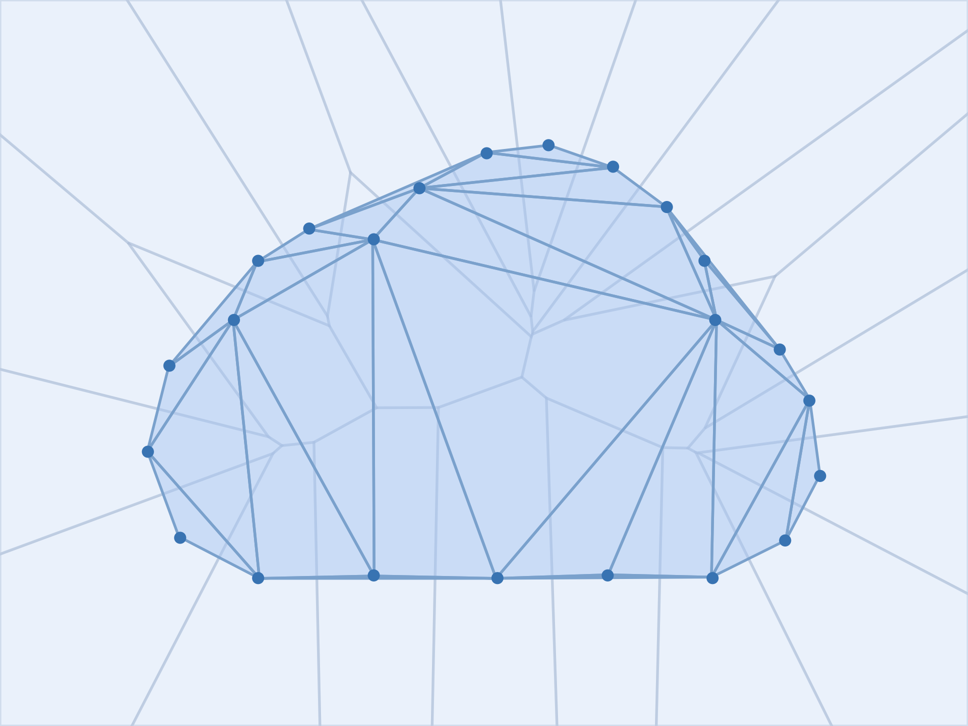

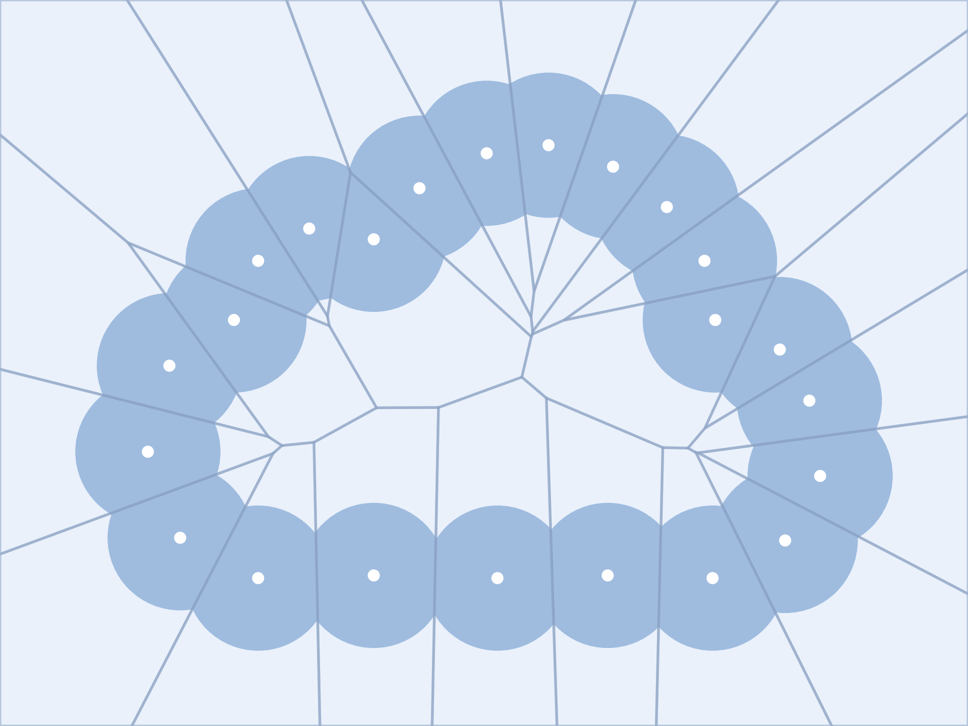

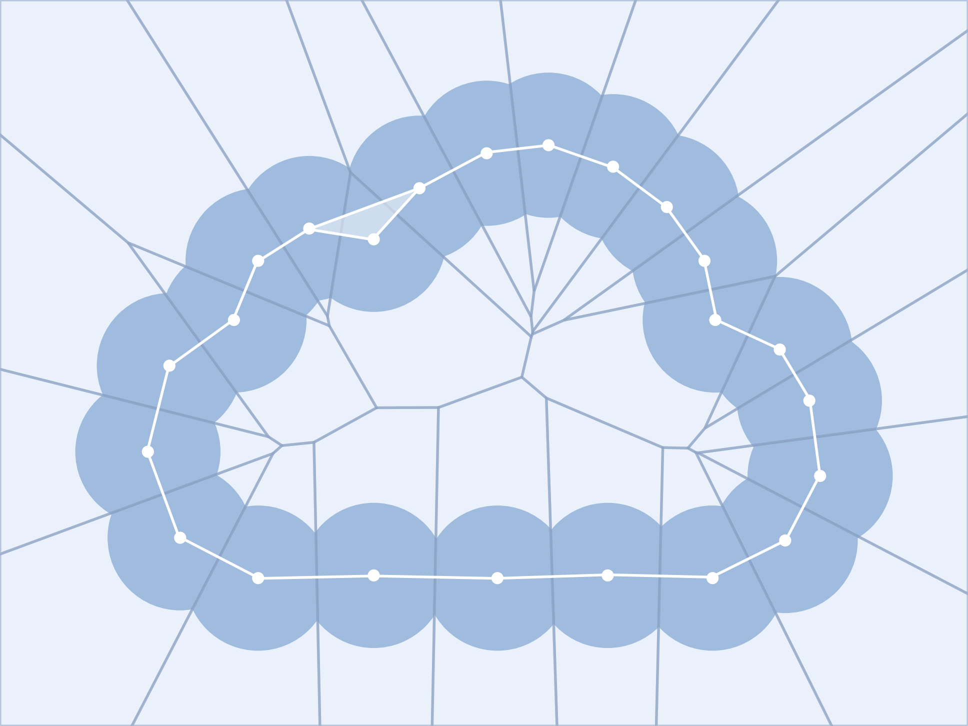

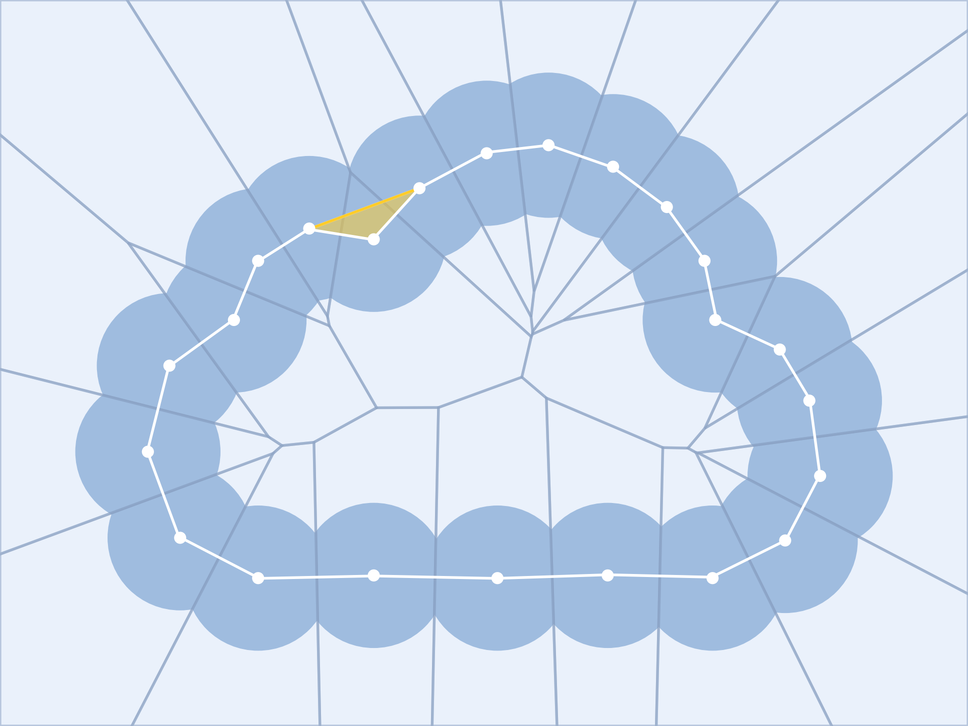

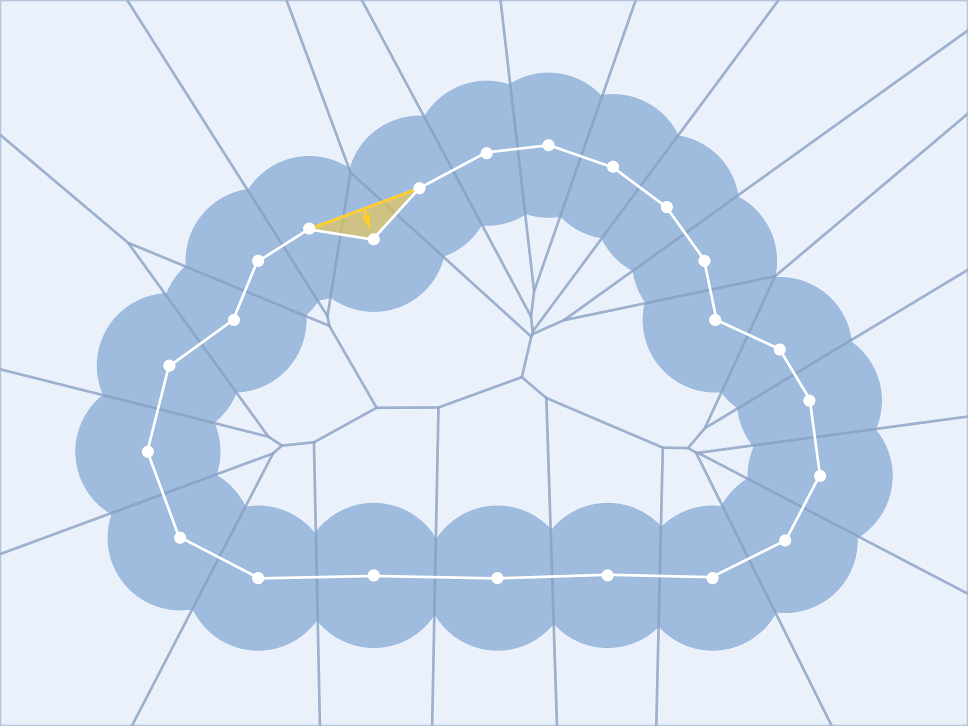

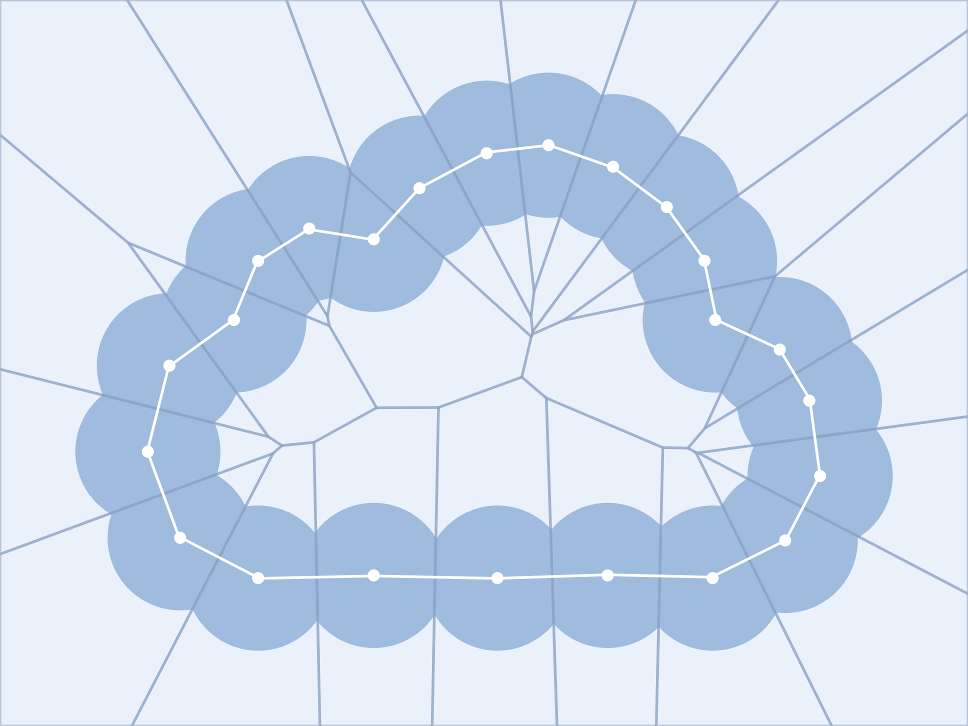

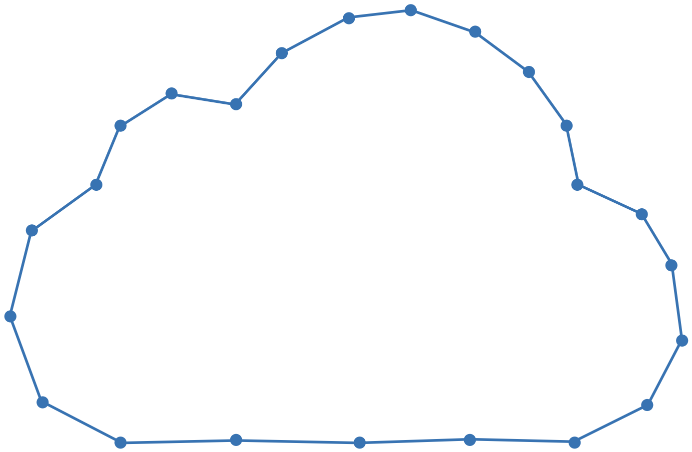

Connecting Čech and Delaunay complexes

By the *Nerve theorem* (Borsuk 1947):

$$\mathrm{Del}_r(X) \simeq \mathrm{Cech}_r(X) \simeq B_r(X).$$

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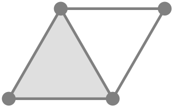
- Are all three complexes homotopy equivalent?
- Are they related by a sequence of *simplicial collapses*?

Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.



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Definition (Whitehead 1938)

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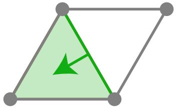
- A *free face* of K is a simplex that has a unique proper coface.



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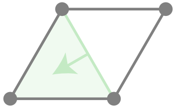

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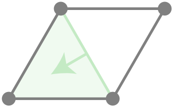

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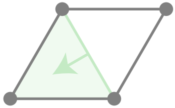

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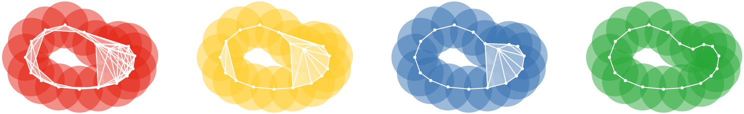
If there is a sequence of such elementary collapses from K to M, we say that K *collapses* to M (written as $K \searrow M$).

Main result: a sequence of collapses

Theorem (B, Edelsbrunner 2017)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent through a sequence of collapses

$$\mathrm{Cech}_r X \searrow \mathrm{Cech}_r X \cap \mathrm{Del}\, X \searrow \mathrm{Del}_r X \searrow \mathrm{Wrap}_r X.$$



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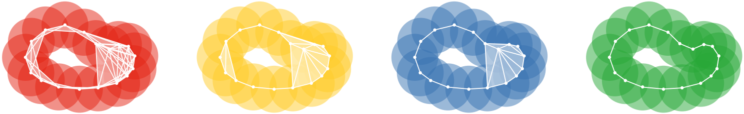
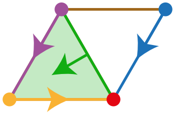


- Extends to weighted Delaunay triagulations
- All collapses are induced by a single *discrete gradient field*
- This yields explicit chain maps, inducing isomorphims in persistent homology

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a simplicial complex is a partition of the simplices into singletons and pairs $\{L, U\}$, with L a facet of U.



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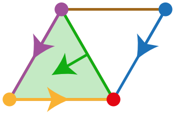

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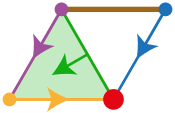

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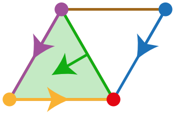
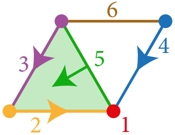


A function $f : K \to \mathbb{R}$ on a simplicial complex is a *discrete Morse function* if for all $t \in \mathbb{R}$:

- sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes



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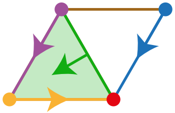

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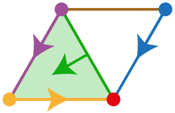

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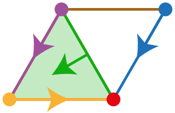

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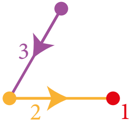

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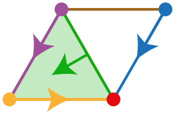

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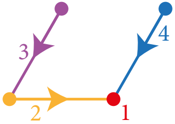

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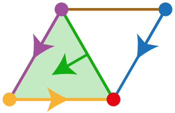

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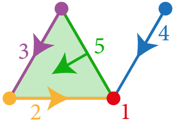

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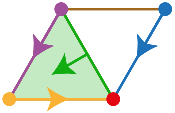

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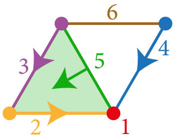

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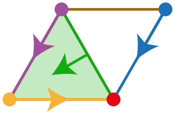

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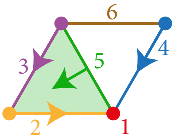

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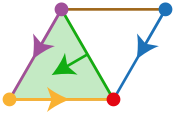

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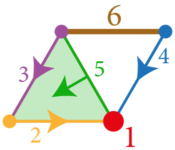

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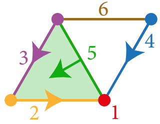
If $f^{-1}(t) = \{Q\}$ then t is a *critical value*.

Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.



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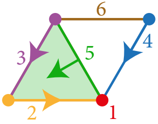
Theorem (Forman 1998)

If $(s, t]$ *contains no critical value of* f*, then* $K_t \searrow K_s$.



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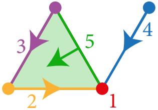

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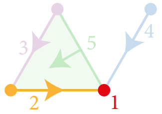

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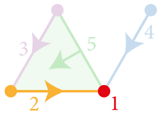

Corollary

K is homotopy-equivalent to a CW complex with one d-cell for each critical d-simplex.

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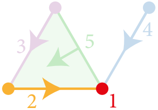
Theorem (Forman 1998)

If $(s, t]$ contains no critical value of f, then $K_t \searrow K_s$.



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K is homotopy-equivalent to a CW complex with one d-cell for each critical d-simplex.

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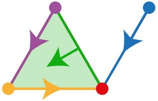
If $K \setminus L$ is a union of pairs of V, then $K \searrow L$.



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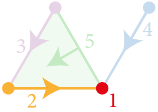

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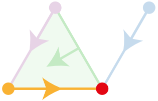

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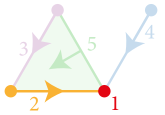

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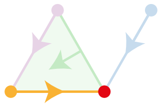
We say that V *induces the collapse $K \searrow L$.*



Čech and Delaunay functions

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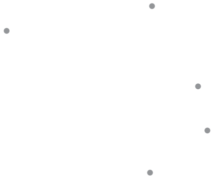
$X \subset \mathbb{R}^d$: finite point set (in general position)



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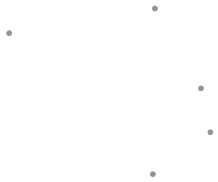

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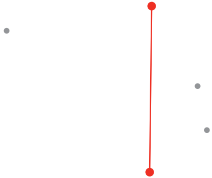

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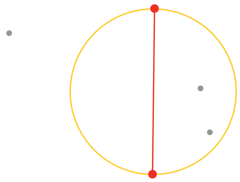

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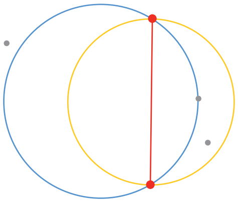
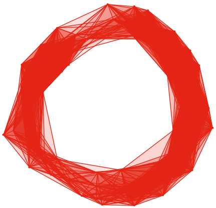
  radius of *smallest empty circumsphere*
  - defined only on $\mathrm{Del}(X)$



Čech and Delaunay complexes from functions

For any radius r:

- $\mathrm{Cech}_r(X) = f_{\mathrm{Cech}}{}^{-1}(-\infty, r]$
  - simplices with smallest enclosing radius $\leq r$



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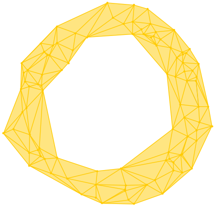

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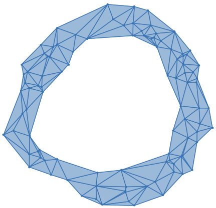

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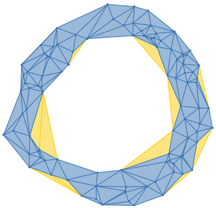

Unfortunately…

Neither the Čech nor the Delaunay functions
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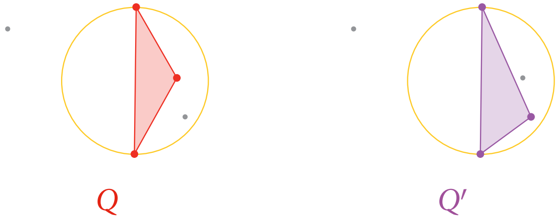
- Example: two simplices Q, Q' with $f_C(Q) = f_C(Q')$ where neither is a face of the other:

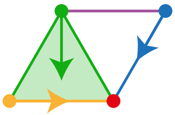


Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex K is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}$$



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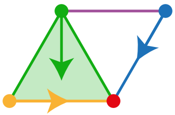
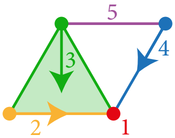


- indicated by an arrow from L to U

A *generalized discrete Morse function* $f : K \to \mathbb{R}$ satisfies:

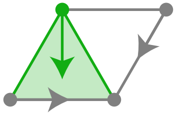
- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes (for all $t \in \mathbb{R}$)
- the level sets $f^{-1}(t)$ form a generalized vector field (the *discrete gradient* of f)



Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:



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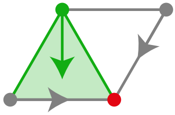

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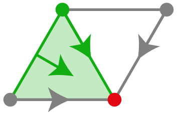

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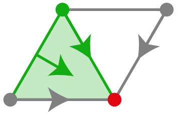

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Therefore the collapsing theorems also hold for generalized discrete Morse functions.

Morse theory of Čech and Delaunay complexes

Proposition

The Čech function and the Delaunay function are generalized discrete Morse functions.

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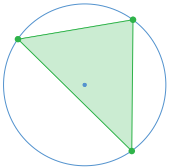
The following are equivalent:

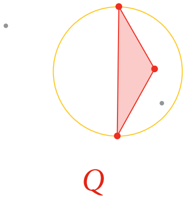
- *Q is a critical simplex of* f_{Cech}
- *Q is a critical simplex of* f_{Del}
- $f_{\mathrm{Cech}}(Q) = f_{\mathrm{Del}}(Q)$
- *Q is a* centered *Delaunay simplex (containing the circumcenter in the interior)*



Čech intervals

Lemma

Let $Q \subseteq X$ be a simplex with smallest enclosing sphere S.
Then $Q' \subseteq X$ has the same smallest enclosing sphere S iff



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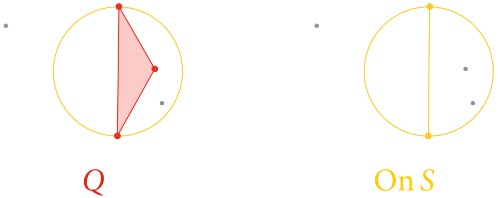

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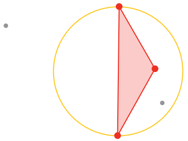
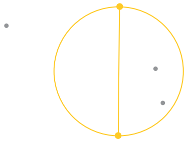
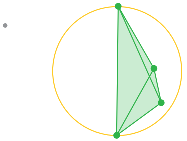
$$\mathrm{On}\, S \subseteq Q' \subseteq \mathrm{Encl}\, S.$$



Q



$\mathrm{On}\, S$



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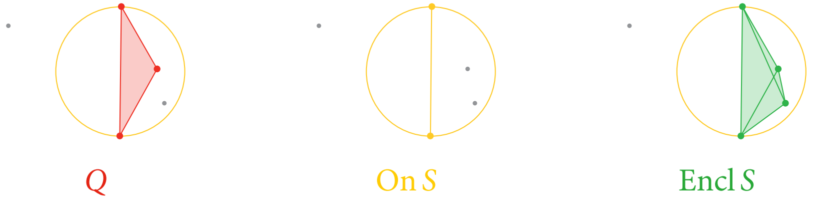

The front and back faces of a simplex

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$$\mathrm{Front}\, S = \{x \in \mathrm{On}\, S \mid \mu_x > 0\},$$

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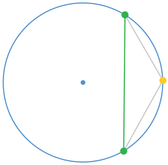

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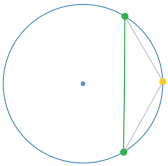

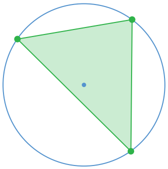

Delaunay intervals

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Let $Q \subseteq X$ be a simplex with smallest empty circumsphere S.
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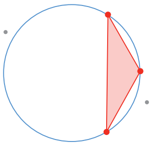

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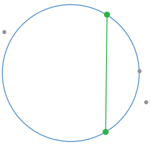
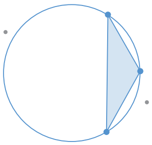


Front S



On S

Selective Delaunay complexes

Sphere optimization problems

Both Čech and Delaunay functions are defined using the smallest sphere S satisfying certain constraints:

$$\begin{array}{lll} \underset{r,z}{\text{minimize}} & r & \\ \text{subject to} & \|z - q\| \leq r, & q \in Q, \\ & \|z - e\| \geq r, & e \in E. \end{array}$$

Here r is the radius of the sphere S, and z is the center of S.

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\begin{aligned}
&\underset{r,z}{\text{minimize}} && r^2 && \\
&\text{subject to} && \|z-q\|^2 \le r^2, && q \in Q, \\
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\begin{array}{lll}
\underset{a,z}{\text{minimize}} & \|z\|^2 - a & (r^2 = \|z\|^2 - a) \\
\text{subject to} & \|z - q\|^2 \leq \|z\|^2 - a, & q \in Q, \\
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The Karush–Kuhn–Tucker optimality conditions

Consider an optimization problem of the form

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & f(x) & \\
\text{subject to} \quad & g_j(x) \geq 0, & \forall j \in J, \\
& g_k(x) = 0, & \forall k \in K, \\
& g_l(x) \leq 0, & \forall l \in L.
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where the function f is convex and g_i are affine ($i \in I = J \cup K \cup L$).

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where the function f is convex and g_i are affine ($i \in I = J \cup K \cup L$).

Theorem (Karush 1939, Kuhn–Tucker 1951)

A feasible x is optimal iff there exist $(\lambda_i)_{i \in I}$ such that

$$\nabla f(x) + \sum_{i \in I} \lambda_i \nabla g_i(x) = 0, \qquad \text{(stationarity)}$$

$$\lambda_i g_i(x) = 0, \quad \forall i \in I, \qquad \text{(complementary slackness)}$$

$$
\left.
\begin{aligned}
\lambda_j \leq 0, & \quad \forall j \in J, \\
\lambda_l \geq 0, & \quad \forall l \in L.
\end{aligned}
\right\} \qquad \text{(dual feasibility)}
$$

KKT conditions for the smallest sphere problem

The KKT conditions for our sphere optimization problem are:

Proposition

A sphere S enclosing Q and excluding E is minimal iff its center z can be written as an affine combination

$$z = \sum_{x \in Q \cup E} \lambda_x x, \qquad 1 = \sum_{x \in Q \cup E} \lambda_x$$

such that

- $\lambda_x = 0$ *whenever* x *does not lie on* S,
- $\lambda_x \leq 0$ *whenever* $x \in E \setminus Q$, *and*
- $\lambda_x \geq 0$ *whenever* $x \in Q \setminus E$.

Čech and Delaunay intervals from KKT

The *Karush–Kuhn–Tucker* optimality conditions yield:

Proposition (Geometric KKT conditions)

A sphere S enclosing Q and excluding E is minimal iff

- *S is the smallest circumsphere of $\mathrm{On}\, S$,*
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The Čech intervals are of the form $[\mathrm{On}\, S, \mathrm{Encl}\, S]$.

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The Čech intervals are of the form $[\mathrm{On}\, S, \mathrm{Encl}\, S]$.
The Delaunay intervals are of the form $[\mathrm{Front}\, S, \mathrm{On}\, S]$.

Selective Delaunay complexes

Define for finite point sets $X, E \subset \mathbb{R}^d$:

- *E-Delaunay function* $f_E(Q)$:
  radius of smallest sphere enclosing Q and excluding E
  - defined only if such a sphere exists: $Q \in \mathrm{Del}(X, E)$

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Theorem (B, Edelsbrunner 2017)

Let $E \subseteq F \subseteq X$. Then

$$\mathrm{Del}_r(X, E) \searrow \mathrm{Del}_r(X, E) \cap \mathrm{Del}\,(X, F) \searrow \mathrm{Del}_r(X, F).$$

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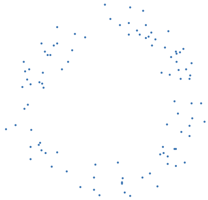
Note: choosing $E = \varnothing$ and $F = X$ yields

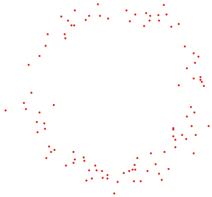
$$\mathrm{Cech}_r(X) \searrow \mathrm{DelCech}_r(X) \searrow \mathrm{Del}_r(X).$$

Connecting different Delaunay complexes

X



Y



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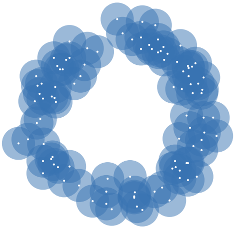
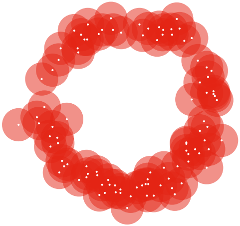
$B_r(X)$



$B_r(Y)$



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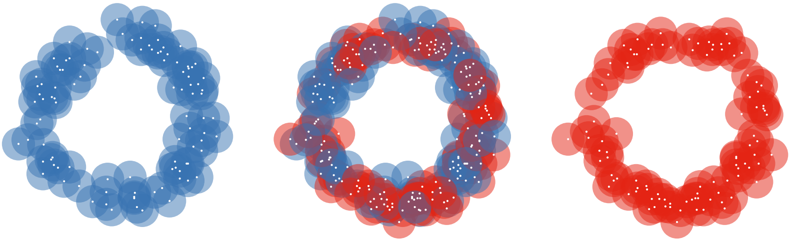
$$B_r(X) \hookrightarrow B_r(X \cup Y) \hookleftarrow B_r(Y)$$



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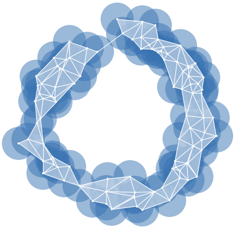
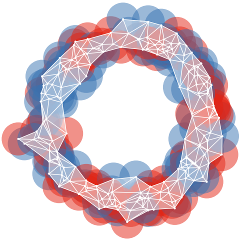


$\mathrm{Del}_r(X)$



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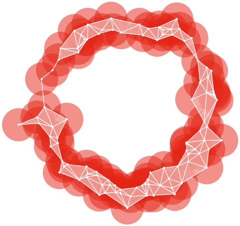

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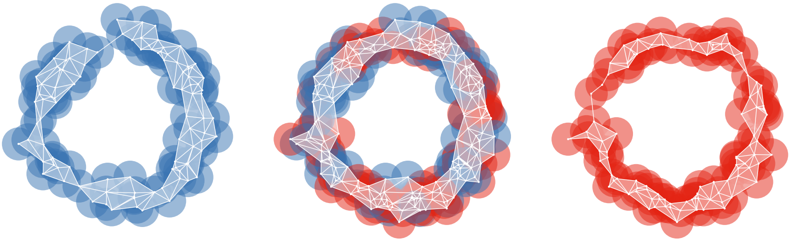

$$\begin{tikzcd}
\mathrm{Del}_r(X) \arrow[dr, hook] & & \mathrm{Del}_r(X \cup Y) \arrow[dl, hook] \arrow[dr, hook] & & \mathrm{Del}_r(Y) \arrow[dl, hook] \\
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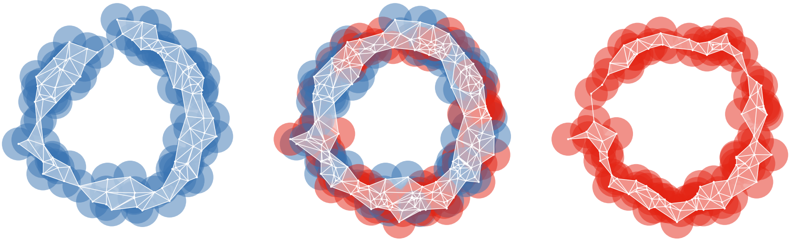

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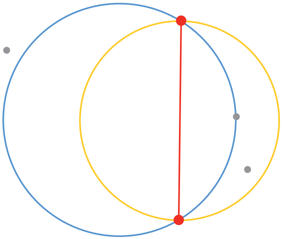
Collapsing from Čech to Delaunay

Collapsing the Delaunay–Čech complex

Construct gradient pairs inducing the collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$:

- Consider a non-critical Delaunay simplex Q



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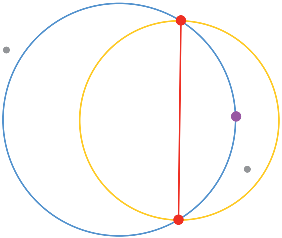

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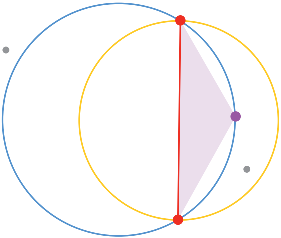

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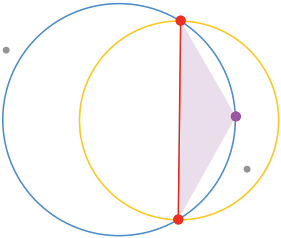

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Lemma

The pairs (Q', Q'') *yield a discrete gradient, inducing a collapse* $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$ *(for any radius r).*

Collapsing non-Delaunay simplices

Construct gradient pairs for the collapse $\mathrm{Cech}_r \searrow \mathrm{DelCech}_r$:

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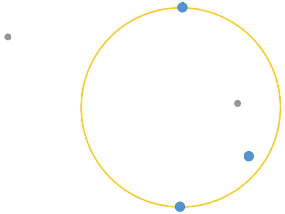

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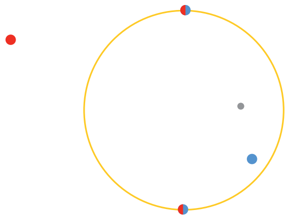

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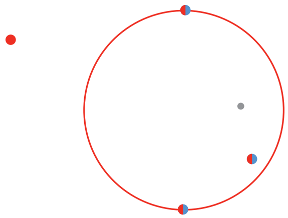

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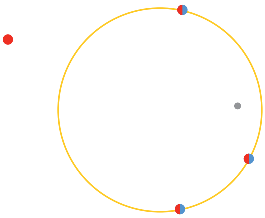

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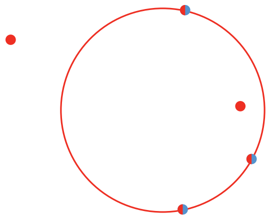

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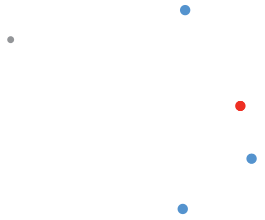

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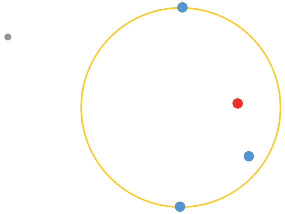

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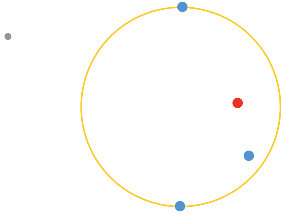

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Generalize and simplify the surface reconstruction method

Wrap (Edelsbrunner 1995, Geomagic)

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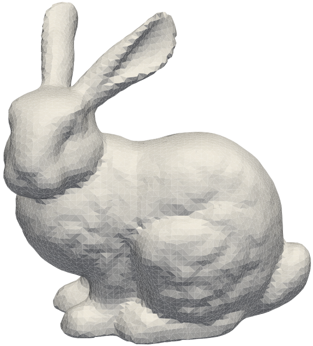


Wrap complexes

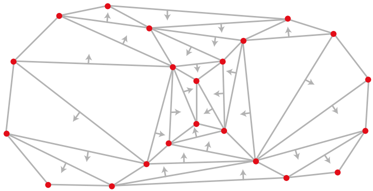
Generalize and simplify the surface reconstruction method *Wrap* (Edelsbrunner 1995, Geomagic)

"It would be interesting to elucidate the connection between the two approaches to a discrete Morse theory."

Herbert Edelsbrunner, 2002

Wrap complexes

Consider the Delaunay gradient V_D of $X \subset \mathbb{R}^D$ (intervals $[\mathrm{Front}\, S, \mathrm{On}\, S]$ of simplices with same Delaunay sphere S).



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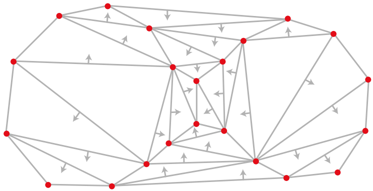

Define $\mathrm{Wrap}_r(X)$ as the smallest complex

- consisting of intervals of V_D and
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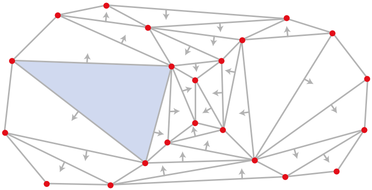

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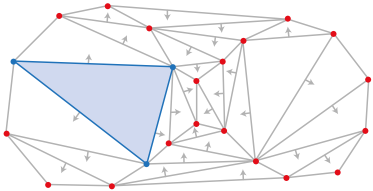

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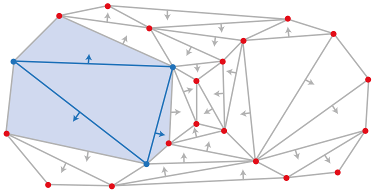

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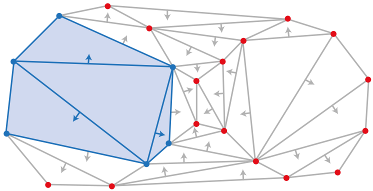

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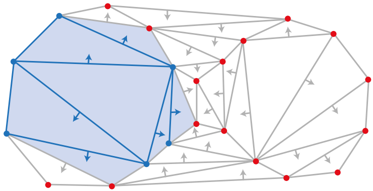

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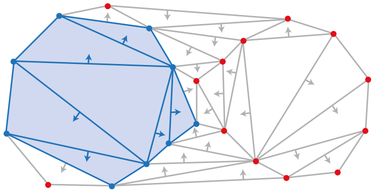

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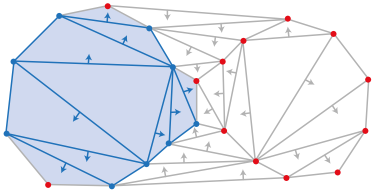

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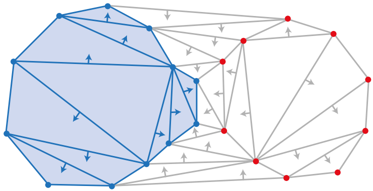

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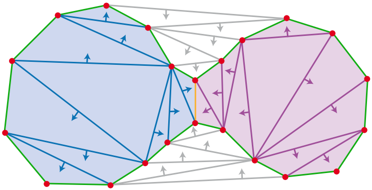

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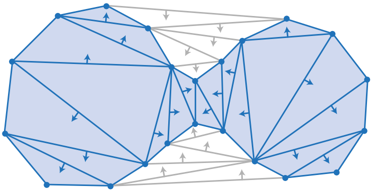

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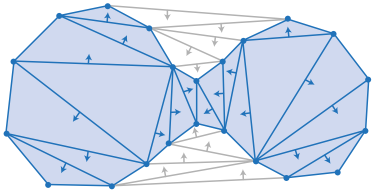
- containing all critical simplices with circumradius $\leq r$.

Wrap complexes

Consider the Delaunay gradient V_D of $X \subset \mathbb{R}^D$ (intervals $[\mathrm{Front}\, S, \mathrm{On}\, S]$ of simplices with same Delaunay sphere S).



Define $\mathrm{Wrap}_r(X)$ as the smallest complex

- consisting of intervals of V_D and
- containing all critical simplices with circumradius $\leq r$.

Corollary

The Delaunay intervals induce a collapse $\mathrm{Del}_r \searrow \mathrm{Wrap}_r$.

Wrapping up

What we learned in this talk

- Čech and Delaunay complexes from Morse functions
- Explicit homotopy equivalence by simplicial collapses
- Simple definition and generalization of *Wrap* complexes