

# The Morse theory of Čech and Delaunay filtrations

## Colloquium

SFB *Discretization in Geometry and Dynamics*

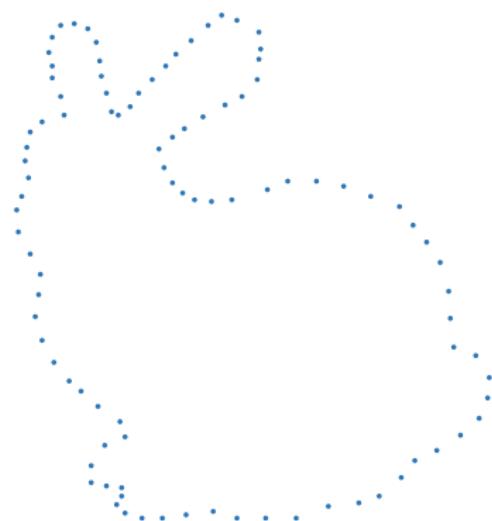
*Ulrich Bauer*<sup>1,2</sup>   *Herbert Edelsbrunner*<sup>2</sup>

<sup>1</sup>TU München

<sup>2</sup>IST Austria

November 18, 2014

# Connect the dots: topology from geometry



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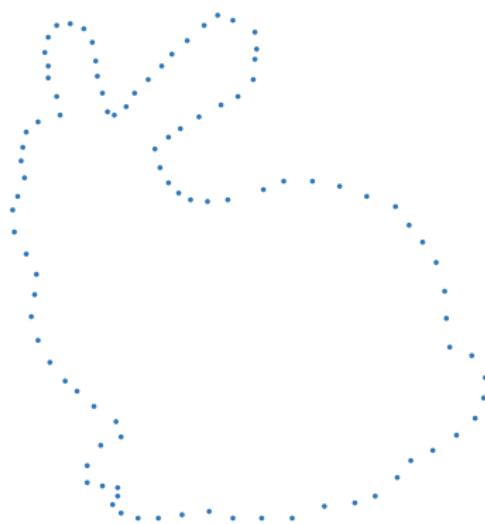


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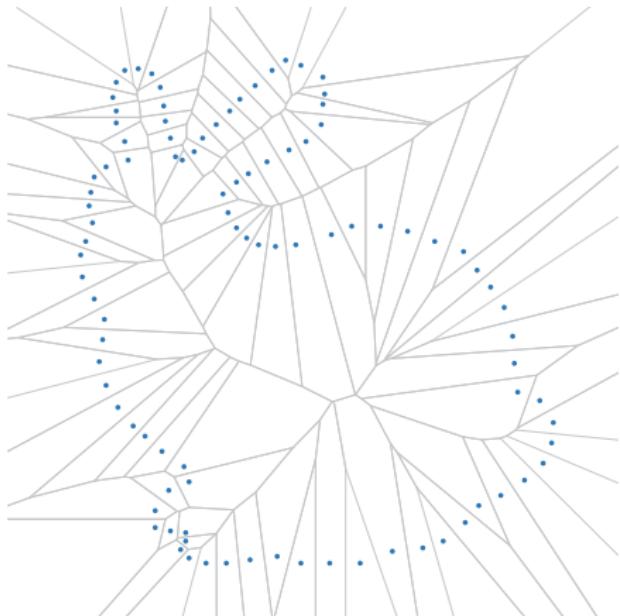
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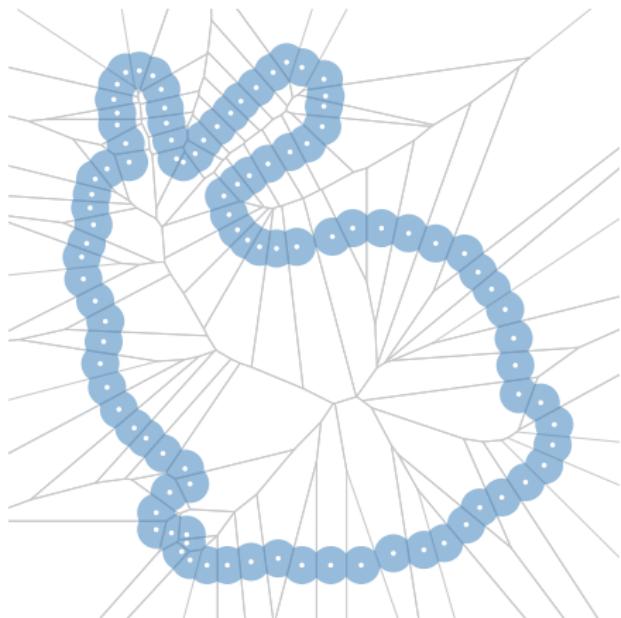
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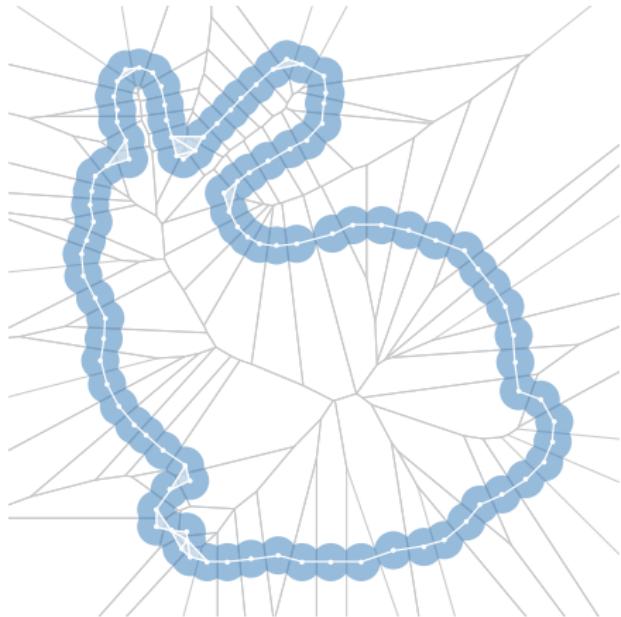
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# Čech and Delaunay functions

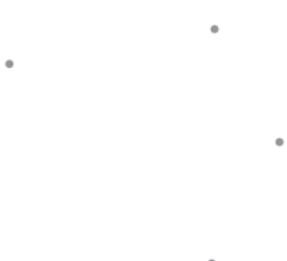
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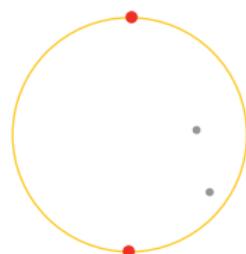
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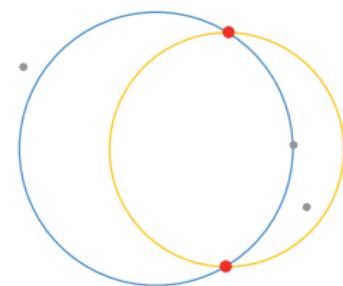
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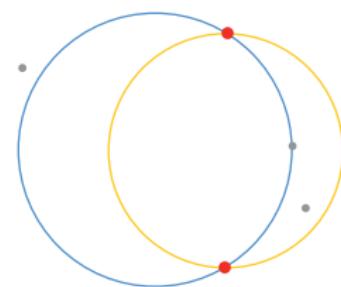
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- defined only if  $Q$  has an empty circumsphere:  $Q \in \text{Del}(X)$

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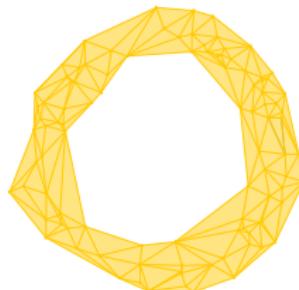
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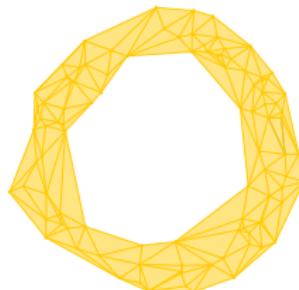
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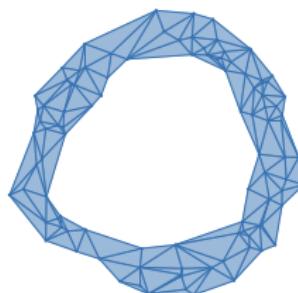
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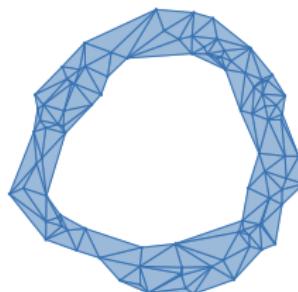
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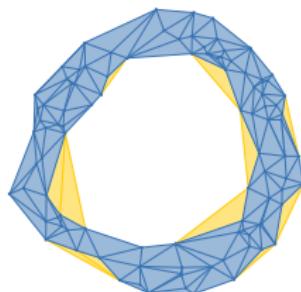
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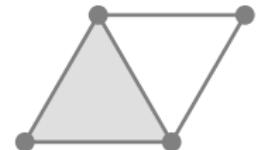
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- Are all three complexes homotopy equivalent?
- Are they related by a sequence of simplicial collapses?

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Definition (Whitehead 1938)

Let  $K$  be a simplicial complex.

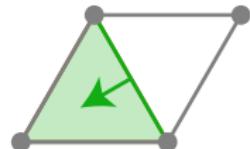


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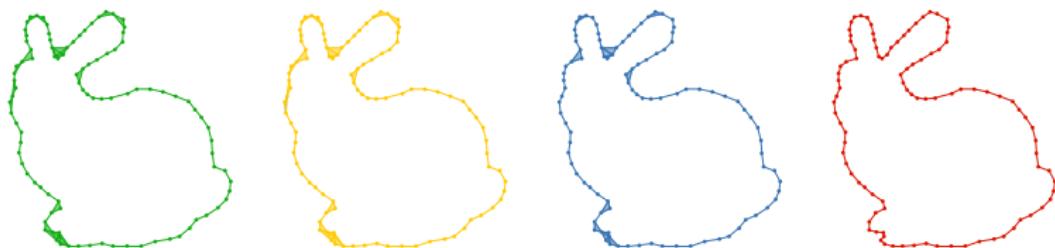
If there is a sequence of such elementary collapses from  $K$  to  $M$ , we say that  $K$  *collapses* to  $M$  (written as  $K \searrow M$ ).

# Main result: a sequence of collapses

Theorem (B, Edelsbrunner 2014)

*Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent. In particular,*

$$\text{Cech}_r \searrow \text{DelCech}_r \searrow \text{Del}_r \searrow \text{Wrap}_r.$$

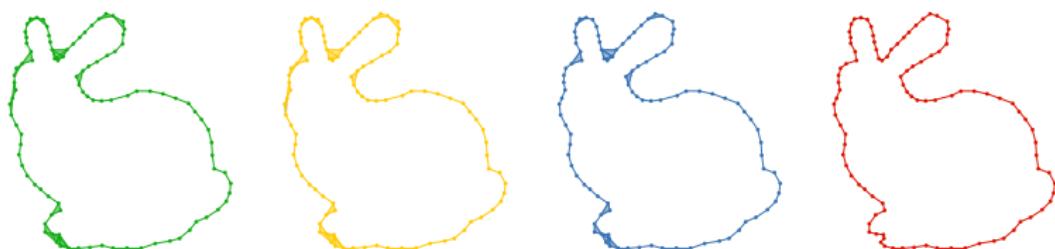


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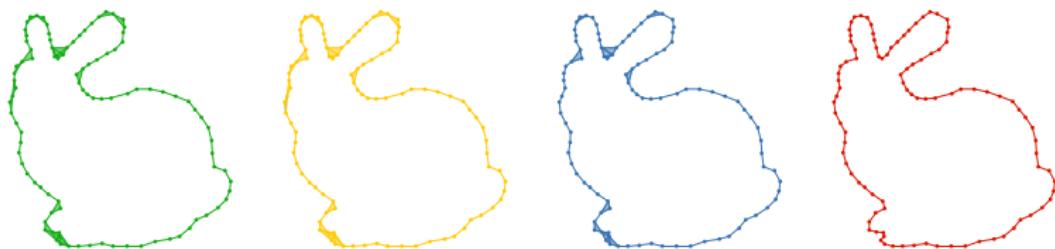
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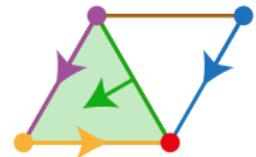


- All collapses are induced by a single *discrete gradient field*
- Also works for weighted point sets

# Discrete Morse theory

## Definition (Forman 1998)

A *discrete vector field* on a simplicial complex is a partition of the simplices into singletons and pairs  $\{L, U\}$ , where  $L$  is a face of  $U$  with codimension 1.

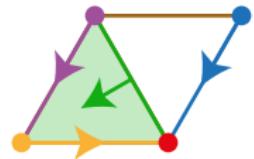


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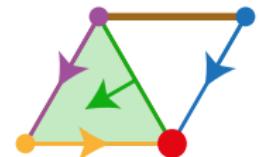


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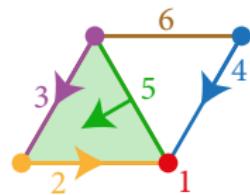
The singletons are called *critical simplices*.

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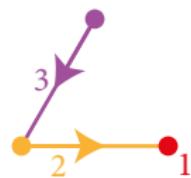


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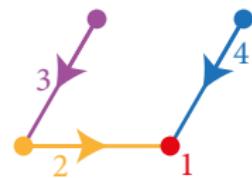


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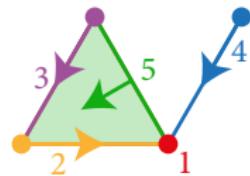


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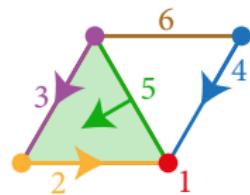


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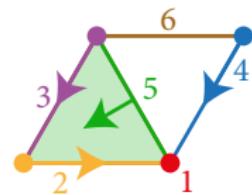


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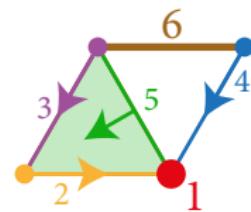


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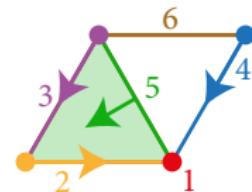
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If  $f^{-1}(t) = \{Q\}$  then  $t$  is a *critical value*.

# Collapses from Morse functions and gradients

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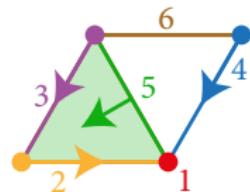


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**Theorem (Forman 1998)**

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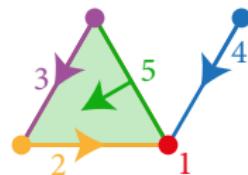


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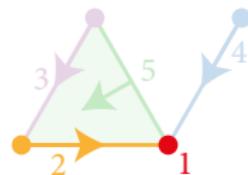


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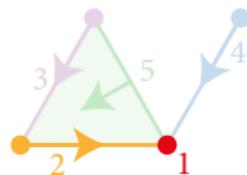


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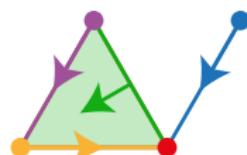
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Let  $V$  be a discrete gradient field on a simplicial complex  $K$ ,  
and let  $L$  be a subcomplex of  $K$ .

**Corollary**

*If  $K \setminus L$  is the union of some pairs of  $V$ ,  
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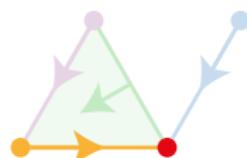
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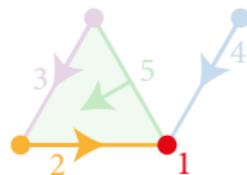


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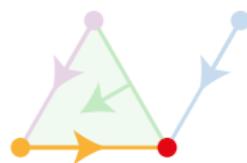
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then  $K_t \searrow K_s$ .*



Let  $V$  be a discrete gradient field on a simplicial complex  $K$ ,  
and let  $L$  be a subcomplex of  $K$ .

**Corollary**

*If  $K \setminus L$  is the union of some pairs of  $V$ ,  
then  $K \searrow L$ .*



We say that  $V$  induces the collapse  $K \searrow L$ .

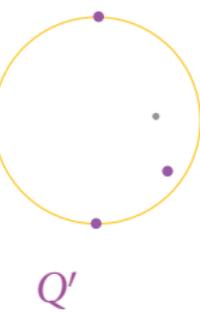
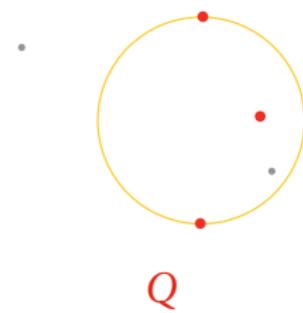
## Unfortunately...

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- Example: two simplices  $Q, Q'$  with  $f_C(Q) = f_C(Q')$  such that neither is a face of the other:

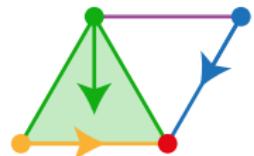


# Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}$$

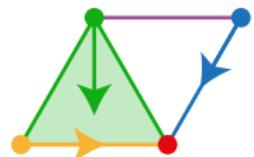


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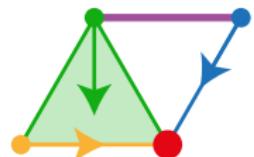
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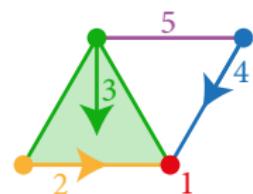
The singletons are called *critical simplices*.

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## Definition

A function  $f : K \rightarrow \mathbb{R}$  on a simplicial complex is a *generalized discrete Morse function* if for  $t \in \mathbb{R}$ :

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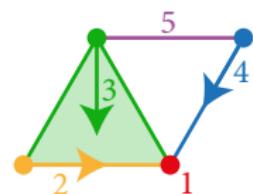


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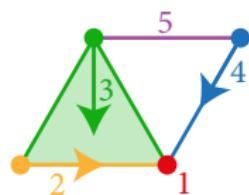


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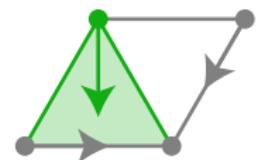
- the sublevel sets  $K_t = f^{-1}(-\infty, t]$  are subcomplexes
- the level sets  $f^{-1}(t)$  form a generalized vector field (the *discrete gradient* of  $f$ )



## Refining generalized vector fields

A generalized vector field  $V$  can be refined to a vector field.

For each non-critical interval  $[L, U] \in V$ :

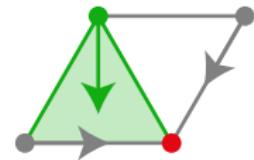


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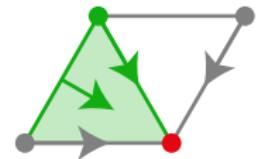


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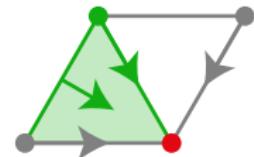


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Therefore the collapsing theorems also hold for generalized discrete Morse functions.

# Morse theory of Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

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- *$Q$  is a critical simplex of  $f_D$*

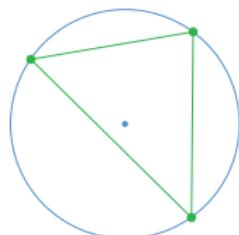
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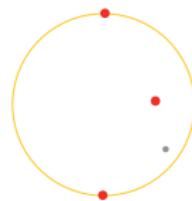
- $f_D(Q) = f_C(Q)$
- $Q$  is a critical simplex of  $f_C$
- $Q$  is a critical simplex of  $f_D$
- $Q$  is a centered Delaunay simplex  
*(containing the circumcenter in the interior)*



# Čech intervals

## Lemma

Let  $Q \subseteq X$  be a simplex with smallest enclosing sphere  $S$ .  
Then  $Q' \subseteq X$  has the same smallest enclosing sphere  $S$  iff



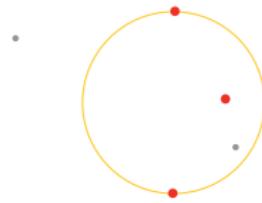
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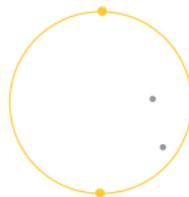
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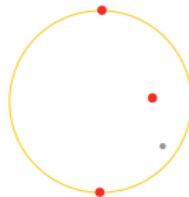
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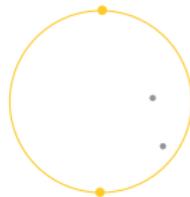
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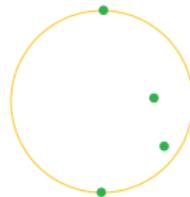
on  $S \subseteq Q' \subseteq \text{encl } S$ .



$Q$



on  $S$



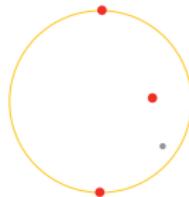
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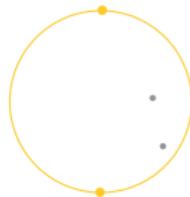
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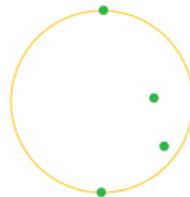
$$Q' \in [\text{on } S, \text{encl } S].$$



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## The front face of a simplex

Let on  $S$  denote the points of  $X$  on the sphere  $S$ .  
Write the center  $z$  of  $S$  as an affine combination

$$z = \sum_{x \in T} \mu_x x, \quad 1 = \sum_{x \in T} \mu_x.$$

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We define

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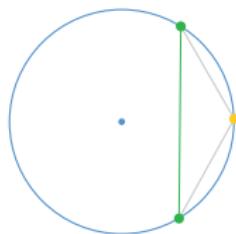
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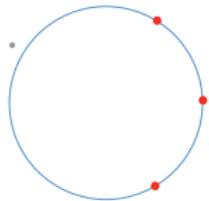


# Delaunay intervals

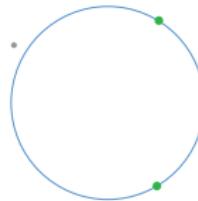
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Let  $Q \subseteq X$  be a simplex with smallest empty circumsphere  $S$ . Then  $Q' \subseteq X$  has the same smallest empty circumsphere  $S$  iff

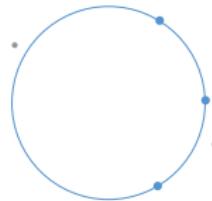
$$Q' \in [\text{front } S, \text{on } S].$$



$Q$



front  $S$



on  $S$

## Sphere optimization problems

Both Čech and Delaunay functions are defined using the smallest sphere  $S$  satisfying certain constraints:

minimize  
 $r, z$

$r$

subject to

$$\|z - q\| \leq r, \quad q \in Q,$$

$$\|z - e\| \geq r, \quad e \in E.$$

Here  $r$  is the radius and  $z$  is the center of the sphere  $S$ .

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# The Karush–Kuhn–Tucker optimality conditions

Consider an optimization problem of the form

minimize<sub>x</sub>

$$f(x)$$

subject to

$$g_i(x) \leq 0, \quad i \in I,$$

$$h_j(x) = 0, \quad j \in J,$$

where the function  $f$  is convex and  $g_i, h_j$  are affine.

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where the function  $f$  is convex and  $g_i, h_j$  are affine.

**Theorem (Karush 1939, Kuhn–Tucker 1951)**

A feasible point  $x$  to the above problem is an optimal solution iff there exist Lagrange multipliers  $(\nu_i)_{i \in I}$  and  $(\lambda_j)_{j \in J}$  such that

$$\begin{aligned}\nabla f(x) + \sum_{i \in I} \nu_i \nabla g_i(x) + \sum_{j \in J} \lambda_j \nabla h_j(x) &= 0, \\ \nu_i g_i(x) &= 0, \quad i \in I, \\ \lambda_j &\geq 0, \quad j \in J.\end{aligned}$$

# KKT conditions for the smallest sphere problem

The KKT conditions for our sphere optimization problem are:

## Proposition

*A sphere  $S$  enclosing  $Q$  and excluding  $E$  is minimal iff its center  $z$  can be written as an affine combination*

$$z = \sum_{x \in Q \cup E} \mu_x x, \quad 1 = \sum_{x \in Q \cup E} \mu_x$$

*such that*

- $\mu_x \geq 0$  for  $x \notin E$ ,
- $\mu_x \leq 0$  for  $x \notin Q$ , and
- $\mu_x = 0$  for  $x \in S$ .

# Čech and Delaunay intervals from KKT

## Proposition

*A sphere  $S$  enclosing  $Q$  and excluding  $E$  is minimal iff*

$$z \in \text{aff}(\text{on } S),$$

$$Q \in [\text{front } S, \text{encl } S], \text{ and}$$

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# $\check{\text{C}}\text{ech}$ and Delaunay intervals from KKT

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## Corollary

*The  $\check{\text{C}}\text{ech}$  intervals are of the form  $[\text{on } S, \text{encl } S]$ .*

*The Delaunay intervals are of the form  $[\text{front } S, \text{on } S]$ .*

## Selective Delaunay complexes

Define for finite point sets  $X, E \subset \mathbb{R}^d$ :

- *E-Delaunay function*  $f_E(Q)$ :  
radius of smallest sphere enclosing  $Q$  and excluding  $E$

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## Theorem (B, Edelsbrunner 2014)

Let  $E \subseteq E'$ . Then

$$\text{Del}_r(X, E) \downarrow \text{Del}_r(X, E) \cap \text{Del}(X, E') \downarrow \text{Del}_r(X, E').$$

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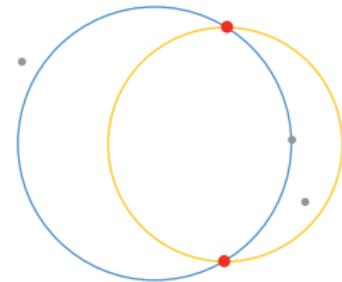
$$\text{Del}_r(X, E) \downarrow \text{Del}_r(X, E) \cap \text{Del}(X, E') \downarrow \text{Del}_r(X, E').$$

Note: choosing  $E = \emptyset$  and  $E' = X$  yields

$$\text{Cech}_r(X) \downarrow \text{DelCech}_r(X) \downarrow \text{Del}_r(X).$$

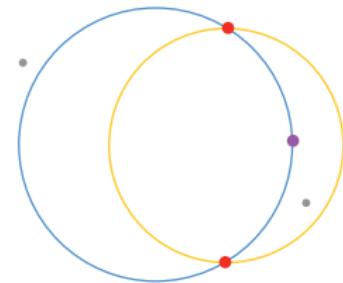
# Collapsing the Delaunay–Čech complex

- Consider a non-critical Delaunay simplex  $Q$



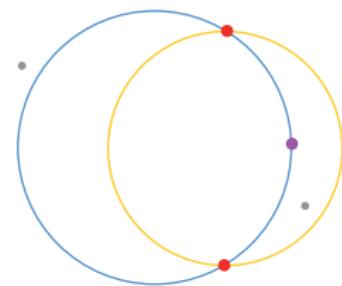
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- Consider a non-critical Delaunay simplex  $Q$
- There is a point  $p$  inside the Čech sphere and on the Delaunay sphere



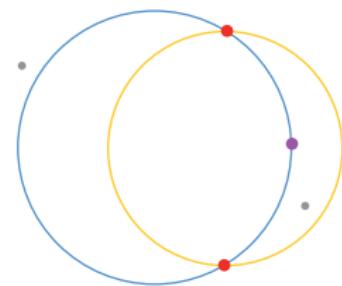
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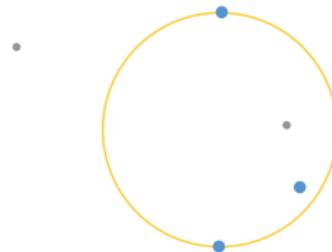


## Lemma

The pairs  $(Q', Q'')$  yield a vector field that induces a collapse  $\text{DelCech}_r \searrow \text{Del}_r$ , for any radius  $r$ .

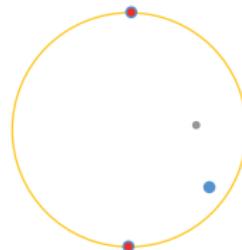
# Collapsing non-Delaunay simplices

- Consider a non-Delaunay simplex  $Q$  together with its Čech sphere  $S$



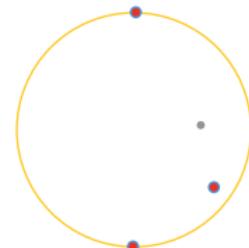
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- Consider a non-Delaunay simplex  $Q$  together with its Čech sphere  $S$
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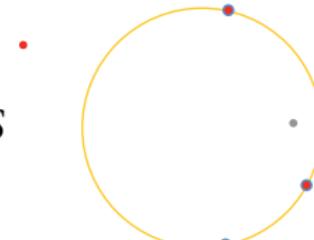
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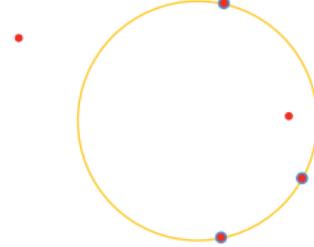
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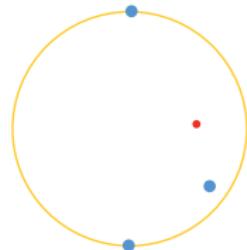
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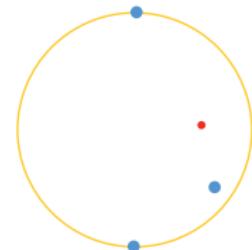
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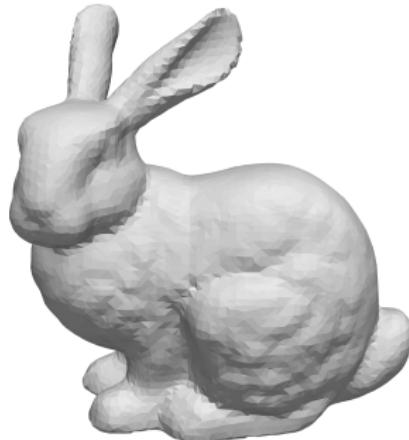


## Lemma

*The pairs  $(Q', Q'')$  yield a vector field that induces a collapse Cech<sub>r</sub> ↴ DelCech<sub>r</sub>, for any radius r.*

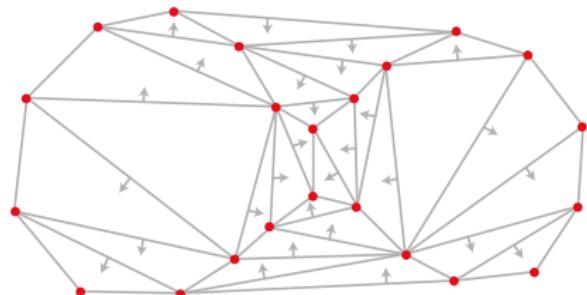
# Wrap complexes

Generalizes and greatly simplifies the surface reconstruction algorithm *Wrap* (Edelsbrunner 1995, Geomagic)



# Wrap complexes

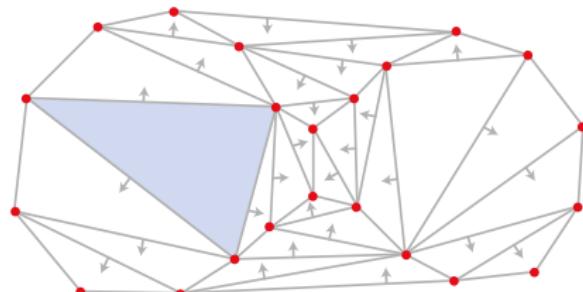
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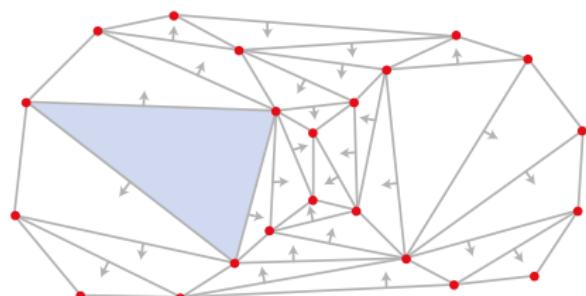
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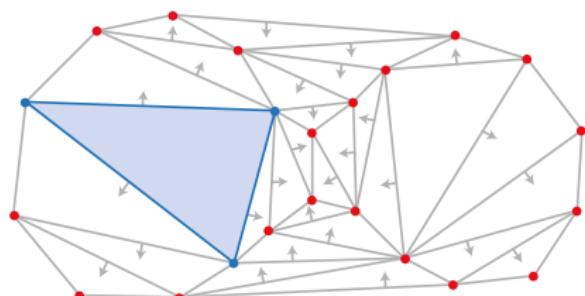
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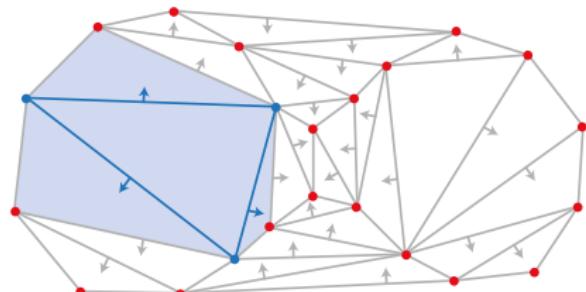
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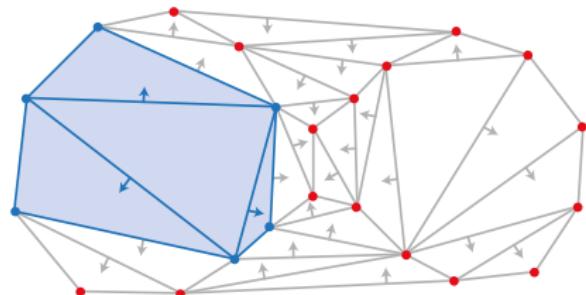
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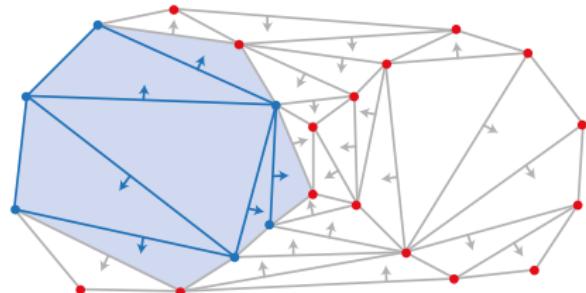
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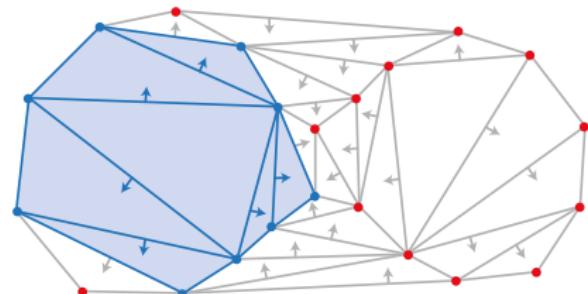
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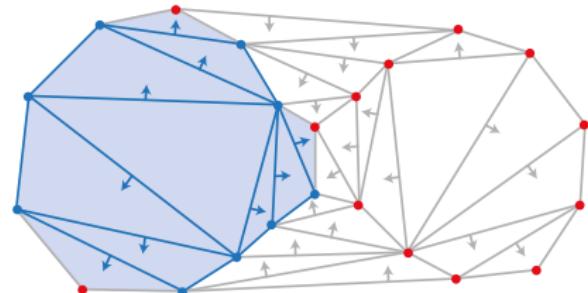
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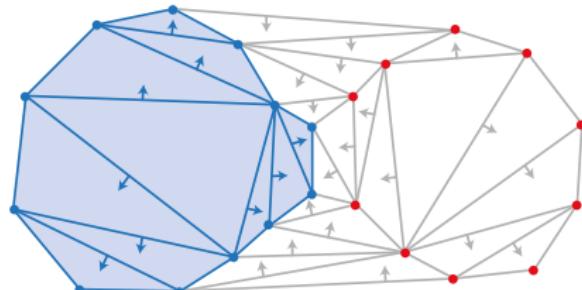
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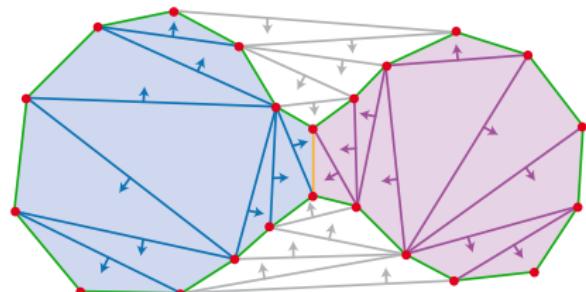
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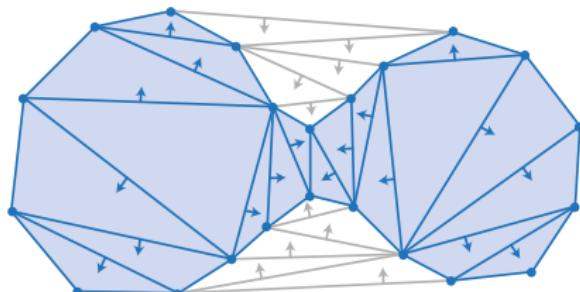
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Define

$$\text{Wrap}_r = \bigcup_{C \in \text{Crit}_r} \downarrow C$$



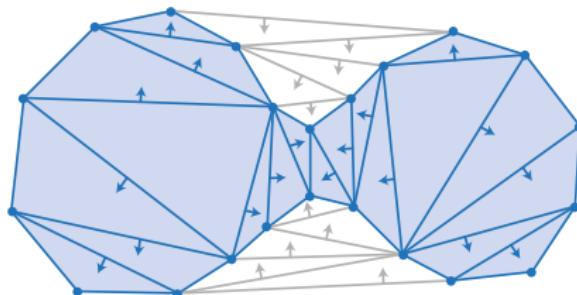
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The Delaunay intervals induce a collapse  $\text{Del}_r \searrow \text{Wrap}_r$ .

# Wrapping up

- Čech and Delaunay complexes from Morse functions
- Explicit construction of simplicial collapses
- Simple definition and generalization of *Wrap* complexes