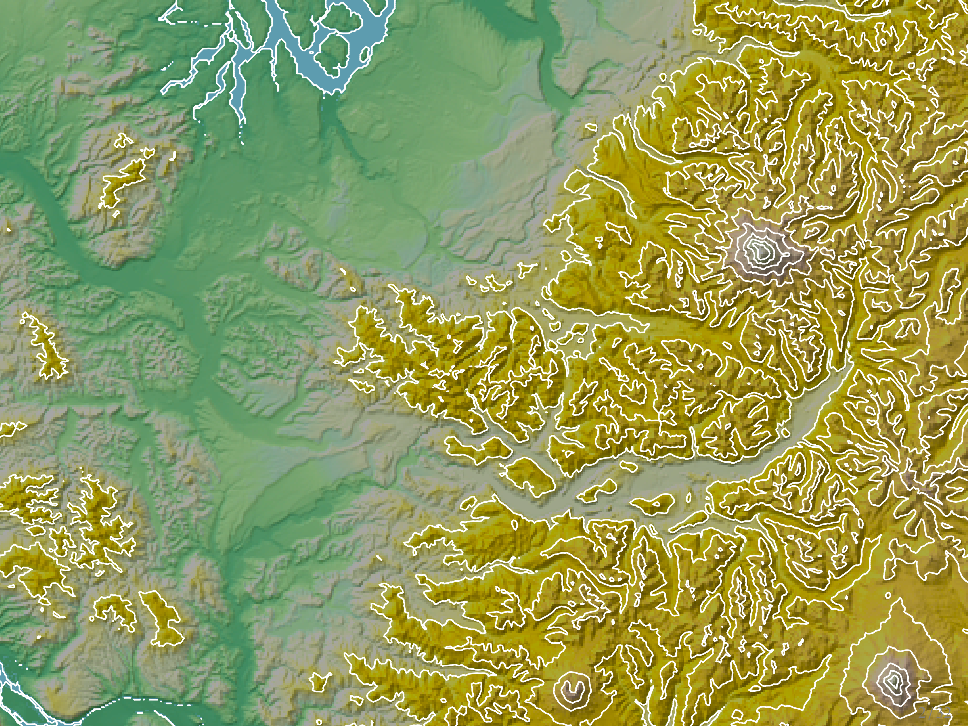
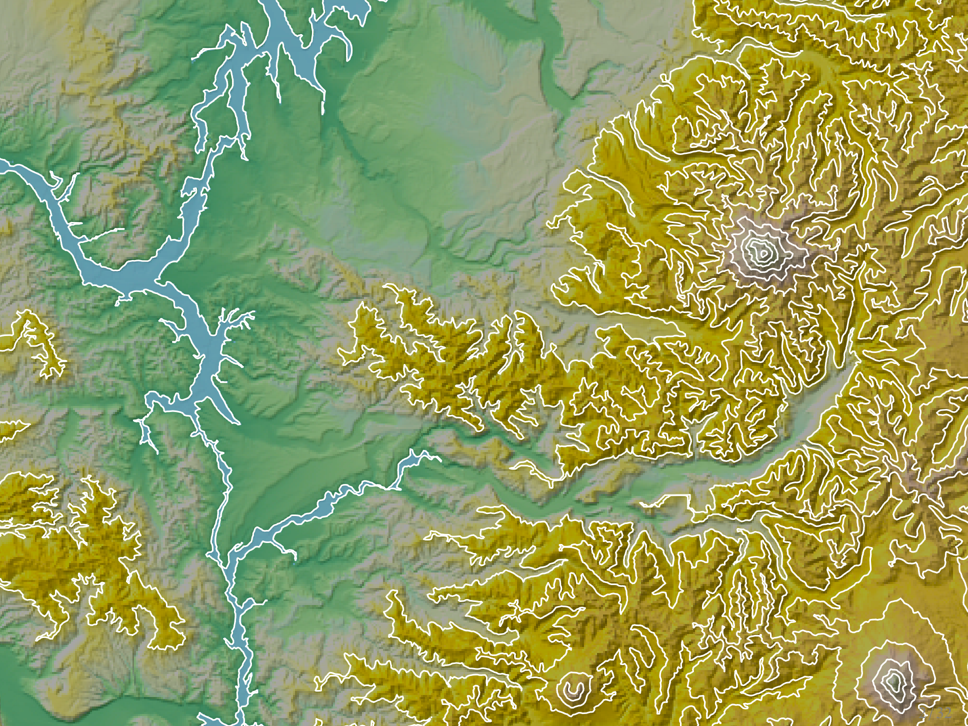
Persistence in discrete Morse theory

Ulrich Bauer

Georg-August-Universität Göttingen

April 14, 2011





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Given a function f on a surface and $\delta > 0$, find a function f_δ that:

- minimizes number of critical points
- stays close to input function: $\|f_\delta - f\|_\infty \leq \delta$

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Using:

- Discrete Morse theory [Forman 1998]
  - provides notion of critical point in the discrete setting
- Homological persistence [Edelsbrunner et al. 2002]
  - quantifies significance of critical points

Regular CW complexes

Think of generalized simplicial complexes
A finite CW complex is constructed inductively:

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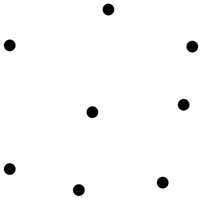

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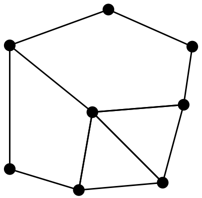

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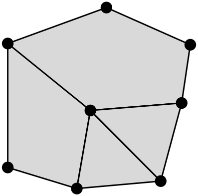

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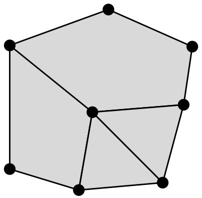
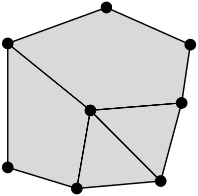


- start with a set of points (0-skeleton)
- construct n-skeleton by attaching n-balls along their boundaries to the $(n-1)$-skeleton
- *regular* CW complex: all attaching maps are topological embeddings

Discrete Morse theory [Forman, 1998]

Finite regular CW complex $\mathcal{K}$



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- Discrete vector field:
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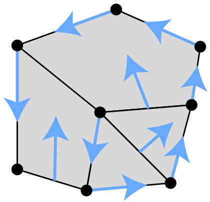

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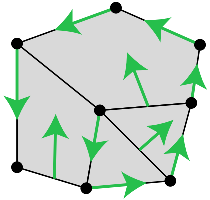

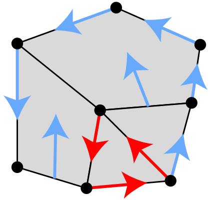

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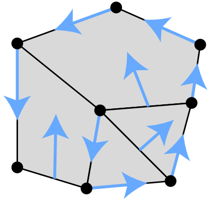

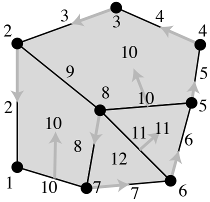

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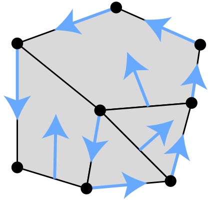

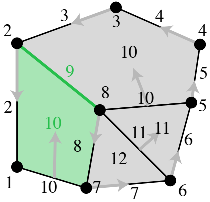

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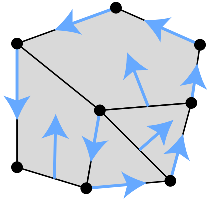

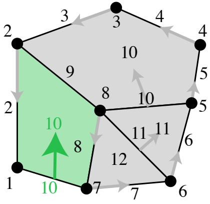

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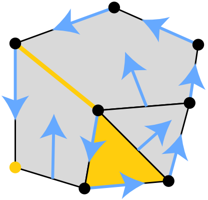

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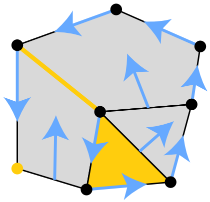

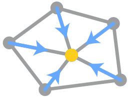

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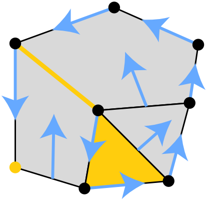

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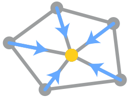

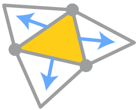

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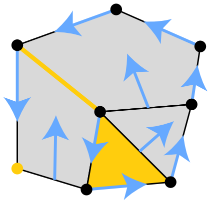

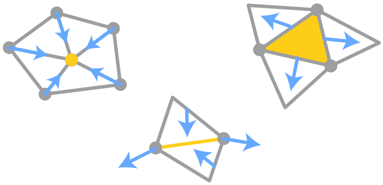

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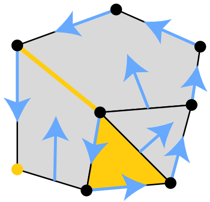

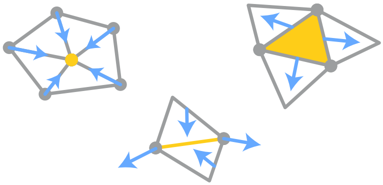

  - No multi-saddle by definition!

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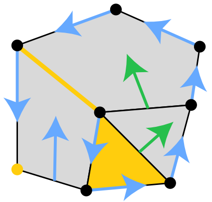
  - Prerequisite: unique path between two critical cells



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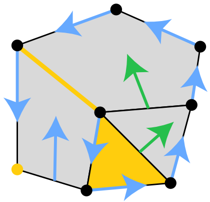

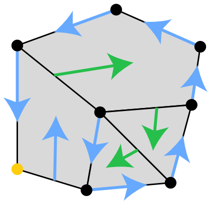

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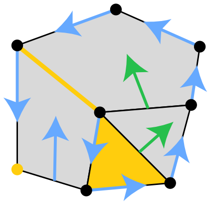
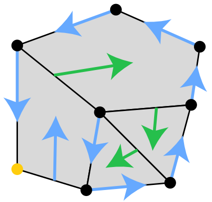
    (of gradient vector fields, *not functions*)





The Morse complex of a gradient vector field

A gradient vector field V on $\mathcal{K}$ defines a *Morse complex* $\mathcal{M}_V$ (homotopy equivalent to $\mathcal{K}$)

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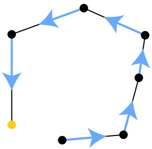
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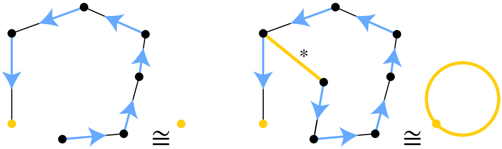

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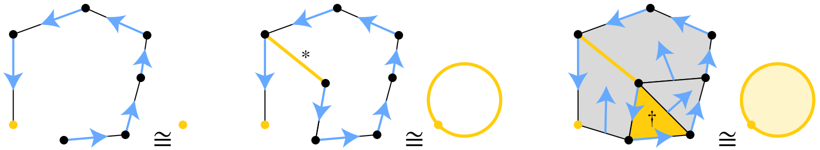

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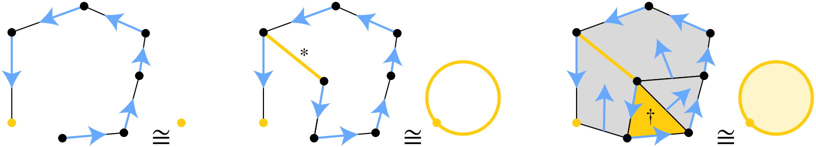
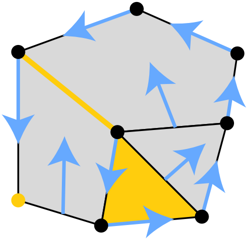


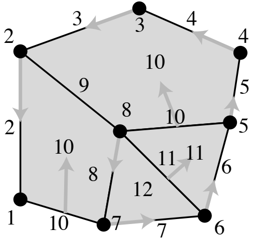
In general, the Morse complex is not regular

- definitions can be extended to non-regular CW complexes

Level subcomplexes

Level subcomplex: union of all closed cells below a certain value





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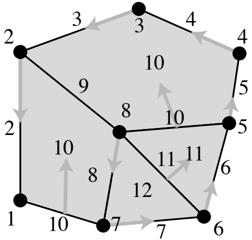

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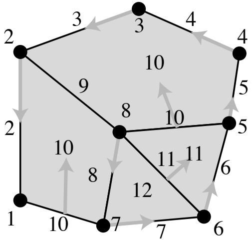

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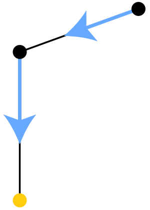

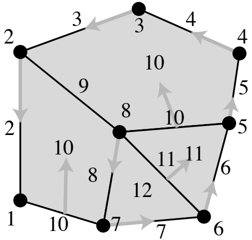

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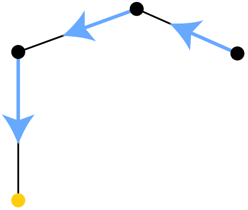

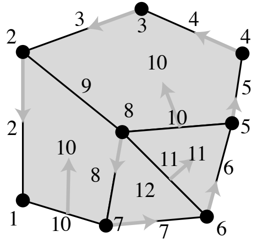

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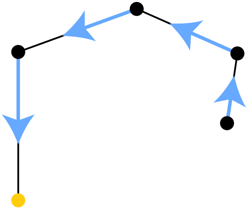

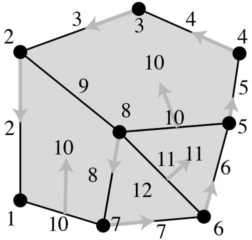

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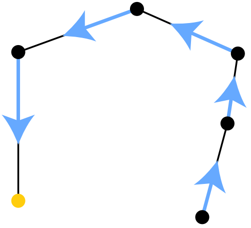

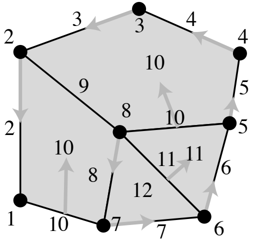

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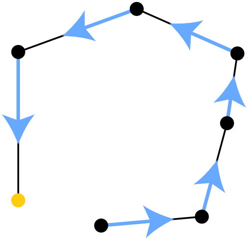

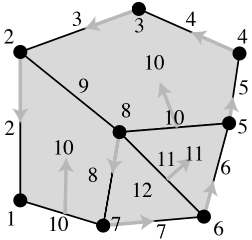

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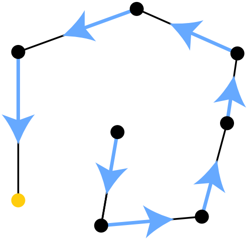

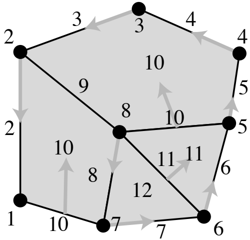

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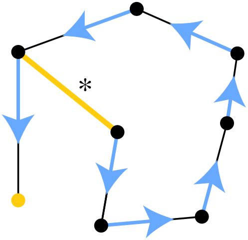

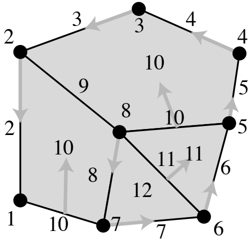

Homotopy type (and hence homology) changes only at critical cells

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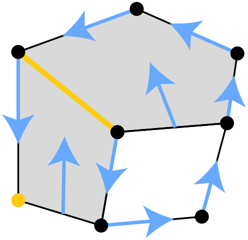

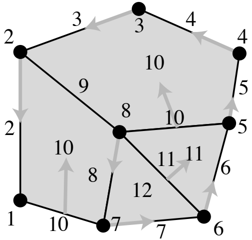

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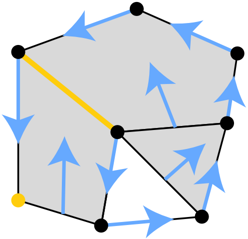

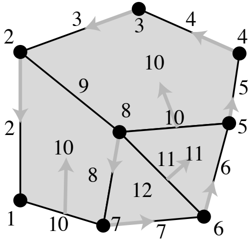

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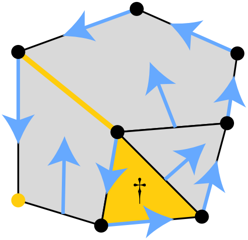

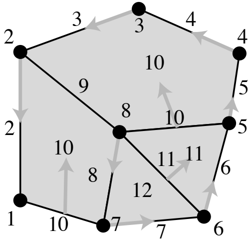

Homotopy type (and hence homology) changes only at critical cells

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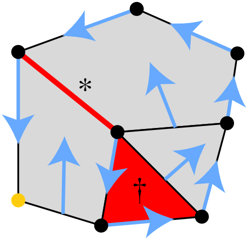

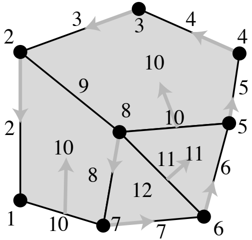

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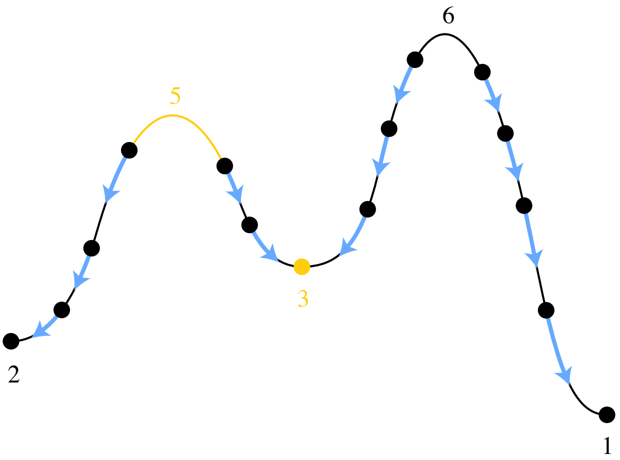
Canceling critical points of a gradient vector field



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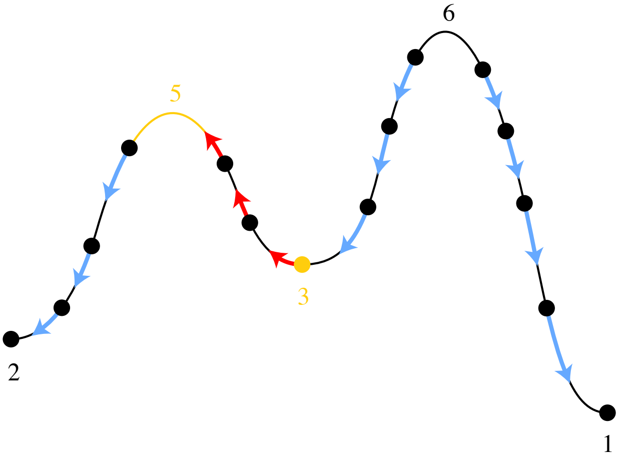
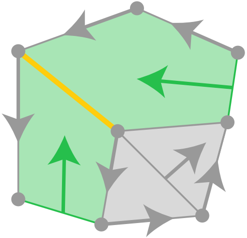


Ascending and descending sets

A gradient vector field V imposes inequalities on values of a consistent function

- *Ascending set* of a cell σ:
  $\{\rho : g(\rho) \geq g(\sigma)$ for *any* g consistent with $V\}$



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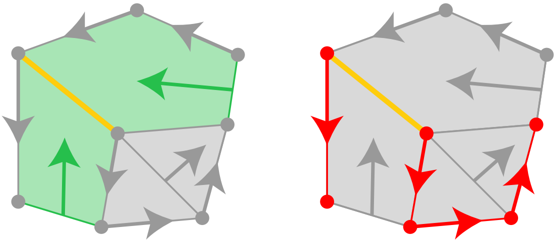

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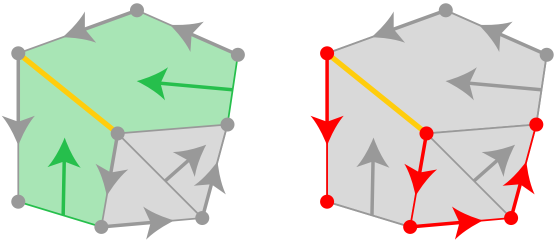
V induces partial order $\prec_V$ on cells: write $\sigma \prec_V \rho$ and $\phi \prec_V \sigma$



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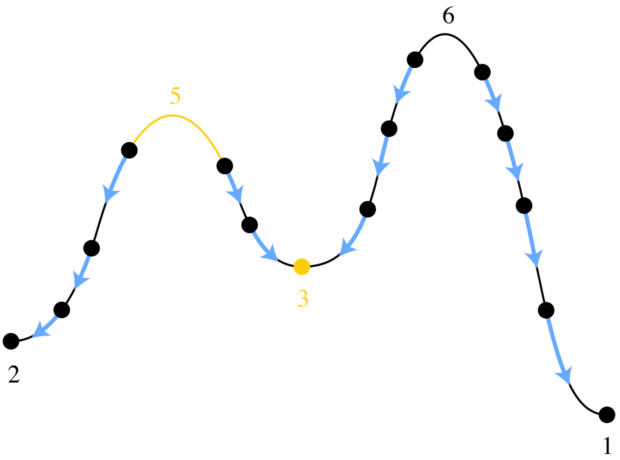

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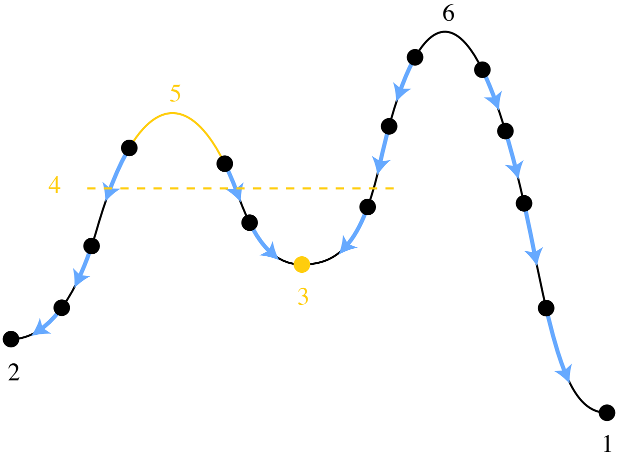

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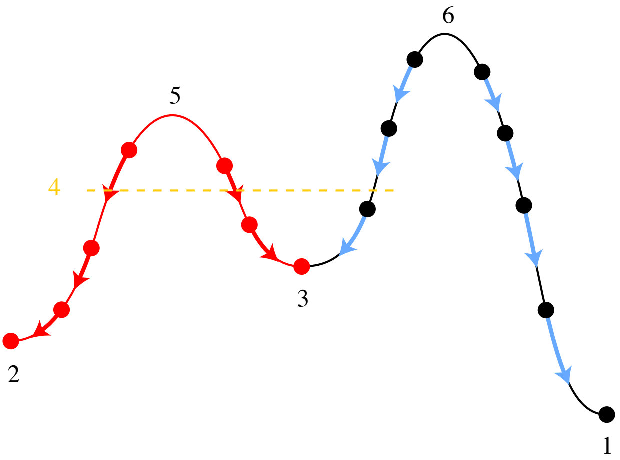

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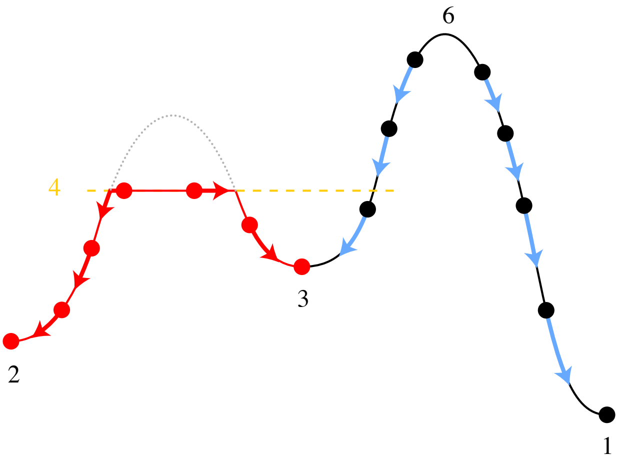

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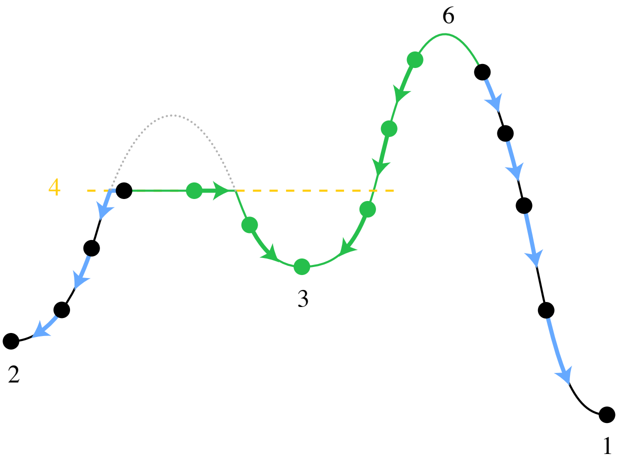

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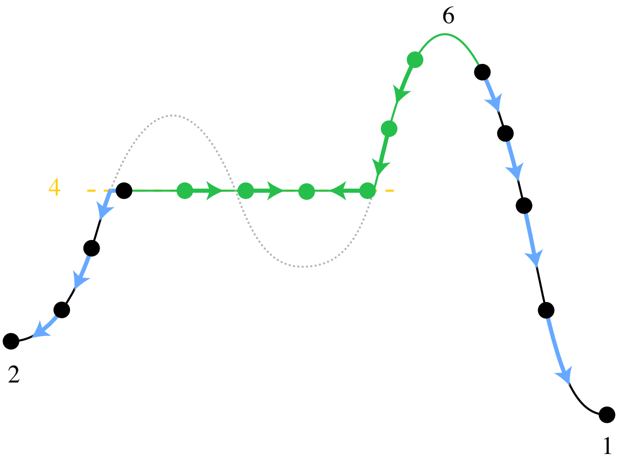

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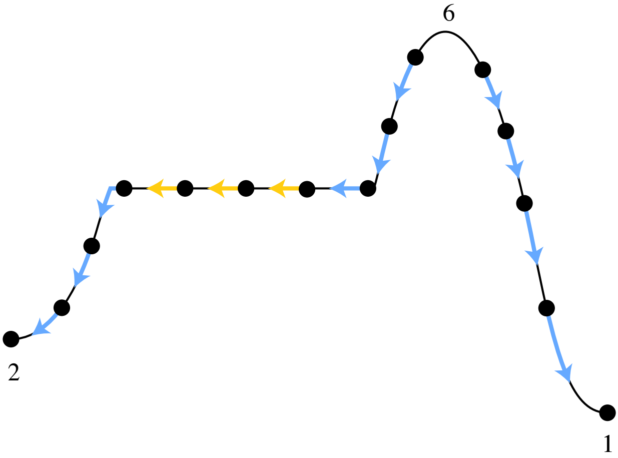
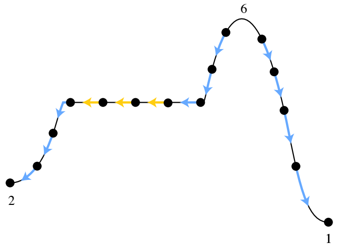


Pseudo-Morse functions



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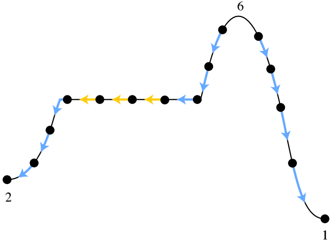

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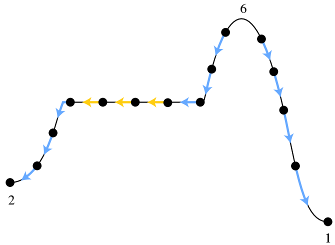

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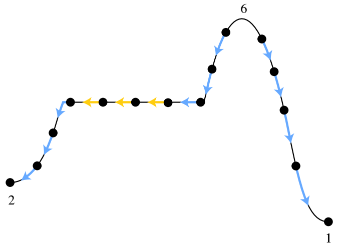

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- Gradient vector field is no longer unique in general

A symbolic perturbation scheme

Lemma

A function f is a pseudo-Morse function consistent with V if and only if for every $\epsilon > 0$ there is a discrete Morse function f_ϵ consistent with V and $\|f_\epsilon - f\|_\infty \leq \epsilon$.

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- can be extended to situations where *unique* critical values are required (choose a consistent *total order*)

Comparing Discrete and PL Morse theory

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We can easily convert one type of function into the other (PL $\to$ pseudo-Morse $\to$ PL: barycentric subdivision)

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- Keep in mind the tolerance

Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for sublevel sets

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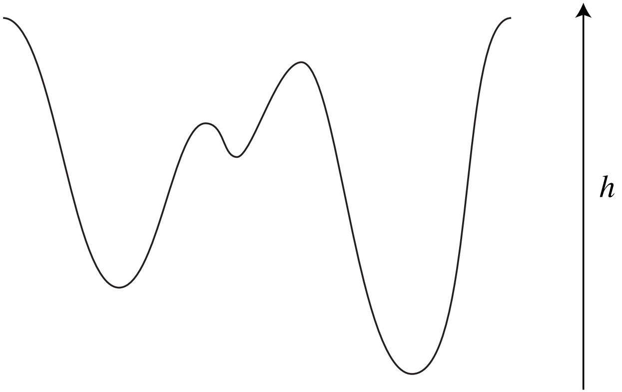
Example: connected components in 1D



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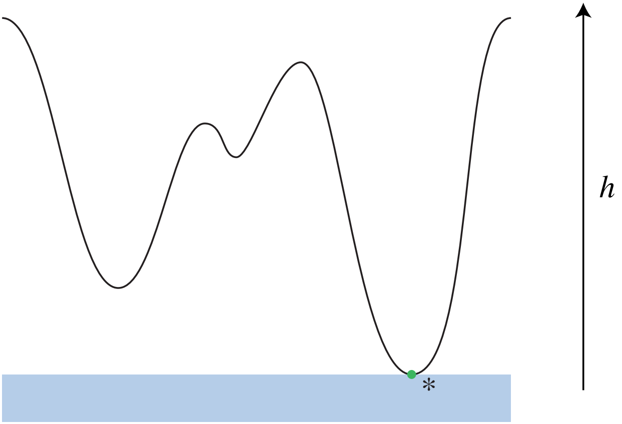

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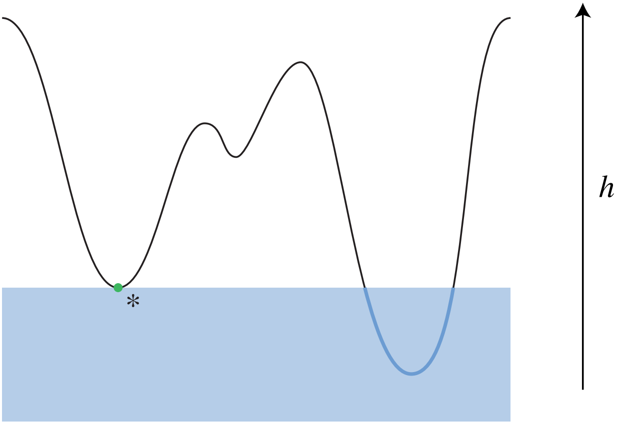

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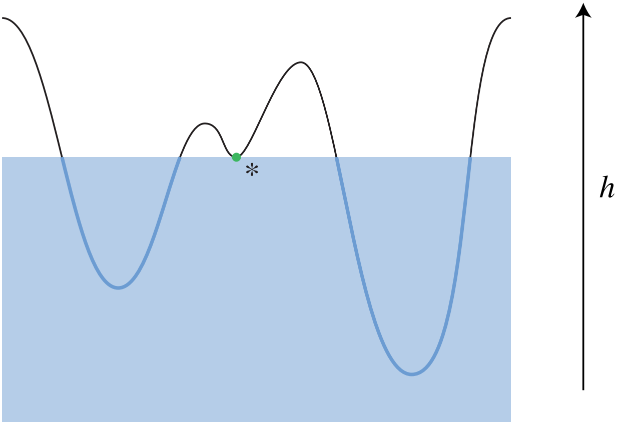

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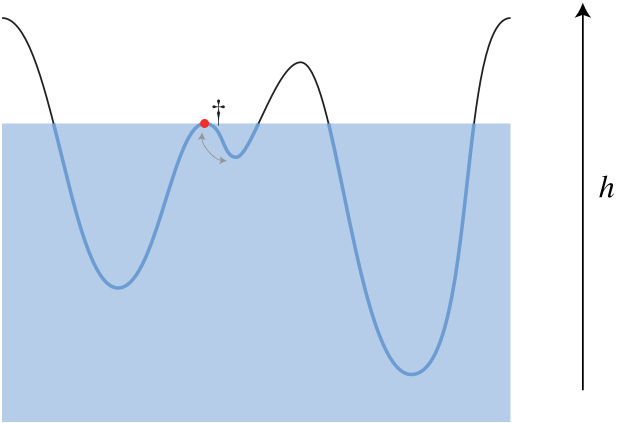

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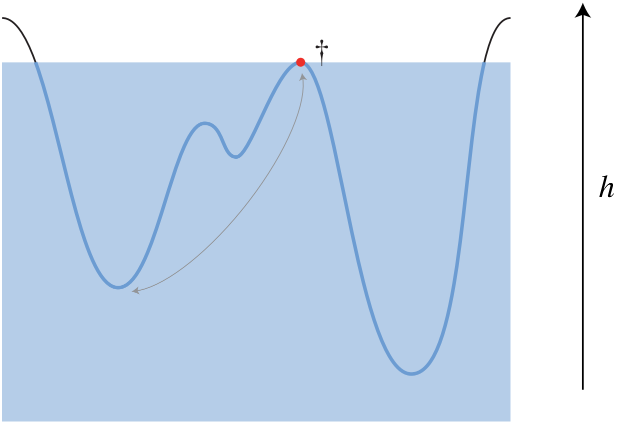
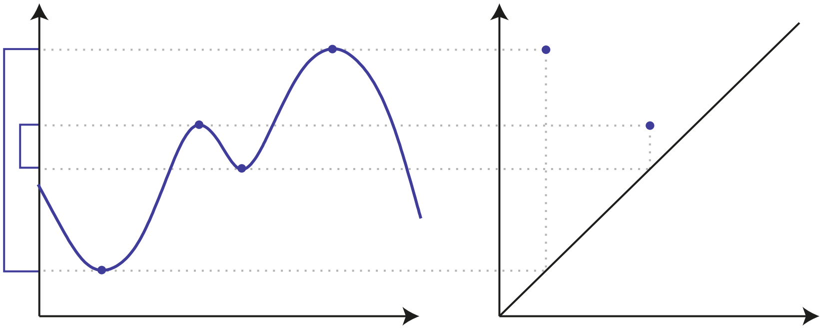


Persistence diagrams [Cohen-Steiner et al., 2005]



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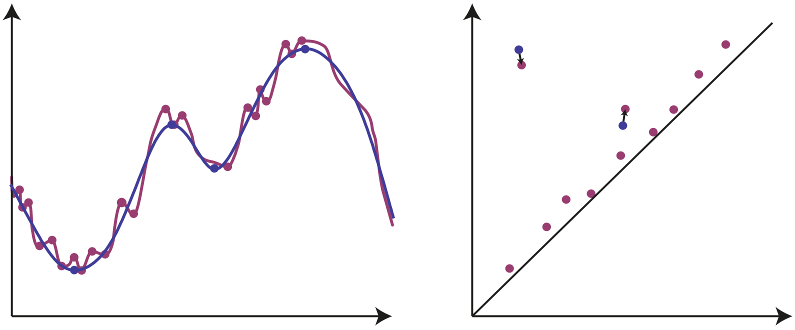
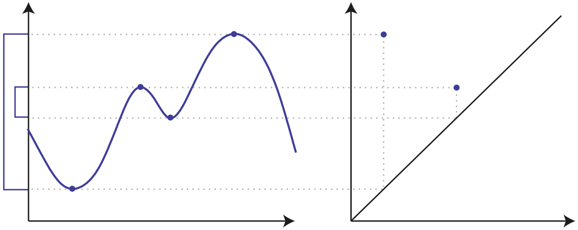


Stability of persistence diagrams

Theorem (Cohen-Steiner et al., 2005)

Let $\|f - g\|_\infty \leq \delta$.
The persistence pairs of f that have persistence $> 2\delta$ can be mapped injectively to the persistence pairs of g such that corresponding points p_f, p_g in the persistence diagrams have distance $\|p_f - p_g\|_\infty \leq \delta$.



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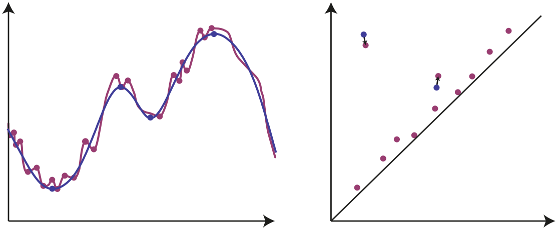

A bound on number of critical points

Corollary

Let f be a discrete Morse function on a surface and let $\delta > 0$.

Then for every function f_δ with $\|f_\delta - f\|_\infty \leq \delta$ we have:

$$\begin{aligned} &\#\text{ critical points of } f_\delta \\ \geq\ &\#\text{ critical points of } f \text{ with persistence} > 2\delta \end{aligned}$$

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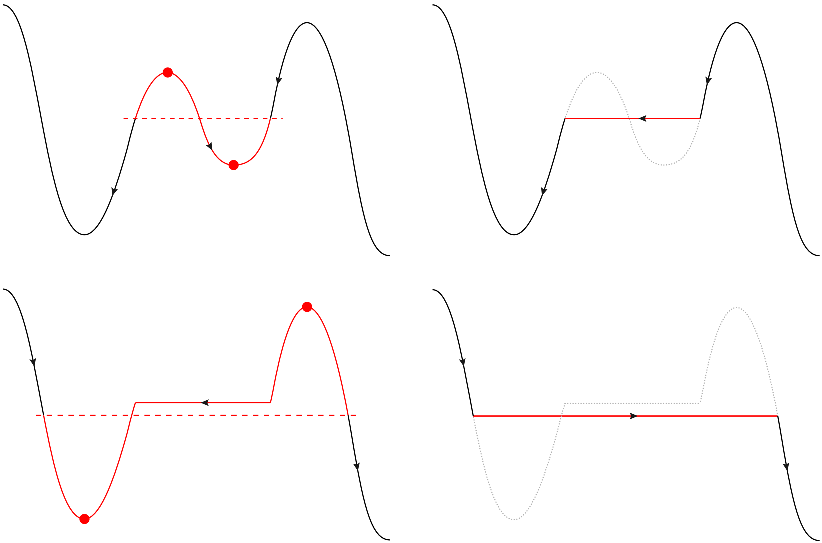
Canceling two nested persistence pairs



Connecting persistence and Morse theory

Theorem (B., Lange, Wardetzky, 2010)

Consider a persistence pair (σ, τ) *of a discrete Morse function on a surface. Then* (σ, τ) *can be canceled after all persistence pairs* $(\tilde{\sigma}, \tilde{\tau})$ *with* $f(\sigma) < f(\tilde{\sigma}) < f(\tilde{\tau}) < f(\tau)$ *have been canceled:*
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What about the tolerance? Does the error accumulate?

Tightness of the stability bound

Theorem (B., Lange, Wardetzky, 2010)

Let f be a pseudo-Morse function on a surface and let $\delta > 0$.

Then a nested cancelation sequence yields a function f_δ with:

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This function achieves the minimal number of critical points.

Proof of the tightness theorem

Lemma

Let V_i be a vector field obtained from nested persistence cancelation and let (σ,τ) be a remaining persistence pair. Then for each cell ρ,

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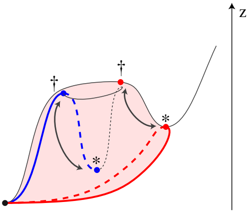
Need to consider interplay between dimensions!

An example in 2D



input

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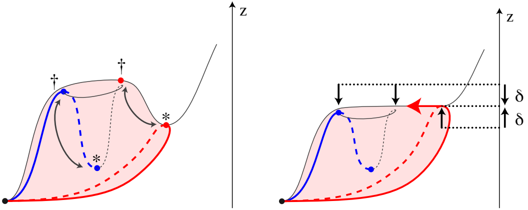

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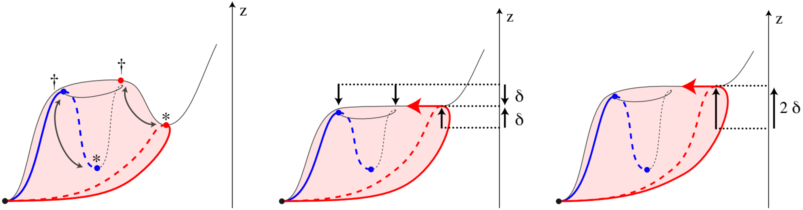

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Proof of correctness requires previous theorem!

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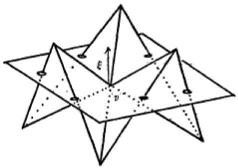

Difficulty comes from multi-saddles

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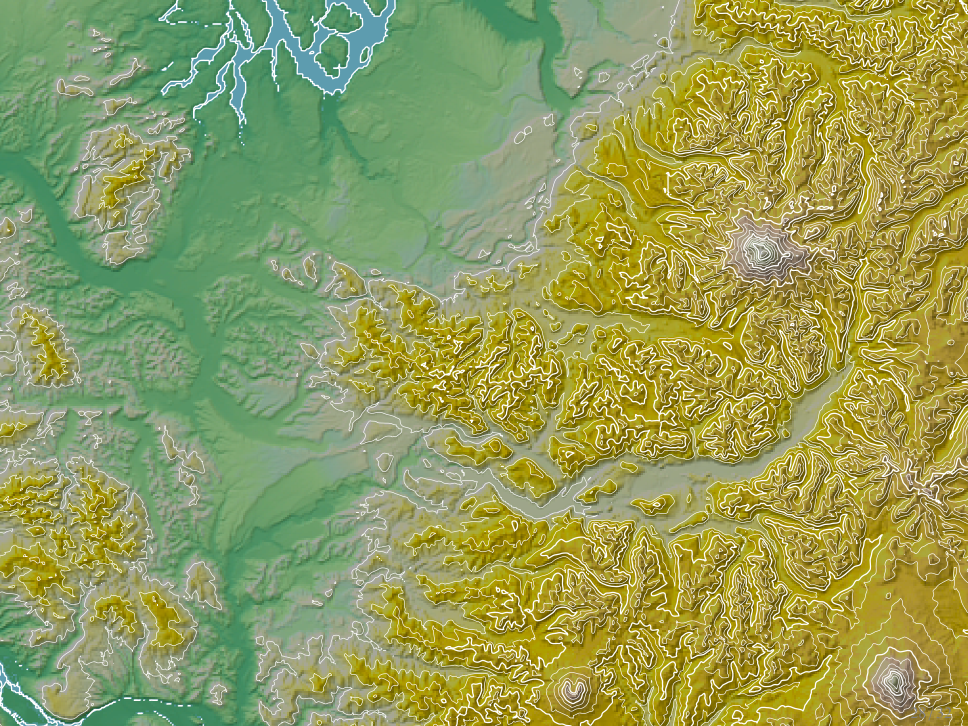
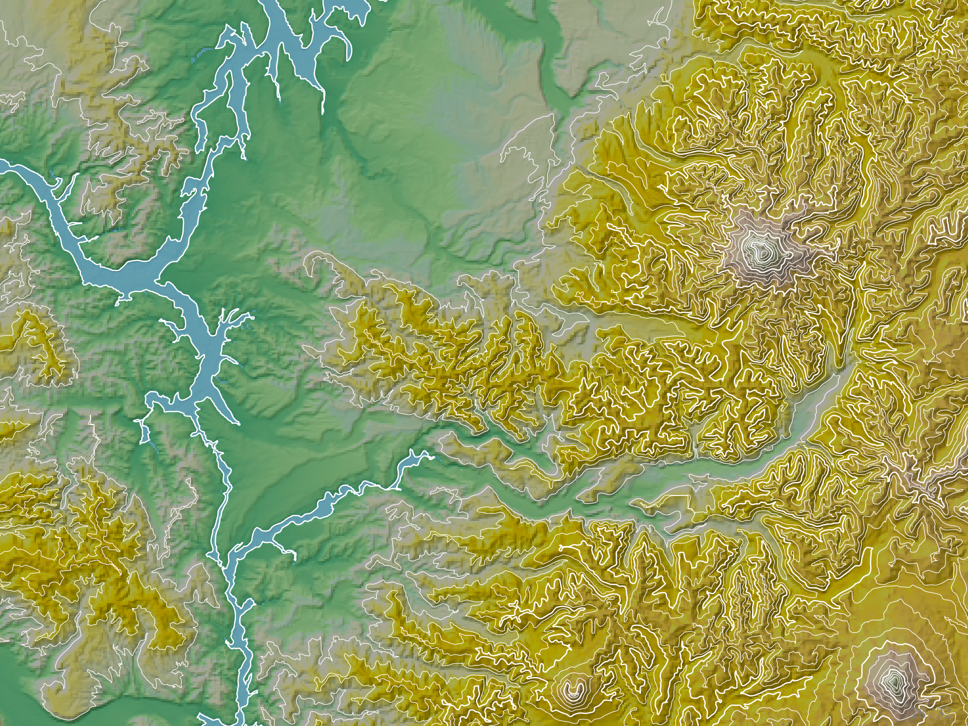
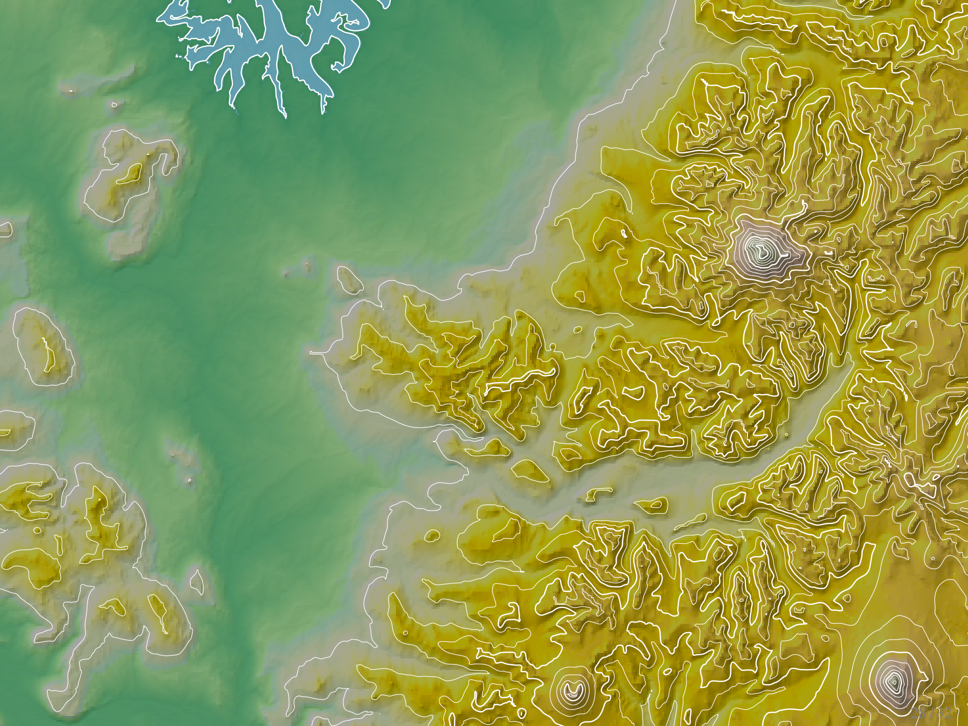
- defines convex set of solutions
- find the "best" solution using your favorite energy functional







Future work

- Connection to total variation denoising
- Extension to higher dimensions (weaker assumptions on critical points)
- Connection to singularity theory?

Present work

U. Bauer.
Persistence in Discrete Morse Theory.
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U. Bauer, C.-B. Schönlieb, and M. Wardetzky.
Total variation meets topological persistence: A first encounter.
Proceedings of ICNAAM 2010, 1022–1026.
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Past work

R. Forman.
A user's guide to discrete Morse theory.
Sém. Loth. de Combinatoire, B48c:1–35, 2002.

H. Edelsbrunner, D. Letscher, and A. Zomorodian.
Topological Persistence and Simplification.
Discr. Comp. Geometry, 28(4):511–533, 2002.

D. Cohen-Steiner, H. Edelsbrunner, and J. Harer.
Stability of Persistence Diagrams.
Discr. Comp. Geometry, 37(1):103–120, 2007.

Thanks for your attention!