Applied Topology

From Theory to Computation

Ulrich Bauer

TUM

February 5, 2016

Inaugural lecture

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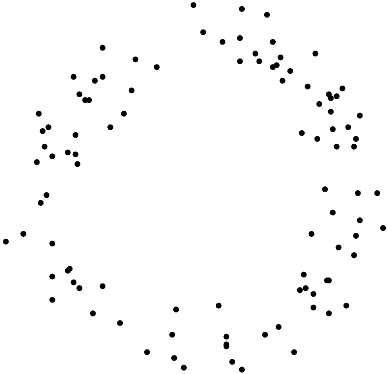
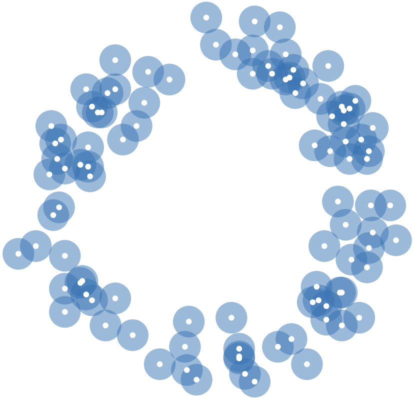
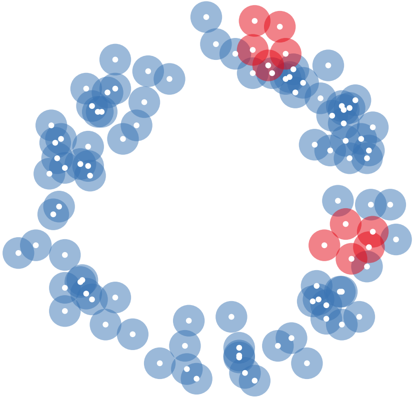
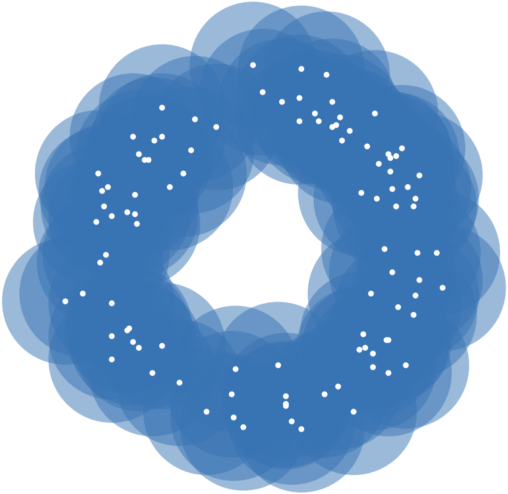
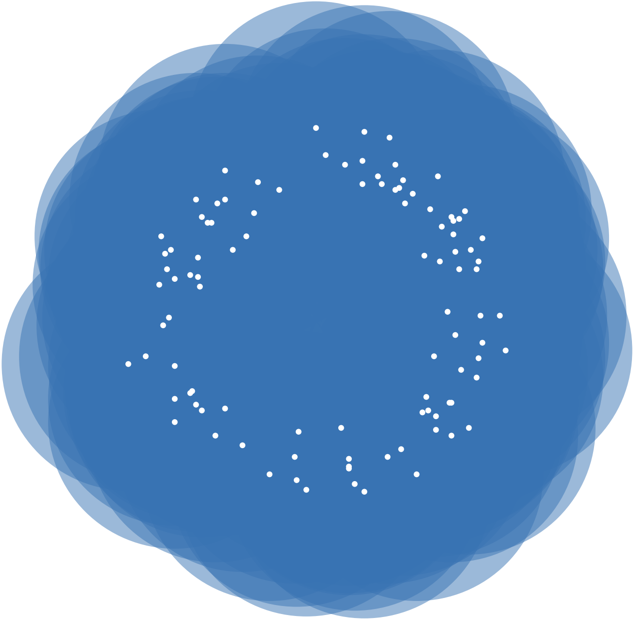
Persistent homology

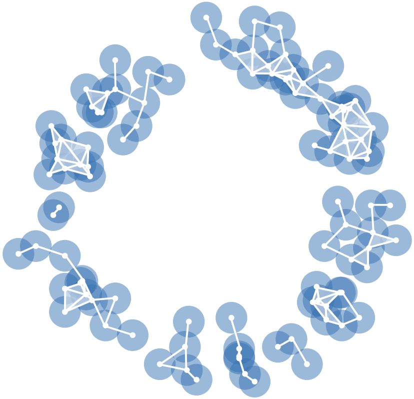


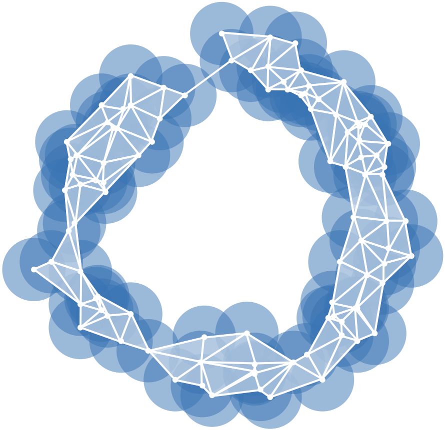


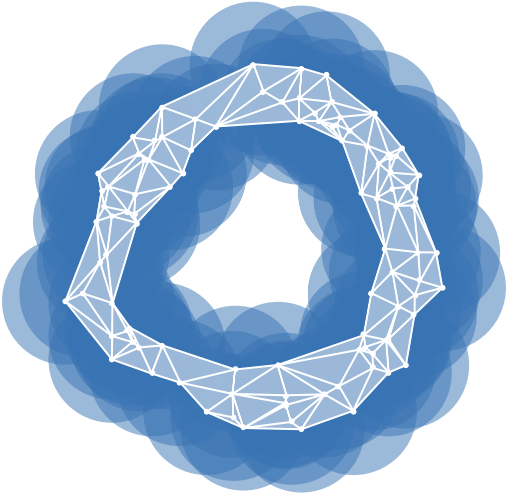


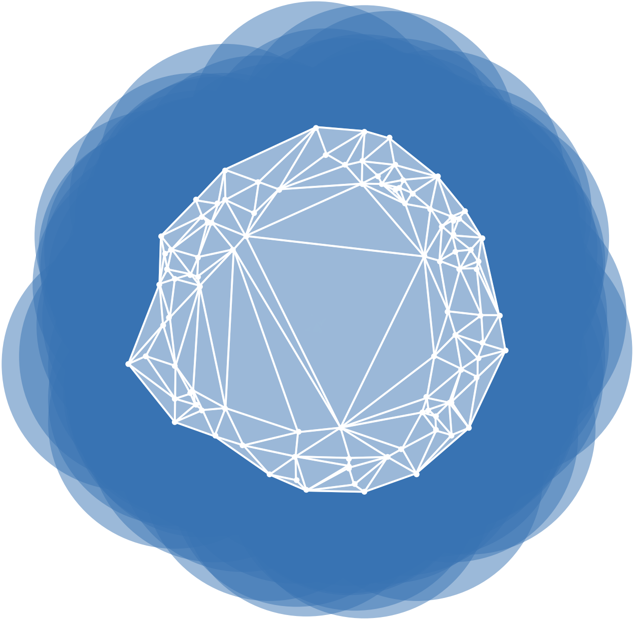












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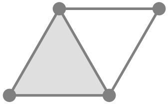

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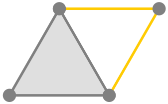

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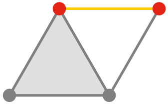

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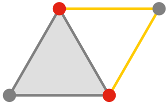

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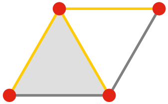

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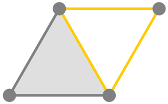

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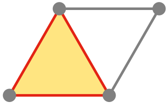

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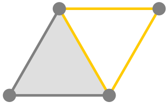

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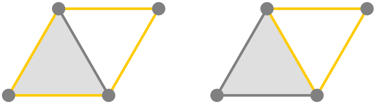

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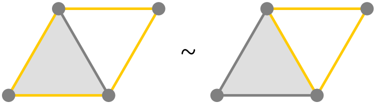

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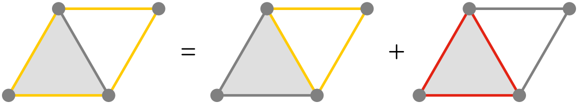

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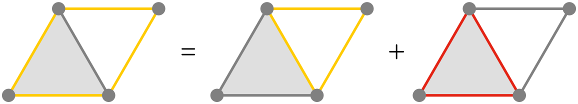

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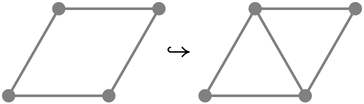
- The equivalence classes form the *homology group* $H_*(K) = Z_*(K)/B_*(K)$

Why is homology algebraic?

Roughly: $\dim H_*(K)$ is the number of holes

- Why not just count holes then?



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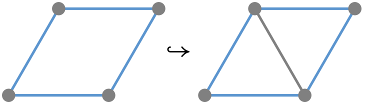

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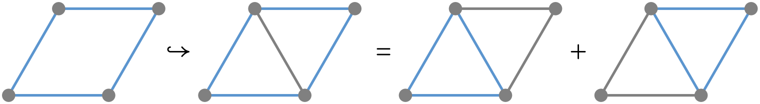

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- A "hole" might get mapped to a linear combination of two "holes"

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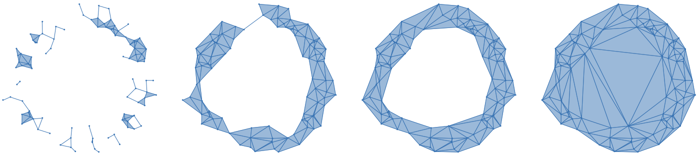

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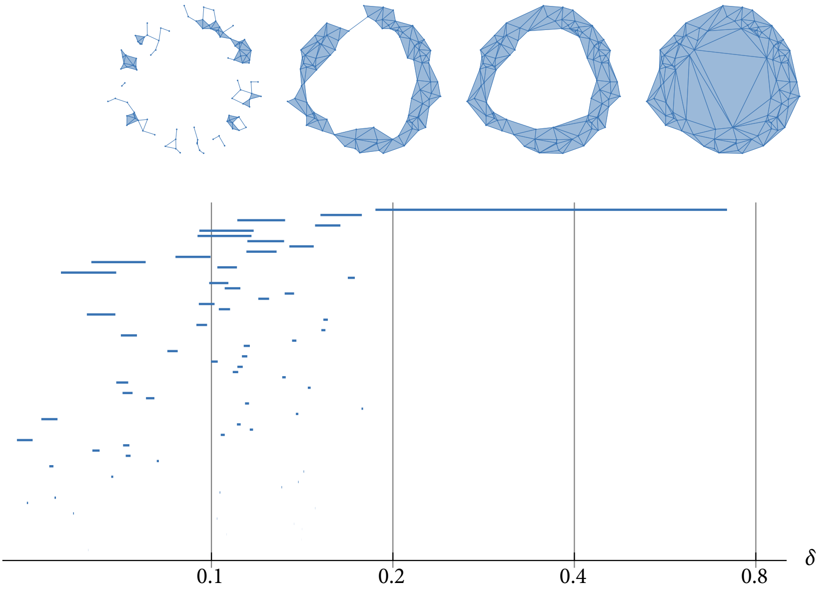

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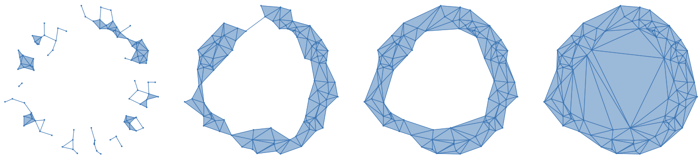

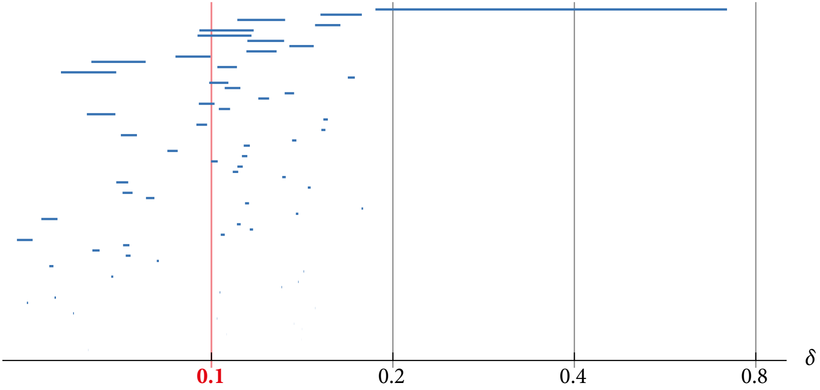

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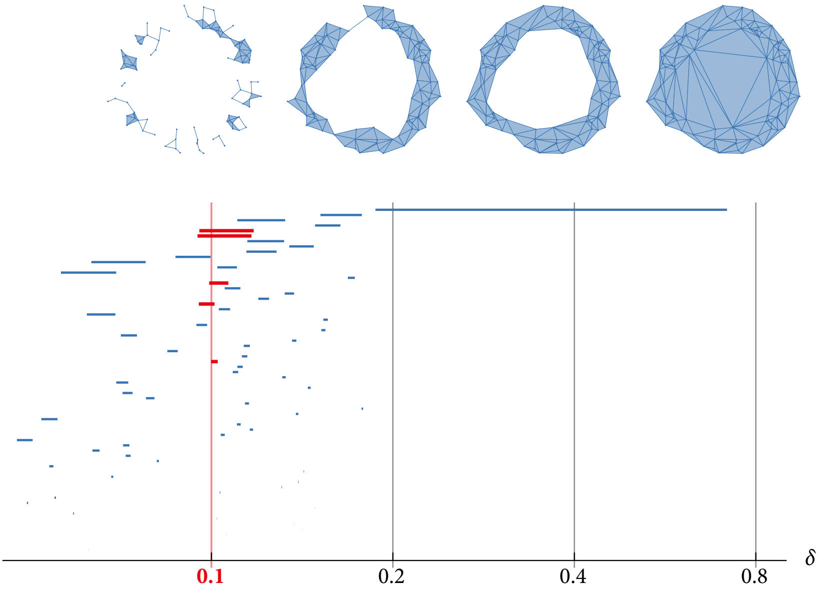

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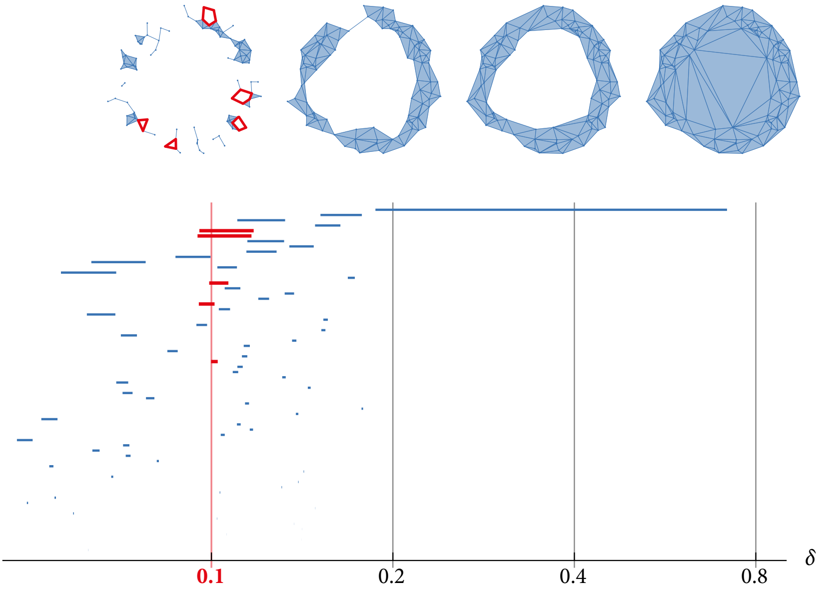

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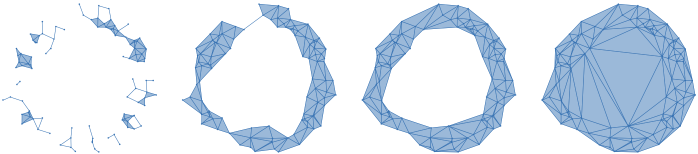

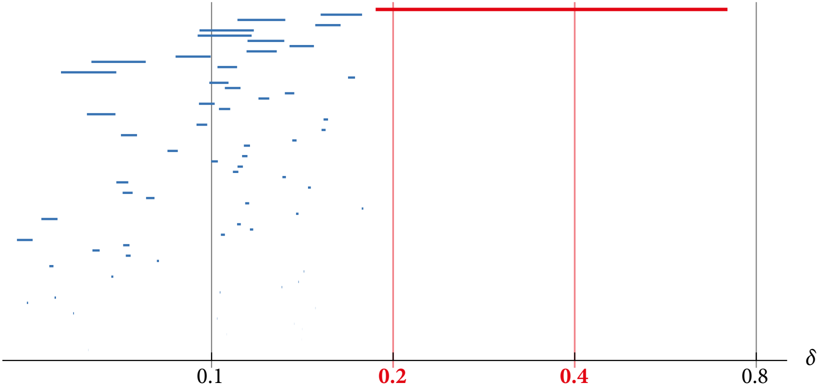

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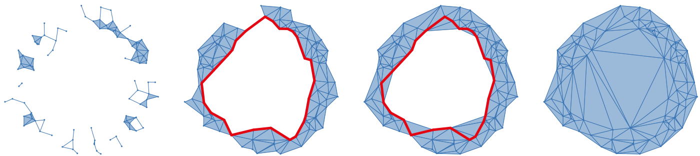

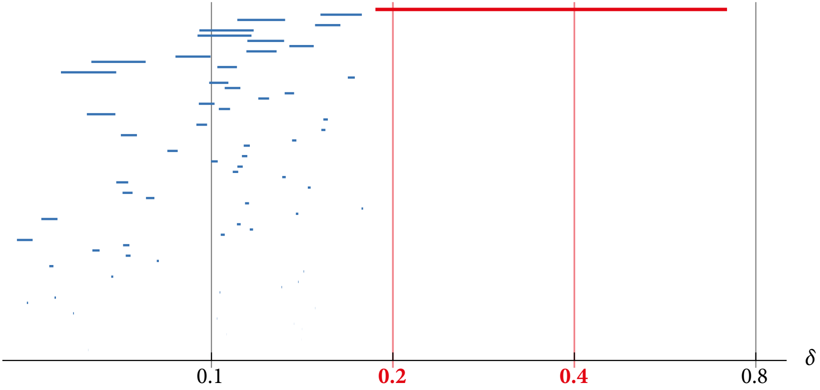

Homology inference & reconstruction

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Given a finite sample $P \subset \Omega$*, construct a shape* X *that is geometrically close to* Ω *and satisfies* $H_*(X) \cong H_*(\Omega)$.

- Approximate the shape by a thickening $B_\delta(P)$ covering Ω
- *Geometrically close* means that the isomorphism is induced by the inclusion $\Omega \hookrightarrow B_\delta(P)$
- Replace $B_\delta(P)$ by an equivalent simplicial complex
    - *Čech complex* $\mathrm{Cech}_\delta(P)$
    - *Delaunay complex* $\mathrm{Del}_\delta(P)$

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Theorem (Niyogi, Smale, Weinberger 2006)

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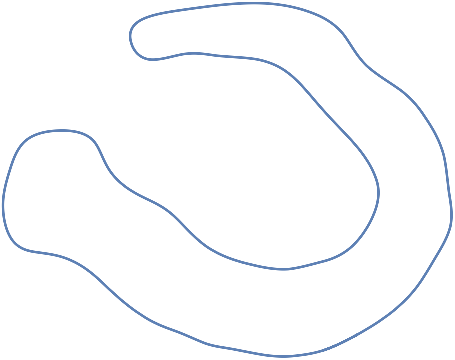
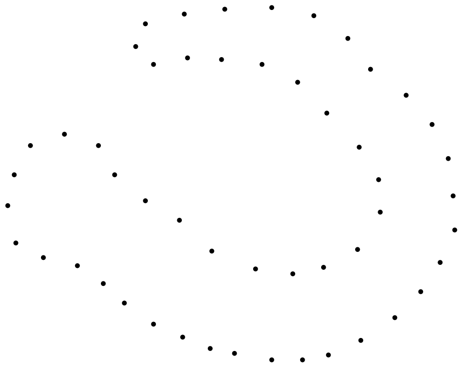
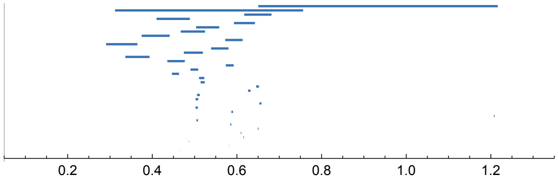
- The isomorphism is induced by the inclusion $M \hookrightarrow P^{2\delta}$.



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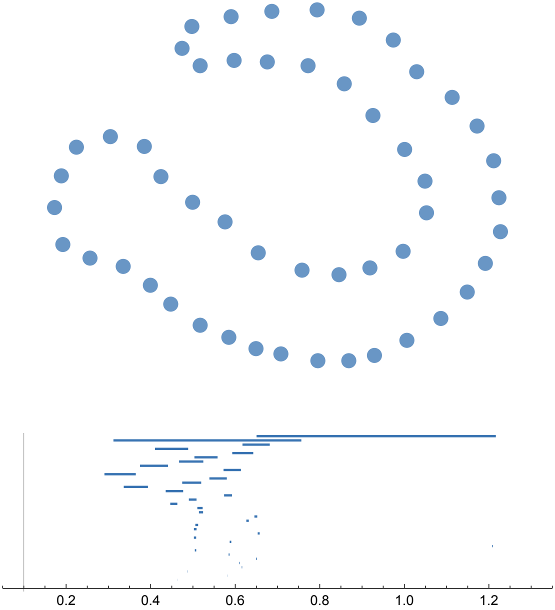
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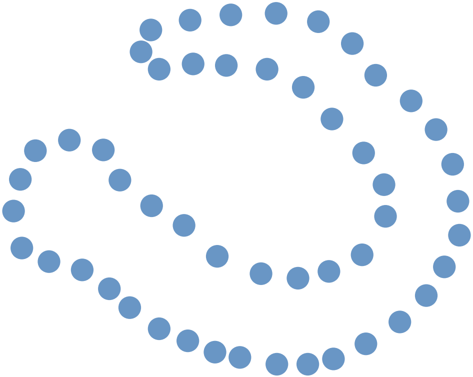

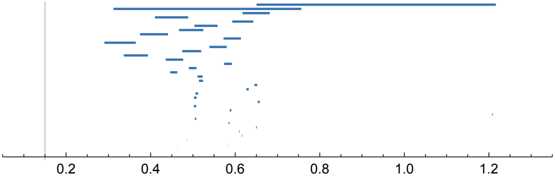
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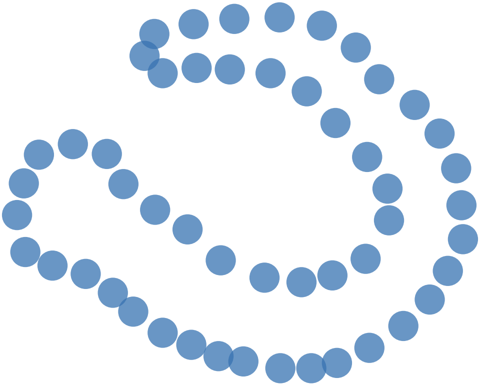

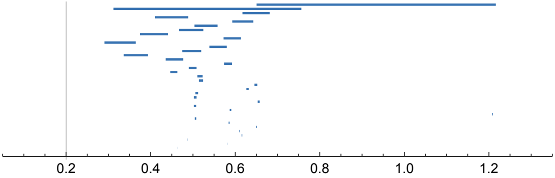
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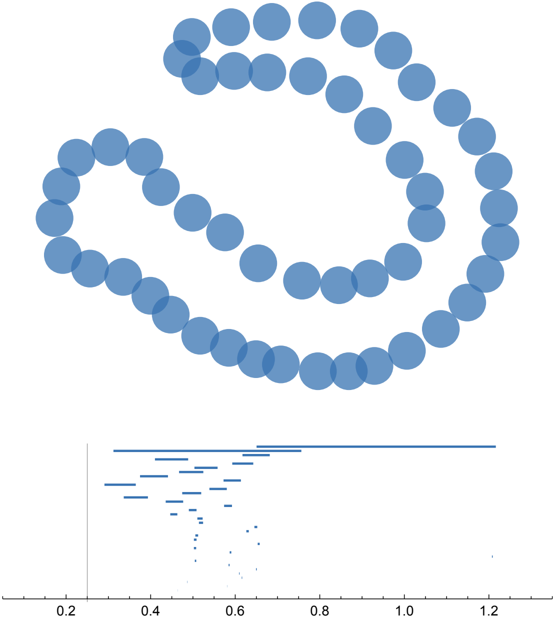
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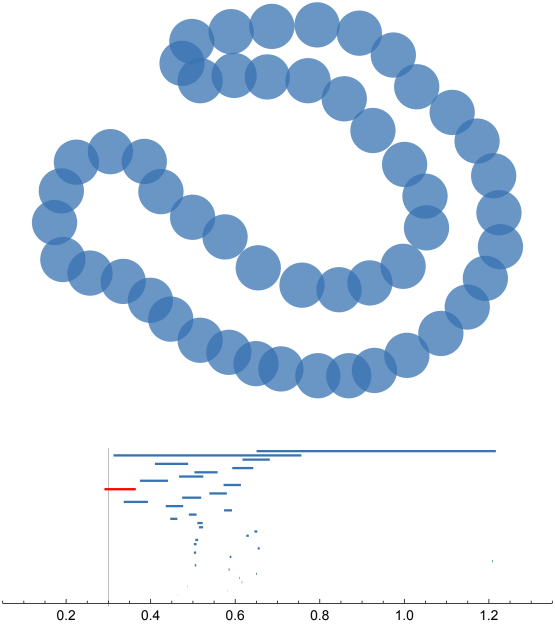
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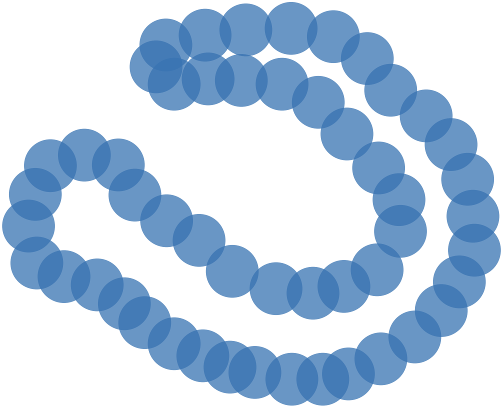

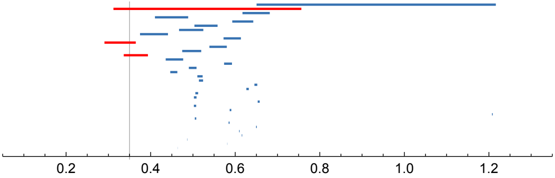
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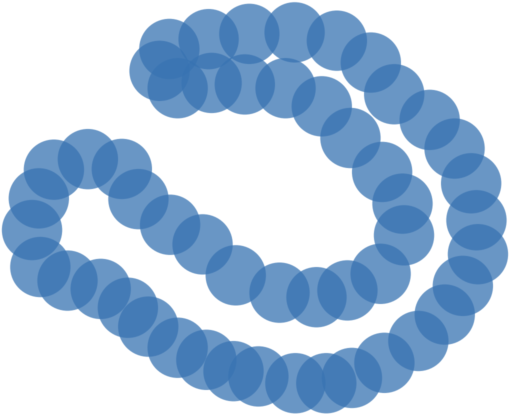

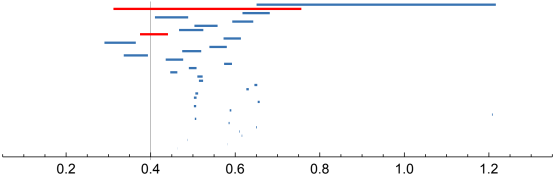
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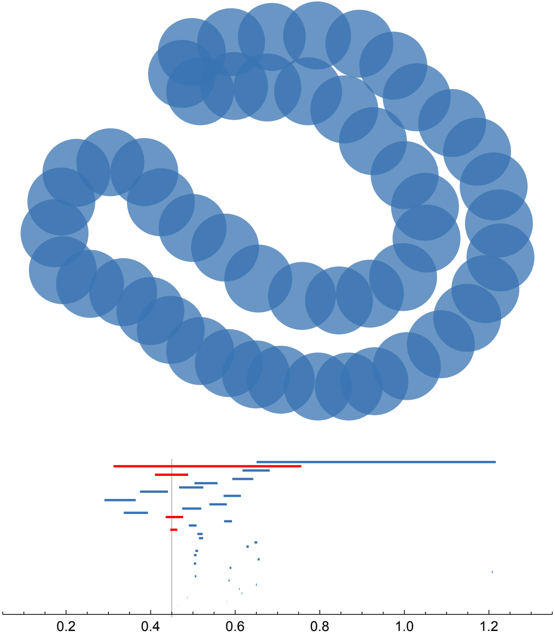
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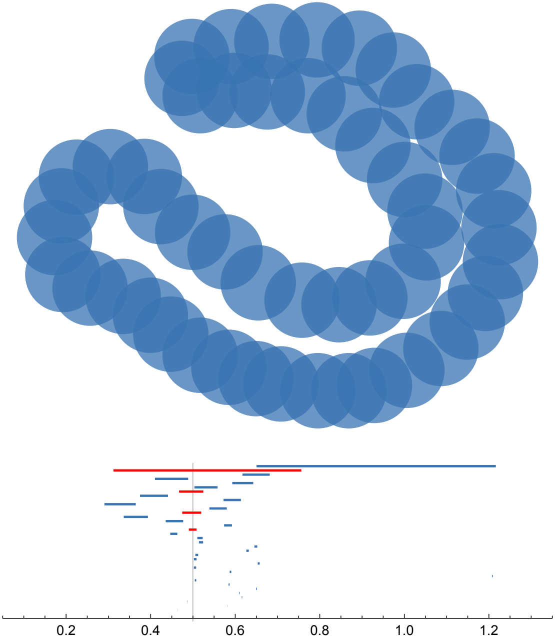
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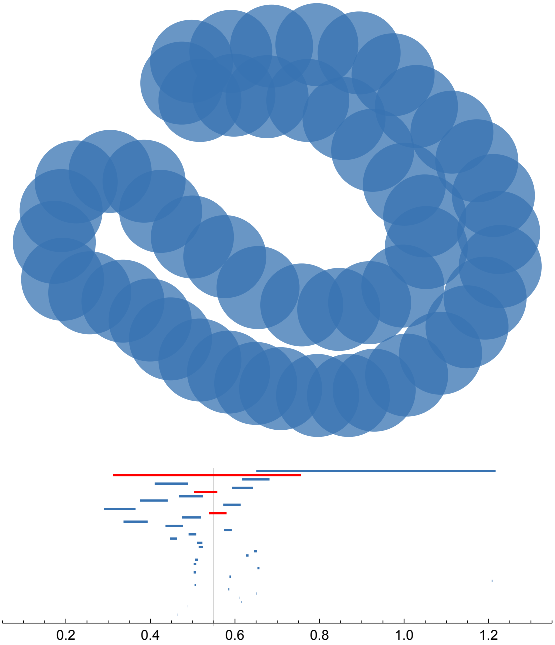
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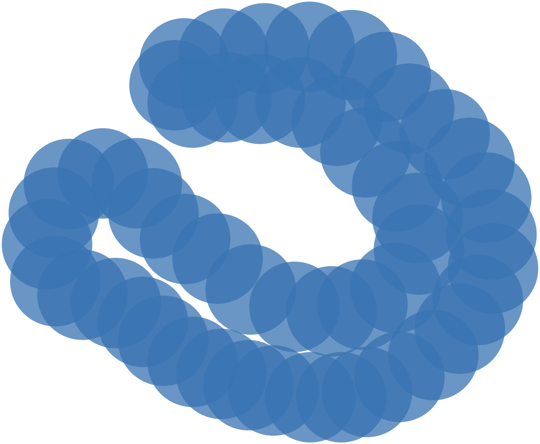

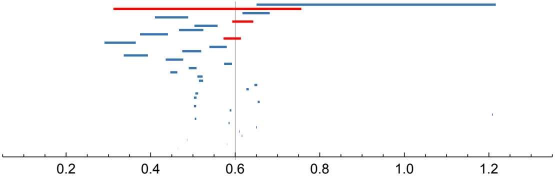
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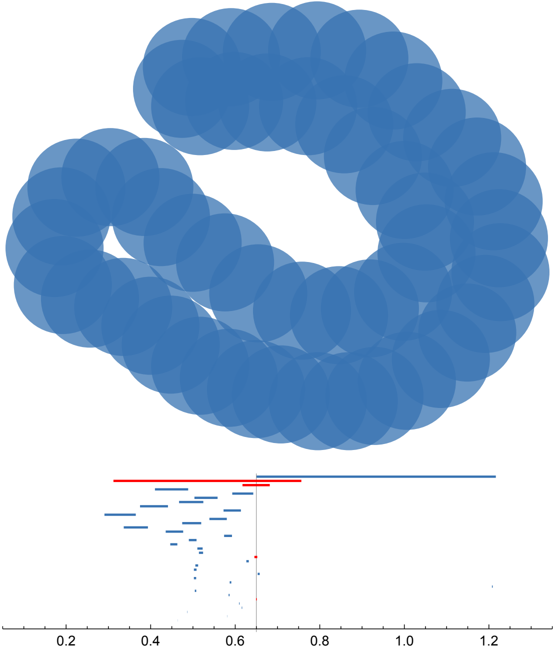
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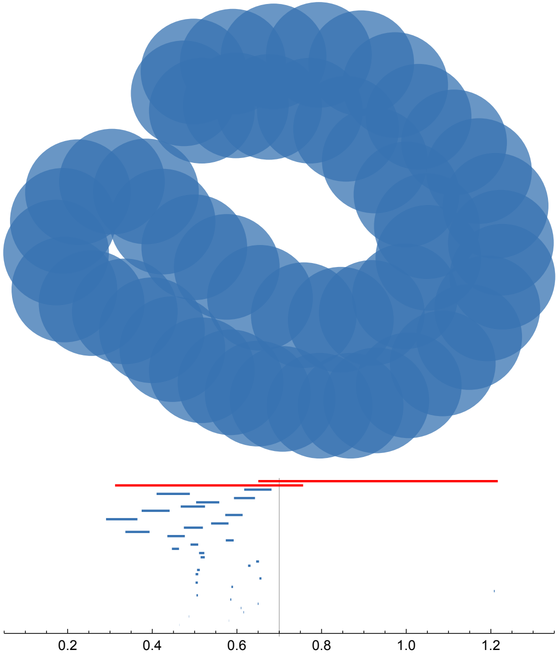

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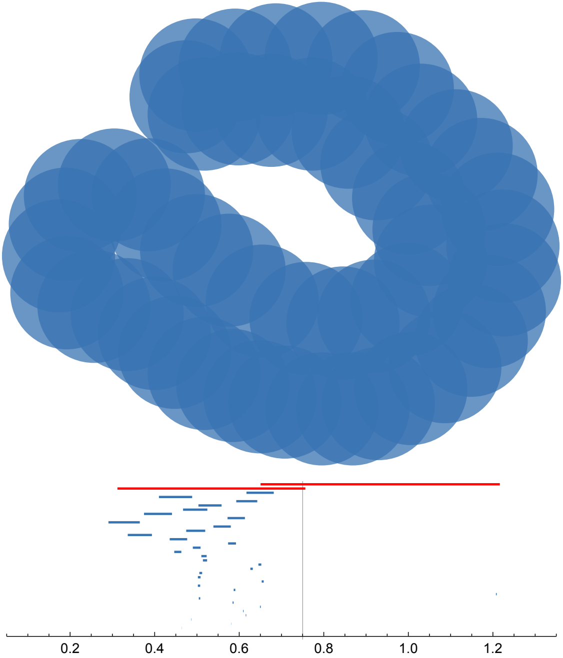
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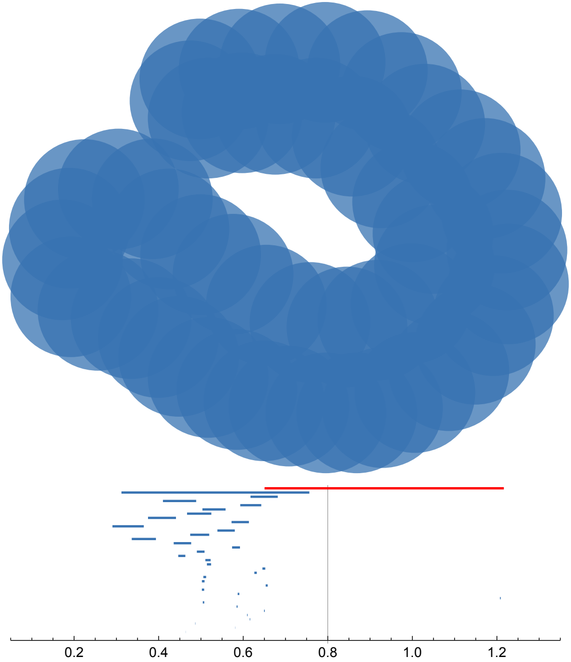
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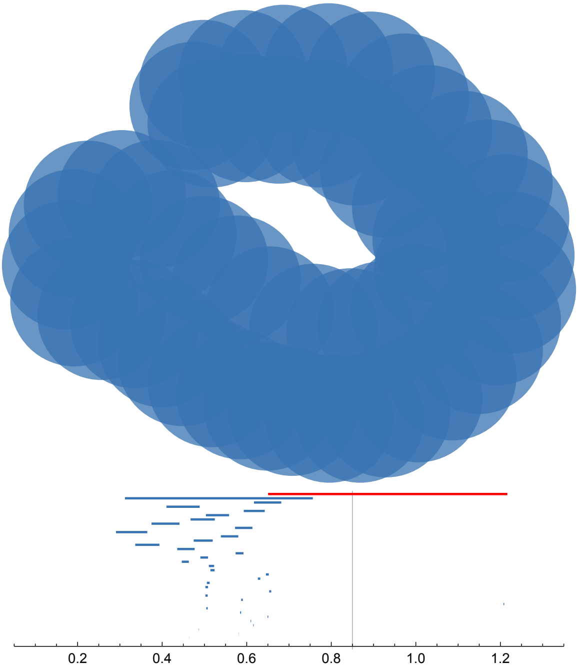
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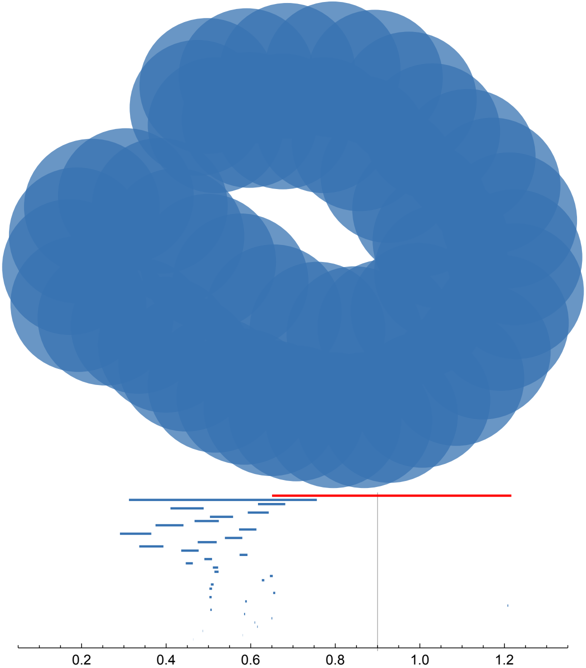

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Homology inference using persistent homology

For $Q \subseteq \mathbb{R}^d$:

$Q_\delta = B_\delta(Q)$: δ-neighborhood (union of balls) around Q

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\Omega \subset \mathbb{R}^d$. Let $P \subset \Omega$ be such that

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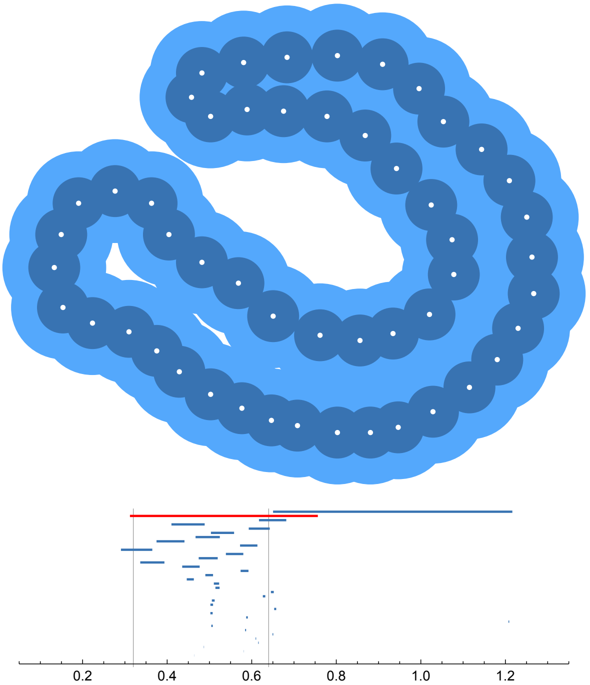
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Proof.

$$\begin{tikzcd}
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\end{tikzcd}$$

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$\square$

Homological realization

This motivates the *homological realization problem*:

Problem

Given a simplicial pair $L \subseteq K$, find X with $L \subseteq X \subseteq K$ such that

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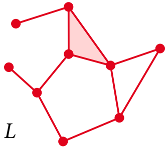

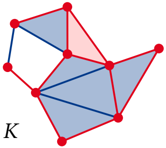

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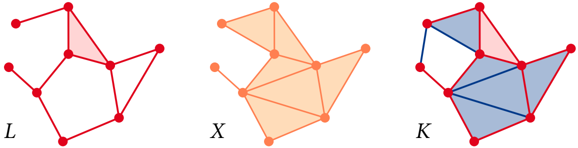

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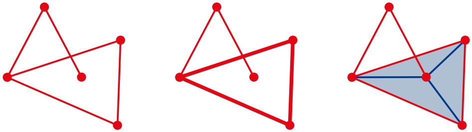

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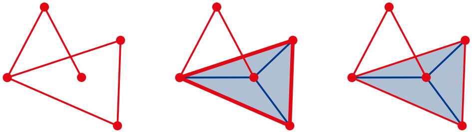

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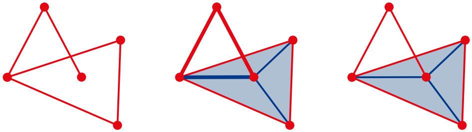

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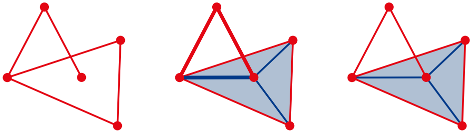

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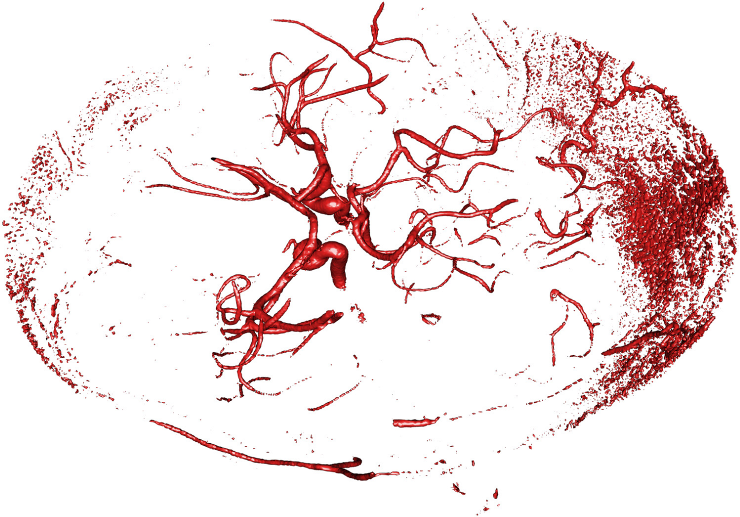
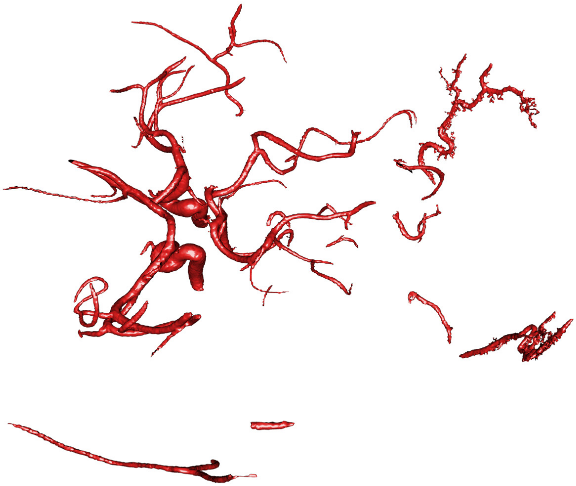
Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

The homological realization problem is NP-hard, even in $\mathbb{R}^3$.

Simplification





Sublevel set simplification

Let $F_{\leq t} = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem (Sublevel set simplification)

Given a function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ *and* $t \in \mathbb{R}$, $\delta > 0$,
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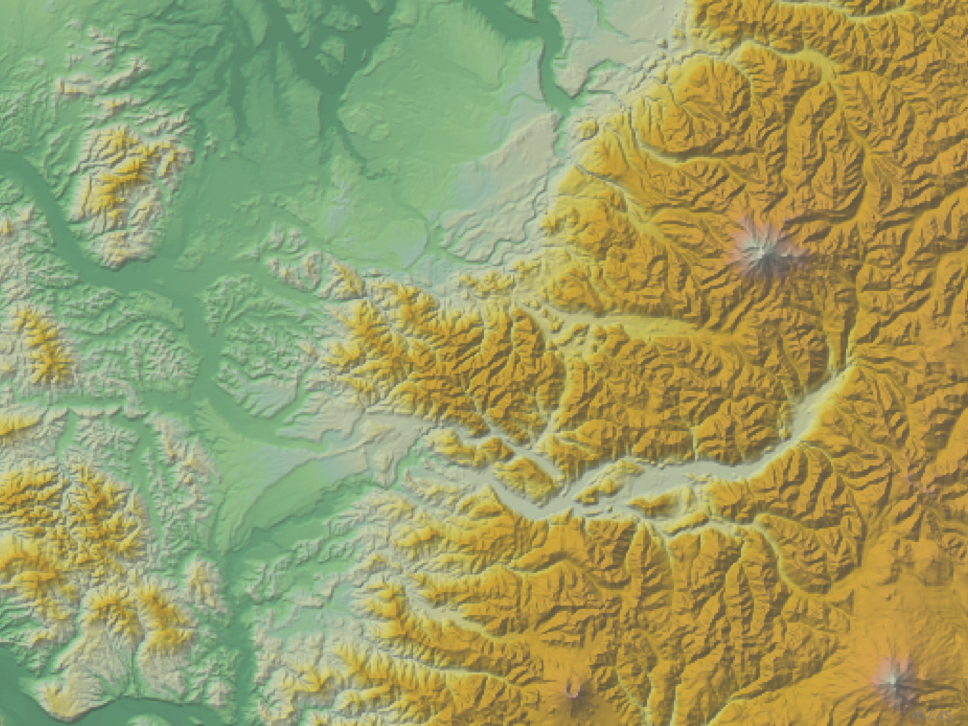
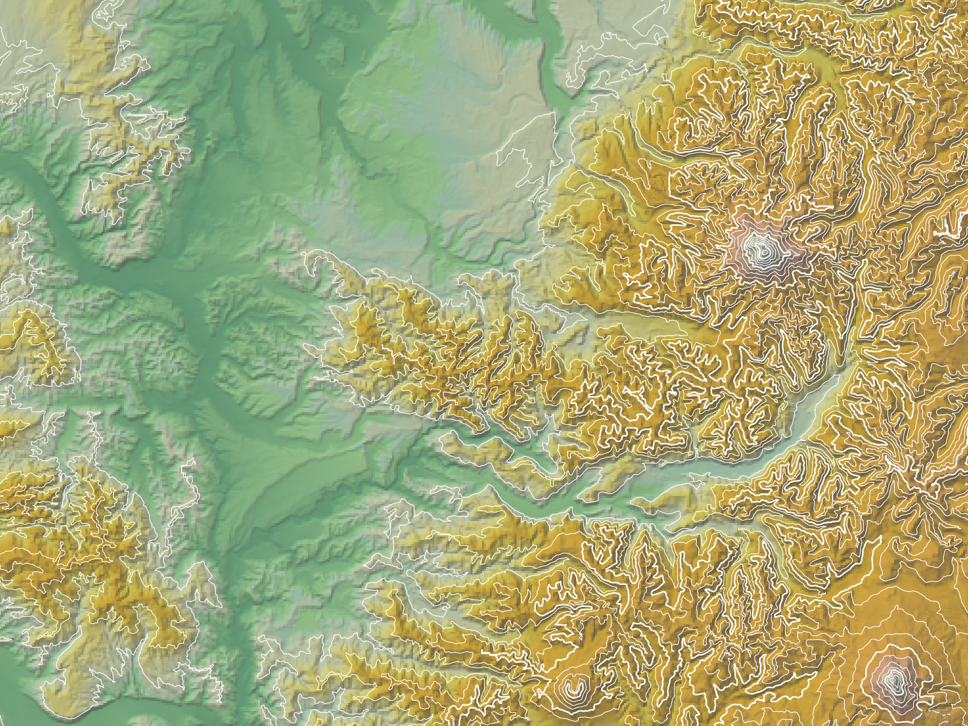
Sublevel set simplification in $\mathbb{R}^3$ *is NP-hard.*

Simplification of functions





Topological simplification of functions

Consider the following problem:

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Given a function f and a real number $\delta \geq 0$,
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Persistent homology:

- Relates homology of different sublevel set
- Identifies pairs of critical points (birth and death of homological feature) and quantifies their *persistence*

Persistence and discrete Morse theory

By stability of persistence barcodes:

Proposition

The critical points with persistence $> 2\delta$ *provide a lower bound on the number of critical points of any function* g *with* $\|g - f\|_\infty \leq \delta$.

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Let f *be a function on a surface and let* $\delta > 0$.
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Theorem (B, Lange, Wardetzky, 2011)

Let f be a function on a surface and let $\delta > 0$.
Construct a function f_δ from f by canceling all persistence pairs with persistence $\leq 2\delta$. Then f_δ satisfies

$$\|f_\delta - f\|_\infty \leq \delta$$

and achieves the lower bound on the number of critical points.

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The critical points with persistence $> 2\delta$ *provide a lower bound on the number of critical points of any function* g *with* $\|g - f\|_\infty \leq \delta$.

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Let f *be a function on a surface and let* $\delta > 0$.
Construct a function f_δ *from* f *by canceling all persistence pairs with persistence* $\leq 2\delta$. *Then* f_δ *satisfies*

$$\|f_\delta - f\|_\infty \leq \delta$$

and achieves the lower bound on the number of critical points.

- Does not generalize to higher-dimensional manifolds!

Morse theory of geometric complexes

Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)

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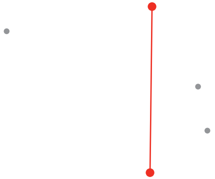

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- *Čech function* $f_C(Q)$:
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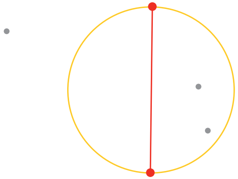

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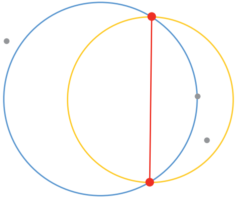

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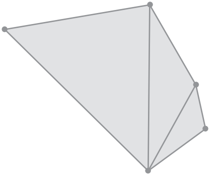

- *Čech function* $f_C(Q)$:
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  - defined only for the *Delaunay triangulation* $\mathrm{Del}(X)$:
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Čech and Delaunay complexes from functions

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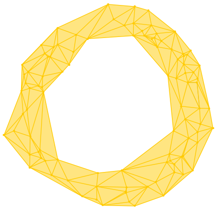

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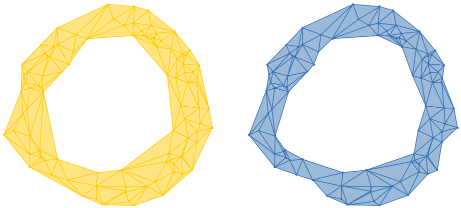
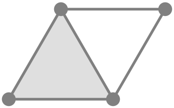


Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.



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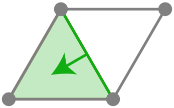
Let K be a simplicial complex.

- A *free face* of K is a simplex contained in a unique higher-dimensional simplex.



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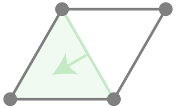

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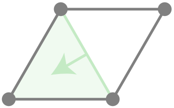

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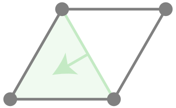

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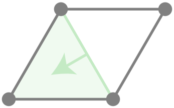

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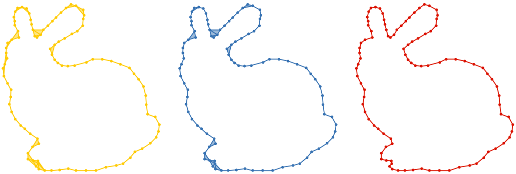
If there is a sequence of elementary collapses from K to M, we say that K *collapses* to M (written as $K \searrow M$).

Main result: a sequence of collapses

Theorem (B, Edelsbrunner 2014, 2015)

Čech, Delaunay, and Wrap complexes are homotopy equivalent by simplicial collapses:

$$\mathrm{Cech}_r \searrow \mathrm{Del}_r \searrow \mathrm{Wrap}_r\,.$$



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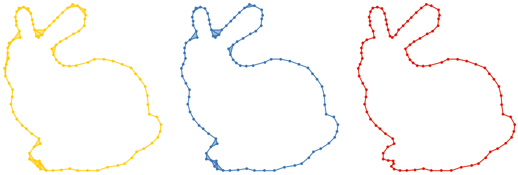

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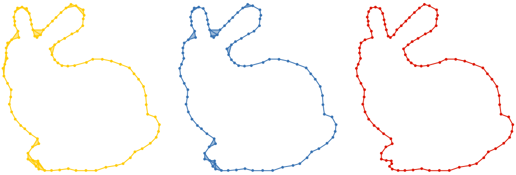

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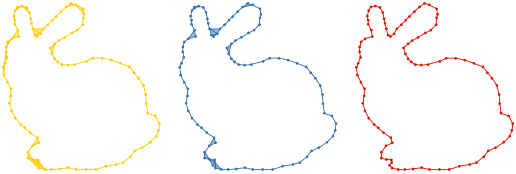

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- Explicit isomorphims in homology

Discrete Morse theory

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Definition (Forman 1998)

A *discrete vector field* on a simplicial complex
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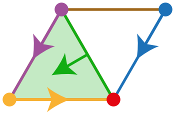

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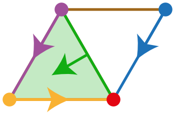

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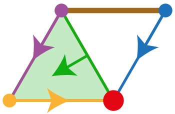

The singletons are called *critical simplices.*

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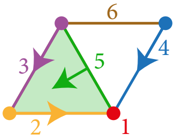

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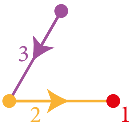

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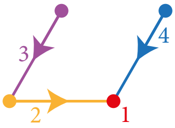

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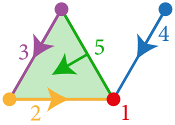

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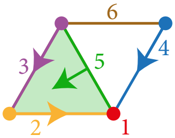

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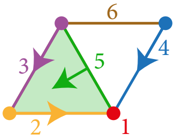

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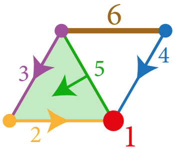

If $f^{-1}(t) = \{C\}$ is a singleton set, then t is a *critical value*.

Collapses from Morse functions and gradients

Theorem (Forman 1998)

Let f be a discrete Morse function on a simplicial complex K.
If $(s, t]$ contains no critical value of f,
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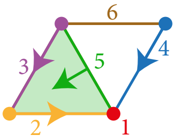

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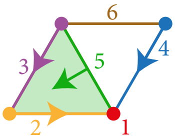

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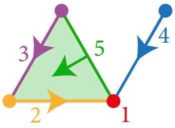

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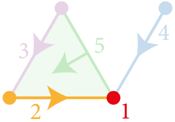

Corollary

Let V be a discrete gradient field on a simplicial complex K,
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If $K \smallsetminus L$ is a union of pairs of V,
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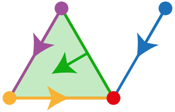

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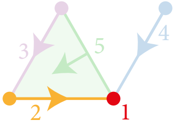

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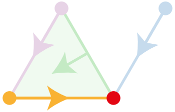

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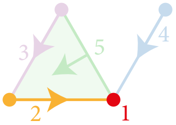

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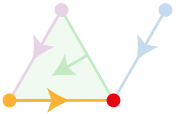

We say that V *induces the collapse $K \searrow L$.*

Unfortunately…

Neither the Čech nor the Delaunay functions are discrete Morse functions!

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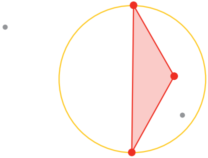
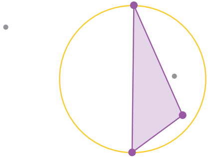
- Example: two simplices Q, Q' with $f_C(Q) = f_C(Q')$ such that neither is a face of the other:



Q



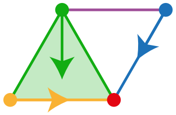
Q'

Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex is a partition of the set of simplices into clusters of simplices containing a simplex L (common face) and contained in a simplex U (common coface):

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}$$



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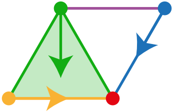

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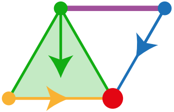

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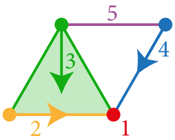

The collapsing theorems carry over to generalized discrete Morse functions.

Generalized discrete Morse theory

Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex
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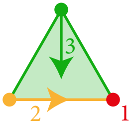

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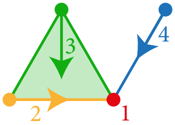

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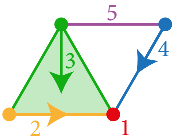

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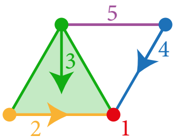
- the level sets $f^{-1}(t)$ partition the set of simplices into a generalized discrete vector field (the *discrete gradient* of f)



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Čech and Delaunay intervals

Morse theory of Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

The Čech function and the Delaunay function are generalized discrete Morse functions.

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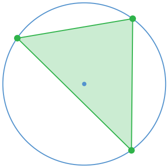
The critical simplices are the centered *Delaunay simplices (containing the circumcenter in the interior)*



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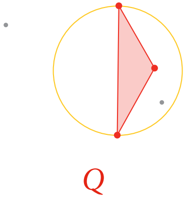
Lemma

Let $Q \subseteq X$ be a simplex with smallest enclosing sphere S.
Then $Q' \subseteq X$ has the same smallest enclosing sphere S iff



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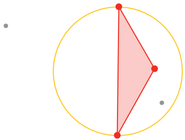

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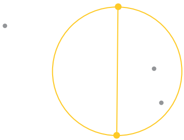

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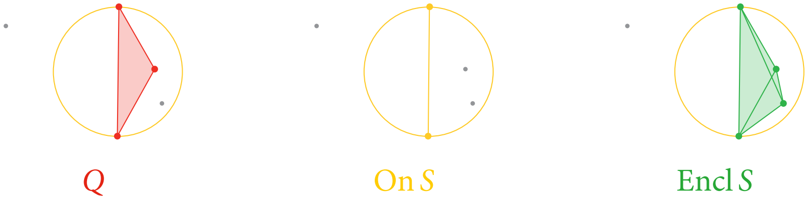

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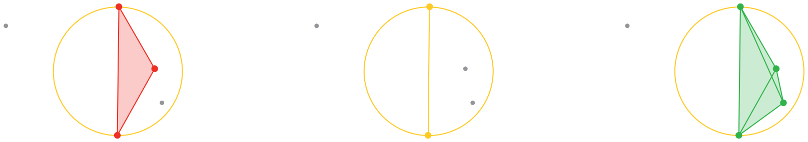

Q $\quad$ $\mathrm{On}\, S$ $\quad$ $\mathrm{Encl}\, S$

The front and back faces of a simplex

Let S be the smallest circumsphere of a simplex $Q = \mathrm{On}\, S \subseteq X$.

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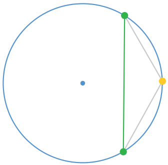

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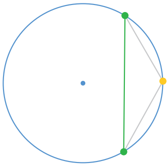

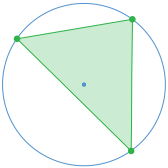

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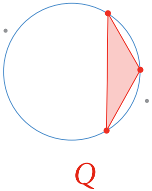

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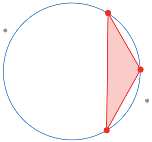

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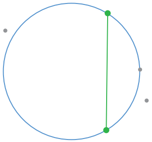

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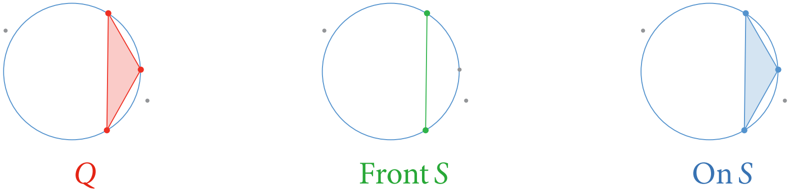

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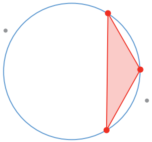

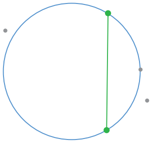

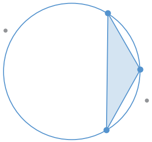

Selective Delaunay complexes

Sphere optimization problems

Both Čech and Delaunay functions are defined using the smallest sphere S satisfying certain constraints:

$$\begin{aligned} \underset{r,z}{\text{minimize}} \quad & r \\ \text{subject to} \quad & \|z - q\| \leq r, \qquad q \in Q, \\ & \|z - e\| \geq r, \qquad e \in E. \end{aligned}$$

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- Čech function: choose $E = \varnothing$
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Čech and Delaunay intervals from KKT

The *Karush–Kuhn–Tucker* optimality conditions yield:

Proposition

A sphere S enclosing Q and excluding E is minimal iff

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Define for finite point sets $X, E \subset \mathbb{R}^d$:

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Let $E \subseteq F \subseteq X$. Then

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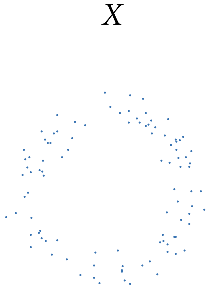
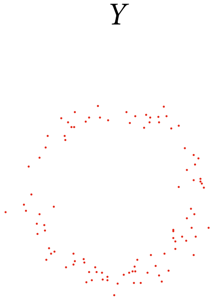
Connecting different Delaunay complexes





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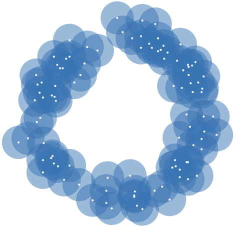

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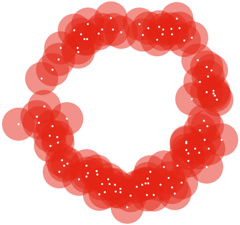

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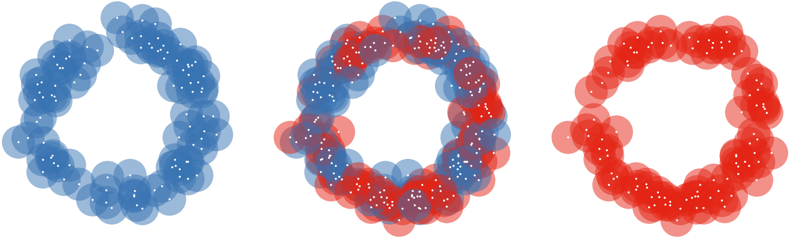

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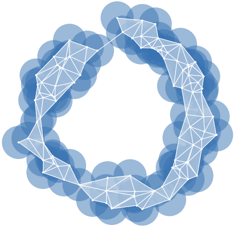

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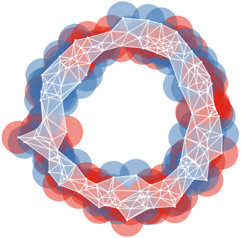

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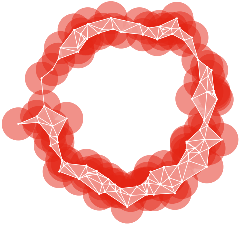

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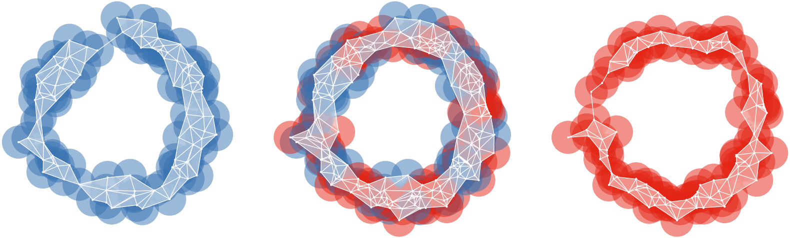

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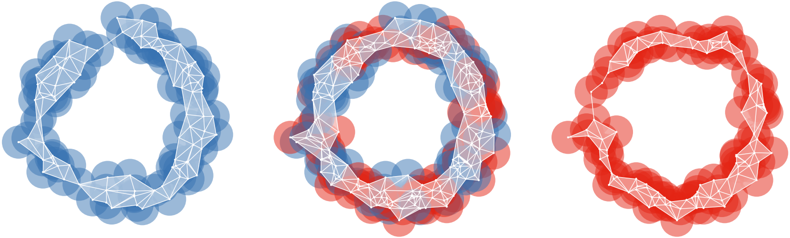

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Stability

The pipeline of topological data analysis

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\text{Data} \arrow[d] & \text{point cloud} \arrow[d, "\text{distance}"] \\
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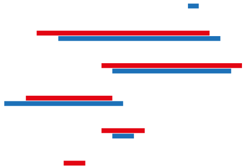
Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

If two functions $f, g : K \to \mathbb{R}$ have distance $\|f - g\|_\infty \leq \delta$, then there exists a δ-matching of their barcodes.



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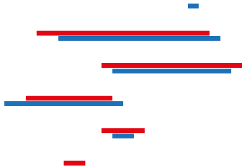

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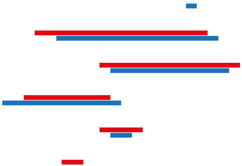

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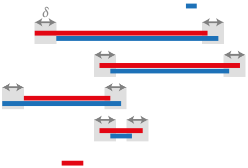

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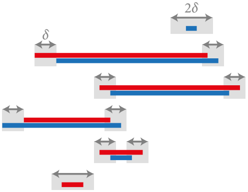

- *matching* $A \nrightarrow B$: bijection of subsets $A' \subseteq A$, $B' \subseteq B$
- *δ-matching of barcodes*:
  - matched intervals have endpoints within distance $\leq \delta$
  - unmatched intervals have length $\leq 2\delta$

Stability revisited

$$
\begin{array}{ccc}
\text{Data} & & \text{point cloud} \\
\downarrow & & \downarrow \text{distance} \\
\text{Geometry} & & \text{function} \\
\downarrow & & \downarrow \text{sublevel sets} \\
\text{Topology} & & \text{topological spaces} \\
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\text{Data} \arrow[d] & \text{point cloud} \arrow[d, "\text{distance}"] \\
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So the sublevel sets are δ-*interleaved:*

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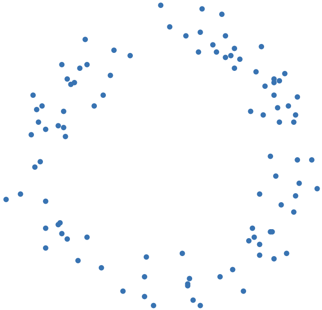
Homology is a *functor:* homology groups are interleaved too.

Geometric interleavings



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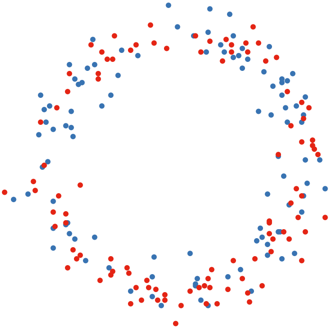

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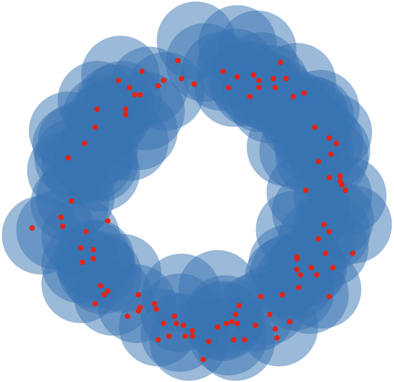

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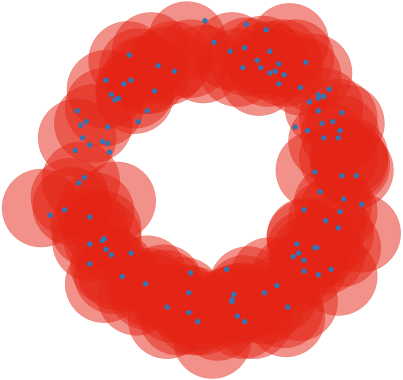

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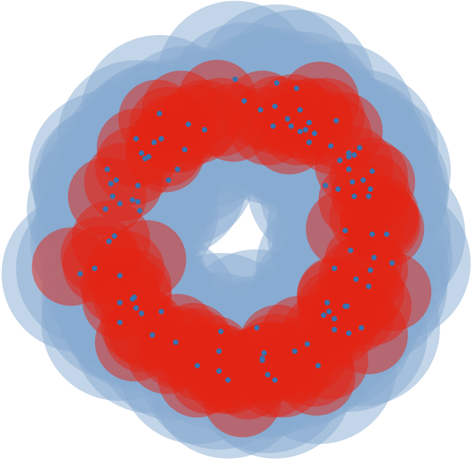

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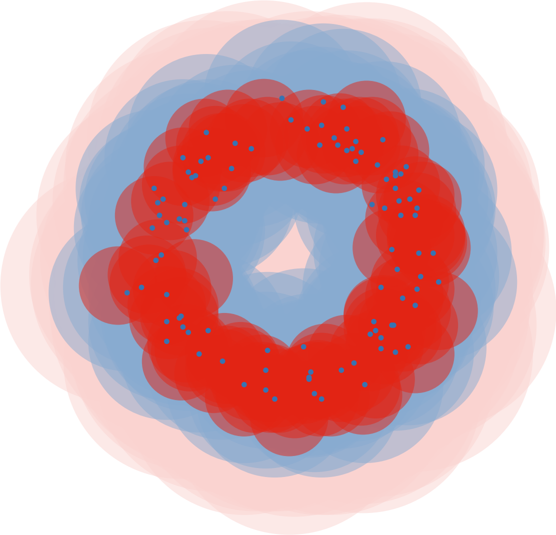

Barcodes: the structure of persistence

Theorem (Crawley-Boewey 2012)

Let $M = (M_t)_{t \in \mathbb{R}}$ be a diagram of vector spaces ($\dim M_t < \infty$) indexed over the reals (with linear maps $M_s \to M_t$ for each $s \leq t$).

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- Motivates use of homology with field coefficients

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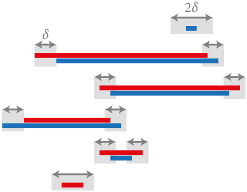

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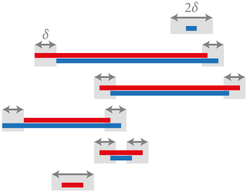

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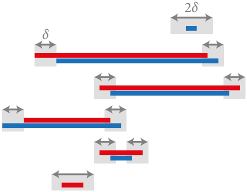

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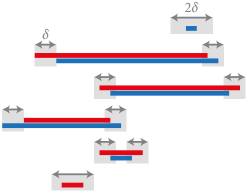

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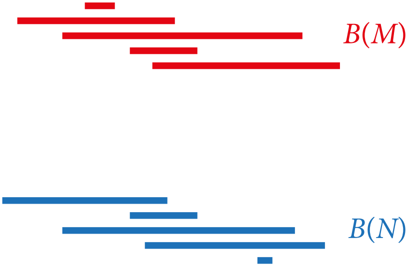
- our proof: constructs matching directly from interleaving maps

Stability via induced matchings



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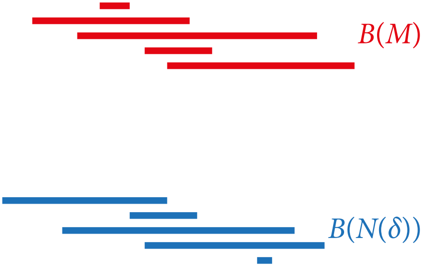

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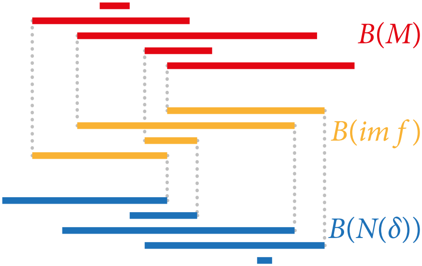

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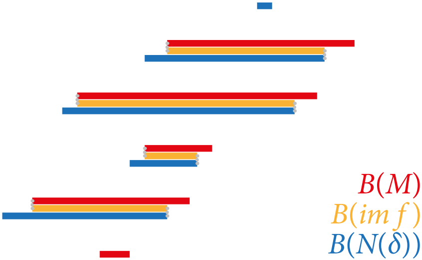

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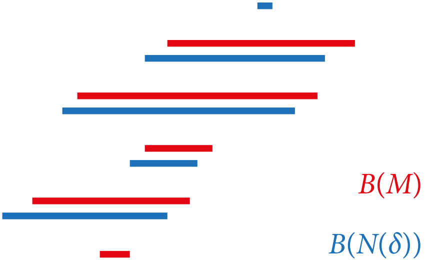

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