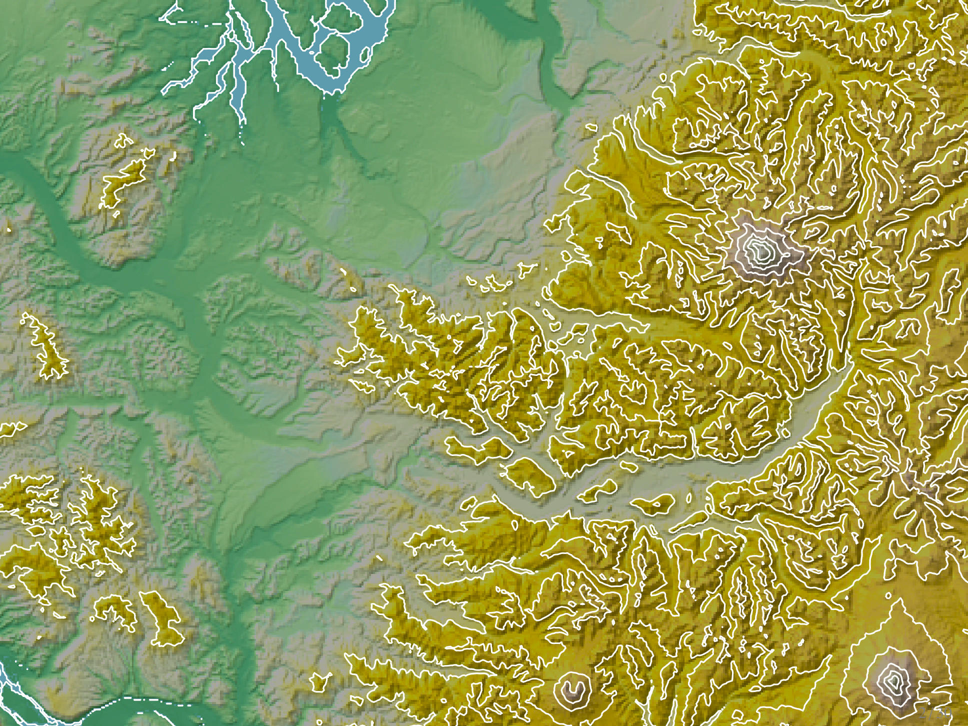
Topological denoising: Persistence meets Total Variation

Ulrich Bauer[1]   Carsten Lange[2]
Carola-Bibiane Schönlieb[1,3]   Max Wardetzky[1]

[1]Georg-August-Universität Göttingen

[2]Freie Universität Berlin

[3]University of Cambridge



Goal

Given a function f on a surface and $\delta > 0$, find a function f_δ that:

- minimizes number of critical points
- stays close to input function: $\|f - f_\delta\|_\infty < \delta$

Goal

Given a function f on a surface and $\delta > 0$, find a function f_δ that:

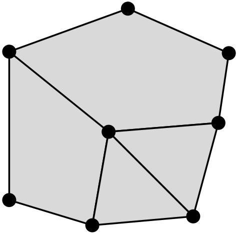
- minimizes number of critical points
- stays close to input function: $\|f - f_\delta\|_\infty < \delta$

Using:

- Discrete Morse theory [Forman 1998]
  - provides notion of critical point in the discrete setting
- Homological persistence [Edelsbrunner et al. 2002]
  - quantifies significance of critical points

Discrete Morse theory [Forman, 1998]

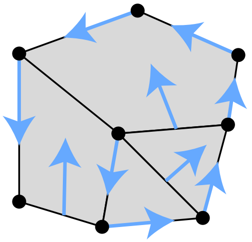
Finite CW complex $\mathcal{K}$



Discrete Morse theory [Forman, 1998]

Finite CW complex $\mathcal{K}$

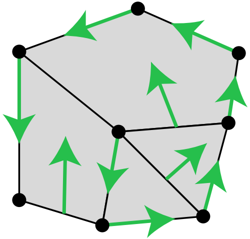
- Discrete vector field:
  - *pairs* of cells (σ, τ) (arrows), σ is a regular facet of τ
  - each cell in at most one pair

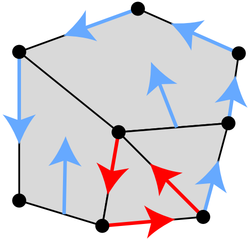


Discrete Morse theory [Forman, 1998]

Finite CW complex $\mathcal{K}$

- Discrete vector field:
  - *pairs* of cells (σ, τ) (arrows), σ is a regular facet of τ
  - each cell in at most one pair
- Discrete *gradient* vector field:
  - no closed paths

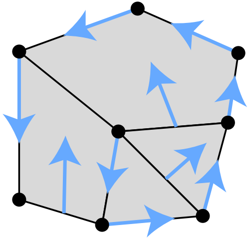


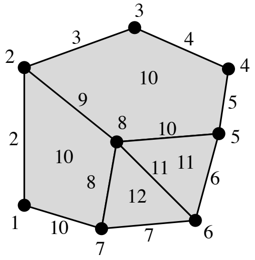


Discrete Morse theory [Forman, 1998]

Finite CW complex $\mathcal{K}$

- Discrete vector field:
  - *pairs* of cells (σ, τ) (arrows), σ is a regular facet of τ
  - each cell in at most one pair
- Discrete *gradient* vector field:
  - no closed paths
- Discrete Morse function:

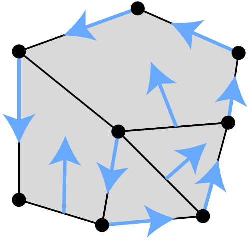


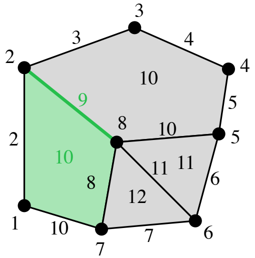


Discrete Morse theory [Forman, 1998]

Finite CW complex $\mathcal{K}$

- Discrete vector field:
  - *pairs* of cells (σ, τ) (arrows), σ is a regular facet of τ
  - each cell in at most one pair
- Discrete *gradient* vector field:
  - no closed paths
- Discrete Morse function:
  - *consistent* with vector field: each cell has greater value than its facets…

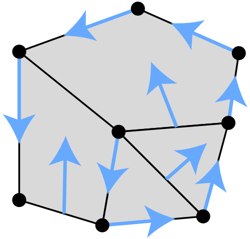


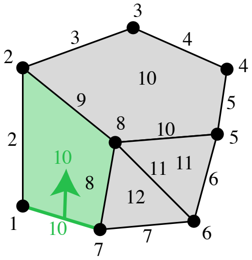


Discrete Morse theory [Forman, 1998]

Finite CW complex $\mathcal{K}$

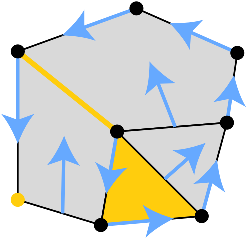
- Discrete vector field:
  - *pairs* of cells (σ, τ) (arrows), σ is a regular facet of τ
  - each cell in at most one pair
- Discrete *gradient* vector field:
  - no closed paths
- Discrete Morse function:
  - *consistent* with vector field: each cell has greater value than its facets… except at arrows





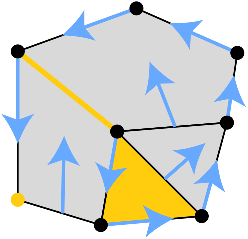
Discrete Morse theory [Forman, 1998]

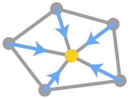
- Critical cell:
  - not contained in vector field



Discrete Morse theory [Forman, 1998]

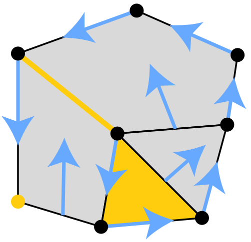
- Critical cell:
  - not contained in vector field
  - Examples in 2D:
    minimum (0-cell)

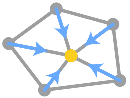


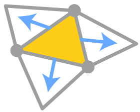


Discrete Morse theory [Forman, 1998]

- Critical cell:
    - not contained in vector field
    - Examples in 2D:
      minimum (0-cell),
      maximum (2-cell)

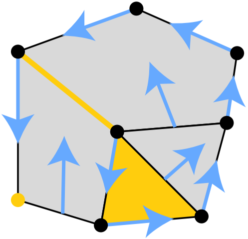


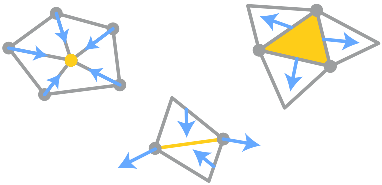




Discrete Morse theory [Forman, 1998]

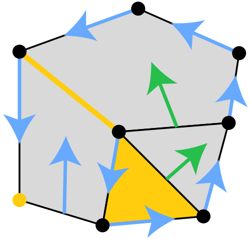
- Critical cell:
  - not contained in vector field
  - Examples in 2D:
    minimum (0-cell),
    maximum (2-cell),
    saddle (1-cell)





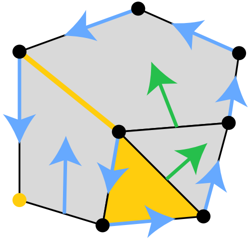
Discrete Morse theory [Forman, 1998]

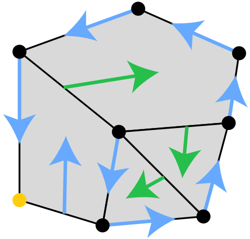
- Critical cell:
  - not contained in vector field
- Cancellation of critical cells:
  - Prerequisite: unique path between two critical cells



Discrete Morse theory [Forman, 1998]

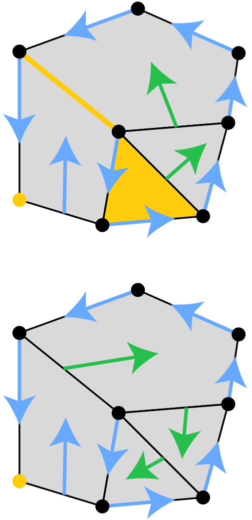
- Critical cell:
  - not contained in vector field
- Cancellation of critical cells:
  - Prerequisite: unique path between two critical cells
  - Reversing vector field along path cancels critical cells





Discrete Morse theory [Forman, 1998]

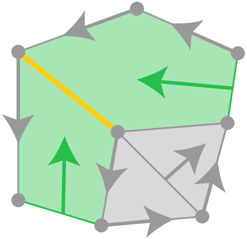
- Critical cell:
  - not contained in vector field
- Cancellation of critical cells:
  - Prerequisite: unique path between two critical cells
  - Reversing vector field along path cancels critical cells (of gradient vector fields, *not functions*)



Attracting and repelling sets

A gradient vector field V imposes inequalities on values of a consistent function

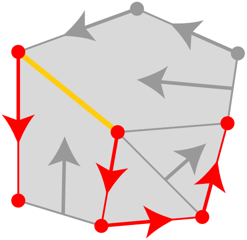
- *Attracting set* of a (critical) cell σ:
  all cells ρ with $g(\rho) \geq g(\sigma)$ for *any* g consistent with V



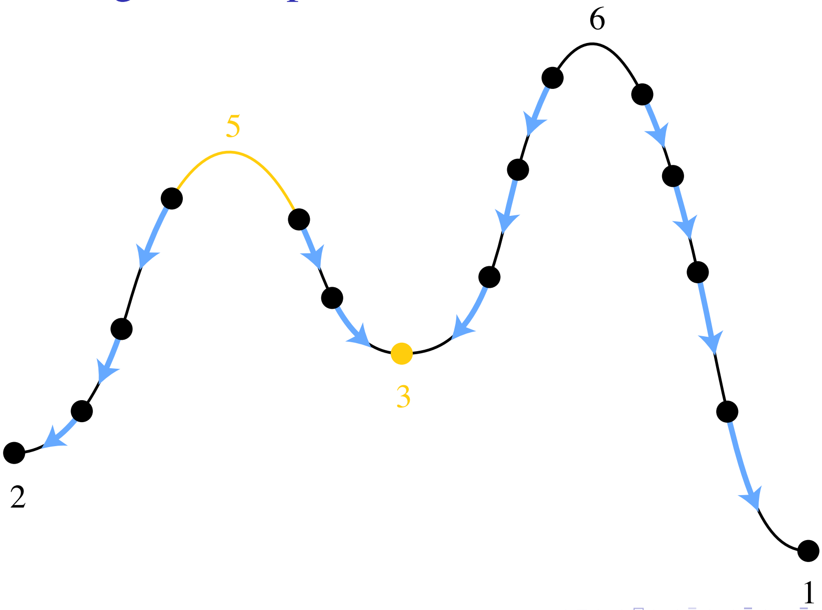
Attracting and repelling sets

A gradient vector field V imposes inequalities on values of a consistent function

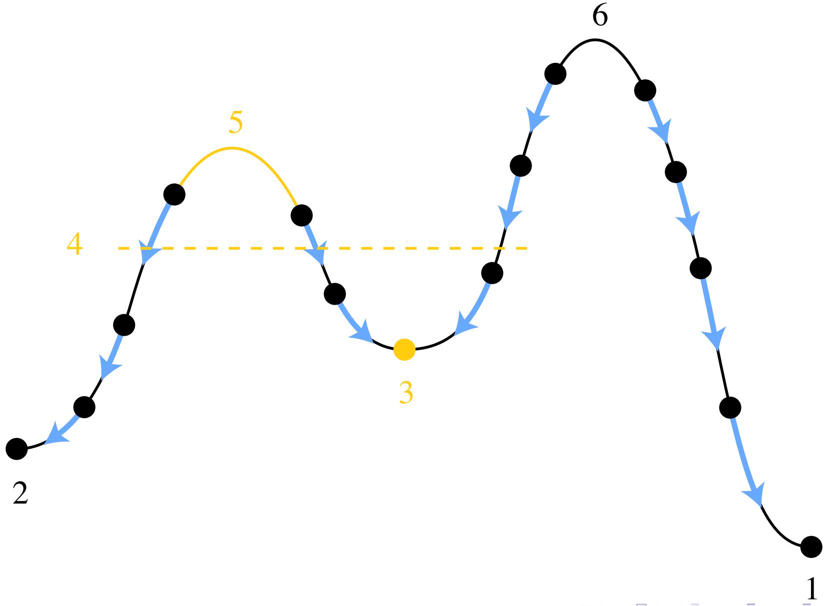
- *Attracting set* of a (critical) cell σ:
  all cells ρ with $g(\rho) \geq g(\sigma)$ for *any* g consistent with V
- *Repelling set:* analogously for $g(\rho) \leq g(\sigma)$



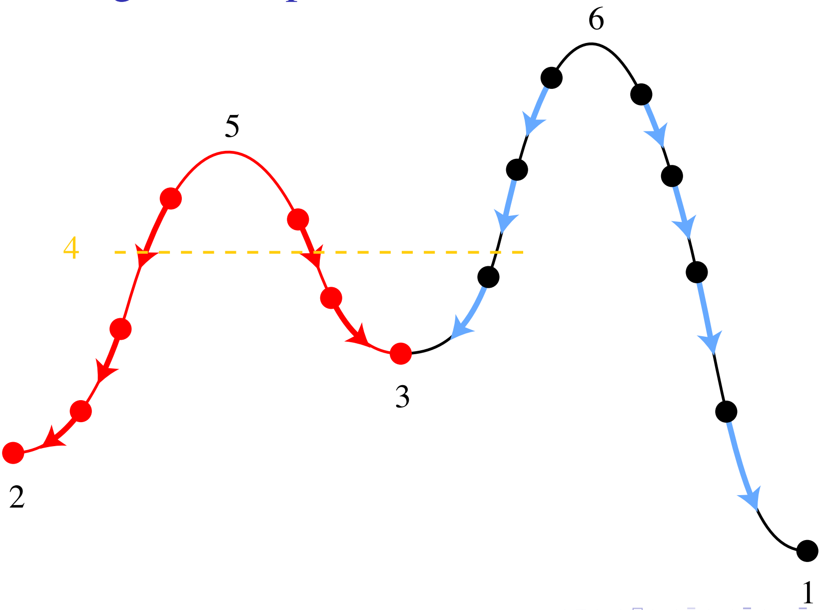
Canceling critical points of a function



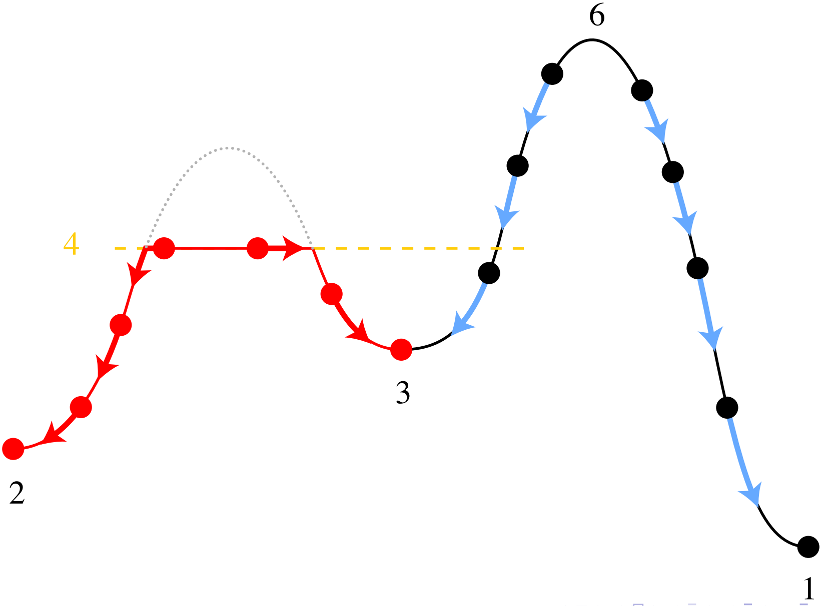
Canceling critical points of a function



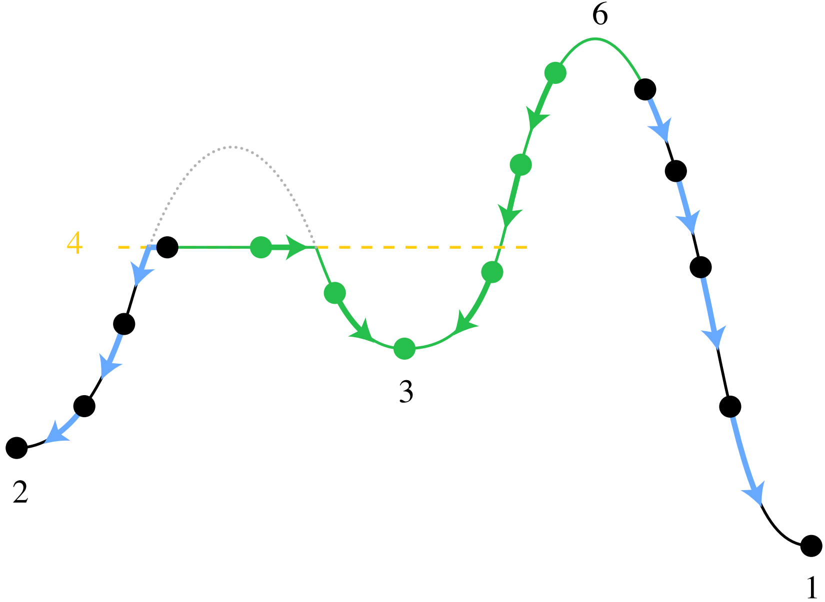
Canceling critical points of a function



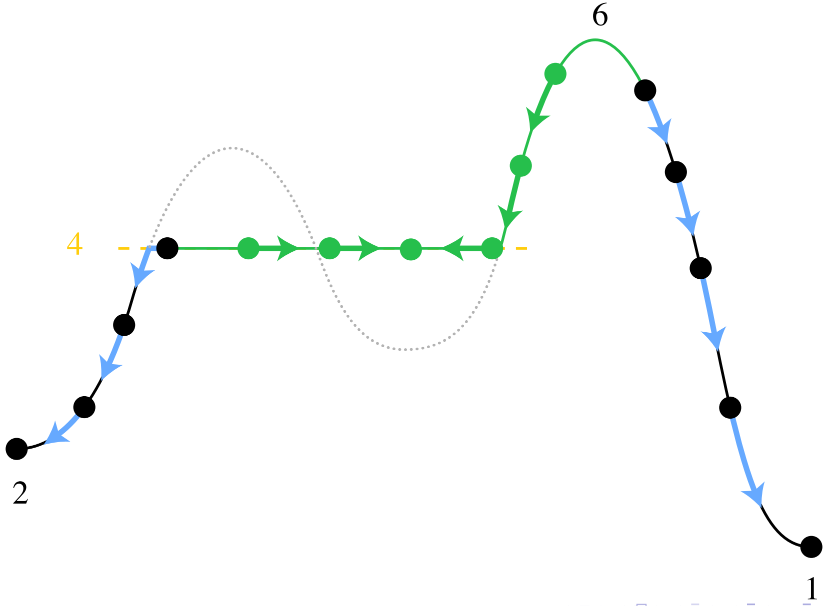
Canceling critical points of a function



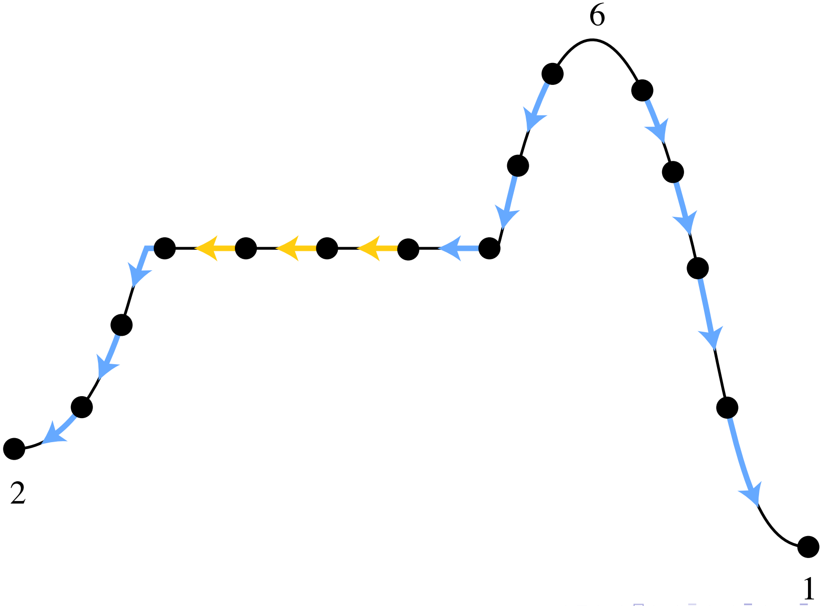
Canceling critical points of a function



Canceling critical points of a function



Canceling critical points of a function



Comparing Discrete and PL Morse theory

Discrete Morse theory is often a more convenient and efficient language

Comparing Discrete and PL Morse theory

Discrete Morse theory is often a more convenient and efficient language

- critical points are always non-degenerate (no multiple saddles)

Comparing Discrete and PL Morse theory

Discrete Morse theory is often a more convenient and efficient language

- critical points are always non-degenerate (no multiple saddles)
- clear and concise notion of vector fields, gradient paths, attracting/repelling sets

Comparing Discrete and PL Morse theory

Discrete Morse theory is often a more convenient and efficient language

- critical points are always non-degenerate (no multiple saddles)
- clear and concise notion of vector fields, gradient paths, attracting/repelling sets
- captures the combinatorial essence of Morse theory (instead of only discretizing)

Comparing Discrete and PL Morse theory

Discrete Morse theory is often a more convenient and efficient language

- critical points are always non-degenerate (no multiple saddles)
- clear and concise notion of vector fields, gradient paths, attracting/repelling sets
- captures the combinatorial essence of Morse theory (instead of only discretizing)

We can convert one type of function to the other

Back to our problem

We have seen how to cancel 2 critical points from a function

Back to our problem

We have seen how to cancel 2 critical points from a function

Strategy: Cancel several pairs of critical points sequentially

Back to our problem

We have seen how to cancel 2 critical points from a function
Strategy: Cancel several pairs of critical points sequentially

- Several possible choices at each step

Back to our problem

We have seen how to cancel 2 critical points from a function
Strategy: Cancel several pairs of critical points sequentially

- Several possible choices at each step
- New choices may appear after each step

Back to our problem

We have seen how to cancel 2 critical points from a function
Strategy: Cancel several pairs of critical points sequentially

- Several possible choices at each step
- New choices may appear after each step
- Keep in mind the tolerance

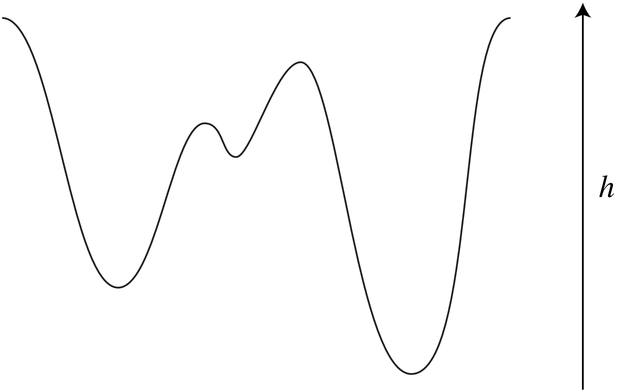
Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for sublevel sets

Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for sublevel sets

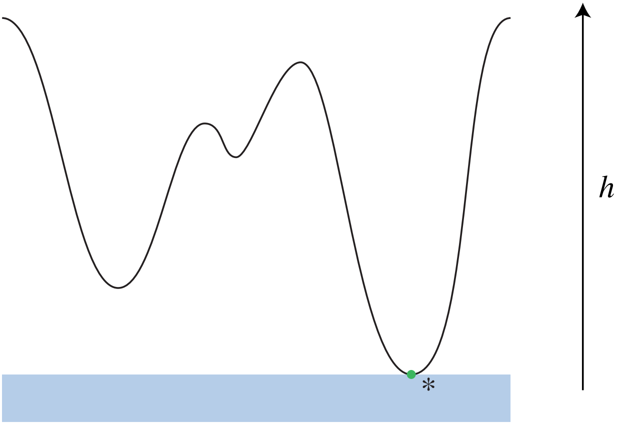
Example: connected components in 1D



Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for sublevel sets

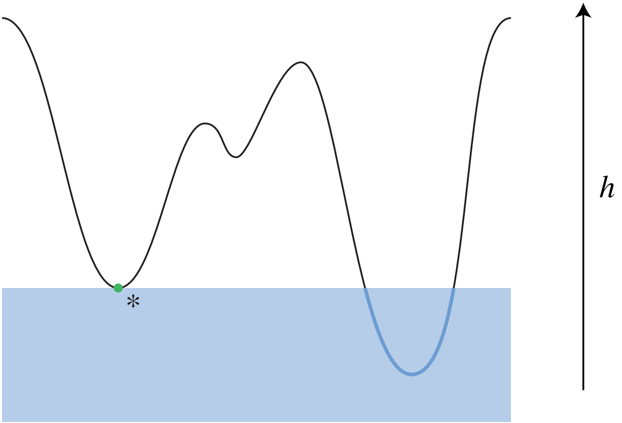
Example: connected components in 1D



Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for sublevel sets

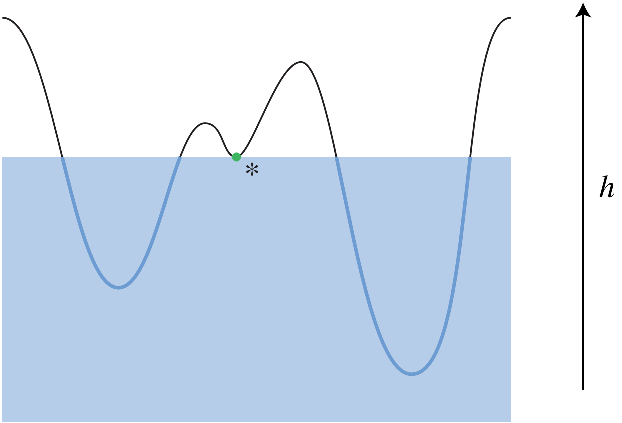
Example: connected components in 1D



Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for sublevel sets

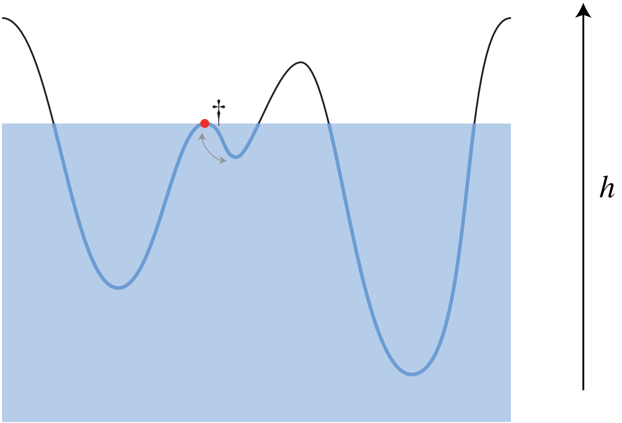
Example: connected components in 1D



Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for sublevel sets

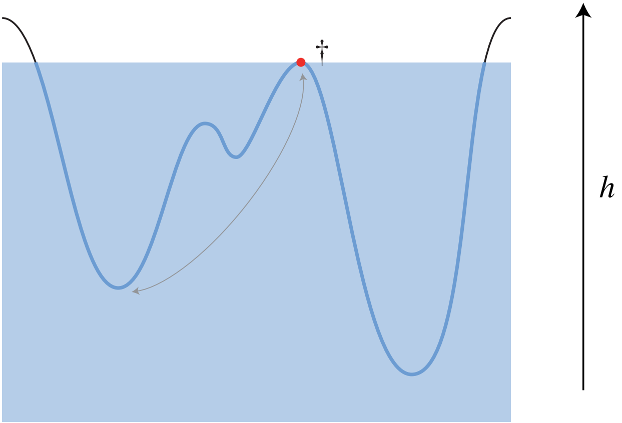
Example: connected components in 1D



Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for sublevel sets

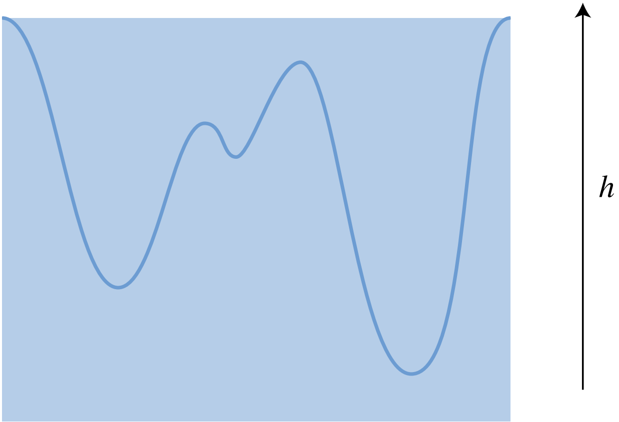
Example: connected components in 1D



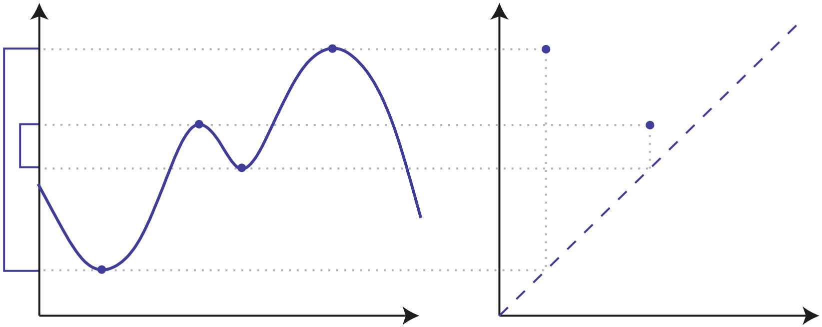
Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for sublevel sets

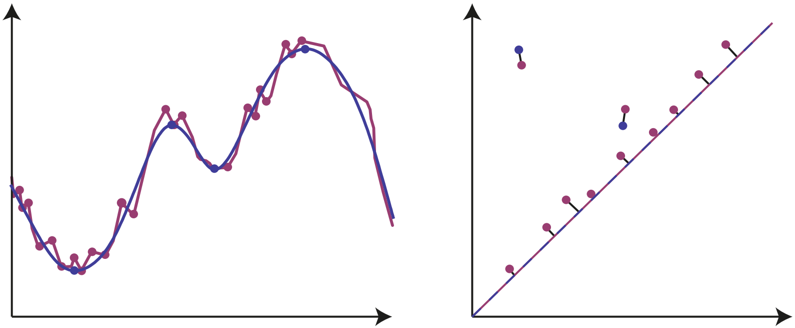
Example: connected components in 1D



Persistence diagrams [Cohen-Steiner et al., 2005]



Persistence diagrams [Cohen-Steiner et al., 2005]



Stability of persistence diagrams

Theorem (Cohen-Steiner et al., 2005)

Let $\|f - g\|_\infty < \delta$. The persistence pairs of f with persistence $\geq 2\delta$ can be mapped injectively to the persistence pairs of g, such that the corresponding points in the persistence diagrams have pairwise distance $\|p_f - p_g\|_\infty < \delta$.

A bound on number of critical points

Corollary

Let f be a discrete Morse function on a surface and let $\delta > 0$.

Then for every function f_δ with $\|f - f_\delta\|_\infty < \delta$ we have:

$$\begin{aligned} &\#\,\textit{critical points of } f_\delta \\ \geq\ &\#\,\textit{critical points of } f \textit{ with persistence} \geq 2\delta \end{aligned}$$

A bound on number of critical points

Corollary

Let f be a discrete Morse function on a surface and let $\delta > 0$.

Then for every function f_δ with $\|f - f_\delta\|_\infty < \delta$ we have:

$$\begin{aligned} &\#\,\textit{critical points of } f_\delta \\ \geq\ &\#\,\textit{critical points of } f \textit{ with persistence} \geq 2\delta \end{aligned}$$

Strategy: Cancel all persistence pairs of f with persistence $< 2\delta$

- Is cancelation always allowed?

A bound on number of critical points

Corollary

Let f be a discrete Morse function on a surface and let $\delta > 0$.

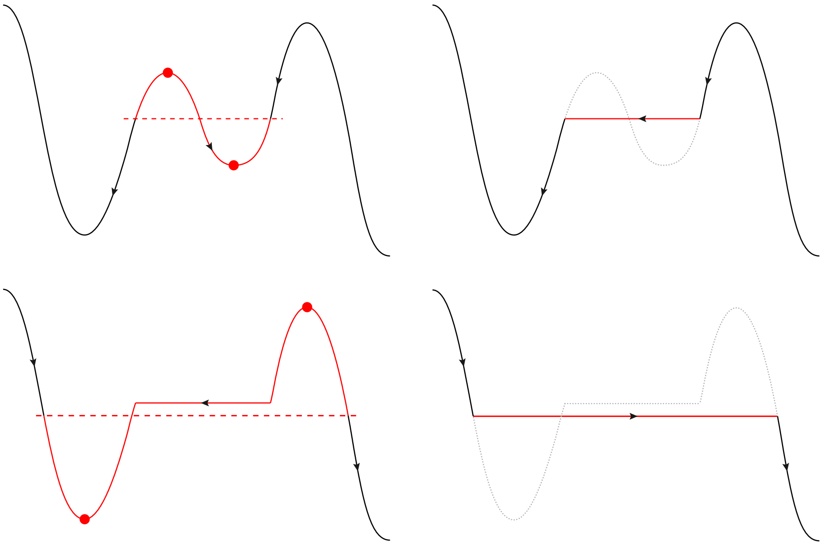
Then for every function f_δ with $\|f - f_\delta\|_\infty < \delta$ we have:

$$\begin{aligned} & \#\,\textit{critical points of } f_\delta \\ \geq & \#\,\textit{critical points of } f \textit{ with persistence} \geq 2\delta \end{aligned}$$

Strategy: Cancel all persistence pairs of f with persistence $< 2\delta$

- Is cancelation always allowed?
- Order of cancelation?

Canceling two nested persistence pairs



Connecting persistence and Morse theory

Theorem (B., Lange, Wardetzky, 2010)

Consider a persistence pair (σ, τ) *of a discrete Morse function on a surface. Then* (σ, τ) *can be canceled after all persistence pairs* $(\tilde{\sigma}, \tilde{\tau})$ *with* $f(\sigma) < f(\tilde{\sigma}) < f(\tilde{\tau}) < f(\tau)$ *have been canceled.*

Connecting persistence and Morse theory

Theorem (B., Lange, Wardetzky, 2010)

Consider a persistence pair (σ, τ) *of a discrete Morse function on a surface. Then* (σ, τ) *can be canceled after all persistence pairs* $(\tilde{\sigma}, \tilde{\tau})$ *with* $f(\sigma) < f(\tilde{\sigma}) < f(\tilde{\tau}) < f(\tau)$ *have been canceled.*

Corollary

On a surface, it is possible to cancel just the persistence pairs with persistence $< 2\delta$ *(without canceling the other pairs).*

Connecting persistence and Morse theory

Theorem (B., Lange, Wardetzky, 2010)

Consider a persistence pair (σ, τ) *of a discrete Morse function on a surface. Then* (σ, τ) *can be canceled after all persistence pairs* $(\tilde{\sigma}, \tilde{\tau})$ *with* $f(\sigma) < f(\tilde{\sigma}) < f(\tilde{\tau}) < f(\tau)$ *have been canceled.*

Corollary

On a surface, it is possible to cancel just the persistence pairs with persistence $< 2\delta$ *(without canceling the other pairs).*

What about the tolerance? Does the error accumulate?

Main result

Theorem (B., Lange, Wardetzky, 2010)

Let f be a discrete Morse function on a surface and let $\delta > 0$.

Then there exists a function f_δ with:

- $\|f - f_\delta\|_\infty < \delta$.
- *All persistence pairs of f with persistence $< 2\delta$ are canceled.*

Main result

Theorem (B., Lange, Wardetzky, 2010)

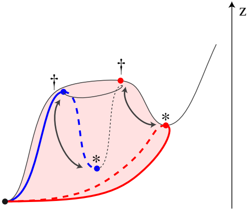
Let f be a discrete Morse function on a surface and let $\delta > 0$.

Then there exists a function f_δ with:

- $\|f - f_\delta\|_\infty < \delta$.
- *All persistence pairs of f with persistence $< 2\delta$ are canceled.*

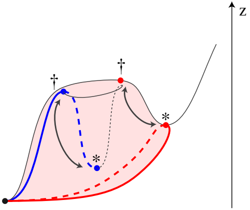
This function achieves the minimal number of critical points.

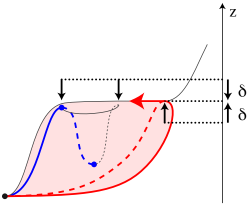
An example in 2D



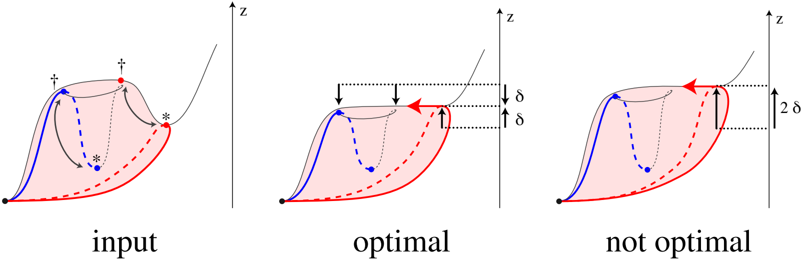
input

An example in 2D





An example in 2D



On surfaces, cancelation may affect other critical values!

An efficient algorithm

A simplified function f_δ can be computed efficiently:

An efficient algorithm

A simplified function f_δ can be computed efficiently:

- Computing persistence pairs: $O(\mathrm{sort}(n))$
  (modified Kruskal's MST algorithm)
- Computing simplified gradient vector field: $O(n)$
  (DFS graph traversal)
- Computing simplified function: $O(n)$
  (topological sort by DFS graph traversal)

An efficient algorithm

A simplified function f_δ can be computed efficiently:

- Computing persistence pairs: $O(\mathrm{sort}(n))$
  (modified Kruskal's MST algorithm)
- Computing simplified gradient vector field: $O(n)$
  (DFS graph traversal)
- Computing simplified function: $O(n)$
  (topological sort by DFS graph traversal)

Proof of correctness requires previous theorem!

Combination with energy methods

Recall: simplified vector field V_δ imposes inequalities on simplified function
$\|f - f_\delta\|_\infty < \delta$: another set of linear inequalities

Combination with energy methods

Recall: simplified vector field V_δ imposes inequalities on simplified function

$\|f - f_\delta\|_\infty < \delta$: another set of linear inequalities

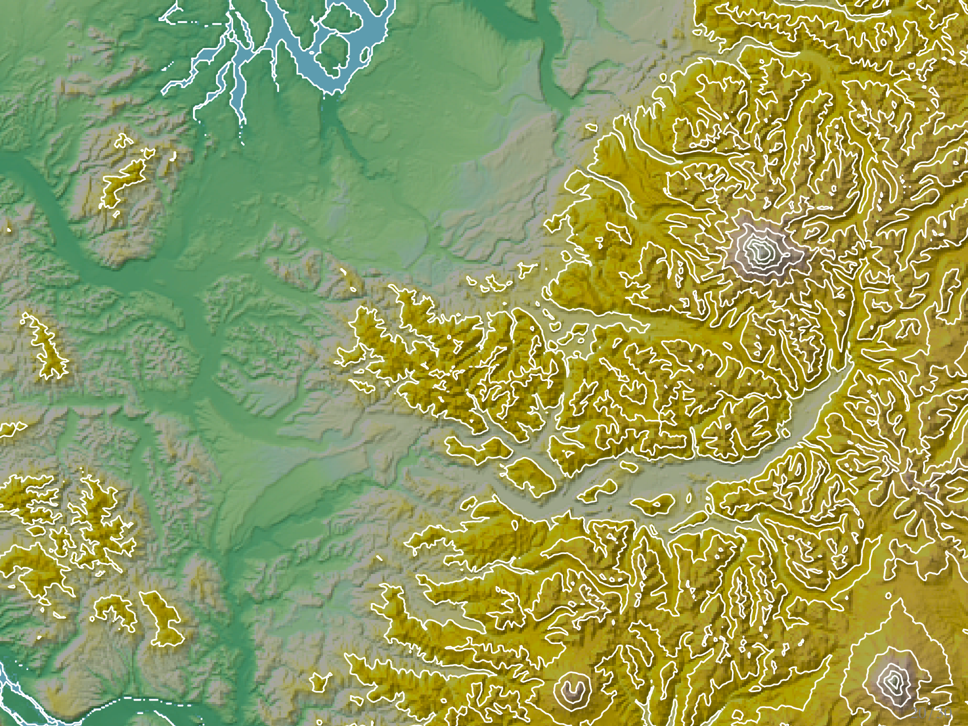
- defines convex set of solutions

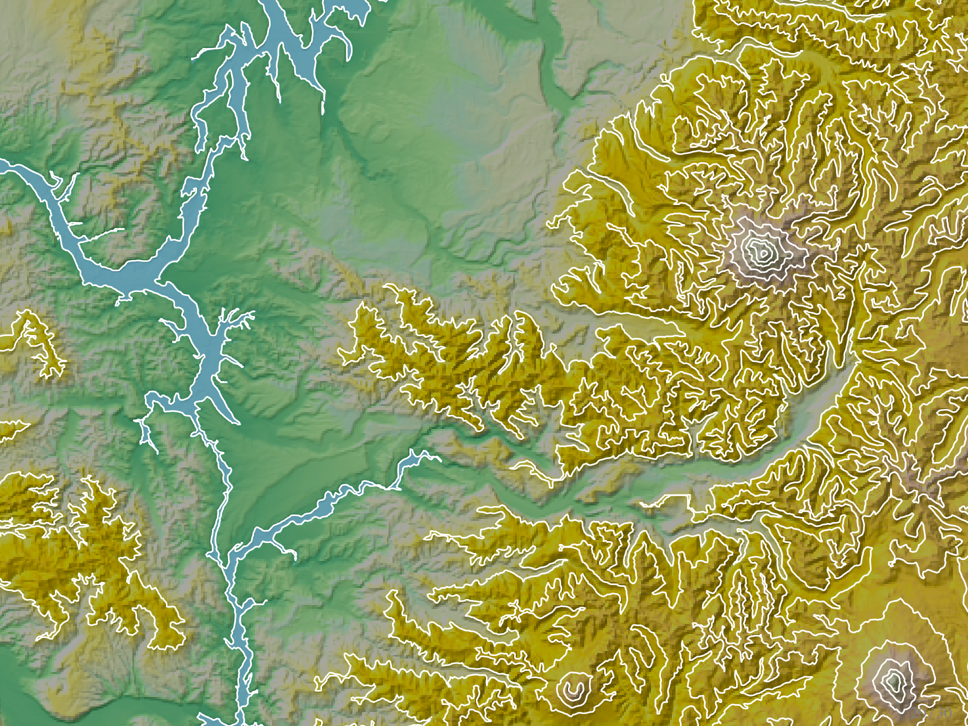
Combination with energy methods

Recall: simplified vector field V_δ imposes inequalities on simplified function

$\|f - f_\delta\|_\infty < \delta$: another set of linear inequalities

- defines convex set of solutions
- find the “best” solution using your favorite energy functional





Robustness

What about other tolerance norms than $\|\cdot\|_\infty$?

- Persistence is not robust to outliers!

A robust denoising method

A well-known denoising method (Rudin, Osher, Fatemi, 1992):
Given $f \in L^1([a,b])$, $\alpha \geq 0$. Find u minimizing

$$2\alpha \, \mathrm{TV}(u) + \|f - u\|_2 \tag{ROF}$$

A robust denoising method

A well-known denoising method (Rudin, Osher, Fatemi, 1992):
Given $f \in L^1([a,b])$, $\alpha \geq 0$. Find u minimizing

$$2\alpha \, \mathrm{TV}(u) + \|f - u\|_2 \qquad \text{(ROF)}$$

Recall the definition of TV *(total variation)*:

- For differentiable u: $\mathrm{TV}(u) = \int_a^b |u'(t)| \, \mathrm{d}t$
- For $u \in L^1([a,b])$:
  $\mathrm{TV}(u) = \sup \left\{ \int_a^b u(t)\phi'(t) \, \mathrm{d}t : \phi \in C_0^1([a,b]), \ \|\phi\|_\infty \leq 1 \right\}$

Equivalence with the taut string problem

Theorem (Grasmair, 2006)

Let u_α be a minimizer of the ROF functional. Then $u_\alpha = U'_\alpha$, where U_α minimizes

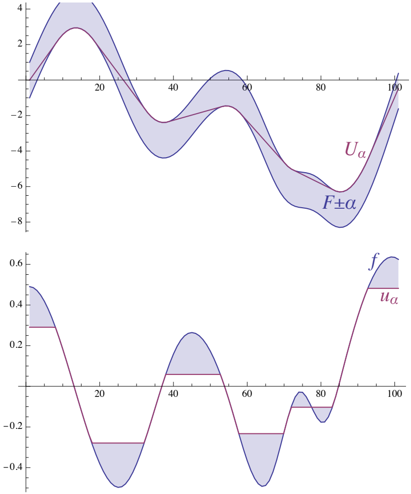
$$\int_a^b \sqrt{1 + U'(t)^2}\,\mathrm{d}t$$

subject to

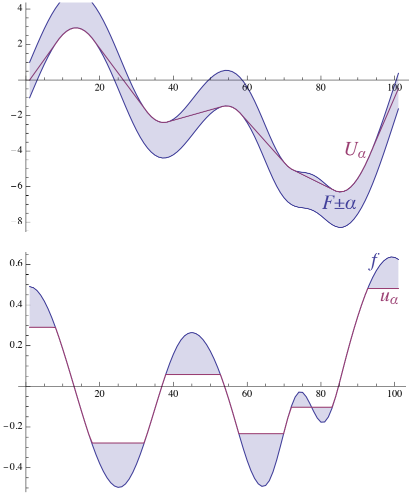
$$U(a) = F(a), \quad U(b) = F(b), \quad \|U - F\|_\infty \le \alpha,$$

where $F(t) = \int_a^t f(\tau)\,\mathrm{d}\tau$.

An example

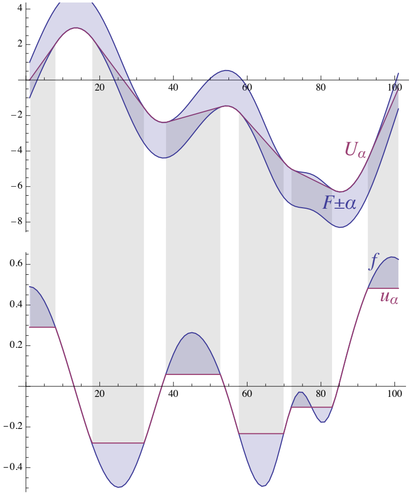


An example



Observations:

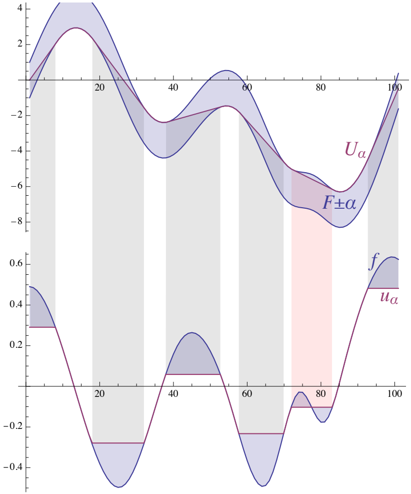
An example



Observations:

- u_α coincides with f apart from some intervals, on which it is constant

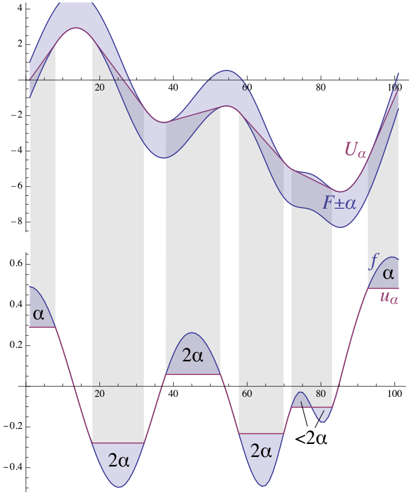
An example



Observations:

- u_α coincides with f apart from some intervals, on which it is constant
- Some but not all constant intervals cancel critical points

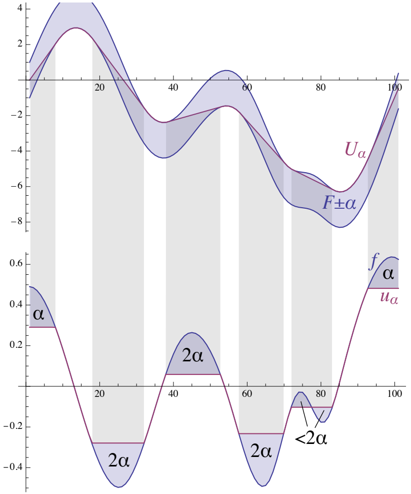
An example



Observations:

- u_α coincides with f apart from some intervals, on which it is constant
- Some but not all constant intervals cancel critical points

An example

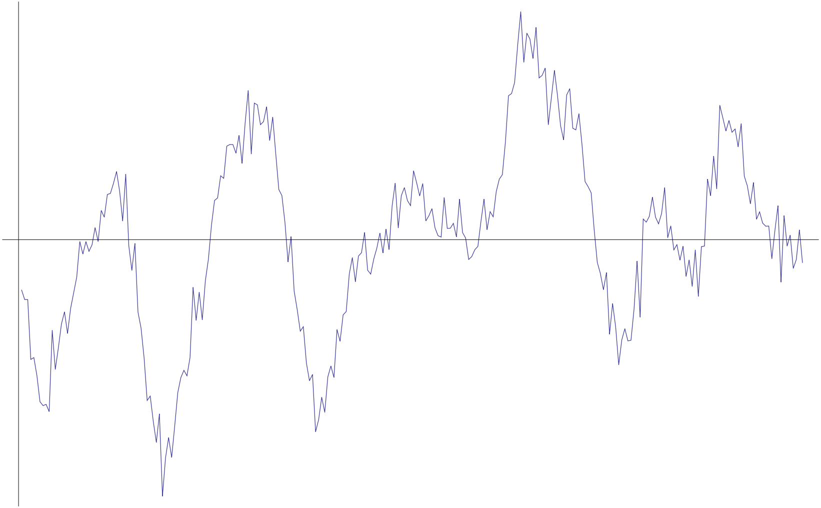


Observations:

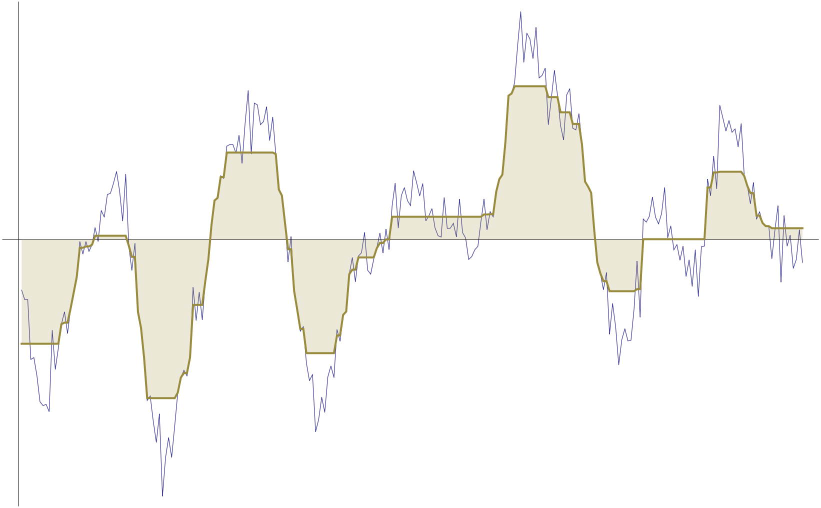
- u_α coincides with f apart from some intervals, on which it is constant
- Some but not all constant intervals cancel critical points

Filling significance of a critical point: $2\alpha^*$
(α^*: smallest α that cancels the critical point in u_α)

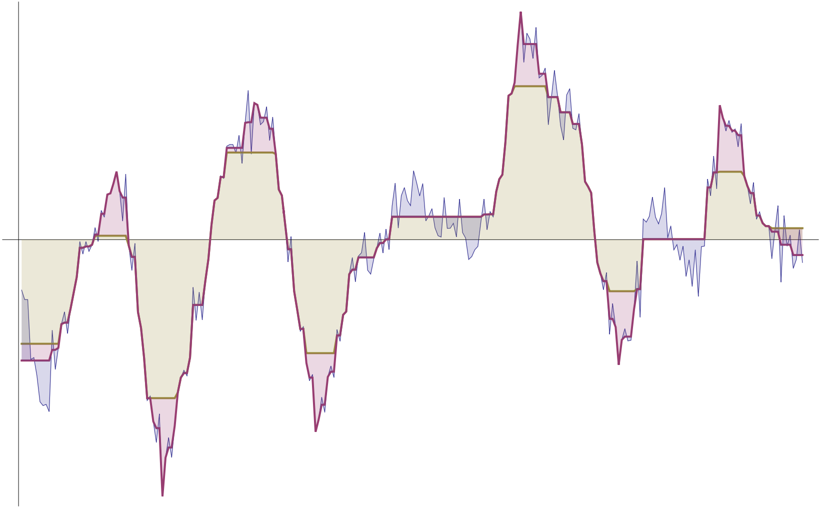
The “best of both worlds”



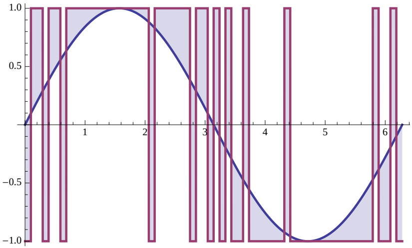
The “best of both worlds”



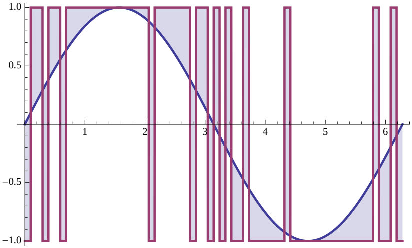
The “best of both worlds”

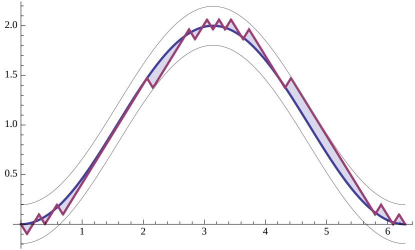


Another example: Delta-Sigma conversion

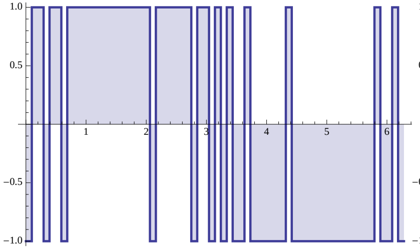


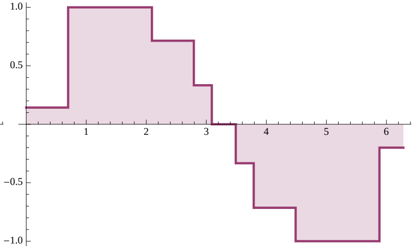
Another example: Delta-Sigma conversion



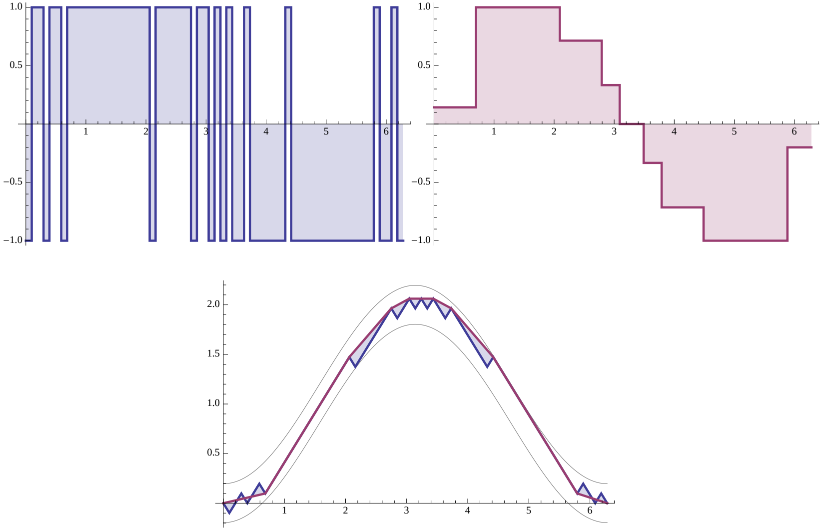


Another example: Delta-Sigma conversion





Another example: Delta-Sigma conversion



Future work

- Stability of descriptors (analogous to persistence diagrams)
- Extension or different approach for higher dimensional domains

Present work

U. Bauer, C. Lange, and M. Wardetzky.
Optimal topological simplification of discrete functions on surfaces.
arXiv preprint, 2010.
arXiv:1001.1269

U. Bauer, C.-B. Schönlieb, and M. Wardetzky.
Total Variation Meets Topological Persistence: A First Encounter.
Proceedings of ICNAAM 2010, 1022–1026.
doi:10.1063/1.3497795

Past work

R. Forman.
A user's guide to discrete Morse theory.
Sém. Loth. de Combinatoire, B48c:1–35, 2002.

H. Edelsbrunner, D. Letscher, and A. Zomorodian.
Topological Persistence and Simplification.
Discr. Comp. Geometry, 28(4):511–533, 2002.

D. Cohen-Steiner, H. Edelsbrunner, and J. Harer.
Stability of Persistence Diagrams.
Discr. Comp. Geometry, 37(1):103–120, 2007.

Thanks for your attention!