

Geometric complexes and topological persistence

A panoramic view

Ulrich Bauer

Technical University of Munich (TUM)

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TSVP Thematic Program

TDA PARTI: Topological Data Analysis, Persistence And Representation Theory Intertwined

OIST, Okinawa

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Deutsche
Forschungsgemeinschaft
German Research Foundation



SFB
TRR
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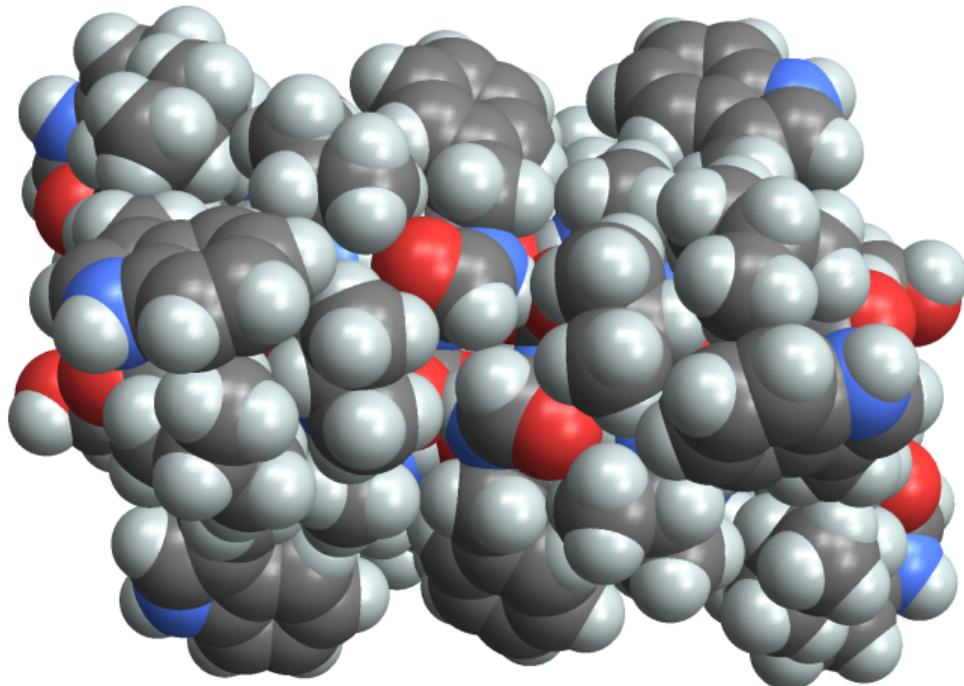
Discretization
In Geometry
and DynamIcs

Technical
University
of Munich



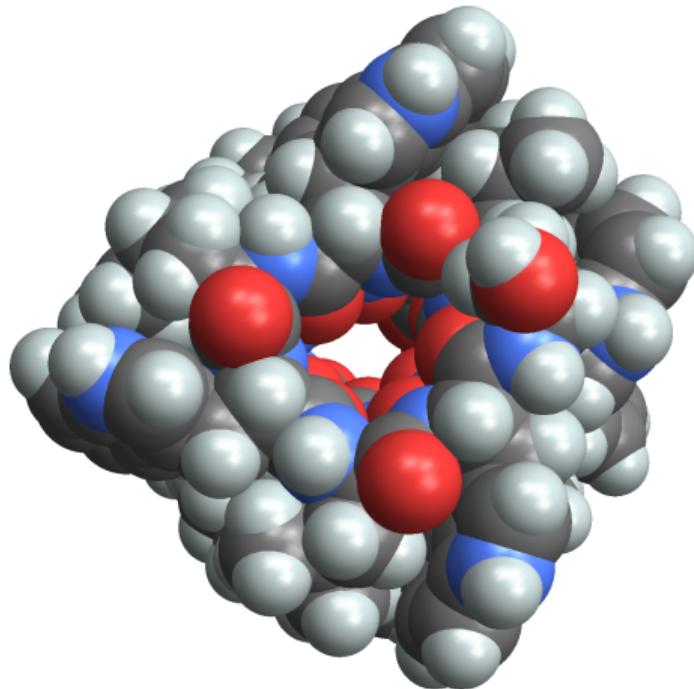
mcmL
Munich Center for Machine Learning

Biogeometry



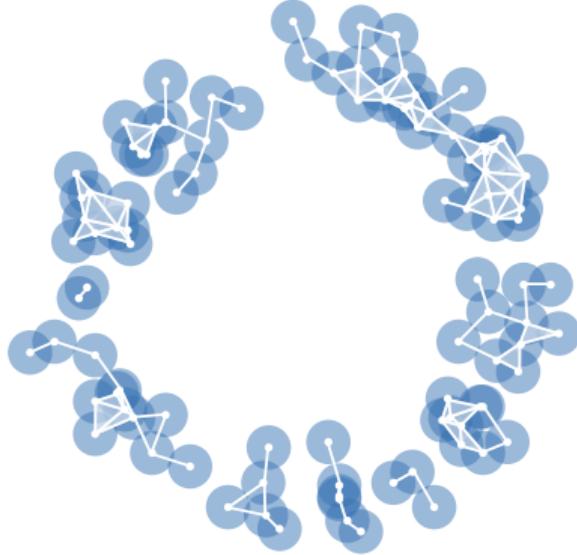
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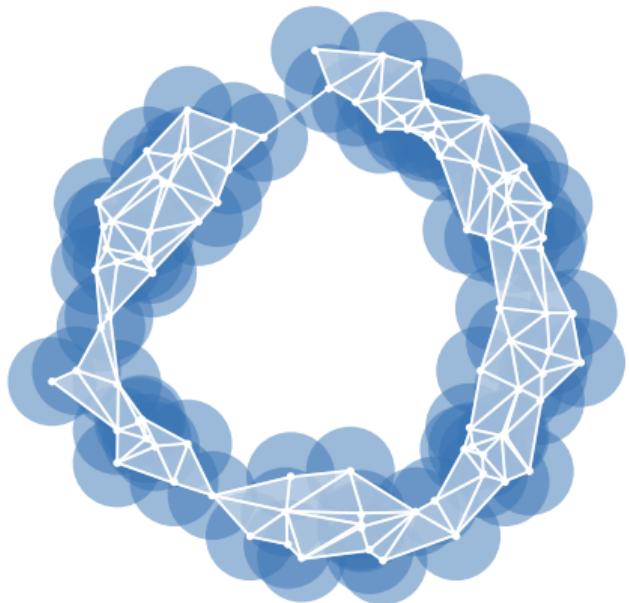
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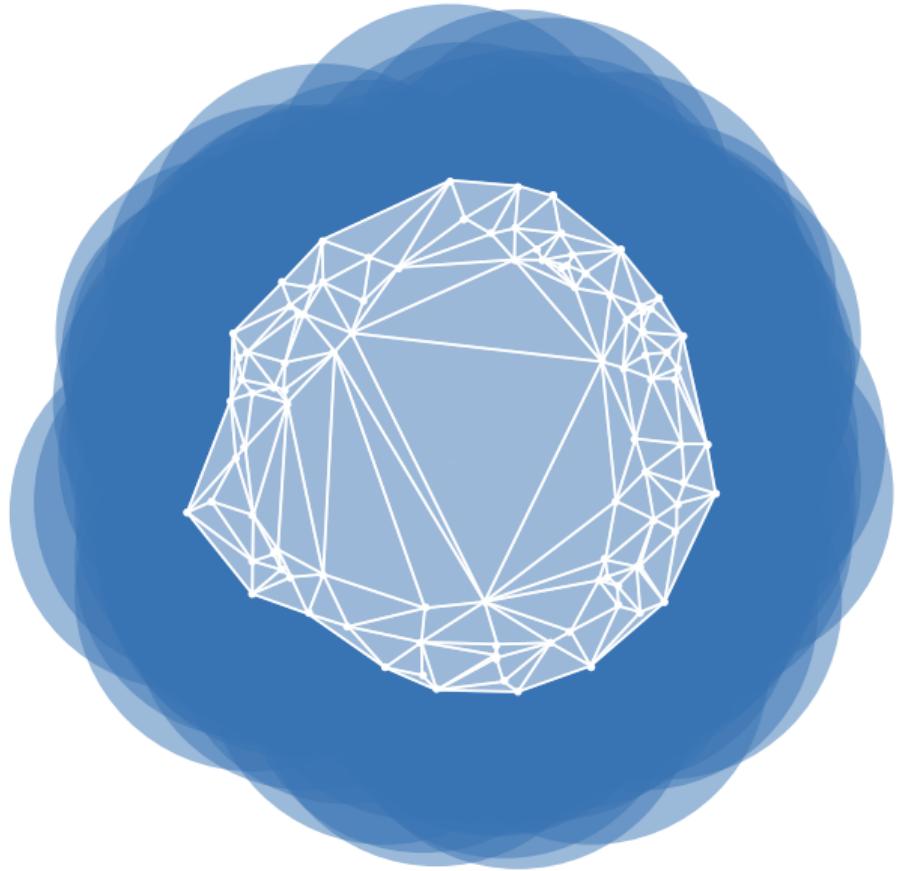
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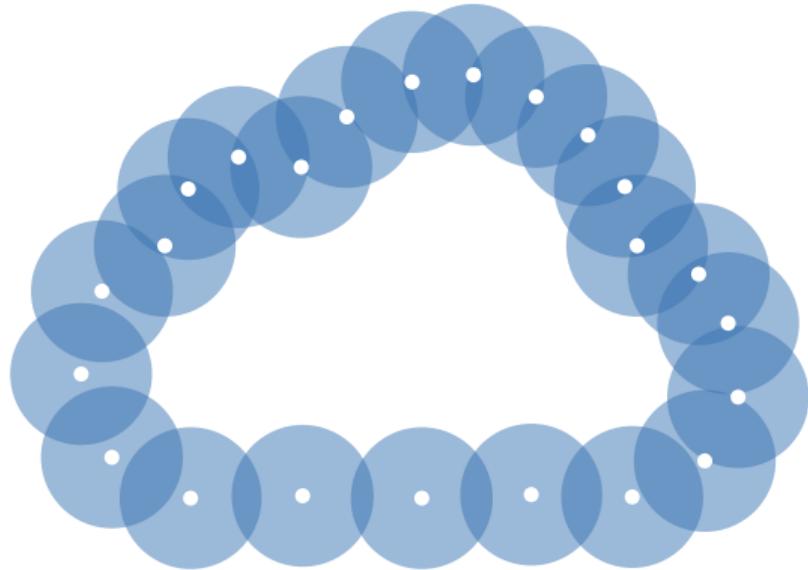




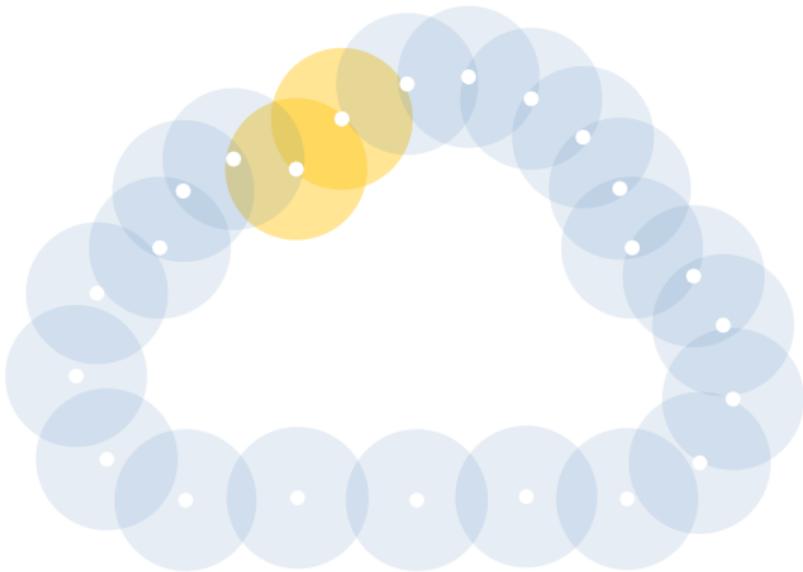


Geometric complexes

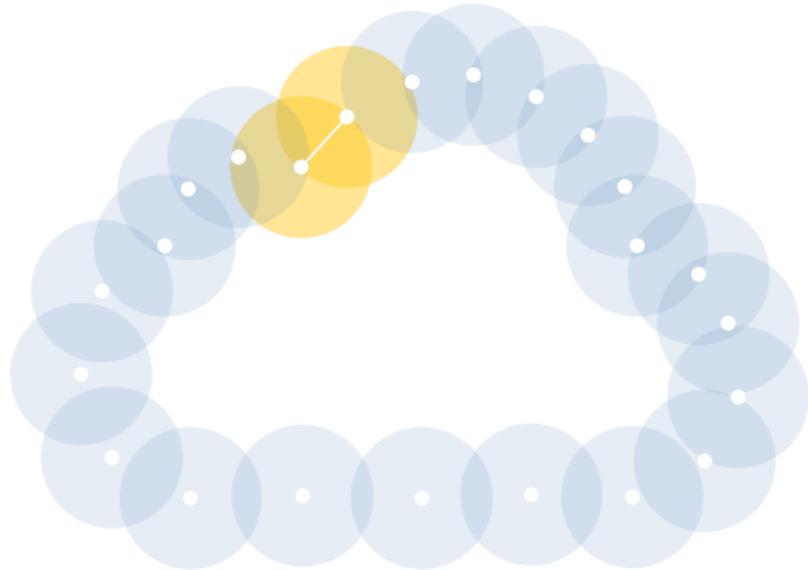
Čech complexes



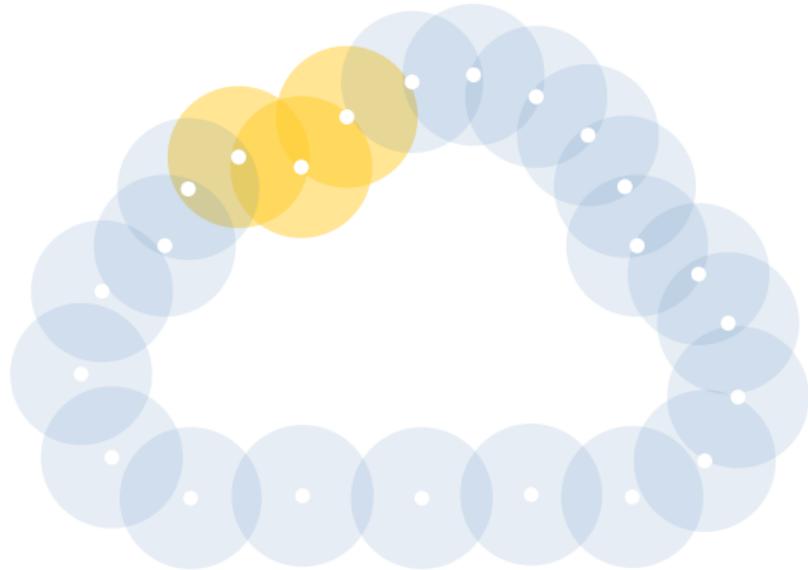
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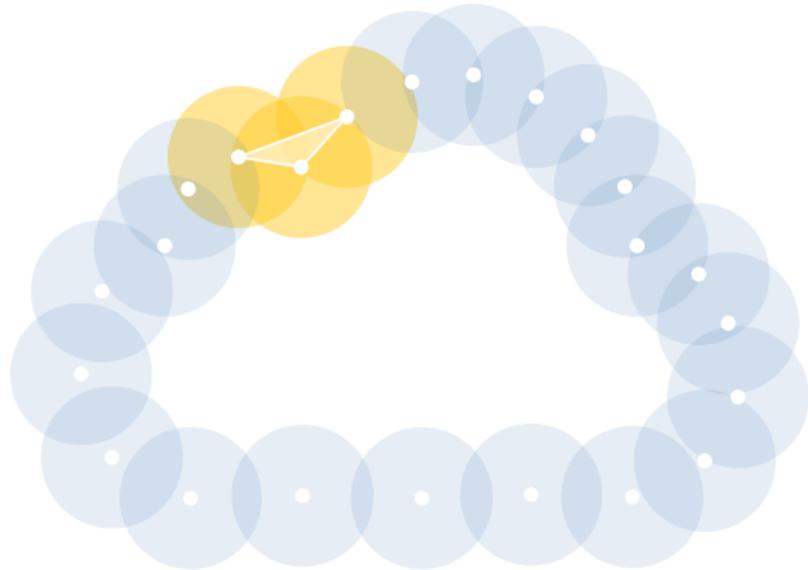
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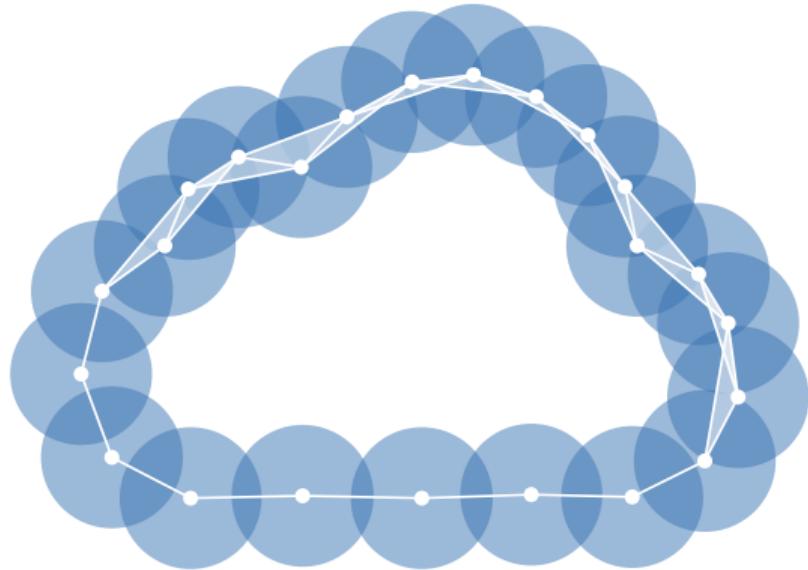
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Nerves

Definition (Alexandrov 1928)

Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space X . The *nerve* of \mathcal{U} is the simplicial complex

$$\text{Nrv}(\mathcal{U}) = \{J \subseteq I \mid 0 < |J| < \infty \text{ and } \bigcap_{j \in J} U_j \neq \emptyset\}.$$

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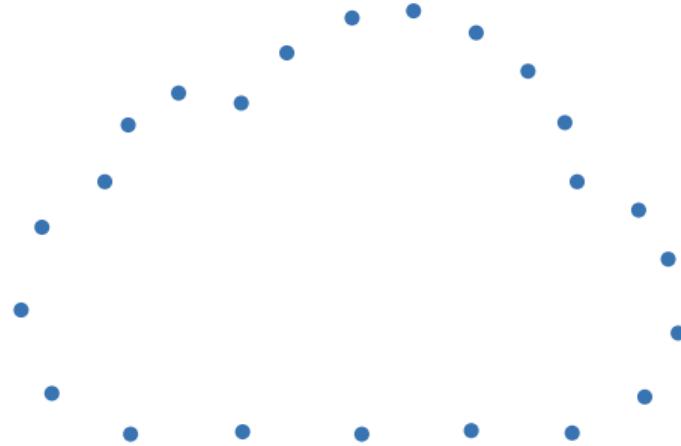


U. Bauer, M. Kerber, F. Roll, and A. Rolle

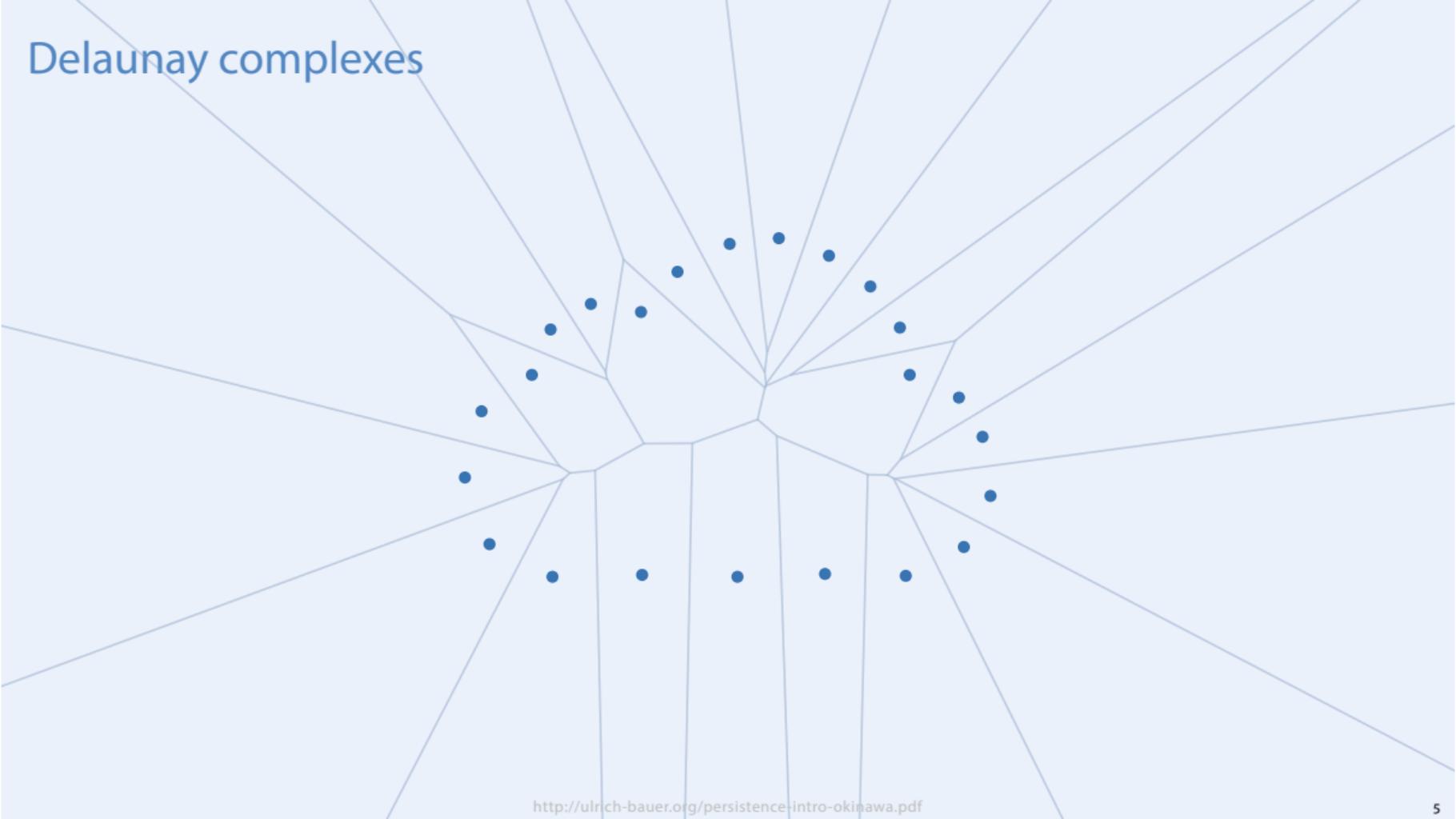
A Unified View on the Functorial Nerve Theorem and its Variations

Expositiones Mathematicae, 2023. doi:10.1016/j.exmath.2023.04.005

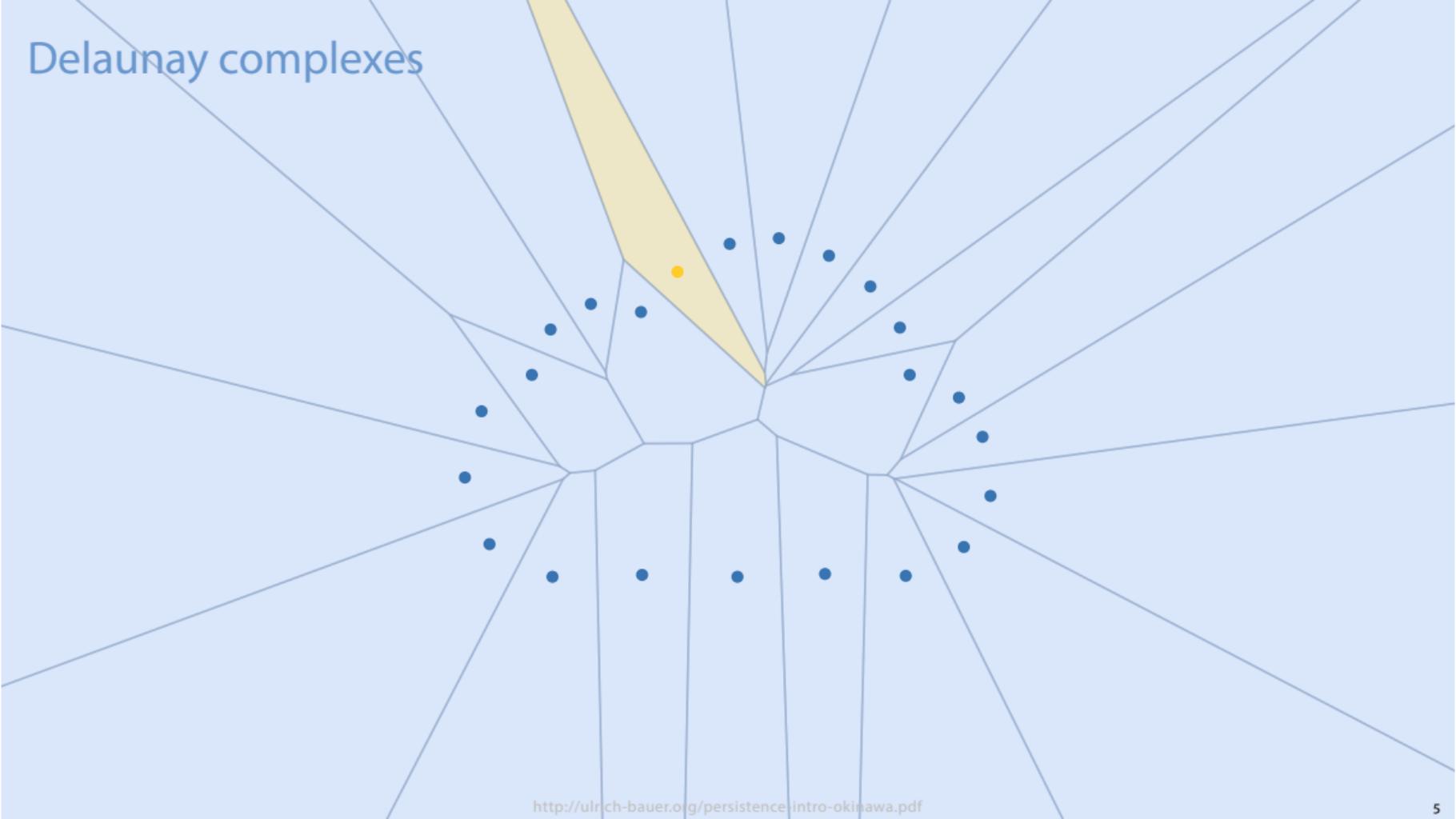
Delaunay complexes



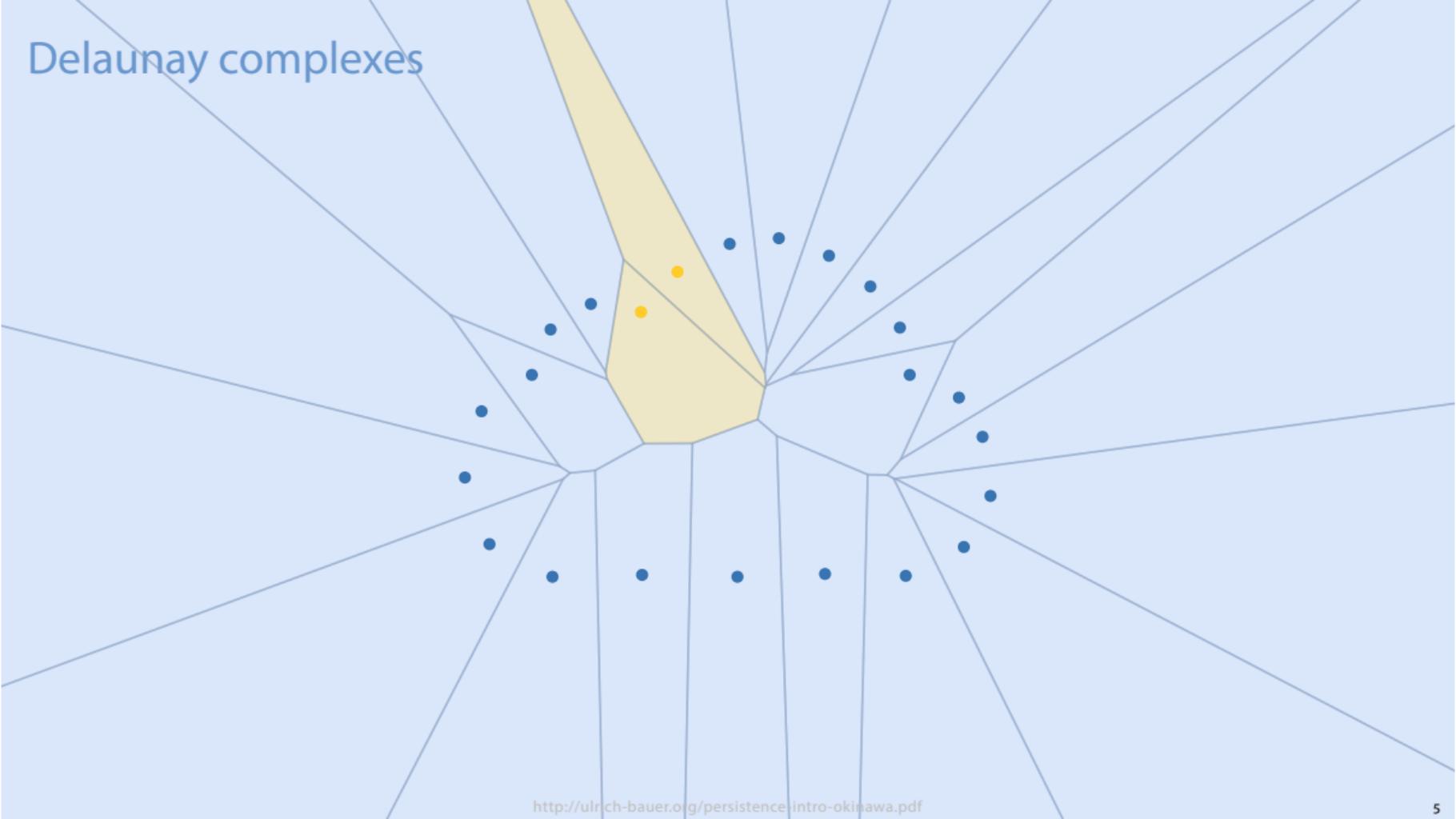
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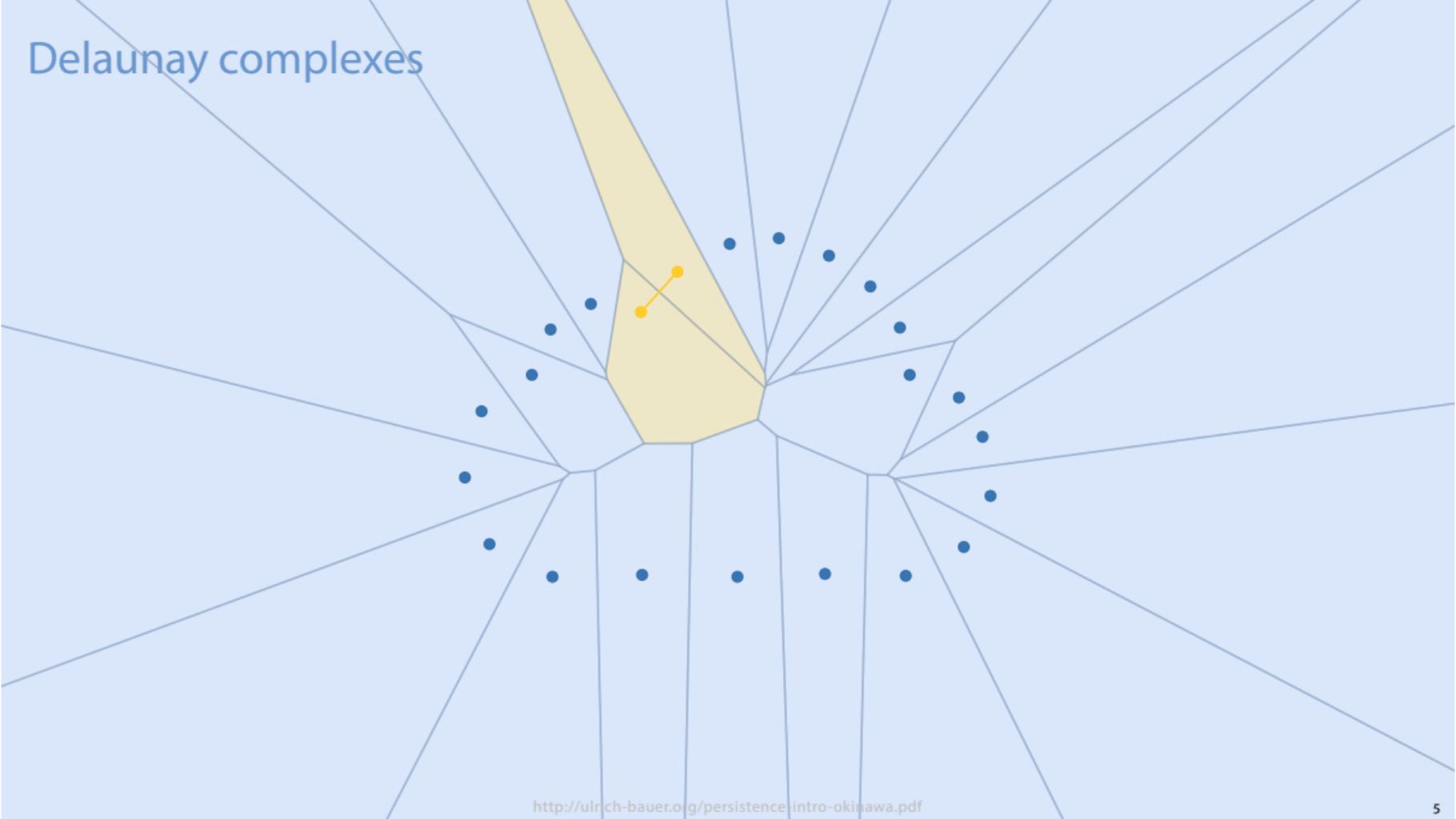
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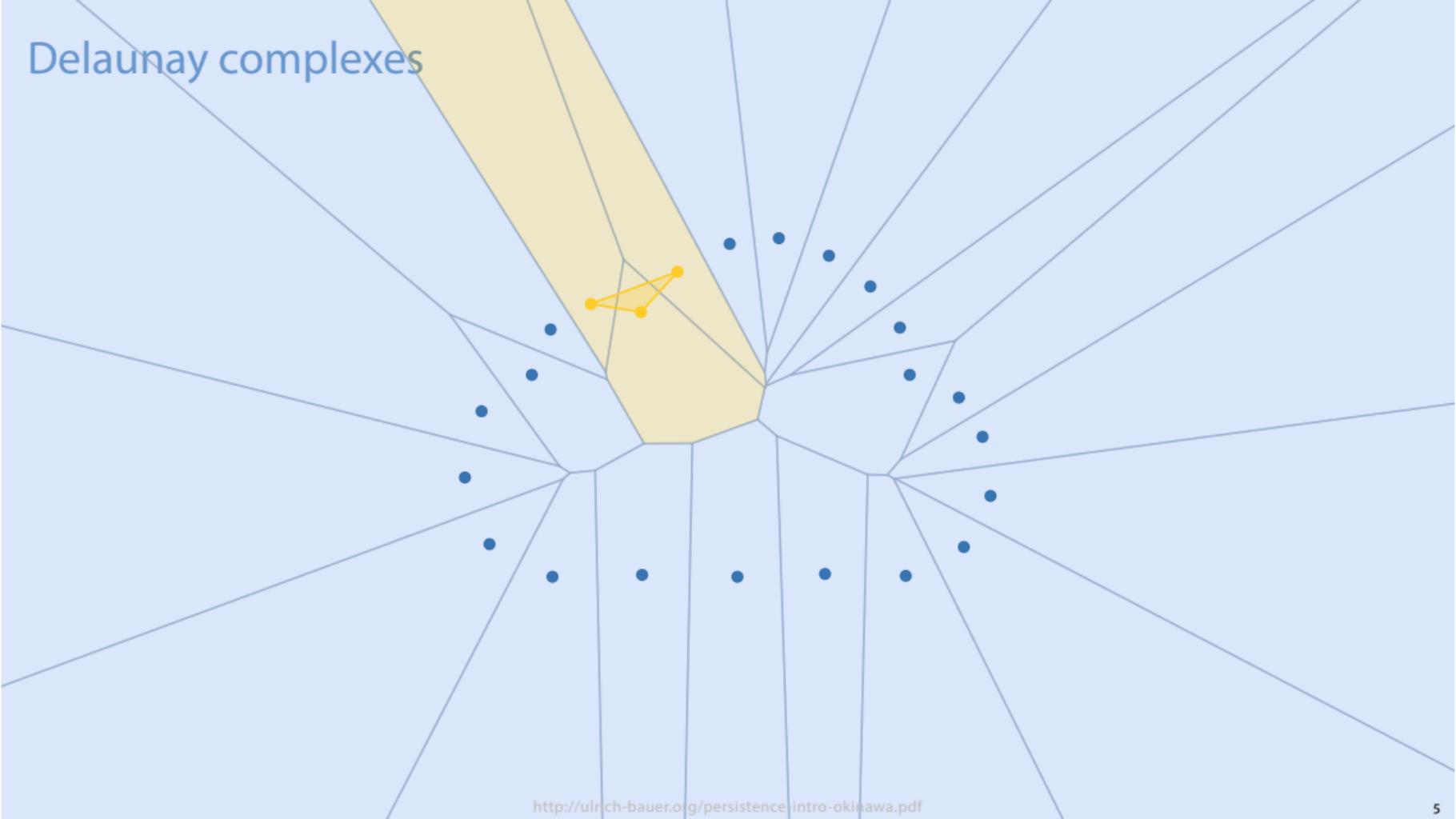
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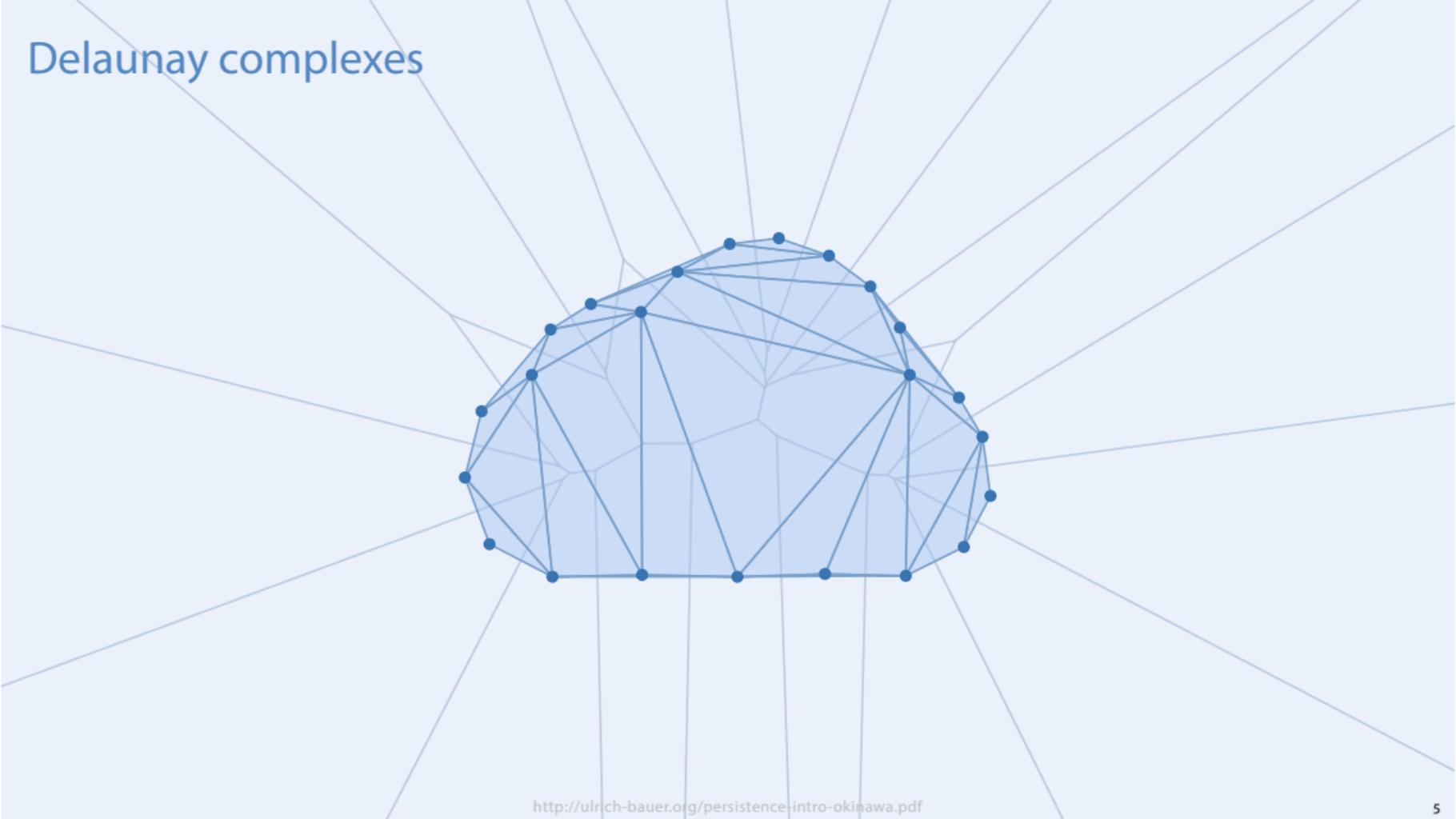
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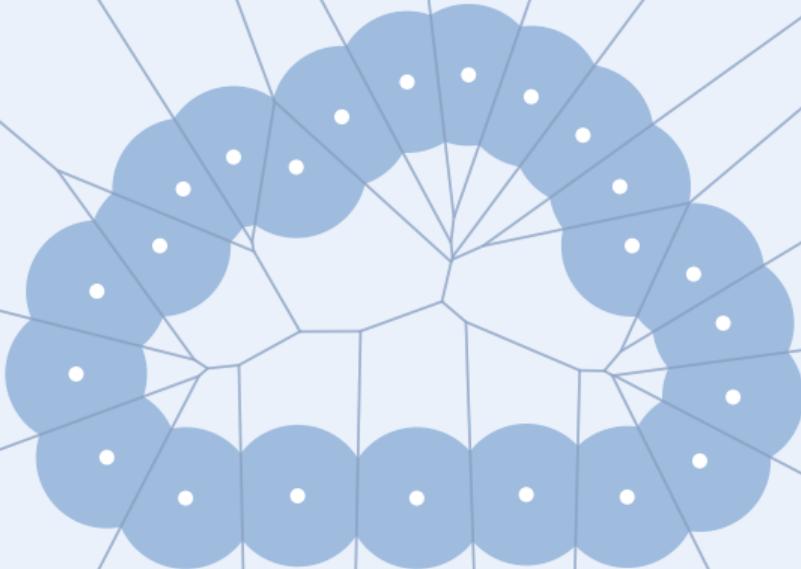
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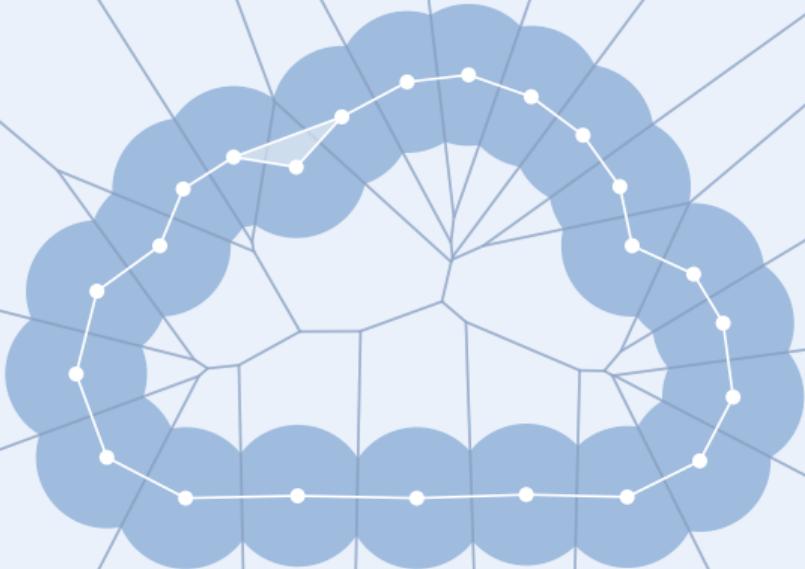
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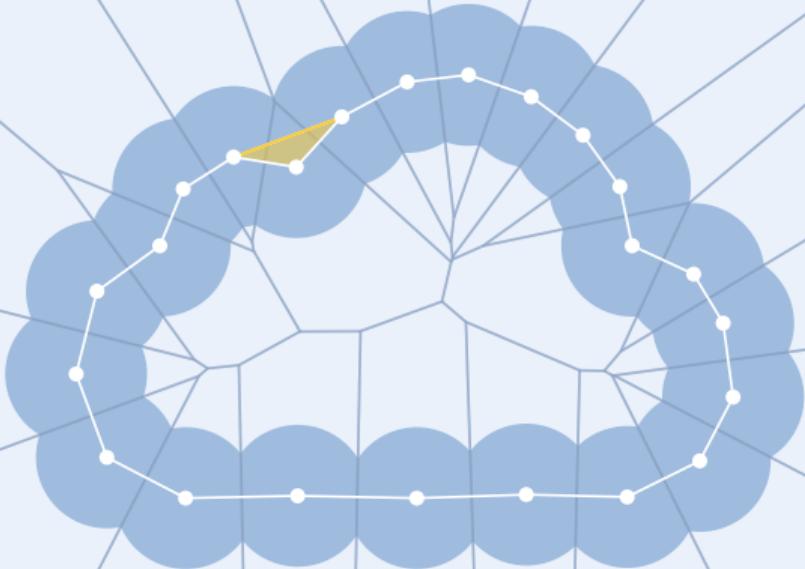
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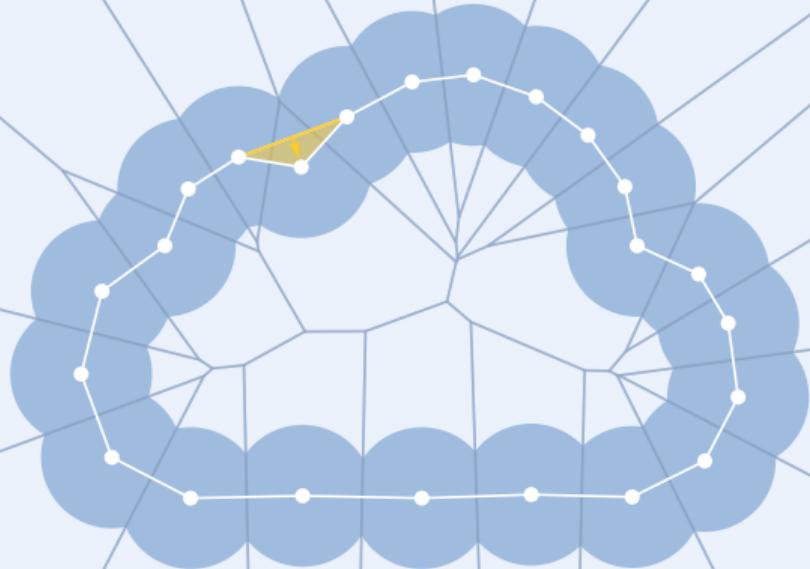
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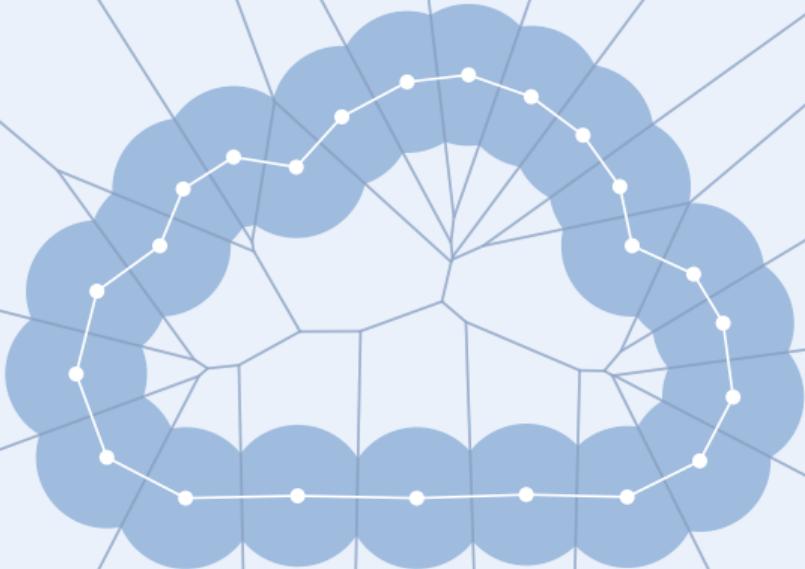
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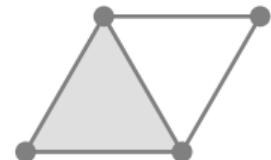
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Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

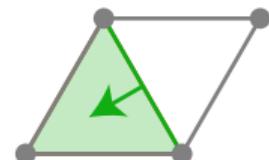


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If there is a sequence of such elementary collapses from K to M , we say that K *collapses* to M (written as $K \searrow M$).

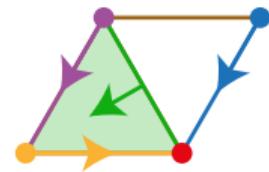


Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ (*critical cells*), and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ .

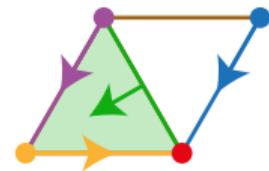


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A function $f : K \rightarrow \mathbb{R}$ on a cell complex is a *discrete Morse function* if

- the sublevel sets $f^{-1}(-\infty, t]$ are subcomplexes, and

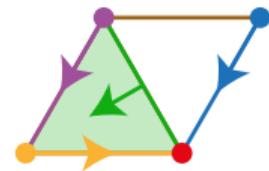
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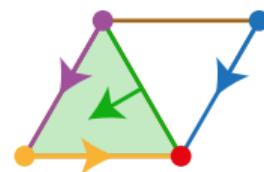


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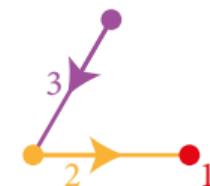
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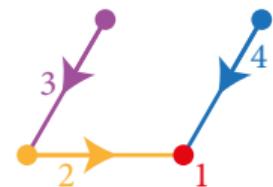
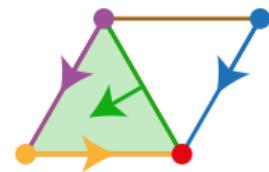
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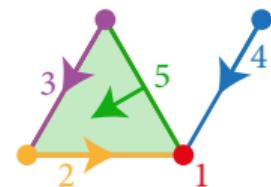
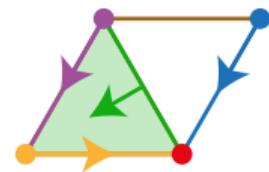
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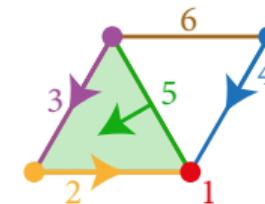
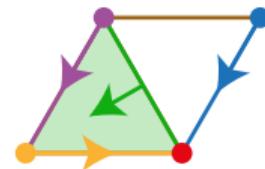
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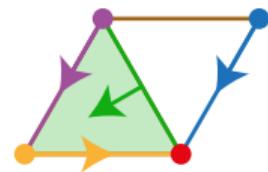


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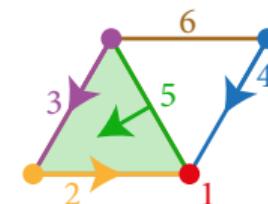
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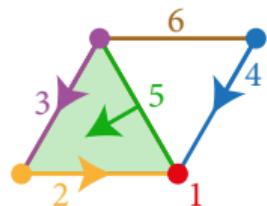


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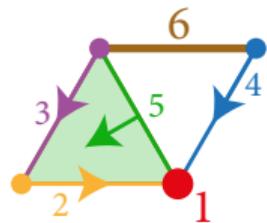
- the sublevel sets $f^{-1}(-\infty, t]$ are subcomplexes, and
- the level sets $f^{-1}(t)$ form a discrete vector field.



Fundamental theorem of discrete Morse theory



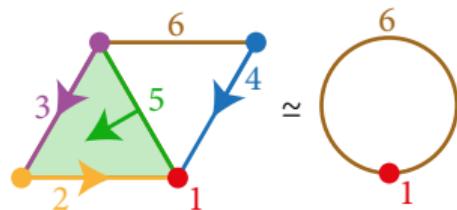
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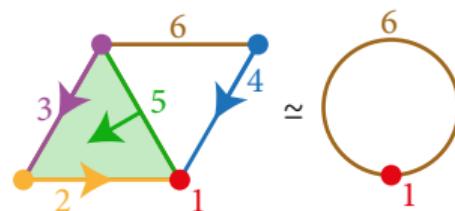
A simplicial complex with a discrete Morse function f is homotopy equivalent to a space (a CW complex) built from the critical simplices off.



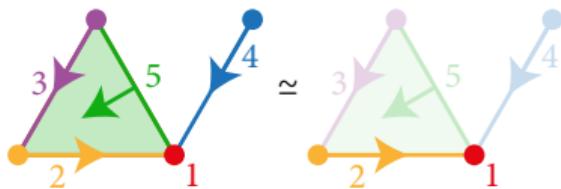
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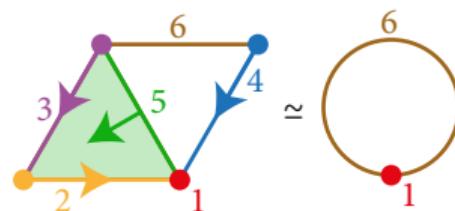
Discrete Morse functions – and their gradients – encode collapses of sublevel sets:



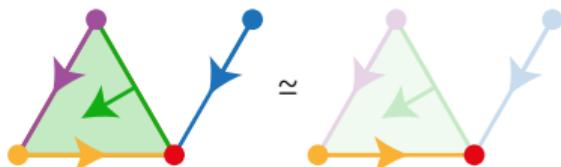
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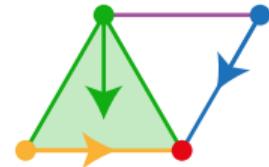
Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex K is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}$$

- indicated by an arrow from L to U



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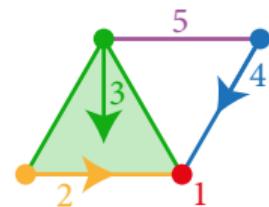
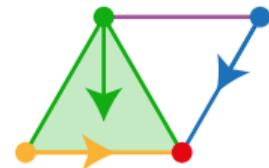
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A *generalized discrete Morse function* $f : K \rightarrow \mathbb{R}$ satisfies:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes (for all $t \in \mathbb{R}$)
- the level sets $f^{-1}(t)$ form a generalized vector field (the *discrete gradient* of f)



Morse theory for Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

The Čech complexes and the Delaunay complexes (alpha shapes) are sublevel sets of (generalized) discrete Morse functions.

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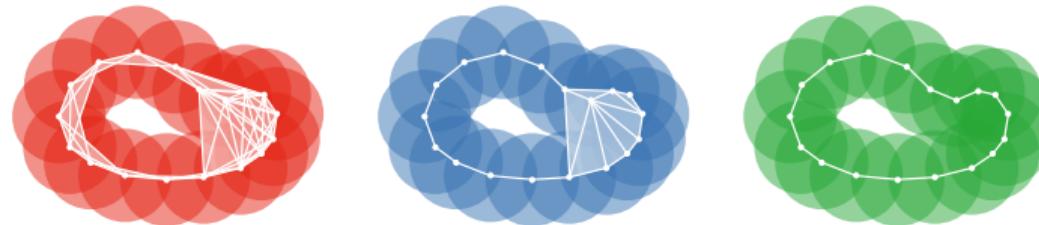
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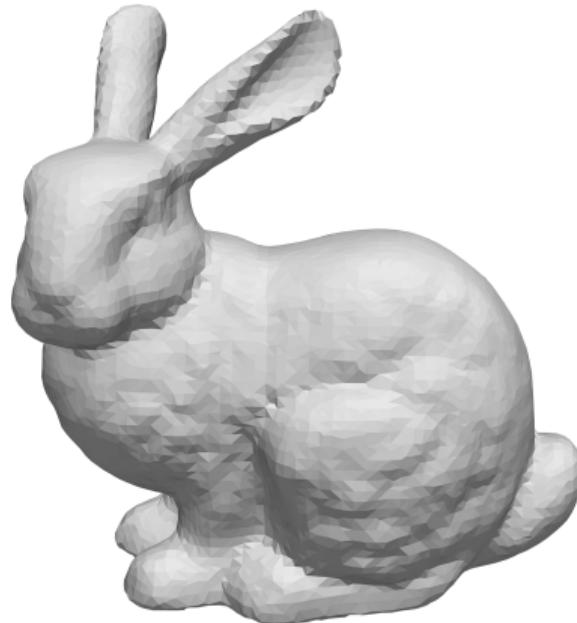
Čech, Delaunay, and Wrap complexes (at any scale r) of a point set $X \subset \mathbb{R}^d$ in general position are related by collapses encoded by a single discrete gradient field:

$$\text{Cech}_r X \searrow \text{Del}_r X \searrow \text{Wrap}_r X.$$



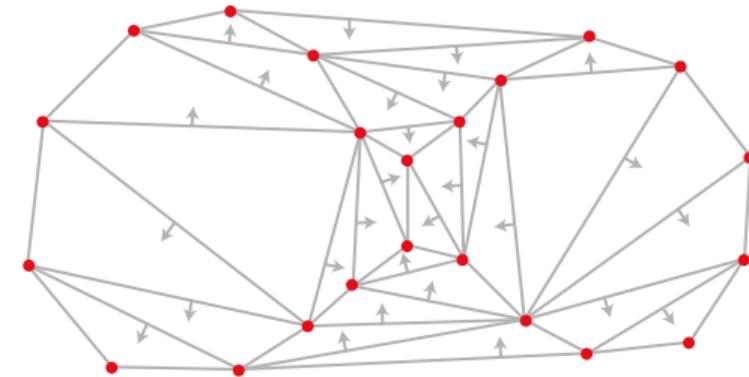
Wrap complexes

Foundation of the surface reconstruction software *Wrap* (Edelsbrunner 1995, Geomagic)



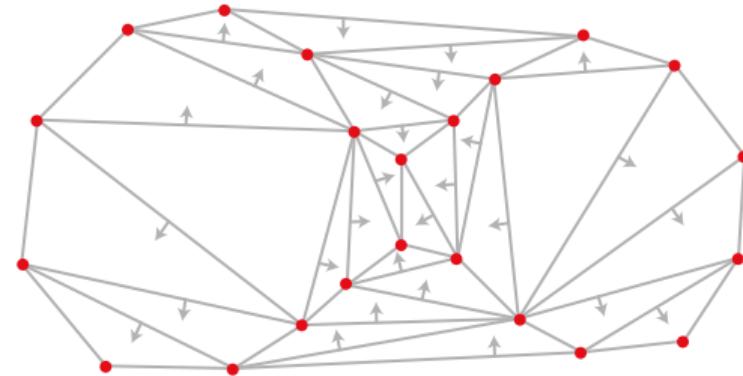
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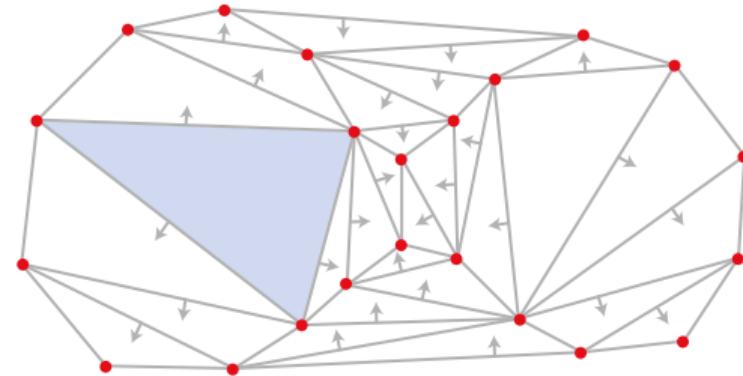
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\text{Wrap}_r(X)$ is the *descending complex* of V on $\text{Del}_r X$:

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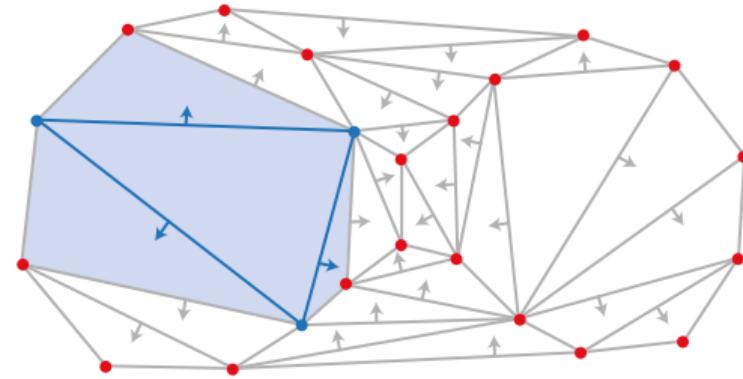
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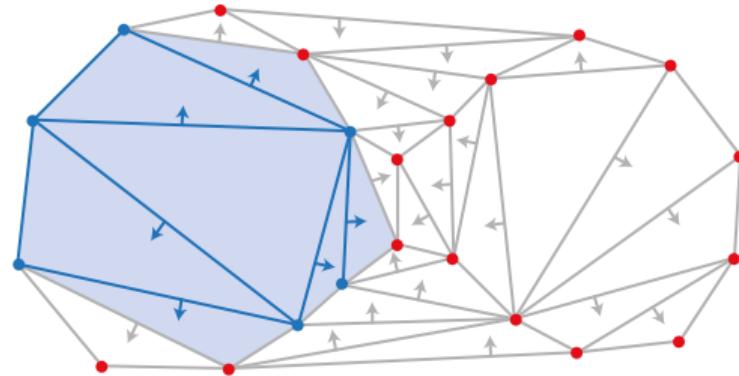
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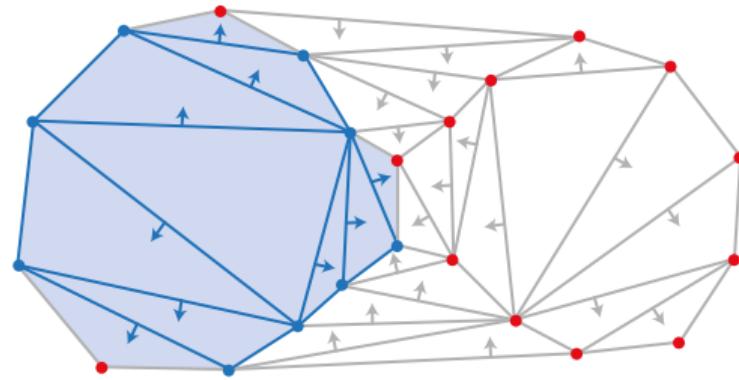
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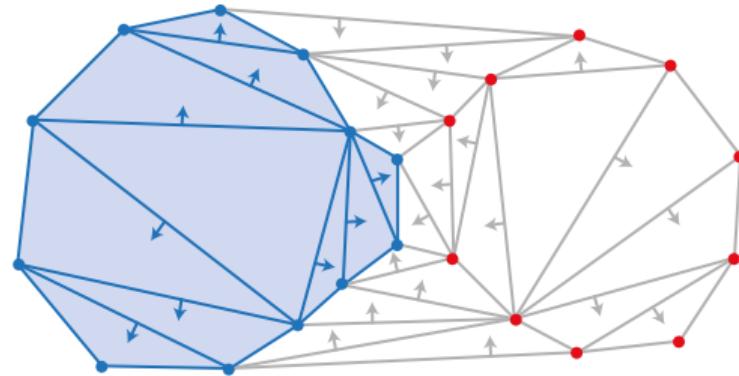
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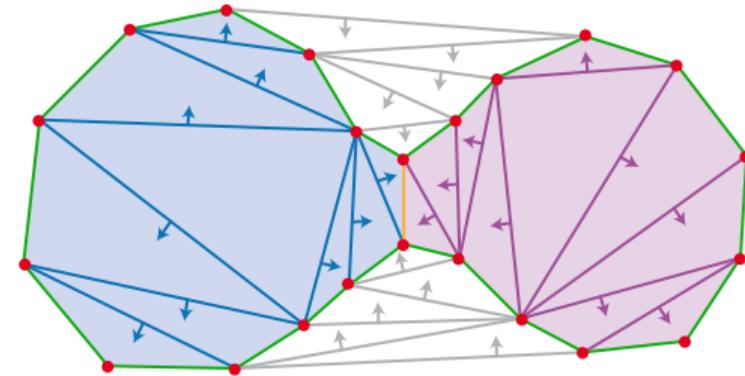
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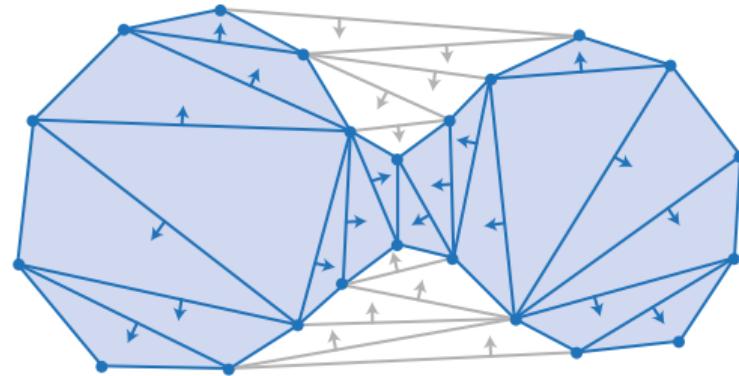
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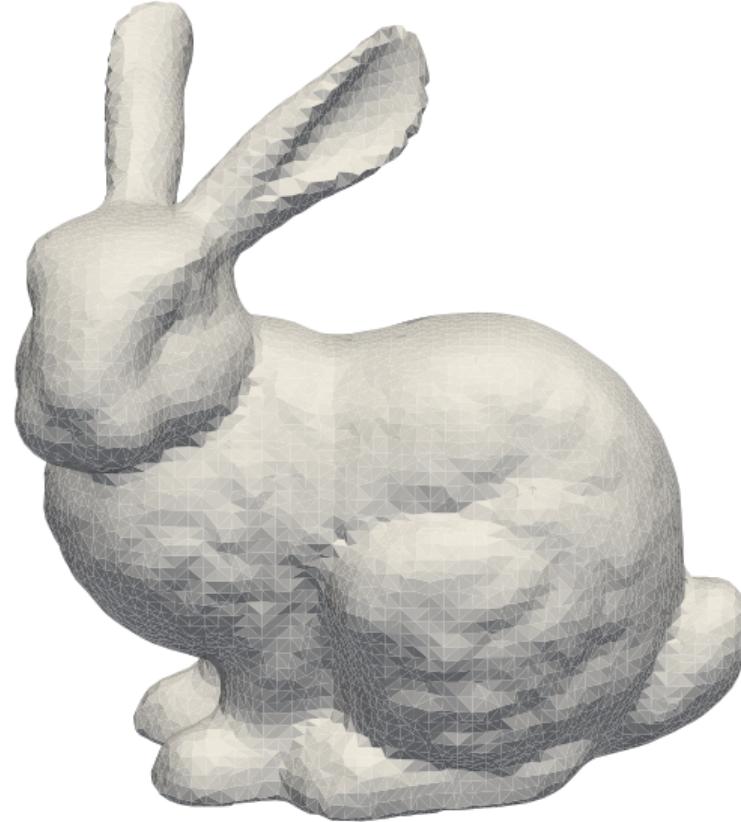
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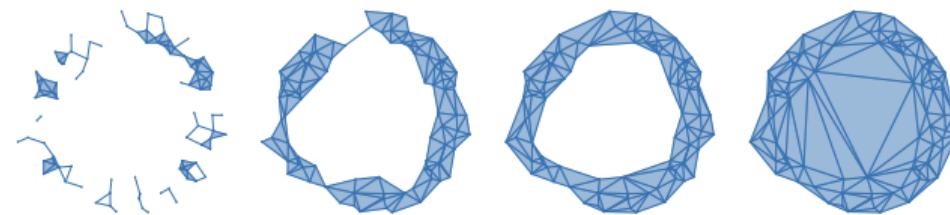


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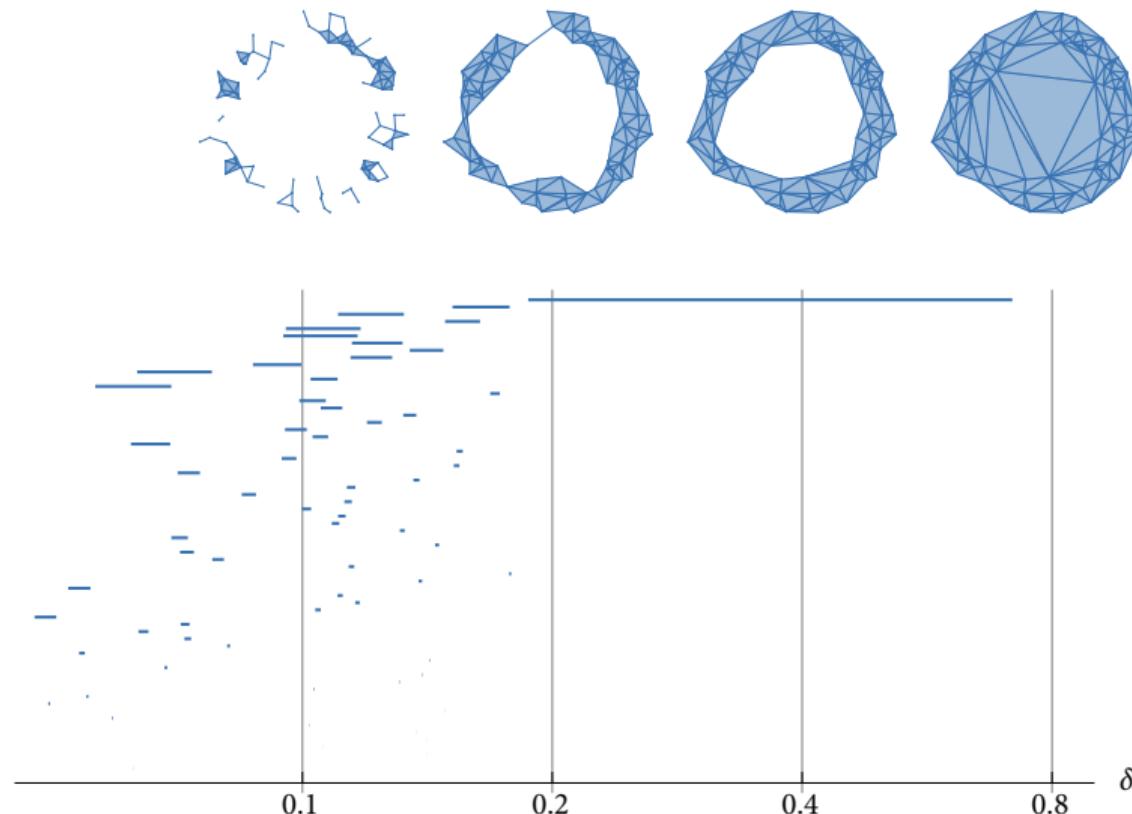


Persistent homology

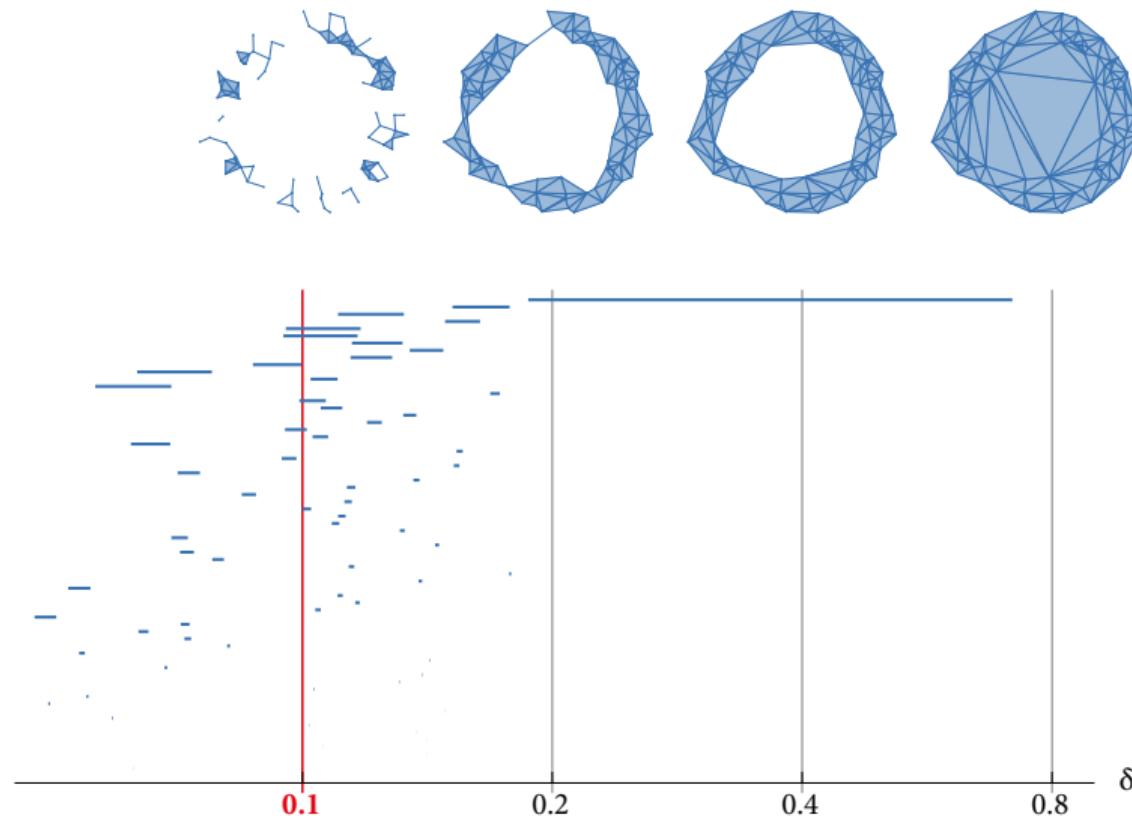
Persistence barcodes



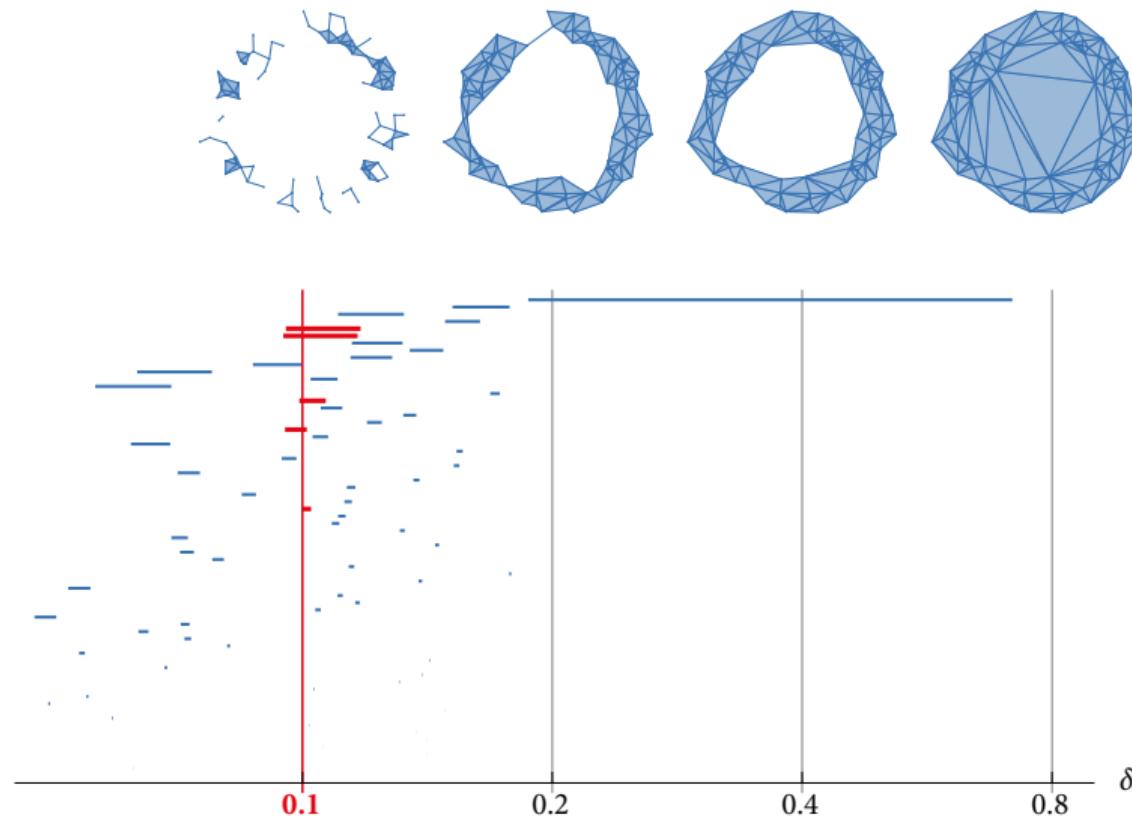
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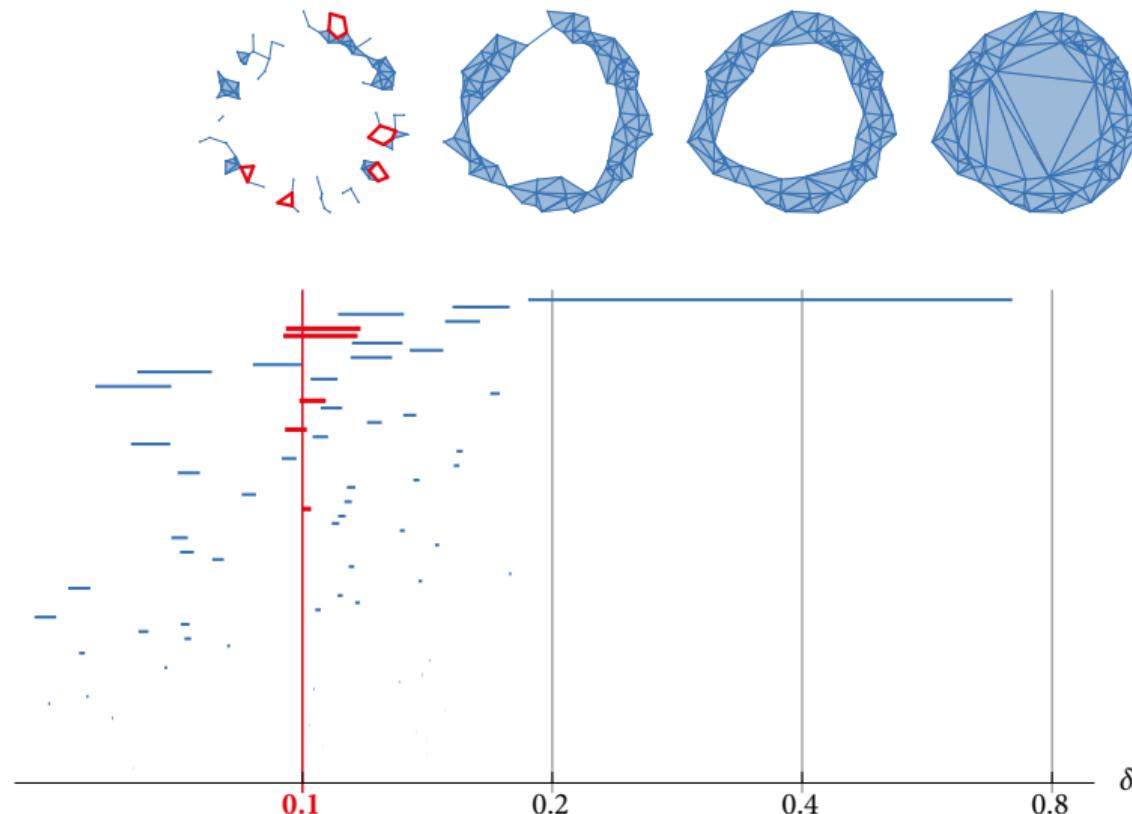
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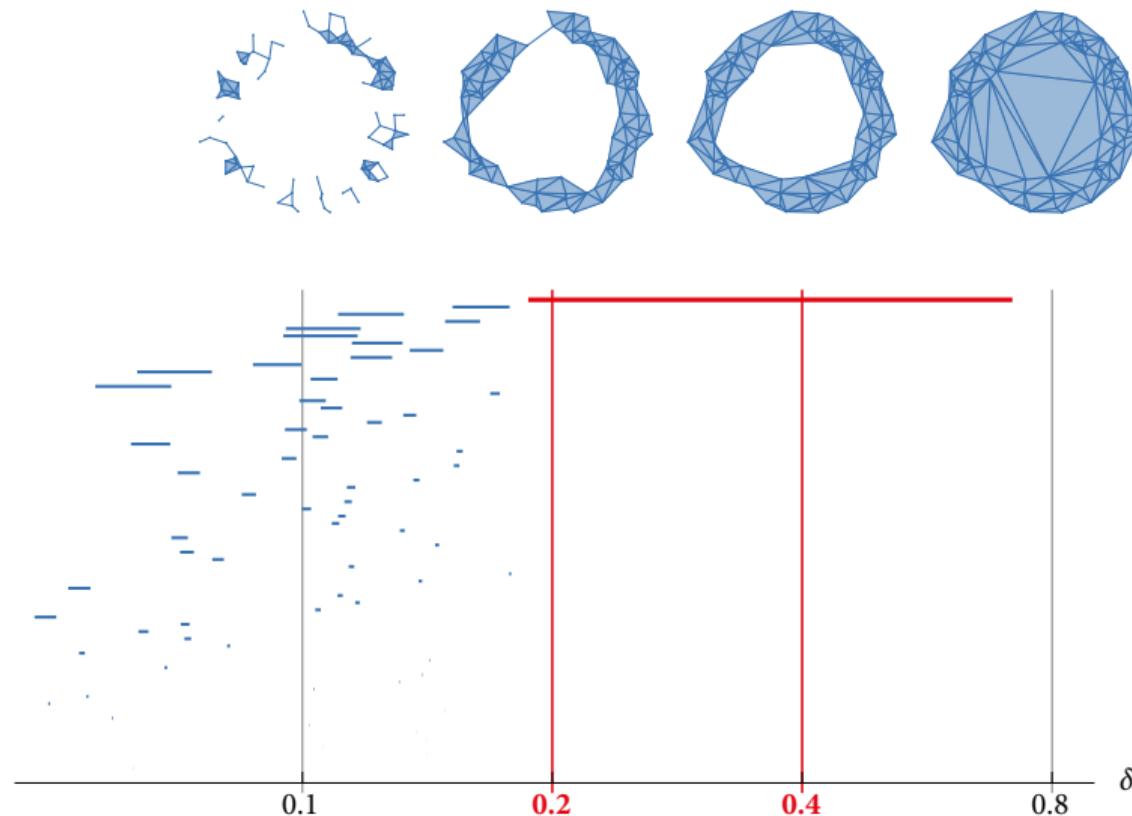
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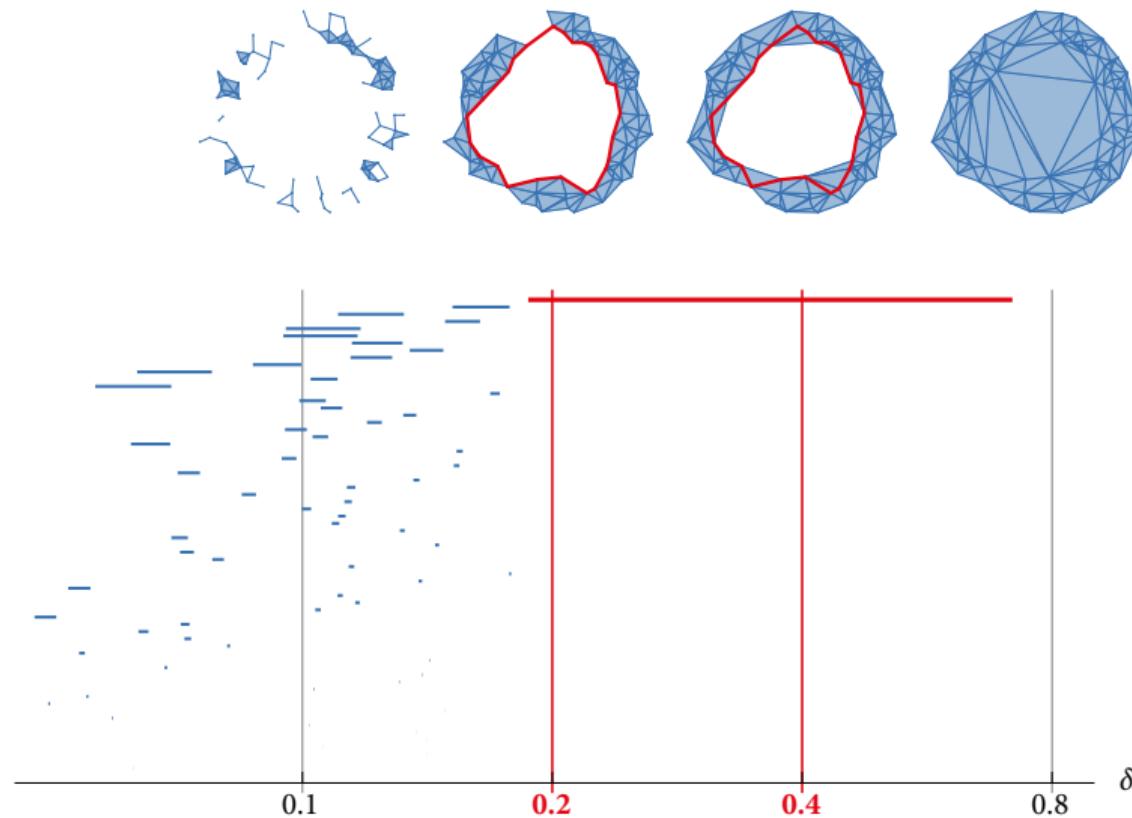
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Persistence barcodes



Computing persistent homology via matrix reduction

Algorithm (matrix reduction; a variant of Gauss elimination)

Require: $D: m \times n$ matrix

Ensure: V is full rank upper triangular, $R = D \cdot V$ has unique column pivots

function Reduce(D)

$R = D$

$V = I(n)$

while there exist $i < j$ such that pivot $R_i =$ pivot R_j **do**

add column R_i to column R_j ▷ eliminate the nonzero entry in row pivot R_i

add column V_i to column V_j

return R, V

Computing persistent homology via matrix reduction

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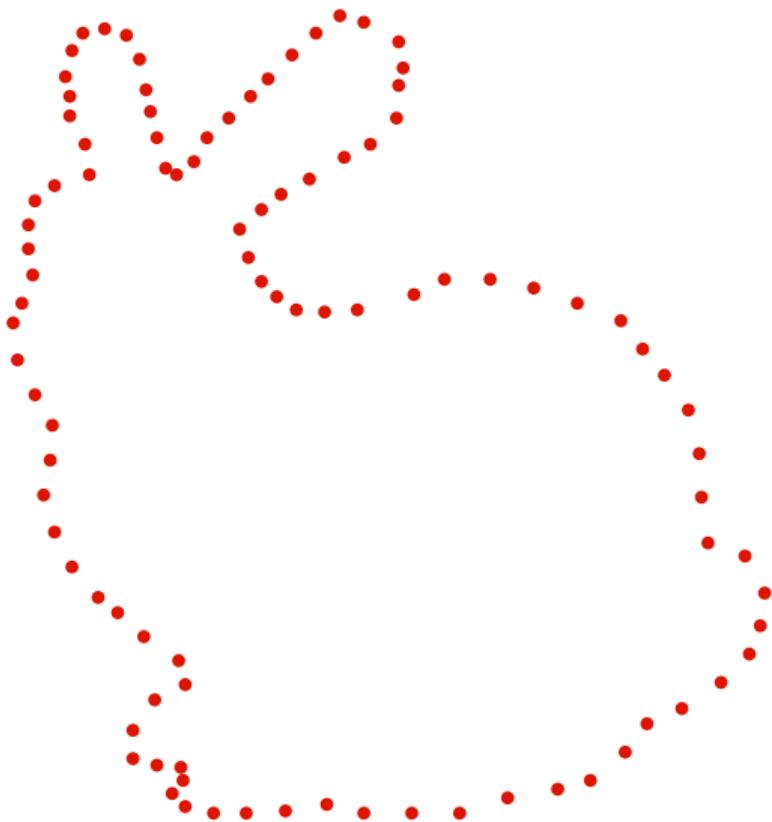
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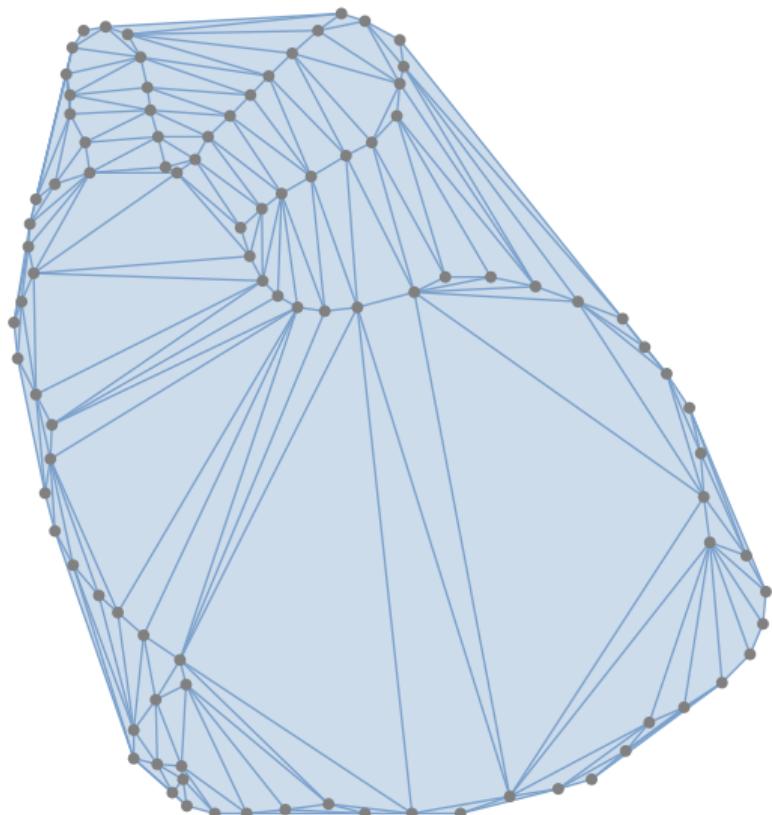
Proposition

The resulting columns R_j are minimal (in a lexicographic order) within their homology class (in K_{j-1}).

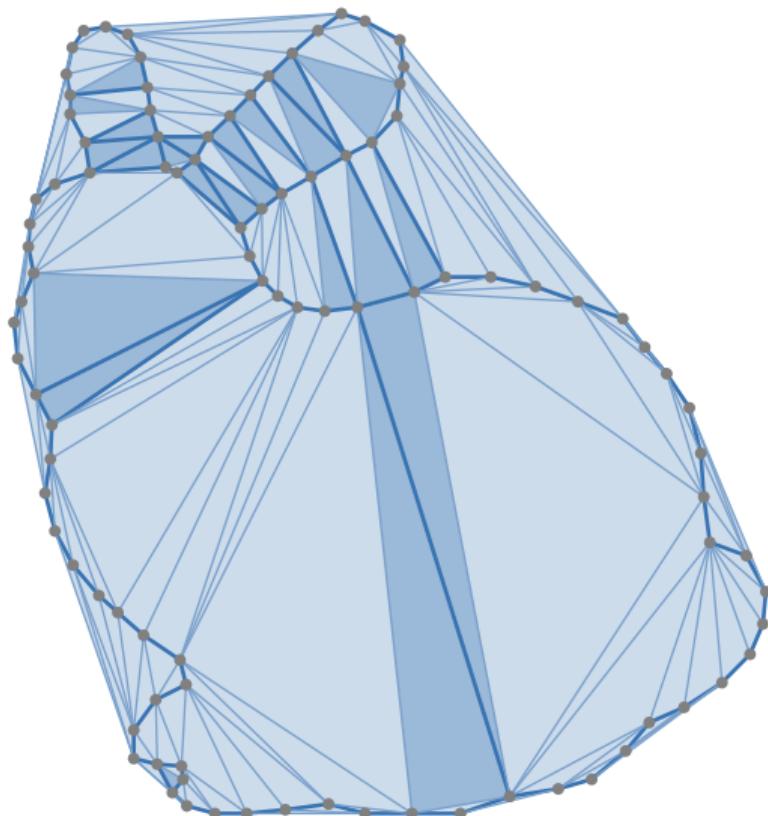
Wrap complexes and lexicographically minimal cycles



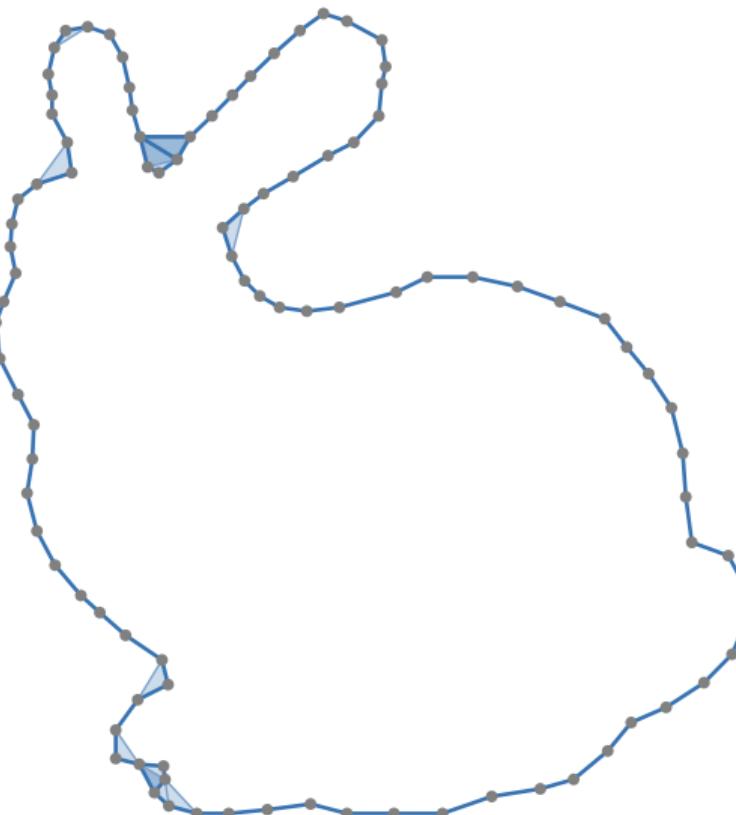
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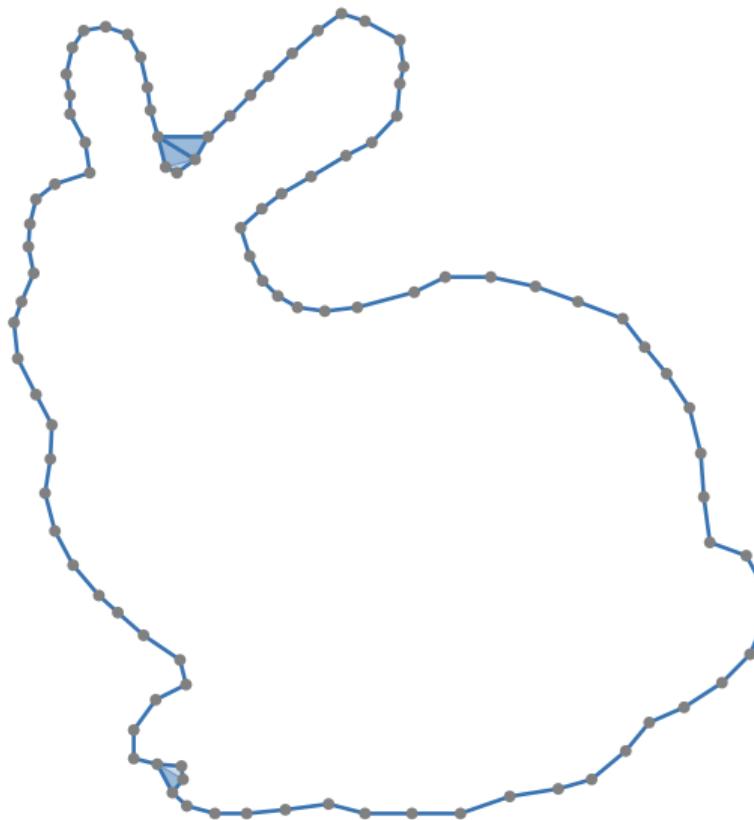
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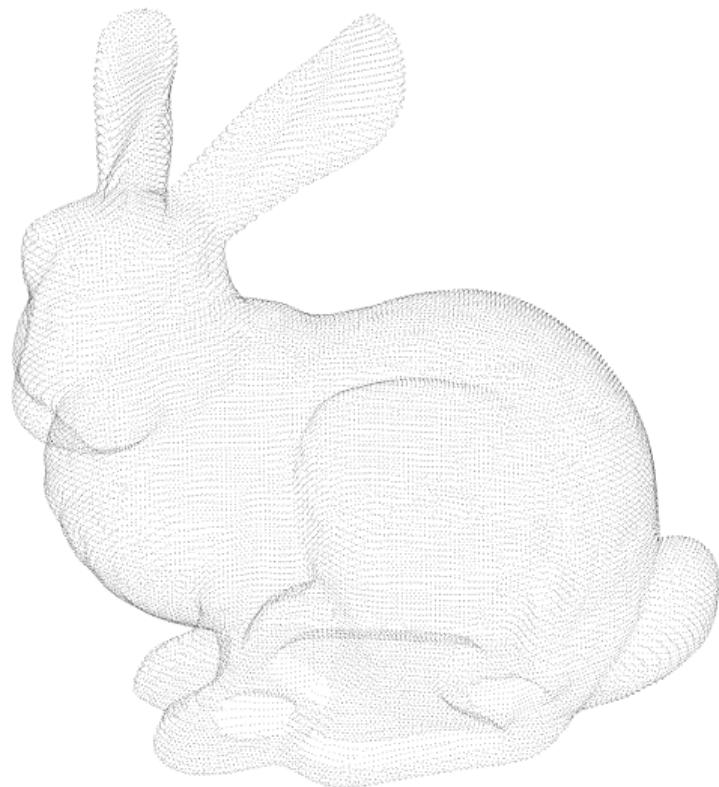
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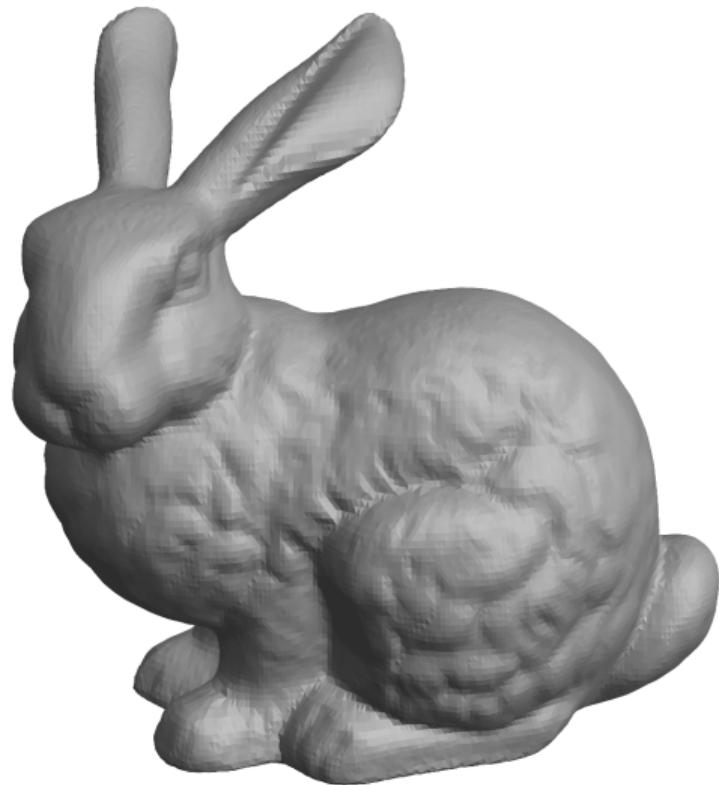
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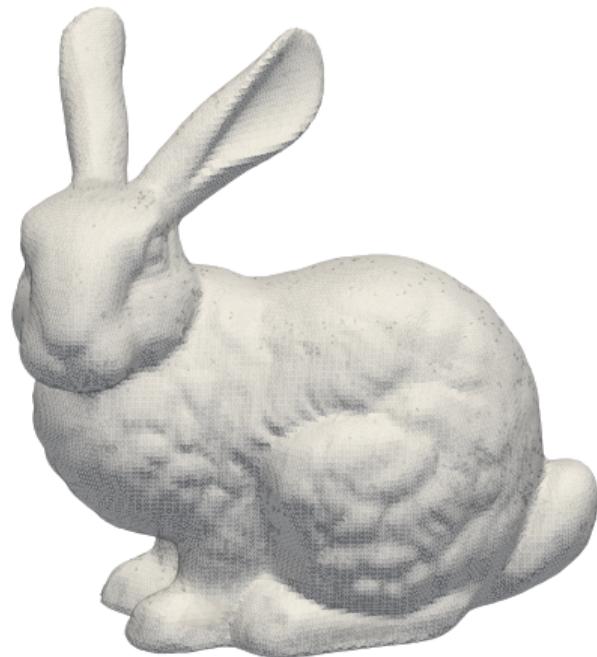
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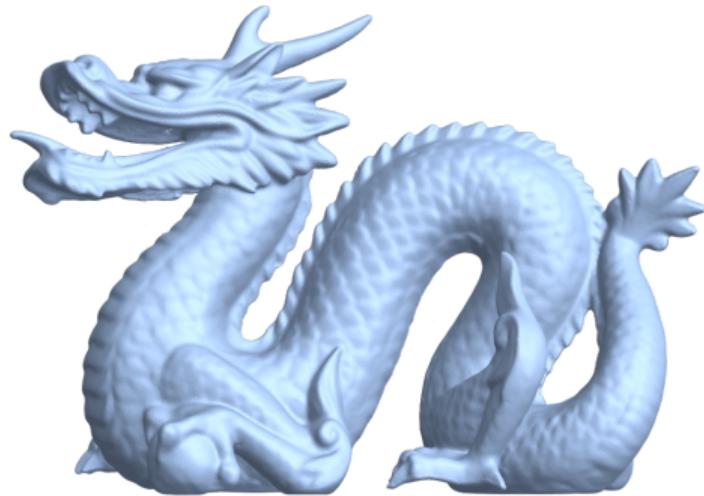
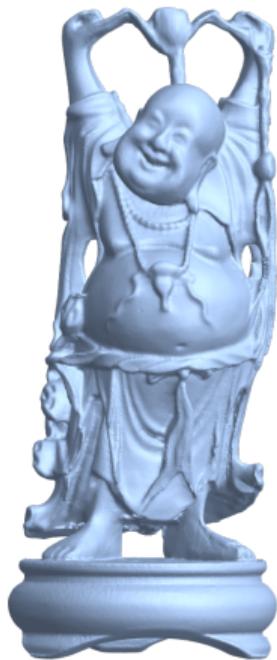
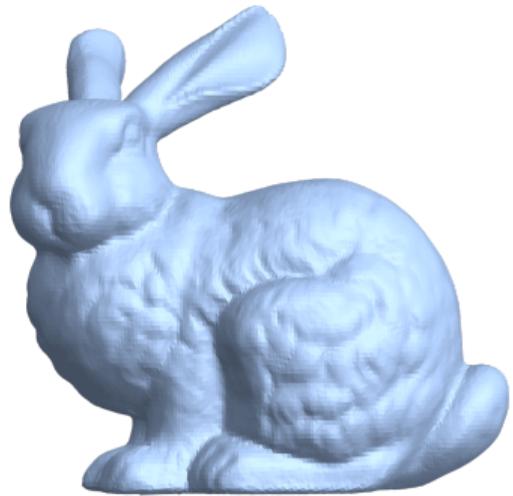
Standard reduction and exhaustive reduction



Standard reduction and exhaustive reduction



Point cloud reconstruction with minimal cycles

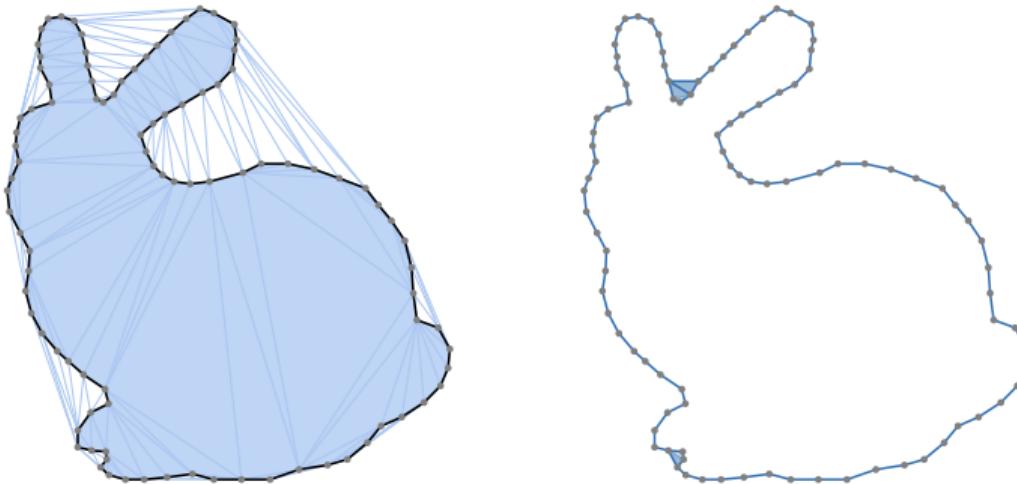


Wrap complexes support minimal cycles

Theorem (B, Roll 2024)

Let $X \subset \mathbb{R}$ be a finite subset in general position and let $r \in \mathbb{R}$.

- Exhaustive matrix reduction computes the minimal cycles homologous to a simplex boundary.
- Any lexicographically minimal cycle of $\text{Del}_r(X)$ is supported on $\text{Wrap}_r(X)$.



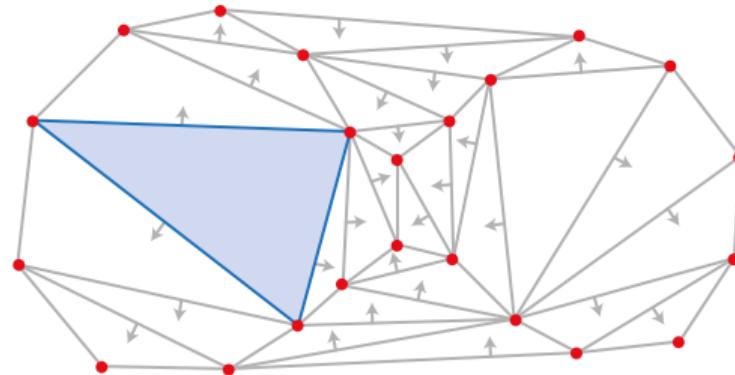
Algebraic gradient flows and persistent homology

Ingredients for relating the two approaches:

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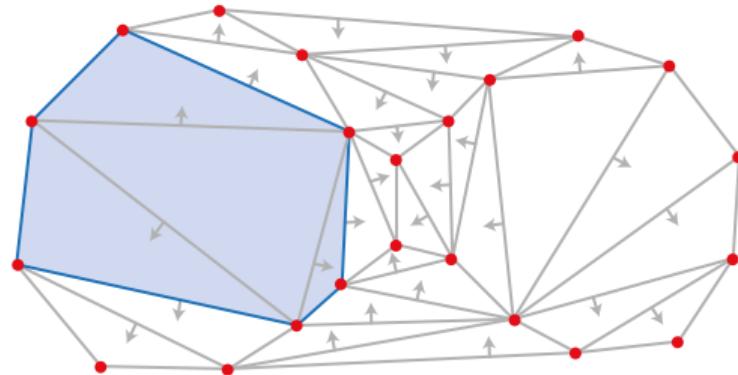
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Algebraic gradient flows and persistent homology

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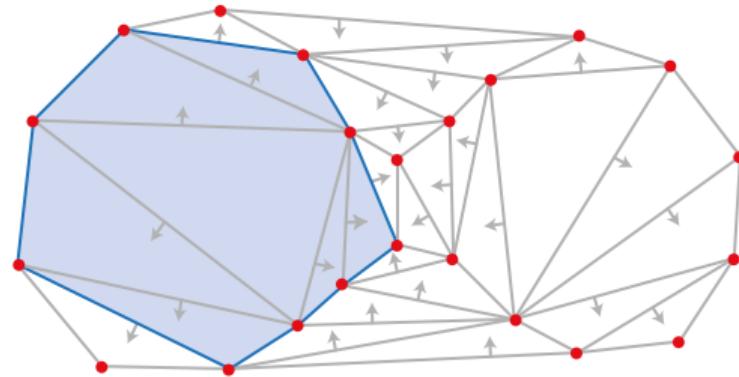
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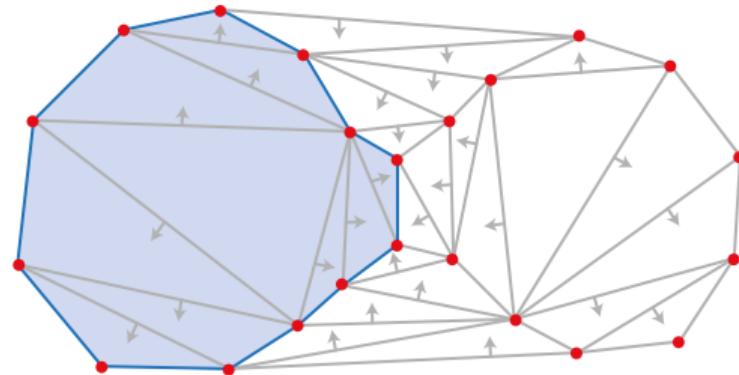
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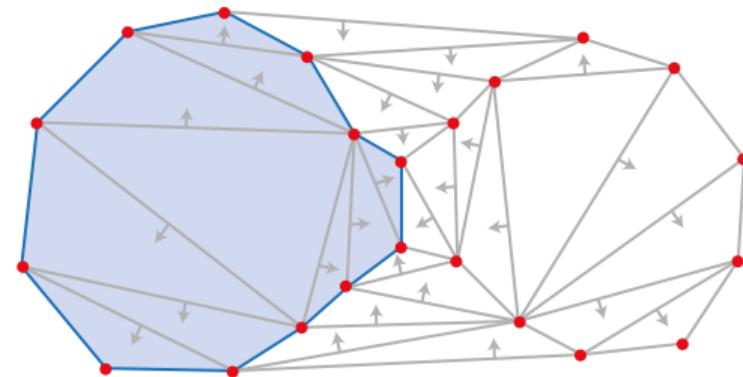
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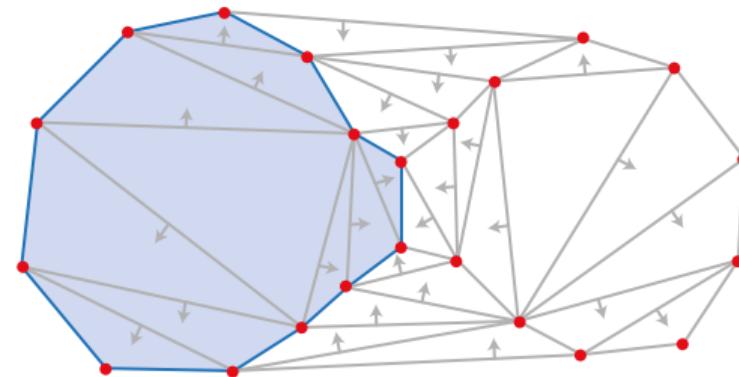


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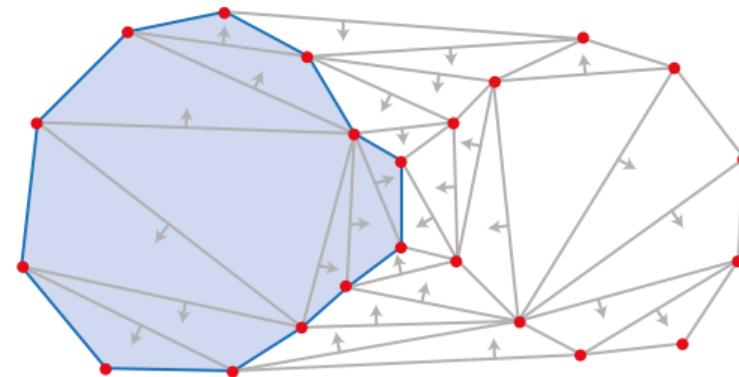
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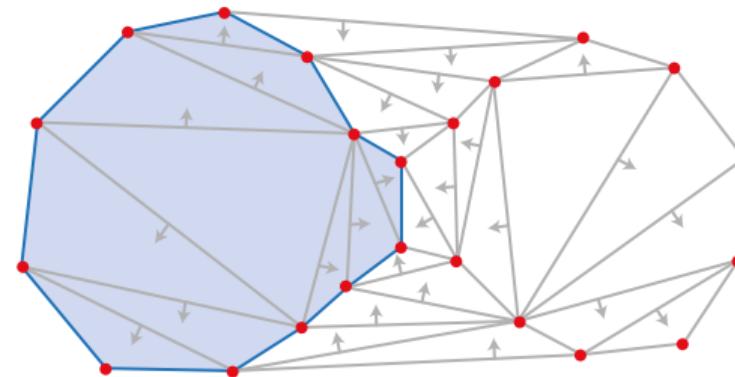
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- Persistence pairs correspond to algebraic gradient pairs

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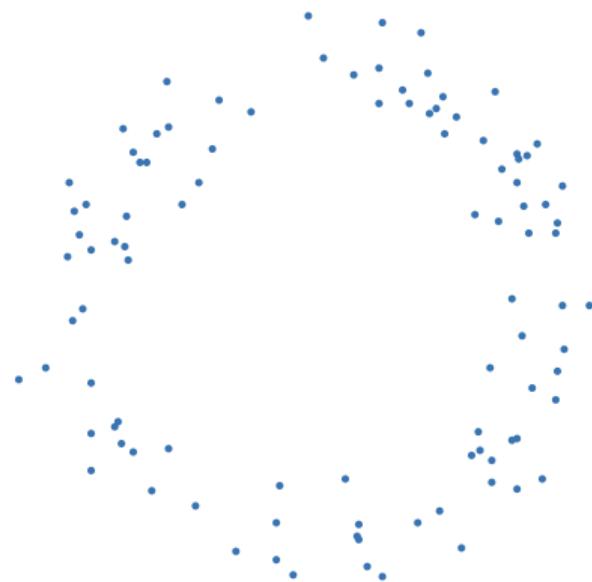
- Persistence pairs correspond to algebraic gradient pairs
- Matrix reduction (exhaustive) corresponds to gradient flow

Vietoris–Rips persistence

Vietoris–Rips complexes

For a metric space X , the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

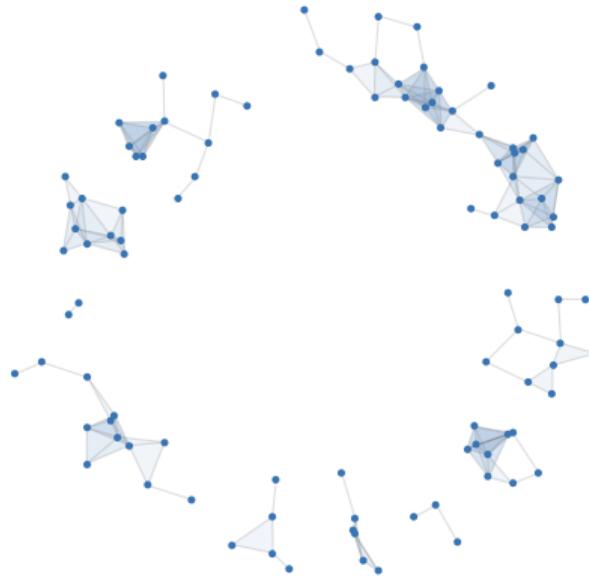
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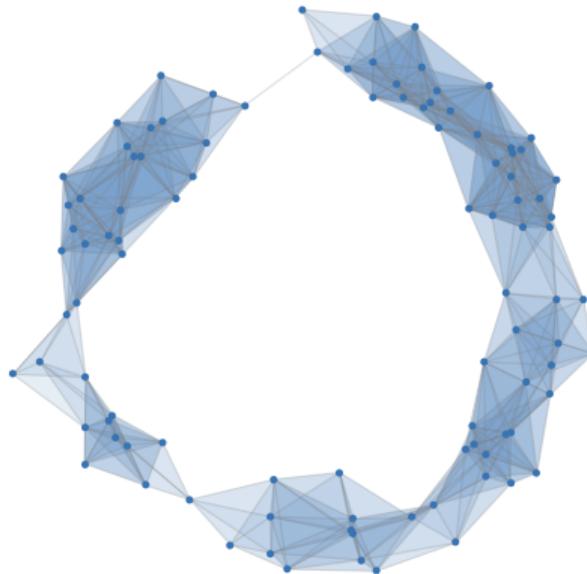
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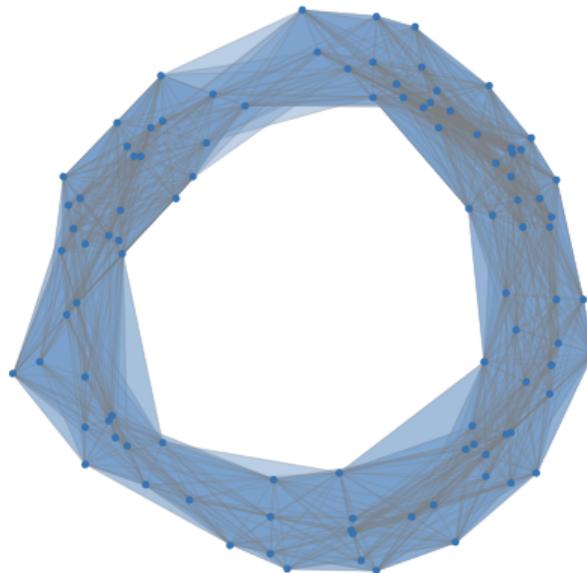
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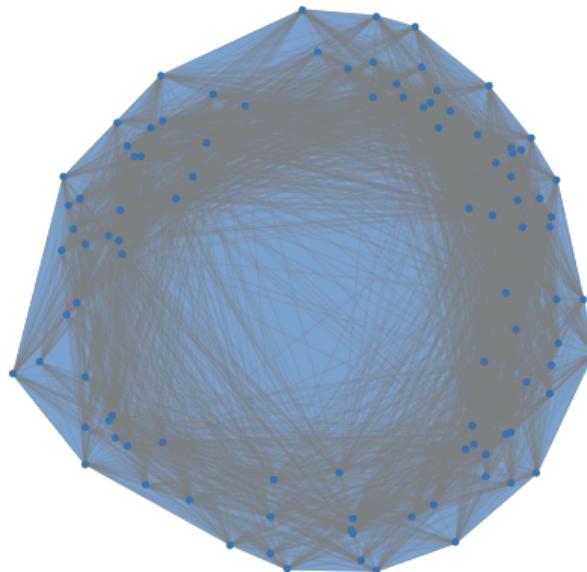
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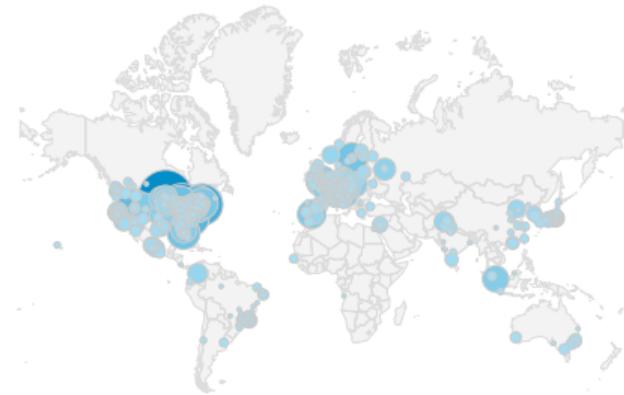
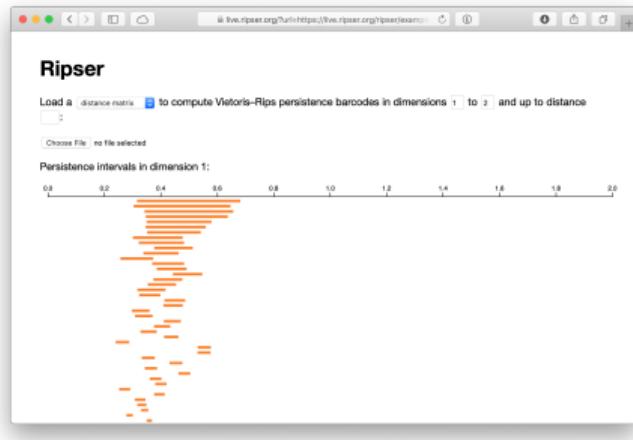
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Ripser: software for computing Vietoris–Rips persistence barcodes

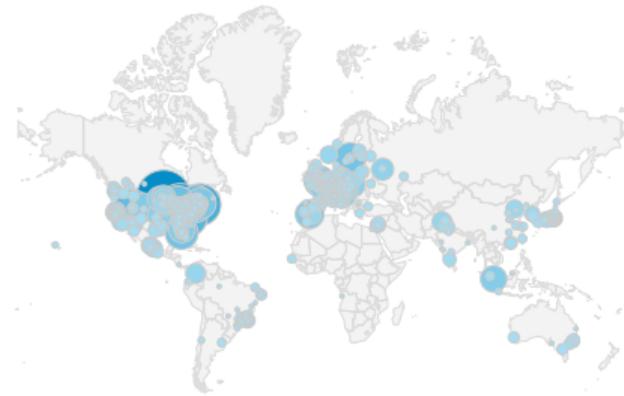
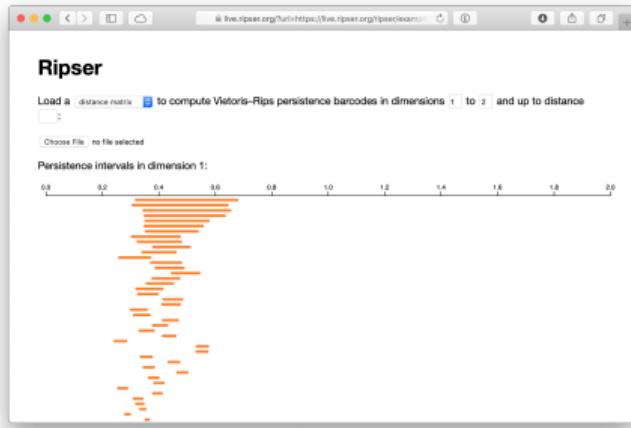
Open source software (ripser.org)



Ripser users across the globe

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Computational improvements based on

- *implicit matrix representations*
- *apparent pairs*, connecting persistence to discrete Morse theory

Apparent pairs

Ripser uses the following pairing of simplices (breaking ties in the filtration lexicographically):

Definition (B 2016, 2021)

In a simplexwise filtration ($K_i = \{\sigma_1, \dots, \sigma_i\}_i$), two simplices (σ_i, σ_j) form an *apparent pair* if

- σ_i is the latest proper face of σ_j , and
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Special cases and equivalent definitions have been (re)discovered independently multiple times

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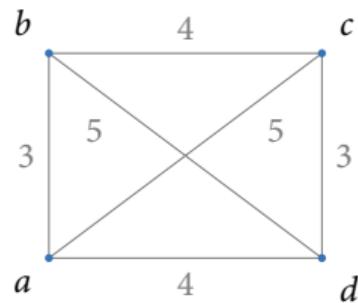
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Proposition (B 2021)

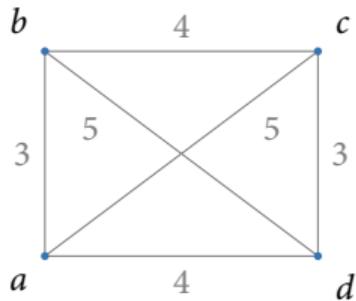
The apparent pairs are both

- *persistence pairs (creating/destroying a feature in homology) and*
- *gradient pairs (in the sense of discrete Morse theory).*

Apparent pairs of the diameter-lexicographic filtration



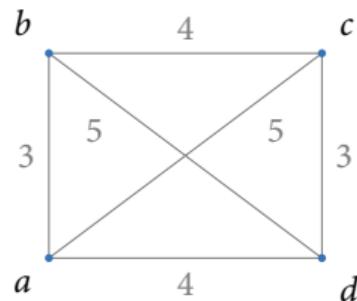
Apparent pairs of the diameter-lexicographic filtration



$$\partial_1 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{matrix} (a) \\ (b) \\ (c) \\ (d) \end{matrix}$$

The matrix has 5 columns corresponding to the vertices a, b, c, d, and a row vector. The first column has a circled '1' in the first position. The second column has a circled '1' in the second position. The third column has a circled '1' in the third position. The fourth column has a circled '1' in the fourth position. The fifth column has a circled '1' in the fifth position.

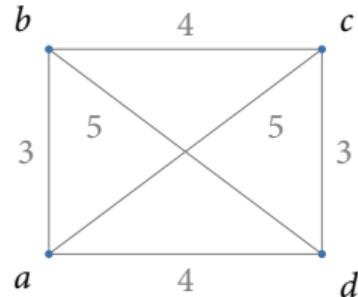
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$$\partial_1 = \begin{pmatrix} 1 & & (a,b):3 & & \\ & 1 & & (c,d):3 & \\ & & 1 & & (a,d):4 \\ & & & 1 & (b,c):4 \\ 1 & & & & (a,c):5 \\ & 1 & & & (b,d):5 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{matrix} (a) \\ (b) \\ (c) \\ (d) \end{matrix}$$

$$\partial_2 = \begin{pmatrix} (a,b,c):5 & & & & \\ & (a,b,d):5 & & & \\ & & (a,c,d):5 & & \\ & & & (b,c,d):5 & \\ 1 & 1 & 1 & 1 & (a,b):3 \\ & 1 & 1 & 1 & (c,d):3 \\ 1 & & & 1 & (a,d):4 \\ & 1 & & & (b,c):4 \\ & & 1 & & (a,c):5 \\ & & & 1 & (b,d):5 \\ & & & & 1 \end{pmatrix}$$

Apparent pairs of the diameter-lexicographic filtration



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$$\partial_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{matrix} (a,b,c) \\ (a,b,d) \\ (a,c,d) \\ (b,c,d) \end{matrix}$$

A shortcut for finding pivots

$$\begin{array}{c} (a,b,c) \\ \left(\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right) \begin{array}{l} (a,b) \\ (a,c) \\ (a,d) \\ (b,c) \\ (b,d) \\ (c,d) \end{array} \end{array} \quad \begin{array}{c} (a,b,c):5 \\ \left(\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right) \begin{array}{l} (a,b):3 \\ (c,d):3 \\ (a,d):4 \\ (b,c):4 \\ (a,c):5 \\ (b,d):5 \end{array} \end{array}$$

Shortcut for finding the pivot (latest) facet of a simplex τ :

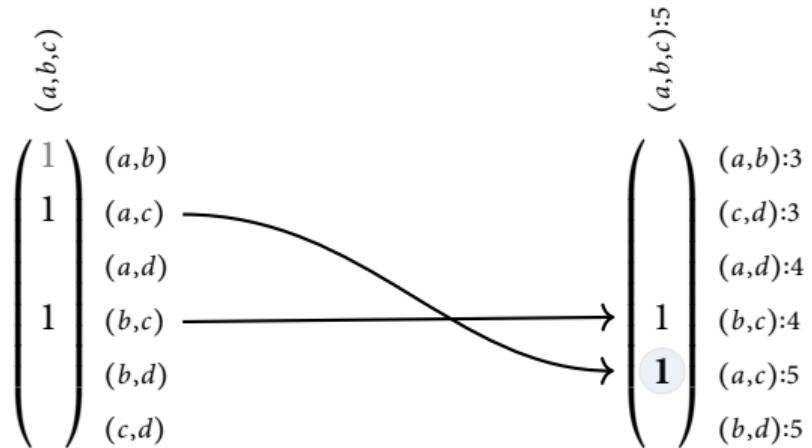
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$$\begin{array}{c} (a,b,c) \\ \left(\begin{array}{l} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right) \begin{array}{l} (a,b) \\ (a,c) \\ (a,d) \\ (b,c) \\ (b,d) \\ (c,d) \end{array} \end{array} \xrightarrow{\hspace{1cm}} \begin{array}{c} (a,b,c):5 \\ \left(\begin{array}{l} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right) \begin{array}{l} (a,b):3 \\ (c,d):3 \\ (a,d):4 \\ (b,c):4 \\ (a,c):5 \\ (b,d):5 \end{array} \end{array}$$

Shortcut for finding the pivot (latest) facet of a simplex τ :

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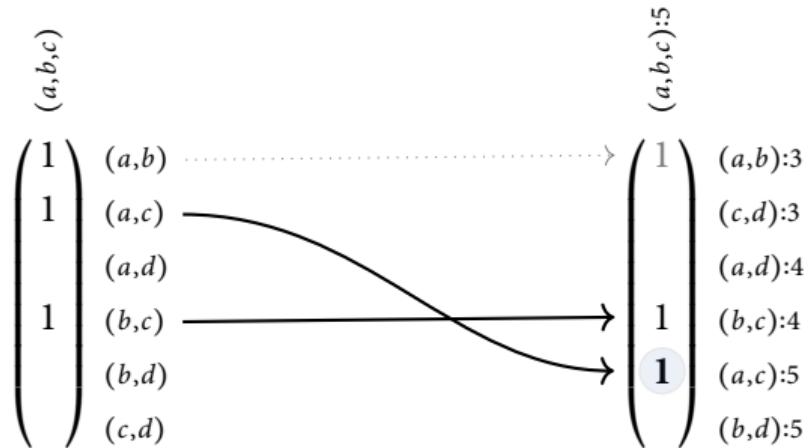
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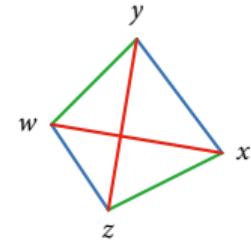
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Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ -hyperbolic (for $\delta \geq 0$) if for all $w, x, y, z \in X$ we have

$$d(w, x) + d(y, z) \leq \max\{d(w, y) + d(x, z), d(w, z) + d(x, y)\} + 2\delta.$$



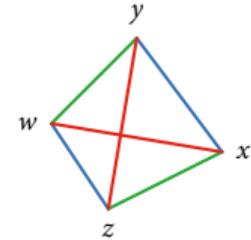
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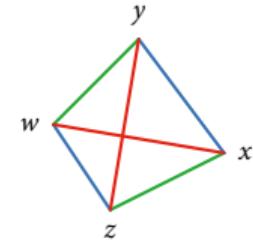
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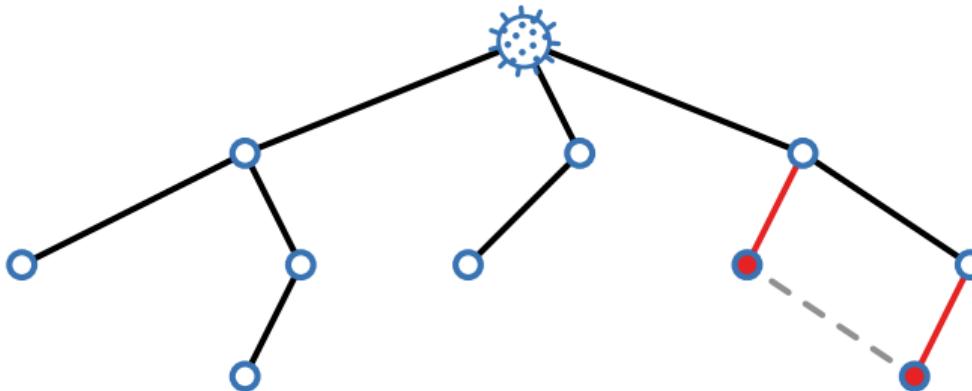
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- Metric trees and their subspaces are precisely the 0-hyperbolic spaces.

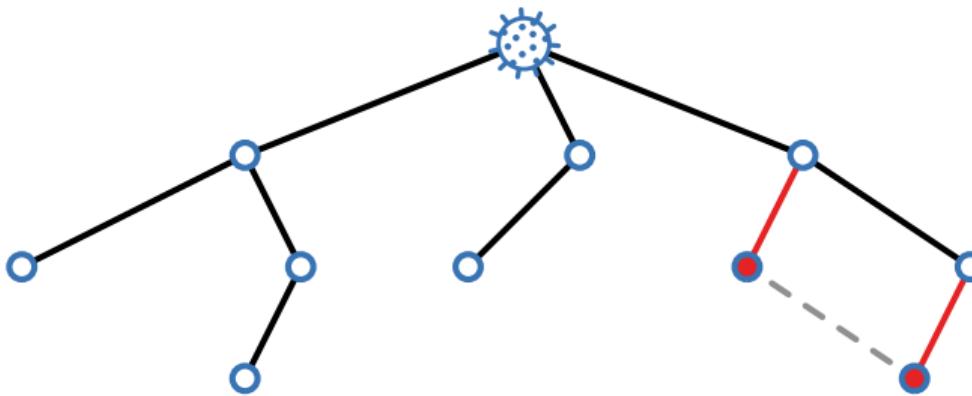


Topology of viral evolution



Joint work with: A. Ott, M. Bleher, L. Hahn (Heidelberg), R. Rabadian, J. Patiño-Galindo (Columbia), M. Carrière (INRIA)

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Observation: Ripser runs unusually fast on genetic distance data

- SARS-CoV2 RNA sequences (spike protein)
- 25556 data points (2.8×10^{12} simplices in 2-skeleton)
- 120 s computation time (with data points ordered appropriately)

Rips complexes of hyperbolic spaces

Theorem (Rips; Gromov 1988)

Let X be a δ -hyperbolic geodesic metric space. Then $\text{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

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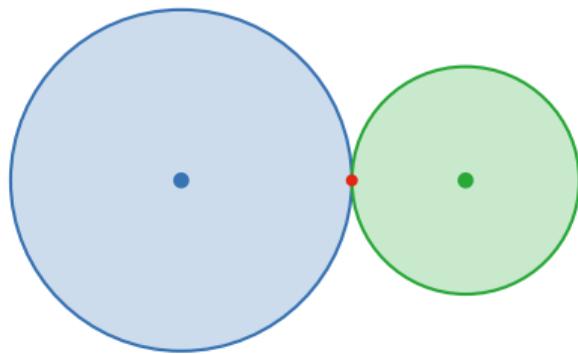
Theorem (B, Roll 2022)

Let X be a finite δ -hyperbolic space. Then there is a single discrete gradient encoding the collapses

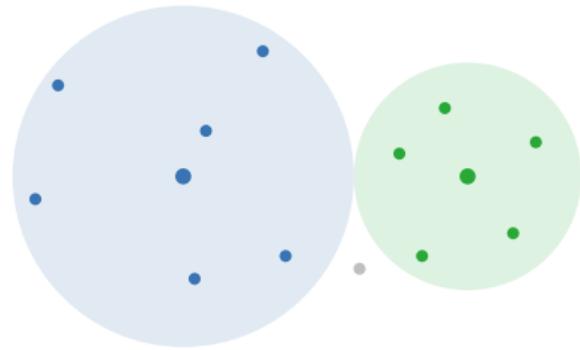
$$\text{Rips}_u(X) \searrow \text{Rips}_t(X) \searrow \{\ast\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect of X .

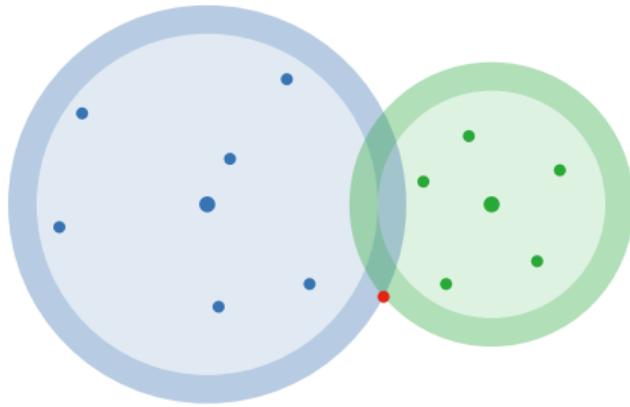
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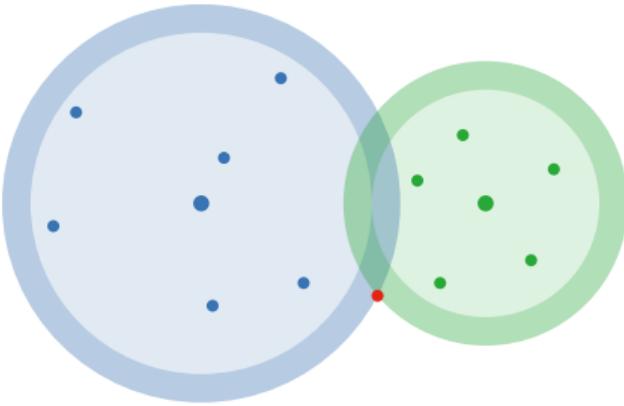
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Geodesic defect



Definition (Bonk, Schramm 2000)

A metric space X is ν -geodesic if for all points $x, y \in X$ and all $r, s \geq 0$ with $r + s = d(x, y)$ we have

$$B_{r+\nu}(x) \cap B_{s+\nu}(y) \neq \emptyset.$$

The infimum of all such ν is the *geodesic defect* of X .

The diameter function of generic trees

Proposition (B, Roll 2022)

Consider a finite weighted tree (V, E) with a generic path length metric (distinct pairwise distances). Then the diameter function $\text{diam}: \Delta(V) \rightarrow \mathbb{R}$ is a generalized discrete Morse function.

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In particular, the persistent homology is trivial in degrees > 0 .

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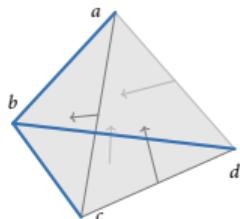
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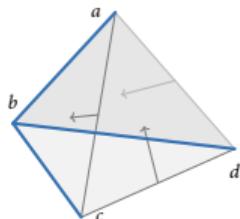
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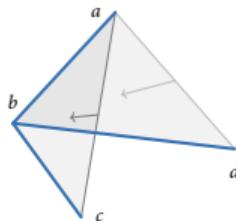
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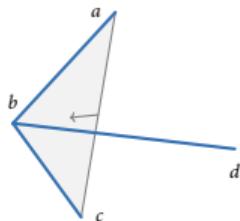
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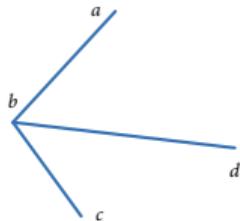
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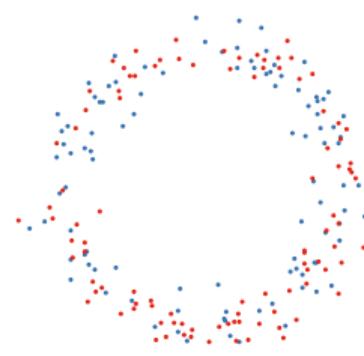
Stability

Stability of persistence barcodes

If two point clouds are close, then their barcodes are also close:

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $X, Y \subset \mathbb{R}^d$ be finite point sets with Hausdorff distance $d_H(X, Y) = \delta$.



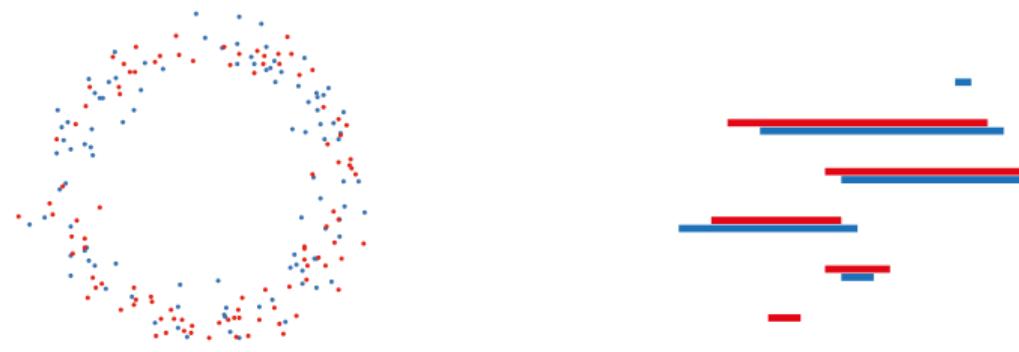
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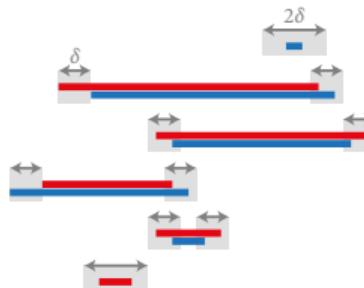
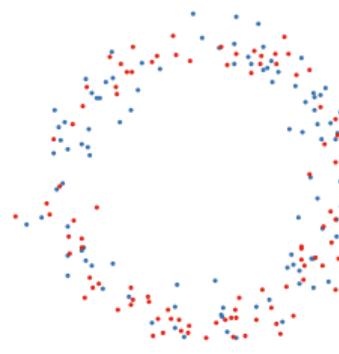
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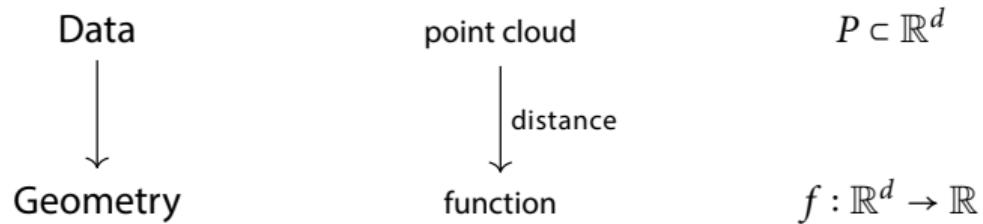
Persistence and stability: the big picture

Data

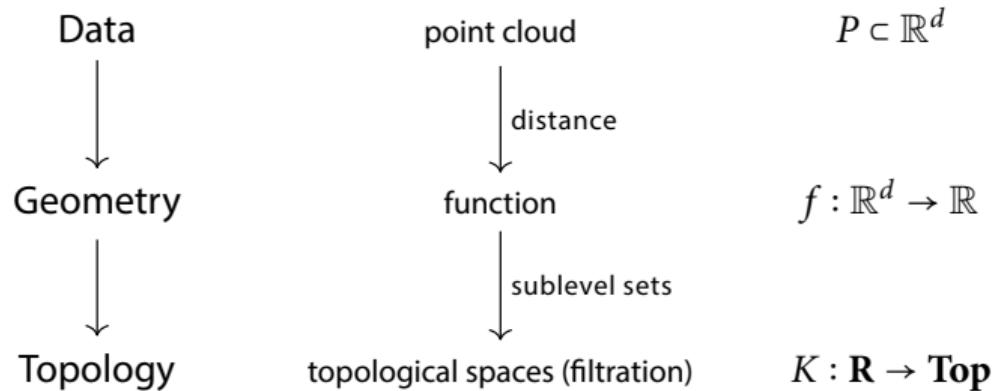
point cloud

$$P \subset \mathbb{R}^d$$

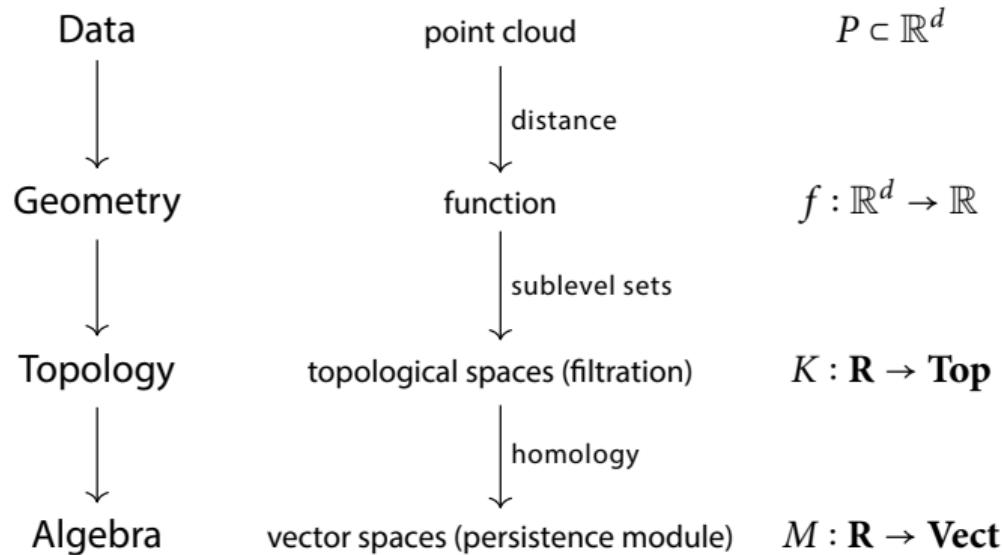
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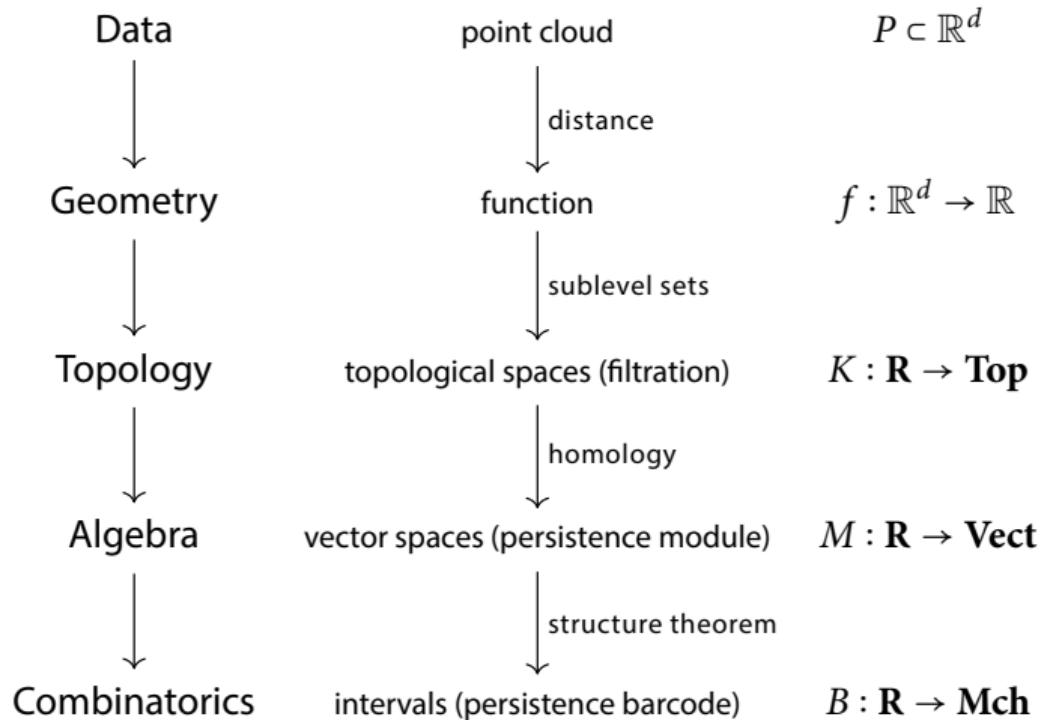
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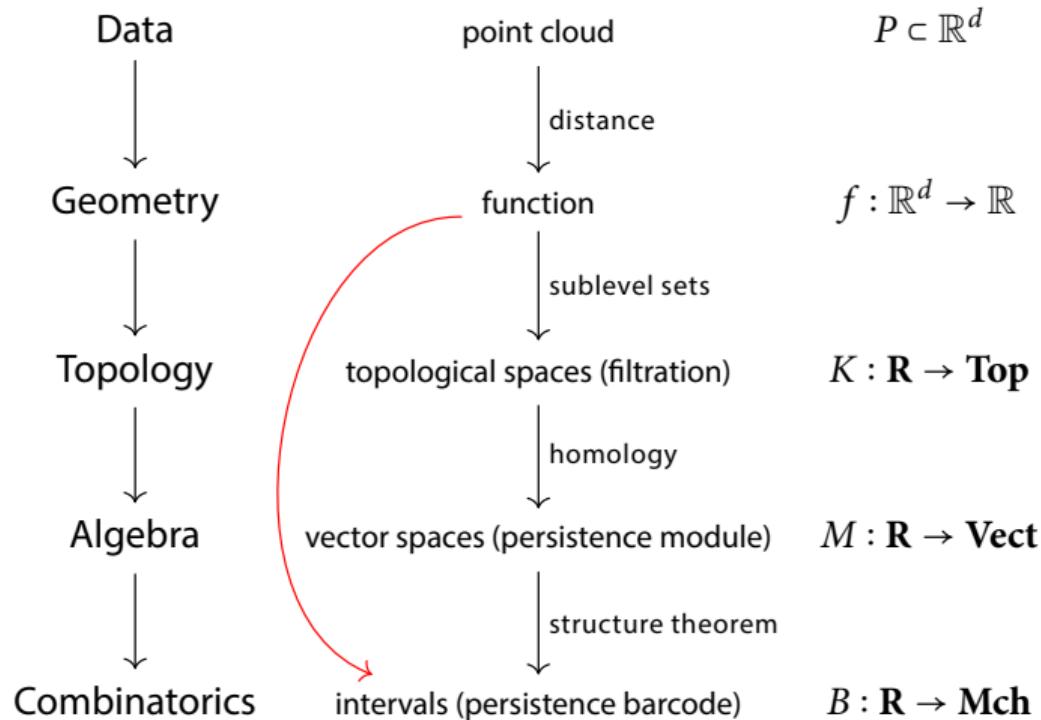
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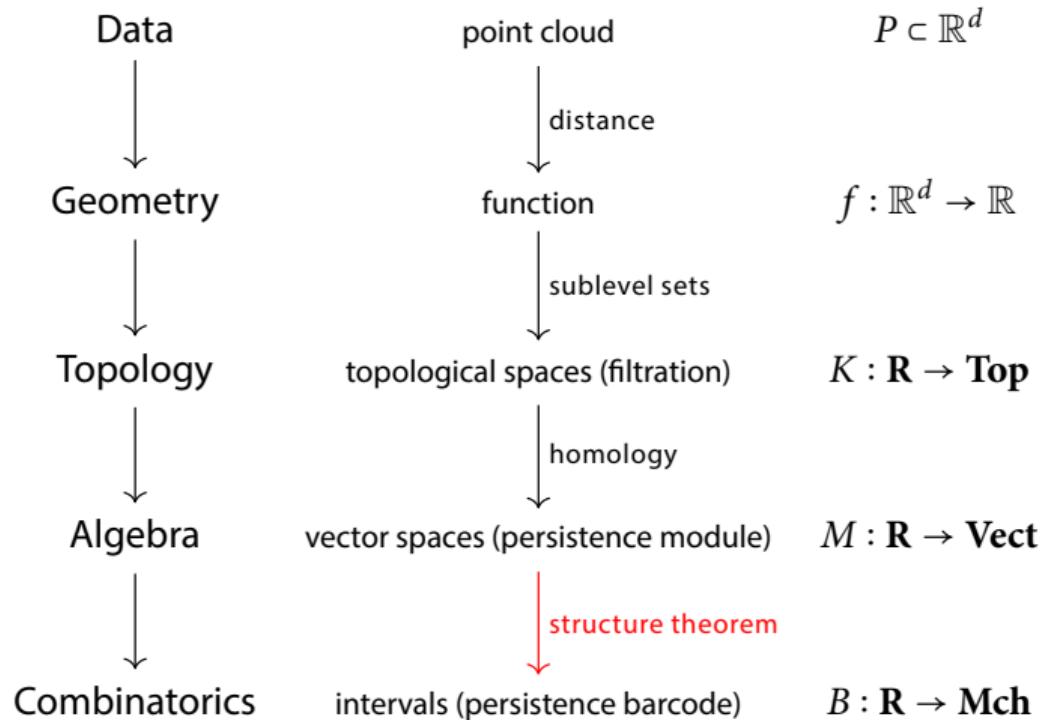
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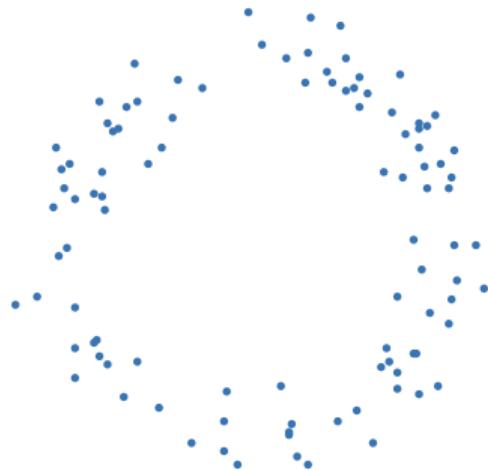
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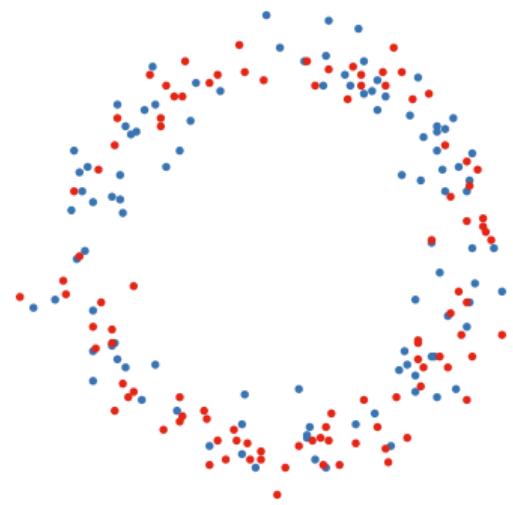
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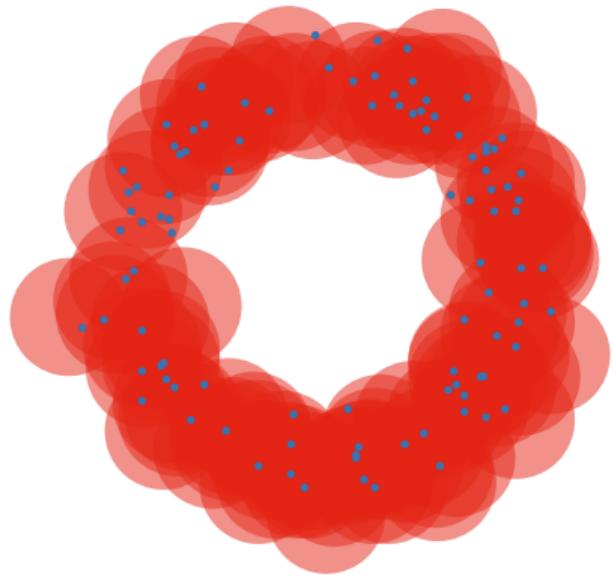
Geometric interleavings



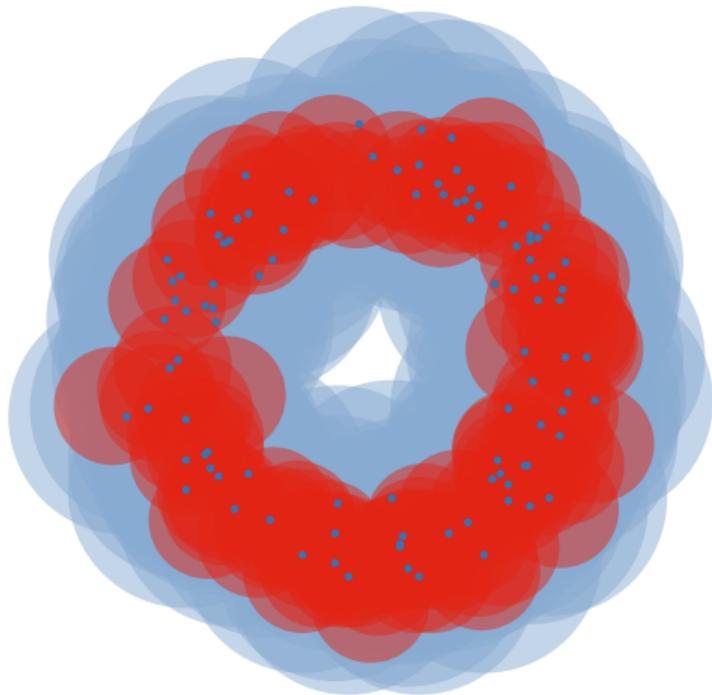
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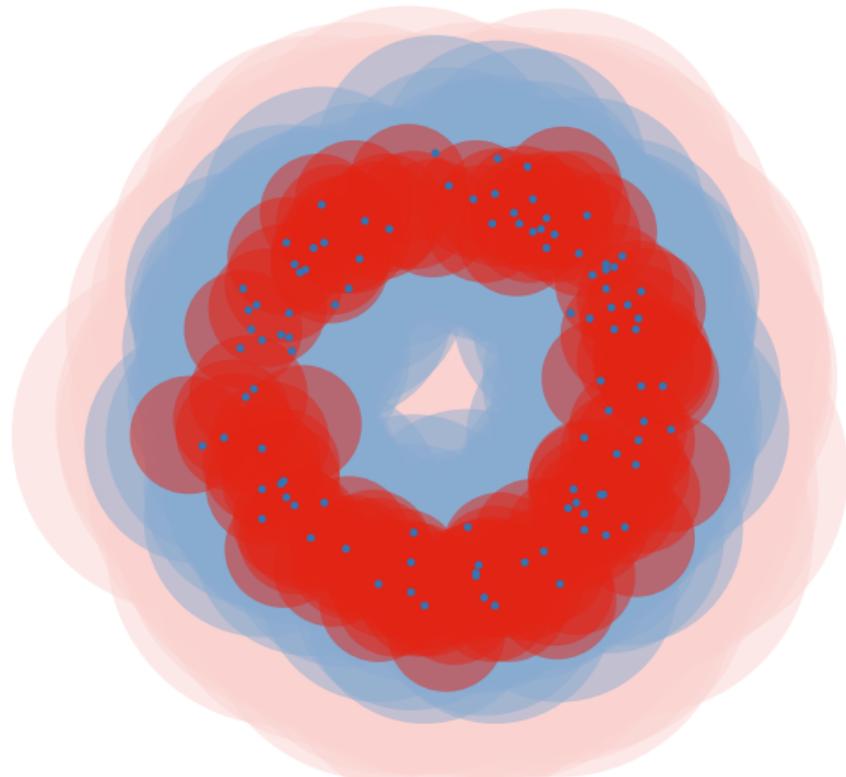
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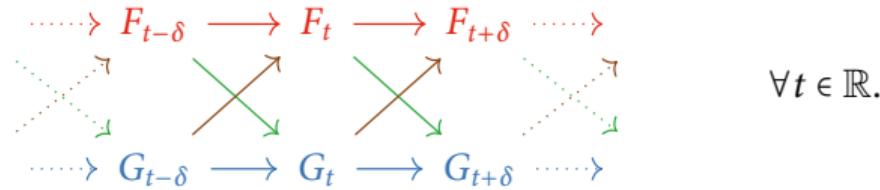
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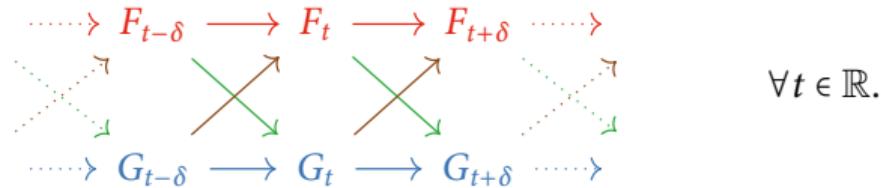
Then the sublevel set filtrations $F, G : \mathbf{R} \rightarrow \mathbf{Top}$ are δ -interleaved:



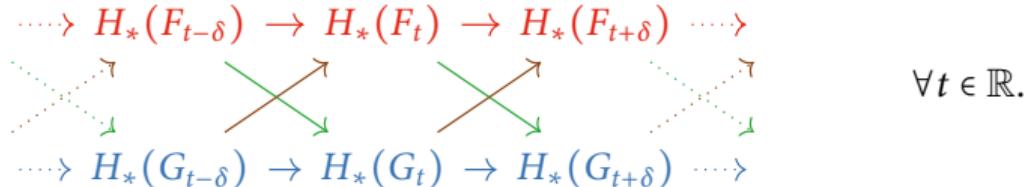
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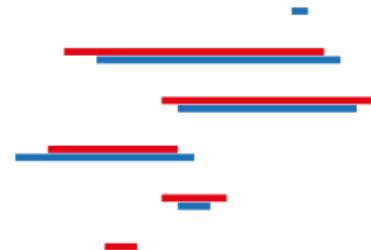
Applying homology, the persistence modules $H_*(F), H_*(G) : \mathbf{R} \rightarrow \mathbf{Vect}$ are δ -interleaved:



Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2016; B, Lesnick 2015)

*If two persistence modules are δ -interleaved,
then their barcodes admit a δ -matching:*

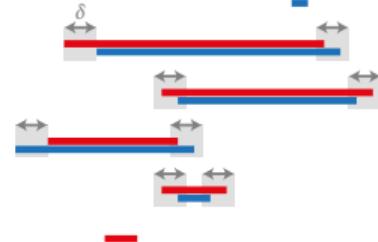


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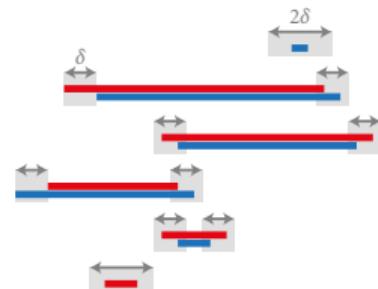


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Structure of persistence sub-/quotient modules

Proposition (B, Lesnick 2015)

Let $M \twoheadrightarrow N$ be an epimorphism of persistence modules.

Then there is an injection of barcodes $B(N) \hookrightarrow B(M)$ such that

if J is mapped to I , then

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Dually, for a monomorphism $N \hookrightarrow O$ there is an injection $B(N) \hookrightarrow B(O)$.



Induced matchings

Any morphism of pfd persistence modules $f : M \rightarrow N$ factors through its image as an epimorphism followed by a monomorphism:

$$M \twoheadrightarrow \text{im } f \hookrightarrow N$$

This gives an *induced matching* between their barcodes:

Induced matchings

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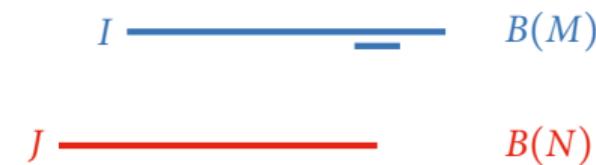
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Warning:

- The corresponding morphism of persistence modules may not be isomorphic to f

The induced matching theorem



The induced matching theorem



The induced matching theorem



The induced matching theorem



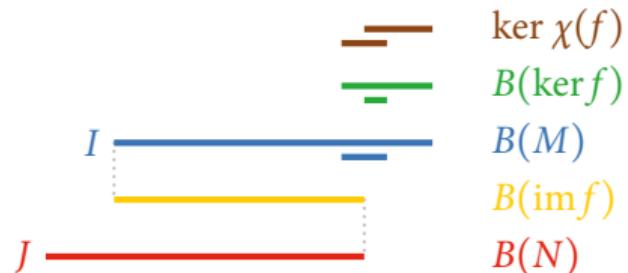
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The induced matching theorem

Theorem (B, Lesnick, 2015, 2020)

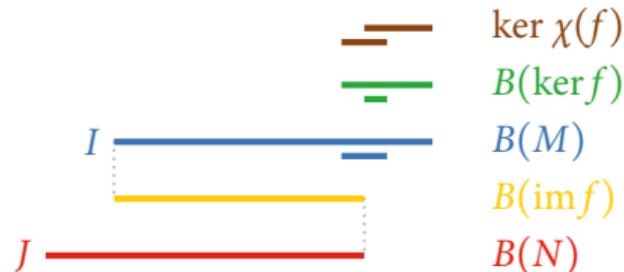
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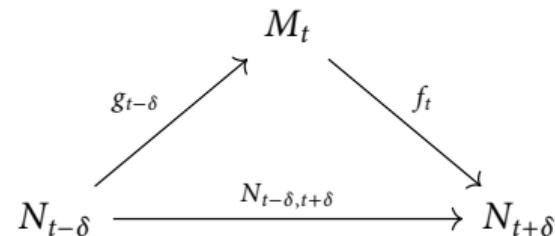


Corollary

If $f : M \rightarrow N(\delta)$ is a δ -interleaving morphism, then this induced matching is a δ -matching.

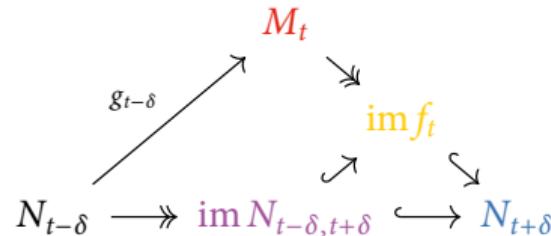
Algebraic stability via induced matchings

Consider an interleaving $f_t : M_t \rightarrow N_{t+\delta}$, $g_t : N_t \rightarrow M_{t+\delta}$ ($t \in \mathbb{R}$):



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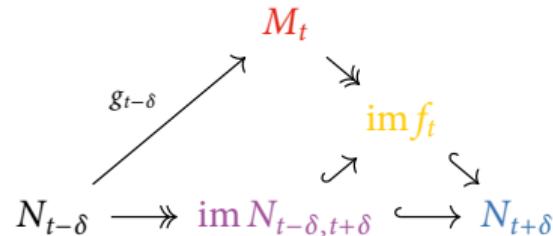
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$B(N)$

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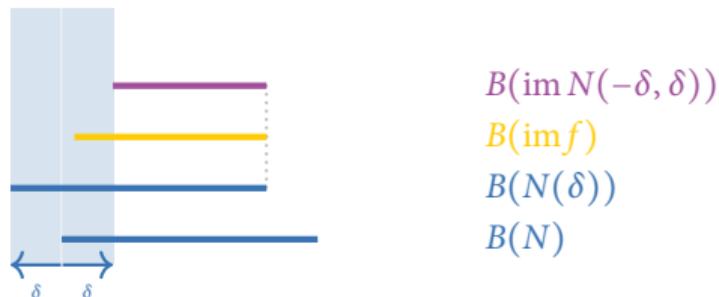
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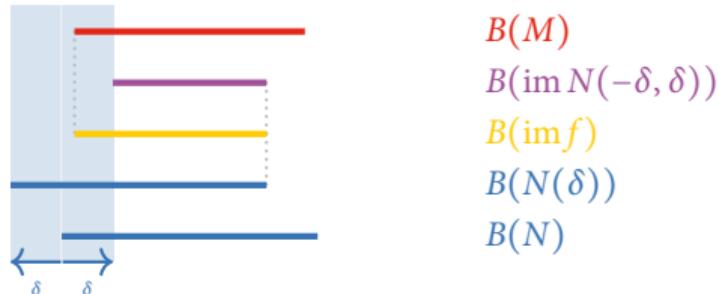
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Instability

Indecomposables abound in multi-parameter persistence

Theorem (B, Scoccola 2022)

For $n > 1$, almost every finitely presentable n -parameter persistence module is indecomposable.

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- Indecomposability is generic (in the closure of the finitely presented persistence modules).

Bottleneck distance and structural stability

The definition of bottleneck (or matching) distance extends to multi-parameter persistence.

Definition

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Bottleneck distance and structural stability

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1-parameter and multi-parameter persistence modules differ in their different structural stability:

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1-parameter and multi-parameter persistence modules differ in their different structural stability:

- *Every 1-parameter pfd persistence module is structurally stable.*
- *A finitely presentable n -parameter persistence module (with $n > 1$) is structurally stable if and only if it is indecomposable.*

Origins

When was persistent homology discovered first?

When was persistent homology discovered first?

ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k -cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k -limits suitably counted is called the k^{th} type number m_k of F . The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M . The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k .

The theory has been able to proceed provided one of the following hypotheses is satisfied. The critical limits cluster at most at p_k ; the critical points are

When was persistent homology discovered first?

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Rank and span in functional topology

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Exact homomorphism sequences in homology theory

[ed.ac.uk \[PDF\]](#)

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JL Kelley, E Pitcher - *Annals of Mathematics*, 1947 - JSTOR

Any time

The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] ...)

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Marston Morse and his mathematical works

[ams.org \[PDF\]](#)

R Bott - *Bulletin of the American Mathematical Society*, 1980 - ams.org

American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters² as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to ...

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Unstable minimal surfaces of higher topological structure

include citations

M Morse, CB Tompkins - *Duke Math. J.* 1941 - projecteuclid.org

1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. The problem of the existence of minimal surfaces of unstable type contains many of the typical difficulties, especially those of a topological nature. Having studied this ...

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When was persistent homology discovered first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT¹

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters² as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

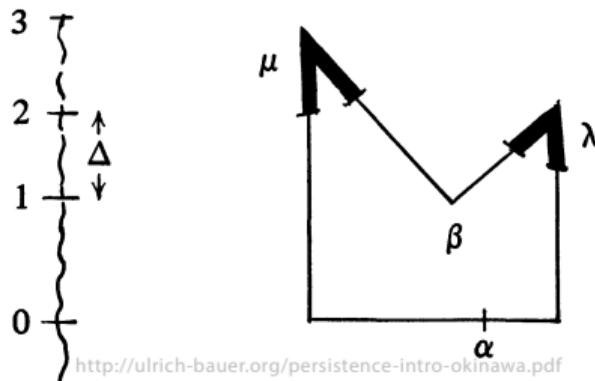
In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was

When was persistent homology discovered first?

inequalities pertain between the dimensions of the A_i and those of $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the “Spectral Sequence magic”, and a modern and tremendously general account of Morse’s work on rank and span in the framework of Leray’s theory was developed by Deheuvels [D] in the 50’s.

Unfortunately both Morse’s and Deheuvel’s papers are not easy reading. On the other hand there is no question in my mind that the papers [36] and [44] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

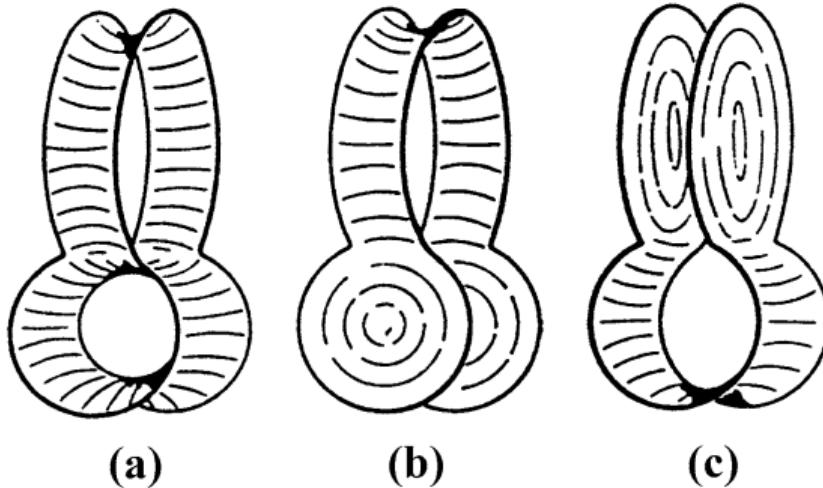
In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M .



Motivation and application: minimal surfaces

Problem (Plateau's problem)

Find an immersed disk of least area spanned by a given closed Jordan curve.

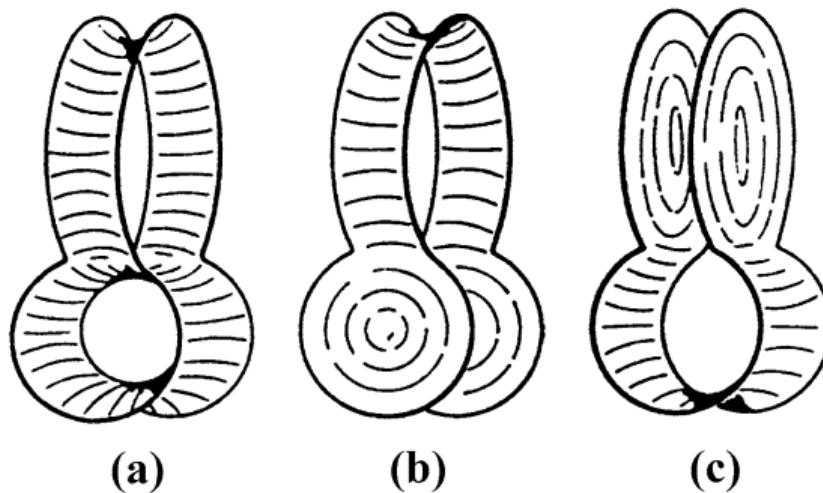


(from Dierkes et al.: *Minimal Surfaces*, 2010)

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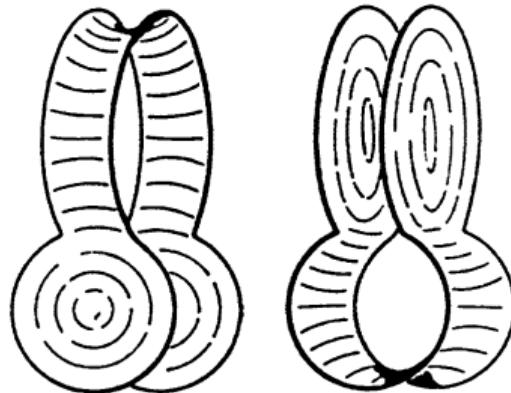
Solution by Douglas (1930; first Fields Medal 1936):

- identifies minimal surfaces with critical points of the *Douglas functional*

Existence of unstable minimal surfaces

Theorem (Morse, Tompkins 1939; Shiffman 1939)

Assume that a given curve bounds two separate stable minimal surfaces.

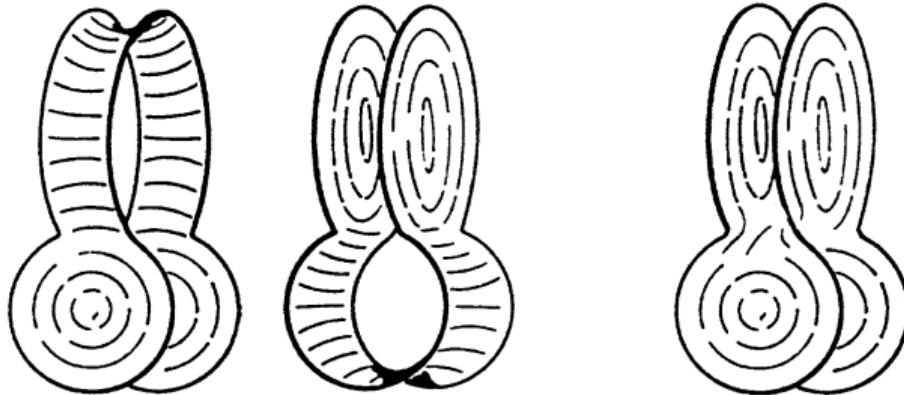


Existence of unstable minimal surfaces

Theorem (Morse, Tompkins 1939; Shiffman 1939)

Assume that a given curve bounds two separate stable minimal surfaces.

Then that curve also bounds an unstable minimal surface (a critical point that is not a local minimum).



Whatever happened to functional topology?

Whatever happened to functional topology?

PLATEAU'S PROBLEM
AND THE
CALCULUS OF VARIATIONS

BY

MICHAEL STRUWE

Whatever happened to functional topology?

82

A. The classical Plateau Problem for disc - type minimal surfaces.

The technical complexity and the use of a sophisticated topological machinery (which is not shadowed in our presentation) moreover tend to make Morse-Tompkins' original paper unreadable and inaccessible for the non-specialist, cf. Hildebrandt [4, p. 324].

Confronting Morse-Tompkins' and Shiffman's approach with that given in Chapter 4 we see how much can be gained in simplicity and strength by merely replacing the C^0 -topology by the $H^{1/2,2}$ -topology and verifying the Palais - Smale - type condition stated in Lemma 2.10.

However, in 1964/65 when Palais and Smale introduced this condition in the calculus of variations it was not clear that it could be meaningful for analyzing the geometry of surfaces, cf. Hildebrandt [4, p. 323 f.].

Q-tame persistence modules

Definition (Chazal et al. 2009)

A persistence module $M : \mathbf{R} \rightarrow \mathbf{vect}$ is *q-tame* if all structure maps $M_s \rightarrow M_t$ ($s < t$) have finite rank.

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Morse's main technical result, in modern language:

- The minimal surface functional has q-tame persistent homology.

Structure of q-tame persistence modules

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

The radical of a q-tame persistence module M , $(\text{rad } M)_t = \sum_{s < t} \text{im } M_{s,t}$,

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Theorem (Chazal, Crawley-Boevey, de Silva 2016)

Every q-tame persistence module has a unique persistence diagram that completely describes its isomorphism type in the observable category.

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- *The observable category is the category of persistence modules, modulo ephemeral modules.*

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- Morse and Tompkins used this idea to show the existence of an unstable minimal surface.

Vietoris vs singular homology

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- Therefore, persistent Vietoris/Čech homology is also continuous from above:

$$\check{H}_*(f_{\leq t}) = \lim_{u > t} \check{H}_*(f_{\leq u})$$

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Definition (Morse 1937; paraphrased)

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such that the inclusion $U \rightarrow V$ is homotopic to a constant map.

Q-tameness from local connectivity

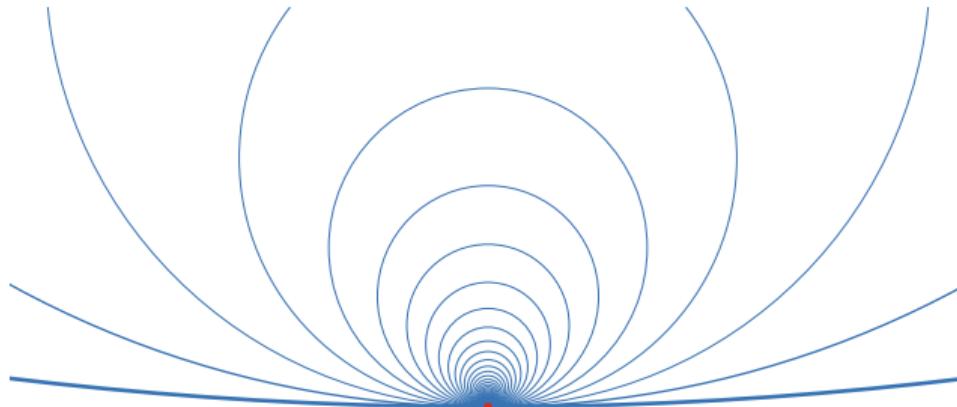
Theorem (Morse, 1937)

If the sublevel set filtration of $f: X \rightarrow \mathbb{R}$ is compact and weakly locally connected, then it has q -tame persistent Vietoris homology.

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Homologically locally small filtrations

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The sublevel set filtration of a function $f: X \rightarrow \mathbb{R}$ is *homologically locally connected (HLC)* if

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Theorem (B, Medina-Mardones, Schmahl 2021)

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- Conditions are satisfied by the Douglas functional
- Fixes the gap in Morse/Tompkins' proof

Simplification

Topological simplification of functions

Consider the following problem:

Problem (Topological simplification)

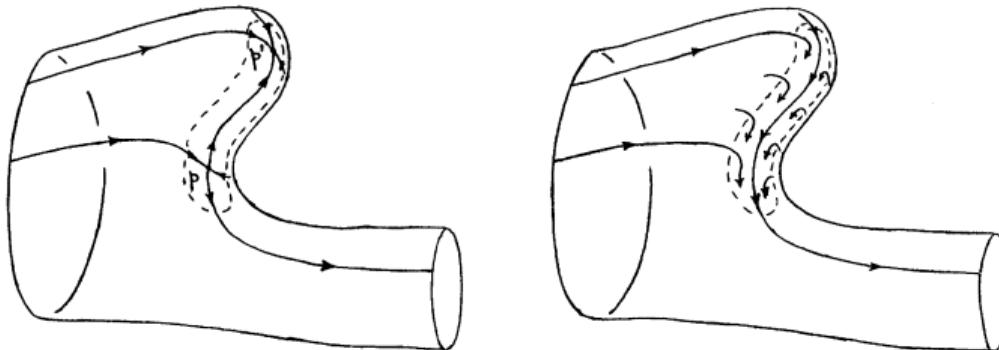
Given a function $f: X \rightarrow \mathbb{R}$ and a real number $\delta \geq 0$, find a function f_δ

- *with the minimal number of critical points*
- *subject to $\|f_\delta - f\|_\infty \leq \delta$.*

Persistence and Morse theory

Morse theory (smooth or discrete):

- Relates critical points to homology of sublevel sets
- Provides a method for *cancelling* pairs of critical points

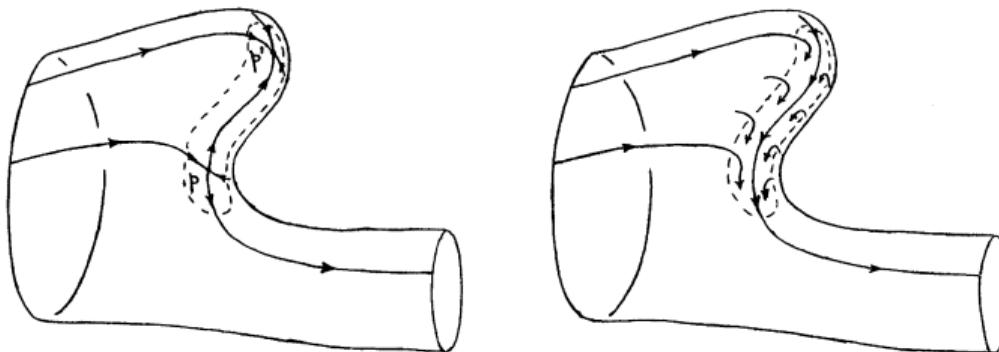


(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

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Persistent homology of a Morse function:

- critical points correspond to endpoints of barcode intervals

Canceling persistence pairs

By stability of persistence barcodes:

Proposition

The critical points of f with persistence $> 2\delta$ provide a lower bound on the number of critical points of any function g with $\|g - f\|_\infty \leq \delta$.

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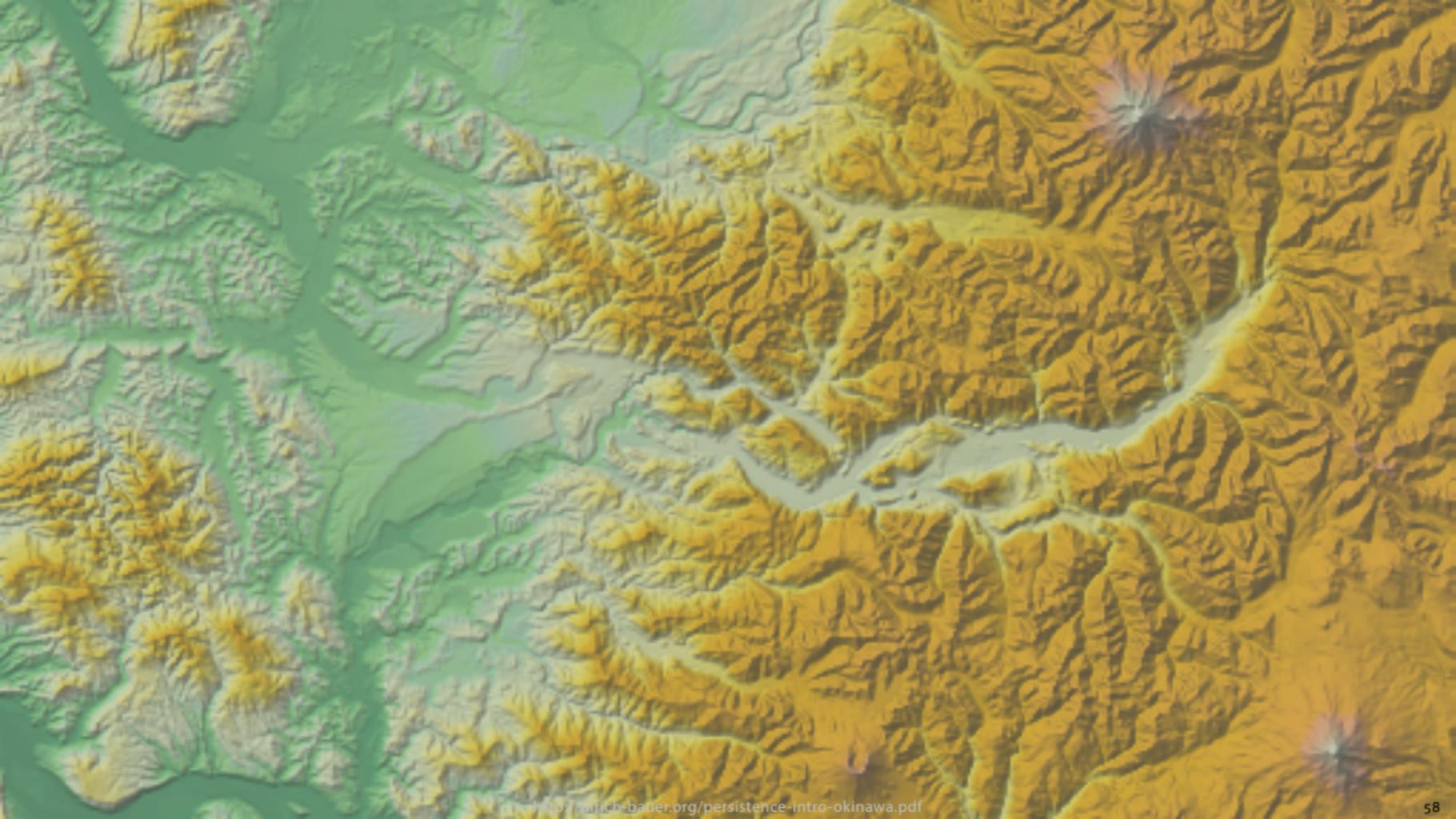
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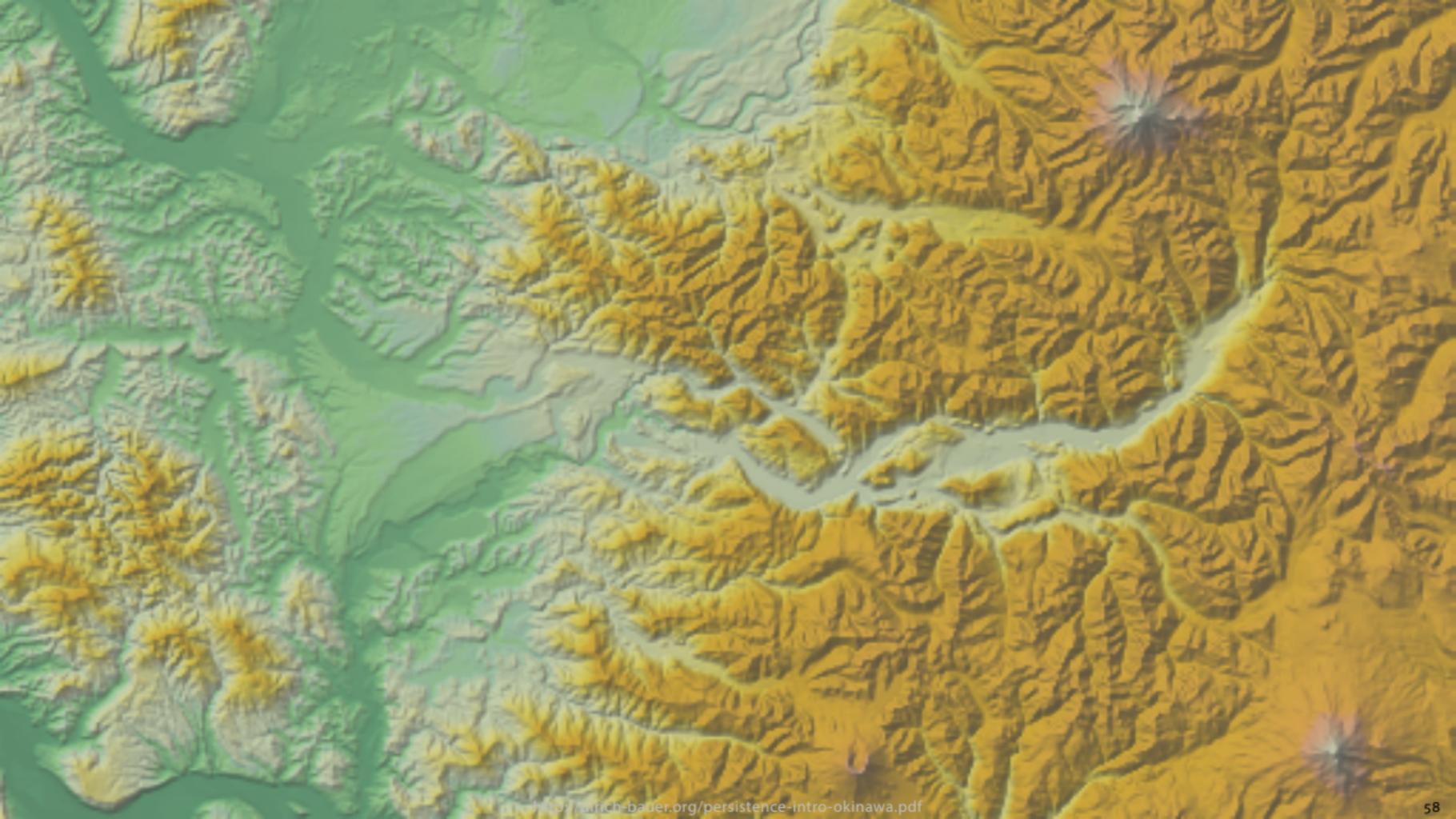
Theorem (B, Lange, Wardetzky, 2011)

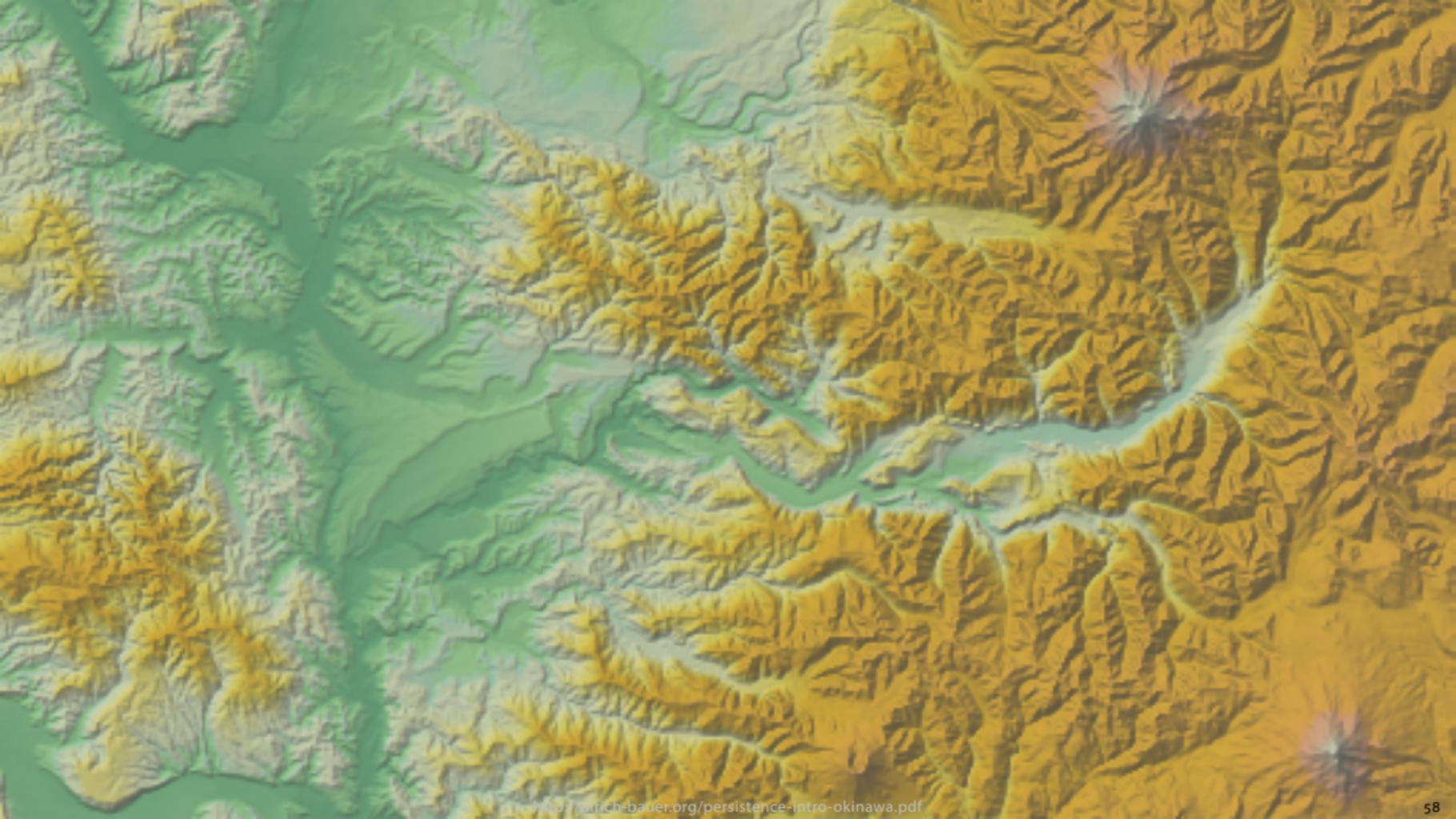
This lower bound can be achieved for a function on a surface

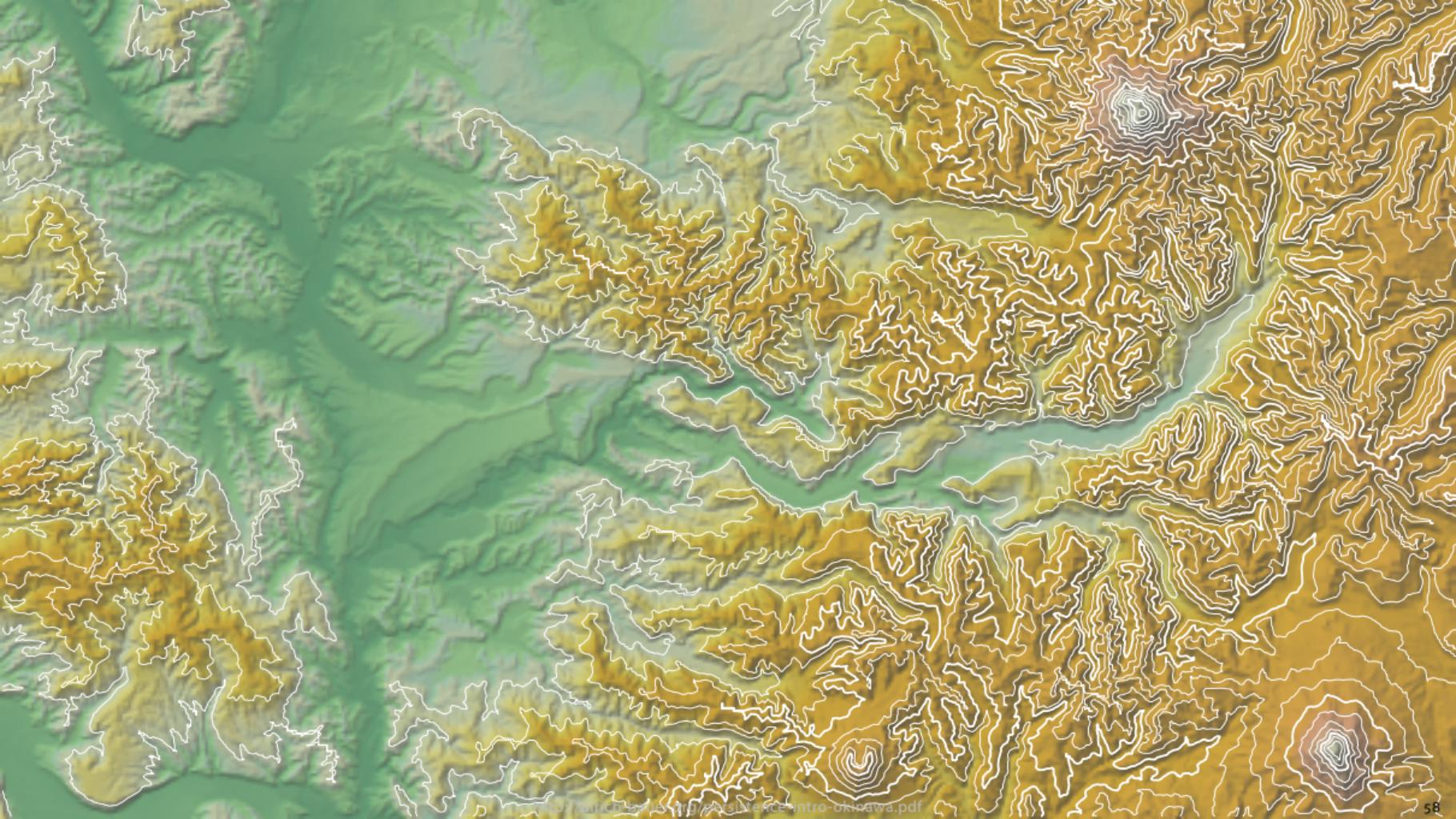
- by canceling all critical points with persistence $\leq 2\delta$.

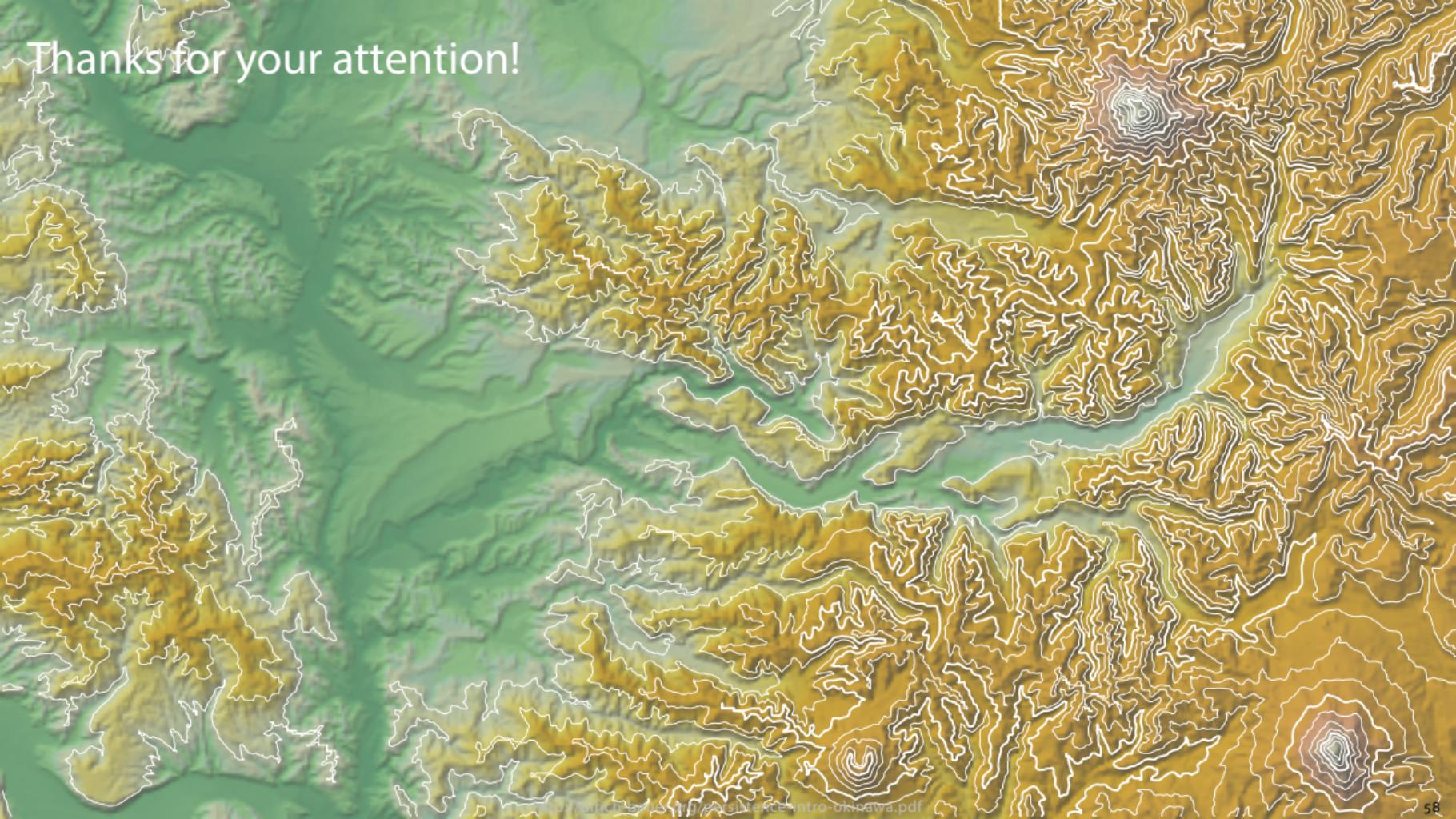












Thanks for your attention!