Persistence is Morse theory

Ulrich Bauer

TUM

January 26, 2017

Geometry & Algebra seminar

University of Bologna

Persistence is (not) Morse theory

Ulrich Bauer

TUM

January 26, 2017

Geometry & Algebra seminar

University of Bologna

Persistence is Morse theory. . . NOT!

Ulrich Bauer

TUM

January 26, 2017

Geometry & Algebra seminar

University of Bologna

Archaeology of persistence

When was persistent homology invented?

- [Edelsbrunner/Letscher/Zomorodian 2000]

When was persistent homology invented?

- [Edelsbrunner/Letscher/Zomorodian 2000]
- [Robbins 1999]

When was persistent homology invented?

- [Edelsbrunner/Letscher/Zomorodian 2000]
- [Robbins 1999]
- [Frosini 1990]

When was persistent homology invented?

- [Edelsbrunner/Letscher/Zomorodian 2000]
- [Robbins 1999]
- [Frosini 1990]
- [Leray 1946]?

When was persistent homology invented first?

When was persistent homology invented first?

ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses is satisfied. The critical limits cluster at most at $+\infty$; the critical points are isolated;[1] the problem is defined by analytic functions; the critical limits taken in their natural order are well-ordered. These conditions are not generally fulfilled. The generality of the theory rested upon the fact that the cases treated approximate in a certain sense the most general problems which it is

When was persistent homology invented first?

Web Images More... Sign in

Google

Scholar 9 results (**0.02** sec) My Citations

- All citations
- Articles
- Case law
- My library

- Any time
- Since 2016
- Since 2015
- Since 2012
- Custom range...

- Sort by relevance
- Sort by date

- include citations

- Create alert

Rank and span in functional topology

Search within citing articles

Exact homomorphism sequences in homology theory

ed.ac.uk [PDF]

JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR

The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] ...

Cited by 46 Related articles All 3 versions Cite Save More

Marston Morse and his mathematical works

ams.org [PDF]

R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org

American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to ...

Cited by 24 Related articles All 4 versions Cite Save More

Unstable minimal surfaces of higher topological structure

M Morse, CB Tompkins - Duke Math. J, 1941 - projecteuclid.org

1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. Theproblem of the existenceof minimal surfaces of unstable type contains many of the typical difficulties, especially those ofa topological nature. Having studied this ...

Cited by 19 Related articles All 2 versions Cite Save

[PDF] Persistence in discrete Morse theory

psu.edu [PDF]

U Bauer - 2011 - Citeseer

The goal of this thesis is to bring together two different theories about critical points of a scalar function and their relation to topology: Discrete Morse theory and Persistent homology. While the goals and fundamental techniques are different, there are certain

When was persistent homology invented first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was in the 1920's to go from the mini-max principle of Birkhoff to the Morse inequalities, let alone extend these inequalities to function spaces, so that by the early 30's Morse could establish the theorem that for any Riemann

When was persistent homology invented first?

inequalities pertain between the dimensions of the A_i and those of $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels [**D**] in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers [**36**] and [**44**] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.

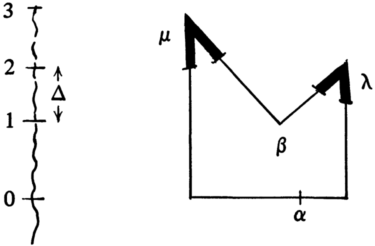


FIGURE 8

The values a where $H(a, a^-) \neq 0$ are indicated on the left, and correspond-

When was persistent homology invented first?

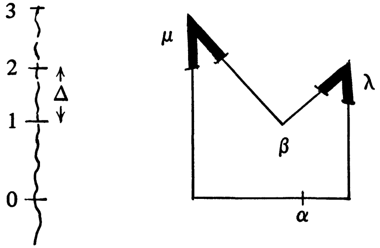


FIGURE 8

The values a where $H(a, a^-) \neq 0$ are indicated on the left, and corresponding to each of these *critical values* a generator of $H(a, a^-)$ is drawn on M, using the singular theory for simplicity. Morse calls such generators "caps". Thus α and β are two "0-caps" and μ and λ two "1-caps". Notice that every cap u defines a definite boundary element ∂u in

$$H(a^-) = \lim_{\varepsilon \to 0^+} H(F < a - \varepsilon);$$

Morse calls a cap u linkable iff $\partial u = 0$. Otherwise it is called *nonlinkable*.

In our example, α, β and μ are linkable while λ is *not*.

Next Morse defines the *span* of a cap u associated to the critical level a in the following manner.

Concepts of Morse's *functional topology*

Early precursors of persistence and spectral sequences:

- F*-homology classes*: similar to persistent homology
  - *inferior/superior cycle limits*: birth/death
- k*-caps*: related to elements of spectral sequence
  - *cap span*: persistence

Real-indexed spectral sequences

Roughly:

- Approximate cycles
- modulo approximate boundaries
- localized at a certain filtration level

Real-indexed spectral sequences

Roughly:

- Approximate cycles
- modulo approximate boundaries
- localized at a certain filtration level

Less roughly:

- Chains at level $\leq p$ with boundaries at level $\leq p - r$
- modulo boundaries at level $\leq p$ of chains at level $< p + r$
- modulo chains at level $< p$

Real-indexed spectral sequences

Roughly:

- Approximate cycles
- modulo approximate boundaries
- localized at a certain filtration level

Less roughly:

- Chains at level $\leq p$ with boundaries at level $\leq p - r$
- modulo boundaries at level $\leq p$ of chains at level $< p + r$
- modulo chains at level $< p$

Even less roughly [Matschke 2013], with $F_{\leq p} = C_*(f^{-1}(-\infty, p])$:

$$E_p^r = \frac{F_{\leq p} \cap \partial^{-1}(F_{\leq p-r})}{\partial(F_{<p+r}) + F_{<p}}$$

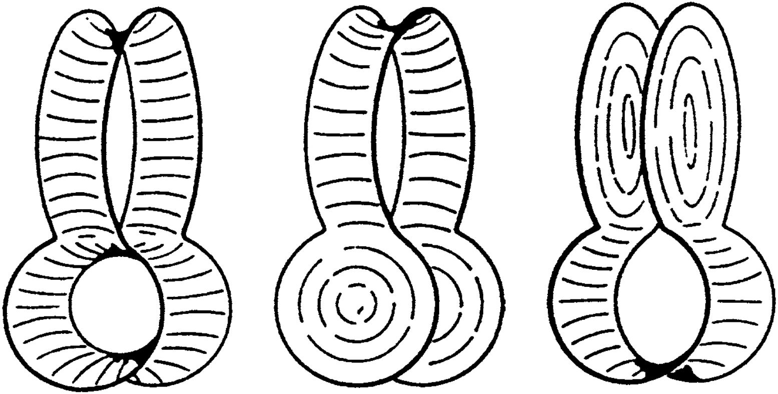
- The E_p^r are equivalence classes of Morse's *caps*

Morse's theory of *functional topology*

Key aspects:

- anticipates persistent homology and spectral sequences
- uses Vietoris homology with field coefficients
- applies to a broad class of functions on metric spaces (not necessarily continuous)
- inclusions of sublevel sets have finite rank homology (*q-tame* persistent homology)
- focus on controlled behavior in pathological cases

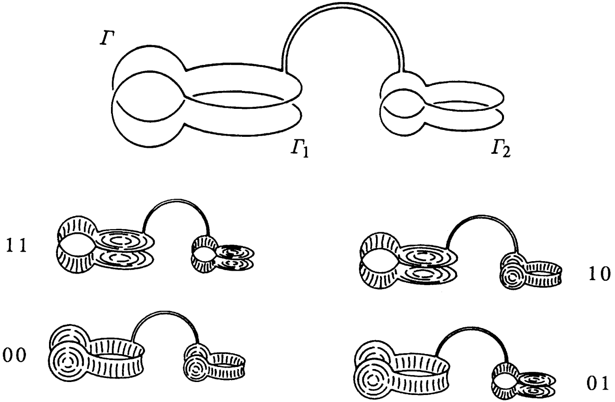
Motivation and application: minimal surfaces



(a) (b) (c)

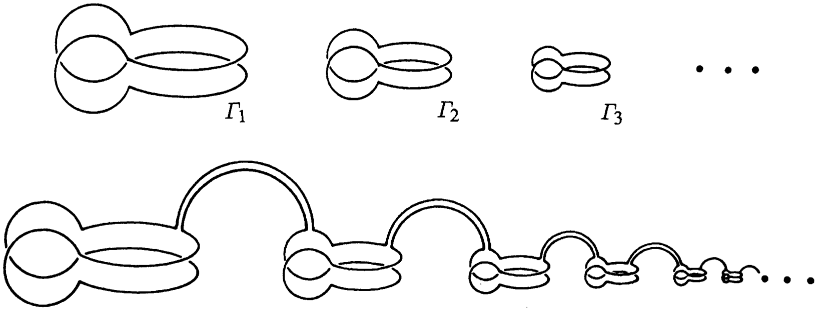
(from Dierkes et al.: Minimal Surfaces, Springer 2010)

Motivation and application: minimal surfaces



(from Dierkes et al.: Minimal Surfaces, Springer 2010)

Motivation and application: minimal surfaces



(from Dierkes et al.: Minimal Surfaces, Springer 2010)

Existence of unstable minimal surfaces

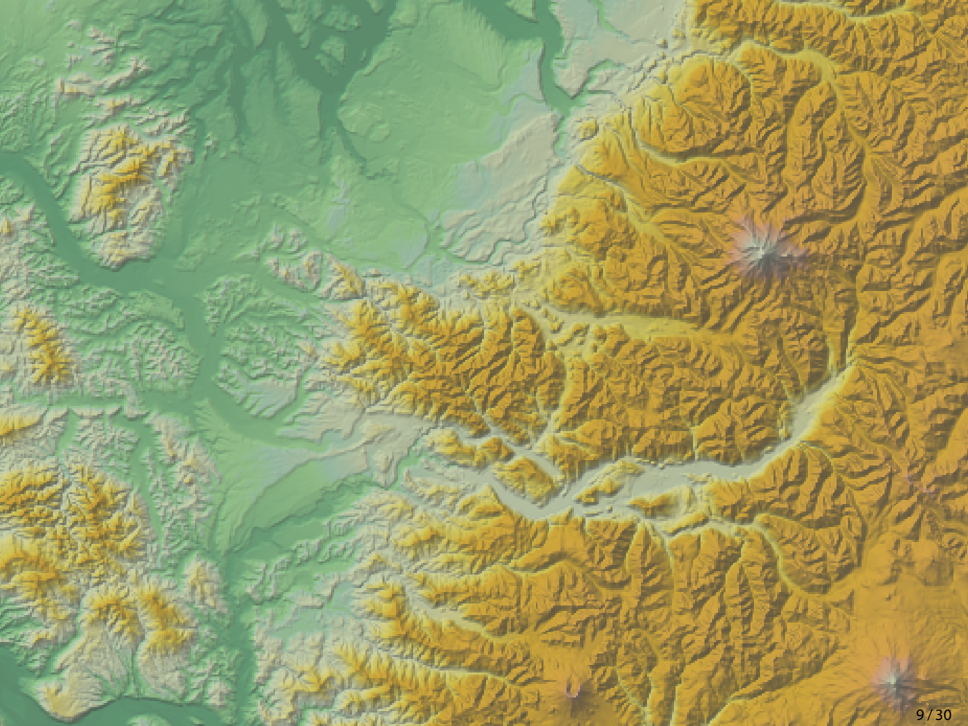
Using persistent homology:

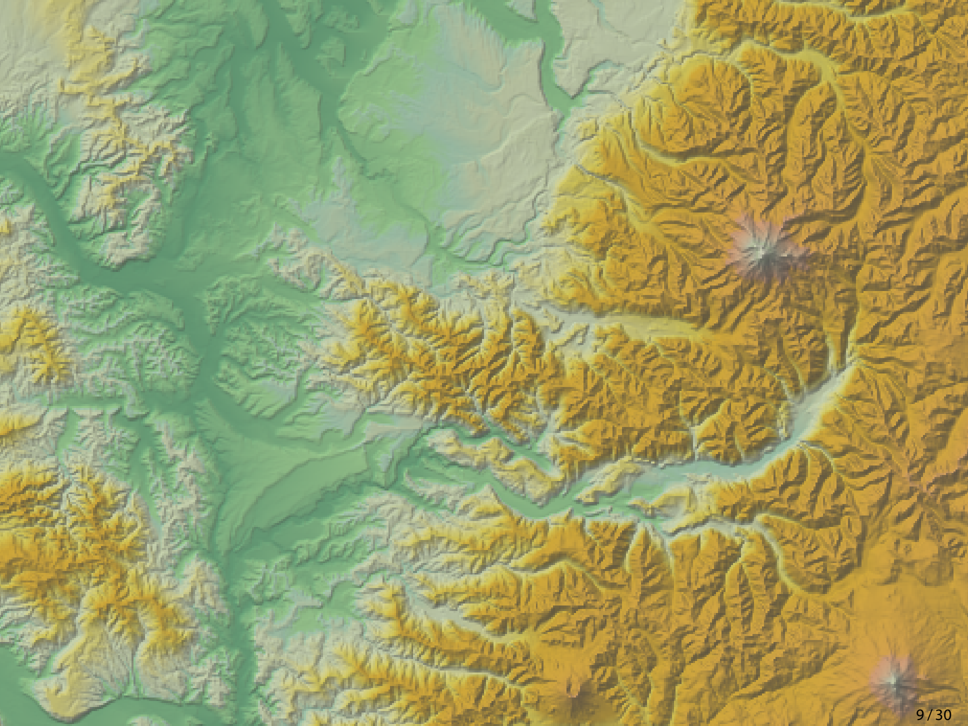
- Number of ϵ-persistent critical points (minimal surfaces) is finite for any $\epsilon > 0$
- Morse inequalities for ϵ-persistent critical points

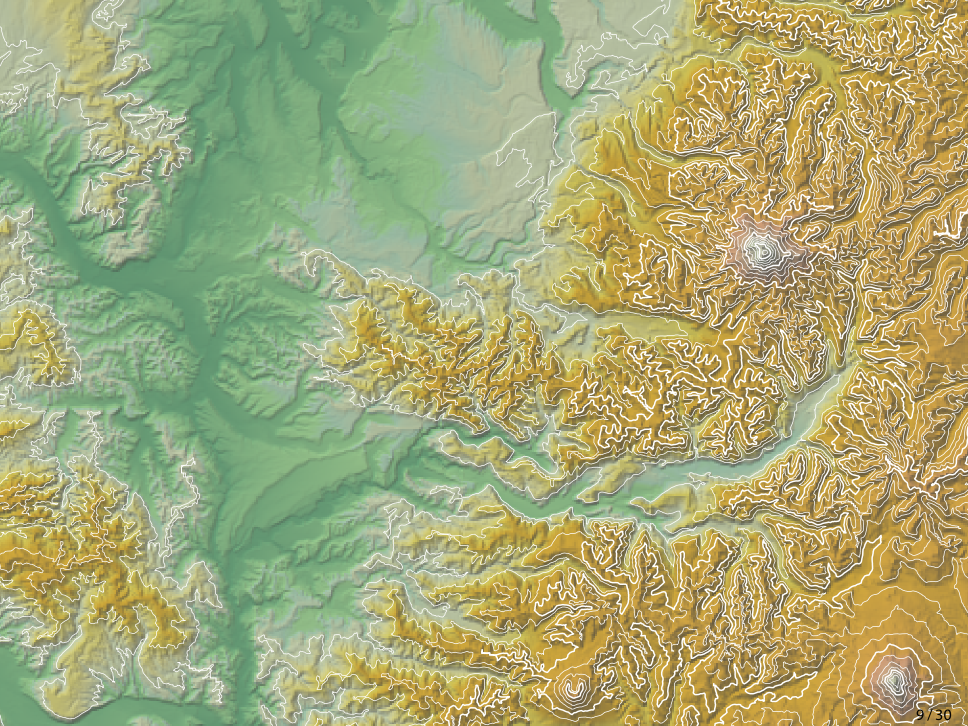
Theorem (Morse, Tompkins 1939)

There is a C_1 curve bounding an unstable minimal surface (a critical point of the area functional with index 1).

Simplification of functions







Topological simplification of functions

Consider the following problem:

Problem (Topological simplification)

Given a function f and a real number $\delta \geq 0$,
find a function g subject to $\|g - f\|_\infty \leq \delta$
with minimal number of critical points.

Topological simplification of functions

Consider the following problem:

Problem (Topological simplification)

Given a function f and a real number $\delta \geq 0$,
find a function g subject to $\|g - f\|_\infty \leq \delta$
with minimal number of critical points.

(Discrete) Morse theory:

- Relates critical points to homology of sublevel set
- Provides method for canceling pairs of critical points

Topological simplification of functions

Consider the following problem:

Problem (Topological simplification)

Given a function f and a real number $\delta \geq 0$,
find a function g subject to $\|g - f\|_\infty \leq \delta$
with minimal number of critical points.

(Discrete) Morse theory:

- Relates critical points to homology of sublevel set
- Provides method for canceling pairs of critical points

Persistent homology:

- Relates homology of two arbitrary sublevel set
- Identifies pairs of critical points (birth and death of homological feature) and quantifies their *persistence*

Persistence pairs and Morse cancelations

By stability of persistence barcodes:

Proposition

The number of critical points of any function g *with* $\|g - f\|_\infty \leq \delta$
is bounded from below by
the number of critical points of f *with persistence* $> 2\delta$*.*

Persistence pairs and Morse cancelations

By stability of persistence barcodes:

Proposition

The number of critical points of any function g with $\|g - f\|_\infty \leq \delta$ is bounded from below by the number of critical points of f with persistence $> 2\delta$.

Theorem (B, Lange, Wardetzky, 2011)

Let f be a function on a surface and let $\delta > 0$. Then canceling all persistence pairs with persistence $\leq 2\delta$ yields a function g satisfying $\|g - f\|_\infty \leq \delta$, achieving the lower bound.

Persistence pairs and Morse cancelations

By stability of persistence barcodes:

Proposition

The number of critical points of any function g with $\|g - f\|_\infty \leq \delta$ is bounded from below by the number of critical points of f with persistence $> 2\delta$.

Theorem (B, Lange, Wardetzky, 2011)

Let f be a function on a surface and let $\delta > 0$.
Then canceling all persistence pairs with persistence $\leq 2\delta$ yields a function g satisfying $\|g - f\|_\infty \leq \delta$, achieving the lower bound.

- Does not generalize to higher-dimensional manifolds!
  (Simplest example: nontrivial knot complement)

Numerics

Cauchy's differentiation formula

Differentiation by integration:

- Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be holomorphic,
- $D \subseteq U$ a closed disk, and
- $a \in \operatorname{int} D$.

Then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

for any cycle γ in the homology class $[\partial D] \in H_1(U \setminus \{a\})$.

Cauchy's differentiation formula

Differentiation by integration:

- Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be holomorphic,
- $D \subseteq U$ a closed disk, and
- $a \in \operatorname{int} D$.

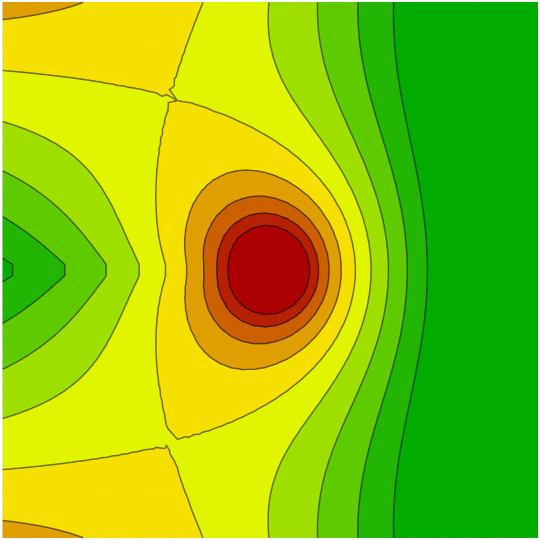
Then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

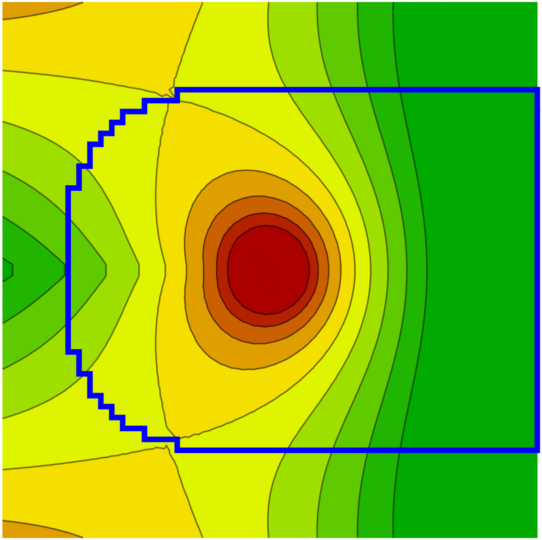
for any cycle γ in the homology class $[\partial D] \in H_1(U \setminus \{a\})$.

- But not all cycles are created equal
  (for numerical integration)!

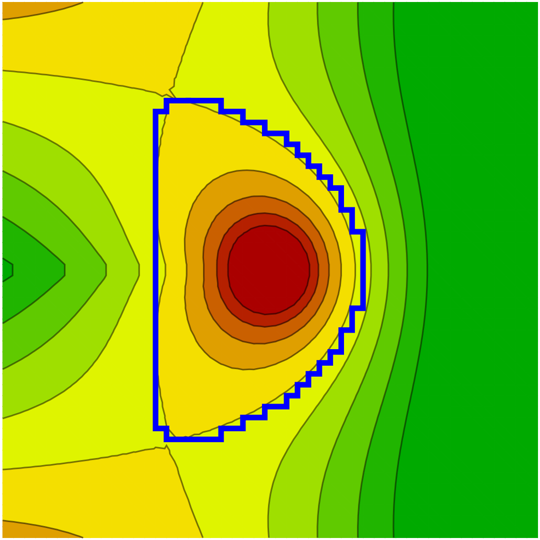
Optimizing condition of contour integration



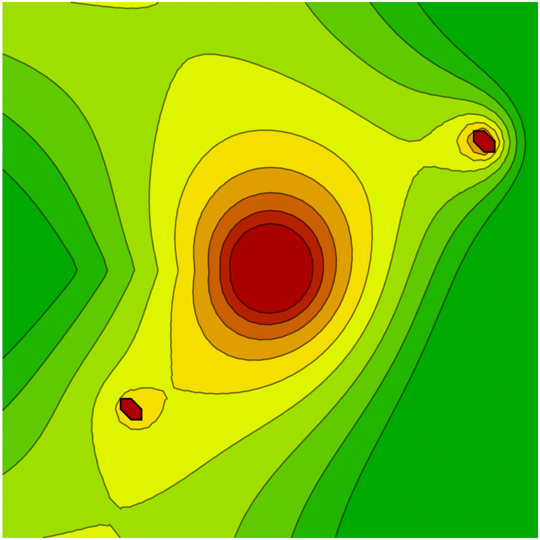
Optimizing condition of contour integration



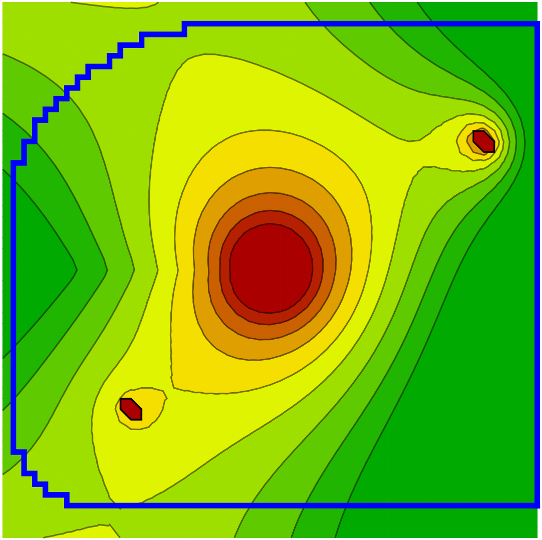
Optimizing condition of contour integration



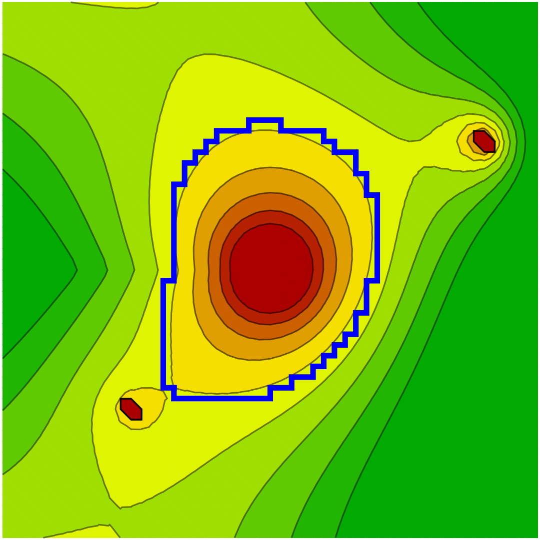
Optimizing condition of contour integration



Optimizing condition of contour integration

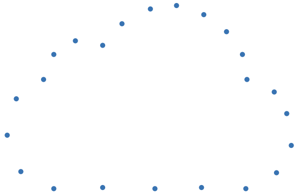


Optimizing condition of contour integration

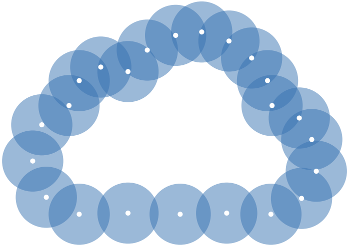


Discrete Morse theory of geometric complexes

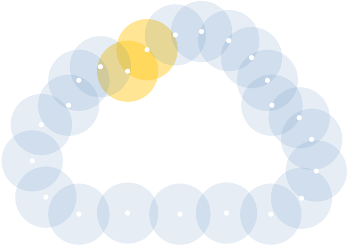
Čech and Delaunay complexes



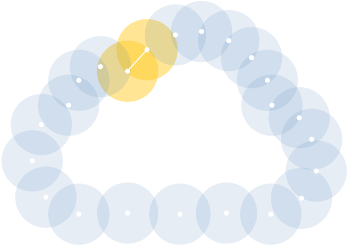
Čech and Delaunay complexes



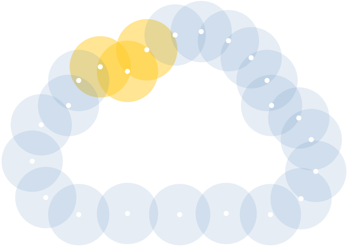
Čech and Delaunay complexes



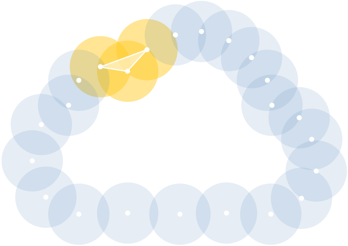
Čech and Delaunay complexes



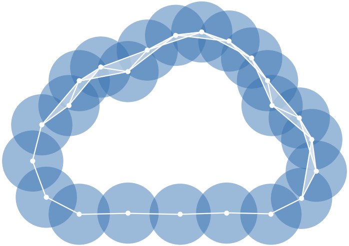
Čech and Delaunay complexes



Čech and Delaunay complexes

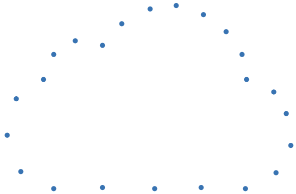


Čech and Delaunay complexes

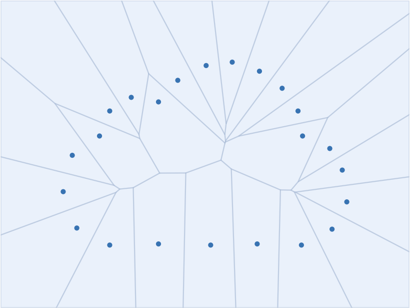


$$\mathrm{Cech}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} B_r(p) \neq \emptyset \right\}$$

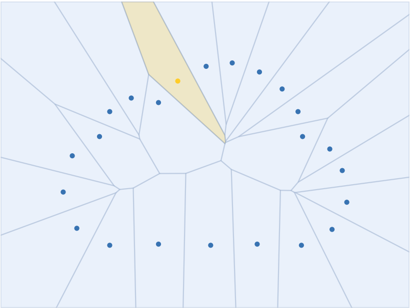
Čech and Delaunay complexes



Čech and Delaunay complexes

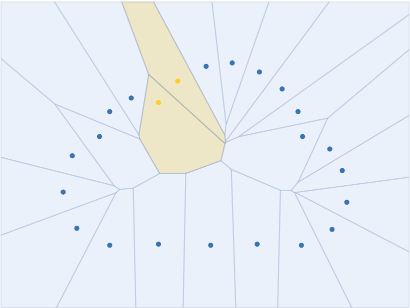


Čech and Delaunay complexes



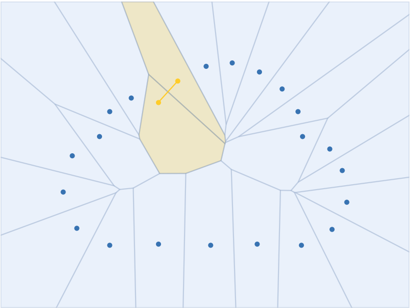
$$\mathrm{Vor}(p, X) = \left\{a \in \mathbb{R}^n \mid d(a,p) = d(a,X)\right\}$$

Čech and Delaunay complexes



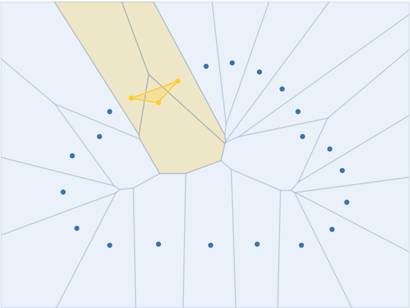
$$\mathrm{Vor}(p, X) = \{a \in \mathbb{R}^n \mid d(a, p) = d(a, X)\}$$

Čech and Delaunay complexes



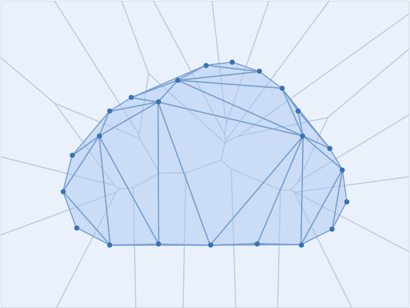
$$\mathrm{Vor}(p, X) = \left\{ a \in \mathbb{R}^n \mid d(a, p) = d(a, X) \right\}$$

Čech and Delaunay complexes



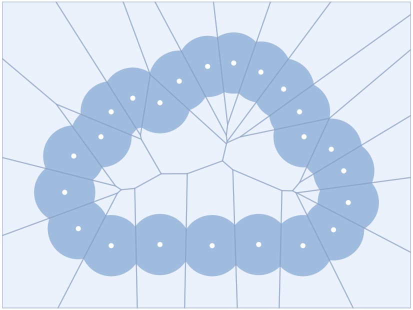
$$\mathrm{Vor}(p, X) = \left\{a \in \mathbb{R}^n \mid d(a, p) = d(a, X)\right\}$$

Čech and Delaunay complexes



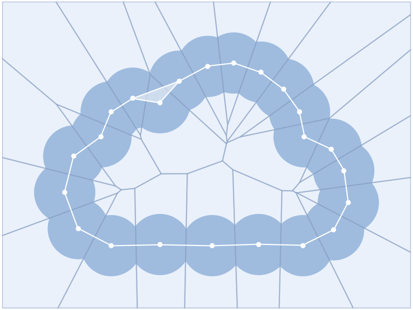
$$\mathrm{Vor}(p, X) = \left\{a \in \mathbb{R}^n \mid d(a, p) = d(a, X)\right\}$$
$$\mathrm{Del}(X) = \left\{Q \subseteq X \mid \bigcap_{p \in Q} \mathrm{Vor}(p, X) \neq \emptyset\right\}$$

Čech and Delaunay complexes



$$\mathrm{Vor}_r(p, X) = B_r(p) \cap \mathrm{Vor}(p, X)$$

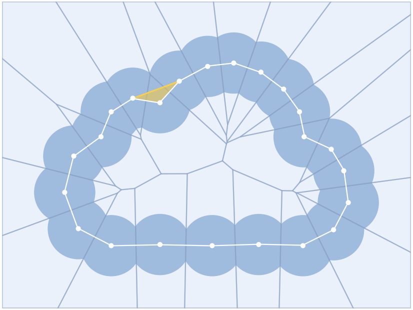
Čech and Delaunay complexes



$$\mathrm{Vor}_r(p, X) = B_r(p) \cap \mathrm{Vor}(p, X)$$

$$\mathrm{Del}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} \mathrm{Vor}_r(p, X) \neq \varnothing \right\}$$

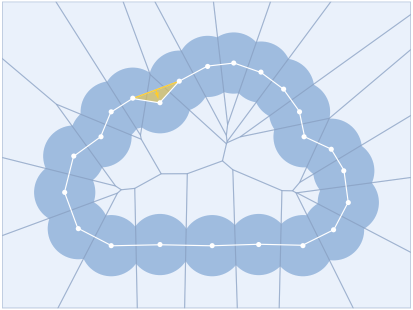
Čech and Delaunay complexes



$$\mathrm{Vor}_r(p, X) = B_r(p) \cap \mathrm{Vor}(p, X)$$

$$\mathrm{Del}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} \mathrm{Vor}_r(p, X) \neq \emptyset \right\}$$

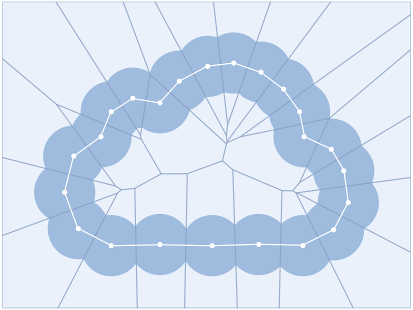
Čech and Delaunay complexes



$$\operatorname{Vor}_r(p, X) = B_r(p) \cap \operatorname{Vor}(p, X)$$

$$\operatorname{Del}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} \operatorname{Vor}_r(p, X) \neq \emptyset \right\}$$

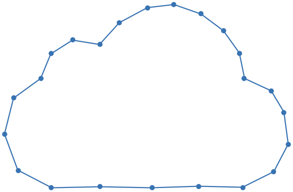
Čech and Delaunay complexes



$$\mathrm{Vor}_r(p, X) = B_r(p) \cap \mathrm{Vor}(p, X)$$

$$\mathrm{Del}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} \mathrm{Vor}_r(p, X) \neq \emptyset \right\}$$

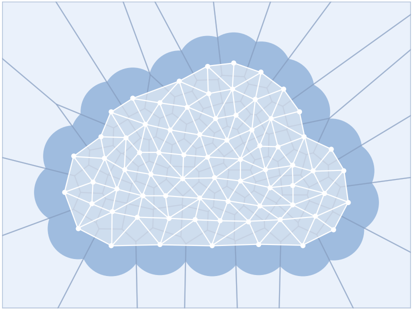
Čech and Delaunay complexes



$$\mathrm{Vor}_r(p, X) = B_r(p) \cap \mathrm{Vor}(p, X)$$

$$\mathrm{Del}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} \mathrm{Vor}_r(p, X) \neq \emptyset \right\}$$

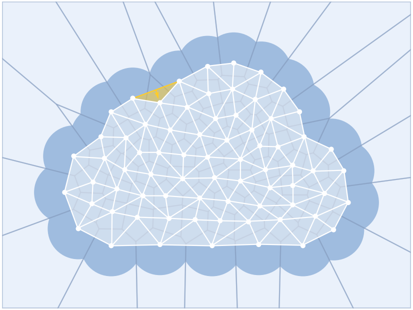
Čech and Delaunay complexes



$$\mathrm{Vor}_r(p, X) = B_r(p) \cap \mathrm{Vor}(p, X)$$

$$\mathrm{Del}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} \mathrm{Vor}_r(p, X) \neq \varnothing \right\}$$

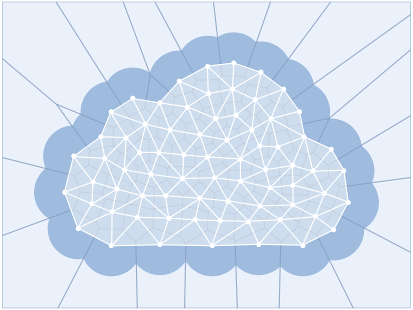
Čech and Delaunay complexes



$$\mathrm{Vor}_r(p, X) = B_r(p) \cap \mathrm{Vor}(p, X)$$

$$\mathrm{Del}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} \mathrm{Vor}_r(p, X) \neq \emptyset \right\}$$

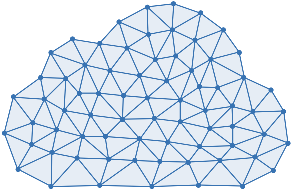
Čech and Delaunay complexes



$$\mathrm{Vor}_r(p, X) = B_r(p) \cap \mathrm{Vor}(p, X)$$

$$\mathrm{Del}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} \mathrm{Vor}_r(p, X) \neq \varnothing \right\}$$

Čech and Delaunay complexes



$$\mathrm{Vor}_r(p, X) = B_r(p) \cap \mathrm{Vor}(p, X)$$

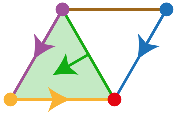
$$\mathrm{Del}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} \mathrm{Vor}_r(p, X) \neq \emptyset \right\}$$

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.

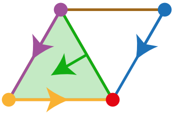


Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

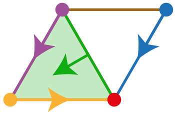
- sublevel sets are subcomplexes, and

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

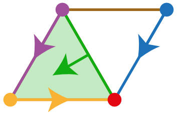
- sublevel sets are subcomplexes, and

Discrete Morse theory

Definition (Forman 1998)

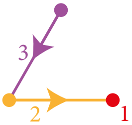
A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

- sublevel sets are subcomplexes, and

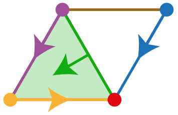


Discrete Morse theory

Definition (Forman 1998)

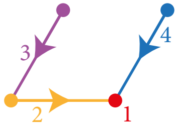
A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

- sublevel sets are subcomplexes, and

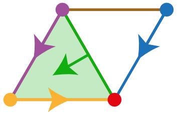


Discrete Morse theory

Definition (Forman 1998)

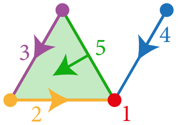
A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

- sublevel sets are subcomplexes, and

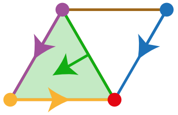


Discrete Morse theory

Definition (Forman 1998)

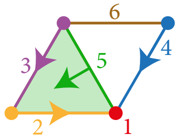
A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

- sublevel sets are subcomplexes, and

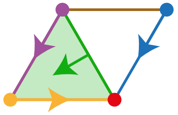


Discrete Morse theory

Definition (Forman 1998)

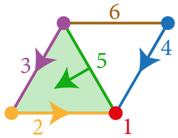
A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

- sublevel sets are subcomplexes, and
- level sets form a discrete vector field.



Fundamental theorem of discrete Morse theory

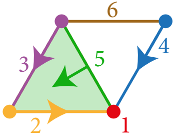
Let f be a discrete Morse function on a cell complex K.

Fundamental theorem of discrete Morse theory

Let f be a discrete Morse function on a cell complex K.

Theorem (Forman 1998)

If $(s, t]$ *contains no critical value of* f,
then the sublevel set K_t collapses to K_s
(written as $K_t \searrow K_s$*).*

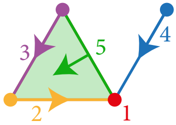


Fundamental theorem of discrete Morse theory

Let f be a discrete Morse function on a cell complex K.

Theorem (Forman 1998)

If $(s, t]$ *contains no critical value of* f*,*
then the sublevel set K_t collapses to K_s
(written as $K_t \searrow K_s$*).*

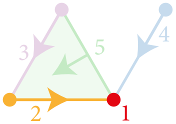


Fundamental theorem of discrete Morse theory

Let f be a discrete Morse function on a cell complex K.

Theorem (Forman 1998)

If $(s, t]$ contains no critical value of f, then the sublevel set K_t collapses to K_s *(written as $K_t \searrow K_s$).*

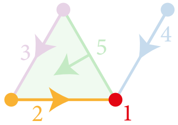


Fundamental theorem of discrete Morse theory

Let f be a discrete Morse function on a cell complex K.

Theorem (Forman 1998)

If $(s, t]$ contains no critical value of f, then the sublevel set K_t collapses to K_s *(written as $K_t \searrow K_s$).*



Corollary

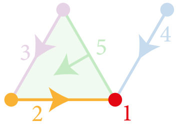
A cell complex $M \simeq K$ can be built from the critical cells of f.

Fundamental theorem of discrete Morse theory

Let f be a discrete Morse function on a cell complex K.

Theorem (Forman 1998)

If $(s, t]$ contains no critical value of f, then the sublevel set K_t collapses to K_s *(written as $K_t \searrow K_s$).*



Corollary

A cell complex $M \simeq K$ can be built from the critical cells of f.

This homotopy equivalence is compatible with the filtration.

Corollary

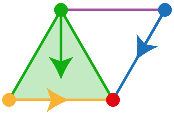
M and K have isomorphic persistent homology (with regard to f).

Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* is a partition of the simplices into clusters of the form

$$\{\rho \mid \sigma \subseteq \rho \subseteq \tau\}$$



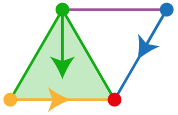
(intervals $[\sigma, \tau]$ of the face poset).

Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* is a partition of the simplices into clusters of the form

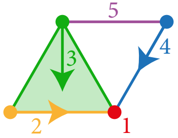
$$\{\rho \mid \sigma \subseteq \rho \subseteq \tau\}$$



(intervals $[\sigma, \tau]$ of the face poset).

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *generalized discrete Morse function* if

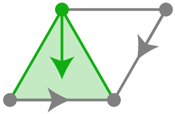
- sublevel sets are subcomplexes
- level sets form a discrete vector field



Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:

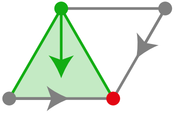


Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:

- choose an arbitrary vertex $x \in U \setminus L$

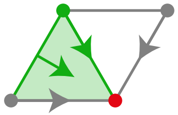


Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:

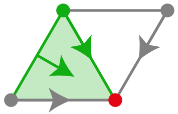
- choose an arbitrary vertex $x \in U \setminus L$
- partition $[L, U]$ into pairs $(Q \setminus \{x\}, Q \cup \{x\})$ for all $Q \in [L, U]$.



Refining generalized vector fields

A generalized vector field V can be refined to a vector field.
For each non-critical interval $[L, U] \in V$:

- choose an arbitrary vertex $x \in U \setminus L$
- partition $[L, U]$ into pairs $(Q \setminus \{x\}, Q \cup \{x\})$ for all $Q \in [L, U]$.



Therefore the collapsing theorems also hold for generalized discrete Morse functions.

Morse theory for Čech and Delaunay complexes

Proposition

The Čech complexes and the Delaunay (alpha) complexes are sublevel set filtrations of generalized discrete Morse functions.

Morse theory for Čech and Delaunay complexes

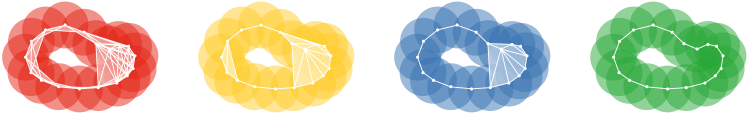
Proposition

The Čech complexes and the Delaunay (alpha) complexes are sublevel set filtrations of generalized discrete Morse functions.

Theorem (B., Edelsbrunner 2016, Trans. AMS)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are naturally homotopy equivalent through a sequence of collapses

$$\mathrm{Cech}_r X \searrow \mathrm{Cech}_r X \cap \mathrm{Del}\, X \searrow \mathrm{Del}_r X \searrow \mathrm{Wrap}_r X.$$



Induced by a single discrete gradient field!

Vietoris–Rips complexes

Consider a finite metric space (X, d).

The *Vietoris–Rips complex* is the simplicial complex

$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid \operatorname{diam} S \leq t\}$$

- 1-skeleton: all edges with pairwise distance $\leq t$
- all possible higher simplices (flag complex)
- Vietoris–Rips complex are *not* generically sublevel sets of discrete Morse functions

Vietoris–Rips complexes

Consider a finite metric space (X, d).
The *Vietoris–Rips complex* is the simplicial complex

$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid \operatorname{diam} S \leq t\}$$

- 1-skeleton: all edges with pairwise distance $\leq t$
- all possible higher simplices (flag complex)
- Vietoris–Rips complex are *not* generically sublevel sets of discrete Morse functions

Note: for $t \geq \operatorname{diam} X$, $\mathrm{Rips}_t(X)$ is the full simplex with vertices X

- Number of simplices grows exponentially in dimension
- Computation is one of the most important challenges in applied topology!

Computation

Ripser

A software for computing Vietoris–Rips persistence barcodes

- around 1000 lines of C++ code, no external dependencies

Ripser

A software for computing Vietoris–Rips persistence barcodes

- around 1000 lines of C++ code, no external dependencies
- support for
  - coefficients in a prime field $\mathbb{Z}/p\mathbb{Z}$

Ripser

A software for computing Vietoris–Rips persistence barcodes

- around 1000 lines of C++ code, no external dependencies
- support for
  - coefficients in a prime field $\mathbb{Z}/p\mathbb{Z}$
  - sparse distance matrices for distance threshold

Ripser

A software for computing Vietoris–Rips persistence barcodes

- around 1000 lines of C++ code, no external dependencies
- support for
  - coefficients in a prime field $\mathbb{Z}/p\mathbb{Z}$
  - sparse distance matrices for distance threshold
- open source (`http://ripser.org`)
  - released in July 2016

Ripser

A software for computing Vietoris–Rips persistence barcodes

- around 1000 lines of C++ code, no external dependencies
- support for
  - coefficients in a prime field $\mathbb{Z}/p\mathbb{Z}$
  - sparse distance matrices for distance threshold
- open source (`http://ripser.org`)
  - released in July 2016
- online version (`http://live.ripser.org`)
  - launched in August 2016

Ripser

A software for computing Vietoris–Rips persistence barcodes

- around 1000 lines of C++ code, no external dependencies
- support for
  - coefficients in a prime field $\mathbb{Z}/p\mathbb{Z}$
  - sparse distance matrices for distance threshold
- open source (`http://ripser.org`)
  - released in July 2016
- online version (`http://live.ripser.org`)
  - launched in August 2016
- most efficient software for Vietoris–Rips persistence by far
  - computes H^2 barcode for 50000 random points on a torus in 136 seconds / 9 GB

Ripser

A software for computing Vietoris–Rips persistence barcodes

- around 1000 lines of C++ code, no external dependencies
- support for
  - coefficients in a prime field $\mathbb{Z}/p\mathbb{Z}$
  - sparse distance matrices for distance threshold
- open source (`http://ripser.org`)
  - released in July 2016
- online version (`http://live.ripser.org`)
  - launched in August 2016
- most efficient software for Vietoris–Rips persistence by far
  - computes H^2 barcode for 50000 random points on a torus in 136 seconds / 9 GB
- (co-)winner of 2016 ATMCS Best New Software Award

Performance of persistence software

Example data set:

- 192 points on $\mathbb{S}^2$
- persistent homology barcodes up to dimension 2
- over 56 mio. simplices in 3-skeleton

Performance of persistence software

Example data set:

- 192 points on $\mathbb{S}^2$
- persistent homology barcodes up to dimension 2
- over 56 mio. simplices in 3-skeleton

Some previous software:

- javaplex (Stanford): 3200 seconds, 12 GB
- Dionysus (Duke): 840 seconds, 5 GB
- DIPHA (IST Austria): 50 seconds, 6 GB
- GUDHI (INRIA): 60 seconds, 3 GB

Performance of persistence software

Example data set:

- 192 points on $\mathbb{S}^2$
- persistent homology barcodes up to dimension 2
- over 56 mio. simplices in 3-skeleton

Some previous software:

- javaplex (Stanford): 3200 seconds, 12 GB
- Dionysus (Duke): 840 seconds, 5 GB
- DIPHA (IST Austria): 50 seconds, 6 GB
- GUDHI (INRIA): 60 seconds, 3 GB

Ripser: 1.2 seconds, 160 MB

The four special ingredients of Ripser

The improved performance is based on 4 techniques:

- Skip inessential columns [Chen, Kerber 2011]
- Compute cohomology [de Silva et al. 2011]
- Implicit boundary matrix
- Apparent pairs

Apparent pairs

Apparent pairs

- provide a close connection between persistence and discrete Morse theory
- are both persistence pairs and Morse pairs
- typically cover almost all simplices in the Rips filtration
- provide a shortcut for computation

Morse pairs and persistence pairs

Natural filtration settings

Typical assumptions on the filtration:

• general filtration	persistence (in theory)
• filtration by singletons or pairs	discrete Morse theory
• simplexwise filtration	persistence (computation)

Natural filtration settings

Typical assumptions on the filtration:

• general filtration	persistence (in theory)
• filtration by singletons or pairs	discrete Morse theory
• simplexwise filtration	persistence (computation)

Conclusion:

- Discrete Morse theory sits in the middle
  between persistence and persistence (hah!)

From Morse theory to persistence and back

Proposition (from Morse to persistence)

The pairs of a Morse filtration are apparent 0-persistence pairs *for the simplexwise refinement of the filtration.*

Apparent persistence pair (σ, τ):

- σ is the youngest face of τ
- τ is the oldest coface of σ

From Morse theory to persistence and back

Proposition (from Morse to persistence)

The pairs of a Morse filtration are apparent 0-persistence pairs *for the simplexwise refinement of the filtration.*

Apparent persistence pair (σ, τ):

- σ is the youngest face of τ
- τ is the oldest coface of σ

Proposition (from persistence to Morse)

Consider an arbitrary filtration with a simplexwise refinement. The apparent 0-persistence pairs yield a Morse filtration

- *refining the original one, and*
- *coarsening the simplexwise one.*

Discrete Morse functions from total orders

The upshot:

- A total order on simplices induces a discrete gradient (apparent pairs)
- Simple and important example: lexicographic order, given by total order on vertices
- Special cases have appeared in literature before
  - random Rips complex in supercritical regime [Kahle 2010]
- Often, the apparent gradient has good properties (few critical points)
- Provides a shortcut to speed up computation in Ripser (by a factor of about 3)

Algebraic Morse theory

Building blocks:

[Kozlov 2005, Sköldberg 2006, Jöllenbeck/Welker 2009]

- free chain complex, with a basis (of your choice)

Algebraic Morse theory

Building blocks:

[Kozlov 2005, Sköldberg 2006, Jöllenbeck/Welker 2009]

- free chain complex, with a basis (of your choice)
- admissible Morse pairs: basis elements (σ, τ) such that the coefficient of σ in $\partial\tau$ is invertible

Algebraic Morse theory

Building blocks:

[Kozlov 2005, Sköldberg 2006, Jöllenbeck/Welker 2009]

- free chain complex, with a basis (of your choice)
- admissible Morse pairs: basis elements (σ, τ) such that the coefficient of σ in $\partial\tau$ is invertible

An important special case: *persistence basis*

- cycles σ creating homology

Algebraic Morse theory

Building blocks:

[Kozlov 2005, Sköldberg 2006, Jöllenbeck/Welker 2009]

- free chain complex, with a basis (of your choice)
- admissible Morse pairs: basis elements (σ, τ) such that the coefficient of σ in $\partial\tau$ is invertible

An important special case: *persistence basis*

- cycles σ creating homology
- chains τ bounding those cycles

Algebraic Morse theory

Building blocks:

[Kozlov 2005, Sköldberg 2006, Jöllenbeck/Welker 2009]

- free chain complex, with a basis (of your choice)
- admissible Morse pairs: basis elements (σ, τ) such that the coefficient of σ in $\partial\tau$ is invertible

An important special case: *persistence basis*

- cycles σ creating homology
- chains τ bounding those cycles

Proposition (from persistence to algebraic Morse)

Any persistence pair of a simplexwise filtration corresponds to an algebraic Morse pair for a persistence basis.

Morse reductions for persistence computation

Morse reduction methods [Mrozek/Wanner 2010, Robbins et al. 2010, Günther et al. 2012, Mischaikow/Nanda 2013, Dłotko/Wagner 2014, Allili et al. 2015]

- find some Morse pairs
- remove them from the complex (reductions)
- compute persistence of remaining complex

Morse reductions for persistence computation

Morse reduction methods [Mrozek/Wanner 2010, Robbins et al. 2010, Günther et al. 2012, Mischaikow/Nanda 2013, Dłotko/Wagner 2014, Allili et al. 2015]

- find some Morse pairs
- remove them from the complex (reductions)
- compute persistence of remaining complex

Variant: Clear & compress [B/Kerber/Reininghaus 2013]

- using local persistence pairs (algebraic Morse pairs)
- can persistence speed up persistence computation?

Morse reductions for persistence computation

Morse reduction methods [Mrozek/Wanner 2010, Robbins et al. 2010, Günther et al. 2012, Mischaikow/Nanda 2013, Dłotko/Wagner 2014, Allili et al. 2015]

- find some Morse pairs
- remove them from the complex (reductions)
- compute persistence of remaining complex

Variant: Clear & compress [B/Kerber/Reininghaus 2013]

- using local persistence pairs (algebraic Morse pairs)
- can persistence speed up persistence computation?

Observed performance:

- previous implementations: significant speedups
- with recent improvements: not so significant anymore
  - mostly for difficult instances (large total persistence)
  - benefits from parallelization
- Morse-theoretic alternative: apparent pairs gradient

Conclusion

Persistence is not quite the same as discrete Morse theory

- Persistence was already discovered by Morse

Conclusion

Persistence is not quite the same as discrete Morse theory

- Persistence was already discovered by Morse
- Čech complexes, Delaunay (alpha) complexes, and Wrap are connected by discrete Morse theory

Conclusion

Persistence is not quite the same as discrete Morse theory

- Persistence was already discovered by Morse
- Čech complexes, Delaunay (alpha) complexes, and Wrap are connected by discrete Morse theory
- All Morse pairs can be seen as persistence pairs

Conclusion

Persistence is not quite the same as discrete Morse theory

- Persistence was already discovered by Morse
- Čech complexes, Delaunay (alpha) complexes, and Wrap are connected by discrete Morse theory
- All Morse pairs can be seen as persistence pairs
- Many persistence pairs can be seen as Morse pairs

Conclusion

Persistence is not quite the same as discrete Morse theory

- Persistence was already discovered by Morse
- Čech complexes, Delaunay (alpha) complexes, and Wrap are connected by discrete Morse theory
- All Morse pairs can be seen as persistence pairs
- Many persistence pairs can be seen as Morse pairs
- These apparent Morse/persistence pairs are very useful, both in proofs and in computation