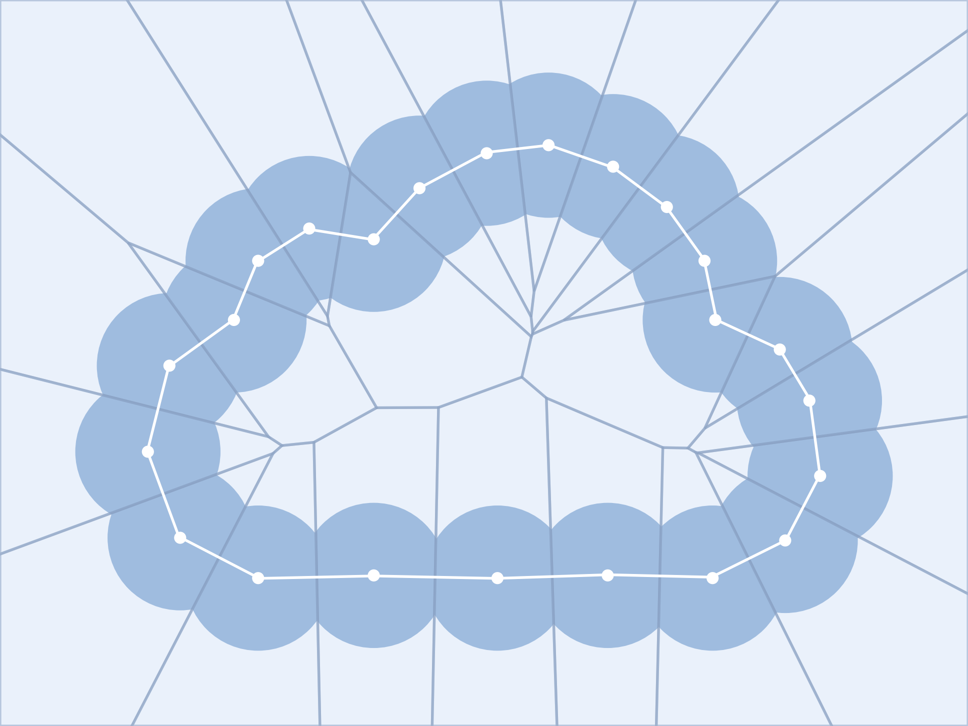


Wrapping clouds

Ulrich Bauer

TUM

June 21, 2018

Joint work with Herbert Edelsbrunner (IST Austria)

Wrapping clouds

With Morse from Čech through Alpha to Wrap

Ulrich Bauer

TUM

June 21, 2018

Joint work with Herbert Edelsbrunner (IST Austria)

From: **Herbert Edelsbrunner** edels@ist.ac.at
Subject: talk in Vienna
Date: 9. January 2014 at 13:25
To: Ulrich.Bauer@ist.ac.at

Hi Ulrich,

I did not find my talk at the suspected place, but I found it in my stack to be discarded. For some reason, I crossed out the talk, and I do not remember why I did that. It is possible that this is not the final version and that I made another, better version from this talk, but if I did I cannot find it anywhere. Or, I made a mistake and thought I am crossing out something else.

In any case, this is all I have at the moment. I hope it is helpful.

I arrived ok in Vienna. I hope you arrived well in Columbus.
Herbert

THE MORSE THEORY OF ČECH AND DELAUNAY FILTRATIONS

I. HISTORY
II. FILTRATIONS
III. DISCRETE MORSE THEORY
IV. RADIUS FUNCTIONS
~~V. COLLAPSIBILITY~~

I.1 ALPHA + WRAP CPLXS

1937 : Delaunay tri. [D]
1983 : α-shapes in $\mathbb{R}^2$ [EKS]
1994 : α-shapes in $\mathbb{R}^3$ with graphics [EM]
1995 : α-shapes in $\mathbb{R}^n$ also weighted [E]
1994 : Bellitti lecture. [DE]
1996 : Wrap + Geomagic

I.2 DISCRETE MORSE THEORY

19... : Morse theory [M]
1963 : in book [Milnor]
1998 : discrete MT [F]
2009 : gen. DMT [Freij]

MAIN THEOREM. $X \subseteq \mathbb{R}^n$ in g.p. then
$D\check{C}_r(X) \searrow D_r(X) \searrow W_r(X)$.
⟹ these complexes are simple-homotopy equivalent.

I.3 TOPOL. DATA ANALYSIS

1945 : Nerve Thm (Leray)
199 : Čech complexes [CdE]
2013 : Delaunay-Čech ⊆ Del. compl. in $\mathbb{R}^2$ [CKim]

IV.5 WRAP COMPLEX

Q = crit. cx of f_D.
↓Q = lower set

$W_r(X) = \bigcup_{f_D(Q) \leq r^2} \downarrow Q$
wrap complex
$W_r(X) = f_W^{-1}[0, r^2]$

II.1 ČECH FILTRATION

$X \subseteq \mathbb{R}^n$, finite

$\check{C}_r(X) = \{Q \subseteq X \mid \bigcap_{x \in Q} B_r(x) \neq \emptyset\}$
Čech complex of X for r.

$\emptyset \subseteq \ldots \subseteq \check{C}_r(X) \subseteq \check{C}_s(X) \subseteq \ldots \subseteq \Delta(X)$ whenever $r \leq s$.

$DC_r(X) = C_r(X) \cap D(X)$

II.2 SIMPLICIAL COMPLEXES

$\boxed{\text{geometric}}$:
p-simplex = conv. hull of p+1 a.i. points.
simpl. complex = collection of simplices s.t.
(i) σ∈K ⟹ faces ∈K.
(ii) σ∩τ = ∅ or face of both.

$\boxed{\text{abstract}}$
set of vertices, V.
ab. simplex = coll in V.
a.s.c. = system of collections closed under taking subsets.

II.3 DELAUNAY TRIANGULATION

$X \subseteq \mathbb{R}^n$

$D(X) = \{Q \subseteq X \mid \bigcap_{x \in Q} V(x) \neq \emptyset\}$
Delaunay tri.
(needs a gen. pos. ass.)

$D_r(X) = \{Q \subseteq X \mid \bigcap_{x \in Q} (B_r(x) \cap V(x)) \neq \emptyset\}$
Delaunay complex of X for r

$\emptyset \subseteq \ldots \subseteq D_r(X) \subseteq D_s(X) \subseteq \ldots \subseteq D(X)$ whenever $r \leq s$.

II.4 GENERAL POSITION.

DEF. A finite $X \subseteq \mathbb{R}^n$ is in g.p. if
(i) ∀ 0≤p<n, the max. # points in a p-plane is p+1; and
(ii) ∀ Q⊆X with dim Q ≤ n, the smallest circumsphere does not contain points of X−Q.

III.1 COLLAPSING

σ∈K is free if contained in unique max. simplex, τ

To collapse we remove $[\sigma, \tau] = \{\upsilon \in K \mid \sigma \subseteq \upsilon \subseteq \tau\}$.
$K \searrow K - [\sigma, \tau]$
elementary collapse if dim σ = dim τ − 1.

III.2 DISCRETE MORSE THEORY

Hasse diagram of

dis. vec. field = matching in HD

is acyclic or dis. grad. field. if no cycle after contraction

LEMMA. f: K→ℝ with f(σ) = f(τ) in V and f(σ) ≠ f(τ) ∉ V.
⟹ V is acyclic.
crit. simplex ∉ V.

III.3 GEN. DISC. MORSE THEORY

pairs → intervals [σ, τ]
l.b. u.b.

gen. disc. vec. field. = partition of K into intervals

is acyclic or g.d.g. field. if no cycle after contraction.

g.d.g.f. can be refined to d.g.f. by forming perfect matchings inside intervals.

IV.6 INTERVAL INTERSECTION

EXCLUDED SINGLETON LEMMA.
$X \subseteq \mathbb{R}^n$ in gen. pos.
Then the intersection of
a non-sing. Del. int. with
a non-sing. Čech int. is
∅ or a non-sing. interval.

IV.1 DELAUNAY FUNCTION

Q⊆X centers of empty circumspheres in $\bigcap_{x \in Q} V(x)$.

$S_D(Q)$ = unique smallest empty circumsphere.

$f_D: D(X) \to \mathbb{R}$ defd by
$f_D(Q)$ = sq. radius of $S_D(Q)$.
Delaunay Function

LEMMA. $D_r(X) = f_D^{-1}[0, r^2]$.

IV.2 ČECH FUNCTION

$S_{\check{C}}(Q)$ = smallest enclosing sphere.

$f_{\check{C}}: \Delta(X) \to \mathbb{R}$ defd by
$f_{\check{C}}(Q)$ = sq. rad. of $S_{\check{C}}(Q)$
Čech Function

LEMMA $\check{C}_r(X) = f_{\check{C}}^{-1}[0, r^2]$.

Note: $f_{\check{C}}(Q) \leq f_D(Q)$.
⟹ $D_r(X) \subseteq D\check{C}_r(X) \subseteq \check{C}_r(X)$.

IV.3 ČECH INTERVALS

$S = S_{\check{C}}(Q)$, $\bar{S}$ = conv S.

$L = L_{\check{C}}(Q) = Q \cap S$
$U = U_{\check{C}}(Q) = X \cap \bar{S}$.

[L, U] consists of Q′⊆X with $L = L_{\check{C}}(Q')$.

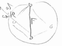
LEMMA. $f_{\check{C}}$ is g.d.M.f. with gradient $V_{\check{C}} = \{[L, U]\}$.

IV.4 DELAUNAY INTERVALS

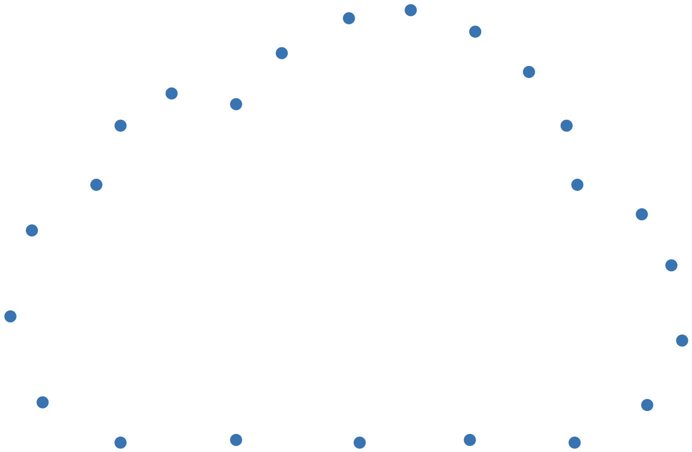
$S = S_D(Q)$.

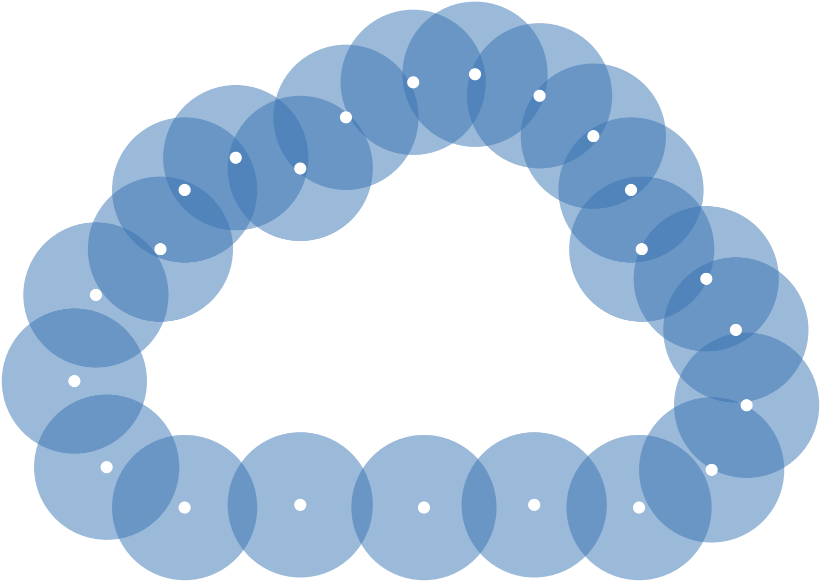
$P = L_D(Q) = F$ — intersection of all front facets
$R = U_D(Q) = X \cap S$

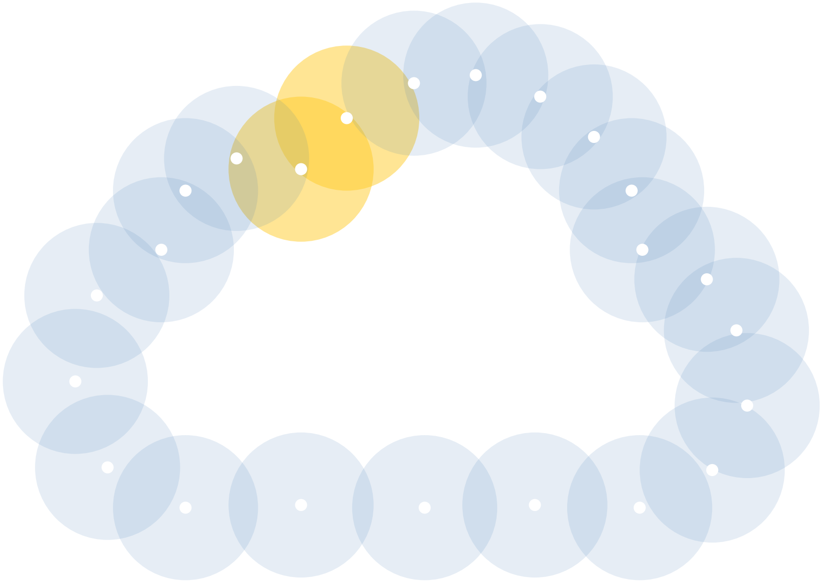
[P, R] consists of all Q′∈D(X) with $P = L_D(Q')$.

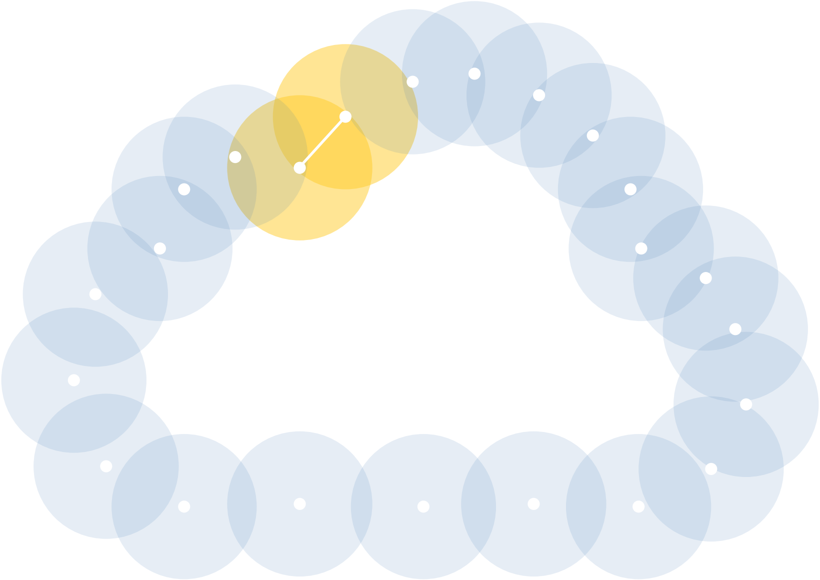


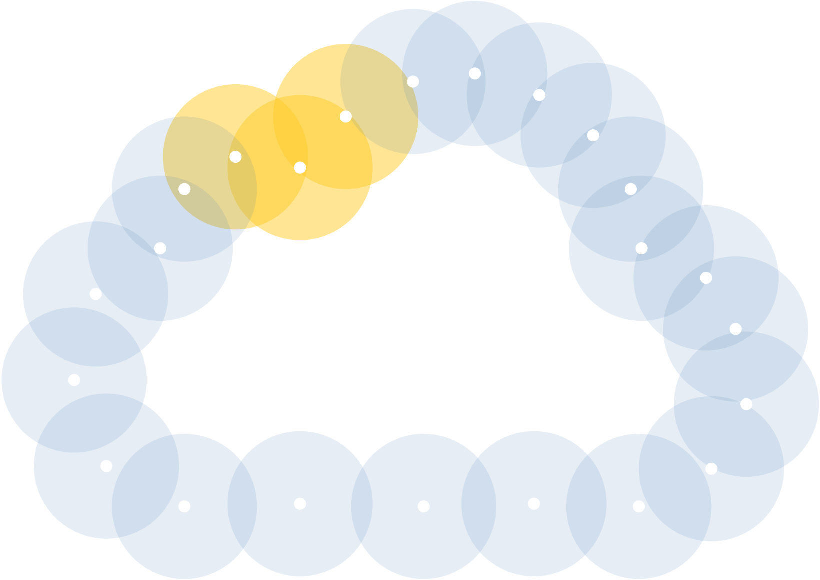
LEMMA. f_D is g.d.M.f. with gradient $V_D = \{[P, R]\}$.

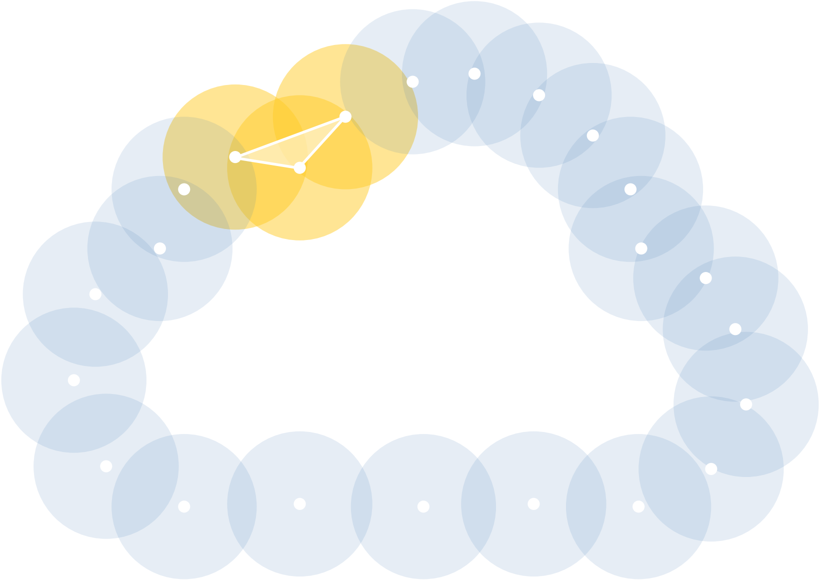


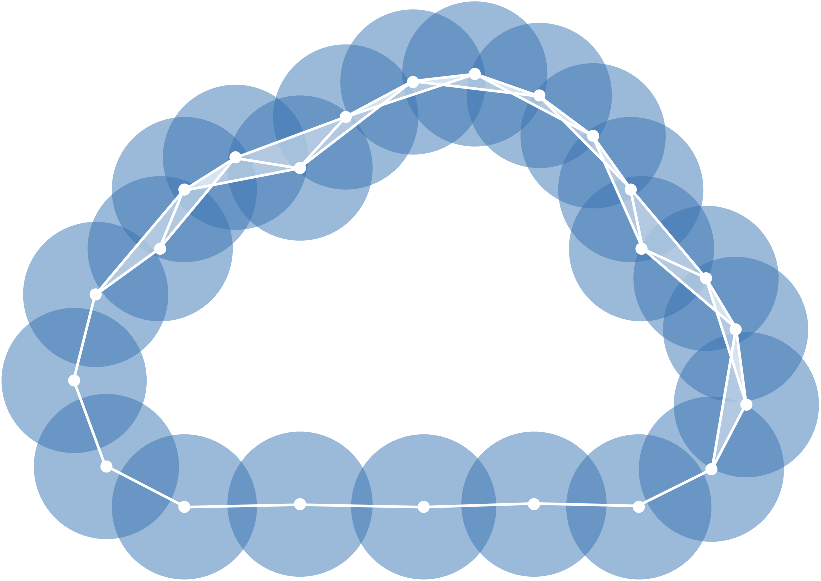


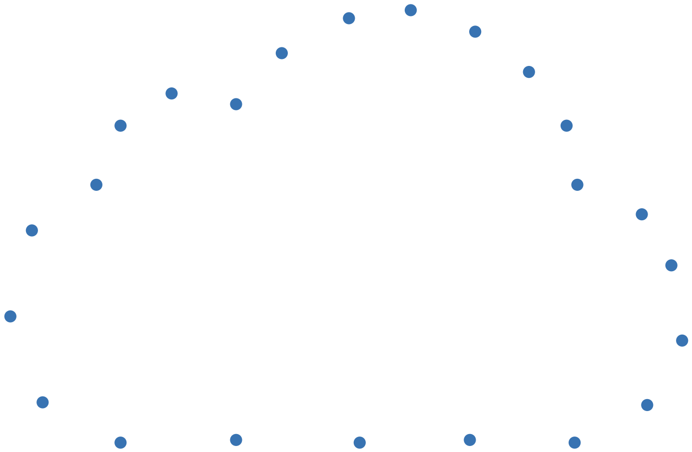


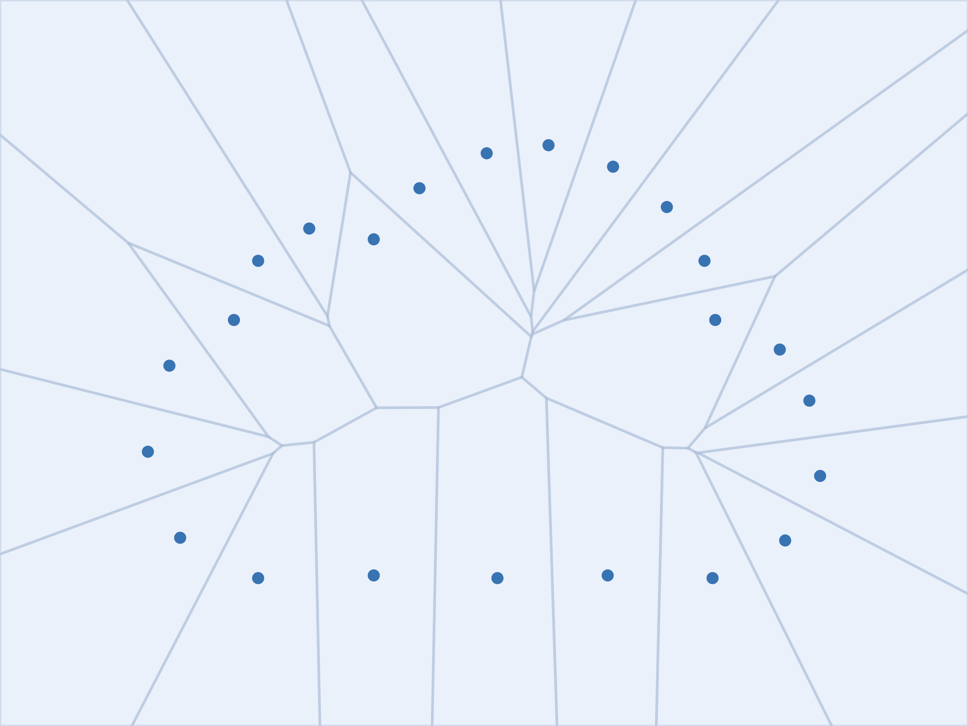


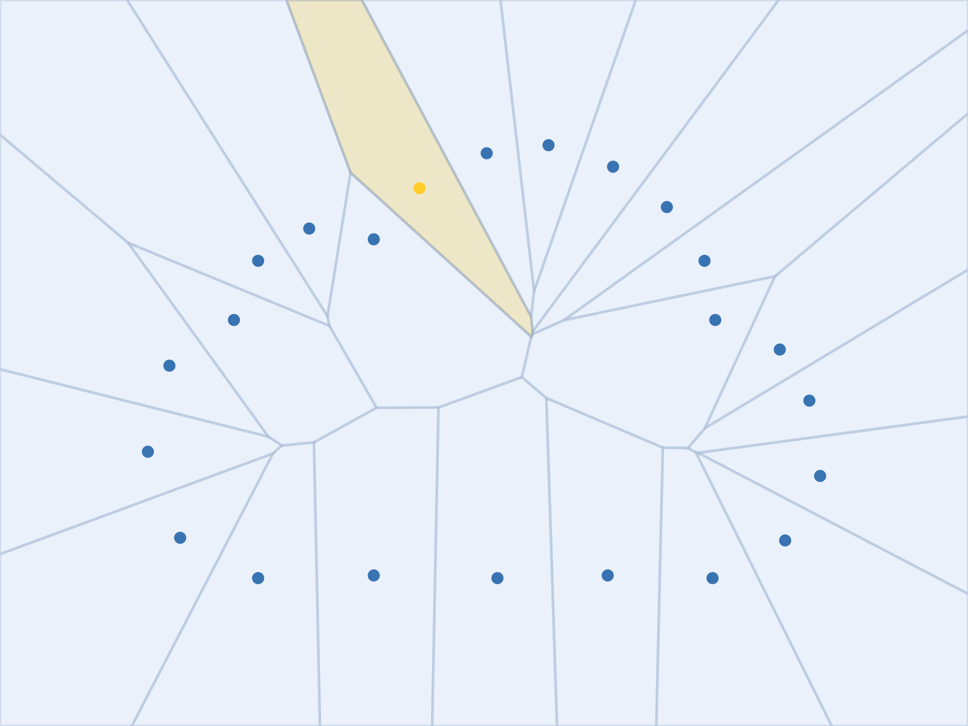


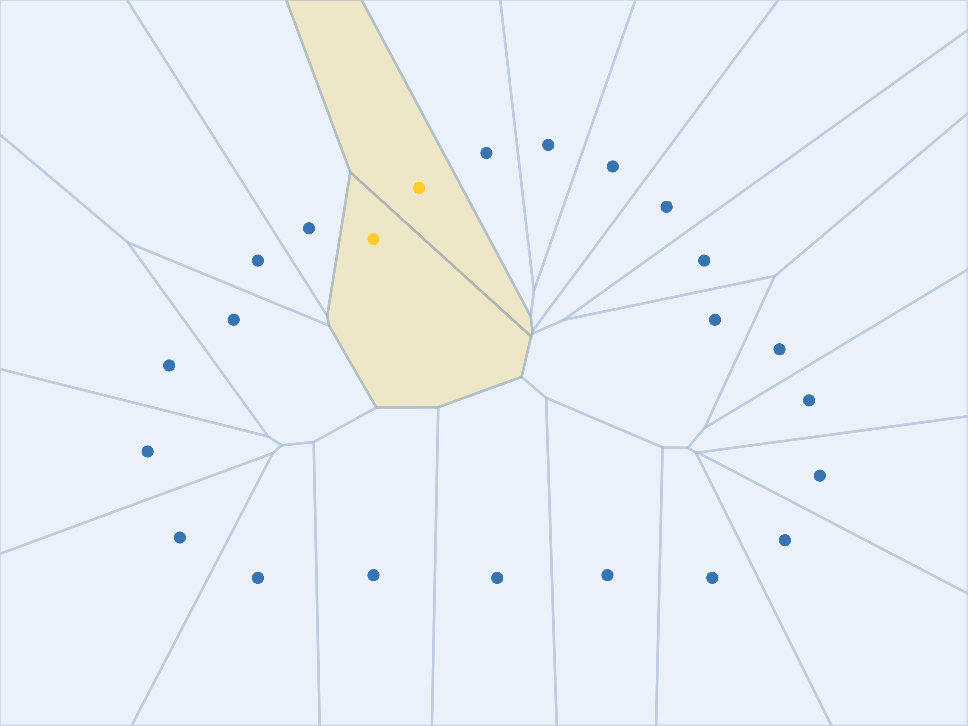


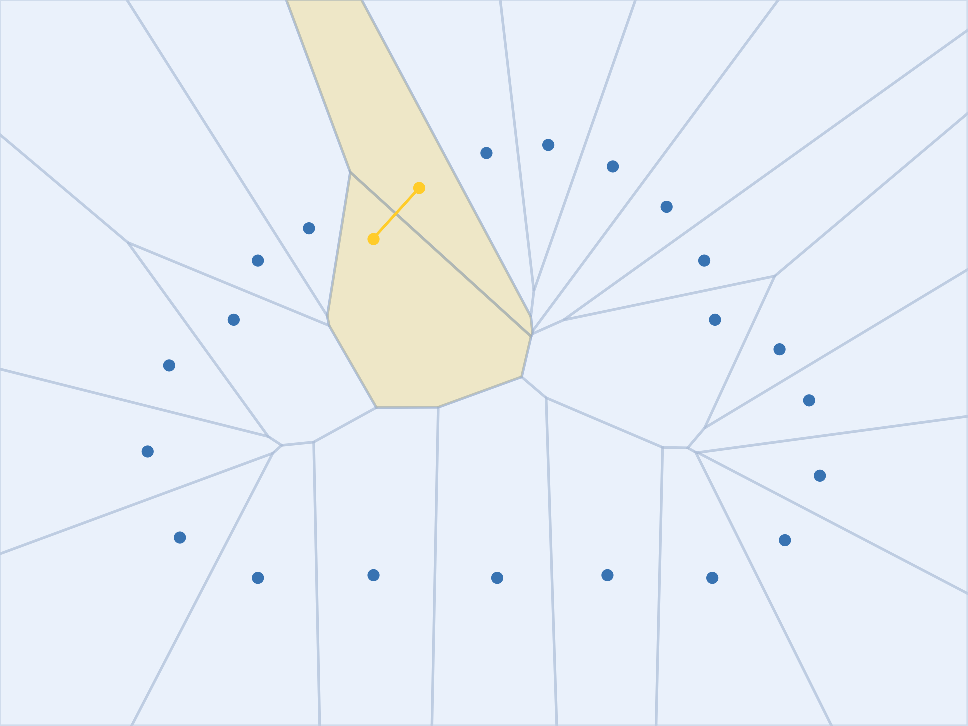


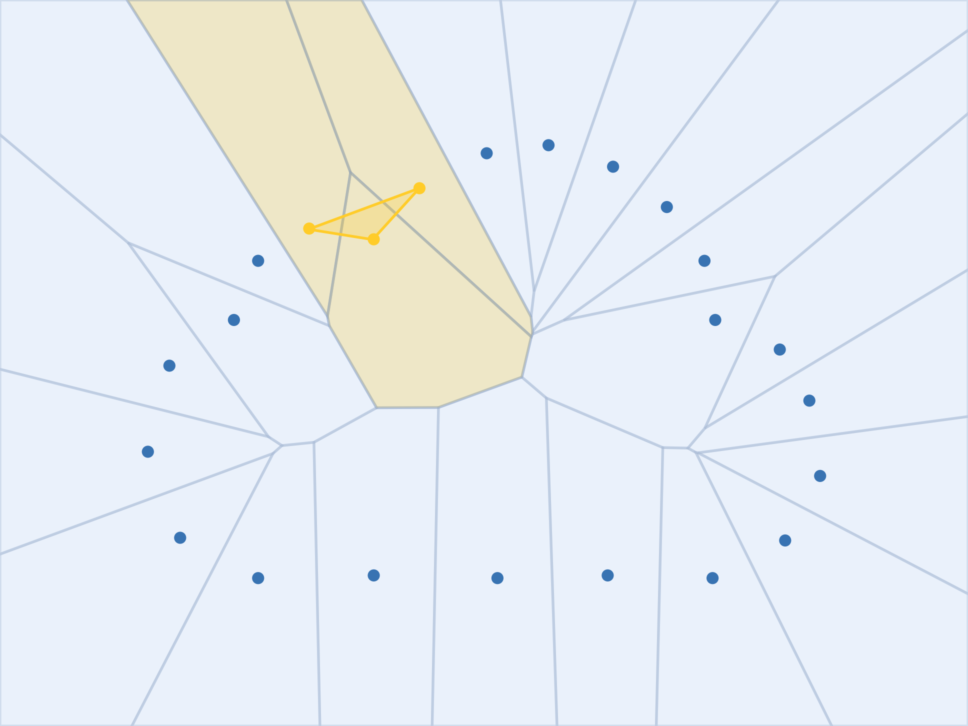


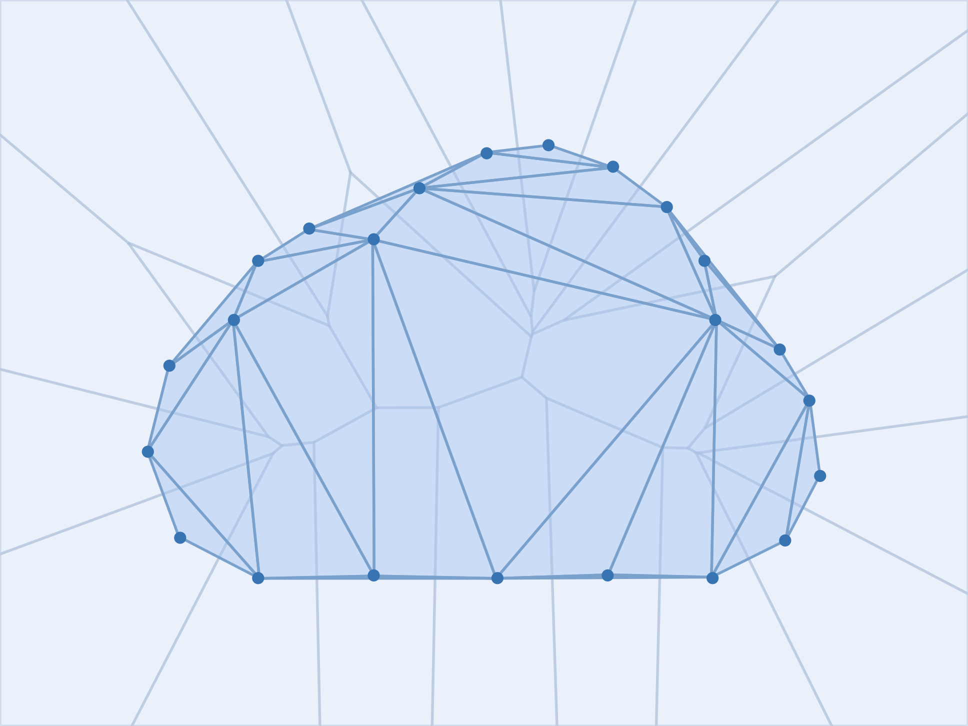


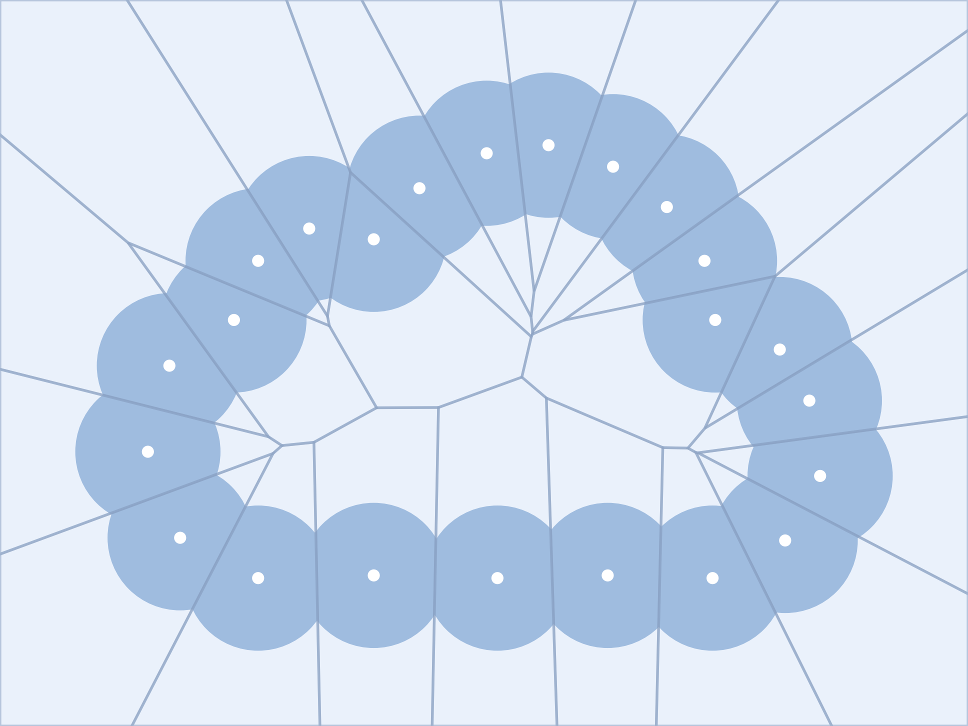


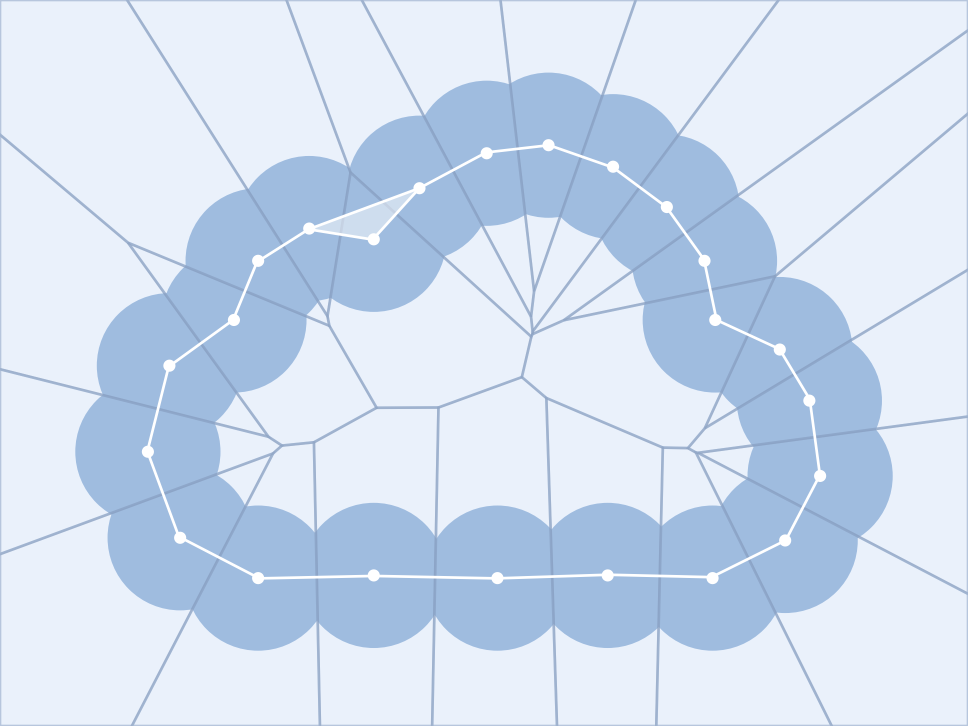


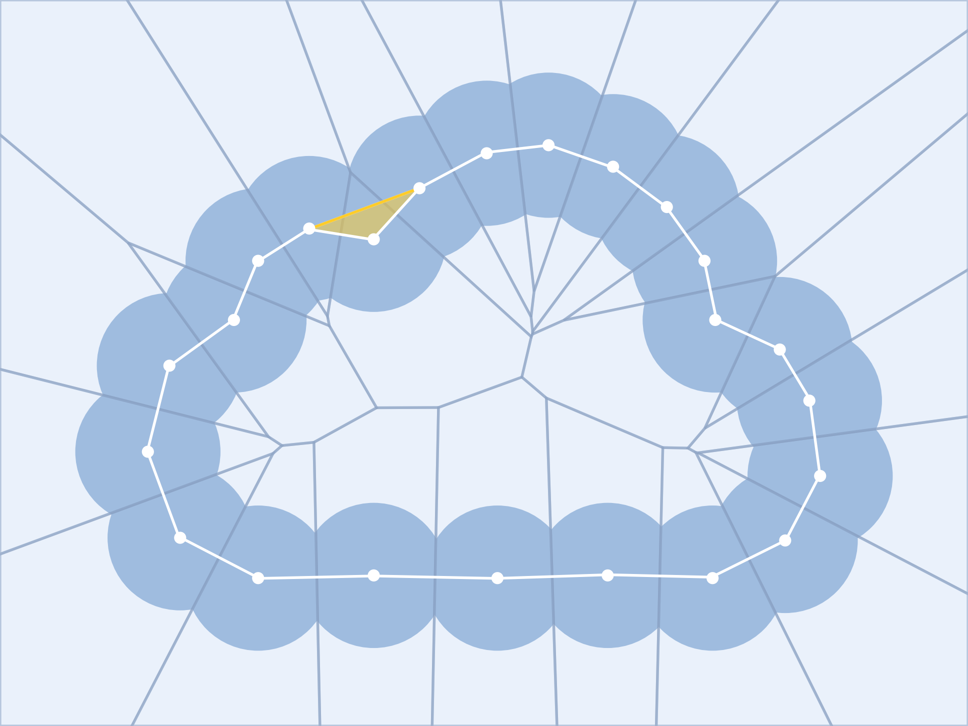


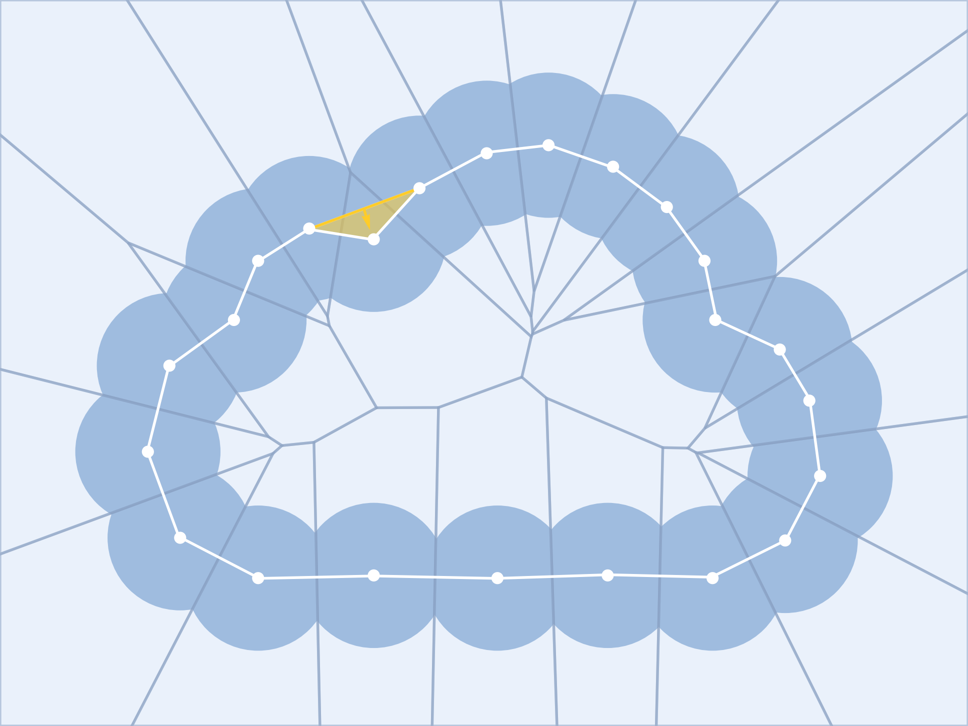


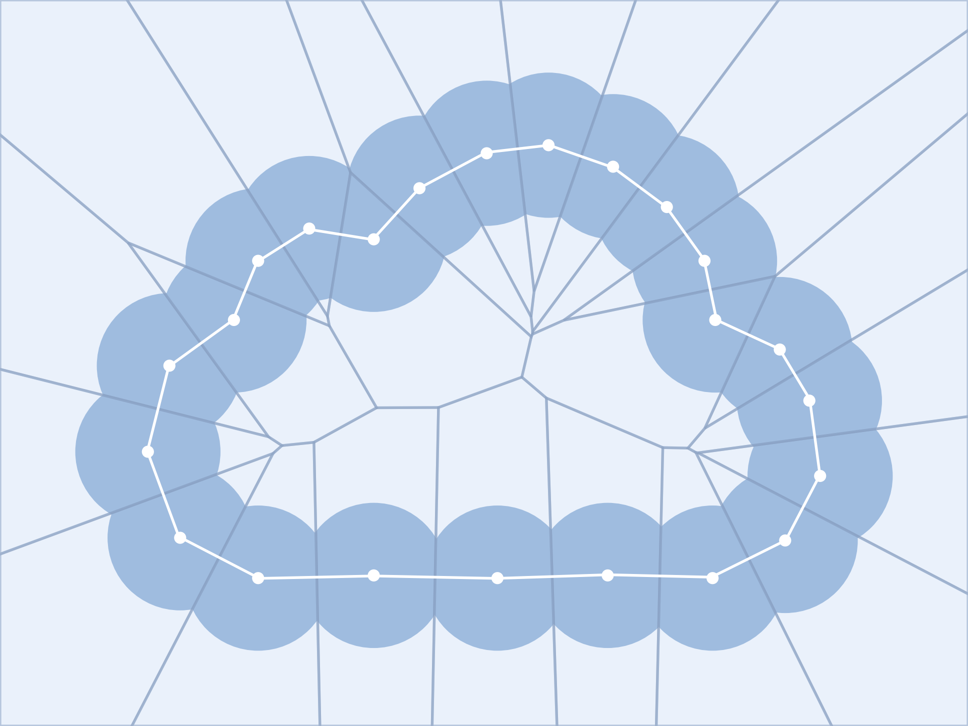


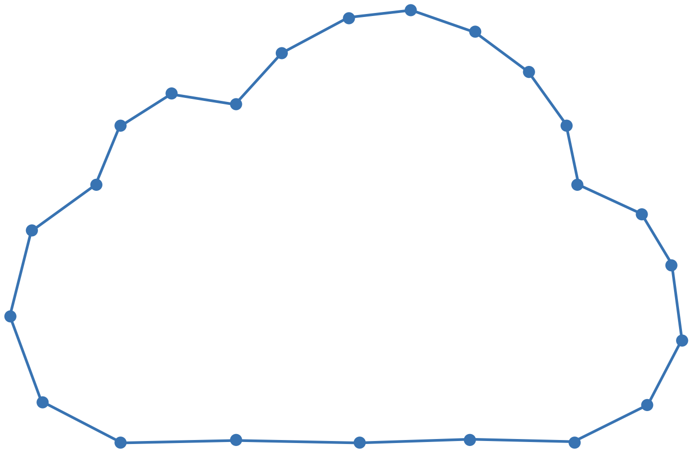












Connecting Čech and Delaunay complexes

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$$\mathrm{Del}_r(X) \simeq \mathrm{Cech}_r(X) \simeq B_r(X).$$

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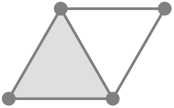
- Are all three complexes homotopy equivalent?
- Are they related by a sequence of *simplicial collapses*?

Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

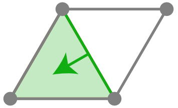


Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

- A *free face* of K is a simplex that has a unique proper coface.



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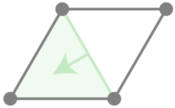

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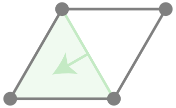

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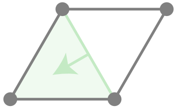

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  In particular, they have isomorphic homology.

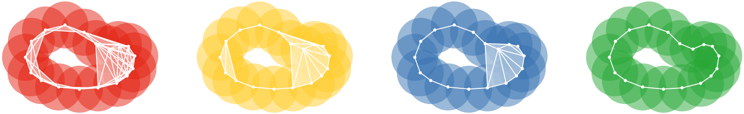
If there is a sequence of such elementary collapses from K to M, we say that K *collapses* to M (written as $K \searrow M$).

Main result: a sequence of collapses

Theorem (B, Edelsbrunner 2017)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent through a sequence of collapses

$$\operatorname{Cech}_r X \searrow \operatorname{Cech}_r X \cap \operatorname{Del} X \searrow \operatorname{Del}_r X \searrow \operatorname{Wrap}_r X.$$



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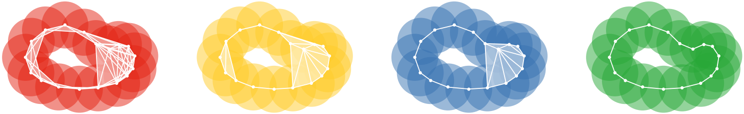


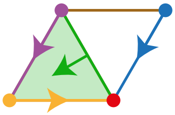
- Extends to weighted Delaunay triagulations
- All collapses are induced by a single *discrete gradient field*
- This yields explicit chain maps, inducing isomorphims in persistent homology

Discrete Morse theory

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a simplicial complex
is a partition of the simplices
into singletons and pairs $\{L, U\}$,
where L is a face of U with codimension 1.



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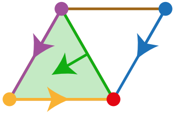

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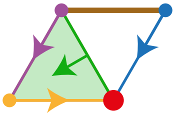


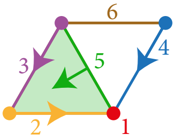
The singletons are called *critical simplices.*

Discrete Morse theory

Definition (Forman 1998)

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *discrete Morse function* if for all $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes



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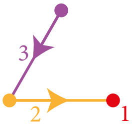

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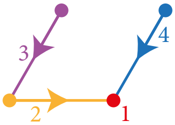

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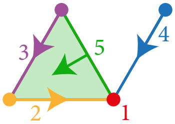

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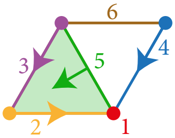

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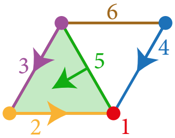

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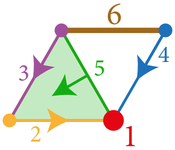

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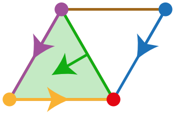

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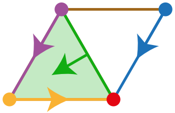

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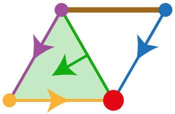

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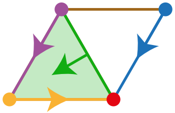

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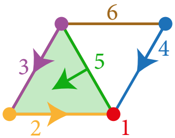

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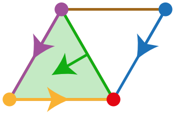

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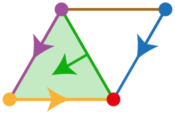

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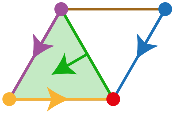

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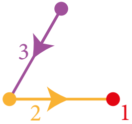

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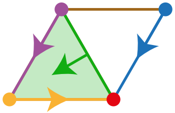

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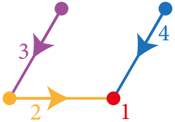

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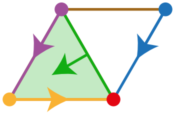

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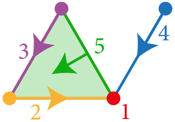

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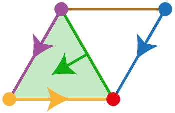

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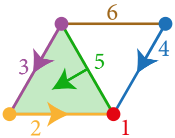

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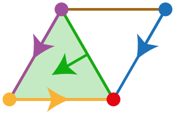

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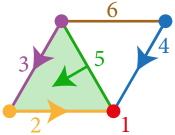

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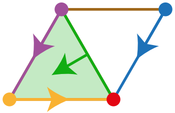

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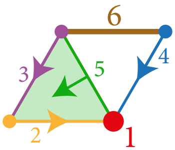

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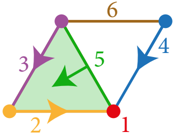

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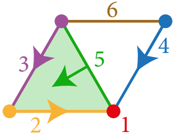

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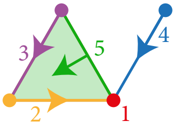

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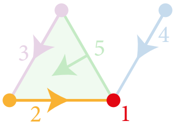

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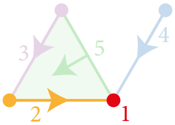

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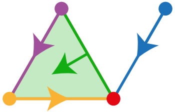

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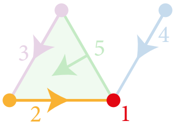

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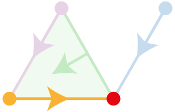

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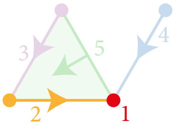

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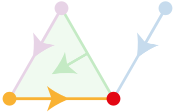

We say that V *induces the collapse* $K \searrow L$.

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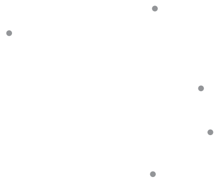
$X \subset \mathbb{R}^d$: finite point set (in general position)



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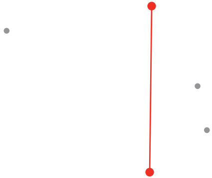

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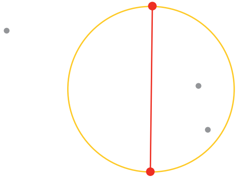

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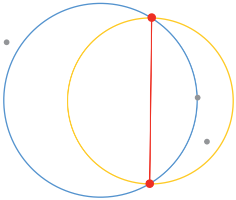
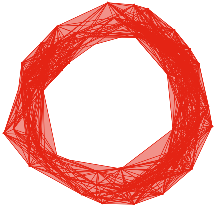


- *Čech radius function* $f_{\mathrm{Cech}}(Q)$:
  radius of *smallest enclosing sphere* of Q
- *Delaunay radius function* $f_{\mathrm{Del}}(Q)$:
  radius of *smallest empty circumsphere*
  - defined only on $\mathrm{Del}(X)$

Čech and Delaunay complexes from functions

For any radius r:

- $\mathrm{Cech}_r(X) = f_{\mathrm{Cech}}{}^{-1}(-\infty, r]$
  - simplices with smallest enclosing radius $\leq r$



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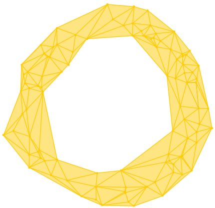

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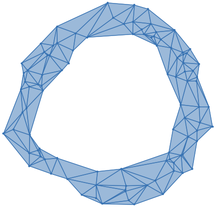

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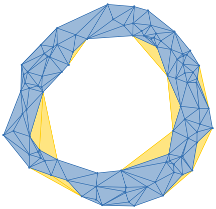

Unfortunately…

Neither the Čech nor the Delaunay functions are discrete Morse functions!

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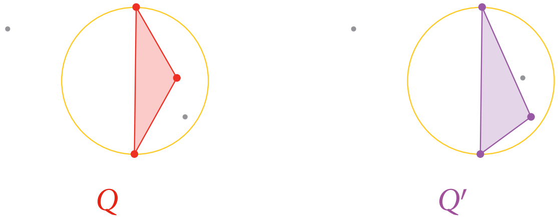
- Example: two simplices Q, Q' with $f_C(Q) = f_C(Q')$ where neither is a face of the other:

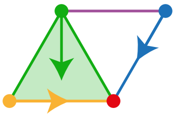


Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex K is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}$$



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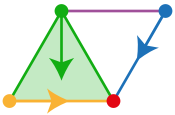

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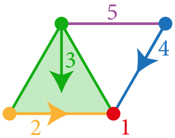

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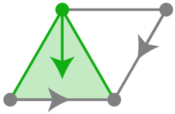
A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:



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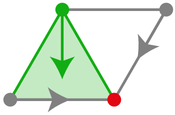

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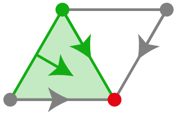

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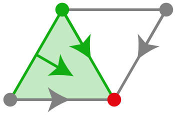

Therefore the collapsing theorems also hold for generalized discrete Morse functions.

Morse theory of Čech and Delaunay complexes

Proposition

The Čech function and the Delaunay function are generalized discrete Morse functions.

Morse theory of Čech and Delaunay complexes

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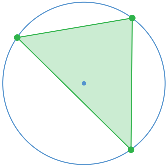
The Čech function and the Delaunay function are generalized discrete Morse functions.

The following are equivalent:

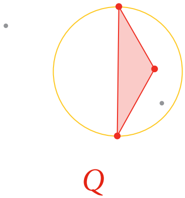
- *Q is a critical simplex of f_{Cech}*
- *Q is a critical simplex of f_{Del}*
- $f_{\mathrm{Cech}}(Q) = f_{\mathrm{Del}}(Q)$
- *Q is a* centered *Delaunay simplex (containing the circumcenter in the interior)*



Čech intervals

Lemma

Let $Q \subseteq X$ be a simplex with smallest enclosing sphere S.
Then $Q' \subseteq X$ has the same smallest enclosing sphere S iff



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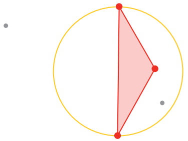
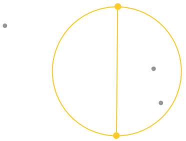


Q



$\mathrm{On}\, S$

Čech intervals

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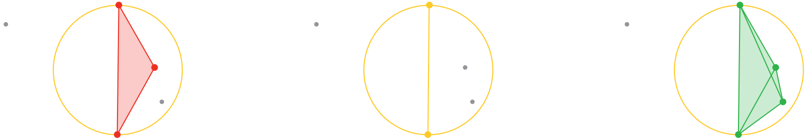

Q $\qquad$ $\operatorname{On} S$ $\qquad$ $\operatorname{Encl} S$

Čech intervals

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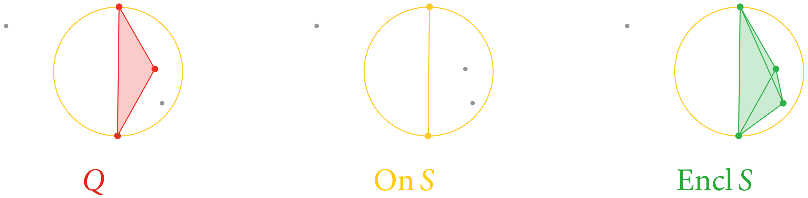

The front and back faces of a simplex

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We define

$$\mathrm{Front}\, S = \{x \in \mathrm{On}\, S \mid \mu_x > 0\},$$

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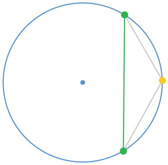

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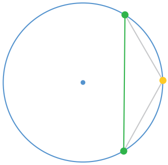
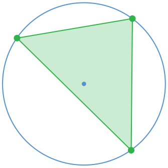
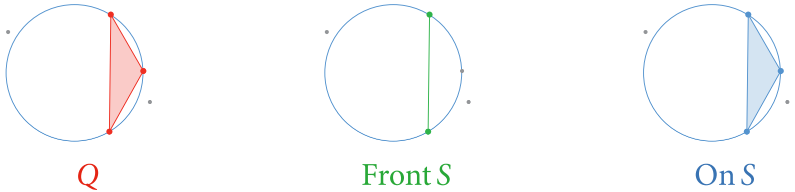
 

Delaunay intervals

Lemma

Let $Q \subseteq X$ be a simplex with smallest empty circumsphere S. Then $Q' \subseteq X$ has the same smallest empty circumsphere S iff

$$Q' \in [\mathrm{Front}\, S, \mathrm{On}\, S].$$



Selective Delaunay complexes

Sphere optimization problems

Both Čech and Delaunay functions are defined using the smallest sphere S satisfying certain constraints:

$$\begin{aligned} &\underset{r,z}{\text{minimize}} && r && \\ &\text{subject to} && \|z - q\| \le r, && q \in Q, \\ & && \|z - e\| \ge r, && e \in E. \end{aligned}$$

Here r is the radius of the sphere S, and z is the center of S.

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Sphere optimization problems

Both Čech and Delaunay functions are defined using the smallest sphere S satisfying certain constraints:

$$
\begin{aligned}
\underset{r,z}{\text{minimize}} \quad & r^2 \\
\text{subject to} \quad & \|z - q\|^2 \le r^2, \qquad q \in Q, \\
& \|z - e\|^2 \ge r^2, \qquad e \in E.
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- Čech function: choose $E = \varnothing$
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The Karush–Kuhn–Tucker optimality conditions

Consider an optimization problem of the form

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f(x) & \\ \text{subject to} & g_j(x) \geq 0, & \forall j \in J, \\ & g_k(x) = 0, & \forall k \in K, \\ & g_l(x) \leq 0, & \forall l \in L. \end{array}$$

where the function f is convex and g_i are affine ($i \in I = J \cup K \cup L$).

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where the function f is convex and g_i are affine ($i \in I = J \cup K \cup L$).

Theorem (Karush 1939, Kuhn–Tucker 1951)

A feasible x is optimal iff there exist $(\lambda_i)_{i \in I}$ such that

$$
\begin{aligned}
\nabla f(x) + \sum_{i \in I} \lambda_i \nabla g_i(x) &= 0, && && \text{(stationarity)} \\
\lambda_i g_i(x) &= 0, && \forall i \in I, && \text{(complementary slackness)} \\
\lambda_j &\leq 0, && \forall j \in J, && \\
\lambda_l &\geq 0, && \forall l \in L. && \text{(dual feasibility)}
\end{aligned}
$$

KKT conditions for the smallest sphere problem

The KKT conditions for our sphere optimization problem are:

Proposition

A sphere S enclosing Q and excluding E is minimal iff its center z can be written as an affine combination

$$z = \sum_{x \in Q \cup E} \lambda_x x, \qquad 1 = \sum_{x \in Q \cup E} \lambda_x$$

such that

- $\lambda_x = 0$ *whenever* x *does not lie on* S,
- $\lambda_x \leq 0$ *whenever* $x \in E \setminus Q$, *and*
- $\lambda_x \geq 0$ *whenever* $x \in Q \setminus E$.

Čech and Delaunay intervals from KKT

The *Karush–Kuhn–Tucker* optimality conditions yield:

Proposition (Geometric KKT conditions)

A sphere S enclosing Q and excluding E is minimal iff

- *S is the smallest circumsphere of $\mathrm{On}\, S$,*
- $Q \in [\mathrm{Front}\, S, \mathrm{Encl}\, S]$*, and*
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Corollary

The Čech intervals are of the form $[\operatorname{On} S, \operatorname{Encl} S]$.

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The Delaunay intervals are of the form $[\mathrm{Front}\, S, \mathrm{On}\, S]$.

Selective Delaunay complexes

Define for finite point sets $X, E \subset \mathbb{R}^d$:

- *E-Delaunay function* $f_E(Q)$:
  radius of smallest sphere enclosing Q and excluding E
  - defined only if such a sphere exists: $Q \in \mathrm{Del}(X, E)$

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Theorem (B, Edelsbrunner 2017)

Let $E \subseteq F \subseteq X$. Then

$$\mathrm{Del}_r(X, E) \searrow \mathrm{Del}_r(X, E) \cap \mathrm{Del}(X, F) \searrow \mathrm{Del}_r(X, F).$$

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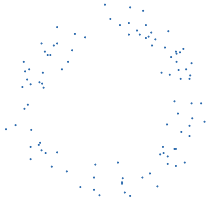
Note: choosing $E = \varnothing$ and $F = X$ yields

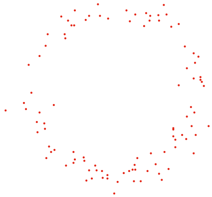
$$\mathrm{Cech}_r(X) \searrow \mathrm{DelCech}_r(X) \searrow \mathrm{Del}_r(X).$$

Connecting different Delaunay complexes

X

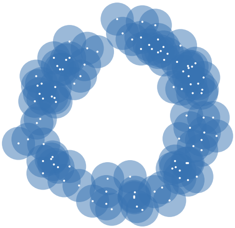


Y

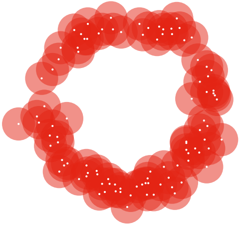


Connecting different Delaunay complexes

$B_r(X)$

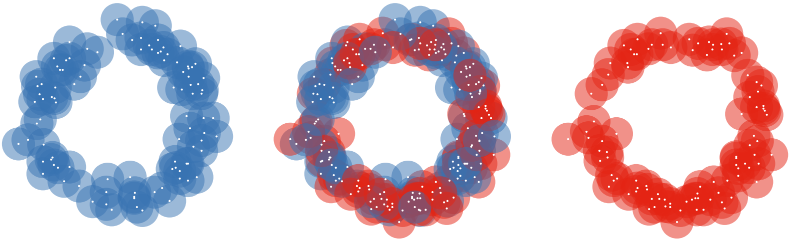


$B_r(Y)$



Connecting different Delaunay complexes

$$B_r(X) \hookrightarrow B_r(X \cup Y) \hookleftarrow B_r(Y)$$



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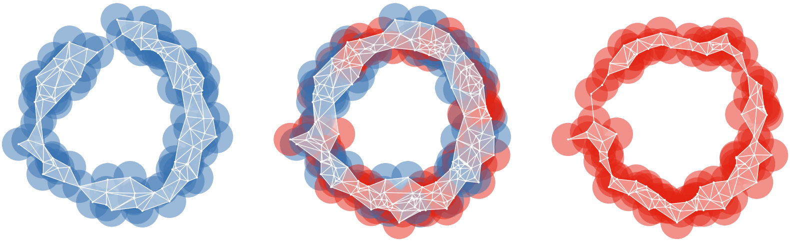
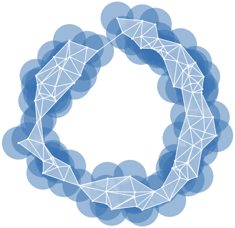
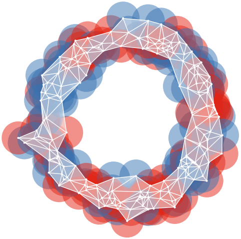
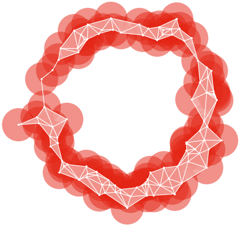


$\mathrm{Del}_r(X)$ $\mathrm{Del}_r(X \cup Y)$ $\mathrm{Del}_r(Y)$

Connecting different Delaunay complexes

$$\begin{tikzcd} B_r(X) \arrow[r, hook] & B_r(X \cup Y) & B_r(Y) \arrow[l, hook'] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] \mathrm{Del}_r(X) \arrow[dr, hook] & & \mathrm{Del}_r(X \cup Y) \arrow[dl, hook'] \arrow[dr, hook] & & \mathrm{Del}_r(Y) \arrow[dl, hook'] \\ & \mathrm{Del}_r(X \cup Y, X) & & \mathrm{Del}_r(X \cup Y, Y) & \end{tikzcd}$$

Connecting different Delaunay complexes

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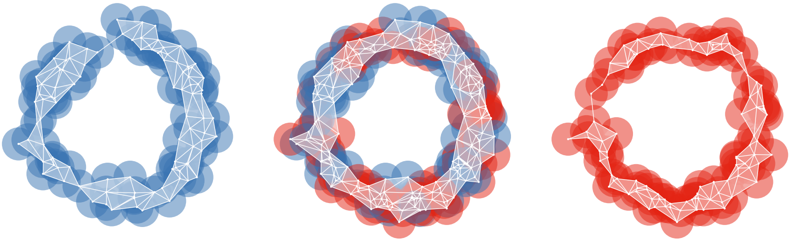


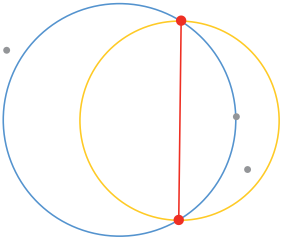
$$\begin{tikzcd}[column sep=small]
\mathrm{Del}_r(X) \arrow[dr, hook] & & \mathrm{Del}_r(X \cup Y) \arrow[dl, hook, "\simeq"'] \arrow[dr, hook, "\simeq"] & & \mathrm{Del}_r(Y) \arrow[dl, hook] \\
& \mathrm{Del}_r(X \cup Y, X) & & \mathrm{Del}_r(X \cup Y, Y) &
\end{tikzcd}$$

Collapsing from Čech to Delaunay

Collapsing the Delaunay–Čech complex

Construct gradient pairs inducing the collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$:

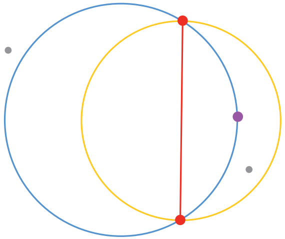
- Consider a non-critical Delaunay simplex Q



Collapsing the Delaunay–Čech complex

Construct gradient pairs inducing the collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$:

- Consider a non-critical Delaunay simplex Q
- There must be a point p *inside* the Čech and *on* the Delaunay sphere



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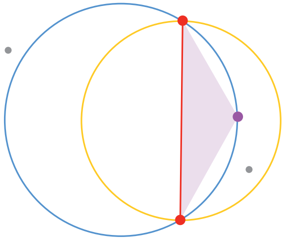

- Consider a non-critical Delaunay simplex Q
- There must be a point p *inside* the Čech and *on* the Delaunay sphere
- Now by the KKT conditions:
  $Q' = Q \setminus \{p\}$ and $Q'' = Q \cup \{p\}$
  have the same Čech and Delaunay sphere

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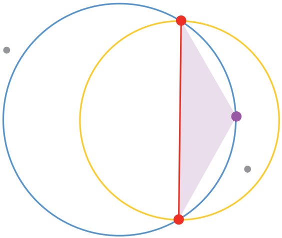

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- Now by the KKT conditions: $Q' = Q \setminus \{p\}$ and $Q'' = Q \cup \{p\}$ have the same Čech and Delaunay sphere

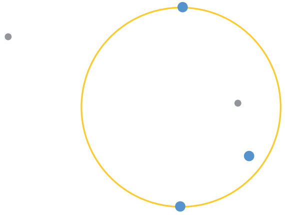
Lemma

The pairs (Q', Q'') *yield a discrete gradient, inducing a collapse* $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$ *(for any radius r).*

Collapsing non-Delaunay simplices

Construct gradient pairs for the collapse $\mathrm{Cech}_r \searrow \mathrm{DelCech}_r$:

- Consider a non-Delaunay simplex Q together with its Čech sphere S



Collapsing non-Delaunay simplices

Construct gradient pairs for the collapse $\mathrm{Cech}_r \searrow \mathrm{DelCech}_r$:

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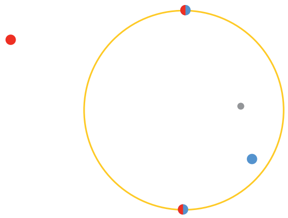
- By the KKT conditions, S is the smallest sphere enclosing Q and excluding $\mathrm{Excl}\, S$



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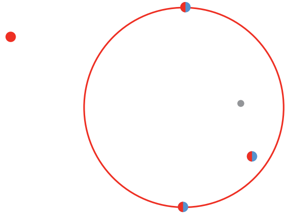
- Exclude more and more points until no feasible sphere exists



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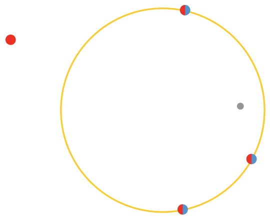

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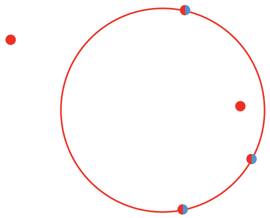

Collapsing non-Delaunay simplices

Construct gradient pairs for the collapse $\mathrm{Cech}_r \searrow \mathrm{DelCech}_r$:

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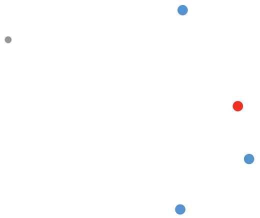

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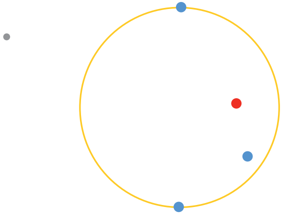

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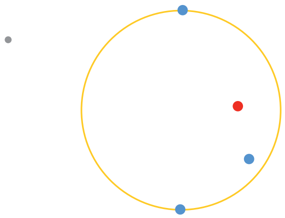

Lemma

The pairs (Q', Q'') yield a discrete gradient, inducing a collapse $\mathrm{Cech}_r \searrow \mathrm{DelCech}_r$ (for any radius r).

Wrap complexes

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Generalize and simplify the surface reconstruction method

Wrap (Edelsbrunner 1995, Geomagic)

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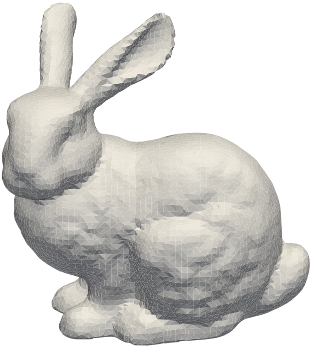

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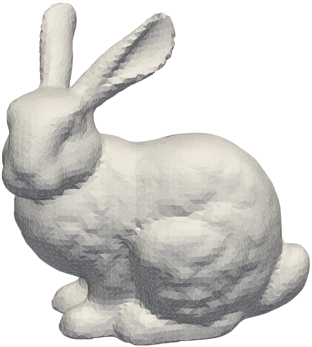
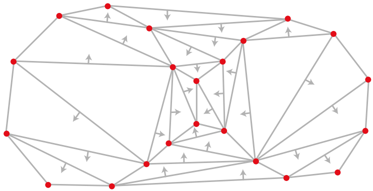
Generalize and simplify the surface reconstruction method *Wrap* (Edelsbrunner 1995, Geomagic)



"It would be interesting to elucidate the connection between the two approaches to a discrete Morse theory." Herbert Edelsbrunner, 2002

Wrap complexes

Consider the Delaunay gradient V_D of $X \subset \mathbb{R}^D$ (intervals $[\mathrm{Front}\, S, \mathrm{On}\, S]$ of simplices with same Delaunay sphere S).



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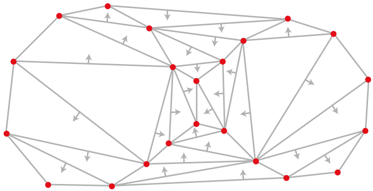

Define $\mathrm{Wrap}_r(X)$ as the smallest complex

- consisting of intervals of V_D and
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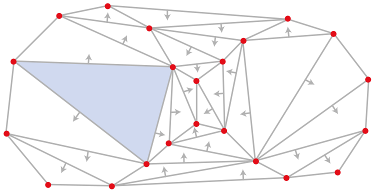

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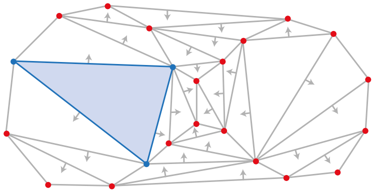

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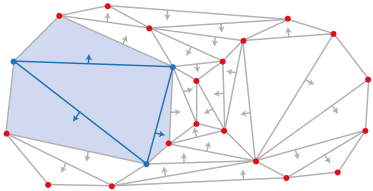

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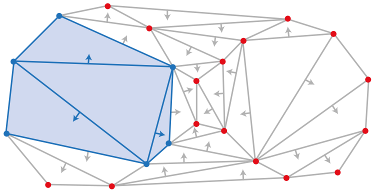

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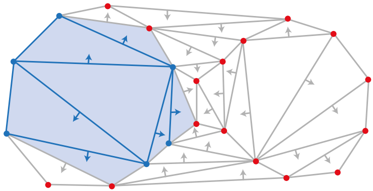

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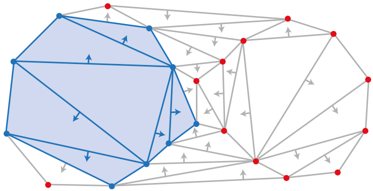

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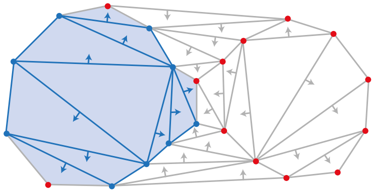

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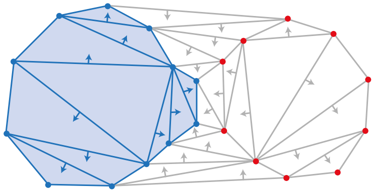

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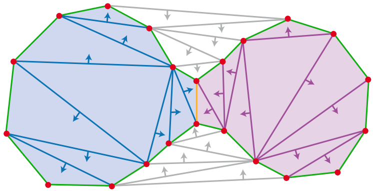

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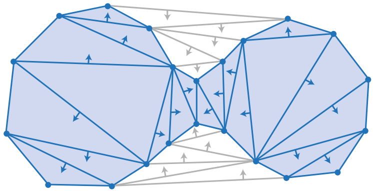

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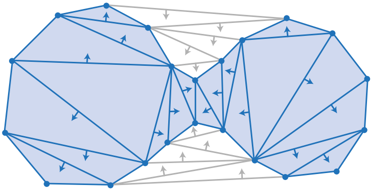

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Corollary

The Delaunay intervals induce a collapse $\mathrm{Del}_r \searrow \mathrm{Wrap}_r$.

Wrapping up

What we learned in this talk

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- Explicit homotopy equivalence by simplicial collapses
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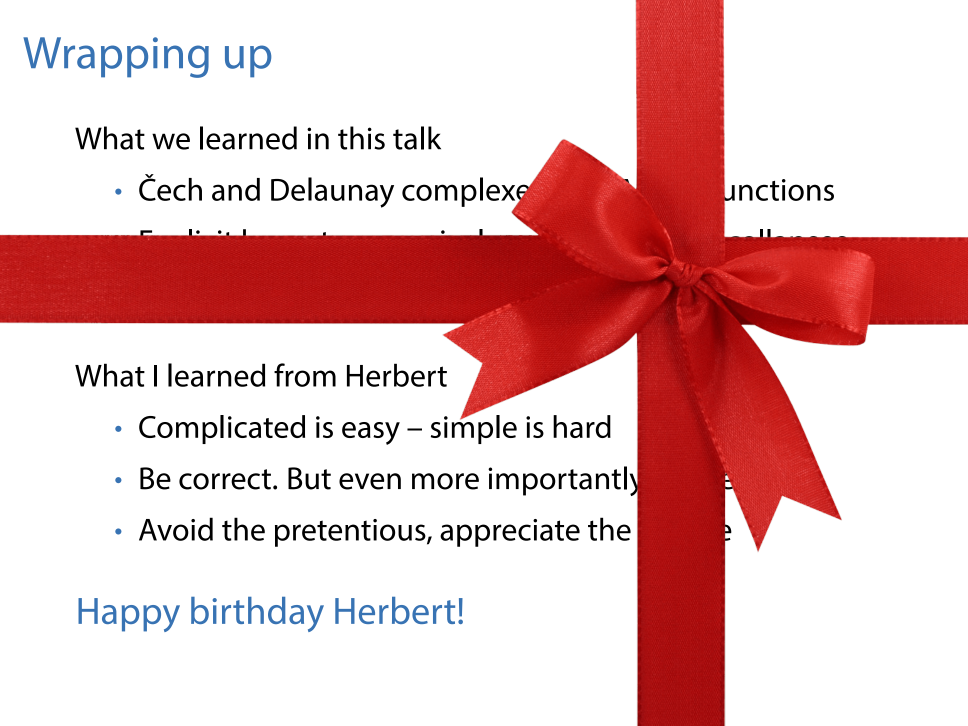

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Happy birthday Herbert!