Delaunay complexes, persistence diagrams, and induced matchings

Ulrich Bauer

TUM

September 17, 2015

Shape Up, TU Berlin

Joint work with
Herbert Edelsbrunner (IST Austria)
and
Michael Lesnick (Columbia, IMA)

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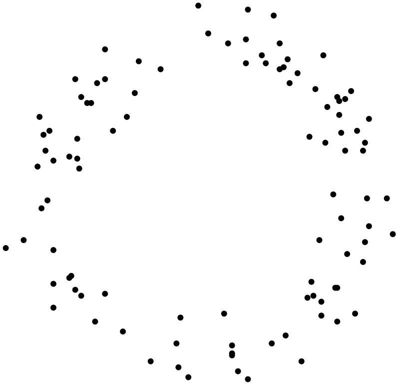

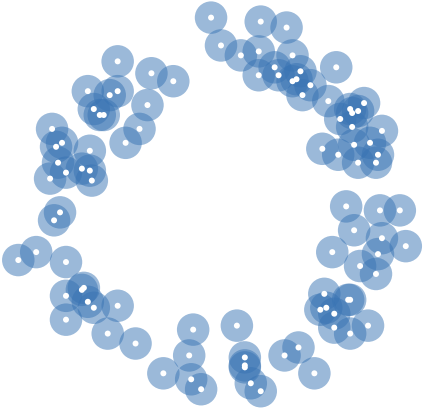

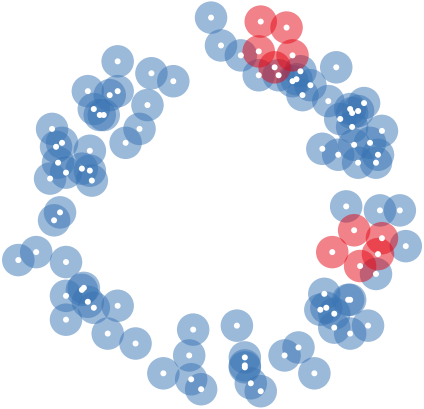

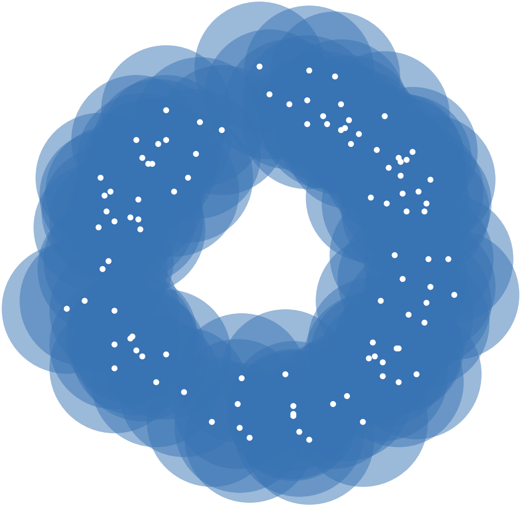

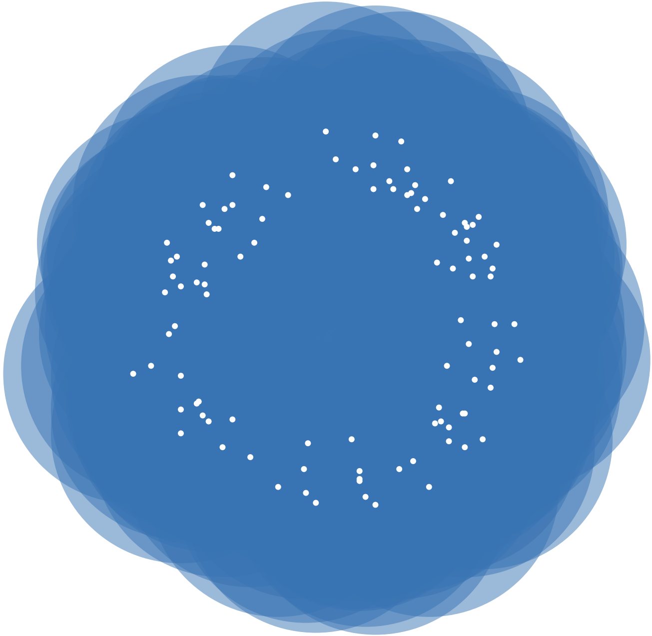

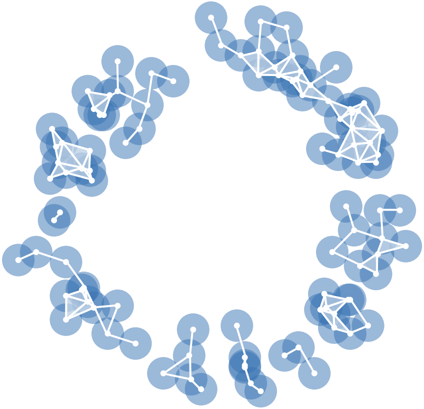

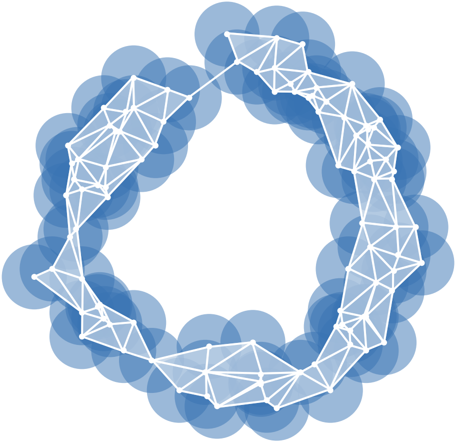

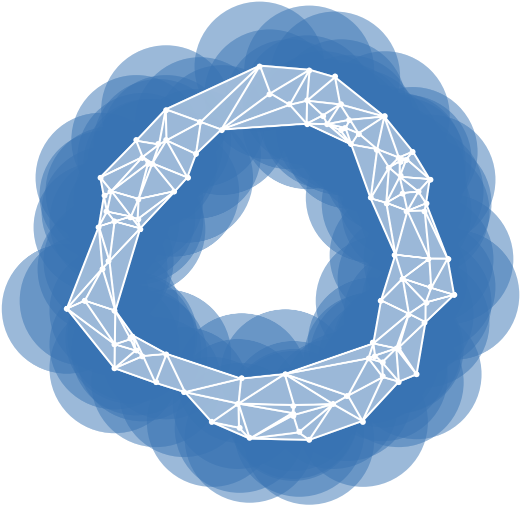

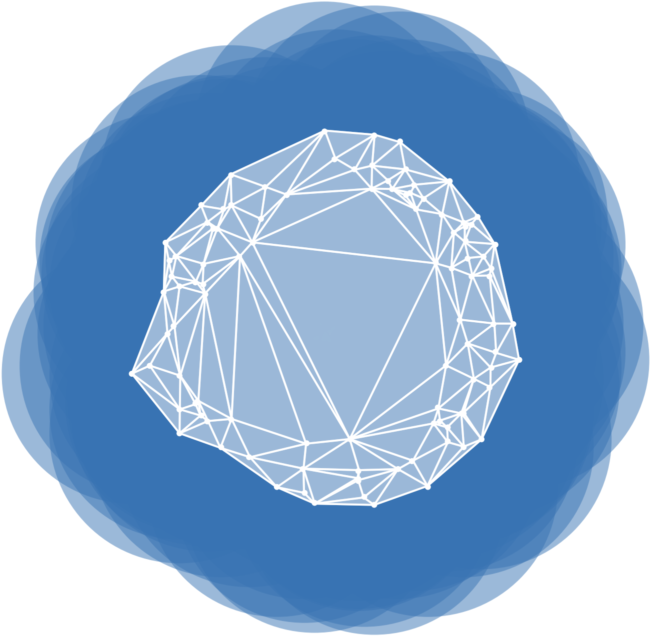

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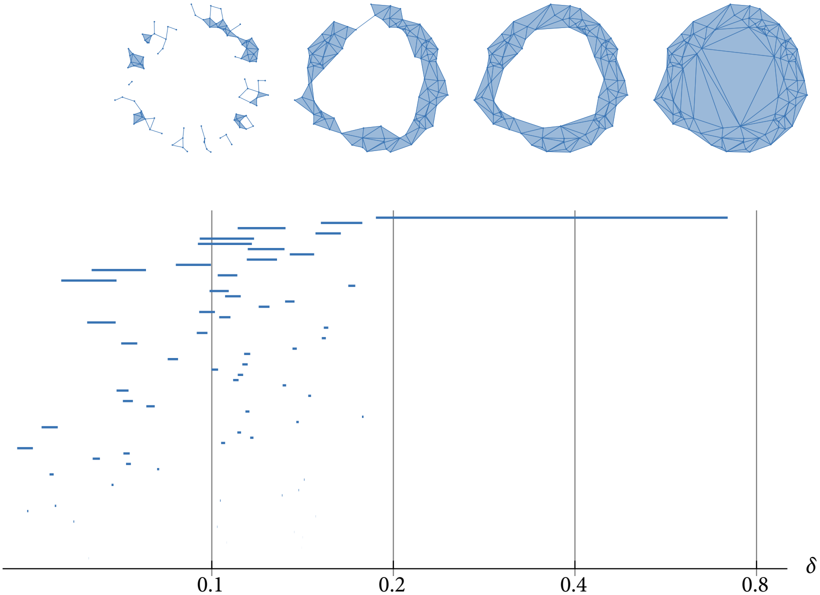

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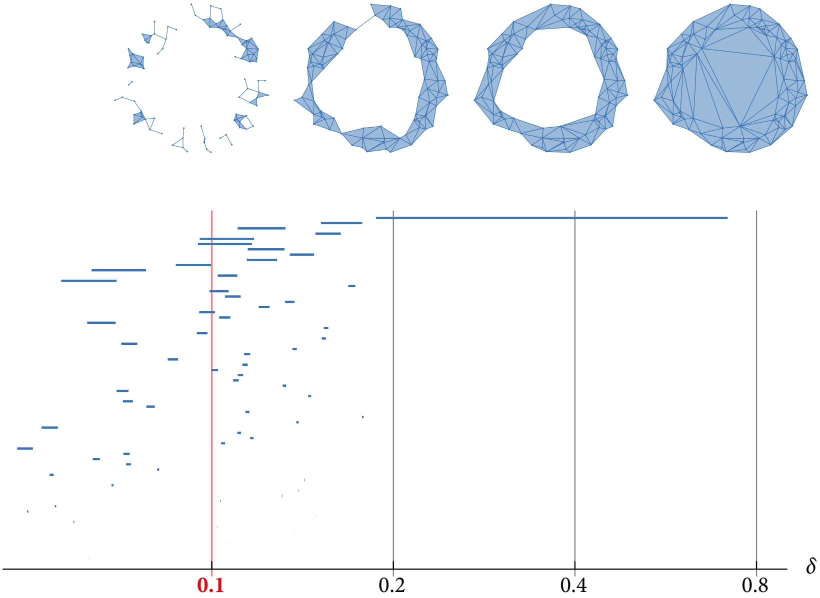

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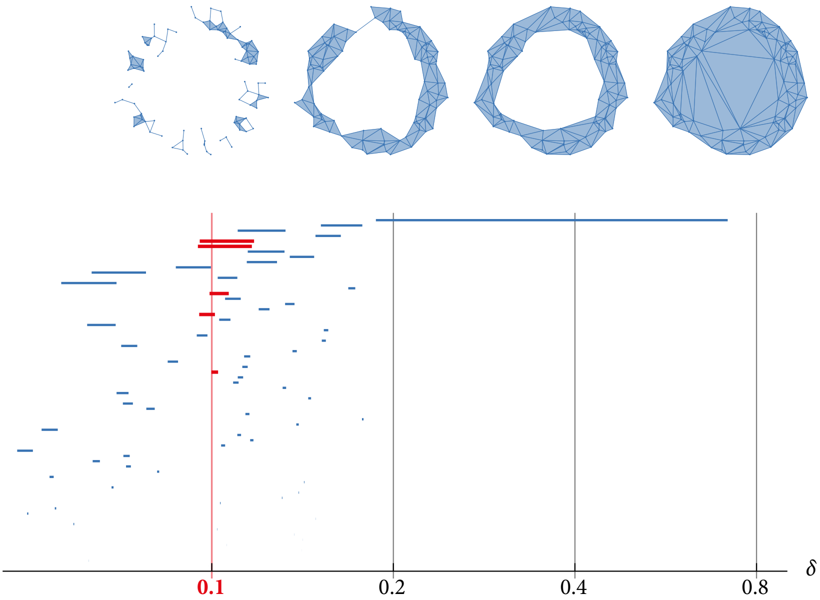

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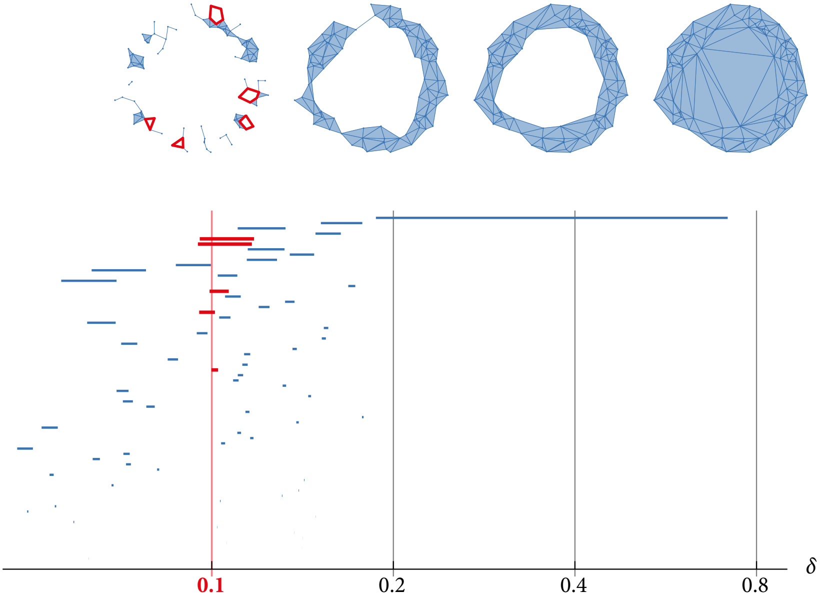

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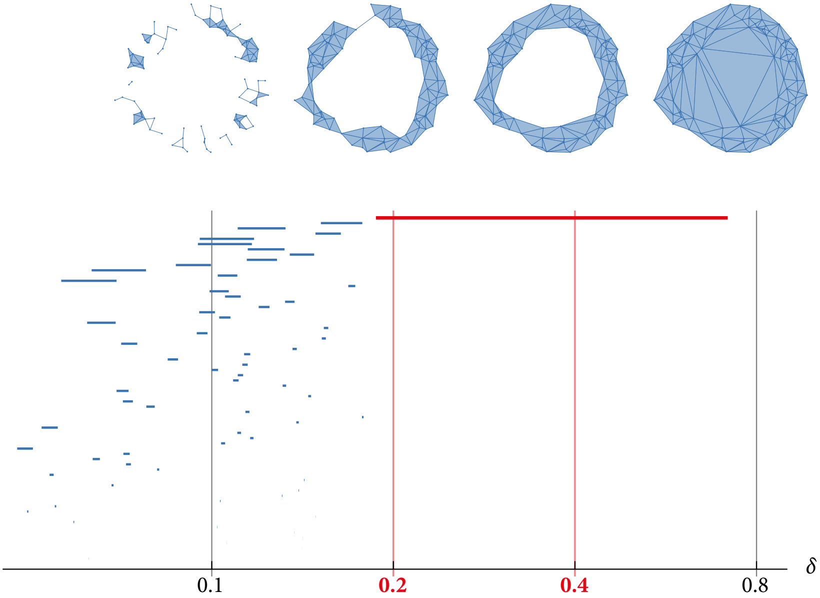

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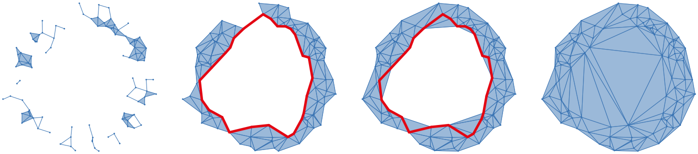

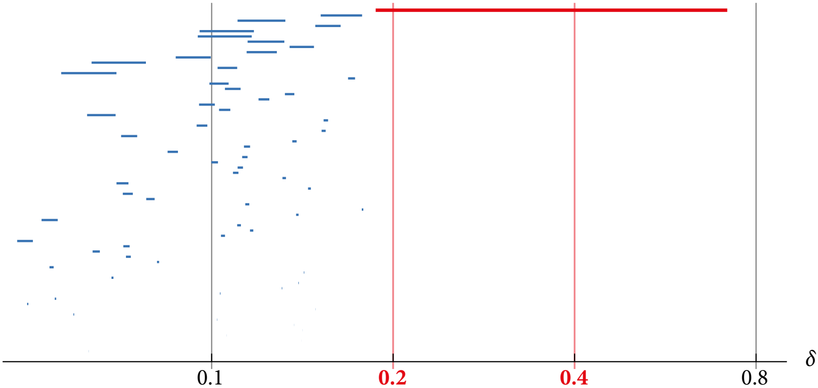

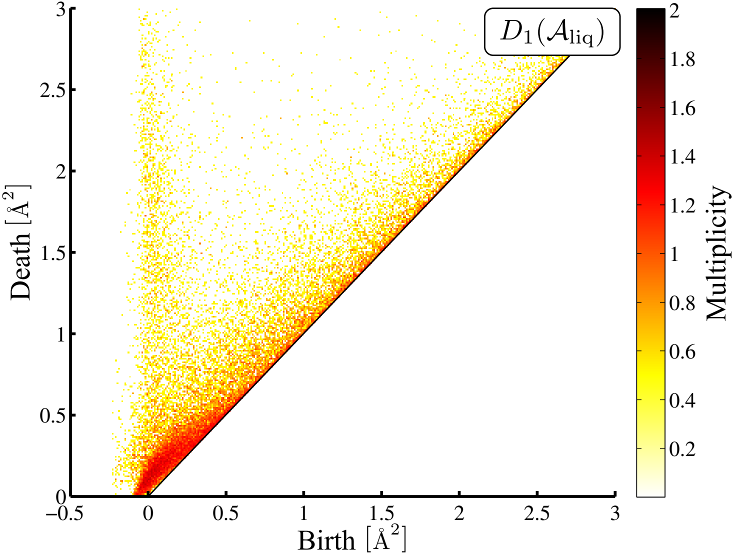

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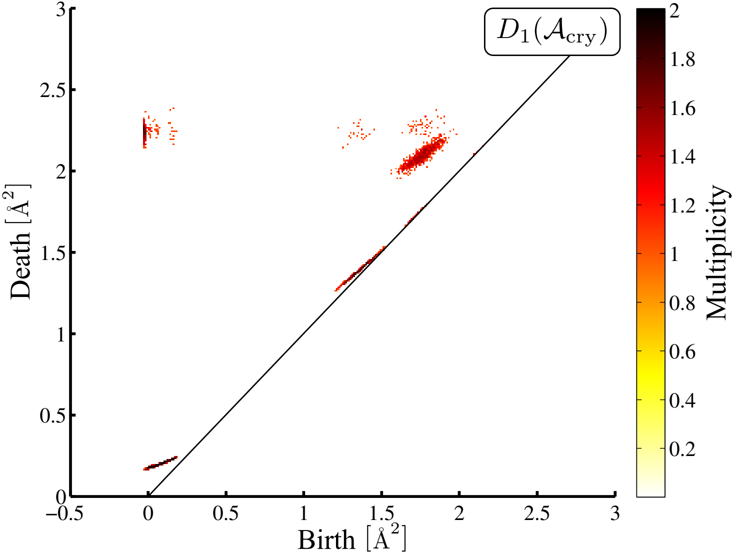

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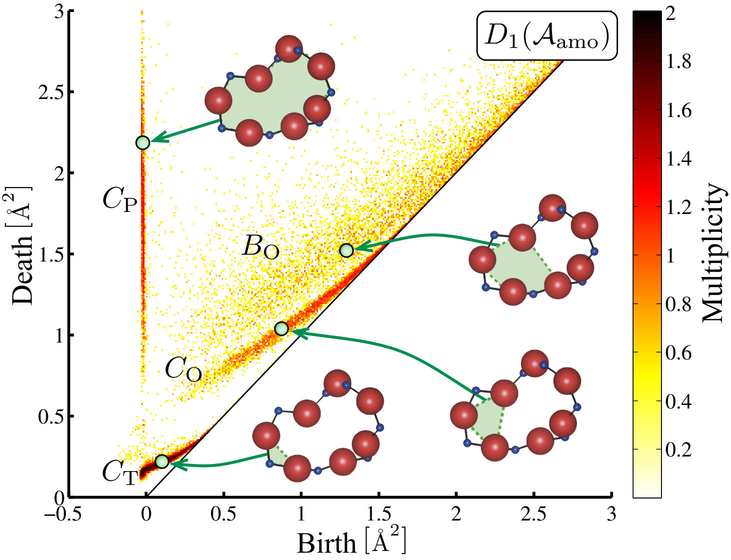

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The pipeline of topological data analysis

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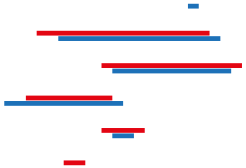
Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

If two functions $f, g : K \to \mathbb{R}$ have distance $\|f - g\|_\infty \leq \delta$, then there exists a δ-matching of their barcodes.



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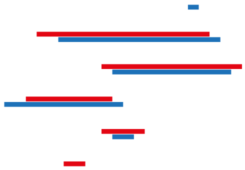

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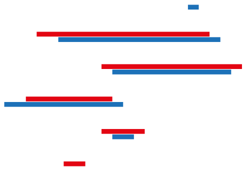

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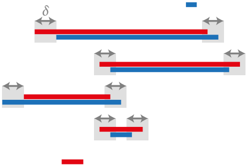

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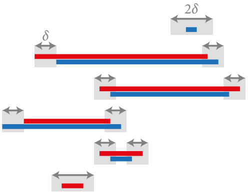

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Stability for functions in the big picture

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Interleavings of sublevel sets

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So the sublevel sets are δ-*interleaved:*

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F_t \arrow[rr, hook] \arrow[dr, hook] & & F_{t+2\delta} \arrow[dr, hook] & \\
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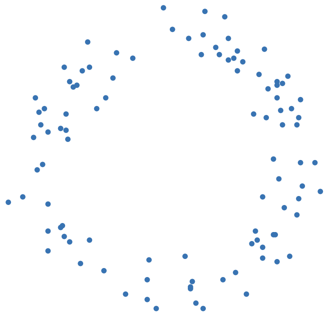
\end{tikzcd}$$

Homology is a *functor:* homology groups are interleaved too.

Geometric interleavings



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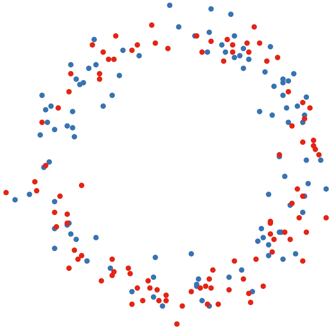

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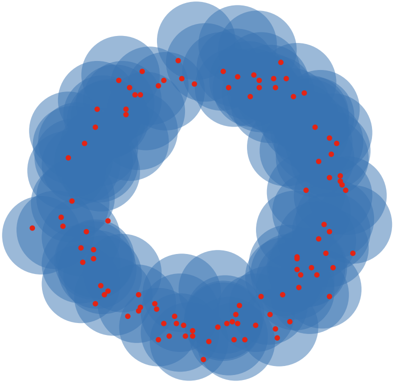

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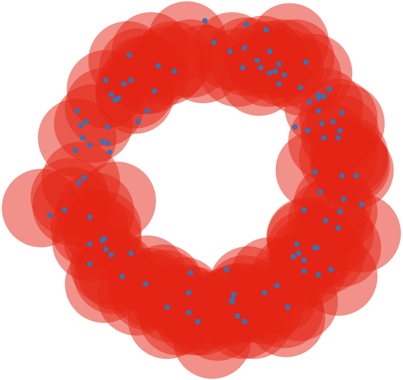

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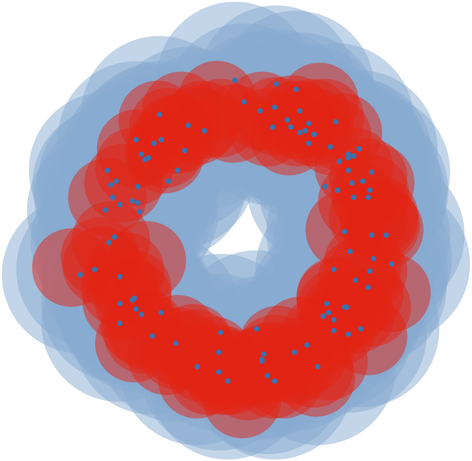

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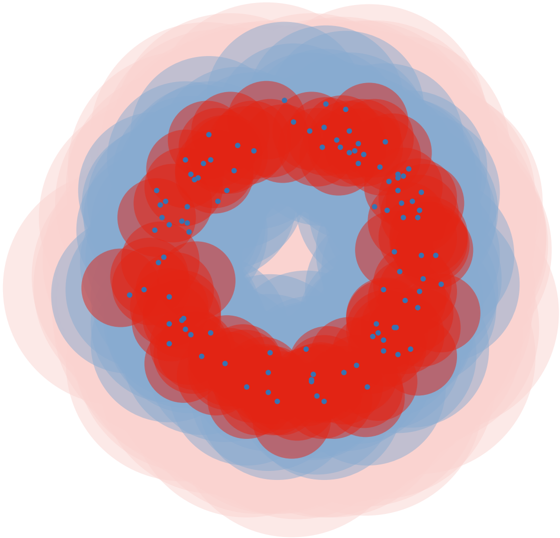

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Theorem (Chazal et al. 2009, 2012)

If two 1-parameter families of vector spaces are δ-interleaved, then there exists a δ-matching of their barcodes.

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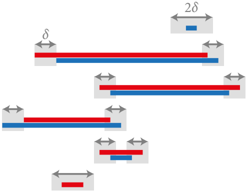

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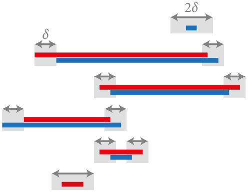

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Such a δ-matching is directly induced by a δ-interleaving.

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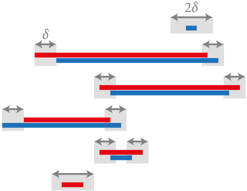
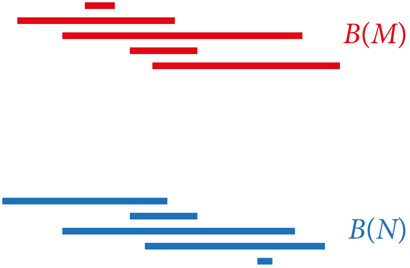


Theorem (B, Lesnick 2014)

Such a δ-matching is directly induced by a δ-interleaving.

Specifically: by any of the two 1-parameter families of linear maps in a δ-interleaving (though the barcode of their images).

Stability via induced matchings



Stability via induced matchings

$B(M)$

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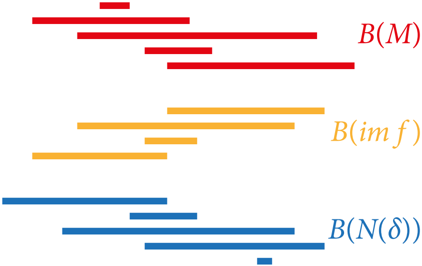

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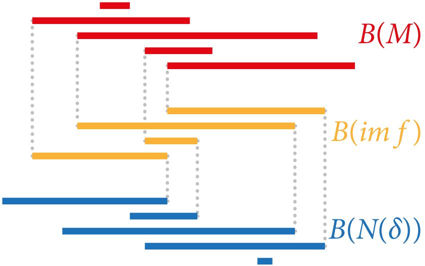

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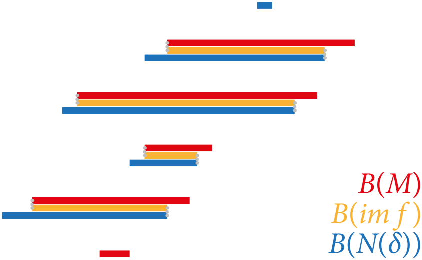

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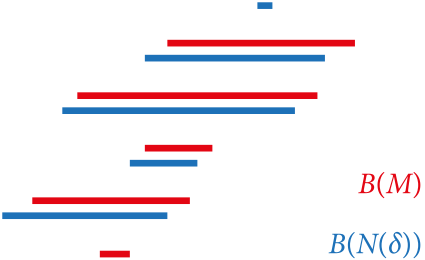

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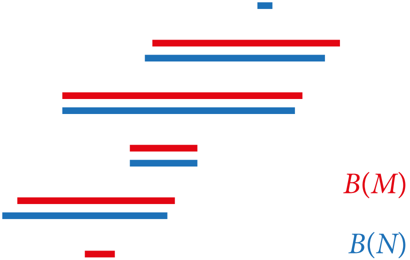
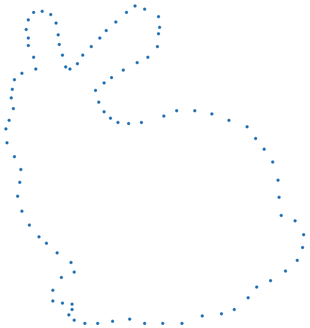


Connect the dots: topology from geometry



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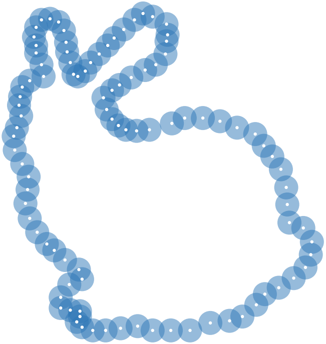

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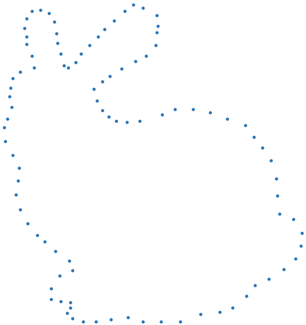

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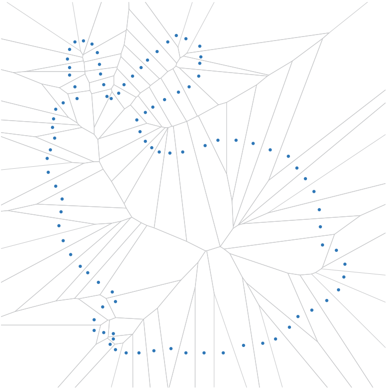

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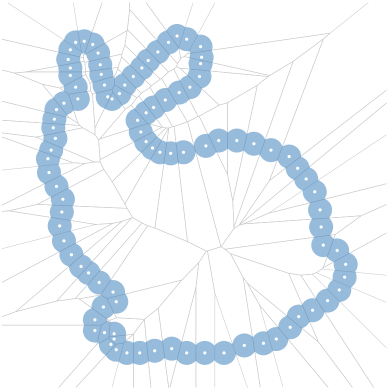

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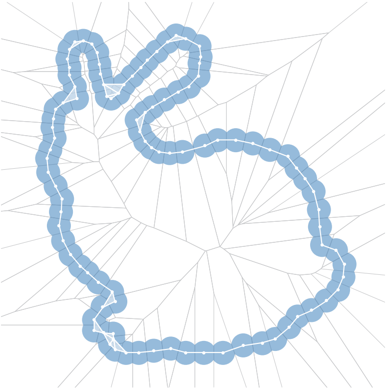
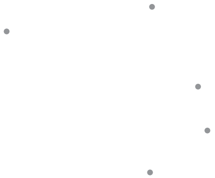


Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)



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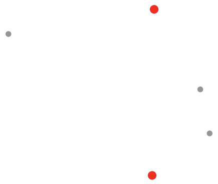

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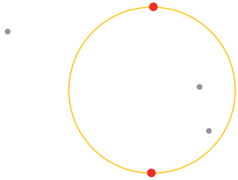

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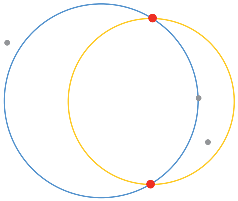

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Delaunay function $f_D(Q)$:
radius of smallest empty circumsphere

Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)

Simplices $\Delta(X)$: nonempty subsets of X

Two functions on simplices $Q \subseteq X$:

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radius of smallest enclosing sphere of Q

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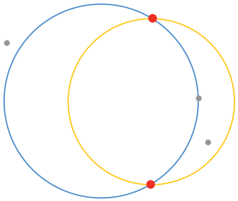
radius of smallest empty circumsphere

- defined only if Q has an empty circumsphere: $Q \in \mathrm{Del}(X)$



Čech and Delaunay complexes from functions

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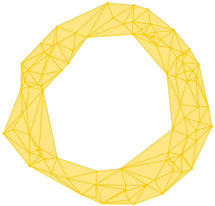

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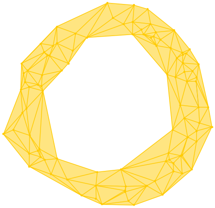

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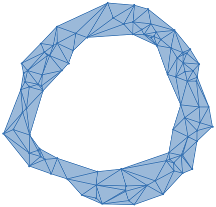

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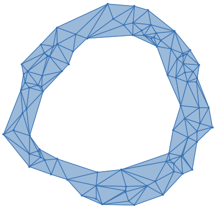

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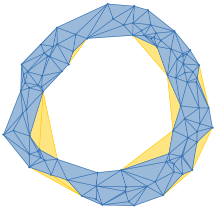

A sequence of collapses

Theorem (B, Edelsbrunner 2015)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent. In particular,

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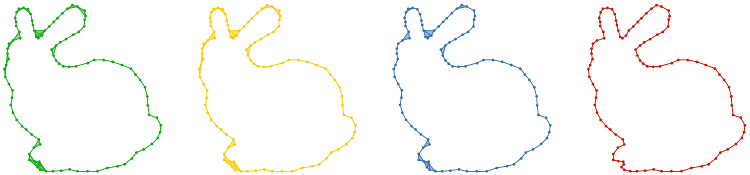

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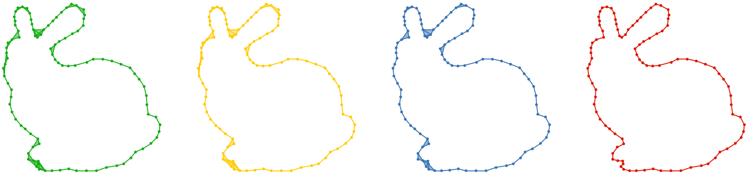

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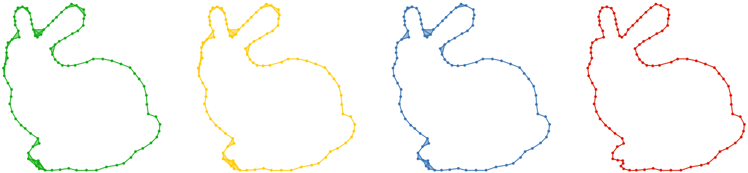

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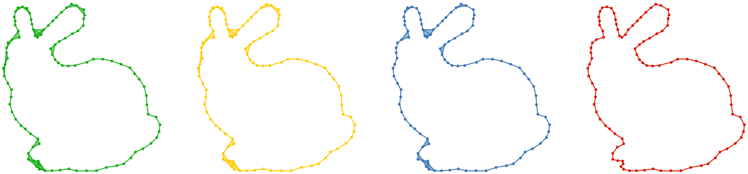

- All collapses are induced by a single *discrete gradient field*
- Explicit chain maps inducing isomorphims in homology
- Also works for weighted point sets

Sphere optimization problems

Both Čech and Delaunay functions are defined using the smallest sphere S satisfying certain constraints:

$$\begin{aligned} &\underset{r,z}{\text{minimize}} && r && \\ &\text{subject to} && \|z - q\| \leq r, && q \in Q, \\ & && \|z - e\| \geq r, && e \in E. \end{aligned}$$

Here r is the radius of the sphere S, and z is the center of S.

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- Čech function: choose $E = \emptyset$
- Delaunay function: choose $E = X$

Selective Delaunay complexes

Define for finite point sets $X, E \subset \mathbb{R}^d$:

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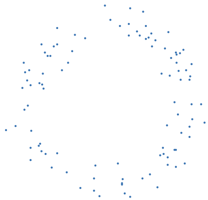
Note: choosing $E = \varnothing$ and $F = X$ yields

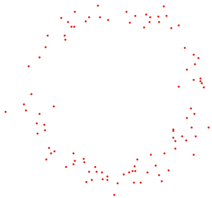
$$\mathrm{Cech}_r(X) \searrow \mathrm{DelCech}_r(X) \searrow \mathrm{Del}_r(X).$$

Connecting different Delaunay complexes

X

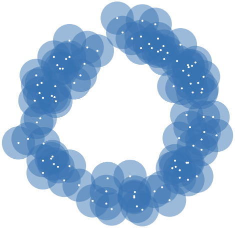


Y

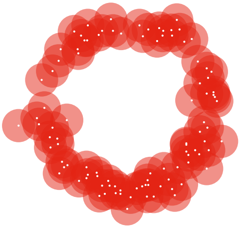


Connecting different Delaunay complexes

$B_r(X)$

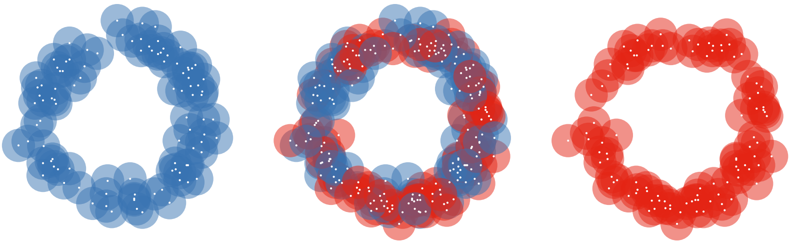


$B_r(Y)$



Connecting different Delaunay complexes

$$B_r(X) \hookrightarrow B_r(X \cup Y) \hookleftarrow B_r(Y)$$



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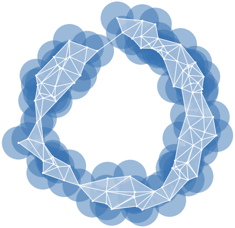
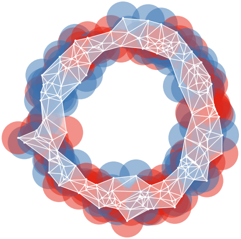
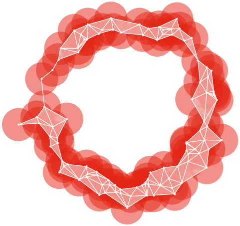


$\mathrm{Del}_r(X)$



$\mathrm{Del}_r(X \cup Y)$



$\mathrm{Del}_r(Y)$

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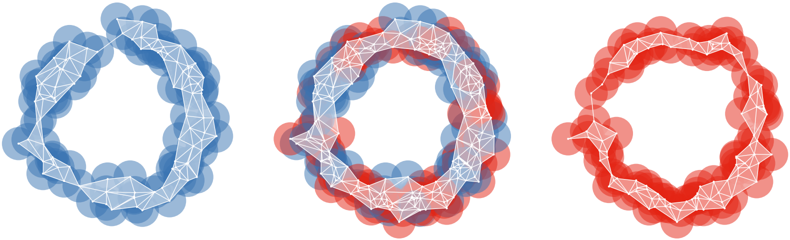

$$\begin{tikzcd}
\mathrm{Del}_r(X) \arrow[dr, hook] & & \mathrm{Del}_r(X \cup Y) \arrow[dl, hook] \arrow[dr, hook] & & \mathrm{Del}_r(Y) \arrow[dl, hook] \\
& \mathrm{Del}_r(X \cup Y, X) & & \mathrm{Del}_r(X \cup Y, Y) &
\end{tikzcd}$$

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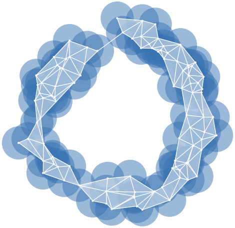
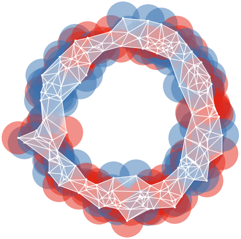
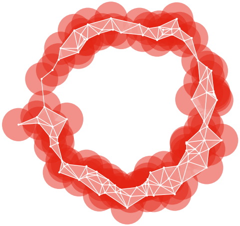

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\end{tikzcd}$$

Wrapping up

- Persistent homology can describe connectedness of shapes at different scales
- Delaunay complexes enable efficient computations
- Induced matchings provide an identification of topological features for similar shapes