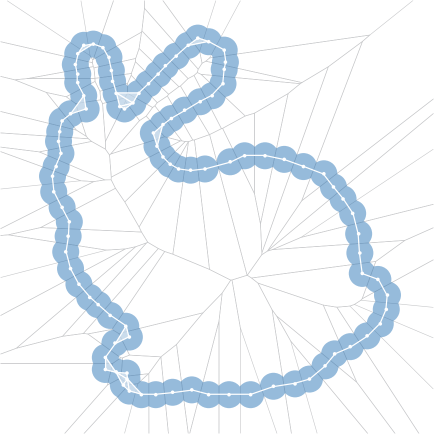


The Morse theory of Čech and Delaunay complexes

Workshop "Computational geometry and topology"
FoCM 2014

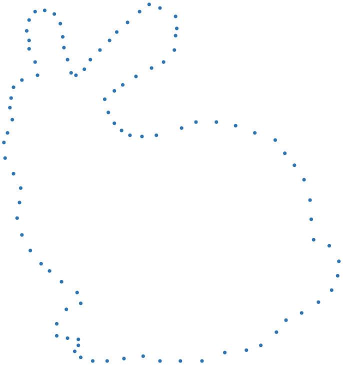
Ulrich Bauer[1] Herbert Edelsbrunner[2]

[1]TU München

[2]IST Austria

Dec 17, 2014

Connect the dots: topology from geometry



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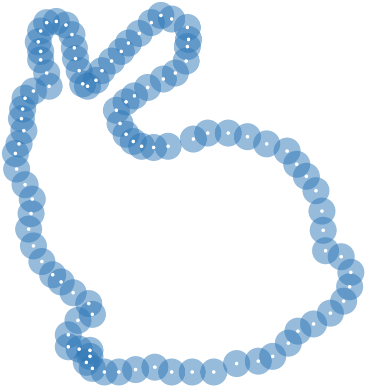

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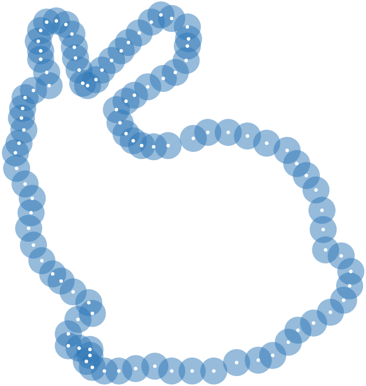

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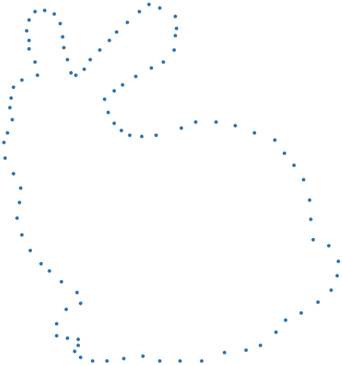

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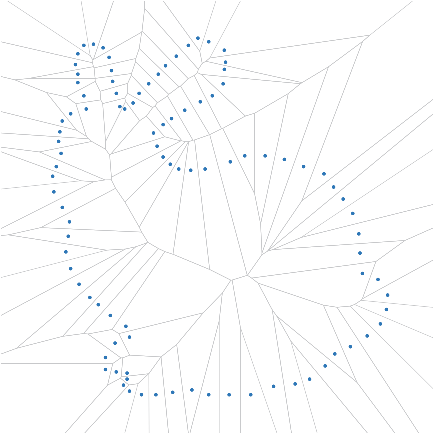

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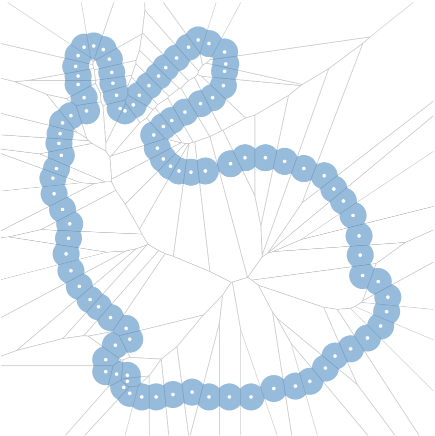

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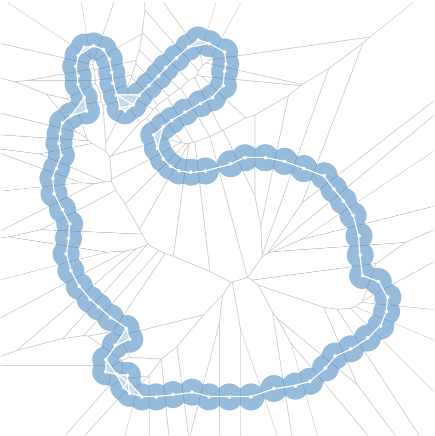
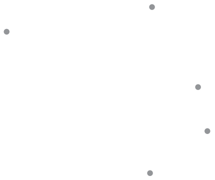


$$\mathrm{Cech}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} B_r(p) \neq \emptyset \right\}$$

$$\mathrm{Del}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} (B_r(p) \cap \mathrm{Vor}_X(p)) \neq \emptyset \right\}$$

Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)



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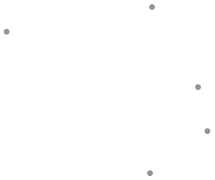

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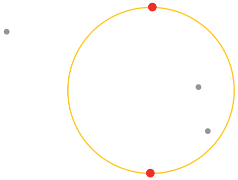

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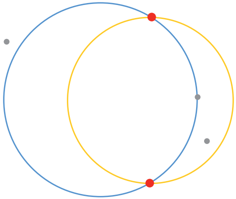

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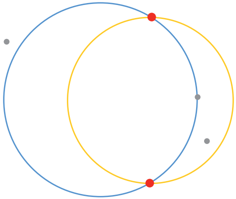
- defined only if Q has an empty circumsphere: $Q \in \mathrm{Del}(X)$



Čech and Delaunay complexes from functions

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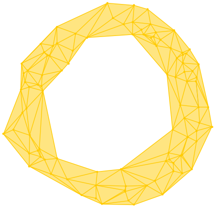

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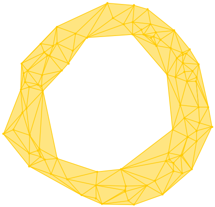

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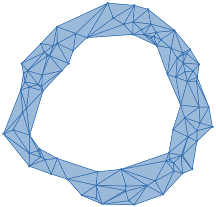

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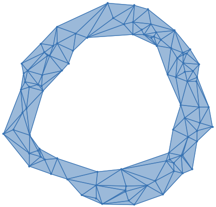

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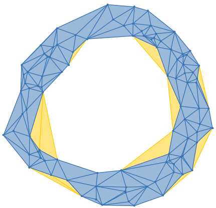

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By the *Nerve theorem* (Borsuk 1947):

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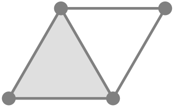
- Are they related by a sequence of simplicial collapses?

Discrete Morse theory

Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.



Simplicial collapses

Definition (Whitehead 1938)

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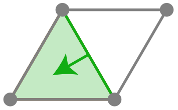
- A *free face* of K is a simplex that has a unique proper coface.



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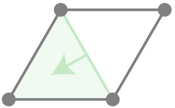

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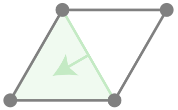

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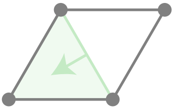

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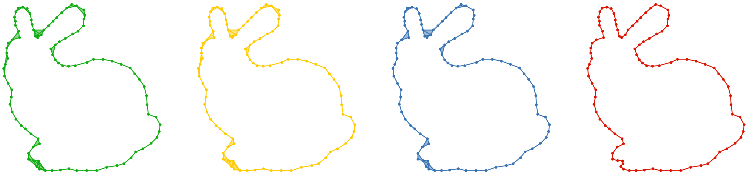
If there is a sequence of such elementary collapses from K to M, we say that K *collapses* to M (written as $K \searrow M$).

Main result: a sequence of collapses

Theorem (B, Edelsbrunner 2014)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent. In particular,

$$\mathrm{Cech}_r \searrow \mathrm{DelCech}_r \searrow \mathrm{Del}_r \searrow \mathrm{Wrap}_r\,.$$



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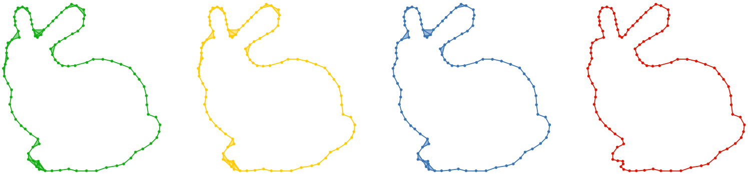

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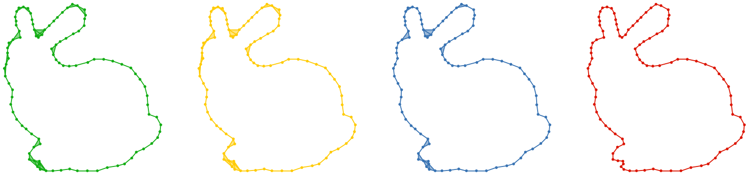
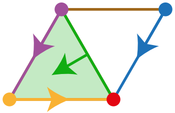


- All collapses are induced by a single *discrete gradient field*
- Also works for weighted point sets

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a simplicial complex
is a partition of the simplices
into singletons and pairs $\{L, U\}$,
where L is a face of U with codimension 1.



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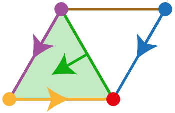

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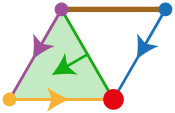

The singletons are called *critical simplices*.

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A function $f : K \to \mathbb{R}$ on a simplicial complex is a *discrete Morse function* if for all $t \in \mathbb{R}$:

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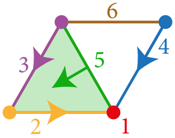

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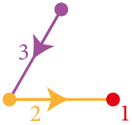

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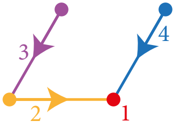

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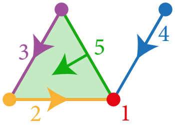

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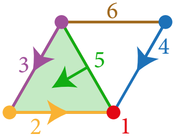

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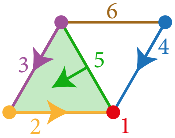

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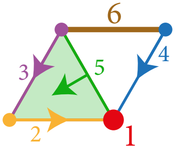
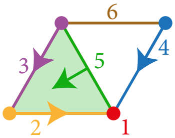


If $f^{-1}(t) = \{Q\}$ then t is a *critical value*.

Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.



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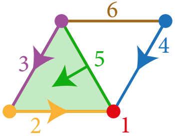

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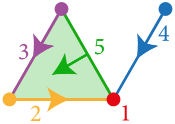

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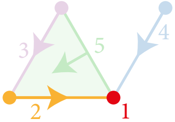

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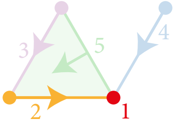

Let V be a discrete gradient field on a simplicial complex K,
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If $K \setminus L$ is the union of some pairs of V,
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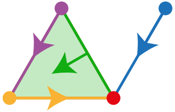

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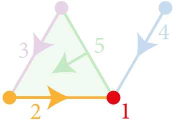

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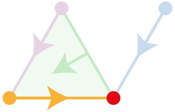

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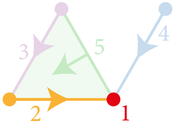

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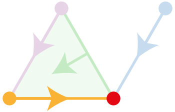

We say that V *induces the collapse $K \searrow L$.*

Unfortunately…

Neither the Čech nor the Delaunay functions
are discrete Morse functions!

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- Example: two simplices Q, Q' with $f_C(Q) = f_C(Q')$ such that neither is a face of the other:

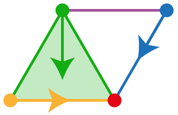
Q $\qquad$ Q'

Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}$$



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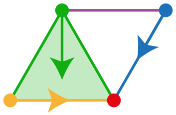

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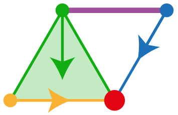

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The singletons are called *critical simplices.*

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Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex
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- the sublevel sets $K_t = f^{-1}(-\infty, t]$
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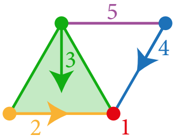

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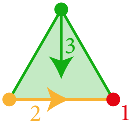

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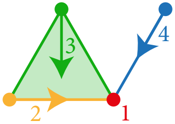

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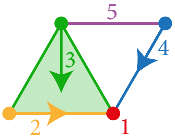

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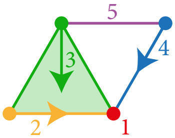
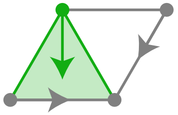


Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:



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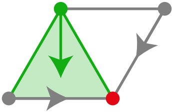

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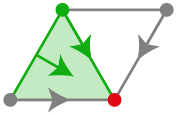

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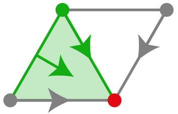

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Therefore the collapsing theorems also hold for generalized discrete Morse functions.

Čech and Delaunay functions

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Proposition (B, Edelsbrunner 2014)

The Čech function and the Delaunay function are generalized discrete Morse functions.

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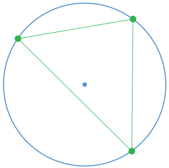

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Let $Q \subseteq X$ be a simplex with smallest enclosing sphere S.
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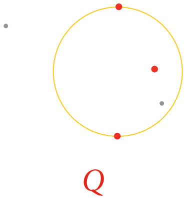

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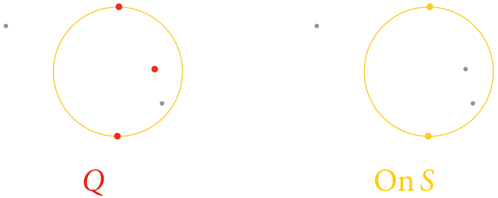

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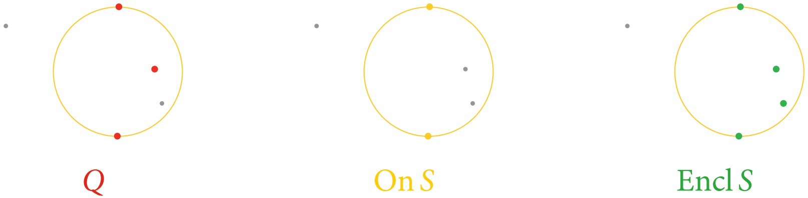

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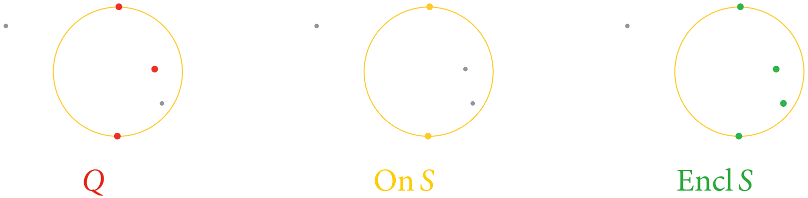

The front and back faces of a simplex

Let $\mathrm{On}\, S$ denote the points of X on the sphere S.

Write the center z of S as an affine combination

$$z = \sum_{x \in \mathrm{On}\, S} \mu_x x, \qquad 1 = \sum_{x \in \mathrm{On}\, S} \mu_x.$$

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$$\operatorname{Front} S = \{x \in \mathrm{On}\,S \mid \mu_x > 0\},$$
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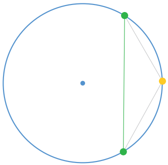

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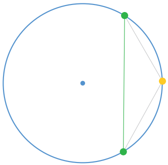

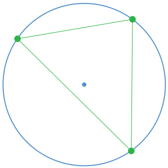

Delaunay intervals

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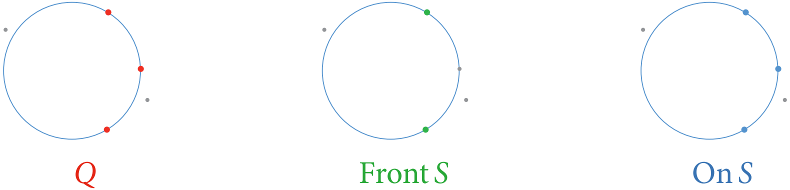

Sphere optimization problems

Both Čech and Delaunay functions are defined using the smallest sphere S satisfying certain constraints:

$$\begin{aligned} \underset{r,z}{\text{minimize}} \quad & r \\ \text{subject to} \quad & \|z - q\| \le r, & q \in Q, \\ & \|z - e\| \ge r, & e \in E. \end{aligned}$$

Here r is the radius and z is the center of the sphere S.

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The Karush–Kuhn–Tucker optimality conditions

Consider an optimization problem of the form

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) & \\ \text{subject to} \quad & g_i(x) \leq 0, & i \in I, \\ & h_j(x) = 0, & j \in J, \end{aligned}$$

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Theorem (Karush 1939, Kuhn–Tucker 1951)

A feasible point x to the above problem is an optimal solution iff there exist Lagrange multipliers $(\nu_i)_{i\in I}$ *and* $(\lambda_j)_{j\in J}$ *such that*

$$\begin{aligned} \nabla f(x) + \sum_{i\in I} \nu_i \nabla g_i(x) + \sum_{j\in J} \lambda_j \nabla h_j(x) &= 0, && \textit{(stationarity)} \\ \nu_i\, g_i(x) &= 0, \quad i \in I, && \textit{(complementary slackness)} \\ \lambda_j &\geq 0, \quad j \in J. && \textit{(dual feasibility)} \end{aligned}$$

KKT conditions for the smallest sphere problem

The KKT conditions for our sphere optimization problem are:

Proposition

A sphere S enclosing Q and excluding E is minimal iff its center z can be written as an affine combination

$$z = \sum_{x \in Q \cup E} \mu_x x, \qquad 1 = \sum_{x \in Q \cup E} \mu_x$$

such that

- $\mu_x = 0$ *for* $x \notin \mathrm{On}\, S$,
- $\mu_x \geq 0$ *for* $x \notin E$, *and*
- $\mu_x \leq 0$ *for* $x \notin Q$.

Čech and Delaunay intervals from KKT

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A sphere S enclosing Q and excluding E is minimal iff

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Theorem (B, Edelsbrunner 2014)

Let $E \subseteq E'$. Then

$$\mathrm{Del}_r(X, E) \searrow \mathrm{Del}_r(X, E) \cap \mathrm{Del}\,(X, E') \searrow \mathrm{Del}_r(X, E').$$

Selective Delaunay complexes

Define for finite point sets $X, E \subset \mathbb{R}^d$:

- *E-Delaunay function* $f_E(Q)$:
  radius of smallest sphere enclosing Q and excluding E
  - defined only if Q has an E-empty enclosing sphere:
    $Q \in \mathrm{Del}(X, E)$
- *E-Delaunay complex* $\mathrm{Del}_r(X, E) = f_D^{-1}(-\infty, r]$

Theorem (B, Edelsbrunner 2014)

Let $E \subseteq E'$. Then

$$\mathrm{Del}_r(X, E) \searrow \mathrm{Del}_r(X, E) \cap \mathrm{Del}\,(X, E') \searrow \mathrm{Del}_r(X, E').$$

Note: choosing $E = \varnothing$ and $E' = X$ yields

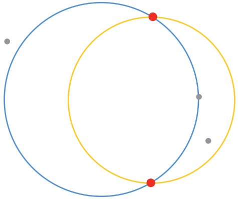
$$\mathrm{Cech}_r(X) \searrow \mathrm{DelCech}_r(X) \searrow \mathrm{Del}_r(X).$$

Collapsing from Čech to Delaunay

Collapsing the Delaunay–Čech complex

To construct the collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$:

- Consider a non-critical Delaunay simplex Q



Collapsing the Delaunay–Čech complex

To construct the collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$:

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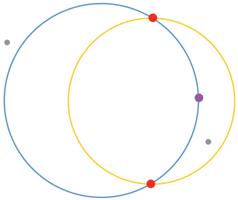

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To construct the collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$:

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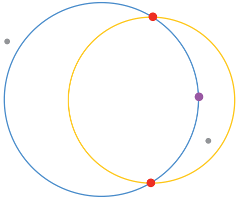

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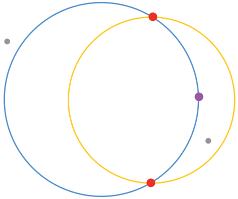


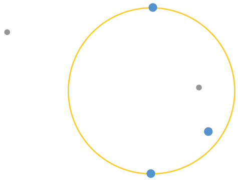
Lemma

The pairs (Q', Q'') yield a vector field that induces a collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$, for any radius r.

Collapsing non-Delaunay simplices

To construct the collapse $\mathrm{Cech}_r \searrow \mathrm{DelCech}_r$:

- Consider a non-Delaunay simplex Q together with its Čech sphere S



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To construct the collapse $\mathrm{Cech}_r \searrow \mathrm{DelCech}_r$:

- Consider a non-Delaunay simplex Q together with its Čech sphere S
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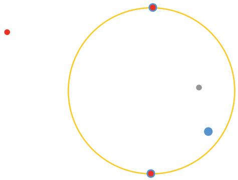

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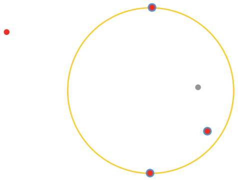

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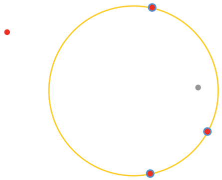

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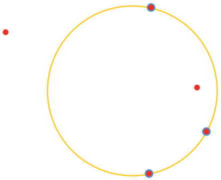

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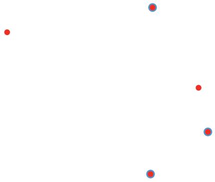

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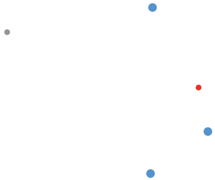

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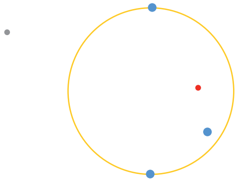

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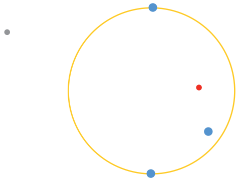


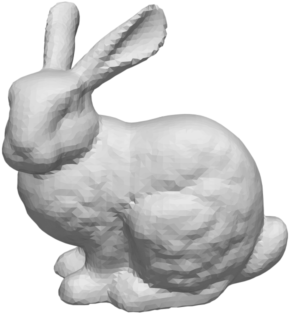
Lemma

The pairs (Q', Q'') yield a vector field that induces a collapse $\mathrm{Cech}_r \searrow \mathrm{DelCech}_r$, for any radius r.

Wrap complexes

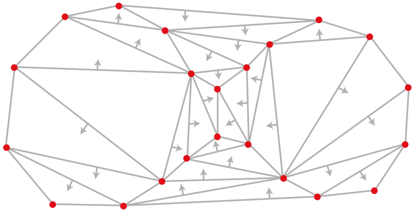
Wrap complexes

Generalizes and greatly simplifies the surface reconstruction algorithm *Wrap* (Edelsbrunner 1995, Geomagic)



Wrap complexes

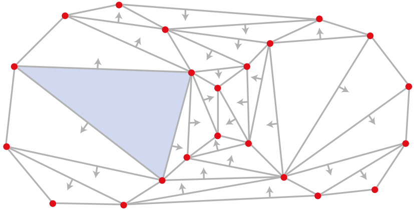
Consider the Delaunay function f_D of X.



Wrap complexes

Consider the Delaunay function f_D of X.

- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$



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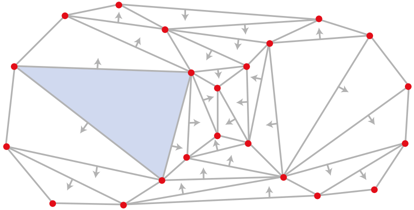

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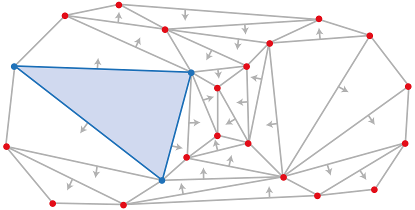

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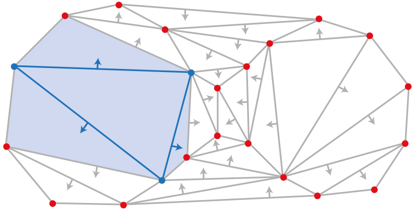

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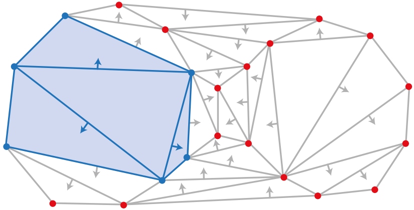

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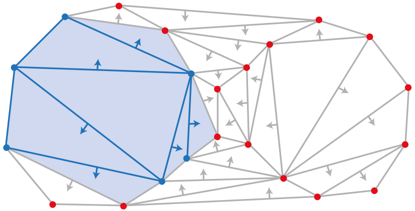

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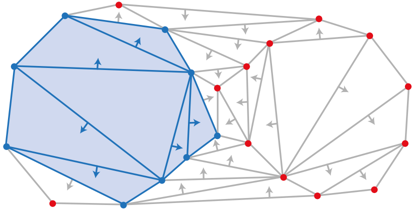

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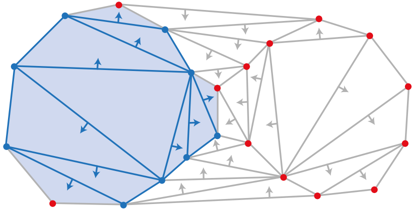

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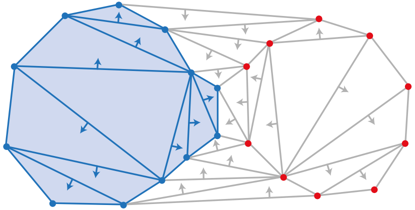

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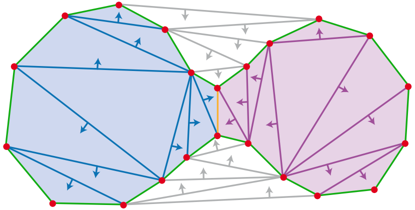

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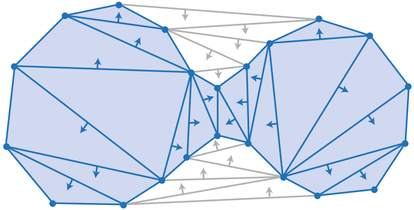
Define

$$\mathrm{Wrap}_r = \bigcup_{C \in \mathrm{Crit}_r} \downarrow C$$



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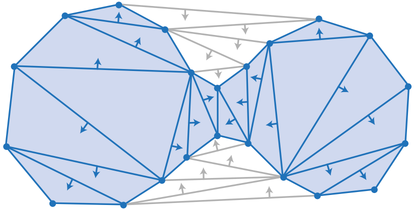

The Delaunay intervals induce a collapse $\mathrm{Del}_r \searrow \mathrm{Wrap}_r$.

Wrapping up

- Čech and Delaunay complexes from Morse functions
- Explicit homotopy equivalence by simplicial collapses
- Simple definition and generalization of *Wrap* complexes