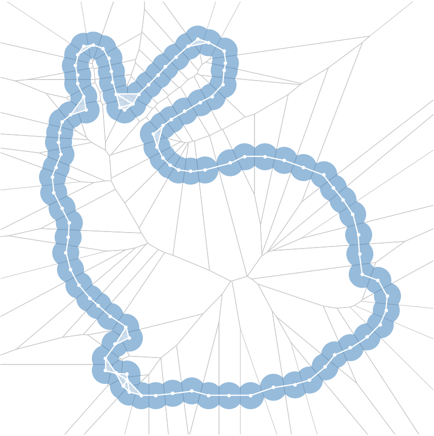


Topological Data Analysis

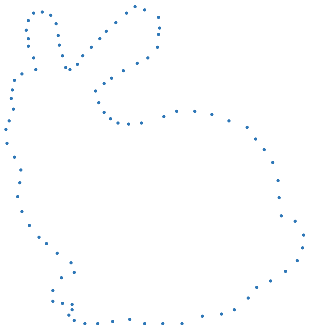
Part III: The Morse theory of Čech and Delaunay complexes

Ulrich Bauer

TUM

February 7, 2015

Connect the dots: topology from geometry



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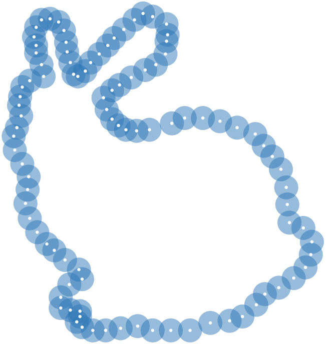

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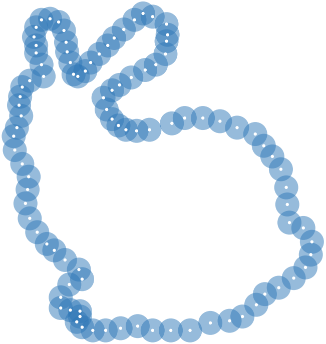

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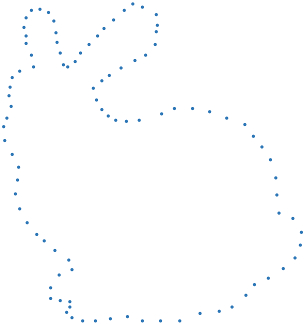

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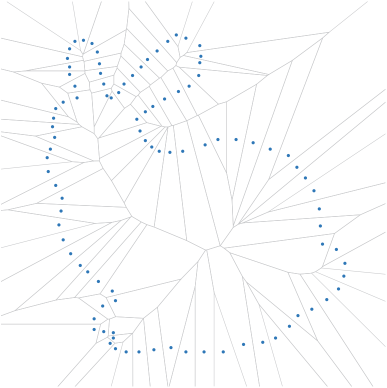

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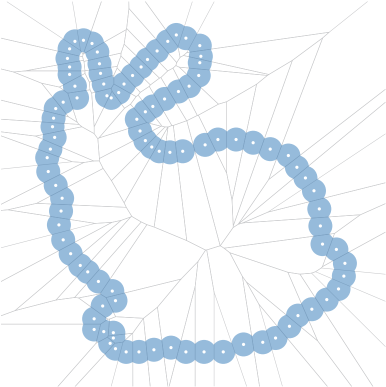

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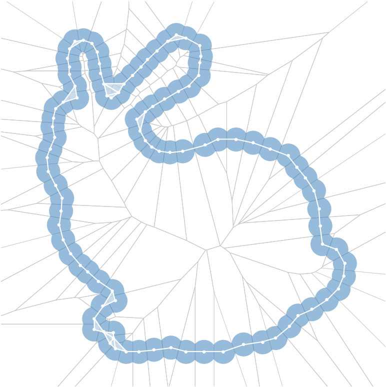

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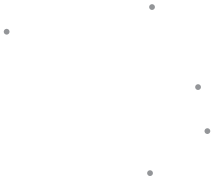
Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)



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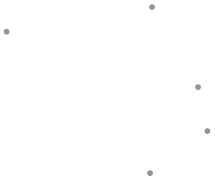

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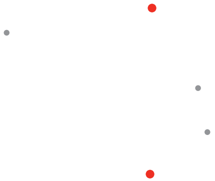

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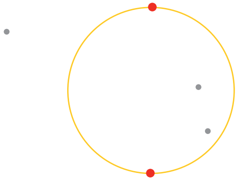

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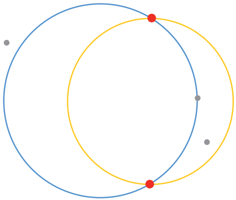

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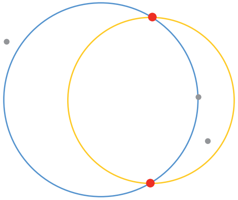

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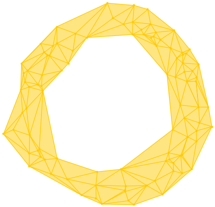

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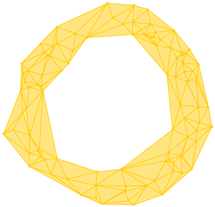

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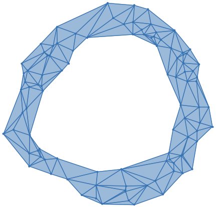

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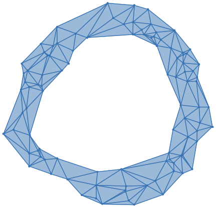

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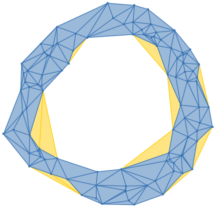

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By the *Nerve theorem* (Borsuk 1947):

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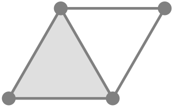
- Are they related by a sequence of simplicial collapses?

Discrete Morse theory

Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.



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Definition (Whitehead 1938)

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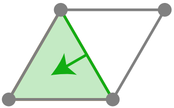
- A *free face* of K is a simplex that has a unique proper coface.



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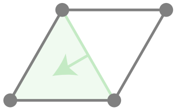

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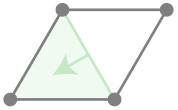

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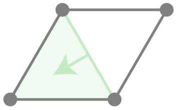

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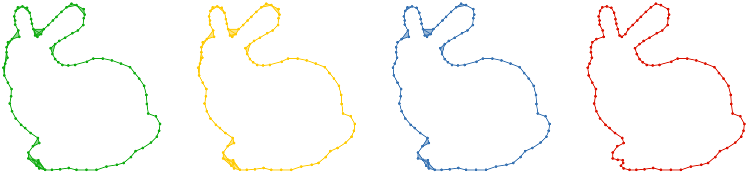
If there is a sequence of such elementary collapses from K to M, we say that K *collapses* to M (written as $K \searrow M$).

Main result: a sequence of collapses

Theorem (B, Edelsbrunner 2014)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent. In particular,

$$\mathrm{Cech}_r \searrow \mathrm{DelCech}_r \searrow \mathrm{Del}_r \searrow \mathrm{Wrap}_r\,.$$



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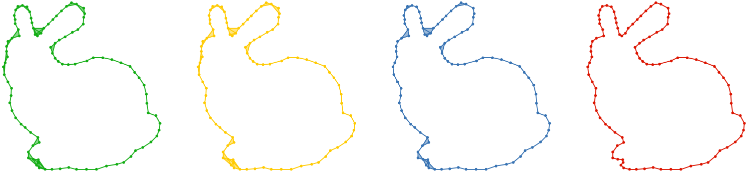

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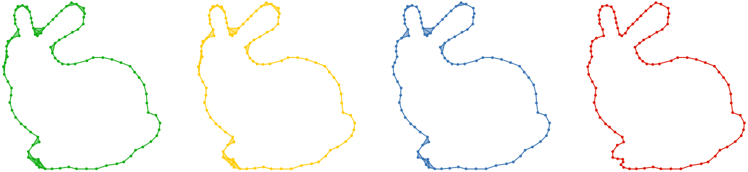
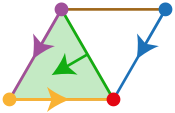


- All collapses are induced by a single *discrete gradient field*
- Also works for weighted point sets

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a simplicial complex
is a partition of the simplices
into singletons and pairs $\{L, U\}$,
where L is a face of U with codimension 1.



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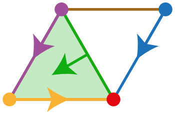

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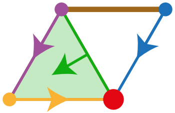

The singletons are called *critical simplices.*

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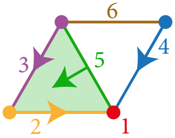

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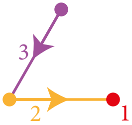

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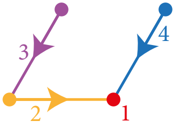

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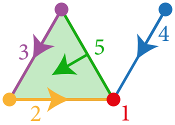

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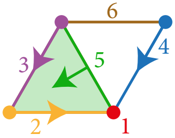

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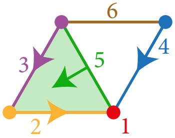

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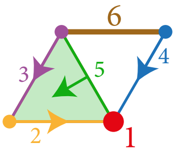
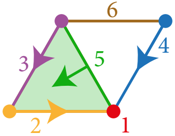


If $f^{-1}(t) = \{Q\}$ then t is a *critical value*.

Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.



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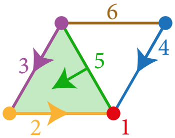

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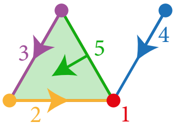

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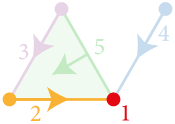

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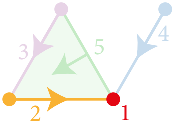

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Corollary

If $K \setminus L$ is the union of some pairs of V,
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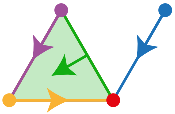

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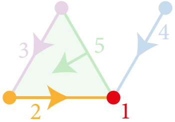

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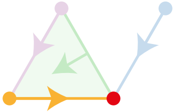

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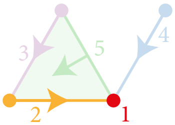

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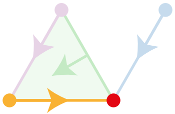

We say that V *induces the collapse $K \searrow L$.*

Unfortunately…

Neither the Čech nor the Delaunay functions are discrete Morse functions!

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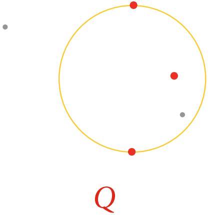
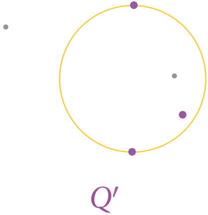
- Example: two simplices Q, Q' with $f_C(Q) = f_C(Q')$ such that neither is a face of the other:



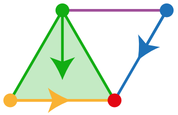


Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}$$



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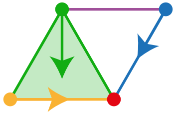

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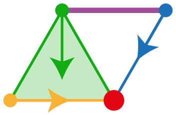

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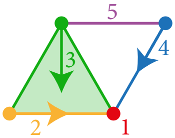
The singletons are called *critical simplices.*

Generalized discrete Morse theory

Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes



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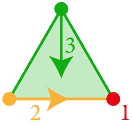

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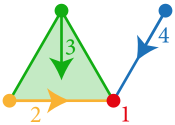

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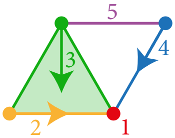

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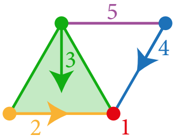
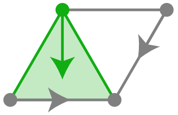


Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:



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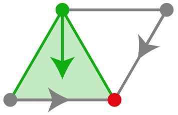

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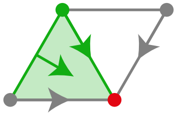

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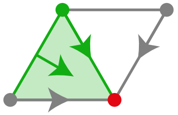

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Therefore the collapsing theorems also hold for generalized discrete Morse functions.

Čech and Delaunay functions

Morse theory of Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

The Čech function and the Delaunay function are generalized discrete Morse functions.

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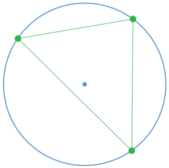
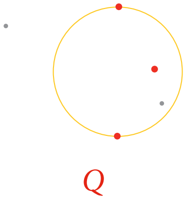


Čech intervals

Lemma

Let $Q \subseteq X$ be a simplex with smallest enclosing sphere S. Then $Q' \subseteq X$ has the same smallest enclosing sphere S iff



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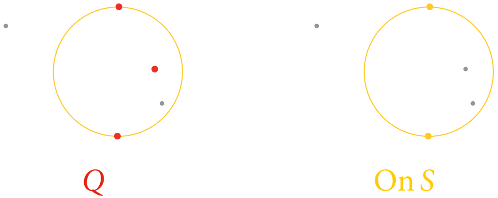

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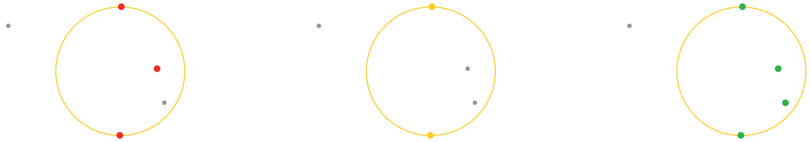

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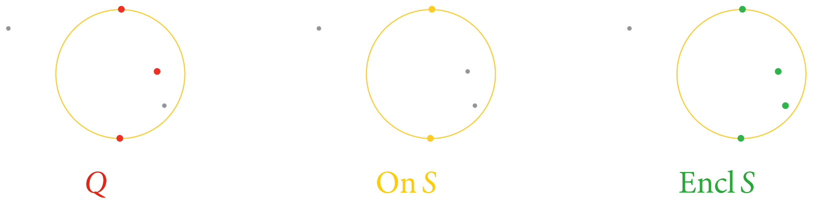

The front and back faces of a simplex

Let $\mathrm{On}\,S$ denote the points of X on the sphere S.

Write the center z of S as an affine combination

$$z = \sum_{x \in \mathrm{On}\,S} \mu_x x, \qquad 1 = \sum_{x \in \mathrm{On}\,S} \mu_x.$$

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$$\mathrm{Front}\, S = \{x \in \mathrm{On}\, S \mid \mu_x > 0\},$$
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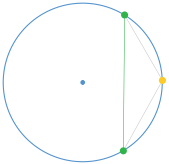

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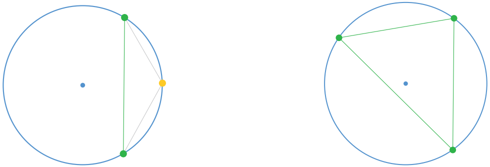

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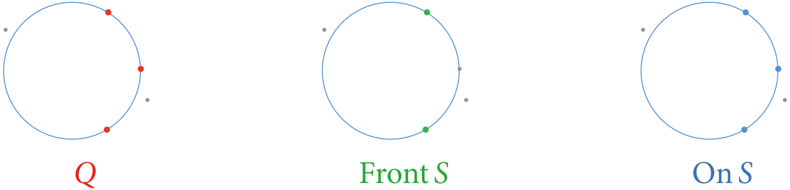

Sphere optimization problems

Both Čech and Delaunay functions are defined using the smallest sphere S satisfying certain constraints:

$$\begin{aligned} \underset{r,z}{\text{minimize}} \quad & r \\ \text{subject to} \quad & \|z - q\| \leq r, \qquad q \in Q, \\ & \|z - e\| \geq r, \qquad e \in E. \end{aligned}$$

Here r is the radius of the sphere S, and z is the center of S.

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The Karush–Kuhn–Tucker optimality conditions

Consider an optimization problem of the form

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f(x) & \\ \text{subject to} & g_i(x) \leq 0, & i \in I, \\ & h_j(x) = 0, & j \in J, \end{array}$$

where the function f is convex and g_i, h_j are affine.

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Theorem (Karush 1939, Kuhn–Tucker 1951)

A feasible point x to the above problem is an optimal solution iff there exist Lagrange multipliers $(\nu_i)_{i \in I}$ *and* $(\lambda_j)_{j \in J}$ *such that*

$$\nabla f(x) + \sum_{i \in I} \nu_i \nabla g_i(x) + \sum_{j \in J} \lambda_j \nabla h_j(x) = 0, \qquad \textit{(stationarity)}$$

$$\nu_i \, g_i(x) = 0, \quad i \in I, \qquad \textit{(complementary slackness)}$$

$$\lambda_j \geq 0, \quad j \in J. \qquad \textit{(dual feasibility)}$$

KKT conditions for the smallest sphere problem

The KKT conditions for our sphere optimization problem are:

Proposition

A sphere S enclosing Q and excluding E is minimal iff its center z can be written as an affine combination

$$z = \sum_{x \in Q \cup E} \mu_x x, \qquad 1 = \sum_{x \in Q \cup E} \mu_x$$

such that

- $\mu_x = 0$ *for* $x \notin \mathrm{On}\, S$,
- $\mu_x \geq 0$ *for* $x \notin E$, *and*
- $\mu_x \leq 0$ *for* $x \notin Q$.

Čech and Delaunay intervals from KKT

Proposition

A sphere S enclosing Q and excluding E is minimal iff

$$\begin{aligned} z &\in \operatorname{Aff}(\operatorname{On} S), \\ Q &\in [\operatorname{Front} S, \operatorname{Encl} S], \text{ and} \\ E &\in [\operatorname{Back} S, \operatorname{Excl} S]. \end{aligned}$$

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Corollary

The Čech intervals are of the form $[\mathrm{On}\, S, \mathrm{Encl}\, S]$.

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The Delaunay intervals are of the form $[\operatorname{Front} S, \operatorname{On} S]$.

Selective Delaunay complexes

Define for finite point sets $X, E \subset \mathbb{R}^d$:

- *E-Delaunay function* $f_E(Q)$:
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  - defined only if Q has an E-empty enclosing sphere:
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Selective Delaunay complexes

Define for finite point sets $X, E \subset \mathbb{R}^d$:

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Note: choosing $E = \varnothing$ and $E' = X$ yields

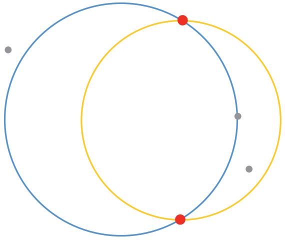
$$\mathrm{Cech}_r(X) \searrow \mathrm{DelCech}_r(X) \searrow \mathrm{Del}_r(X).$$

Collapsing from Čech to Delaunay

Collapsing the Delaunay–Čech complex

To construct the collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$:

- Consider a non-critical Delaunay simplex Q



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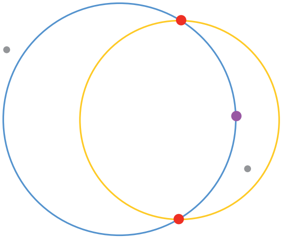

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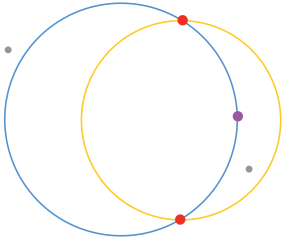

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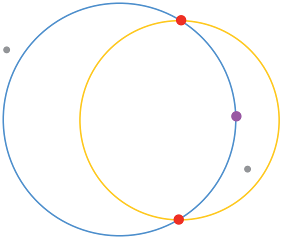

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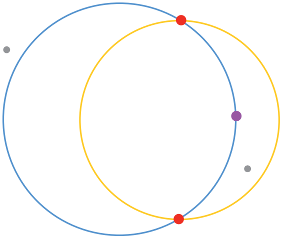

Lemma

The pairs (Q', Q'') *yield a discrete gradient.*
This gradient induces a collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$,
for any radius r.

Collapsing non-Delaunay simplices

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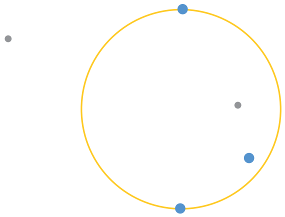

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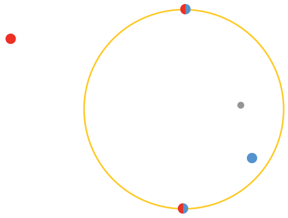

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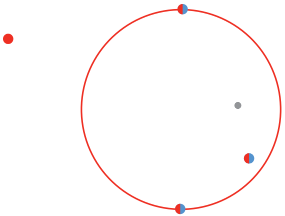

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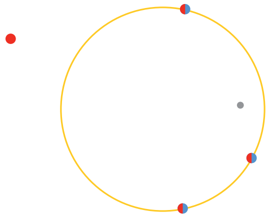

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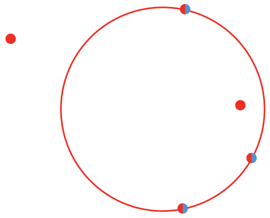

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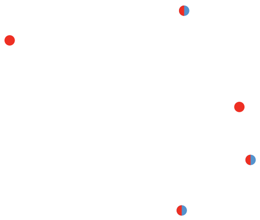

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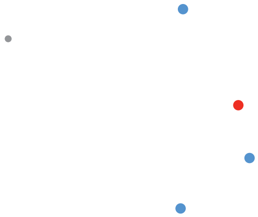

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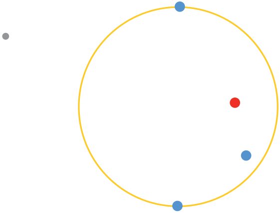

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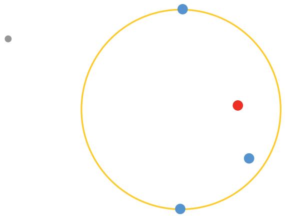

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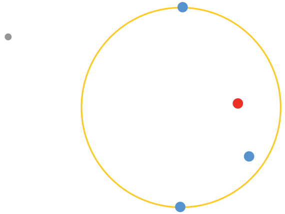


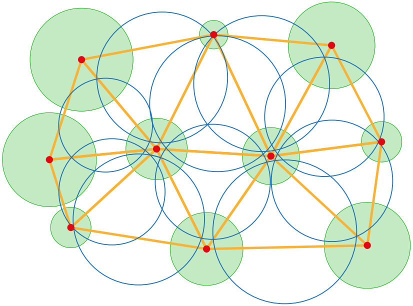
Lemma

The pairs (Q', Q'') yield a discrete gradient.
This gradient induces a collapse $\mathrm{Cech}_r \searrow \mathrm{DelCech}_r$, for any radius r.

Weighted point sets

All constructions can be generalized to weighted point sets

- Intuition: spheres with given squared radius
- Instead of circumspheres, consider *orthogonal spheres*



Wrap complexes

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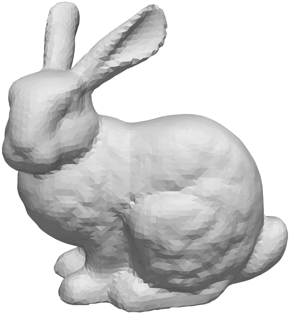
Generalizes and greatly simplifies the surface reconstruction algorithm *Wrap* (Edelsbrunner 1995, Geomagic)



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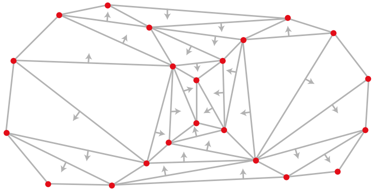
Consider the Delaunay function f_D of X.



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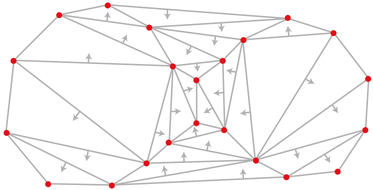

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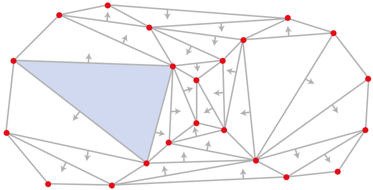

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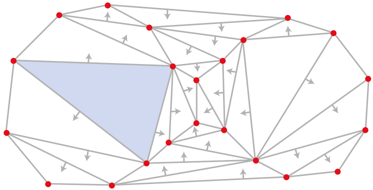

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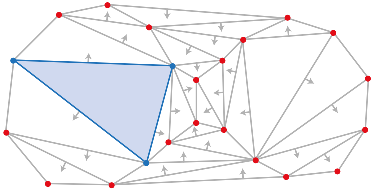

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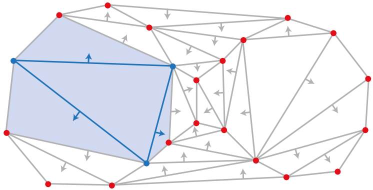

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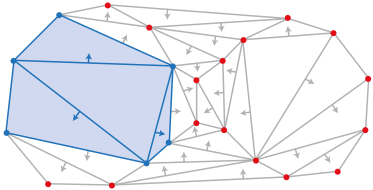

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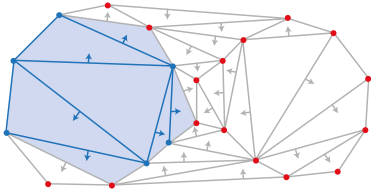

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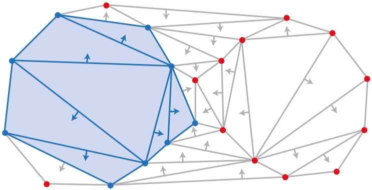

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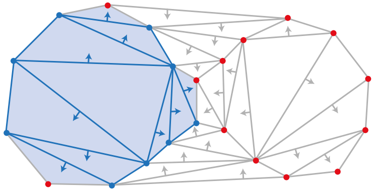

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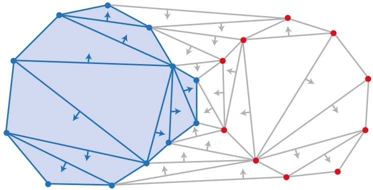

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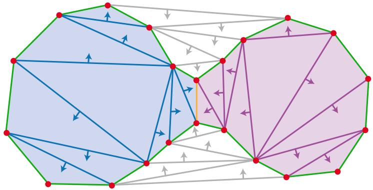

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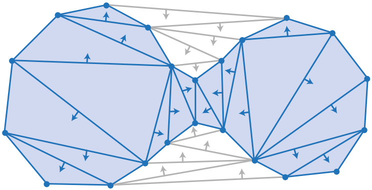

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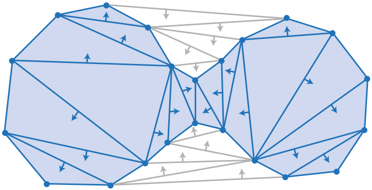

The Delaunay intervals induce a collapse $\mathrm{Del}_r \searrow \mathrm{Wrap}_r$.

Wrapping up

- Čech and Delaunay complexes from Morse functions
- Explicit homotopy equivalence by simplicial collapses
- Simple definition and generalization of *Wrap* complexes