Persistent Homology and the Stability Theorem

Ulrich Bauer

TUM

February 26, 2018

Workshop *Persistence, Representations, and Computation*, Raitenhaslach

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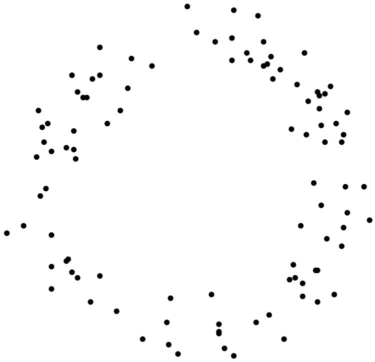

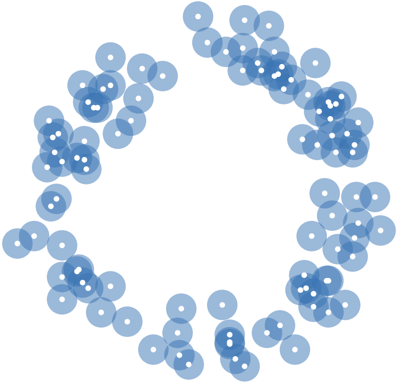

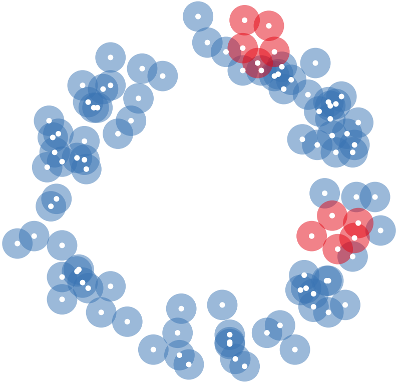

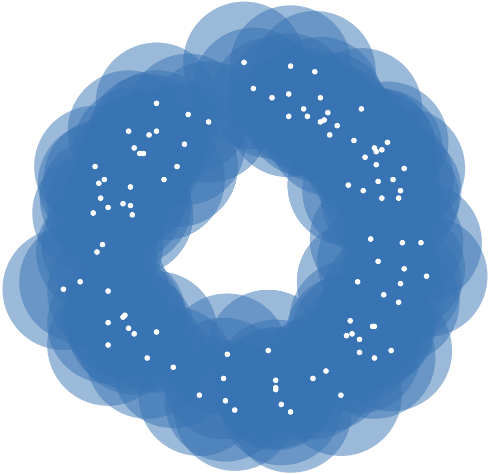

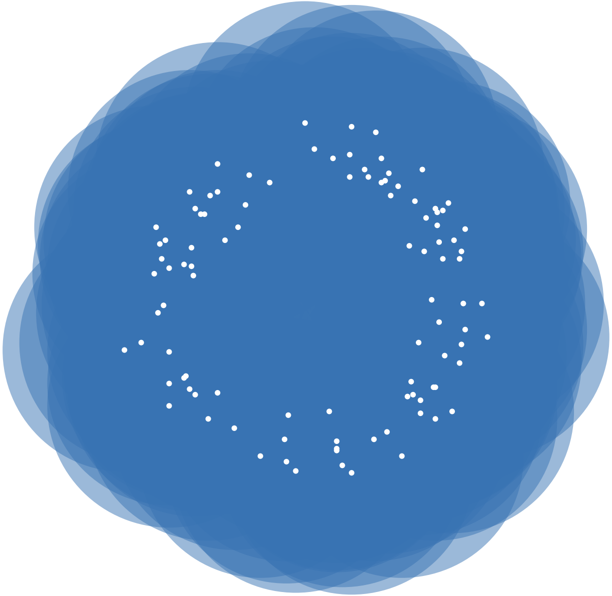

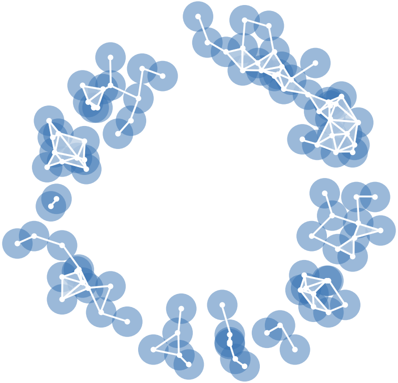

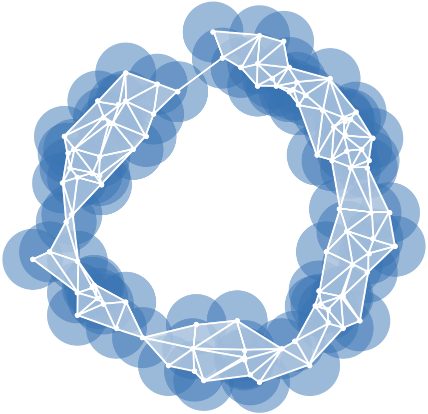

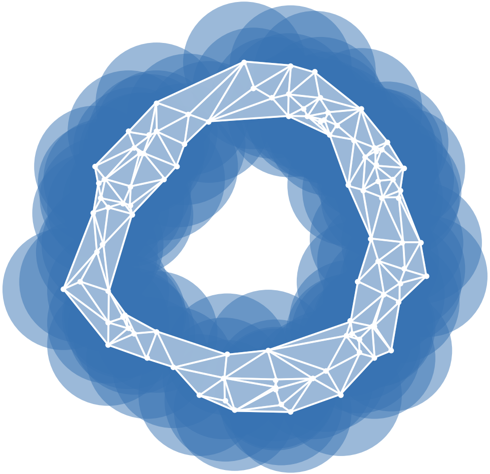

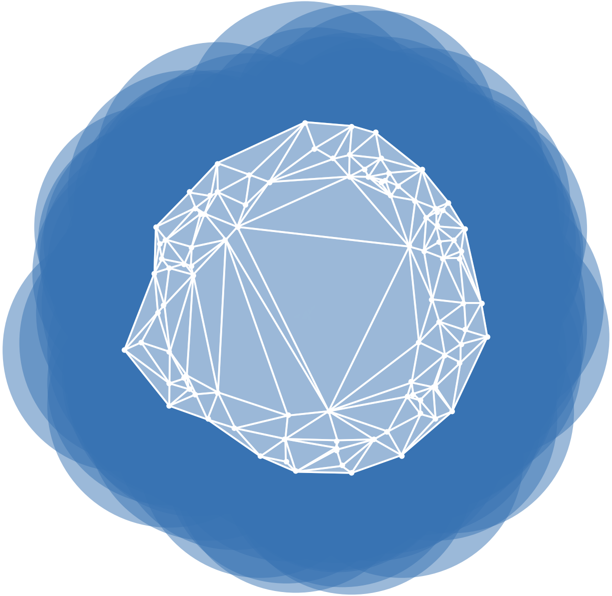

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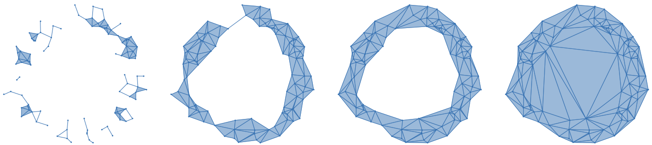

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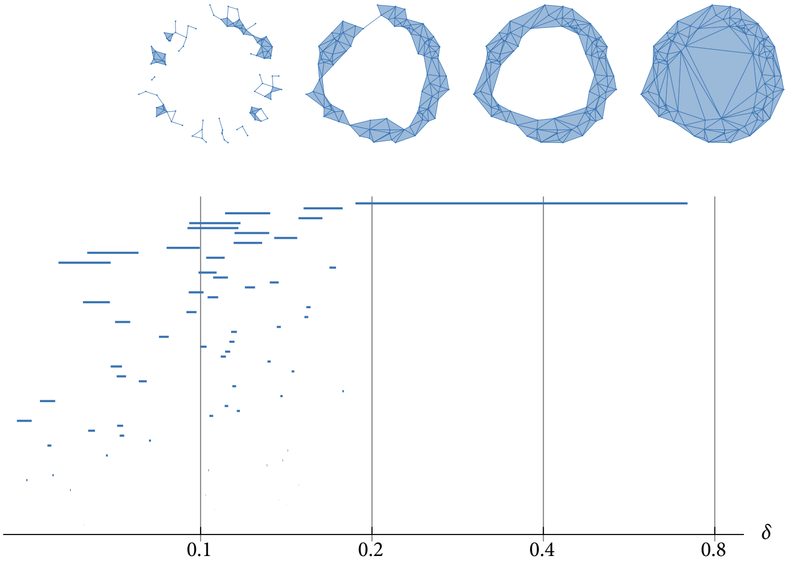

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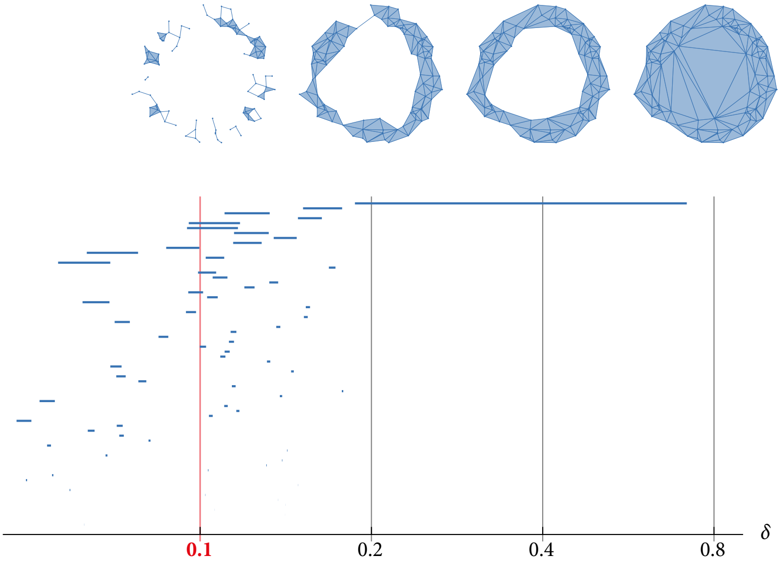

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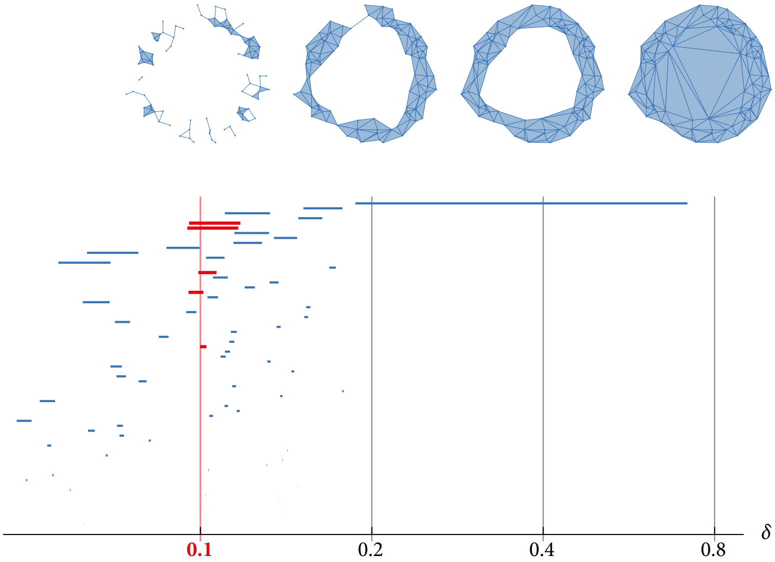

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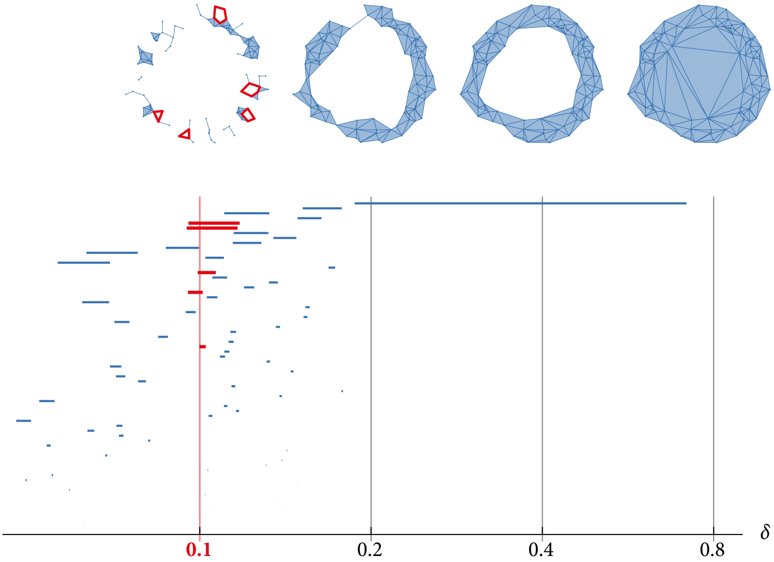

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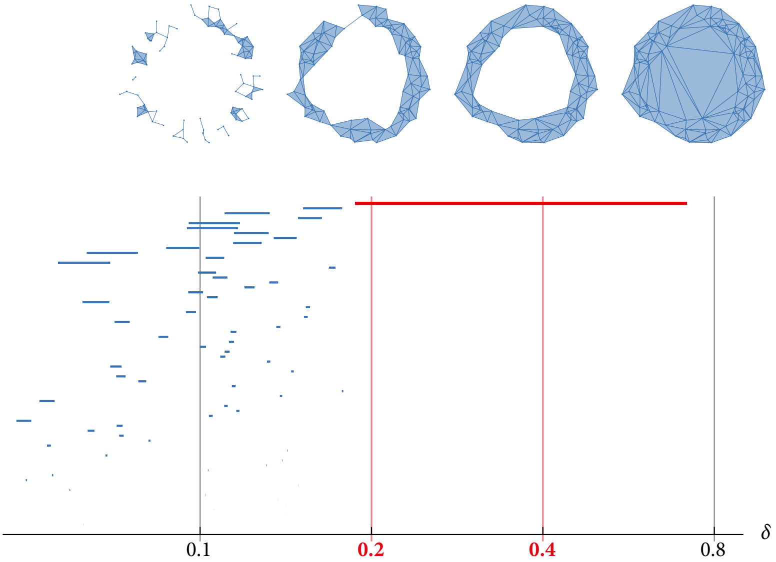

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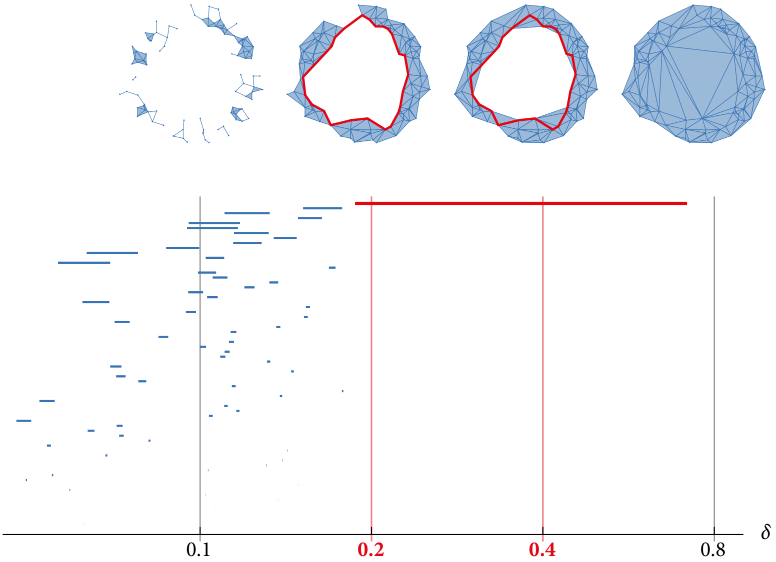

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  - $\mathbf{R}$ is the poset category of $(\mathbb{R}, \leq)$
  - A topological space K_t for each $t \in \mathbb{R}$
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- Consider homology with coefficients in a field (often $\mathbb{Z}_2$) $H_* : \mathbf{Top} \to \mathbf{Vect}$
- Persistent homology is a diagram $M : \mathbf{R} \to \mathbf{Vect}$ *(persistence module)*

Homology inference

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Requires strong assumptions:

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Theorem (Niyogi, Smale, Weinberger 2006)

Let M be a submanifold of $\mathbb{R}^d$. Let $P \subset M$, $\delta > 0$ be such that

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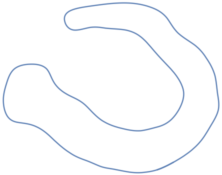

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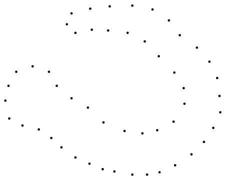

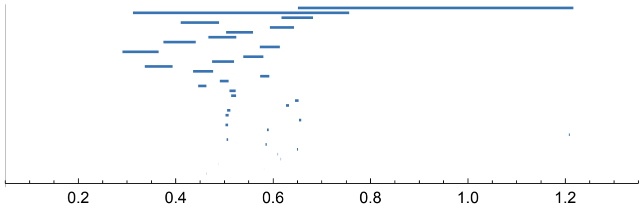

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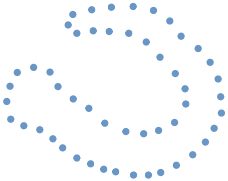

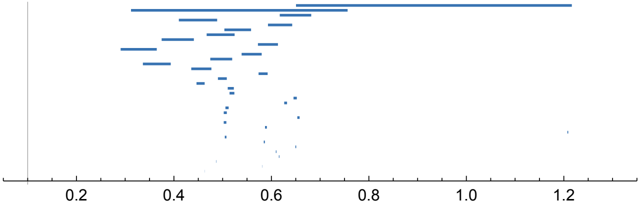

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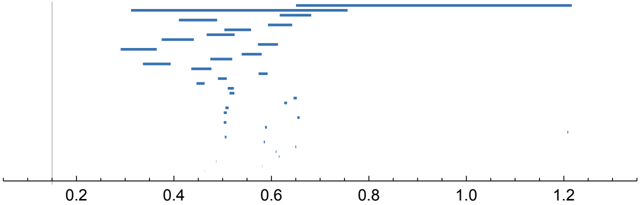

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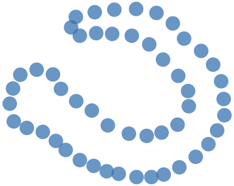

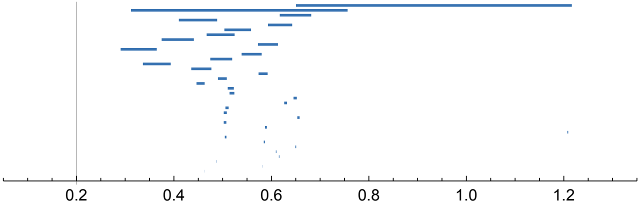

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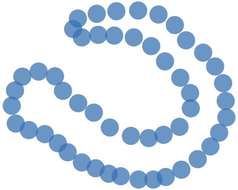

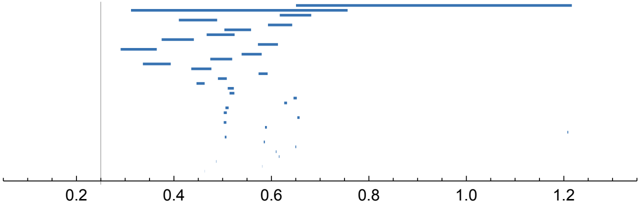

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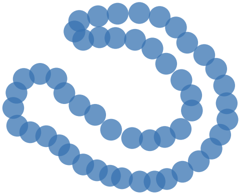

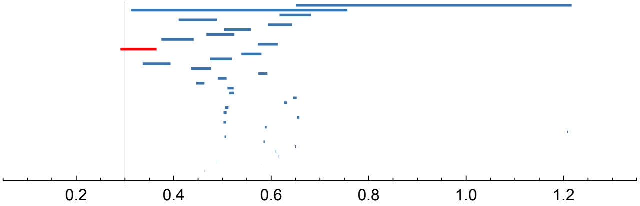

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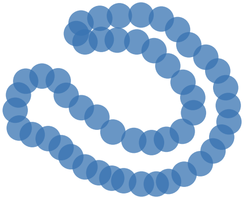

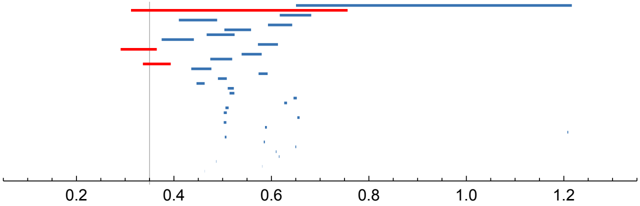

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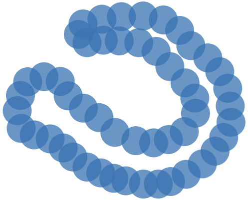

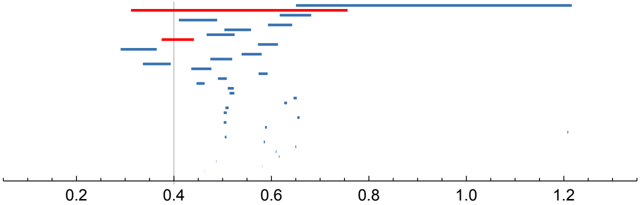

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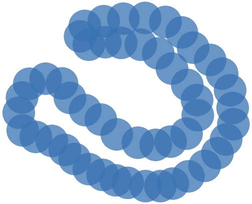

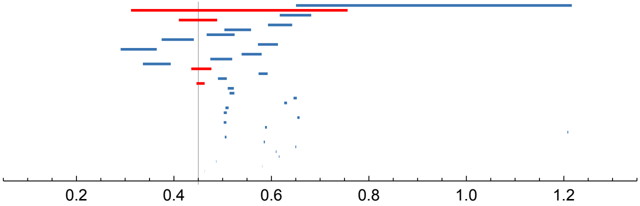

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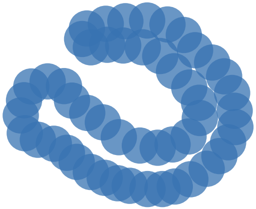

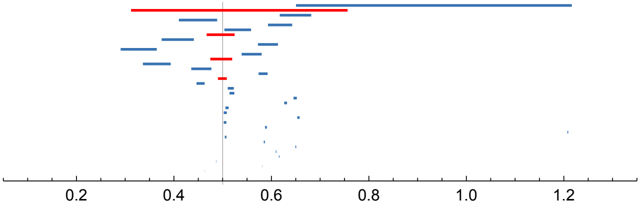

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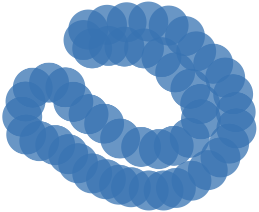

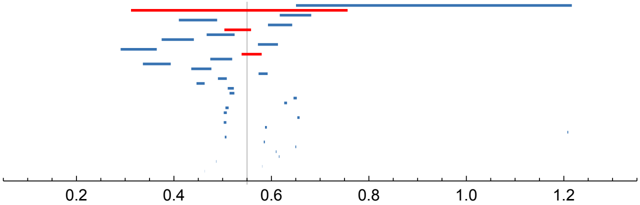

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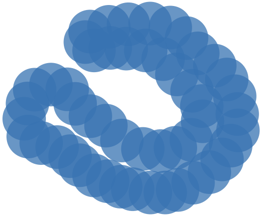

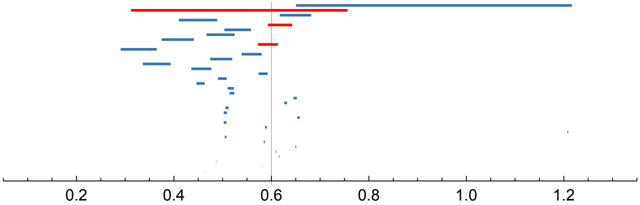

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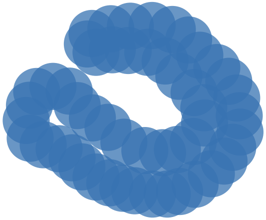

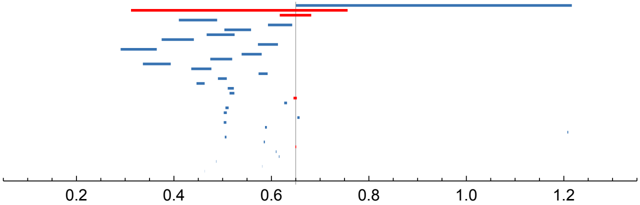

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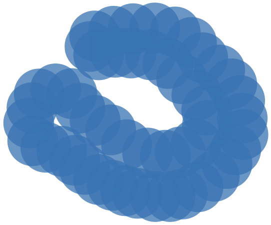

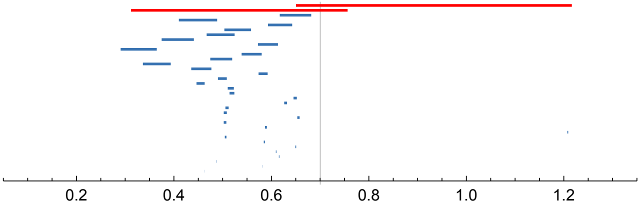

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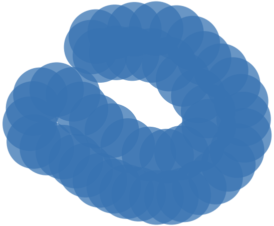

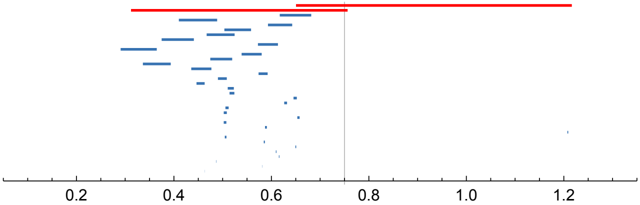

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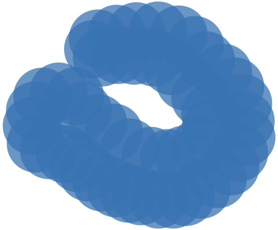

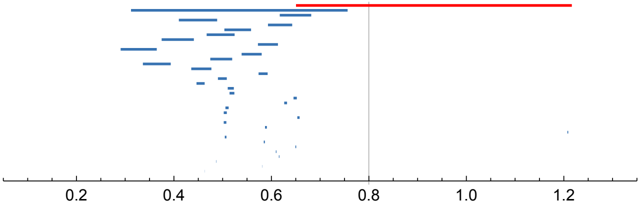

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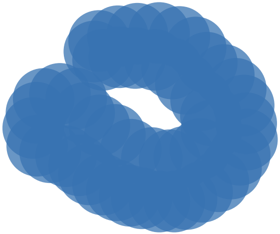
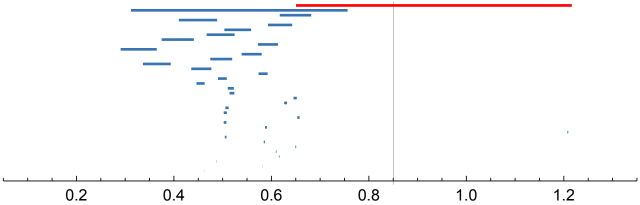




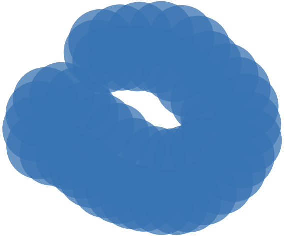
Homology reconstruction by thickening

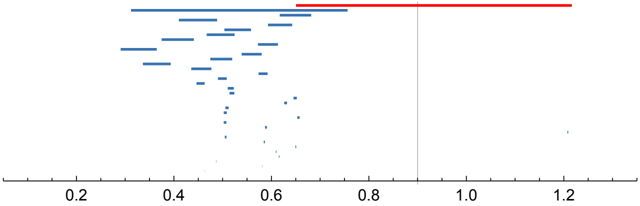
Theorem (Niyogi, Smale, Weinberger 2006)

Let M be a submanifold of $\mathbb{R}^d$. Let $P \subset M$, $\delta > 0$ be such that

- *$B_\delta(P)$ covers Ω, and*
- $\delta < \sqrt{3/20}\,\mathrm{reach}(M)$.

Then $H_*(M) \cong H_*(B_{2\delta}(P))$.





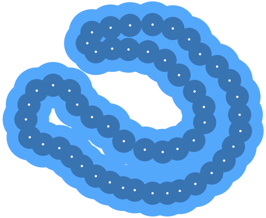
Homology inference using persistence

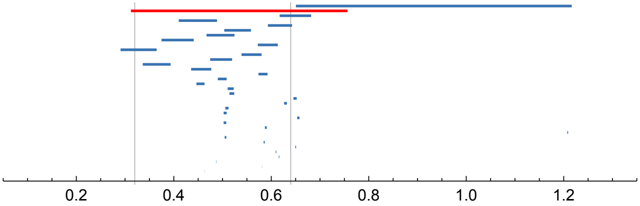
Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\Omega \subset \mathbb{R}^d$. Let $P \subset \Omega$, $\delta > 0$ be such that

- $B_\delta(P)$ *covers* Ω*, and*
- *the inclusions* $\Omega \hookrightarrow B_\delta(\Omega) \hookrightarrow B_{2\delta}(\Omega)$ *preserve homology.*

Then $H_*(\Omega) \cong \operatorname{im} H_*(B_\delta(P) \hookrightarrow B_{2\delta}(P))$.





Homological realization

This motivates the *homological realization problem*:

Problem

Given a simplicial pair $L \subseteq K$, find X with $L \subseteq X \subseteq K$ such that

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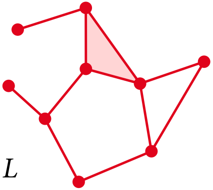

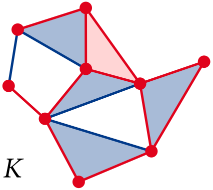

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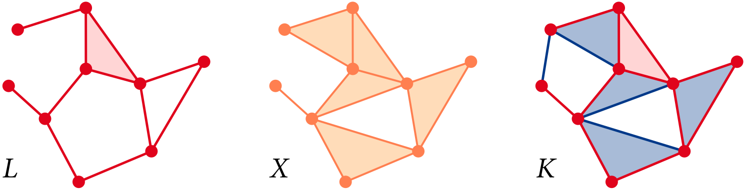

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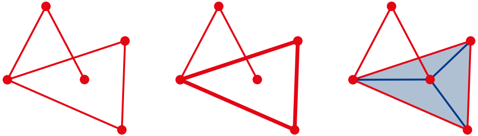

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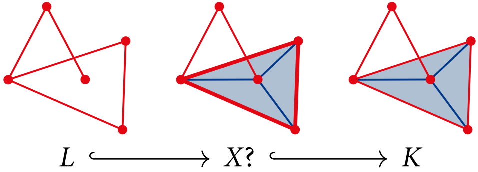

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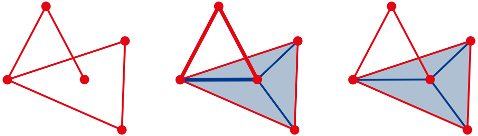

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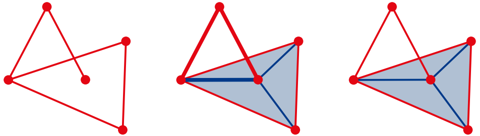

$$L \hookrightarrow X? \hookrightarrow K$$

Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

The homological realization problem is NP-hard, even in $\mathbb{R}^3$.

Stability

Persistence and stability: the big picture

Data	point cloud	$P \subset \mathbb{R}^d$

Persistence and stability: the big picture

$$\begin{tikzcd} \text{Data} \arrow[d] & \text{point cloud} \arrow[d, "\text{distance}"] & P \subset \mathbb{R}^d \\ \text{Geometry} & \text{function} & f : \mathbb{R}^d \to \mathbb{R} \end{tikzcd}$$

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\text{Combinatorics} & \text{intervals (persistence barcode)} & B : \mathbf{R} \to \mathbf{Mch}
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Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\|f - g\|_\infty = \delta$*. Then there exists a matching between the intervals of the persistence barcodes of* f *and* g *such that*

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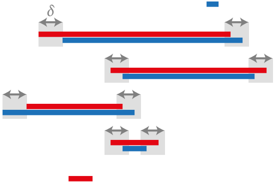

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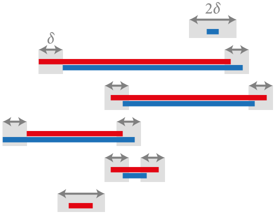

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Let $\delta = \|f - g\|_\infty$. Write $F_t = f^{-1}(-\infty, t]$ and $G_t = g^{-1}(-\infty, t]$.

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Then the sublevel set filtrations $F, G : \mathbf{R} \to \mathbf{Top}$ are *δ-interleaved:*

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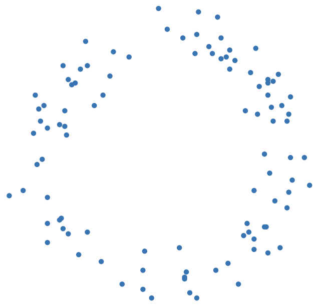
Applying homology (functor) preserves commutativity

- persistent homology of f, g yields
  δ-interleaved persistence modules $\mathbf{R} \to \mathbf{Vect}$

Geometric interleavings



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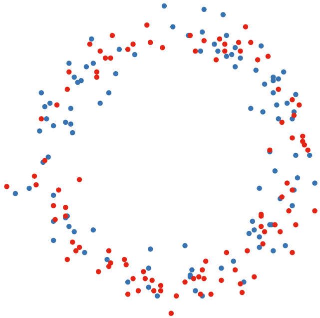

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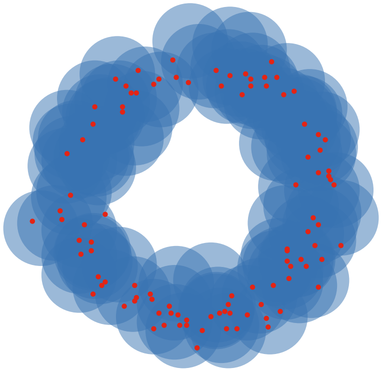

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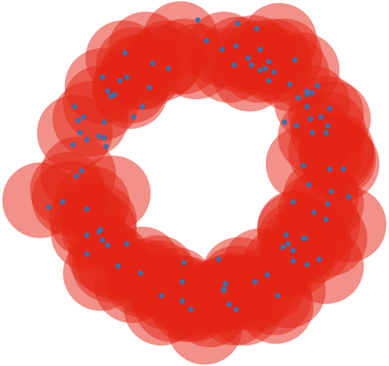

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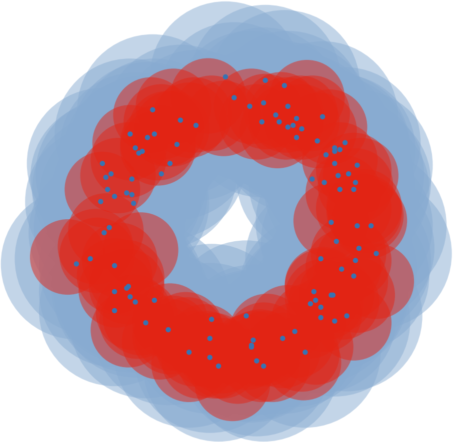

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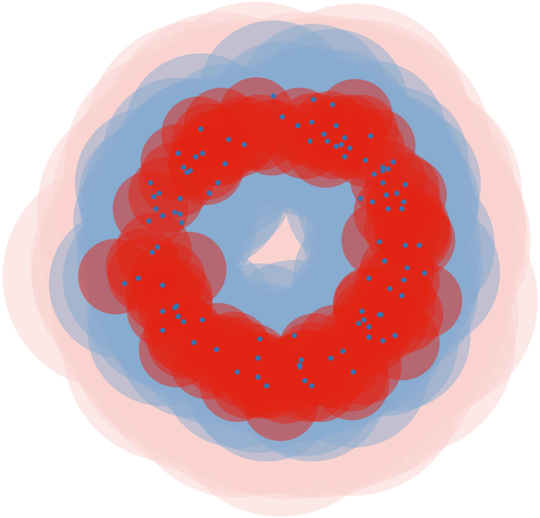

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- morphism and transition maps commute:

$$\begin{tikzcd} {} \arrow[r, dashed] & M_s \arrow[r, "m_s^t"] \arrow[d, "f_s"'] & M_t \arrow[r, dashed] \arrow[d, "f_t"] & {} \\ {} \arrow[r, dashed] & N_s \arrow[r, "n_s^t"'] & N_t \arrow[r, dashed] & {} \end{tikzcd}$$

Interval Persistence Modules

Let $\mathbb{K}$ be a field. For an arbitrary interval $I \subseteq \mathbb{R}$, define the *interval persistence module* $\mathbb{K}(I)$ by

$$\mathbb{K}(I)_t = \begin{cases} \mathbb{K} & \text{if } t \in I, \\ 0 & \text{otherwise,} \end{cases}$$

with transition maps of maximal rank.

Schematic example:

$$\cdots\dashrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{K} \longrightarrow \mathbb{K} \dashrightarrow \mathbb{K} \longrightarrow 0 \longrightarrow 0 \dashrightarrow$$

Barcodes: the structure of persistence modules

Theorem (Krull–Schmidt; Crawley-Boewey 2015)

Let M be a pointwise finite-dimensional persistence module.

Barcodes: the structure of persistence modules

Theorem (Krull–Schmidt; Crawley-Boewey 2015)

Let M be a pointwise finite-dimensional persistence module.
Then M is interval-decomposable:
there exists a unique collection of intervals $B(M)$ such that

$$M \cong \bigoplus_{I \in B(M)} \mathbb{K}(I).$$

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$B(M)$ is called the *barcode* of M.

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- The decomposition itself is not unique.
- This is why we use homology with coefficients in a field.

Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2012; B, Lesnick 2015)

If two persistence modules are δ-interleaved,
then there exists a δ-matching of their barcodes:

- matched intervals have endpoints within distance $\leq \delta$,
- unmatched intervals have length $\leq 2\delta$.

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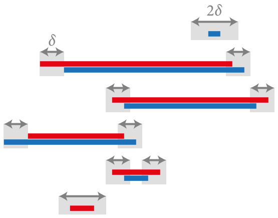
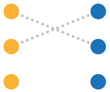


Barcodes as diagrams

The matching category

A *matching* $\sigma : S \nrightarrow T$ is a bijection $S' \to T'$, where $S' \subseteq S$, $T' \subseteq T$.



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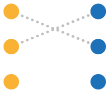
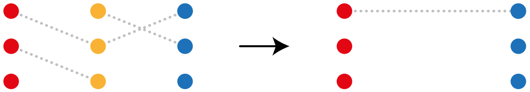


Composition of matchings $\sigma : S \nrightarrow T$ and $\tau : T \nrightarrow U$:



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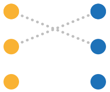

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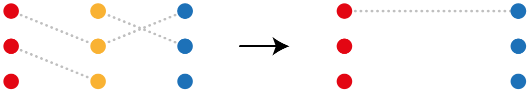
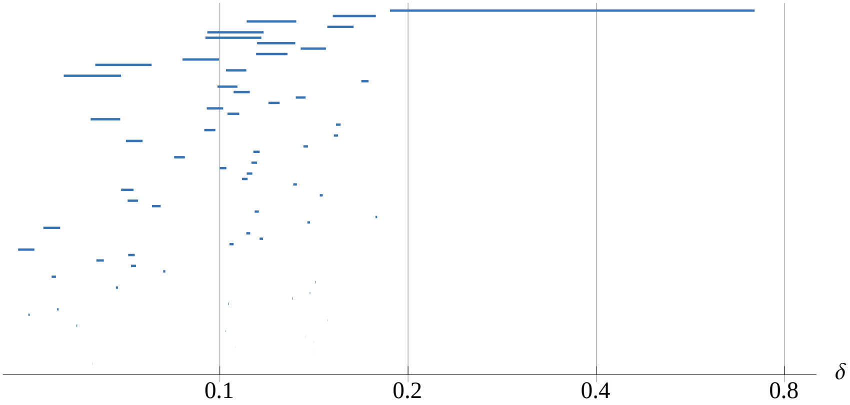


Matchings form a category $\mathbf{Mch}$

- objects: sets
- morphisms: matchings

Barcodes as matching diagrams

We can regard a barcode B as a functor $\mathbf{R} \to \mathbf{Mch}$:



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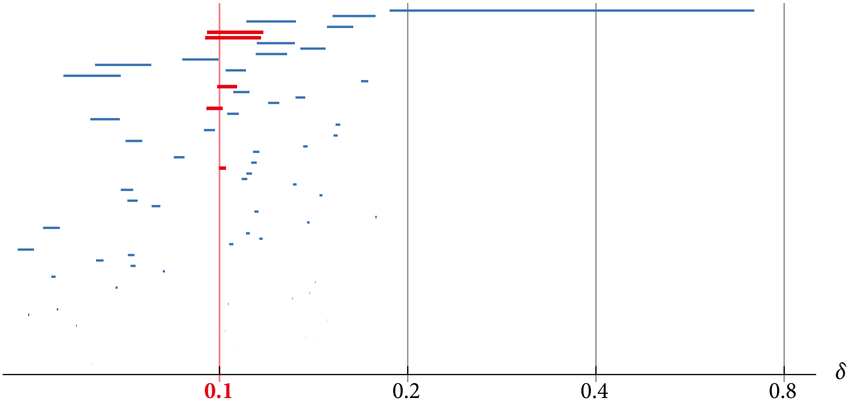
- For each real number t, let B_t be the set of intervals of B that contain t, and



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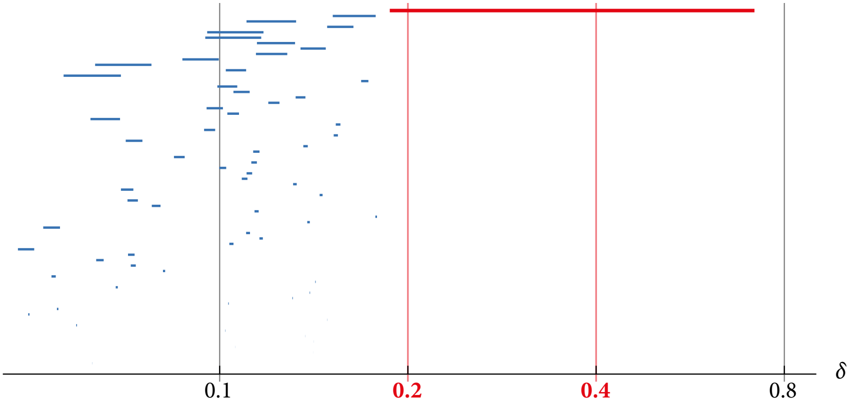
- For each real number t, let B_t be the set of intervals of B that contain t, and
- for each $s \leq t$, define the matching $B_s \nrightarrow B_t$
  to be the identity on $B_s \cap B_t$.



Stability via functoriality?

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B(H_*(F_t)) \arrow[rr] \arrow[dr] & & B(H_*(F_{t+2\delta})) \arrow[dr] & \\
& B(H_*(G_{t+\delta})) \arrow[ur] \arrow[rr] & & B(H_*(G_{t+3\delta}))
\end{tikzcd}$$

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Non-functoriality of the persistence barcode

Theorem (B, Lesnick 2015)

There exists no functor $\mathbf{Vect}^{\mathbf{R}} \to \mathbf{Mch}^{\mathbf{R}}$ *sending each persistence module to its barcode.*

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Proposition

There exists no functor $\mathbf{Vect} \to \mathbf{Mch}$ *sending each vector space of dimension* d *to a set of cardinality* d*.*

Induced barcode matchings

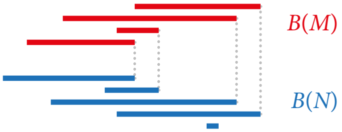
Structure of persistence submodules / quotients

Proposition

Let $f : M \to N$ be a monomorphism of persistence modules: each $f_t : M_t \to N_t$ is injective.

Then f induces an injective map $B(M) \to B(N)$ mapping each $I \in B(M)$ to some $J \in B(N)$ with larger or same left and same right endpoint.

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & M_s \arrow[r] \arrow[d, hook] & M_t \arrow[r, dotted] \arrow[d, hook] & \cdots \\
\cdots \arrow[r, dotted] & N_s \arrow[r] & N_t \arrow[r, dotted] & \cdots
\end{tikzcd}$$



Dually for epimorphisms (left and right exchanged).

Induced matchings

For a general morphism $f : M \to N$ of persistence modules: consider *epi-mono factorization*

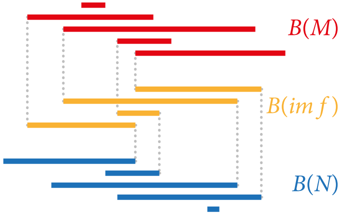
$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N.$$

Induced matchings

For a general morphism $f : M \to N$ of persistence modules: consider *epi-mono factorization*

$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N.$$

- $\operatorname{im} f \hookrightarrow N$ induces injection $B(\operatorname{im} f) \hookrightarrow B(N)$
- $M \twoheadrightarrow \operatorname{im} f$ induces injection $B(\operatorname{im} f) \hookrightarrow B(M)$
- compose to a matching $B(M) \nrightarrow B(N)$:



Stability from interleavings

Consider interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(\forall t \in \mathbb{R})$:

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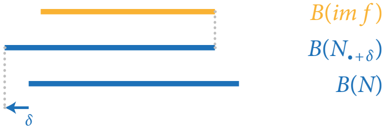

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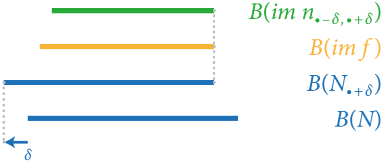

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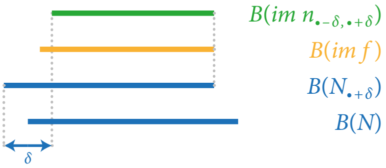

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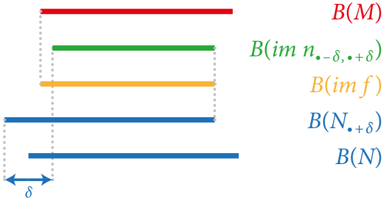

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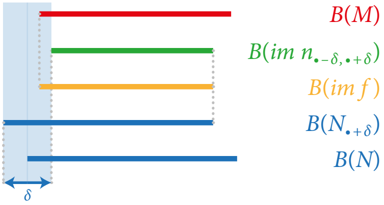

Computation

Demo: Ripser

Example data set:

- 192 points on $\mathbb{S}^2$
- persistent homology barcodes up to dimension 2
- over 56 mio. simplices in 3-skeleton

Some previous software:

- javaplex (Stanford): 3200 seconds, 12 GB
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Ripser: 1.2 seconds, 160 MB

Ripser

A software for computing Vietoris–Rips persistence barcodes

- around 1000 lines of C++ code, no external dependencies
- support for
  - coefficients in a prime field $\mathbb{Z}/p\mathbb{Z}$
  - sparse distance matrices (for distance threshold)
- open source (`http://ripser.org`)
  - released in July 2016
- online version (`http://live.ripser.org`)
  - launched in August 2016
- (co-)winner of 2016 ATMCS Best New Software Award

Computing homology

Computing homology $H_* = Z_*/B_*$ (recall: $B_* \subseteq Z_* \subseteq C_*$):

- compute basis for boundaries $B_* = \operatorname{im} \partial_*$
- extend to basis for cycles $Z_* = \ker \partial_*$
- new (non-boundary) basis cycles generate quotient Z_*/B_*

Homology by matrix reduction

Notation:

- D: boundary matrix (with $\mathbb{Z}_2$ coefficients)
- R_i: ith column of R

Matrix reduction algorithm (variant of Gaussian elimination):

- $R = D, V = I$
- while $\exists i < j$ with $\operatorname{pivot} R_i = \operatorname{pivot} R_j$
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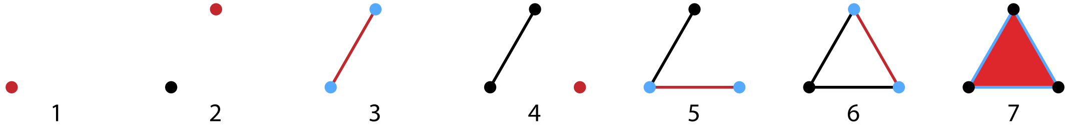
Result:

- $R = D \cdot V$ is reduced (each column has a unique pivot)
- V is full rank upper triangular

Matrix reduction



$$\underbrace{\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & & & 1 & & 1 & & \\ 2 & & & 1 & & & 1 & \\ 3 & & & & & & & 1 \\ 4 & & & & & 1 & 1 & \\ 5 & & & & & & & 1 \\ 6 & & & & & & & 1 \\ 7 & & & & & & & \end{array}}_{R} = D \cdot \underbrace{\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & & & & & & \\ 2 & & 1 & & & & & \\ 3 & & & 1 & & & & \\ 4 & & & & 1 & & & \\ 5 & & & & & 1 & & \\ 6 & & & & & & 1 & \\ 7 & & & & & & & 1 \end{array}}_{V}$$

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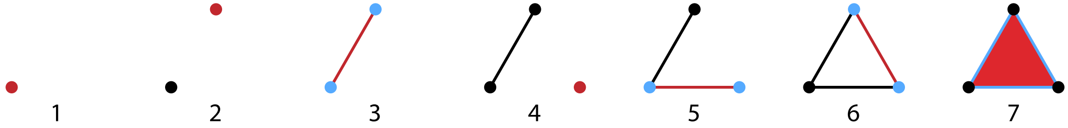
Matrix reduction



	1	2	3	4	5	6	7
1			1		1		
2			1			1	
3							1
4					1	1	
5							1
6							1
7							

R

$= D \cdot$

	1	2	3	4	5	6	7
1	1						
2		1					
3			1				
4				1			
5					1		
6						1	
7							1

V

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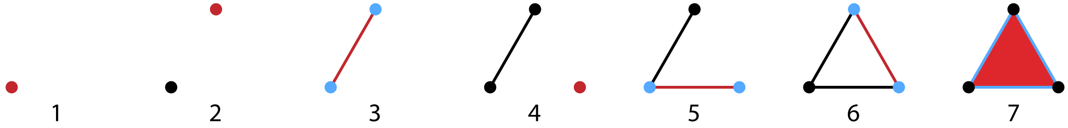
Matrix reduction



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1			1		1		
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3							1
4					1	1	
5							1
6							1
7							

R

$= D \cdot$

	1	2	3	4	5	6	7
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2		1					
3			1				
4				1			
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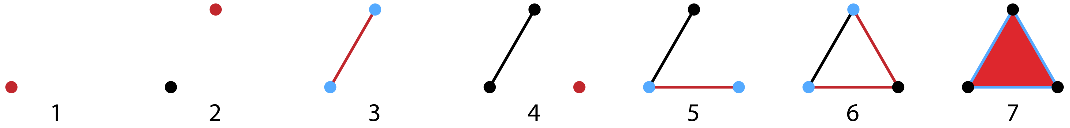
Matrix reduction



$$
\underbrace{\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & & & 1 & & 1 & \color{red}{1} & \\
2 & & & 1 & & & 1 & \\
3 & & & & & & & 1 \\
4 & & & & & 1 & \color{red}{0} & \\
5 & & & & & & & 1 \\
6 & & & & & & & 1 \\
7 & & & & & & &
\end{array}}_{R}
= D \cdot
\underbrace{\begin{array}{c|ccccccc}
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1 & 1 & & & & & & \\
2 & & 1 & & & & & \\
3 & & & 1 & & & & \\
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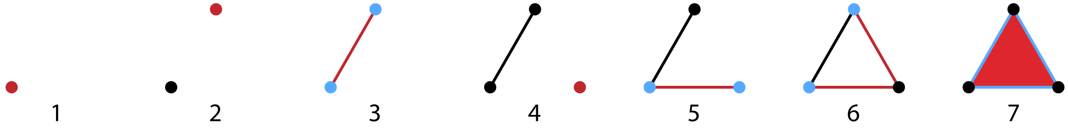
Matrix reduction



	1	2	3	4	5	6	7
1			1		1	1	
2			1			1	
3							1
4					1		
5							1
6							1
7							

R

$= D \cdot$

	1	2	3	4	5	6	7
1	1						
2		1					
3			1				
4				1			
5					1	1	
6						1	
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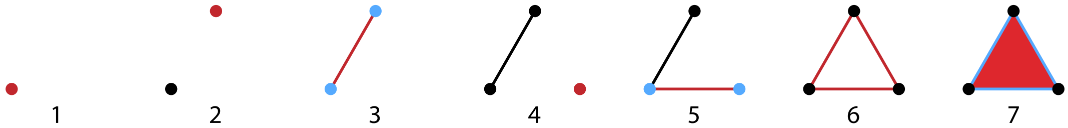

$$\underbrace{\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline
1 & & & 1 & & 1 & 0 & \\
2 & & & 1 & & & 0 & \\
3 & & & & & & & 1 \\
4 & & & & & 1 & & \\
5 & & & & & & & 1 \\
6 & & & & & & & 1 \\
7 & & & & & & &
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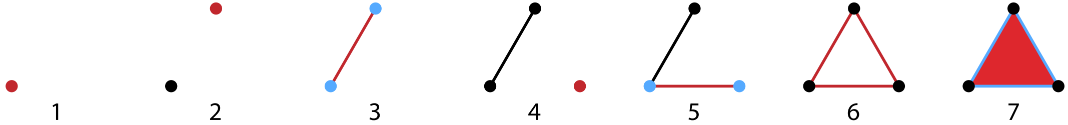

	1	2	3	4	5	6	7
1			1		1		
2			1				
3							1
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7							

R

$= D \cdot$

	1	2	3	4	5	6	7
1	1						
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3			1			1	
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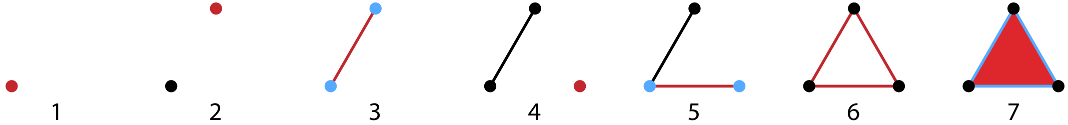

R

	1	2	3	4	5	6	7
1			1		1		
2			1				
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4					1		
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6							1
7							

$= D \cdot$

V

	1	2	3	4	5	6	7
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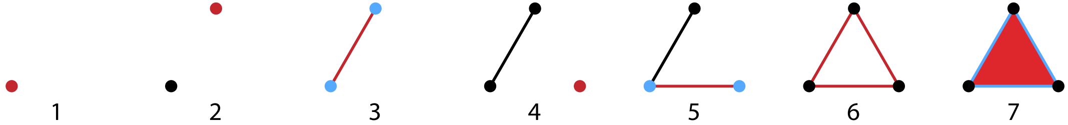

R:

	1	2	3	4	5	6	7
1			1		1		
2			1				
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4					1		
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6							1
7							

$= D \cdot$

V:

	1	2	3	4	5	6	7
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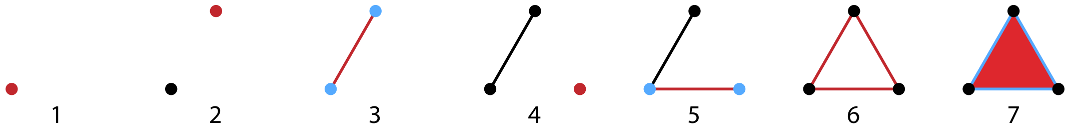

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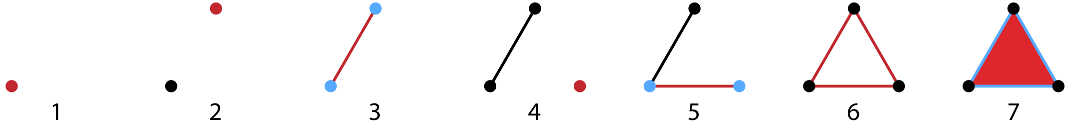

	1	2	3	4	5	6	7
1			1		1		
2			1				
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R

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	1	2	3	4	5	6	7
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The four special ingredients of Ripser

The improved performance of Ripser is based on 4 insights:

- Compute cohomology [de Silva et al. 2011]
  - reduce transposed matrix
- Skip inessential columns [Chen, Kerber 2011]
  - many columns are redundant for homology
- Implicit boundary matrix
  - don't store the matrix $R = D \cdot V$ in memory
- Apparent pairs
  - find column pivot without constructing entire column

Sending persistence into Hilbert space

Extending the TDA pipeline

Mapping barcodes into a Hilbert space?

- desirable for (kernel-based) machine learning methods and statistics
- stability (Lipschitz continuity): important for reliable predictions
- inverse stability (bi-Lipschitz): avoid loss of information

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Can we hope for something better?

No bi-Lipschitz feature maps for persistence

Theorem (B, Carrière 2018)

There is no bi-Lipschitz map from the persistence diagrams
(with the interleaving or any p–Wasserstein distance)
into any finite-dimensional Hilbert space,
even when restricting to bounded range or number of bars.

No bi-Lipschitz feature maps for persistence

Theorem (B, Carrière 2018)

There is no bi-Lipschitz map from the persistence diagrams
(with the interleaving or any $p-$Wasserstein distance)
into any finite-dimensional Hilbert space,
even when restricting to bounded range or number of bars.

Theorem (B, Carrière 2018)

If there was such a bi-Lipschitz map into some Hilbert space,
the ratio of the Lipschitz constants would have to go to ∞
together with the bounds on number or range of bars.

History

When was persistent homology invented?

- [Edelsbrunner/Letscher/Zomorodian 2000]

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- [Robbins 1999]
- [Frosini 1990]
- [Leray 1946]?

When was persistent homology invented first?

When was persistent homology invented first?

ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the

When was persistent homology invented first?

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Rank and span in functional topology

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Exact homomorphism sequences in homology theory
ed.ac.uk **[PDF]**

JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR

The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] **...**

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Marston Morse and his mathematical works
ams.org **[PDF]**

R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org

American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to **...**

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Unstable minimal surfaces of higher topological structure

When was persistent homology invented first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

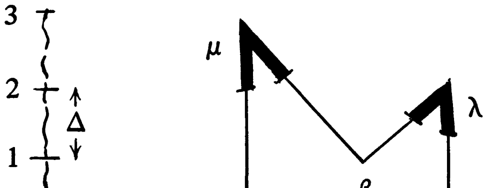
1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

When was persistent homology invented first?

inequalities pertain between the dimensions of the A_i and those of $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels [D] in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers [36] and [44] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.

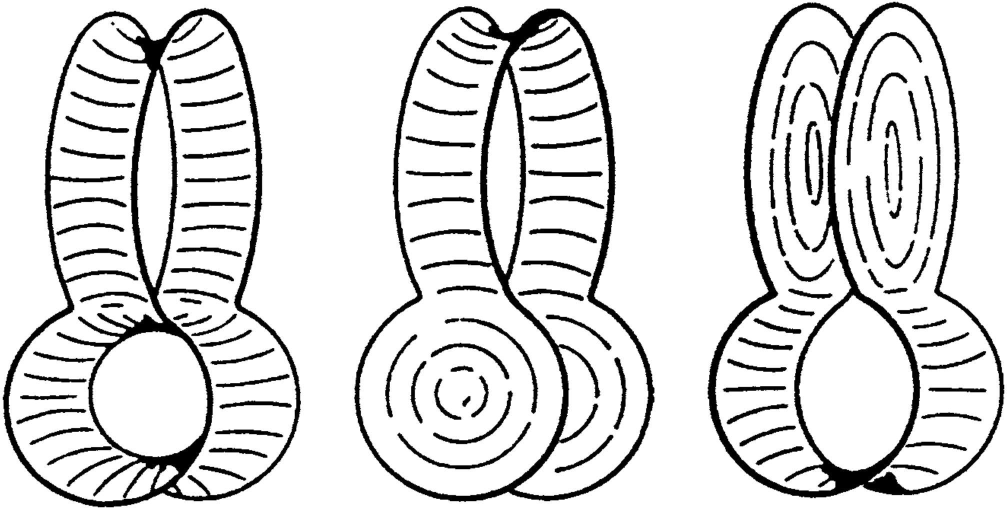


Morse's *functional topology*

Key aspects:

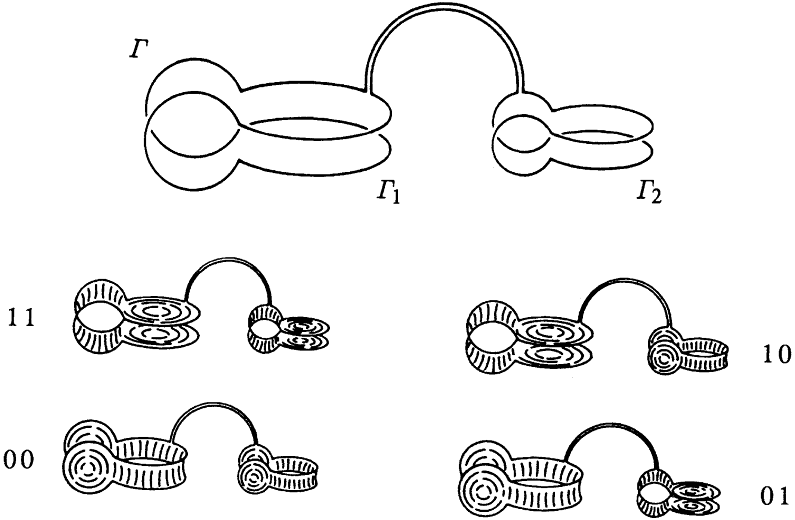
- early precursor of persistence and spectral sequences
- uses Vietoris homology with field coefficients
- applies to a broad class of functions on metric spaces (not necessarily continuous)
- inclusions of sublevel sets have finite rank homology (*q-tame* persistent homology)
- focus on controlled behavior in pathological cases

Motivation and application: minimal surfaces



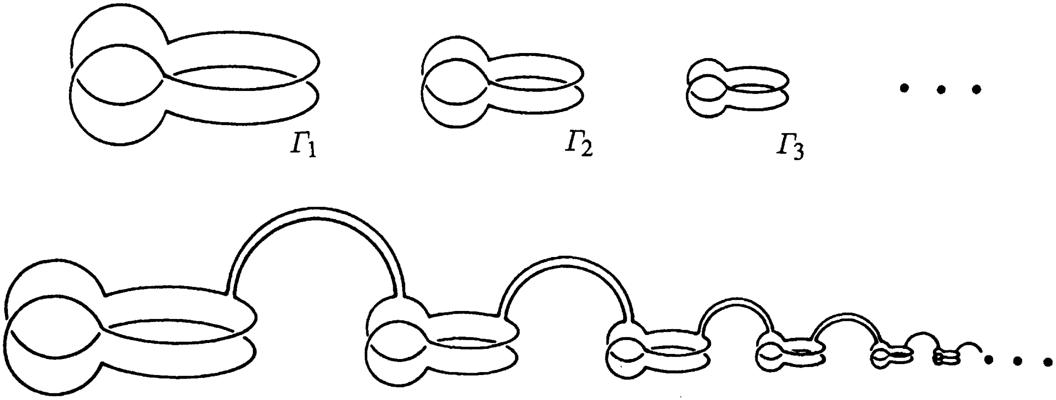
(a) (b) (c)

Motivation and application: minimal surfaces



(from Dierkes et al.: Minimal Surfaces, Springer 2010)

Motivation and application: minimal surfaces



(from Dierkes et al.: Minimal Surfaces, Springer 2010)

Existence of unstable minimal surfaces

Using persistent homology:

- Number of ϵ-persistent critical points (minimal surfaces) is finite for any $\epsilon > 0$
- Morse inequalities for ϵ-persistent critical points

Theorem (Morse, Tompkins 1939)

There is a C_1 curve bounding an unstable minimal surface (an index 1 critical point of the area functional).

Thanks for your attention!

Thanks for your attention!