Topological simplification of functions on surfaces

Ulrich Bauer[1] Carsten Lange[2] Max Wardetzky[1]

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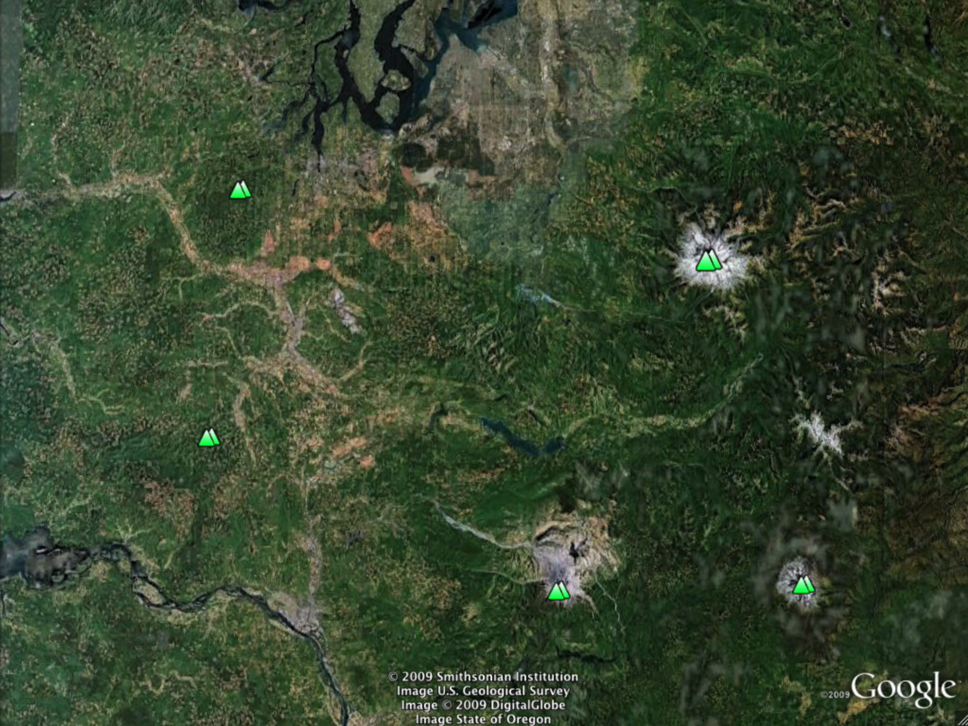

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Goal

Given a function f on a surface and $\delta > 0$, find a function f_δ that:

- minimizes number of critical points
- stays close to input function: $\|f - f_\delta\|_\infty \leq \delta$

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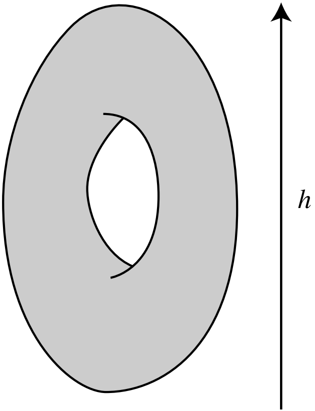
Using:

- Discrete Morse theory [Forman 1998]
  - provides notion of critical point in the discrete setting
- Homological persistence [Edelsbrunner et al. 2002]
  - quantifies significance of critical points

Morse theory at a glance



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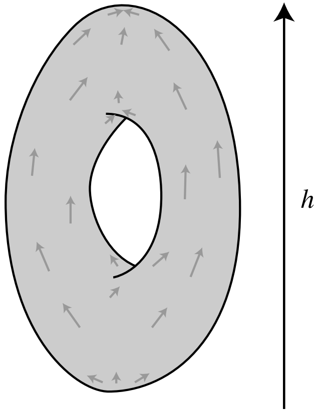

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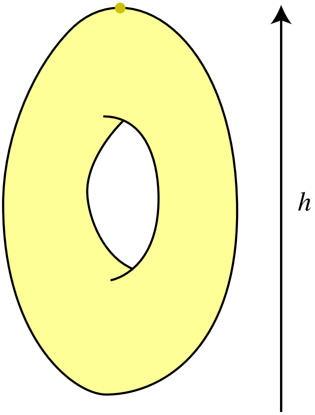

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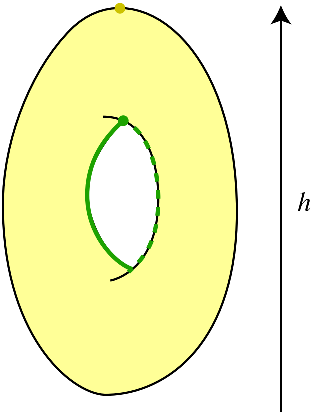

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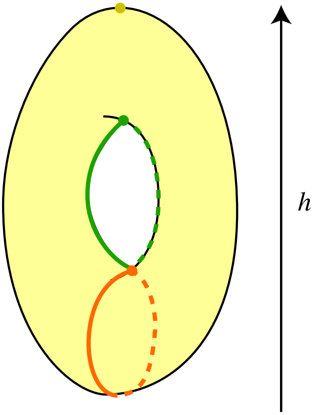

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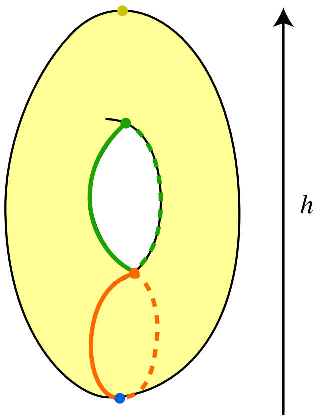

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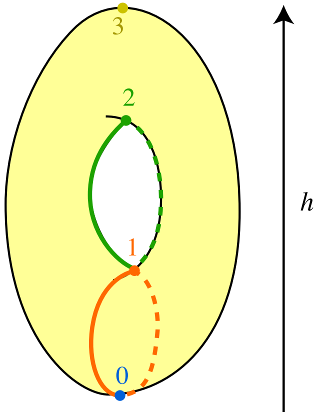

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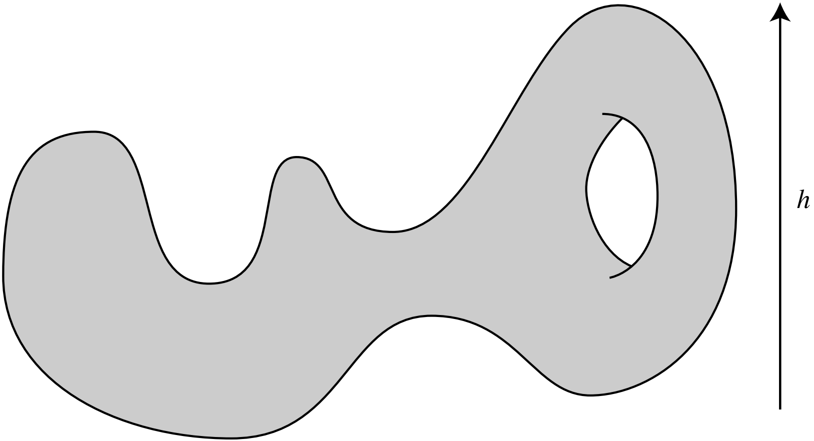

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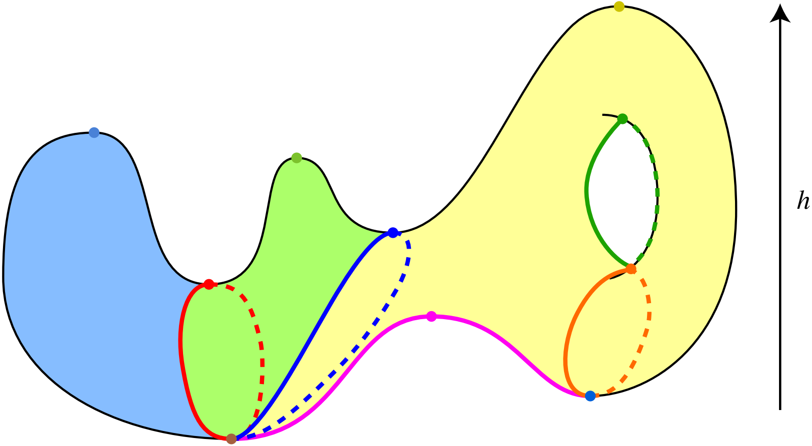

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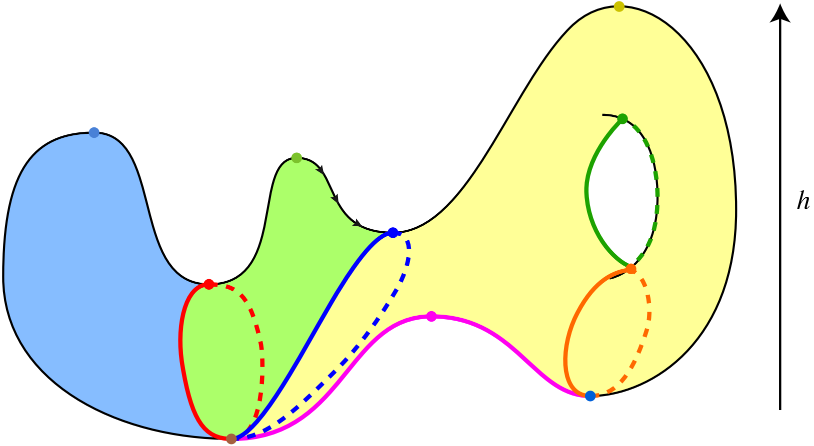

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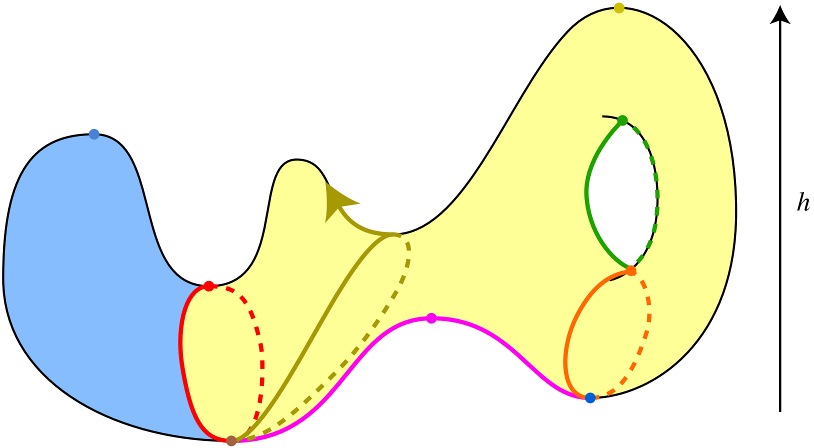

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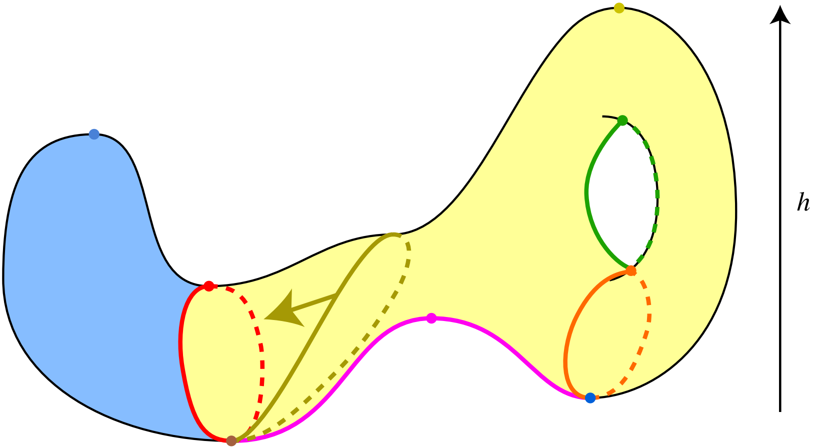

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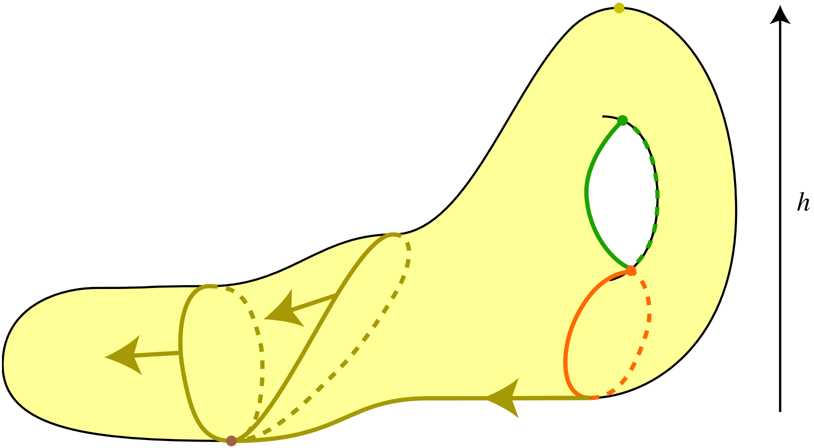
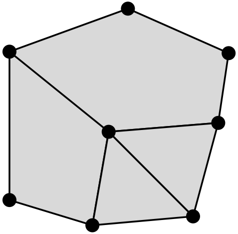


Discrete Morse theory [Forman, 1998]

Finite CW complex $\mathcal{K}$



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- Discrete vector field:
  - *pairs* of cells (σ, τ) (arrows), σ is a regular facet of τ
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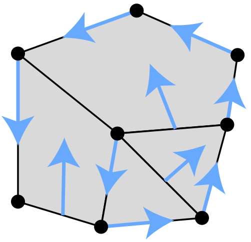

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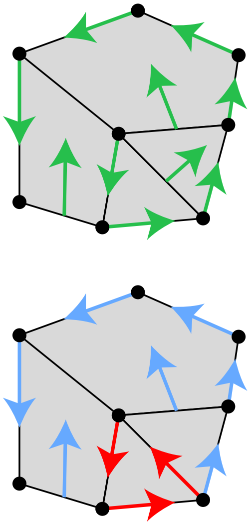

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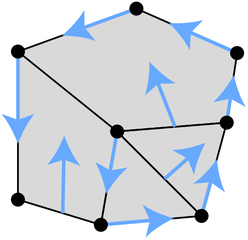

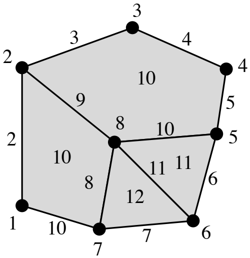

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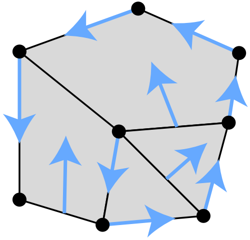

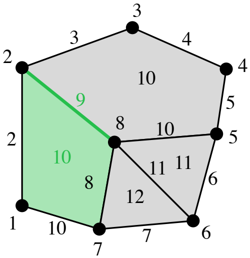

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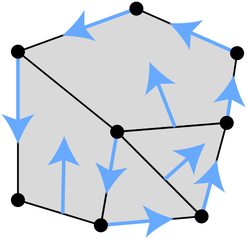
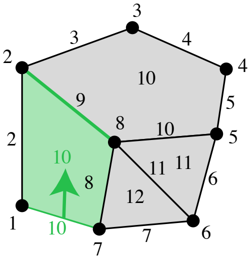
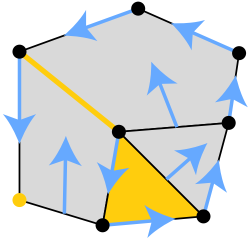
- Discrete Morse function:
  - facets have lower value…
    except when there is an arrow





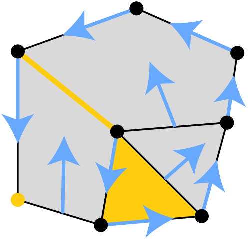
Discrete Morse theory [Forman, 1998]

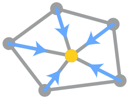
- Critical cell:
  - not contained in any pair



Discrete Morse theory [Forman, 1998]

- Critical cell:
  - not contained in any pair
  - Examples: minimum





Discrete Morse theory [Forman, 1998]

- Critical cell:
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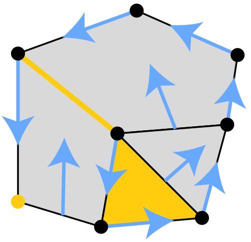
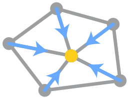
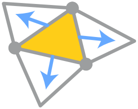
  - Examples: minimum, maximum

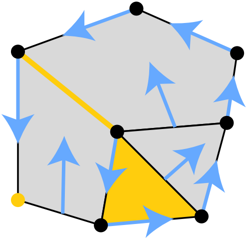


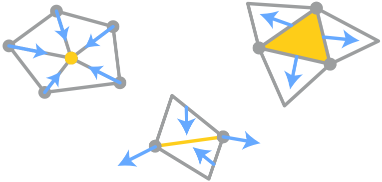




Discrete Morse theory [Forman, 1998]

- Critical cell:
  - not contained in any pair
  - Examples: minimum, maximum, saddle





Discrete Morse theory [Forman, 1998]

- Critical cell:
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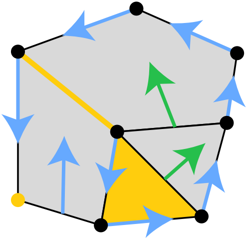
- Cancellation of critical cells:
  - Prerequisite: unique path between two critical cells



Discrete Morse theory [Forman, 1998]

- Critical cell:
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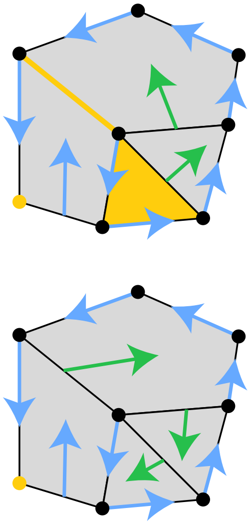
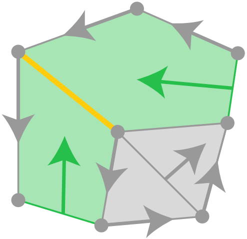
  - Prerequisite: unique path between two critical cells
  - Reversing vector field along path cancels critical cells



Attracting and repelling sets

A gradient vector field V enforces inequalities on cells

- Attracting set of a (critical) cell σ:
  all cells ρ with $g(\rho) \geq g(\sigma)$ for *any* g consistent with V



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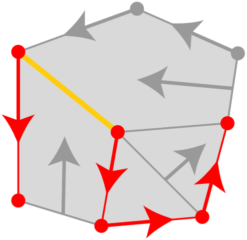
- Repelling set: analogously for $g(\rho) \leq g(\sigma)$



Comparing Discrete and PL Morse theory

Discrete Morse theory is often a more convenient and efficient language

- critical points are always non-degenerate (no multiple saddles)
- clear and concise notion of vector fields, gradient paths, attracting/repelling sets
- captures the combinatorial essence of Morse theory (instead of only discretizing)

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We can convert one type of function to the other

Back to our problem

Aim: Cancel critical points from pseudo-Morse function

To do: Cancelation requires three steps:

- Identify pairs for cancellation (which pairs?)
- Reverse gradient vector field
- Make function consistent to new vector field (how?)

Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for growing spaces

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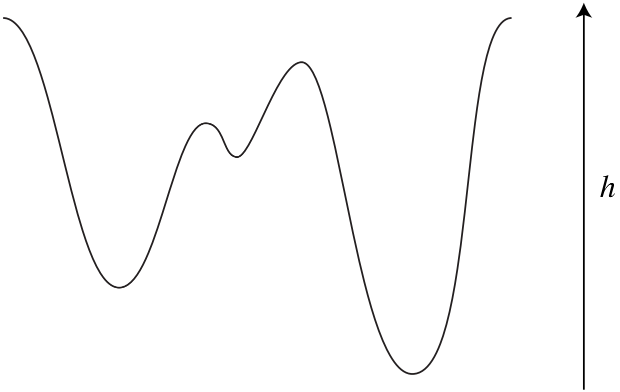
Investigate change of homology for growing spaces

Example: connected components of sublevel sets



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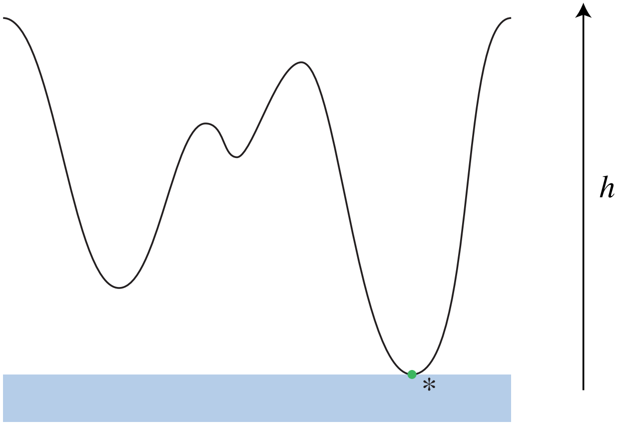

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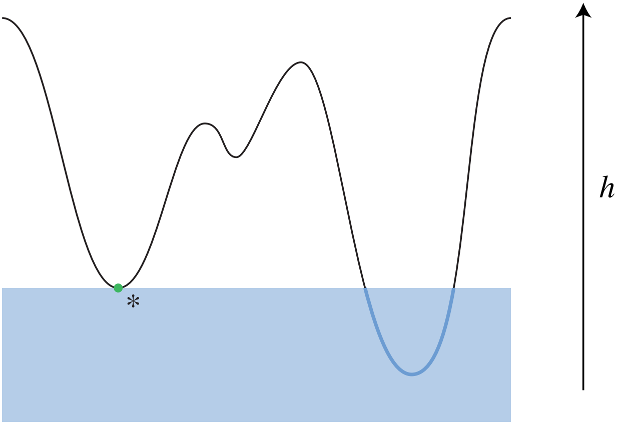

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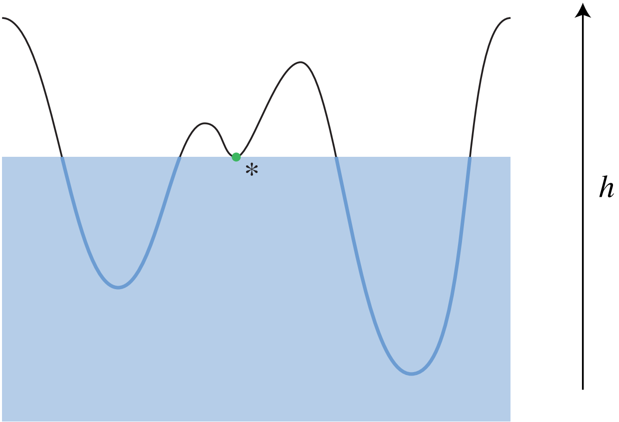

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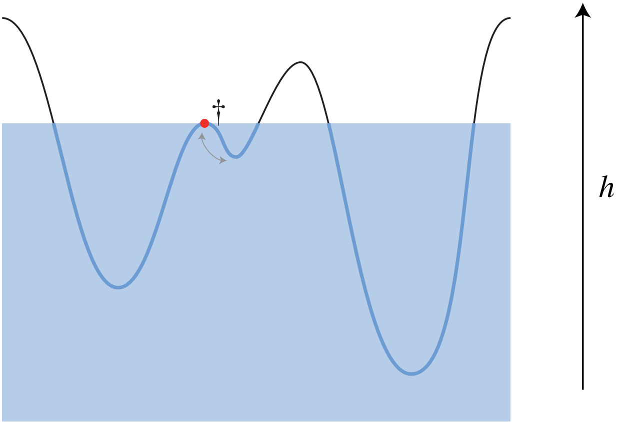

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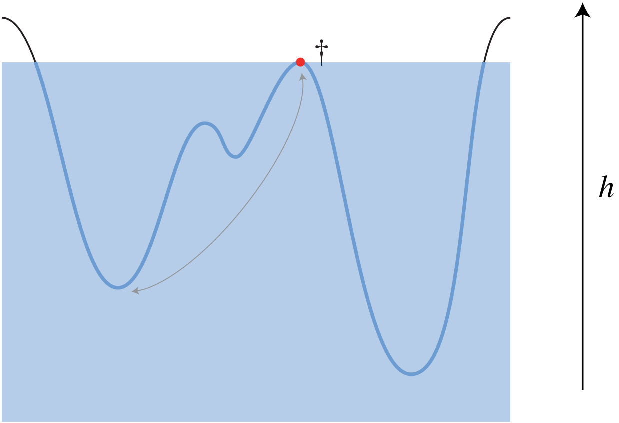

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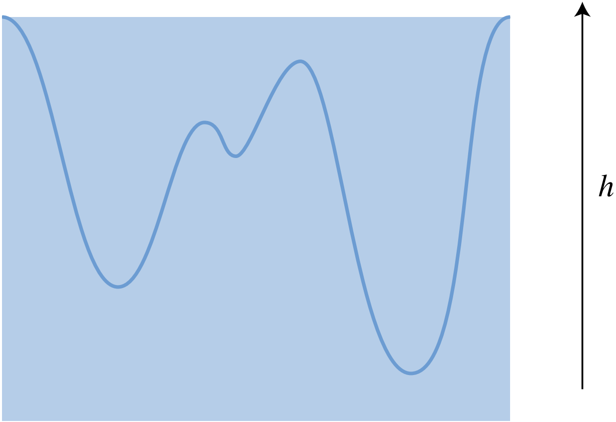

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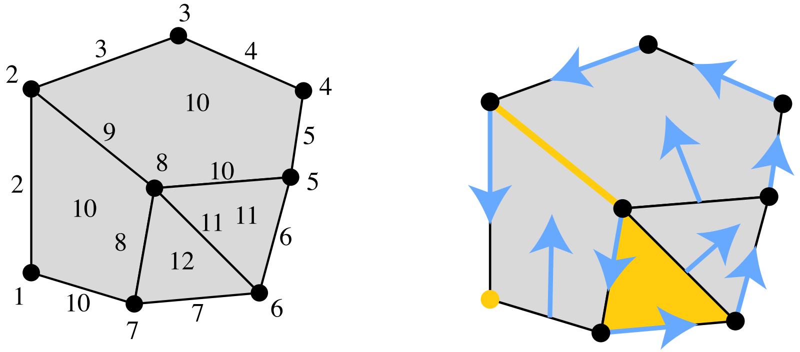

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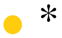

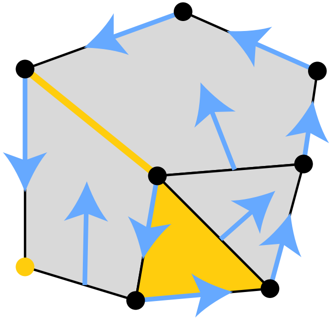

birth (connected component created)

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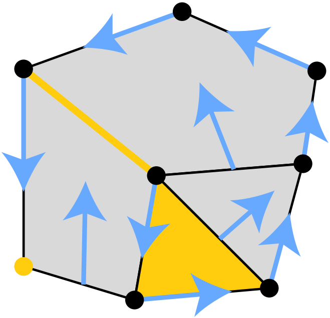

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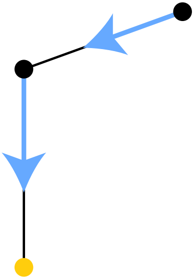

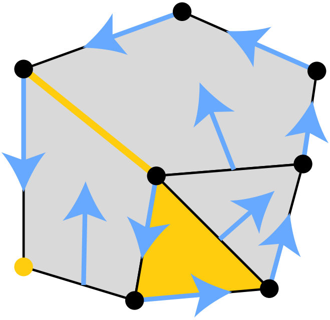

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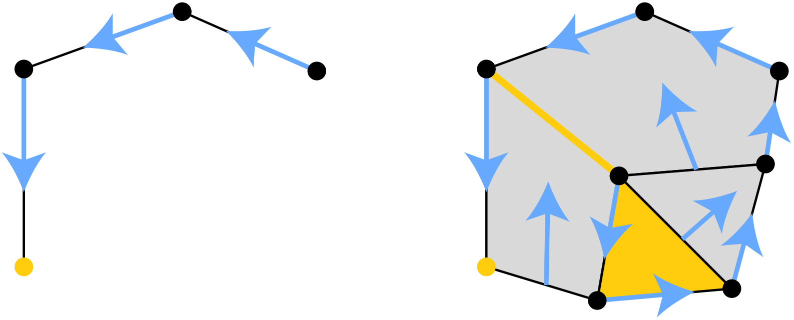

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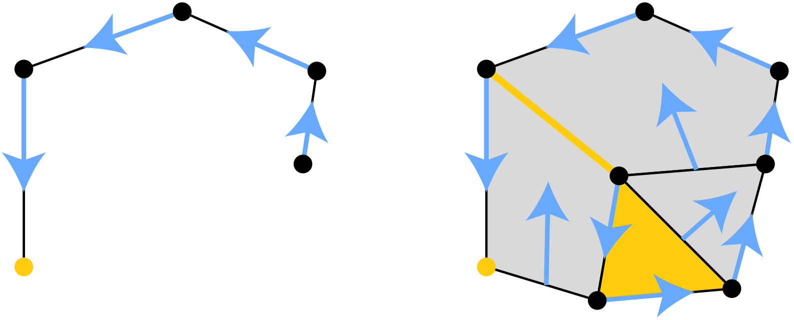

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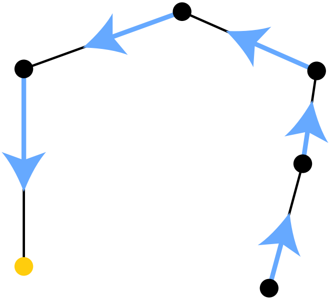

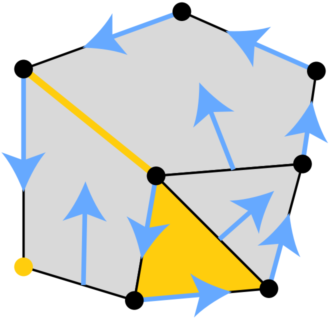

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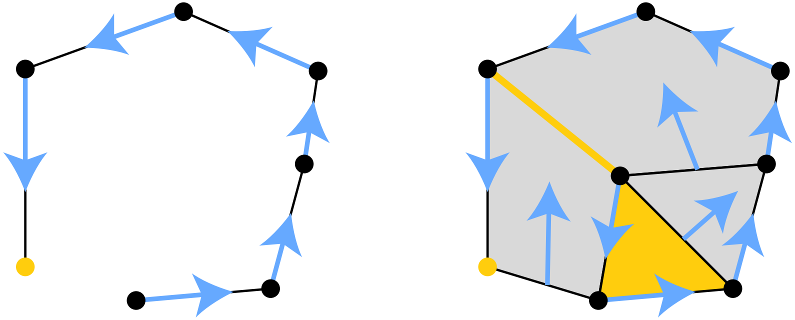

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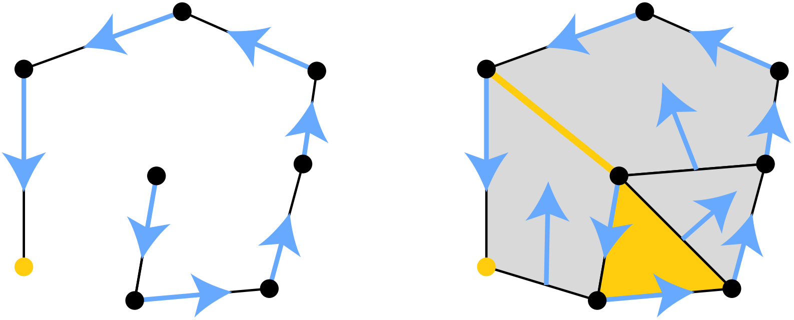

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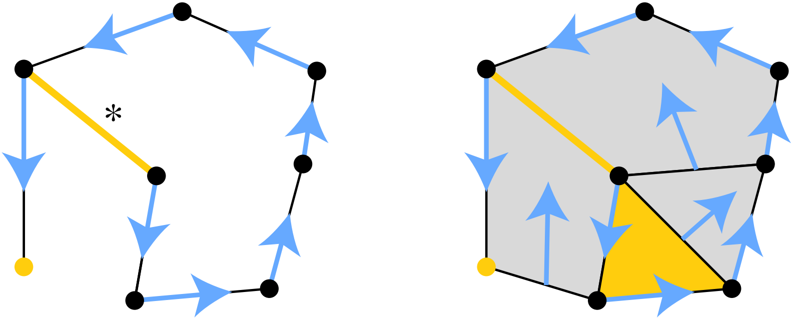

birth (1-cycle created)

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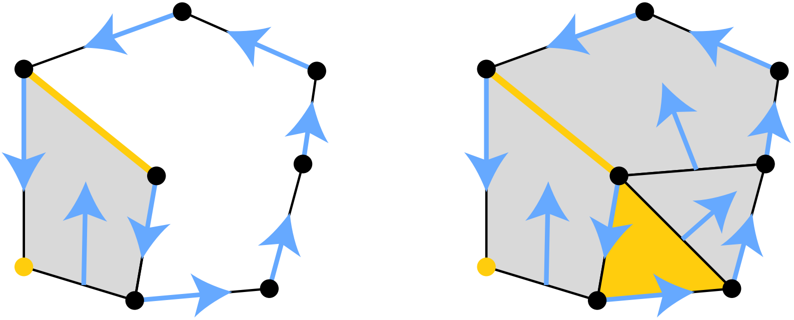

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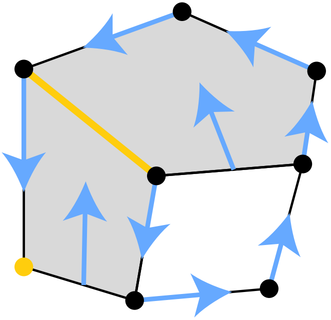

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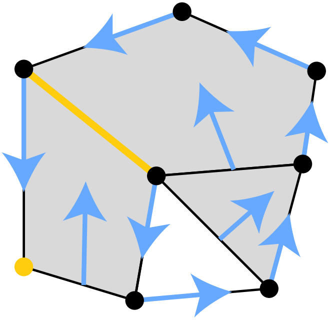

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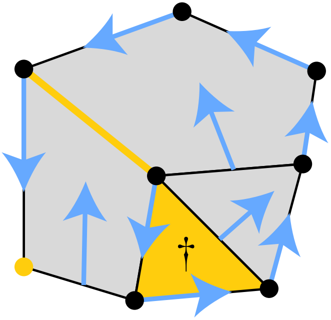
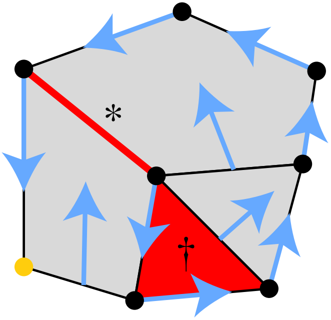
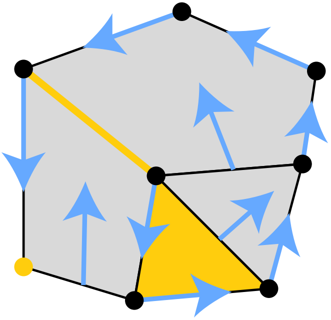


death (1-cycle killed)

Example: level subcomplexes





Persistence pair: homology *born* at value 9 *dies* at 12

The lifespan is called *persistence* of the pair

Connecting persistence and Morse theory

How do the two theories fit together?

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Consider a discrete Morse function on a surface. Assume that all persistence pairs $(\tilde{\sigma}, \tilde{\tau})$ *with* $\sigma \prec \tilde{\sigma} \prec \tilde{\tau} \prec \tau$ *have been canceled. Then* (σ, τ) *can be canceled too.*

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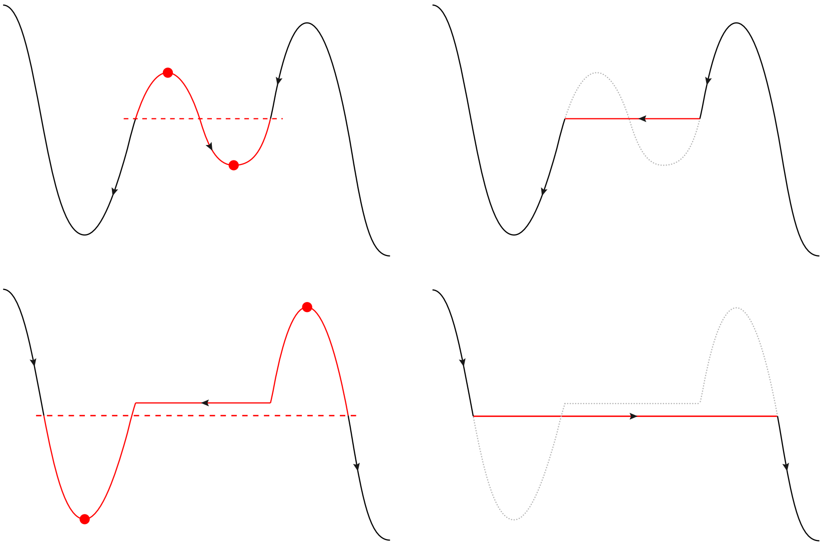
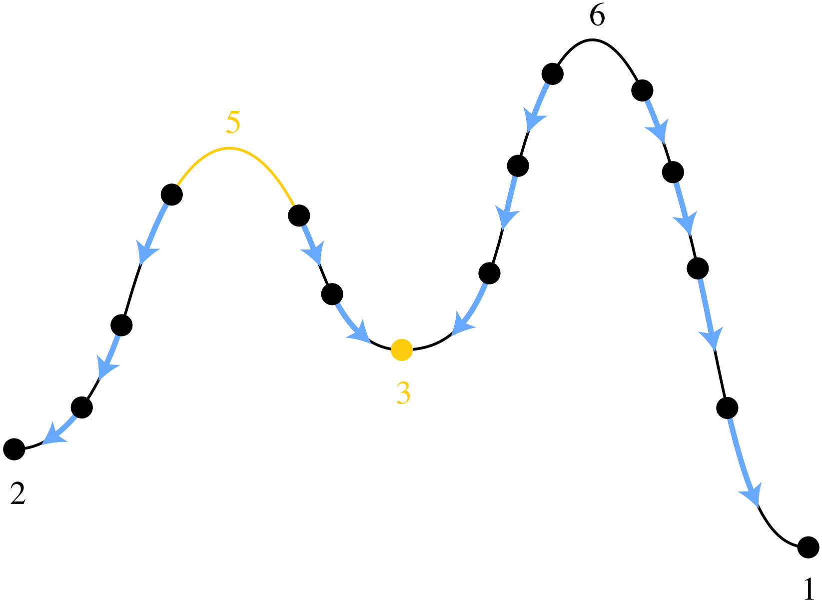
Next: how can we change the function accordingly?

Canceling two nested persistence pairs



Canceling a persistence pair



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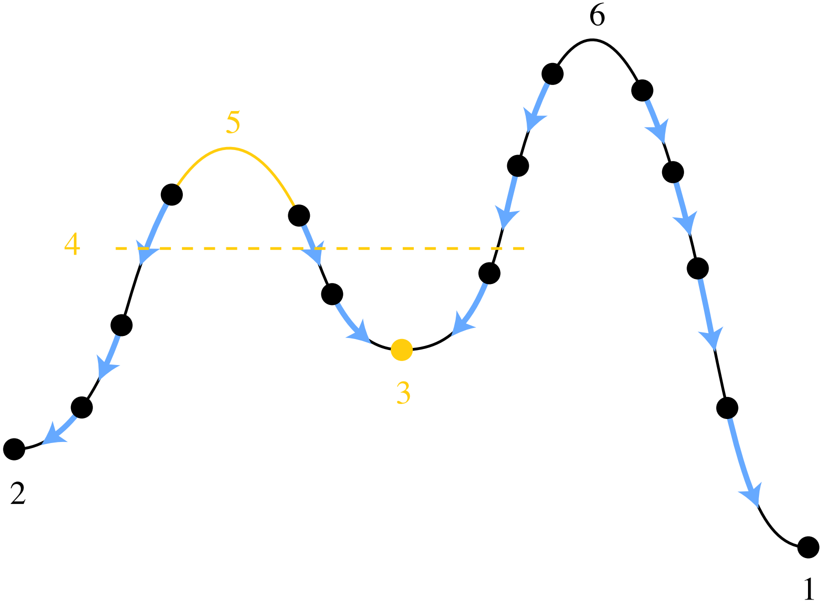

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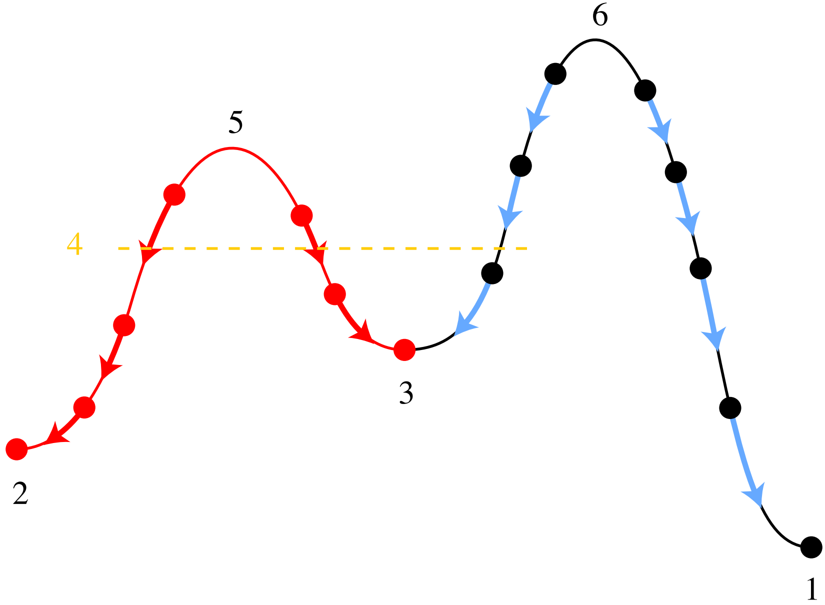

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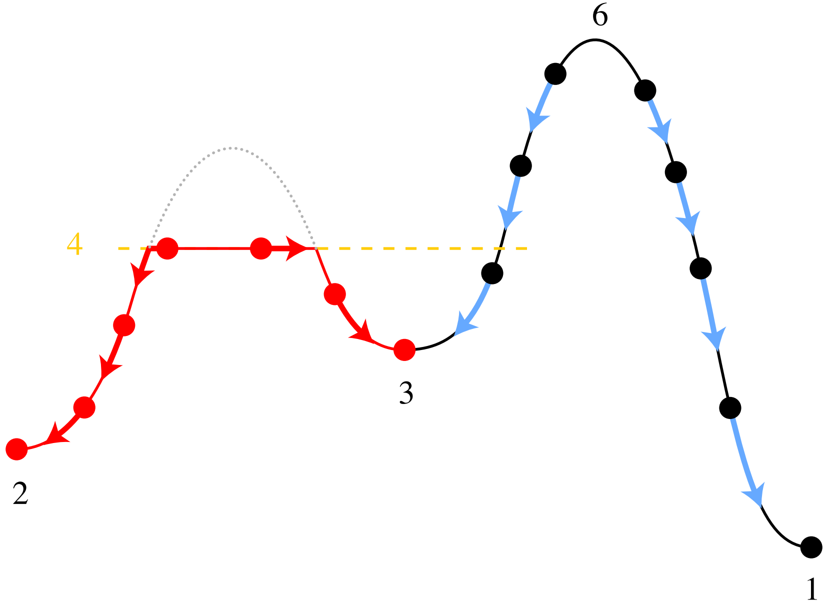
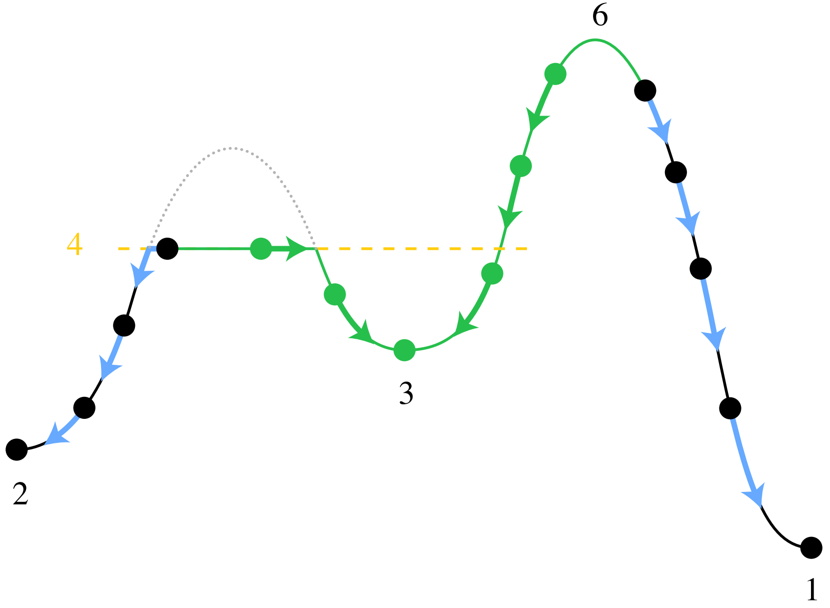


Canceling a persistence pair



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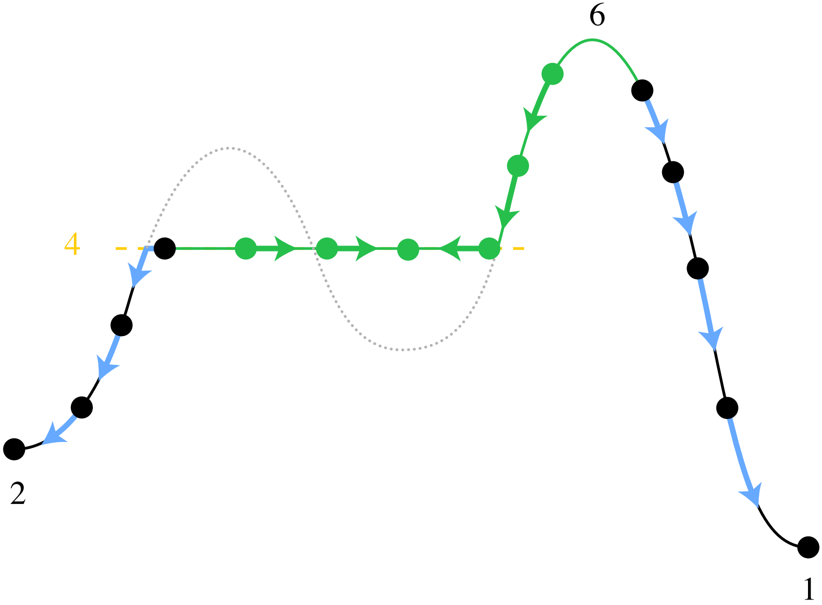
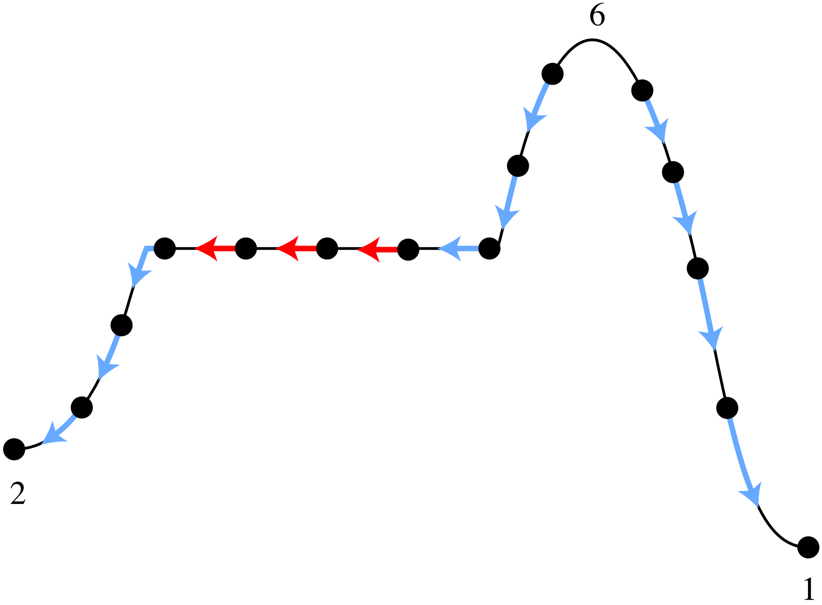


Canceling a persistence pair



Main result

Theorem

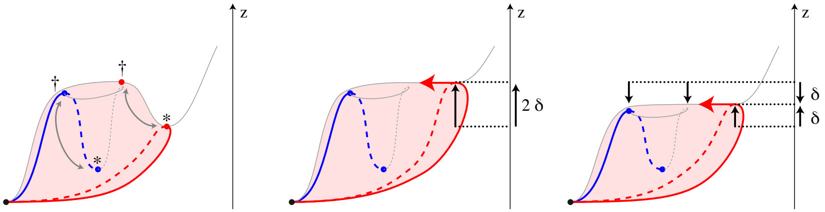
Let f be a discrete Morse function on a surface and let $\delta > 0$.

Then there exists a function f_δ with:

- $\|f - f_\delta\|_\infty \leq \delta$.
- *All persistence pairs of f with persistence $\leq 2\delta$ are canceled.*

This function achieves the minimal number of critical points.

Comparison with other methods



Original function

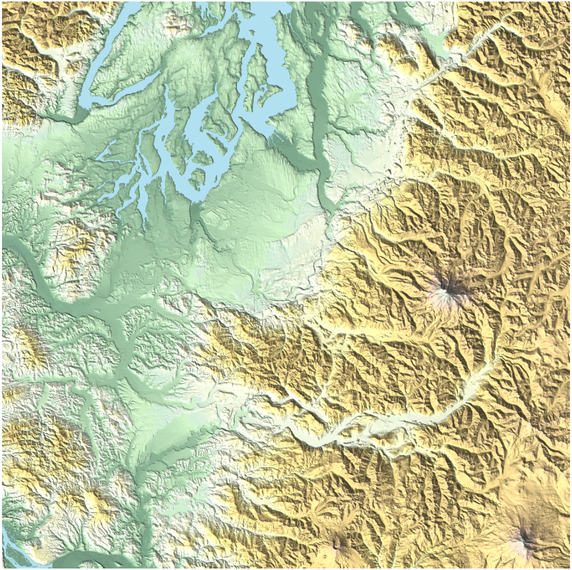
after [Edelsbrunner et al. 2006, Attali et al. 2009]

our method

An efficient algorithm

- Computing persistence pairs: $O(\mathrm{sort}(n))$ (Kruskal's algorithm)
- Computing simplified gradient vector field: $O(n)$ (graph traversal)
- Computing simplified function: $O(n)$ (topological sort)

Example: Simplification of terrain



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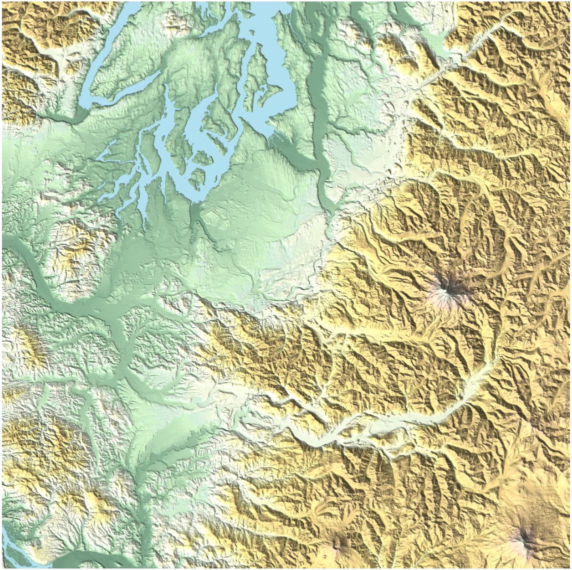
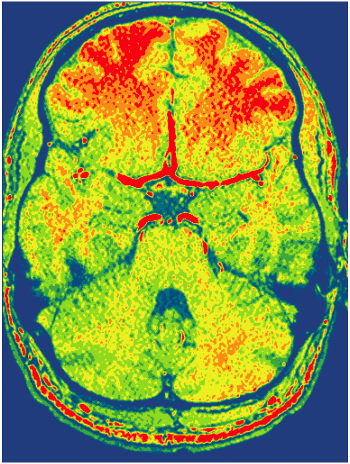
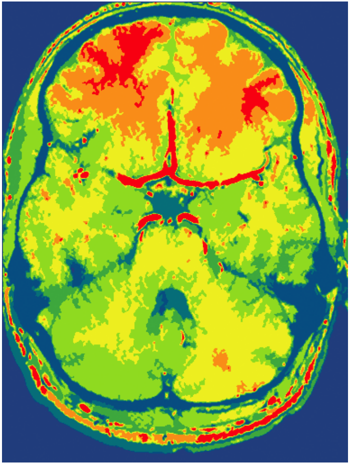


Example: Medical images





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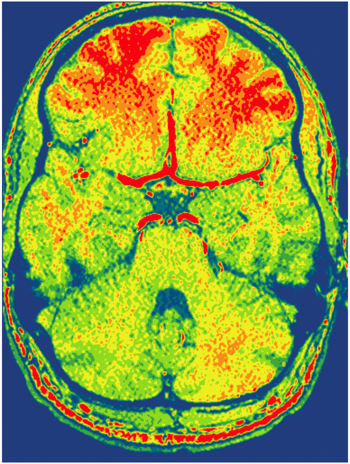

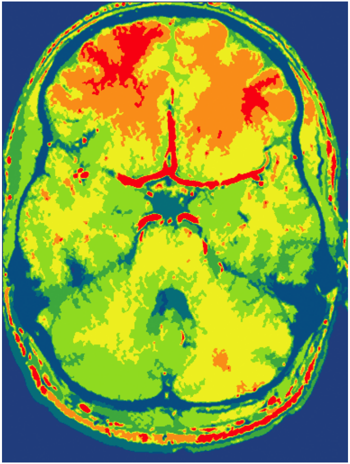

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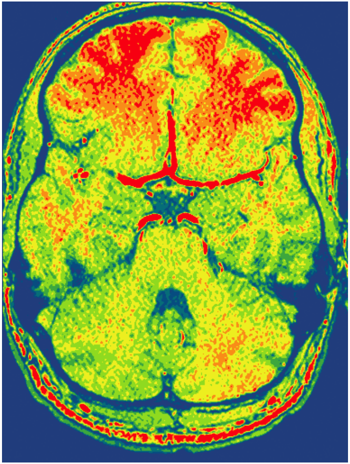 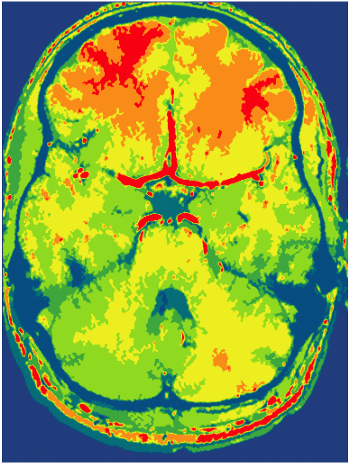

...thanks for your attention!