PERSISTENT MATCHMAKING

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TUM

WORKSHOP ON TOPOLOGY

IDENTIFYING ORDER IN COMPLEX SYSTEMS

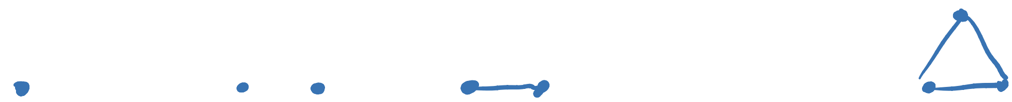
MAR 5, 2021

RIPSER: COMPUTING VIETORIS-RIPS PERSISTENCE

new version released last week (v 1.2)

- roughly 2x faster / less memory
- handles H^1 of a data set (COVID genomes) with 93k points (136 trillion simplices)

Persistent Homology



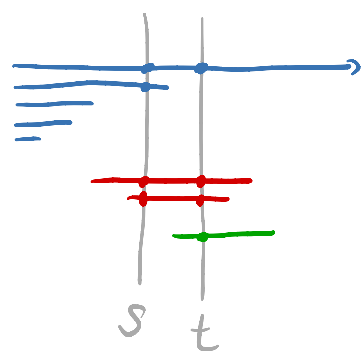
Filtration $K_\bullet$: $K_1 \hookrightarrow K_2 \hookrightarrow K_3 \hookrightarrow \cdots \hookrightarrow K_s \hookrightarrow \cdots$

Homology H_*

Persistence module $H_*(K_\bullet)$: $H_*(K_1) \to \cdots \quad \cdots \to H_*(K_s) \to \cdots$

Barcode $\mathcal{B}$

Matching diagram



INDUCED MATCHINGS

Two filtrations:

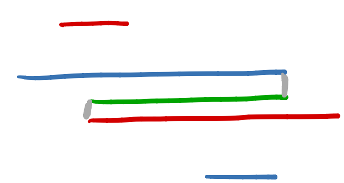
$$
\begin{tikzcd}
\cdots \arrow[rr] & & K_t \arrow[rr] \arrow[dr] & & K_{t+2\delta} \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[r] & L_{t-\delta} \arrow[ur] \arrow[rr] & & L_{t+\delta} \arrow[ur] \arrow[r] & \cdots & {}
\end{tikzcd}
$$

Interleaving of persistent homology:

$$
\begin{tikzcd}
\cdots \arrow[rr] & & H_*(K_t) \arrow[rr] \arrow[dr] & & H_*(K_{t+2\delta}) \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[r] & H_*(L_{t-\delta}) \arrow[ur] \arrow[rr] & & H_*(L_{t+\delta}) \arrow[ur] \arrow[r] & \cdots & {}
\end{tikzcd}
$$

$H_*(L_\bullet) \xrightarrow{\varphi} H_*(K_{\bullet+\delta})$ induces matching of barcodes

$B(H_*(L_\bullet)) \hookleftarrow B(\operatorname{im} \varphi) \hookrightarrow B(H_*(K_\bullet))$



ALGEBRAIC STABILITY OF BARCODES

Theorem [Chazal et al. '09] [B, Lesnick '14]

A δ-interleaving of persistence modules ~~admits~~ induces a δ-matching of their barcodes.

Applications: (extending classical stability)

- weaker tameness assumptions
- filtrations of two different domains
- Gromov-Hausdorff stability (Vietoris-Rips barcode)

Persistence Computation: The Basics

(X, d) : metric space

$\mathrm{Rips}_t(X) = \{\emptyset \neq \sigma \subseteq X \mid \operatorname{diam} \sigma \leq t\}$

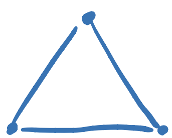
$\rightsquigarrow$ Rips filtration

$\rightsquigarrow$ refine (lexicographically) to simplexwise

D : p- boundary matrix (wrt. ordered simplices)

Matrix reduction: compute $R = D \cdot V$

- R reduced: unique pivots
- V upper triangular, full rank



$$\underset{R}{\begin{pmatrix} 1 & & \\ 1 & 1 & \\ & 1 & \end{pmatrix}} = \underset{D}{\begin{pmatrix} 1 & & 1 \\ 1 & 1 & \\ & 1 & 1 \end{pmatrix}} \cdot \underset{V}{\begin{pmatrix} 1 & & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}} \qquad (\text{in } \mathbb{Z}/(2))$$

Columns of R and some of V form a compatible basis for persistent homology:

R_j generates homology (at index $i = \text{pivot } R_j$)

V_j kills homology (at index j)

If $R_i = 0$:

- V_i generates homology (at index i)
- if $i = \text{pivot } R_j$: we don't need V_i
- else: V_i is essential cycle

Clearing (Chen, Kerber 2011):

- avoid unneeded computations
- requires reducing boundary matrices in decreasing dimensions
- unavailable in top dimension

Persistent cohomology

- vector space dual to homology
- same barcode
- computed by reducing coboundary matrices (transpose boundary matrix & reverse order) in increasing dimension
- unavailable in dimension 0 (where Kruskal's MST algorithm is used)

Filtration $K_1 \hookrightarrow K_2 \hookrightarrow \dots \hookrightarrow K_m = A$

Cochains $C^*(K_1) \twoheadleftarrow C^*(K_2) \twoheadleftarrow \dots \twoheadleftarrow C^*(K_m)$

Relative filt. $C^*(A, K_1) \hookleftarrow C^*(A, K_2) \hookleftarrow \dots \hookleftarrow C^*(A, K_m)$

Reducing the coboundary matrix actually computes persistent relative cohomology!

[dSMV11]: absolute/relative barcodes determine each other.

Consider short exact sequence

$$C_*(K_\bullet) \hookrightarrow C_*(A) \twoheadrightarrow C_*(A, K_\bullet)$$

of filtered chain complexes.

$\rightsquigarrow$ long exact sequence

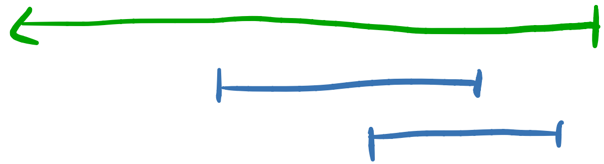
$$\begin{tikzcd} \cdots \arrow[r] & H_d(K_\bullet) \arrow[r] & H_d(A) \arrow[r] & H_d(A, K_\bullet) \arrow[dll, "\partial"'] \\ & H_{d-1}(K_\bullet) \arrow[r, "\eta"'] & H_{d-1}(A) \arrow[r] & \cdots \end{tikzcd}$$

• splits at $H_{d-1}(K_\bullet) = \underbrace{H_{d-1}(K_\bullet)^\infty}_{\mathrm{im}\,\eta} \oplus \underbrace{H_{d-1}(K_\bullet)^\dagger}_{\mathrm{im}\,\partial}$

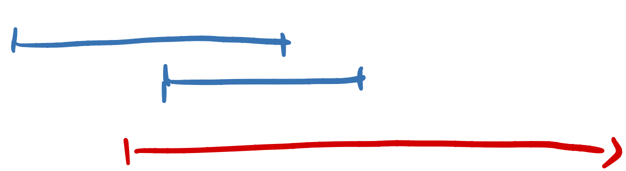
and at $H_d(A, K_\bullet) = \overbrace{H_d(A, K_\bullet)^*}^{\mathrm{im}\,\partial} \oplus H_d(A, K_\bullet)^{-\infty}$

Example

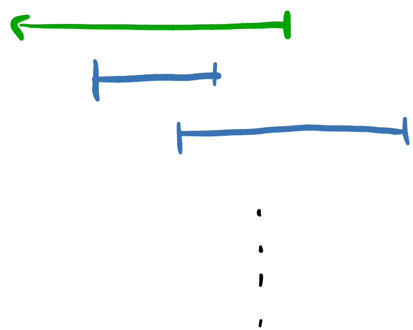
$H_d(A, K_\cdot)$



$H_{d-1}(K_\cdot)$



$H_{d-1}(A, K_\cdot)$



COMPUTING IMAGE BARCODES

Now consider a pair of filtrations:

$$\begin{tikzcd} K_1 \arrow[r, hook] & K_2 \arrow[r, hook] & \cdots \arrow[r, hook] & K_m = A \\ L_1 \arrow[r, hook] \arrow[u, hook] & L_2 \arrow[r, hook] \arrow[u, hook] & \cdots \arrow[r, hook] & L_m = A \end{tikzcd}$$

$f : L_\bullet \hookrightarrow K_\bullet$

Compute $\operatorname{im} H_*(f)$?

- Long exact sequence is functorial
- But image may not be exact

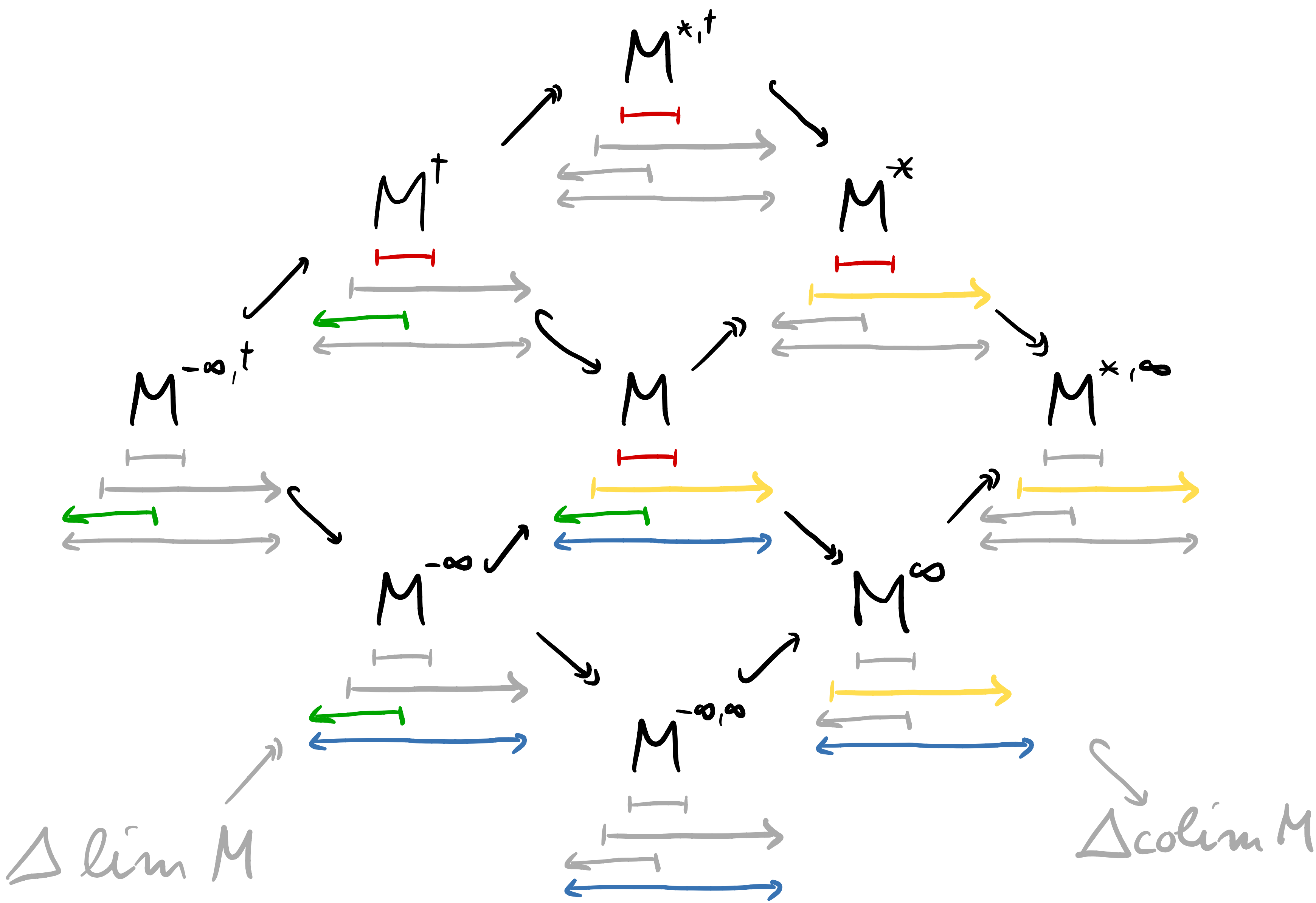
Previous work:

- [Edels., Harer, Morozov '09]:
  - $L_t = K_t \cap L$ (one function)
  - no clearing
- [Skraba, Vej.-Joh '13]
  - for presentations of persistence modules

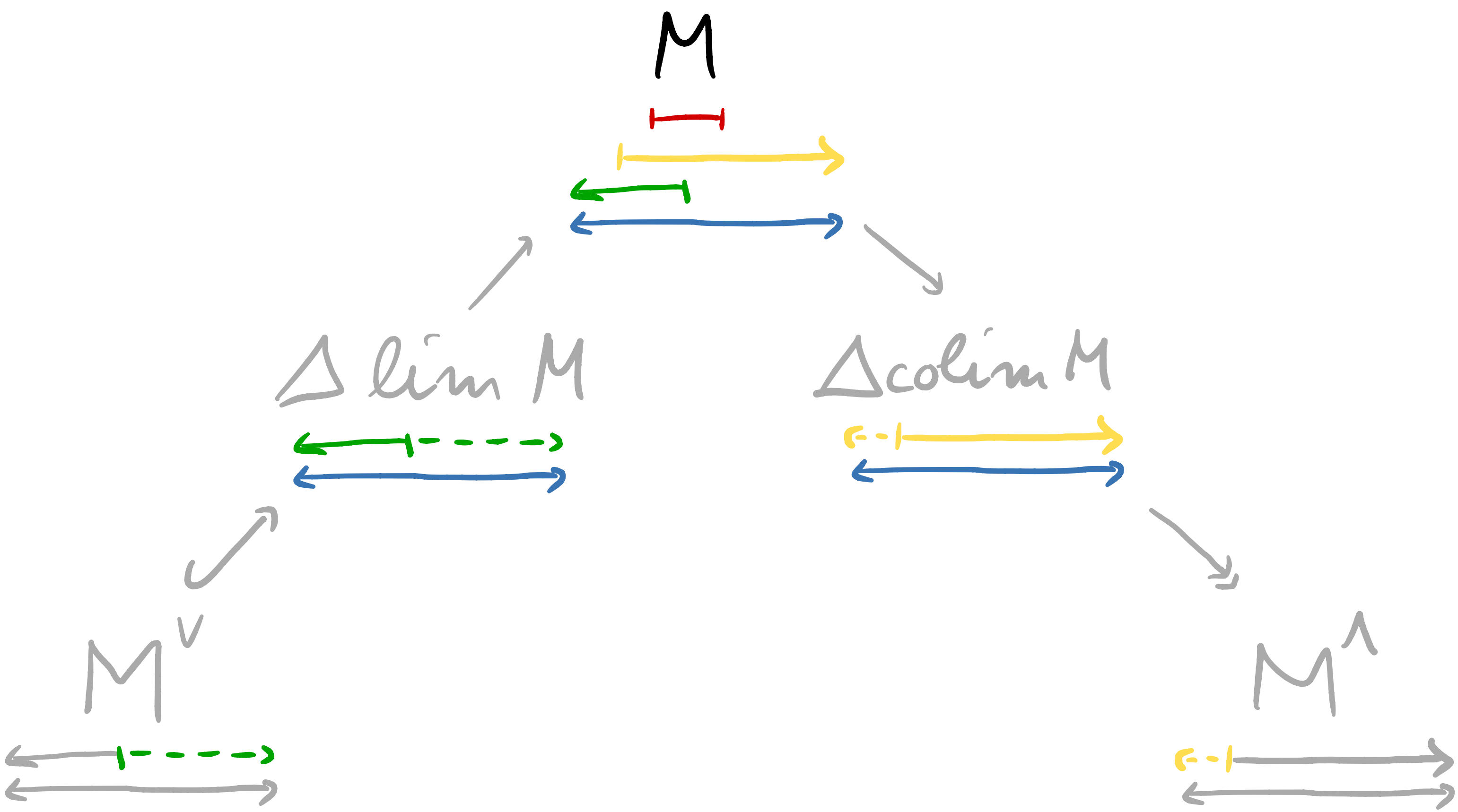
This talk:

- [B, Schmahl '20]
  - $K_\bullet$, $L_\bullet$ arbitrary filtrations of Δ
  - Clearing & cohomology

LIFESPAN FUNCTORS



COMPLEMENTS



interval decomposition not natural

$$
\begin{tikzcd}
H_d(K_\bullet) \arrow[r, phantom, "\cong"] & H_d(K_\bullet)^\dagger \arrow[d, phantom, "\cong" rotate=90] \arrow[r, phantom, "\oplus"] & H_d(K_\bullet)^\infty \arrow[d, phantom, "\cong" rotate=90] \\
& H_{d+1}(A, K_\bullet)^* \arrow[dd, leftrightarrow] & H_d(A, K_\bullet)^\vee \arrow[d, leftrightarrow] \\
& & H_d(A, K_\bullet)^{-\infty} \arrow[d, leftrightarrow] \\
& H^{d+1}(A, K_\bullet)_* & H^d(A, K_\bullet)_{-\infty}
\end{tikzcd}
$$

relative l.e.s.

complement

vector space duality

$f : L_\bullet \hookrightarrow K_\bullet \qquad \phi : (A, L_\bullet) \hookrightarrow (A, K_\bullet)$

$$
\begin{tikzcd}
\operatorname{im} H_d(f) \cong & (\operatorname{im} H_d(f))^{\dagger} \arrow[d, "\cong"', "\text{requires colim } H_d(f) \text{ mono}" left=12pt, no head] & \oplus & (\operatorname{im} H_d(f))^{\infty} \arrow[d, "\cong", no head] \\
& \operatorname{im}(H_d(f)^{\dagger}) \arrow[d, "\cong"', "\text{relative l.e.s.}" left=12pt, no head] & & H_d(L_\bullet)^{\infty} \arrow[d, "\cong", no head] \\
& \operatorname{im}(H_{d+1}(\phi)^{*}) \arrow[d, "\cong"', "\text{requires lim } H_{d+1}(\phi) \text{ epi}" left=12pt, no head] & & H_d(A, L)^{\vee} \arrow[d, leftrightarrow] \\
& (\operatorname{im} H_{d+1}(\phi))^{*} & & H_d(A, L_\bullet)^{-\infty}
\end{tikzcd}
$$

COMPUTING IMAGE PERSISTENCE

$$
\begin{array}{cc} & \xrightarrow{L} \\ \downarrow L & D^L \end{array}
\qquad\qquad
\begin{array}{cc} & \xrightarrow{K} \\ \downarrow L & D^{im} \end{array}
$$

Reduce to $\quad R^L = D^L V^L \qquad R^{im} = D^{im} V^{im}$

Theorem [B, Schmahl]

$$
\begin{aligned}
B(\operatorname{im} H_*(L_\bullet \hookrightarrow K_\bullet)) = \\
&\{[i, j) \mid i = \operatorname{pivot} R^{im}_j\} \\
\cup &\{[i, \infty) \mid R^L_i = 0,\ i \in \operatorname{pivots} R^L\}
\end{aligned}
$$

Clearing for Image Persistence

Homology:

$$R^L = D^L V^L \qquad R^{im} = D^{im} V^{im} \qquad R^K = D^K V^K$$

$$H_*(f)^\infty \qquad H_*(f)^+ \qquad H_*(\emptyset)^\infty$$

& clearing R^{im}

Cohomology:

$$S^L = W^L D^L \qquad S^{im} = W^{im} D^{im} \qquad S^K = W^K D^K$$

$$H^*(\emptyset)_\infty \qquad H^*(f)_+ \qquad H^*(f)_\infty$$

& clearing S^{im}

IMPLICIT REDUCTION

$$R = D \cdot V$$

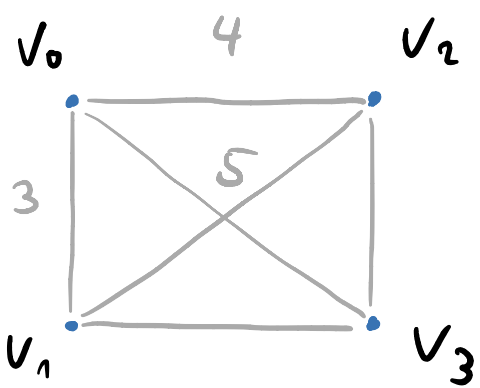
store

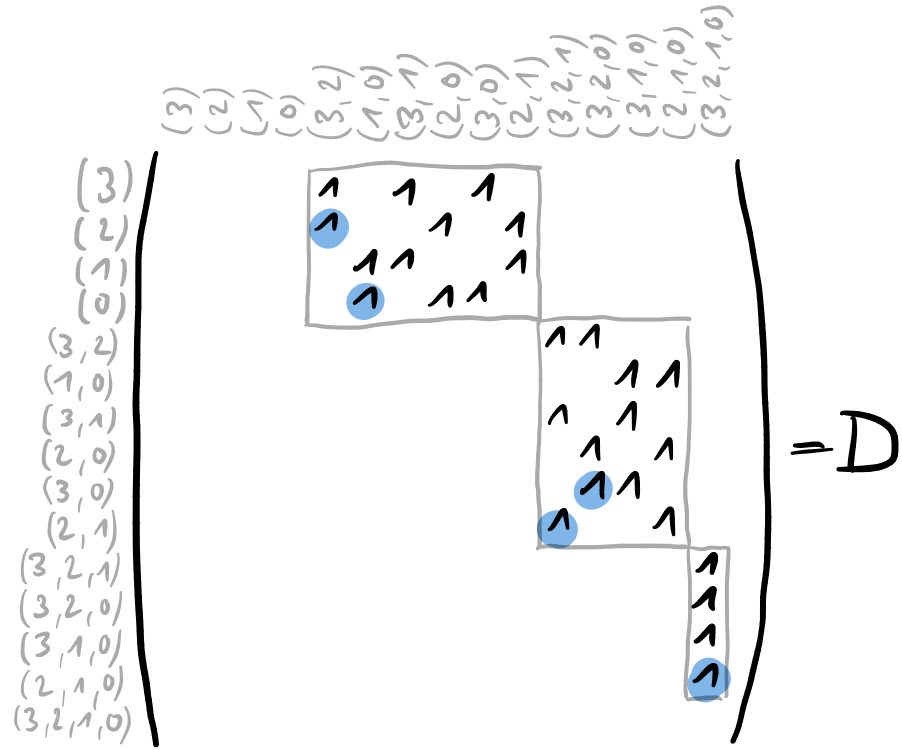
recompute

store pivots (unless apparent)
recompute other entries

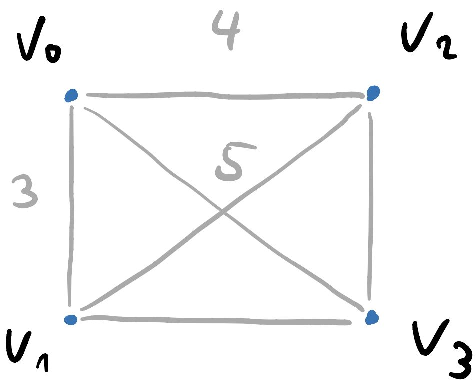
APPARENT PAIRS

(how simplices find the perfect partner)





APPARENT ZERO PAIRS



$(3,2,0)$: diam $= 5$

facets (lex order):
$(2,0)$, $(3,0)$, $(3,2)$

1	$(3,2)$:	3
	$(1,0)$:	3
	$(3,1)$:	4
1	$(2,0)$:	4
1	$(3,0)$:	5
	$(2,1)$:	5

An Excursion into the Wilderness

Given a finite indexing poset P.

Indecomposable diagrams of K-vector spaces with shape P? (K algebr. closed)

3 cases (representation types):

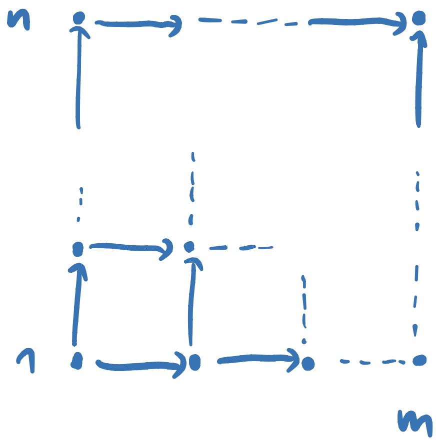
ⓐ A finite list — finite type

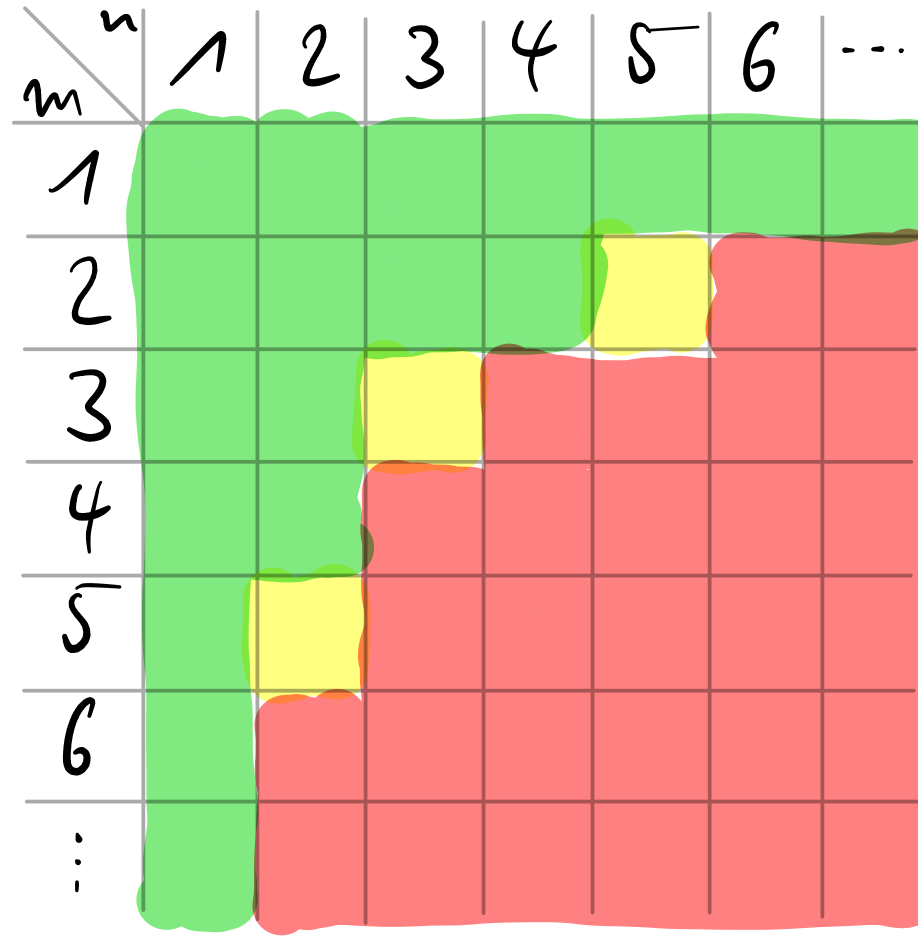
ⓑ A finite list (of 1-param. families) — tame

ⓒ It's complicated. — wild

(as complicated as modules over any finite-dim. algebra; undecidable theory)

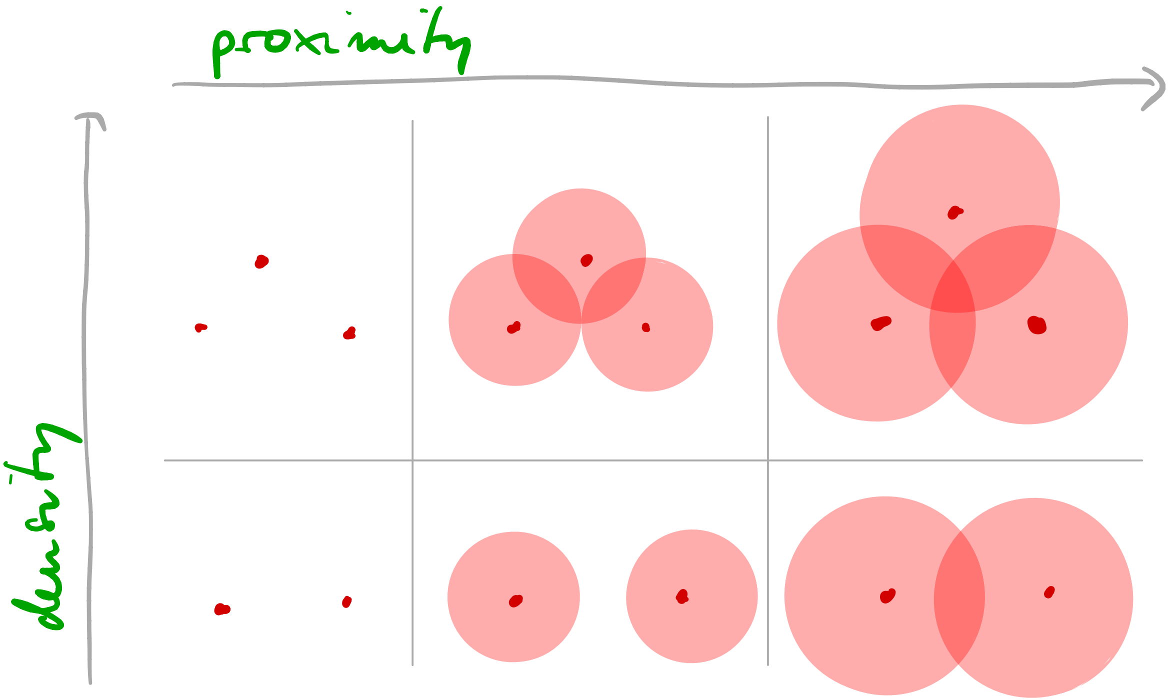
Representation Types of Commutative Grids



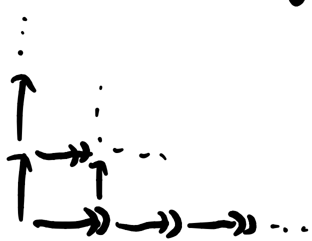


$$\left.\begin{array}{l}\text{finite}\\\text{tame}\\\text{wild}\end{array}\right\}:(m-1)(n-1)\left\{\begin{array}{c}<\\=\\>\end{array}\right\}4$$

GRID DIAGRAMS FROM CLUSTERING



Apply $H_0 \rightsquigarrow$ diagram of the form



(horizontal : surjective)

$$\begin{tikzcd}
K^3 \arrow[r, two heads, "(1\ 1\ 1)"] & K \arrow[r, two heads] & K \\
K^2 \arrow[u, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}"] \arrow[r, two heads] & K^2 \arrow[u, "(1\ 1)"'] \arrow[r, two heads, "(1\ 1)"] & K \arrow[u]
\end{tikzcd}$$

$$\cong \begin{tikzcd}
K \arrow[r, two heads] & K \arrow[r, two heads] & K \\
K \arrow[u] \arrow[r, two heads] & K \arrow[u] \arrow[r, two heads] & K \arrow[u]
\end{tikzcd} \oplus \begin{tikzcd}
K \arrow[r, two heads] & 0 \arrow[r, two heads] & 0 \\
K \arrow[u] \arrow[r, two heads] & K \arrow[u] \arrow[r, two heads] & 0 \arrow[u]
\end{tikzcd} \oplus \begin{tikzcd}
K \arrow[r, two heads] & 0 \arrow[r, two heads] & 0 \\
0 \arrow[u] \arrow[r, two heads] & 0 \arrow[u] \arrow[r, two heads] & 0 \arrow[u]
\end{tikzcd}$$

Lemma $\mathrm{Rep}^{\twoheadrightarrow}(m,2)$ is finite type.

EPIC GRIDS & WILD THINGS

Theorem [B, Botnan, Oppermann, Steen '19]

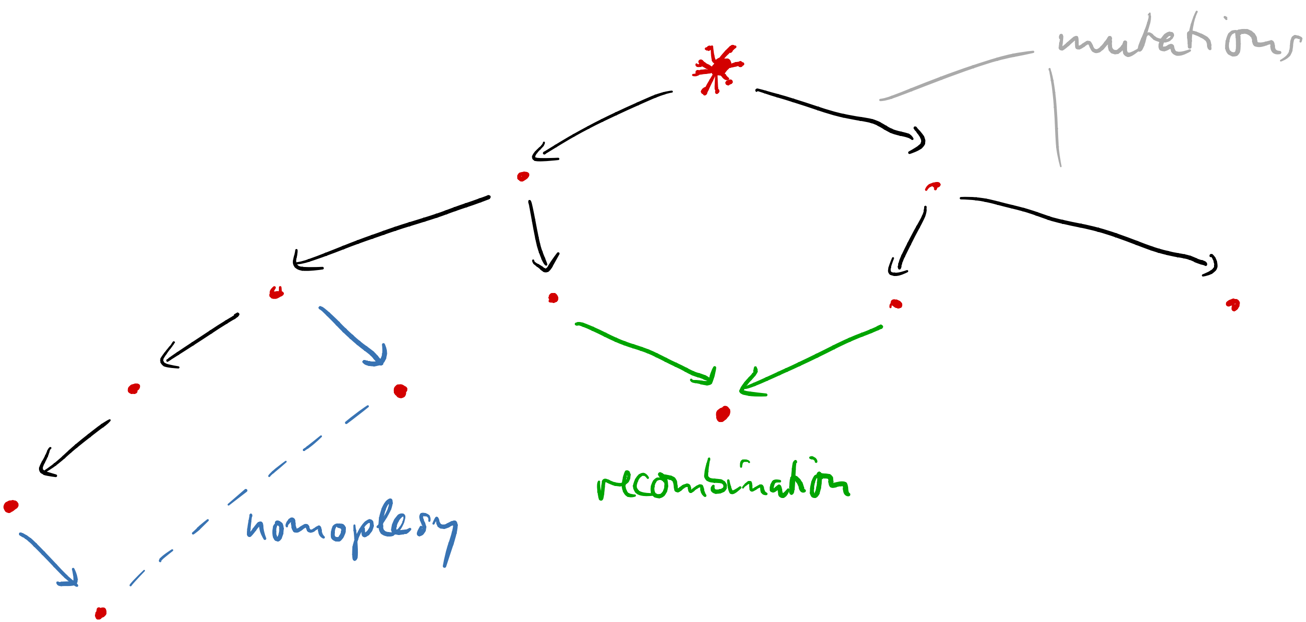
$$\begin{array}{ccc} \mathrm{Rep}^{\hat{\uparrow}\twoheadrightarrow}(m,n) & \sim & \mathrm{Rep}^{\twoheadrightarrow}(m-1,n) \\ \wr & & \wr \\ \mathrm{Rep}^{\twoheadrightarrow}(m-1,n) & \sim & \mathrm{Rep}(m-1,n-1) \end{array}$$

Corollary $\mathrm{Rep}^{\twoheadrightarrow}(m,n)$ is

$$\left.\begin{array}{l} \text{finite type} \\ \text{tame} \\ \text{wild} \end{array}\right\} \text{ for } (m-1)(n-2) \left\{\begin{array}{c} < \\ = \\ > \end{array}\right\} 4.$$

TOPOLOGY OF VIRAL EVOLUTION

[Chen et al. 2013]



Example (Mutation E484K):

- prevalence still low (15.1.21: < 1%)
- but appears in several persistent cycles