Persistent homology

From Theory to Computation

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TUM

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Mathematical Signal Processing and Data Analysis – University of Bremen

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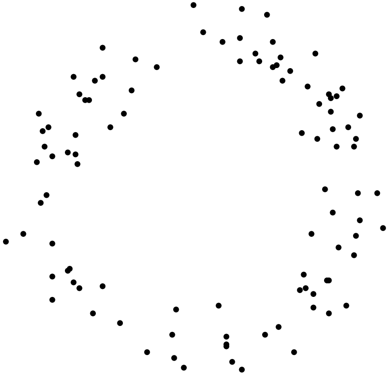
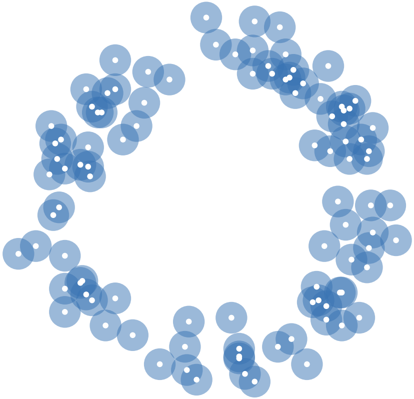
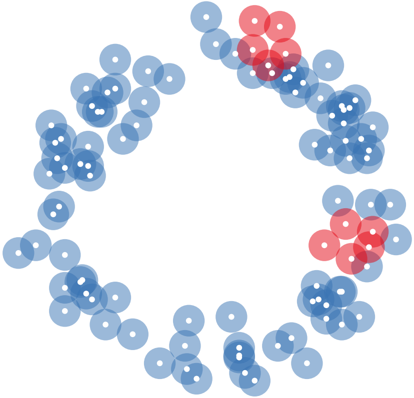
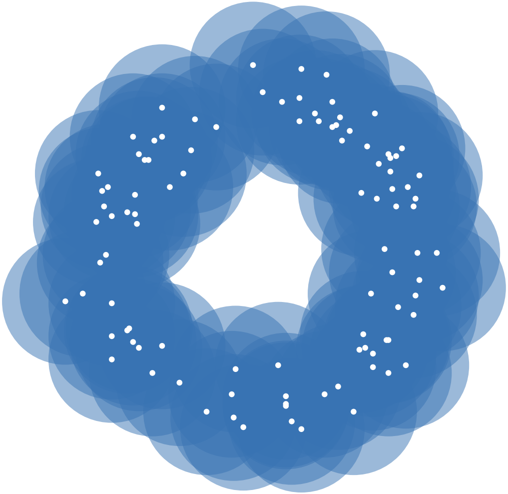
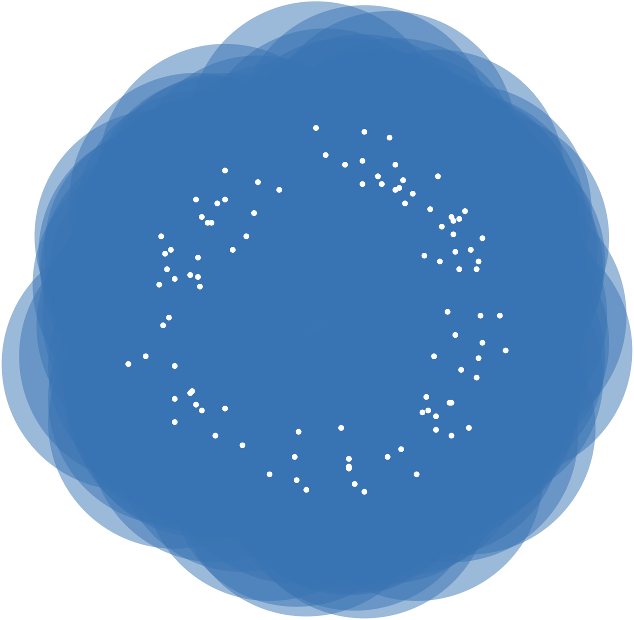
Persistent homology

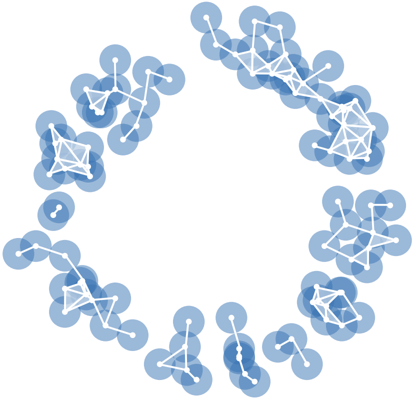


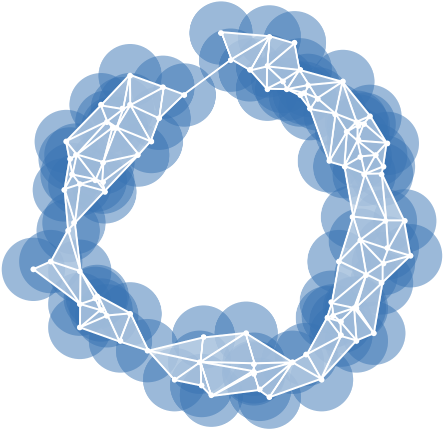


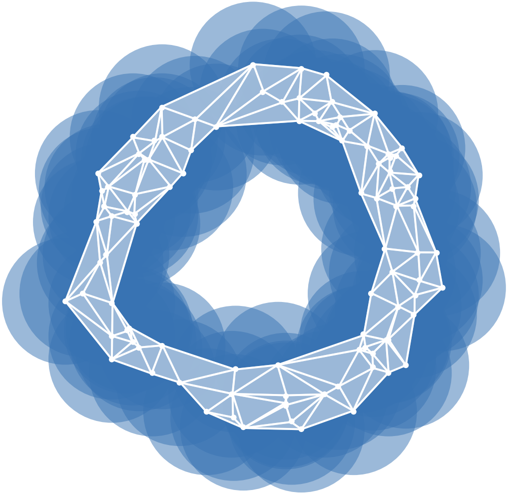


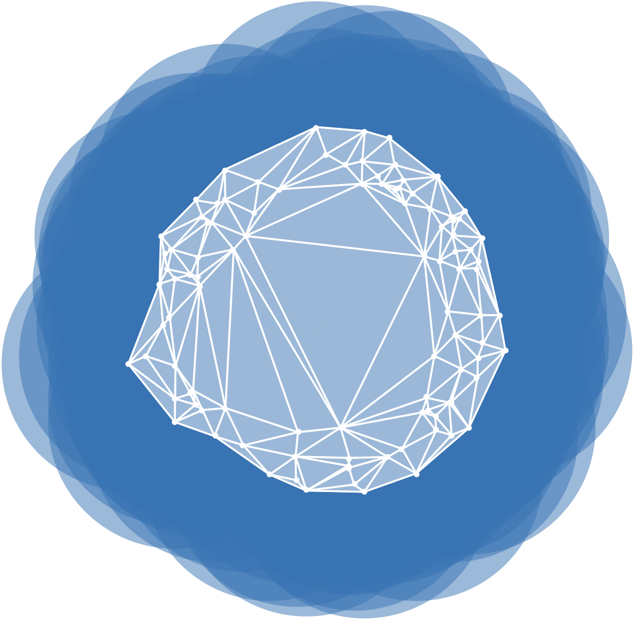












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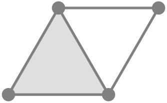

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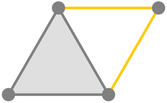

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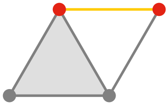

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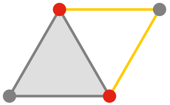

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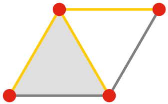

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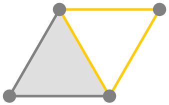

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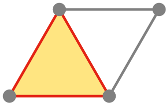

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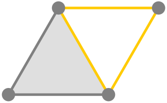

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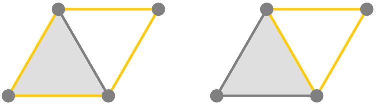

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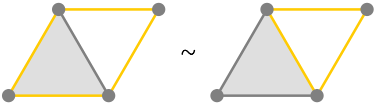

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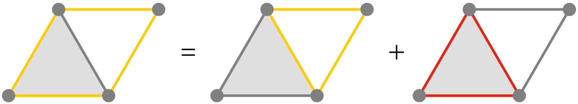

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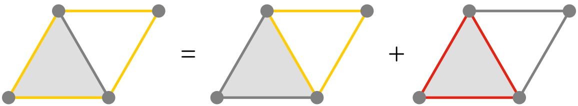

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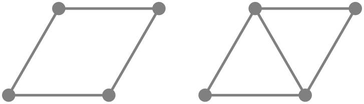
- The equivalence classes form the *homology group* $H_*(K) = Z_*(K)/B_*(K)$ (a vector space over $\mathbb{Z}_2$)

Why is homology algebraic?

Roughly: $\dim H_*(K)$ is the number of holes



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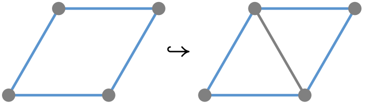

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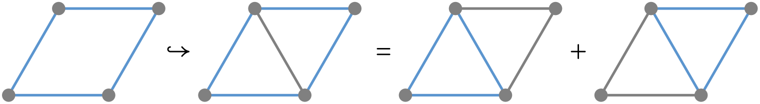

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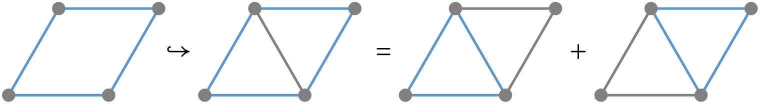

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Homology does not have a canonical basis

- Computing (persistent) homology is all about choosing (compatible) bases

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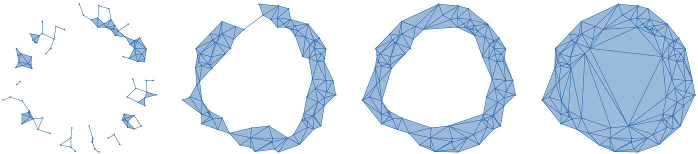

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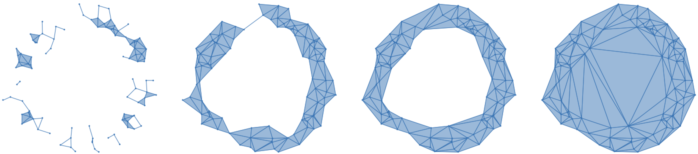

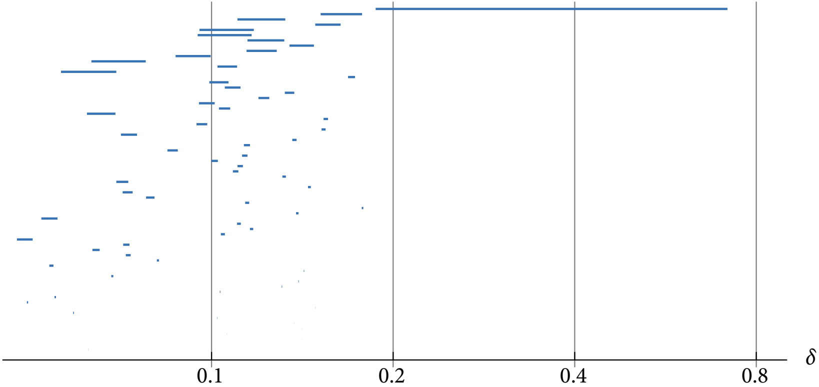

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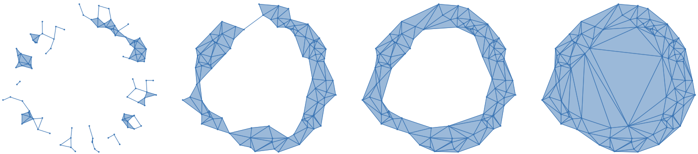

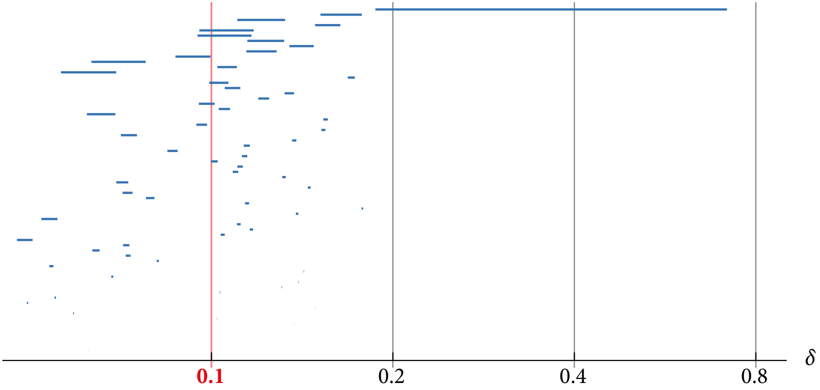

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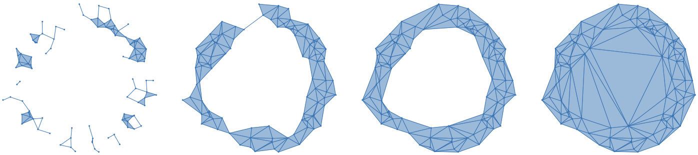

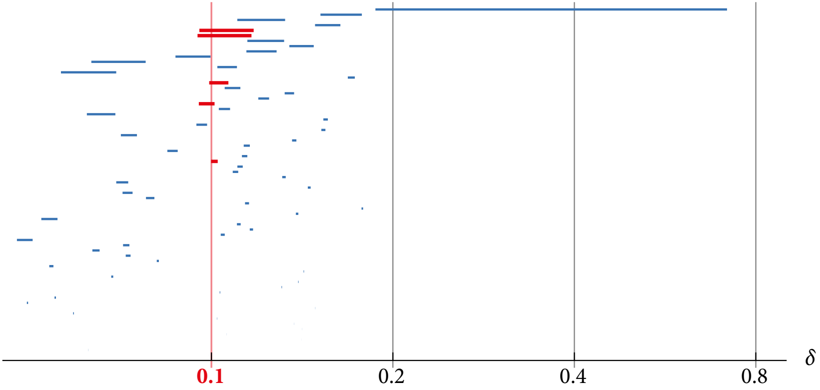

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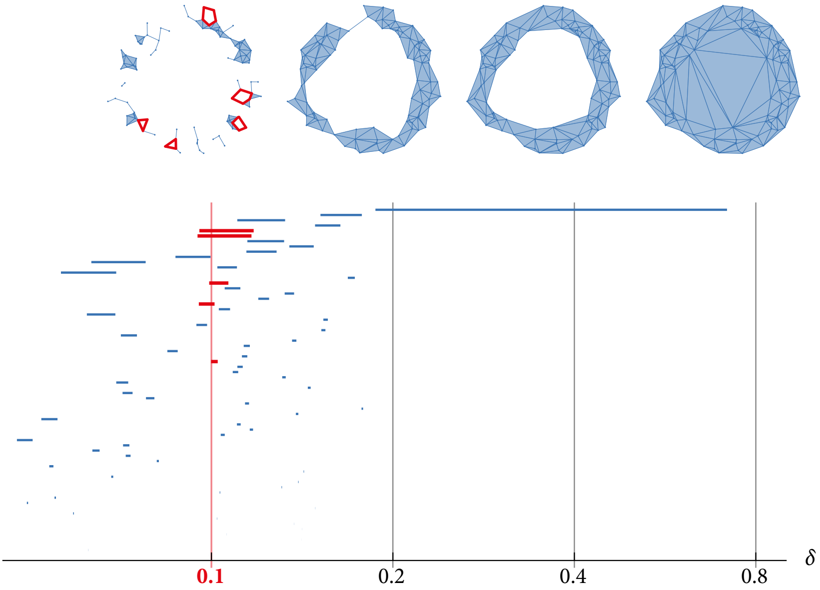

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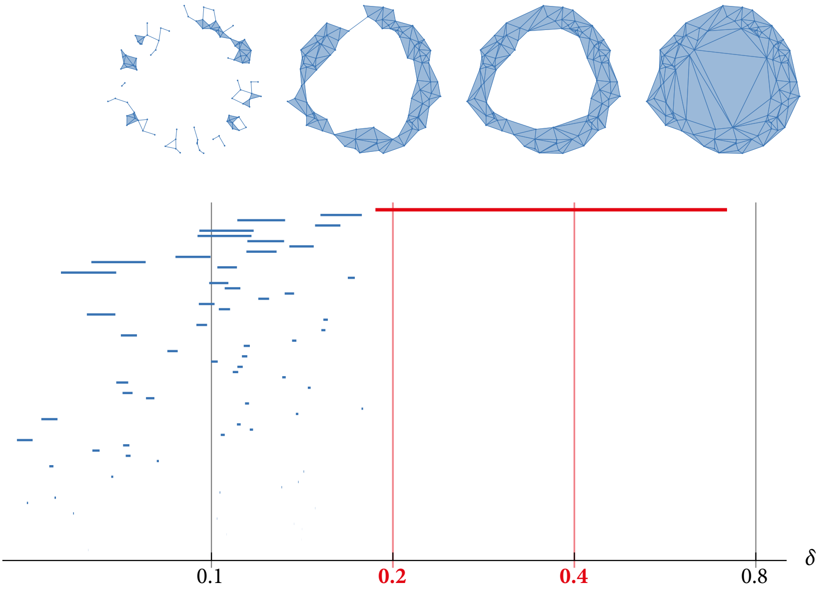

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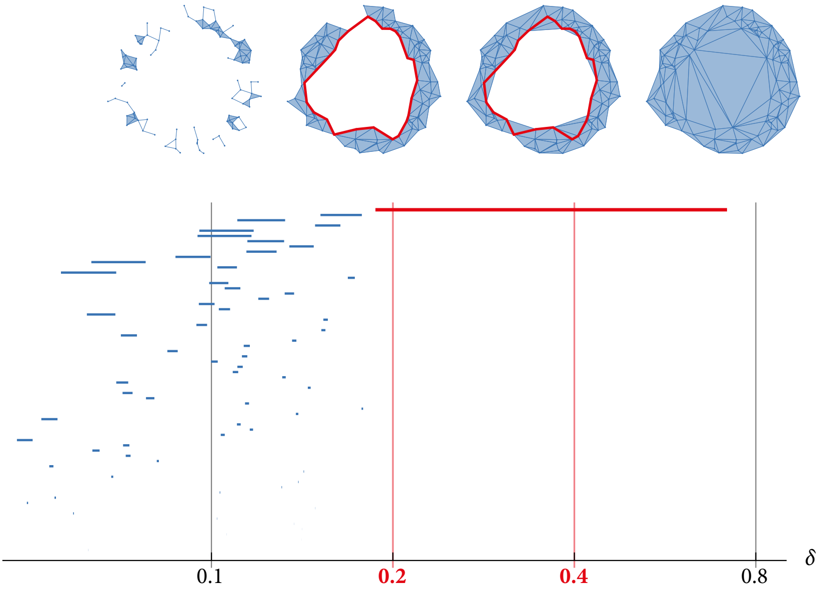

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- A filtration is a certain diagram $K : \mathbf{R} \to \mathbf{Top}$
  - $\mathbf{R}$ is the poset category of $(\mathbb{R}, \leq)$
  - A topological space K_t for each $t \in \mathbb{R}$
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- Consider homology $H_* : \mathbf{Top} \to \mathbf{Vect}$
- Persistent homology is a diagram $M : \mathbf{R} \to \mathbf{Vect}$ *(persistence module)*

Barcodes: the structure of persistence modules

Theorem (Krull–Schmidt–Remak; Crawley-Boewey 2015)

Any persistence module M of vector spaces over the field $\mathbb{F}$ is a direct sum of interval modules

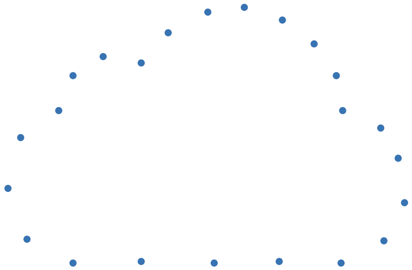
$$0 \to \cdots \to 0 \to \mathbb{F} \to \cdots \to \mathbb{F} \to 0 \to \cdots \to 0$$

(in an essentially unique way).

- The support of these intervals is the *persistence barcode*.
- It completely describes the persistence module (up to isomorphism).
- This is why we use homology with coefficients in a field.

Complexes

Čech and Delaunay complexes



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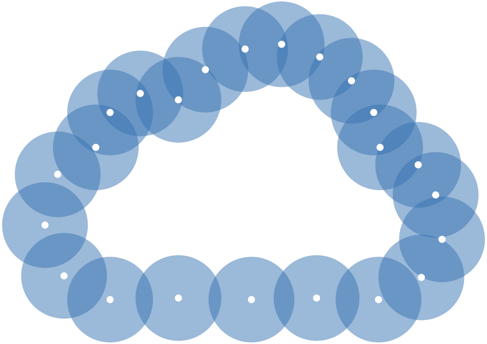

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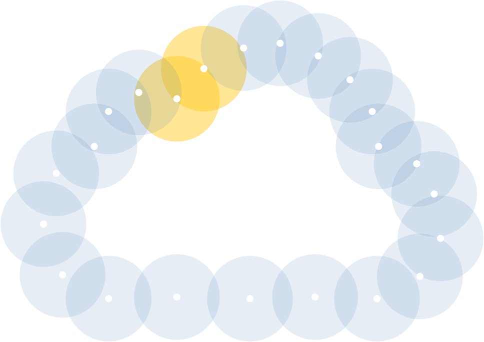

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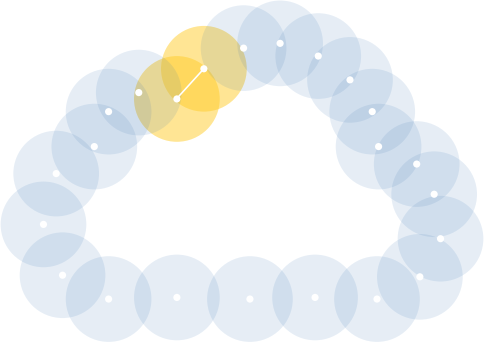

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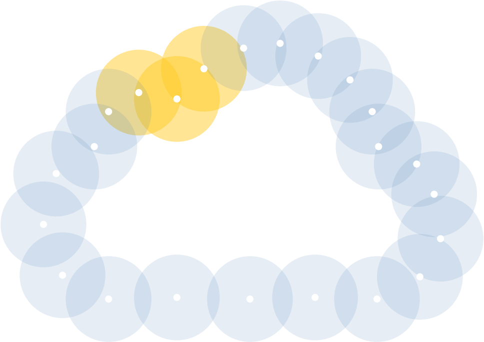

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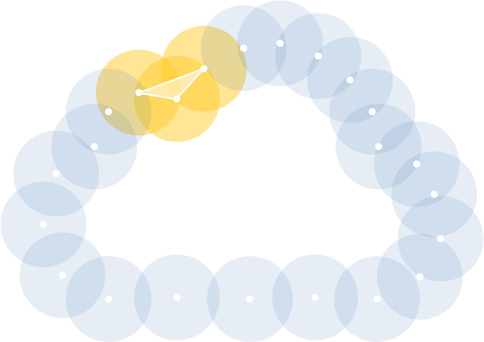

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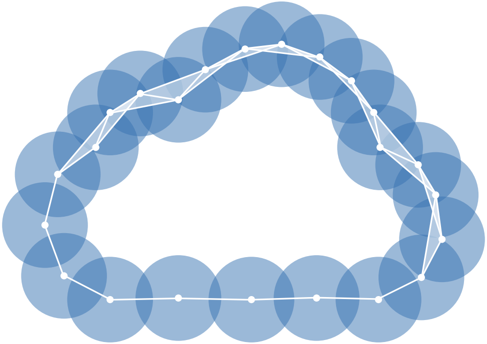

$$\mathrm{Cech}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} B_r(p) \neq \emptyset \right\}$$

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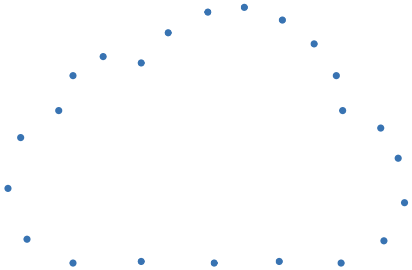

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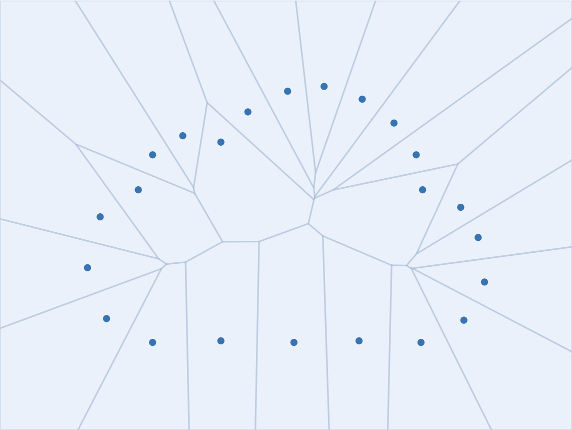

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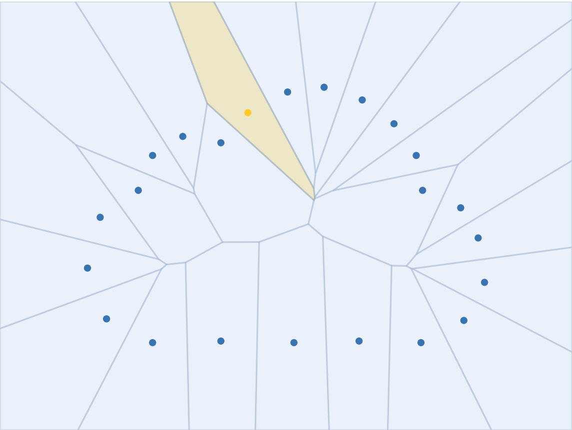

$$\mathrm{Vor}(p, X) = \left\{ a \in \mathbb{R}^n \mid d(a,p) = d(a,X) \right\}$$

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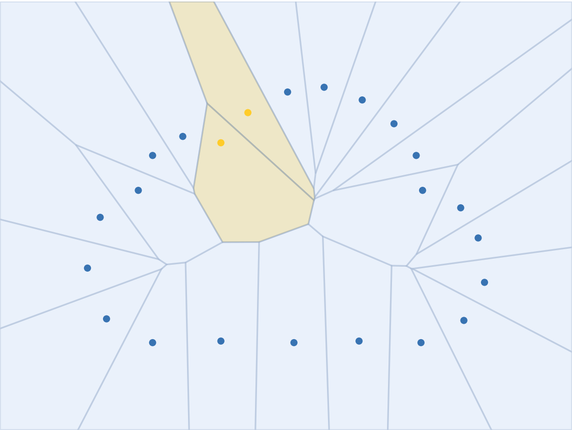

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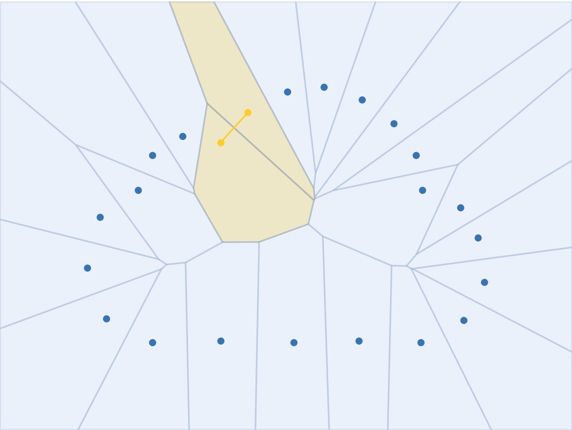

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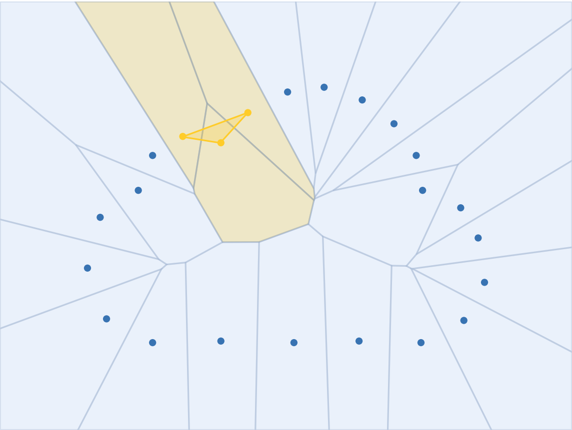

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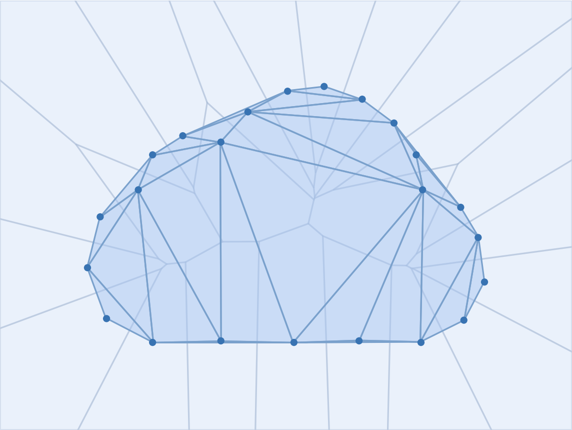

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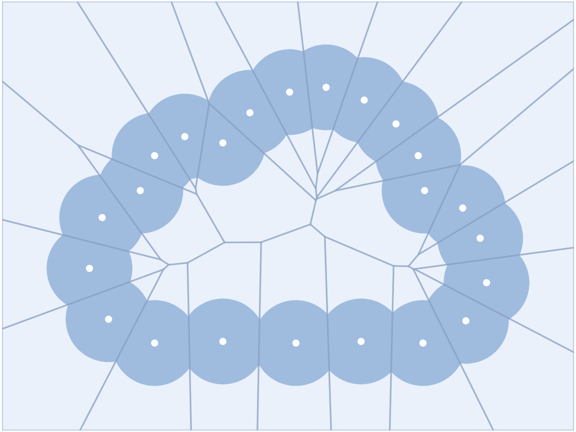
Čech and Delaunay complexes



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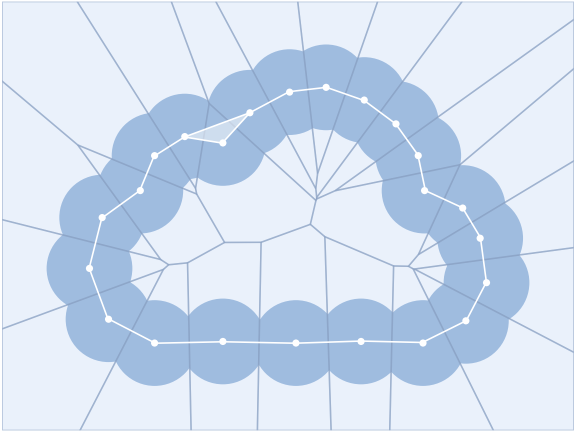

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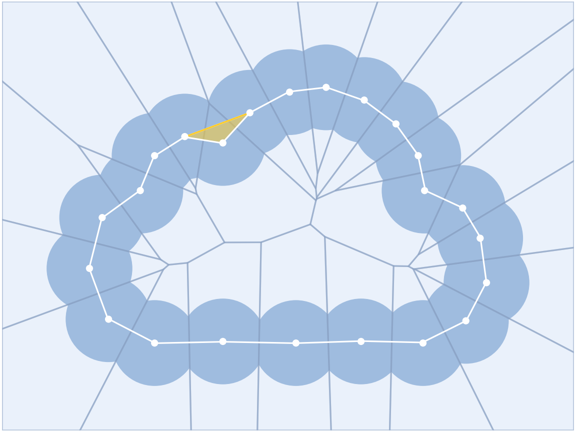

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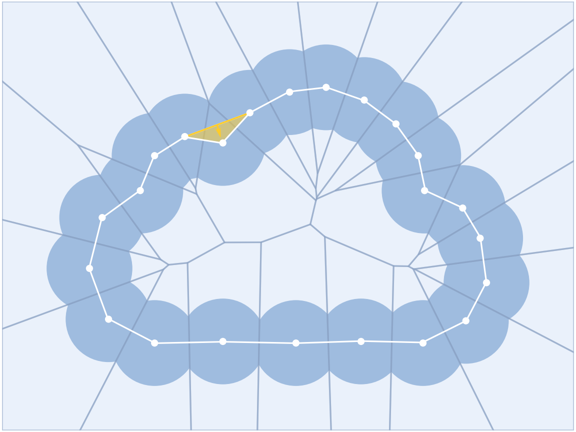

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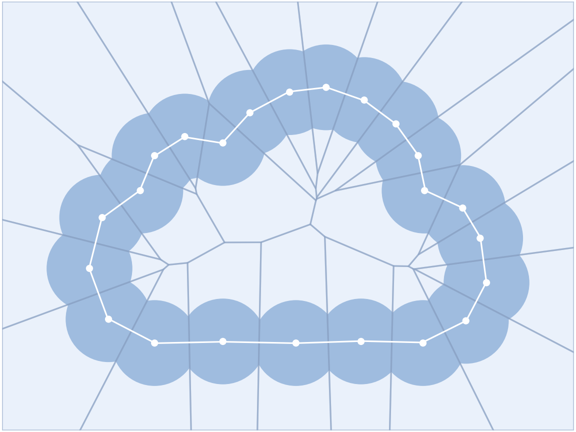

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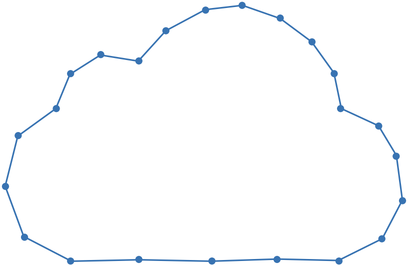
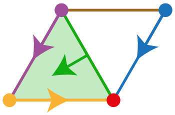


Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



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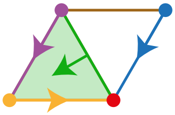

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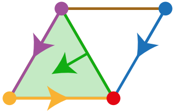

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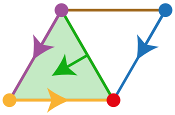

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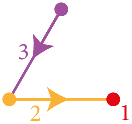

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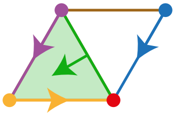

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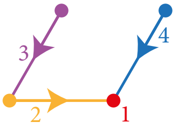

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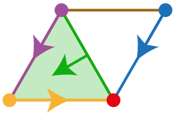

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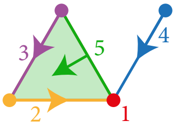

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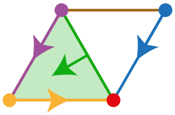

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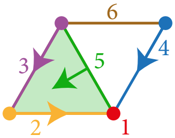

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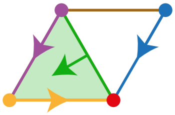

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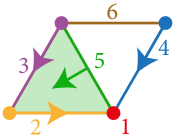

Fundamental theorem of discrete Morse theory

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If $(s, t]$ *contains no critical value of* f*,*
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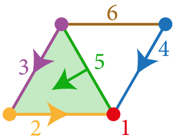

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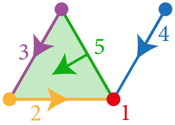

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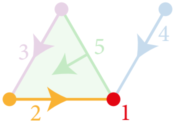

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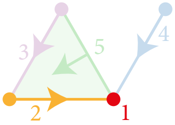

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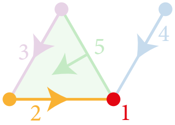

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This homotopy equivalence is compatible with the filtration.

Corollary

K and M have isomorphic persistent homology (with regard to f).

Morse theory for Čech and Delaunay complexes

Proposition

The Čech complexes and the Delaunay complexes (alpha shapes) are sublevel sets of (generalized) discrete Morse functions.

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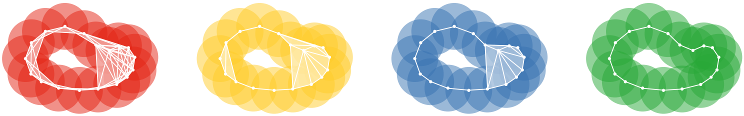
Theorem (B., Edelsbrunner 2017)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent through a sequence of collapses

$$\mathrm{Cech}_r X \searrow \mathrm{Cech}_r X \cap \mathrm{Del}\, X \searrow \mathrm{Del}_r X \searrow \mathrm{Wrap}_r X,$$

induced by a single discrete gradient field.



Delaunay and Wrap complexes

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Vietoris–Rips complexes

Consider a finite metric space (X, d).

The *Vietoris–Rips complex* is the simplicial complex

$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid \operatorname{diam} S \leq t\}$$

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For large t, $\mathrm{Rips}_t(X)$ is the full simplex with vertices X

- Number of simplices grows exponentially in dimension
- Computation is one of the most important challenges in applied topology!

Computation

Demo: Ripser

Example data set:

- 192 points on $\mathbb{S}^2$
- persistent homology barcodes up to dimension 2
- over 56 mio. simplices in 3-skeleton

Some previous software:

- javaplex (Stanford): 3200 seconds, 12 GB
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Ripser: 1.2 seconds, 160 MB

Ripser

A software for computing Vietoris–Rips persistence barcodes

- around 1000 lines of C++ code, no external dependencies
- support for
  - coefficients in a prime field $\mathbb{Z}/p\mathbb{Z}$
  - sparse distance matrices (for distance threshold)
- open source (`http://ripser.org`)
  - released in July 2016
- online version (`http://live.ripser.org`)
  - launched in August 2016
- (co-)winner of 2016 ATMCS Best New Software Award

Computing homology

Computing homology $H_* = Z_*/B_*$ (recall: $B_* \subseteq Z_* \subseteq C_*$):

- compute basis for boundaries $B_* = \operatorname{im} \partial_*$
- extend to basis for cycles $Z_* = \ker \partial_*$
- new (non-boundary) basis cycles generate quotient Z_*/B_*

Homology by matrix reduction

Notation:

- D: boundary matrix (with $\mathbb{Z}_2$ coefficients)
- R_i: ith column of R

Matrix reduction algorithm (variant of Gaussian elimination):

- $R = D, V = I$
- while $\exists i < j$ with $\operatorname{pivot} R_i = \operatorname{pivot} R_j$
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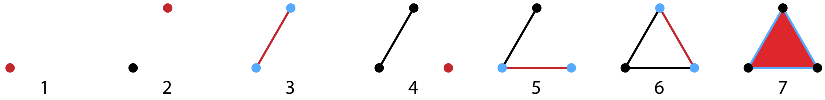
Result:

- $R = D \cdot V$ is reduced (each column has a unique pivot)
- V is full rank upper triangular

Matrix reduction



$$
\underbrace{\begin{array}{c|ccccccc}
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6 & & & & & & & 1 \\
7 & & & & & & &
\end{array}}_{R}
= D \cdot
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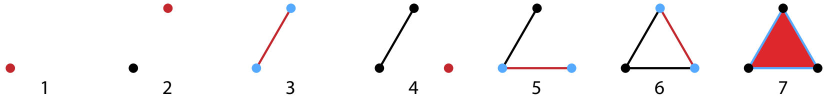

	1	2	3	4	5	6	7
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3							1
4					1	1	
5							1
6							1
7							

R

$= D \cdot$

	1	2	3	4	5	6	7
1	1						
2		1					
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4				1			
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7							1

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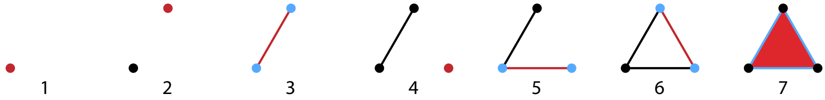

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2			1			1	
3							1
4					1	1	
5							1
6							1
7							

R

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	1	2	3	4	5	6	7
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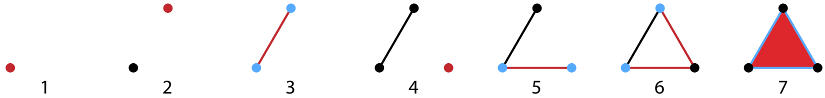

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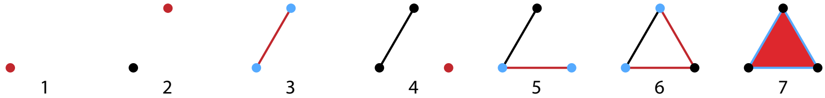

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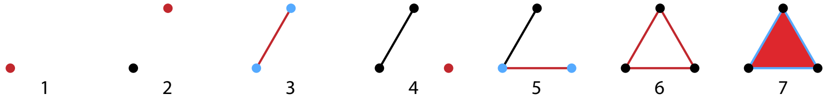

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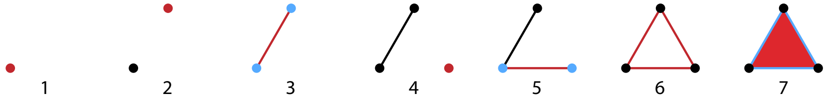

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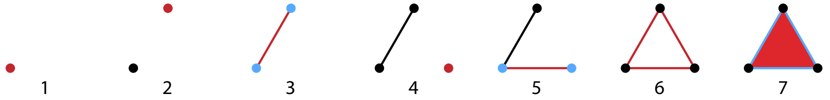

$$\underbrace{\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & & & 1 & & 1 & & \\ 2 & & & 1 & & & & \\ 3 & & & & & & & 1 \\ 4 & & & & & \color{red}{1} & & \\ 5 & & & & & & & 1 \\ 6 & & & & & & & 1 \\ 7 & & & & & & & \end{array}}_{R} = D \cdot \underbrace{\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & & & & & & \\ 2 & & 1 & & & & & \\ 3 & & & 1 & & & 1 & \\ 4 & & & & 1 & & & \\ 5 & & & & & 1 & 1 & \\ 6 & & & & & & 1 & \\ 7 & & & & & & & 1 \end{array}}_{V}$$

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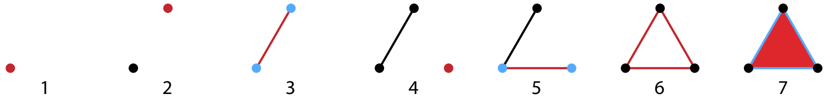

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\hline
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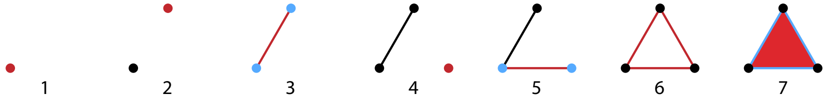

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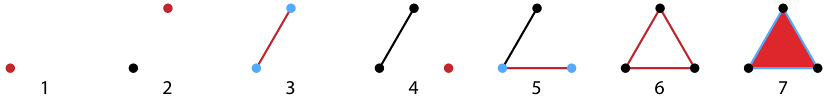

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The four special ingredients of Ripser

The improved performance of Ripser is based on 4 insights:

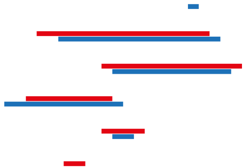
- Compute cohomology [de Silva et al. 2011]
  - reduce transposed matrix
- Skip inessential columns [Chen, Kerber 2011]
  - many columns are redundant for homology
- Implicit boundary matrix
  - don't store the matrix $R = D \cdot V$ in memory
- Apparent pairs
  - find column pivot without constructing entire column

Stability

Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\|f - g\|_\infty = \delta$*. Then there exists a matching between the intervals of the persistence barcodes of* f *and* g *such that*



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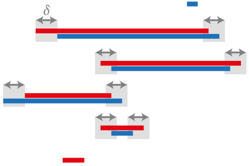

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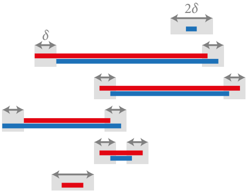

Persistence and stability: the big picture

Data	point cloud	$P \subset \mathbb{R}^d$

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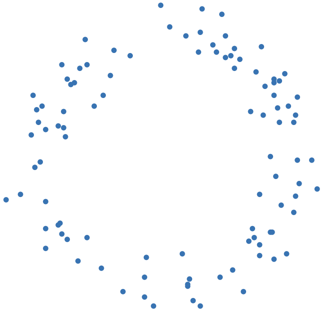
Combinatorics	intervals (persistence barcode)	$B : \mathbf{R} \to \mathbf{Mch}$

Geometric interleavings



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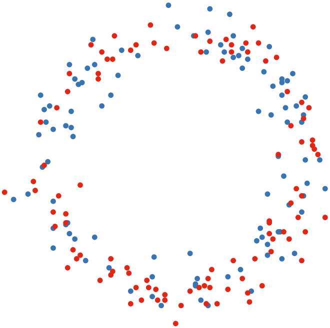

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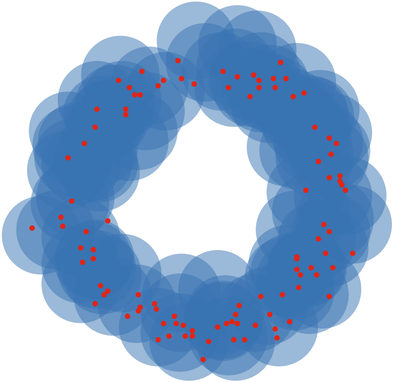

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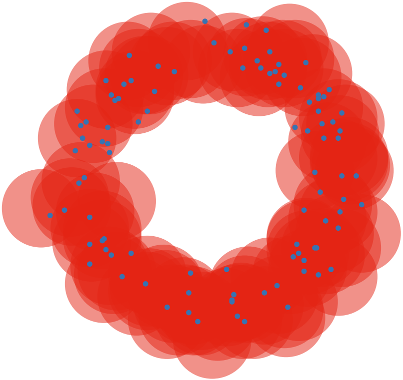

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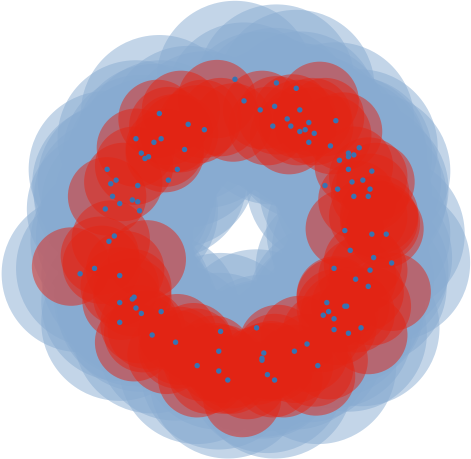

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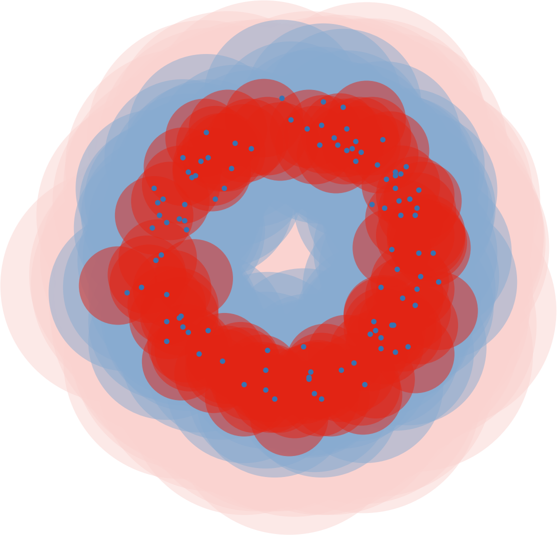

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F_t \arrow[r, hook] \arrow[dr, hook] & F_{t+\delta} \arrow[r, hook] \arrow[dr, hook] & F_{t+2\delta} \arrow[r, dotted] & {} \\
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Applying homology (a functor) preserves commutativity

- persistent homology of f, g yields δ-interleaved persistence modules $\mathbf{R} \to \mathbf{Vect}$

Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2012; B, Lesnick 2015)

If two persistence modules are δ-interleaved,
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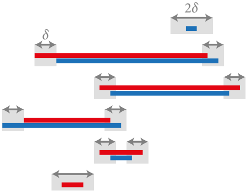

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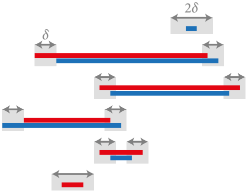

An interleaving morphism and its image

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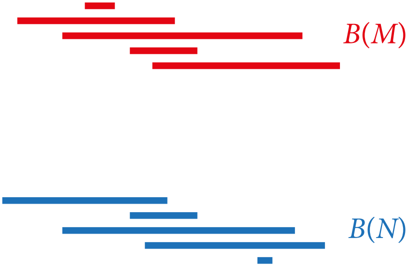

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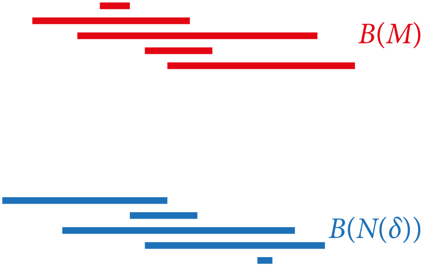

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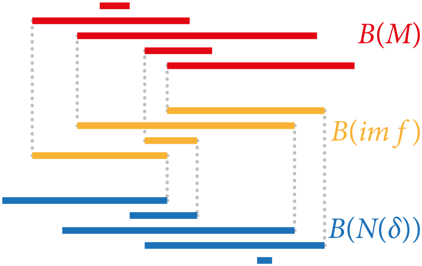

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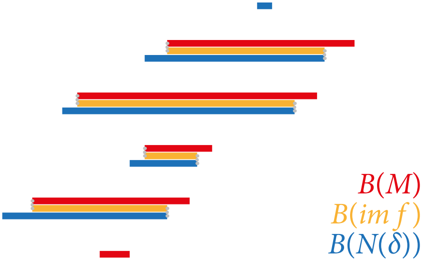

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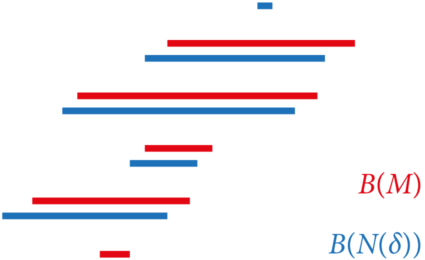

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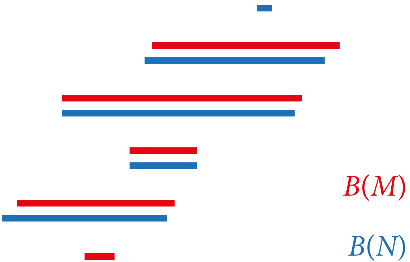

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Requires strong assumptions:

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Theorem (Niyogi, Smale, Weinberger 2006)

Let M be a submanifold of $\mathbb{R}^d$. Let $P \subset M$, $\delta > 0$ be such that

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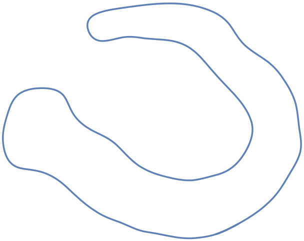

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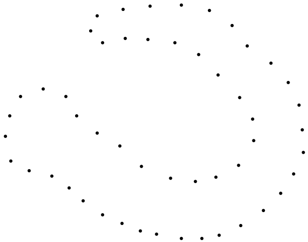

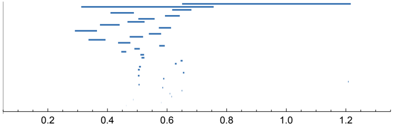

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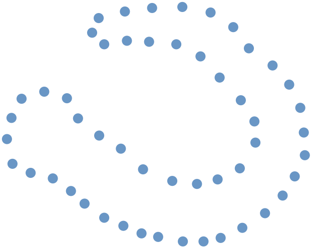

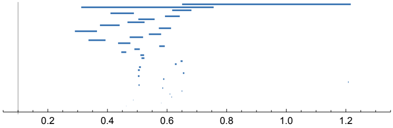

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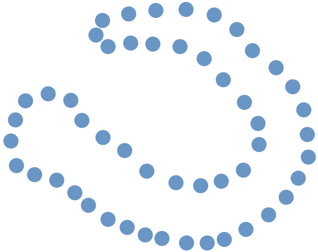

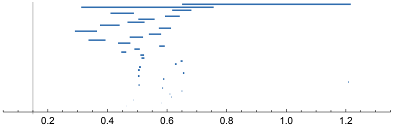

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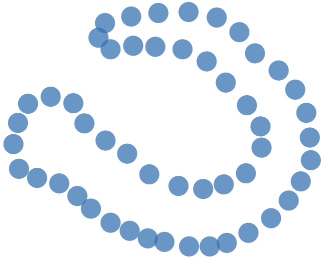

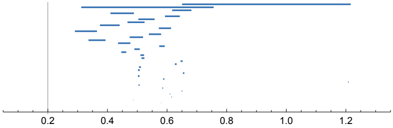

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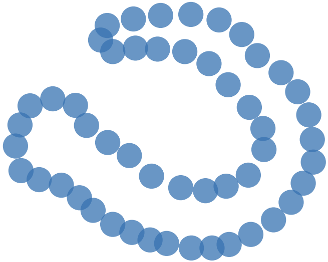

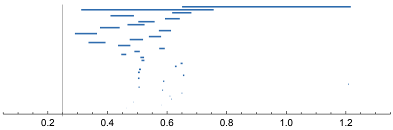

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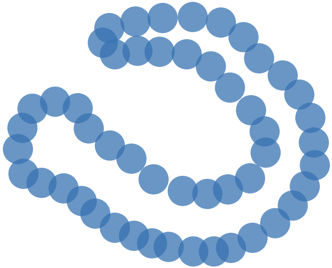

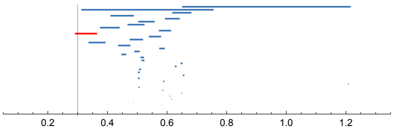

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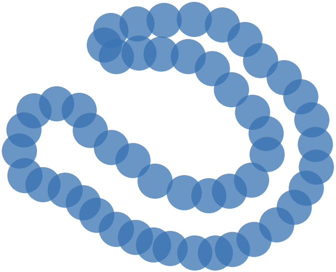

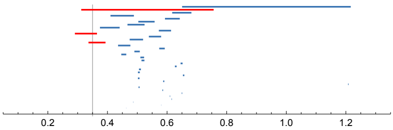

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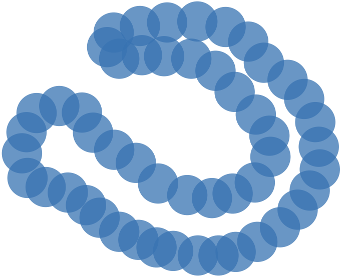

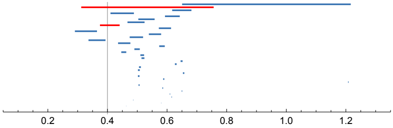

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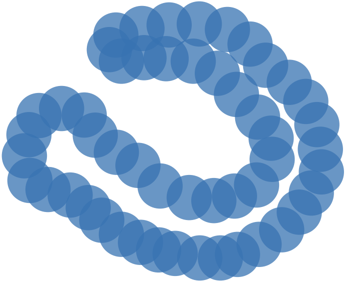

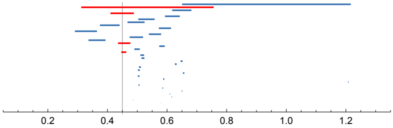

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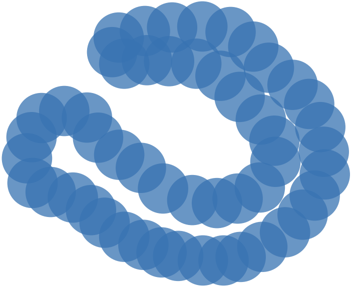

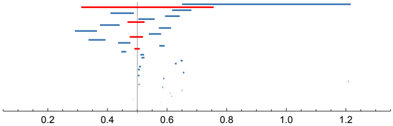

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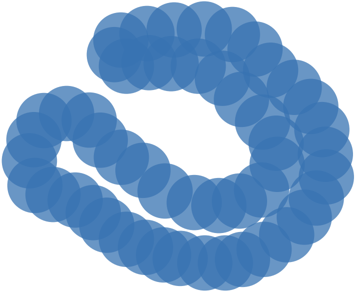

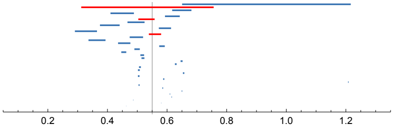

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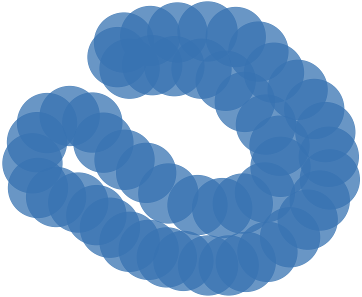

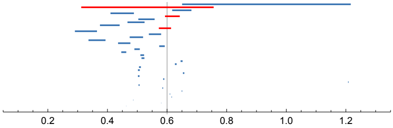

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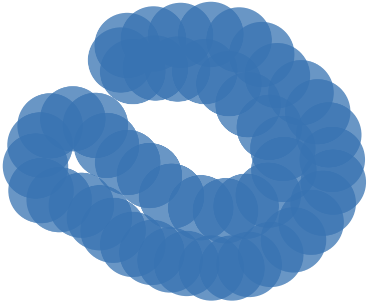

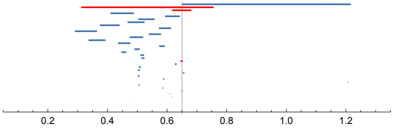

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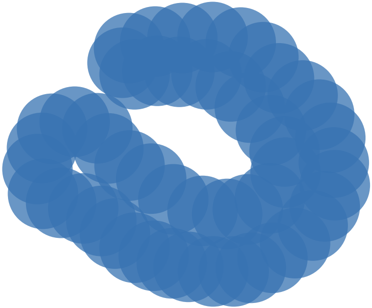

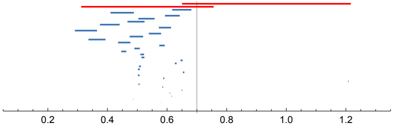

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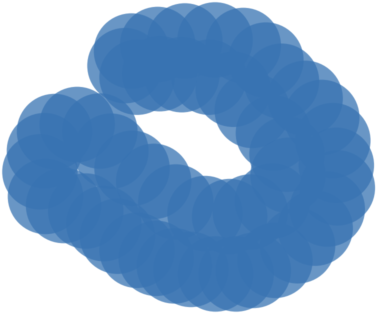

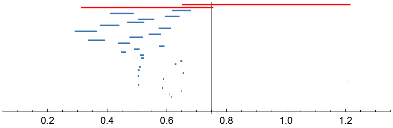

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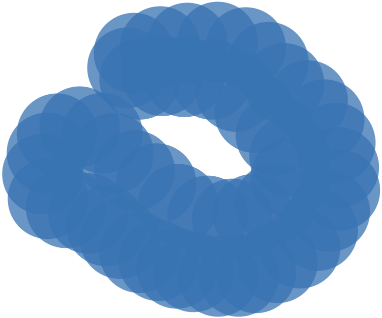

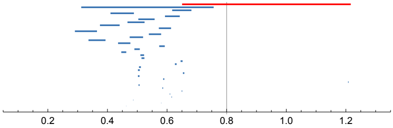

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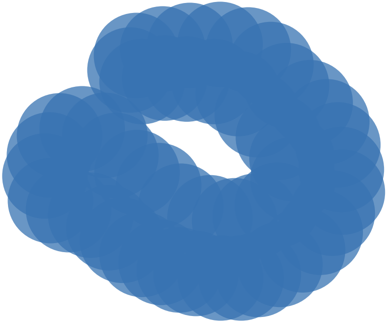

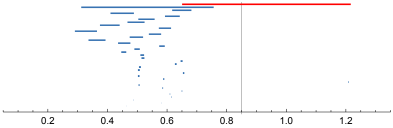

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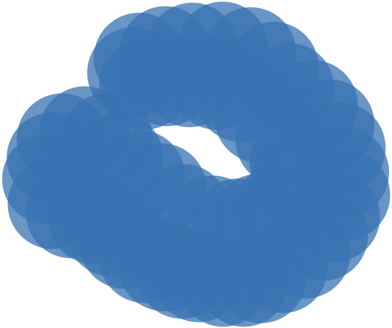
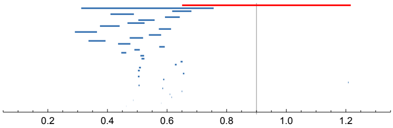




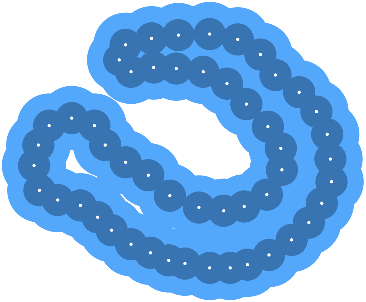
Homology inference using persistence

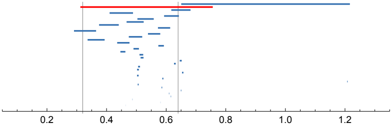
Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\Omega \subset \mathbb{R}^d$. Let $P \subset \Omega$, $\delta > 0$ be such that

- *$B_\delta(P)$ covers Ω, and*
- *the inclusions $\Omega \hookrightarrow B_\delta(\Omega) \hookrightarrow B_{2\delta}(\Omega)$ preserve homology.*

Then $H_*(\Omega) \cong \operatorname{im} H_*(B_\delta(P) \hookrightarrow B_{2\delta}(P))$.





Homological realization

This motivates the *homological realization problem*:

Problem

Given a simplicial pair $L \subseteq K$*, find* X *with* $L \subseteq X \subseteq K$ *such that*

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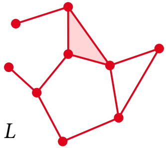

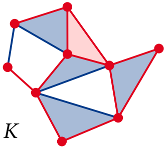

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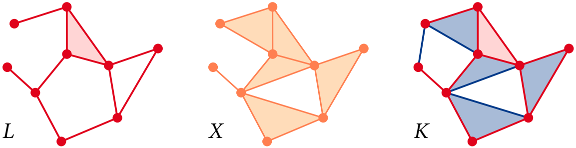

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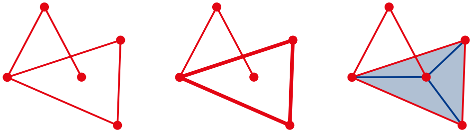

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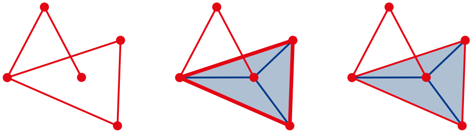

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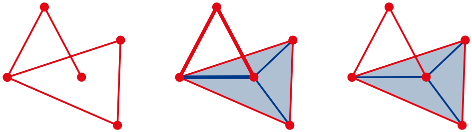

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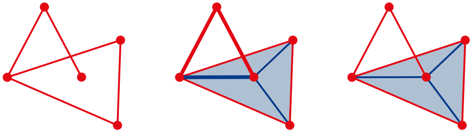

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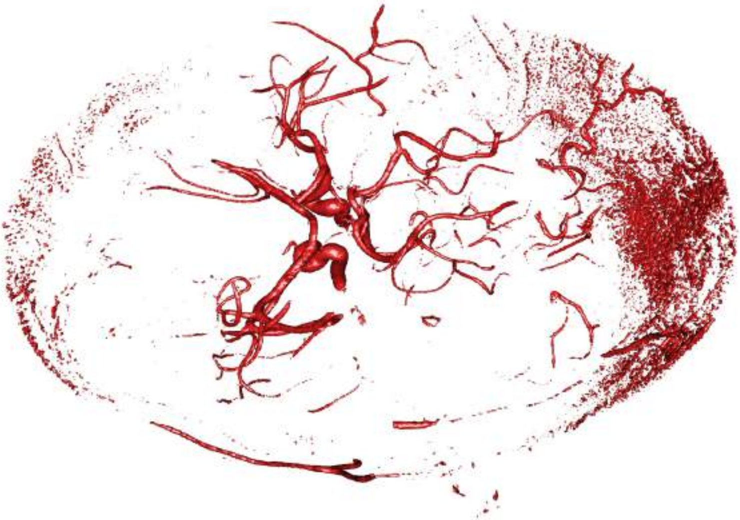
Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

The homological realization problem is NP-hard, even in $\mathbb{R}^3$.

Simplification



Sublevel set simplification

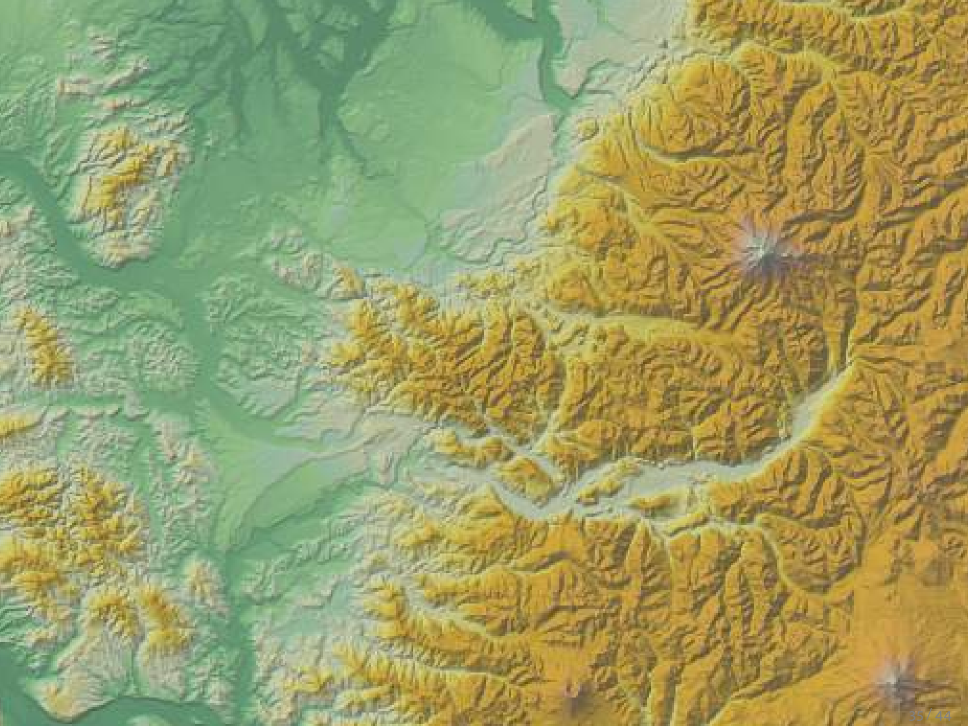
Let $F_{\leq t} = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem (Sublevel set simplification)

Given a function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ and $t \in \mathbb{R}$, $\delta > 0$,
find a function g with $\|g - f\|_\infty \leq \delta$ minimizing $\dim H_*(G_{\leq t})$.

Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

Sublevel set simplification in $\mathbb{R}^3$ is NP-hard.



Topological simplification of functions

Consider the following problem:

Problem (Topological simplification)

Given a function f and a real number $\delta \geq 0$,
find a function f_δ subject to $\|f_\delta - f\|_\infty \leq \delta$
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Persistent homology:

- Relates homology of different sublevel set
- Identifies pairs of critical points (birth and death of homological feature) and quantifies their *persistence*

(Discrete) Morse theory:

- Relates critical points to homology of sublevel sets
- Provides method for canceling pairs of critical points

Persistence and discrete Morse theory

By stability of persistence barcodes:

Proposition

The intervals with persistence $> 2\delta$ *provide a lower bound on the number of critical points of any function* g *with* $\|g - f\|_\infty \leq \delta$.

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Theorem (B, Lange, Wardetzky, 2011)

Let f *be a function on a surface and let* $\delta > 0$.
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and achieves the lower bound on the number of critical points.

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- Does not generalize to higher-dimensional manifolds!

History

When was persistent homology invented?

- [Edelsbrunner/Letscher/Zomorodian 2000]

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- [Leray 1946]?

When was persistent homology invented first?

When was persistent homology invented first?

Annals of Mathematics
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

By Marston Morse

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses is satisfied. The critical limits cluster at most at $+\infty$; the critical points are isolated;[1] the problem is defined by analytic functions; the critical limits taken in their natural order are well-ordered. These conditions are not generally fulfilled. The generality of the theory rested upon the fact that the cases treated approximate in a certain sense the most general problems which it is

When was persistent homology invented first?

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Rank and span in functional topology

Search within citing articles

Exact homomorphism sequences in homology theory — ed.ac.uk [PDF]

JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR

The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] ...

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Marston Morse and his mathematical works — ams.org [PDF]

R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org

American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to ...

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Unstable minimal surfaces of higher topological structure

M Morse, CB Tompkins - Duke Math. J, 1941 - projecteuclid.org

1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. Theproblem of the existenceof minimal surfaces of unstable type contains many of the typical difficulties, especially those ofa topological nature. Having studied this ...

Cited by 19 Related articles All 2 versions Cite Save

[PDF] **Persistence in discrete Morse theory** — psu.edu [PDF]

U Bauer - 2011 - Citeseer

The goal of this thesis is to bring together two different theories about critical points of a scalar function and their relation to topology: Discrete Morse theory and Persistent

When was persistent homology invented first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was in the 1920's to go from the mini-max principle of Birkhoff to the Morse inequalities, let alone extend these inequalities to function spaces, so that by the early 30's Morse could establish the theorem that for any Riemann

When was persistent homology invented first?

inequalities pertain between the dimensions of the A_i and those of $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels [D] in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers [36] and [44] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.

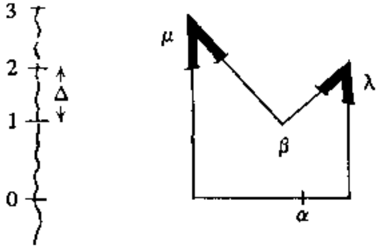


FIGURE 8

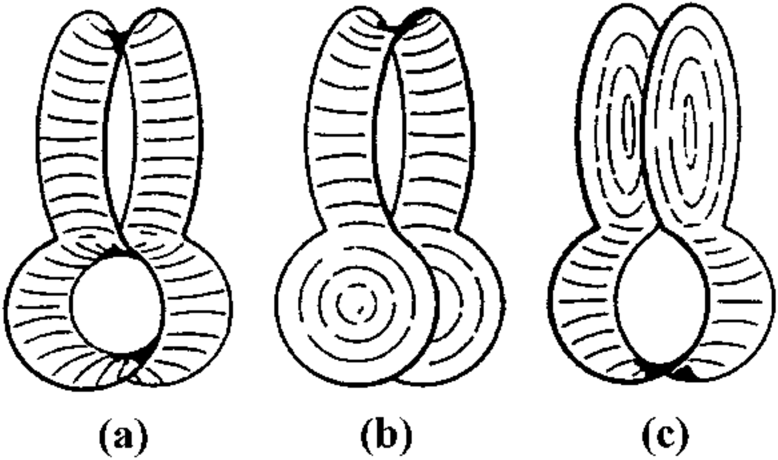
The values a where $H(a, a^-) \neq 0$ are indicated on the left, and correspond

Morse's *functional topology*

Key aspects:

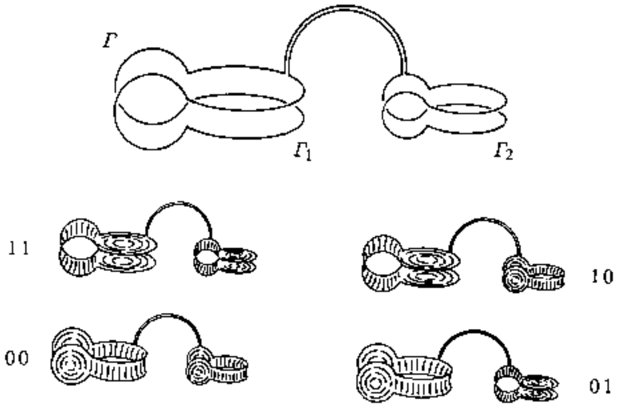
- early precursor of persistence and spectral sequences
- uses Vietoris homology with field coefficients
- applies to a broad class of functions on metric spaces (not necessarily continuous)
- inclusions of sublevel sets have finite rank homology (*q-tame* persistent homology)
- focus on controlled behavior in pathological cases

Motivation and application: minimal surfaces



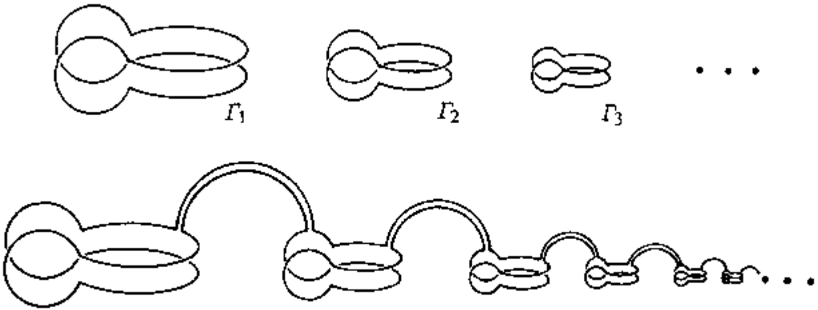
(from Dierkes et al.: Minimal Surfaces, Springer 2010)

Motivation and application: minimal surfaces



(from Dierkes et al.: Minimal Surfaces, Springer 2010)

Motivation and application: minimal surfaces



(from Dierkes et al.: Minimal Surfaces, Springer 2010)

Existence of unstable minimal surfaces

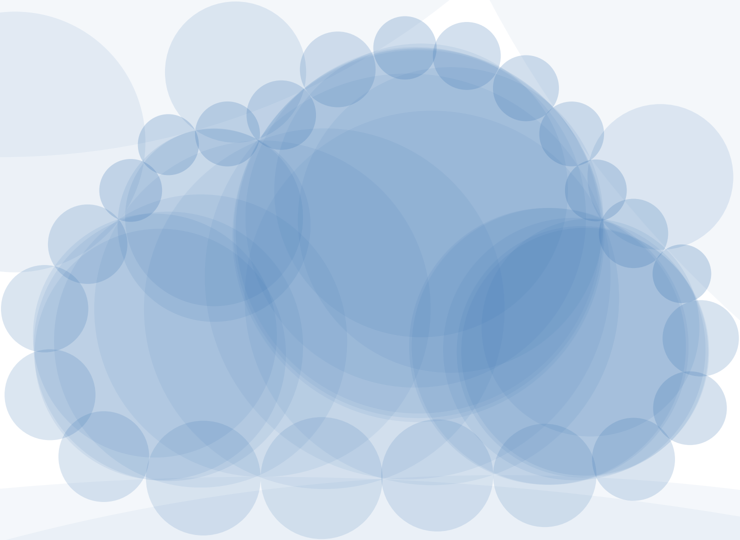
Using persistent homology:

- Number of ϵ-persistent critical points (minimal surfaces) is finite for any $\epsilon > 0$
- Morse inequalities for ϵ-persistent critical points

Theorem (Morse, Tompkins 1939)

There is a C_1 curve bounding an unstable minimal surface (an index 1 critical point of the area functional).

Thanks for your attention!



Thanks for your attention!