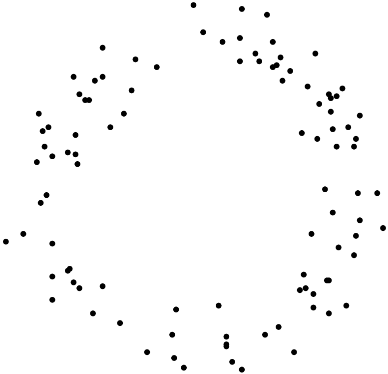
Persistent connections

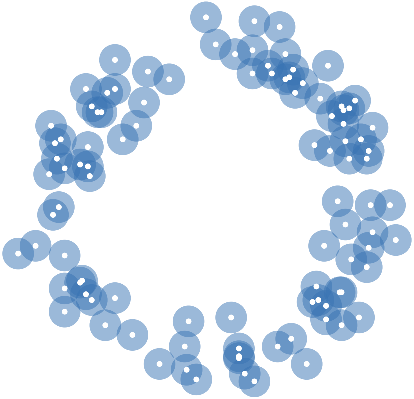
Ulrich Bauer

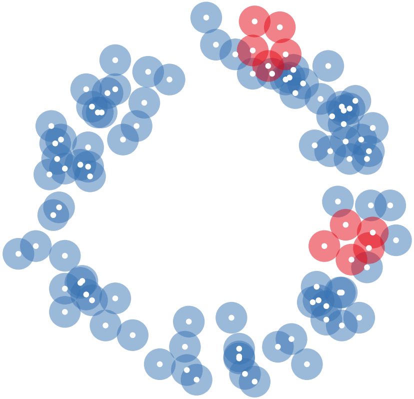
TUM

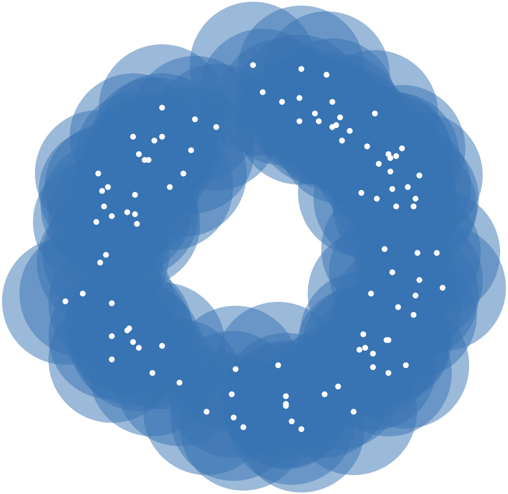
January 9, 2017

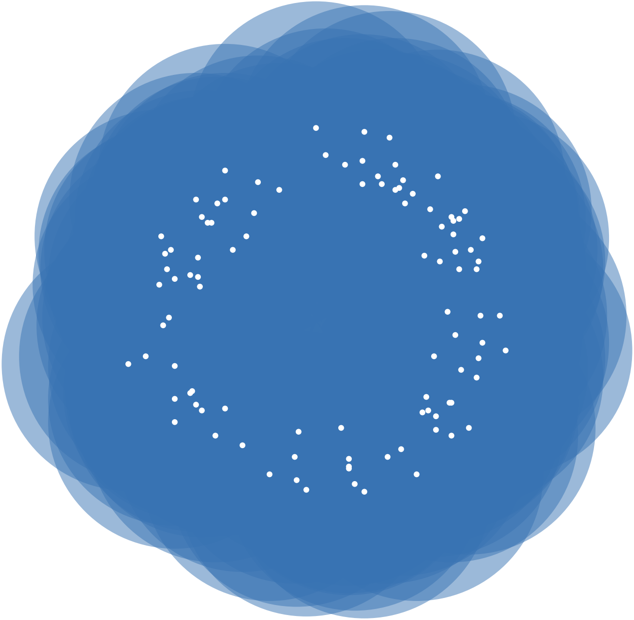
Persistent homology

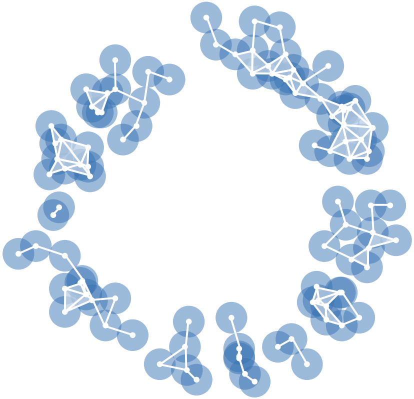


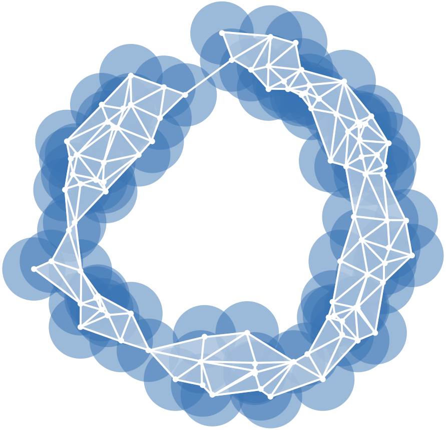


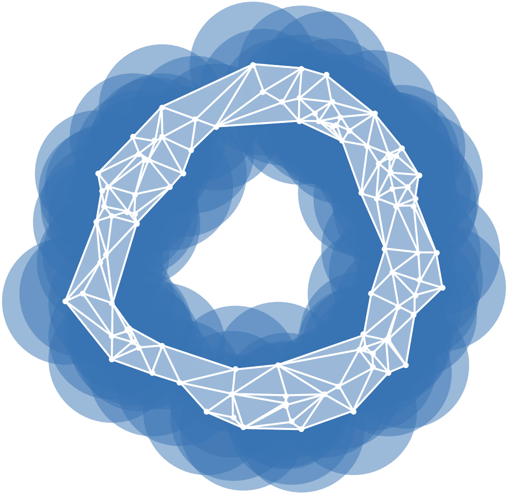


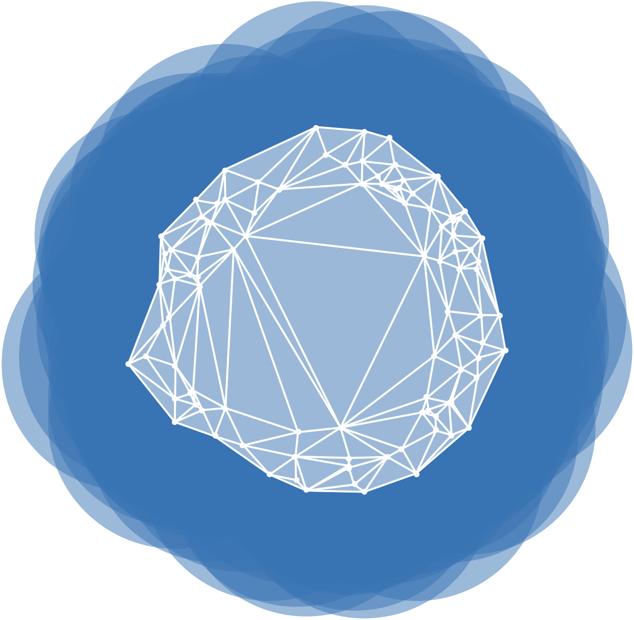




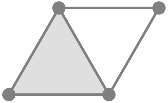








What is homology?



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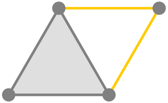

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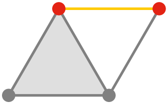

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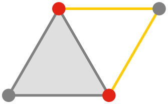

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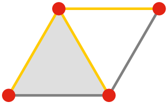

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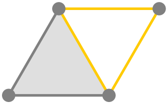

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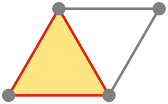

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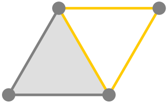

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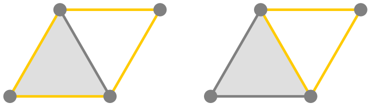

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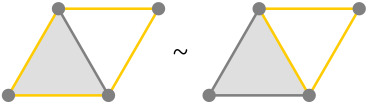

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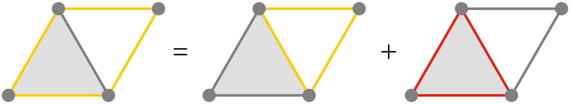

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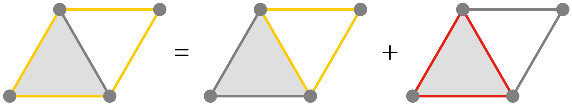

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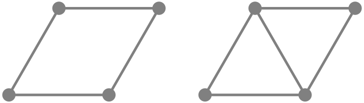
- The equivalence classes form the *homology group* $H_*(K) = Z_*(K)/B_*(K)$

What is homology?

Roughly: $\dim H_*(K)$ is the number of holes



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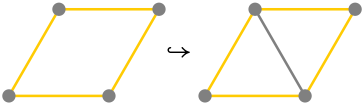

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- An inclusion map $L \hookrightarrow K$ induces a linear map $H_*(L) \to H_*(K)$

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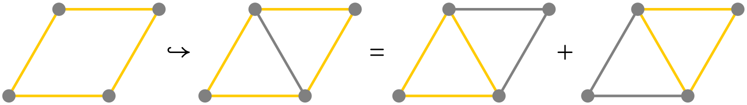

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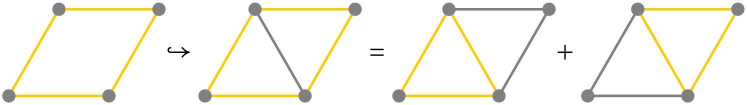

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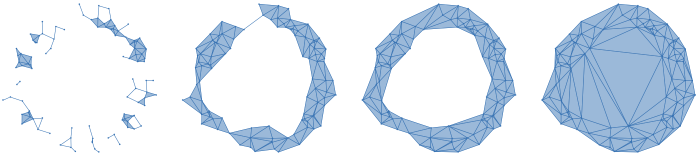
Homology does not have a canonical basis

- Computing (persistent) homology is all about choosing (compatible) bases

What is persistent homology?



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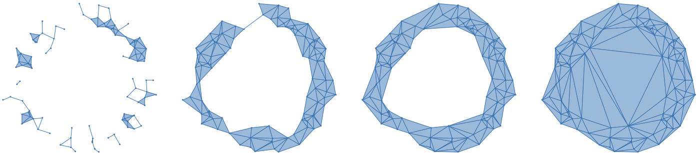

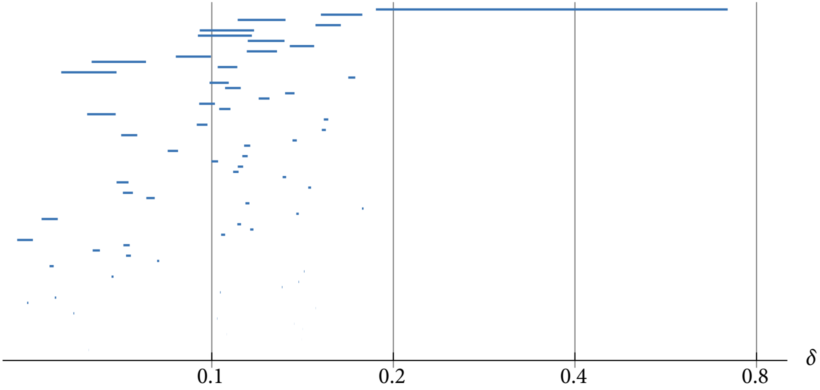

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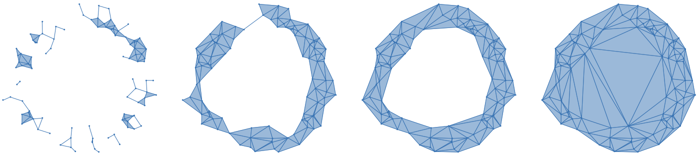

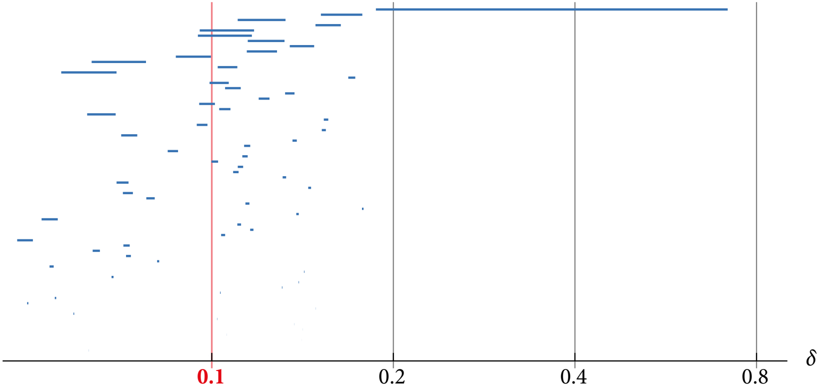

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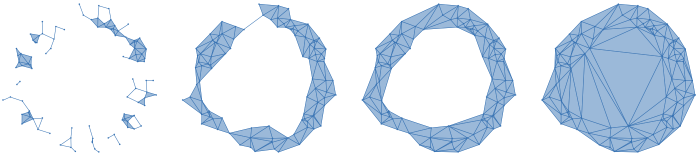

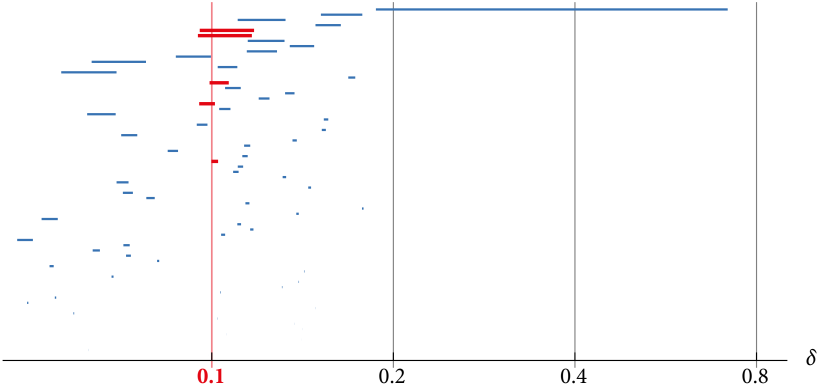

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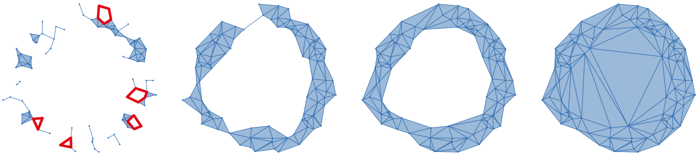

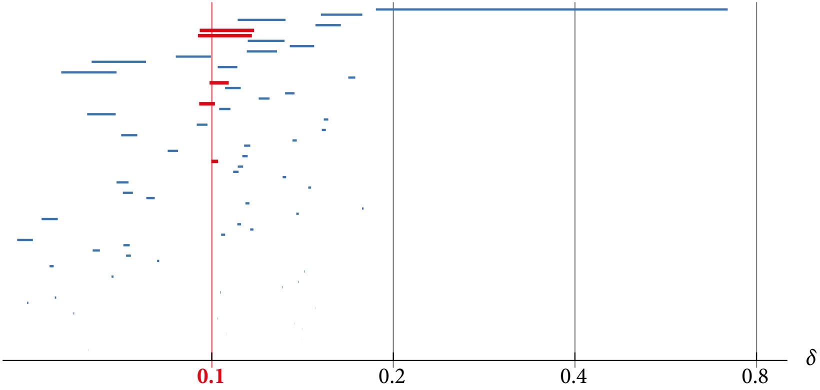

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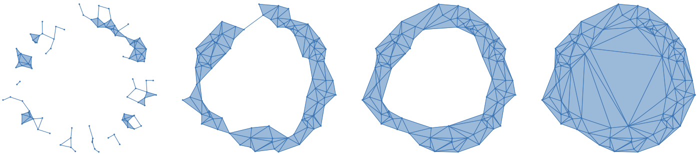

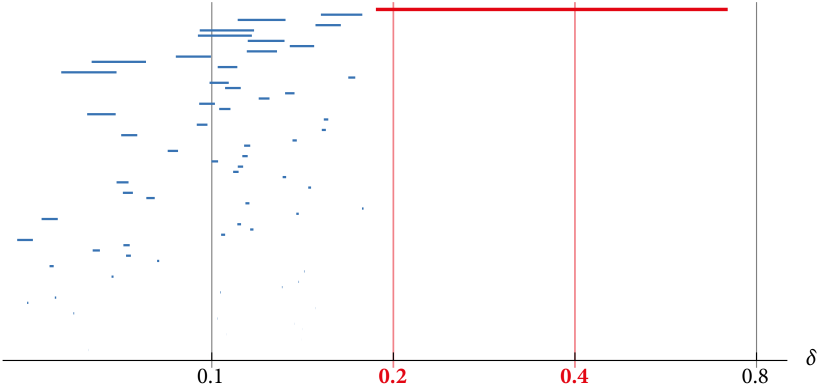

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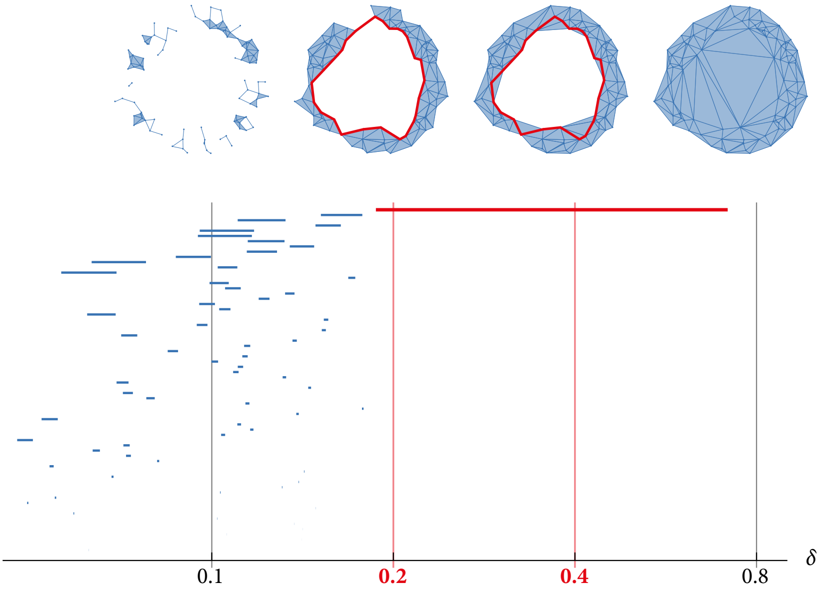

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- A filtration is a certain diagram $K : \mathbf{R} \to \mathbf{Top}$, where $\mathbf{R}$ is the poset category of $(\mathbb{R}, \leq)$:
  - A topological space K_t for each $t \in \mathbb{R}$
  - A continuous map $K_s \to K_t$ for each $s \leq t \in \mathbb{R}$

Stability

The pipeline of topological data analysis

$$
\begin{tikzcd}
\text{point cloud} \arrow[d, "\text{distance}"] \\
\text{function} \arrow[d, "\text{sublevel sets}"] \\
\text{topological spaces} \arrow[d, "\text{homology}"] \\
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$P \subset \mathbb{R}^d$

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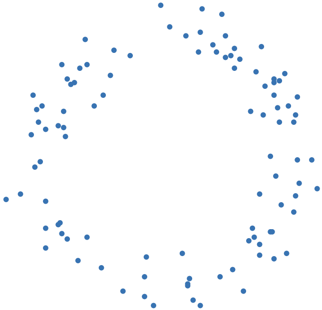
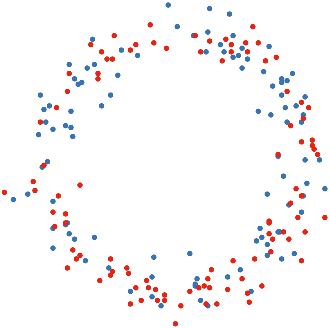
Geometric interleavings



Geometric interleavings



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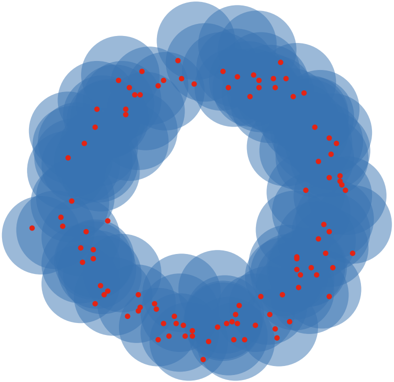

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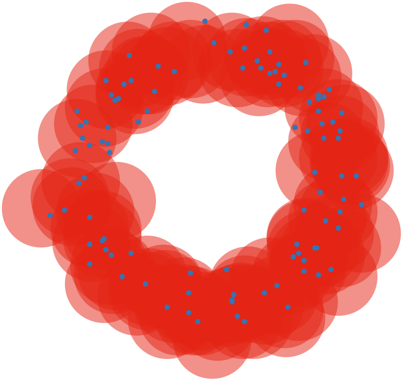

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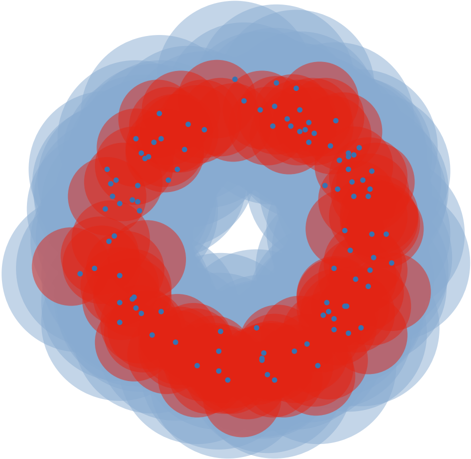

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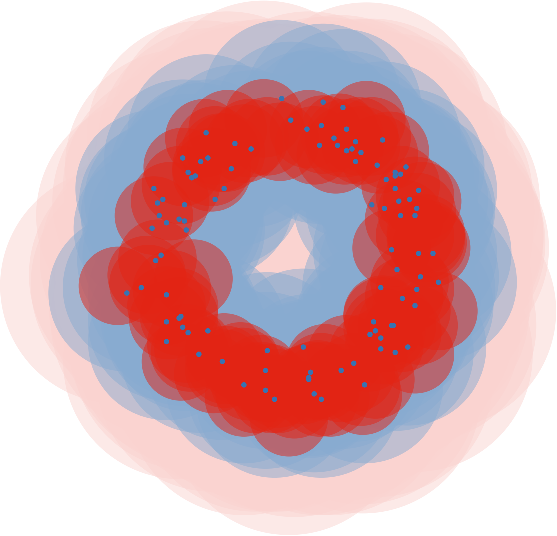

Interleavings of sublevel sets

Let

- $F_t = f^{-1}(-\infty, t]$,
- $G_t = g^{-1}(-\infty, t]$.

If $\|f - g\|_\infty \leq \delta$ then $F_t \subseteq G_{t+\delta}$ and $G_t \subseteq F_{t+\delta}$.

So the sublevel sets are δ-*interleaved:*

$$\begin{tikzcd}
F_t \arrow[rr, hook] \arrow[dr, hook] & & F_{t+2\delta} \arrow[dr, hook] & \\
& G_{t+\delta} \arrow[ur, hook] \arrow[rr, hook] & & G_{t+3\delta}
\end{tikzcd}$$

Interleavings

Definition

Two diagrams (functors) M, N indexed over the reals (as a poset category) are δ-*interleaved* if there are natural transformations

$$\mu_t : M_t \to N_{t+\delta}, \qquad \nu_t : N_t \to M_{t+\delta} \qquad \forall t \in \mathbb{R}$$

such that this diagrams commutes for all t:

$$\begin{tikzcd}
M_t \arrow[rr] \arrow[dr, "\mu_t"] & & M_{t+2\delta} \arrow[dr, "\mu_{t+2\delta}"] & \\
& N_{t+\delta} \arrow[ur, "\nu_{t+\delta}"] \arrow[rr] & & N_{t+3\delta}
\end{tikzcd}$$

- Interleavings are approximate isomorphisms, up to an index shift of δ
- Interleaving distance: infimum δ admitting an interleaving

Stability via functoriality?

$$\begin{tikzcd}
F_t \arrow[rr, hook] \arrow[dr, hook] & & F_{t+2\delta} \arrow[dr, hook] & \\
& G_{t+\delta} \arrow[ur, hook] \arrow[rr, hook] & & G_{t+3\delta}
\end{tikzcd}$$

Stability via functoriality?

$$\begin{tikzcd}
H_*(F_t) \arrow[rr] \arrow[dr] & & H_*(F_{t+2\delta}) \arrow[dr] & \\
& H_*(G_{t+\delta}) \arrow[ur] \arrow[rr] & & H_*(G_{t+3\delta})
\end{tikzcd}$$

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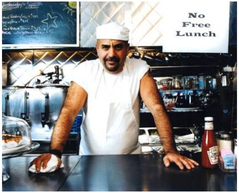
Stability via functoriality?

$$\begin{tikzcd}[column sep=small]
B(H_*(F_t)) \arrow[rr] \arrow[dr] & & B(H_*(F_{t+2\delta})) \arrow[dr] & \\
& B(H_*(G_{t+\delta})) \arrow[ur] \arrow[rr] & & B(H_*(G_{t+3\delta}))
\end{tikzcd}$$

Stability via functoriality?

$$\begin{tikzcd}
B(H_*(F_t)) \arrow[r] \arrow[dr] & B(H_*(F_{t+2\delta})) \arrow[dr] & \\
& B(H_*(G_{t+\delta})) \arrow[u] \arrow[r] & B(H_*(G_{t+3\delta}))
\end{tikzcd}$$



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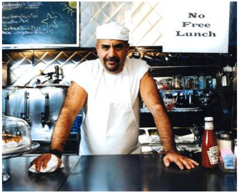


Proposition (B, Lesnick 2015)

There exists no functor $\mathbf{Vect} \to \mathbf{Mch}$ *sending each vector space of dimension* d *to a set of cardinality* d.

Interleavings and (co)kernels

Consider a δ-interleaving in $\mathbf{Vect}$ or $\mathbf{Mch}$:

$$\begin{tikzcd}
M_t \arrow[rr] \arrow[dr, "\mu_t"] & & M_{t+2\delta} \arrow[dr, "\mu_{t+2\delta}"] & \\
& N_{t+\delta} \arrow[ur, "\nu_{t+\delta}"] \arrow[rr] & & N_{t+3\delta}
\end{tikzcd}$$

Proposition

The diagrams $\ker \mu$ *and* $\operatorname{coker} \mu$ *are* 2δ*-trivial.*

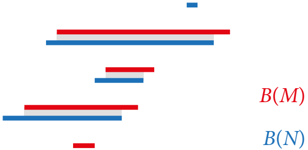
- A diagram K is ϵ-*trivial* if the maps $K_t \to K_{t+\epsilon}$ are zero.
- For example, if a barcode $B : \mathbf{R} \to \mathbf{Mch}$ is ϵ-trivial, then the intervals of B have length at most ϵ.

The induced matching theorem

Theorem (B, Lesnick 2015, 2016)

Let μ be a natural transformation between $M, N : \mathbf{R} \to \mathbf{Vect}$. Then

- *μ induces a natural transformation $X(\mu)$ between the barcodes $B(M), B(N) : \mathbf{R} \to \mathbf{Mch}$.*
- *If $\ker \mu$ is ϵ-trivial then so is $\ker X(\mu)$, and likewise for coker.*



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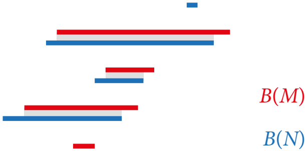

This implies the algebraic stability for persistence barcodes:

Theorem (Chazal et al. 2009; B, Lesnick 2015, 2016)

Two diagrams $M, N : \mathbf{R} \to \mathbf{Vect}$ are δ-interleaved if and only if their barcodes $B(M), B(N)$ are.

History

When was persistent homology invented?

- [Edelsbrunner/Letscher/Zomorodian 2000]

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- [Robbins 1999]

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- [Robbins 1999]
- [Frosini 1990]
- [Leray 1946]?

When was persistent homology invented first?

When was persistent homology invented first?

ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses is satisfied. The critical limits cluster at most at $+\infty$; the critical points are isolated;[1] the problem is defined by analytic functions; the critical limits taken in their natural order are well-ordered. These conditions are not generally fulfilled. The generality of the theory rested upon the fact that the cases treated approximate in a certain sense the most general problems which it is

When was persistent homology invented first?

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Rank and span in functional topology

Search within citing articles

Exact homomorphism sequences in homology theory
ed.ac.uk **[PDF]**

JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR

The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] **...**

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Marston Morse and his mathematical works
ams.org **[PDF]**

R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org

American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to **...**

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Unstable minimal surfaces of higher topological structure

M Morse, CB Tompkins - Duke Math. J, 1941 - projecteuclid.org

1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. Theproblem of the existenceof minimal surfaces of unstable type contains many of the typical difficulties, especially those ofa topological nature. Having studied this **...**

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[PDF] Persistence in discrete Morse theory
psu.edu **[PDF]**

U Bauer - 2011 - Citeseer

The goal of this thesis is to bring together two different theories about critical points of a scalar function and their relation to topology: Discrete Morse theory and Persistent

When was persistent homology invented first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was in the 1920's to go from the mini-max principle of Birkhoff to the Morse inequalities, let alone extend these inequalities to function spaces, so that by the early 30's Morse could establish the theorem that for any Riemann

When was persistent homology invented first?

inequalities pertain between the dimensions of the A_i and those of $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels [**D**] in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers [**36**] and [**44**] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.

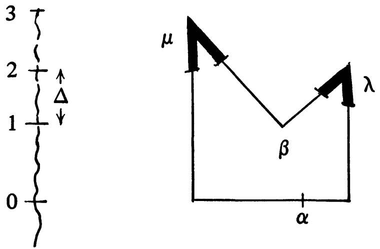

FIGURE 8

The values a where $H(a, a^-) \neq 0$ are indicated on the left, and correspond

Morse's *functional topology*

Key aspects:

- early precursor of persistence and spectral sequences

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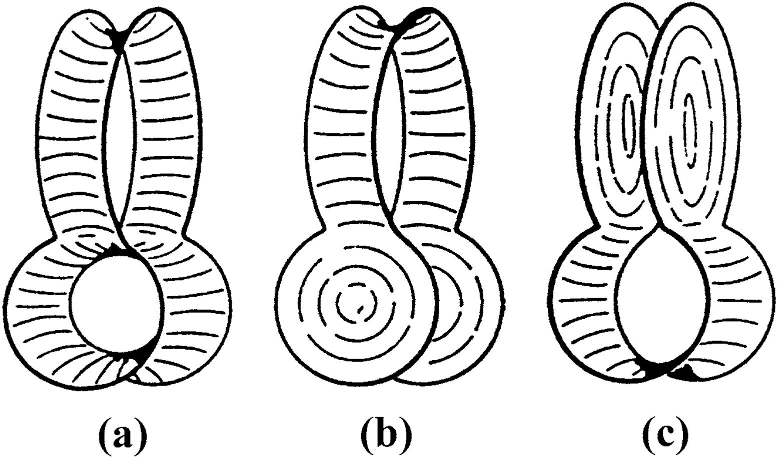
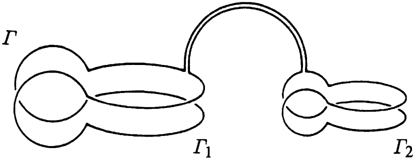
- inclusions of sublevel sets have finite rank homology (*q-tame* persistent homology)
- focus on controlled behavior in pathological cases

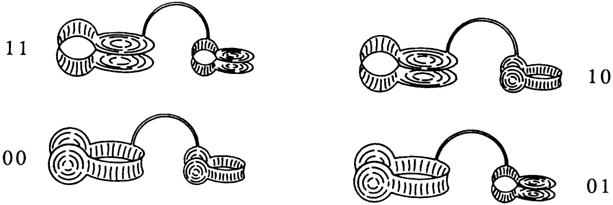
Motivation and application: minimal surfaces



(from Dierkes et al.: Minimal Surfaces, Springer 2010)

Motivation and application: minimal surfaces





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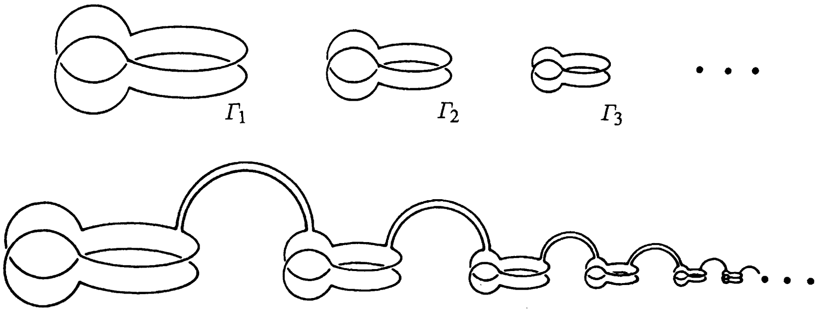


(from Dierkes et al.: Minimal Surfaces, Springer 2010)

Existence of unstable minimal surfaces

Using persistent homology:

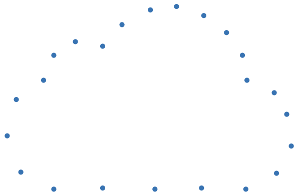
- Number of ϵ-persistent critical points (minimal surfaces) is finite for any $\epsilon > 0$
- Morse inequalities for ϵ-persistent critical points

Theorem (Morse, Tompkins 1939)

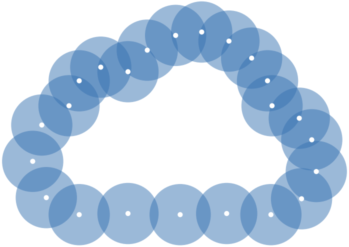
There is a C_1 curve bounding an unstable minimal surface (a critical point of the area functional with index 1).

Complexes

Čech and Delaunay complexes



Čech and Delaunay complexes



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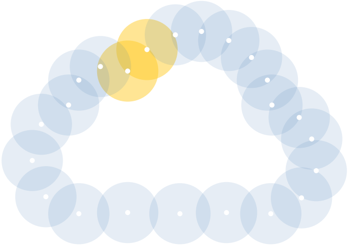

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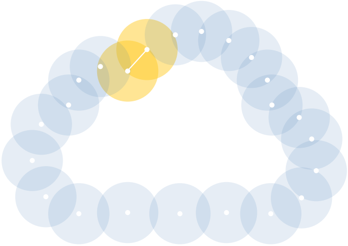

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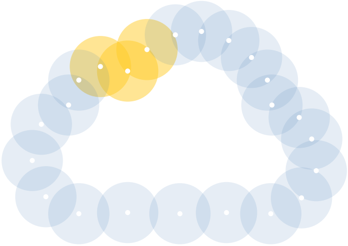

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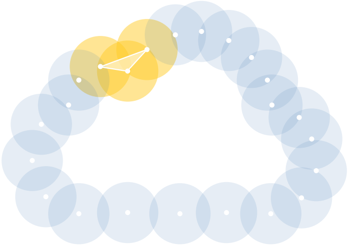

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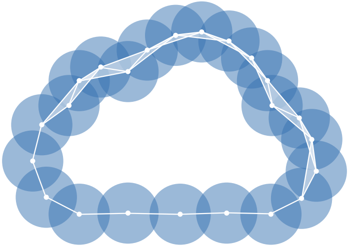
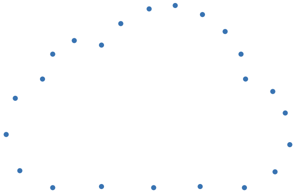


$$\mathrm{Cech}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} B_r(p) \neq \emptyset \right\}$$

Čech and Delaunay complexes



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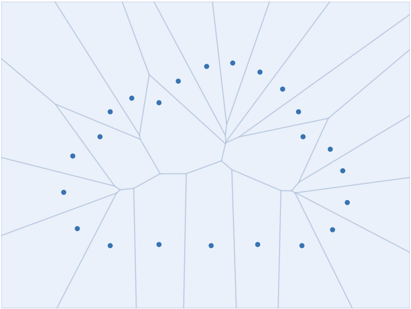

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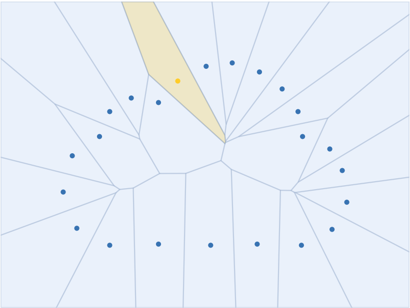
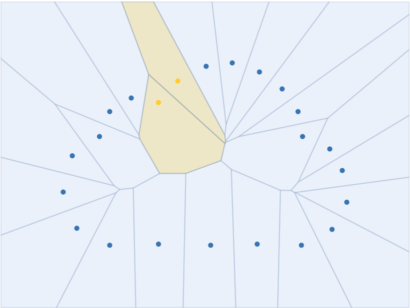


$$\mathrm{Vor}(p, X) = \left\{a \in \mathbb{R}^n \mid d(a,p) = d(a,X)\right\}$$

Čech and Delaunay complexes



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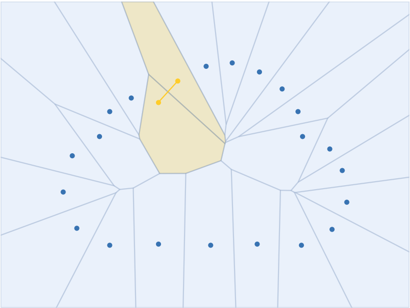

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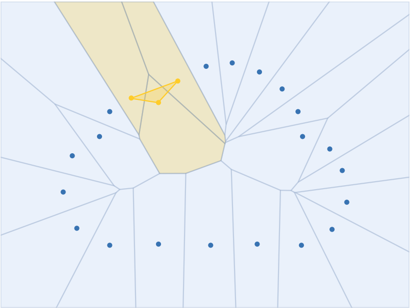

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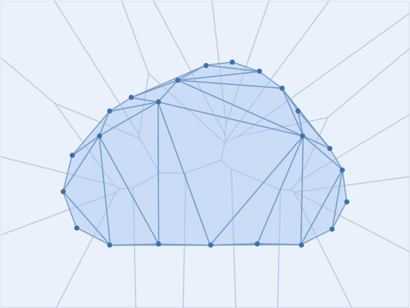
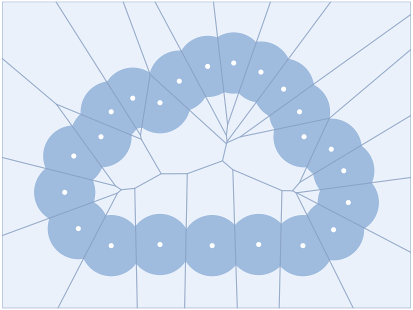


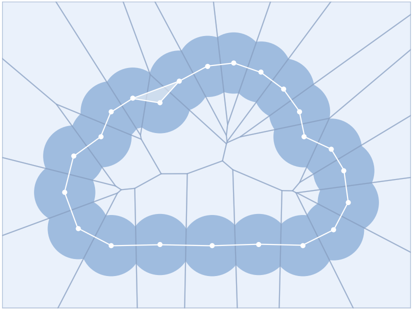
$$\mathrm{Vor}(p, X) = \left\{a \in \mathbb{R}^n \mid d(a, p) = d(a, X)\right\}$$
$$\mathrm{Del}(X) = \left\{Q \subseteq X \mid \bigcap_{p \in Q} \mathrm{Vor}(p, X) \neq \varnothing\right\}$$

Čech and Delaunay complexes



$$\mathrm{Vor}_r(p, X) = B_r(p) \cap \mathrm{Vor}(p, X)$$

Čech and Delaunay complexes



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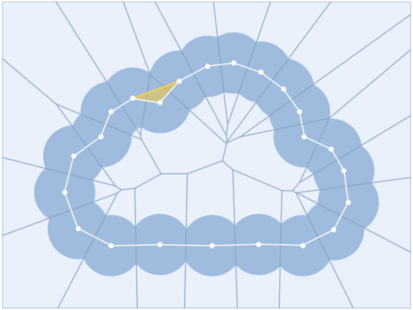

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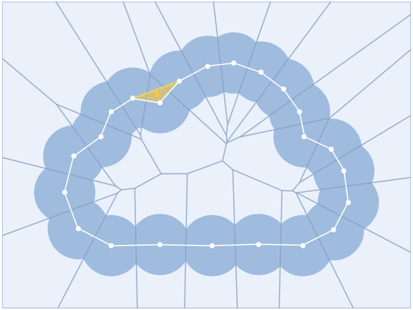

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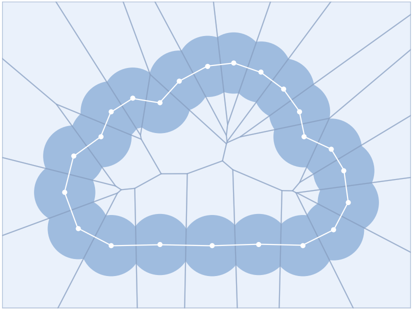

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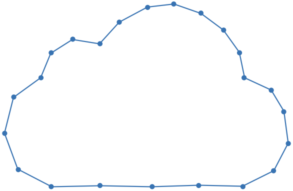

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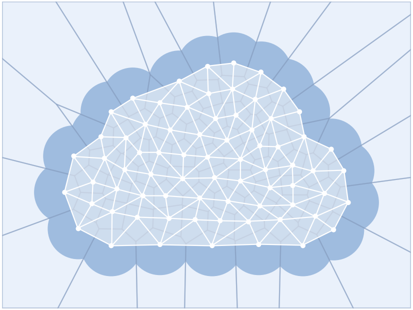

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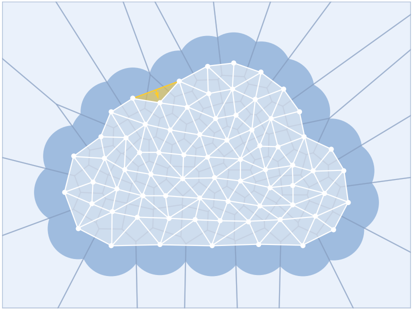

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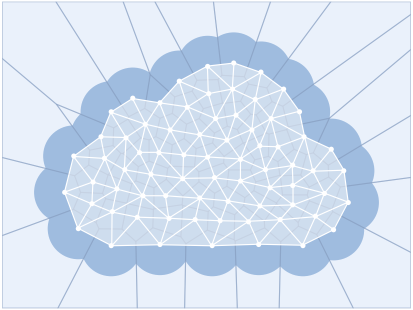

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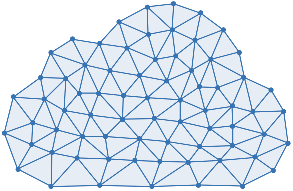

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Definition (Forman 1998)

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- singleton sets $\{\phi\}$ *(critical cells)*, and
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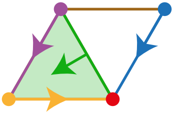

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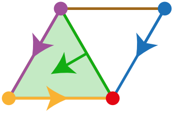

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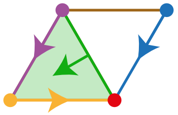

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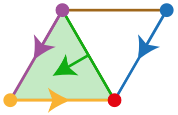

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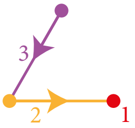

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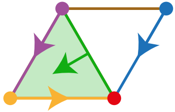

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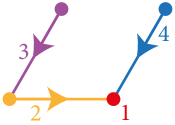

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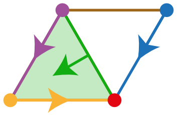

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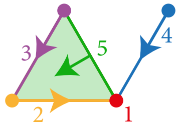

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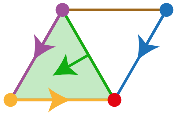

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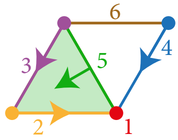

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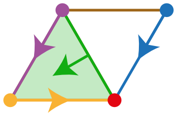

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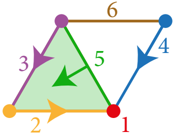

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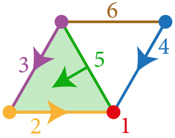
If $(s, t]$ contains no critical value of f, then the sublevel set K_t collapses to K_s *(written as $K_t \searrow K_s$).*



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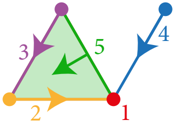

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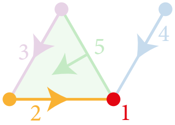

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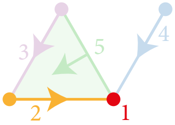

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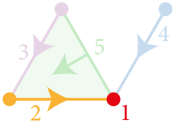

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This homotopy equivalence is compatible with the filtration.

Corollary

K and M have isomorphic persistent homology (with regard to f).

Morse theory for Čech and Delaunay complexes

Proposition

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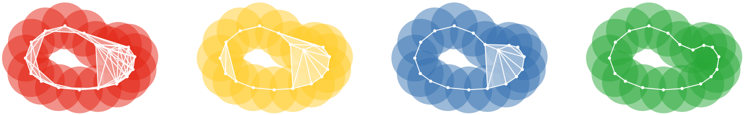
Theorem (B., Edelsbrunner 2016, Trans. AMS)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are naturally homotopy equivalent through a sequence of collapses

$$\mathrm{Cech}_r X \searrow \mathrm{Cech}_r X \cap \mathrm{Del}\, X \searrow \mathrm{Del}_r X \searrow \mathrm{Wrap}_r X.$$



Vietoris–Rips complexes

Consider a finite metric space (X, d).

The *Vietoris–Rips complex* is the simplicial complex

$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid \operatorname{diam} S \leq t\}$$

- 1-skeleton: all edges with pairwise distance $\leq t$
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Note: for $t \geq \operatorname{diam} X$, $\mathrm{Rips}_t(X)$ is the full simplex with vertices X

- Number of simplices grows exponentially in dimension
- Computation is one of the most important challenges in applied topology!

Computation

Computing Vietoris–Rips persistence

Goal:

- compute homology $H_d(\mathrm{Rips}_t(X))$
  (for all t and $0 \leq d < k$)
- together with induced maps $H_d(\mathrm{Rips}_s(X) \hookrightarrow \mathrm{Rips}_t(X))$
  (for all $s \leq t$)

Demo: Ripser

Example data set:

- 192 points on $\mathbb{S}^2$
- persistent homology barcodes up to dimension 2
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The four special ingredients

The improved performance is based on 4 insights:

- Skip inessential columns
- Compute cohomology
- Implicit boundary matrix
- Apparent pairs

Computing homology

Computing homology $H_* = Z_*/B_*$ (recall: $B_* \subseteq Z_* \subseteq C_*$):

- compute basis for boundaries $B_* = \operatorname{im} \partial_*$
- extend to basis for cycles $Z_* = \ker \partial_*$
- new (non-boundary) basis cycles generate quotient Z_*/B_*

Homology by matrix reduction

Notation:

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- $R = D, V = I$
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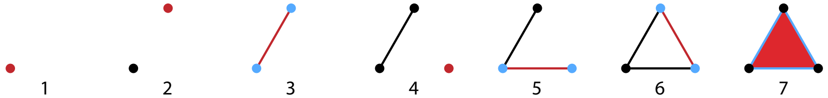
Result:

- $R = D \cdot V$ is reduced (each column has a unique pivot)
- V is full rank upper triangular

Matrix reduction



	1	2	3	4	5	6	7
1			1		1		
2			1			1	
3							1
4					1	1	
5							1
6							1
7							

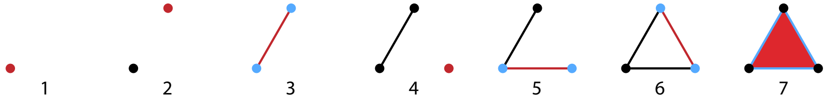
R

$= D \cdot$

	1	2	3	4	5	6	7
1	1						
2		1					
3			1				
4				1			
5					1		
6						1	
7							1

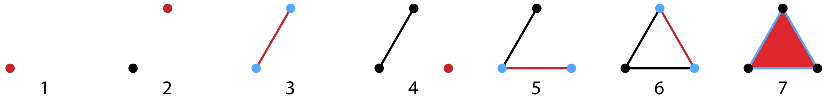
V

Matrix reduction



$$
\underbrace{\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & & & 1 & & 1 & & \\
2 & & & {\color{red}1} & & & 1 & \\
3 & & & & & & & 1 \\
4 & & & & & {\color{red}1} & {\color{red}1} & \\
5 & & & & & & & 1 \\
6 & & & & & & & {\color{red}1} \\
7 & & & & & & &
\end{array}}_{R}
= D \cdot
\underbrace{\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 1 & & & & & & \\
2 & & 1 & & & & & \\
3 & & & 1 & & & & \\
4 & & & & 1 & & & \\
5 & & & & & 1 & & \\
6 & & & & & & 1 & \\
7 & & & & & & & 1
\end{array}}_{V}
$$

Matrix reduction



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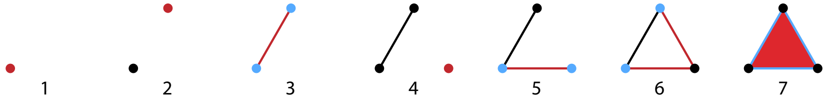

R =

	1	2	3	4	5	6	7
1			1		1	1	
2			1			1	
3							1
4					1	0	
5							1
6							1
7							

$= D \cdot$ **V**, where **V** =

	1	2	3	4	5	6	7
1	1						
2		1					
3			1				
4				1			
5					1	1	
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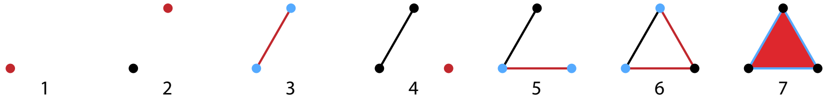
7							1

Matrix reduction



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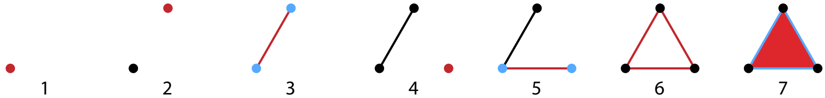

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5							1
6							1
7							

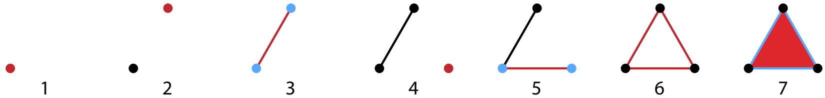
R

$= D \cdot$

	1	2	3	4	5	6	7
1	1						
2		1					
3			1			1	
4				1			
5					1	1	
6						1	
7							1

V

Matrix reduction



$$\underbrace{\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & & & 1 & & 1 & & \\
2 & & & 1 & & & & \\
3 & & & & & & & 1 \\
4 & & & & & 1 & & \\
5 & & & & & & & 1 \\
6 & & & & & & & 1 \\
7 & & & & & & &
\end{array}}_{R} = D \cdot \underbrace{\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 1 & & & & & & \\
2 & & 1 & & & & & \\
3 & & & 1 & & & 1 & \\
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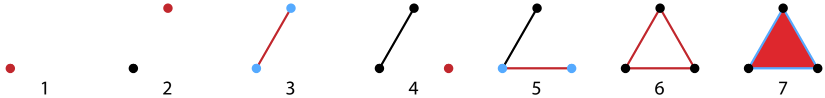
5 & & & & & 1 & 1 & \\
6 & & & & & & 1 & \\
7 & & & & & & & 1
\end{array}}_{V}$$

Matrix reduction



$$
\underbrace{\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & & & 1 & & 1 & & \\
2 & & & 1 & & & & \\
3 & & & & & & & 1 \\
4 & & & & & \color{red}{1} & & \\
5 & & & & & & & 1 \\
6 & & & & & & & 1 \\
7 & & & & & & &
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= D \cdot
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$$

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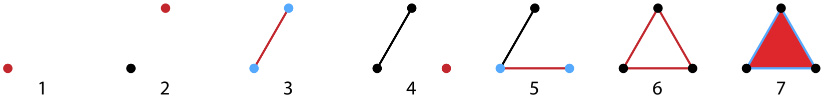

$$\underbrace{\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & \color{red}{5} & 6 & 7 \\ \hline 1 & & & 1 & & 1 & & \\ 2 & & & 1 & & & & \\ 3 & & & & & & & 1 \\ 4 & & & & & \color{red}{1} & & \\ 5 & & & & & & & 1 \\ 6 & & & & & & & 1 \\ 7 & & & & & & & \end{array}}_{R} = D \cdot \underbrace{\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & & & & & & \\ 2 & & 1 & & & & & \\ 3 & & & 1 & & & 1 & \\ 4 & & & & 1 & & & \\ 5 & & & & & 1 & 1 & \\ 6 & & & & & & 1 & \\ 7 & & & & & & & 1 \end{array}}_{V}$$

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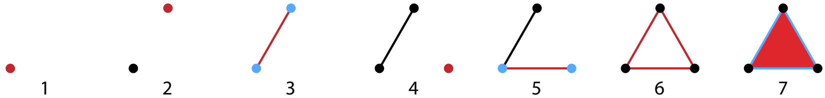

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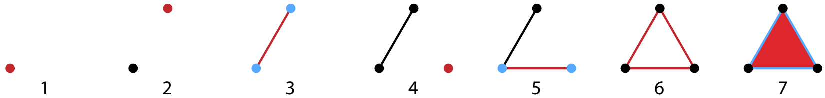

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\end{array}}_{V}$$

Skipping inessential columns

Assume that $R = D \cdot V$ is reduced, V is full rank upper triangular. Then

$$\Sigma_B = \{R_j \mid R_j \neq 0\} \quad \text{is a basis for } B_* = \operatorname{im} D \text{ (boundaries),}$$

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Homology Z_*/B_* is generated by the extra basis cycles Σ_E

- The columns with indices $i = \operatorname{pivot} R_j$ are never used!

Persistent cohomology

We have seen: many columns of $R = D \cdot V$ are not needed

- Skip those inessential columns in matrix reduction

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For persistence barcodes in low dimensions $d < k$:

- Number of skipped indices for reducing D^T (cohomology) is much larger than for D (homology)
  - reducing boundary matrix produces basis for $H_k(K_k)$, which is not needed
- The resulting persistence barcode is the same [de Silva et al. 2011]

Implicit boundary matrix

Typical approach:

- Boundary matrix D with respect to filtration-ordered basis
- Explicitly generated and stored in memory

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- Implicitly defined by number of vertices, dimension
- Rows/columns of D *recomputed* instead of stored

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Matrix reduction in Ripser: store only coefficient matrix V

- recompute columns of $R = D \cdot V$ from D, V
    - Typically, V is much sparser and smaller than R (fewer k-simplices than $(k+1)$-simplices)

Apparent pairs

Apparent pairs

- provide a close connection between persistence and discrete Morse theory
- are both persistence pairs and Morse pairs
- typically cover almost all simplices in the Rips filtration
- provide a shortcut for computation

Natural filtration settings

Typical assumptions on the filtration:

• general filtration	persistence (in theory)
• filtration by singletons or pairs	discrete Morse theory
• simplexwise filtration	persistence (computation)

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Conclusion:

- Discrete Morse theory sits in the middle between persistence and persistence (hah!)

Morse pairs and persistence pairs

Consider a *Morse filtration* (one or two simplices at a time).

Morse pair (σ, τ):

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Consider a *simplexwise filtration* (one simplex at a time).

Persistence pair (σ, τ):

- inserting simplex σ creates a new *homological* feature
- inserting τ destroys that feature again

From Morse theory to persistence and back

Proposition (from Morse to persistence)

The pairs of a Morse filtration are apparent 0-persistence pairs *for the simplexwise refinement of the filtration.*

Apparent persistence pair (σ, τ):

- σ is the youngest face of τ
- τ is the oldest coface of σ

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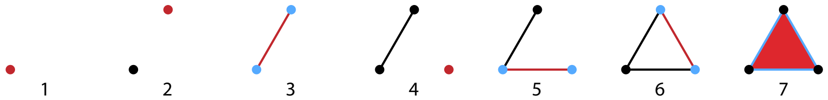
Proposition (from persistence to Morse)

Consider an arbitrary filtration with a simplexwise refinement. The apparent 0*-persistence pairs yield a Morse filtration*

- *refining the original one, and*
- *coarsening the simplexwise one.*

Example: apparent pairs



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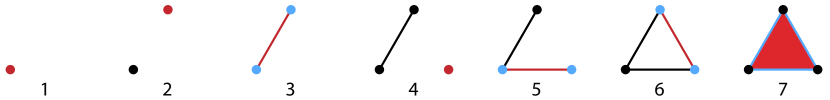

$D =$

	1	2	3	4	5	6	7
1			1		1		
2			1			1	
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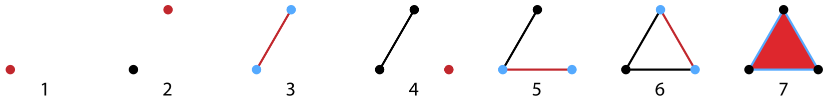

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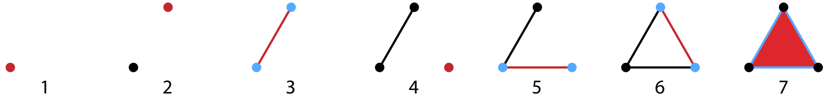

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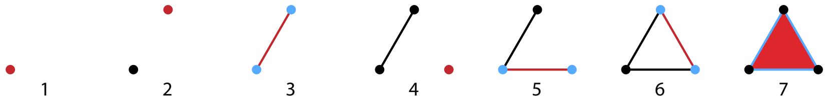

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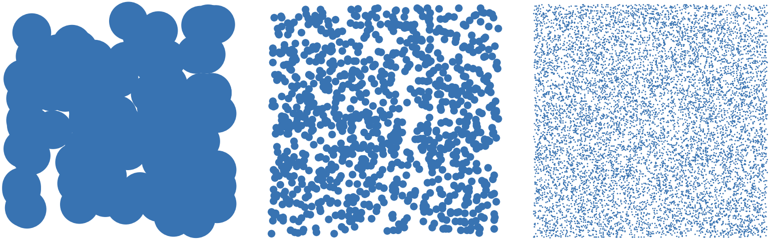
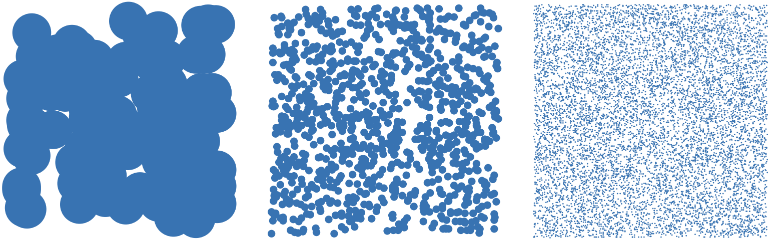
Discrete Morse functions from total orders

The upshot:

- Total order on simplices induces discrete Morse function
- Simple and important example: lexicographic order, given by total order on vertices
- Often, this discrete Morse function has good properties (few critical points)
- Special cases have appeared in literature before
  - random Rips complex in supercritical regime [Kahle 2010]

Stochastics



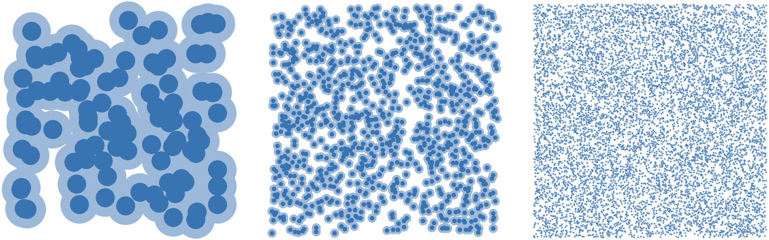


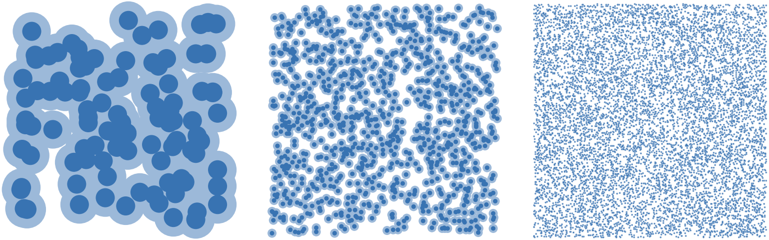
Theorem (Kahle 2011)

The expectation of the kth Betti number of a random Čech complex $\mathrm{Cech}_r(\mathcal{P}_n)$ *satisfies*

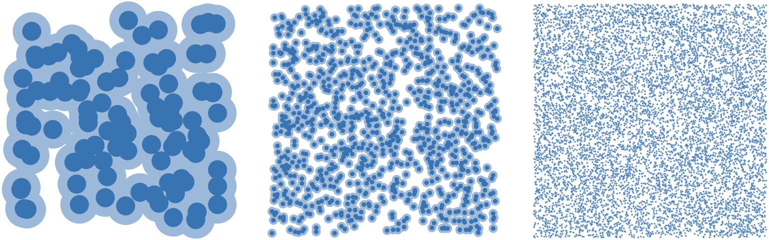
$$E(\beta_k(\mathcal{P}_n, r_n)) \in \Theta\left(n({r_n}^d n)^{k+1}\right)$$

in the subcritical regime ${r_n}^d n \to 0$ *as* $n \to \infty$.





Let $m(\theta, k)$ be the smallest number of points required to form a persistent cycle with death/birth ratio $\geq \theta$.



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Theorem (B, Pausinger 2016)

The expectation of the kth θ-persistent Betti number of random Čech complexes $\mathrm{Cech}_r(\mathcal{P}_n) \subseteq \mathrm{Cech}_{\theta r}(\mathcal{P}_n)$ satisfies

$$E(\beta_k(\mathcal{P}_n, r_n)) \in \Theta\left(n(r_n{}^d n)^{m(\theta,k)-1}\right)$$

in the subcritical regime $r_n{}^d n \to 0$ as $n \to \infty$.

Numerics

Cauchy's differentiation formula

Differentiation by integration:

- Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be holomorphic,
- $D \subseteq U$ a closed disk, and
- $a \in \operatorname{int} D$.

Then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

for any cycle γ in the homology class $[\partial D] \in H_1(U \setminus \{a\})$.

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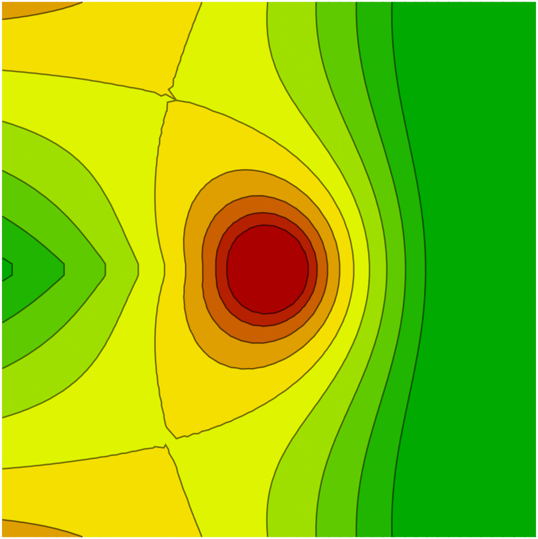
- But not all cycles are created equal
  (for numerical integration)!

Optimizing condition of contour integration



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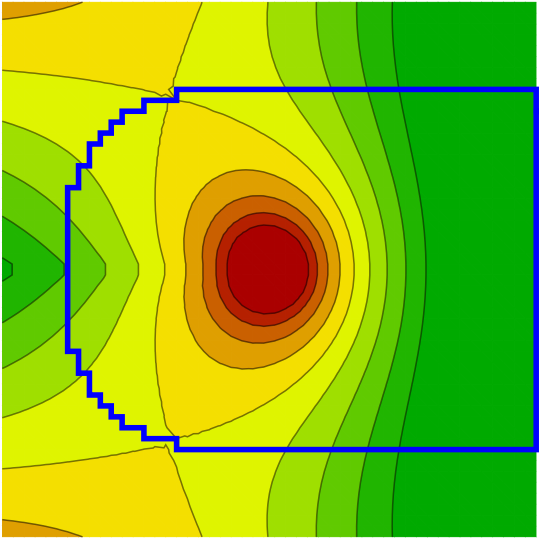

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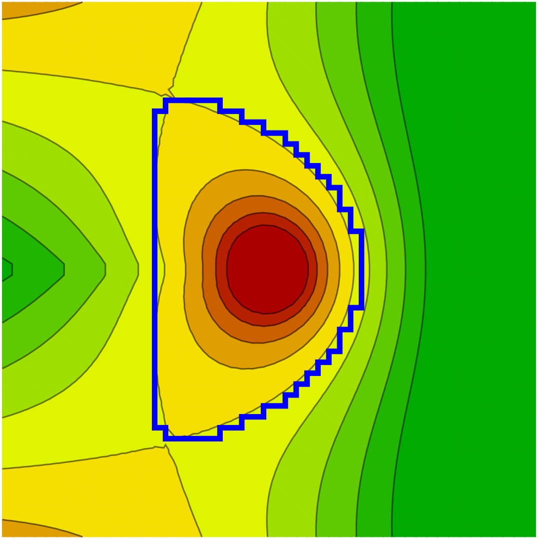

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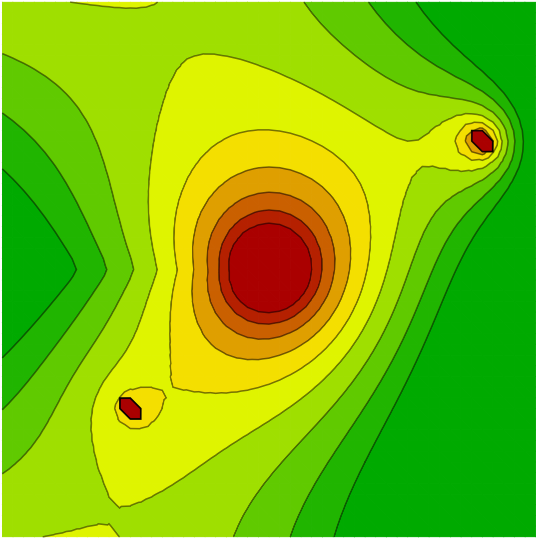

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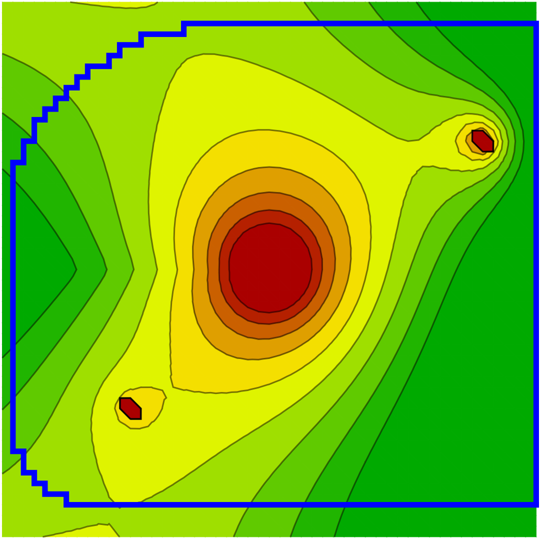

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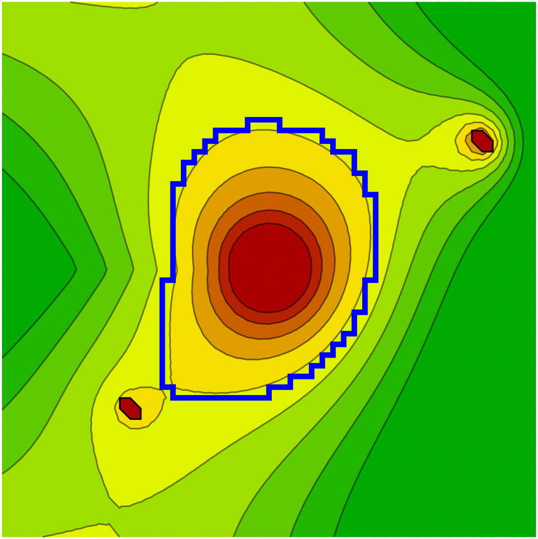
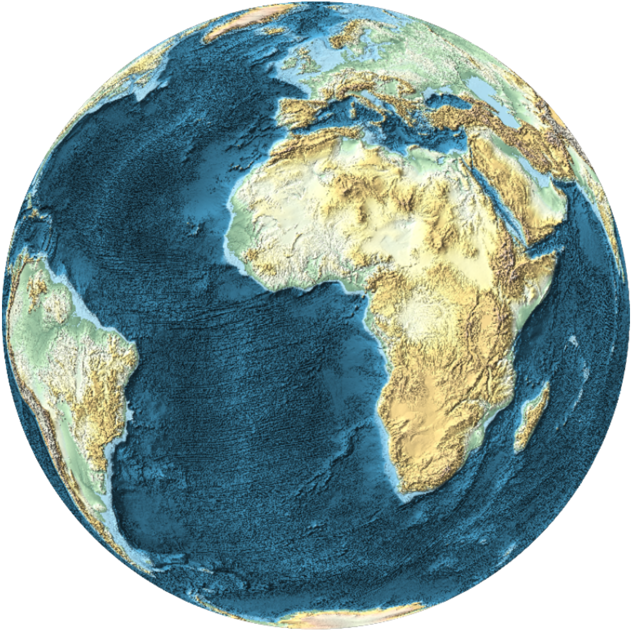
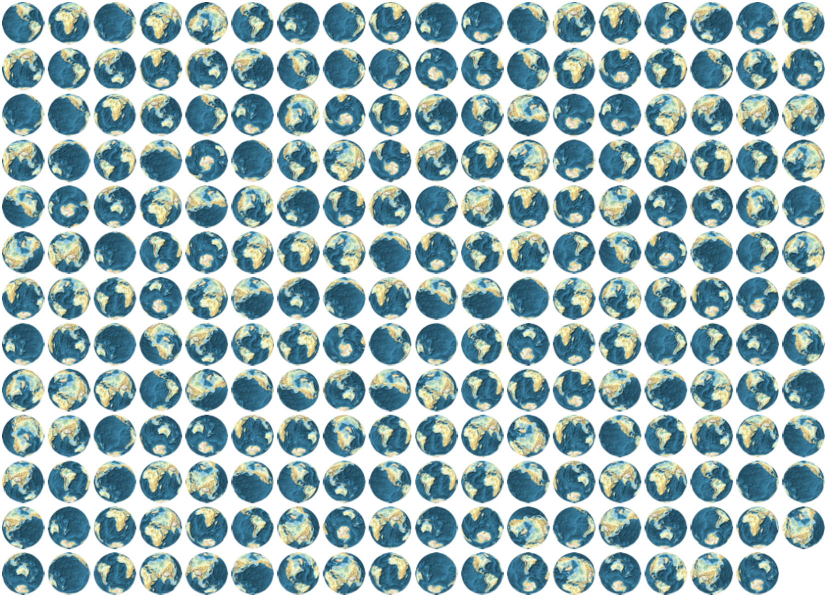
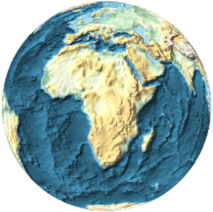
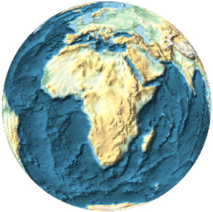


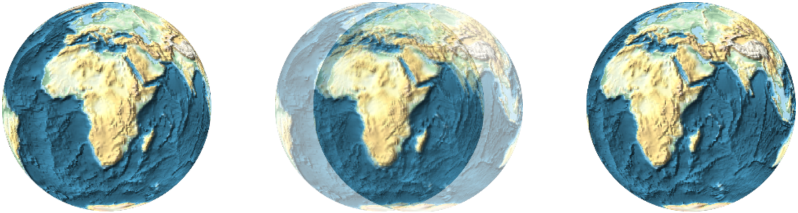
Around the world

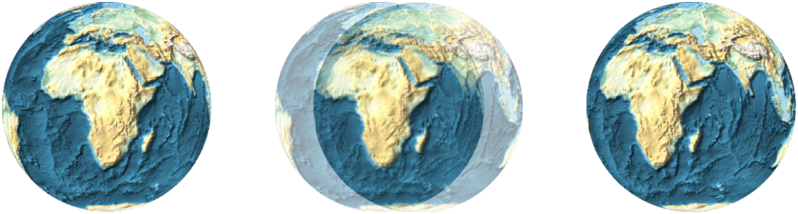




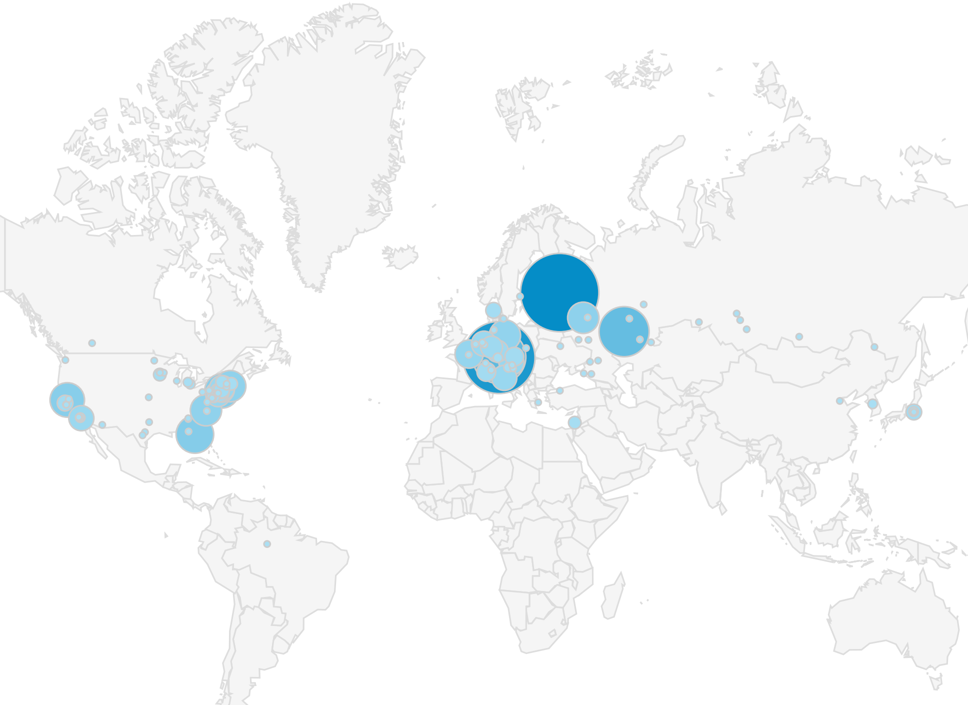








Demo (Ripser)



Users of Ripser Live