Persistent Homology

From Theory to Computation

Ulrich Bauer

TUM

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Mathematical Colloquium, Bielefeld University

Workshop Computational Applications of Quiver Representations: TDA and QPA

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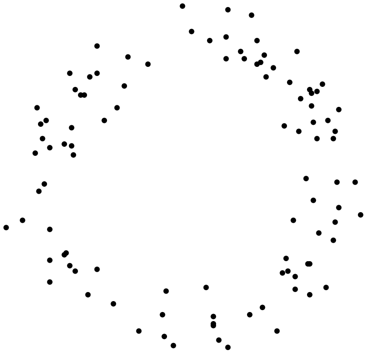
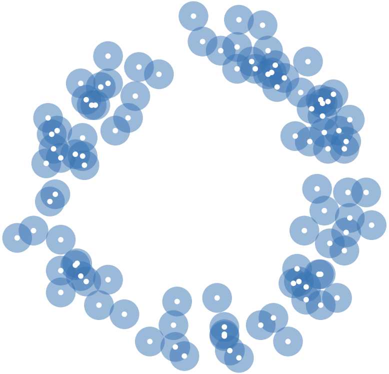
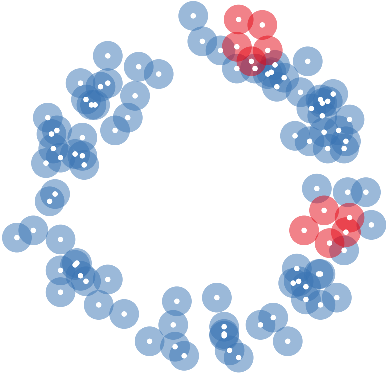
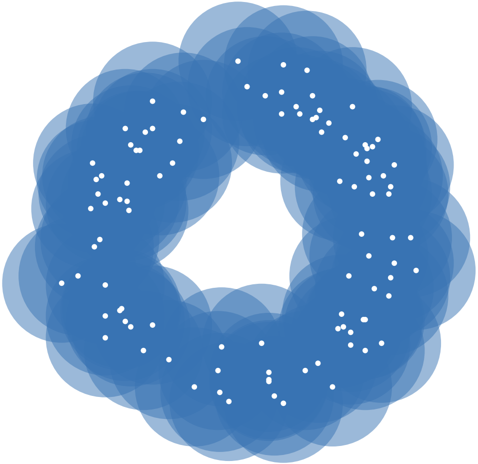
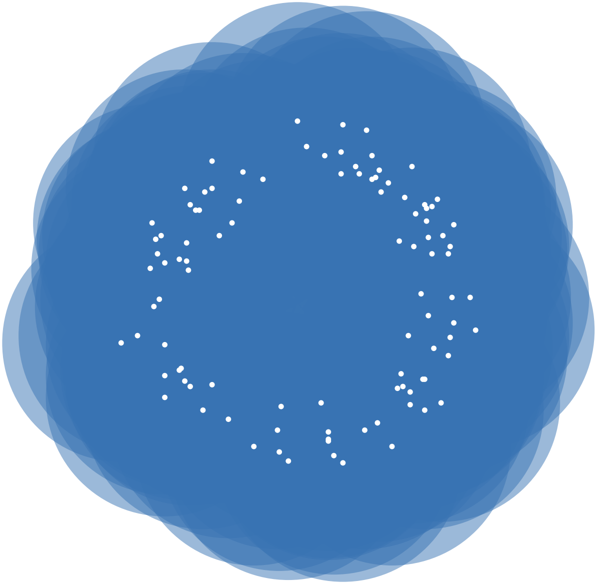
Persistent homology

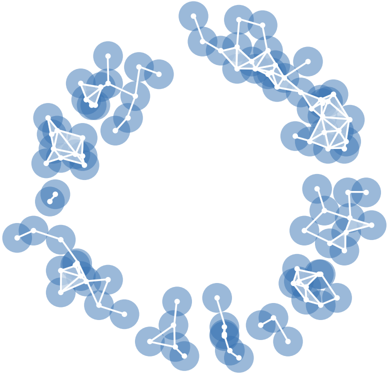


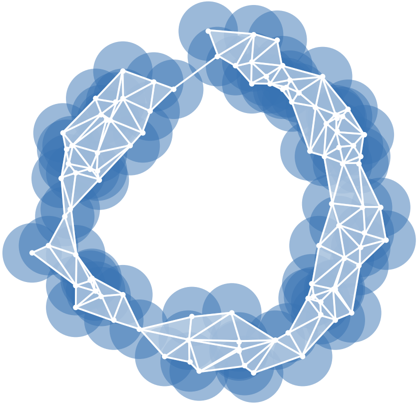


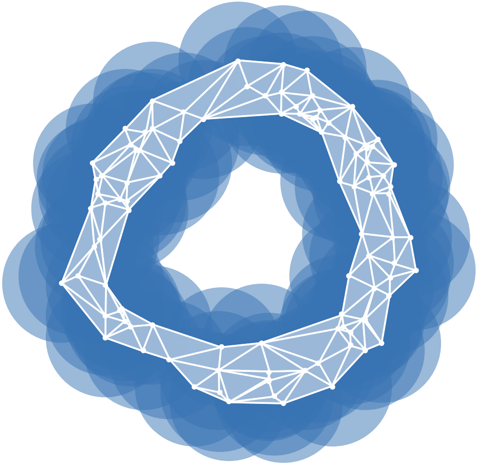


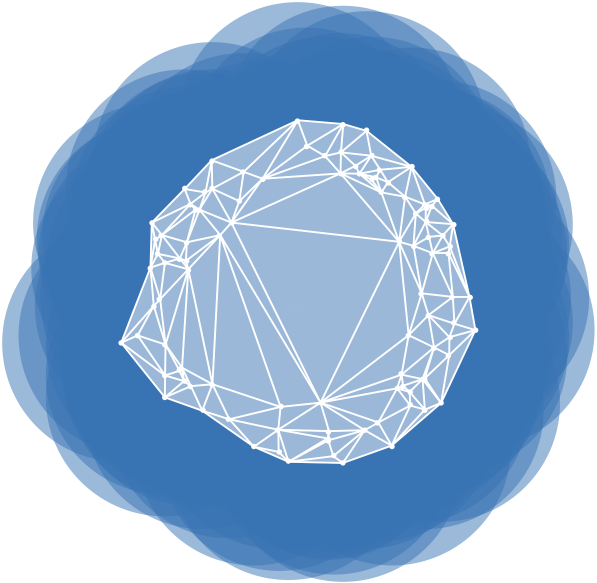




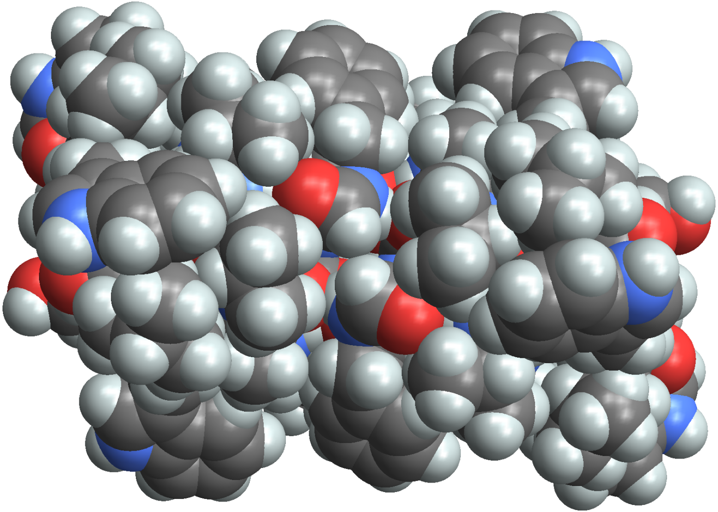








Gramicidin: an antibiotic functioning as ion channel



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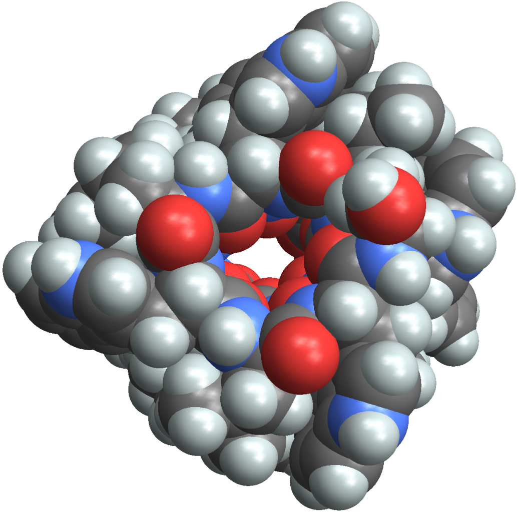

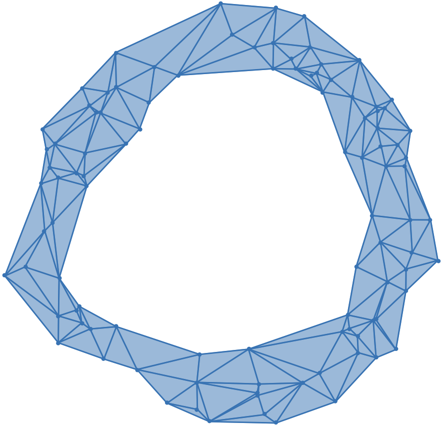

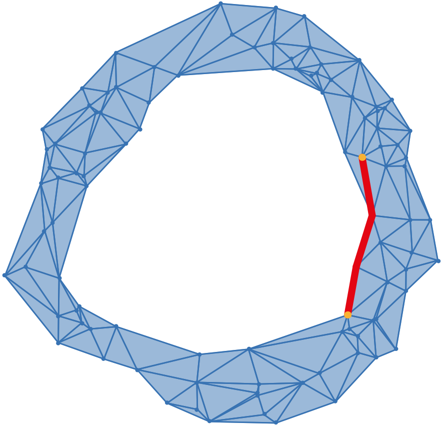

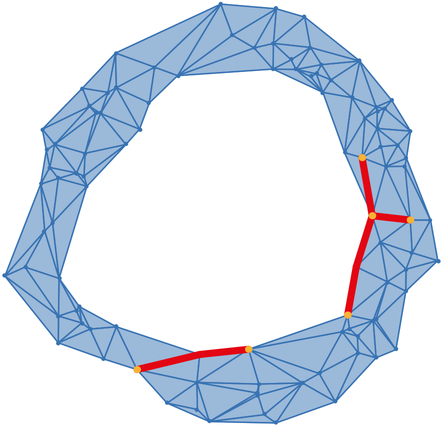

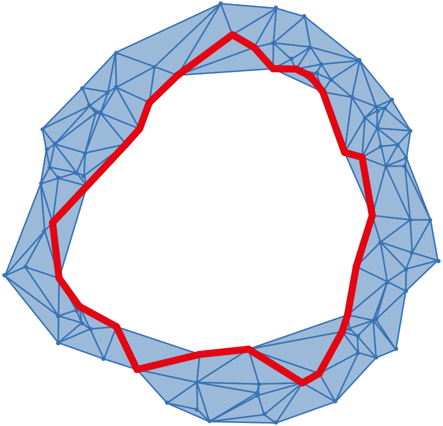

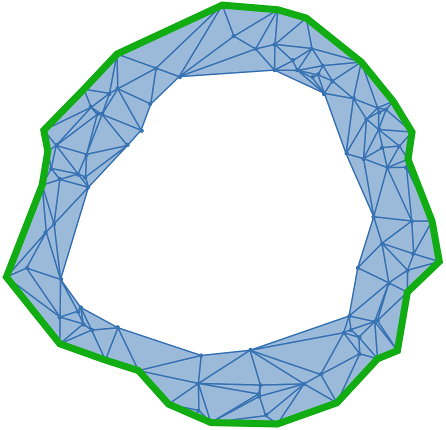

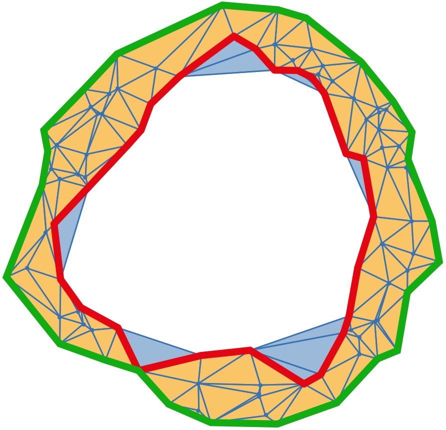

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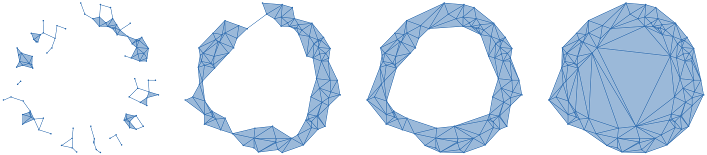

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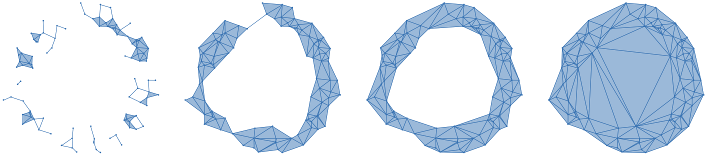

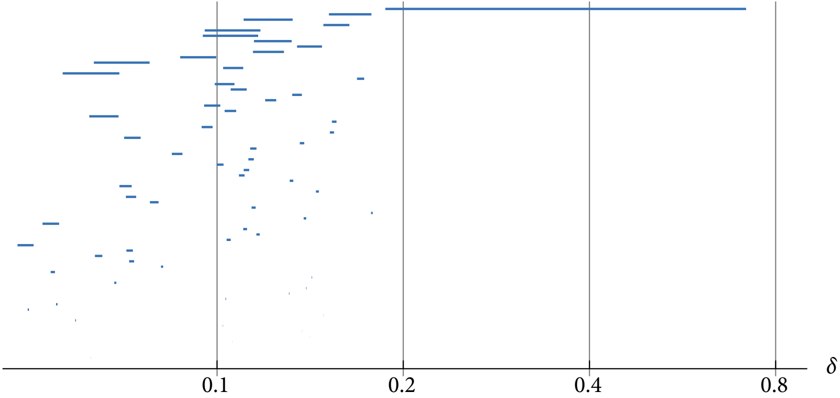

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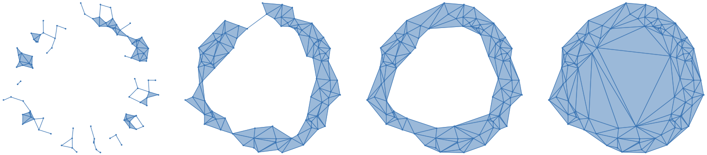

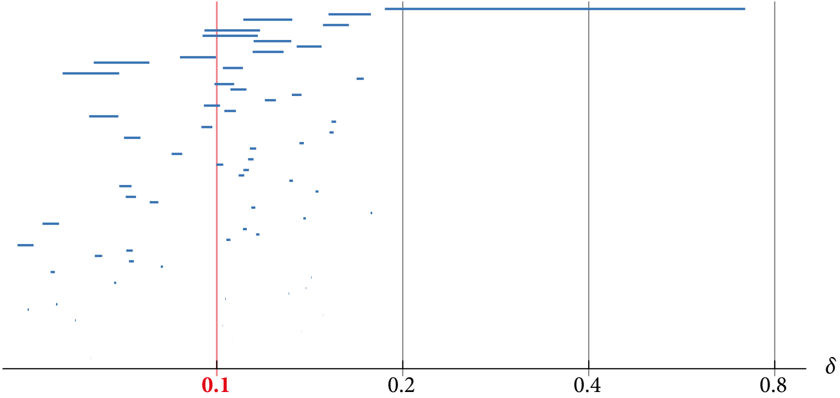

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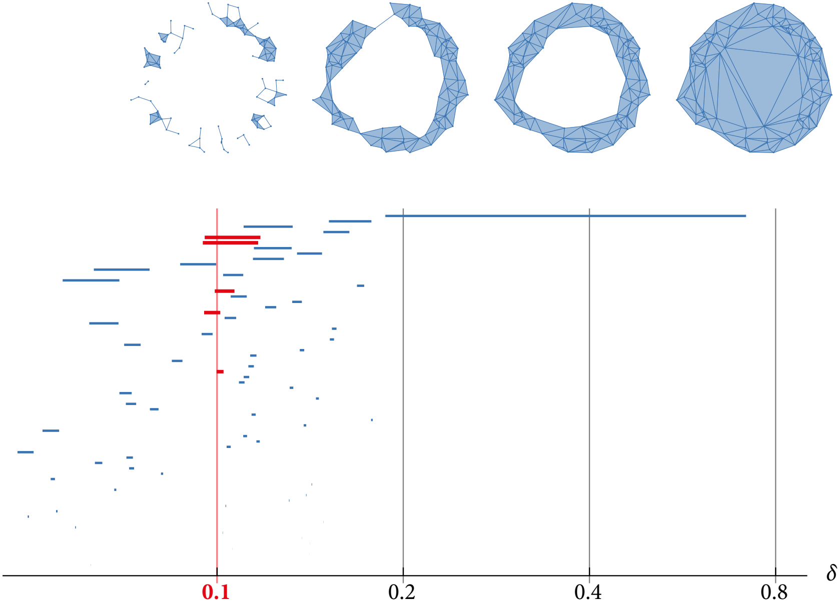

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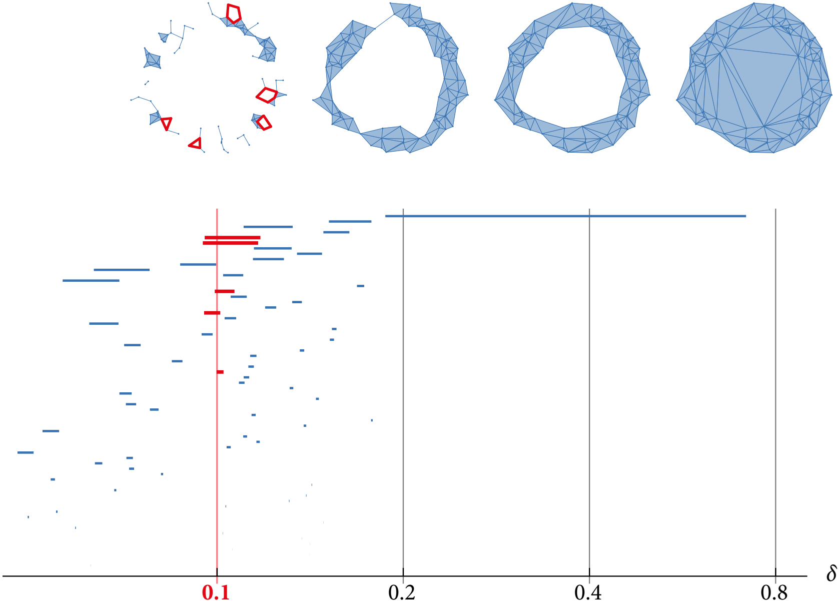

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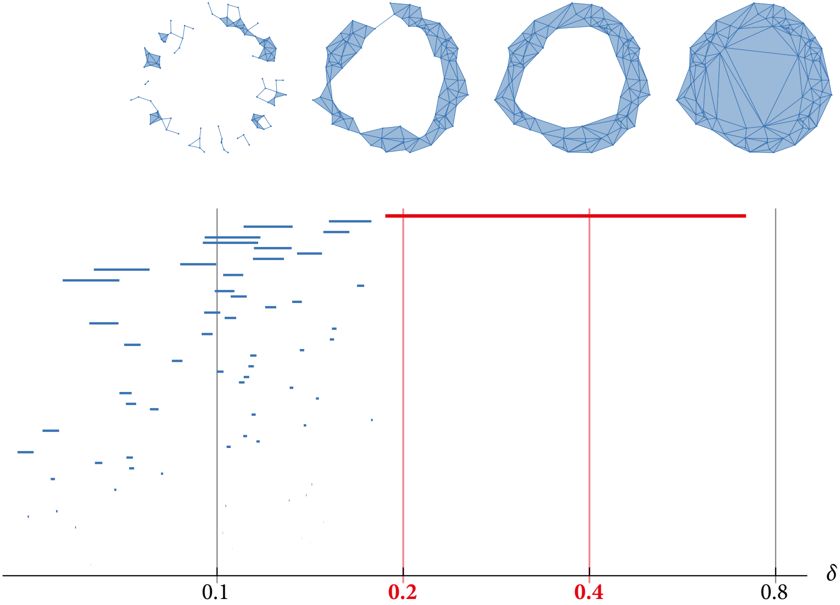

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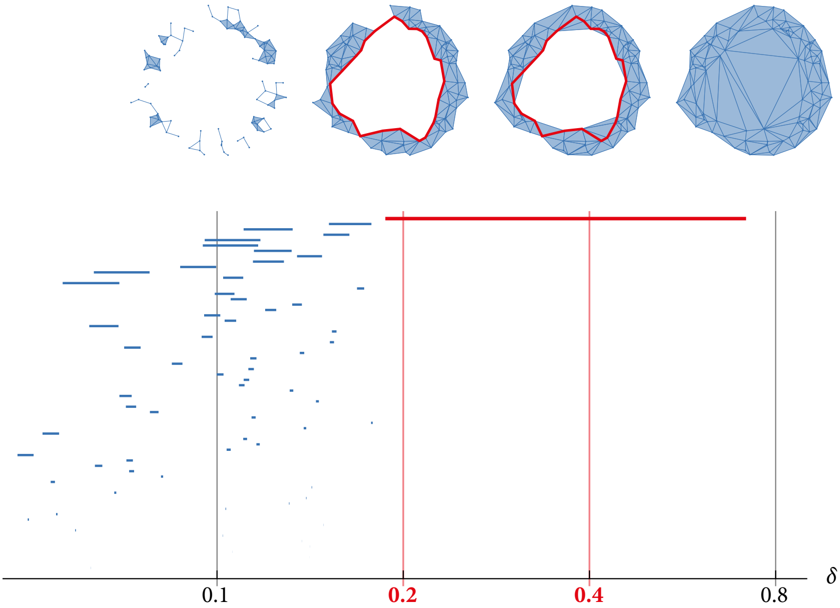

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- A filtration is a certain diagram $K : \mathbf{R} \to \mathbf{Top}$
  - $\mathbf{R}$ is the poset $(\mathbb{R}, \leq)$
  - A topological space K_t for each $t \in \mathbb{R}$
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In this talk, all vector spaces will be finite dimensional.

Barcodes: the structure of persistence modules

Theorem (Crawley-Boevey 2015)

Any persistence module $M : \mathbf{R} \to \mathbf{vect}$ *(of finite dim. vector spaces over some field* $\mathbb{F}$*) decomposes as a direct sum of* interval modules

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- The supporting intervals form the *persistence barcode* $B(M)$.

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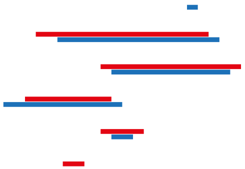
- The barcode completely describes the persistence module (up to isomorphism).
- This is why we use homology with coefficients in a field.
- We rarely have a similarly clean structure for other indexing posets, like $\mathbf{R}^2 \to \mathbf{vect}$ (two-parameter persistence modules)

Stability

Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $f, g : X \to \mathbb{R}$ *with* $\|f - g\|_\infty = \delta$ *(and some regularity assumptions).*
Consider the persistence barcodes of (sublevel set filtrations of) f *and* g*.*
Then there exists a matching between their intervals such that



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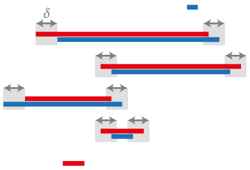

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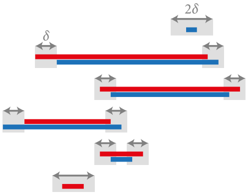

Persistence and stability: the big picture

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$\downarrow$	$\downarrow$ distance	
Geometry	function	$f : \mathbb{R}^d \to \mathbb{R}$

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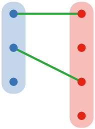
The category of matchings

Consider the category $\mathbf{Mch}$ with

- objects: sets,
- morphisms: matchings (bijections between subsets).



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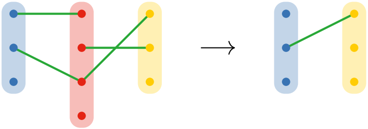

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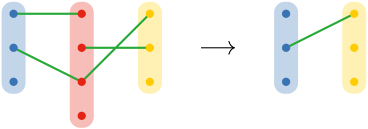
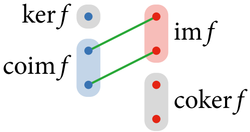
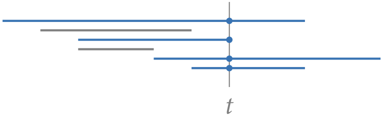


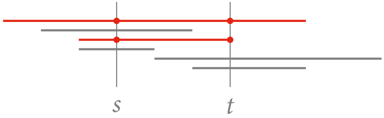
(Co)kernel/(co)image:



From barcodes to matching diagrams (and back)

- A barcode (collection of intervals) defines a diagram $\mathbf{R} \to \mathbf{Mch}$:





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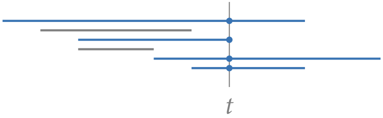

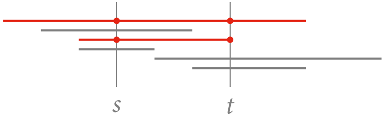

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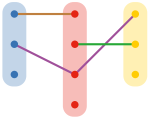

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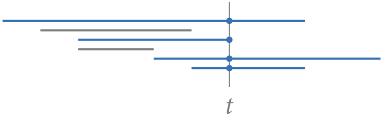

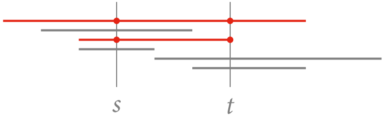

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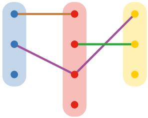

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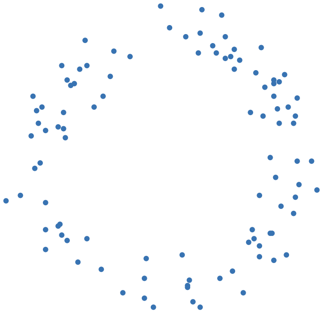
We can turn barcodes into a category $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbf{R}}$

Geometric interleavings



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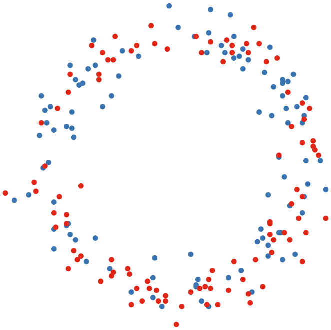

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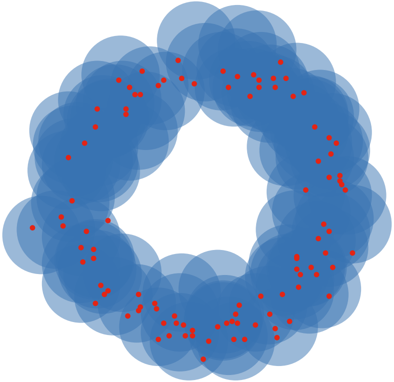

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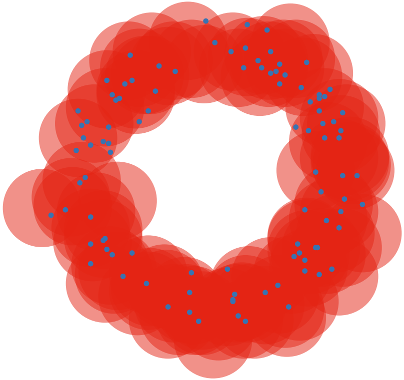

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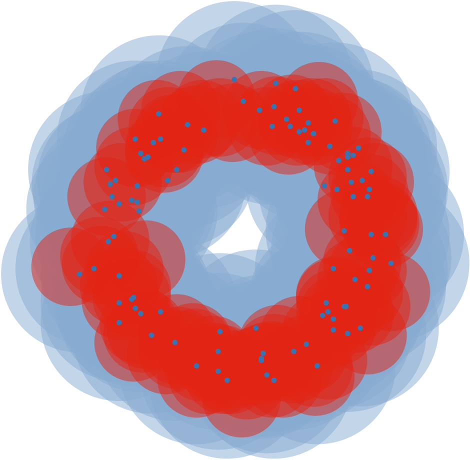

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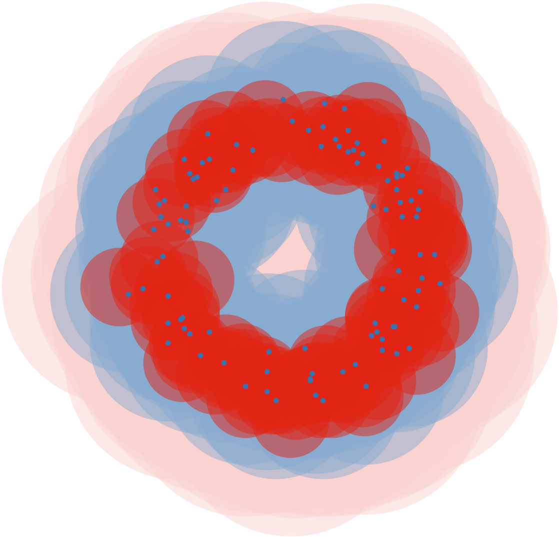

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Applying homology (a functor) preserves commutativity

- persistent homology of f, g yields δ-interleaved persistence modules $\mathbf{R} \to \mathbf{Vect}$

Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2012; B, Lesnick 2015)

If two persistence modules are δ-interleaved, then there exists a δ-matching of their barcodes:

- matched intervals have endpoints within distance $\leq \delta$,
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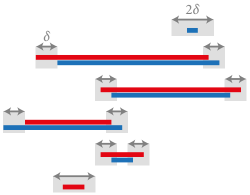

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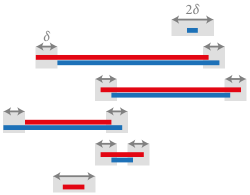

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Equivalently: there exists a δ-interleaving of their barcodes (as diagrams $\mathbf{R} \to \mathbf{Mch}$).

Non-functoriality of persistence barcodes

Can a persistence module M be mapped to its barcode $B(M)$ by a functor $B : \mathbf{vect} \to \mathbf{Mch}$?

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But:

- there is a barcode functor for monos/epis of persistence modules $\mathbf{vect}^{\mathbf{R}}$:

Structure of persistence sub-/quotient modules

Proposition

Let $M \twoheadrightarrow N$ be an epimorphism.

Then there is an injection of barcodes $B(N) \hookrightarrow B(M)$ such that if J is mapped to I, then

- *I and J are aligned below, and*
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Dually, there is an injection $B(M) \hookrightarrow B(N)$ for monomorphisms $M \hookrightarrow N$.

Induced matchings

For $f : M \to N$ a morphism of pfd persistence modules, the epi-mono factorization

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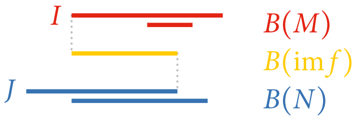

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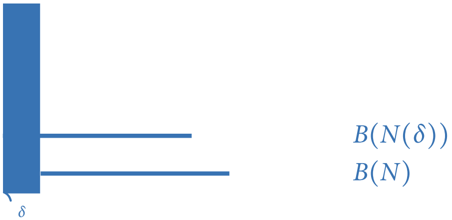

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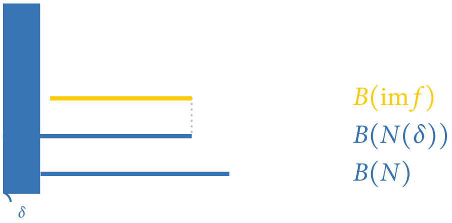

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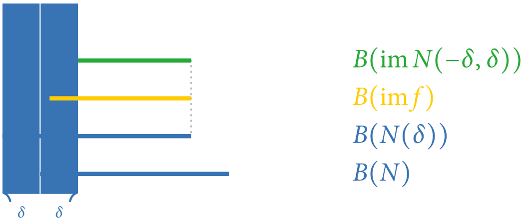

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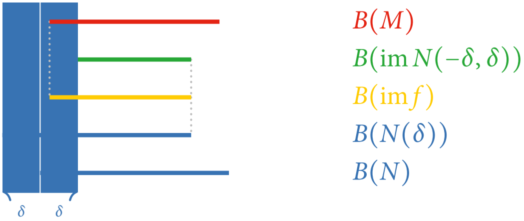

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Requires strong assumptions:

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Theorem (Niyogi, Smale, Weinberger 2006)

Let X be a submanifold of $\mathbb{R}^d$. Let $P \subset X$ and $\delta > 0$ be such that

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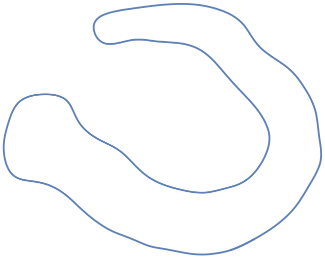

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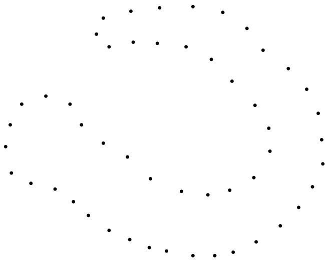

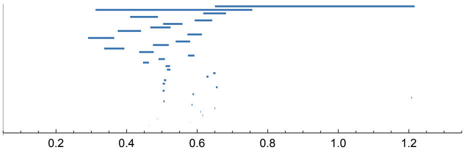

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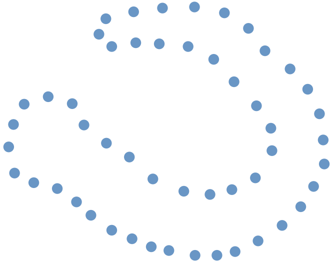

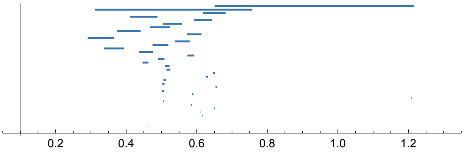

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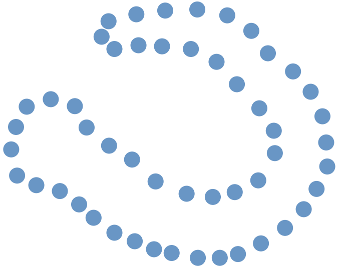

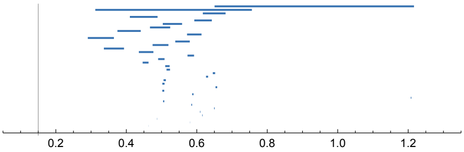

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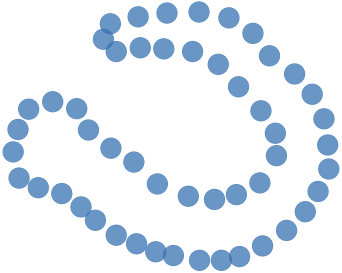

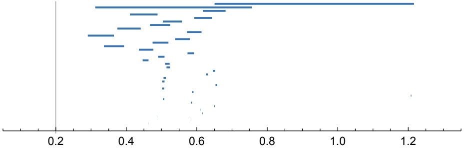

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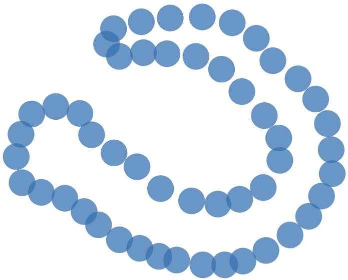

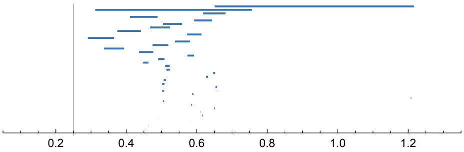

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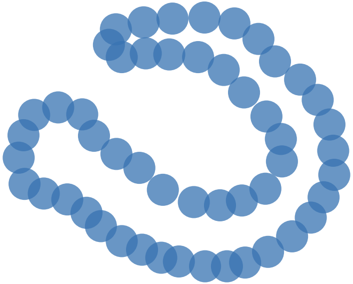

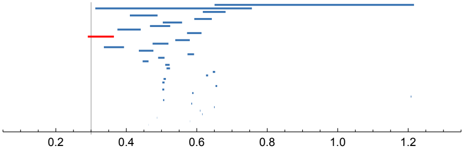

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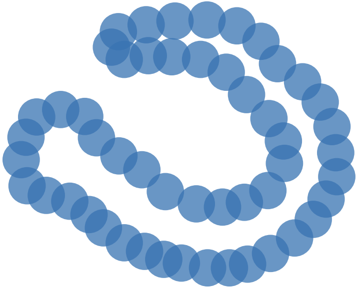

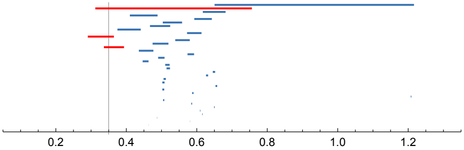

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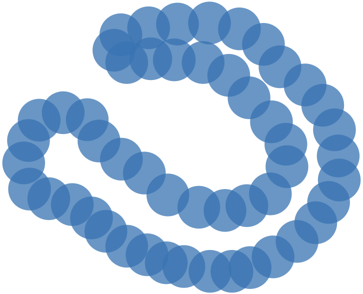

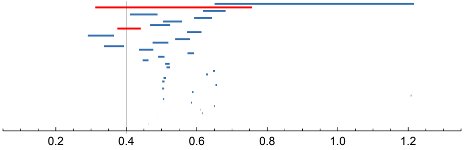

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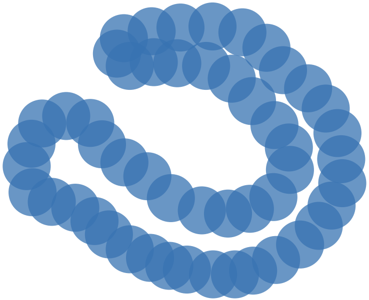

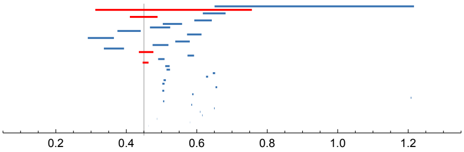

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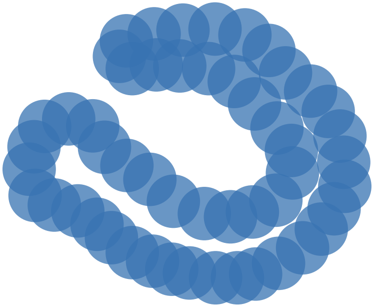

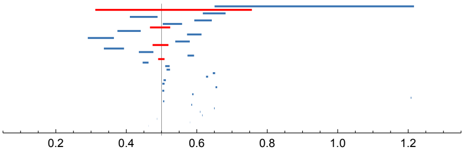

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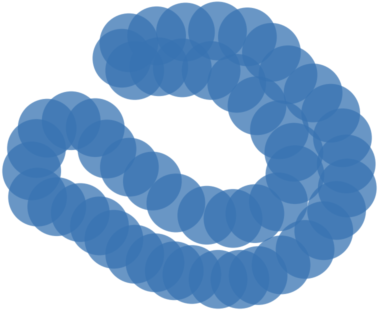

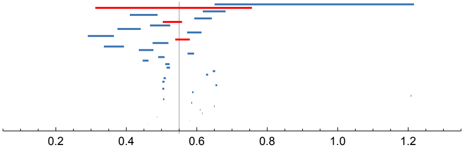

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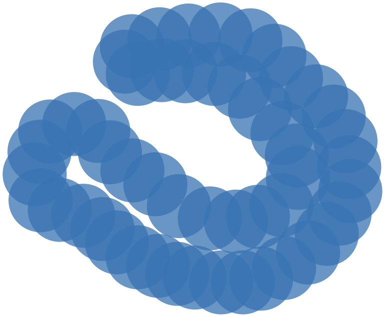

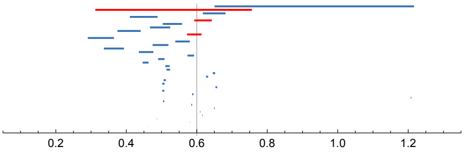

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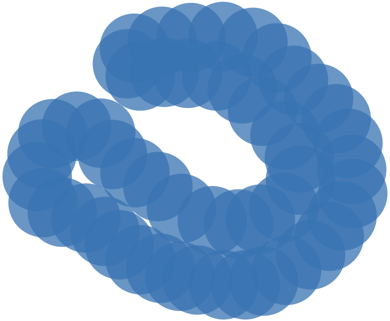

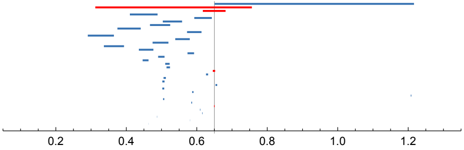

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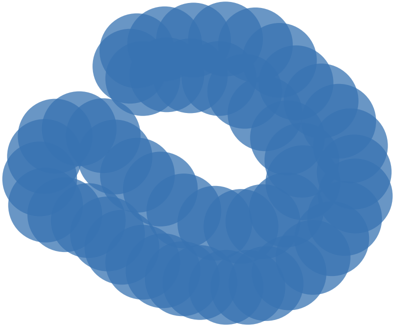

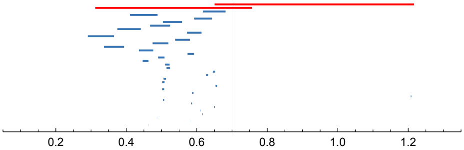

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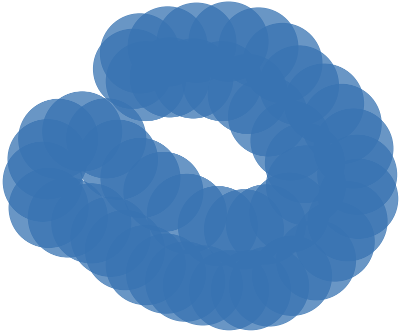

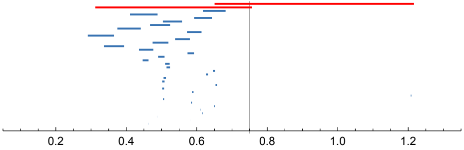

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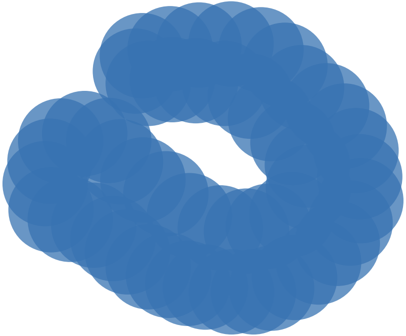

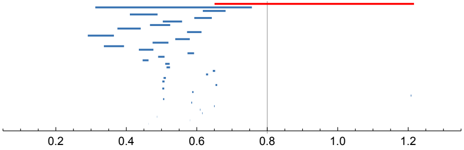

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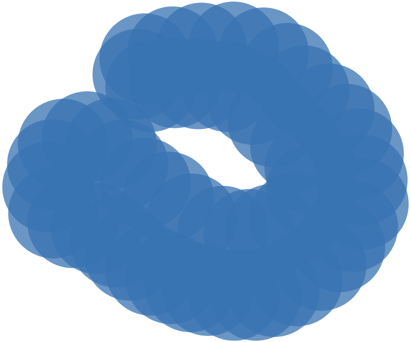

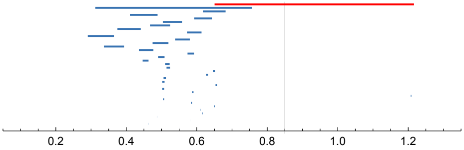

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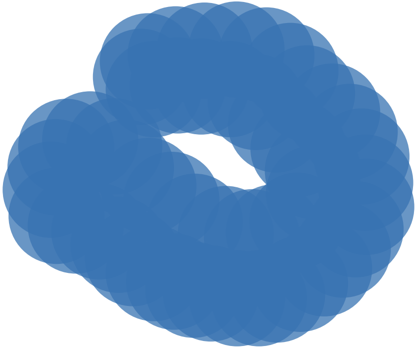

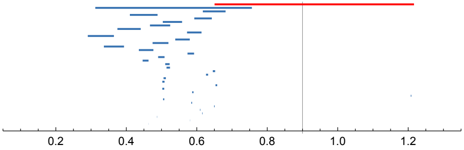

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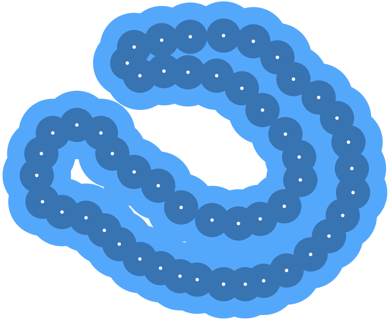

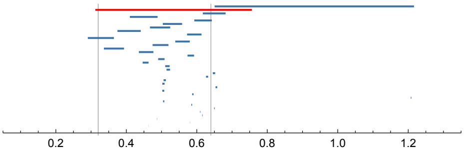

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Proof.

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X \arrow[dr, hook] & & X_\delta \arrow[dr, hook] & & X_{2\delta} \\
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This motivates the *homological realization problem*:

Problem

Given a simplicial pair $L \subseteq K$, find X with $L \subseteq X \subseteq K$ such that

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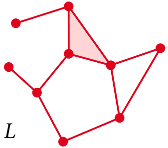

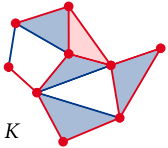

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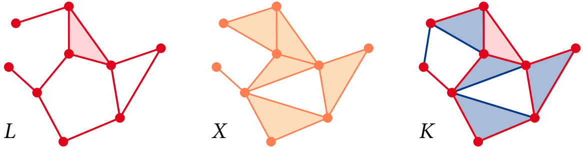

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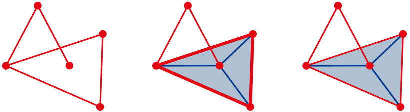

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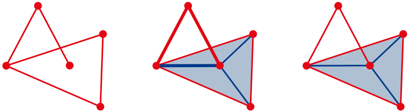

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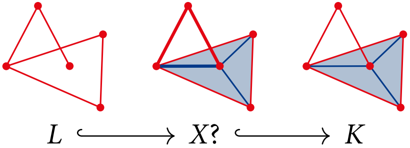
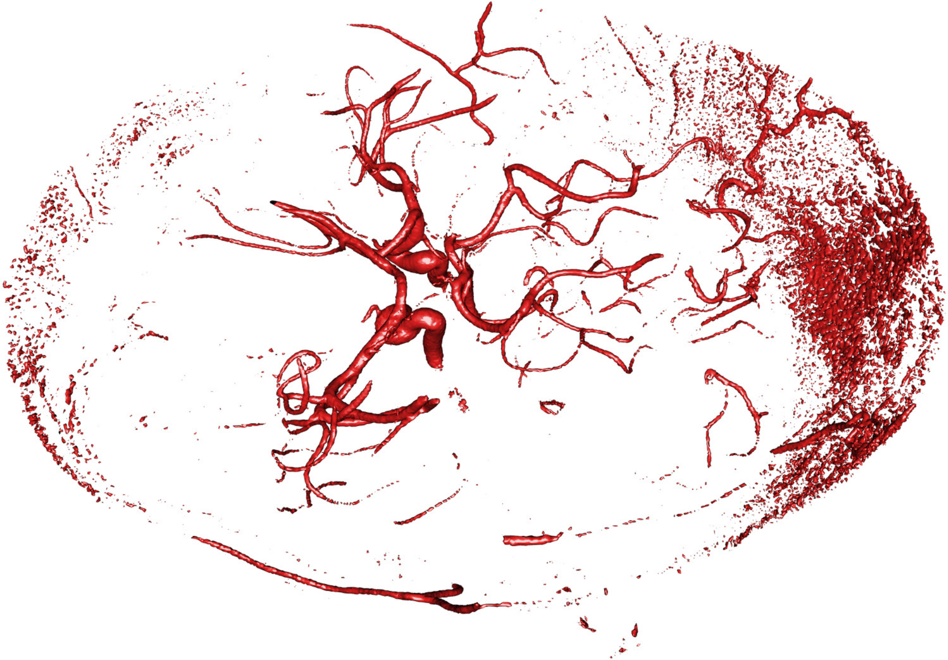
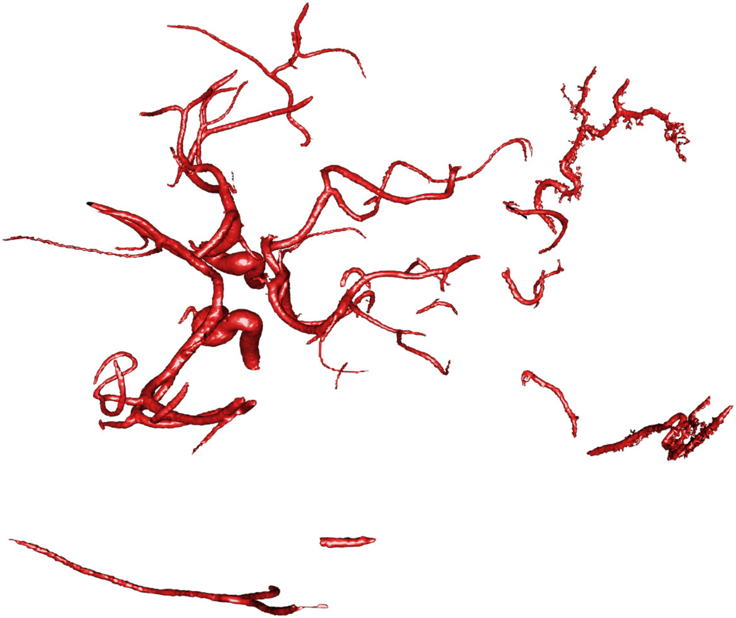


Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

The homological realization problem is NP-hard, even in $\mathbb{R}^3$.

Simplification





Sublevel set simplification

Let $F_t = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem (Sublevel set simplification)

Given a function $f : X \subset \mathbb{R}^3 \to \mathbb{R}$ and $t \in \mathbb{R}$, $\delta > 0$,
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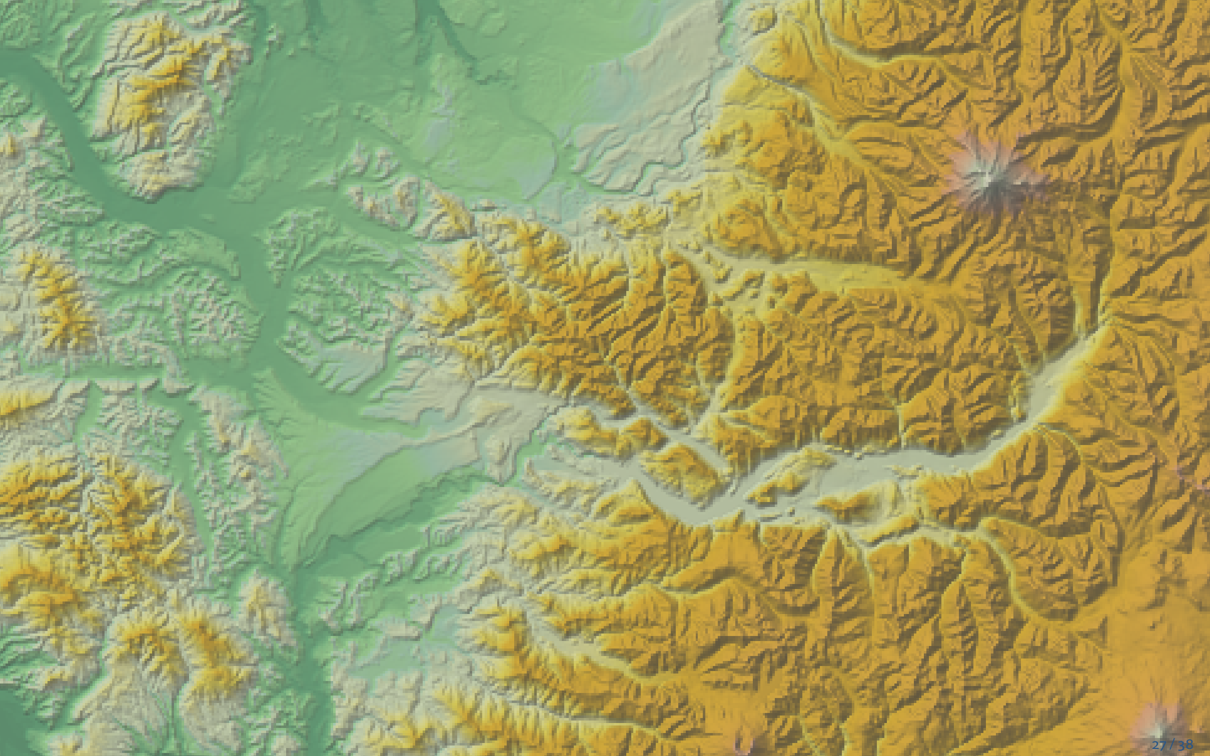
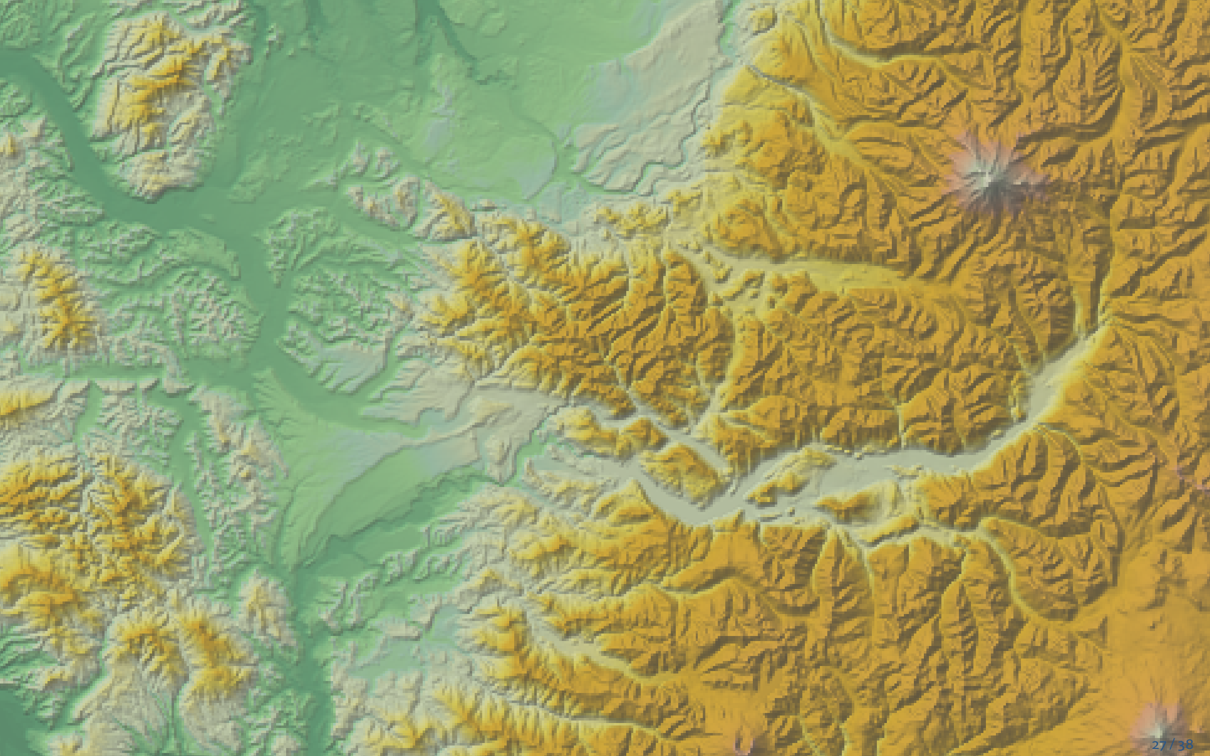
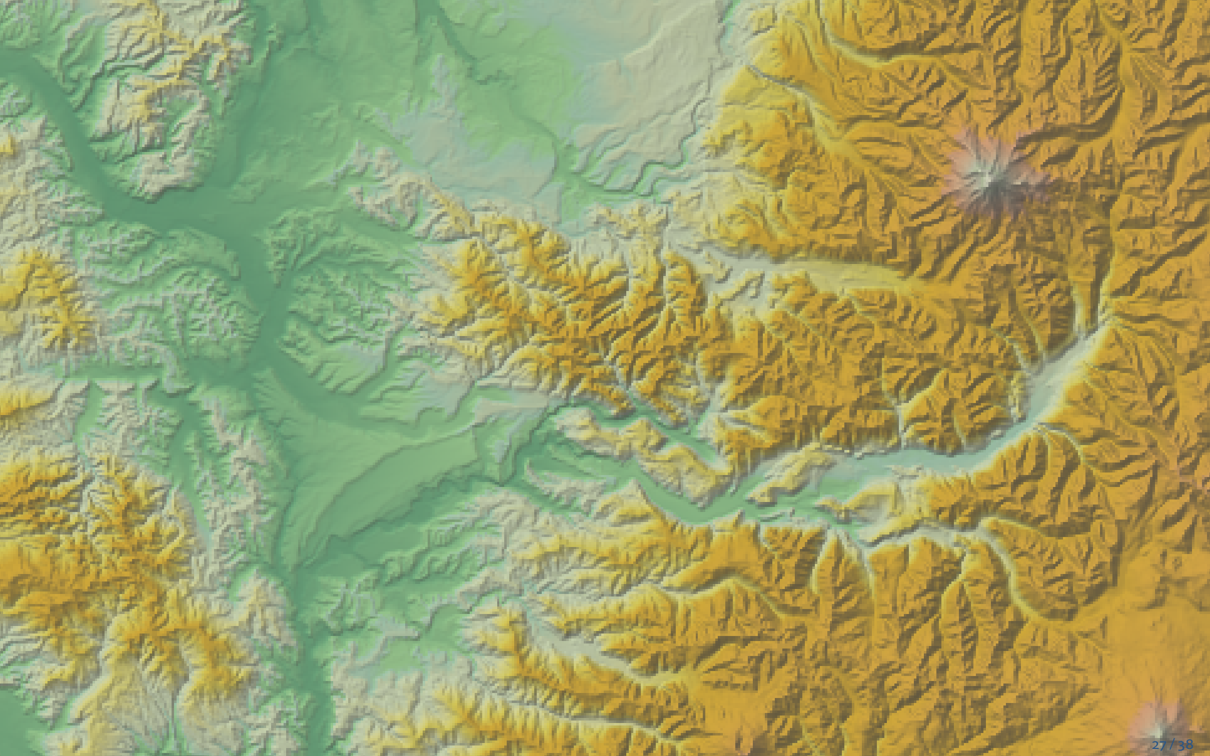
Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

Sublevel set simplification in $\mathbb{R}^3$ is NP-hard.







Topological simplification of functions

Consider the following problem:

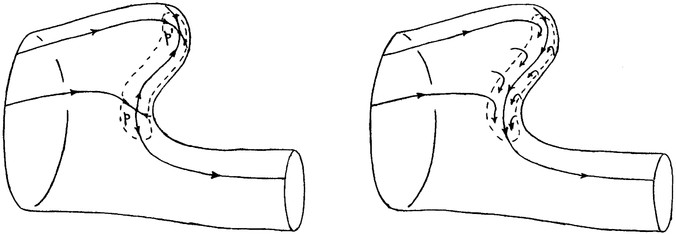
Problem (Topological simplification)

Given a function f and a real number $\delta \geq 0$, find a function f_δ subject to $\|f_\delta - f\|_\infty \leq \delta$ with the minimal number of critical points.

Persistence and Morse theory

Morse theory (smooth or discrete):

- Relates critical points to homology of sublevel sets
- Provides a method for *canceling* pairs of critical points



(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

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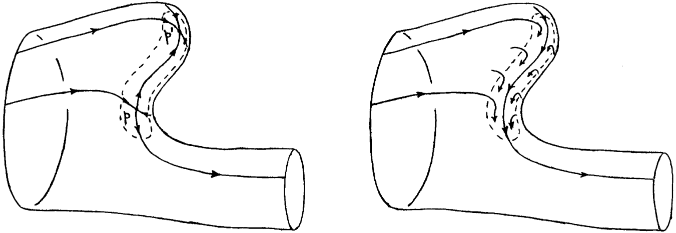
- Provides a method for *canceling* pairs of critical points



(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

Persistent homology:

- Relates homology of different sublevel set
- Identifies pairs of critical points (birth and death of homological feature) and quantifies their *persistence*

Persistence and discrete Morse theory

By stability of persistence barcodes:

Proposition

The intervals in the barcode of f with persistence $> 2\delta$ provide a lower bound on the number of critical points of any function g with $\|g - f\|_\infty \leq \delta$.

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Theorem (B, Lange, Wardetzky, 2011)

Let f be a function on a surface and let $\delta > 0$.
Then canceling all persistence pairs with persistence $\leq 2\delta$ yields a function f_δ with

$$\|f_\delta - f\|_\infty \leq \delta$$

that achieves the lower bound on the number of critical points.

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that achieves the lower bound on the number of critical points.

- Does not generalize to higher-dimensional manifolds!

History

When was persistent homology invented?

H. Edelsbrunner, D. Letscher, and A. Zomorodian
Topological persistence and simplification
Foundations of Computer Science, 2000

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P. Frosini
A distance for similarity classes of submanifolds of a Euclidean space
Bulletin of the Australian Mathematical Society, 1990.

When was persistent homology invented first?

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ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses is satisfied. The critical limits cluster at most at $+\infty$; the critical points are isolated;[1] the problem is defined by analytic functions; the critical limits taken in their natural order are well-ordered. These conditions are not generally

When was persistent homology invented first?

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Rank and span in functional topology

Search within citing articles

Exact homomorphism sequences in homology theory ed.ac.uk [PDF]
JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR
The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] ...
Cited by 46 Related articles All 3 versions Cite Save More

Marston Morse and his mathematical works ams.org [PDF]
R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org
American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to ...
Cited by 24 Related articles All 4 versions Cite Save More

Unstable minimal surfaces of higher topological structure
M Morse, CB Tompkins - Duke Math. J, 1941 - projecteuclid.org
1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. Theproblem of the existenceof minimal surfaces of unstable type contains many of the typical difficulties, especially those ofa topological nature. Having studied this ...
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[PDF] **Persistence in discrete Morse theory** psu.edu [PDF]
U Bauer - 2011 - Citeseer
The goal of this thesis is to bring together two different theories about critical points of a scalar function and their relation to topology: Discrete Morse theory and Persistent homology. While the goals and fundamental techniques are different, there are certain ...
Cited by 8 Related articles All 4 versions Cite Save More

When was persistent homology invented first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was in the 1920's to go from the mini-max principle of Birkhoff to the Morse inequalities, let alone extend these inequalities to function spaces, so that by the early 30's Morse could establish the theorem that for any Riemann

When was persistent homology invented first?

inequalities pertain between the dimensions of the A_i and those of $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels **[D]** in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers **[36]** and **[44]** constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.

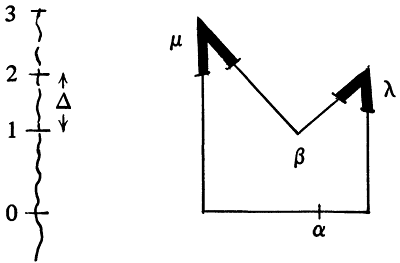
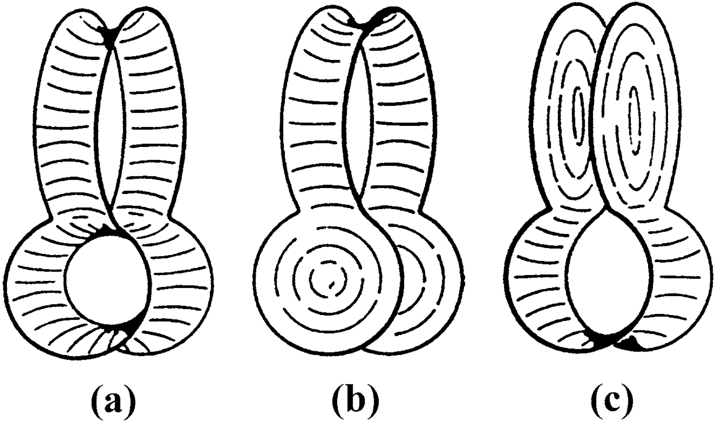


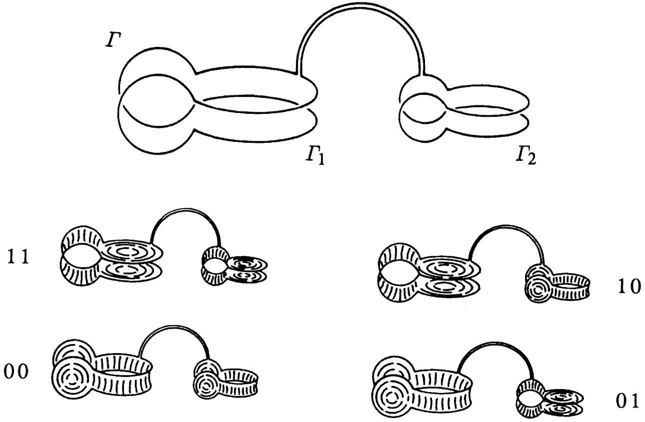
FIGURE 8

Motivation and application: minimal surfaces



(from Dierkes et al.: *Minimal Surfaces*, 2010)

Motivation and application: minimal surfaces



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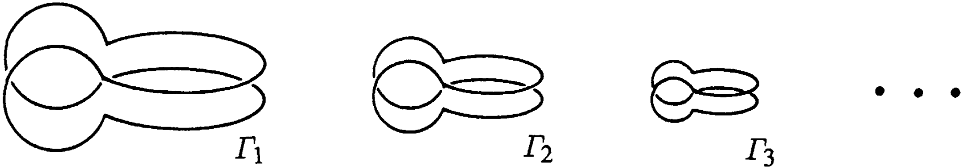

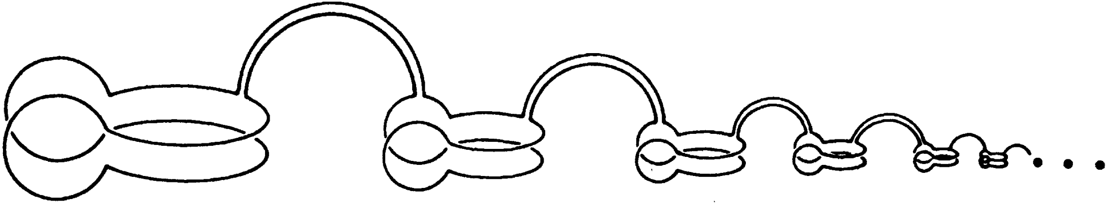

(from Dierkes et al.: *Minimal Surfaces*, 2010)

Existence of unstable minimal surfaces

Using persistent homology:

- Number of ϵ-persistent critical points (minimal surfaces) is finite for any $\epsilon > 0$
- Morse inequalities for ϵ-persistent critical points

Theorem (Morse, Tompkins 1939)

There is a C_1 curve bounding an unstable minimal surface (an index 1 critical point of the area functional).

Computation

Vietoris–Rips complexes

Consider a finite metric space (X, d).
The *Vietoris–Rips complex* is the simplicial complex

$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid \operatorname{diam} S \leq t\}$$

- all edges with pairwise distance $\leq t$
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- all edges with pairwise distance $\leq t$
- all possible higher simplices

For large t, $\mathrm{Rips}_t(X)$ is the full simplex with vertices X

- Number of d-simplices is $\binom{|X|}{d+1}$
- Computation is one of the most important challenges in applied topology!

An example computation

Example data set:

- 192 points on $\mathbb{S}^2$
- persistent homology barcodes up to dimension 2
- over 56 mio. simplices in 3-skeleton

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Some previous software:

- javaplex (Stanford): 3200 seconds, 12 GB
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Demo: live.ripser.org

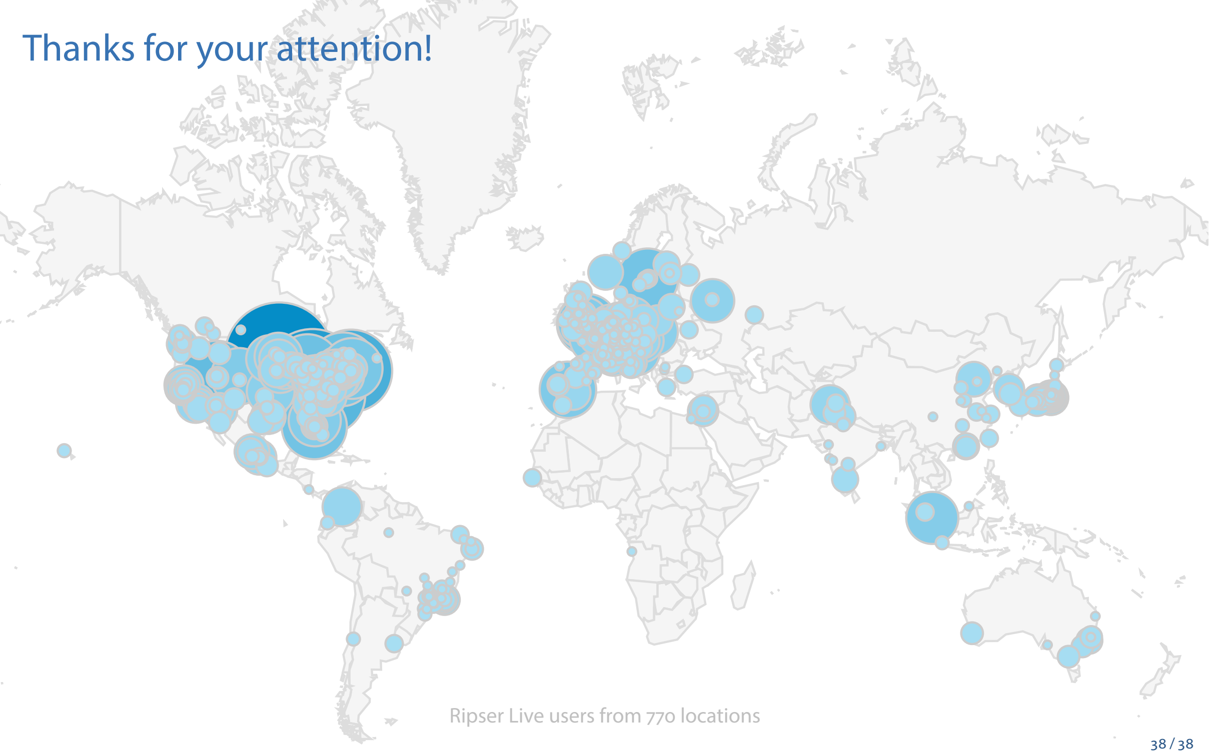
- Ripser: 1.2 seconds, 160 MB

Ripser

A software for computing Vietoris–Rips persistence barcodes

- around 1000 lines of C++ code, no external dependencies
- support for
  - coefficients in a prime field $\mathbb{Z}/p\mathbb{Z}$
  - sparse distance matrices (for distance threshold)
- open source (`http://ripser.org`)
- online version (`http://live.ripser.org`)
- 2016 ATMCS Best New Software Award (joint with RIVET by M. Lesnick and M. Wright)

Thanks for your attention!



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