Persistence in geometry and analysis

The functional topology of Plateau's minimal surface problem

Ulrich Bauer

TUM

Sep 16, 2021

DGD Days 2021

Morse inequalities

Theorem (Morse 1925)

Let $f : M \to \mathbb{R}$ be a Morse function on a compact manifold M. The Betti numbers β_i of M and the numbers m_j of index j critical points of f satisfy:

$$\begin{aligned} m_0 &\geq \beta_0 \\ m_1 - m_0 &\geq \beta_1 - \beta_0 \\ &\vdots \\ m_d - m_{d-1} + \cdots \pm m_0 &\geq \beta_d - \beta_{d-1} + \cdots \pm \beta_0 \end{aligned}$$

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\end{aligned}
$$

Corollary ("Mountain pass lemma")

If M is connected ($\beta_0 = 1$) and f has two minima ($m_0 = 2$), then it also has a critical point of index 1 ($m_1 \geq \beta_1 - \beta_0 + m_0 = \beta_1 + 1 \geq 1$).

Morse inequalities through the lens of persistence

$$\sum_{i=0}^{d}(-1)^{d-i}(m_i - \beta_i) \geq 0$$

Morse inequalities through the lens of persistence

What is persistent homology?

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- A filtration is a certain diagram $K : \mathbf{R} \to \mathbf{Top}$ of topological spaces, indexed over the poset of real numbers $\mathbf{R} := (\mathbb{R}, \leq)$

$$\cdots\cdots\rightarrow K_s \hookrightarrow K_t \cdots\cdots\rightarrow$$

  - a topological space K_t for each $t \in \mathbb{R}$
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- Apply the homology functor (with coefficients in some field) $H_* : \mathbf{Top} \to \mathbf{Vect}$
- Persistent homology is a diagram $M = H_* \circ K : \mathbf{R} \to \mathbf{Vect}$ *(persistence module)*:

$$\cdots\cdots\rightarrow M_s \longrightarrow M_t \cdots\cdots\rightarrow$$

Barcodes: the structure of persistence modules

Theorem (Crawley-Boevey 2015)

Any persistence module $M : \mathbf{R} \to \mathbf{vect}$ *(of finite dim. vector spaces over some field* $\mathbb{F}$*) decomposes as a direct sum of* interval modules

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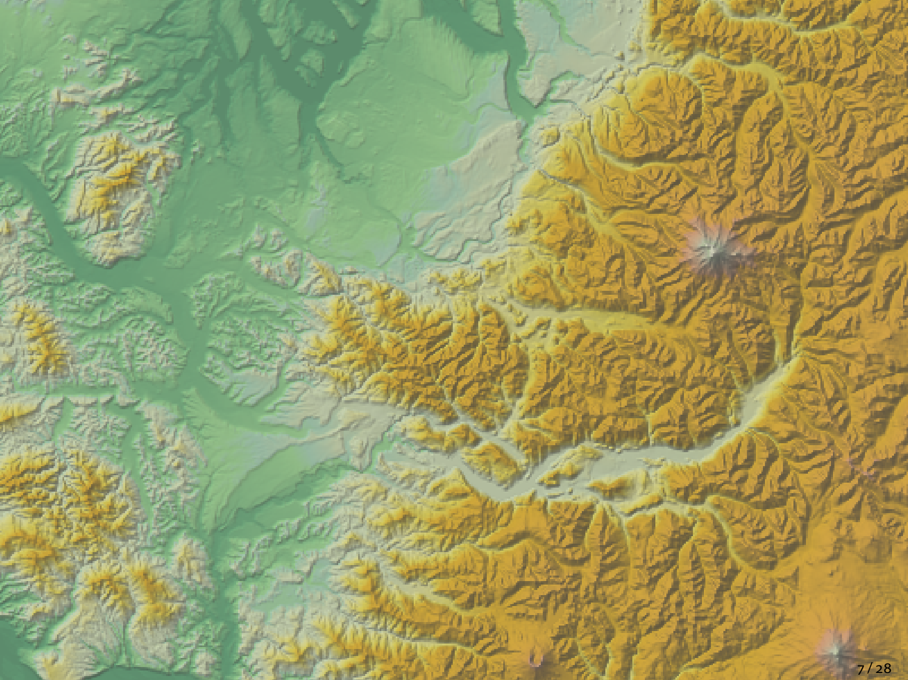
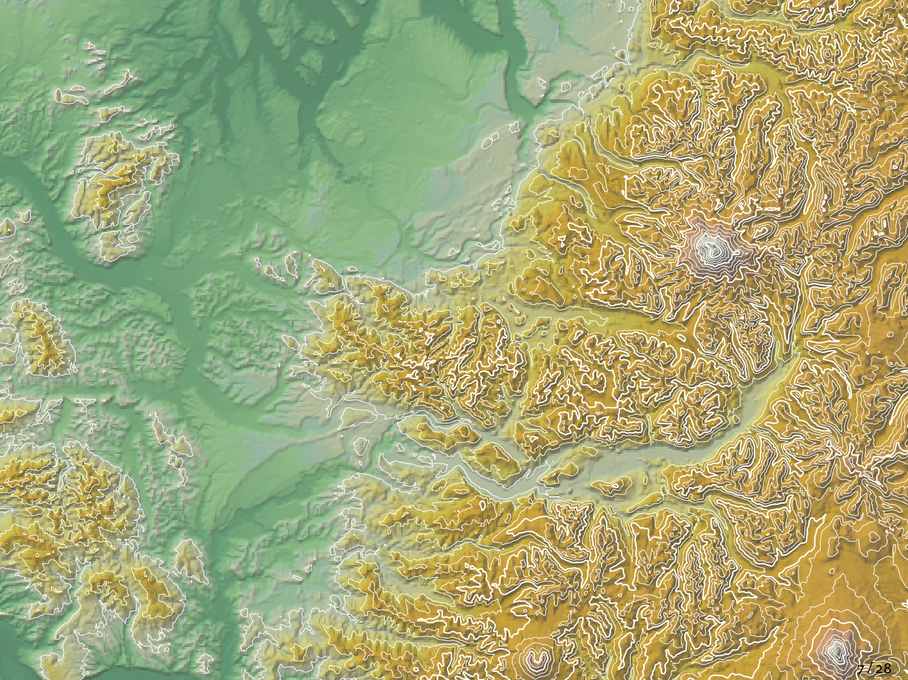
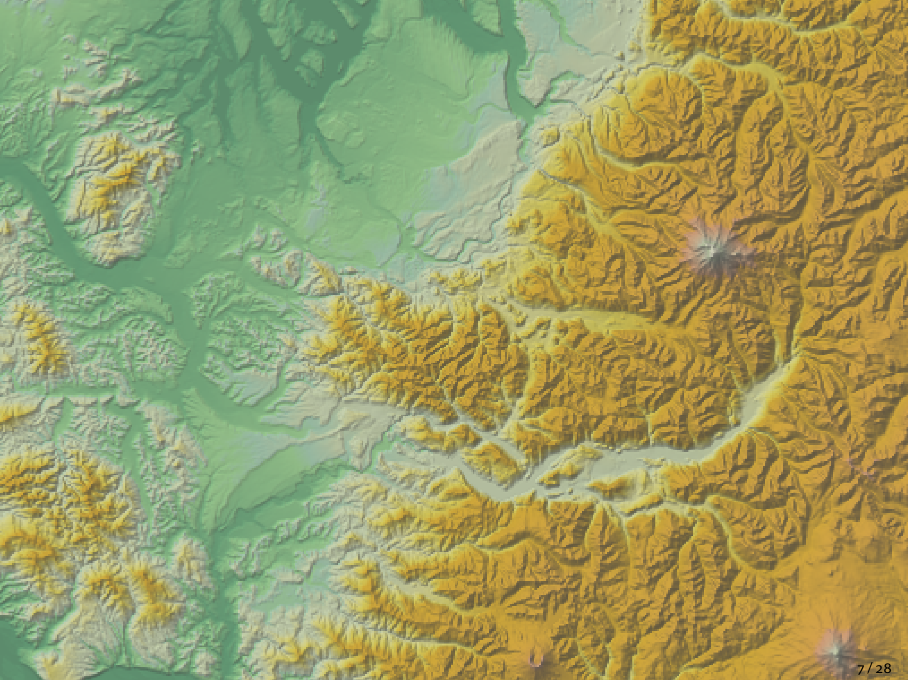
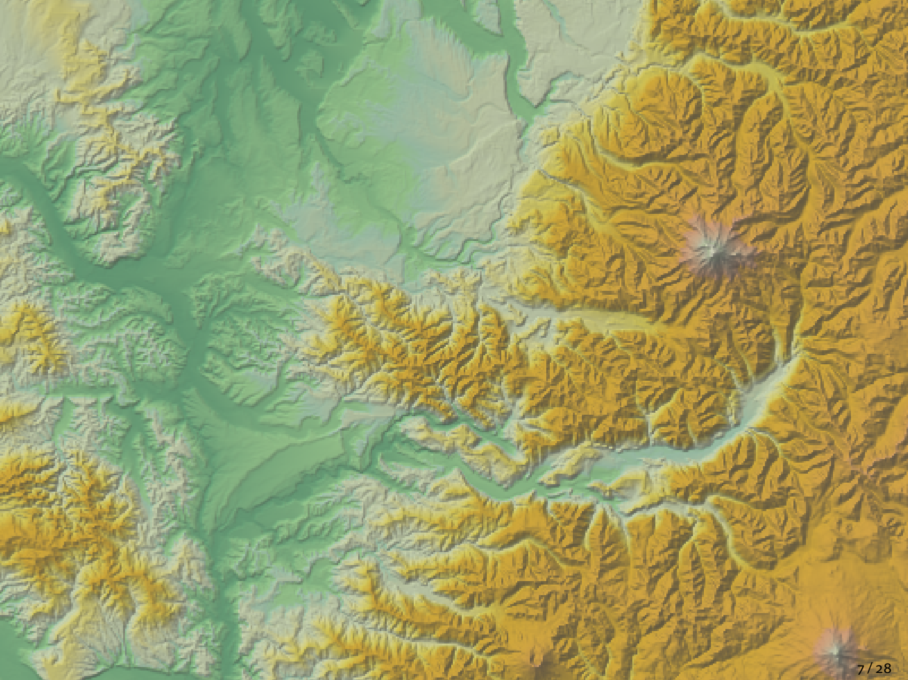
- This is why we use homology with coefficients in a field.

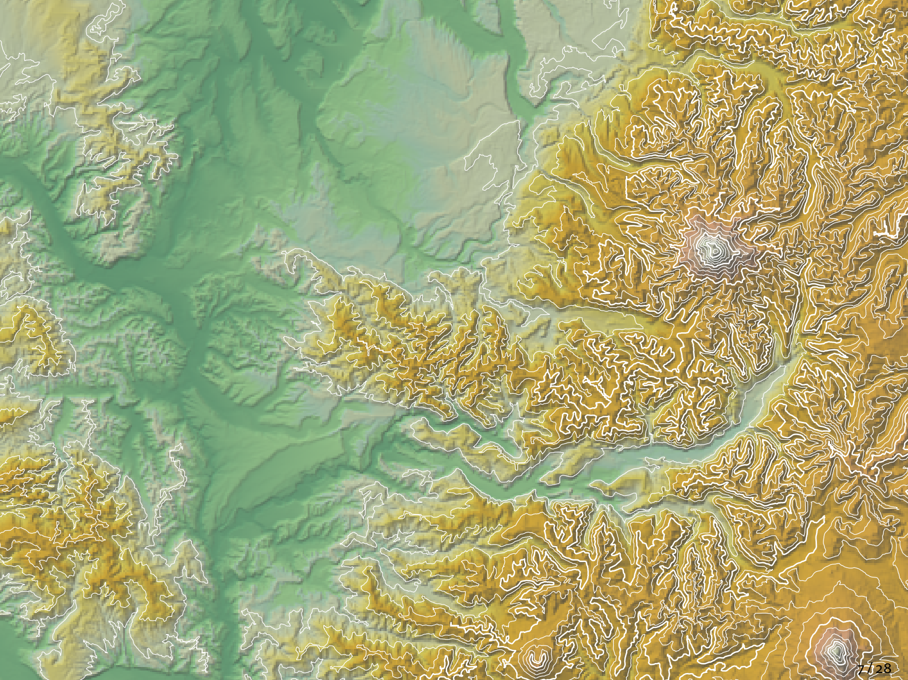
Simplification











Topological simplification of functions

Consider the following problem:

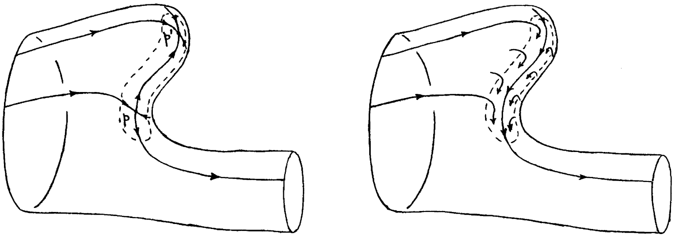
Problem (Topological simplification)

Given a function f and a real number $\delta \geq 0$, find a function f_δ with the minimal number of critical points subject to $\|f_\delta - f\|_\infty \leq \delta$.

Persistence and Morse theory

Morse theory (smooth or discrete):

- Relates critical points to homology of sublevel sets
- Provides a method for *canceling* pairs of critical points



(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

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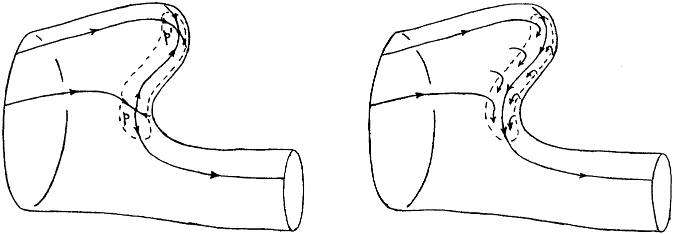

(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

Persistent homology:

- Relates homology of different sublevel set
- Identifies pairs of critical points (birth and death of homology) and quantifies their *persistence*

Persistence and Morse theory

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Proposition

The critical points of f with persistence $> 2\delta$ provide a lower bound on the number of critical points of any function g with $\|g - f\|_\infty \leq \delta$.

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Theorem (B, Lange, Wardetzky, 2011)

Let f be a function on a surface and let $\delta > 0$.
Canceling all pairs with persistence $\leq 2\delta$ yields a function f_δ

- *satisfying $\|f_\delta - f\|_\infty \leq \delta$ and*
- *achieving the lower bound on the number of critical points.*

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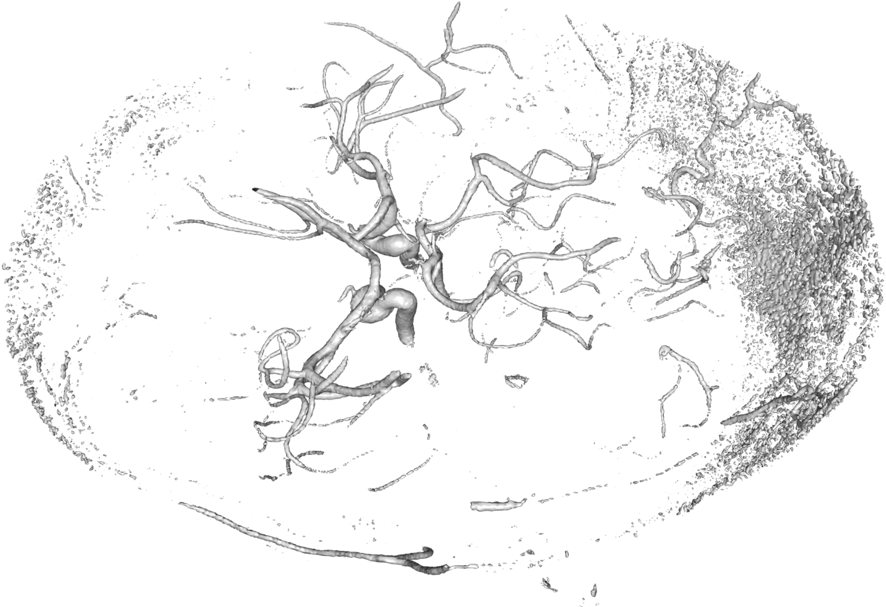
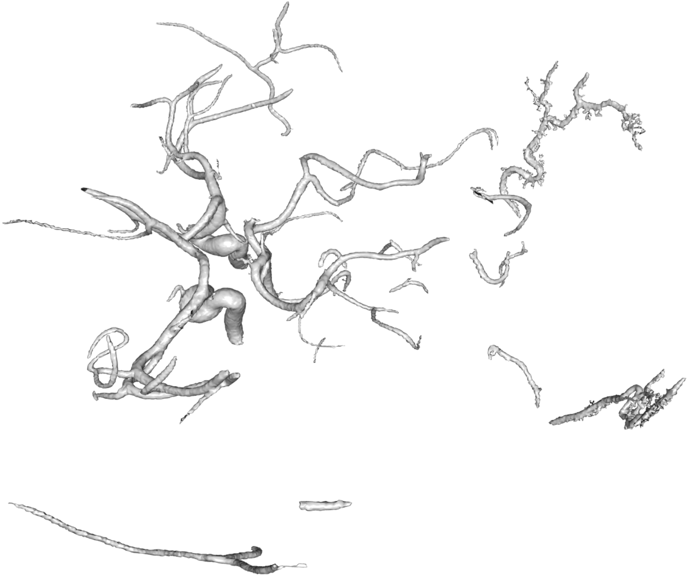
Does not generalize to higher-dimensional manifolds!





Sublevel set simplification

Let $F_t = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem (Sublevel set simplification)

Given a function $f : X \subset \mathbb{R}^3 \to \mathbb{R}$ *and* $t \in \mathbb{R}$, $\delta > 0$,
find a function g *with* $\|g - f\|_\infty \leq \delta$ *minimizing* $\dim H_*(G_t)$.

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Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

Sublevel set simplification in $\mathbb{R}^3$ is NP-hard.

Functional topology

When was persistent homology discovered?

H. Edelsbrunner, D. Letscher, and A. Zomorodian
Topological persistence and simplification
Foundations of Computer Science, 2000

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S. A. Barannikov.
The framed Morse complex and its invariants.
In *Singularities and bifurcations, Adv. Soviet Math. (vol. 21)*, 1994.

When was persistent homology discovered first?

When was persistent homology discovered first?

Annals of Mathematics
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

By Marston Morse

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses

When was persistent homology discovered first?

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Search within citing articles

Exact homomorphism sequences in homology theory
ed.ac.uk **[PDF]**

JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR

The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] **...**

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Marston Morse and his mathematical works
ams.org **[PDF]**

R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org

American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to **...**

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Unstable minimal surfaces of higher topological structure

M Morse, CB Tompkins - Duke Math. J, 1941 - projecteuclid.org

1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. Theproblem of the existenceof minimal surfaces of unstable type contains many of the typical difficulties, especially those ofa topological nature. Having studied this **...**

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[PDF] Persistence in discrete Morse theory
psu.edu **[PDF]**

U Bauer - 2011 - Citeseer

When was persistent homology discovered first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

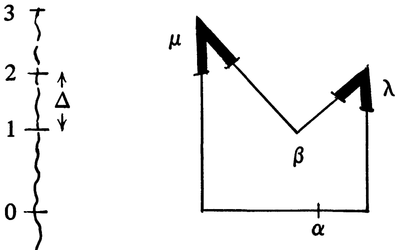
In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was

When was persistent homology discovered first?

inequalities pertain between the dimensions of the A_i *and those of* $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels [**D**] in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers [**36**] and [**44**] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.



When was persistent homology discovered first?

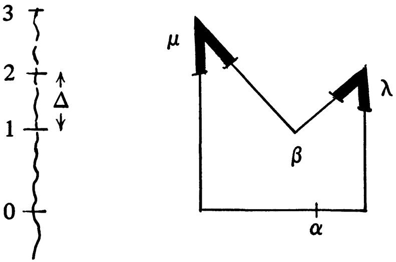


FIGURE 8

The values a where $H(a, a^-) \neq 0$ are indicated on the left, and corresponding to each of these *critical values* a generator of $H(a, a^-)$ is drawn on M, using the singular theory for simplicity. Morse calls such generators "caps". Thus α and β are two "0-caps" and μ and λ two "1-caps". Notice that every cap u defines a definite boundary element ∂u in

$$H(a^-) = \lim_{\varepsilon \to 0^+} H(F < a - \varepsilon);$$

Morse calls a cap u linkable iff $\partial u = 0$. Otherwise it is called *nonlinkable*.

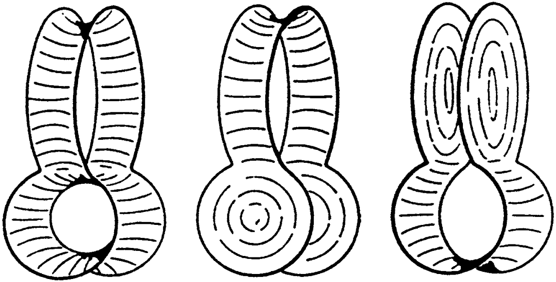
In our example, α, β and μ are linkable while λ is *not*.

Next Morse defines the *span* of a cap u associated to the critical level a in

Motivation and application: minimal surfaces

Problem (Plateau's problem)

Find a surface of least area spanned by a given closed Jordan curve.



(a) (b) (c)

(from Dierkes et al.: *Minimal Surfaces*, 2010)

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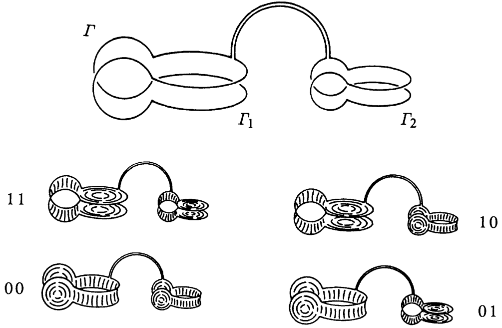

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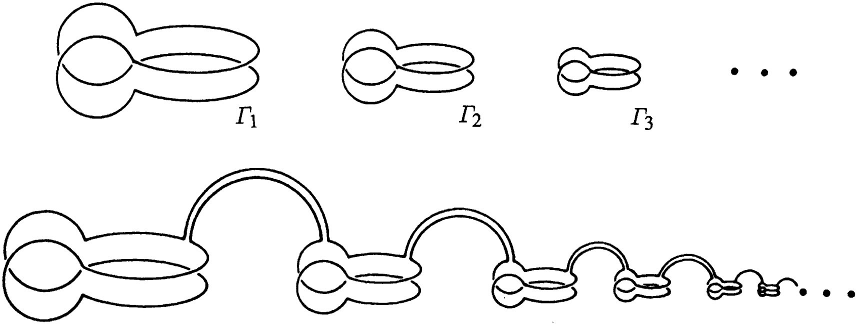

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The Douglas functional

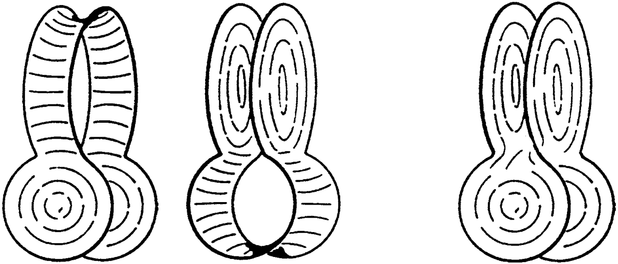
Theorem (Douglas 1930)

Given a Jordan curve $\Gamma : S^1 \to \mathbb{R}^3$*, there is a functional* A_γ *on the space of reparametrizations* $S^1 \to S^1$ *fixing three arbitrary points* $q_1, q_2, q_3 \in S^1$*, whose critical points correspond to the minimal surfaces of disk type bounded by* Γ*.*

Existence of unstable minimal surfaces

Theorem (Morse, Tompkins 1939; Shiffman 1939)

If there are two separate stable minimal surfaces with a given boundary curve, then there also exists an unstable *minimal surface bounding that curve (a critical point that is not a local minimum).*



Whatever happened to functional topology?

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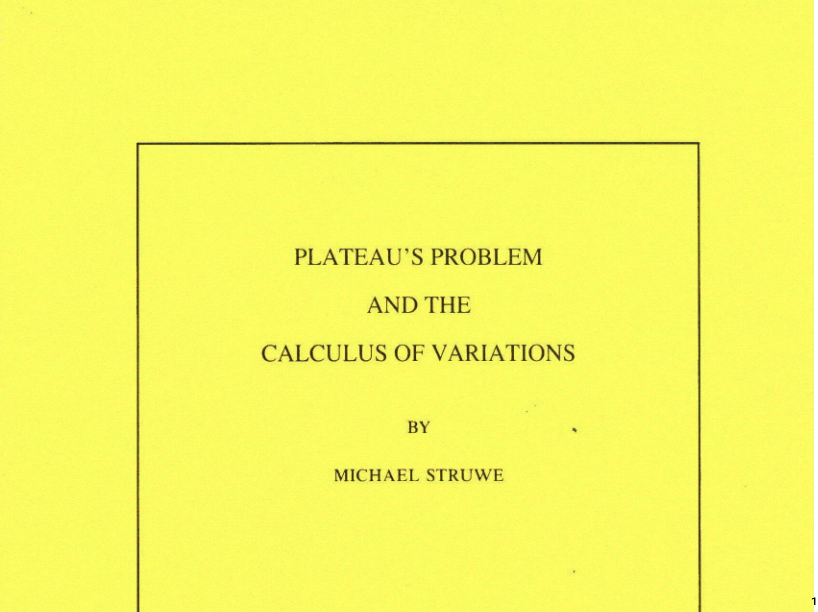

PLATEAU'S PROBLEM

AND THE

CALCULUS OF VARIATIONS

BY

MICHAEL STRUWE

Whatever happened to functional topology?

82 A. The classical Plateau Problem for disc - type minimal surfaces.

The technical complexity and the use of a sophisticated topological machinery (which is not shadowed in our presentation) moreover tend to make Morse-Tompkins' original paper unreadable and inaccessible for the non-specialist, cf. Hildebrandt [4, p. 324].

Confronting Morse-Tompkins' and Shiffman's approach with that given in Chapter 4 we see how much can be gained in simplicity and strength by merely replacing the C^o–topology by the $H^{1/2,2}$–topology and verifying the Palais - Smale - type condition stated in Lemma 2.10.

However, in 1964/65 when Palais and Smale introduced this condition in the calculus of variations it was not clear that it could be meaningful for analyzing the geometry of surfaces, cf. Hildebrandt [4, p. 323 f.].

Instead, a completely new approach was taken by Böhme and Tromba [1] to tackle the problem of understanding the global structure of the set of minimal surfaces spanning a wire.

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In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was in the 1920's to go from the mini-max principle of Birkhoff to the Morse

Whatever happened to functional topology?

Legion of Honor of France, ...

Nevertheless, when I first met Marston in 1949 he was in a sense a solitary figure, battling the *algebraic topology*, into which his beloved Analysis Situs had grown. For Marston always saw topology from the side of Analysis, Mechanics, and Differential geometry. The unsolved problems he proposed had to do with dynamics–the three body problem, the billiard ball problem, and so on. The development of the algebraic tools of topology, or the project of bringing order into the vast number of homology theories which had sprung up in the thirties–and which was eventually accomplished by the Eilenberg-Steenrod axioms–these had little interest for him. "*The battle between algebra and geometry has been waged from antiquity to the present*" he wrote in his address *Mathematics and the Arts* at Kenyon College in 1949, and

Received by the editors April 15, 1980.

[1]This work was supported in part through funds provided by the National Science Foundation under the grant 33-966-7566-2.

[2]Poincaré was born in 1854, the others all in the 1880's.

Q-tame persistence modules

Definition (Chazal et al. 2009)

A persistence module $M : \mathbf{R} \to \mathbf{vect}$ is *q-tame* if for every $s < t$ the structure map $M_s \to M_t$ has finite rank.

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- A q-tame persistence module does not necessarily have a barcode decomposition.

Structure of q-tame persistence modules

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

The radical *of a q-tame persistence module* M, $(\operatorname{rad} M)_t = \sum_{s<t} \operatorname{im} M_{s,t}$,

- *admits a barcode decomposition, and*

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- *A* persistence diagram *describes the intervals in a barcode, modulo the endpoints.*
- *The* observable category *is the category of persistence modules, modulo ephemeral persistence modules (localization).*

Generalized Morse inequalities

Assume that the sublevel sets of a bounded function $f : X \to \mathbb{R}$ have q-tame persistent homology.

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- Morse and Tompkins used this idea to show the existence of a minimal surface.

Weakly πLC filtrations

Definition (Morse 1937; paraphrased)

The sublevel set filtration of a function $f: X \to \mathbb{R}$ is said to be *weakly homotopically locally connected*, or *weakly πLC*, if for

- any point $x \in X$,
- any neighborhood V of x, and
- any value $t > f(x)$,

there is

- a value s with $f(x) < s < t$ and
- a neighborhood U of x with $U \subseteq V$

such that the inclusion $U \cap f_{\leq s} \to V \cap f_{\leq t}$ induces trivial maps on all homotopy groups.

Vietoris vs singular homology

- A compact sublevel set filtration $(f_{\leq t})_{t \in \mathbb{R}}$ is continuous from above:

$$f_{\leq t} = \bigcap_{u>t} f_{\leq u} = \lim_{u>t} f_{\leq u}$$

- Vietoris (equivalently, Čech) homology (with field coefficients) preserves limits:

$$\lim \circ \check{H}_* = \check{H}_* \circ \lim$$

- Therefore, persistent Vietoris/Čech homology is also continuous from above:

$$\check{H}_*(f_{\leq t}) = \lim_{u>t} \check{H}_*(f_{\leq u})$$

Q-tameness from local connectivity

Theorem (Morse, 1937)

If $f\colon X \to \mathbb{R}$ on a metric space X is bounded below and the sublevel sets are compact and weakly πLC, then it has q-tame persistent Vietoris homology.

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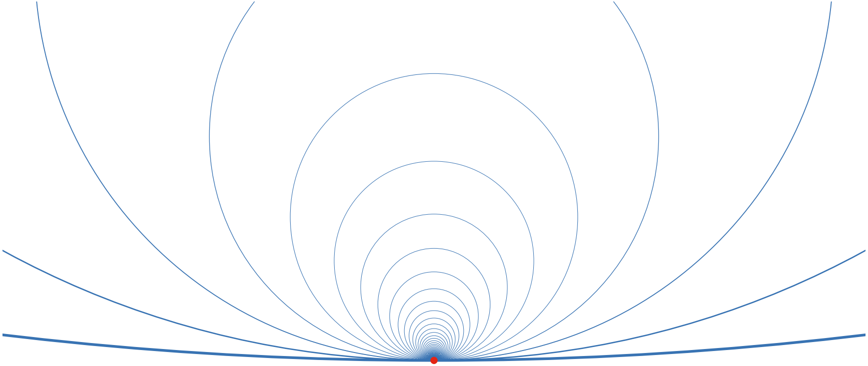

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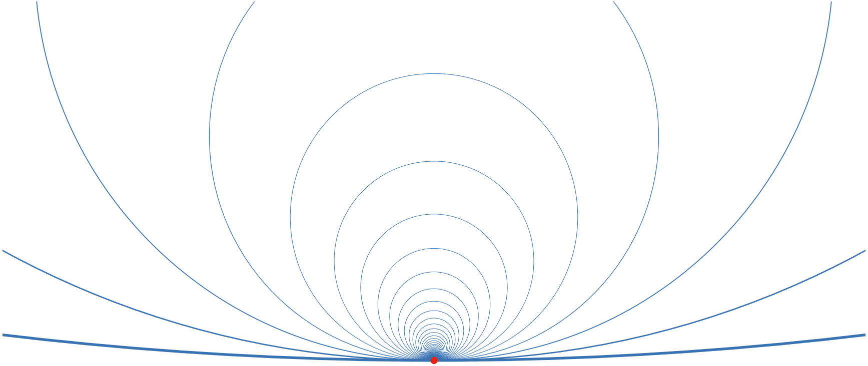

Homologically locally small filtrations

Definition (B, Medina-Mardones, Schmahl)

The sublevel set filtration of a function $f: X \to \mathbb{R}$ is called *homologically locally small* or *HLS* if for

- any point $x \in X$,
- any neighborhood V of x, and
- any pair of values s, t with $f(x) < s < t$

there is

- a neighborhood U of x with $U \subseteq V$

such that the inclusion $U \cap f_{\leq s} \to V \cap f_{\leq t}$ induces maps of finite rank on homology.

A sufficient condition for q-tame persistence

Theorem (B, Medina-Mardones, Schmahl 2021)

If the sublevel set filtration of a (not necessarily continuous) function $f\colon X \to \mathbb{R}$ is compact and HLS, then its persistent homology is q-tame.

- Applies to Vietoris/Čech as well as singular homology

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- Persistence diagrams explain the topology of functionals
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- Local homological smallness ensures q-tameness
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