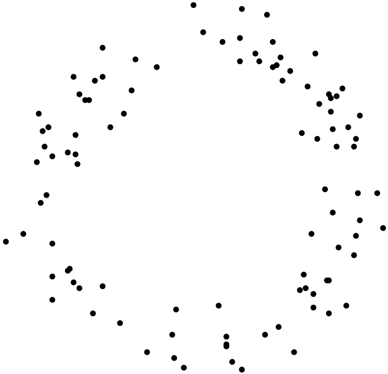
Topological Data Analysis

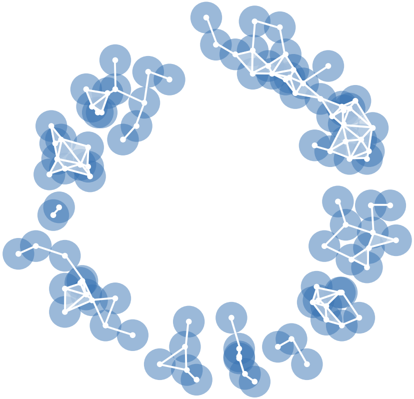
Part II: Stability of persistence barcodes

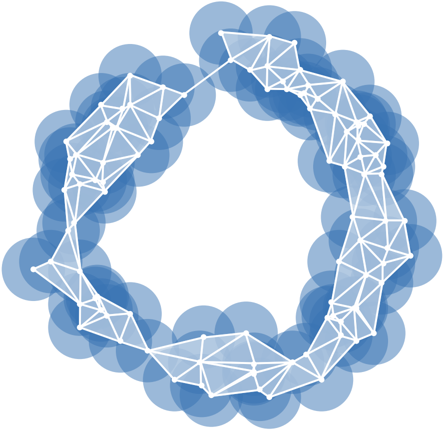
Ulrich Bauer

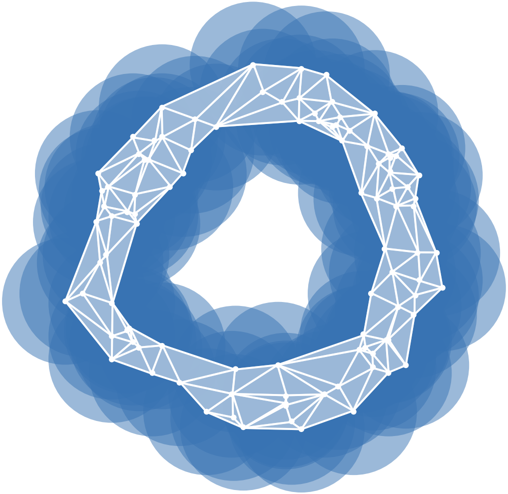
TUM

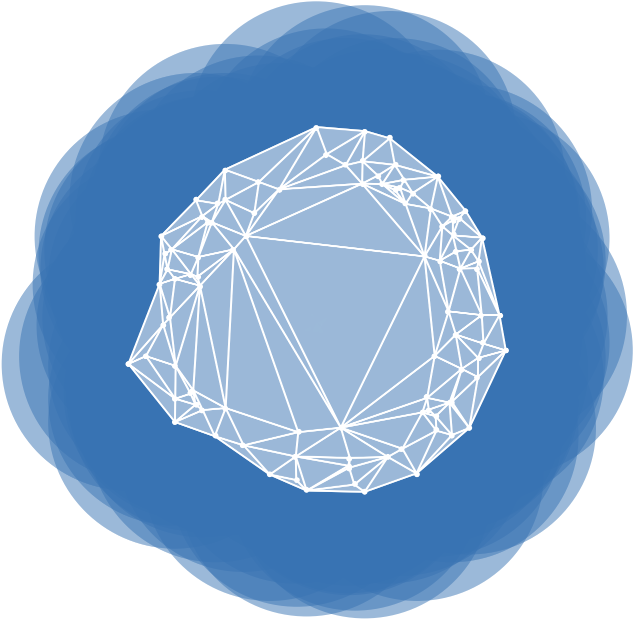
February 5, 2015



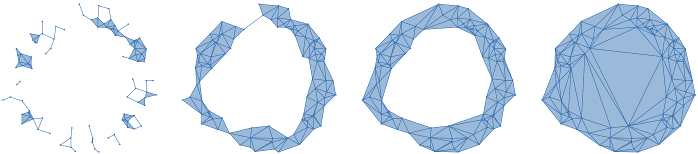


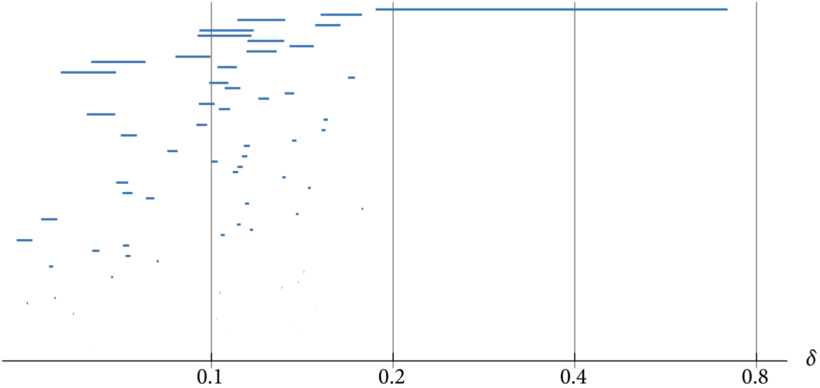






What is persistent homology?





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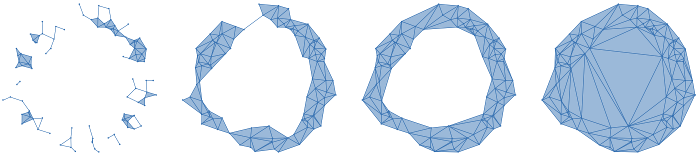

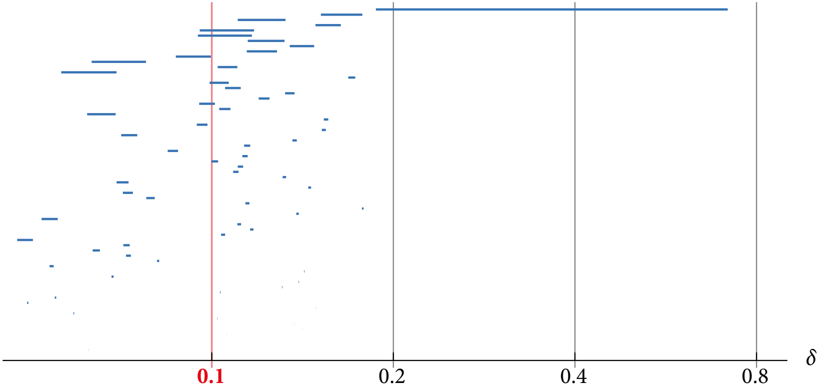

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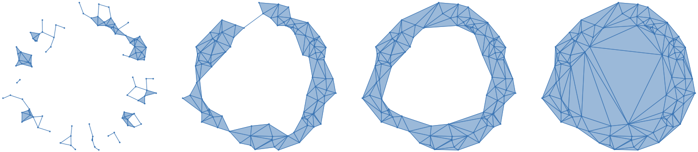

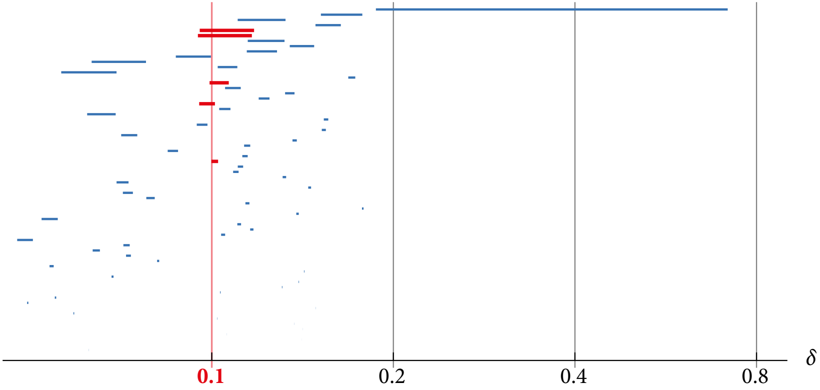

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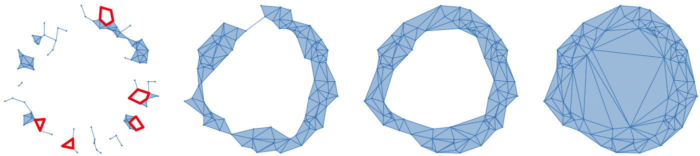

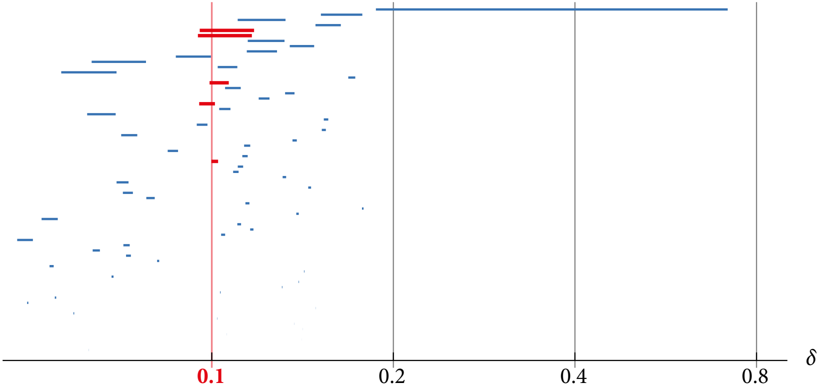

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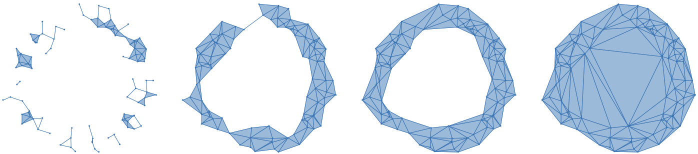

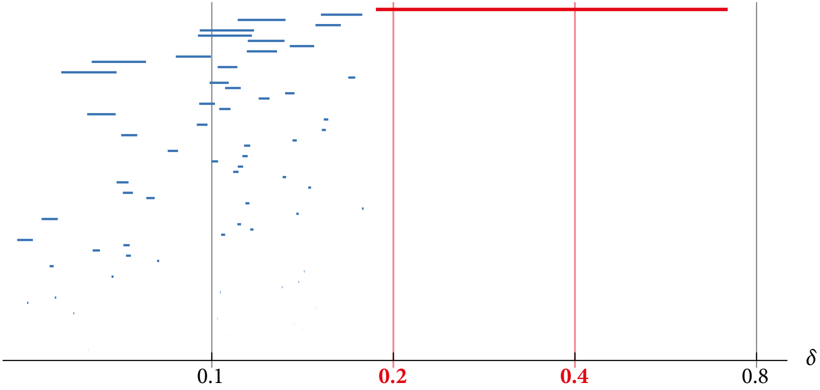

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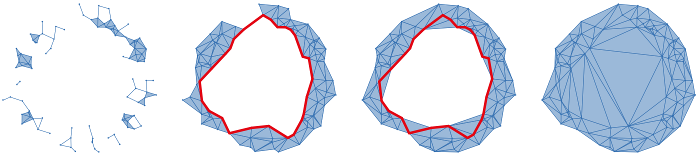

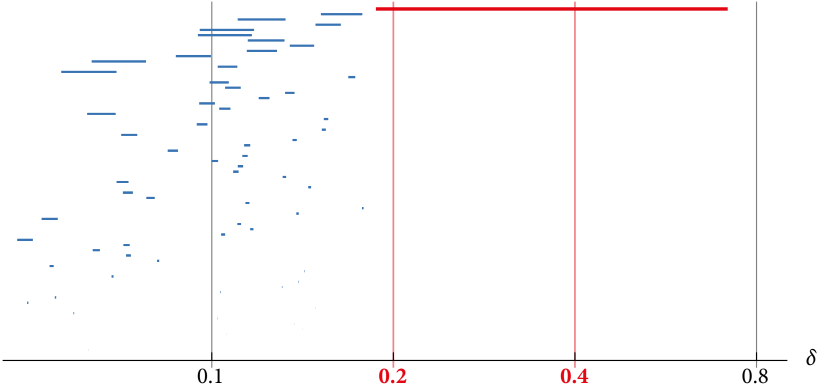

The pipeline of topological data analysis

$$
\begin{tikzcd}
\text{point cloud} \arrow[d, "\text{distance}"] \\
\text{function} \arrow[d, "\text{sublevel sets}"] \\
\text{topological spaces} \arrow[d, "\text{homology}"] \\
\text{vector spaces} \arrow[d, "\text{barcode}"] \\
\text{intervals}
\end{tikzcd}
$$

The pipeline of topological data analysis

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\begin{array}{cc}
\text{point cloud} & P \subset \mathbb{R}^d \\
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$$\begin{array}{ccc} \text{point cloud} & & P \subset \mathbb{R}^d \\ \Big\downarrow \text{distance} & & \Big\downarrow x \mapsto d(x,P) \\ \text{function} & & \\ \Big\downarrow \text{sublevel sets} & & \\ \text{topological spaces} & & \\ \Big\downarrow \text{homology} & & \\ \text{vector spaces} & & \\ \Big\downarrow \text{barcode} & & \\ \text{intervals} & & \end{array}$$

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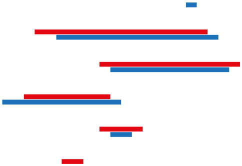
Stability of persistence barcodes

Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

If two functions $f, g : K \to \mathbb{R}$ *have distance* $\|f - g\|_\infty \leq \delta$ *then there exists a* δ*-matching of their barcodes.*



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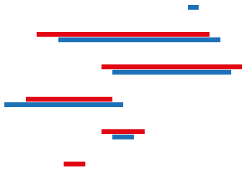

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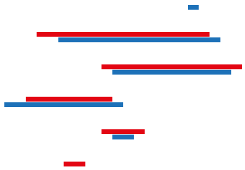

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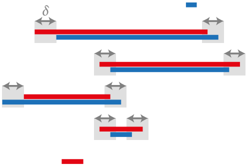

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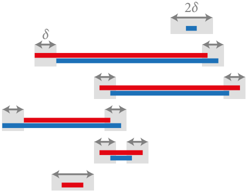

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Stability for functions in the big picture

$$\begin{tikzcd}
\text{Data} \arrow[d] & \text{point cloud} \arrow[d, "\text{distance}"] \\
\text{Geometry} \arrow[d] & \text{function} \arrow[d, "\text{sublevel sets}"] \\
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So the sublevel sets are δ-*interleaved:*

$$\begin{tikzcd} F_t \arrow[rr, hook] \arrow[dr, hook] & & F_{t+2\delta} \arrow[dr, hook] & \\ & G_{t+\delta} \arrow[ur, hook] \arrow[rr, hook] & & G_{t+3\delta} \end{tikzcd}$$

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Homology is a *functor:* homology groups are interleaved too.

Persistence modules

A *persistence module* M is a *diagram* (functor) $\mathbf{R} \to \mathbf{Vect}$:

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A *morphism* $f : M \to N$ is a *natural transformation*:

- a linear map $f_t : M_t \to N_t$ for each $t \in \mathbb{R}$
- morphism and transition maps commute:

$$\begin{tikzcd} M_s \arrow[r] \arrow[d, "f_s"'] & M_t \arrow[d, "f_t"] \\ N_s \arrow[r] & N_t \end{tikzcd}$$

Interval Persistence Modules

Let $\mathbb{K}$ be a field. For an arbitrary interval $I \subseteq \mathbb{R}$, define the *interval persistence module* $C(I)$ by

$$C(I)_t = \begin{cases} \mathbb{K} & \text{if } t \in I, \\ 0 & \text{otherwise;} \end{cases}$$

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The structure of persistence modules

Theorem (Crawley-Boewey 2012)

Let M be a persistence module with $\dim M_t < \infty$ for all t.

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$B(M)$ is called the *barcode* of M.

The structure of persistence modules

Theorem (Crawley-Boewey 2012)

Let M *be a persistence module with* $\dim M_t < \infty$ *for all* t.
Then M *is* interval-decomposable:
there exists a unique collection of intervals $B(M)$ *such that*

$$M \cong \bigoplus_{I \in B(M)} C(I).$$

$B(M)$ is called the *barcode* of M.

- Motivates use of homology with field coefficients

Interleavings of persistence modules

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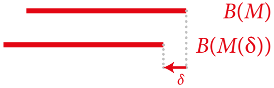
- define $M(\delta)$ by $M(\delta)_t = M_{t+\delta}$ (shift barcode to the left by δ)



Algebraic stability of persistence barcodes

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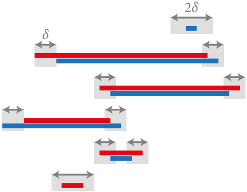

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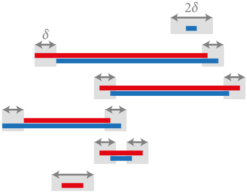

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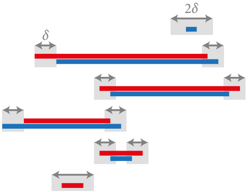

- converse statement also holds (isometry theorem)
- indirect proof, 80 page paper (Chazal et al. 2012)

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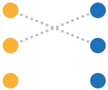
- stability follows from properties of a *single morphism*, not just from a pair of morphisms
- relies on *partial functoriality* of the induced matching

The matching category

A *matching* $\sigma : S \nrightarrow T$ is a bijection $S' \to T'$, where $S' \subseteq S$, $T' \subseteq T$.



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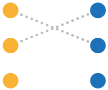
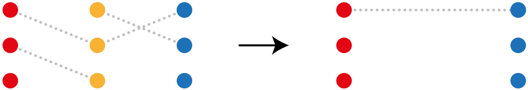


Composition of matchings $\sigma : S \nrightarrow T$ and $\tau : T \nrightarrow U$:



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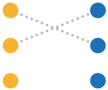

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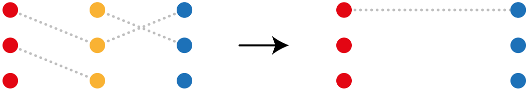
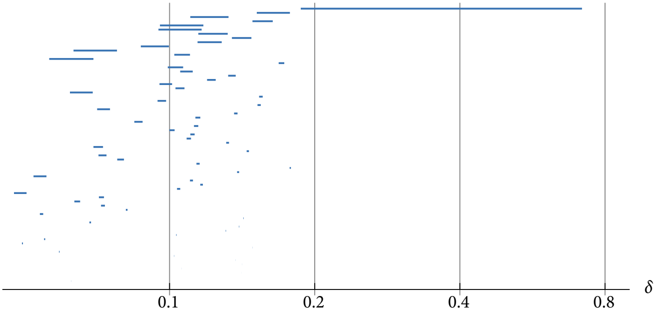


Matchings form a category $\mathbf{Mch}$

- objects: sets
- morphisms: matchings

Barcodes as matching diagrams

We can regard a barcode $\mathcal{D}$ as a functor $\mathbf{R} \to \mathbf{Mch}$:



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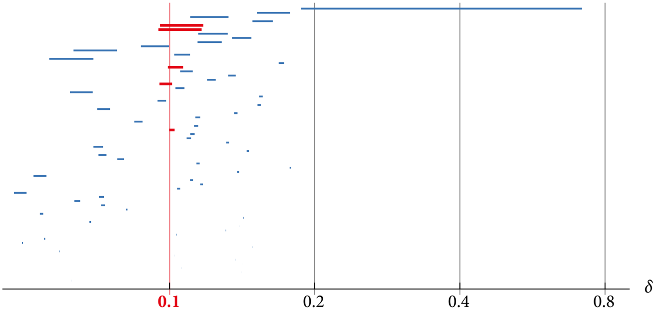

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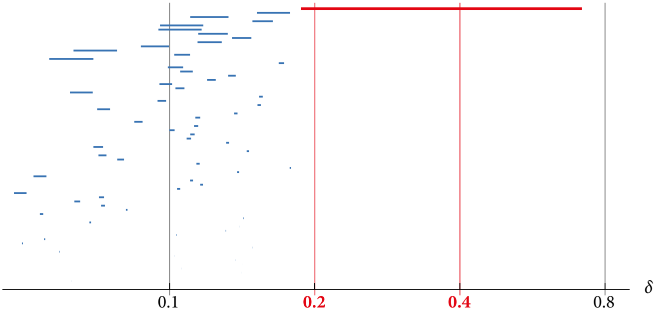
- For each real number t, let $\mathcal{D}_t$ be those intervals of $\mathcal{D}$ that contain t, and
- for each $s \leq t$, define the matching $\mathcal{D}_s \nrightarrow \mathcal{D}_t$ to be the identity on $\mathcal{D}_s \cap \mathcal{D}_t$.



Barcode matchings as interleavings

We can regard matchings of barcodes $\sigma : \mathcal{C} \nrightarrow \mathcal{D}$ as natural transformations.

- consider restrictions $\sigma_t : \mathcal{C}_t \nrightarrow \mathcal{D}_t$ of σ to $\mathcal{C}_t \times \mathcal{D}_t$:

$$\begin{tikzcd}
\mathcal{C}_s \arrow[r, "+" marking] \arrow[d, "\sigma_s"', "+" marking] & \mathcal{C}_t \arrow[d, "\sigma_t", "+" marking] \\
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This way, we can regard a δ-matching of barcodes $\mathcal{C} \nrightarrow \mathcal{D}$ as a δ-interleaving of functors $\mathbf{R} \to \mathbf{Mch}$:

$$\begin{tikzcd}
\mathcal{C}_t \arrow[rr, "\shortmid" marking] \arrow[dr, "\shortmid" marking] & & \mathcal{C}_{t+2\delta} \arrow[dr, "\shortmid" marking] & \\
& \mathcal{D}_{t+\delta} \arrow[ur, "\shortmid" marking] \arrow[rr, "\shortmid" marking] & & \mathcal{D}_{t+3\delta}
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Stability via functoriality?

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F_t \arrow[rr, hook] \arrow[dr, hook] & & F_{t+2\delta} \arrow[dr, hook] & \\
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Theorem (B, Lesnick 2014)

There exists no functor $\mathbf{Vect}^{\mathbf{R}} \to \mathbf{Mch}$ *sending each persistence module to its barcode.*

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Proposition

There exists no functor $\mathbf{Vect} \to \mathbf{Mch}$ *sending each vector space of dimension* d *to a set of cardinality* d*.*

Structure of submodules and quotient modules

Proposition (B, Lesnick 2013)

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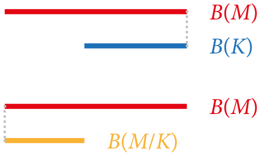

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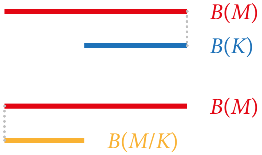

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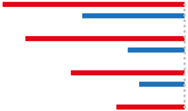
- If multiple bars have same endpoint: match in order of decreasing length



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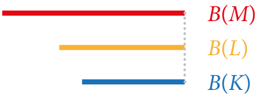
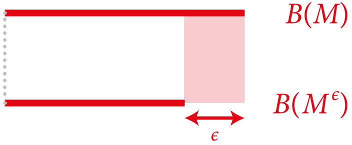


Similar for surjections.

The induced matching theorem

Define M^ϵ by shrinking bars of $B(M)$ from the right by ϵ.

Say K is *ϵ-trivial* if all bars of $B(K)$ are shorter than ϵ.



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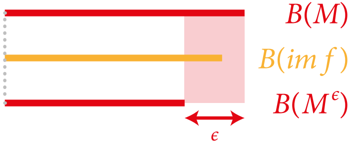
Lemma

Let $f : M \to N$ be a morphism such that $\ker f$ is ϵ-trivial.

Then M^ϵ is a quotient module of $\operatorname{im} f$.



$$\begin{tikzcd}
 & M \arrow[dl, two heads] \arrow[dr, two heads] & \\
M^\epsilon & & \operatorname{im} f \arrow[ll, two heads]
\end{tikzcd}$$

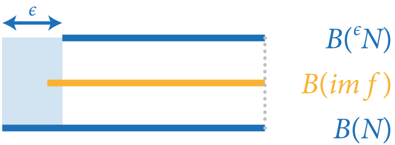
The induced matching theorem

Define ${}^{\epsilon}N$ by shrinking bars of $B(N)$ from the left by ϵ.

Say K is ϵ-*trivial* if all bars of $B(K)$ are shorter than ϵ.

Lemma

Let $f : M \to N$ be a morphism such that $\operatorname{coker} f$ *is ϵ-trivial.*
Then ${}^{\epsilon}N$ is a submodule of $\operatorname{im} f$.



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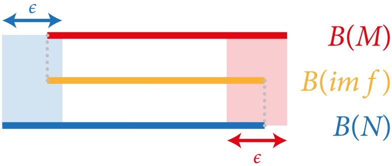

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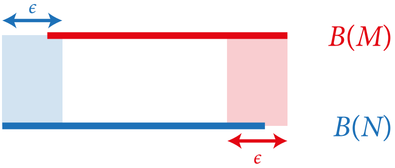

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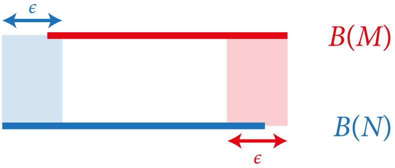

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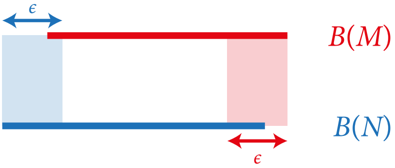
Let $f : M \to N$ *be a morphism with* $\ker f$ *and* $\operatorname{coker} f$ ϵ*-trivial.*
Then each interval of length $\geq \epsilon$ *is matched by* $B(f)$*.*



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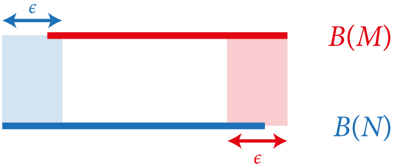
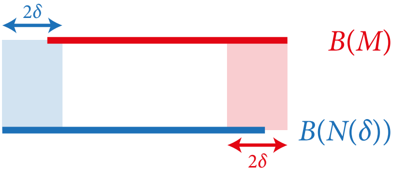
If $B(f)$ *matches* $[b, d) \in B(M)$ *to* $[b', d') \in B(N)$*, then*
$b' \leq b \leq b' + \epsilon$ *and* $d - \epsilon \leq d' \leq d$.



The induced matching theorem

Let $f : M \to N(\delta)$ be an interleaving morphism.

Then $\ker f$ and $\operatorname{coker} f$ are 2δ-trivial.



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Let $f : M \to N(\delta)$ be an interleaving morphism.

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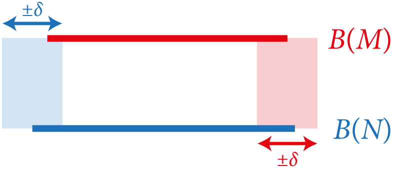
Corollary (Algebraic stability via induced matchings)

A δ-interleaving between persistence modules induces a δ-matching of their persistence barcodes.



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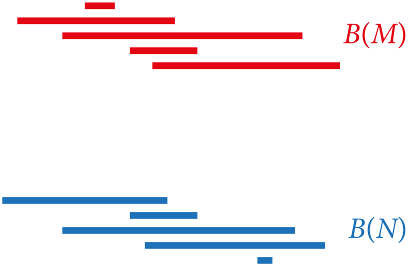

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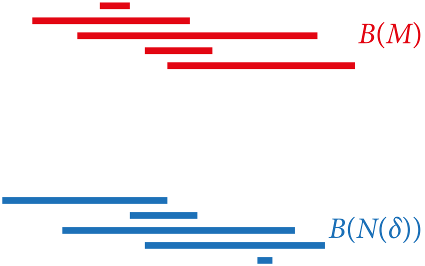

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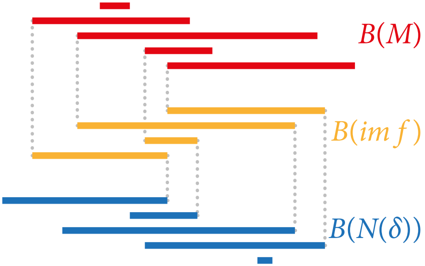

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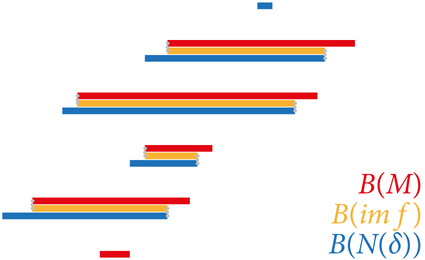

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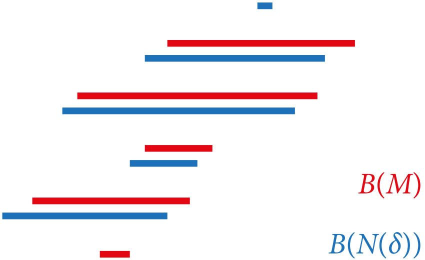

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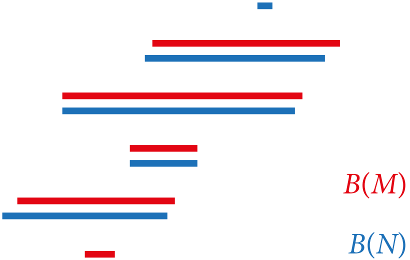
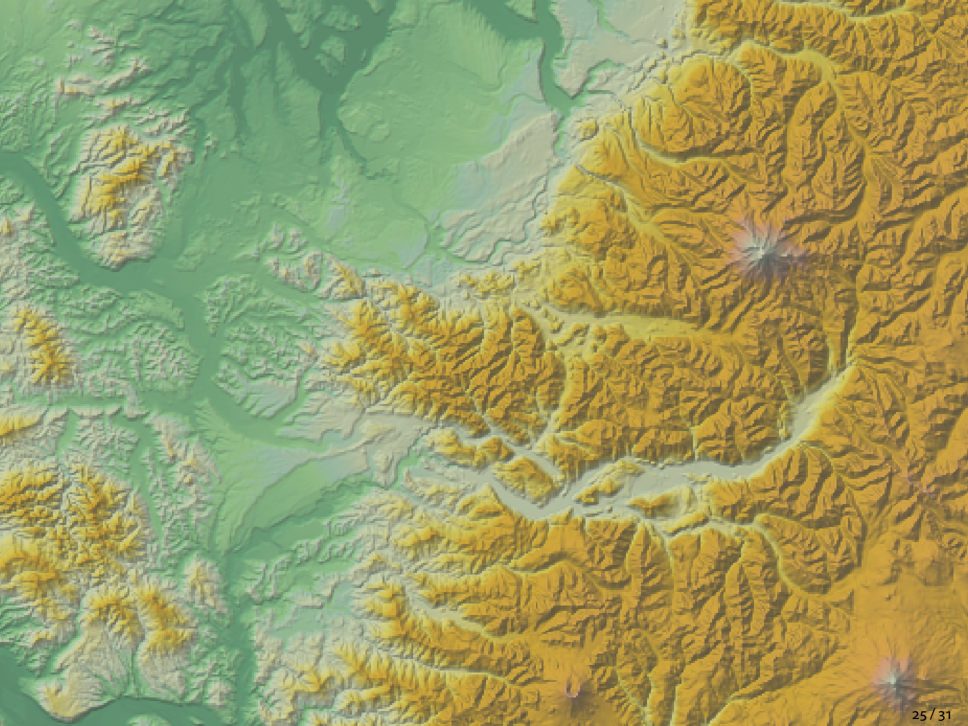
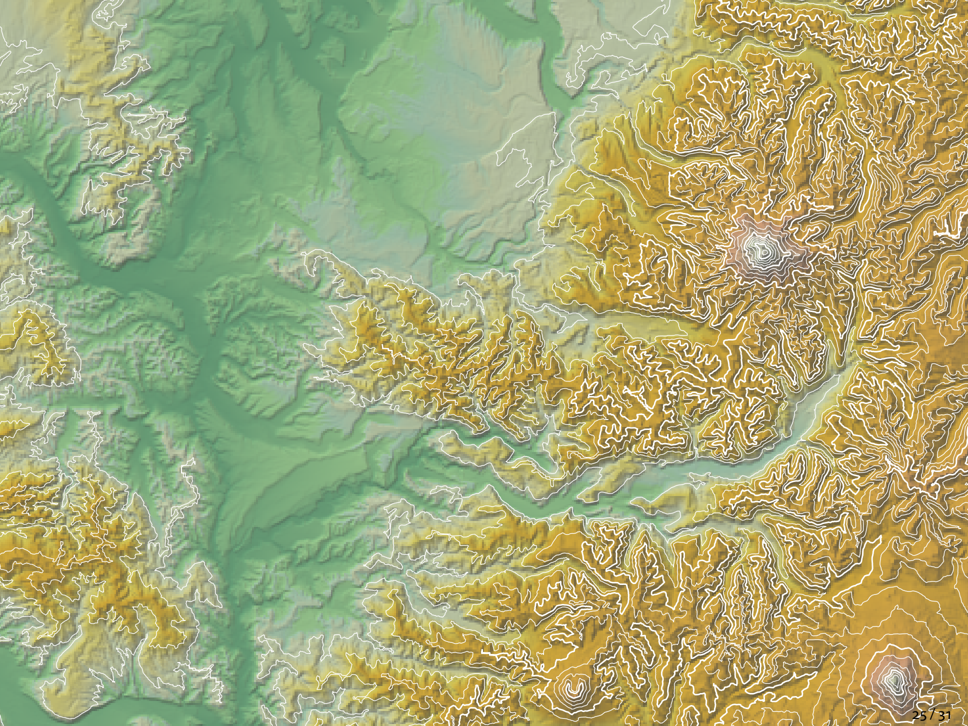


Simplification of functions





Topological simplification of functions

Consider the following problem:

Problem (Topological simplification)

Given a function f *and a real number* $\delta \geq 0$*, find a function* f_δ *subject to* $\|f_\delta - f\|_\infty \leq \delta$ *with minimal number of critical points.*

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Persistent homology:

- Relates homology of different sublevel set
- Identifies pairs of critical points (birth and death of homological feature)

Persistence and discrete Morse theory

By stability of persistence barcodes:

Proposition

The critical points with persistence $> 2\delta$ *provide a lower bound on the number of critical points of any function* g *with* $\|g - f\|_\infty \leq \delta$.

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- Does not hold for general complexes

Geometric stability

Čech and Vietoris–Rips complexes

Let $X \subset \mathbb{R}^d$. Given $\delta > 0$, consider

$$\mathrm{Cech}_\delta(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} B_\delta(p) \neq \varnothing \right\}$$

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Interleaving of Čech and Rips complexes

Using the logarithmic scale $t = \log \delta$:

$$\mathrm{Cech}_{\exp t} \subseteq \mathrm{Rips}_{2\exp(t+\log \vartheta_d)} \subseteq \mathrm{Cech}_{\exp(t+2\log \vartheta_d)}$$

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Corollary

The persistence modules $H_*(\mathrm{Cech}_{\exp t}(P))$ *and* $H_*(\mathrm{Rips}_{2\exp t}(P))$ *are* $(\log \vartheta_d)$*-interleaved.*

Stability of Čech and Rips persistence

Proposition

Let $P, \Omega \subset \mathbb{R}^n$. Assume that P and Ω have Hausdorff distance $< \delta$ (i.e., $P \subseteq B_\delta(\Omega)$ and $\Omega \subseteq B_\delta(P)$).

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