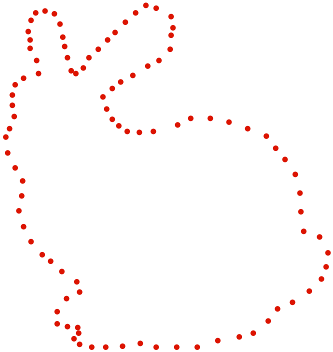
The Morse theory of Čech and Delaunay filtrations

Ulrich Bauer Herbert Edelsbrunner

IST Austria

SoCG 2014

Connect the dots: topology from geometry



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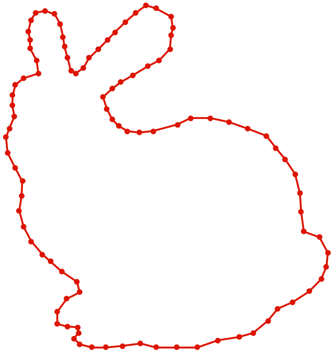

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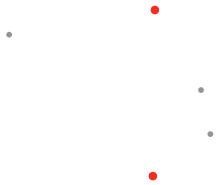

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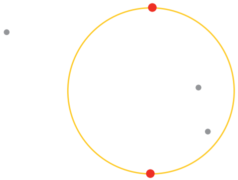

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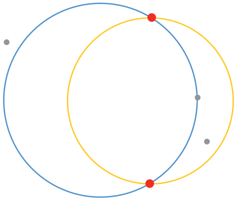

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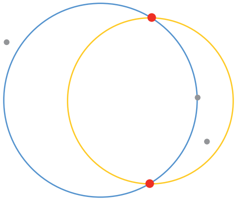

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- defined only if Q has an empty circumsphere: $Q \in \mathrm{Del}(X)$

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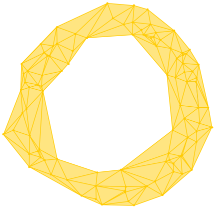

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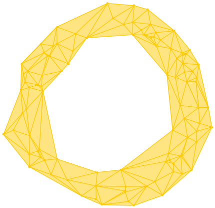

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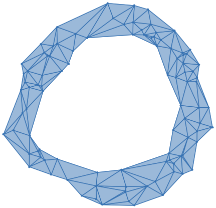

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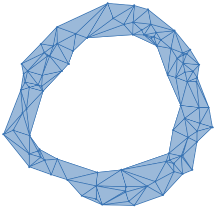

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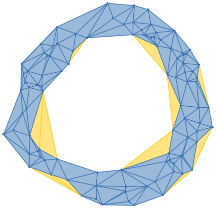

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By the *Nerve theorem* (Borsuk 1947):

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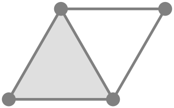
- Are they related by a sequence of simplicial collapses?

Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.



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Definition (Whitehead 1938)

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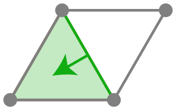
- A *free face* of K is a simplex which is contained in a unique proper coface.



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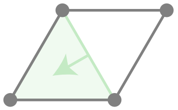

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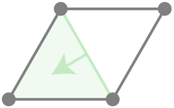

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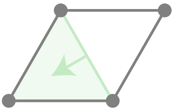

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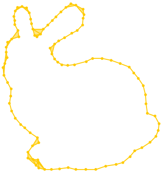
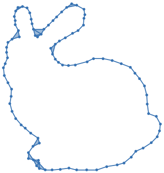
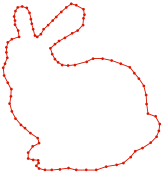
If there is a sequence of these elementary collapses from K to L, we say that K *collapses* to L (written as $K \searrow L$).

Main result: a sequence of collapses

Theorem (B, Edelsbrunner)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent. In particular,

$$\mathrm{DelCech}_r \searrow \mathrm{Del}_r \searrow \mathrm{Wrap}_r .$$

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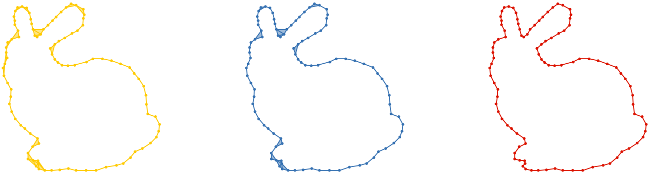

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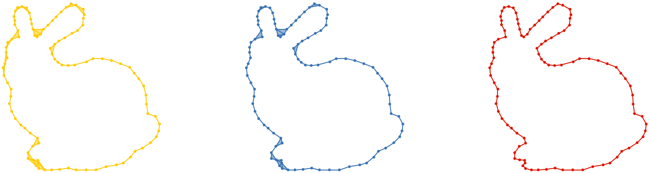

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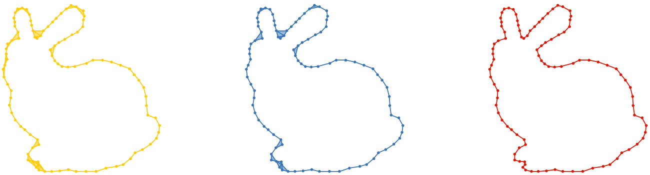

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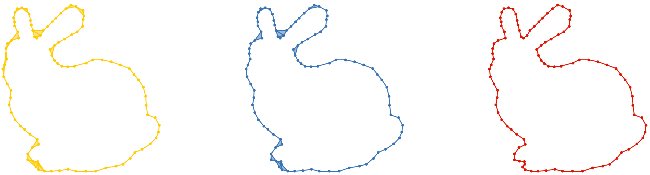

- All collapses are induced by a single *discrete gradient field*
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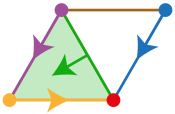
- Also works for weights

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a simplicial complex
is a partition of the simplices
into singletons and *facet pairs* (Q, R)
(Q is a face of R with codimension 1).



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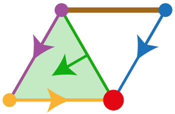

The singletons are called *critical simplices*.

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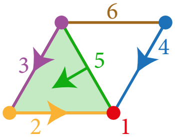
A function $f : K \to \mathbb{R}$ on a simplicial complex is a *discrete Morse function* if for all $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes



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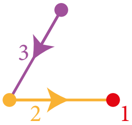

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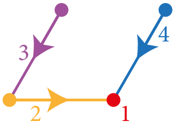

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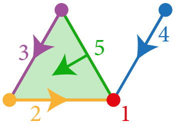

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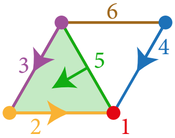

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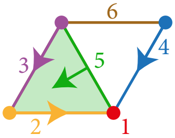

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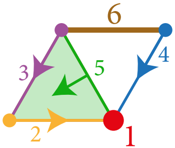
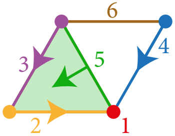


If $f^{-1}(t) = \{Q\}$ then t is a *critical value*.

Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.

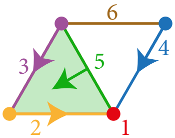


Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.

Theorem (Forman 1998)

If $(s,t]$ contains no critical value of f,
then K_t collapses to K_s *(written as $K_t \searrow K_s$).*



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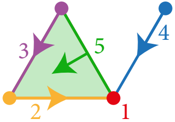

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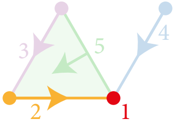

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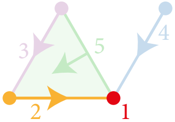
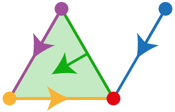


Let V be a discrete gradient field on a simplicial complex K, and let L be a subcomplex of K.

Corollary

If $K \setminus L$ is the union of facet pairs of V, then $K \searrow L$.



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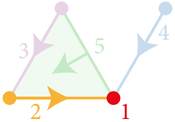

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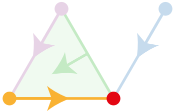
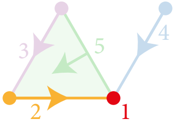


Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.

Theorem (Forman 1998)

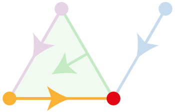
If $(s,t]$ contains no critical value of f, then K_t collapses to K_s (written as $K_t \searrow K_s$).



Let V be a discrete gradient field on a simplicial complex K, and let L be a subcomplex of K.

Corollary

If $K \smallsetminus L$ is the union of facet pairs of V, then $K \searrow L$.



We say that V *induces the collapse $K \searrow L$.*

Unfortunately…

Neither the Čech nor the Delaunay functions
are discrete Morse functions!

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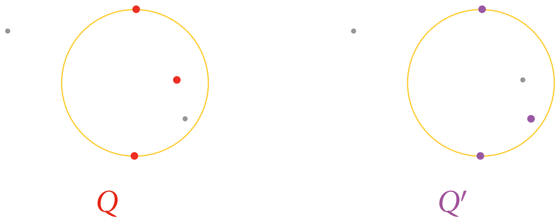
- Example: two simplices Q, Q' with $f_C(Q) = f_C(Q')$ that do not form a facet pair:

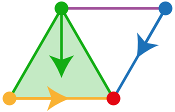


Generalized discrete Morse theory

Definition (Freij 2009)

A *generalized discrete vector field* on a simplicial complex is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}.$$



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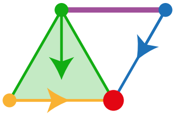


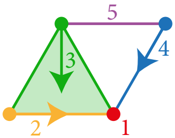
The singletons are called *critical simplices.*

Generalized discrete Morse theory

Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes



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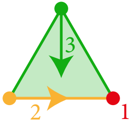

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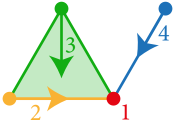

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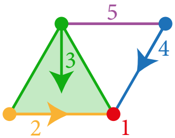

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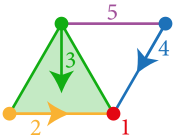
is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

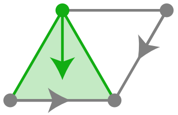
- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes
- the level sets $f^{-1}(t)$ form a generalized vector field (the *discrete gradient* of f)



Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:



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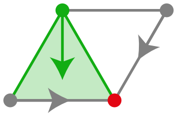

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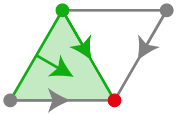

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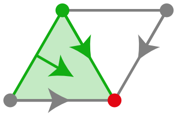

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Therefore the collapsing theorems also hold for generalized discrete Morse functions.

Morse theory of Čech and Delaunay complexes

Proposition

The Čech function and the Delaunay function are generalized discrete Morse functions.

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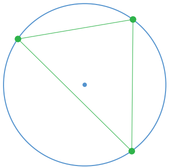
- $f_D(Q) = f_C(Q)$
- *Q is a critical simplex of f_C*
- *Q is a critical simplex of f_D*
- *Q is a* centered *Delaunay simplex (containing the circumcenter in the interior)*



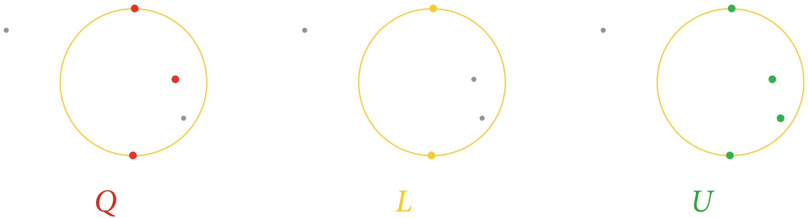
Čech intervals

Lemma

Let $Q \subseteq X$ be a simplex with smallest enclosing sphere S. Then $Q' \subseteq X$ has the same smallest enclosing sphere as Q iff $L \subseteq Q' \subseteq U$, where

$$L = X \cap S,$$

$$U = X \cap \operatorname{conv} S.$$



The front face of a simplex

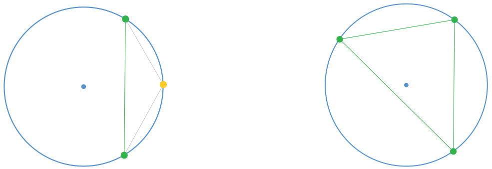
Let $T \subseteq X$ be a simplex with smallest circumsphere S.
Write the center z of S as an affine combination

$$z = \sum_{x \in T} \mu_x x, \qquad 1 = \sum_{x \in T} \mu_x .$$

We call

$$\operatorname{front} T = \{x \in T \mid \mu_x > 0\}$$

the *front face* of T.



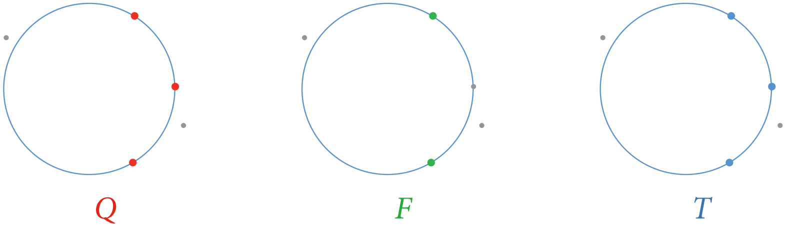
Delaunay intervals

Lemma

Let $Q \subseteq X$ be a simplex with smallest empty circumsphere S. Then $Q' \subseteq X$ has the same smallest empty circumsphere as Q iff $F \subseteq Q' \subseteq T$, where

$$T = X \cap S,$$

$$F = \operatorname{front} T.$$

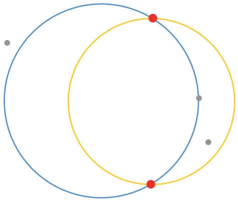


Intersections of Čech and Delaunay intervals

Lemma (Excluded Singularity)

The intersection of non-singular Čech and Delaunay intervals is a non-singular interval.

- Consider a non-critical Delaunay simplex Q



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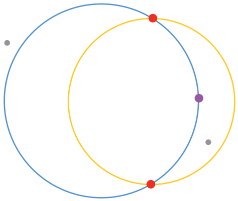

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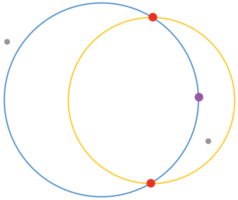

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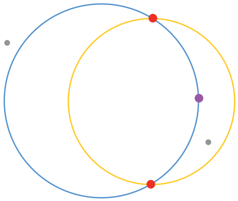
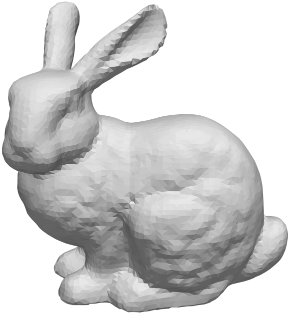


Corollary

The pairs (Q', Q'') yield a vector field that induces a collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$.

Wrap complexes

Generalizes and greatly simplifies the surface reconstruction algorithm *Wrap* (Edelsbrunner 1995, Geomagic)



Wrap complexes

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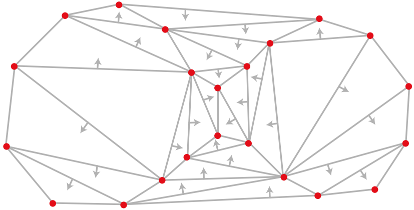

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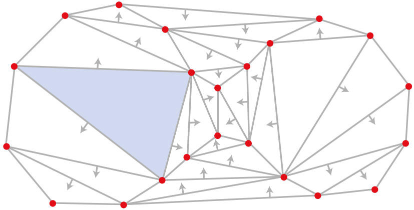

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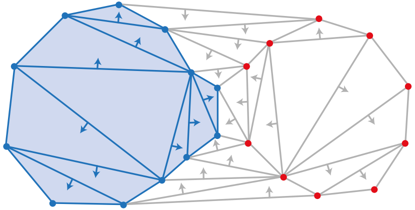

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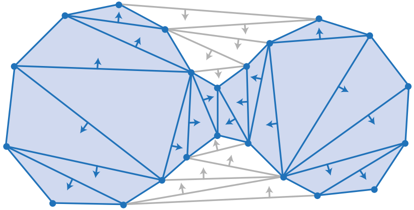

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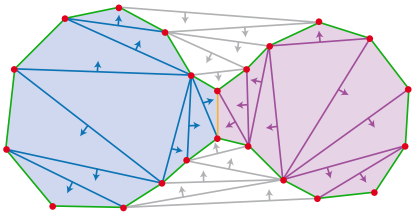

Equivalent to stable manifolds from smooth Morse theory

Wrapping up

- Čech and Delaunay complexes from Morse functions
- Explicit construction of simplicial collapses
- Simple definition and generalization of *Wrap* complexes