Topological simplification problems

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Joint work with:
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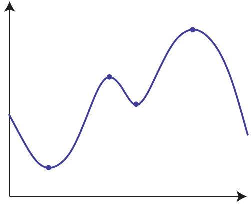
Nina Amenta, Dominique Attali, Olivier Devillers, Marc Glisse, André Lieutier

Topological denoising of a function



Noise (even small) can create lots of critical points

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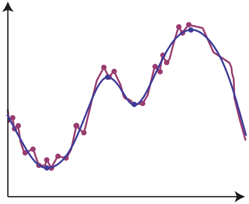

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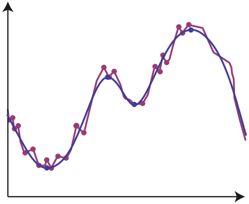

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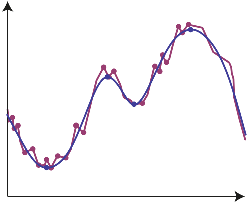

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Topological denoising by simplification of critical set:

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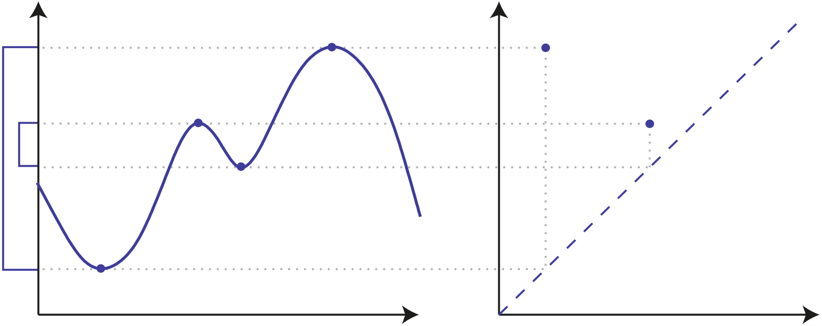
Problem

Given a function f and $\delta > 0$, find a function f_δ that:

- *minimizes number of critical points*
- *stays close to input function: $\|f_\delta - f\|_\infty \leq \delta$*

Persistence diagrams [Cohen-Steiner et al., 2005]



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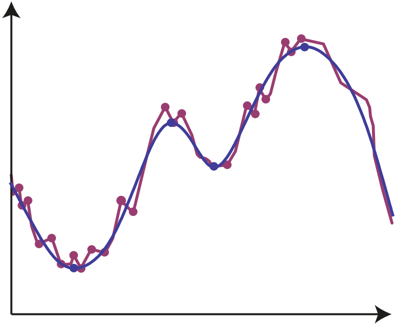
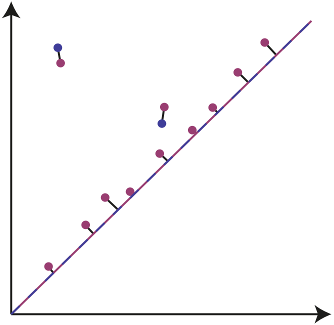


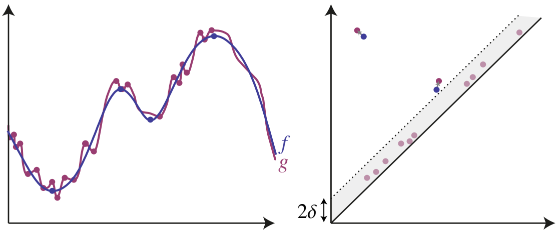


Stability of persistence diagrams

Theorem (Cohen-Steiner et al., 2005)

Let $\|f - g\|_\infty \leq \delta$.

- *The persistence pairs of* f *that have persistence* $> 2\delta$ *can be mapped injectively to the persistence pairs of* g.



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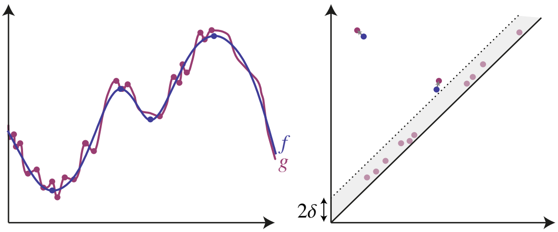
- *Corresponding points* p_f, p_g *in the persistence diagrams have distance* $\|p_f - p_g\|_\infty \leq \delta$.



A bound on number of critical points

Corollary

Let f be a discrete Morse function and let $\delta > 0$.

Then for every function f_δ with $\|f_\delta - f\|_\infty \leq \delta$ we have:

$$\begin{aligned} &\#\ \textit{critical points of } f_\delta \\ \geq\ &\#\ \textit{critical points of } f \textit{ with persistence} > 2\delta. \end{aligned}$$

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Side-effects of elimination

Idea for simplifying critical points [Edelsbrunner et al. 2006, Attali et al. 2009]:

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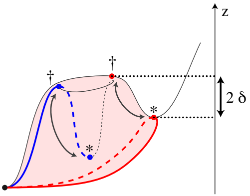

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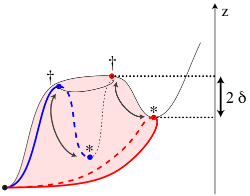

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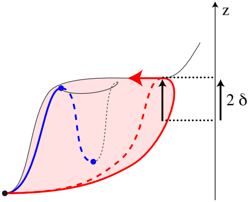

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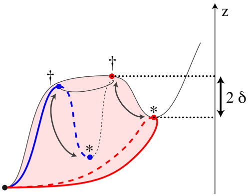

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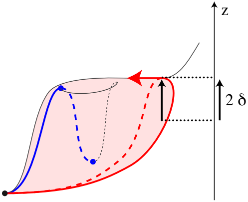

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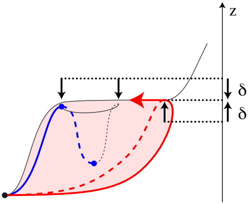

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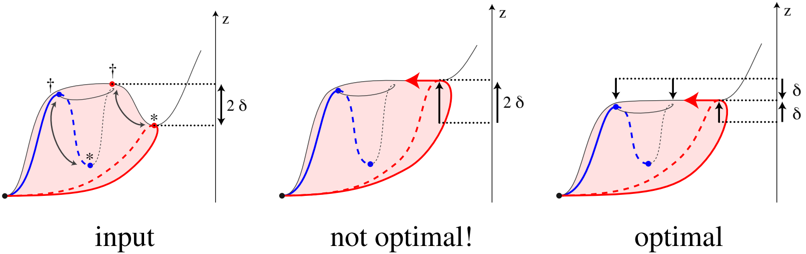

For an optimal solution, we must allow the critical values to change!

Interlude: discrete Morse theory

Since Vidit Nanda's talk today was canceled...

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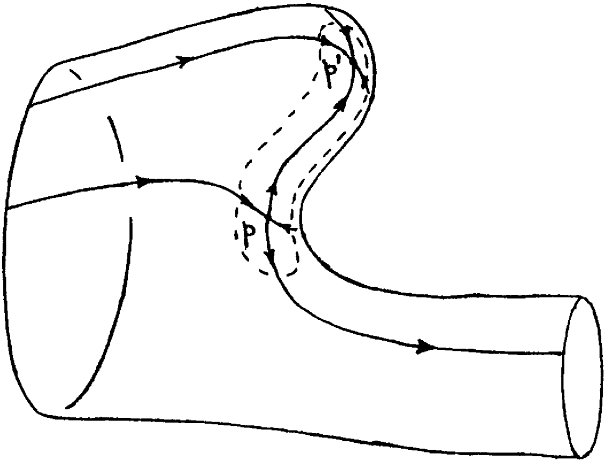
- Morse functions are generic, nondegenerate functions (isolated critical points)
- discrete vector field: set of inequality constraints on function values
- homotopy type of sublevel sets changes only at critical values

Canceling critical points of a gradient field



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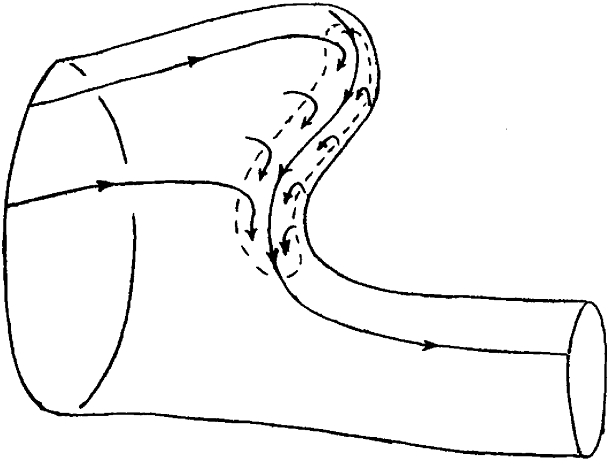

Persistence pairs and Morse cancellations

Theorem (B., Lange, Wardetzky, 2011)

Let f be an excellent discrete Morse function on a surface (distinct critical values) with gradient field V. Any persistence pair (σ, τ) can be canceled in V after all persistence pairs $(\tilde{\sigma}, \tilde{\tau})$ with

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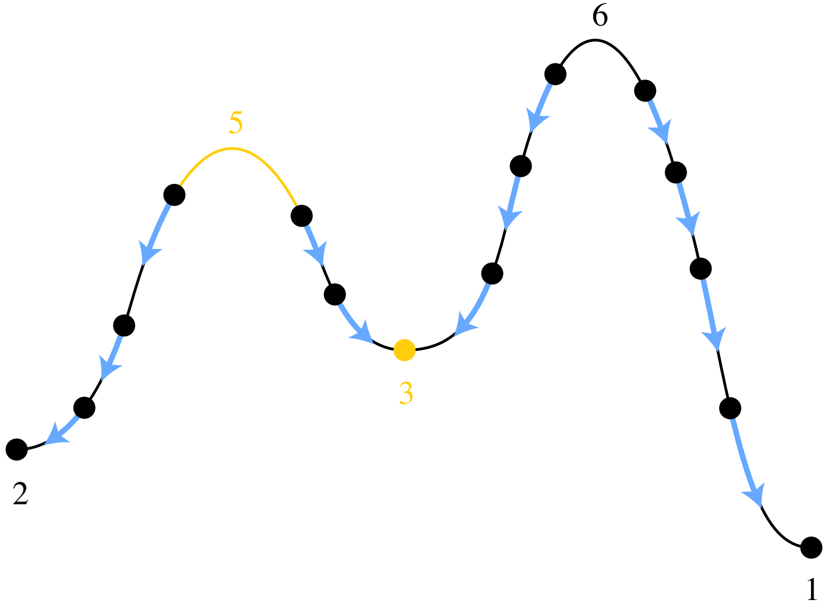
Does not hold in higher dimensions!

Canceling critical points of a function



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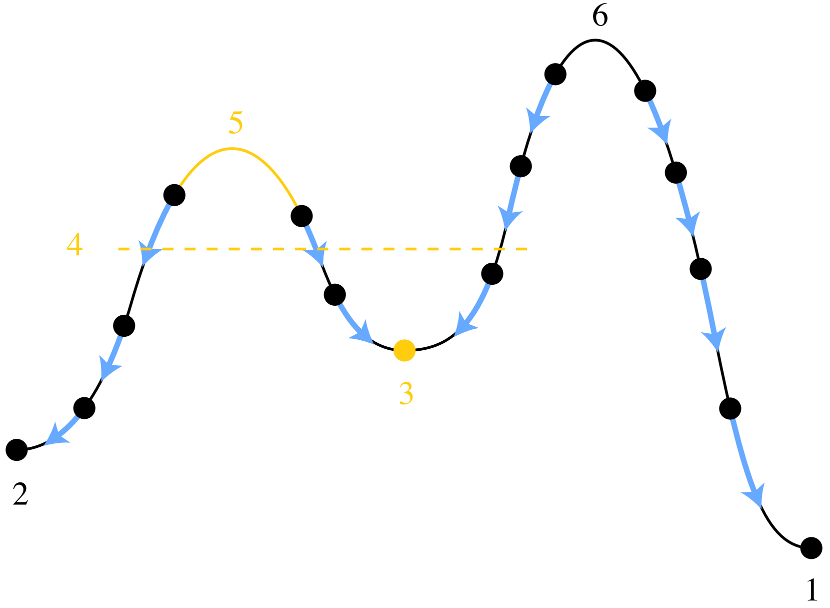

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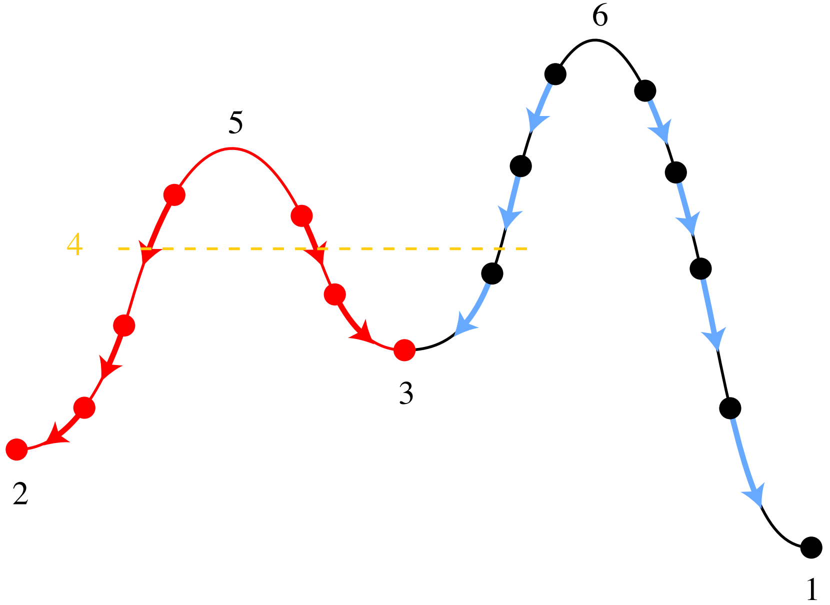

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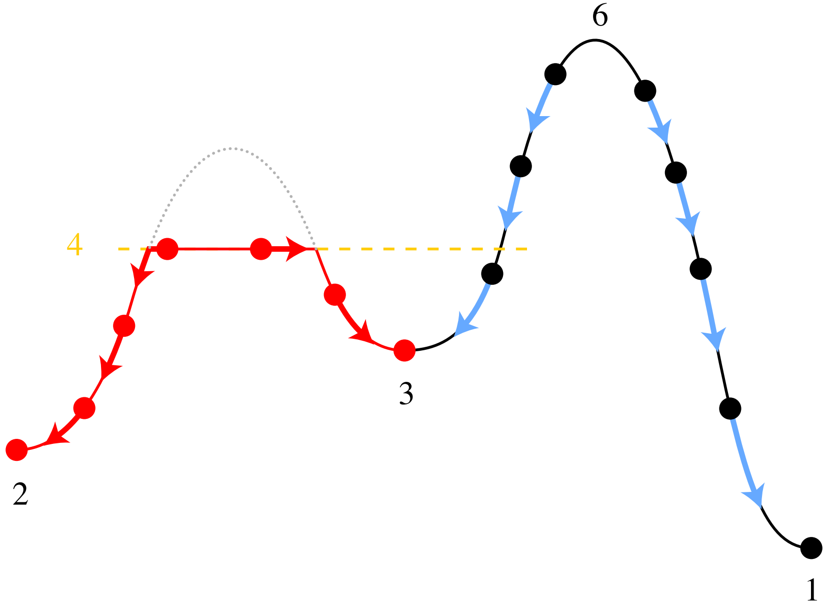

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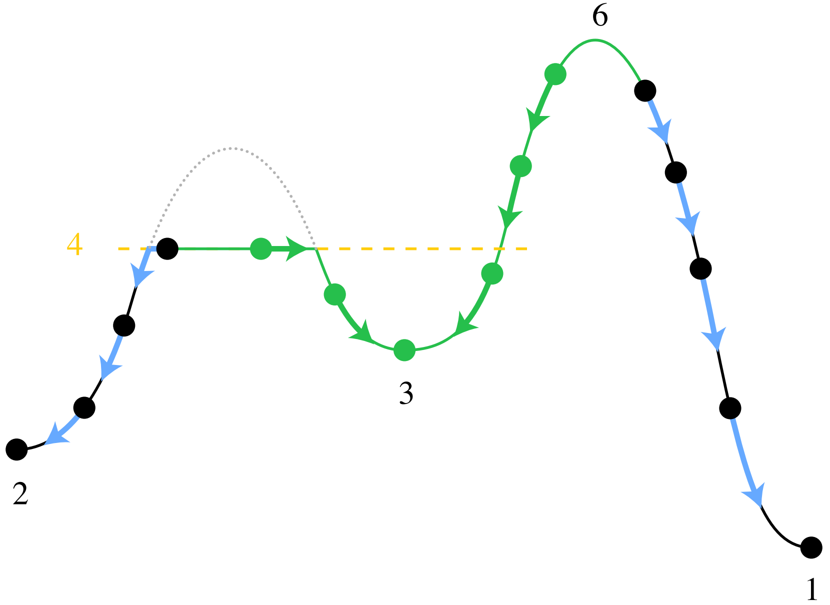

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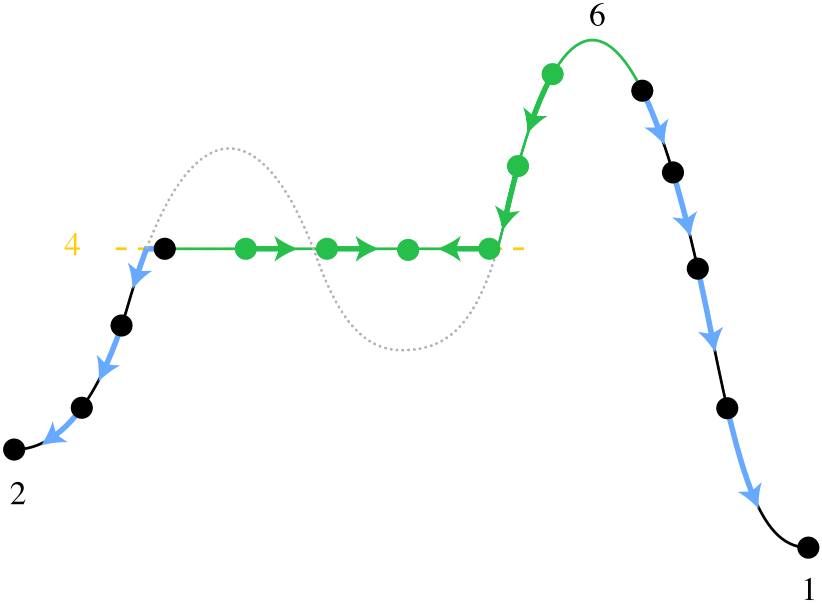

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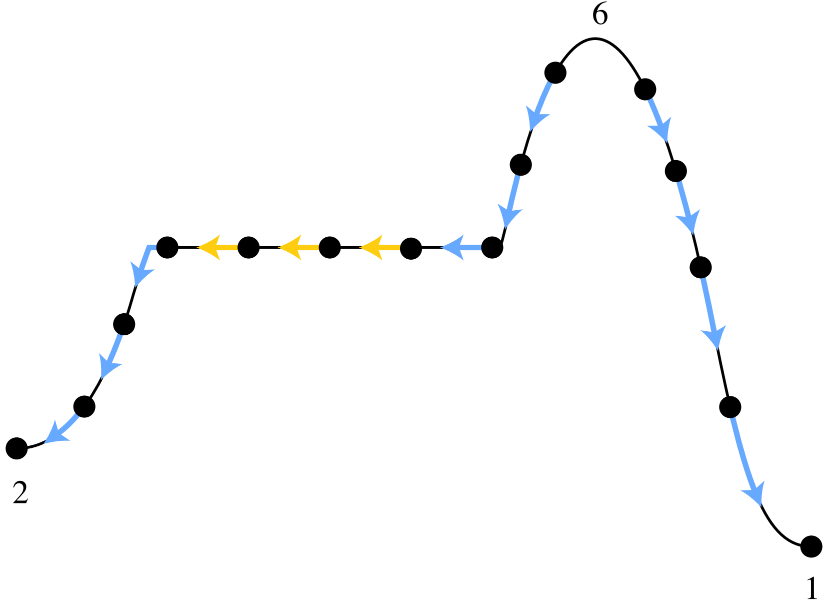

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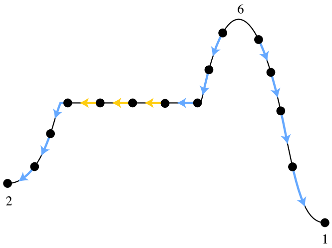

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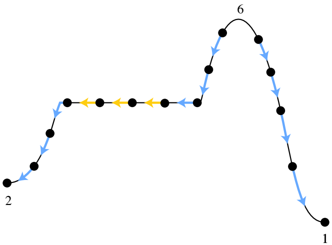

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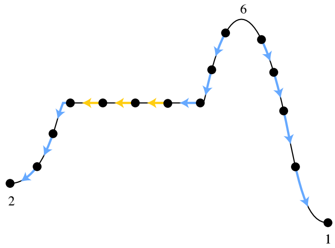

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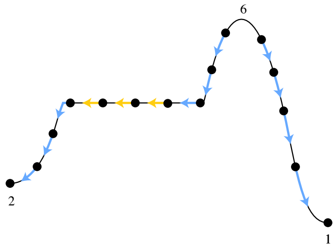

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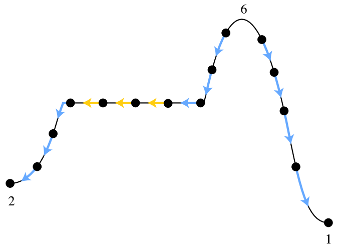

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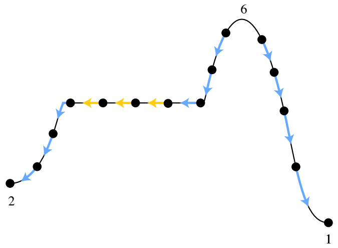

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Most important statements allow passing to the limit $\epsilon \to 0$!

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- Does not hold for non-manifold 2-complexes or higher dimensions (in general, simplification is NP-hard)
- Solution can be found in linear time after computation of persistence pairs

Combination with energy methods

Recall: simplified vector field V_δ imposes inequalities on simplified function consistent with V_δ

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- defines convex polytope of solutions:
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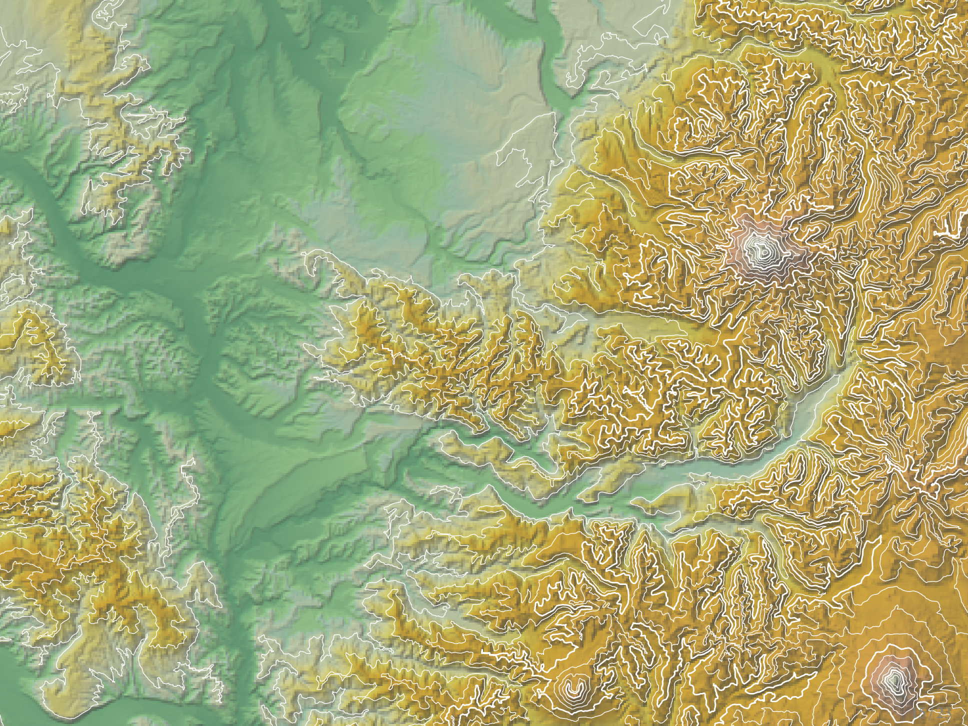
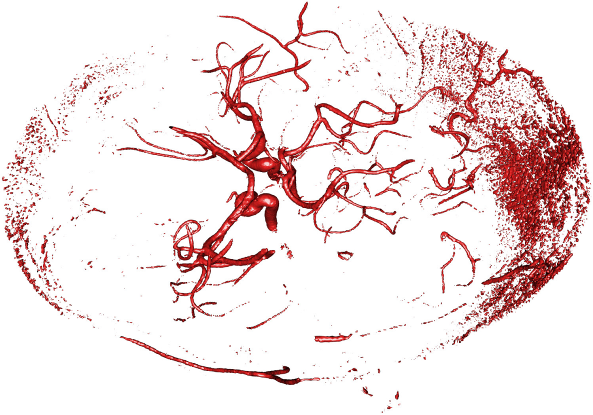
- find the “best” solution using your favorite energy functional



Removing local extrema from 3D data



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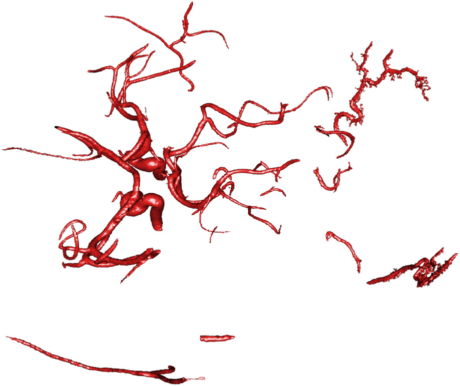

In 3D:

- simplifying critical points is hard
- simplifying only extrema is easy

Sublevel set simplification

Let $F_{\leq t} = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem

Given a PL function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ *and* $t \in \mathbb{R}$, $\delta > 0$,
find a PL function g *with* $\|g - f\|_\infty \leq \delta$
minimizing $\beta_*(G_{\leq t})$.

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- For any g, we have $L \subset G_{\leq t} \subset K$.

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Sublevel set simplification

Let $F_{\leq t} = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem

Given a PL function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ and $t \in \mathbb{R}$, $\delta > 0$, find a PL function g with $\|g - f\|_\infty \leq \delta$ minimizing $\beta_*(G_{\leq t})$.

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Thus we are looking for X with $L \subset X \subset K$ minimizing $\beta_*(X)$.

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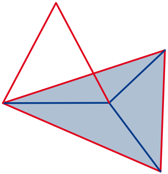

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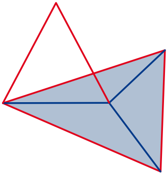

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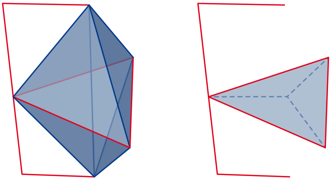

a homological factorization may exist, but not as a subcomplex of K

Homological factorizability in $\mathbb{R}^3$ is NP-complete

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Deciding whether (K, L) has a homological factorization as a subcomplex is NP-complete.

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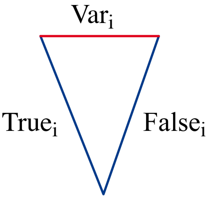
- X is homological factorization $\Leftrightarrow$ X is acyclic, $L \subset X \subset K$

Reduction from 3-SAT: the variable gadget



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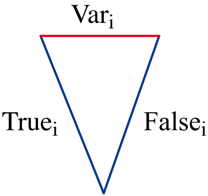

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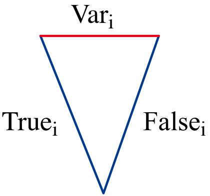

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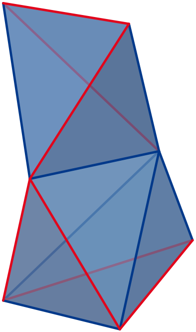

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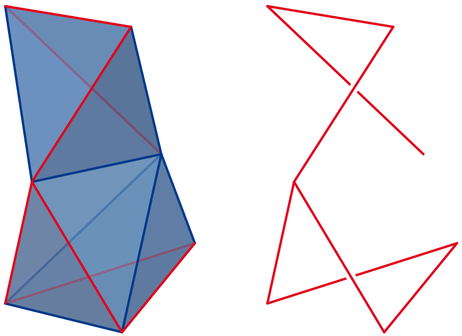

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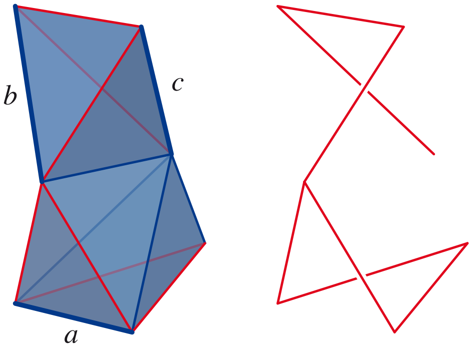

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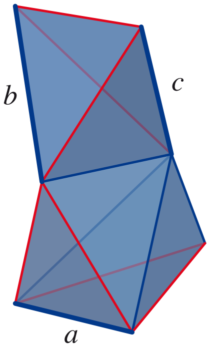

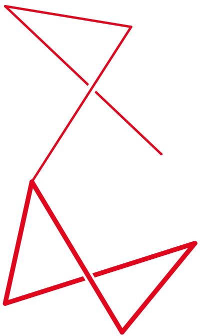

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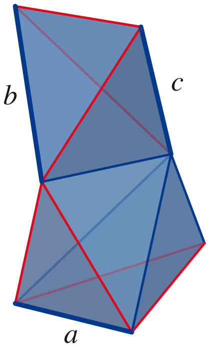

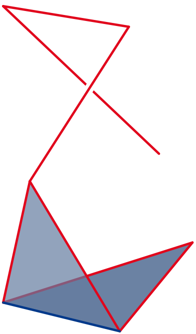

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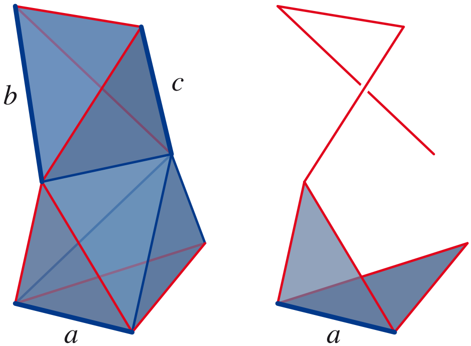

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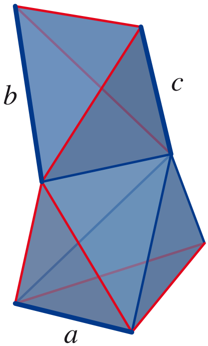

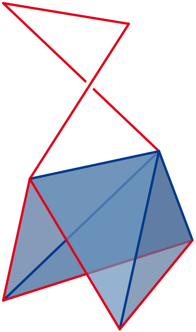

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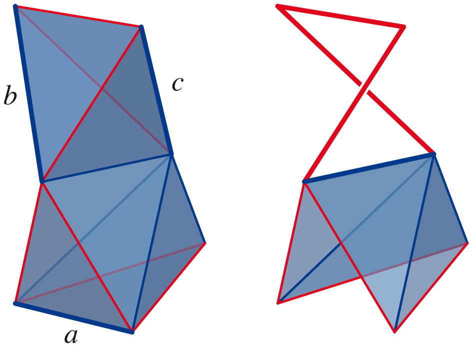

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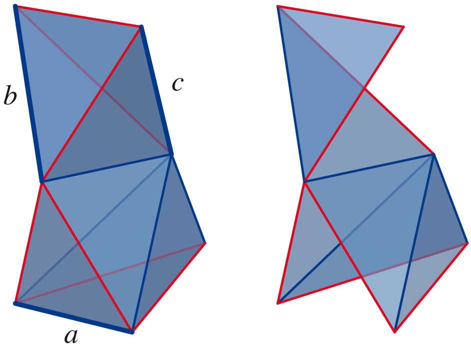

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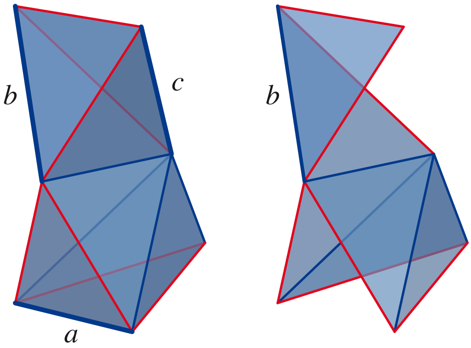

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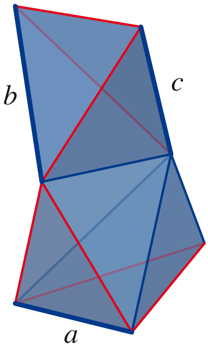

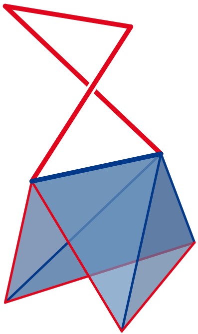

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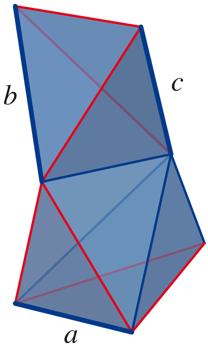

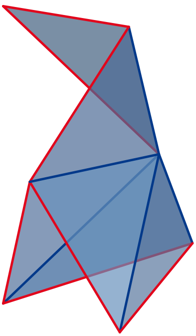

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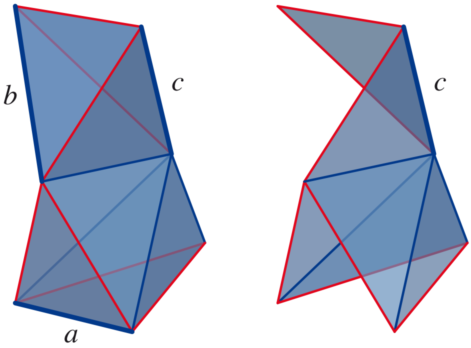

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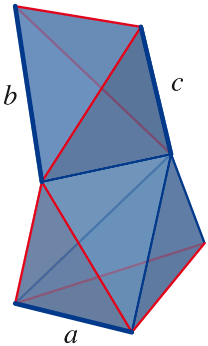

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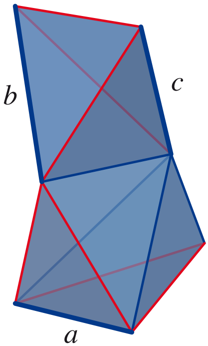

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Thanks for your attention!