Homological reconstruction and simplification

Ulrich Bauer

IST Austria

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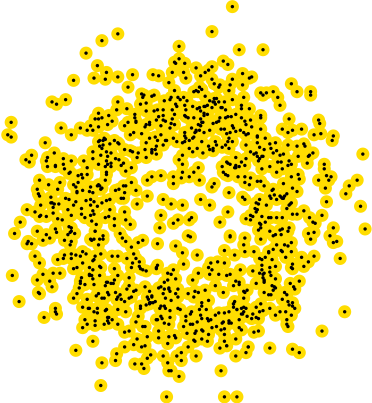
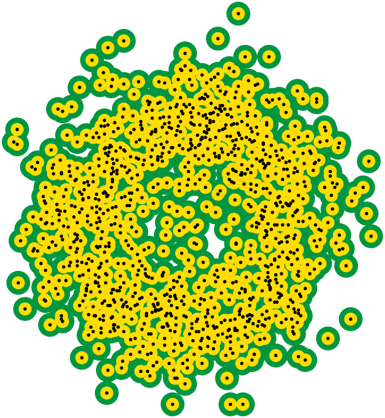
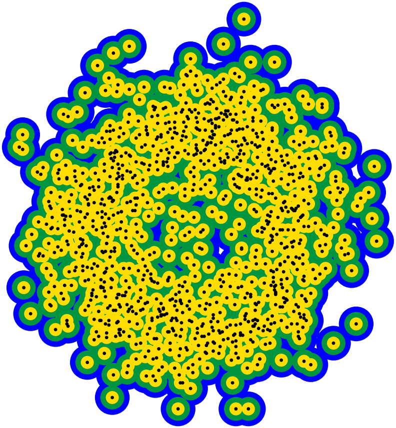
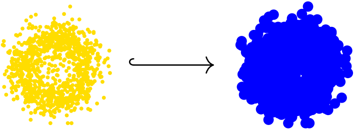
EMS/DMF Joint Mathematical Weekend

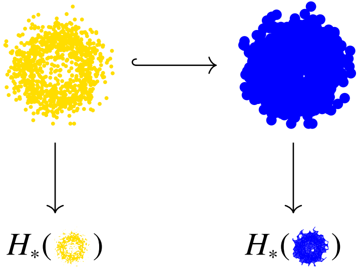
Joint work with:
Dominique Attali, Olivier Devillers, Marc Glisse, André Lieutier



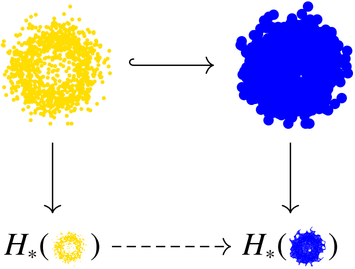








- Homology transforms topological spaces to groups



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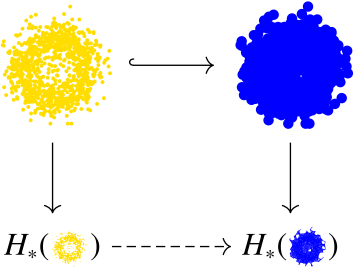

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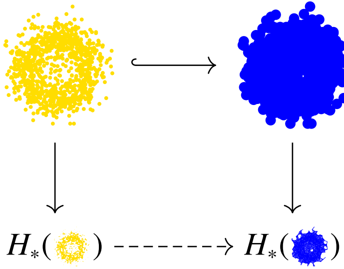
- *Persistent homology group*: $\operatorname{im} H_*($ $\hookrightarrow$ $)$



- Homology transforms topological spaces to groups
  … and continuous maps into group homomorphisms
- *Persistent homology group*: $\operatorname{im} H_*($ $\hookrightarrow$ $)$
- Under mild sampling conditions on :

$\operatorname{im} H_*($ $\hookrightarrow$ $) \sim H_*($$)$

[Edelsbrunner et al. 2005, Chazal et al. 2005]

$$
\begin{tikzcd}
A \arrow[dr, hook] & & B \arrow[dr, hook] & & C \\
& X \arrow[ur, hook] & & Y \arrow[ur, hook] &
\end{tikzcd}
$$

- $A \subset B \subset C$: thickenings of 
- $X \subset Y$: thickenings of 

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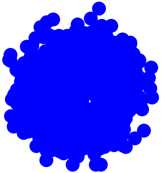
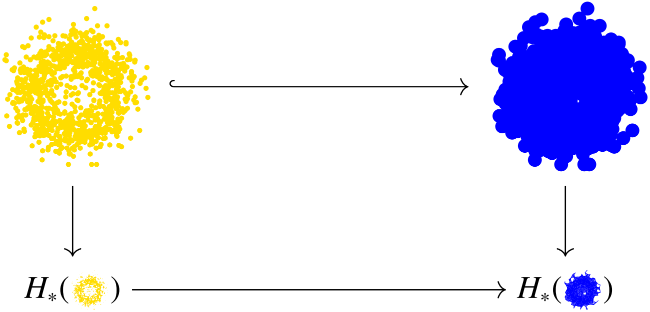
$\operatorname{im} H_*(X \hookrightarrow Y) \sim H_*(B) \sim H_*($  $)$

 $\hookrightarrow$ 

$\downarrow$ $\downarrow$

$H_*($$)$ $H_*($$)$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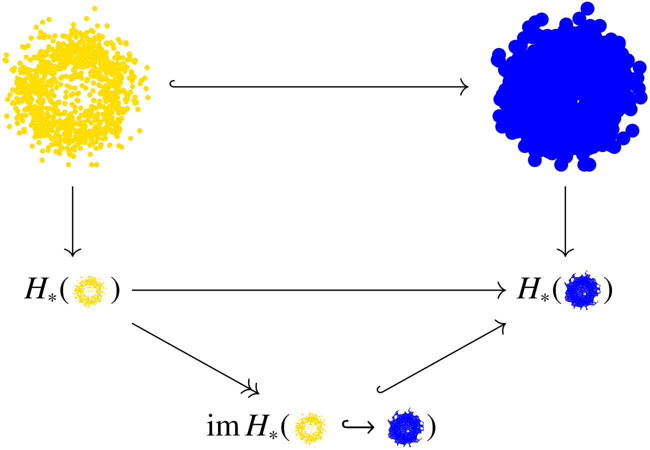
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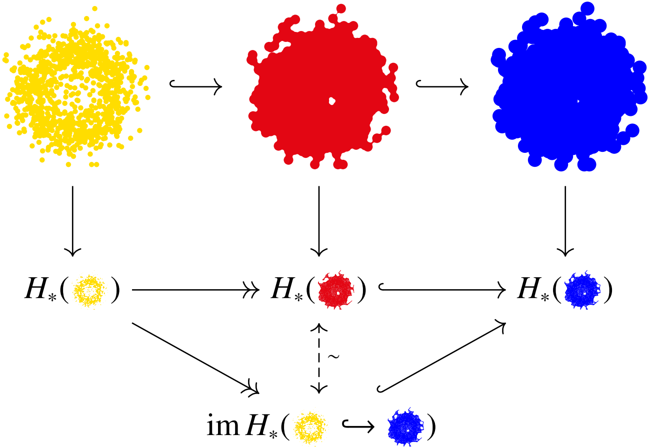
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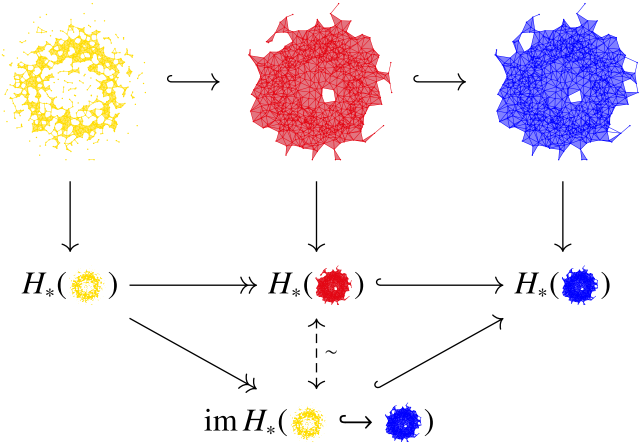
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Homological reconstruction

(Throughout this talk: homology with field coefficients)

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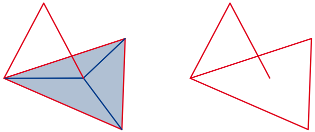
- If $L \subset X \subset K$ then $\beta_*(X) \geq \operatorname{rank} H_*(L \hookrightarrow K)$
- Equality iff X is a homological reconstruction

Do homological reconstructions always exist?

- Homological reconstructions do not always exists



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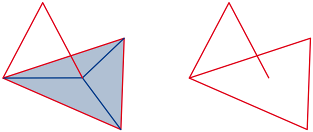
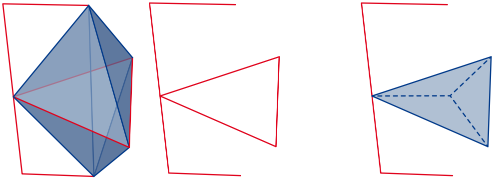


- A homological reconstruction may exists, but not as a subcomplex of X



Homological reconstruction in $\mathbb{R}^3$ is NP-hard

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Let (K,L) *be a simplicial pair in* $\mathbb{R}^3$. *Deciding whether* (K,L) *has a homological reconstruction is NP-hard.*

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3-SAT: satisfiability of Boolean formulas in 3-CNF *(3-conjunctive normal form)*

- $\Phi = (\neg t \wedge u \wedge v) \vee (t \wedge \neg v \wedge \neg w) \vee (\neg u \wedge \neg v \wedge w)$

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- $(\neg t \wedge u \wedge v), \ldots$: *clauses* (disjunction of 3 literals)

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Outline of proof: reduction from 3-SAT

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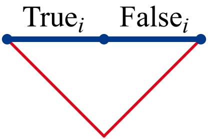
  - (K, L) has a reconstruction as a subcomplex of K

Reduction from 3-SAT: the variable gadget



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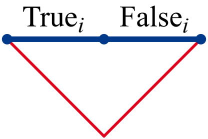

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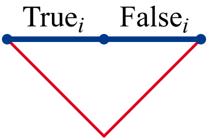

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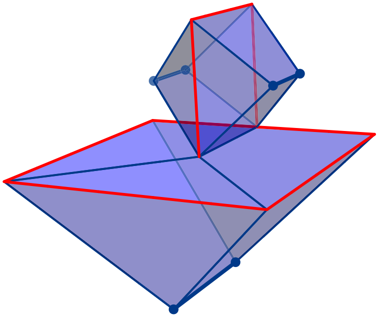
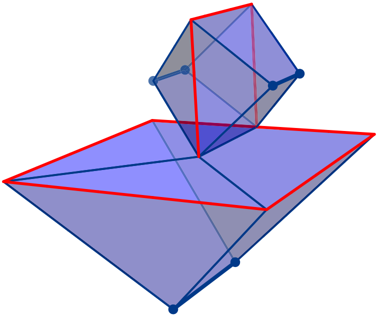
- edges $\mathrm{True}_i, \mathrm{False}_i$ correspond to possible values of variable x_i

Reduction from 3-SAT: the clause gadget





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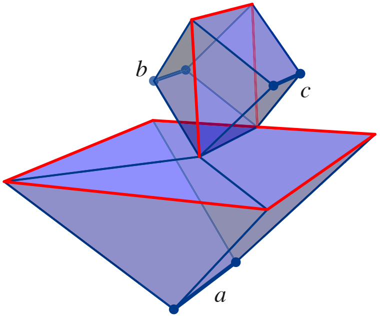

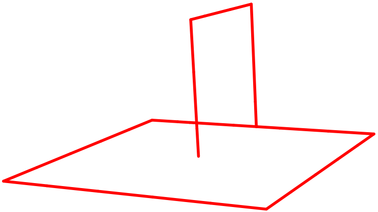

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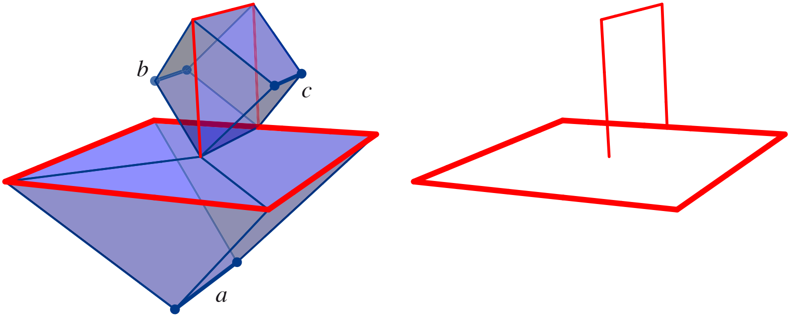

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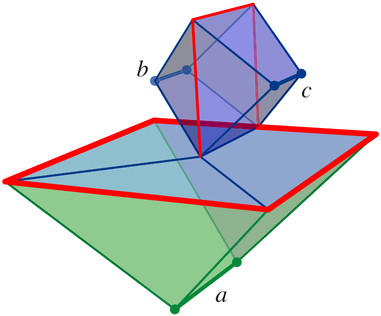

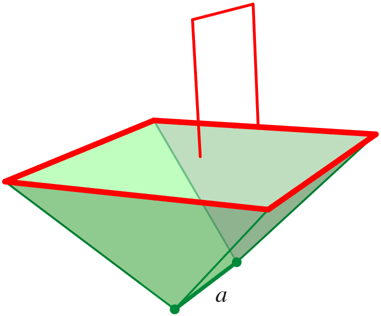

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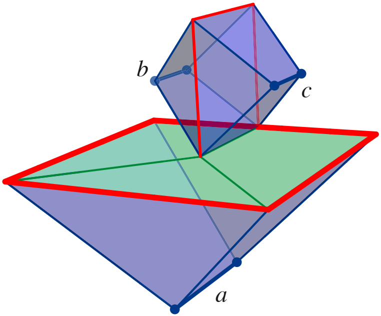

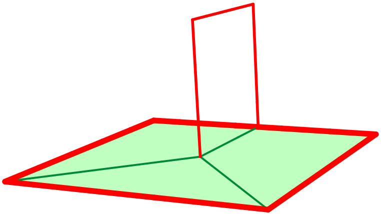

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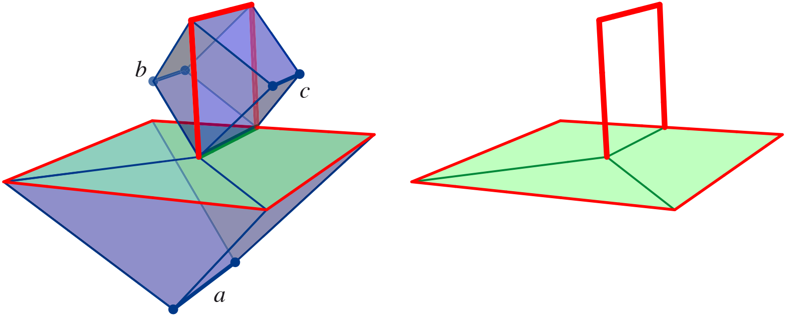

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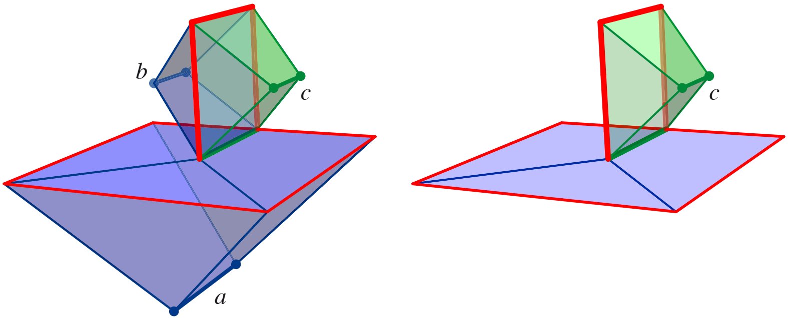

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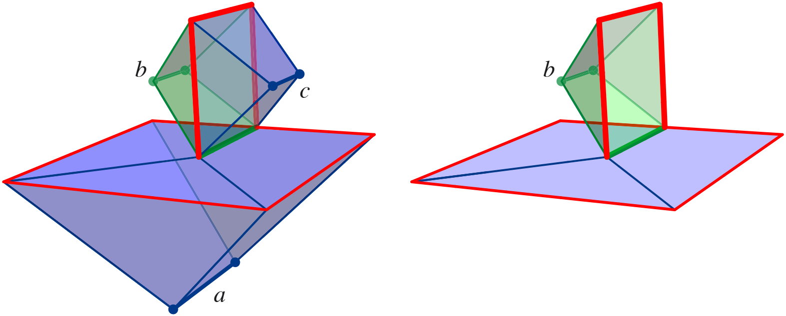

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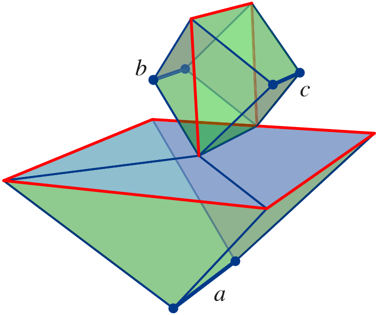

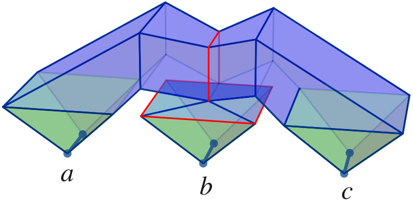

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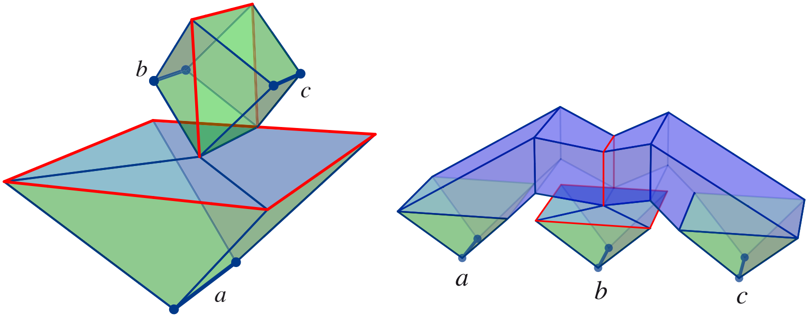
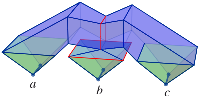
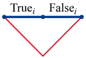
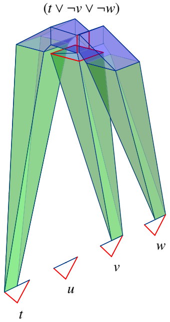


- red: contained in L, blue: $K \setminus L$
- X must contain one of the edges a, b, c
- edges a, b, c correspond to literals in a clause

Reduction from 3-SAT: an example







- For every clause: identify edges a, b, c of clause gadget with edges of variable gadgets corresponding to the literals

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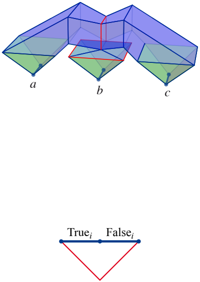

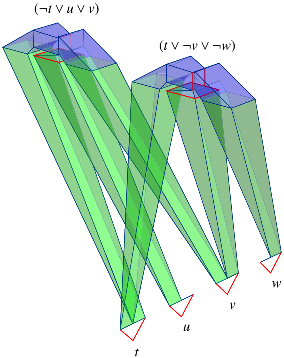

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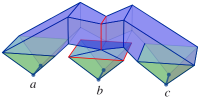

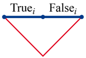

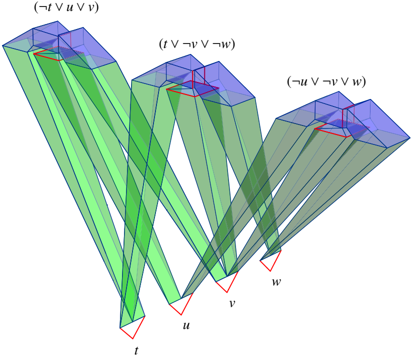

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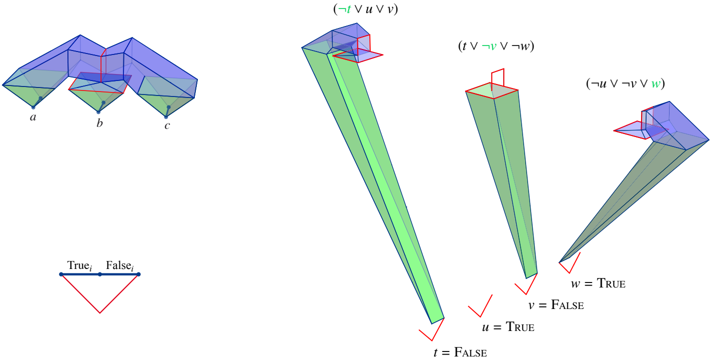

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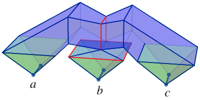

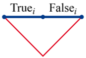

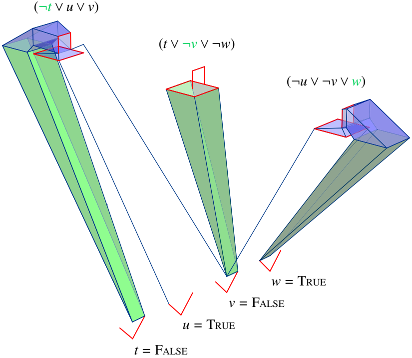

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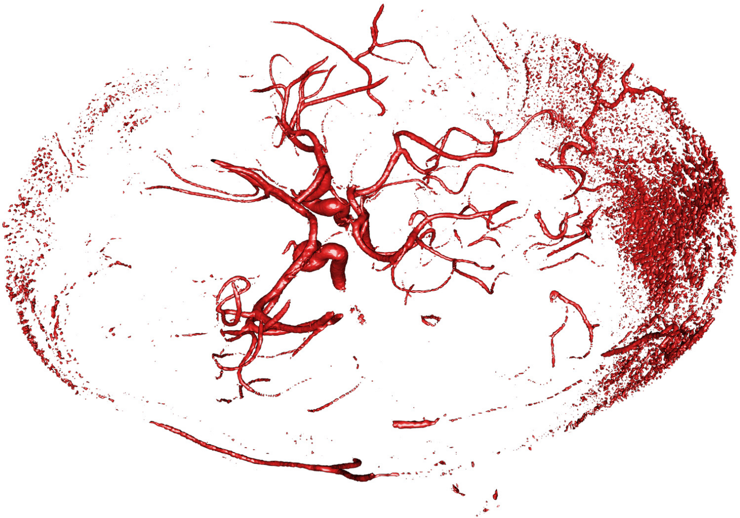

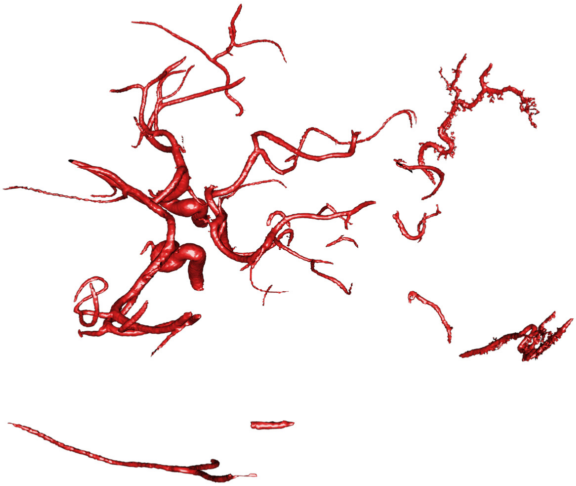

Sublevel set simplification

Let $F_{\leq t} = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem (Sublevel set simplification)

Given a PL function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ *and* $t \in \mathbb{R}$, $\delta > 0$,
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Thus we are looking for X with $L \subset X \subset K$ minimizing $\beta(X)$.

Simplification of sublevel sets

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- Then (K, L) has a homological reconstruction $\iff$ g is a *sublevel set reconstruction*, i.e., $\beta(G_{\leq t}) = \operatorname{rank} H_*(L \hookrightarrow K)$.

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Problem

Given a PL function $f : \mathbb{S}^3 \to \mathbb{R}$ *and* $t \in \mathbb{R}$, $\delta > 0$,
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- g can be shown to have regular value t, so
  $\beta_*(G_t) = 2\beta_*(G_{\leq t})$
- Hence g also minimizes $\beta_*(G_{\leq t})$
- Conversely, a minimizer of $\beta_*(G_{\leq t})$ also minimizes $\beta_*(G_t)$

Well groups [Edelsbrunner, Morozov, Patel 2011]

Let f be a PL function and $t \in \mathbb{R}$, $\delta \geq 0$. Let $F_{[a,b]} = f^{-1}[a,b]$.

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The (t,δ)-*well group* of f is

$$W_*(f,t,\delta) = \bigcap_{g:\|g-f\|_\infty \leq \delta} \operatorname{im} H_*(G_t \hookrightarrow F_{[t-\delta,t+\delta]})$$

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- $\beta(G_t) \geq \operatorname{rank} H_*(G_t \hookrightarrow F_{[t-\delta,t+\delta]}) \geq \operatorname{rank} W_*(f,t,\delta)$
- *Realization* of well group: a function g with $\|g-f\|_\infty \leq \delta$ and

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Well group realization is level set reconstruction

Theorem (Attali, B., Devillers, Glisse, Lieutier)

Let f be a PL function on $\mathbb{S}^n$ with $t \pm \delta \in (\operatorname{im} f)$.
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Corollary

Realization of a well group on $\mathbb{S}^3$ is NP-hard.

Simplification and reconstruction in 2D

Things look brighter in 2D (on surfaces) . . .

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Given a PL function f and $\delta \geq 0$, find a PL function g with $\|g - f\|_\infty \leq \delta$ minimizing the number of critical points.

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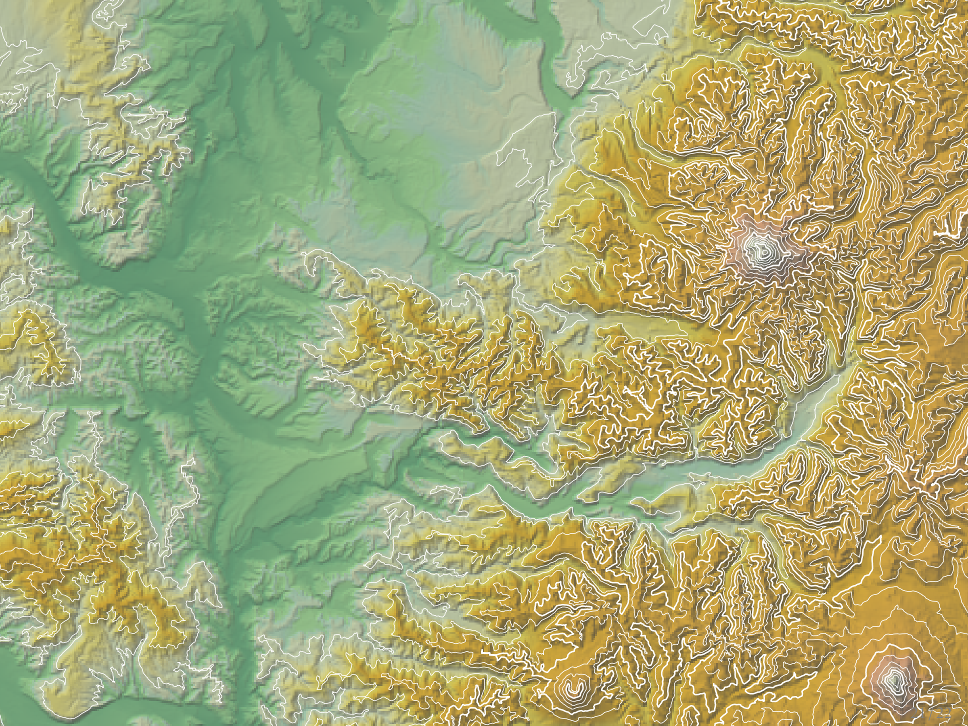
- On a surface homological reconstructions always exist



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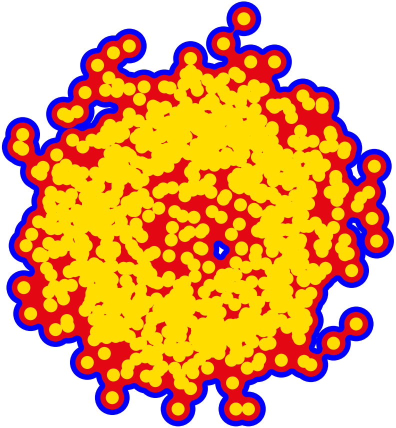

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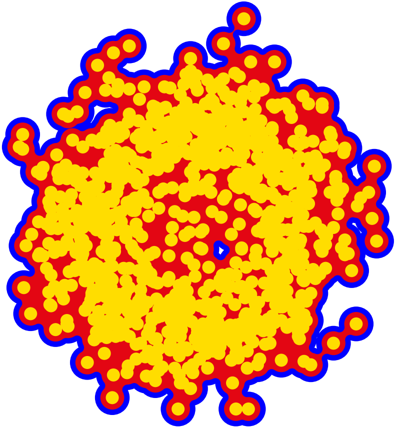

… and have a safe trip home!