The structure of persistence

(An introspection)

Ulrich Bauer

TUM

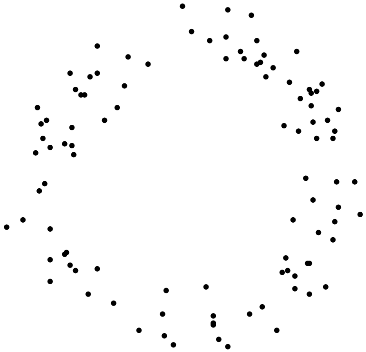
Apr 24, 2019

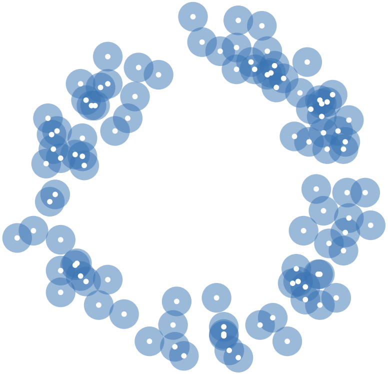
TopApp workshop 2019, IST Austria

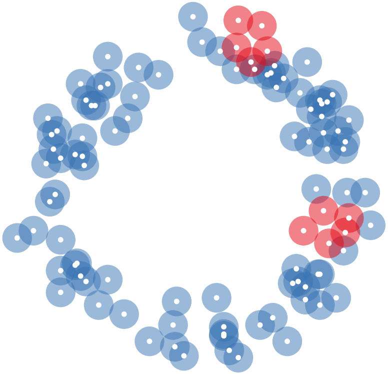
Part 1:
1-parameter persistence

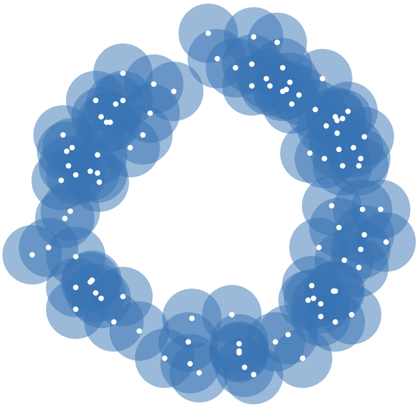
(Barcodes & persistence diagrams)

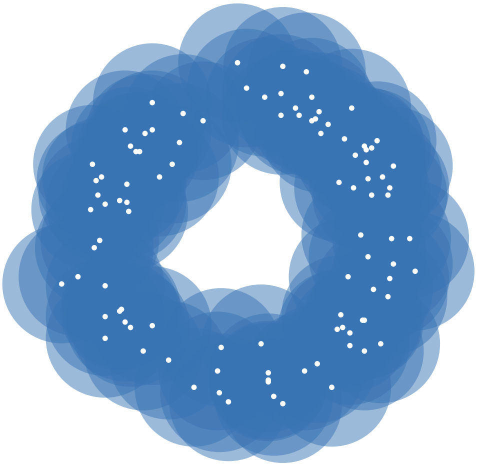
Joint work with Michael Lesnick (Albany)

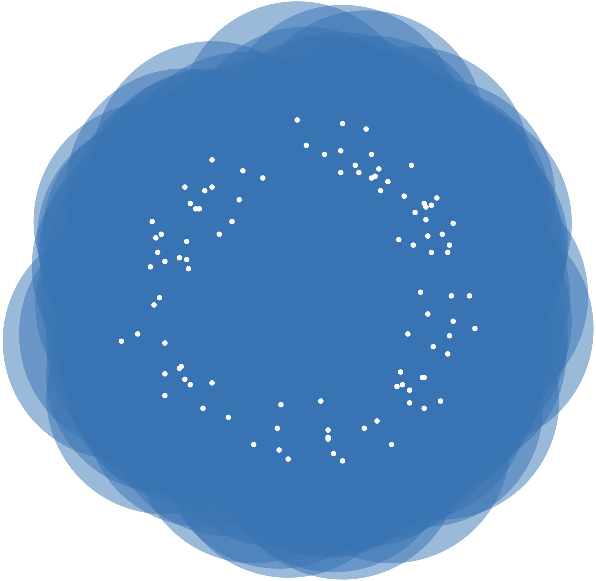




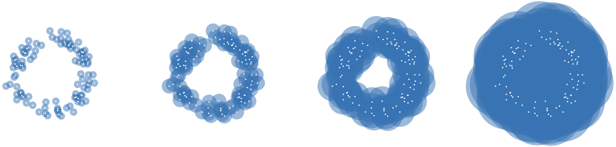








What is persistent homology?



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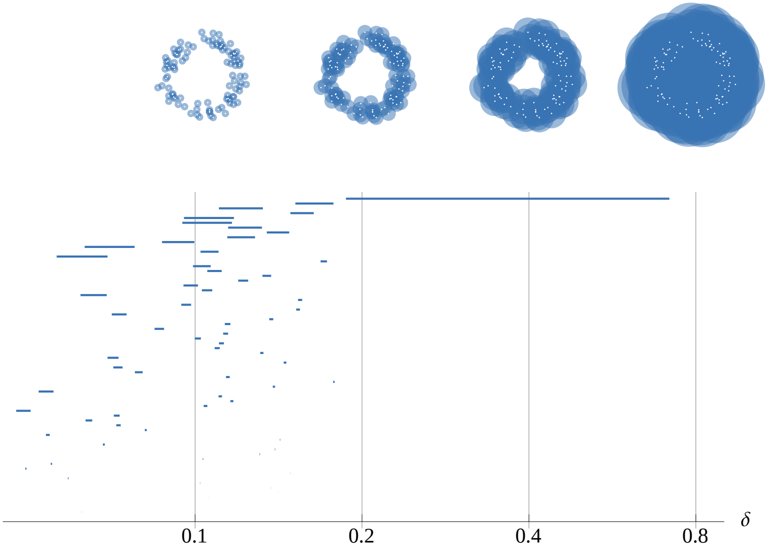

Inerval decompositions and persistence modules

Theorem (Crawley-Boewey 2015)

Any pointwise finite-dimensional (pfd) persistence module (a diagam $M : \mathbb{R} \to \mathbf{vect}$) has an essentially unique decomposition as a direct sum of indecomposable interval modules, *isomorphic to*

$$0 \to \cdots \to 0 \to \underbrace{\mathbb{K} \to \cdots \to \mathbb{K}}_{\textit{supported by an interval } I \subseteq \mathbb{R}} \to 0 \to \cdots$$

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- The points in the *persistence diagram* are the endpoints of the intervals in the barcode.
- This is not a diagram in the sense of category theory (functor)!

Persistence and stability: the big picture

point cloud	$P \subset \mathbb{R}^d$	Hausdorff distance

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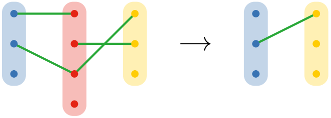
$\big\downarrow$ decomposition		
intervals (persistence barcode)	$B : \mathbf{R} \to \mathbf{Mch}$	Bottleneck/interleaving distance

The category of matchings

Consider the category $\mathbf{Mch}$ (a subcategory of the category $\mathbf{Rel}$ of sets and relations) with

- objects: sets,
- morphisms: matchings (partial bijections).

Composition:



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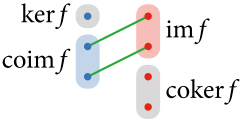

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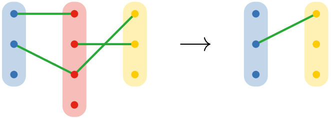
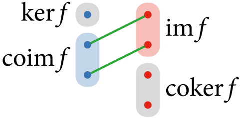
Composition: 

(Co)kernel/(co)image: 

Mch is a *Puppe-exact (p-exact)* category:

- it has a zero object ($\varnothing$)
- it has all (co)kernels
- every mono (epi) is (co)kernel
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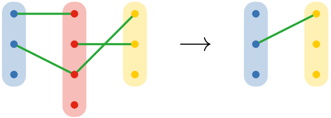

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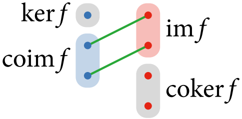

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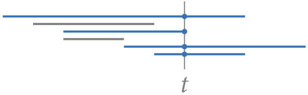
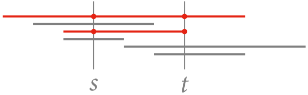
but not additive:

- it does not have all (co)products

From barcodes to matching diagrams (and back)

- A barcode (collection of intervals) can be read as a diagram $\mathbb{R} \to \mathbf{Mch}$:





$t \mapsto \{$intervals in barcode containing $t\}$ $\quad$ $(s \leq t) \mapsto \{$intervals containing both $s, t\}$

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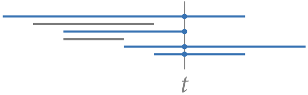

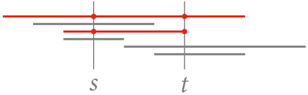

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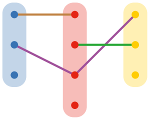

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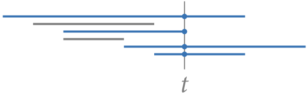

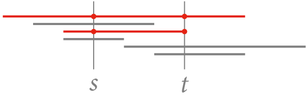

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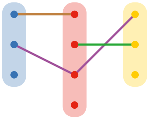

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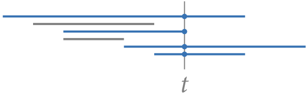

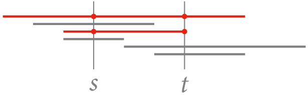

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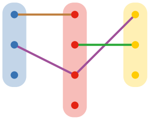

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Turn this into an equivalence of categories $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbb{R}}$

A category of barcodes

Proposition

The functor category $\mathbf{Mch}^{\mathbb{R}}$ *is equivalent to* $\mathbf{Barc}$*, the category with*

- *objects: barcodes (as a disjoint union of intervals),*
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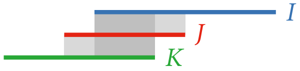
- *composition of overlap matchings:* $\tau \bullet \sigma = \{(I, K) \in \tau \circ \sigma \mid I \text{ overlaps } K \text{ above}\}$
  (where $\tau \circ \sigma$ *is the standard composition of matchings)*



$(I, K) \in \tau \bullet \sigma$ *(overlap)*

$(I, K) \notin \tau \bullet \sigma$ *(no overlap)*

Bottleneck distance as an interleaving distance

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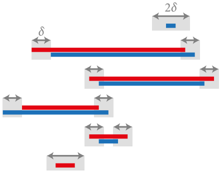

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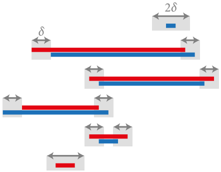

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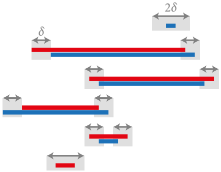

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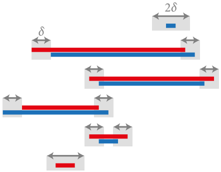

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Proposition

$d_I = d_B$ *(using the equivalence* $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbb{R}}$).

Non-functoriality of persistence barcodes

Can a pfd persistence module $M : \mathbf{vect}^{\mathbb{R}}$ be mapped to its barcode $B(M) : \mathbf{Mch}^{\mathbb{R}}$ by a functor $B : \mathbf{vect} \to \mathbf{Mch}$ (or $\mathbf{vect}^{\mathbb{R}} \to \mathbf{Mch}^{\mathbb{R}}$)?

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There is no functor $\mathbf{vect} \to \mathbf{Mch}$ *sending every vector space* V *to a set of cardinality* $\dim V$ *(equivalently: sending a linear map* f *to a matching of cardinality* $\operatorname{rank} f$*).*

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But: there is a barcode functor for subcategories of monos/epis of persistence modules $\mathbf{vect}^{\mathbb{R}}$:

Structure of persistence sub-/quotient modules

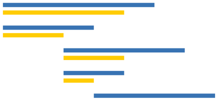
Proposition

Let $M \twoheadrightarrow N$ be an epimorphism.

Then there is an injection of barcodes $B(N) \hookrightarrow B(M)$ such that if J is mapped to I, then

- *I and J are aligned below, and*
- *I bounds J above.*

This construction is functorial.



Dually, there is an injection $B(M) \hookrightarrow B(N)$ for monomorphisms $M \hookrightarrow N$.

Persistence sub-/quotient modules and their matching diagrams

Structure of persistence sub-/quotient modules, rephrased for $\mathbf{Mch}^{\mathbb{R}}$:

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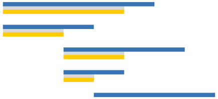
There is a functor from epimorphisms of persistence modules to epimorphisms of matching diagrams.

Dually, there is a functor from monos to monos.



Algebraic stability via induced matchings

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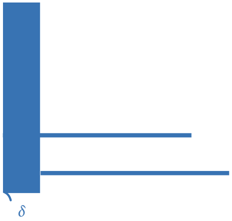

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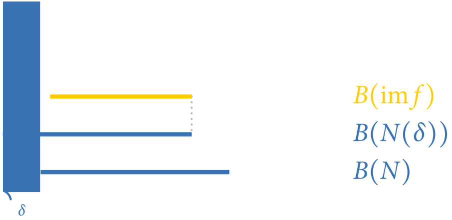

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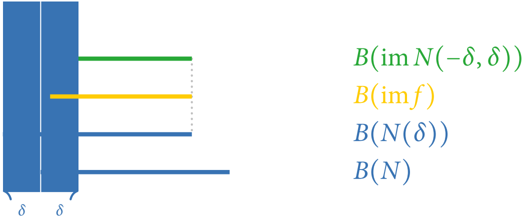

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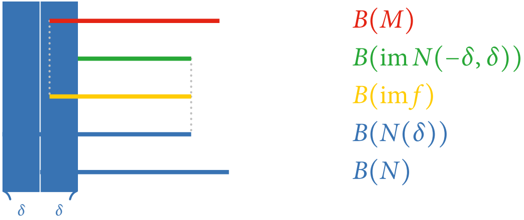

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For $f : M \to N$ a morphism of pfd persistence modules, the epi-mono factorization

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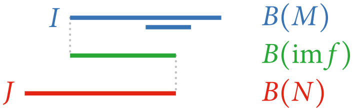

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Theorem

Assume that $\ker f$ is δ-trivial. If I is matched to J, then

(i) *I overlaps J, and J overlaps $I(\delta)$.*

(ii) *Any unmatched interval of $B(M)$ is δ-trivial.*

There is a dual statement for $\operatorname{coker} f$ δ-trivial.

The categorified induced matching theorem

Induced matching theorem, rephrased in $\mathbf{Mch}^{\mathbb{R}}$:

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If $f : M \to N$ has δ-trivial (co)kernel, then so does the induced matching $\chi(f)\colon B(M) \nrightarrow B(N)$.

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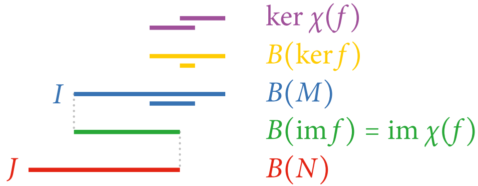

Note:

- We always have $B(\operatorname{im} f) = \operatorname{im} \chi(f)$ by construction.
- But $\ker \chi(f)$ may differ from $B(\ker f)$.
- The induced matching may strictly decrease the triviality of the kernel.

A general criterion for δ-trivial (co)kernels

Lemma

For a natural transformation $f : M \to N$ between diagrams $M, N : \mathbf{R} \to \mathbf{A}$ in a Puppe-exact $\mathbf{A}$, consider the epi-mono factorization

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By a diagram chase, the following are equivalent:

(i) $\ker f$ *is δ-trivial;*

(ii) *the image epimorphism $M \twoheadrightarrow \operatorname{im} s$ factors (through q) as*

$$M \overset{q}{\twoheadrightarrow} \operatorname{im} f \twoheadrightarrow \operatorname{im} s.$$

A dual statement holds for $\operatorname{coker} f$.

The induced matching theorem for (monos and) epis

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Let $f : M \twoheadrightarrow N$ be an epimorphism. If $\ker f$ is δ-trivial, then so is $\ker \chi(f)$. (Dually for monos.)

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Converse direction:

- Apply the free functor $\mathbf{Mch} \to \mathbf{Vect}$. $\square$

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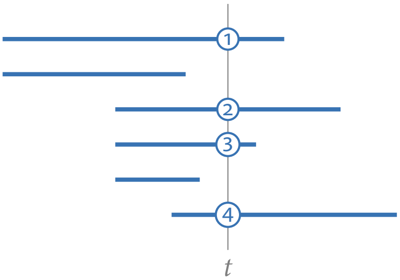

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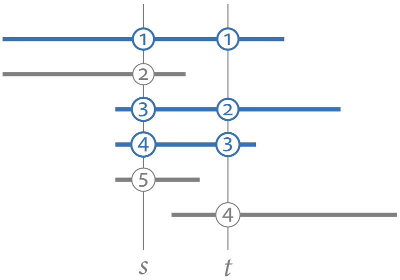

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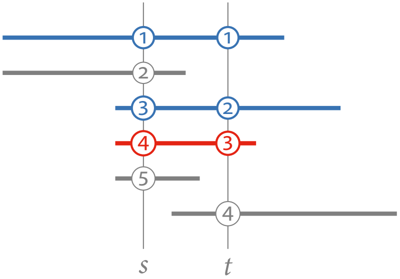

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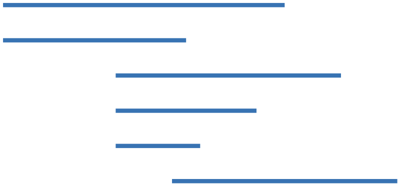

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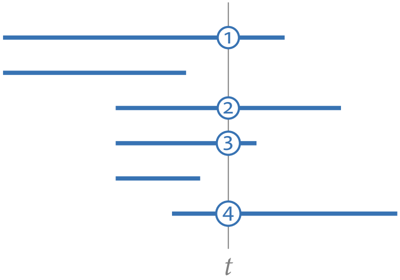

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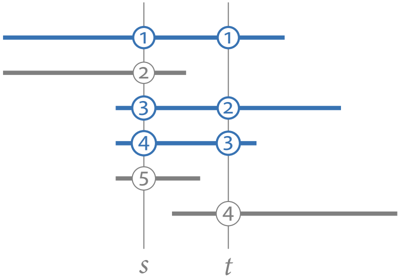

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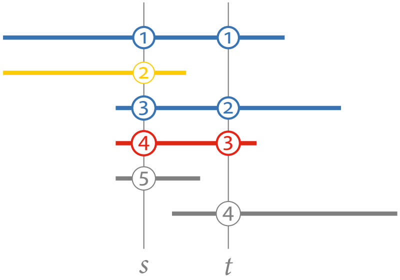

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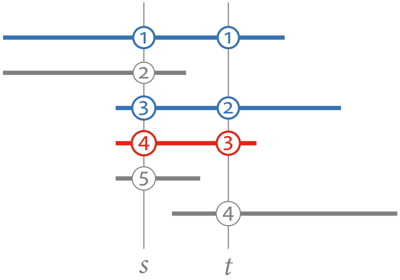

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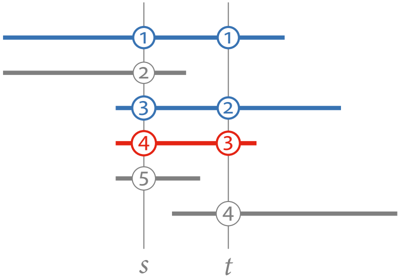

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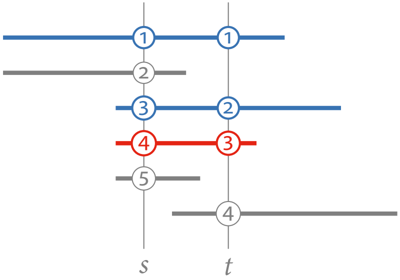

This specifies the barcode of M (as a matching diagram) based on ranks only.

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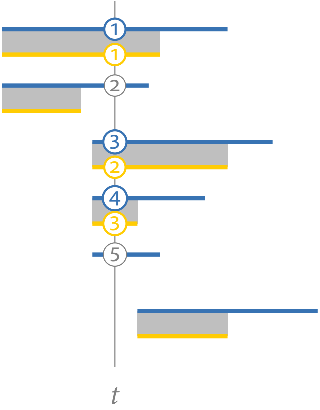

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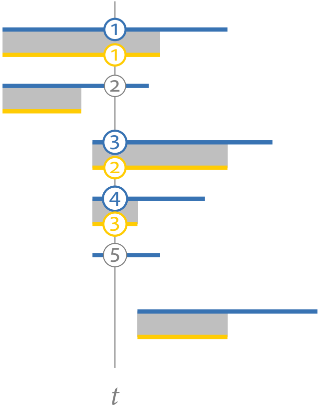

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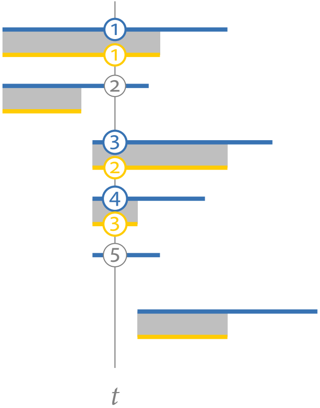

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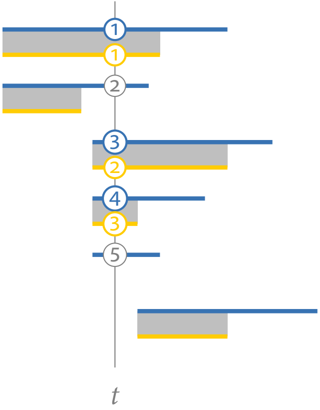

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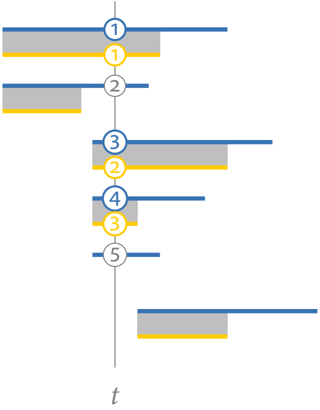
- For the record: $i \in D(M)_t$ is matched to $j \in D(N)_t$ iff

$$\max \{ \operatorname{rank} M_{s,t} - \operatorname{rank} N_{s,t} \mid s < t, \operatorname{rank} N_{s,t} < j \}.$$

- Dually for monomorphisms
- Epi-mono factorization yields an induced matching
- Coincides with previous induced matching (under the equivalence $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbb{R}}$)



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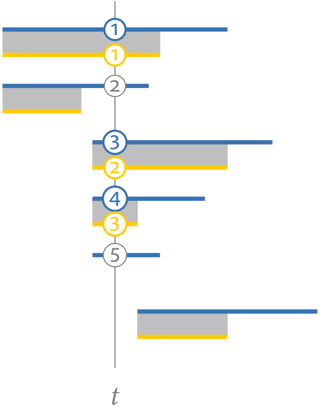

Obtain induced matching and algebraic stability theorems without an interval decomposition