Persistence is Morse theory

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July 26, 2016

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Persistence is (not) Morse theory

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Persistence is Morse theory. . . NOT!

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Archaeology of persistence

When was persistent homology invented?

- [Edelsbrunner/Letscher/Zomorodian 2000]

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- [Leray 1946]?

When was persistent homology invented first?

When was persistent homology invented first?

ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses is satisfied. The critical limits cluster at most at $+\infty$; the critical points are isolated;[1] the problem is defined by analytic functions; the critical limits taken in their natural order are well-ordered. These conditions are not generally fulfilled. The generality of the theory rested upon the fact that the cases treated approximate in a certain sense the most general problems which it is

When was persistent homology invented first?

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Rank and span in functional topology

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Exact homomorphism sequences in homology theory — ed.ac.uk [PDF]
JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR
The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] ...
Cited by 46   Related articles   All 3 versions   Cite   Save   More

Marston Morse and his mathematical works — ams.org [PDF]
R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org
American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to ...
Cited by 24   Related articles   All 4 versions   Cite   Save   More

Unstable minimal surfaces of higher topological structure
M Morse, CB Tompkins - Duke Math. J, 1941 - projecteuclid.org
1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. Theproblem of the existenceof minimal surfaces of unstable type contains many of the typical difficulties, especially those ofa topological nature. Having studied this ...
Cited by 19   Related articles   All 2 versions   Cite   Save

[PDF] **Persistence in discrete Morse theory** — psu.edu [PDF]
U Bauer - 2011 - Citeseer
The goal of this thesis is to bring together two different theories about critical points of a scalar function and their relation to topology: Discrete Morse theory and Persistent

When was persistent homology invented first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was in the 1920's to go from the mini-max principle of Birkhoff to the Morse inequalities, let alone extend these inequalities to function spaces, so that by the early 30's Morse could establish the theorem that for any Riemann

When was persistent homology invented first?

inequalities pertain between the dimensions of the A_i and those of $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels [**D**] in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers [**36**] and [**44**] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.

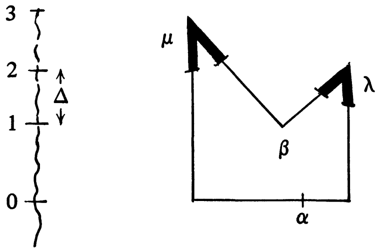

FIGURE 8

The values a where $H(a, a^-) \neq 0$ are indicated on the left, and correspond

When was persistent homology invented first?

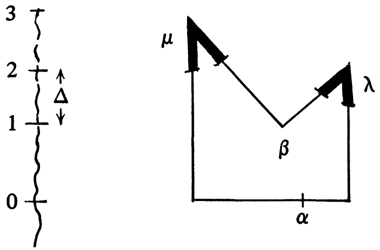

FIGURE 8

The values a where $H(a, a^-) \neq 0$ are indicated on the left, and corresponding to each of these *critical values* a generator of $H(a, a^-)$ is drawn on M, using the singular theory for simplicity. Morse calls such generators "caps". Thus α and β are two "0-caps" and μ and λ two "1-caps". Notice that every cap u defines a definite boundary element ∂u in

$$H(a^-) = \lim_{\varepsilon \to 0^+} H(F < a - \varepsilon);$$

Morse calls a cap u linkable iff $\partial u = 0$. Otherwise it is called *nonlinkable*.

In our example, α, β and μ are linkable while λ is *not*.

Next Morse defines the *span* of a cap u associated to the critical level a in the following manner.

Morse's *functional topology*

Early precursors of persistence and spectral sequences:

- F-*homology classes*: similar to persistent homology
  - *inferior/superior cycle limits*: birth/death
- k-*caps*: related to elements of spectral sequence
  - *cap span*: persistence

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Key aspects:

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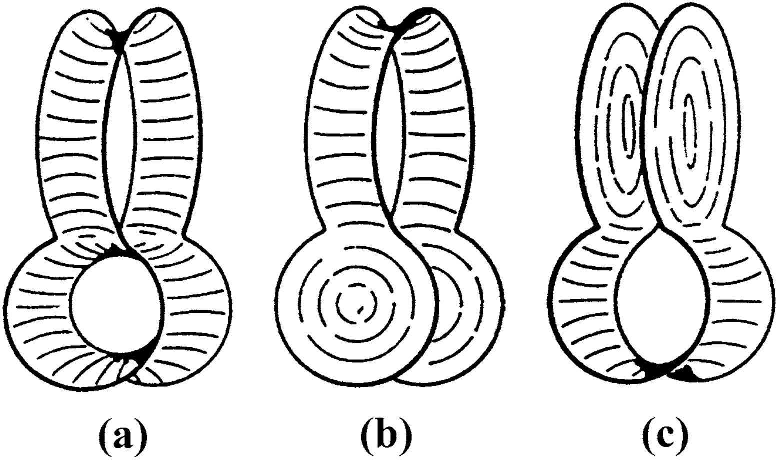
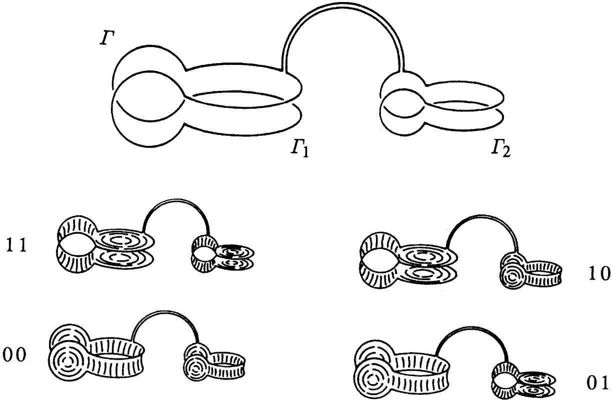
- focus on controlled behavior in pathological cases

Motivation and application: minimal surfaces



(from Dierkes et al.: Minimal Surfaces, Springer 2010)

Motivation and application: minimal surfaces



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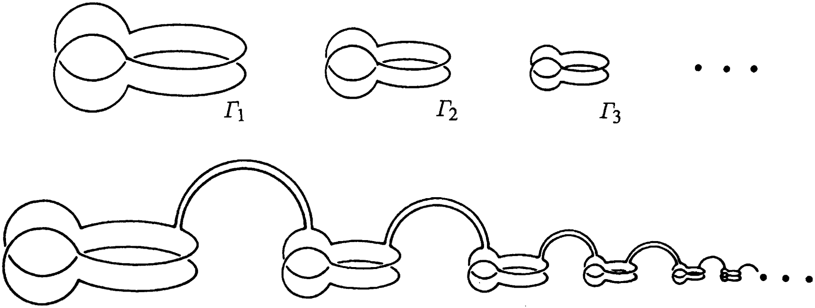


(from Dierkes et al.: Minimal Surfaces, Springer 2010)

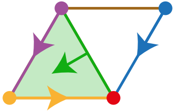
Discrete Morse theory of geometric complexes

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a cell complex is a partition of the set of simplices into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



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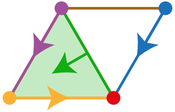

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- sublevel sets are subcomplexes, and

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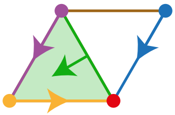

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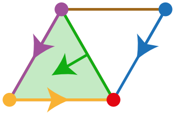

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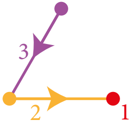

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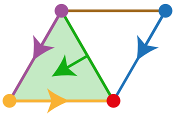

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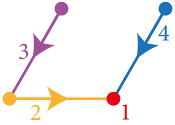

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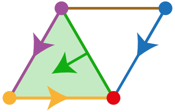

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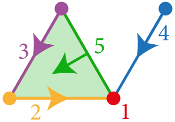

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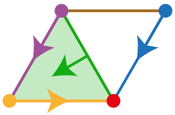

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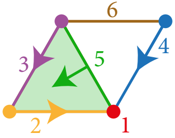

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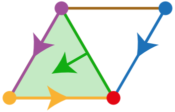

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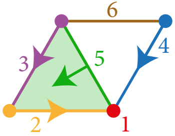
- sublevel sets are subcomplexes, and
- level sets form a discrete vector field.



Fundamental theorem of discrete Morse theory

Let f be a discrete Morse function on a cell complex K.

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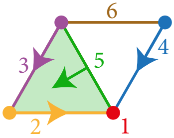
Let f be a discrete Morse function on a cell complex K.

Theorem (Forman 1998)

If $(s,t]$ *contains no critical value of* f*,*
then the sublevel set K_t collapses to K_s
(written as $K_t \searrow K_s$*).*



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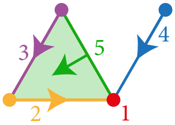

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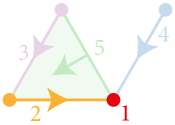

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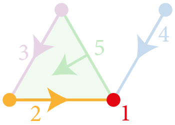

Corollary

$K \simeq M$ for some cell complex M built from the critical cells of f.

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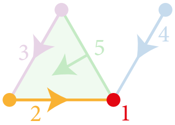


Corollary

$K \simeq M$ for some cell complex M built from the critical cells of f.

This homotopy equivalence is compatible with the filtration.

Corollary

K and M have isomorphic persistent homology (with regard to the sublevel set filtration of f).

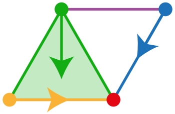
Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* is a partition of the simplices into clusters of the form

$$\{\rho \mid \sigma \subseteq \rho \subseteq \tau\}$$

(intervals $[\sigma, \tau]$ of the face poset).



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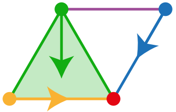

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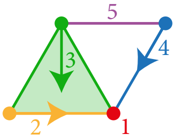

Čech and Delaunay complexes

Proposition

The Čech complexes and the Delaunay (alpha) complexes are sublevel sets filtrations of generalized discrete Morse functions.

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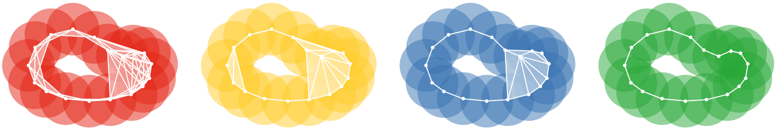
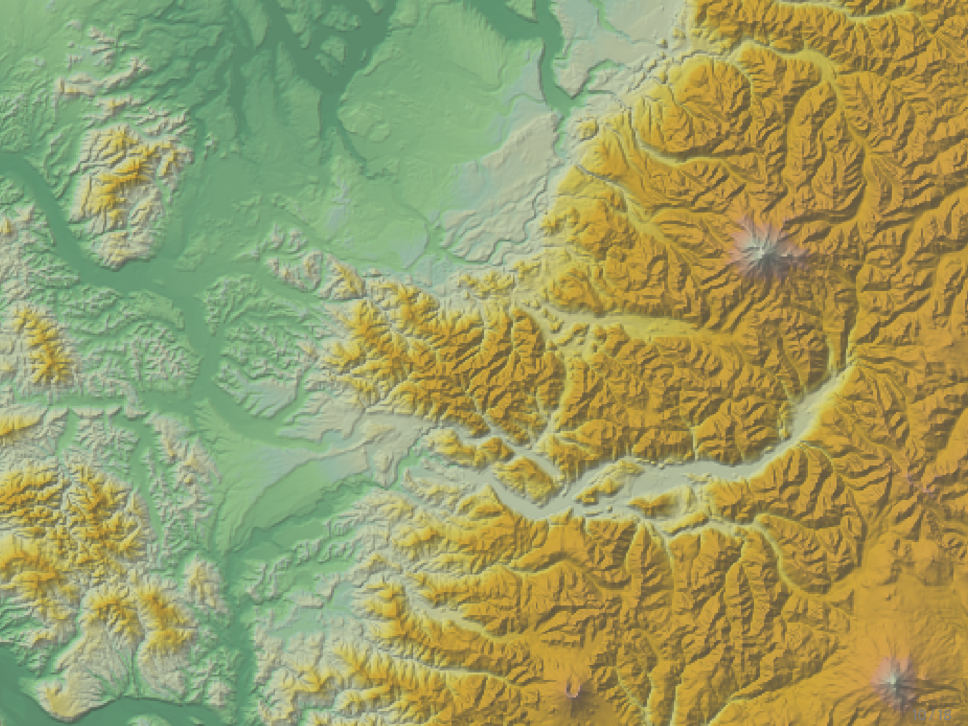
Theorem (B, Edelsbrunner 2015)

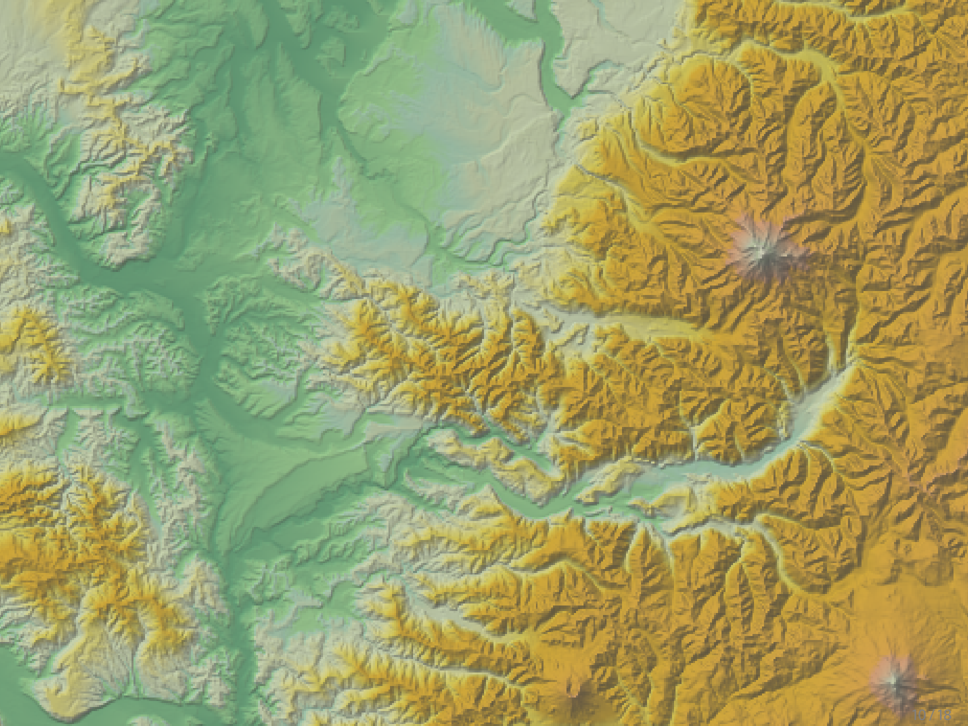
Čech, Delaunay–Čech, Delaunay, and Wrap complexes are naturally homotopy equivalent through a sequence of collapses

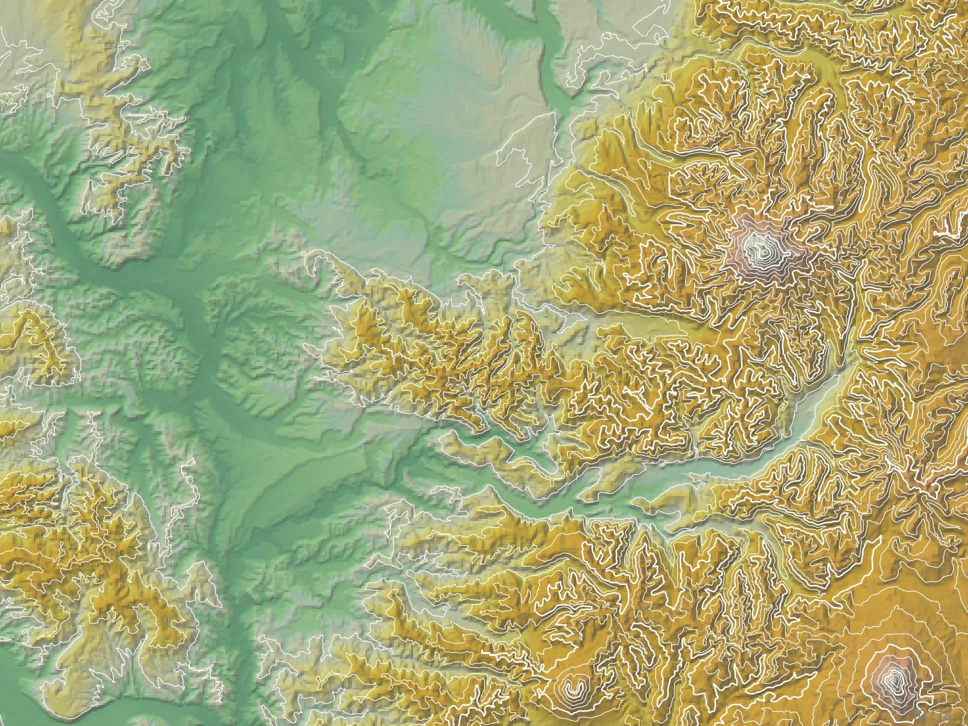
$$\mathrm{Cech}_r \searrow \mathrm{DelCech}_r \searrow \mathrm{Del}_r \searrow \mathrm{Wrap}_r \,.$$



Simplification of functions







Topological simplification of functions

Consider the following problem:

Problem (Topological simplification)

Given a function f and a real number $\delta \geq 0$,
find a function g subject to $\|g - f\|_\infty \leq \delta$
with minimal number of critical points.

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- Provides method for canceling pairs of critical points

Persistent homology:

- Relates homology of two arbitrary sublevel set
- Identifies pairs of critical points (birth and death of homological feature) and quantifies their *persistence*

Persistence pairs and Morse cancelations

By stability of persistence barcodes:

Proposition

The number of critical points of any function g *with* $\|g - f\|_\infty \leq \delta$
is bounded from below by
the number of critical points of f *with persistence* $> 2\delta$.

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Theorem (B, Lange, Wardetzky, 2011)

Let f be a function on a surface and let $\delta > 0$.
Then canceling all persistence pairs with persistence $\leq 2\delta$ yields a function g satisfying $\|g - f\|_\infty \leq \delta$, achieving the lower bound.

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Then canceling all persistence pairs with persistence $\leq 2\delta$ yields a function g satisfying $\|g - f\|_\infty \leq \delta$, achieving the lower bound.

- Does not generalize to higher-dimensional manifolds!
  (Simplest example: nontrivial knot complement)

Morse pairs and persistence pairs

Natural filtration settings

Typical assumptions on the filtration:

• general filtration	persistence (in theory)
• filtration by singletons or pairs	discrete Morse theory
• simplexwise filtration	persistence (computation)

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Conclusion:

- Discrete Morse theory sits in the middle
  between persistence and persistence (hah!)

From Morse theory to persistence and back

Proposition (from Morse to persistence)

The pairs of a Morse filtration are apparent 0-persistence pairs.

Apparent persistence pair (σ, τ):

- σ is the youngest face of τ
- τ is the oldest coface of σ

From Morse theory to persistence and back

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You may have seen this before!

- Gunnar's talk
- Ripser software demo
- Matt Kahle, random Rips complex in supercritical regime

Algebraic Morse theory

Building blocks:

[Kozlov 2005, Sköldberg 2006, Jöllenbeck/Welker 2009]

- free chain complex, with a basis (of your choice)

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Proposition (from persistence to algebraic Morse)

Any persistence pair of a simplexwise filtration corresponds to an algebraic Morse pair for a persistence basis.

Morse reductions for persistence computation

Morse reduction methods

[Mrozek/Wanner 2010, Robbins et al. 2010, Günther et al. 2012, Mischaikow/Nanda 2013]

- find some Morse pairs
- remove them from the complex (reductions)
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- using local persistence pairs (algebraic Morse pairs)
- can persistence speed up persistence computation?

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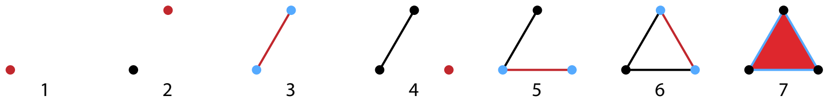
Observed performance:

- previous implementations: significant speedups
- with recent improvements (PHAT): not anymore
  - mostly for difficult instances (large total persistence)
  - benefit of parallelizability
  - negligible for easy instances (Rips filtrations)

Persistence without reduction: apparent pairs



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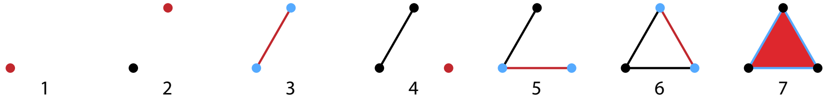

$D =$

	1	2	3	4	5	6	7
1			1		1		
2			1			1	
3							1
4					1	1	
5							1
6							1
7							

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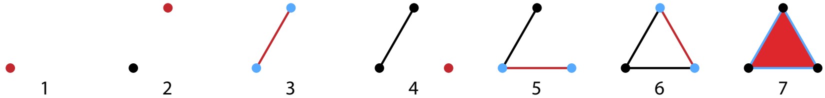

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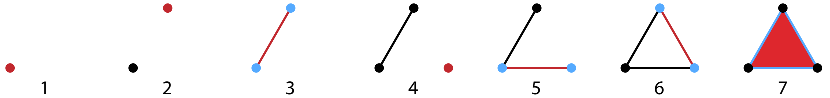

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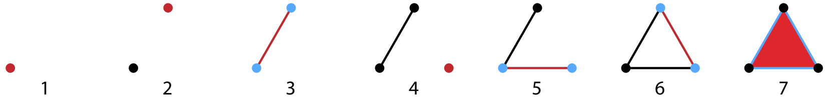

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- All Morse pairs can be seen as persistence pairs
- Many persistence pairs can be seen as Morse pairs
- These apparent Morse/persistence pairs are very useful, both in proofs and in computation