Geometry and Topology of Data

From point clouds through complexes to persistent homology

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Technical University of Munich (TUM)

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SFB TRR 109 Discretization in Geometry and Dynamics

Technical University of Munich

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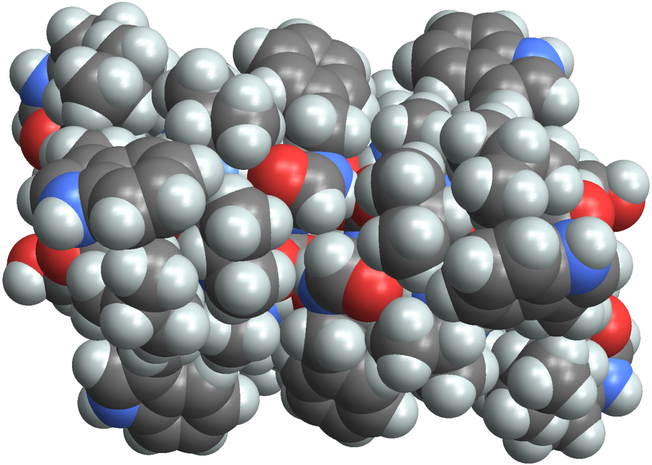
Biogeometry



Gramicidin (an antibiotic functioning as an ion channel)

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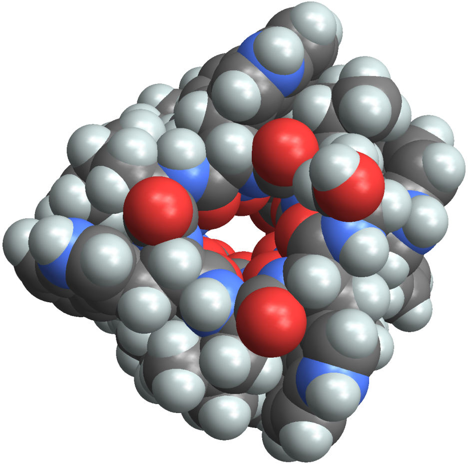

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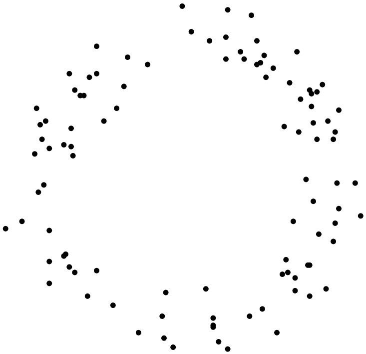
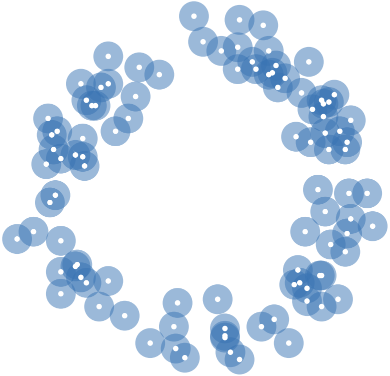
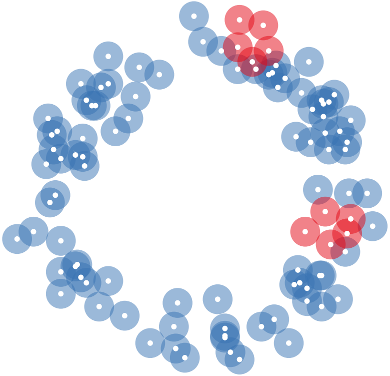
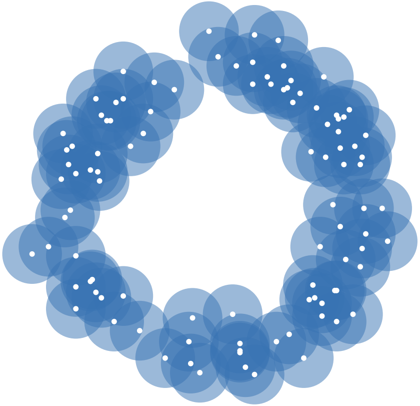
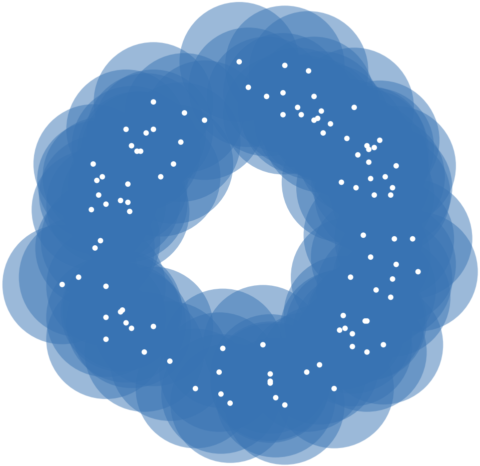
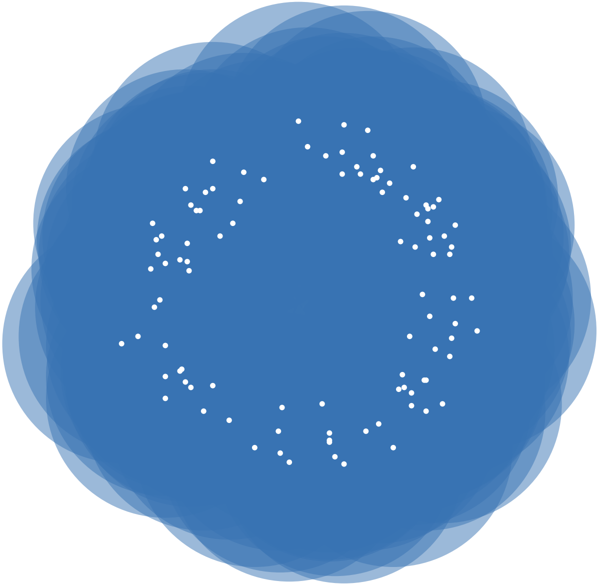


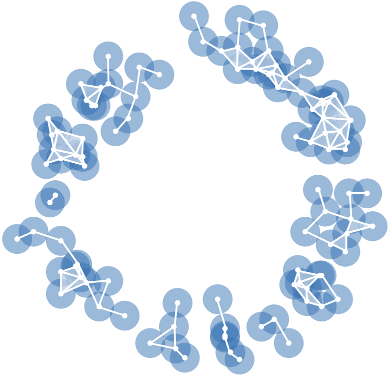


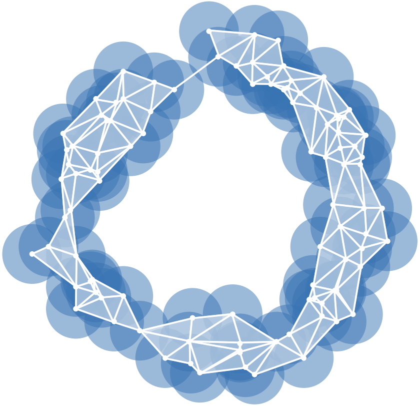


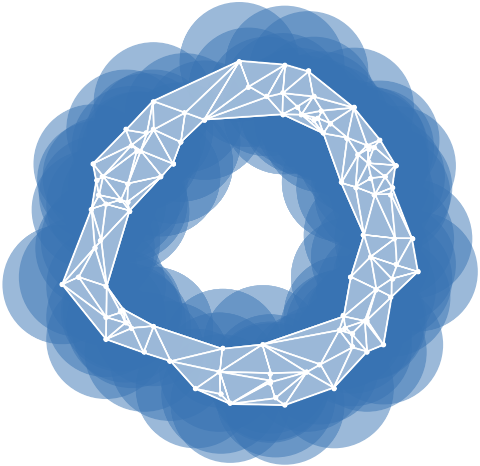


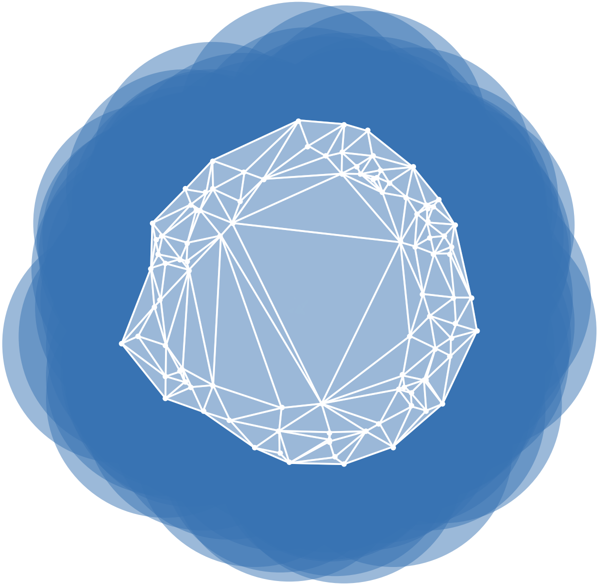






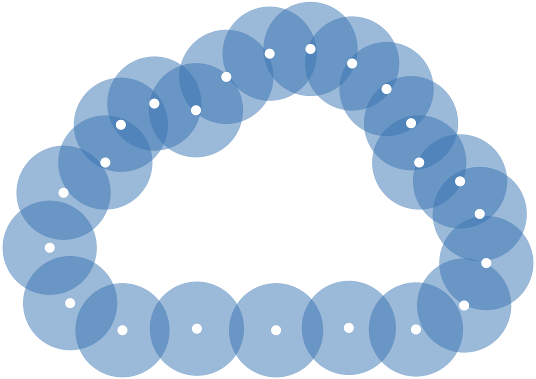






Geometric complexes

Čech complexes



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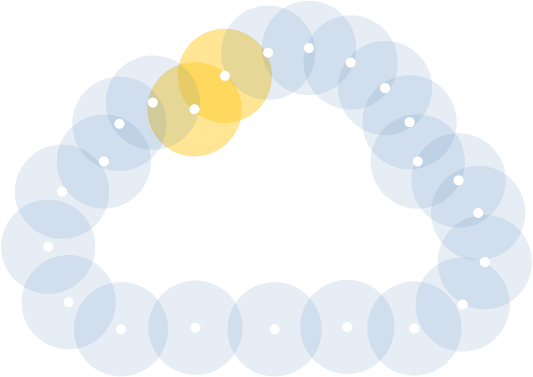

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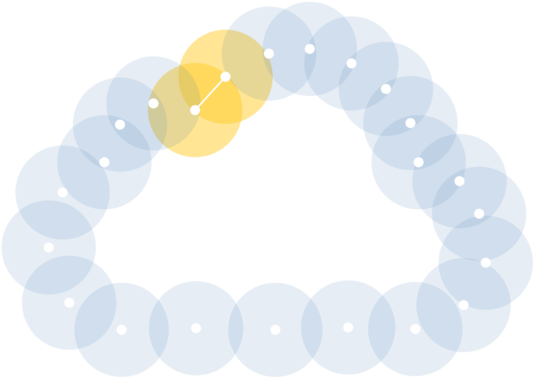

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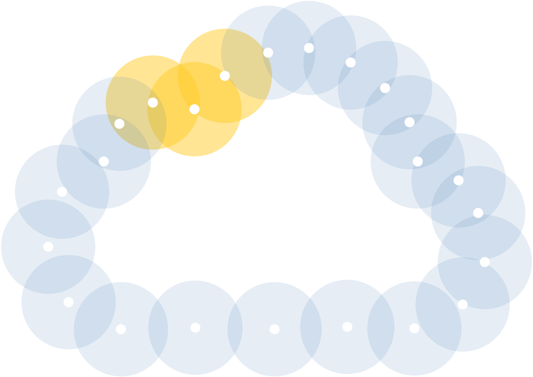

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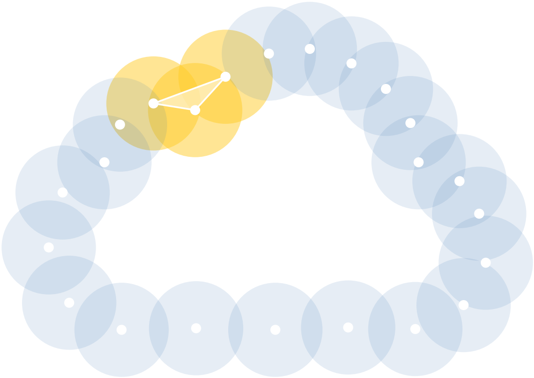

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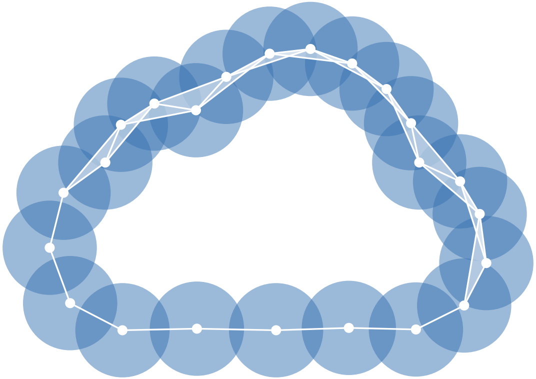
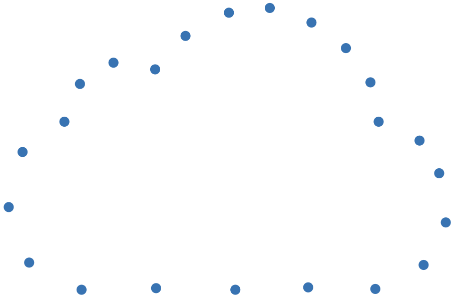


Delaunay complexes



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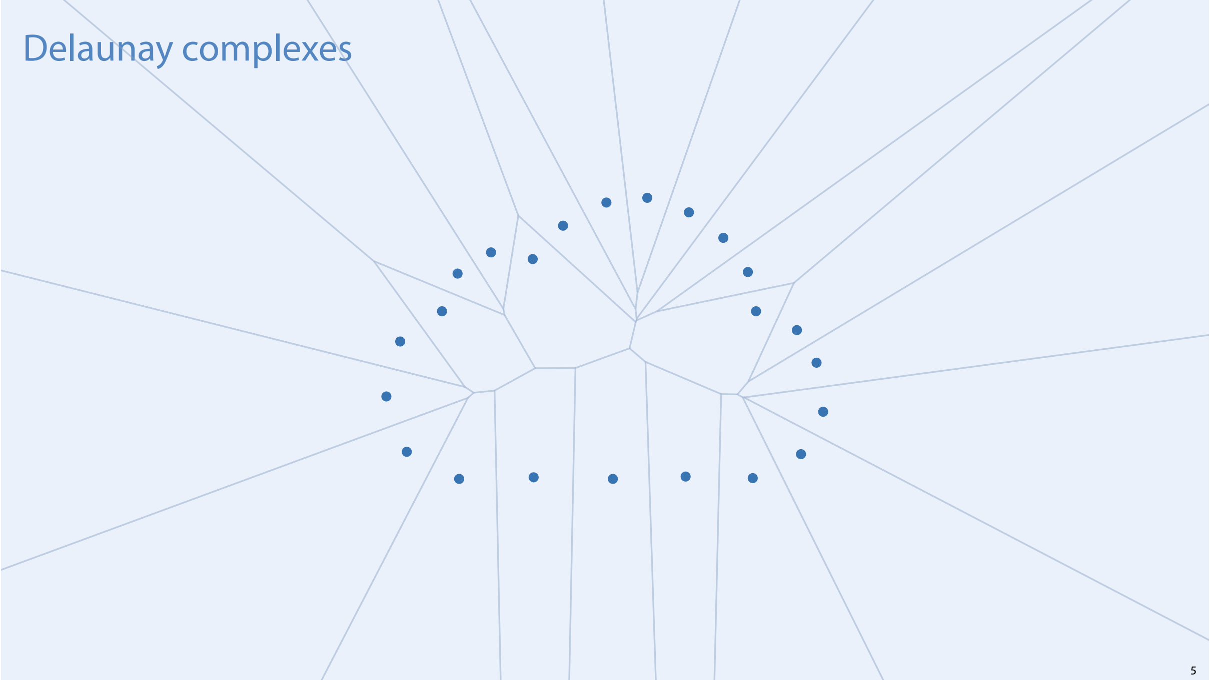

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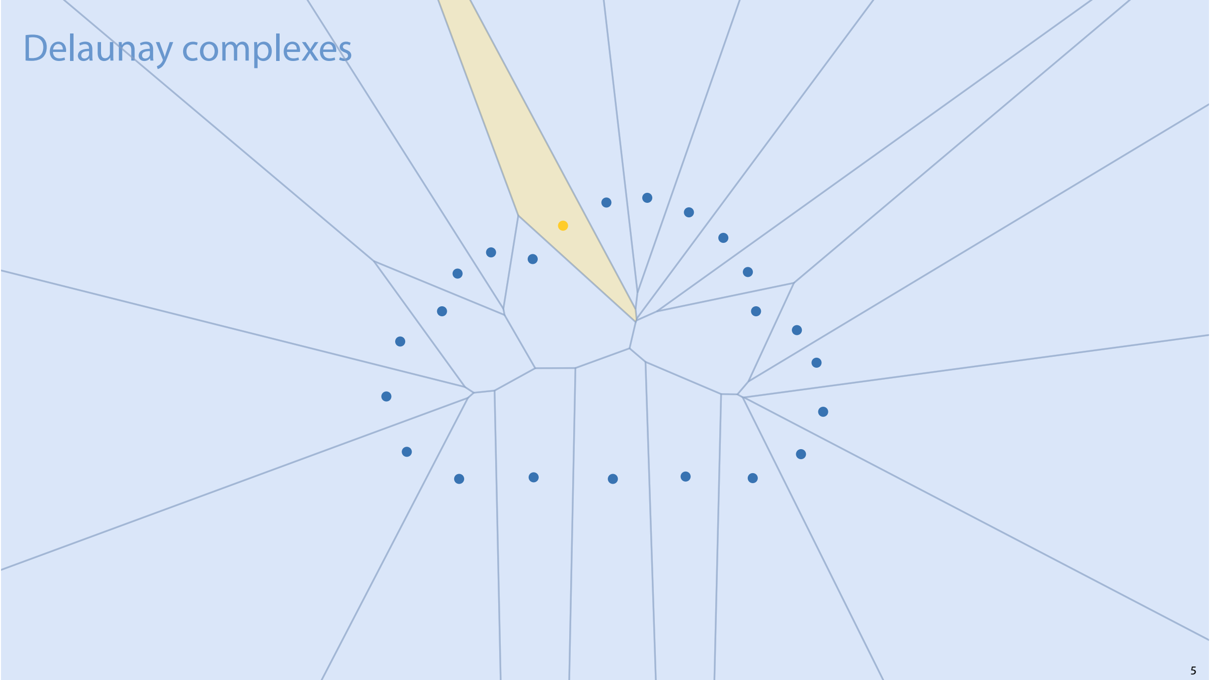

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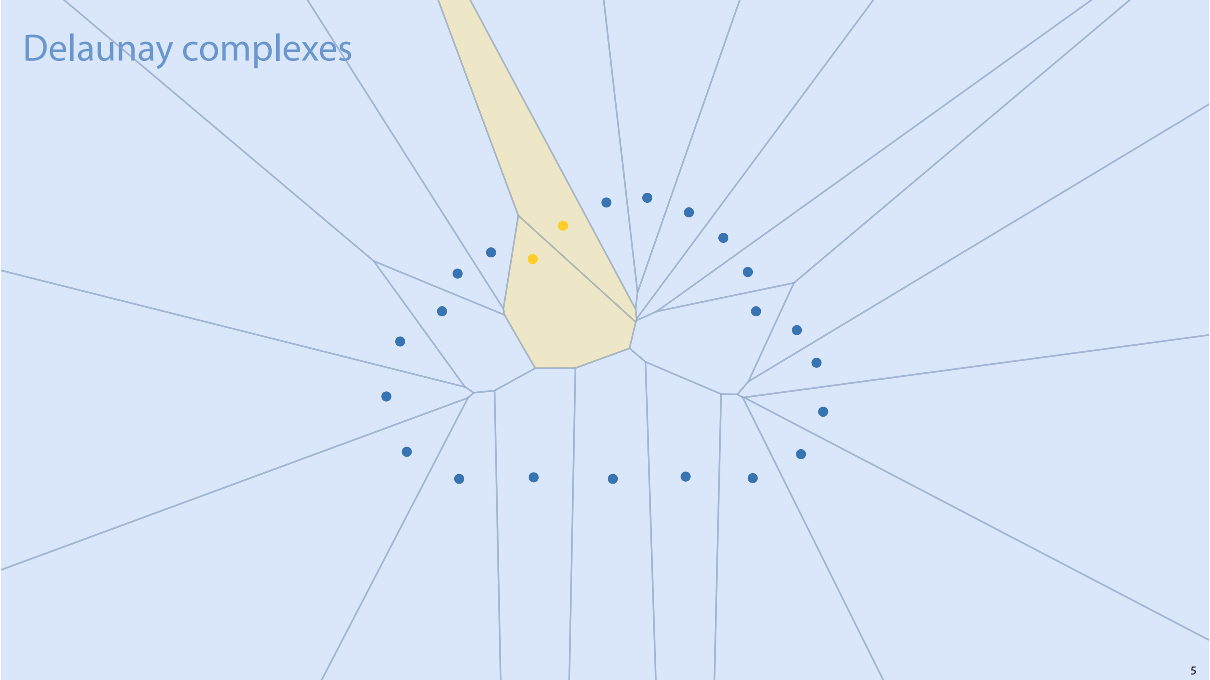

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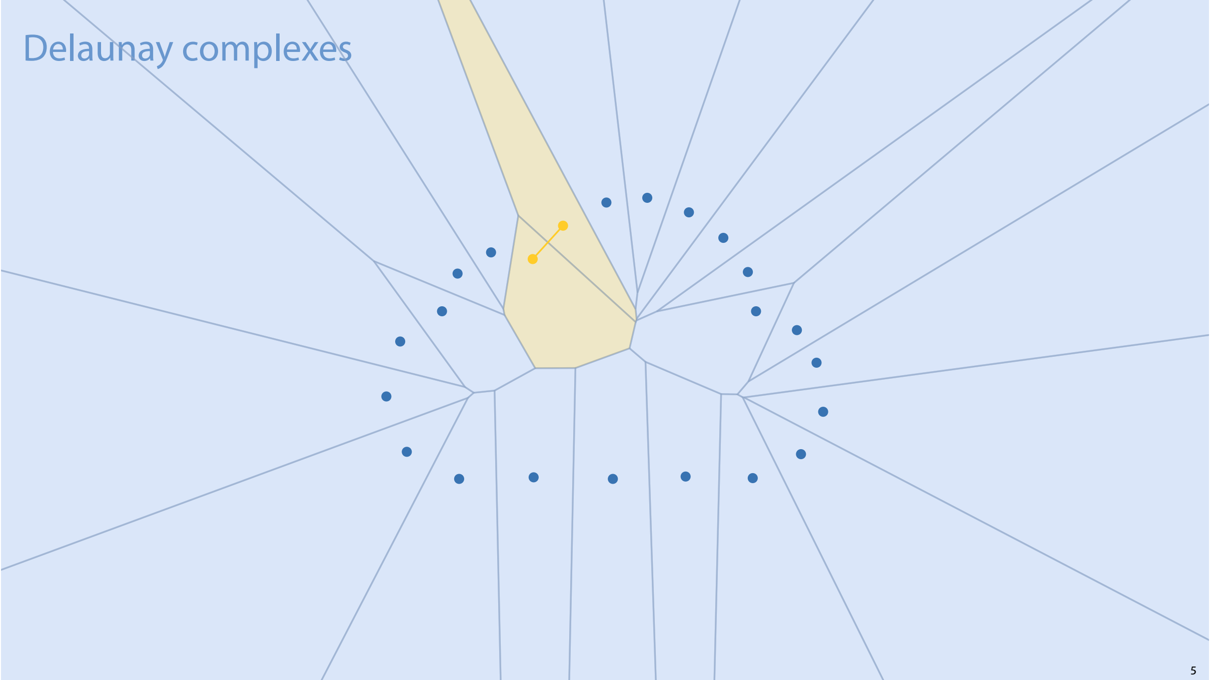

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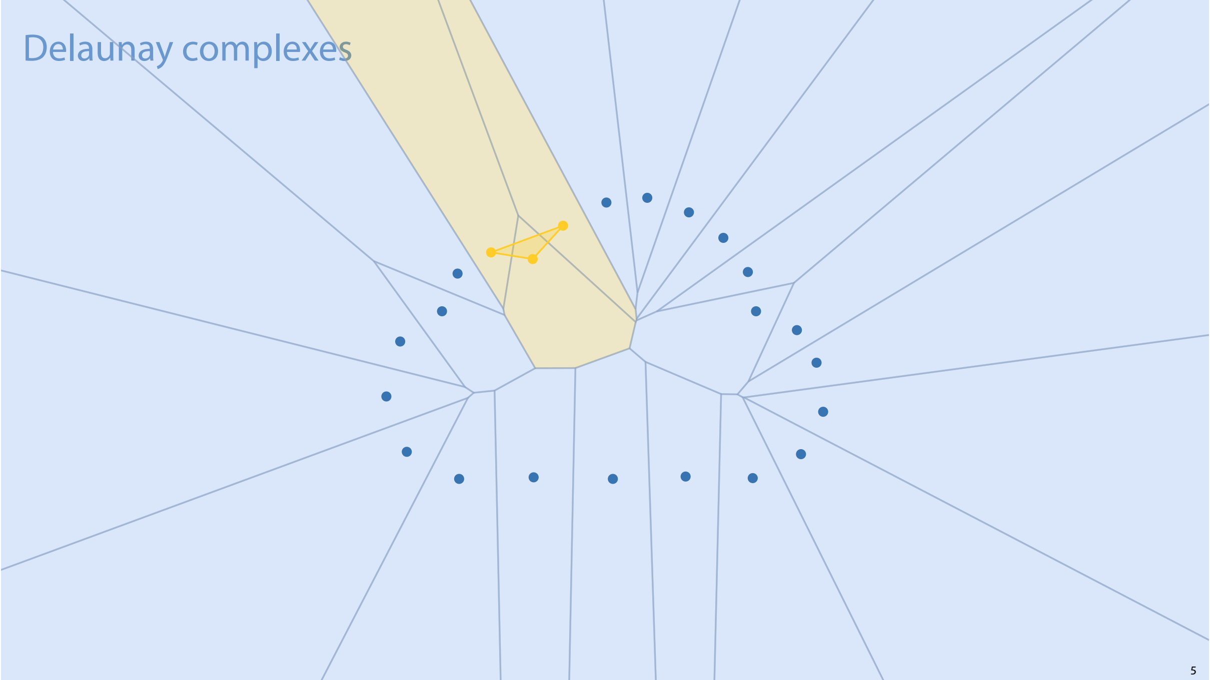

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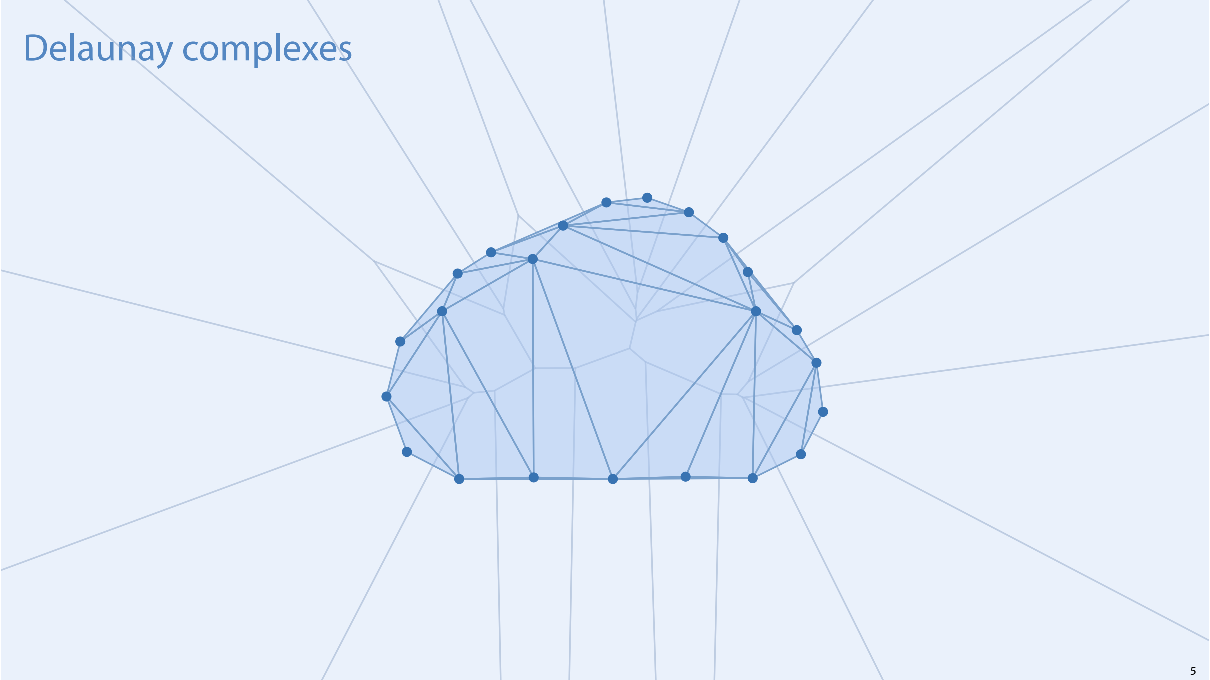

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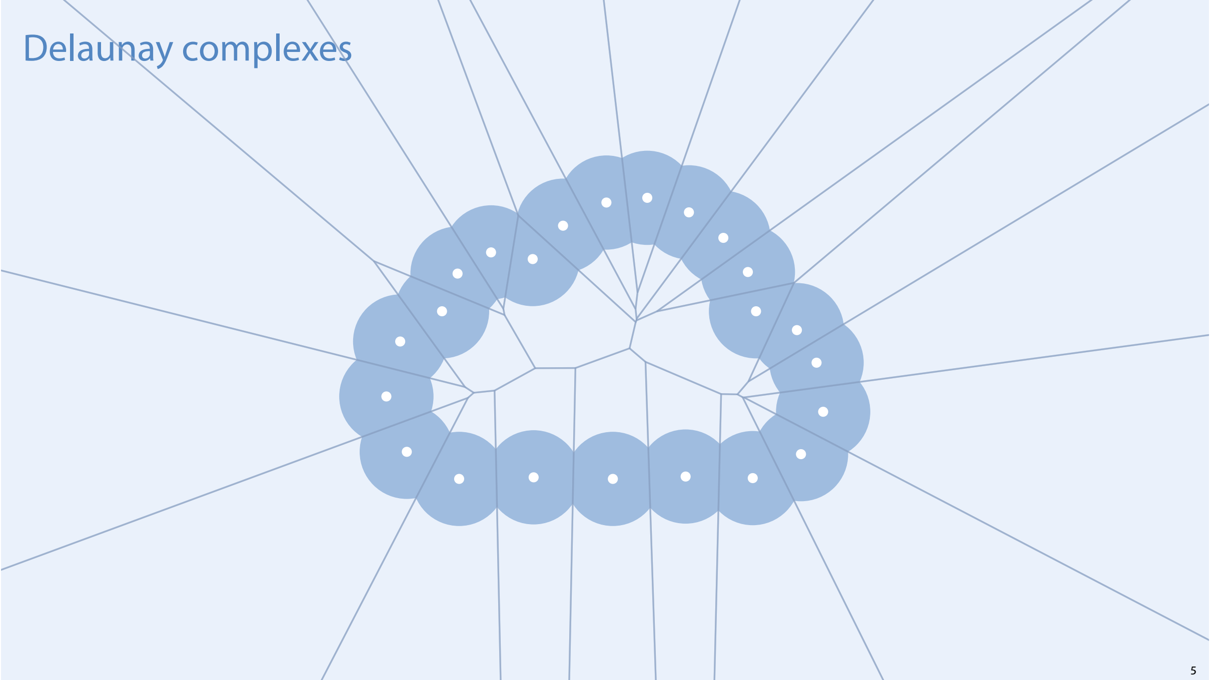

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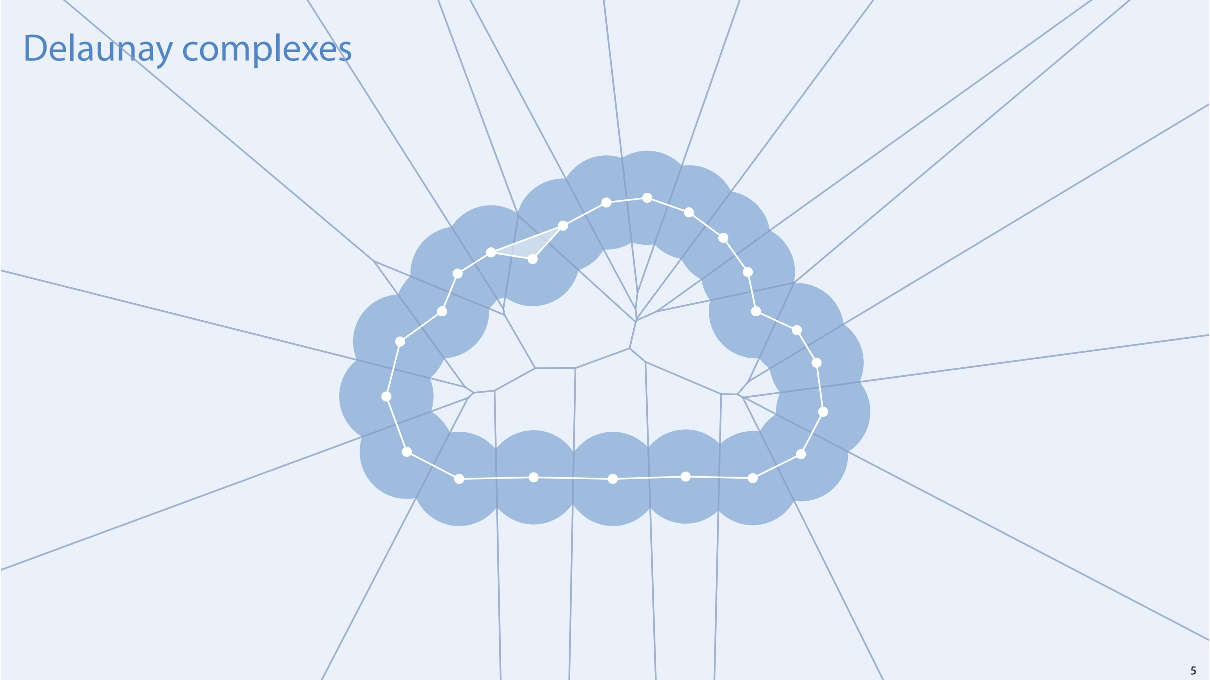

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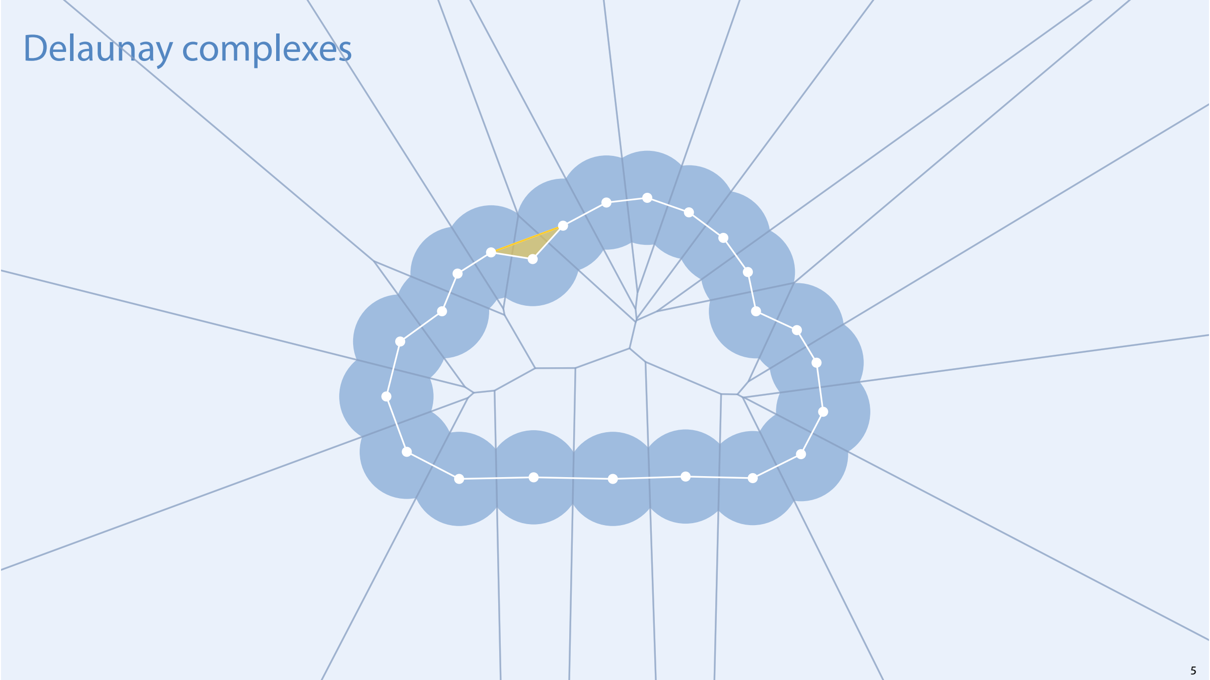

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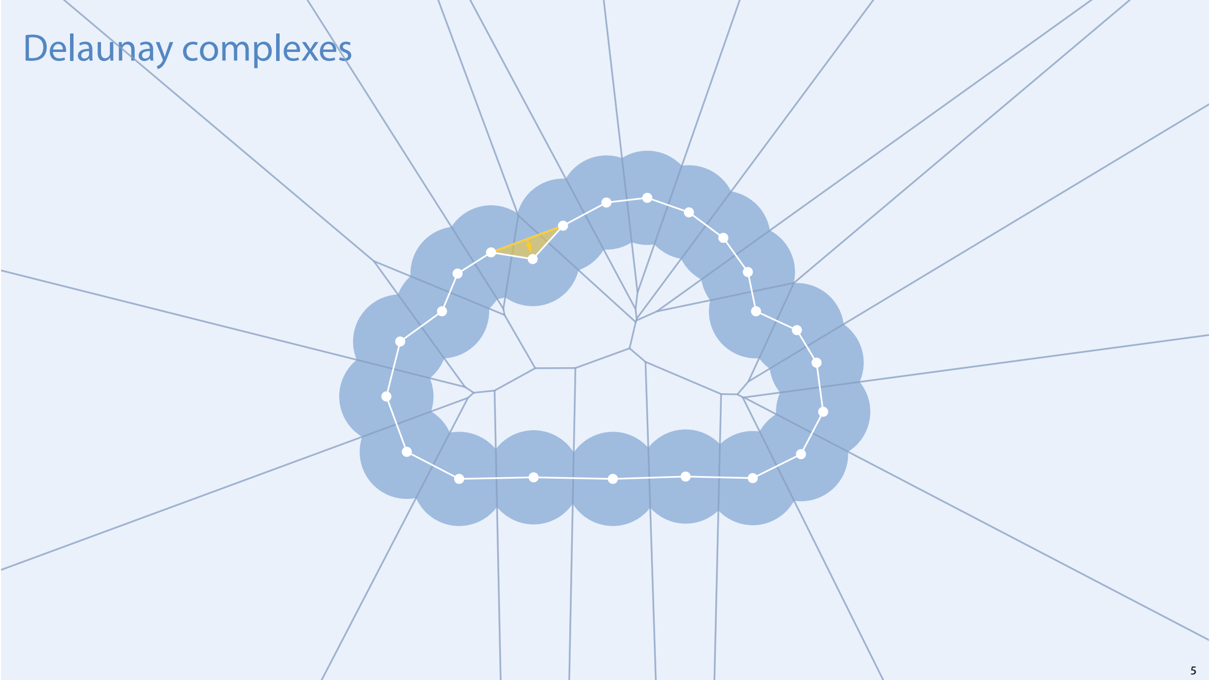

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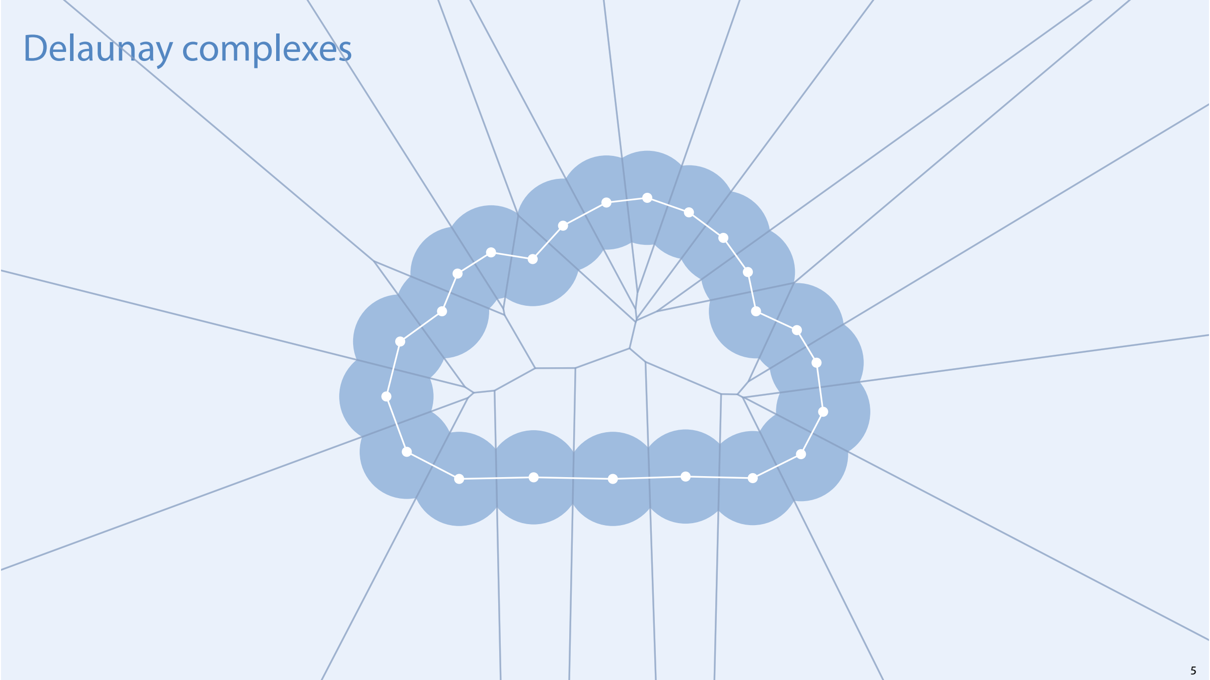
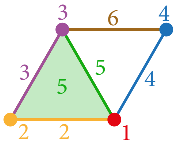


Discrete Morse theory



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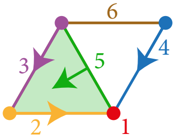
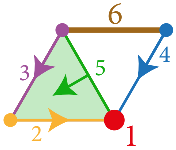


Discrete Morse theory

Theorem (Forman 1998)

A simplicial complex with a discrete Morse function f is homotopy equivalent to a space (a cell complex*) built from the critical simplices of f.*



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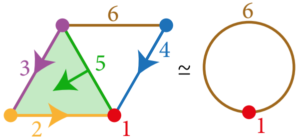
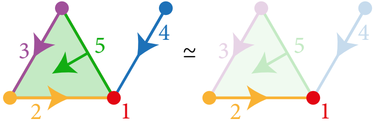


Discrete Morse functions – and their gradients – encode *collapses* of sublevel sets:



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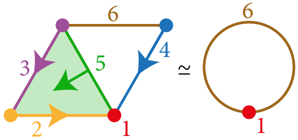

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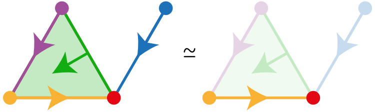
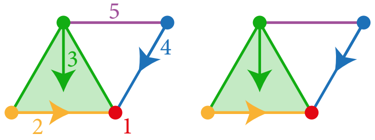


Generalizing discrete Morse theory

Generalized gradients are a partition into intervals in the face poset (instead of just facet pairs):



Morse theory for Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

For a point set in general position, the Čech and Delaunay complexes are sublevel sets of (generalized) discrete Morse functions.

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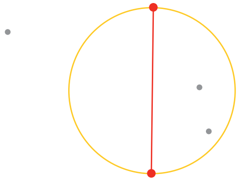
- *Čech: radius of smallest enclosing ball*



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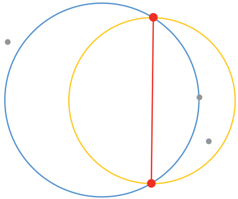
- *Čech: radius of smallest enclosing ball*
- *Delaunay: radius of smallest empty circumsphere*

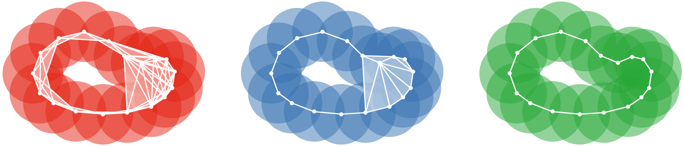


From Cech to Delaunay to Wrap complexes

Theorem (B, Edelsbrunner 2017)

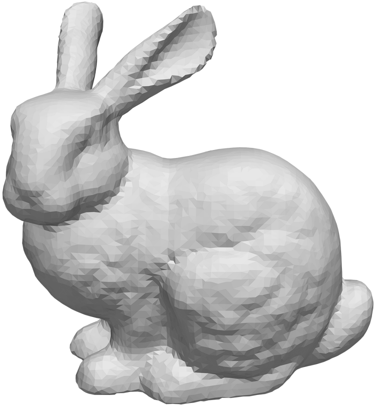
Čech, Delaunay, and Wrap complexes (at any scale r) of a point set $X \subset \mathbb{R}^d$ in general position are related by collapses encoded by a single discrete gradient field:

$$\mathrm{Cech}_r X \searrow \mathrm{Del}_r X \searrow \mathrm{Wrap}_r X.$$



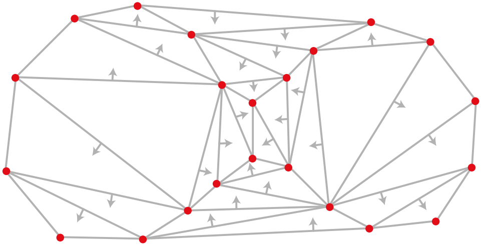
Wrap complexes

Foundation of the surface reconstruction software *Wrap* (Edelsbrunner 1995, Geomagic)



Wrap complexes

Consider the gradient V of the Delaunay radius function $\mathrm{Del}(X) \to \mathbb{R}$.



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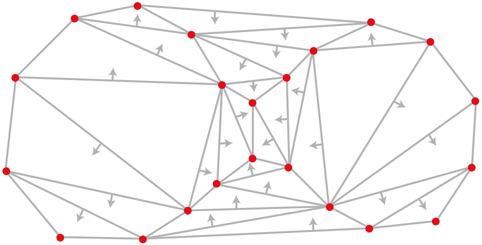

Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

$\mathrm{Wrap}_r(X)$ is the *descending complex* of V on $\mathrm{Del}_r X$:

- the smallest subcomplex of $\mathrm{Del}_r X$ that
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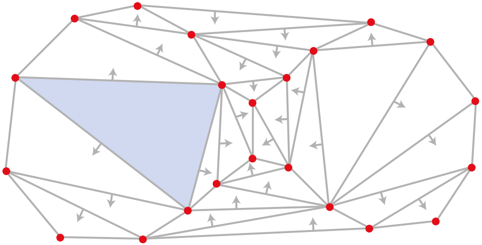

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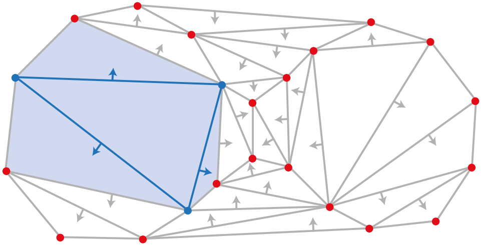

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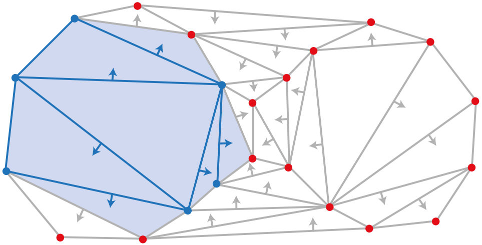

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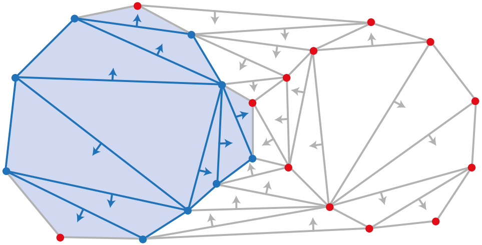

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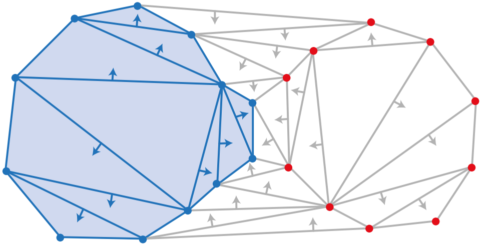

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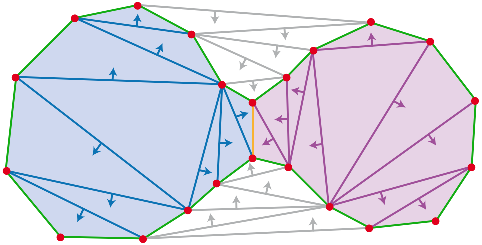

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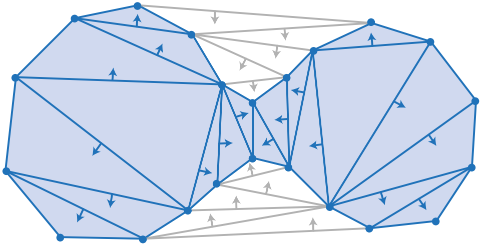

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Delaunay and Wrap complexes

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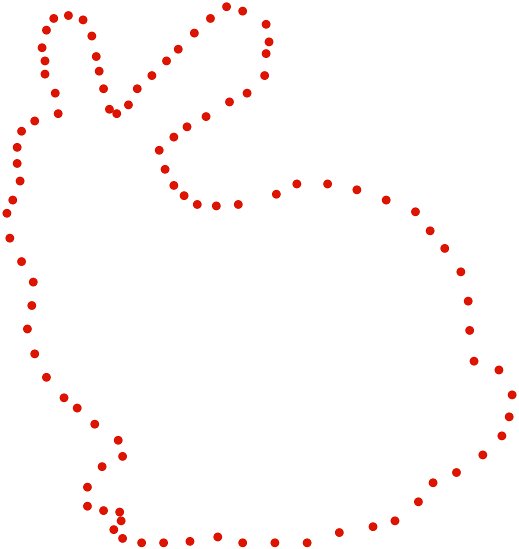
Wrap complexes and lexicographically minimal cycles



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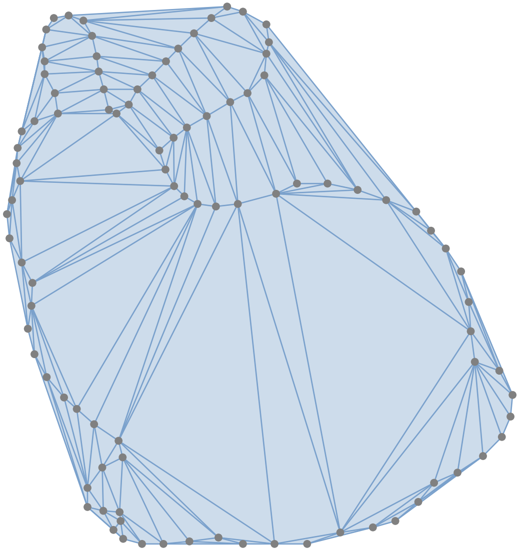

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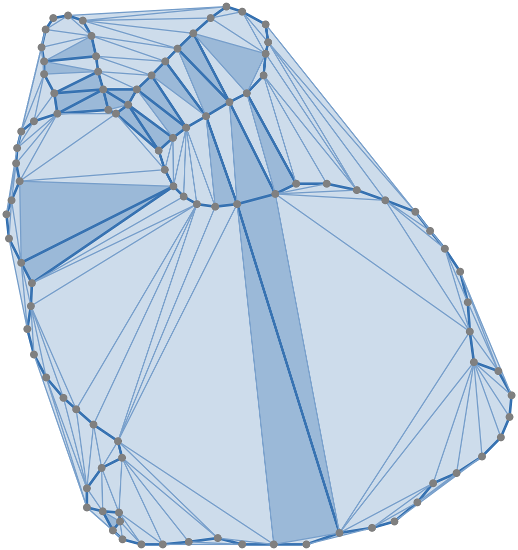

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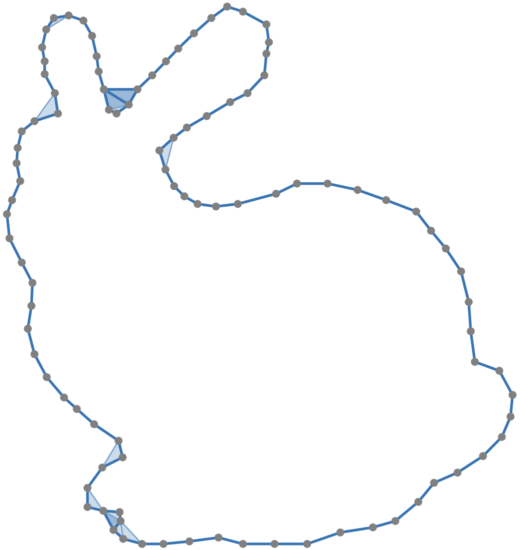

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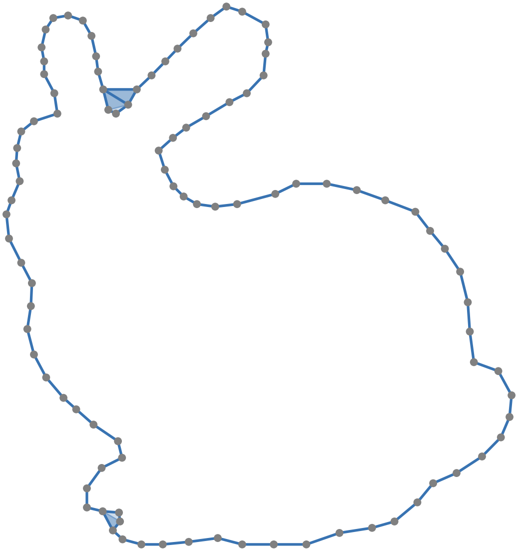

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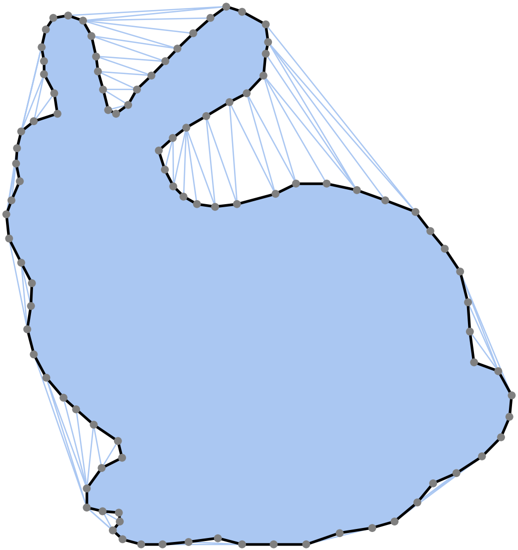

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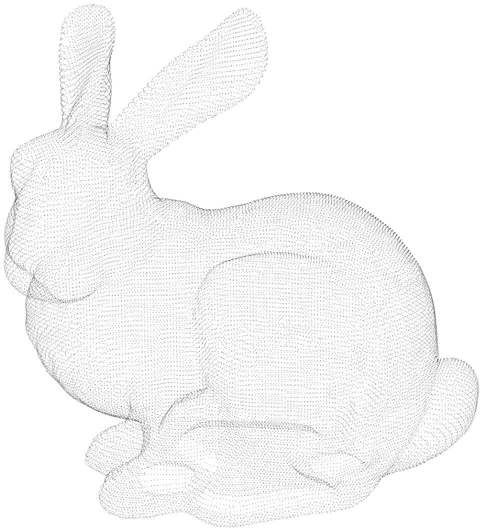

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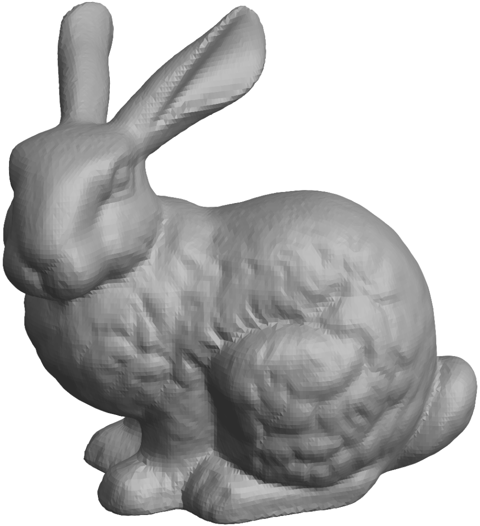
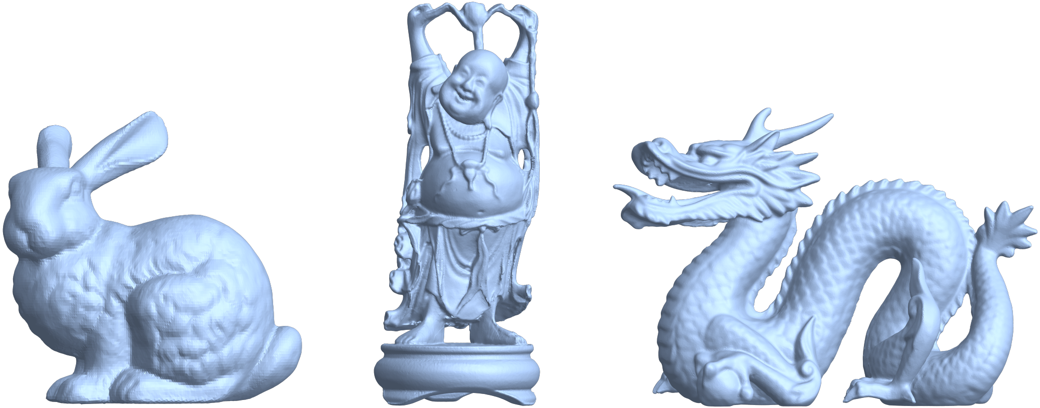


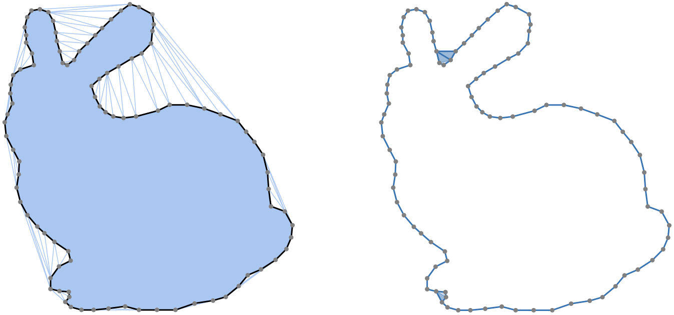
Point cloud reconstruction with minimal cycles



Minimal cycles within Wrap complexes

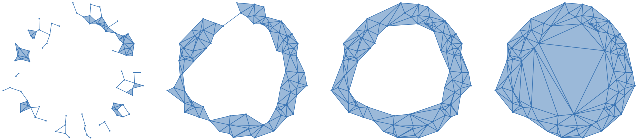
Theorem (B, Roll 2024)

Let $X \subset \mathbb{R}$ *be a finite subset in general position and let* $r \in \mathbb{R}$*. Then the lexicographically minimal cycles of* $\mathrm{Del}_r(X)$ *are supported on* $\mathrm{Wrap}_r(X)$.



Persistent homology

What is persistent homology?



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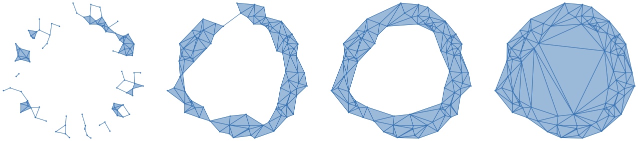

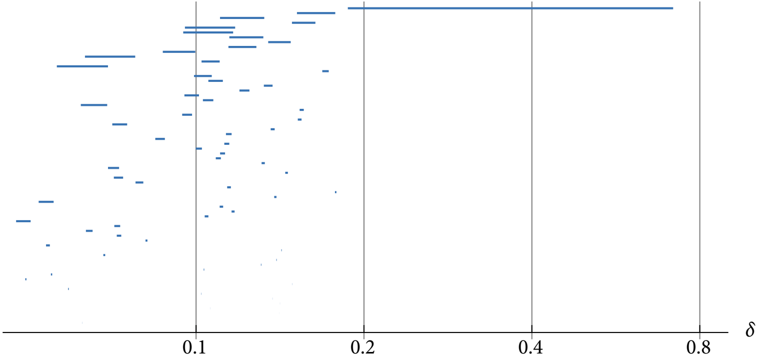

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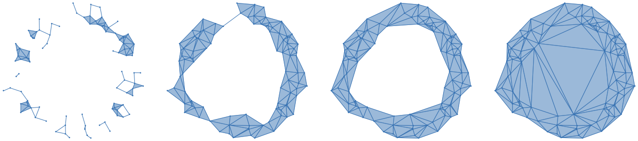

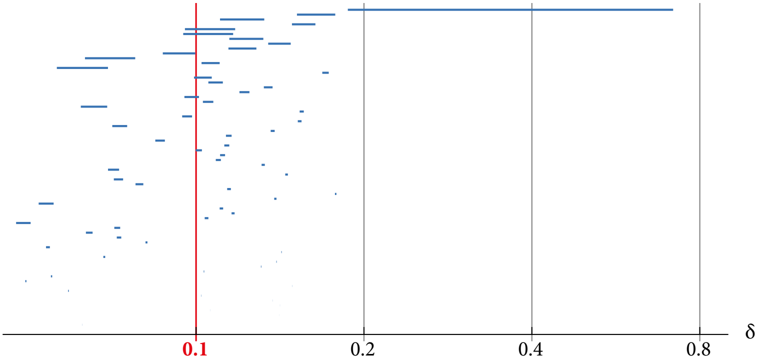

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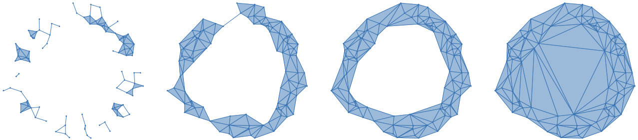

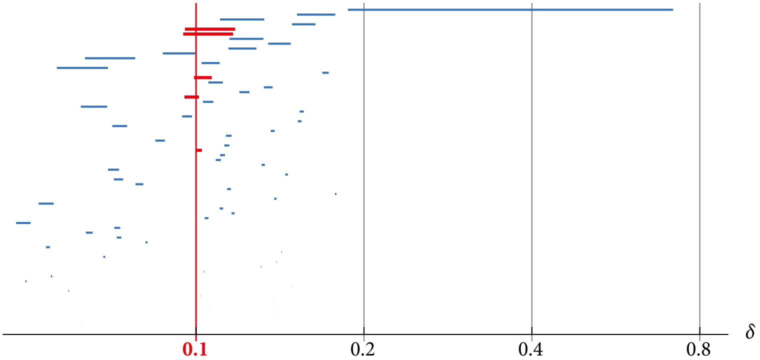

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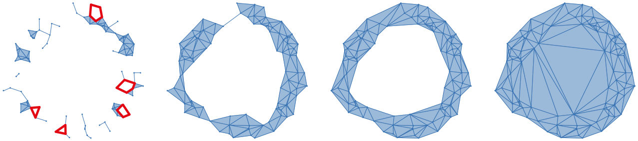

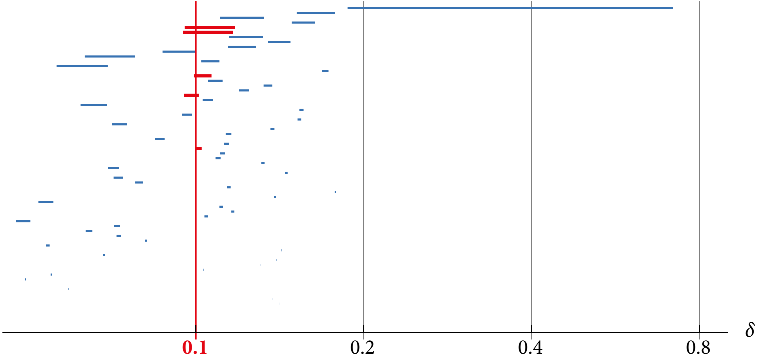

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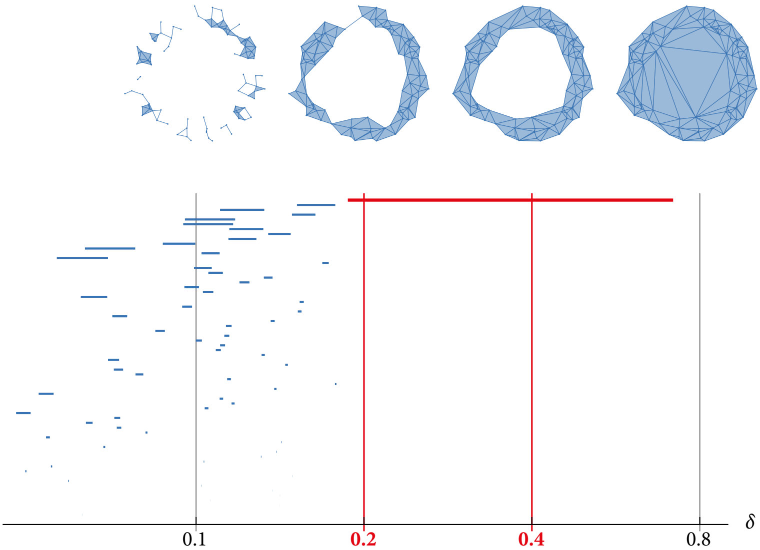

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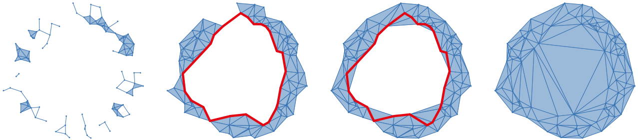

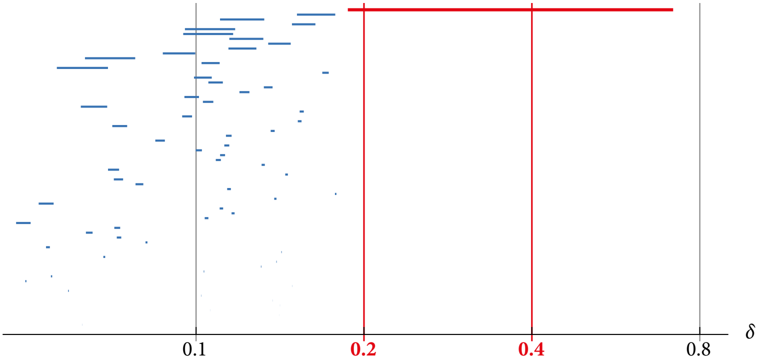

Structure

The formalism of persistent homology

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Persistent homology is the homology of a filtration $K : \mathbf{R} \to \mathbf{Top}$.

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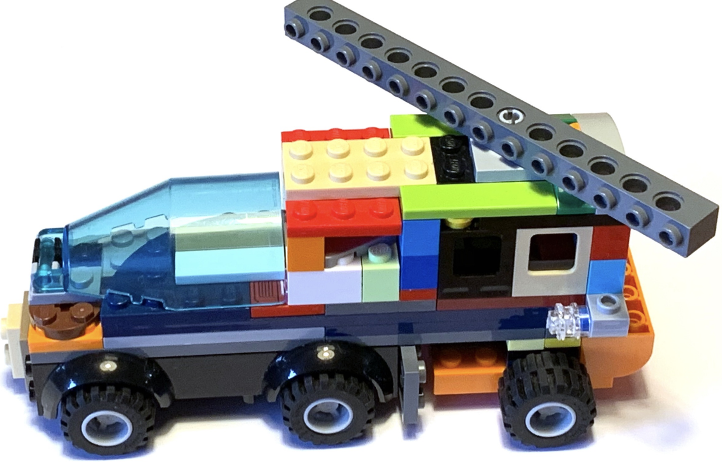
- The maps $K_s \hookrightarrow K_t$ for $s \leq t \in \mathbb{R}$ are inclusions.
- Applying homology $H_* : \mathbf{Top} \to \mathbf{Vect}$ yields a *persistence module*

$$M = H_* \circ K : \mathbf{R} \to \mathbf{Vect}$$



Barcodes: the structure of persistence modules

Theorem (Crawley-Boevey 2015)

Any persistence module $M : \mathbf{R} \to \mathbf{vect}$ *(of finite dim. vector spaces over some field* $\mathbb{F}$*) decomposes as a direct sum of* interval modules

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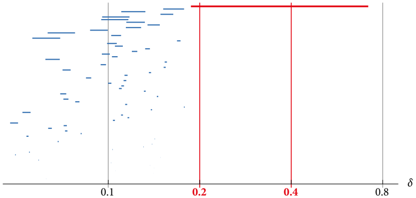

- The supporting intervals form the *persistence barcode*. It describes M up to isomorphism.

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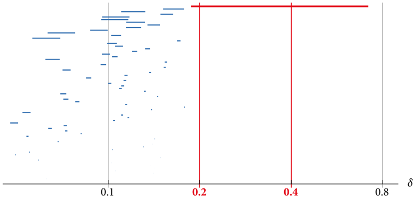

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- We rarely have such a simple structure for other diagrams, like $\mathbf{R}^2 \to \mathbf{vect}$ (2 parameters).

Indecomposables abound in multi-parameter persistence

Theorem (B, Scoccola 2022)

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In other words:

- Indecomposability is a generic property.

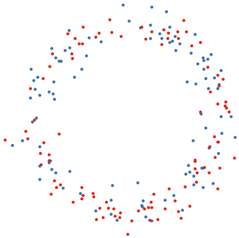
Stability

Stability of persistence barcodes

If two point clouds are close, then their barcodes are also close:

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $X, Y \subset \mathbb{R}^d$ be finite point sets with Hausdorff distance $d_H(X, Y) = \delta$.



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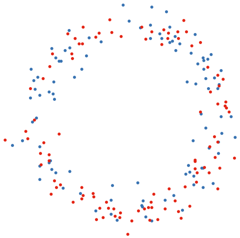

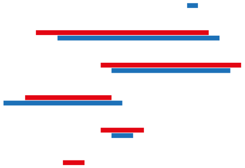

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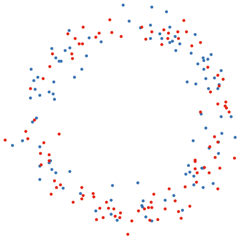

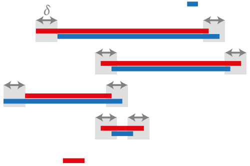

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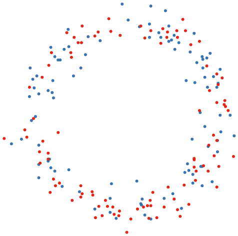

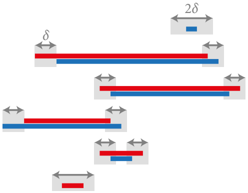

Persistence and stability: the big picture

Data point cloud $P \subset \mathbb{R}^d$

Persistence and stability: the big picture

$$\begin{array}{ccccc}
\text{Data} & \qquad & \text{point cloud} & \qquad & P \subset \mathbb{R}^d \\
\downarrow & & \big\downarrow{\scriptstyle \text{distance}} & & \\
\text{Geometry} & & \text{function} & & f : \mathbb{R}^d \to \mathbb{R}
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Persistence and stability: the big picture

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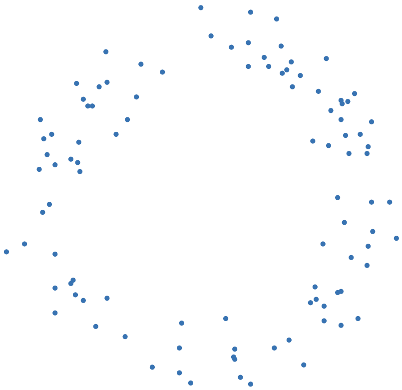
Vietoris–Rips persistence

Vietoris–Rips complexes

For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

$$\operatorname{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$



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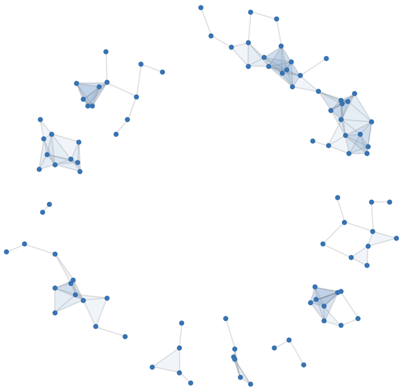

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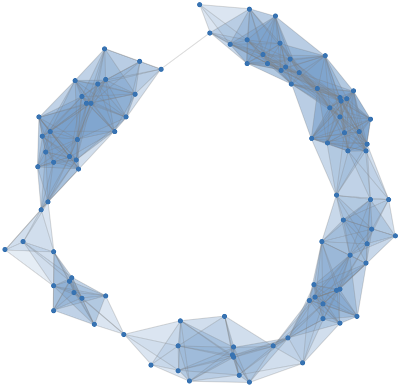

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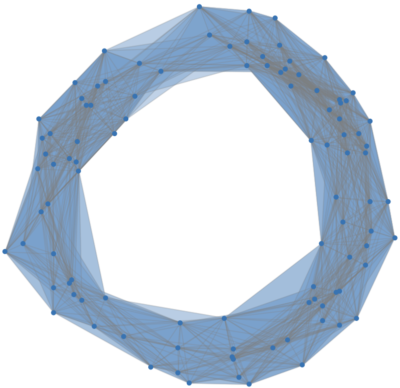

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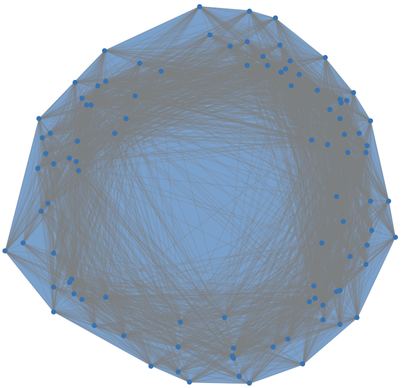
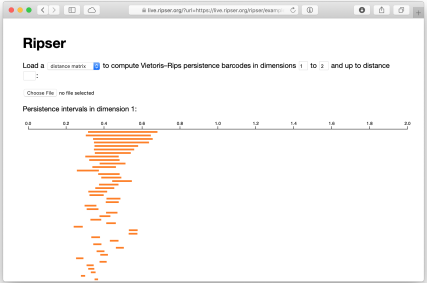
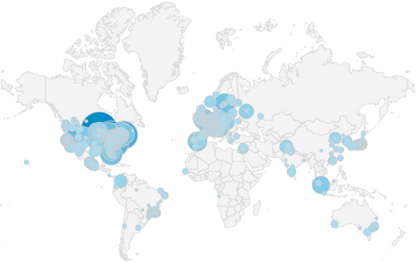


Ripser: software for computing Vietoris–Rips persistence barcodes

Open source software (`ripser.org`)





Ripser users across the globe

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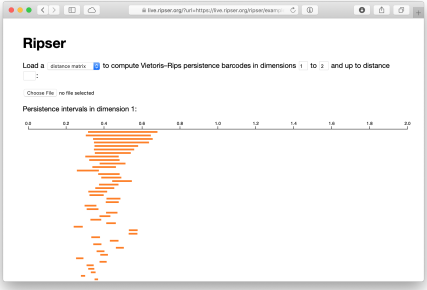
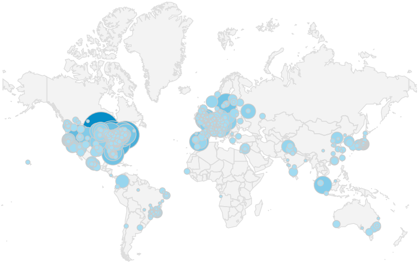


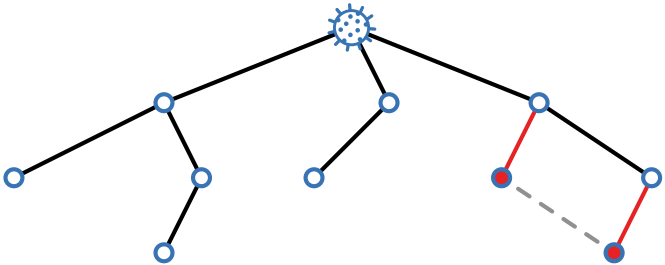


Ripser users across the globe

Computational improvements based on

- *implicit matrix representations*
- *apparent pairs*, connecting persistence to discrete Morse theory

Topology of viral evolution



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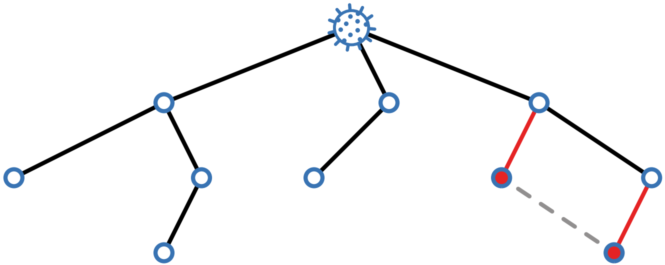

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Observation: Ripser runs unusually fast on genetic distance data

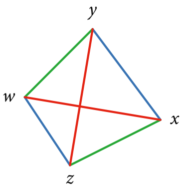
- SARS-CoV2 RNA sequences (spike protein)
- 25556 data points (2.8×10^{12} simplices in 2-skeleton)
- $120\,s$ computation time (with data points ordered appropriately)

Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ-*hyperbolic* (for $\delta \geq 0$) if for all $w, x, y, z \in X$ we have

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$



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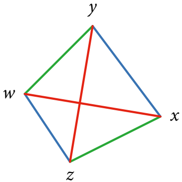

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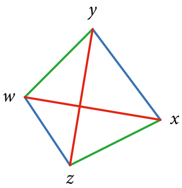
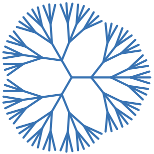


- The hyperbolic plane is $(\ln 2)$-hyperbolic.

- The 0-hyperbolic spaces are precisely the metric trees and their subspaces.



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Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

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Let X be a finite δ-hyperbolic space. Then there is a single discrete gradient encoding the collapses

$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect *of X.*

Rips Contractibility

Theorem (Rips; Gromov 1988)

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for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect *of X.*

If X is a sample of a metric tree, then $H_d(\mathrm{Rips}_t(X))$ is trivial for $d > 0$.

Origins

When was persistent homology discovered first?

When was persistent homology discovered first?

ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses is satisfied. The critical limits cluster at most at $+\infty$; the critical points are

When was persistent homology discovered first?

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Rank and span in functional topology

Search within citing articles

Exact homomorphism sequences in homology theory
ed.ac.uk **[PDF]**

JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR

The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] **...**

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Marston Morse and his mathematical works
ams.org **[PDF]**

R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org

American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to **...**

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Unstable minimal surfaces of higher topological structure

M Morse, CB Tompkins - Duke Math. J, 1941 - projecteuclid.org

1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. Theproblem of the existenceof minimal surfaces of unstable type contains many of the typical difficulties, especially those ofa topological nature. Having studied this **...**

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[PDF] Persistence in discrete Morse theory
psu.edu **[PDF]**

U Bauer - 2011 - Citeseer

When was persistent homology discovered first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

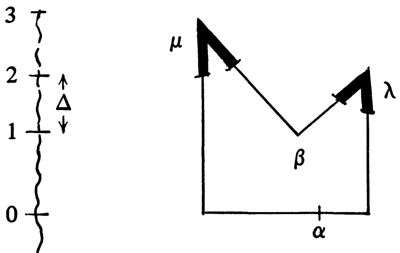
In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was

When was persistent homology discovered first?

inequalities pertain between the dimensions of the A_i and those of $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels [**D**] in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers [**36**] and [**44**] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

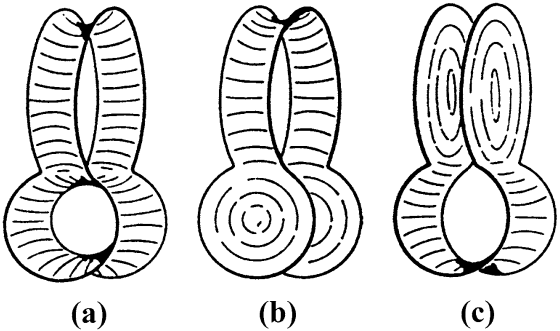
In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.



Motivation and application: minimal surfaces

Problem (Plateau's problem)

Find an immersed disk of least area spanned by a given closed Jordan curve.



(from Dierkes et al.: *Minimal Surfaces*, 2010)

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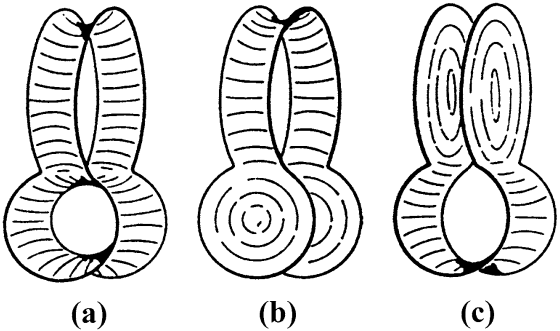


(from Dierkes et al.: *Minimal Surfaces*, 2010)

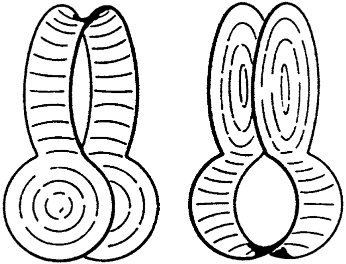
Solution by Douglas (1930; first Fields Medal 1936):

- identifies minimal surfaces with critical points of the *Douglas functional*

Existence of unstable minimal surfaces

Theorem (Morse, Tompkins 1939; Shiffman 1939)

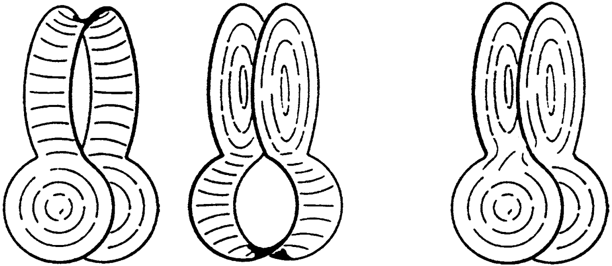
Assume that a given curve bounds two separate stable minimal surfaces.



Existence of unstable minimal surfaces

Theorem (Morse, Tompkins 1939; Shiffman 1939)

Assume that a given curve bounds two separate stable minimal surfaces.
Then that curve also bounds an unstable *minimal surface (a critical point that is not a local minimum).*



Q-tame persistence modules

Definition (Chazal et al. 2009)

A persistence module $M : \mathbf{R} \to \mathbf{vect}$ is *q-tame* if all structure maps $M_s \to M_t$ $(s < t)$ have finite rank.

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- A q-tame persistence module still admits a persistence barcode.

Morse's main technical result, in modern language:

- The minimal surface functional has q-tame persistent Vietoris homology.

Q-tameness from local connectivity

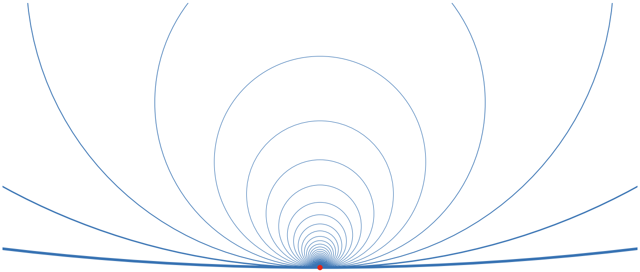
Theorem (Morse, 1937)

If the sublevel set filtration of $f : X \to \mathbb{R}$ is compact and weakly locally connected*, then it has q-tame persistent Vietoris homology.*

Q-tameness from local connectivity

Theorem (Morse, 1937; incorrect)

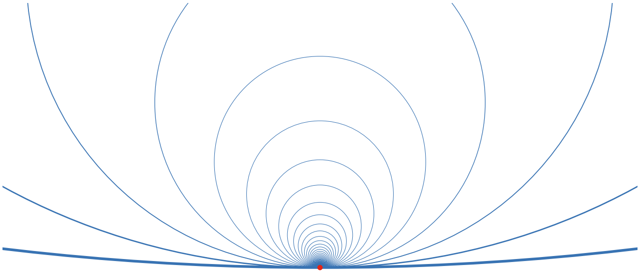
If the sublevel set filtration of $f\colon X \to \mathbb{R}$ is compact and weakly locally connected*, then it has q-tame persistent Vietoris homology.*



Q-tameness from local connectivity

Theorem (Morse, 1937; incorrect)

If the sublevel set filtration of $f: X \to \mathbb{R}$ is compact and weakly locally connected, *then it has q-tame persistent Vietoris homology.*



Theorem (B, Medina-Mardones, Schmahl 2021)

If the sublevel set filtration of $f: X \to \mathbb{R}$ is compact and homologically locally connected, *then it has q-tame persistent homology.*

Simplification

Topological simplification of functions

Consider the following problem:

Problem (Topological simplification)

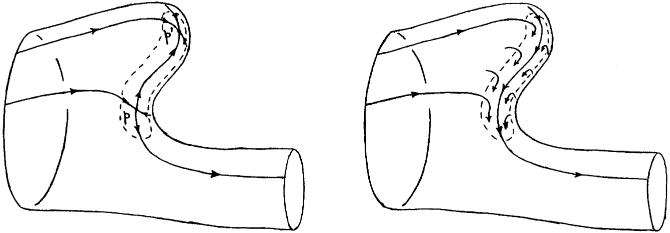
Given a function $f: X \to \mathbb{R}$ *and a real number* $\delta \geq 0$, *find a function* f_δ

- *with the minimal number of critical points*
- *subject to* $\|f_\delta - f\|_\infty \leq \delta$.

Persistence and Morse theory

Morse theory (smooth or discrete):

- Relates critical points to homology of sublevel sets
- Provides a method for *canceling* pairs of critical points



(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

Persistence and Morse theory

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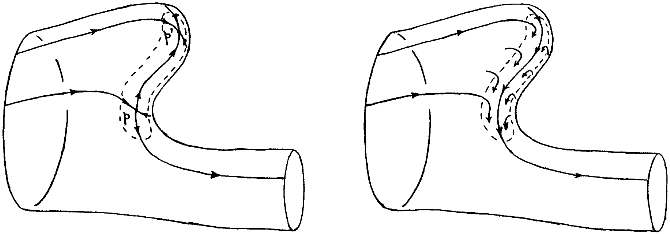


(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

Persistent homology of a Morse function:

- critical points correspond to endpoints of barcode intervals

Canceling persistence pairs

By stability of persistence barcodes:

Proposition

The critical points of f *with persistence* $> 2\delta$ *provide a lower bound on the number of critical points of any function* g *with* $\|g - f\|_\infty \leq \delta$.

Canceling persistence pairs

By stability of persistence barcodes:

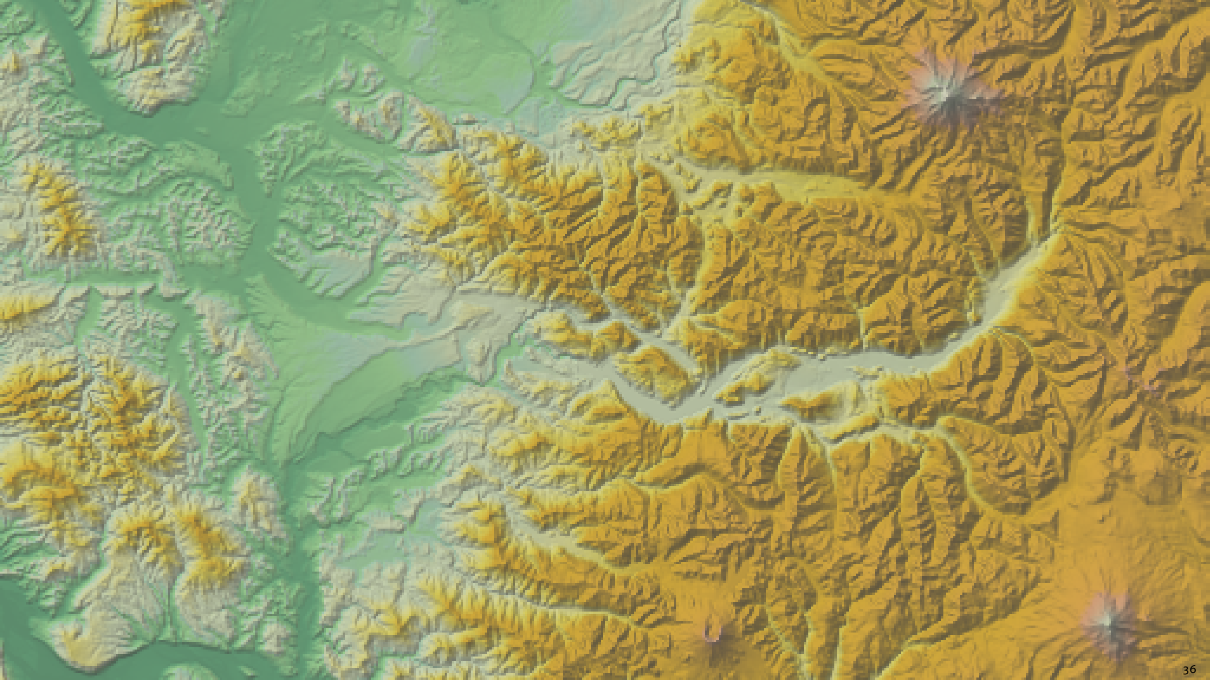
Proposition

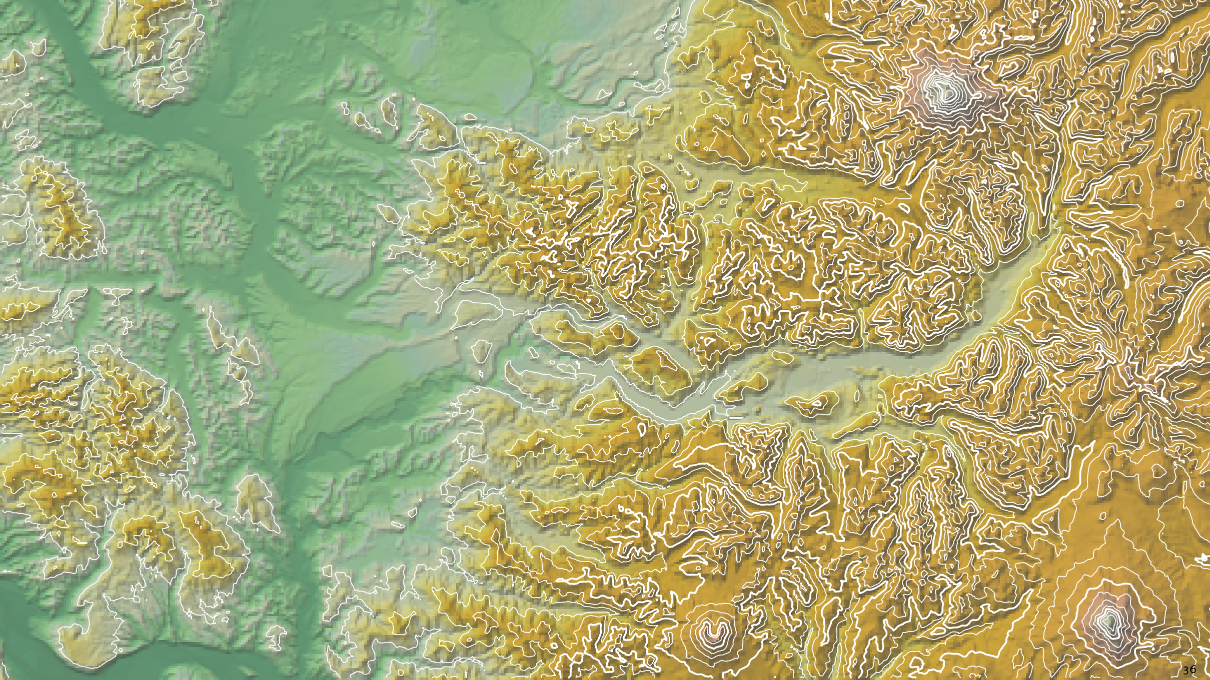
The critical points of f with persistence $> 2\delta$ provide a lower bound on the number of critical points of any function g with $\|g - f\|_\infty \leq \delta$.

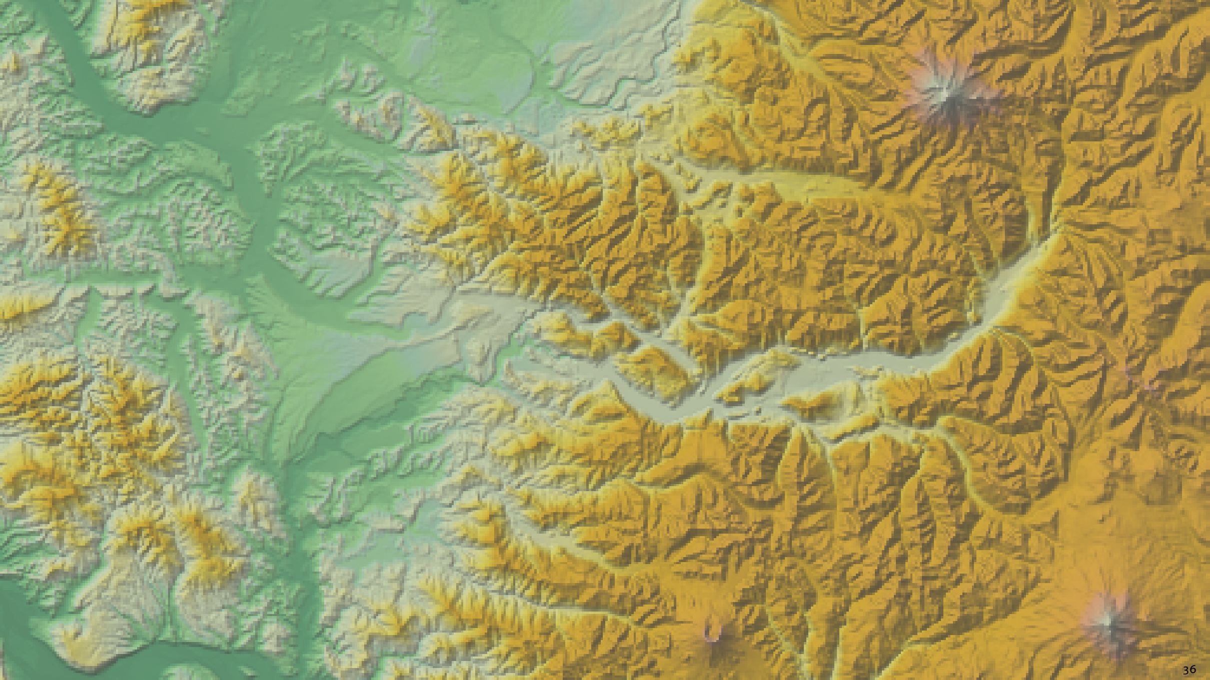
Theorem (B, Lange, Wardetzky, 2011)

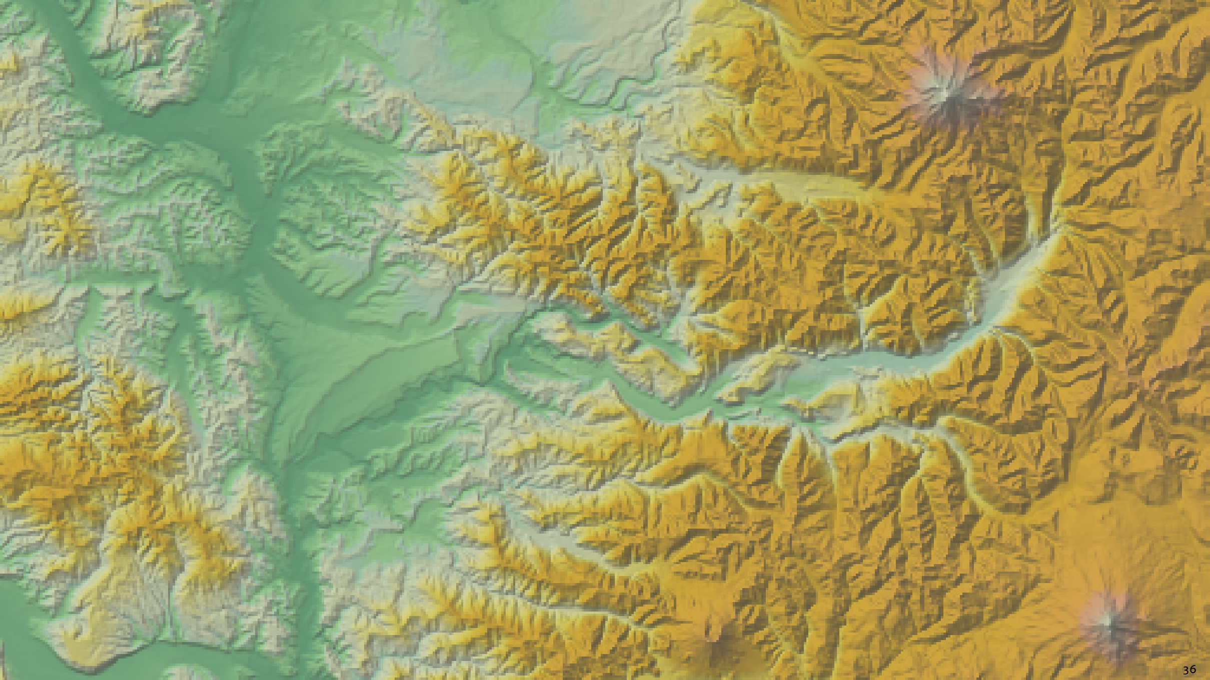
This lower bound can be achieved for a function on a surface

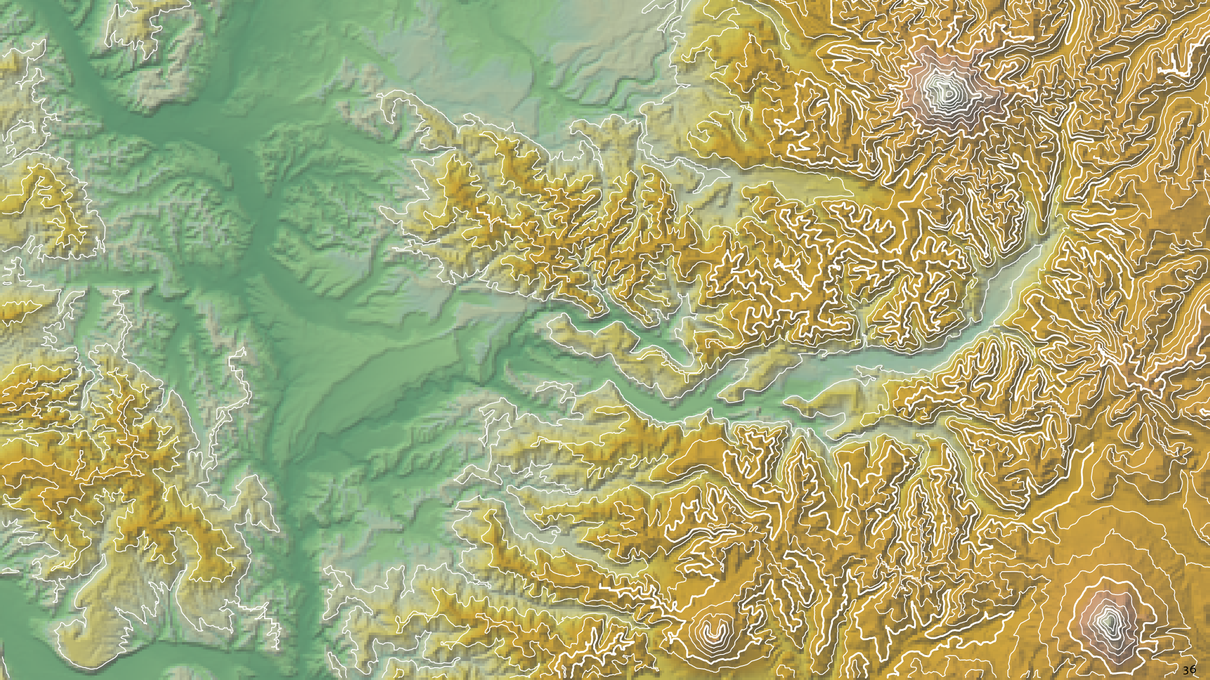
- *by canceling all critical points with persistence $\leq 2\delta$.*











Thanks for your attention!