Induced matchings and the algebraic stability of persistence barcodes

Ulrich Bauer

Technische Universität München (TUM)

December 7, 2015

2ª Escuela/Conferencia de Análisis Topológico de Datos

Joint work with Michael Lesnick (IMA)

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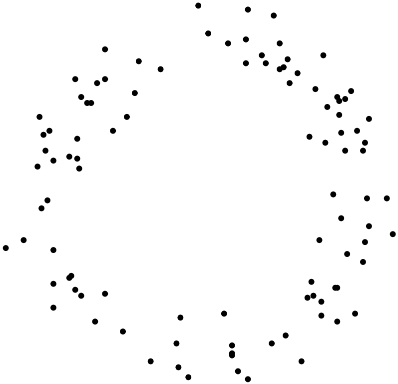
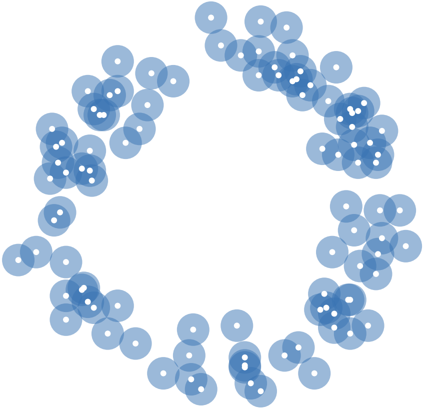
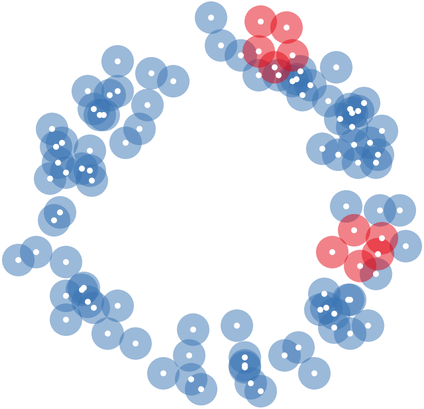
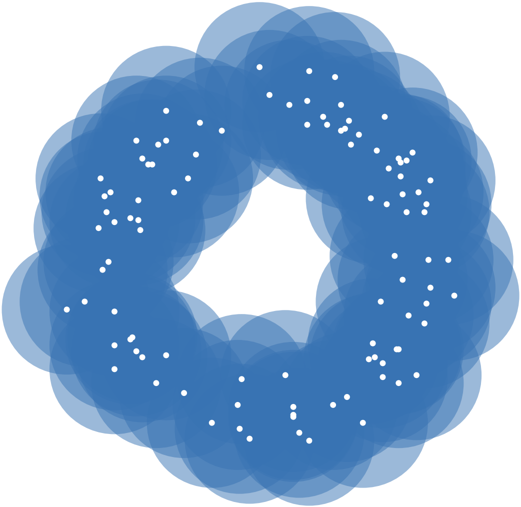
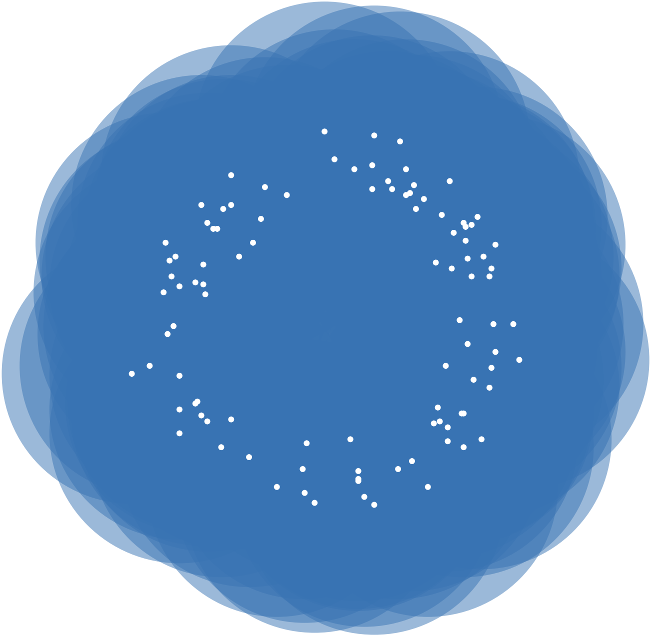
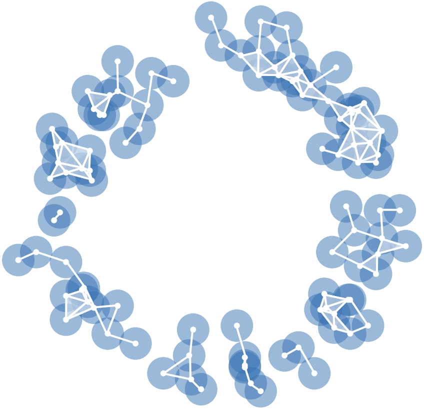


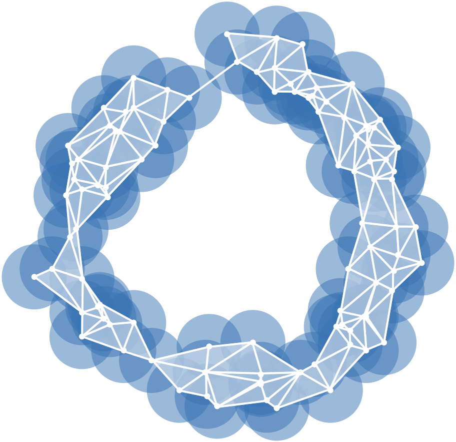


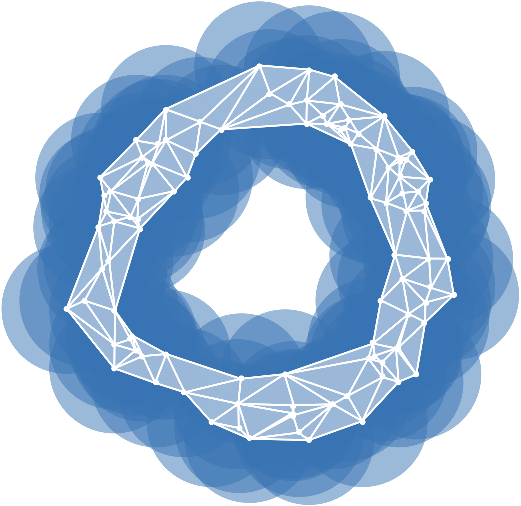


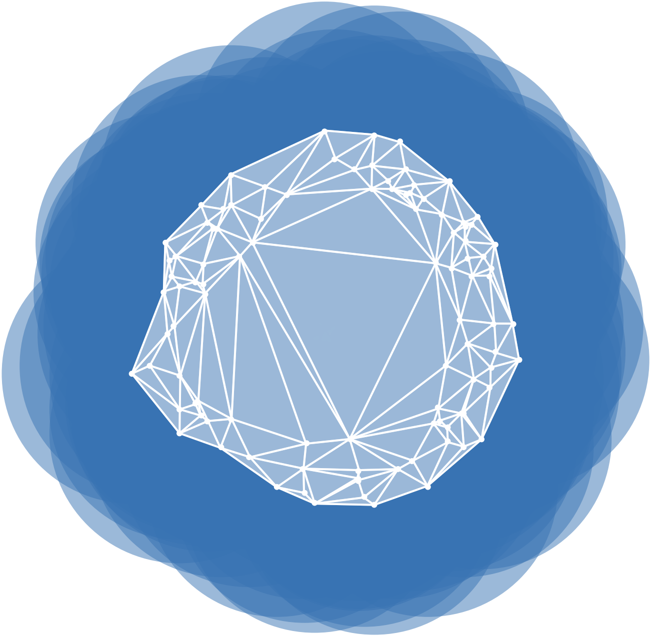




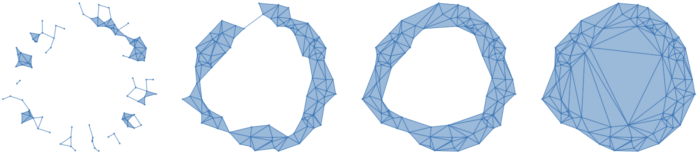


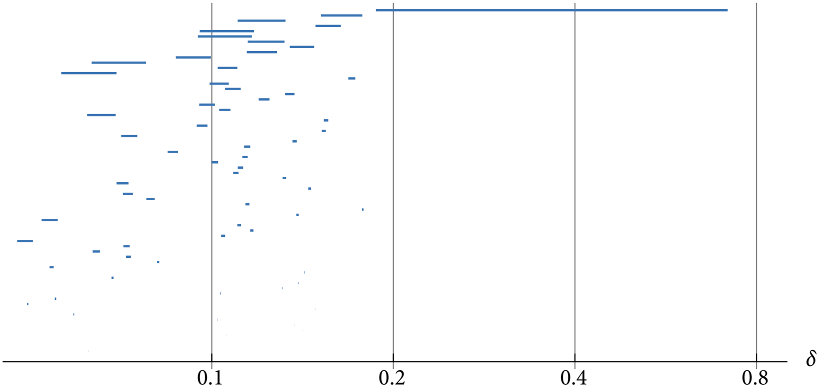






What is persistent homology?





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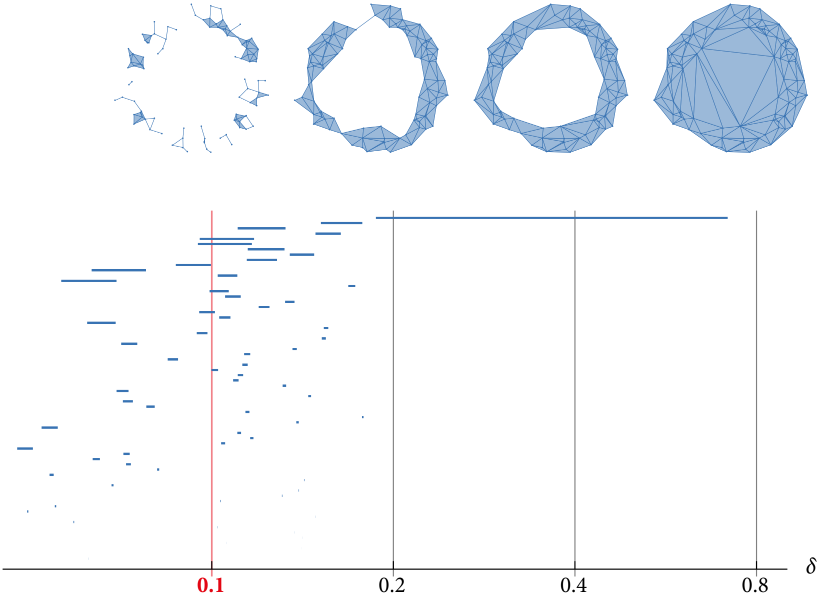

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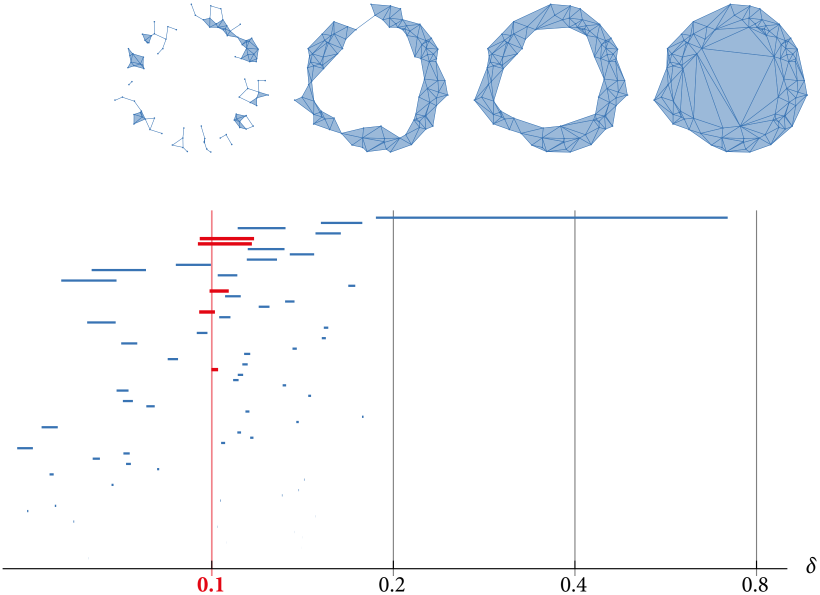

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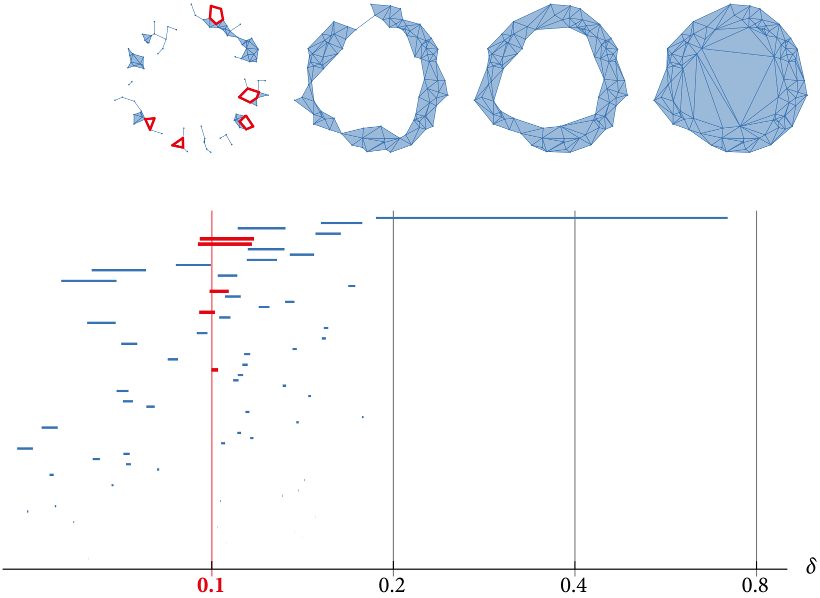

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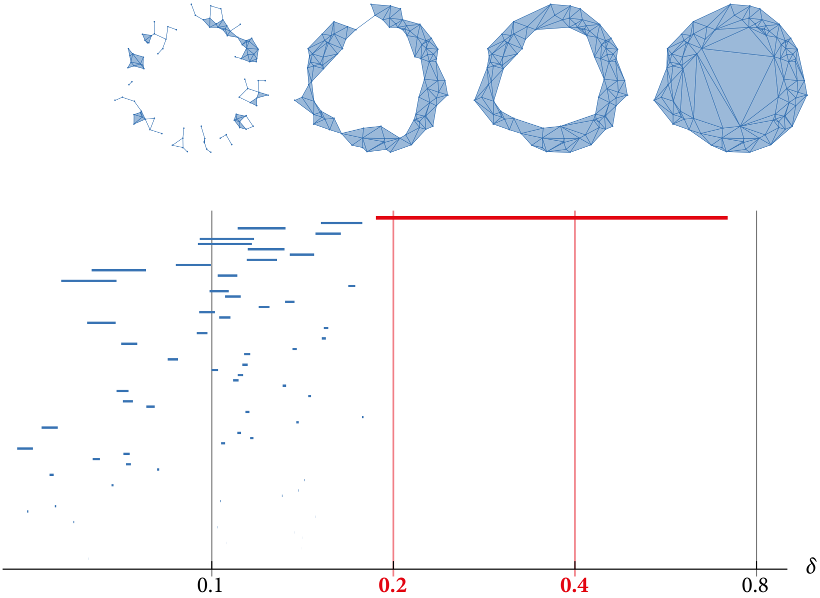

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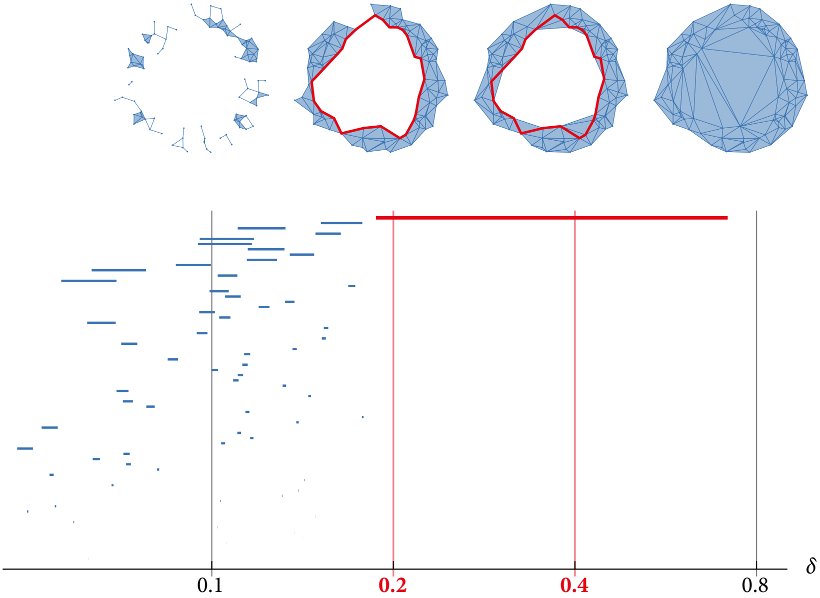

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  - A topological space K_t for each $t \in \mathbb{R}$
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- $\mathbf{R}$ is the poset category of $(\mathbb{R}, \leq)$

When was persistent homology invented?

When was persistent homology invented first?

ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses is satisfied. The critical limits cluster at most at $+\infty$; the critical points are

When was persistent homology invented first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However

When was persistent homology invented first?

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers **[36]** and **[44]** constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.

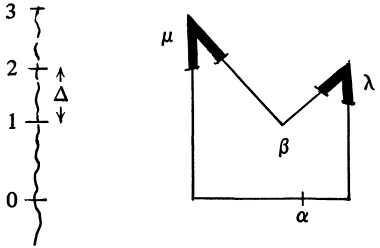
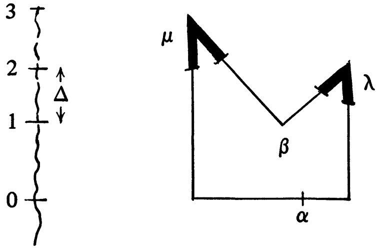


FIGURE 8

The values a where $H(a, a^-) \neq 0$ are indicated on the left, and corresponding to each of these *critical values* a generator of $H(a, a^-)$ is drawn on M, using the singular theory for simplicity. Morse calls such generators "caps". Thus α and β are two "0-caps" and μ and λ two "1-caps". Notice that every cap μ defines a definite boundary element $\partial\mu$ in

When was persistent homology invented first?



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$$H(a^-) = \lim_{\varepsilon \to 0^+} H(F < a - \varepsilon);$$

Morse calls a cap u linkable iff $\partial u = 0$. Otherwise it is called *nonlinkable*.

In our example, α, β and μ are linkable while λ is *not*.

Next Morse defines the *span* of a cap u associated to the critical level a in the following manner.

Inference

Homology inference

Problem (Homology inference)

Determine the homology $H_*(\Omega)$ of a shape $\Omega \subset \mathbb{R}^d$ from a finite approximation P close to Ω.

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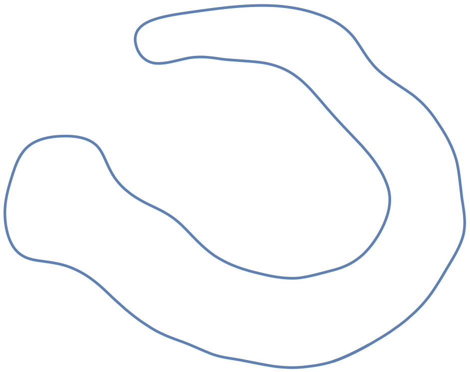
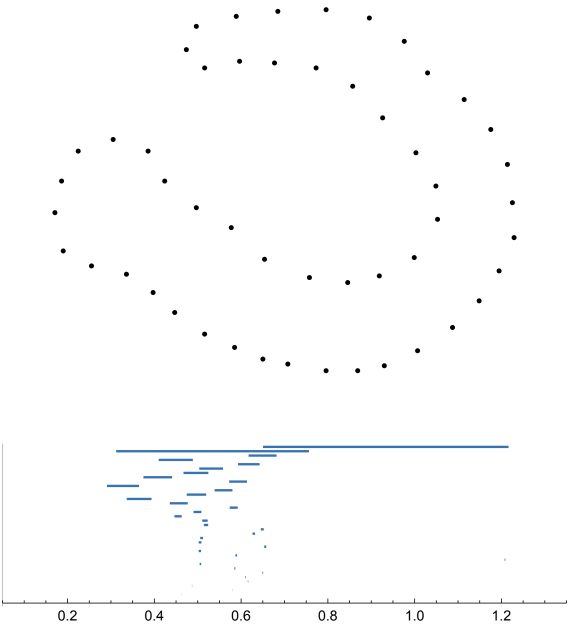
Idea:

- approximate the shape by a thickening $B_\delta(P)$ covering Ω

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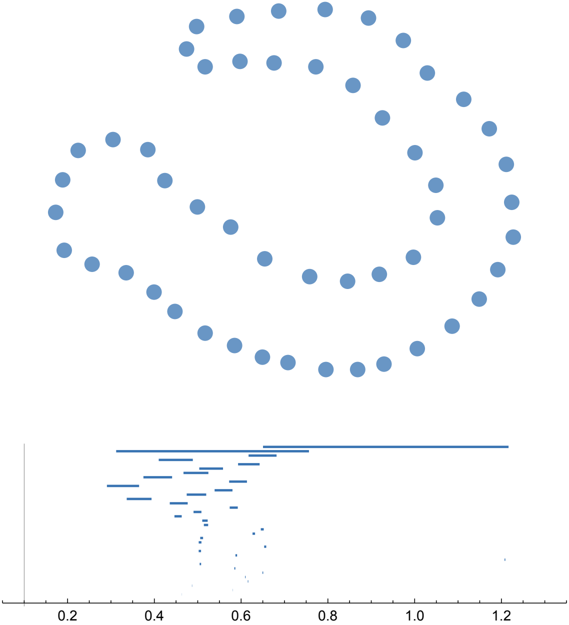
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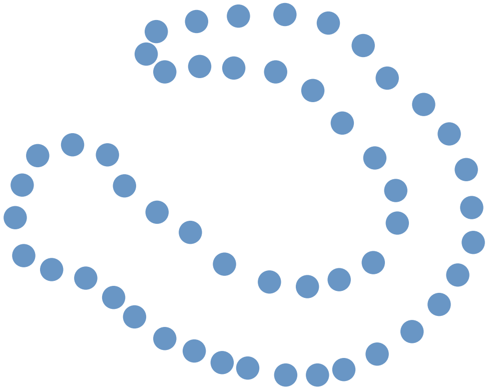

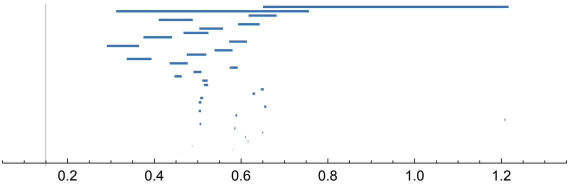
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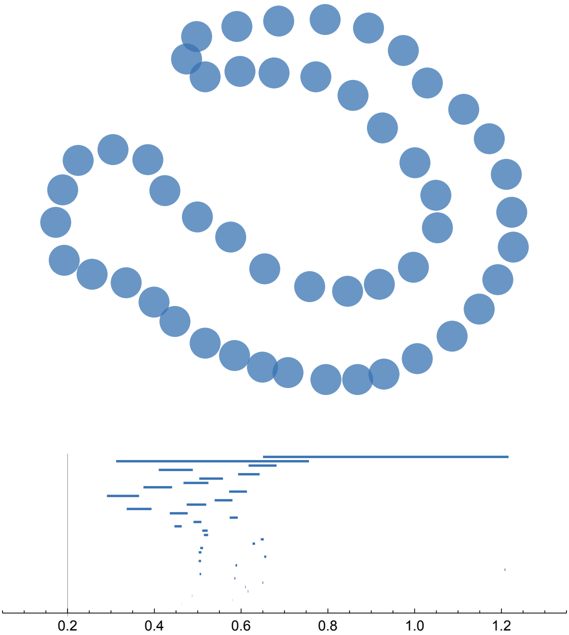
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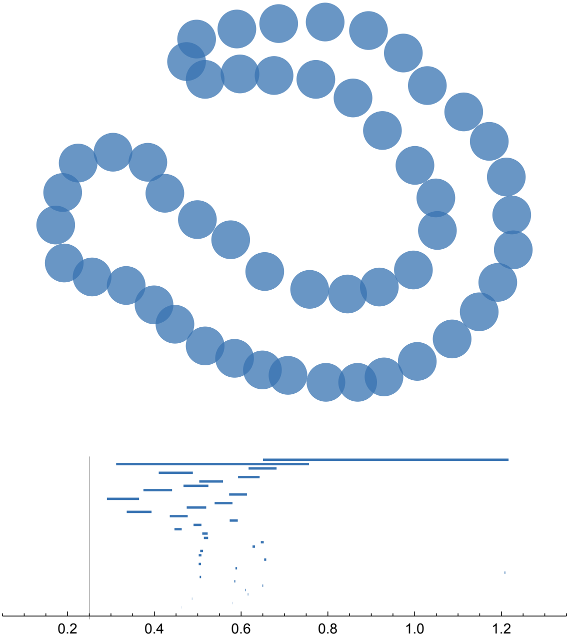
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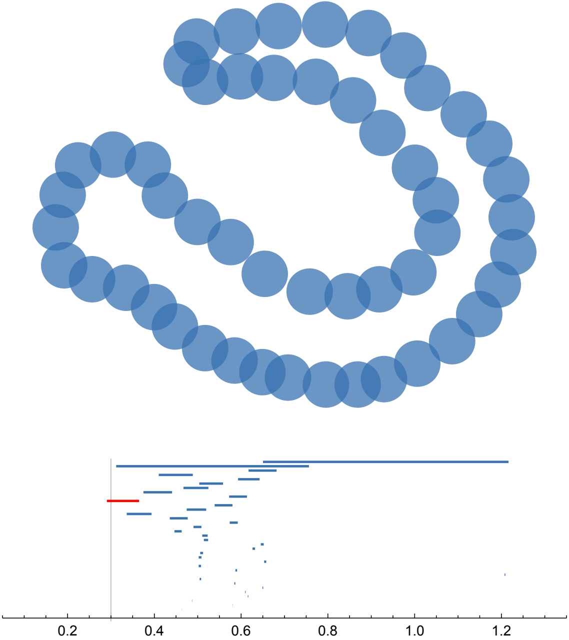

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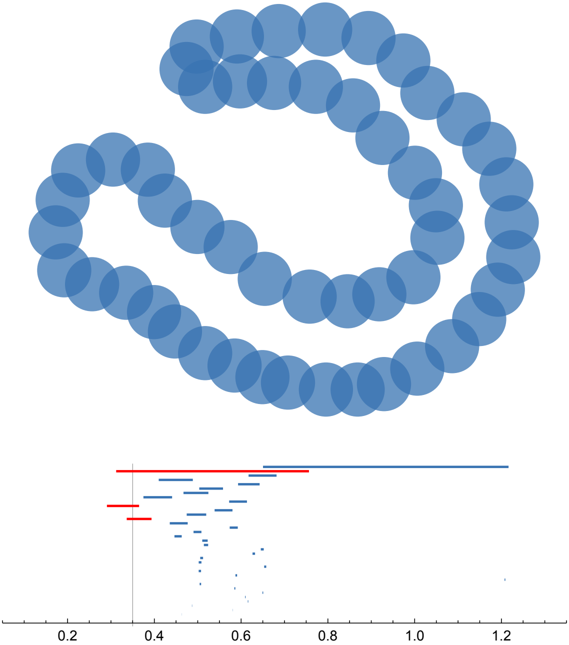

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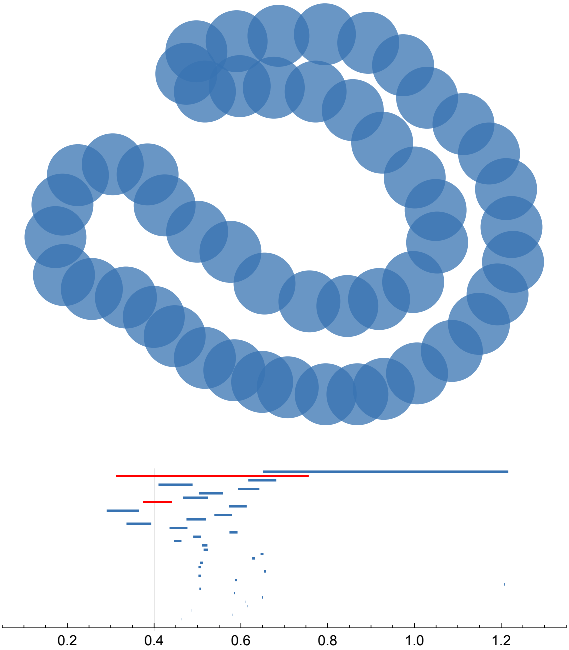
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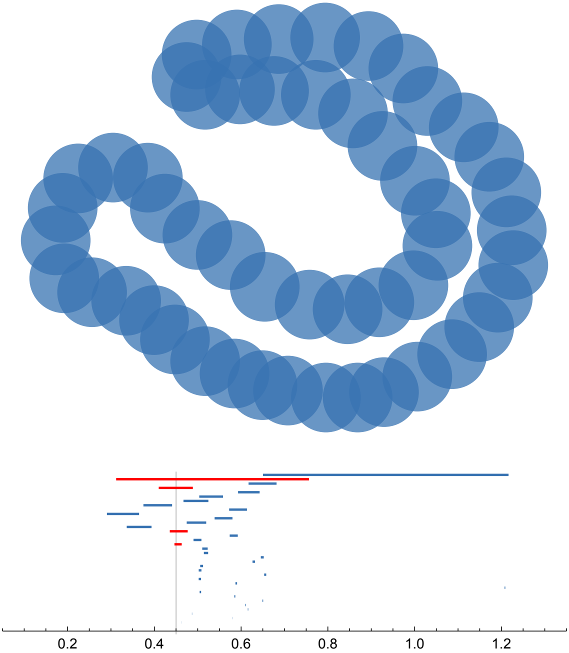
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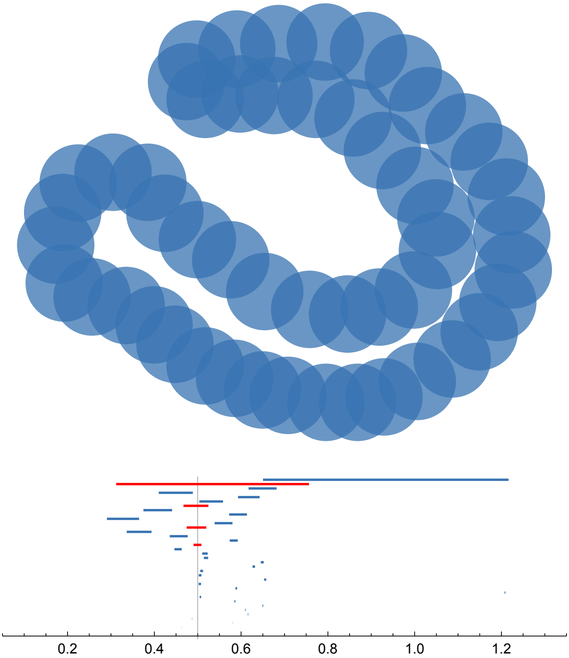
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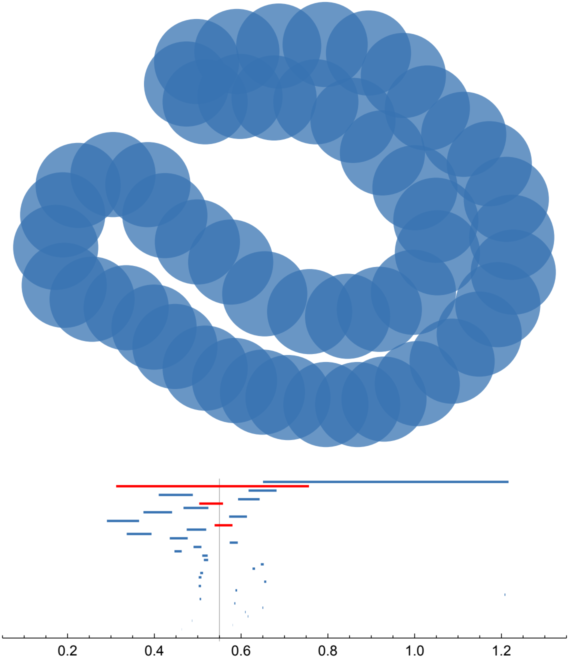
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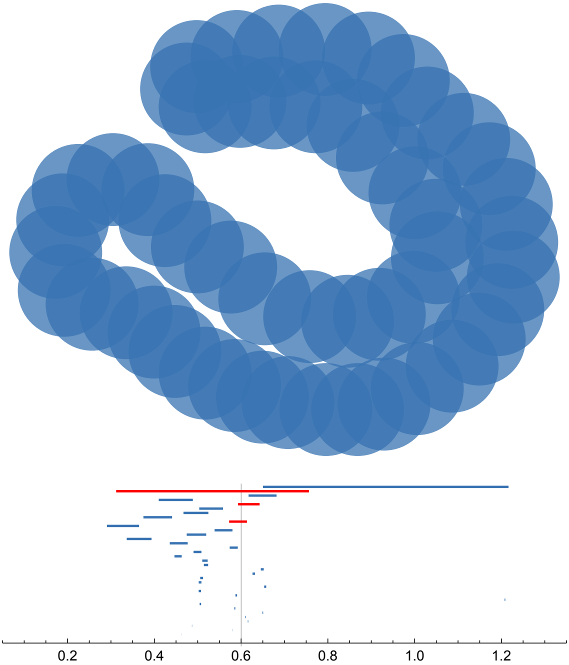
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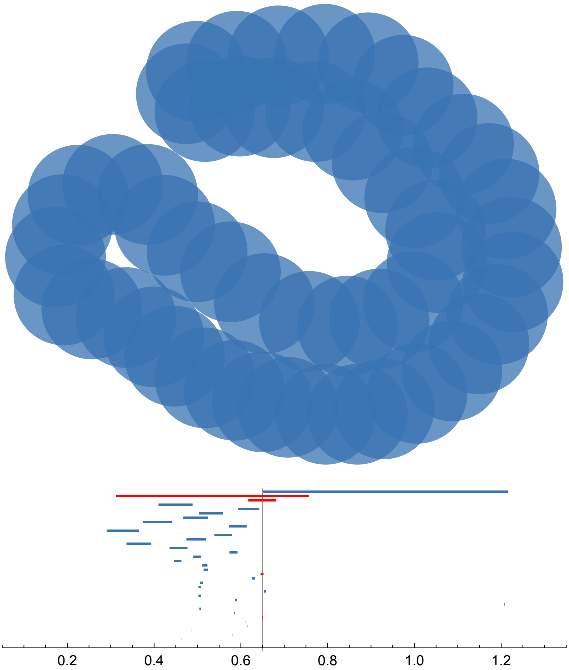
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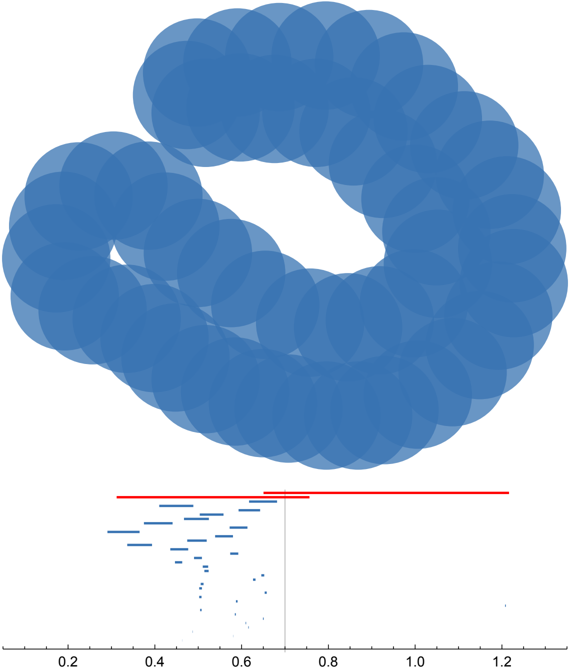

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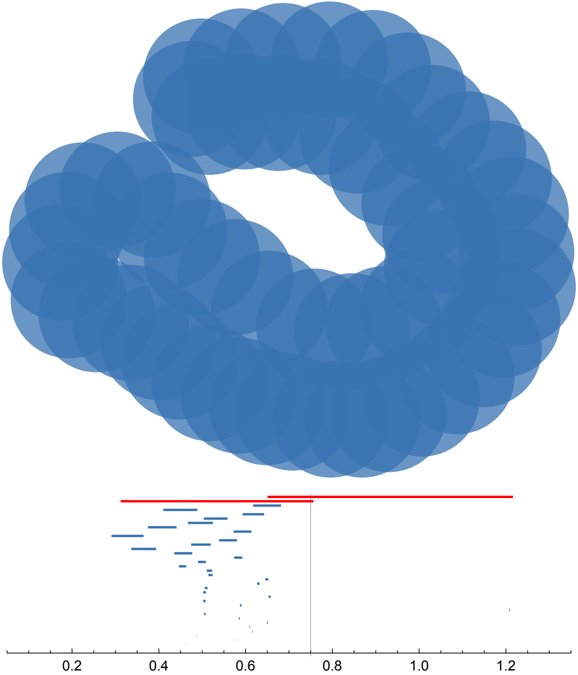
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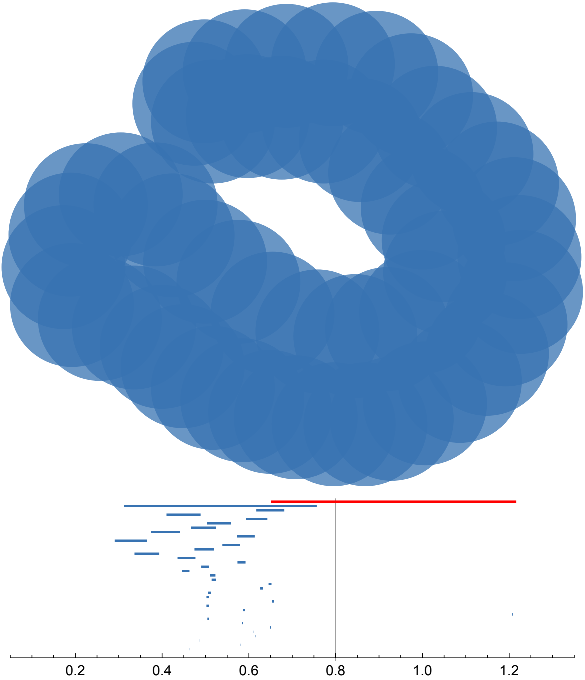
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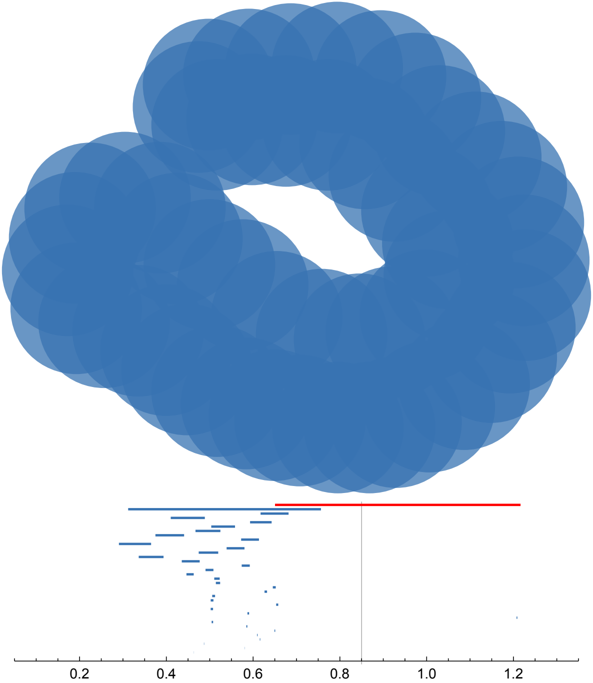

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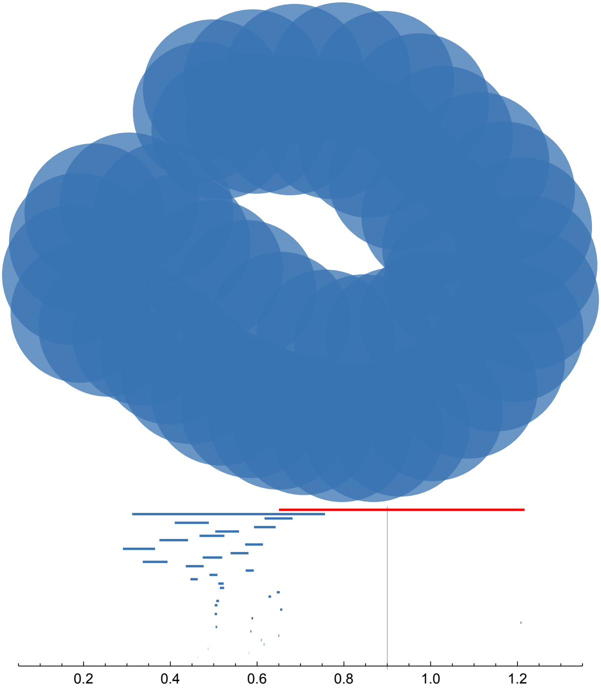

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Homology inference using persistent homology

$P_\delta = B_\delta(P)$: δ-neighborhood (union of balls) around P

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\Omega \subset \mathbb{R}^d$. Let $P \subset \Omega$ be such that

- $\Omega \subseteq P_\delta$ *for some* $\delta > 0$, *and*
- *both* $H_*(\Omega \hookrightarrow \Omega_\delta)$ *and* $H_*(\Omega_\delta \hookrightarrow \Omega_{2\delta})$ *are isomorphisms.*

Then

$$H_*(\Omega) \cong \operatorname{im} H_*(P_\delta \hookrightarrow P_{2\delta}).$$

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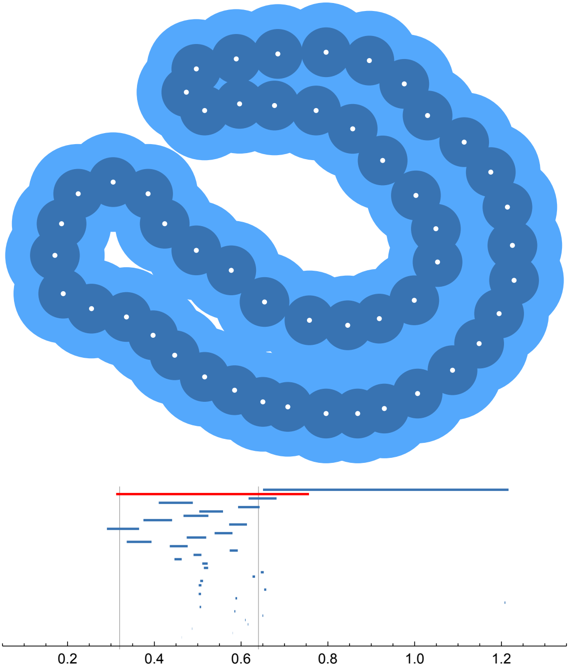
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Proof.

$$
\begin{tikzcd}
\Omega \arrow[dr, hook] & & \Omega_\delta \arrow[dr, hook] & & \Omega_{2\delta} \\
& P_\delta \arrow[ur, hook] & & P_{2\delta} \arrow[ur, hook] &
\end{tikzcd}
$$

Proof.

$$
\begin{tikzcd}[column sep=small]
H_*(\Omega) \arrow[dr] & & H_*(\Omega_\delta) \arrow[dr] & & H_*(\Omega_{2\delta}) \\
& H_*(P_\delta) \arrow[ur] & & H_*(P_{2\delta}) \arrow[ur] &
\end{tikzcd}
$$

Proof.

$$\begin{tikzcd}
H_*(\Omega) \arrow[rr, hook, two heads] \arrow[dr] & & H_*(\Omega_\delta) \arrow[rr, hook] \arrow[dr] & & H_*(\Omega_{2\delta}) \\
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$$
\begin{tikzcd}
H_*(\Omega) \arrow[rr, hook, two heads] \arrow[dr] & & H_*(\Omega_\delta) \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, dashed, leftrightarrow, "\cong" description] & & H_*(\Omega_{2\delta}) \\
& H_*(P_\delta) \arrow[ur, two heads] \arrow[dr, two heads] & & H_*(P_{2\delta}) \arrow[ur] & \\
& & \operatorname{im} H_*(P_\delta \hookrightarrow P_{2\delta}) \arrow[ur, hook] & &
\end{tikzcd}
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Proof.

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\begin{tikzcd}
H_*(\Omega) \arrow[rr, hook, two heads, "\cong"] \arrow[dr] & & H_*(\Omega_\delta) \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, dashed, leftrightarrow, "\cong"] & & H_*(\Omega_{2\delta}) \\
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& & \operatorname{im} H_*(P_\delta \hookrightarrow P_{2\delta}) \arrow[ur, hook] & &
\end{tikzcd}
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□

Stability

The pipeline of topological data analysis

$$
\begin{tikzcd}
\text{point cloud} \arrow[d, "\text{distance}"] \\
\text{function} \arrow[d, "\text{sublevel sets}"] \\
\text{topological spaces} \arrow[d, "\text{homology}"] \\
\text{vector spaces} \arrow[d, "\text{barcode}"] \\
\text{intervals}
\end{tikzcd}
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\text{vector spaces} \arrow[d, "\text{barcode}"] \\
\text{intervals}
\end{tikzcd}
$$

$P \subset \mathbb{R}^d$

The pipeline of topological data analysis

$$\begin{tikzcd}
\text{point cloud} \arrow[d, "\text{distance}"] & P \subset \mathbb{R}^d \arrow[d, "x \mapsto d(x,P)"] \\
\text{function} \arrow[d, "\text{sublevel sets}"] & {} \\
\text{topological spaces} \arrow[d, "\text{homology}"] & \\
\text{vector spaces} \arrow[d, "\text{barcode}"] & \\
\text{intervals} &
\end{tikzcd}$$

The pipeline of topological data analysis

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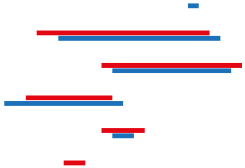
Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

If two functions $f, g : K \to \mathbb{R}$ have distance $\|f - g\|_\infty \leq \delta$, then there exists a δ-matching of their barcodes.



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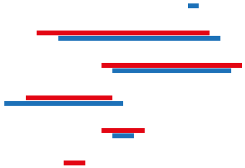

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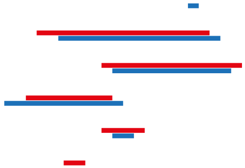

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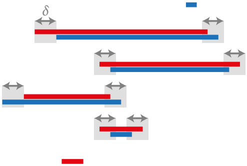

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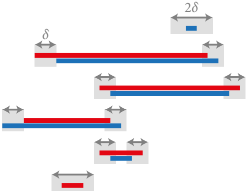

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Stability for functions in the big picture

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Interleavings of sublevel sets

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So the sublevel sets are δ*-interleaved:*

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F_t \arrow[rr, hook] \arrow[dr, hook] & & F_{t+2\delta} \arrow[dr, hook] & \\
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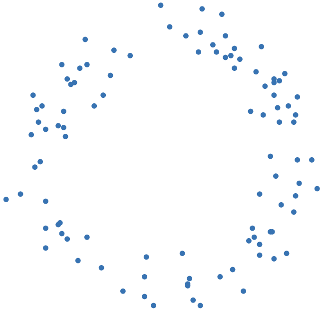
Homology is a *functor:* homology groups are interleaved too.

Geometric interleavings



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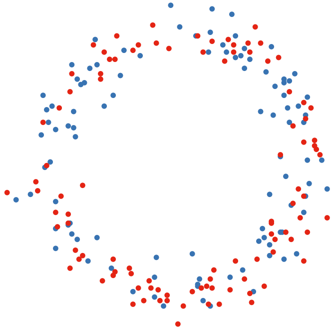

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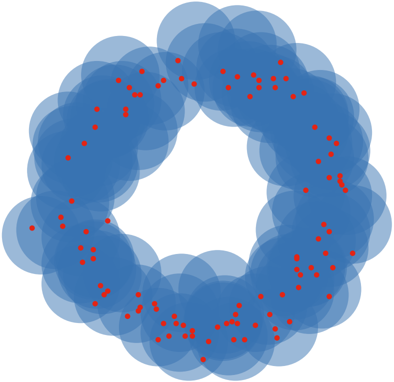

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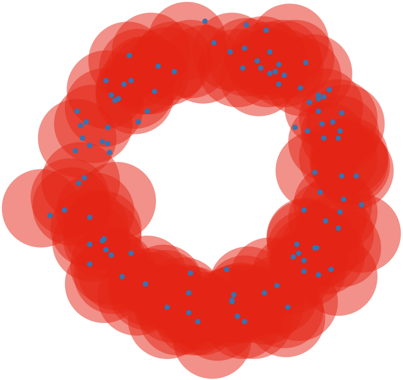

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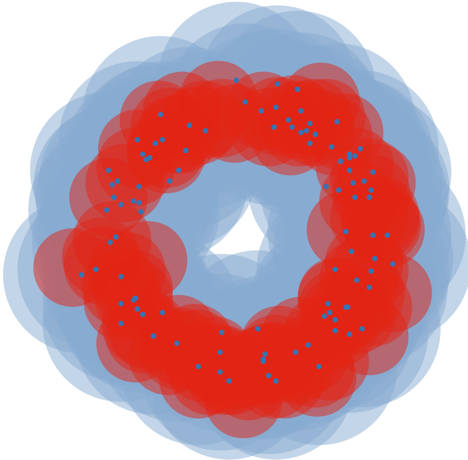

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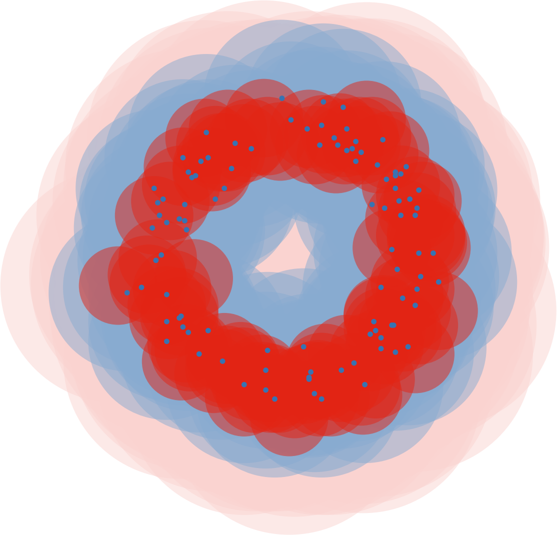

Persistence modules

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- morphism and transition maps commute:

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N_s \arrow[r, "n_s^t"'] & N_t
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Interval Persistence Modules

Let $\mathbb{K}$ be a field. For an arbitrary interval $I \subseteq \mathbb{R}$, define the *interval persistence module* $C(I)$ by

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- Motivates use of homology with field coefficients
- The decomposition itself is not unique

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Two persistence modules M and N are δ-*interleaved*

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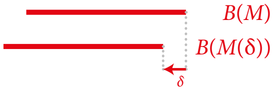
- define $M(\delta)$ by $M(\delta)_t = M_{t+\delta}$ (shift barcode to the left by δ)



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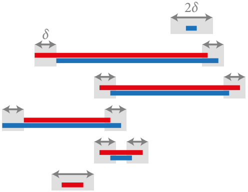

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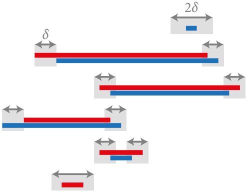

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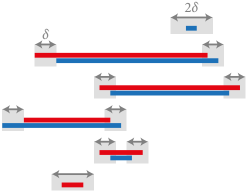

- converse statement also holds (isometry theorem)
- indirect proof, elaborate construction following interpolation argument of original stability theorem

Our approach

Our proof takes a different approach:

- direct proof (no interpolation, matching immediately from interleaving)

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- direct proof (no interpolation, matching immediately from interleaving)
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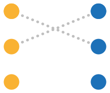
- stability follows from properties of a *single morphism*, not just from a pair of morphisms
- relies on *partial functoriality* of the induced matching

Induced matchings

The matching category

A *matching* $\sigma : S \nrightarrow T$ is a bijection $S' \to T'$, where $S' \subseteq S$, $T' \subseteq T$.



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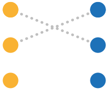
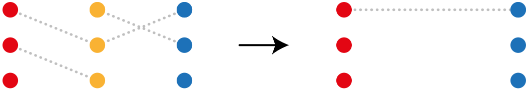


Composition of matchings $\sigma : S \nrightarrow T$ and $\tau : T \nrightarrow U$:



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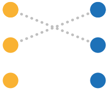

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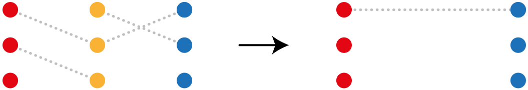
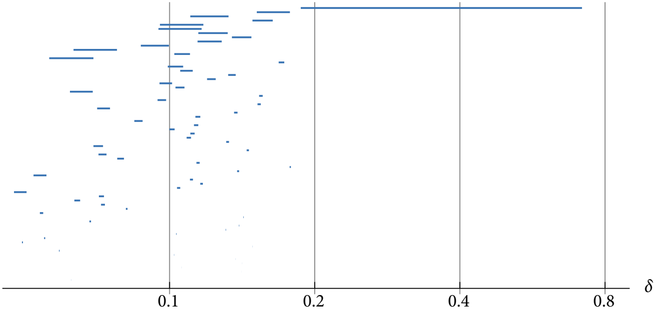


Matchings form a category $\mathbf{Mch}$

- objects: sets
- morphisms: matchings

Barcodes as matching diagrams

We can regard a barcode B as a functor $\mathbf{R} \to \mathbf{Mch}$:



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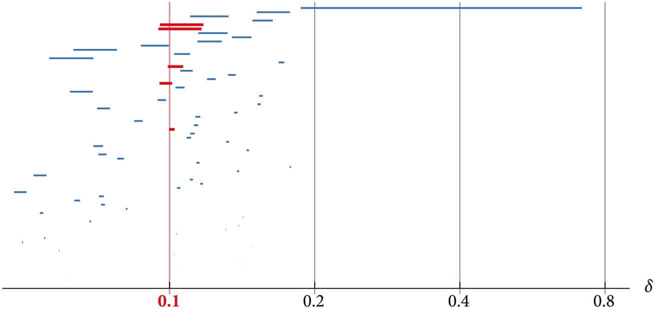
- For each real number t, let B_t be those intervals of B that contain t, and



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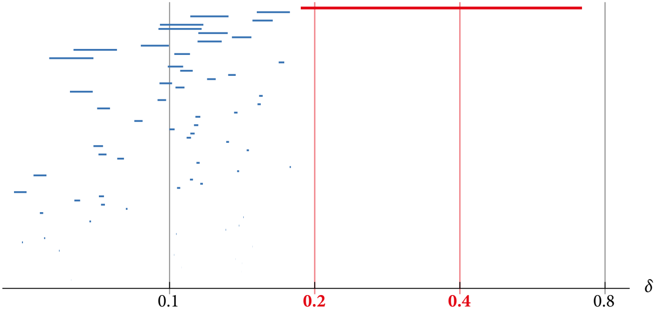
- For each real number t, let B_t be those intervals of B that contain t, and
- for each $s \leq t$, define the matching $B_s \nrightarrow B_t$ to be the identity on $B_s \cap B_t$.



Barcode matchings as natural transformations

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$$\begin{tikzcd} A_s \arrow[r, "\shortmid" marking] \arrow[d, "\sigma_s"', "\shortmid" marking] & A_t \arrow[d, "\sigma_t", "\shortmid" marking] \\ B_s \arrow[r, "\shortmid" marking] & B_t \end{tikzcd}$$

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- requirement on the matching σ:
  if $I \in A$ is matched to $J \in B$, then I overlaps J to the right.

Barcode matchings as interleavings

We can regard a δ-matching of barcodes $\sigma : A \nrightarrow B$
as a δ-interleaving of functors $\mathbf{R} \to \mathbf{Mch}$:

$$
\begin{tikzcd}
A_t \arrow[rr, "\shortmid" marking] \arrow[dr, "\shortmid" marking, "\sigma_t"'] & & A_{t+2\delta} \arrow[dr, "\shortmid" marking, "\sigma_{t+2\delta}"] & \\
& B_{t+\delta} \arrow[rr, "\shortmid" marking] & & B_{t+3\delta}
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- each matching $\sigma_t : A_t \nrightarrow B_{t+\delta}$ is the restriction of σ, yielding a natural transformation $A \to B(\delta)$

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- each matching $\sigma^\dagger_t : B_t \nrightarrow A_{t+\delta}$ is the restriction of the reverse matching $\sigma^\dagger : B \nrightarrow A$

Stability via functoriality?

$$\begin{tikzcd}
F_t \arrow[rr, hook] \arrow[dr, hook] & & F_{t+2\delta} \arrow[dr, hook] & \\
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$$
\begin{tikzcd}
H_*(F_t) \arrow[rr] \arrow[dr] & & H_*(F_{t+2\delta}) \arrow[dr] & \\
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$$\begin{tikzcd}[column sep=small]
B(H_*(F_t)) \arrow[rr] \arrow[dr] & & B(H_*(F_{t+2\delta})) \arrow[dr] & \\
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Non-functoriality of the persistence barcode

Theorem (B, Lesnick 2014)

There exists no functor $\mathbf{Vect}^{\mathbf{R}} \to \mathbf{Mch}$ *sending each persistence module to its barcode.*

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- Such a functor would necessarily send a linear map of rank r to a matching of cardinality r.
- In particular, there is no natural choice of basis for vector spaces

Structure of submodules and quotient modules

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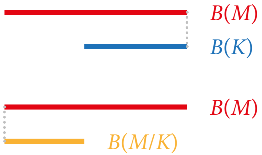

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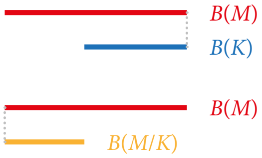

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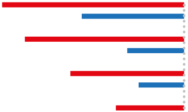
- If multiple bars have same endpoint: match in order of decreasing length



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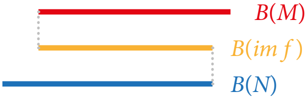

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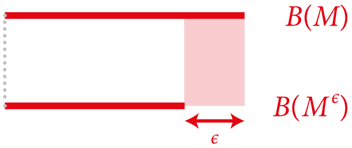
Similar for surjections.

The induced matching theorem

Define M^{ϵ} by shrinking bars of $B(M)$ from the right by ϵ:

$M^{\epsilon}_t = M_t / \ker(M_t \to M_{t+\epsilon})$.



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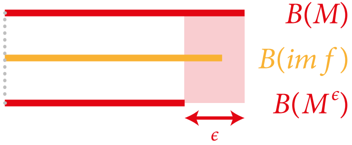
Let $f : M \to N$ *be such that* $\ker f$ *is* ϵ-trivial: $(\ker f)^\epsilon = 0$.

Then M^ϵ *is a quotient module of* $\operatorname{im} f$.



$$\begin{tikzcd} & M \arrow[dl, two heads] \arrow[dr, two heads] & \\ M^\epsilon & & \operatorname{im} f \arrow[ll, two heads] \end{tikzcd}$$

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Define ${}^{\epsilon}N$ by shrinking bars of $B(N)$ from the left by ϵ:

${}^{\epsilon}N_t = \operatorname{im}(N_{t-\epsilon} \to N_t)$.

Lemma

Let $f : M \to N$ *be such that* $\operatorname{coker} f$ *is* ϵ-trivial: $(\operatorname{coker} f)^{\epsilon} = 0$.

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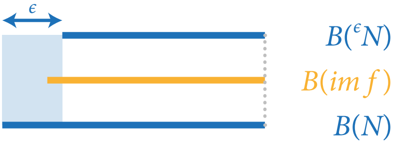

$$\begin{tikzcd}
 & N & \\
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\end{tikzcd}$$

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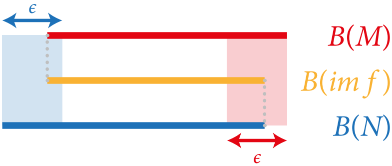

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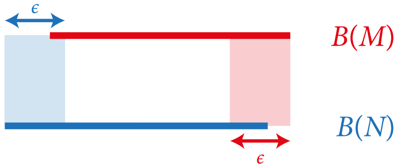

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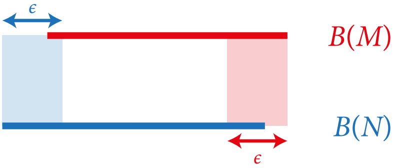

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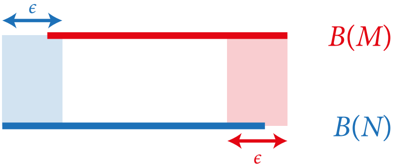
Then each interval of length $\geq \epsilon$ *is matched by* $B(f)$*.*



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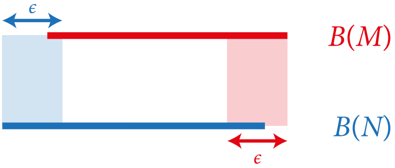
If $B(f)$ *matches* $[b, d) \in B(M)$ *to* $[b', d') \in B(N)$*, then*

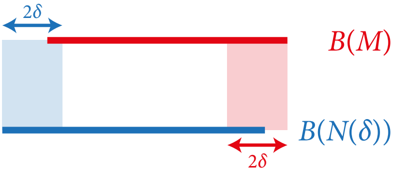
$b' \leq b \leq b' + \epsilon$ *and* $d - \epsilon \leq d' \leq d$.



The induced matching theorem

Let $f : M \to N(\delta)$ be an interleaving morphism.

Then $\ker f$ and $\operatorname{coker} f$ are 2δ-trivial.



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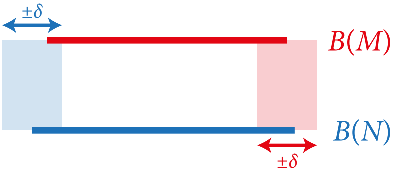
Corollary (Algebraic stability via induced matchings)

A δ-interleaving between persistence modules induces a δ-matching of their persistence barcodes.



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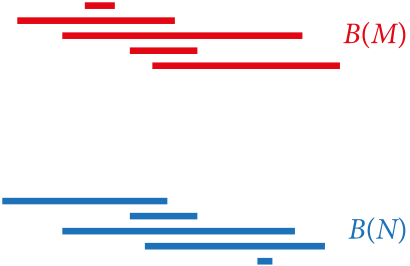

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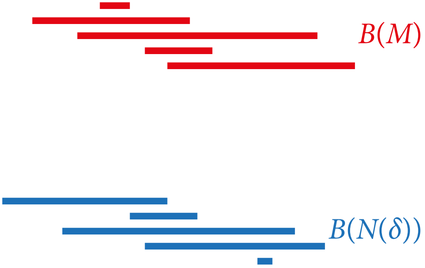

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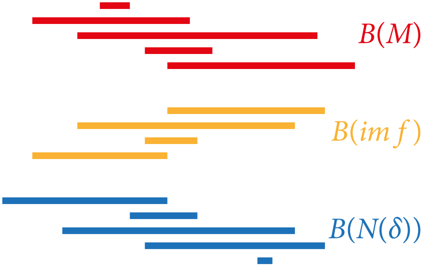

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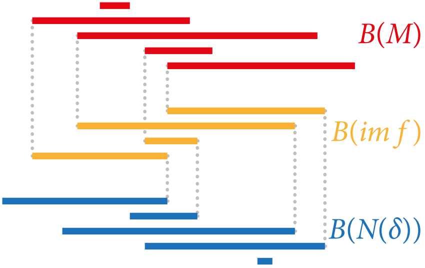

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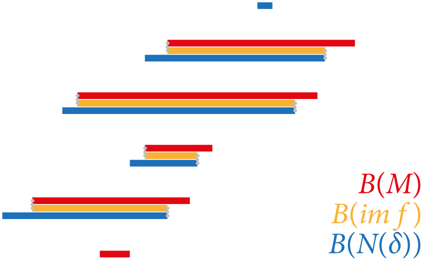

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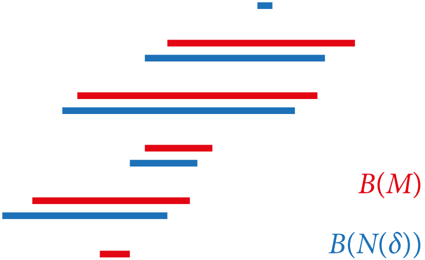

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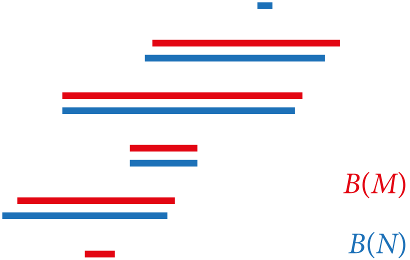
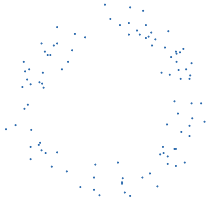


Application: connecting different point clouds

X



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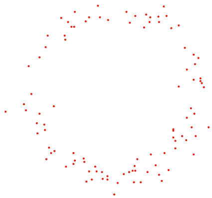

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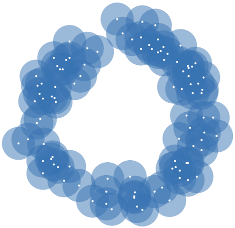
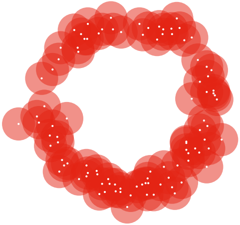


$B_r(Y)$



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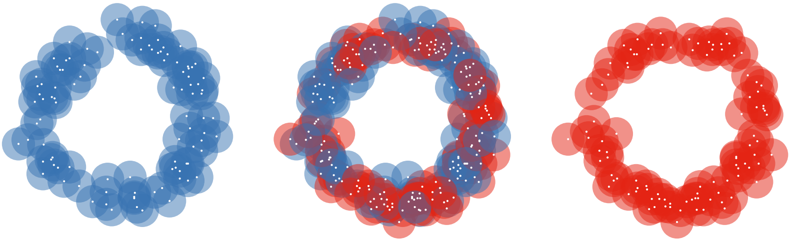
$$B_r(X) \hookrightarrow B_r(X \cup Y) \hookleftarrow B_r(Y)$$



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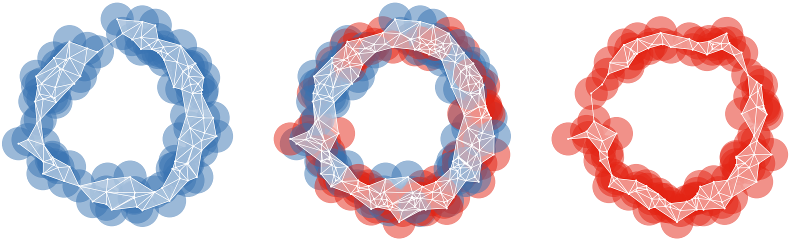
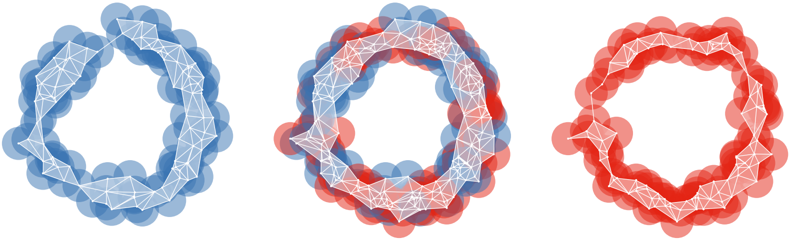


$\mathrm{Del}_r(X)$ $\qquad$ $\mathrm{Del}_r(X \cup Y)$ $\qquad$ $\mathrm{Del}_r(Y)$

Application: connecting different point clouds

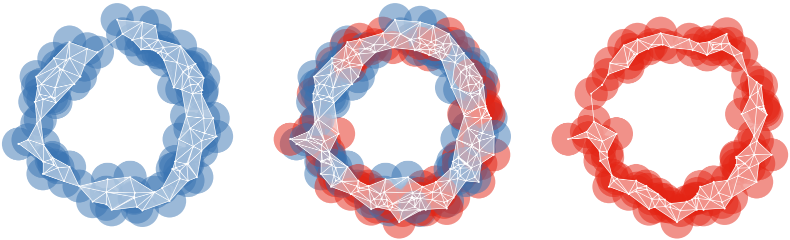
$$B_r(X) \hookrightarrow B_r(X \cup Y) \hookleftarrow B_r(Y)$$



$$\begin{tikzcd}[column sep=small]
\mathrm{Del}_r(X) \arrow[dr, hook] & & \mathrm{Del}_r(X \cup Y) \arrow[dl, hook] \arrow[dr, hook] & & \mathrm{Del}_r(Y) \arrow[dl, hook] \\
& \mathrm{Del}_r(X \cup Y, X) & & \mathrm{Del}_r(X \cup Y, Y) &
\end{tikzcd}$$

Application: connecting different point clouds

$$B_r(X) \hookrightarrow B_r(X \cup Y) \hookleftarrow B_r(Y)$$



$$\begin{tikzcd}
\mathrm{Del}_r(X) \arrow[dr, hook] & & \mathrm{Del}_r(X \cup Y) \arrow[dl, hook, "\simeq"'] \arrow[dr, hook, "\simeq"] & & \mathrm{Del}_r(Y) \arrow[dl, hook] \\
& \mathrm{Del}_r(X \cup Y, X) & & \mathrm{Del}_r(X \cup Y, Y) &
\end{tikzcd}$$