The Reeb Graph Edit Distance is Universal

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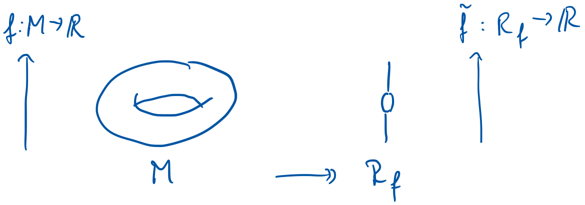
"Here are two things that are reasonably close to each other, and I want to compare them."

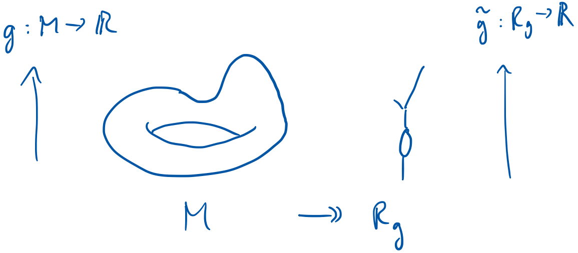
S. WEINBERGER, FoCM 17

REEB GRAPHS



identify components of level sets $f^{-1}(t)$: $\quad R_f = M/\sim_f$, where

$x \sim_f y \iff x, y$ in same component of $f^{-1}(t)$, $t \in \mathbb{R}$.



GOALS

How to compare two Reeb graphs R_f, R_g? ($f, g : M \to \mathbb{R}$ are unknown)

Assign distance (extended pseudo-metric) $d(R_f, R_g)$.

Desirable properties:

<u>Stability</u>: For <u>any</u> space X and $f, g : X \to \mathbb{R}$ yielding Reeb graphs R_f, R_g,

$$d(R_f, R_g) \leq \|f - g\|_\infty .$$

<u>Universality</u>: For any other stable distance d_0,

$$d_0(R_f, R_g) \leq d(R_f, R_g) .$$

FORMAL SETTING

We consider

- locally compact Hausdorff spaces (<u>Reeb domains</u>)
- proper quotient maps with connected fibers (<u>Reeb quotient maps</u>)

This forms a category (which has all finite limits).

Define a <u>Reeb graph</u> as

- a Reeb domain R_f with
- a function $\hat{f} : R_f \to \mathbb{R}$ with discrete fibers (<u>Reeb function</u>)

R_f is <u>the Reeb graph</u> of $f : X \to \mathbb{R}$ if

- $f = \hat{f} \circ p$ for some Reeb quotient map $p : X \twoheadrightarrow R_f$.
- In this case, $R_f \cong X/\sim_f$.

Moreover: let $q : Y \twoheadrightarrow X$ be a Reeb quotient map.

Then R_f is also the Reeb graph of $f \circ q$.

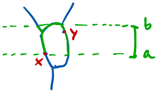
$$\begin{tikzcd} Y \arrow[r, two heads, "q"'] & X \arrow[d, two heads, "p"'] \arrow[dr, "f"] & \\ & R_f \arrow[r, "\tilde{f}"'] & \mathbb{R} \end{tikzcd}$$

- Reeb quotient maps preserve Reeb graphs.

PREVIOUS WORK: FUNCTIONAL DISTORTION DISTANCE [B., Ge, Wang 2014]

- On a Reeb graph R_f with $\tilde{f} : R_f \to \mathbb{R}$, consider the metric

  $d_f : (x, y) \mapsto \inf \{ b - a \mid x, y$ are in same component of $\tilde{f}^{-1}[a, b] \}$.



- Given maps $\phi : R_f \to R_g$, $\psi : R_g \to R_f$, consider

  $$G(\phi, \psi) = \{ (x, \phi(x)) \mid x \in R_f \} \cup \{ (\psi(y), y \mid y \in R_g \}$$

- Define the <u>distortion</u> of (ϕ, ψ) as

  $$D(\phi, \psi) = \sup_{(x,y), (\tilde{x}, \tilde{y}) \in G(\phi, \psi)} \frac{1}{2} \left| d_f(x, \tilde{x}) - d_g(y, \tilde{y}) \right|$$

- Define the <u>functional distortion distance</u> as

  $$d_{FD}(R_f, R_g) = \inf_{\phi, \psi} \left(\max \{ D(\phi, \psi), \|f - g \circ \phi\|_\infty, \|g - f \circ \psi\|_\infty \right).$$

PREVIOUS WORK: INTERLEAVING DISTANCE [Bubenik, de Silva, Scott 2015; de Silva, Munch, Patel 2016]

- Interpret Reeb graph R_f as a functor $F : \mathrm{Int}_{\mathbb{R}} \to \mathrm{Set}$, $I \mapsto \pi_0(\tilde{f}^{-1}(I))$ ($\mathrm{Int}_{\mathbb{R}}$ are the open intervals, as a poset wrt. $\subseteq$)

- A <u>δ-interleaving</u> between F and G is a pair of natural transformations φ, ψ (with components $\varphi_I : F(I) \to G(B_\delta(I)), \ldots$) such that

$$\begin{tikzcd}
F(I) \arrow[r] \arrow[dr, "\varphi_I"'] & F(B_\delta(I)) \arrow[r] \arrow[dr, "\varphi_{B_\delta(I)}"] & F(B_{2\delta}(I)) \\
G(I) \arrow[r] \arrow[ur, "\psi_I"] & G(B_\delta(I)) \arrow[r] \arrow[ur, "\psi_{B_\delta(I)}"'] & G(B_{2\delta}(I))
\end{tikzcd}$$

commutes for all $I \in \mathrm{Int}_{\mathbb{R}}$.

(unlabeled maps induced by inclusion)

- The <u>interleaving distance</u> is

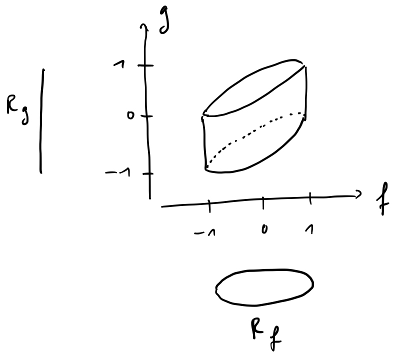
$$d_I(F, G) = \inf \{ \delta \mid \exists\ \delta\text{-interleaving between } F \text{ and } G \}$$

$$d_{I,\pi_0}(R_f, R_g) := d_I(\pi_0 \circ \tilde{f}^{-1}, \pi_0 \circ \tilde{g}^{-1})$$

<u>Thm</u> [B., Munch, Wang 2015] $\frac{1}{3} d_{FD} \le d_{I,\pi_0} \le d_{FD}$.

NON-UNIVERSALITY OF FUNCTIONAL DISTORTION & INTERLEAVING DISTANCES

Consider a cylinder with two functions f, g.



We have:

- $\|f-g\|_\infty = 1$
- $d_{I,H_0}(R_f, R_g) := d_I(H_0 \circ \tilde{f}^{-1}, H_0 \circ \tilde{g}^{-1}) = 1$ (this is a stable distance)
- $d_{I,\pi_0}(R_f, R_g) \leq d_{FD}(R_f, R_g) \leq \frac{1}{2}$:

A CANONICAL UNIVERSAL DISTANCE

Given Reeb graphs R_f, R_g with functions $\tilde{f}$, $\tilde{g}$, define

$$d_u(R_f, R_g) = \inf_{\substack{X \\ p_f \swarrow \quad \searrow p_g \\ R_f \qquad R_g}} \| \tilde{f} \circ p_f - \tilde{g} \circ p_g \|_\infty$$

taken over all Reeb domains X and Reeb quotient maps p_f, p_g.

This is a distance (triangle inequality): consider pullback

$$\begin{tikzcd}
 & & X \times_{R_g} Y \arrow[dl] \arrow[dr] & & \\
 & X \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & \\
R_f \arrow[d, "f"'] & & R_g \arrow[d, "g"'] & & R_h \arrow[d, "h"'] \\
\mathbb{R} & & \mathbb{R} & & \mathbb{R}
\end{tikzcd}$$

- stability and universality immediate from definition
- working with arbitrary spaces X is unfeasible

THE TOPOLOGICAL EDIT DISTANCE

- Consider zig-zag diagrams Z of Reeb quotient maps

$$\begin{tikzcd}
R_f = R_1 & & R_2 & & R_3 & \cdots & R_n = R_g \\
& X_1 \arrow[ul, two heads] \arrow[ur, two heads] & & X_2 \arrow[ul, two heads] \arrow[ur, two heads] & \cdots & X_{n-1} \arrow[ur, two heads] & \\
& & & L \arrow[ull, two heads] \arrow[u, two heads] \arrow[urr, two heads] & & &
\end{tikzcd}$$

and take the limit L (note: all maps are Reeb quotient maps).
Let q_i be the unique map $L \twoheadrightarrow R_i$ in this diagram, and $f_i = \tilde{f}_i \circ q_i$.

- Define the <u>spread</u> of the functions $f_1, \dots, f_n : L \twoheadrightarrow R_n \to \mathbb{R}$ as

$$s_Z : L \to \mathbb{R}, \qquad x \mapsto \max_i f_i(x) - \min_j f_j(x).$$

- Define the <u>topological edit distance</u> as

$$d_{eTop}(R_f, R_g) = \inf_Z \|s_Z\|_\infty .$$

<u>Prop.</u> d_{eTop} is stable and universal.

THE REEB GRAPH EDIT DISTANCE

- Consider zig-zag diagrams Z of Reeb quotient maps

$$\begin{tikzcd}[column sep=small]
R_f = R_1 & & R_2 & & R_3 & \cdots & R_n = R_g \\
& G_1 \arrow[ul, two heads] \arrow[ur, two heads] & & G_2 \arrow[ul, two heads] \arrow[ur, two heads] & \cdots & G_{n-1} \arrow[ur, two heads]
\end{tikzcd}$$

as before, but restrict R_i, G_j to finite graphs.

Interpretation:

$$\begin{tikzcd}
& \mathbb{R} & \\
& R_{i+1} \arrow[u, "\tilde{f}_i"'] & \\
G_i \arrow[ur, two heads] & & G_{i+1} \arrow[ul, two heads]
\end{tikzcd}
\qquad \text{edit}
\qquad\qquad
\begin{tikzcd}
\mathbb{R} & & \mathbb{R} \\
R_i \arrow[u, "\tilde{f}_i"] & & R_{i+1} \arrow[u, "\tilde{f}_{i+1}"'] \\
& G_i \arrow[ul, two heads] \arrow[ur, two heads] &
\end{tikzcd}
\qquad \text{relabel}$$

modify G_i to G_{i+1}, maintaining the Reeb graph R_{i+1}

modify f_i to $f_{i+1} : G_i \to \mathbb{R}$, maintaining the graph G_i

- Define the <u>Reeb graph edit distance</u> analogously (using the limit) as

$$d_{eGraph}(R_f, R_g) = \inf_Z \|s_Z\|_\infty .$$

MAIN RESULT

Thm [B., Landi] The Reeb graph edit distance is stable & universal.

- We restrict to the PL category here.
- The hard part is stability:
  given $f, g : K \to \mathbb{R}$ (PL), how to construct an edit zigzag between R_f and R_g with cost $\leq \|f-g\|_\infty$?
- Idea:
  - consider straight-line homotopy $f_\lambda = \lambda f + (1-\lambda) g$
  - The structure of $R_\lambda = R_{f_\lambda}$ changes only finitely often (say, at parameters $0 = \lambda_0 < \dots < \lambda_n = 1$). Choose $p_i \in (\lambda_i, \lambda_{i+1})$.
  - Construct zigzag $R_f = R_0 \leftarrow \dots \rightarrow R_{\lambda_i} \leftarrow R_{p_i} \rightarrow R_{\lambda_{i+1}} \leftarrow \dots \rightarrow R_n = R_g$
  - But what are the maps here?

THE WORKHORSE

Assume that there is a monotonic PL surjection $\chi : \mathrm{im}\, f \twoheadrightarrow \mathrm{im}\, g$ such that

$$g(v) = \chi \circ f(v) \text{ for every vertex } v \text{ of } K.$$

Then g is not the same as $h = \chi \circ f$, but has the same Reeb graph!

Construction:

$$\begin{tikzcd}
\mathrm{im}\, f \arrow[rr, two heads, "\chi"] & & \mathrm{im}\, g \\
R_f \arrow[u, "\tilde{f}"] \arrow[r, gray, two heads] & R_h \arrow[ur, "\tilde{h}"] \arrow[r, gray] & R_g \arrow[u, "\tilde{g}"'] \\
K \arrow[u, two heads, "q_f"] \arrow[ur, two heads, "q_h"'] & & K \arrow[u, two heads, "q_g"'] \arrow[ul, dotted, two heads, "k"]
\end{tikzcd}$$

Lemma The relation

$$k = q_h \circ \left((h^{-1} \circ g) \cap \mathrm{st}_K \right)$$

is a Reeb quotient map.

This provides the maps $R_{\lambda_i} \twoheadleftarrow R_{p_i} \twoheadrightarrow R_{\lambda_{i+1}}$ in our zigzag.

SUMMARY

- Interleaving and functional distortion distances are not universal
- There is a simple construction of a universal distance between Reeb graphs
- A universal distance in PL can be constructed using graph edit zigzags
- What is the complexity of computing the distance?

- Further questions?