

# Persistence is Morse theory

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July 26, 2016

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# Persistence is (not) Morse theory

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# Persistence is Morse theory... NOT!

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# Archaeology of persistence

# When was persistent homology invented?

- [Edelsbrunner/Letscher/Zomorodian 2000]

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- [Leray 1946]?

When was persistent homology invented first?

# When was persistent homology invented first?

ANNALS OF MATHEMATICS  
Vol. 41, No. 2, April, 1940

## RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

### 1. Introduction.

The analysis of functions  $F$  on metric spaces  $M$  of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative  $k$ -cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of  $k$ -limits suitably counted is called the  $k^{\text{th}}$  type number  $m_k$  of  $F$ . The theory seeks to establish relations between the numbers  $m_k$  and the connectivities  $p_k$  of  $M$ . The numbers  $p_k$  are finite in the most important applications. It is otherwise with the numbers  $m_k$ .

The theory has been able to proceed provided one of the following hypotheses is satisfied. The critical limits cluster at most at  $+\infty$ ; the critical points are isolated;<sup>1</sup> the problem is defined by analytic functions; the critical limits taken in their natural order are well-ordered. These conditions are not generally fulfilled. The generality of the theory rested upon the fact that the cases treated approximate in a certain sense the most general problems which it is

# When was persistent homology invented first?

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Exact homomorphism sequences in homology theory ed.ac.uk [PDF]

JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR

The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] ...)

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Unstable minimal surfaces of higher topological structure ams.org [PDF]

R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org

American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to ...

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Persistence in discrete Morse theory psu.edu [PDF]

U Bauer - 2011 - Citeseer

The goal of this thesis is to bring together two different theories about critical points of a scalar function and their relation to topology: Discrete Morse theory and Persistent

# When was persistent homology invented first?

BULLETIN (New Series) OF THE  
AMERICAN MATHEMATICAL SOCIETY  
Volume 3, Number 3, November 1980

## MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT<sup>1</sup>

**1. Introduction.** Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters<sup>2</sup> as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was in the 1920's to go from the mini-max principle of Birkhoff to the Morse inequalities, let alone extend these inequalities to function spaces, so that by the early 30's Morse could establish the theorem that for any Riemann

# When was persistent homology invented first?

inequalities between the dimensions of the  $H_i$  and those of  $H(A_i)$ . Thus the Morse inequalities already reflect a certain part of the “Spectral Sequence magic”, and a modern and tremendously general account of Morse’s work on rank and span in the framework of Leray’s theory was developed by Deheuvels [D] in the 50’s.

Unfortunately both Morse’s and Deheuvel’s papers are not easy reading. On the other hand there is no question in my mind that the papers [36] and [44] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

In the figure which follows I have drawn a homeomorph of  $M = S^1$  in the plane, and I will be studying the height function  $F = y$  on  $M$ .

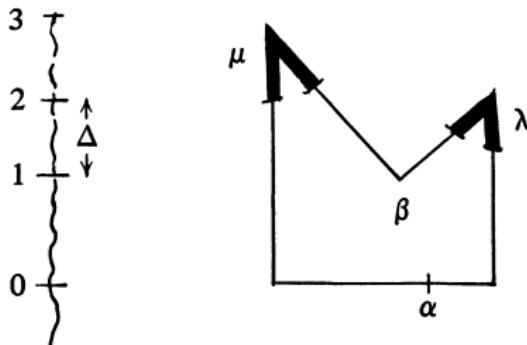


FIGURE 8

The values  $a$  where  $H(a, a^-) \neq 0$  are indicated on the left, and correspond

# When was persistent homology invented first?

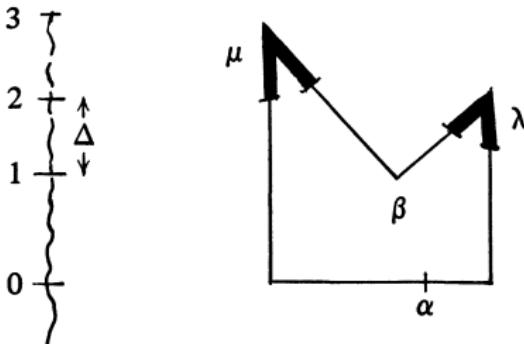


FIGURE 8

The values  $a$  where  $H(a, a^-) \neq 0$  are indicated on the left, and corresponding to each of these *critical values* a generator of  $H(a, a^-)$  is drawn on  $M$ , using the singular theory for simplicity. Morse calls such generators “caps”. Thus  $\alpha$  and  $\beta$  are two “0-caps” and  $\mu$  and  $\lambda$  two “1-caps”. Notice that every cap  $u$  defines a definite boundary element  $\partial u$  in

$$H(a^-) = \lim_{\epsilon \rightarrow 0^+} H(F < a - \epsilon);$$

Morse calls a cap  $u$  linkable iff  $\partial u = 0$ . Otherwise it is called *nonlinkable*.

In our example,  $\alpha$ ,  $\beta$  and  $\mu$  are linkable while  $\lambda$  is *not*.

Next Morse defines the *span* of a cap  $u$  associated to the critical level  $a$  in the following manner.

## Morse's functional topology

Early precursors of persistence and spectral sequences:

- *F-homology classes*: similar to persistent homology
  - *inferior/superior cycle limits*: birth/death
- *k-caps*: related to elements of spectral sequence
  - *cap span*: persistence

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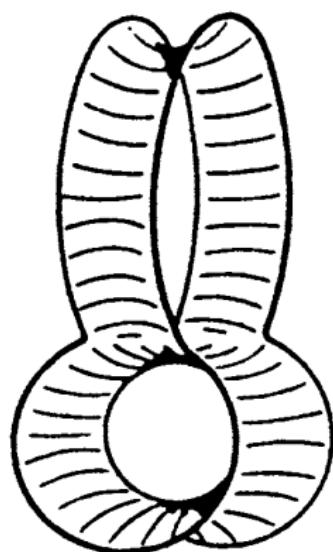
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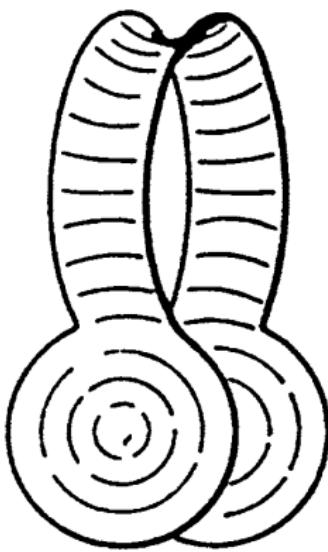
Key aspects:

- applies to a broad class of functions on metric spaces (not necessarily continuous)
- uses Vietoris homology with field coefficients
- inclusions of sublevel sets have homology of finite rank (*q-tame* persistent homology)
- focus on controlled behavior in pathological cases

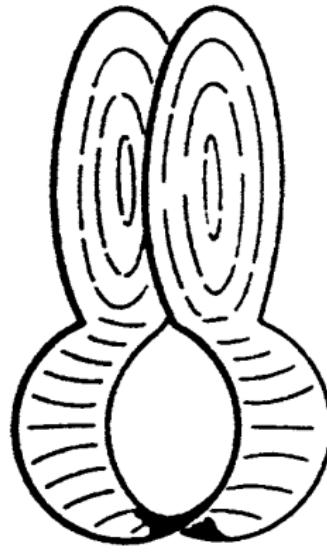
# Motivation and application: minimal surfaces



(a)



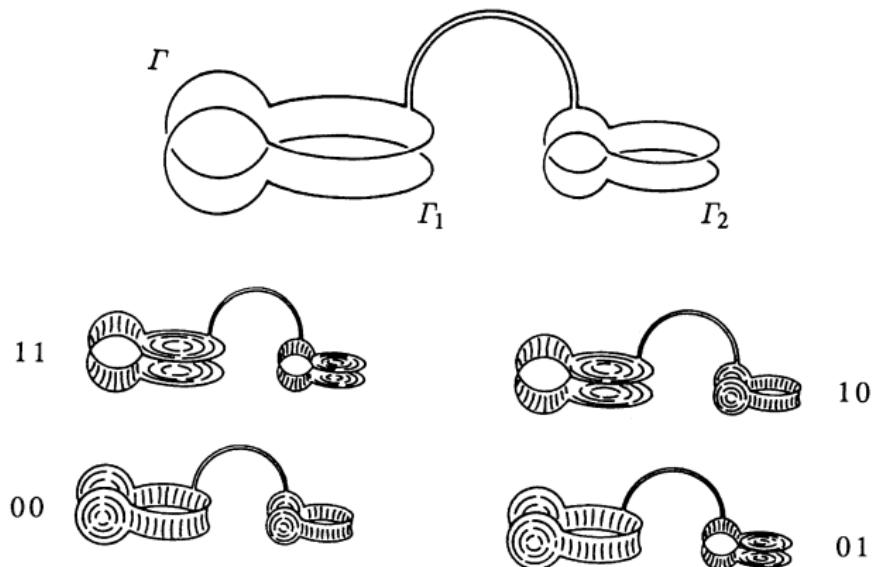
(b)



(c)

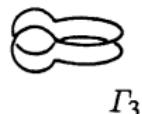
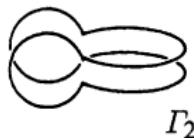
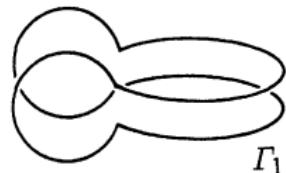
(from Dierkes et al.: Minimal Surfaces, Springer 2010)

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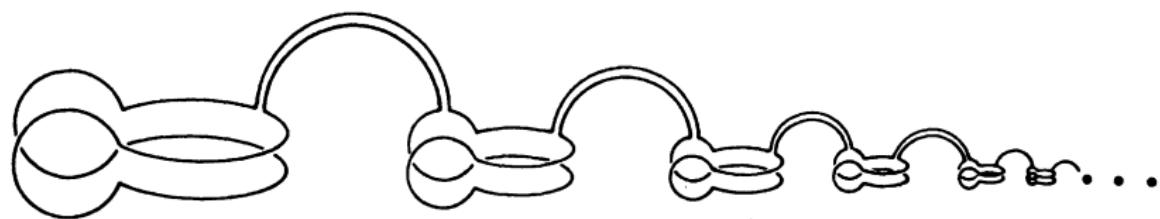


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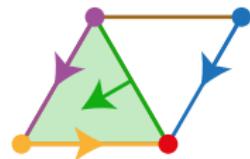
# Discrete Morse theory of geometric complexes

# Discrete Morse theory

## Definition (Forman 1998)

A *discrete vector field* on a cell complex  
is a partition of the set of simplices into

- singleton sets  $\{\phi\}$  (*critical cells*), and
- pairs  $\{\sigma, \tau\}$ , where  $\sigma$  is a facet of  $\tau$ .

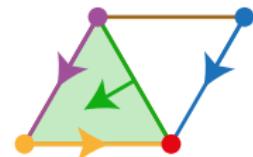


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A function  $f : K \rightarrow \mathbb{R}$  on a cell complex is a *discrete Morse function* if

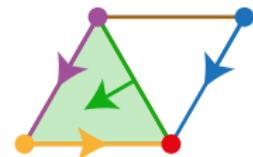
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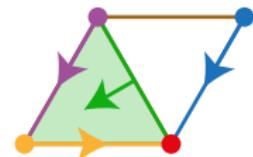


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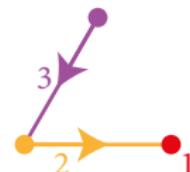
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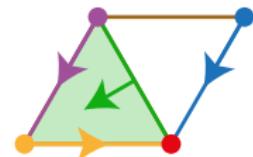


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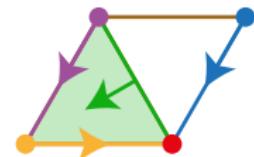


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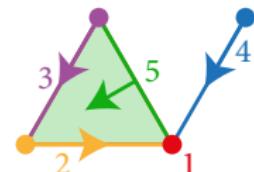
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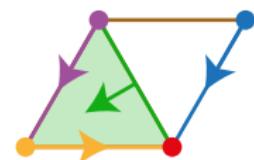


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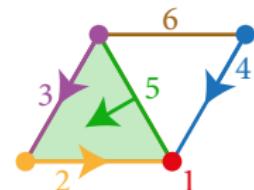
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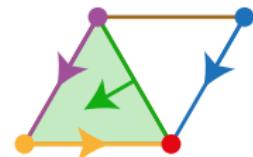


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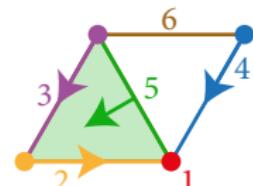
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A function  $f : K \rightarrow \mathbb{R}$  on a cell complex is a *discrete Morse function* if

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- level sets form a discrete vector field.



## Fundamental theorem of discrete Morse theory

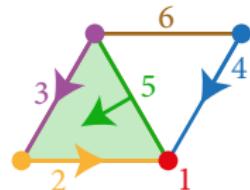
Let  $f$  be a discrete Morse function on a cell complex  $K$ .

# Fundamental theorem of discrete Morse theory

Let  $f$  be a discrete Morse function on a cell complex  $K$ .

## Theorem (Forman 1998)

If  $(s, t]$  contains no critical value of  $f$ ,  
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(written as  $K_t \searrow K_s$ ).

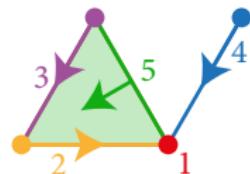


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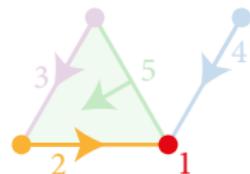


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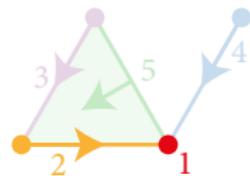


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This homotopy equivalence is compatible with the filtration.

## Corollary

$K$  and  $M$  have isomorphic persistent homology  
(with regard to the sublevel set filtration of  $f$ ).

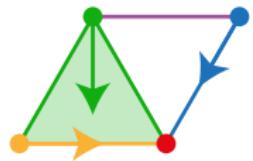
# Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* is a partition of the simplices into clusters of the form

$$\{\rho \mid \sigma \subseteq \rho \subseteq \tau\}$$

(intervals  $[\sigma, \tau]$  of the face poset).



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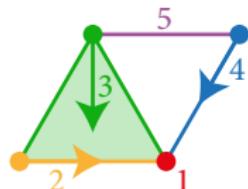
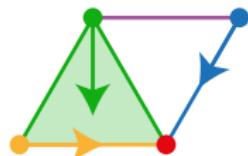
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# $\check{C}$ ech and Delaunay complexes

## Proposition

*The Čech complexes and the Delaunay (alpha) complexes are sublevel sets filtrations of generalized discrete Morse functions.*

# $\check{\text{C}}$ ech and Delaunay complexes

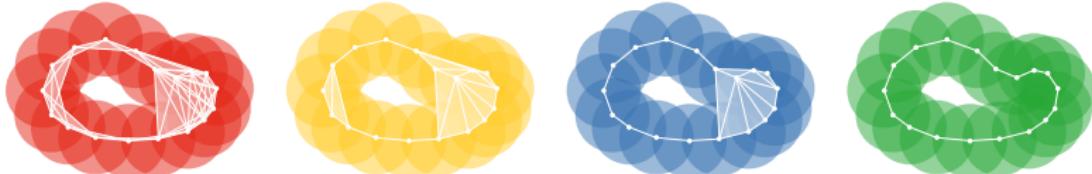
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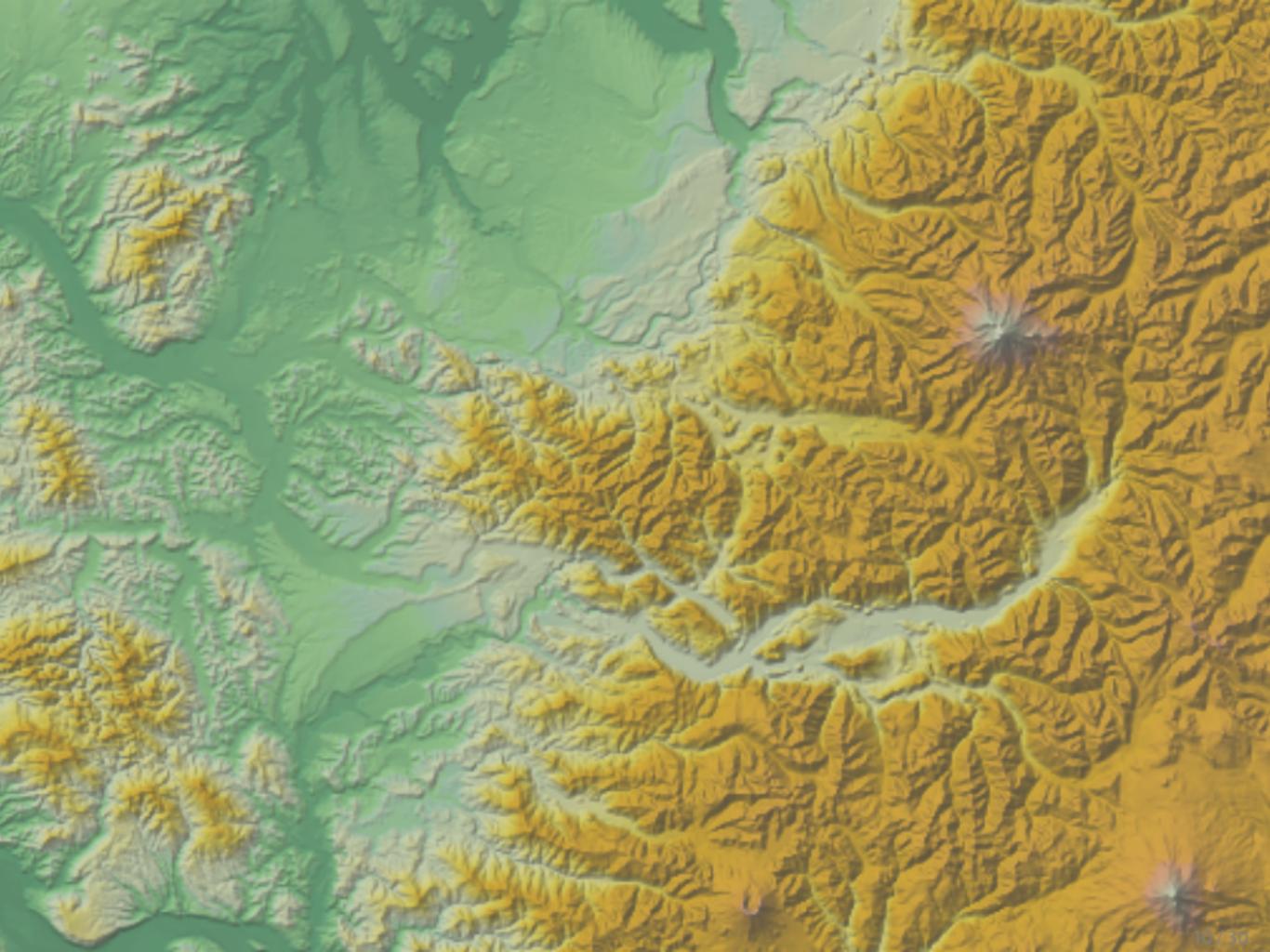
## Theorem (B, Edelsbrunner 2015)

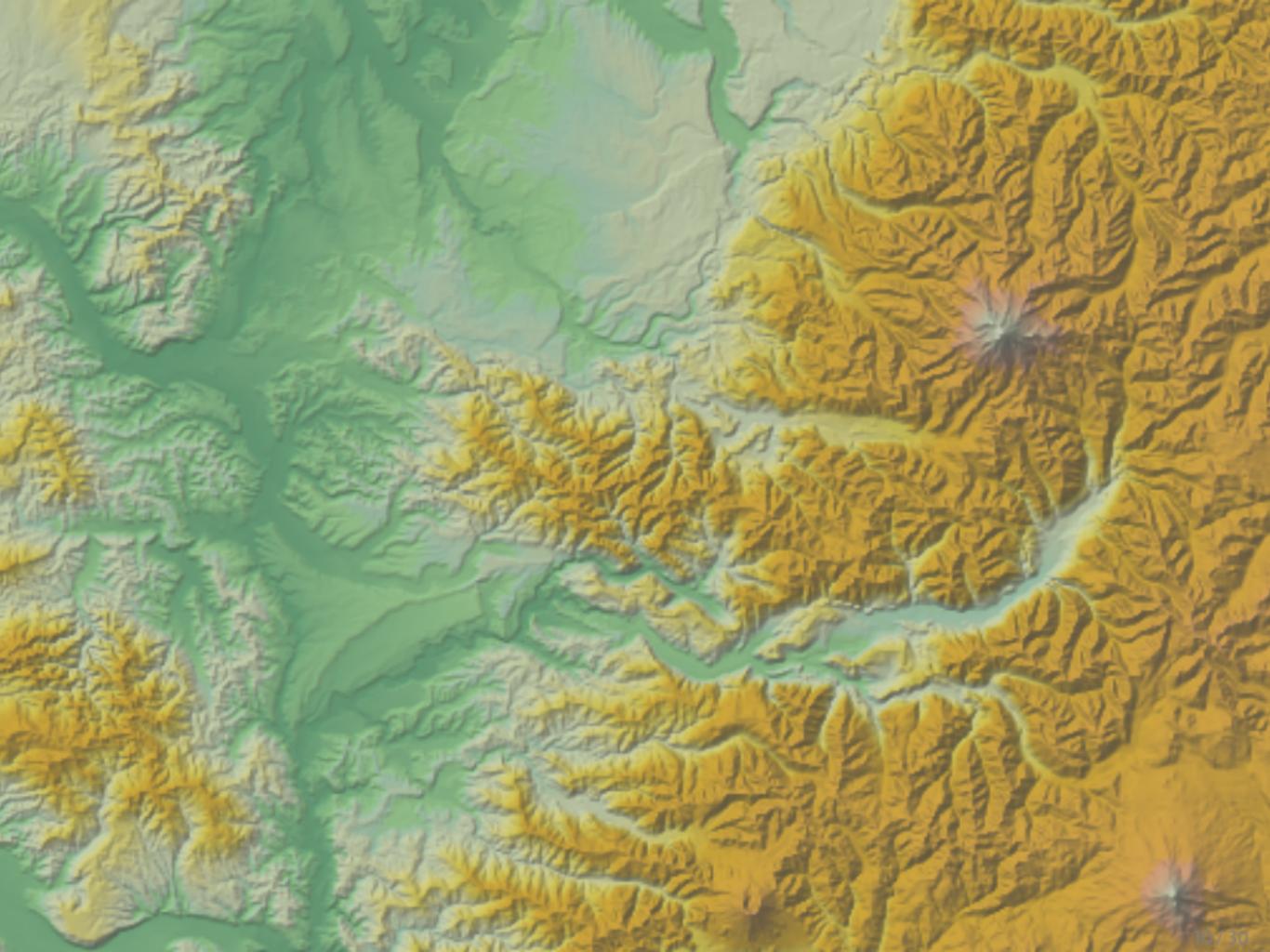
*$\check{\text{C}}$ ech, Delaunay– $\check{\text{C}}$ ech, Delaunay, and Wrap complexes are naturally homotopy equivalent through a sequence of collapses*

$$\text{Cech}_r \searrow \text{DelCech}_r \searrow \text{Del}_r \searrow \text{Wrap}_r.$$

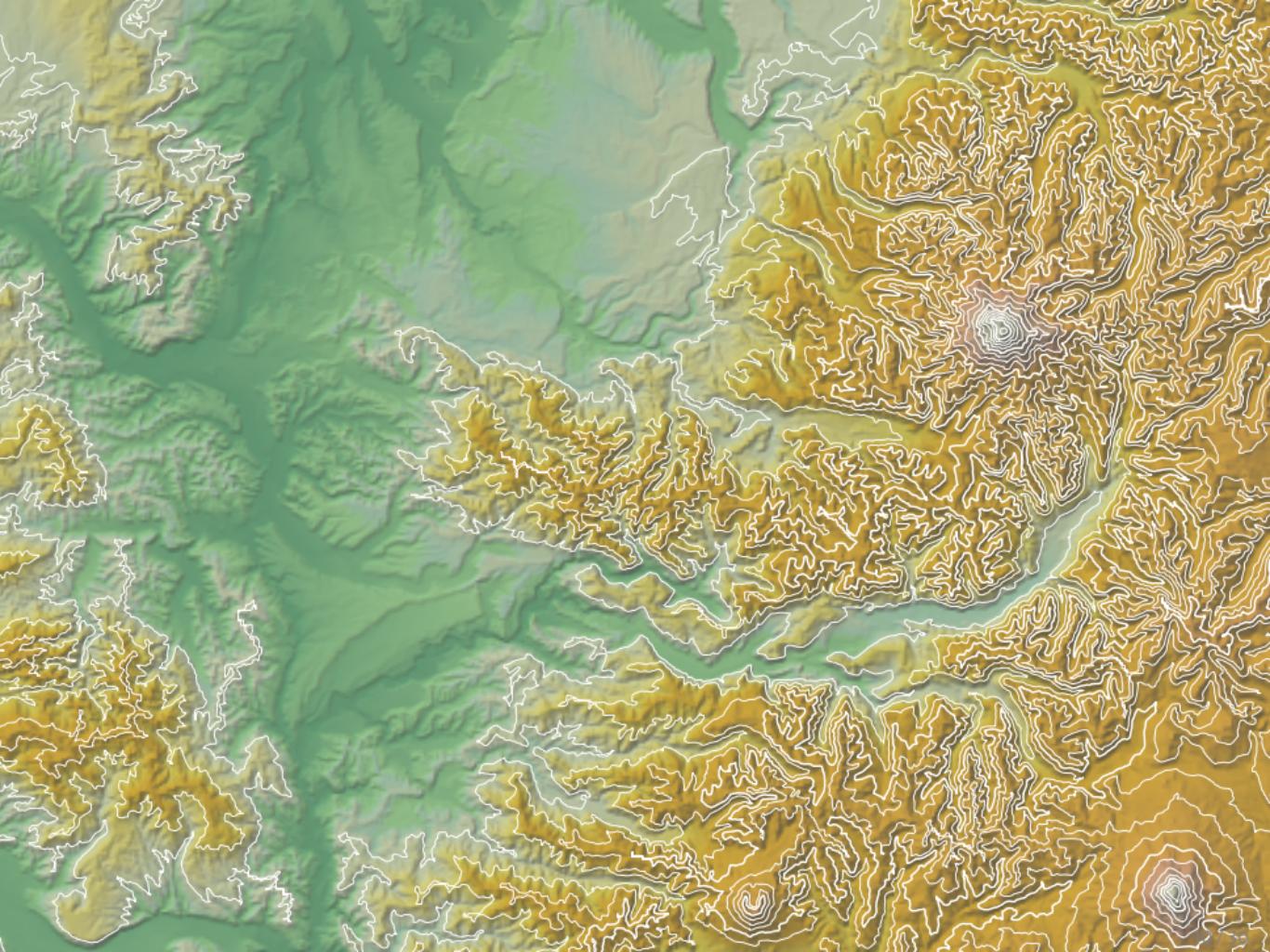


# Simplification of functions









# Topological simplification of functions

Consider the following problem:

## Problem (Topological simplification)

*Given a function  $f$  and a real number  $\delta \geq 0$ ,  
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(Discrete) Morse theory:

- Relates critical points to homology of sublevel set
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Persistent homology:

- Relates homology of two arbitrary sublevel set
- Identifies pairs of critical points (birth and death of homological feature) and quantifies their *persistence*

# Persistence pairs and Morse cancelations

By stability of persistence barcodes:

## Proposition

*The number of critical points of any function  $g$  with  $\|g - f\|_\infty \leq \delta$  is bounded from below by the number of critical points of  $f$  with persistence  $> 2\delta$ .*

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## Theorem (B, Lange, Wardetzky, 2011)

*Let  $f$  be a function on a surface and let  $\delta > 0$ .*

*Then canceling all persistence pairs with persistence  $\leq 2\delta$  yields a function  $g$  satisfying  $\|g - f\|_\infty \leq \delta$ , achieving the lower bound.*

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## Proposition

*The number of critical points of any function  $g$  with  $\|g - f\|_\infty \leq \delta$  is bounded from below by the number of critical points of  $f$  with persistence  $> 2\delta$ .*

## Theorem (B, Lange, Wardetzky, 2011)

*Let  $f$  be a function on a surface and let  $\delta > 0$ .*

*Then canceling all persistence pairs with persistence  $\leq 2\delta$  yields a function  $g$  satisfying  $\|g - f\|_\infty \leq \delta$ , achieving the lower bound.*

- Does not generalize to higher-dimensional manifolds!  
(Simplest example: nontrivial knot complement)

Morse pairs and  
persistence pairs

# Natural filtration settings

Typical assumptions on the filtration:

- general filtration persistence (in theory)
- filtration by singletons or pairs discrete Morse theory
- simplexwise filtration persistence (computation)

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Conclusion:

- Discrete Morse theory sits in the middle between persistence and persistence (hah!)

# From Morse theory to persistence and back

## Proposition (from Morse to persistence)

*The pairs of a Morse filtration are apparent 0-persistence pairs.*

*Apparent persistence pair  $(\sigma, \tau)$ :*

- $\sigma$  is the youngest face of  $\tau$
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You may have seen this before!

- Gunnar's talk
- Ripser software demo
- Matt Kahle, random Rips complex in supercritical regime

# Algebraic Morse theory

Building blocks:

[Kozlov 2005, Sköldberg 2006, Jöllenbeck/Welker 2009]

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**Proposition (from persistence to algebraic Morse)**

*Any persistence pair of a simplexwise filtration corresponds to an algebraic Morse pair for a persistence basis.*

# Morse reductions for persistence computation

## Morse reduction methods

[Mrozek/Wanner 2010, Robbins et al. 2010, Günther et al. 2012, Mischaikow/Nanda 2013]

- find some Morse pairs
- remove them from the complex (reductions)
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Variant: Clear & compress [B/Kerber/Reininghaus 2013]

- using local persistence pairs (algebraic Morse pairs)
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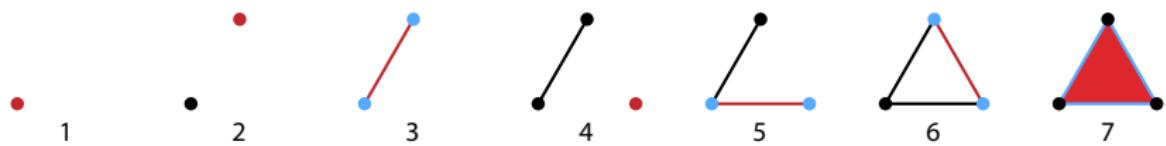
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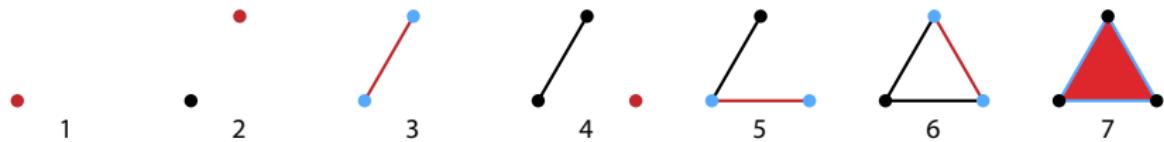
Observed performance:

- previous implementations: significant speedups
- with recent improvements (PHAT): not anymore
  - mostly for difficult instances (large total persistence)
  - benefit of parallelizability
  - negligible for easy instances (Rips filtrations)

## Persistence without reduction: apparent pairs



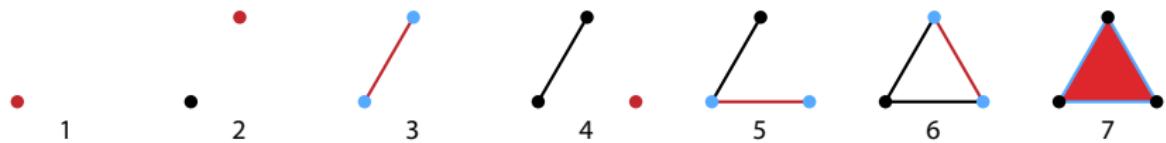
# Persistence without reduction: apparent pairs



$D =$

	1	2	3	4	5	6	7
1			1		1		
2			1			1	
3							1
4					1	1	
5							1
6							1
7							

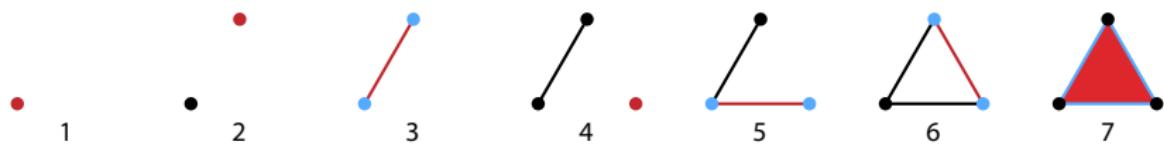
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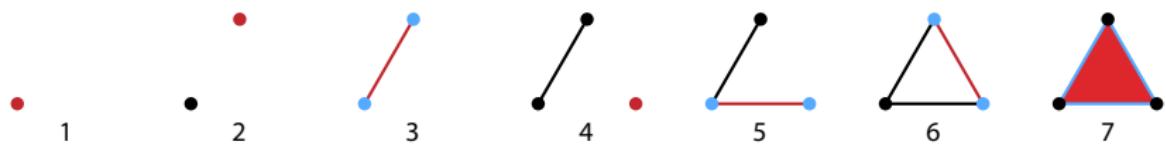
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- All Morse pairs can be seen as persistence pairs
- Many persistence pairs can be seen as Morse pairs
- These apparent Morse/persistence pairs are very useful, both in proofs and in computation