Homological reconstruction and simplification in $\mathbb{R}^3$

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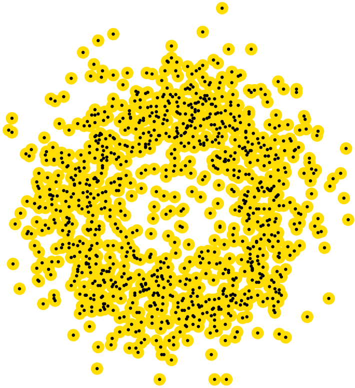
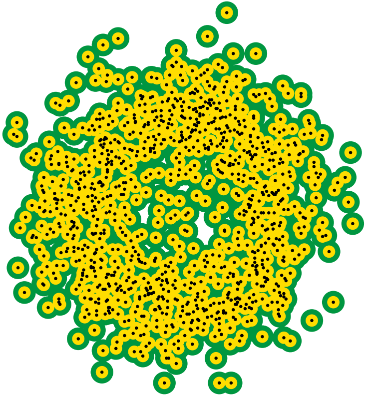
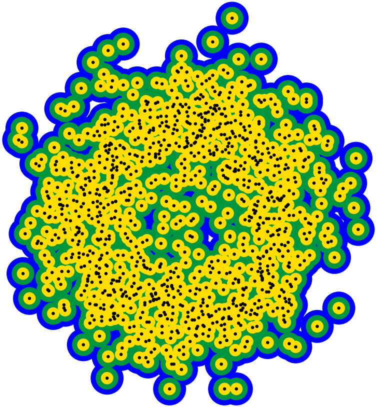
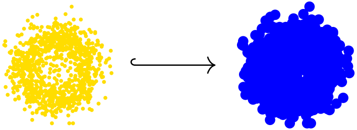
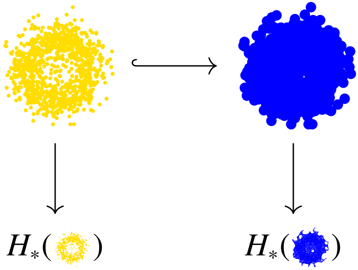
Geometry Workshop Strobl 2013



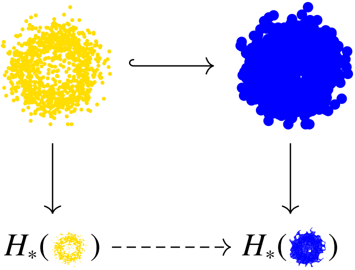








- Homology (with field coefficients) transforms topological spaces to vector spaces



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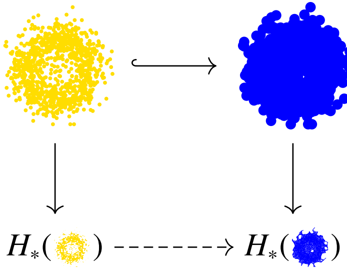

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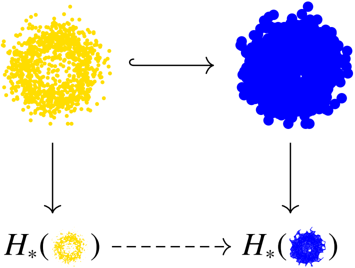
- *Persistent homology group*: $\operatorname{im} H_*($ $\hookrightarrow$  $)$



- Homology (with field coefficients) transforms topological spaces to vector spaces

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- *Persistent homology group*: $\operatorname{im} H_*($ $\hookrightarrow$  $)$
- Under very mild sampling conditions on  :

$\operatorname{im} H_*($ $\hookrightarrow$  $) \cong H_*($  $)$

[Edelsbrunner et al. 2005, Chazal et al. 2005]

$$\begin{tikzcd}
A \arrow[dr, hook] & & B \arrow[dr, hook] & & C \\
& X \arrow[ur, hook] & & Y \arrow[ur, hook] &
\end{tikzcd}$$

- $A \subset B \subset C$: thickenings of 
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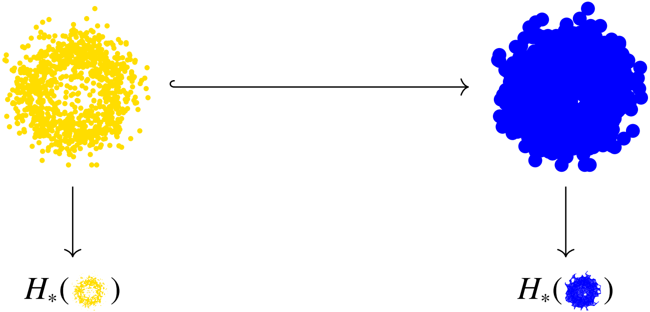
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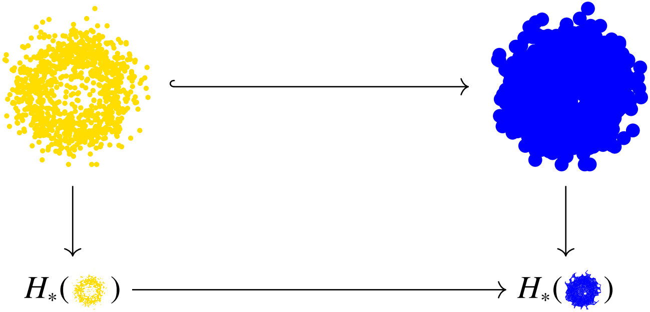
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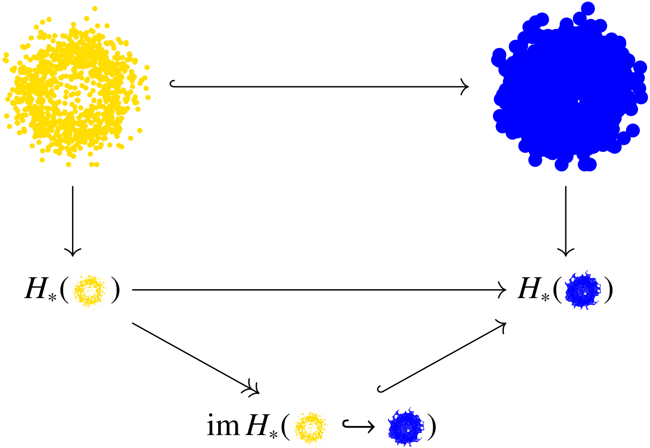
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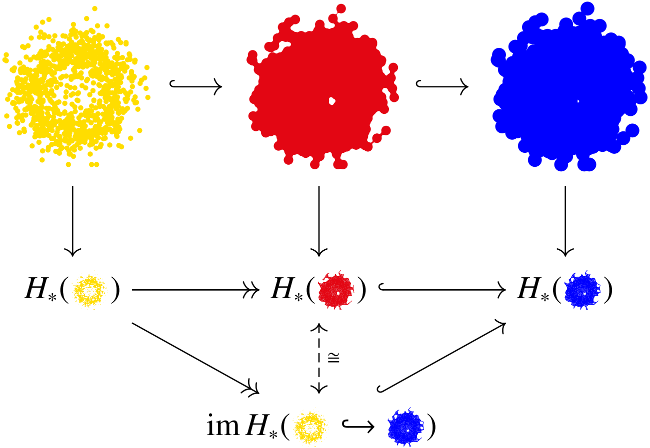
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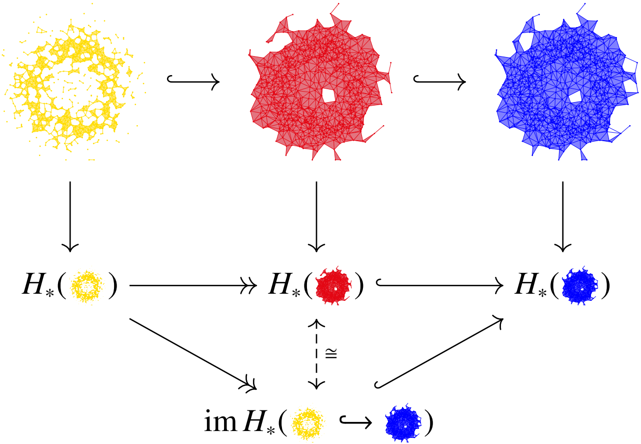
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(Throughout this talk: homology with field coefficients)

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- If $L \subset Y \subset K$ then $\beta(Y) \geq \operatorname{rank} H_*(L \hookrightarrow K)$
- Equality iff Y is a homological reconstruction

Do homological reconstructions always exist?

- Homological reconstructions do not always exists

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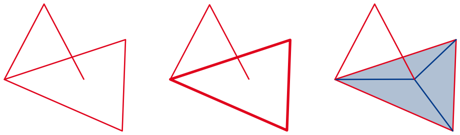

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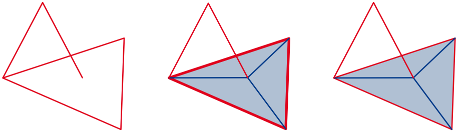

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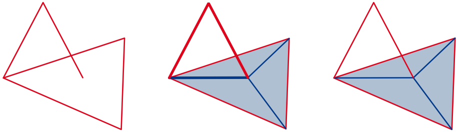

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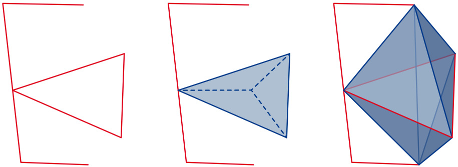

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Homological reconstruction in $\mathbb{R}^3$ is NP-hard

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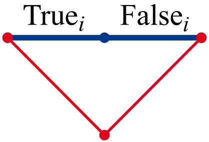
Reduction from 3-SAT: the variable gadget



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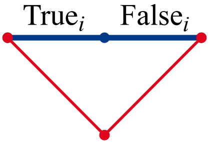

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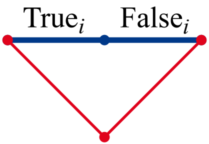

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- edges $\mathrm{True}_i, \mathrm{False}_i$ correspond to possible values of variable x_i

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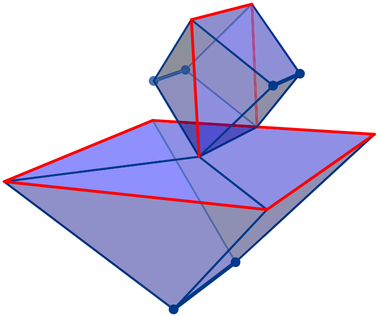

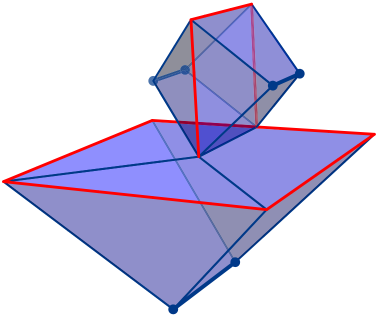

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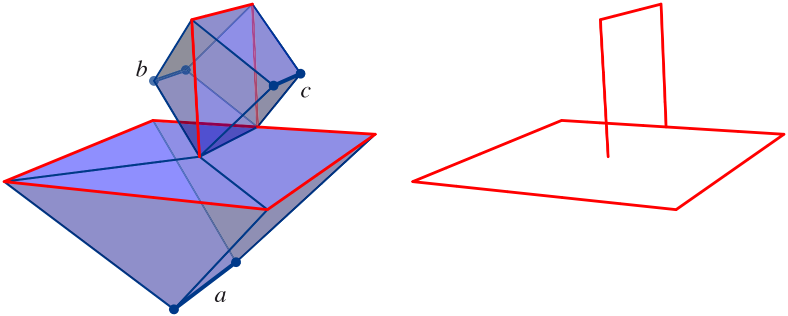

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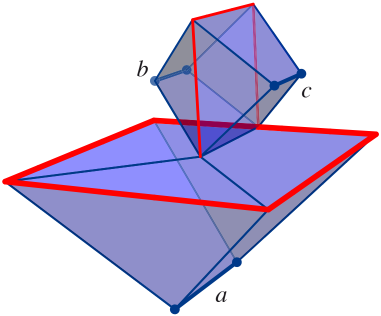

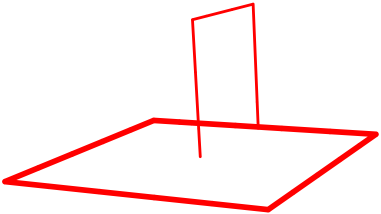

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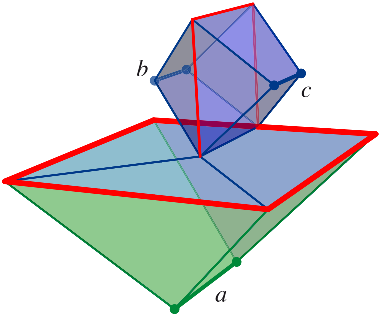

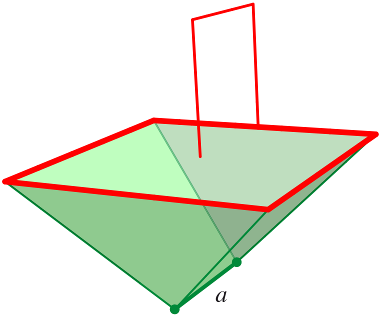

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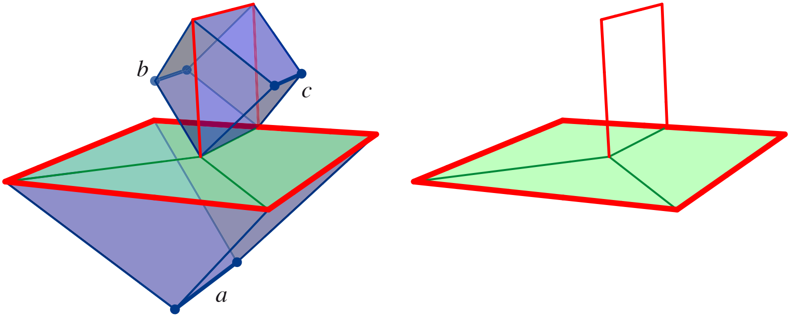

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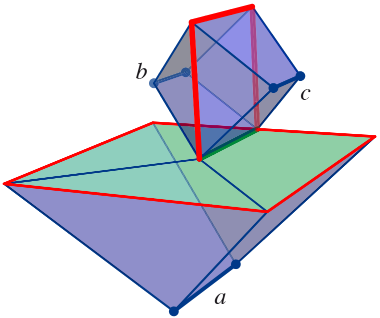

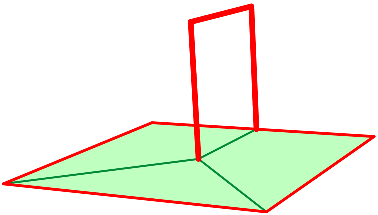

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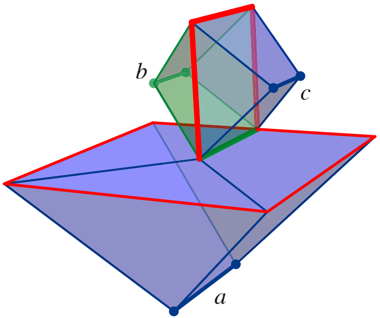

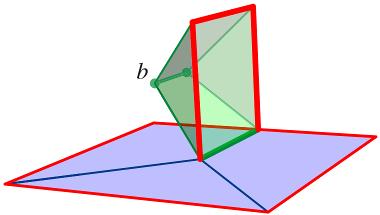

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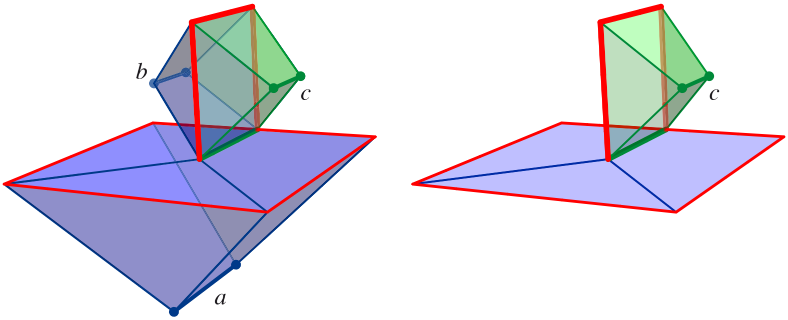

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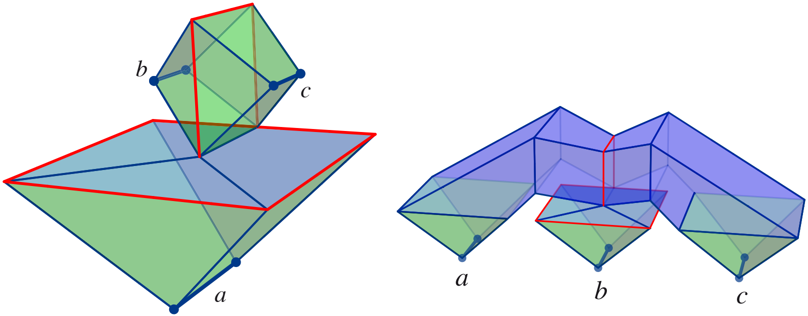

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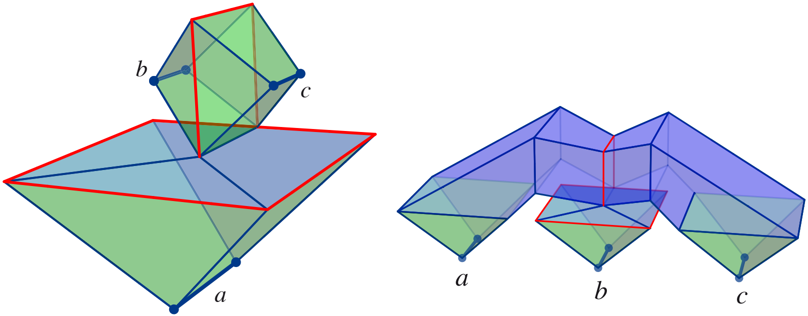

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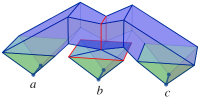
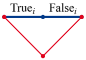
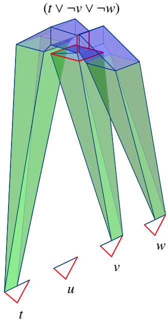
- edges a, b, c correspond to literals in a clause

Reduction from 3-SAT: an example







- For every clause: identify edges a, b, c of clause gadget with edges of variable gadgets corresponding to the literals

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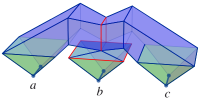

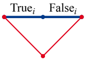

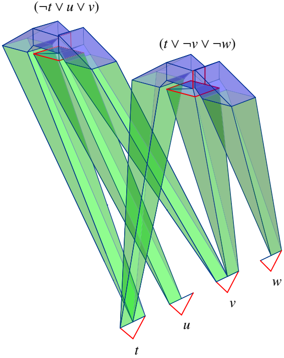

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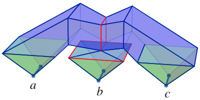

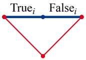

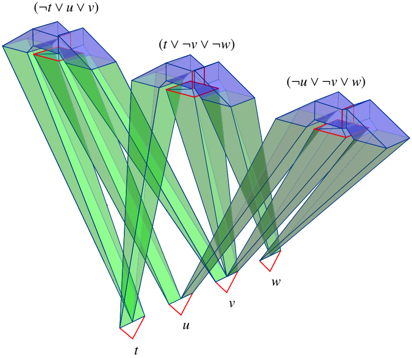

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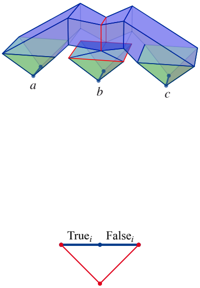

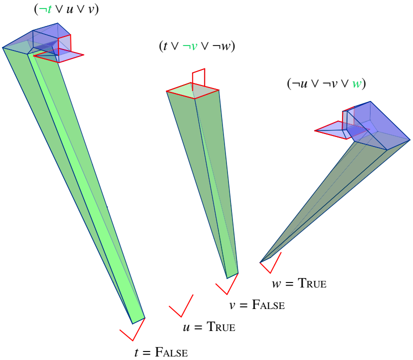

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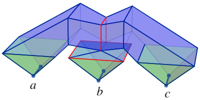

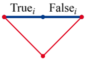

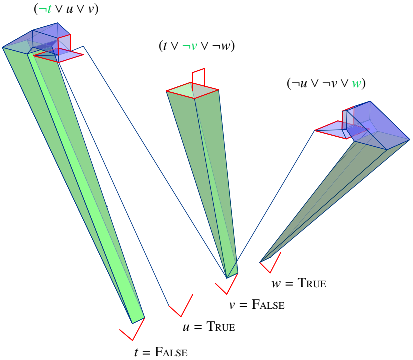

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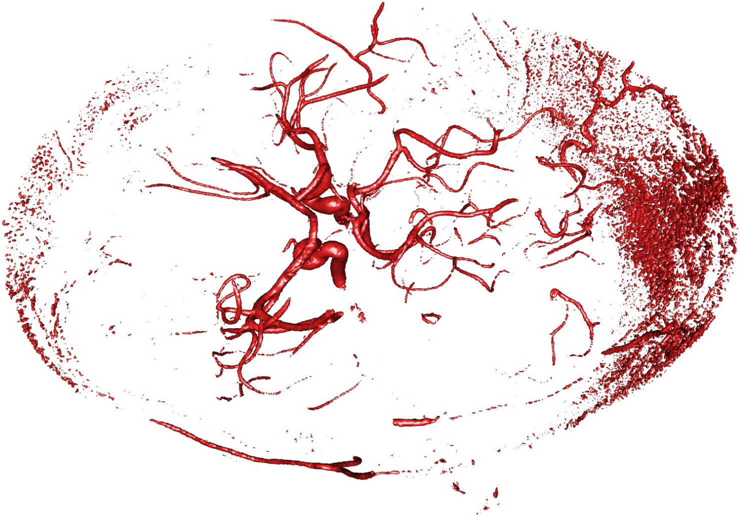

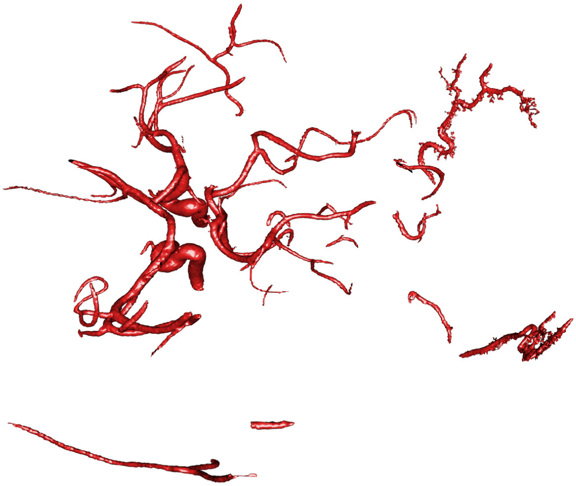

Sublevel set simplification

Let $F_{\leq t} = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem (Sublevel set simplification)

Given a PL function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ *and* $t \in \mathbb{R}$, $\delta > 0$,
find a PL function g *with* $\|g - f\|_\infty \leq \delta$
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Thus we are looking for X with $L \subset X \subset K$ minimizing $\beta(X)$.

- Lower bound $\beta(G_{\leq t}) \geq \operatorname{rank} H_*(L \hookrightarrow K)$
- *Sublevel set reconstruction*: a function g achieving the lower bound

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Sublevel set simplification in $\mathbb{R}^3$ is NP-hard.

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- Construct (K, L) from Φ as before.
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- Extend f to $\tilde{f} : \mathbb{S}^3 \to \mathbb{R}$. Equivalent to the above:
  - (f) $\tilde{f}$ has a simplexwise linear level set reconstruction $\tilde{g}$
  - (g) $\tilde{f}$ has a level set reconstruction $\tilde{g}$

Well groups [Edelsbrunner, Morozov, Patel 2011]

Let f be a PL function and $t \in \mathbb{R}$, $\delta \geq 0$. Let $F_{[a,b]} = f^{-1}[a,b]$.

Definition (Well group)

The (t,δ)-*well group* of f is

$$W_*(f,t,\delta) = \bigcap_{g:\|g-f\|_\infty \leq \delta} \operatorname{im} H_*(G_t \hookrightarrow F_{[t-\delta,t+\delta]})$$

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- $\beta(G_t) \geq \operatorname{rank} H_*(G_t \hookrightarrow F_{[t-\delta,t+\delta]}) \geq \operatorname{rank} W_*(f, t, \delta)$
- *Realization* of well group: a function g with $\|g - f\|_\infty \leq \delta$ achieving the lower bound $\beta(G_t) = \operatorname{rank} W_*(f, t, \delta)$.

Well group realization is level set reconstruction

Theorem

Let f be a PL function on $\mathbb{S}^n$ with $t \pm \delta \in \mathrm{int}(\mathrm{im} f)$.
A PL function g realizes the well group $W_*(f, t, \delta)$ if and only if it is a (t, δ)-level set reconstruction of f.

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Corollary

Realization of a well group on $\mathbb{S}^3$ is NP-hard.

Not all is lost

There is an important special case:

$$\begin{tikzcd}
A \arrow[dr, hook] & & B \arrow[dr, hook] & & C \\
& X \arrow[ur, hook] & & Y \arrow[ur, hook] &
\end{tikzcd}$$

- $A \subset B \subset C$: thickenings of 
- $X \subset Y$: thickenings of 

$$
\begin{tikzcd}
A \arrow[dr, hook] & & B \arrow[dr, hook] & & C \\
& X \arrow[ur, hook] & & Y \arrow[ur, hook] & \\
& L \arrow[u, hook] \arrow[rr, hook] & & K \arrow[u, hook] &
\end{tikzcd}
$$

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- $X \subset Y$: thickenings of 
- $L \subset K$: alpha shapes of 

$$\begin{tikzcd}
H_*(A) \arrow[rr, hook, two heads] \arrow[dr] & & H_*(B) \arrow[rr, hook, two heads] \arrow[dr] & & H_*(C) \\
& H_*(X) \arrow[ur] & & H_*(Y) \arrow[ur] & \\
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- $A \subset B \subset C$: thickenings of  (with same homology)
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H_*(A) \arrow[rr, hook, two heads] \arrow[dr] & & H_*(B) \arrow[rr, hook, two heads] \arrow[drr, hook] \arrow[d, dashed, leftrightarrow, "\cong"] & & H_*(C) \\
& H_*(X) \arrow[ur, two heads] \arrow[r, two heads] & \operatorname{im} H_*(X \hookrightarrow Y) \arrow[r, hook] & H_*(Y) \arrow[ur] & \\
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Theorem

In $\mathbb{R}^3$, a subcomplex reconstruction R of $L \subset K$ can be found by a simple greedy algorithm (if such an R exists).
This complex R reconstructs $H_*($$)$ geometrically.

Thanks for your attention!