INDECOMPOSABLES IN MULTI-PARAMETER PERSISTENCE

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Applied Topology Seminar, Oxford University

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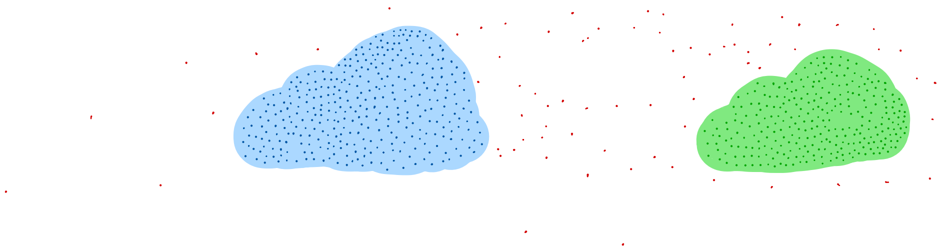
joint work with:

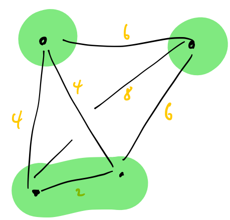
Magnus Botnan / Steffen Oppermann / Johan Steen / Luis Scoccola / Ben Fluhr



CLUSTERING FUNCTIONS

X : finite set

Clustering function φ :
maps a metric $d : X \times X \to \mathbb{R}$ (distance matrix)
to a partition of X.

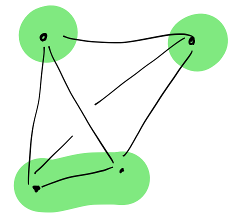


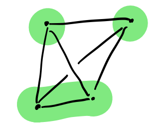
KLEINBERG'S AXIOMS

Desirable properties

- scale invariance :

$$\varphi(d) = \varphi(\lambda \cdot d).$$





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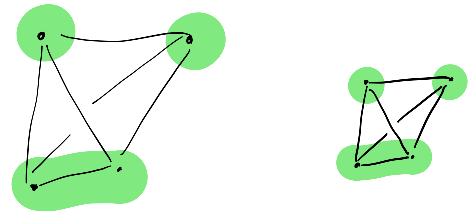

- richness:

every partition is obtained from some d.

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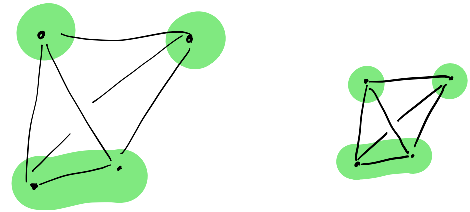


- **scale invariance:**

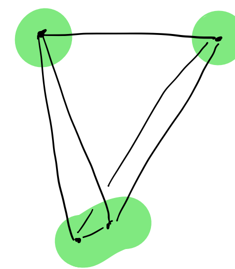
  $$\varphi(d) = \varphi(\lambda \cdot d).$$

- **richness:**

  every partition is obtained from some d.

- **consistency:**

  decreasing d within clusters /
  increasing d across clusters
  does not change the result.



KLEINBERG'S IMPOSSIBILITY THEOREM

Thm [Kleinberg 2002] No clustering function satisfies

- scale-invariant,
- richness, and
- consistency.

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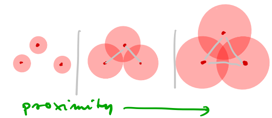
Motivates the use of a scale parameter

⇝ hierarchical clustering

CLUSTERING FROM CONNECTED COMPONENTS

proximity graph

- filter edges by proximity



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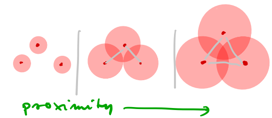
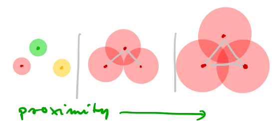


π_0 (connected components)



single-linkage clustering

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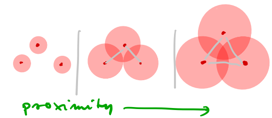

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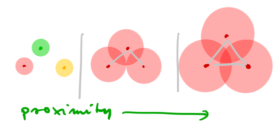


single-linkage clustering

H_0 (homology in deg. 0 with coeffs in K)

$$K^3 \to K \to K$$

proximity $\longrightarrow$

persistent homology

$H_0 = F \circ \pi_0$

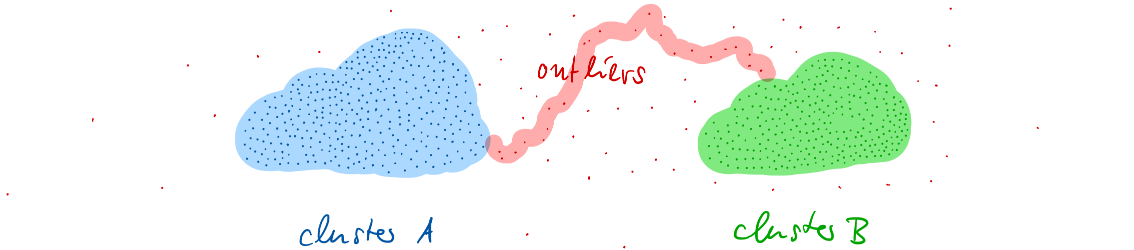
HIERARCHICAL CLUSTERING: EXISTENCE & UNIQUENESS

Thm [Carlsson, Mémoli 2010] Single-linkage clustering is the unique hierarchical clustering method satisfying [... certain axioms similar to Kleinberg's].

But ...

CHAINING EFFECT

Single-linkage clustering is sensitive to outliers

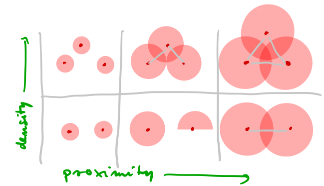


→ not used much in practice!

2-PARAMETER CLUSTERING

density-proximity graph

- filter points by density
- filter edges by proximity



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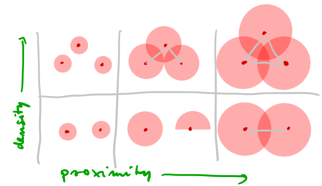

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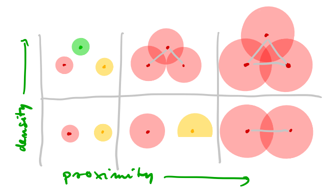

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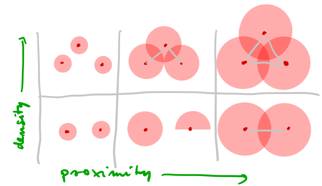

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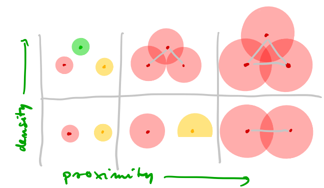

H_0 (homology in deg. 0
with coeffs in K)

$$\begin{tikzcd}
K^3 \arrow[r] & K \arrow[r] & K \\
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density

proximity

TRICHOTOMY OF REPRESENTATION TYPES

Given a finite indexing poset P. (k : algebr. closed)

What are the indecomposable diagrams with shape P ?

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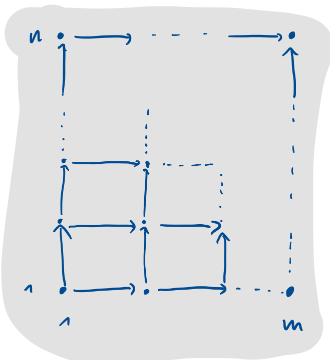
(a) A finite list. finite type

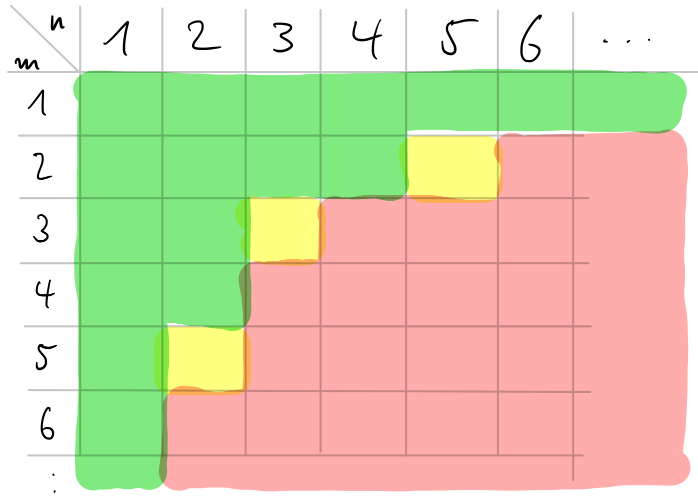
(b) 1-param. families. tame

(c) It's complicated. wild

(as complicated as modules over any finite-dim. algebra; including undecidable problems)

REPRESENTATION TYPES OF COMMUTATIVE GRIDS





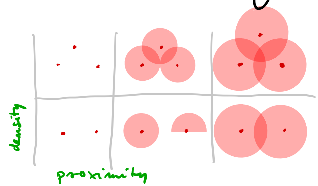
- finite type
- tame
- wild

$\left.\begin{matrix}\text{finite type}\\ \text{tame}\\ \text{wild}\end{matrix}\right\}$ for $(m-1)(n-1) \left\{\begin{matrix} < \\ = \\ > \end{matrix}\right\} 4$.

[Leszczyński '94, Skowroński '2000]

GRID DIAGRAMS FROM CLUSTERING

consider again 2-parameter clustering (proximity / density)



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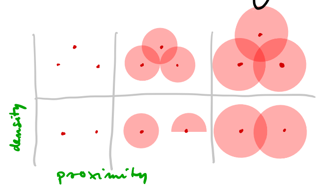

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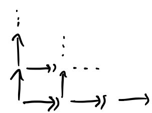

Horizontal maps are surjective !

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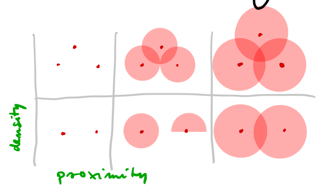


This yields diagrams of the form

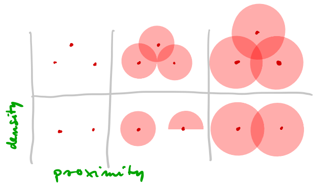
$$\begin{tikzcd}
\vdots & \vdots & \\
{} \arrow[u] \arrow[r, two heads] & {} & \cdots & \\
{} \arrow[u] \arrow[r, two heads] & {} \arrow[u] \arrow[r, two heads] & {} \arrow[r, two heads] & {}
\end{tikzcd}$$

Horizontal maps are surjective !

Does this simplify the picture ?

EPIMORPHISMS

Lemma $\mathrm{Rep}^{\twoheadrightarrow}(m,2)$ is finite type.



$$H_0 \rightsquigarrow \begin{tikzcd} K^3 \arrow[r, two heads, "(1\,1\,1)"] & K \arrow[r, two heads] & K \\ K^2 \arrow[u, "\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}"] \arrow[r, two heads] & K^2 \arrow[u, "(1\,1)"'] \arrow[r, two heads, "(1\,1)"'] & K \arrow[u] \end{tikzcd}$$

$$\cong \begin{tikzcd} K \arrow[r, two heads] & K \arrow[r, two heads] & K \\ K \arrow[u] \arrow[r, two heads] & K \arrow[u] \arrow[r, two heads] & K \arrow[u] \end{tikzcd} \oplus \begin{tikzcd} K \arrow[r, two heads] & 0 \arrow[r, two heads] & 0 \\ K \arrow[u] \arrow[r, two heads] & K \arrow[u] \arrow[r, two heads] & 0 \arrow[u] \end{tikzcd} \oplus \begin{tikzcd} K \arrow[r, two heads] & 0 \arrow[r, two heads] & 0 \\ 0 \arrow[u] \arrow[r, two heads] & 0 \arrow[u] \arrow[r, two heads] & 0 \arrow[u] \end{tikzcd}$$

- $\mathrm{Rep}(m,n)$: all commutative dgms over $m \times n$ grid
- $\mathrm{Rep}^{\twoheadrightarrow}(m,n)$: epis in horizontal direction
- $\mathrm{Rep}^{\twoheaduparrow\twoheadrightarrow}(m,n)$: epis in both directions.

EPIC GRIDS & WILD THINGS

Thm [B, Botnan, Oppermann, Steen 20]

$$\begin{array}{ccc} \mathrm{Rep}^{\twoheaduparrow \twoheadrightarrow}(m,n) & \sim & \mathrm{Rep}^{\twoheaduparrow}(m,n-1) \\ \wr & & \wr \\ \mathrm{Rep}^{\twoheadrightarrow}(m-1,n) & \sim & \mathrm{Rep}(m-1,n-1) \end{array}$$

same representation type

Corollary $\mathrm{Rep}^{\twoheadrightarrow}(m,n)$ is

- finite type
- tame
- wild

for $(m-1)(n-2) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 4$.

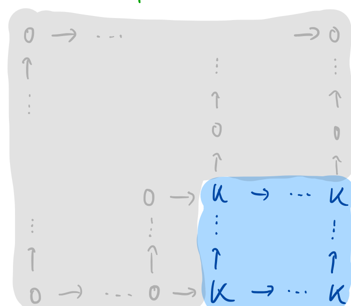
BEHIND THE SCENES

equivalence of categories

$$\frac{\mathrm{Rep}^{\twoheadrightarrow}(m,n)}{\mathrm{Rep}^{\hat{\imath}\twoheadrightarrow}(m,n)} \simeq \mathrm{Rep}(m,n-1)$$

additive quotient $\frac{A}{B}$: identify morphisms in A whose difference factors through B

indecomposables are of the form



$\Rightarrow$ finite type

THE INSTABILITY OF DECOMPOSITIONS

How useful are indecomposables for TDA?

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Thm [B, Scoccola 22] For $n > 1$, among the finitely presented n-parameter persistence modules the indecomposables are dense in interleaving distance.

The Instability of Decompositions

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Thm [B, Scoccola 22] For $n > 1$, among the finitely presented n-parameter persistence modules the indecomposables are dense in interleaving distance.

For every $\varepsilon > 0$, the ε-indecomposables

$$\underbrace{A}_{\text{indecomposable}} \oplus \underbrace{B}_{d_I(B,0) < \varepsilon}$$

form an open & dense subset.

- ε-indecomposability is a generic property

The Instability of Decompositions

Proof.

(a) $\forall \varepsilon, M : \mathbb{R}^n \to \text{vect}$ f.p., ε-indecomposable

$\exists \delta$:

$d_I(M, N) < \delta \implies N$ is ε-indecomposable.

The Instability of Decompositions

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$\exists \delta$:

$d_I(M, N) < \delta \implies N$ is ε-indecomposable.

(b) Indecomposables can be "tacked together" with an arbitrarily small change in interleaving distance.

The Idea of Tacking Indecomposables

$$
\begin{tikzcd}
k \arrow[r] & k \arrow[r] & k \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}
\quad \oplus \quad
\begin{tikzcd}
0 \arrow[r] & k \arrow[r] & k \\
0 \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u]
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0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & k \arrow[u]
\end{tikzcd}
$$

$$
=
\quad
\begin{tikzcd}
k \arrow[r] & k^2 \arrow[r] & k^3 \\
0 \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k^2 \arrow[u] \\
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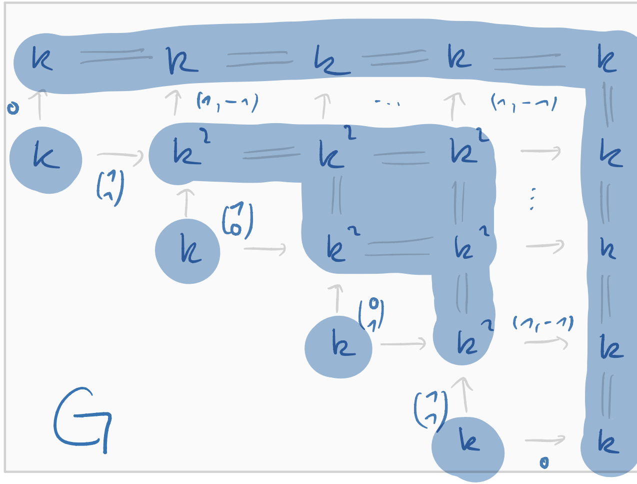
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & k \arrow[u]
\end{tikzcd}
\quad \twoheadrightarrow \quad
\begin{tikzcd}
k \arrow[r, "\binom{1}{0}"] & k^2 \arrow[r] & k^2 \\
0 \arrow[r] \arrow[u] & k \arrow[r, "\binom{0}{1}"] \arrow[u] & k^2 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & k \arrow[u, "\binom{1}{1}"']
\end{tikzcd}
$$

THE TACKING GADGET

$$
\begin{tikzcd}
k \arrow[r, equal] & k \arrow[r, equal] & k \arrow[r, equal] & k \arrow[r, equal] & k \\
k \arrow[u, "0"] \arrow[r, "\binom{1}{1}"'] & k^2 \arrow[u, "{(1,-1)}"'] \arrow[r, equal] & k^2 \arrow[u, "\cdots"'] \arrow[r, equal] \arrow[d, equal] & k^2 \arrow[u, "{(1,-1)}"'] \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\vdots"] & k \arrow[u, equal] \arrow[d, equal] \\
 & k \arrow[u] \arrow[r, "\binom{1}{0}"] & k^2 \arrow[r, equal] & k^2 \arrow[r] \arrow[d, equal] & k \arrow[d, equal] \\
 & & k \arrow[u, "\binom{0}{1}"'] \arrow[r] & k^2 \arrow[r, "{(1,-1)}"] & k \arrow[d, equal] \\
 & & & k \arrow[u, "\binom{1}{1}"] \arrow[r, "0"'] & k
\end{tikzcd}
$$

G

THE TACKING GADGET



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$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & k \arrow[r, equal] & k \arrow[r, equal] & k \arrow[r, equal] & k \arrow[r, equal] & k \\
\cdots \, k \arrow[u] \arrow[r] & k \arrow[u] \arrow[r] & k \arrow[u, "0"] \arrow[r, "\binom{1}{1}"'] & k^2 \arrow[u, "{(1,-1)}"'] \arrow[r, equal] & k^2 \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\cdots"] & k^2 \arrow[u] \arrow[r] \arrow[ur, phantom, "{(1,-1)}"] \arrow[dr, phantom, "\vdots"] & k \arrow[u, equal] \\
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 & & & & & \vdots & \vdots
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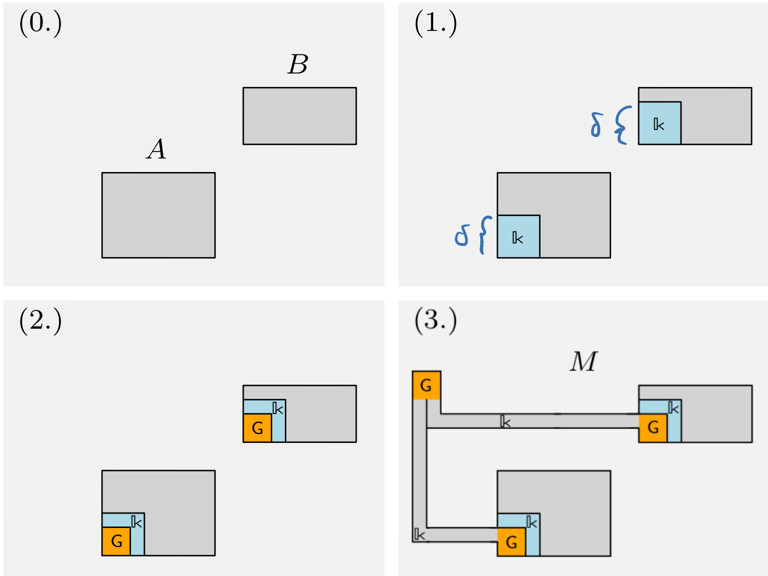
Tacking Indecomposables



$d_I(A \oplus B, M) < \delta$ for $\delta > 0$ arbitrary

THIN-DECOMPOSABLES ARE NOWHERE DENSE

Can we work with classes of simpler indecomposables?
(e.g. thin : pointwise dim ≤ 1)

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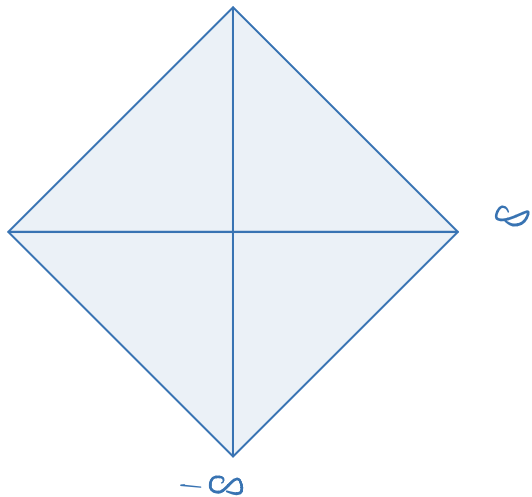
Theorem [B, Scoccola '23] Let $\mathcal{F}$ be a class of indecomposable $(n \geq 2)$-parameter persistence modules with pointwise dimension bounded by some constant. Then the $\mathcal{F}$-decomposable persistence modules are nowhere dense.

THE MAYER-VIETORIS PYRAMID

[Carlsson et al. 2009] $f : X \to \mathbb{R}$



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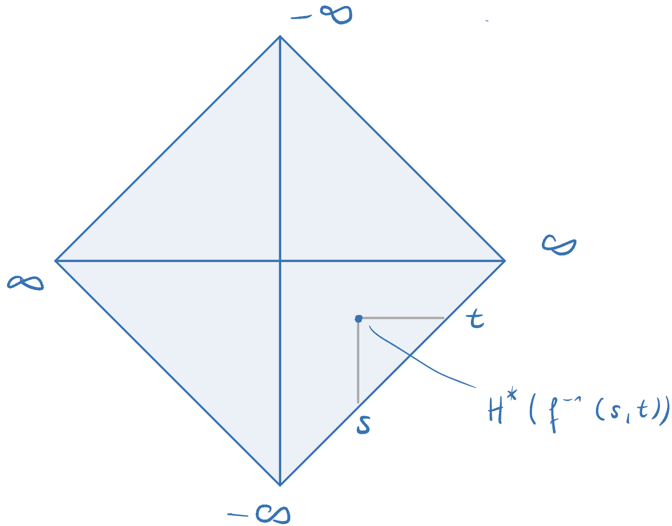

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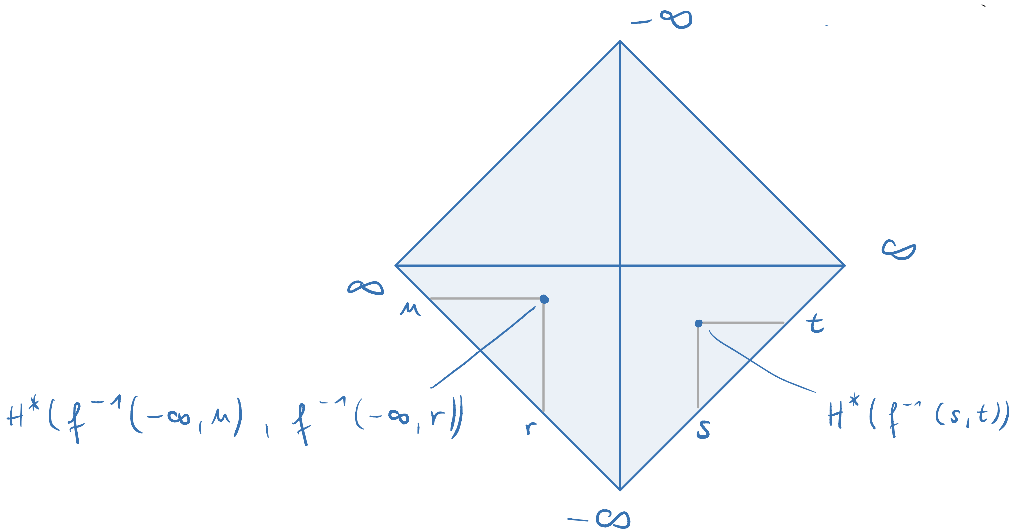

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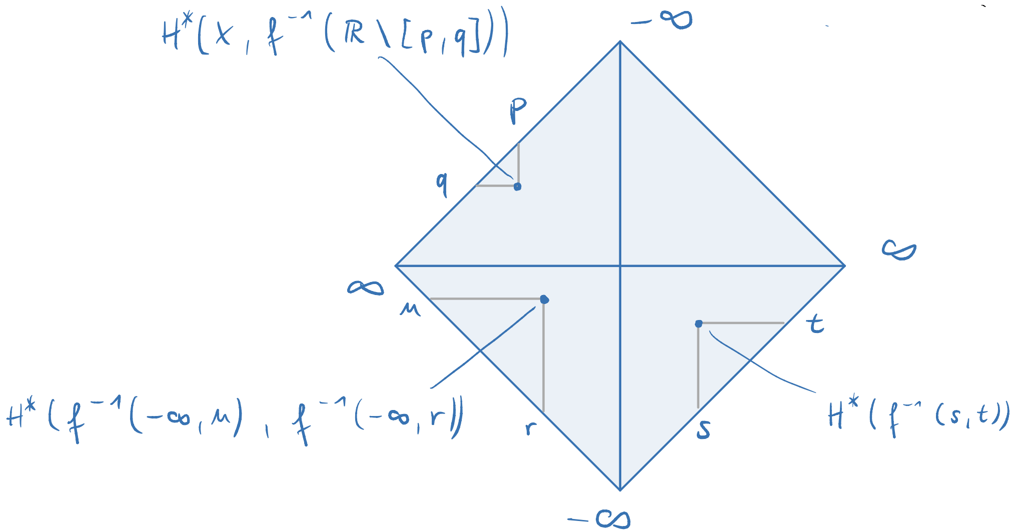

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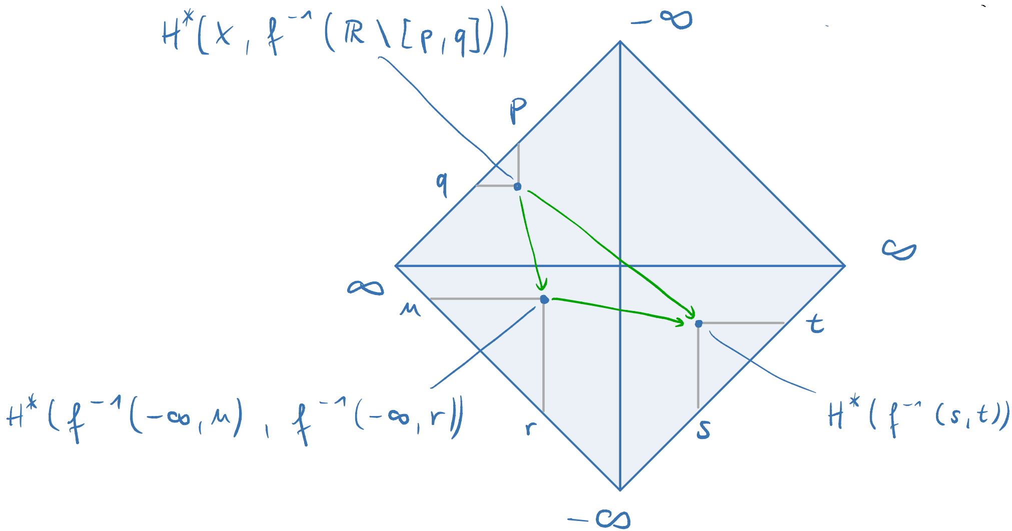

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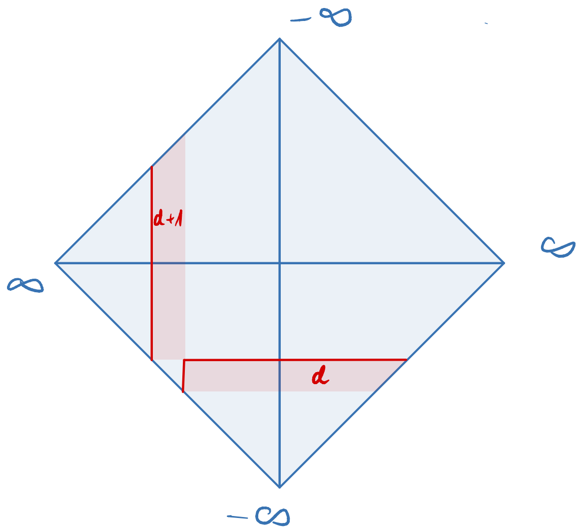

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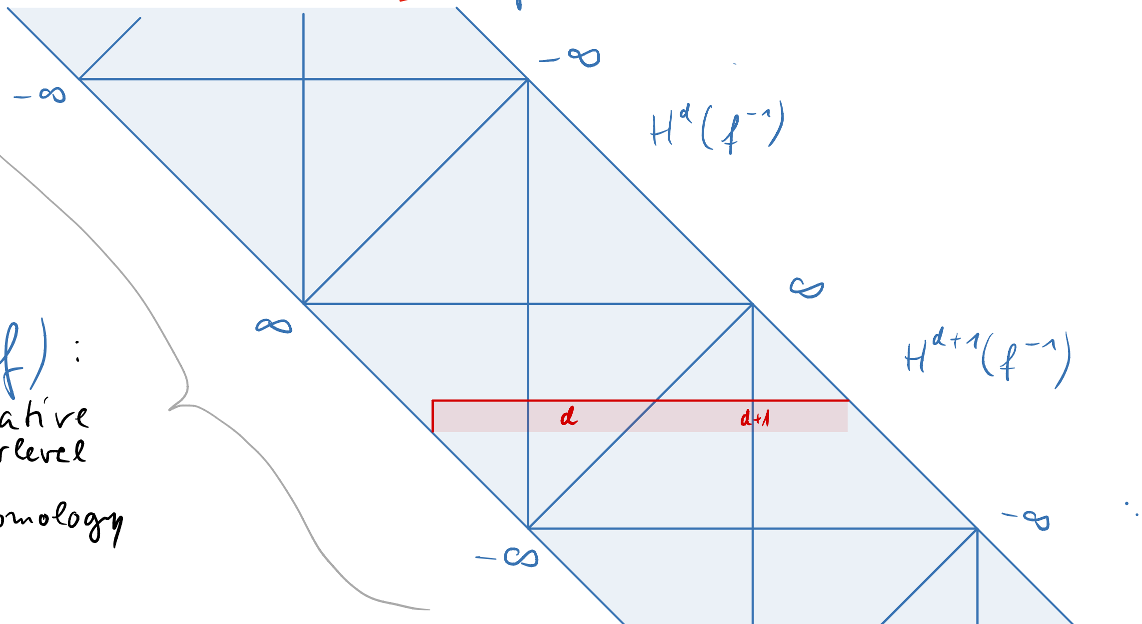

$h(f)$:
Relative
Interlevel
Set
Cohomology

RISC OF A FUNCTION

[B, Botnan, Fluhr 2021] Assume $h(f)$ p.f.d.

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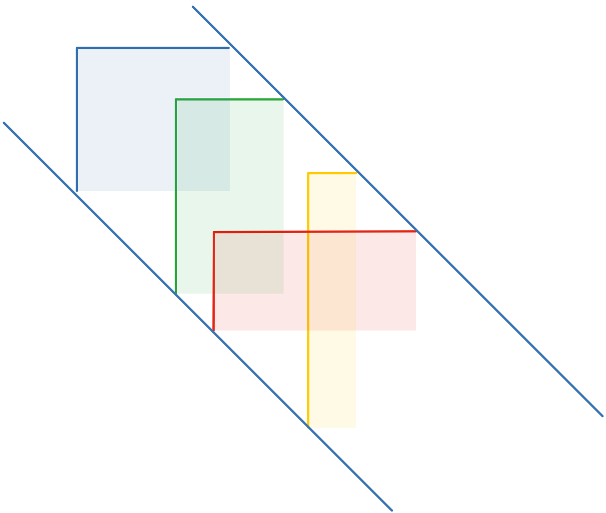
Prop. $h(f)$ is sequentially continuous

(for sequences moving up/left in M)

DECOMPOSITION OF COHOMOLOGICAL FUNCTORS

Theorem [BBF '21] Any cohomological sequentially continuous p.f.d. functor $M \to \mathrm{vect}$ decomposes into rectangle summands.



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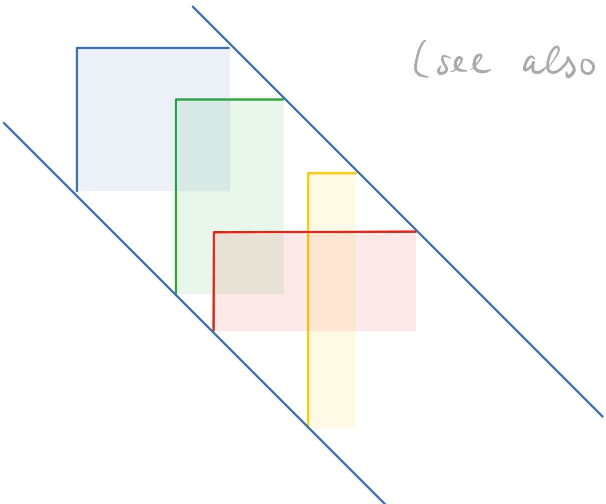

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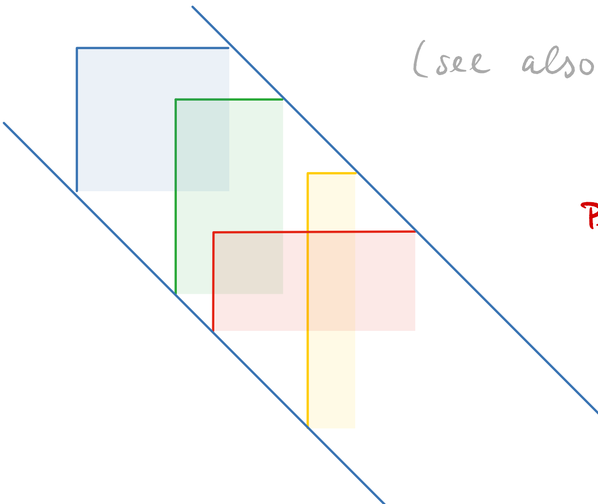

(see also: [Botnan, Lebovici, Oudot '20])

Proposition $\mathbb{M} \to \mathrm{vect}$ q-tame & sequentially continuous $\Rightarrow$ p.f.d.

INDUCED MORPHISMS & INTERLEAVINGS

- A map φ :
$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & \mathbb{R} & \end{tikzcd}$$
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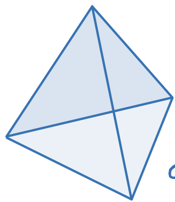
- These induced morphisms are richer than those in standard persistent homology!

WHAT STANDARD PERSISTENCE CAN'T SEE (BUT RISC CAN)



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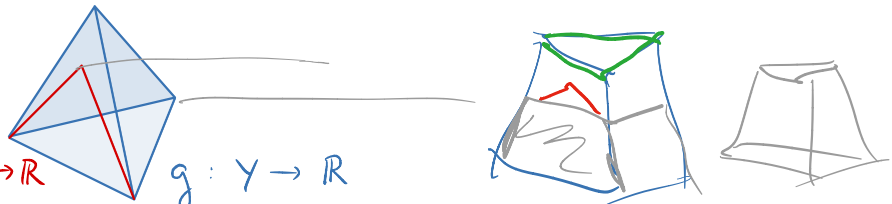
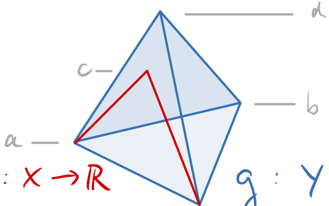


$f : X \to \mathbb{R}$ $g : Y \to \mathbb{R}$

$\varphi : X \hookrightarrow Y$

What standard persistence can't see (but RISC can)



$f : X \to \mathbb{R}$ $\qquad$ $g : Y \to \mathbb{R}$

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- Hence the induced map is zero.

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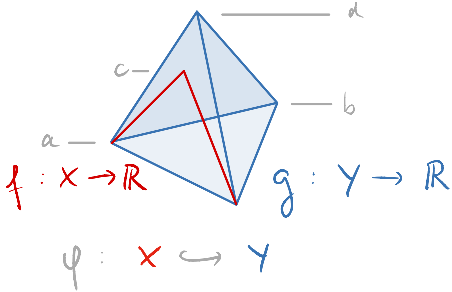

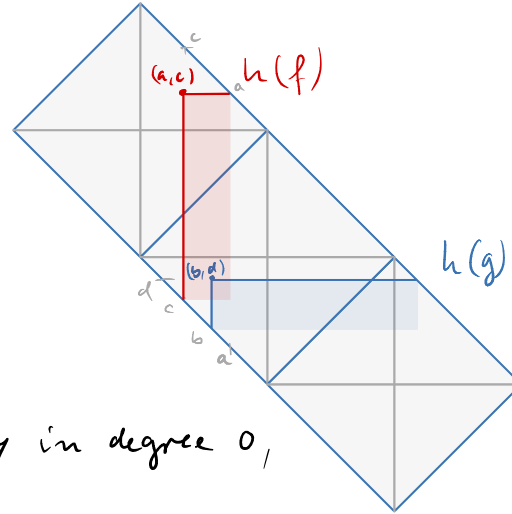

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But $h(\varphi)$ is nonzero!

EXTENDED & LEVEL SET PERSISTENCE

Extended persistence [Cohen-Steiner, Edelsbrunner, Harer '09]

$$\cdots \hookrightarrow H_*(f^{-1}(-\infty, s]) \hookrightarrow \cdots \hookrightarrow H_*(f^{-1}(-\infty, t]) \hookrightarrow \cdots \hookrightarrow H_*(X) \to$$

$$\cdots \hookrightarrow H_*(X, f^{-1}[v, \infty)) \hookrightarrow \cdots \hookrightarrow H_*(X, f^{-1}[u, \infty)) \hookrightarrow \cdots$$

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Theorem [Carlsson, de Silva, Morozov '09] The persistence diagrams of extended pesistence and level set persistence are in bijective correspondence.

- Some features appear in different degrees across this correspondence

FUNCTORIAL EQUIVALENCE

Can we make the correspondence of extended and level set persistence functorial (an equivalence of categories)?

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Yes using RISC

(level set pesistence in a derived category)

RISC ≅ DERIVED LEVEL SET PERSISTENCE

Relative
Interlevel
Set
Cohomology

$$f : X \to \mathbb{R}$$

$$\mathcal{M} \to \text{rect}$$

$h(f)$:

RISC ≅ DERIVED LEVEL SET PERSISTENCE

Derived
Level
Set
Persistence

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$$\begin{tikzcd}
 & f : X \to \mathbb{R} \arrow[dl] \arrow[dr] & \\
Rf_*(k_X) : D^+(Sh(\mathbb{R})) & & h(f) : \mathbb{M} \to \mathrm{vect}
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(X locally contractible)

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\begin{tikzcd}[column sep=small]
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Theorem [B, Fluhr '22] DLSP ≅ RISC.

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Theorem [B, Fluhr '22] DLSP ≅ RISC.

Depends crucially on morphisms across degrees!

CATEGORIFICATION OF EXTENDED PERSISTENCE

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- more structure (in particular : morphisms)

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$\forall$ s.e.s $0 \to A \to B \to C \to 0$

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Categorify (the group of signed) extended persistence diagrams!

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- This map is additive $\Rightarrow$ well-defined on K_0

RISC CATEGORIFIES EXTENDED PERSISTENCE

Theorem [B, Fluhr '22] The signed persistence map categorifies extended persistence diagrams:

$$K_0(\text{pres } J) \xrightarrow{\cong} \{\text{signed ext. PDs}\}$$

TAKE-HOME MESSAGES

Indecomposables in multi-parameter persistence:

- can be complicated, even in H_0

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