Morse theory and persistent homology of geometric complexes

Ulrich Bauer

TUM

February 9, 2023

Funded by

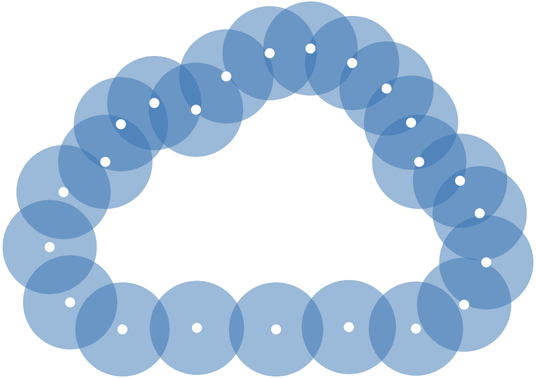
Deutsche Forschungsgemeinschaft

German Research Foundation

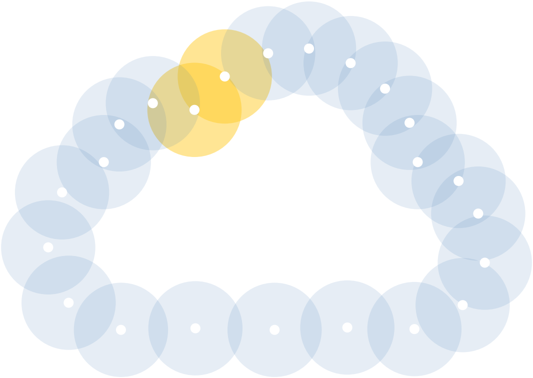
SFB TRR 109

Discretization in Geometry and Dynamics

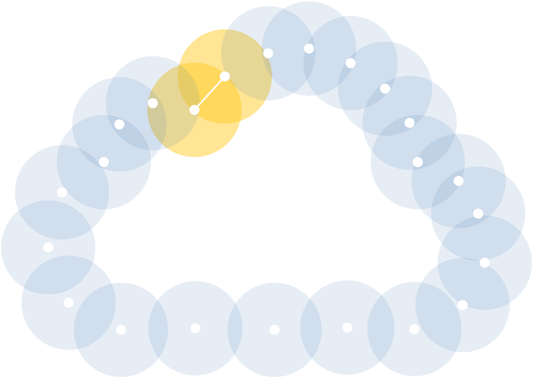
Čech complexes



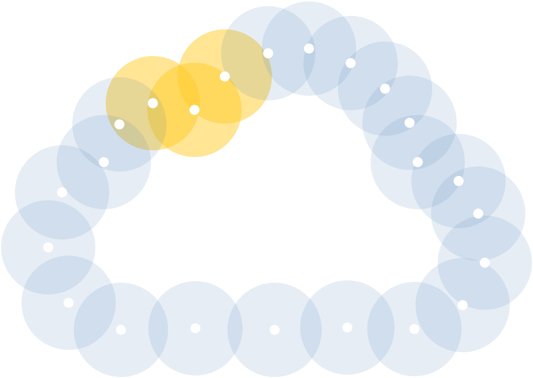
Čech complexes



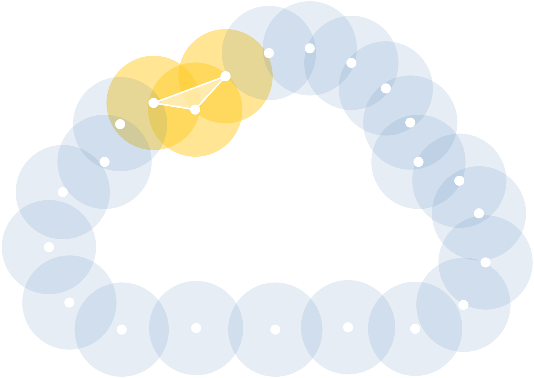
Čech complexes



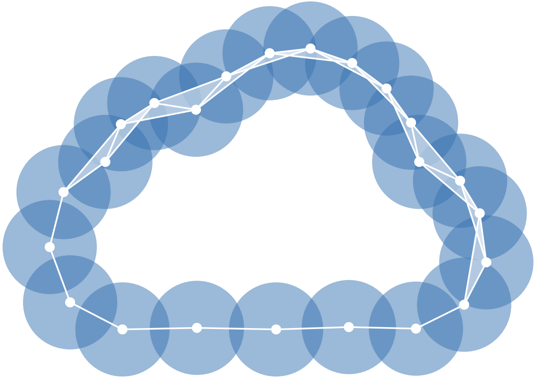
Čech complexes



Čech complexes



Čech complexes



Nerves

Definition (Alexandrov 1928)

Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\operatorname{Nrv}(\mathcal{U}) = \left\{ J \subseteq I \mid |J| < \infty \text{ and } \bigcap_{i \in J} U_i \neq \varnothing \right\} .$$

Nerves

Definition (Alexandrov 1928)

Let $\mathcal{U} = (U_i)_{i\in I}$ be a cover of a space X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \left\{ J \subseteq I \;\middle|\; |J| < \infty \text{ and } \bigcap_{i\in J} U_i \neq \varnothing \right\} .$$

Theorem (Borsuk 1948, and many more)

Let $\mathcal{U}$ be a good cover *of X. Then $\mathrm{Nrv}(\mathcal{U})$ is homotopy equivalent to X.*

Nerves

Definition (Alexandrov 1928)

Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \left\{ J \subseteq I \;\middle|\; |J| < \infty \text{ and } \bigcap_{i \in J} U_i \neq \varnothing \right\} .$$

Theorem (Borsuk 1948, and many more)

Let $\mathcal{U}$ be a good cover *of X. Then $\mathrm{Nrv}(\mathcal{U})$ is homotopy equivalent to X.*

Here *good* can mean different things:

- open, numerable cover, contractible intersections
- closed, convex finite cover

Nerves

Definition (Alexandrov 1928)

Let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of a space X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \left\{ J \subseteq I \mid |J| < \infty \text{ and } \bigcap_{i \in J} U_i \neq \emptyset \right\} .$$

Theorem (Borsuk 1948, and many more)

Let $\mathcal{U}$ be a good cover *of X. Then $\mathrm{Nrv}(\mathcal{U})$ is homotopy equivalent to X.*

Here *good* can mean different things:

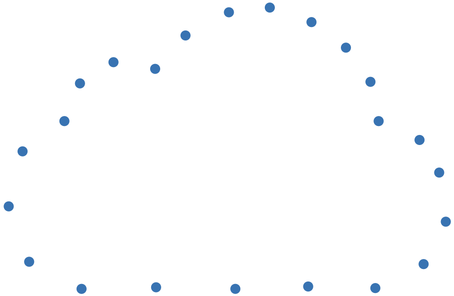
- open, numerable cover, contractible intersections
- closed, convex finite cover

U. Bauer, M. Kerber, F. Roll, and A. Rolle

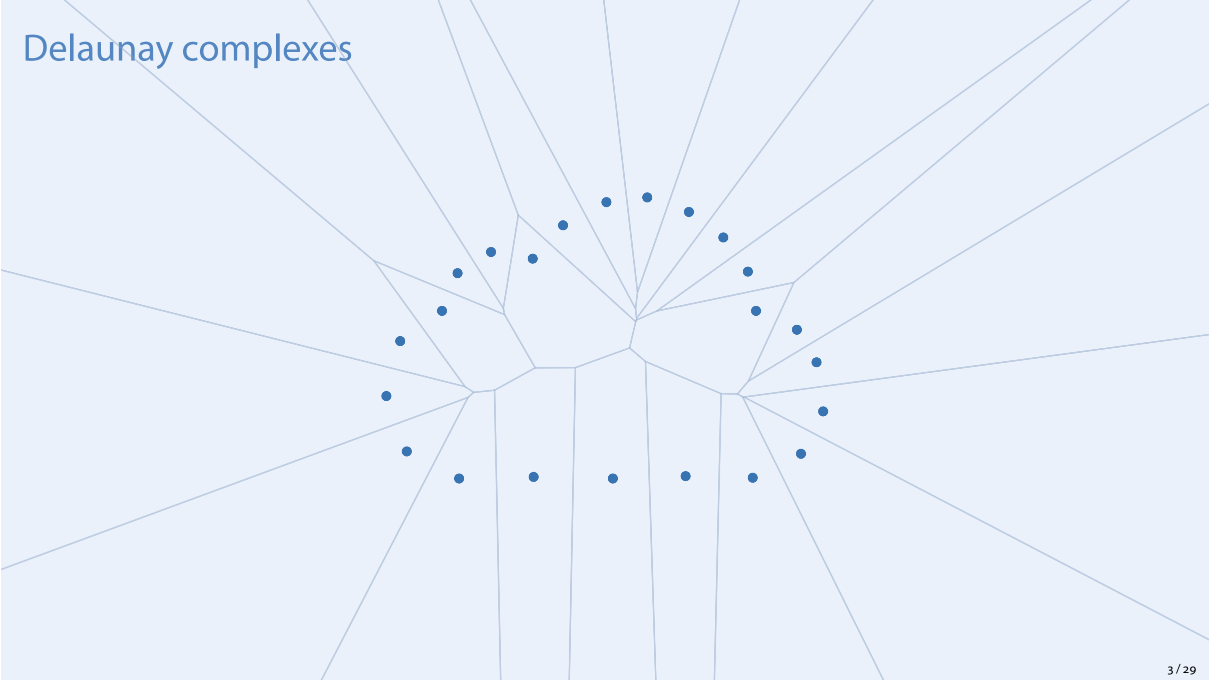
A Unified View on the Functorial Nerve Theorem and its Variations

Preprint, arXiv:2203.03571, 2022

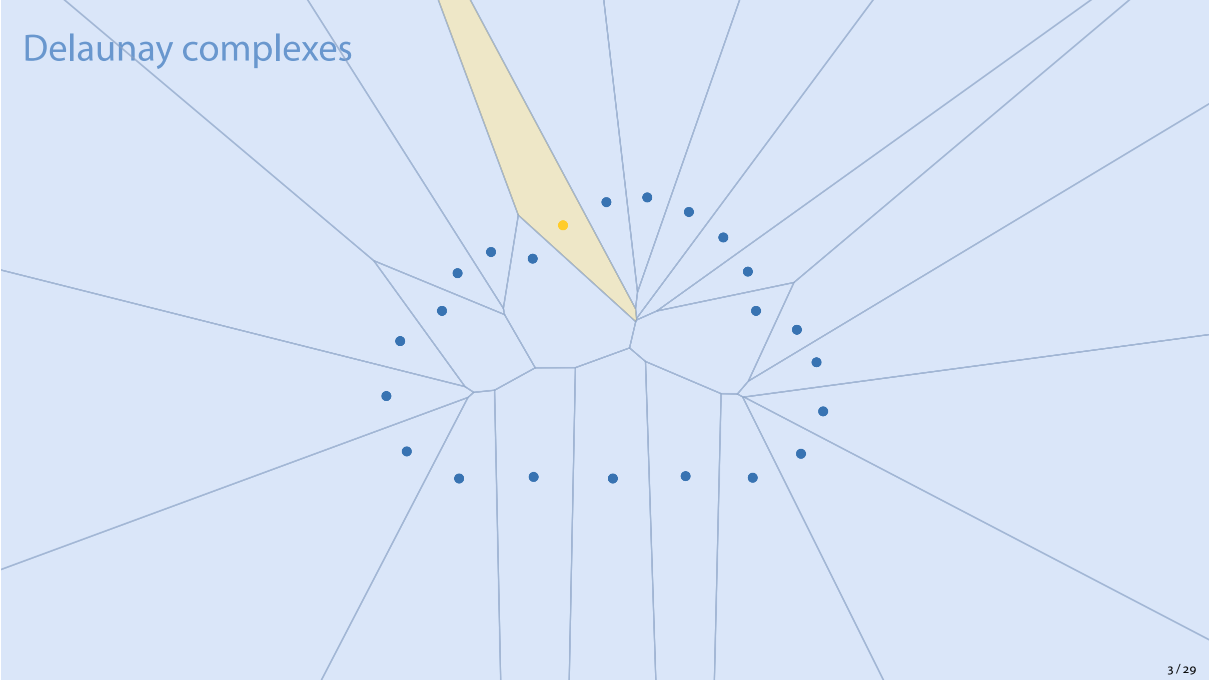
Delaunay complexes



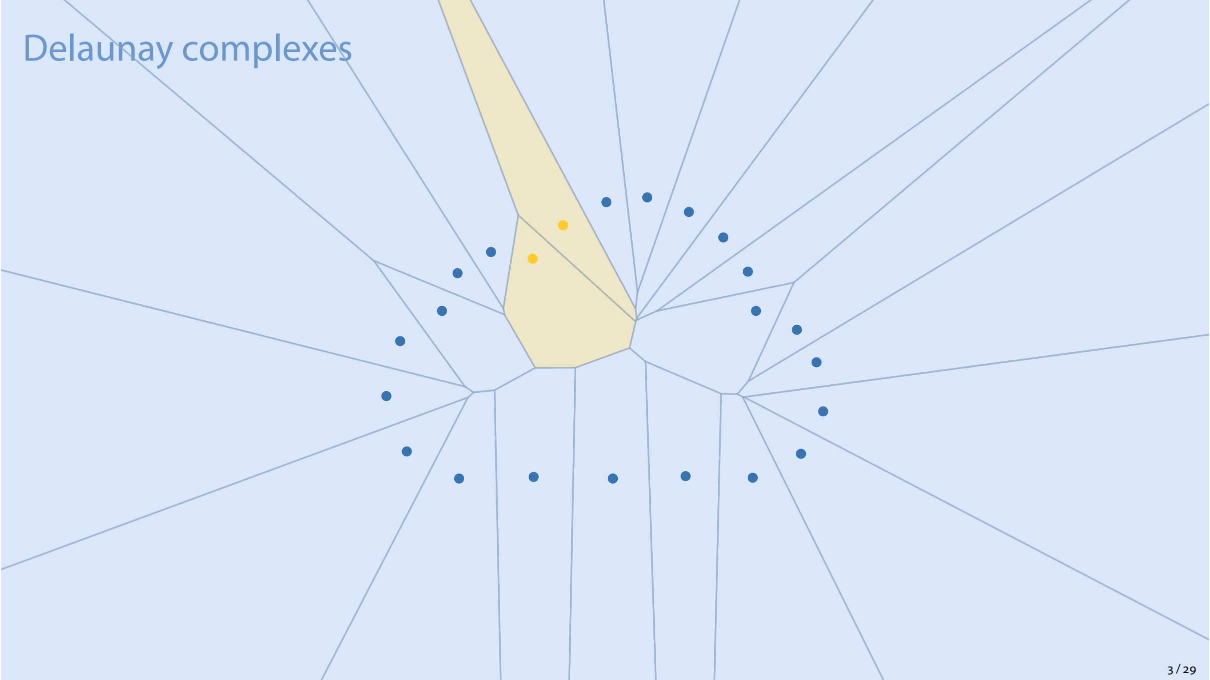
Delaunay complexes



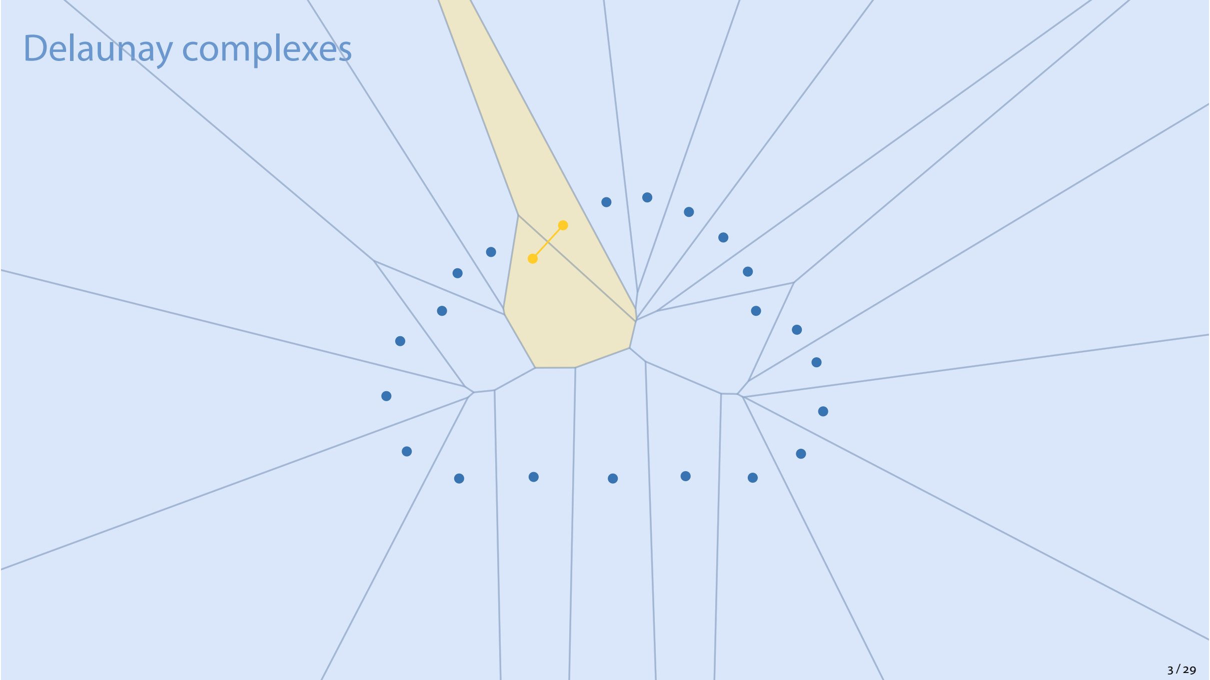
Delaunay complexes



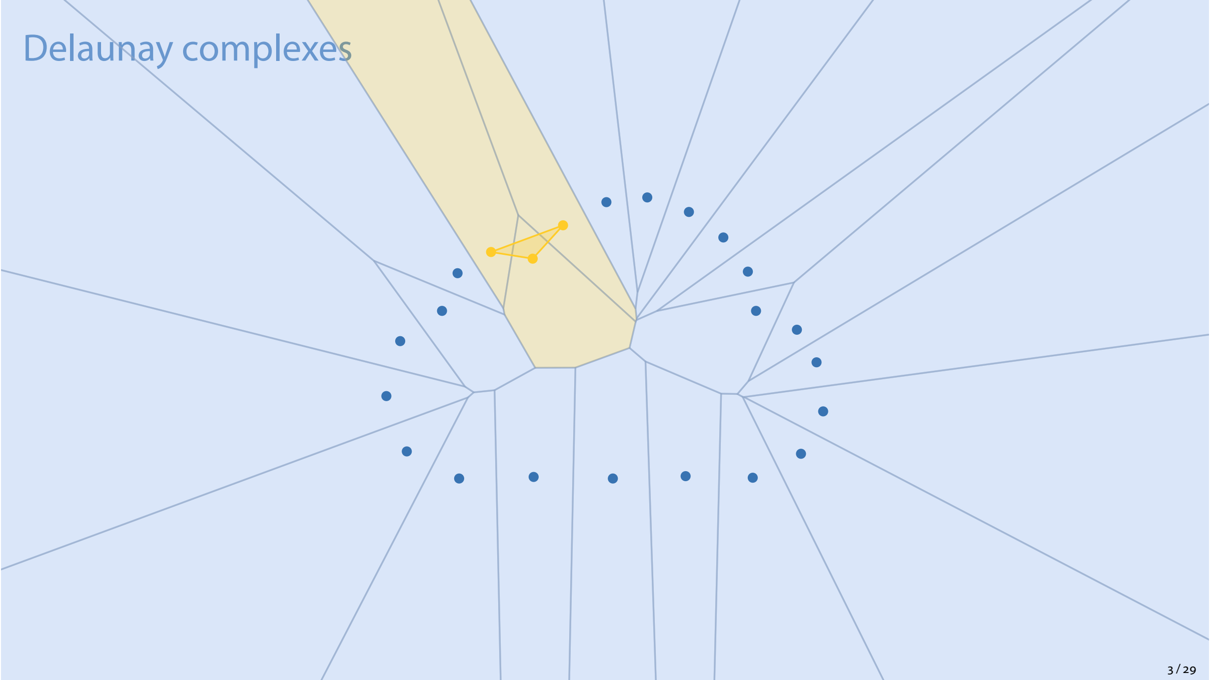
Delaunay complexes



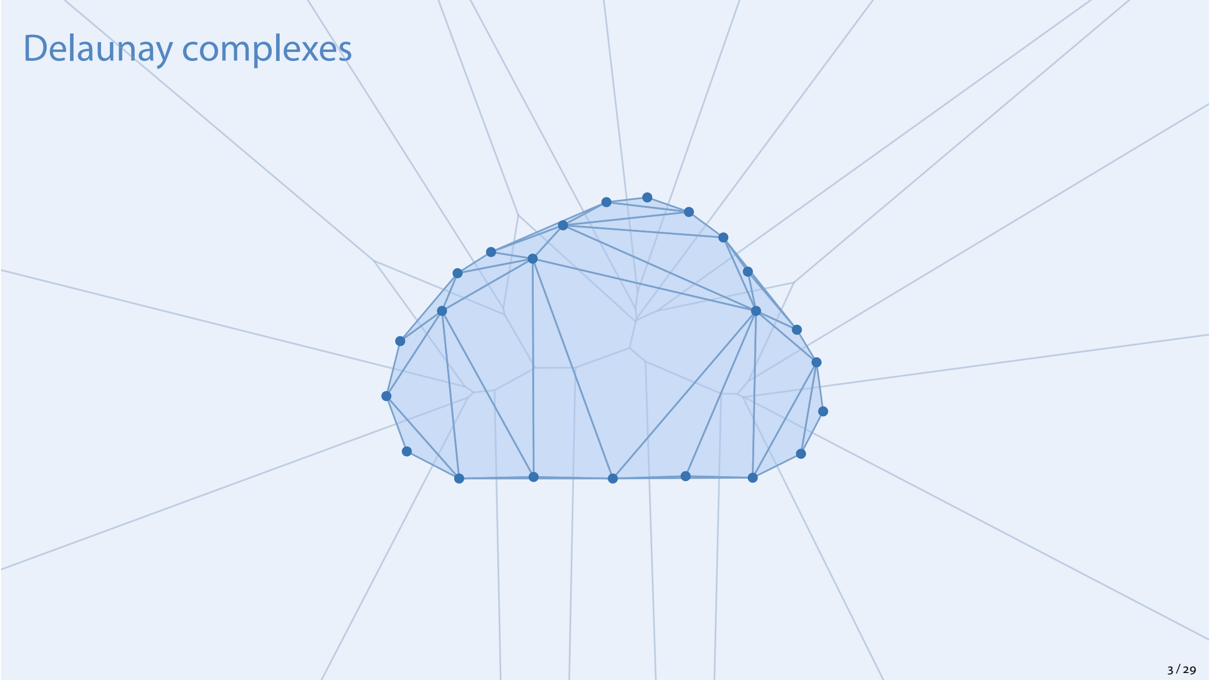
Delaunay complexes



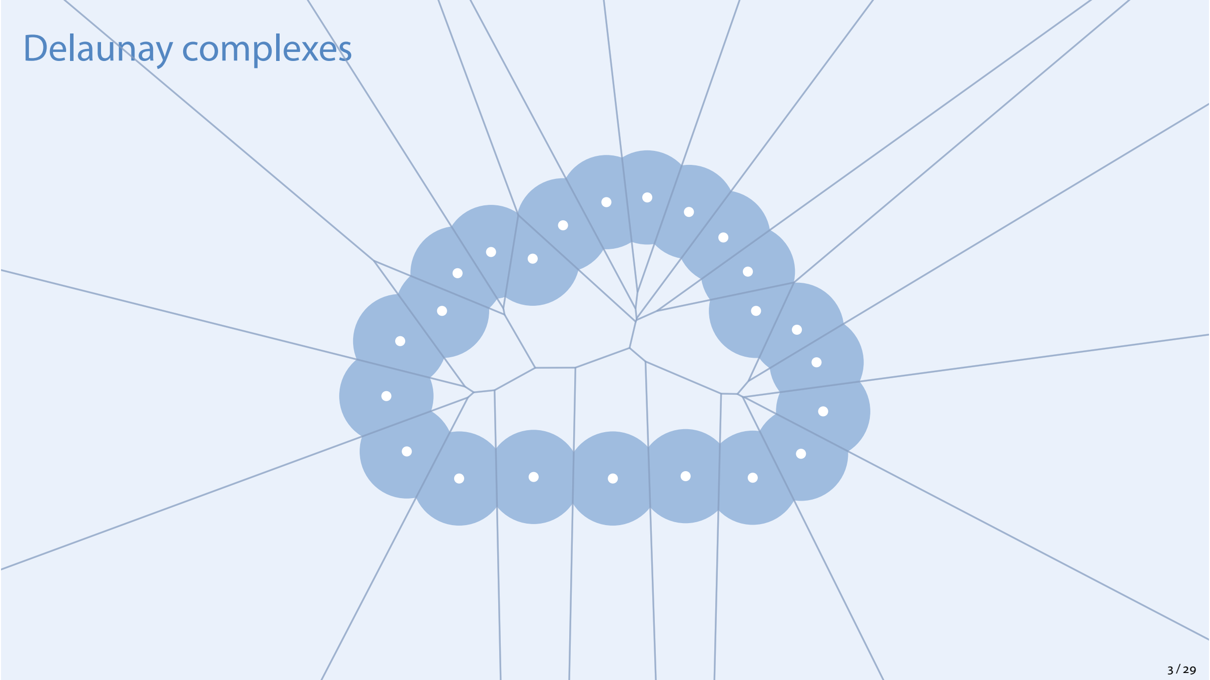
Delaunay complexes



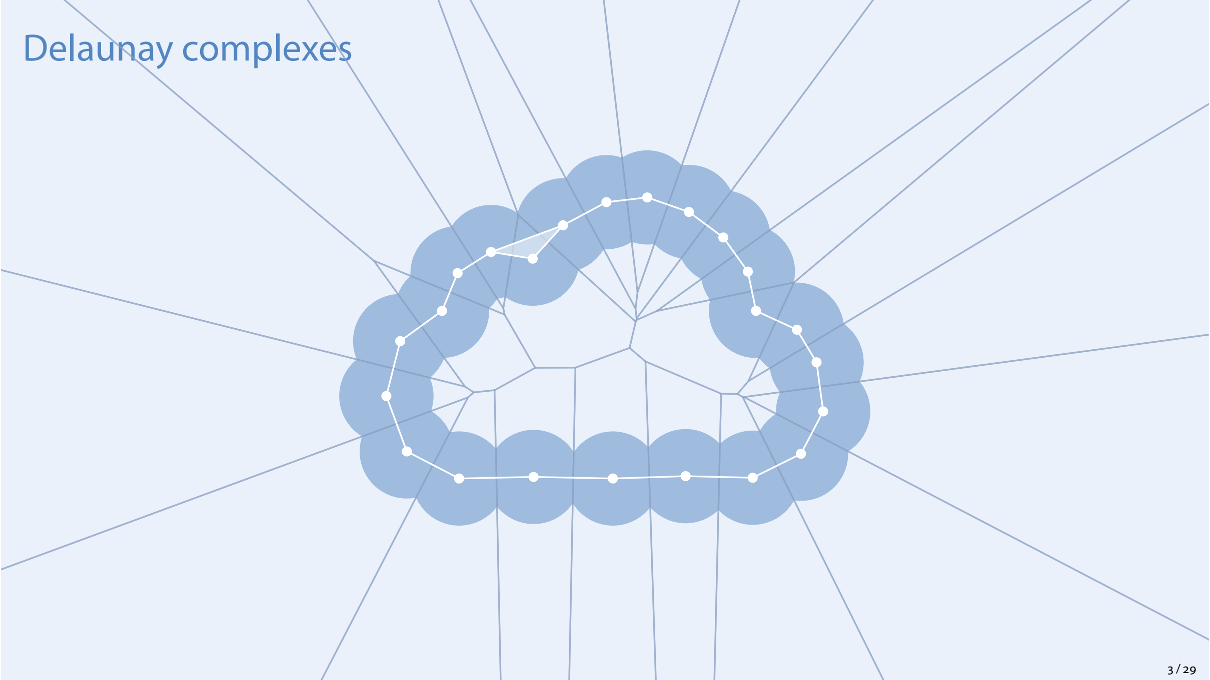
Delaunay complexes



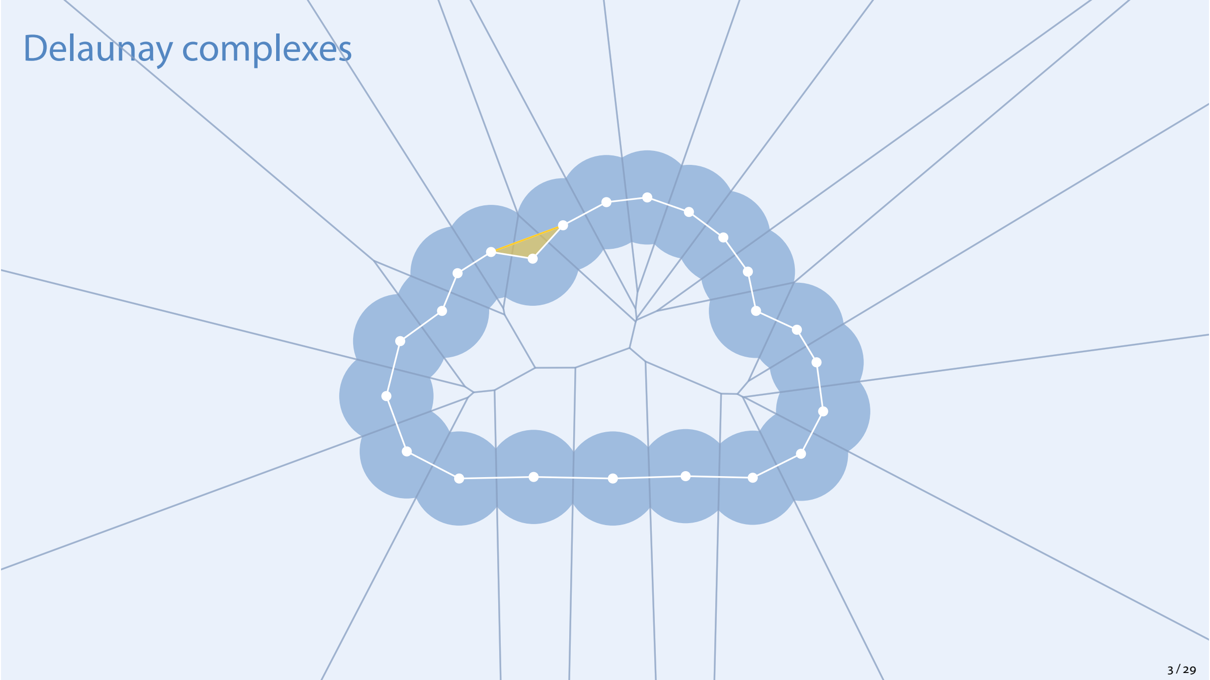
Delaunay complexes



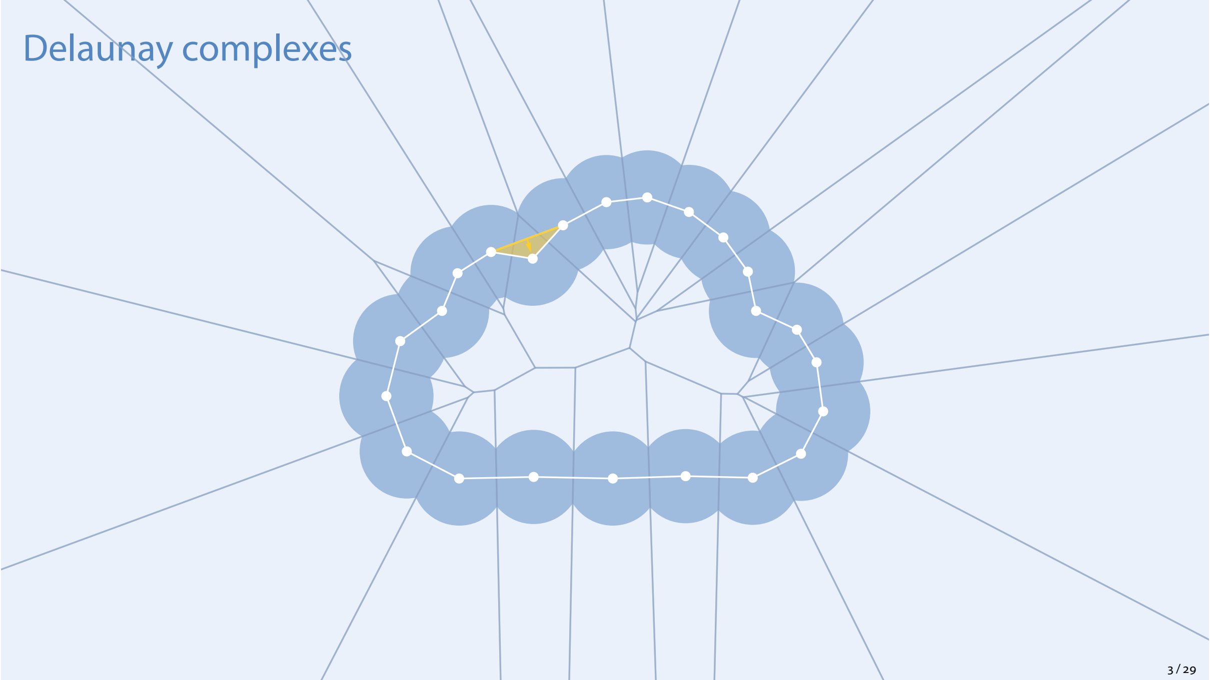
Delaunay complexes



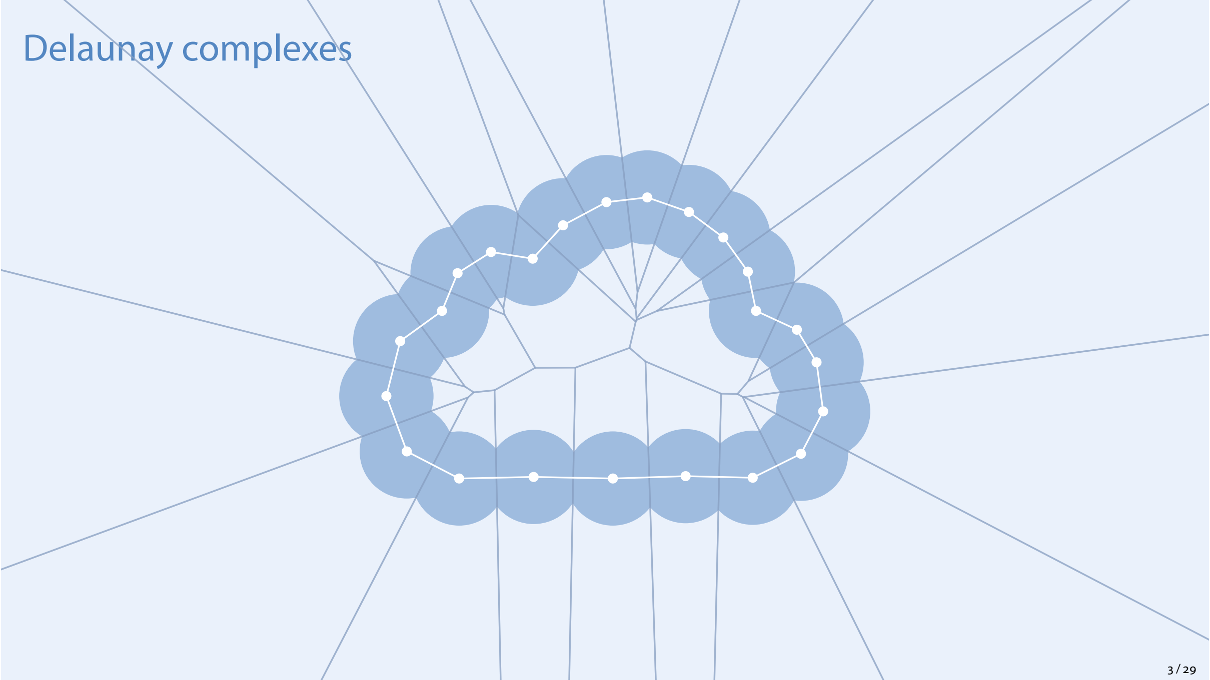
Delaunay complexes



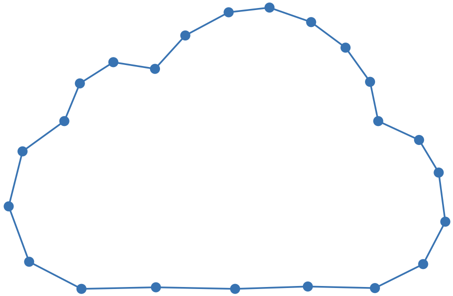
Delaunay complexes



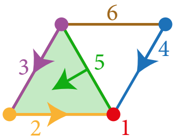
Delaunay complexes



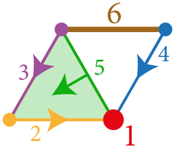
Delaunay complexes



Discrete Morse theory



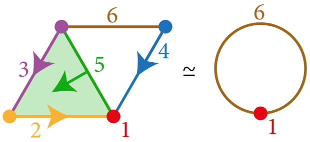
Discrete Morse theory



Discrete Morse theory

Theorem (Forman 1998)

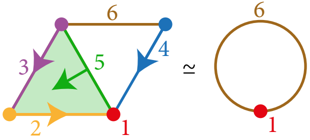
A simplicial complex with a discrete Morse function f is homotopy equivalent to a CW complex build from the critical simplices of f.



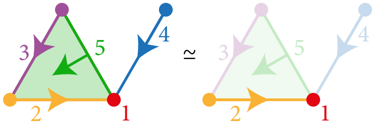
Discrete Morse theory

Theorem (Forman 1998)

A simplicial complex with a discrete Morse function f is homotopy equivalent to a CW complex build from the critical simplices of f.



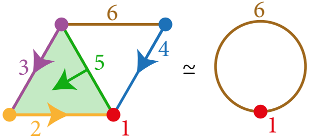
Discrete Morse functions – and their gradients – encode collapses of sublevel sets:



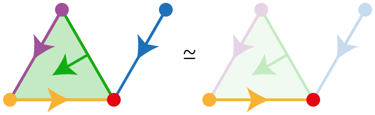
Discrete Morse theory

Theorem (Forman 1998)

A simplicial complex with a discrete Morse function f is homotopy equivalent to a CW complex build from the critical simplices of f.

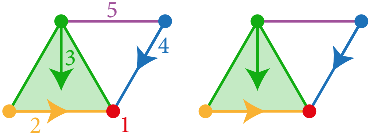


Discrete Morse functions – and their gradients – encode collapses of sublevel sets:



Generalizing discrete Morse theory

Generalized discrete Morse functions/gradients:



Morse theory for Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

The Čech complexes and the Delaunay complexes (alpha shapes) are sublevel sets of (generalized) discrete Morse functions.

Morse theory for Čech and Delaunay complexes

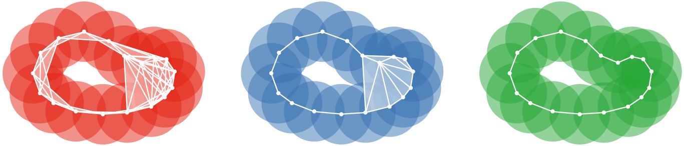
Proposition (B, Edelsbrunner 2014)

The Čech complexes and the Delaunay complexes (alpha shapes) are sublevel sets of (generalized) discrete Morse functions.

Theorem (B., Edelsbrunner 2017)

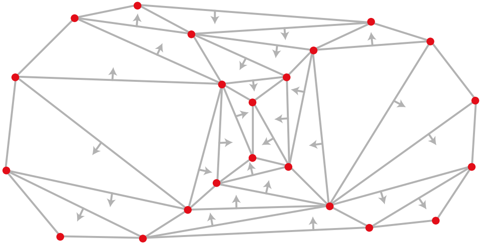
Čech, Delaunay, and Wrap complexes (at any scale r) are related by collapses encoded by a single discrete gradient field:

$$\mathrm{Cech}_r X \searrow \mathrm{Del}_r X \searrow \mathrm{Wrap}_r X.$$



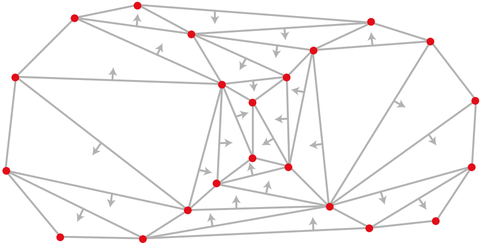
Wrap complexes

Consider the gradient V of the Delaunay function $\mathrm{Del}(X) \to \mathbb{R}$.



Wrap complexes

Consider the gradient V of the Delaunay function $\mathrm{Del}(X) \to \mathbb{R}$.



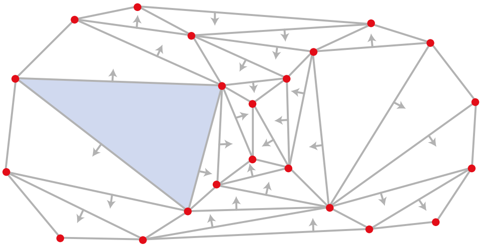
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

Define $\mathrm{Wrap}_r(X)$ as the smallest subcomplex of $\mathrm{Del}\,X$ that

- contains all critical simplices with circumradius $\leq r$ and
- is a union of face intervals of V.

Wrap complexes

Consider the gradient V of the Delaunay function $\mathrm{Del}(X) \to \mathbb{R}$.



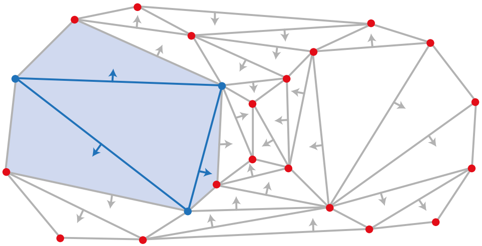
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

Define $\mathrm{Wrap}_r(X)$ as the smallest subcomplex of $\mathrm{Del}\, X$ that

- contains all critical simplices with circumradius $\leq r$ and
- is a union of face intervals of V.

Wrap complexes

Consider the gradient V of the Delaunay function $\mathrm{Del}(X) \to \mathbb{R}$.



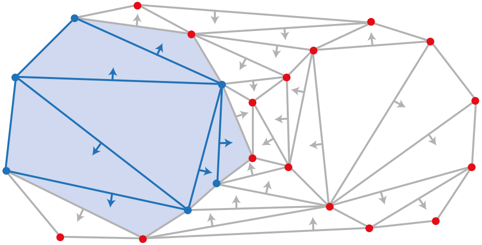
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

Define $\mathrm{Wrap}_r(X)$ as the smallest subcomplex of $\mathrm{Del}\, X$ that

- contains all critical simplices with circumradius $\leq r$ and
- is a union of face intervals of V.

Wrap complexes

Consider the gradient V of the Delaunay function $\mathrm{Del}(X) \to \mathbb{R}$.



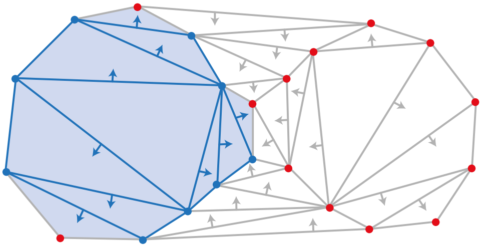
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

Define $\mathrm{Wrap}_r(X)$ as the smallest subcomplex of $\mathrm{Del}\, X$ that

- contains all critical simplices with circumradius $\leq r$ and
- is a union of face intervals of V.

Wrap complexes

Consider the gradient V of the Delaunay function $\mathrm{Del}(X) \to \mathbb{R}$.



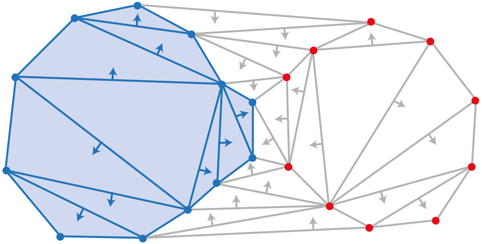
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

Define $\mathrm{Wrap}_r(X)$ as the smallest subcomplex of $\mathrm{Del}\, X$ that

- contains all critical simplices with circumradius $\leq r$ and
- is a union of face intervals of V.

Wrap complexes

Consider the gradient V of the Delaunay function $\mathrm{Del}(X) \to \mathbb{R}$.



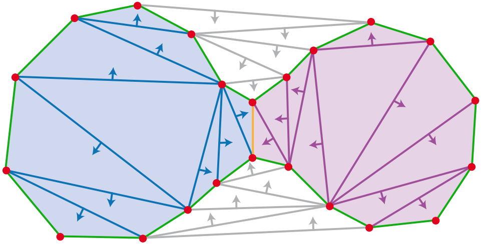
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

Define $\mathrm{Wrap}_r(X)$ as the smallest subcomplex of $\mathrm{Del}\, X$ that

- contains all critical simplices with circumradius $\leq r$ and
- is a union of face intervals of V.

Wrap complexes

Consider the gradient V of the Delaunay function $\mathrm{Del}(X) \to \mathbb{R}$.



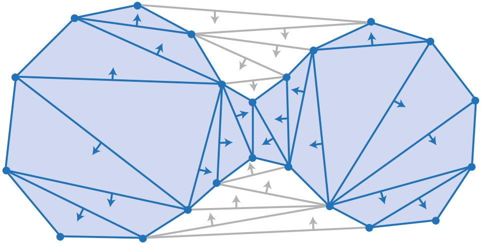
Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

Define $\mathrm{Wrap}_r(X)$ as the smallest subcomplex of $\mathrm{Del}\, X$ that

- contains all critical simplices with circumradius $\leq r$ and
- is a union of face intervals of V.

Wrap complexes

Consider the gradient V of the Delaunay function $\mathrm{Del}(X) \to \mathbb{R}$.



Definition (Edelsbrunner 1995; B, Edelsbrunner 2017)

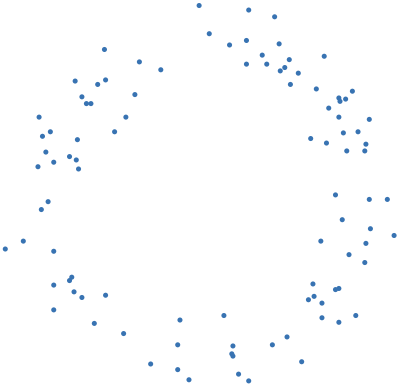
Define $\mathrm{Wrap}_r(X)$ as the smallest subcomplex of $\mathrm{Del}\, X$ that

- contains all critical simplices with circumradius $\leq r$ and
- is a union of face intervals of V.

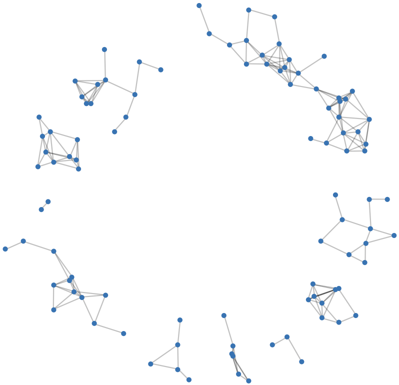
Delaunay and Wrap complexes

Delaunay and Wrap complexes

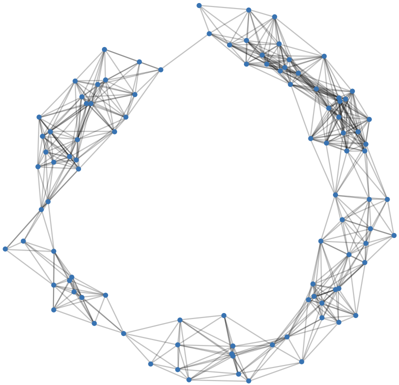
Vietoris–Rips complexes



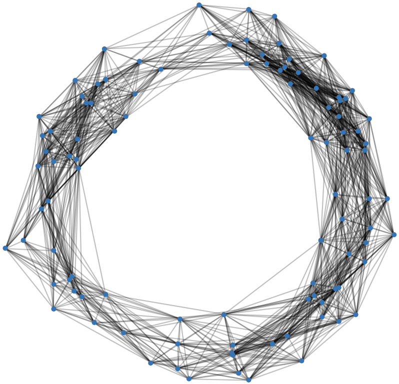
Vietoris–Rips complexes



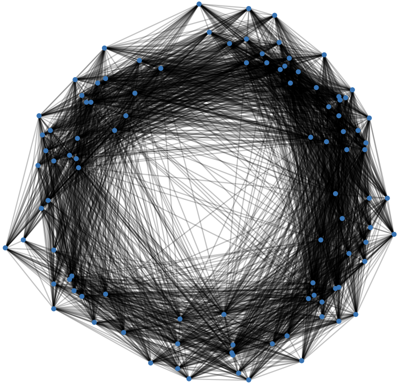
Vietoris–Rips complexes



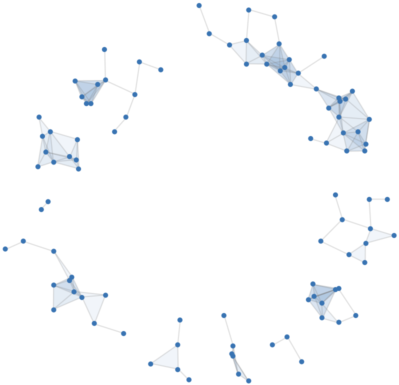
Vietoris–Rips complexes



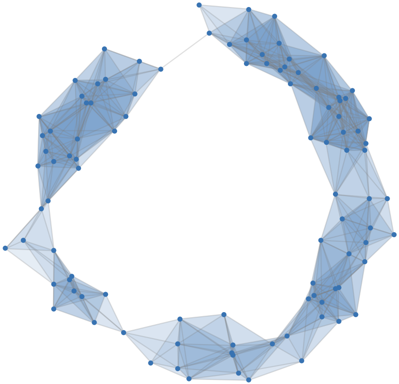
Vietoris–Rips complexes



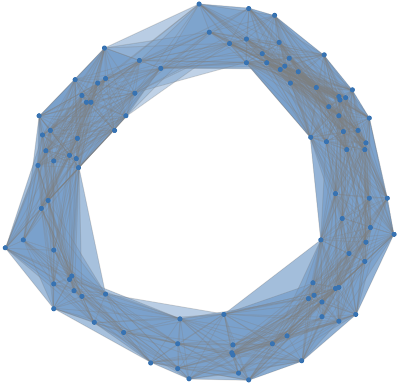
Vietoris–Rips complexes



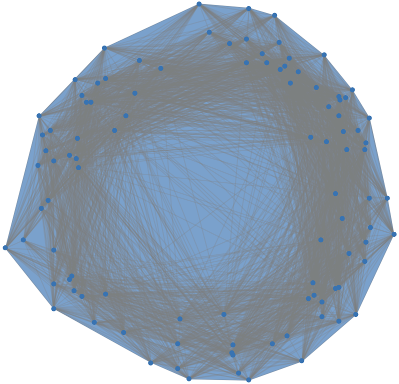
Vietoris–Rips complexes



Vietoris–Rips complexes



Vietoris–Rips complexes



Vietoris–Rips complexes

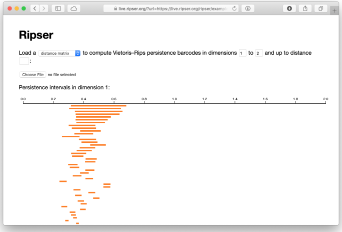
For a metric space X, the *Vietoris–Rips complex* at $t > 0$ is the simplicial complex

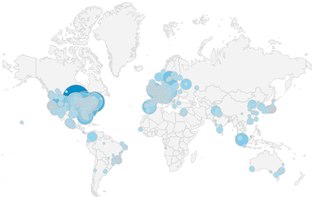
$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid S \neq \varnothing \text{ finite}, \operatorname{diam} S \leq t\}.$$

Ripser: software for computing Vietoris–Rips persistence barcodes

Open source software (`ripser.org`)

- significantly faster / more memory efficient than previous codes
- de facto standard for topological data analysis applications
- most popular persistent homology project on GitHub





Ripser users from 935 different locations

Computational improvements based on

- *implicit matrix representations*
- connecting persistence to *discrete Morse theory*

Apparent pairs

Ripser uses the following construction for a computational shortcut:

Definition

In a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, two simplices (σ_i, σ_j) are an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

Apparent pairs

Ripser uses the following construction for a computational shortcut:

Definition

In a simplexwise filtration $(K_i = \{\sigma_1, \ldots, \sigma_i\})_i$, two simplices (σ_i, σ_j) are an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

For the Vietoris–Rips filtration, we order simplices with same diameter lexicographically.

Apparent pairs

Ripser uses the following construction for a computational shortcut:

Definition

In a simplexwise filtration $(K_i = \{\sigma_1, \dots, \sigma_i\})_i$, two simplices (σ_i, σ_j) are an *apparent pair* if

- σ_i is the latest proper face of σ_j, and
- σ_j is the earliest proper coface of σ_i.

For the Vietoris–Rips filtration, we order simplices with same diameter lexicographically.

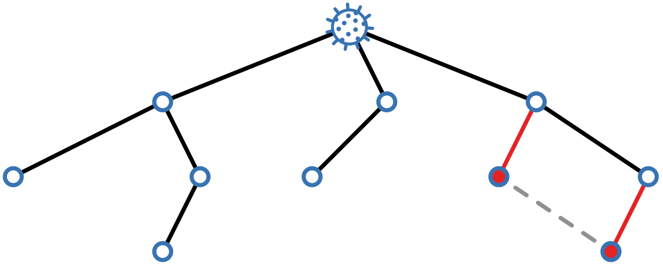
Proposition

The apparent pairs form a discrete gradient.

Proposition

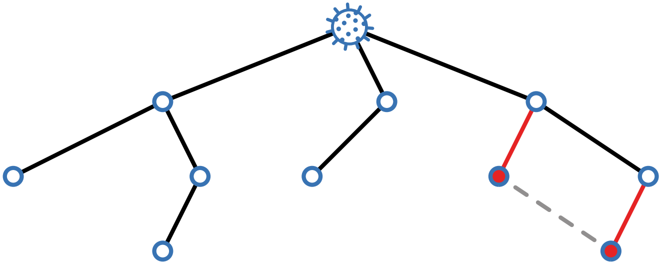
The apparent pairs form persistence pairs.

Topology of viral evolution



Joint work with: A. Ott, M. Bleher, L. Hahn (Heidelberg), R. Rabadan, J. Patiño-Galindo (Columbia), M. Carrière (INRIA)

Topology of viral evolution



Joint work with: A. Ott, M. Bleher, L. Hahn (Heidelberg), R. Rabadan, J. Patiño-Galindo (Columbia), M. Carrière (INRIA)

Observation: Ripser runs unusually fast on genetic distance data

- SARS-CoV2 RNA sequences (spike protein)
- 25556 data points (2.8×10^{12} simplices in 2-skeleton)
- $120\,s$ computation time (with data points ordered appropriately)

The Rips Contractibility Lemma

Theorem (Rips; Gromov 1988)

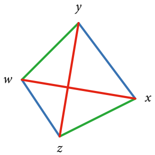
Let X be a δ-hyperbolic geodesic metric space. Then $\mathrm{Rips}_t(X)$ is contractible for all $t \geq 4\delta$.

Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ-*hyperbolic* if for all $w, x, y, z \in X$ we have

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$

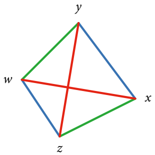


Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ-*hyperbolic* if for all $w, x, y, z \in X$ we have

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$



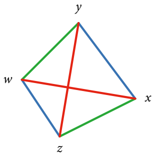
- The hyperbolic plane is $\ln 2$-hyperbolic

Gromov-hyperbolicity

Definition (Gromov 1988)

A metric space X is δ-*hyperbolic* if for all $w, x, y, z \in X$ we have

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(w,z) + d(x,y)\} + 2\delta.$$



- The hyperbolic plane is $\ln 2$-hyperbolic

- 0-hyperbolic spaces are subspaces of trees

Rips contractibility for non-geodesic spaces

- What about non-geodesic spaces? In particular, finite metric spaces?
- Connection to apparent pairs/Ripser?

Rips contractibility for non-geodesic spaces

- What about non-geodesic spaces? In particular, finite metric spaces?
- Connection to apparent pairs/Ripser?

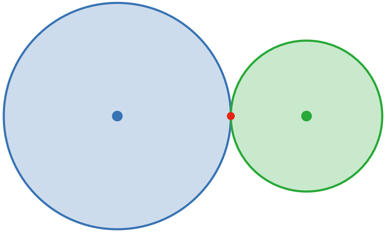
Theorem (B, Roll 2022)

Let X be a finite δ-hyperbolic space. Then there is a single discrete gradient encoding the collapses

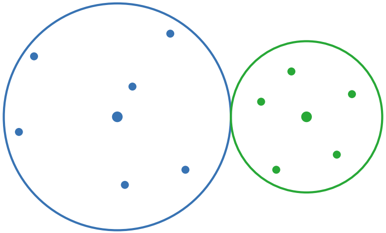
$$\mathrm{Rips}_u(X) \searrow \mathrm{Rips}_t(X) \searrow \{*\}$$

for all $u > t \geq 4\delta + 2\nu$, where ν is the geodesic defect *of X.*

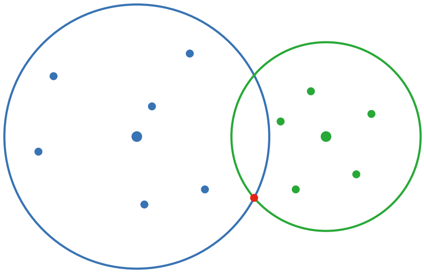
Geodesic defect



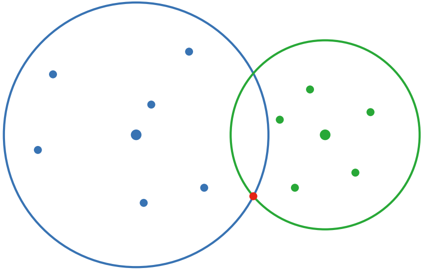
Geodesic defect



Geodesic defect



Geodesic defect



Definition (Bonk, Schramm 2000)

A metric space X is ν-*geodesic* if for all points $x, y \in X$ and all $r, s \geq 0$ with $r + s = d(x, y)$ we have

$$B_{r+\nu}(x) \cap B_{r+\nu}(y) \neq \varnothing.$$

The infimum of all such ν is the *geodesic defect* of X.

Collapsing Vietoris–Rips complexes of generic trees

Consider a *generic* finite metric tree $T = (V, E)$ (distinct distances).

- $\mathrm{diam}\colon 2^V \to \mathbb{R}$ is a generalized discrete Morse function.

Collapsing Vietoris–Rips complexes of generic trees

Consider a *generic* finite metric tree $T = (V, E)$ (distinct distances).

- $\operatorname{diam}: 2^V \to \mathbb{R}$ is a generalized discrete Morse function.
- Its discrete gradient consists of the intervals $[e, \Delta_e]$, where

Collapsing Vietoris–Rips complexes of generic trees

Consider a *generic* finite metric tree $T = (V, E)$ (distinct distances).

- $\operatorname{diam}: 2^V \to \mathbb{R}$ is a generalized discrete Morse function.
- Its discrete gradient consists of the intervals $[e, \Delta_e]$, where
  - e is any non-tree edge, and

Collapsing Vietoris–Rips complexes of generic trees

Consider a *generic* finite metric tree $T = (V, E)$ (distinct distances).

- $\operatorname{diam}\colon 2^V \to \mathbb{R}$ is a generalized discrete Morse function.
- Its discrete gradient consists of the intervals $[e, \Delta_e]$, where
  - e is any non-tree edge, and
  - Δ_e is the unique maximal coface of e with same diameter

Collapsing Vietoris–Rips complexes of generic trees

Consider a *generic* finite metric tree $T = (V, E)$ (distinct distances).

- $\operatorname{diam}: 2^V \to \mathbb{R}$ is a generalized discrete Morse function.
- Its discrete gradient consists of the intervals $[e, \Delta_e]$, where
  - e is any non-tree edge, and
  - Δ_e is the unique maximal coface of e with same diameter
- Only tree simplices V, E are critical

Collapsing Vietoris–Rips complexes of generic trees

Consider a *generic* finite metric tree $T = (V, E)$ (distinct distances).

- $\operatorname{diam}: 2^V \to \mathbb{R}$ is a generalized discrete Morse function.
- Its discrete gradient consists of the intervals $[e, \Delta_e]$, where
  - e is any non-tree edge, and
  - Δ_e is the unique maximal coface of e with same diameter
- Only tree simplices V, E are critical
- The apparent pairs refine this gradient

Collapsing Vietoris–Rips complexes of generic trees

Consider a *generic* finite metric tree $T = (V, E)$ (distinct distances).

- $\operatorname{diam}: 2^V \to \mathbb{R}$ is a generalized discrete Morse function.
- Its discrete gradient consists of the intervals $[e, \Delta_e]$, where
  - e is any non-tree edge, and
  - Δ_e is the unique maximal coface of e with same diameter
- Only tree simplices V, E are critical
- The apparent pairs refine this gradient

Theorem (B, Roll 2022)

This discrete gradient induces the collapses

$$\begin{aligned}
\operatorname{Rips}_t(X) &\searrow T_t && \textit{for all } t \in \mathbb{R}, \textit{ with} \\
\operatorname{Rips}_t(X) &\searrow T \searrow \{*\} && \textit{for all } t \geq \max l(E), \textit{ and} \\
\operatorname{Rips}_u(X) &\searrow \operatorname{Rips}_t(X) && \textit{for all } (t, u] \cap l(E) = \varnothing.
\end{aligned}$$

In particular, the persistent homology is trivial in degrees > 0.

Tree metrics beyond the generic case

Example: phylogenetic trees

- non-generic tree metric
- diam is not a generalized discrete Morse function

Tree metrics beyond the generic case

Example: phylogenetic trees

- non-generic tree metric
- diam is not a generalized discrete Morse function

There is still a compatible gradient:

- independent of choices (*canonical gradient*)
- for generic trees: equals the diam gradient
- induces the same collapses as in the generic case

Tree metrics beyond the generic case

Example: phylogenetic trees

- non-generic tree metric
- diam is not a generalized discrete Morse function

There is still a compatible gradient:

- independent of choices (*canonical gradient*)
- for generic trees: equals the diam gradient
- induces the same collapses as in the generic case

However, the canonical gradient is not compatible with the apparent pairs gradient.

Symbolic perturbation

Tie breaking for non-distinct pairwise distances:

- Choose total order on vertices
- Order edges lexicographically
- Perturb metric symbolically w.r.t. edge order

Symbolic perturbation

Tie breaking for non-distinct pairwise distances:

- Choose total order on vertices
- Order edges lexicographically
- Perturb metric symbolically w.r.t. edge order

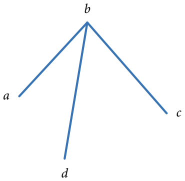
This results in a gradient structure like a generic tree metric

Symbolic perturbation

Tie breaking for non-distinct pairwise distances:

- Choose total order on vertices
- Order edges lexicographically
- Perturb metric symbolically w.r.t. edge order

This results in a gradient structure like a generic tree metric

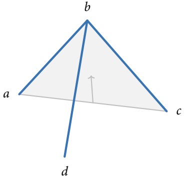


Symbolic perturbation

Tie breaking for non-distinct pairwise distances:

- Choose total order on vertices
- Order edges lexicographically
- Perturb metric symbolically w.r.t. edge order

This results in a gradient structure like a generic tree metric

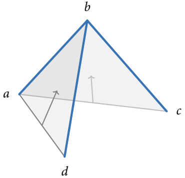


Symbolic perturbation

Tie breaking for non-distinct pairwise distances:

- Choose total order on vertices
- Order edges lexicographically
- Perturb metric symbolically w.r.t. edge order

This results in a gradient structure like a generic tree metric

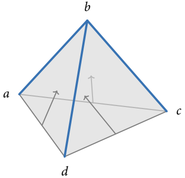


Symbolic perturbation

Tie breaking for non-distinct pairwise distances:

- Choose total order on vertices
- Order edges lexicographically
- Perturb metric symbolically w.r.t. edge order

This results in a gradient structure like a generic tree metric

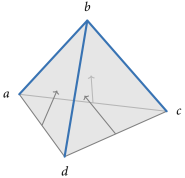


Symbolic perturbation

Tie breaking for non-distinct pairwise distances:

- Choose total order on vertices
- Order edges lexicographically
- Perturb metric symbolically w.r.t. edge order

This results in a gradient structure like a generic tree metric



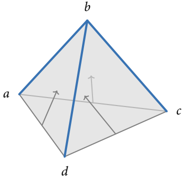
- The canonical gradient (from the previous slide) refines the perturbed gradient (for any choice of total order)

Symbolic perturbation

Tie breaking for non-distinct pairwise distances:

- Choose total order on vertices
- Order edges lexicographically
- Perturb metric symbolically w.r.t. edge order

This results in a gradient structure like a generic tree metric



- The canonical gradient (from the previous slide) refines the perturbed gradient (for any choice of total order)
- Hence, the perturbed gradient induces the same collapses as in the generic case

Collapsing Rips complexes of trees with apparent pairs

Let X be the path length metric space for a weighted tree $T = (X, E)$.

- Choose a root and extend the tree order to a total order.

Collapsing Rips complexes of trees with apparent pairs

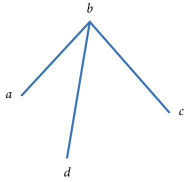
Let X be the path length metric space for a weighted tree $T = (X, E)$.

- Choose a root and extend the tree order to a total order.

Theorem (B, Roll 2022)

The resulting apparent pairs gradient refines the perturbed gradient.

Hence, it induces the same collapses as in the generic case.



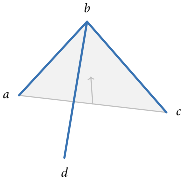
Collapsing Rips complexes of trees with apparent pairs

Let X be the path length metric space for a weighted tree $T = (X, E)$.

- Choose a root and extend the tree order to a total order.

Theorem (B, Roll 2022)

The resulting apparent pairs gradient refines the perturbed gradient.
Hence, it induces the same collapses as in the generic case.



Collapsing Rips complexes of trees with apparent pairs

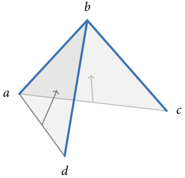
Let X be the path length metric space for a weighted tree $T = (X, E)$.

- Choose a root and extend the tree order to a total order.

Theorem (B, Roll 2022)

The resulting apparent pairs gradient refines the perturbed gradient.

Hence, it induces the same collapses as in the generic case.



Collapsing Rips complexes of trees with apparent pairs

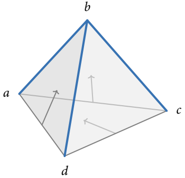
Let X be the path length metric space for a weighted tree $T = (X, E)$.

- Choose a root and extend the tree order to a total order.

Theorem (B, Roll 2022)

The resulting apparent pairs gradient refines the perturbed gradient.

Hence, it induces the same collapses as in the generic case.



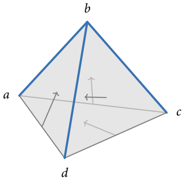
Collapsing Rips complexes of trees with apparent pairs

Let X be the path length metric space for a weighted tree $T = (X, E)$.

- Choose a root and extend the tree order to a total order.

Theorem (B, Roll 2022)

The resulting apparent pairs gradient refines the perturbed gradient.
Hence, it induces the same collapses as in the generic case.



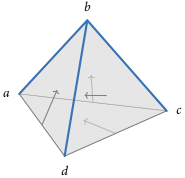
Collapsing Rips complexes of trees with apparent pairs

Let X be the path length metric space for a weighted tree $T = (X, E)$.

- Choose a root and extend the tree order to a total order.

Theorem (B, Roll 2022)

The resulting apparent pairs gradient refines the perturbed gradient.
Hence, it induces the same collapses as in the generic case.



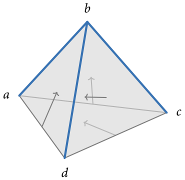
Collapsing Rips complexes of trees with apparent pairs

Let X be the path length metric space for a weighted tree $T = (X, E)$.

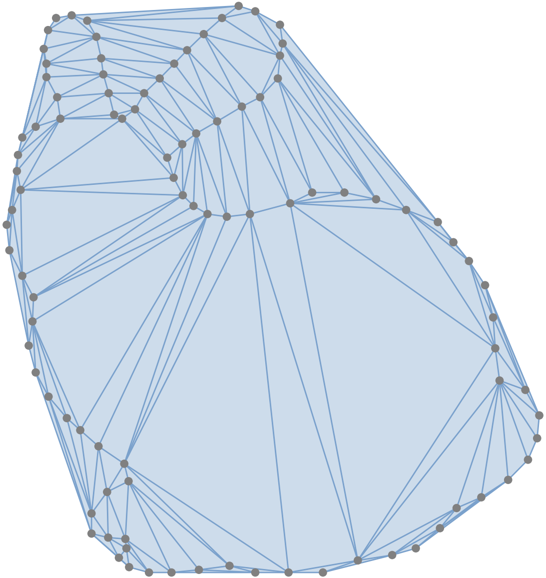
- Choose a root and extend the tree order to a total order.

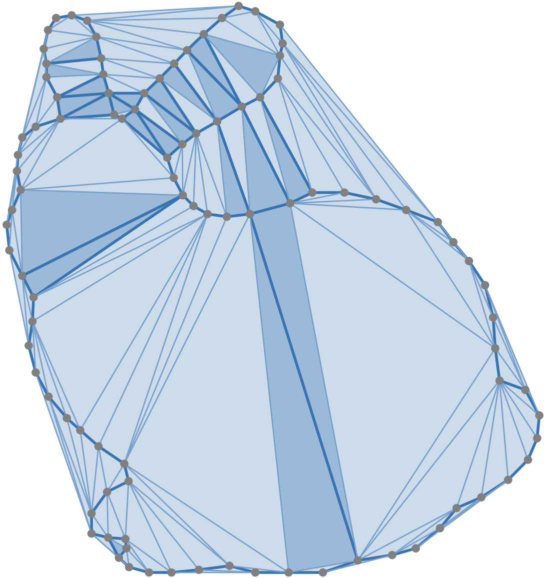
Theorem (B, Roll 2022)

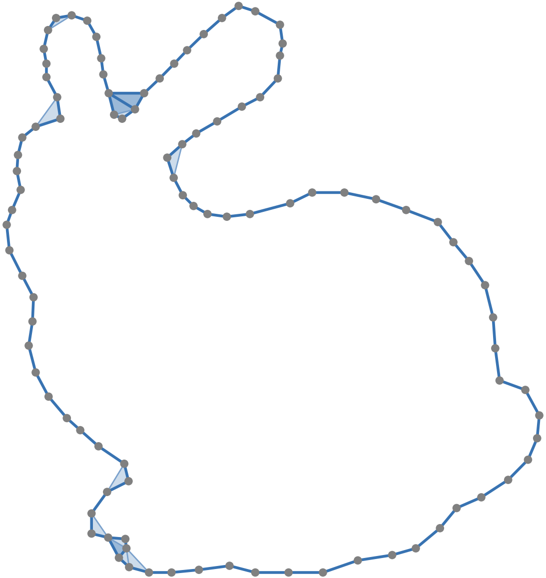
The resulting apparent pairs gradient refines the perturbed gradient.
Hence, it induces the same collapses as in the generic case.

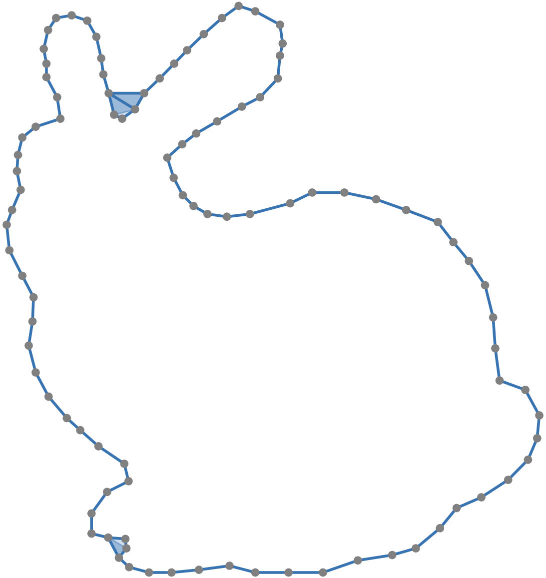


- Explains why Ripser is extraordinarily fast on genetic distances (tree-like)









Algebraic gradient flows and persistent homology

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a discrete gradient

Algebraic gradient flows and persistent homology

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a discrete gradient
- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to chain complexes (*algebraic Morse theory*)

Algebraic gradient flows and persistent homology

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a discrete gradient
- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to chain complexes (*algebraic Morse theory*)
- Cohen-Steiner–Lieutier–Vuillamy (2022) propose lexicographic optimal cycles for reconstruction from point clouds

Algebraic gradient flows and persistent homology

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a discrete gradient
- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to chain complexes (*algebraic Morse theory*)
- Cohen-Steiner–Lieutier–Vuillamy (2022) propose lexicographic optimal cycles for reconstruction from point clouds

Proposition (B, Roll 2022)

Computation of persistent homology can be interpreted in terms of algebraic Morse theory:

- *Persistence pairs correspond to algebraic gradient pairs.*

Algebraic gradient flows and persistent homology

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a discrete gradient
- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to chain complexes (*algebraic Morse theory*)
- Cohen-Steiner–Lieutier–Vuillamy (2022) propose lexicographic optimal cycles for reconstruction from point clouds

Proposition (B, Roll 2022)

Computation of persistent homology can be interpreted in terms of algebraic Morse theory:

- *Persistence pairs correspond to algebraic gradient pairs.*
- *Gradient flow corresponds to matrix reduction (computation of persistent homology)*

Algebraic gradient flows and persistent homology

- Forman (1998) defines a *flow* $C_*(K) \to C_*(K)$ associated to a discrete gradient
- Kozlov/Sköldberg/Jöllenbeck–Welker (2006/08/09) generalize discrete Morse theory to chain complexes (*algebraic Morse theory*)
- Cohen-Steiner–Lieutier–Vuillamy (2022) propose lexicographic optimal cycles for reconstruction from point clouds

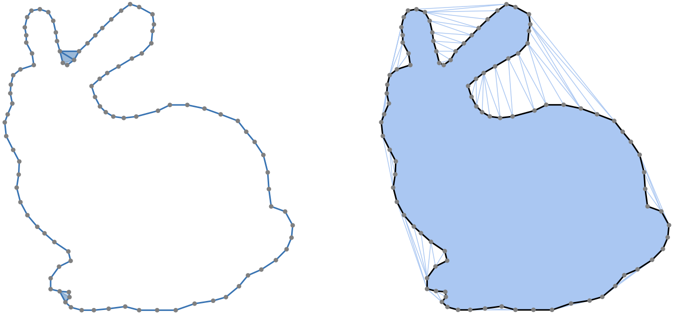
Proposition (B, Roll 2022)

Computation of persistent homology can be interpreted in terms of algebraic Morse theory:

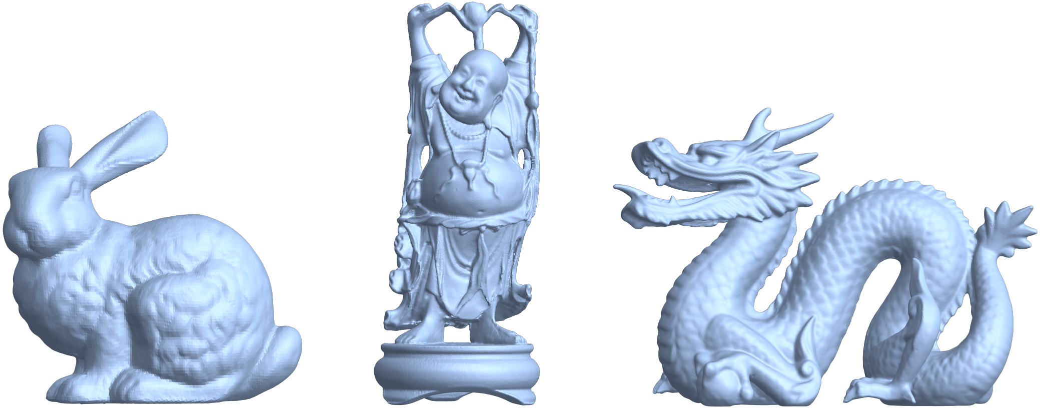
- *Persistence pairs correspond to algebraic gradient pairs.*
- *Gradient flow corresponds to matrix reduction (computation of persistent homology)*
- *The resulting cycle representatives are lexicographically optimal.*

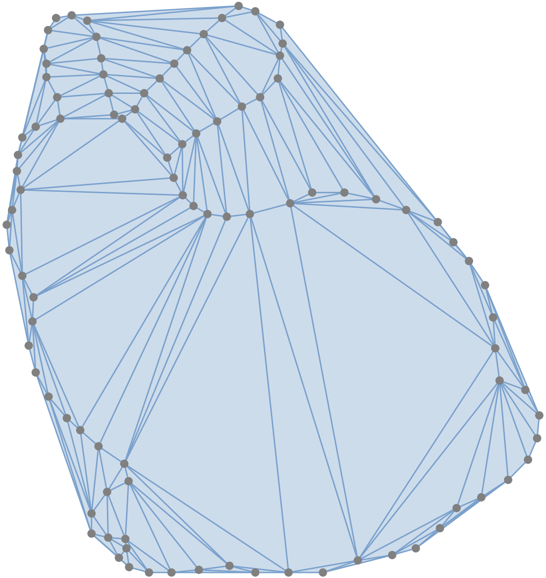
Theorem (B, Roll 2022)

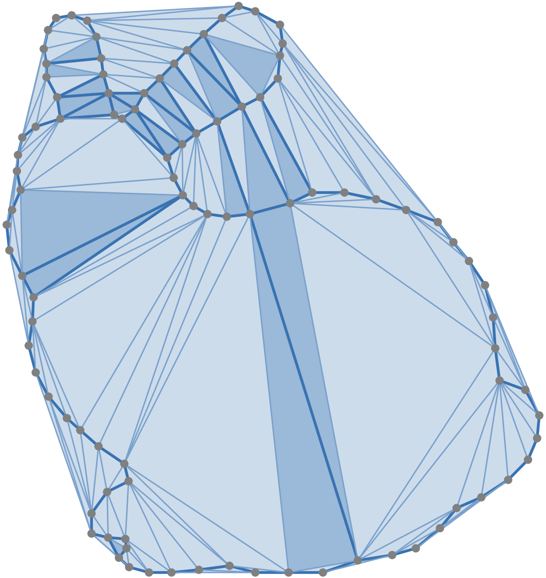
Let $X \subset \mathbb{R}$ *be a finite subset in general position, and let* $r \in \mathbb{R}$*. Then the lexicographic optimal cycles of* $\mathrm{Del}_r(X)$ *are supported on* $\mathrm{Wrap}_r(X)$.

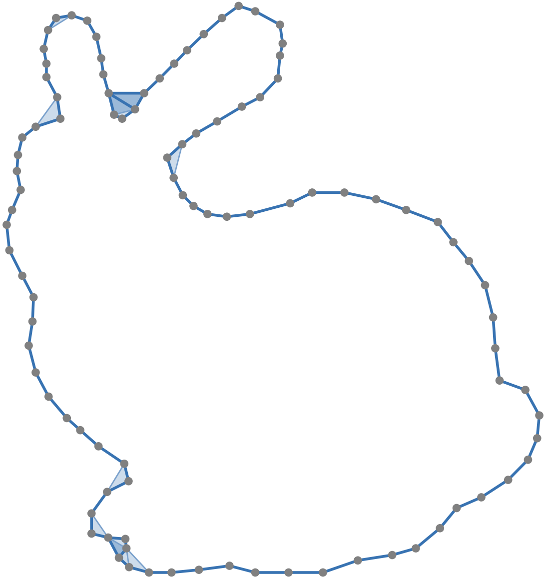


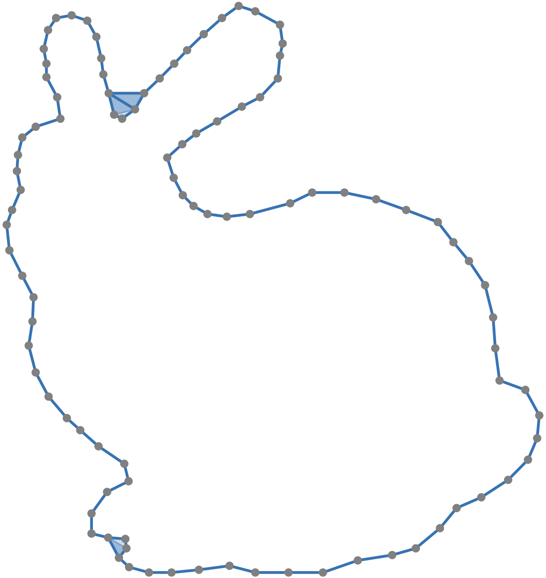
Point cloud reconstruction with lexicographic optimal cycles

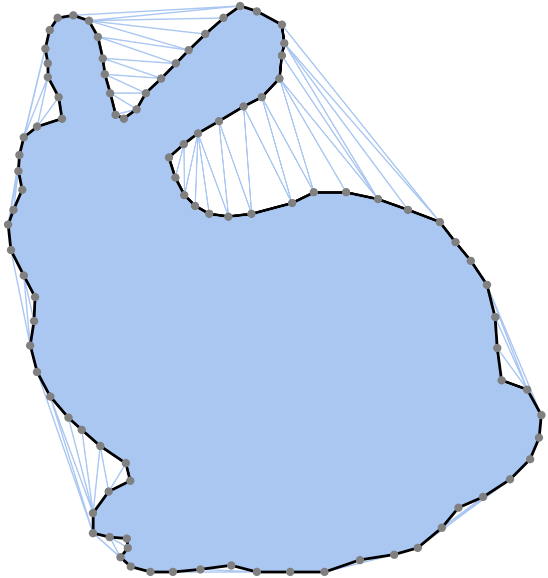


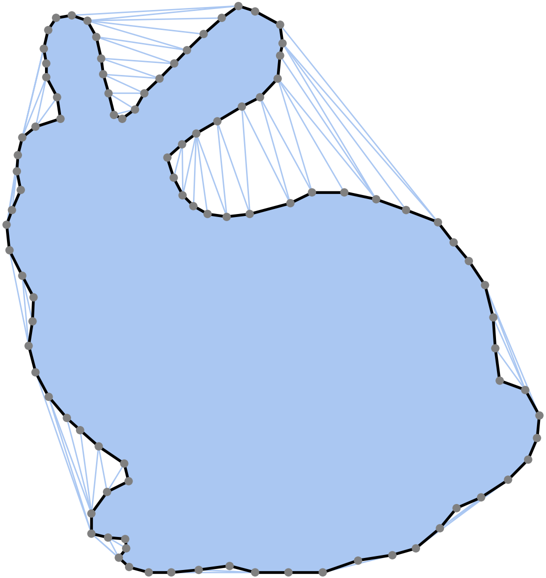


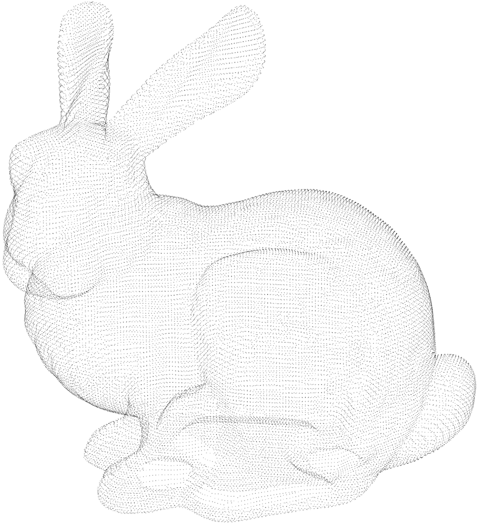


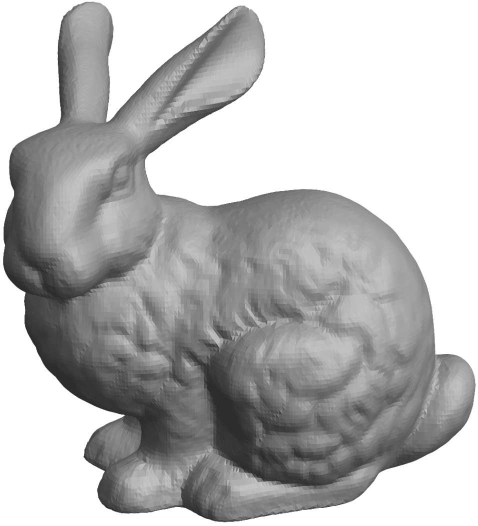




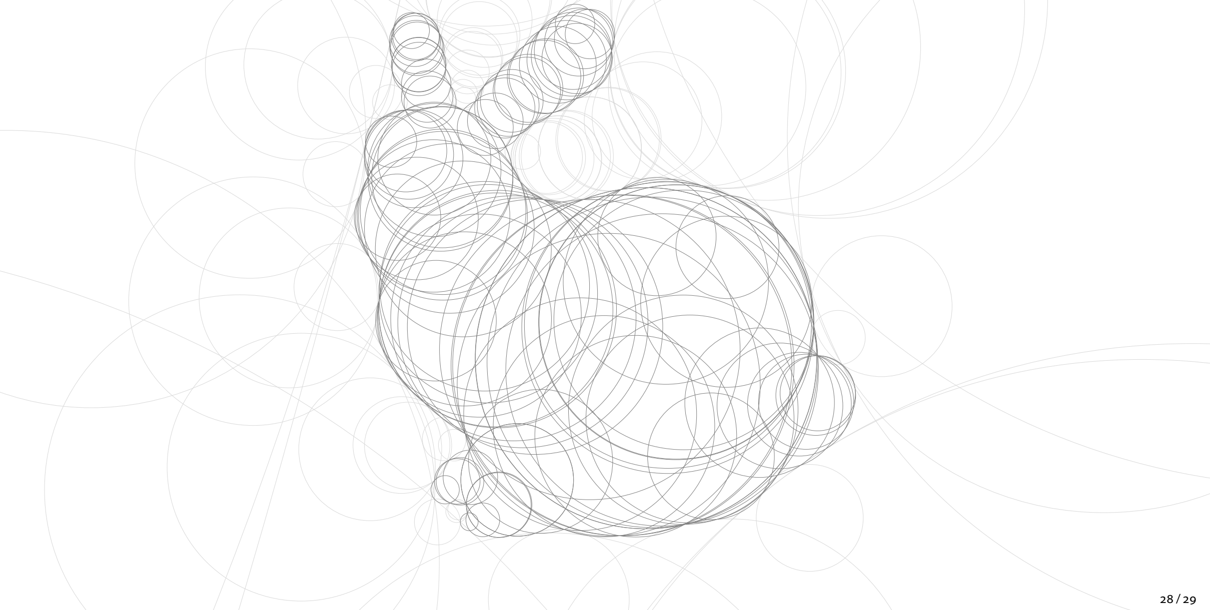








Thanks for your attention!



Further reading

U. Bauer, M. Kerber, F. Roll, and A. Rolle
A Unified View on the Functorial Nerve Theorem and its Variations
Preprint, arXiv:2203.03571, 2022

U. Bauer, H. Edelsbrunner
The Morse Theory of Čech and Delaunay Complexes
Transactions of the AMS, 2017.

U. Bauer, F. Roll
Gromov hyperbolicity, geodesic defect, and apparent pairs in Vietoris-Rips filtrations
Symposium on Computational Geometry, 2022

U. Bauer, F. Roll
Connecting Discrete Morse Theory and Persistence: Wrap Complexes and Lexicographic Optimal Cycles
Preprint, arXiv:2212.02345, 2022