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Geometric complexes and discrete Morse theory

Tutorial (May 18, 2018)

Problem 1. Enclosing Spheres and the Čech Complex.

Let X be a finite subset of $\mathbb{R}^d$. Show that $Q \in \check{\mathrm{C}}\mathrm{ech}_r(X)$ if and only if the smallest enclosing sphere of Q has radius at most r.

Problem 2. Uniqueness of Circumspheres.

Let $Q \neq \emptyset$ be an affinely independent subset of $\mathbb{R}^d$, let H be the affine span of Q, and let $q \in Q$.

a) Show that the orthogonal complement of $H - q$ parametrizes the circumspheres of Q.

b) Show that Q has exactly one circumsphere if and only if the cardinality of Q is $d + 1$.

c) Show that there is a unique smallest circumsphere of Q.

Problem 3. Enclosing Spheres.

Let T and $Q \neq \emptyset$ be finite subsets of $\mathbb{R}^d$ and assume there is a circumsphere of T enclosing Q. Let $M(T, Q)$ denote the smallest such sphere and let $x \in Q$. Show that

$$M(T,Q) = \begin{cases} M(T, Q \setminus \{x\}) & \text{if this sphere exists and encloses } x \\ M(T \cup \{x\}, Q \setminus \{x\}) & \text{otherwise.} \end{cases}$$

Problem 4. Delaunay complex by empty circumspheres.

Let X be a finite subset of some Euclidean space. Show that $Q \subseteq X$ is a simplex of $\mathrm{Del}(X)$ iff Q has a circumsphere S bounding an open ball B with $X \cap B = \emptyset$ (an *empty circumsphere*).

Problem 5. Geometric realization of Čech and Delaunay filtrations.

Let X be a finite subset of some Euclidean space.

a) Show that the Čech and Delaunay complexes for the same radius $r > 0$ have homotopy equivalent geometric realizations: $|\,\mathrm{Del}_r(X)| \simeq |\,\check{\mathrm{C}}\mathrm{ech}_r(X)|$.

b) Is $\mathrm{Del}_r(X) = \mathrm{Del}(X) \cap \check{\mathrm{C}}\mathrm{ech}_r(X)$?

Problem 6. Delaunay filtrations and smallest circumspheres.

Let X be a finite subset of some Euclidean space. Show that $Q \in \mathrm{Del}(X)$ is in $\mathrm{Del}_r(X)$ iff $s(Q, X) \leq r^2$.

Problem 7. Generalized MiniSphere algorithm.

Let X be a finite set of points in general spherical position and let Q and E be subsets of X such that $S = S(Q, E)$ exists (uniquely).

a) Show that if $x \in Q \setminus E$, then

$$S(Q, E) = \begin{cases} S(Q \setminus \{x\}, E) & \text{if this sphere encloses } x, \\ S(Q, E \cup \{x\}) & \text{otherwise.} \end{cases}$$

Analogously, if $y \in E \setminus Q$, then

$$S(Q, E) = \begin{cases} S(Q, E \setminus \{y\}) & \text{if this sphere excludes } y, \\ S(Q \cup \{y\}, E) & \text{otherwise.} \end{cases}$$

b) Generalize the MiniSphere algorithm to compute the sphere $S(Q, E)$.

Problem 8. Minimum Spanning Trees.

Let X be a finite set of points in some Euclidean space. Show that a minimum spanning tree T of X is a subcomplex of $\mathrm{Del}(X)$.

Problem 9. Generalized discrete vector fields.

a) Show that any collapse can be obtained as a sequence of elementary collapes.

b) Let K be a simplicial complex, let $f\colon K \to \mathbb{R}$ be a monotone function, and let V be a generalized gradient of f. Show that there is a refinement of V which is a discrete vector field.

Problem 10. Induced collapses.

a) Let f be a generalized discrete Morse function on K. Show that if $(s, t]$ contains no critical value of f, then $K_t \searrow K_s$.

b) Let K be a simplicial complex with a (generalized) discrete gradient V, and let $L \subseteq K$ be a subcomplex such that $K \setminus L$ is a union of non-critical intervals in V. Show that $K \searrow L$.

Problem 11. Čech and Delaunay functions.

We consider the Čech function $s_\emptyset\colon Q \mapsto s(Q, \emptyset)$ and the Delaunay function $s_X\colon Q \mapsto s(Q, X)$.

a) Show that both are generalized Morse functions.

b) What are the critical simplices of each function?

c) Show that $\mathrm{Del}_r(X) \searrow \mathrm{Wrap}_r(X)$.

Problem 12. Simplex Pairing.

Let $E \subseteq F \subseteq X$, and let $Q \subseteq X$ be such that both $S(Q, E) \neq S(Q, F)$ exist.

a) Show that

(i) $S = S(Q \setminus \{x\}, E) = S(Q \cup \{x\}, E)$ iff $x \in \mathrm{Incl}\, S \setminus \mathrm{Front}\, S$,

(ii) $S = S(Q, E \setminus \{x\}) = S(Q, E \cup \{x\})$ iff $y \in \mathrm{Excl}\, S \setminus \mathrm{Back}\, S$.

b) Show that there exists a point $x \in F \setminus E$ such that

(i) $S(Q \setminus \{x\}, E) = S(Q \cup \{x\}, E)$,

(ii) $S(Q \setminus \{x\}, F) = S(Q \cup \{x\}, F)$.

c) Show that $\mathrm{Cech}_r(X) \cap \mathrm{Del}(X) \searrow \mathrm{Del}_r(X)$.

Problem 13. Apparent pairs.

Consider a simplexwise filtration $(K_i)_{i\in I}$ of a finite simplicial complex K. We call a pair of simplices (σ, τ) of K an *apparent pair* if both

- σ is the youngest proper face of τ, and
- τ is the oldest proper coface of σ.

a) The apparent pairs of a simplexwise filtration form a discrete gradient on K.

b) Given a total order on the vertices $v_1, \dots v_n$ of the simplicial complex K, the *lexicographic gradient* V_L is as follows. Whenever possible, pair a simplex $\sigma = \{v_{i_k}, \dots v_{i_1}\}$, $i_k > \cdots > i_1$, with the simplex $\tau = \{v_{i_k}, \dots v_{i_1}, v_{i_0}\}$ for which $i_0 < i_1$ is minimal.

   Show that the lexicographic gradient V_L is the apparent pairs gradient of the *lexicographic filtration* of K: simplices are ordered by dimension, and simplices of the same dimension are ordered lexicographically according to the given vertex order.