Algebraic perspectives of Persistence

The stability of persistence barcodes

Ulrich Bauer

TUM

April 24, 2017

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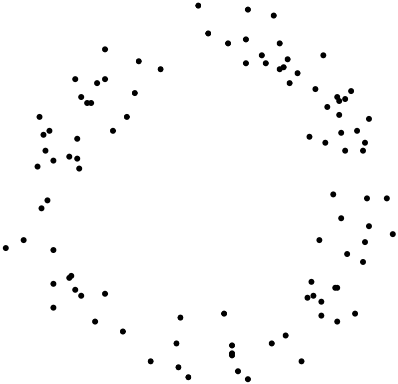

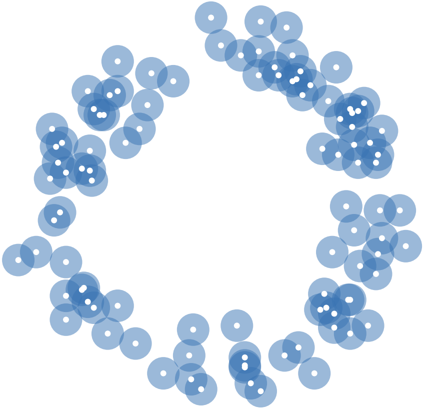

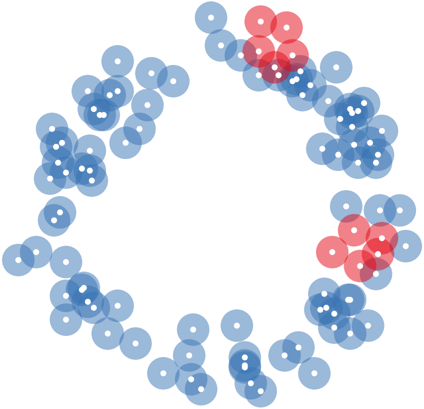

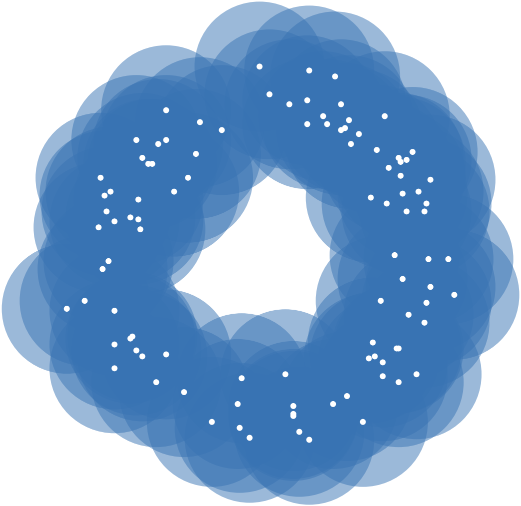

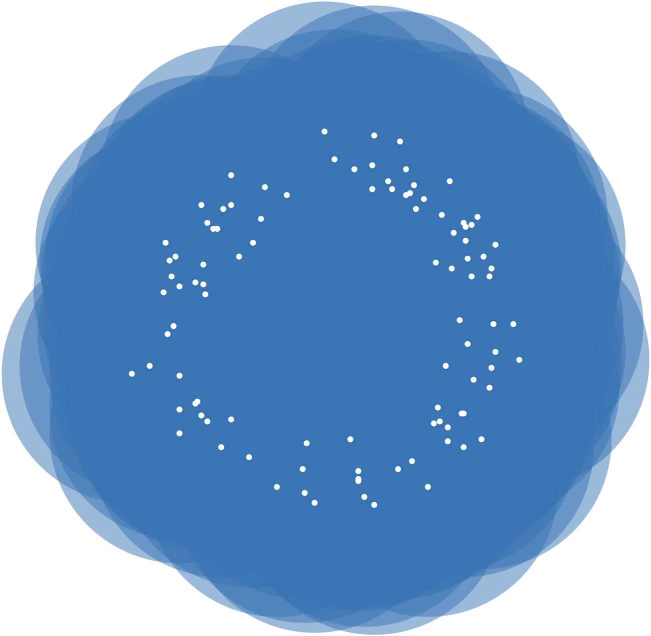

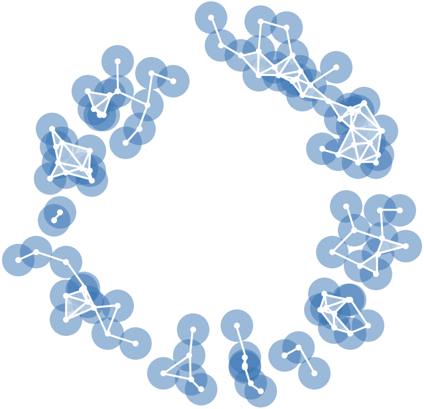

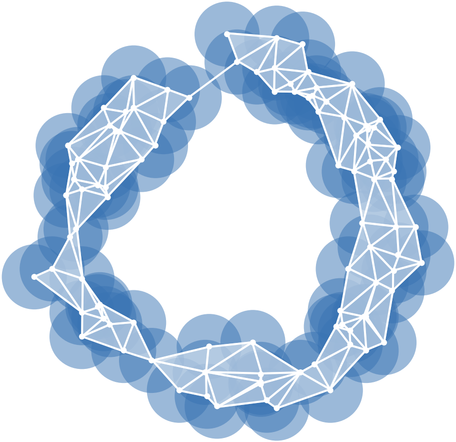

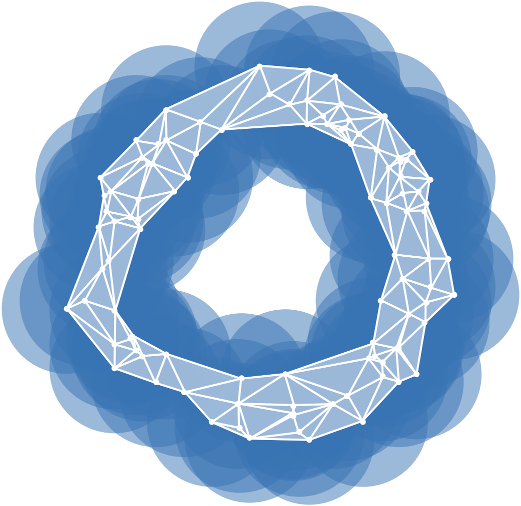

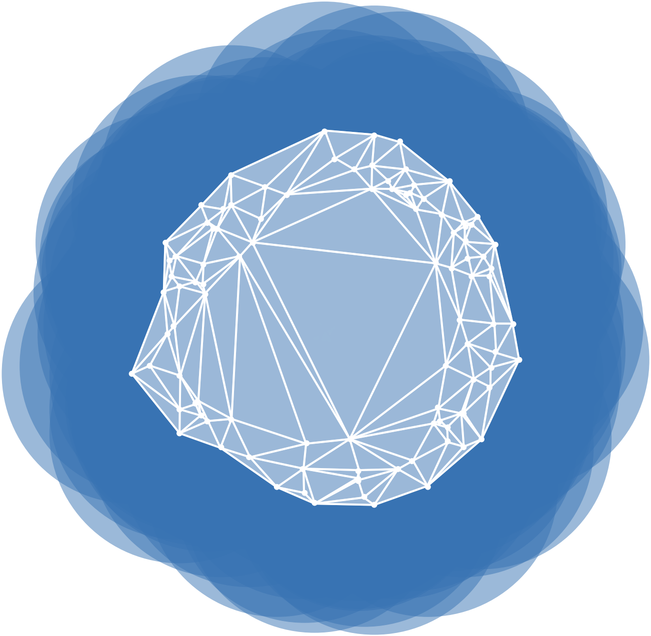

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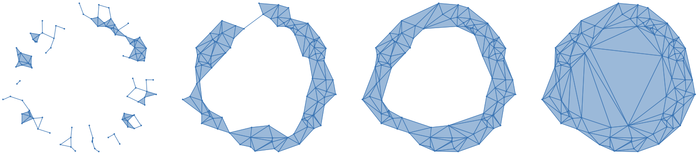

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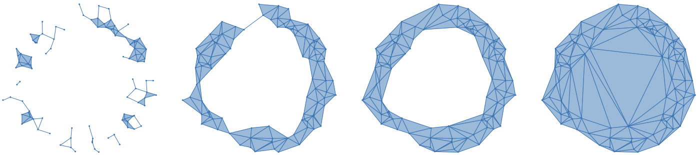

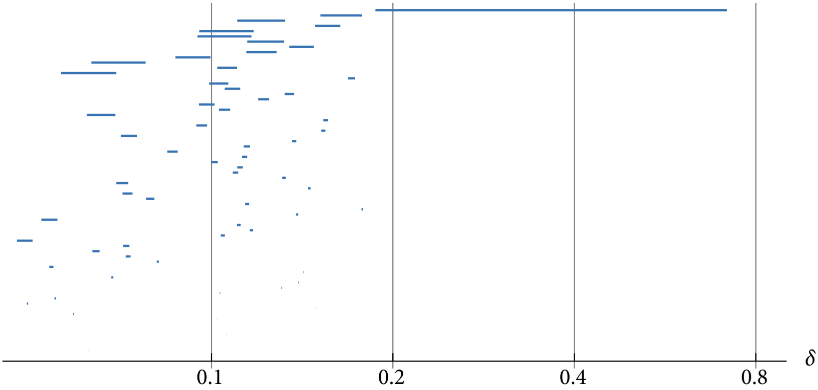

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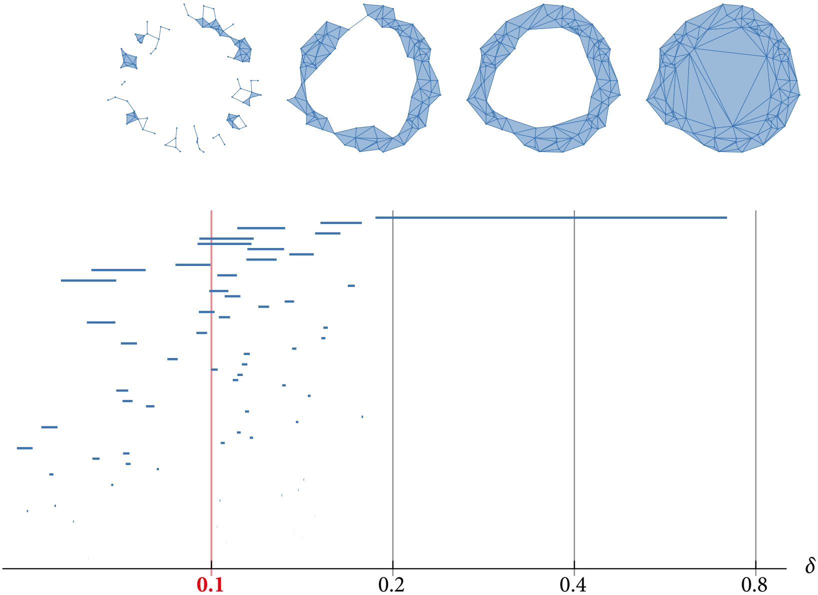

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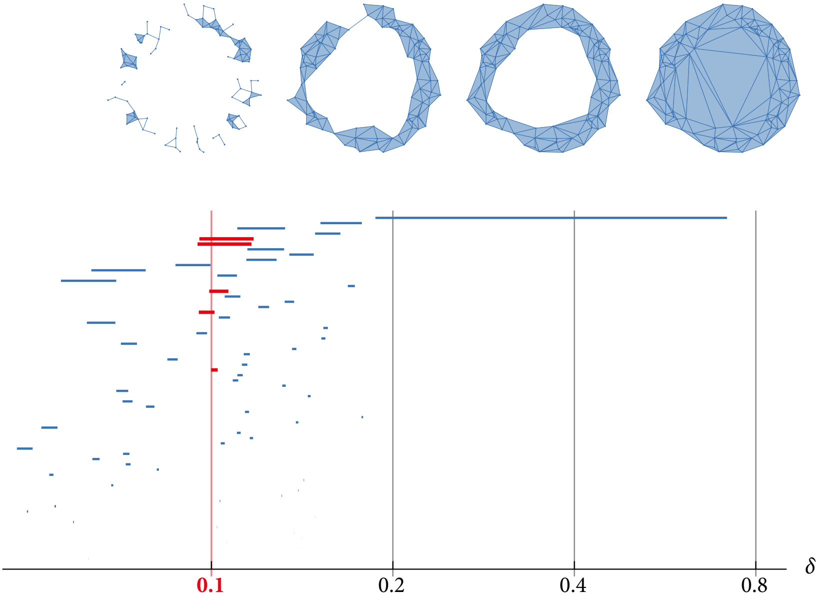

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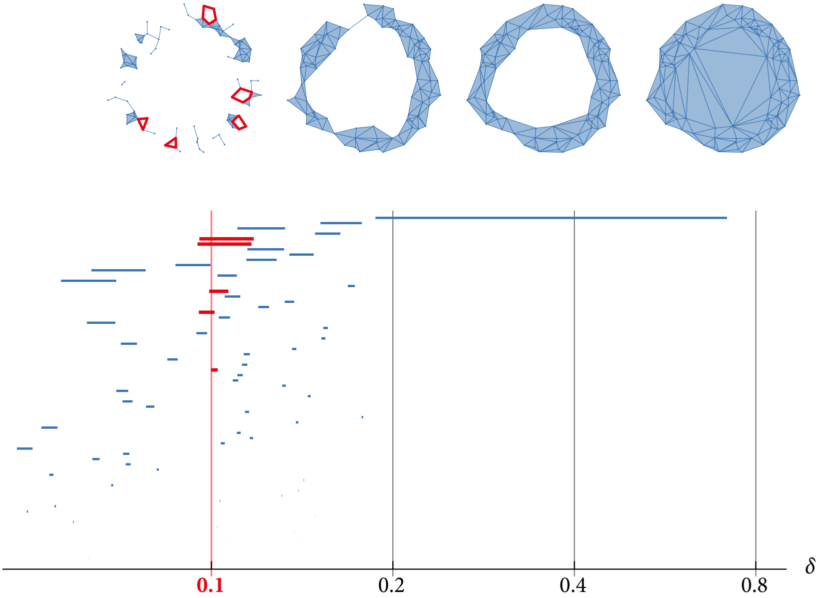

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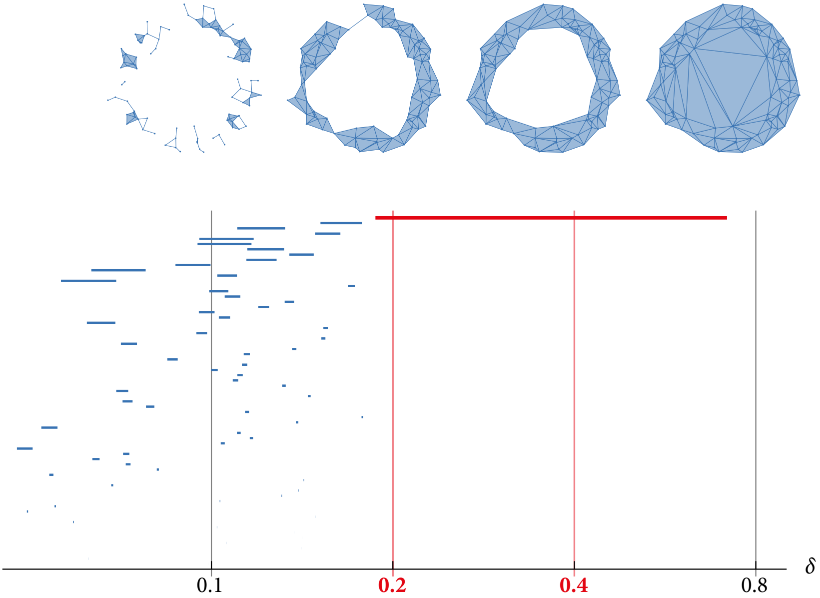

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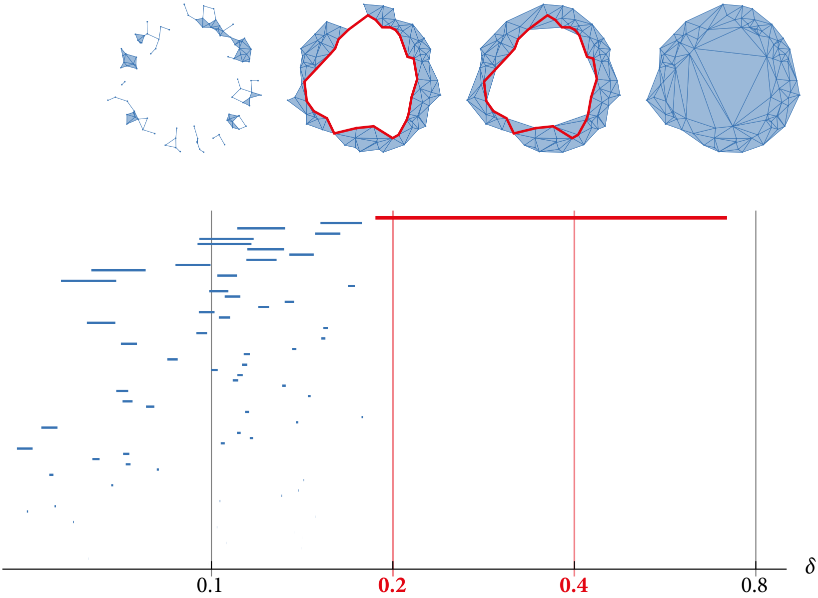

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  - $\mathbf{R}$ is the poset category of $(\mathbb{R}, \leq)$
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- Consider homology with coefficients in a field (often $\mathbb{Z}_2$)
  $H_* : \mathbf{Top} \to \mathbf{Vect}$
- Persistent homology is a diagram $M : \mathbf{R} \to \mathbf{Vect}$
  (persistence module)

Homology inference

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Given: finite sample $P \subset \Omega$ of unknown shape $\Omega \subset \mathbb{R}^d$

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Requires strong assumptions:

Homology reconstruction by thickening

Theorem (Niyogi, Smale, Weinberger 2006)

Let M be a submanifold of $\mathbb{R}^d$. Let $P \subset M$, $\delta > 0$ be such that

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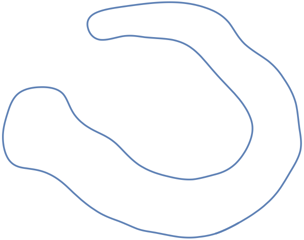

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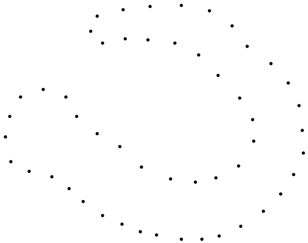

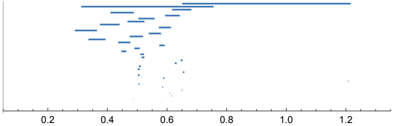

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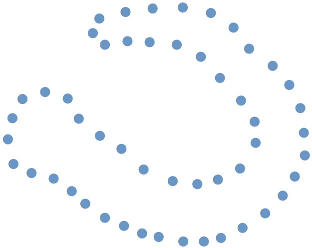

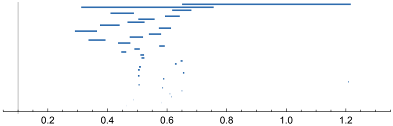

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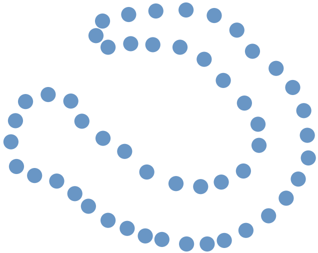

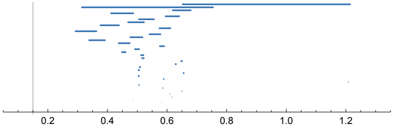

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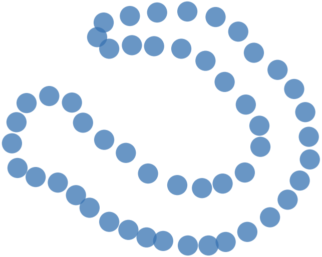

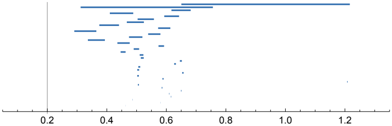

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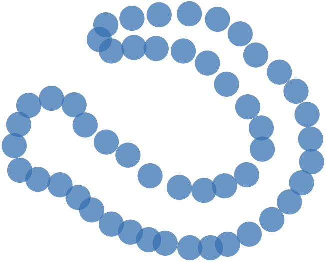

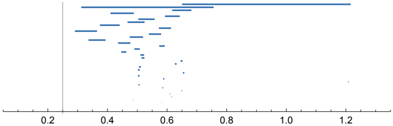

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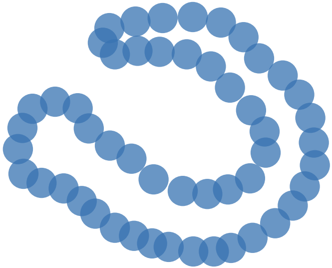

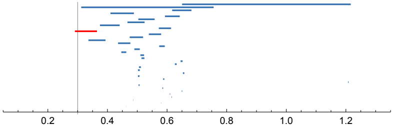

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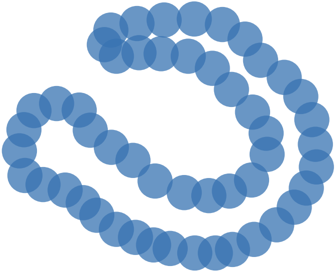

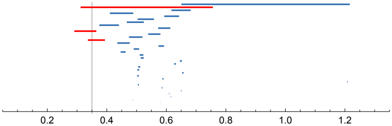

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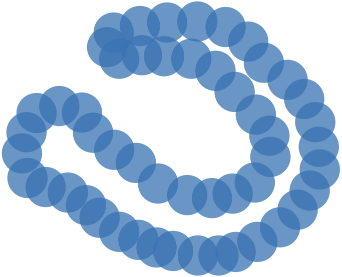

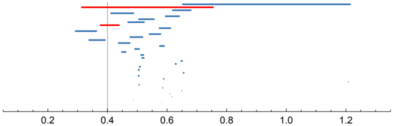

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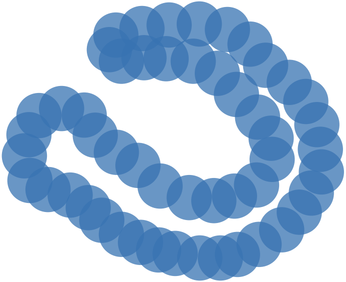

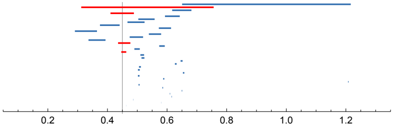

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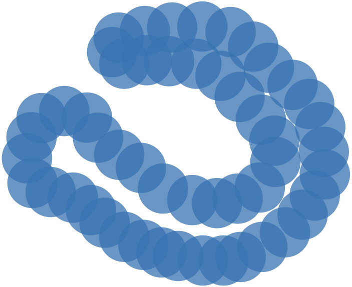

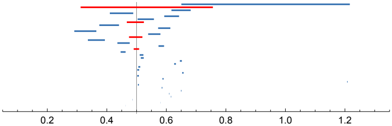

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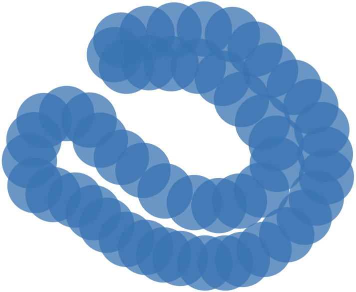

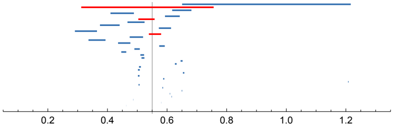

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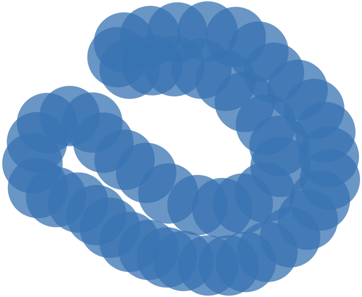

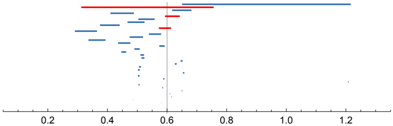

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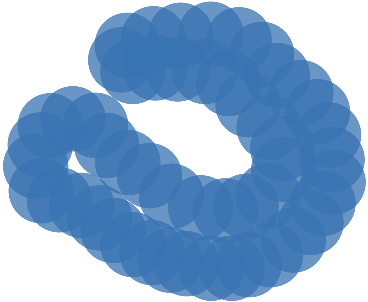

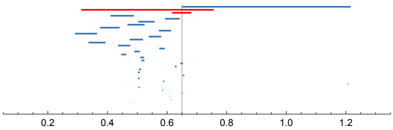

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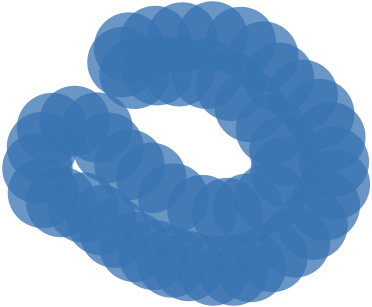

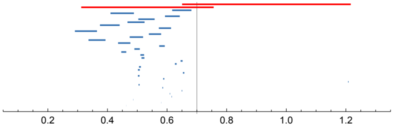

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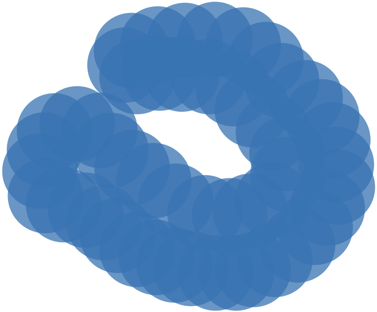

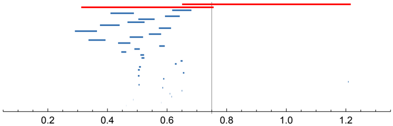

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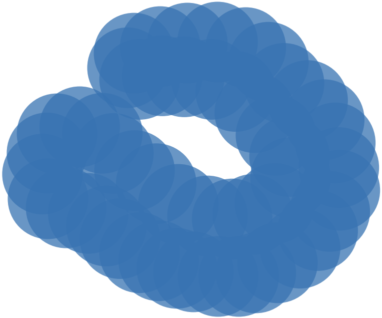

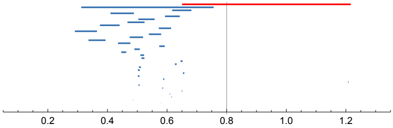

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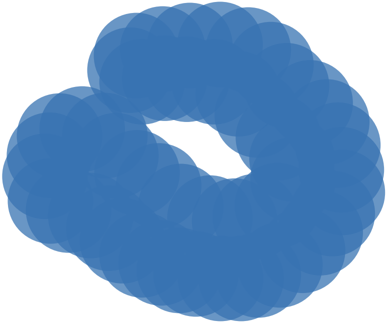

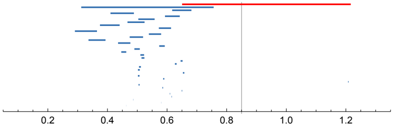

Homology reconstruction by thickening

Theorem (Niyogi, Smale, Weinberger 2006)

Let M be a submanifold of $\mathbb{R}^d$. Let $P \subset M$, $\delta > 0$ be such that

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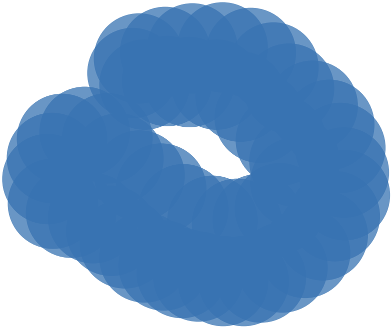
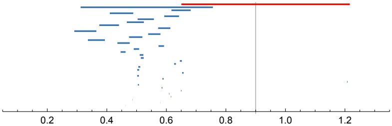




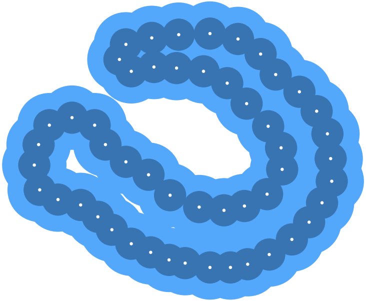
Homology inference using persistence

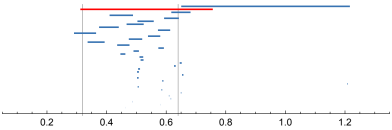
Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\Omega \subset \mathbb{R}^d$. *Let* $P \subset \Omega$, $\delta > 0$ *be such that*

- $B_\delta(P)$ *covers* Ω, *and*
- *the inclusions* $\Omega \hookrightarrow B_\delta(\Omega) \hookrightarrow B_{2\delta}(\Omega)$ *preserve homology.*

Then $H_*(\Omega) \cong \operatorname{im} H_*(B_\delta(P) \hookrightarrow B_{2\delta}(P))$.





Stability

Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\|f - g\|_\infty = \delta$. Then there exists a matching between the intervals of the persistence barcodes of f and g such that

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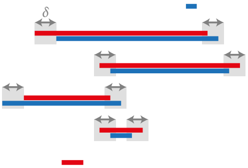

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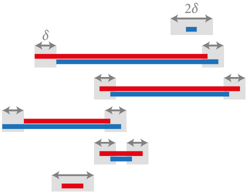

Persistence and stability: the big picture

Data	point cloud	$P \subset \mathbb{R}^d$

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$$\begin{tikzcd} \text{Data} \arrow[d] & \text{point cloud} \arrow[d, "\text{distance}"] & P \subset \mathbb{R}^d \\ \text{Geometry} & \text{function} & f : \mathbb{R}^d \to \mathbb{R} \end{tikzcd}$$

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\text{Combinatorics} & \text{intervals (persistence barcode)} & B : \mathbf{R} \to \mathbf{Mch}
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Interleavings

Let $\delta = \|f - g\|_\infty$. Write $F_t = f^{-1}(-\infty, t]$ and $G_t = g^{-1}(-\infty, t]$.

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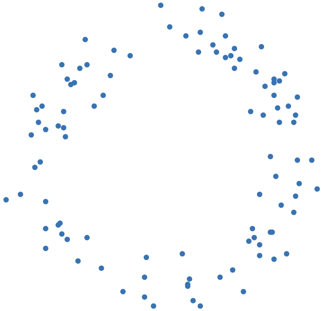
Applying homology (functor) preserves commutativity

- persistent homology of f, g yields δ-interleaved persistence modules $\mathbf{R} \to \mathbf{Vect}$

Geometric interleavings



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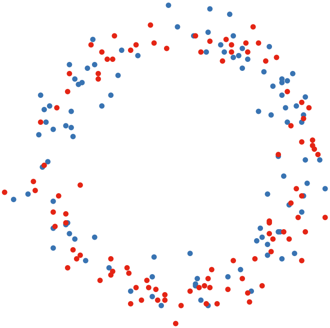

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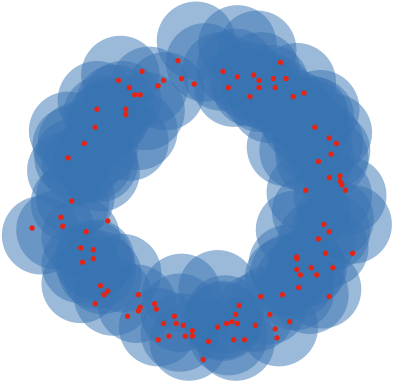

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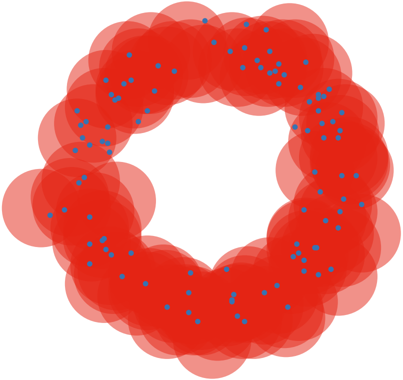

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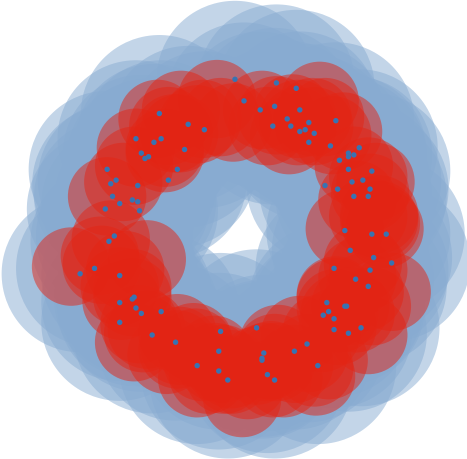

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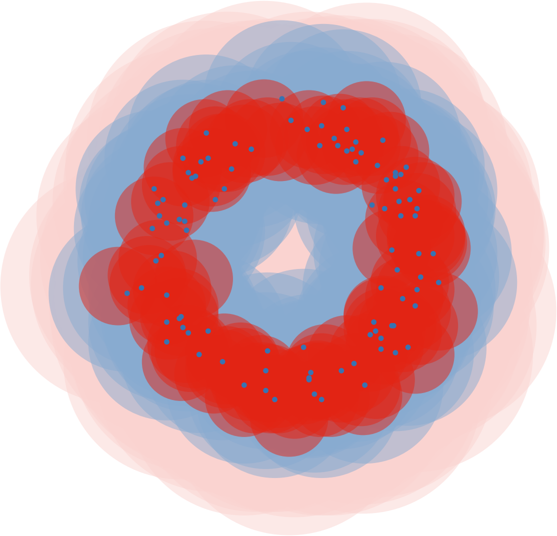

Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2012; B, Lesnick 2015)

If two persistence modules are δ-interleaved,
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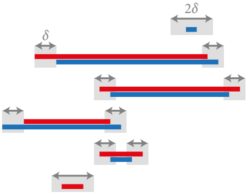

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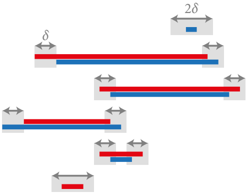
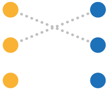


Barcodes as diagrams

The matching category

A *matching* $\sigma : S \nrightarrow T$ is a bijection $S' \to T'$, where $S' \subseteq S$, $T' \subseteq T$.



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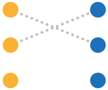
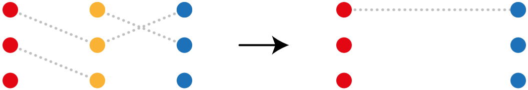


Composition of matchings $\sigma : S \nrightarrow T$ and $\tau : T \nrightarrow U$:



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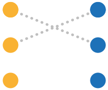

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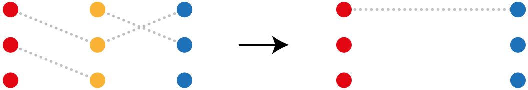
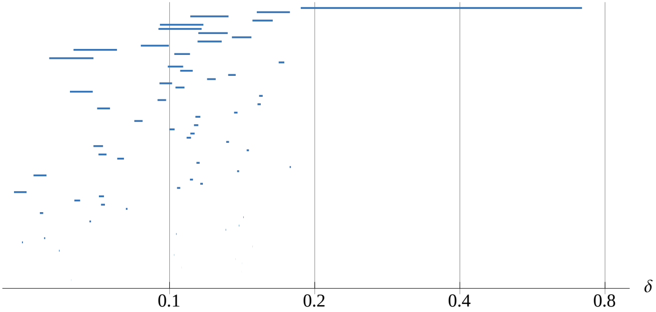


Matchings form a category $\mathbf{Mch}$

- objects: sets
- morphisms: matchings

Barcodes as matching diagrams

We can regard a barcode B as a functor $\mathbf{R} \to \mathbf{Mch}$:



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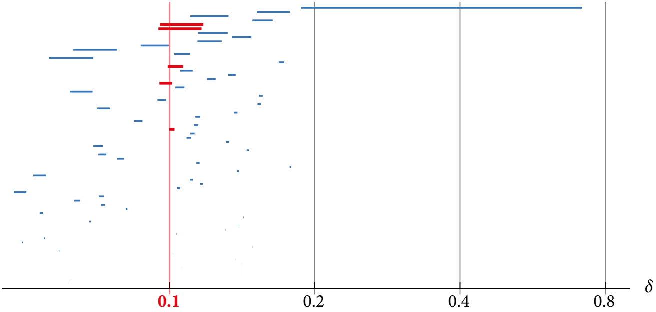

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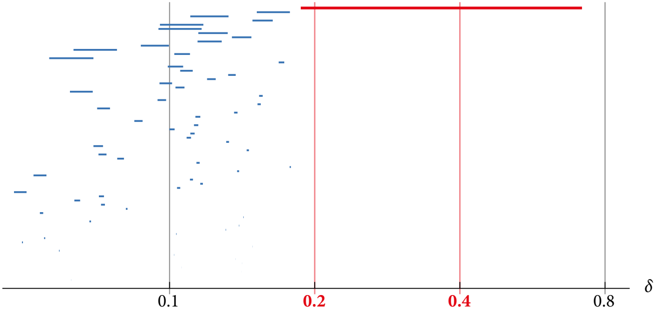
- for each $s \le t$, define the matching $B_s \nrightarrow B_t$ to be the identity on $B_s \cap B_t$.



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- In particular, there is no natural choice of basis for vector spaces