Persistent diagrams as diagrams

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TUM

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Joint work with Michael Lesnick (Princeton/Albany)

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- Matching diagrams: sequence of partial bijections
  (Edelsbrunner et al. 2014)

Inerval decompositions and persistence modules

Theorem (Crawley-Boewey 2015)

Any pointwise finite-dimensional (pfd) persistence module (a diagam $M : \mathbb{R} \to \mathbf{vect}$) has an essentially unique decomposition as a direct sum of indecomposable interval modules, *isomorphic to*

$$0 \to \cdots \to 0 \to \underbrace{\mathbb{K} \to \cdots \to \mathbb{K}}_{\text{supported by an interval } I \subseteq \mathbb{R}} \to 0 \to \cdots$$

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- The points in the *persistence diagram* are the endpoints of the intervals in the barcode.
- This is not a diagram in the sense of category theory (functor)!

Persistence and stability: the big picture

point cloud	$P \subset \mathbb{R}^d$	Hausdorff distance

Persistence and stability: the big picture

$$\begin{array}{ccc} \text{point cloud} & P \subset \mathbb{R}^d & \text{Hausdorff distance} \\ \big\downarrow{\scriptstyle\text{distance}} & & \\ \text{function} & f : \mathbb{R}^d \to \mathbb{R} & L_\infty \text{ distance} \end{array}$$

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$$\begin{tikzcd}
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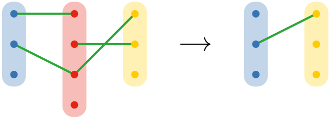
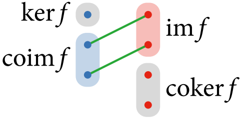
$\downarrow$ decomposition		
intervals (persistence barcode)	$B : \mathbf{R} \to \mathbf{Mch}$	Bottleneck/interleaving distance

The category of matchings

Consider the category $\mathbf{Mch}$ (a subcategory of the category $\mathbf{Rel}$ of sets and relations) with

- objects: sets,
- morphisms: matchings (partial bijections).

Composition: 

(Co)kernel/(co)image: 

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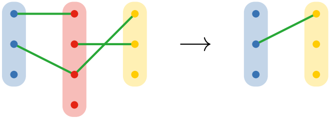
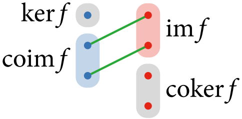
Composition: 

(Co)kernel/(co)image: 

$\mathbf{Mch}$ is *Puppe-exact (p-exact)*:

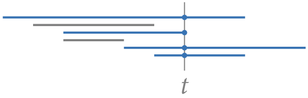
- it has a zero object ($\varnothing$)
- it has all (co)kernels
- every mono (epi) is (co)kernel
- every morphism $f : A \to B$ has an epi-mono factorization $A \twoheadrightarrow \operatorname{im} f \hookrightarrow B$

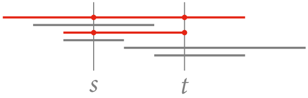
but not additive:

- it does not have all (co)products

From barcodes to matching diagrams (and back)

- A barcode (collection of intervals) can be read as a diagram $\mathbb{R} \to \mathbf{Mch}$:





$t \mapsto \{\text{intervals in barcode containing } t\}$ $\qquad$ $(s \leq t) \mapsto \{\text{intervals containing both } s, t\}$

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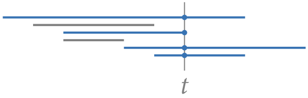

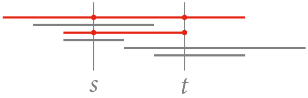

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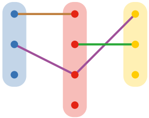

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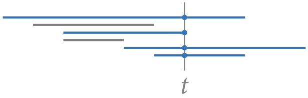

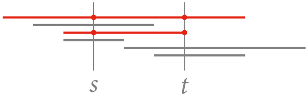

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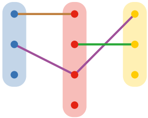

  - equivalence classes $\mathcal{E}(D) := \left(\bigcup_{t \in \mathbb{R}} \{t\} \times D_t \right) / \sim$, where $(s, x) \sim (t, y)$ for all $s \le t,\ x \in D_s,\ y \in D_t$

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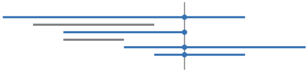

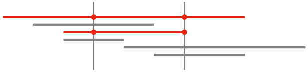

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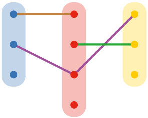

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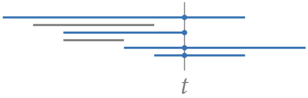

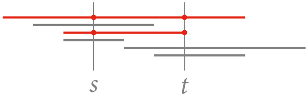

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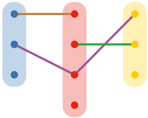

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Turn this into an equivalence of categories $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbb{R}}$

A category of barcodes

Proposition

The functor category is equivalent to $\mathbf{Barc}$*, the category with*

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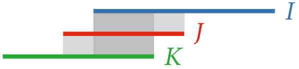
- *composition of overlap matchings:* $\tau \bullet \sigma = \{(I, K) \in \tau \circ \sigma \mid I \text{ overlaps } K \text{ above}\}$
  (where $\tau \circ \sigma$ *is the standard composition of matchings)*



$(I, K) \in \tau \bullet \sigma$ *(overlap)*

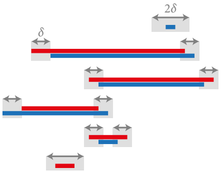
$(I, K) \notin \tau \bullet \sigma$ *(no overlap)*

Bottleneck distance as an interleaving distance

δ-matching between barcodes U, V:

- if I is matched to J, then endpoints are δ-close
- unmatched intervals are 2δ-trivial (shorter than 2δ)

Bottleneck distance: $d_B(U, V) = \inf\{\delta \mid \exists \delta\text{-matching } U \nrightarrow V\}$



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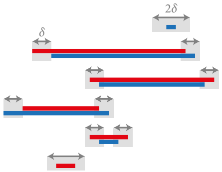

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δ-interleaving between diagrams X, Y indexed over $\mathbb{R}$ (in any category):

natural transformations $f_t : X_t \to Y_{t+\delta}, g_t : Y_t \to X_{t+\delta}$ yielding commutative diagrams

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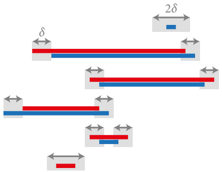

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Proposition

$d_I = d_B$ *(using the equivalence* $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbb{R}}$).

Non-functoriality of persistence barcodes

Can a pfd persistence module $M : \mathbf{vect}^{\mathbb{R}}$ be turned into its barcode $B(M) : \mathbf{Mch}^{\mathbb{R}}$ by a functor $B : \mathbf{vect} \to \mathbf{Mch}$ (or $\mathbf{vect}^{\mathbb{R}} \to \mathbf{Mch}^{\mathbb{R}}$)?

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There is no functor $\mathbf{vect} \to \mathbf{Mch}$ *sending every vector space* V *to a set of cardinality* $\dim V$ *(equivalently, a linear map* f *to a matching of cardinality* $\operatorname{rank} f$*).*

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But: there is a barcode functor for subcategories of monos/epis of persistence modules $\mathbf{vect}^{\mathbb{R}}$:

Structure of persistence sub-/quotient modules

Proposition

Let N be a quotient module of a persistence module M (for $M \twoheadrightarrow N$ an epimorphism).
Then there is an injective map between the barcodes $B(N) \hookrightarrow B(M)$.
If J is mapped to I, then

- *I and J are aligned below, and*
- *I bounds J above.*

This construction is functorial. There is a dual result for submodules.

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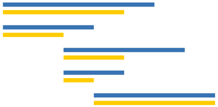
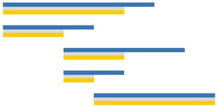


Rephrased for $\mathbf{Mch}^{\mathbb{R}}$:

Proposition

There is a functor from epimorphisms of persistence modules
to epimorphisms of matching diagrams.
(Dually, there is a functor from monos to monos.)



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For $f : M \to N$ a morphism of pfd persistence modules, the epi-mono factorization $M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$ gives an induced matching *$\chi(f)$ between their barcodes. If I is matched to J, then*

(i) *I overlaps J above.*

(ii) *If $\ker f$ is δ-trivial, then*

  (a) *J bounds $I(\delta)$ above, and*

  (b) *any unmatched interval of $B(M)$ is δ-trivial.*

(iii) *If $\operatorname{coker} f$ is δ-trivial, then*

  (a) *$I(\delta)$ bounds J below, and*

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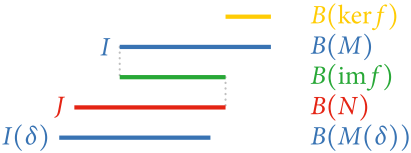

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(i) *I overlaps J above.*

(ii) *If $\ker f$ is δ-trivial, then*

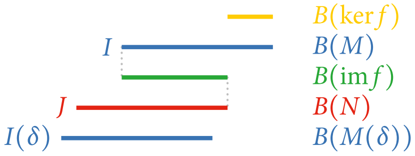
  (a) *J bounds $I(\delta)$ above, and*

  (b) *any unmatched interval of $B(M)$ is δ-trivial.*

(iii) *If $\operatorname{coker} f$ is δ-trivial, then*

  (a) *$I(\delta)$ bounds J below, and*

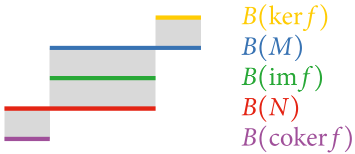
  (b) *any unmatched interval of $B(N)$ is δ-trivial.*



Rephrased in $\mathbf{Mch}^{\mathbb{R}}$:

Theorem

If $f : M \to N$ has δ-trivial (co)kernel, then so does $\chi(f)$.



Algebraic stability via induced matchings

Consider interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(\forall t \in \mathbb{R})$:

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$B(N)$

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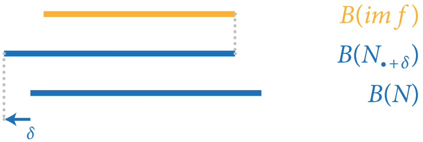

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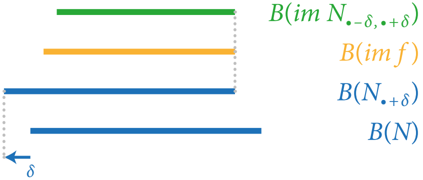

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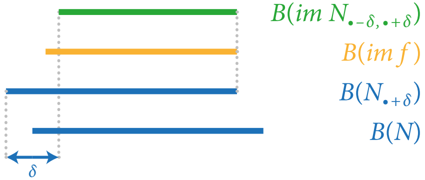

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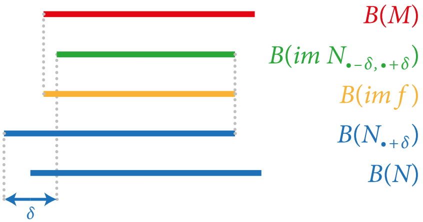

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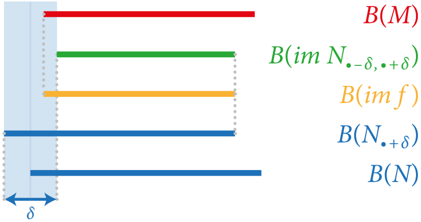
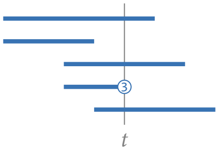


Matching diagrams from persistence modules

Let $M : \mathbb{R} \to \mathbf{vect}$. For $t \in \mathbb{R}$, $i \in \mathbb{N}$, define

$$\operatorname{birthCut}(M, t, i) = \{s < t \mid \operatorname{rank} M_{s,t} < i\},$$



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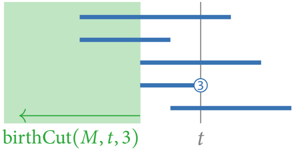

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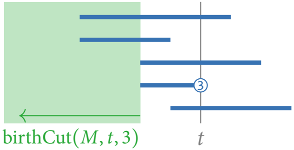

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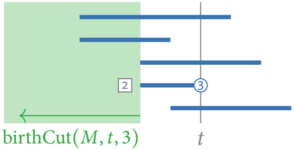

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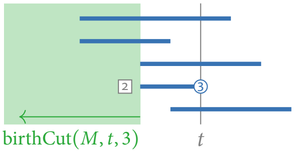

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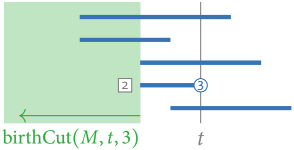

Construct a matching diagram $B(M) : \mathbb{R} \to \mathbf{Mch}$:
for all $t \leq u \in \mathbb{R}$, define

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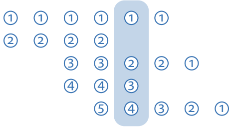

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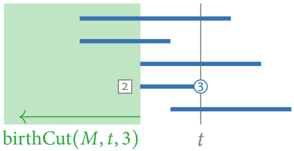

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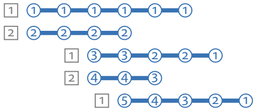

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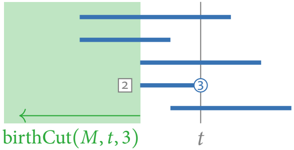

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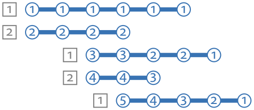

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- Yields a barcode without using interval decomposition!

More about matching diagrams

Proposition

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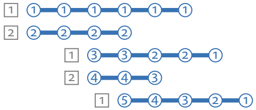

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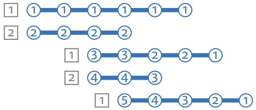

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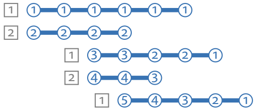

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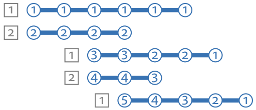

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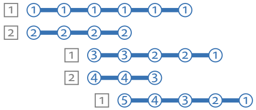

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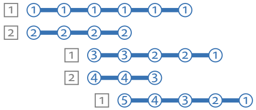

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Applies even to q-tame persistence modules ($\operatorname{rank} M_{t,u} < \infty$ for all $t < u$)!

Induced matchings for matching diagrams

Let N be a quotient module of a persistence module M ($M \twoheadrightarrow N$ an epimorphism). Define

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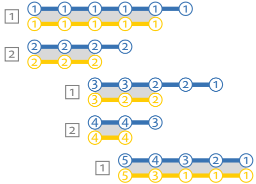

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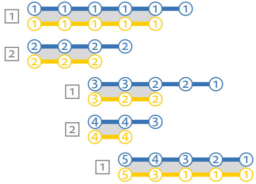

Theorem

B and χ form a functor from epimorphisms of persistence modules to epimorphisms of matching diagrams.
(Dually, there is a functor from monos to monos.)

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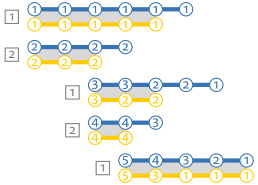

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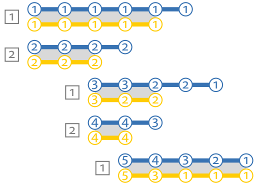

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- This is the structure theorem for sub-/quotient modules, in terms of matching diagrams.
- Using an epi-mono factorization, this yields induced matchings and algebraic stability for q-tame persistence modules.

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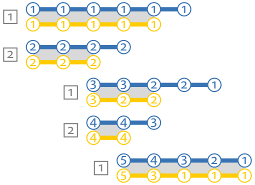
Let N be a quotient module of a persistence module M ($M \twoheadrightarrow N$ an epimorphism). Define

$$\chi(M \twoheadrightarrow N)_t = \{(i,j) \mid \mathrm{birthId}(M,t,i) = \mathrm{birthId}(N,t,j)\}.$$



Theorem

B and χ form a functor from epimorphisms of persistence modules to epimorphisms of matching diagrams.
(Dually, there is a functor from monos to monos.)

- This is the structure theorem for sub-/quotient modules, in terms of matching diagrams.
- Using an epi-mono factorization, this yields induced matchings and algebraic stability for q-tame persistence modules.
- Can be used to guide the construction of a decomposition for pdf modules.

Thanks for your attention!