

THE REEB GRAPH EDIT DISTANCE IS UNIVERSAL

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EINSTEIN WORKSHOP ON
DISCRETE GEOMETRY & TOPOLOGY

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JOINT WORK WITH CLAUDIA LANDI (U MODENA)
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"Here are two things that are
reasonably close to each other,
and I want to compare them." 66

S. WEINBERGER

REEB GRAPHS

$$f: M \rightarrow \mathbb{R}$$



M

$$\tilde{f}: R_f \rightarrow \mathbb{R}$$

$\longrightarrow R_f$

identify components of level sets $f^{-1}(t)$:

$$R_f = M / \sim_f,$$

where $x \sim_f y \Leftrightarrow x, y$ in same component of some $f^{-1}(t)$, $t \in \mathbb{R}$.

$$g: M \rightarrow \mathbb{R}$$



M

$$\tilde{g}: R_g \rightarrow \mathbb{R}$$

$\longrightarrow R_g$

FORMAL SETTING

We consider

- locally compact Hausdorff spaces (**Reeb domains**)
- proper quotient maps with connected fibers (**Reeb quotient maps**)

These maps are closed under composition, and stable under pullbacks.

Define a **Reeb graph** as

- a Reeb domain R_f with
- a function $\tilde{f}: R_f \rightarrow \mathbb{R}$ with discrete fibers (**Reeb function**)

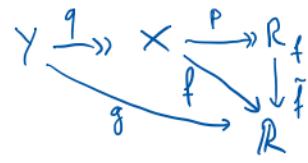
A Reeb graph R_f is the **Reeb graph** of a function $f: X \rightarrow \mathbb{R}$ if

- $f = \tilde{f} \circ p$ for some Reeb quotient map $p: X \rightarrow R_f$.
- In this case, $R_f \cong X / \sim_f$.

Moreover: let $q: Y \rightarrow X$ be a Reeb quotient map.

Then R_f is also the Reeb graph of $g = f \circ q$.

- Reeb quotient maps **preserve Reeb graphs**.



Goals

How to compare two Reeb graphs R_f, R_g ? ($f, g : M \rightarrow \mathbb{R}$ are unknown)

Assign distance (extended pseudo-metric) $d(R_f, R_g)$.

Desirable properties:

Stability: For any space X and $f, g : X \rightarrow \mathbb{R}$ yielding Reeb graphs R_f, R_g ,

$$d(R_f, R_g) \leq \|f - g\|_\infty.$$

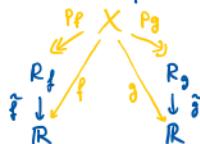
Universality: For any other stable distance d_s ,

$$d_s(R_f, R_g) \leq d(R_f, R_g).$$

A CANONICAL UNIVERSAL DISTANCE

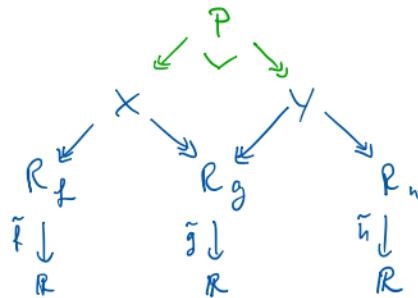
Given Reeb graphs R_f, R_g with functions f, \tilde{f} , define

$$d_u(R_f, R_g) = \inf \|f - g\|_\infty$$



taken over all Reeb domains X and Reeb quotient maps p_f, p_g .

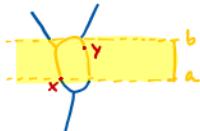
This is a distance (triangle inequality): consider pullbacks



- **stability** and **universality** immediate from definition
- working with arbitrary spaces X is unfeasible

PREVIOUS WORK: FUNCTIONAL DISTORTION DISTANCE [B, Ge, Wang 2014]

- On a Reeb graph R_f with $\tilde{f}: R_f \rightarrow \mathbb{R}$, consider the metric $d_f: (x, y) \mapsto \inf \{b-a \mid x, y \text{ in same component of } \tilde{f}^{-1}[a, b]\}$.



- Given maps $\phi: R_f \rightarrow R_g$, $\psi: R_g \rightarrow R_f$, consider $G(\phi, \psi) = \{(x, \phi(x)) \mid x \in R_f\} \cup \{\psi(\psi(y), y) \mid y \in R_g\}$.

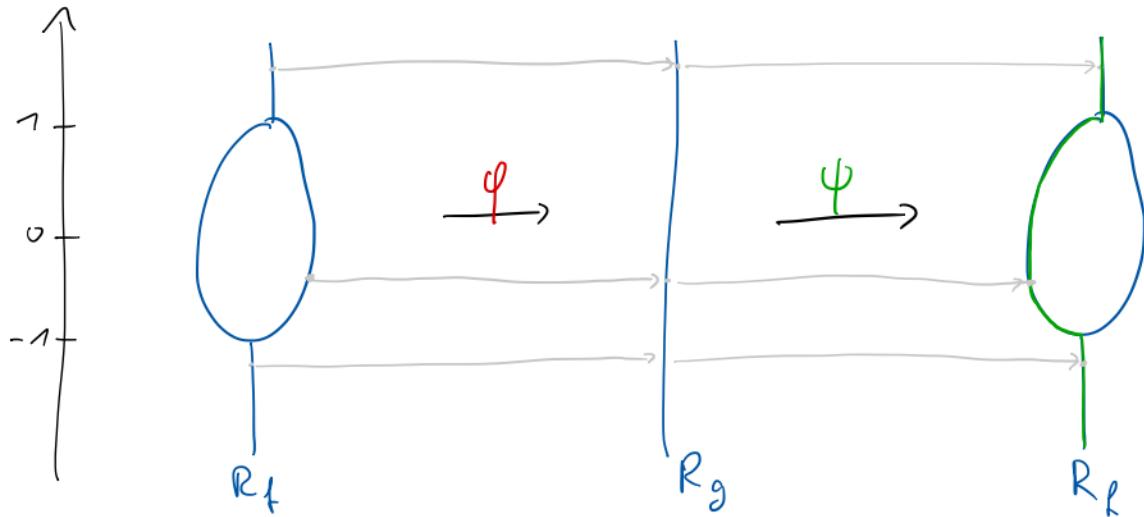
- Define the *distortion* of (ϕ, ψ) as

$$D(\phi, \psi) = \sup_{(x, y), (\tilde{x}, \tilde{y}) \in G(\phi, \psi)} \frac{1}{2} |d_f(x, \tilde{x}) - d_g(y, \tilde{y})|.$$

- Define the *functional distortion distance* as

$$d_{FD}(R_f, R_g) = \inf_{\phi, \psi} (\max \{D(\phi, \psi), \|f - g \circ \phi\|_\infty, \|g - f \circ \psi\|_\infty\})$$

EXAMPLE: FUNCTIONAL DISTORTION DISTANCE



$$D(\varphi, \psi) = \sup \frac{1}{2} (d(x, \tilde{x}) - d(y, \tilde{y})) = \frac{1}{2}$$

where $x, \tilde{x} \in R_f$, $y, \tilde{y} \in R_g$

with $\varphi(x) = y$ or $x = \psi(y)$,

$\varphi(\tilde{x}) = \tilde{y}$ or $\tilde{x} = \psi(\tilde{y})$

PREVIOUS WORK: INTERLEAVING DISTANCE [Bubenik & al. 2015; deSilva & al. 2016]

- Interpret Reeb graph R_f as a functor $F : \text{Int}_{\mathbb{R}} \rightarrow \text{Set}$, $I \mapsto \pi_0(\tilde{f}^{-1}(I))$ ($\text{Int}_{\mathbb{R}}$ are the open intervals, as a poset wrt. \subseteq)
- A δ -interleaving between F and G is a pair of natural transformations φ, ψ (with components $\varphi_I : F(I) \rightarrow G(B_\delta(I)), \dots$) such that

$$\begin{array}{ccccc} F(I) & \longrightarrow & F(B_\delta(I)) & \longrightarrow & F(B_{2\delta}(I)) \\ & \searrow \varphi_I & \nearrow \psi_I & & \nearrow \psi_{B_\delta(I)} \\ G(I) & \longrightarrow & G(B_\delta(I)) & \longrightarrow & G(B_{2\delta}(I)) \end{array}$$

commutes for all $I \in \text{Int}_{\mathbb{R}}$
(unlabelled maps induced by inclusion).

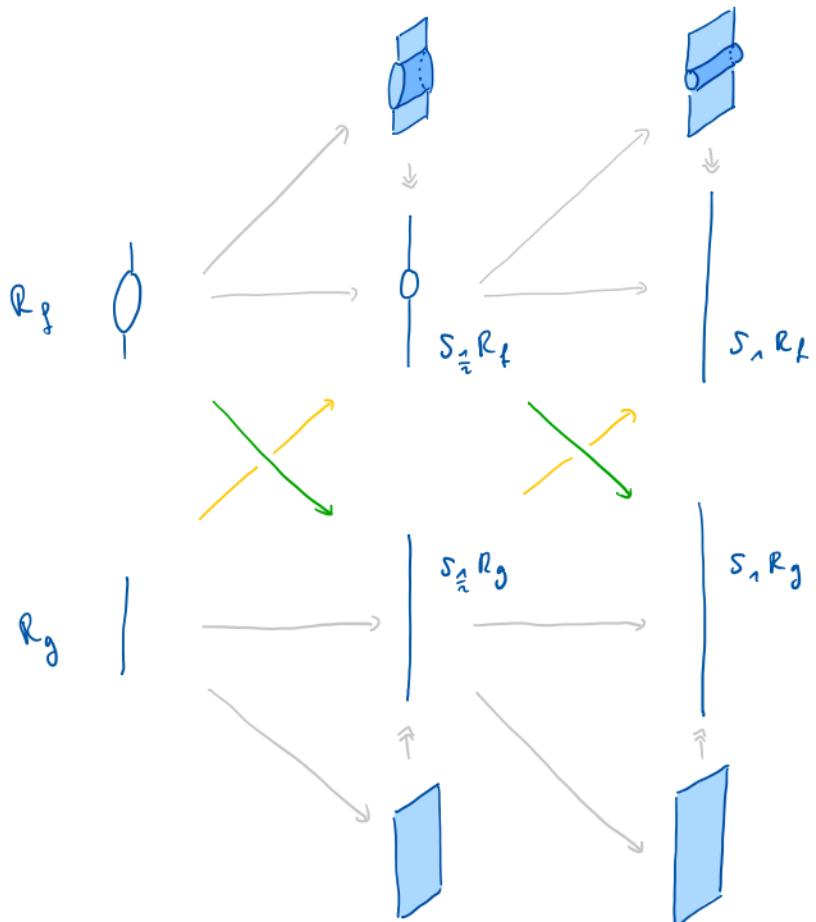
- The interleaving distance is

$$d_I(R_f, R_g) := \inf \{ \delta \mid \exists \text{ } \delta\text{-interleaving between } F \text{ and } G \}$$

Thm [B., Munch, Wang 2015] $\frac{1}{3} d_{FD} \leq d_I \leq d_{FD}$.

Open problem:
is the lower bound tight?

ABSTRACT AND TOPOLOGICAL INTERLEAVINGS



LEVEL SET PERSISTENT HOMOLOGY

Thm [Carlsson, de Silva, Morozov 2009]

Given $f: X \rightarrow \mathbb{R}$ (PL, with X compact):

Homology of level sets $H_*(f^{-1}(t); \mathbb{F})$ (and more generally,

of inclusions $f^{-1}(I) \hookrightarrow f^{-1}(J)$ for intervals $I \subseteq J$)

is encoded (up to isomorphism) by a unique
collection of intervals (level set persistence barcode).

Example for Reeb graphs:

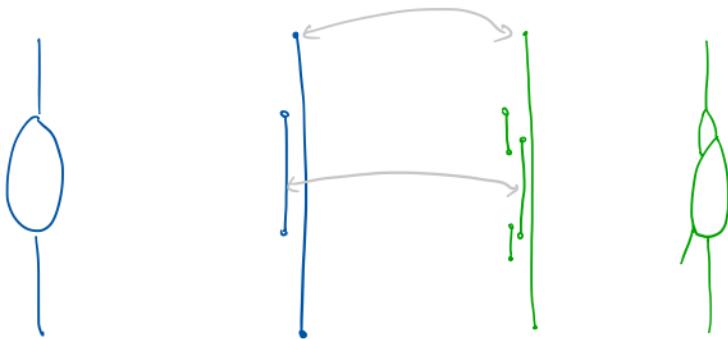


R_f



$\text{Barcode}(\tilde{f})$

THE BOTTLENECK DISTANCE BETWEEN PERSISTENCE BARCODES



A δ -matching between two barcodes $\text{Barc}(f)$, $\text{Barc}(g)$ satisfies:

- matched intervals (I, J) have distance $d_H(I, J) \leq \delta$
- unmatched intervals have length $\leq 2\delta$

The bottleneck distance $d_B(f, g)$ is

$$\inf \delta : \exists \delta\text{-matching between } \text{Barc}(f), \text{Barc}(g)$$

A ZOO OF DISTANCES AND INEQUALITIES

[Carlsson, de Silva, Morozou 2009]

$$d_B(R_f, R_g) \leq \|f - g\|_\infty$$

[B., Yu, Wang 2014]

$$\frac{1}{3} d_B(R_f, R_g) \leq d_{FD}(R_f, R_g) \leq \|f - g\|_\infty$$

[B., Munch, Wang 2015]

$$\frac{1}{3} d_{FD}(R_f, R_g) \leq d_I(R_f, R_g) \leq d_{FD}(R_f, R_g)$$

[Botnan, Lesnick 2016]

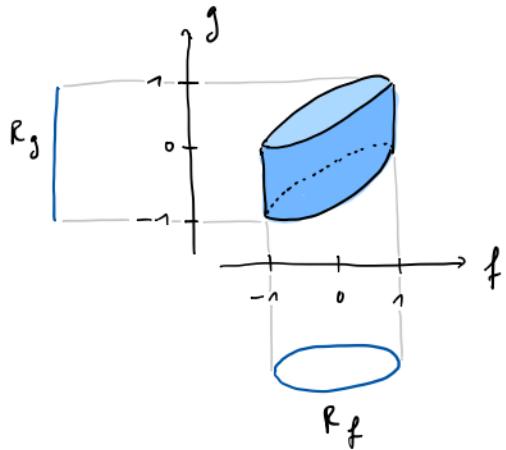
$$\frac{1}{5} d_B(R_f, R_g) \leq d_I(R_f, R_g)$$

[Björkevik 2016]

$$\frac{1}{2} d_B(R_f, R_g) \leq d_I(R_f, R_g)$$

FUNCTIONAL DISTORTION & INTERLEAVING DISTANCES ARE NOT UNIVERSAL

Consider a cylinder with two functions f, g:



- $d_u(R_f, R_g) \leq \|f - g\|_\infty = 1$
- $d_I(R_f, R_g) \leq d_{FD}(R_f, R_g) \leq \frac{1}{2} < d_u(R_f, R_g)$:

$$R_f \circlearrowleft \xrightarrow{\phi} |_{R_g}$$

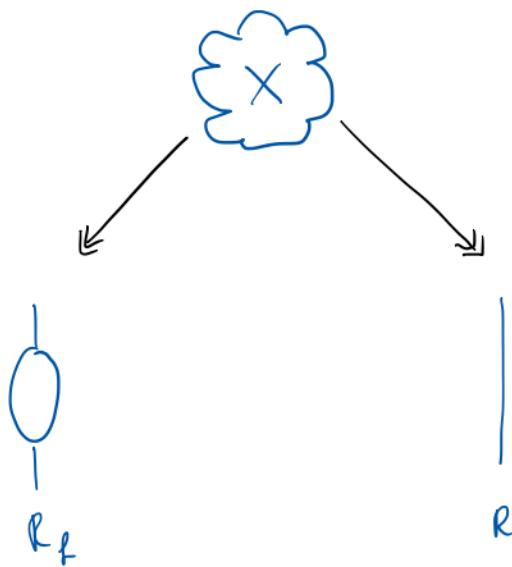
$$R_g \xrightarrow{\psi} \overset{\text{im } \psi}{\circlearrowright} R_f$$

FROM CLOSE REEB GRAPHS TO CLOSE FUNCTIONS

Open problem

Given two Reeb graphs R_f, R_g with $d_I(R_f, R_g) = \delta$.

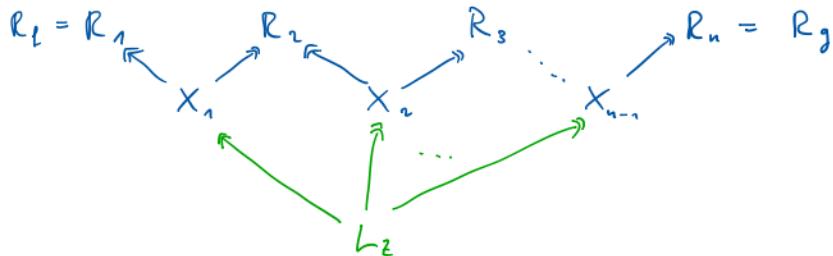
Is there a space X with $f, g : X \rightarrow \mathbb{R}$, $\|f - g\| \leq C \cdot \delta$,
yielding Reeb graphs R_f, R_g , for some fixed constant C ?



By the previous example:
if yes, then $C \geq 2$.

THE TOPOLOGICAL EDIT DISTANCE

- Consider zig-zag diagrams \mathcal{Z} of Reeb quotient maps



and take the limit L_z (note: all maps are Reeb quotient maps).
 Each $\tilde{f}_i : R_i \rightarrow \mathbb{R}$ composes to $f_i : L_z \rightarrow R_i \rightarrow \mathbb{R}$.

- Define the *spread* of the functions $f_1, \dots, f_n : L_z \rightarrow \mathbb{R}$ as

$$S_z : L_z \rightarrow \mathbb{R}, \quad x \mapsto \max_i f_i(x) - \min_j f_j(x).$$

- Define the (topological) edit distance as

$$d_{\text{Top}}(R_f, R_g) = \inf_{\mathcal{Z}} \|S_z\|_\infty.$$

Prop. d_{Top} is stable and universal.

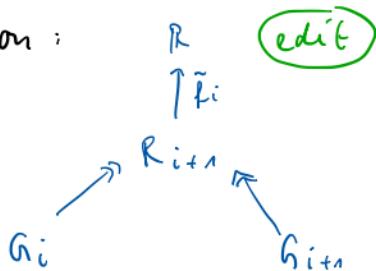
THE REEB GRAPH EDIT DISTANCE

- Consider zig-zag diagrams \mathcal{Z} of Reeb quotient maps

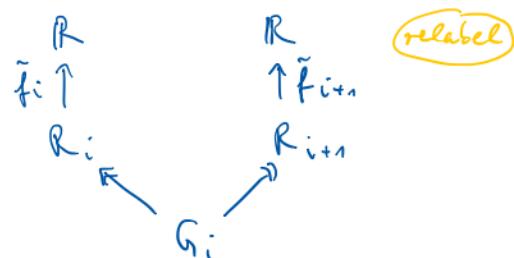


as before, but **restrict** R_i , G_i in \mathcal{Z} to be **finite graphs**.

Interpretation :



modify G_i to G_{i+1} ,
maintaining the Reeb graph R_{i+1}



modify f_i to $f_{i+1} : G_i \rightarrow R_i$,
maintaining the domain G_i

- Define the **Reeb graph edit distance** analogously as

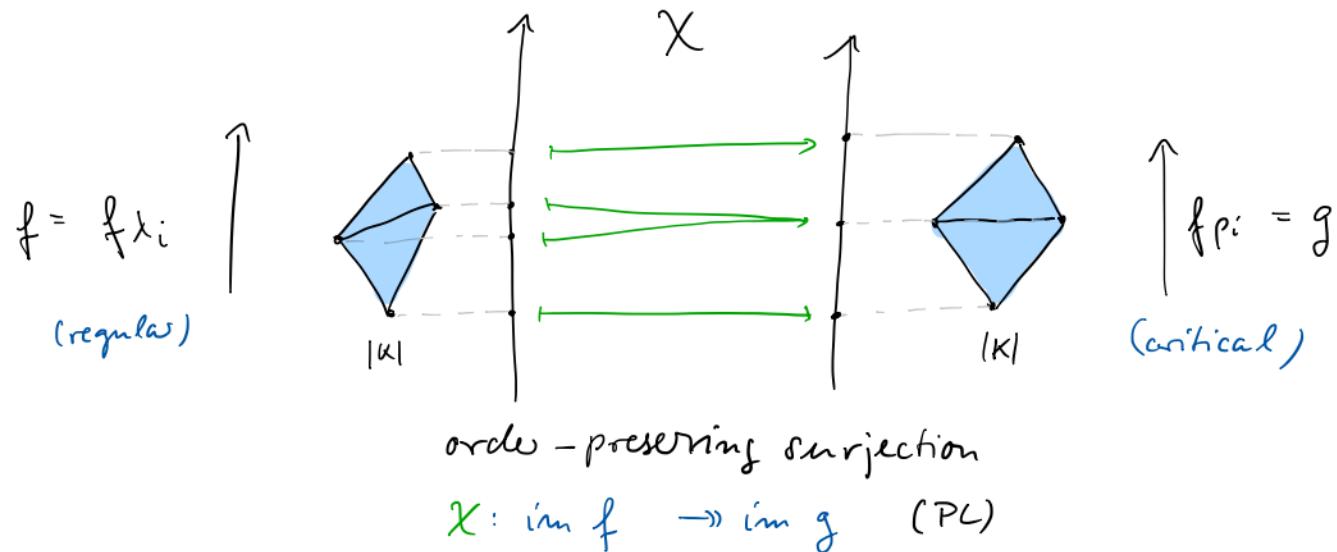
$$d_{\text{Graph}}(R_f, R_g) = \inf_{\mathcal{Z}} \|s_{\mathcal{Z}}\|_{\infty}.$$

MAIN RESULT

Then [B., Landi, Mémoli] The Reeb graph edit distance is stable & universal.

- We restrict to the PL category here.
- The hard part is stability:
given $f, g : X \rightarrow \mathbb{R}$ (PL, for triangulation $X = lkI$), how to construct an edit zigzag between R_f and R_g with spread $\leq \|f - g\|_\infty$?
- Idea:
 - Consider straight-line homotopy $f_\lambda = \lambda f + (1-\lambda)g$
 - The structure of $R_\lambda = R_{f_\lambda}$ changes only finitely often (say, at parameters $0 = \lambda_0 < \dots < \lambda_n = 1$). Choose $\pi_i \in (\lambda_i, \lambda_{i+1})$.
 - Construct zigzag $R_f = R_{\lambda_0} \xrightarrow{R_{\lambda_1}} \dots \xrightarrow{R_{\lambda_i}} \xrightarrow{R_{\lambda_{i+1}}} \xrightarrow{R_{\lambda_n}} R_g$
 - How to get the Reeb quotient maps in this zigzag?

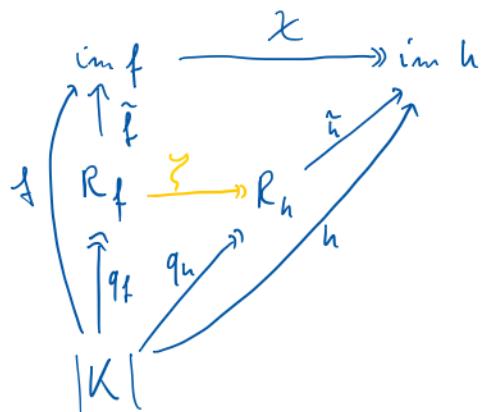
CRITICAL INSTANTS OF A PL STRAIGHT-LINE HOMOTOPY



- We have $X \circ f(v) = g(v) \quad \forall v \in \text{Vert}(K)$
- But $X \circ f \neq g$!
- However : $X \circ f$ and g have the same Reeb graph ...

LIFTING REPARAMETRIZATIONS

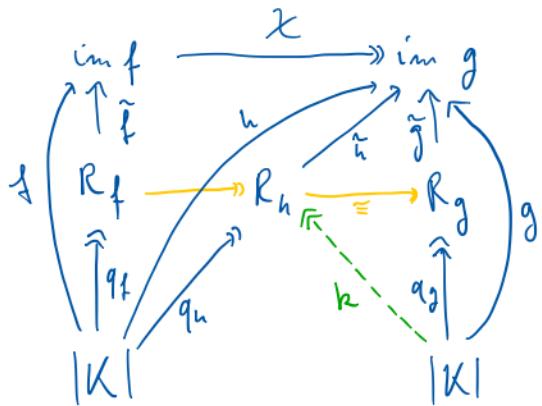
Lemma Let $h = \chi \circ f$. Then χ lifts to a Reeb quotient map $\tilde{\chi} : R_f \rightarrow R_h$.



THE WORKHORSE : REEB QUOTIENT MAPS FROM INTERPOLATION

$\chi : \text{im } f \rightarrow \text{im } g$ (order-preserving PL surjection),

$$g(v) = \underbrace{\chi \circ f(v)}_h \quad \forall v \in \text{Vert } K.$$



Lemma The relation

$$k = q_h \circ ((h^{-1} \circ g) \cap \text{st}_K)$$

is a Reeb quotient map.

Corollary $R_h \cong R_g$, and
 χ lifts to a Reeb quotient map

$$R_f \rightarrow R_g.$$

This provides the maps

$$\begin{matrix} & R_{\lambda_i} \\ R_{\lambda_i} & \swarrow \quad \searrow \\ & R_{\lambda_{i+1}} \end{matrix}$$

$$R_{\lambda_i} \quad R_{\lambda_{i+1}}$$

$$R_{\lambda_i} \quad R_{\lambda_{i+1}}$$

for our interpolation zigzag.

CONCLUSION

- A universal distance is the most discriminative stable distance between Reeb graphs
- There is a simple construction of a universal distance
- Interleaving and functional distortion distances are not universal
- A universal distance in PL can be constructed using graph edit zigzags

Questions:

- What is the complexity of computing the distance?
- Is $d_u \leq C \cdot d_I$ for some constant C ?

Announcement ...