The Reeb Graph Edit Distance is Universal

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Theory & Foundations of TGDA

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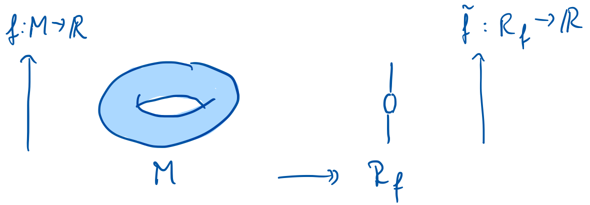
"Here are two things that are reasonably close to each other, and I want to compare them."

S. WEINBERGER

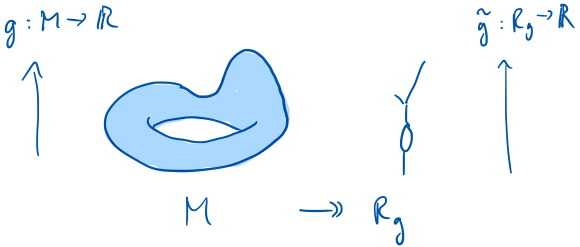
REEB GRAPHS



identify components of level sets $f^{-1}(t)$:

$$R_f = M/\sim_f ,$$

where $x \sim_f y \iff x, y$ in same component of some $f^{-1}(t)$, $t \in \mathbb{R}$.



Formal Setting

We consider

- locally compact Hausdorff spaces (Reeb domains)
- proper quotient maps with connected fibers (Reeb quotient maps)

These maps are closed under composition, and stable under pullbacks.

Define a Reeb graph as

- a Reeb domain R_f with
- a function $\tilde{f} : R_f \to \mathbb{R}$ with discrete fibers (Reeb function)

A Reeb graph R_f is the Reeb graph of a function $f : X \to \mathbb{R}$ if

- $f = \tilde{f} \circ p$ for some Reeb quotient map $p : X \twoheadrightarrow R_f$.

In this case, $R_f \cong X/{\sim_f}$.

Moreover: let $q : Y \twoheadrightarrow X$ be a Reeb quotient map.
Then R_f is also the Reeb graph of $g = f \circ q$.

$$\begin{tikzcd} Y \arrow[r, two heads, "q"] \arrow[drr, bend right, "g"'] & X \arrow[r, two heads, "p"] \arrow[dr, "f"'] & R_f \arrow[d, "\tilde{f}"] \\ & & \mathbb{R} \end{tikzcd}$$

- Reeb quotient maps preserve Reeb graphs.

GOALS

How to compare two Reeb graphs R_f, R_g? ($f, g : M \to \mathbb{R}$ are unknown)

Assign distance (extended pseudo-metric) $d(R_f, R_g)$.

Desirable properties:

Stability: For any space X and $f, g : X \to \mathbb{R}$ yielding Reeb graphs R_f, R_g,

$$d(R_f, R_g) \leq \|f - g\|_\infty .$$

Universality: For any other stable distance d_S,

$$d_S(R_f, R_g) \leq d(R_f, R_g) .$$

A CANONICAL UNIVERSAL DISTANCE

Given Reeb graphs R_f, R_g with functions $\tilde{f}, \tilde{g}$, define

$$d_u(R_f, R_g) = \inf \quad \|f-g\|_\infty$$

$$\begin{tikzcd}
& X \arrow[dl, "p_f"'] \arrow[dr, "p_g"] \arrow[ddl, "f"] \arrow[ddr, "g"'] & \\
R_f \arrow[d, "\tilde{f}"'] & & R_g \arrow[d, "\tilde{g}"] \\
\mathbb{R} & & \mathbb{R}
\end{tikzcd}$$

taken over all Reeb domains X and Reeb quotient maps p_f, p_g.

- stability: infimum is lower bound on all $\|f-g\|_\infty$
- universality: infimum is greatest lower bound

TRIANGLE INEQUALITY FOR THE CANONICAL UNIVERSAL DISTANCE

Triangle inequality: for all

$$\begin{tikzcd}
 & X \arrow[dl, two heads] \arrow[dr, two heads] & & Y \arrow[dl, two heads] \arrow[dr, two heads] & \\
R_f & & R_g & & R_h
\end{tikzcd},$$

consider pullback

$$\begin{tikzcd}
 & & P \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] \arrow[dddll, bend right, "f_P"'] \arrow[ddd, bend left, "g_P"] \arrow[dddrr, bend left, "h_P"] & & \\
 & X \arrow[dl, two heads] \arrow[dr, two heads] \arrow[ddl, "f_X"] \arrow[ddr, "g_X"] & & Y \arrow[dl, two heads] \arrow[dr, two heads] \arrow[ddl, "g_Y"] \arrow[ddr, "h_Y"] & \\
R_f \arrow[d, "\tilde f"'] & & R_g \arrow[d, "\tilde g"'] & & R_h \arrow[d, "\tilde h"'] \\
\mathbb{R} & & \mathbb{R} & & \mathbb{R}
\end{tikzcd}$$

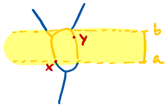
Then
$$\|f_P - h_P\| \leq \|f_P - g_P\| + \|g_P - h_P\|$$
$$= \|f_X - g_X\| + \|g_Y - h_Y\|.$$

- working with arbitrary spaces X is unfeasible

PREVIOUS WORK: FUNCTIONAL DISTORTION DISTANCE [B., Ge, Wang 2014]

- On a Reeb graph R_f with $\tilde{f}: R_f \to \mathbb{R}$, consider the metric

$$d_f : (x,y) \mapsto \inf \{ b-a \mid x,y \text{ in same component of } \tilde{f}^{-1}[a,b] \}.$$



- Given maps $\phi : R_f \to R_g$, $\psi : R_g \to R_f$, consider

$$G(\phi,\psi) = \{ (x, \phi(x)) \mid x \in R_f \} \cup \{ (\psi(y), y) \mid y \in R_g \}.$$

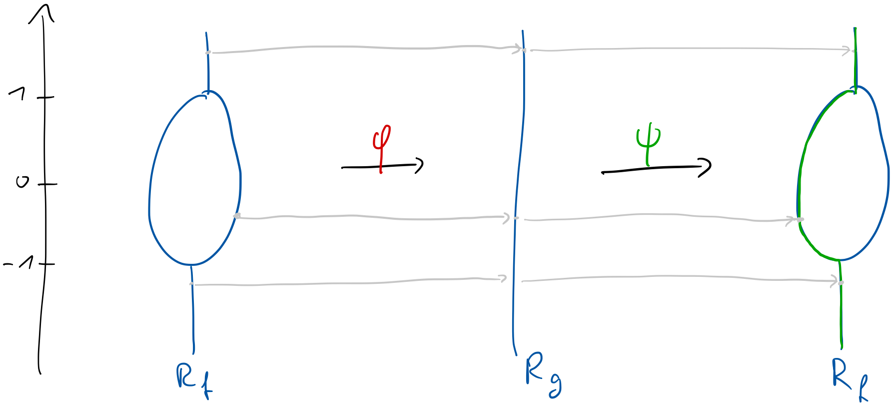
- Define the distortion of (ϕ,ψ) as

$$D(\phi,\psi) = \sup_{(x,y),(\tilde{x},\tilde{y}) \in G(\phi,\psi)} \frac{1}{2} \left| d_f(x,\tilde{x}) - d_g(y,\tilde{y}) \right|.$$

- Define the functional distortion distance as

$$d_{FD}(R_f, R_g) = \inf_{\phi,\psi} \left(\max \{ D(\phi,\psi), \| f - g\circ\phi \|_\infty, \| g - f\circ\psi \|_\infty \} \right).$$

EXAMPLE: FUNCTIONAL DISTORTION DISTANCE



$$D(\varphi, \psi) = \sup \frac{1}{2}\left(d(x, \tilde{x}) - d(y, \tilde{y})\right) = \frac{1}{2}$$

where $x, \tilde{x} \in R_f$, $y, \tilde{y} \in R_g$

with $\varphi(x) = y$ or $x = \psi(y)$,

$\varphi(\tilde{x}) = \tilde{y}$ or $\tilde{x} = \psi(\tilde{y})$

PREVIOUS WORK: INTERLEAVING DISTANCE [Bubenik & al. 2015; de Silva & al. 2016]

- Interpret Reeb graph R_f as a functor (in fact, a cosheaf)

$$F : \mathrm{Int}_{\mathbb{R}} \to \mathbf{Set}, \quad I \mapsto \pi_0(\tilde{f}^{-1}(I))$$

($\mathrm{Int}_{\mathbb{R}}$: open intervals, as a poset wrt. $\subseteq$)

- A δ-interleaving between F and G is a pair of natural transformations φ, ψ (with components $\varphi_I : F(I) \to G(B_\delta(I)), \dots$) such that

$$\begin{tikzcd}
F(I) \arrow[r] \arrow[dr, "\varphi_I"'] & F(B_\delta(I)) \arrow[r] \arrow[dr, "\varphi_{B_\delta(I)}"'] & F(B_{2\delta}(I)) \\
G(I) \arrow[r] \arrow[ur, "\psi_I"] & G(B_\delta(I)) \arrow[r] \arrow[ur, "\psi_{B_\delta(I)}"] & G(B_{2\delta}(I))
\end{tikzcd}$$

commutes for all $I \in \mathrm{Int}_{\mathbb{R}}$.

- $B_\delta(I)$: thickening I by δ
- Unlabeled maps: induced by inclusion

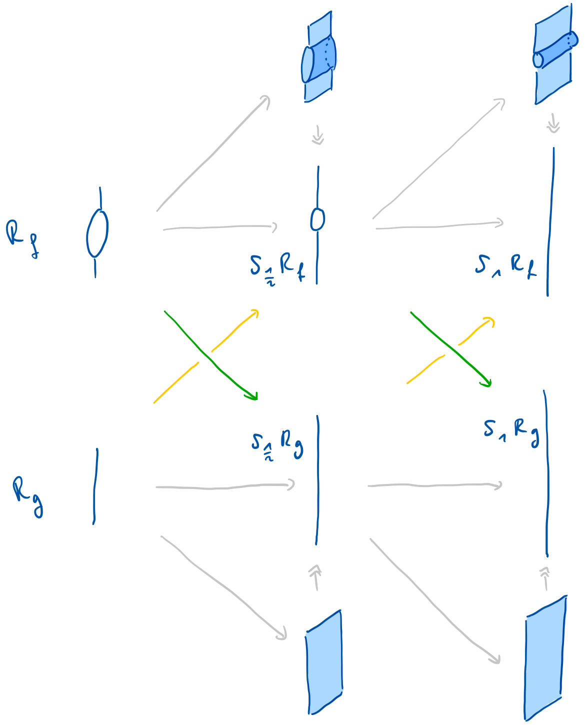
- The interleaving distance is

$$d_I(R_f, R_g) := \inf \{ \delta \mid \exists\ \delta\text{-interleaving between } F \text{ and } G \}$$

Thm [B., Munch, Wang 2015] $\frac{1}{3} d_{FD} \le d_I \le d_{FD}$.

Open problem:
is the lower bound tight?

ABSTRACT AND TOPOLOGICAL INTERLEAVINGS [de Silva & al. 2016]



Reeb graph $\rightsquigarrow$ cosheaf

$$R_f \rightsquigarrow F = \pi_0 \circ f^{-1}$$

$$S_\delta R_f \rightsquigarrow F \circ B_\delta$$

"smoothing" thickening

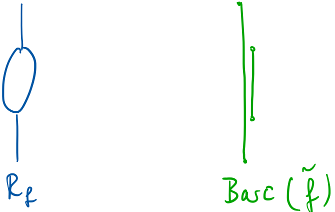
LEVEL SET PERSISTENT HOMOLOGY

Thm [Carlsson, de Silva, Morozov 2009]

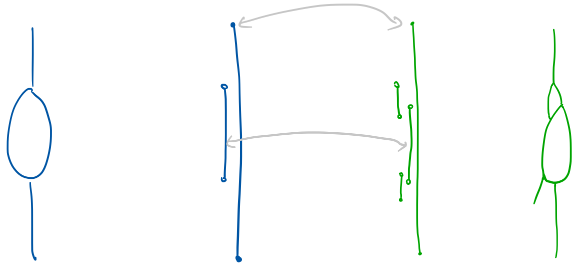
Given $f : X \to \mathbb{R}$ (PL, with X compact):

Homology of level sets $H_*(f^{-1}(t); \mathbb{F})$ (and more generally, of inclusions $f^{-1}(I) \hookrightarrow f^{-1}(J)$ for intervals $I \subseteq J$) is encoded (up to isomorphism) by a unique collection of intervals (level set persistence barcode).

Example for a Reeb graph:



THE BOTTLENECK DISTANCE BETWEEN PERSISTENCE BARCODES



A **δ-matching** between two barcodes $\mathrm{Barc}(f)$, $\mathrm{Barc}(g)$ satisfies:

- matched intervals (I, J) have distance $d_H(I, J) \leq \delta$
- unmatched intervals have length $\leq 2\delta$

The **bottleneck distance** $d_B(f, g)$ is

$$\inf \delta : \exists \ \delta\text{-matching between } \mathrm{Barc}(f), \mathrm{Barc}(g)$$

A Zoo of Distances and Inequalities

[Carlsson, de Silva, Morozov 2009]

$$d_B(R_f, R_g) \leq \|f-g\|_\infty$$

[B., Ye, Wang 2014]

$$\frac{1}{3} d_B(R_f, R_g) \leq d_{FD}(R_f, R_g) \leq \|f-g\|_\infty$$

[B., Munch, Wang 2015]

$$\frac{1}{3} d_{FD}(R_f, R_g) \leq d_I(R_f, R_g) \leq d_{FD}(R_f, R_g)$$

[Botnan, Lesnick 2016]

$$\frac{1}{5} d_B(R_f, R_g) \leq d_I(R_f, R_g)$$

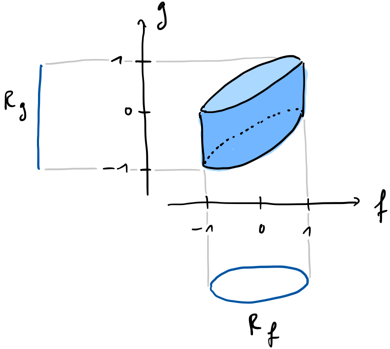
[Bjerkevik 2016]

$$\frac{1}{2} d_B(R_f, R_g) \leq d_I(R_f, R_g)$$

- d_B : bottleneck distance (of level set H_0 barcode)
- d_{FD} : functional distortion distance
- d_I : interleaving distance

FUNCTIONAL DISTORTION & INTERLEAVING DISTANCES ARE NOT UNIVERSAL

Consider a cylinder with two functions f, g:



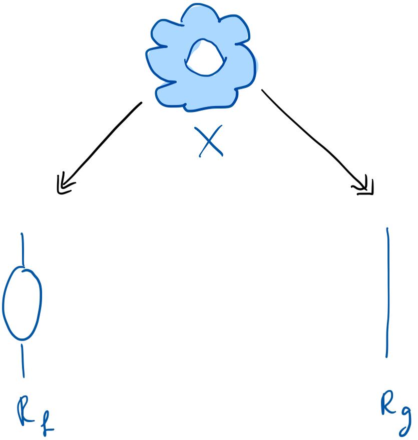
- $d_u(R_f, R_g) \leq \|f-g\|_\infty = 1$

- $d_I(R_f, R_g) \leq d_{FD}(R_f, R_g) \leq \frac{1}{2} < d_u(R_f, R_g)$:

FROM CLOSE REEB GRAPHS TO CLOSE FUNCTIONS

Open problem

Given two Reeb graphs R_f, R_g with $d_I(R_f, R_g) = \delta$.
Is there a space X with $f, g : X \to \mathbb{R}$, $\|f - g\| \leq C \cdot \delta$,
yielding Reeb graphs R_f, R_g, for some fixed constant C?



By the previous example:
if yes, then $C \geq 2$.

The Topological Edit Distance

- Consider zig-zag diagrams Z of Reeb quotient maps

$$\begin{tikzcd}
\mathbb{R} & & \mathbb{R} & & \mathbb{R} & & & \mathbb{R} \\
R_f = R_1 \arrow[u, "\tilde f_1"] & & R_2 \arrow[u, "\tilde f_2"] & & R_3 \arrow[u, "\tilde f_3"] & & & R_n = R_g \arrow[u, "\tilde f_n"] \\
& X_1 \arrow[ul, two heads] \arrow[ur, two heads] & & X_2 \arrow[ul, two heads] \arrow[ur, two heads] & & \cdots & X_{n-1} \arrow[ur, two heads] & \\
& & & L_Z \arrow[ull, two heads] \arrow[u, two heads] \arrow[urrr, two heads] & \cdots & & &
\end{tikzcd}$$

and take the limit L_Z (note: all maps are Reeb quotient maps).

- Define the *spread* of the functions $f_i : L_Z \twoheadrightarrow R_i \to \mathbb{R}$ as

$$s_Z : L_Z \to \mathbb{R}, \qquad x \mapsto \max_i f_i(x) - \min_j f_j(x).$$

(Note: $\|f-g\|_\infty \leq \|s_Z\|_\infty$, with equality for $n=2$)

- Define the (topological) *edit distance* as

$$d_{eTop}(R_f, R_g) = \inf_Z \|s_Z\|_\infty .$$

Prop. d_{eTop} is stable and universal.

THE REEB GRAPH EDIT DISTANCE

- Consider zig-zag diagrams Z of Reeb quotient maps

$$R_f = R_1 \twoheadleftarrow G_1 \twoheadrightarrow R_2 \twoheadleftarrow G_2 \twoheadrightarrow R_3 \quad \cdots \quad G_{n-1} \twoheadrightarrow R_n = R_g$$

as before, but restrict R_i, G_j in Z to be finite graphs.

Interpretation:

$$\begin{tikzcd}[column sep=small] & \mathbb{R} & \\ & R_{i+1} \arrow[u, "\tilde{f}_i"'] & \\ G_i \arrow[ur, two heads] & & G_{i+1} \arrow[ul, two heads] \end{tikzcd} \qquad \text{edit} \qquad \begin{tikzcd}[column sep=small] \mathbb{R} & & \mathbb{R} \\ R_i \arrow[u, "\tilde{f}_i"] & & R_{i+1} \arrow[u, "\tilde{f}_{i+1}"'] \\ & G_i \arrow[ul, two heads] \arrow[ur, two heads] & \end{tikzcd} \qquad \text{relabel}$$

modify G_i to G_{i+1}, maintaining the Reeb graph R_{i+1}

modify f_i to $f_{i+1} : G_i \to \mathbb{R}$, maintaining the domain G_i

- Define the Reeb graph edit distance analogously as

$$d_{eGraph}(R_f, R_g) = \inf_Z \|s_Z\|_\infty .$$

MAIN RESULT

Thm [B., Landi, Mémoli] The Reeb graph edit distance is stable & universal.

- We restrict to the (compact) PL category here.
- The hard part is stability:

  given $f, g : X \to \mathbb{R}$ (PL, for triangulation $X = |K|$),

  construct edit zigzag between R_f and R_g with spread $\leq \|f-g\|_\infty$?

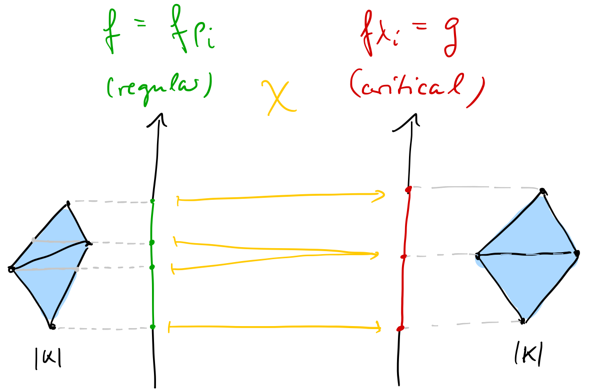
- Idea:
  - Consider straight-line homotopy $f_\lambda = \lambda f + (1-\lambda) g$
  - The structure of $R_\lambda = R_{f_\lambda}$ changes only finitely often (say, at parameters $0 = \lambda_0 < \dots < \lambda_n = 1$). Choose $p_i \in (\lambda_i, \lambda_{i+1})$.
  - Construct zigzag

$$\begin{tikzcd}[column sep=small]
R_f = R_{\lambda_0} & & R_{\lambda_i} & & R_{\lambda_{i+1}} & & R_{\lambda_n} = R_g \\
& \cdots \arrow[ul] \arrow[ur, dotted] & & R_{p_i} \arrow[ul] \arrow[ur] & & \cdots \arrow[ul, dotted] \arrow[ur] &
\end{tikzcd}$$

  - How to get the Reeb quotient maps in this zigzag?

CRITICAL INSTANTS OF A PL STRAIGHT-LINE HOMOTOPY

W.l.o.g.:



order-preserving surjection

$\chi : \operatorname{im} f \twoheadrightarrow \operatorname{im} g$ (PL)

- We have $\chi \circ f(v) = g(v) \quad \forall \; v \in \operatorname{Vert}(K)$
- But $\chi \circ f \neq g$!
- However: $\chi \circ f$ and g have the same Reeb graph...

LIFTING REPARAMETRIZATIONS

Lemma Let $h = \chi \circ f$. Then χ lifts to a Reeb quotient map $\zeta : R_f \to R_h$.

$$\begin{tikzcd}
\operatorname{im} f \arrow[r, two heads, "\chi"] & \operatorname{im} h \\
R_f \arrow[u, "\tilde{f}"] \arrow[r, two heads, "\zeta"] & R_h \arrow[u, "\tilde{h}"'] \\
{|K|} \arrow[u, two heads, "q_f"] \arrow[ur, two heads, "q_h"] \arrow[uu, bend left, "f"] \arrow[uur, bend right, "h"']
\end{tikzcd}$$

THE WORKHORSE: REEB QUOTIENT MAPS FROM INTERPOLATION

$\chi : \operatorname{im} f \to \operatorname{im} g$ (order-preserving PL surjection),

$$g(v) = \underbrace{\chi \circ f}_{h}(v) \qquad \forall\, v \in \operatorname{Vert} K .$$

$$\begin{tikzcd}
\operatorname{im} f \arrow[r, two heads, "\chi"] & \operatorname{im} h \arrow[r, phantom, "="] & \operatorname{im} g \\
R_f \arrow[u, "\tilde f"] \arrow[r, two heads] & R_h \arrow[u, "\tilde h"'] \arrow[r, "\cong"'] & R_g \arrow[u, "\tilde g"] \\
{|K|} \arrow[u, two heads, "q_f"] \arrow[uu, bend left, "f"] \arrow[ur, two heads, "q_h"'] \arrow[uur, bend right, "h"] & & {|K|} \arrow[u, two heads, "q_g"'] \arrow[uu, bend right, "g"'] \arrow[ul, dashed, "k"]
\end{tikzcd}$$

Lemma The relation

$$k = q_h \circ \big((h^{-1} \circ g) \cap st_K \big)$$

is a Reeb quotient map.

Corollary $R_h \cong R_g$, and χ lifts to a Reeb quotient map $R_f \twoheadrightarrow R_g$.

This provides the maps $R_{\lambda_i} \twoheadleftarrow R_{p_i} \twoheadrightarrow R_{\lambda_{i+1}}$ for our interpolation zigzag.

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{|K|} \arrow[u, two heads, "q_f"] \arrow[uu, bend left, "f"] \arrow[uur, "h"] \arrow[ur, two heads, "q_h"'] & & {|K|} \arrow[u, two heads, "q_g"'] \arrow[uu, bend right, "g"'] \arrow[ul, dashed, two heads, "k"]
\end{tikzcd}$$

Lemma The relation

$$k = q_h \circ \left((h^{-1} \circ g) \cap st_K \right)$$

is a Reeb quotient map.

Corollary $R_h \cong R_g$, and χ lifts to a Reeb quotient map

$$R_f \twoheadrightarrow R_g.$$

This provides the maps $R_{\lambda_i} \twoheadleftarrow R_{\rho_i} \twoheadrightarrow R_{\lambda_{i+1}}$ for our interpolation zigzag.

CONCLUSION

- A universal distance is the most discriminative stable distance between Reeb graphs
- There is a simple construction of a universal distance
- Interleaving and functional distortion distances are not universal
- A universal distance in PL can be constructed using graph edit zigzags

Questions:

- What is the complexity of computing the distance?
- Is $d_u \leq C \cdot d_I$ for some constant C?