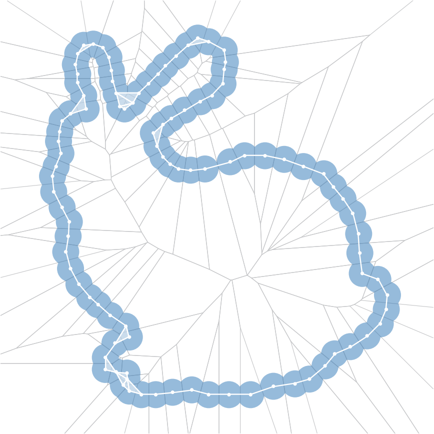


The Morse theory of Čech and Delaunay complexes

3rd ERC *SDModels* workshop

Discrete Models in Geometry and Topology

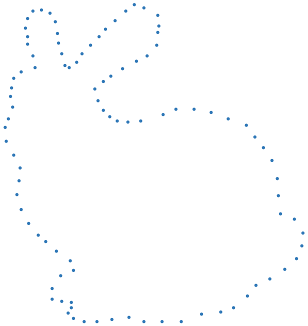
Ulrich Bauer

TUM

March 23, 2015

Joint work with Herbert Edelsbrunner (IST Austria)

Connect the dots: topology from geometry



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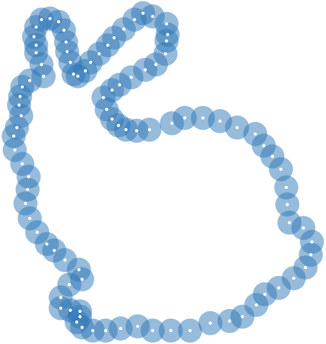

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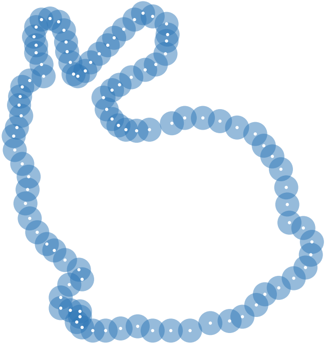

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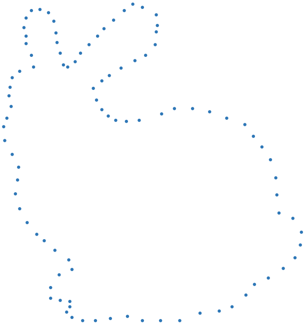

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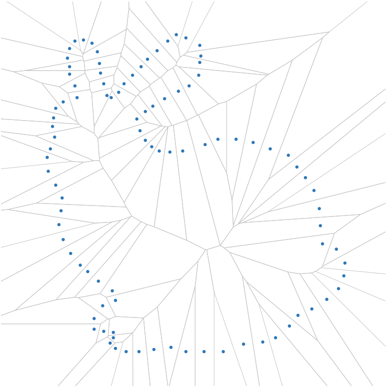

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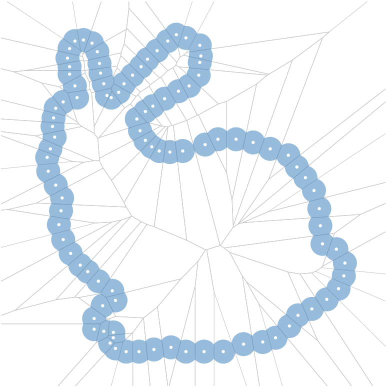

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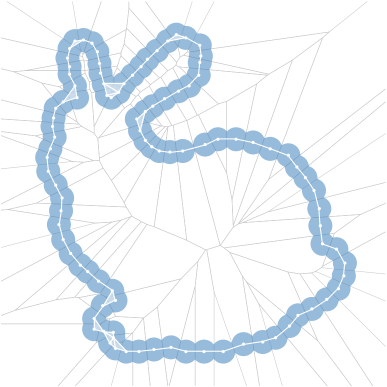
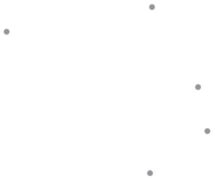


Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)



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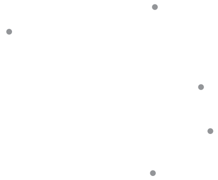

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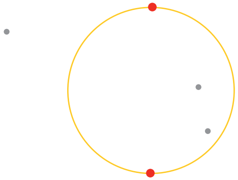

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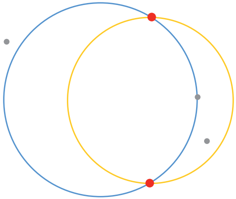

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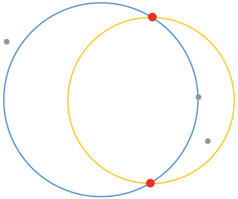

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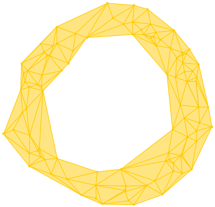

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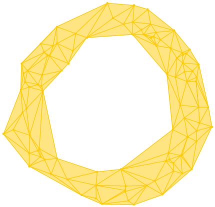

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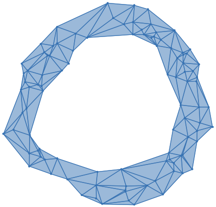

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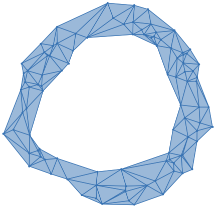

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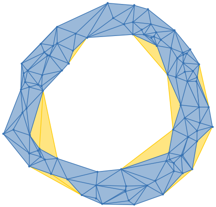

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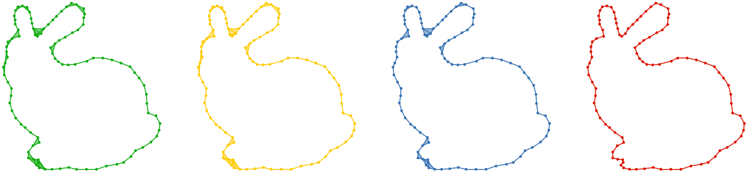
- Are all three complexes homotopy equivalent?
- Are they related by a sequence of simplicial collapses?

Main result: a sequence of collapses

Theorem (B, Edelsbrunner 2014)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent. In particular,

$$\mathrm{Cech}_r \searrow \mathrm{DelCech}_r \searrow \mathrm{Del}_r \searrow \mathrm{Wrap}_r \,.$$



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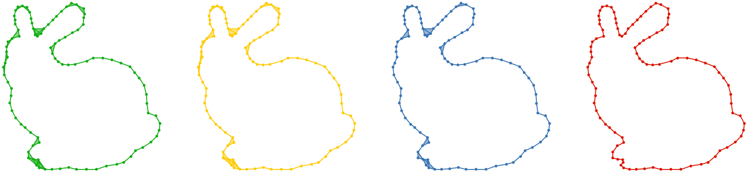

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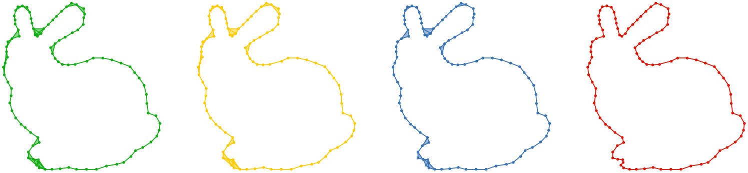

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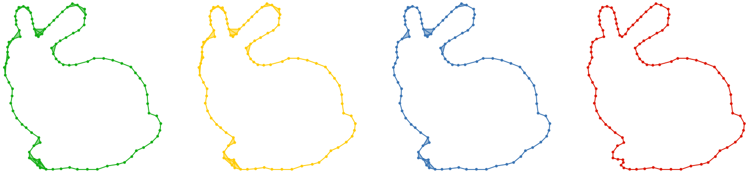

- All collapses are induced by a single *discrete gradient field*
- Explicit chain maps inducing isomorphisms in homology
- Also works for weighted point sets

Discrete Morse theory

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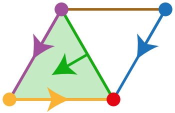
Definition (Forman 1998)

A *discrete vector field* on a simplicial complex
is a partition of the simplices
into singletons and pairs $\{L, U\}$,
where L is a face of U with codimension 1.



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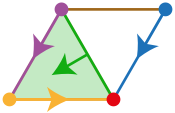

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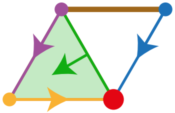

The singletons are called *critical simplices.*

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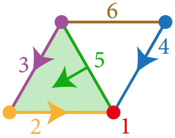

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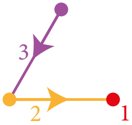

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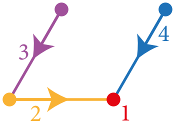

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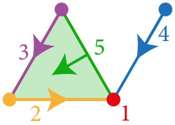

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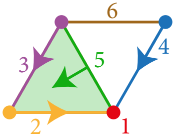

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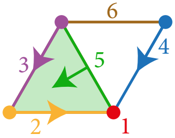

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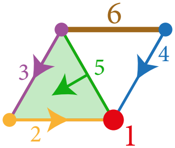
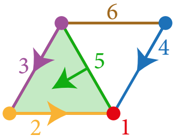


If $f^{-1}(t) = \{Q\}$ then t is a *critical value*.

Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.



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Theorem (Forman 1998)

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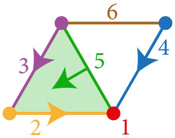

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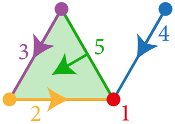

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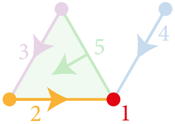

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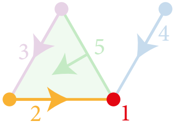

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Corollary

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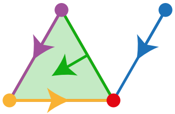

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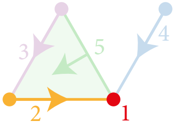

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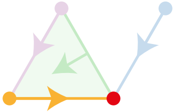

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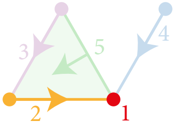

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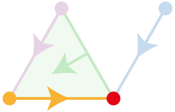

We say that V *induces the collapse $K \searrow L$.*

Unfortunately…

Neither the Čech nor the Delaunay functions are discrete Morse functions!

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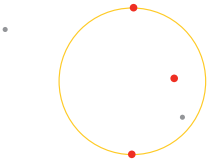
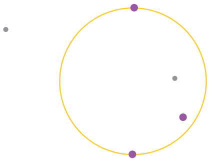
- Example: two simplices Q, Q' with $f_C(Q) = f_C(Q')$ such that neither is a face of the other:



Q



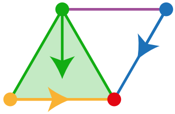
Q'

Generalized discrete Morse theory

Definition (Chari 2000, Freij 2009)

A *generalized discrete vector field* on a simplicial complex is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}$$



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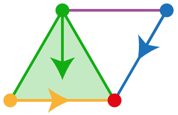

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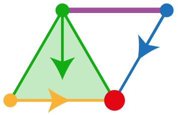

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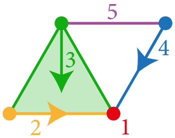
The singletons are called *critical simplices.*

Generalized discrete Morse theory

Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes



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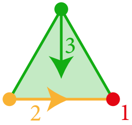

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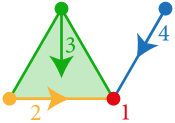

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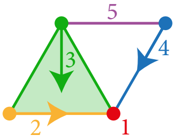

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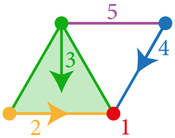
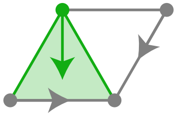
- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes
- the level sets $f^{-1}(t)$ form a generalized vector field (the *discrete gradient* of f)



Refining generalized vector fields

A generalized vector field V can be refined to a vector field.
For each non-critical interval $[L, U] \in V$:



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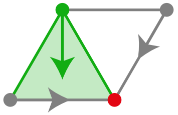

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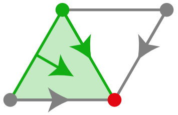

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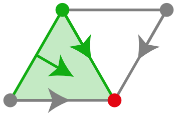

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Therefore the collapsing theorems also hold for generalized discrete Morse functions.

Čech and Delaunay intervals

Morse theory of Čech and Delaunay complexes

Proposition (B, Edelsbrunner 2014)

The Čech function and the Delaunay function are generalized discrete Morse functions.

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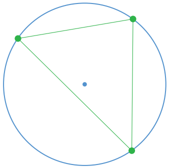
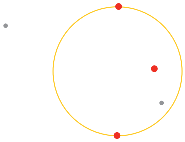
- *Q is a critical simplex of f_D*
- $f_D(Q) = f_C(Q)$
- *Q is a* centered *Delaunay simplex (containing the circumcenter in the interior)*



Čech intervals

Lemma

Let $Q \subseteq X$ be a simplex with smallest enclosing sphere S. Then $Q' \subseteq X$ has the same smallest enclosing sphere S iff



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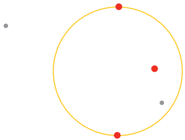

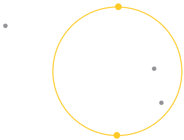

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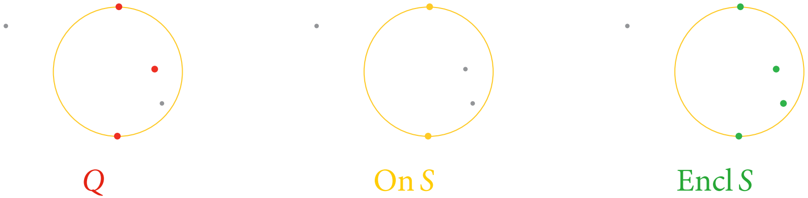

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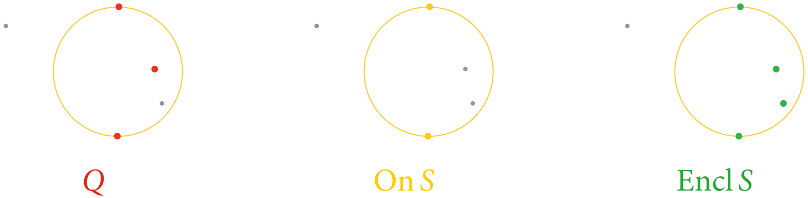

The front and back faces of a simplex

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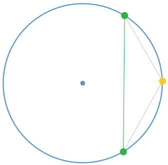

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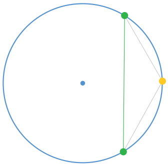

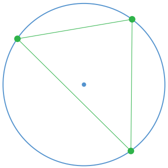

Delaunay intervals

Lemma

Let $Q \subseteq X$ be a simplex with smallest empty circumsphere S. Then $Q' \subseteq X$ has the same smallest empty circumsphere S iff

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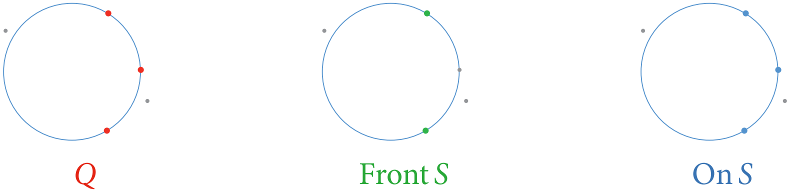

Selective Delaunay complexes

Sphere optimization problems

Both Čech and Delaunay functions are defined using the smallest sphere S satisfying certain constraints:

$$\begin{aligned}
&\underset{r,z}{\text{minimize}} && r && \\
&\text{subject to} && \|z-q\| \leq r, && q \in Q, \\
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- Čech function: choose $E = \varnothing$
- Delaunay function: choose $E = X$

Čech and Delaunay intervals from KKT

The *Karush–Kuhn–Tucker* optimality conditions yield:

Proposition

A sphere S enclosing Q and excluding E is minimal iff

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Let $E \subseteq F \subseteq X$. Then

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Note: choosing $E = \varnothing$ and $F = X$ yields

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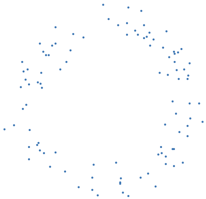
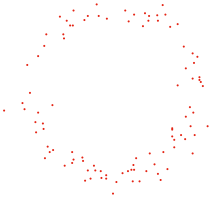
Connecting different Delaunay complexes

X



Y



Connecting different Delaunay complexes

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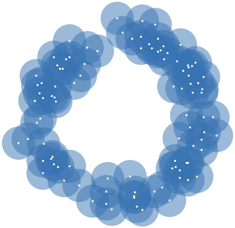
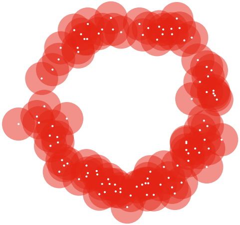
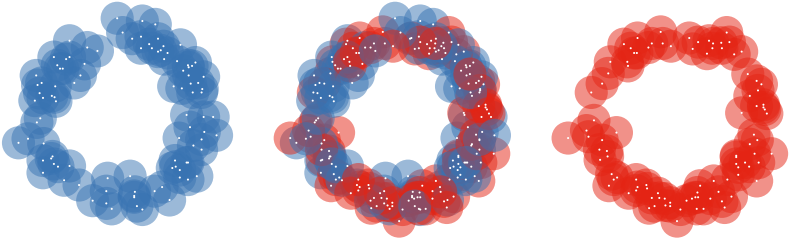


$B_r(Y)$



Connecting different Delaunay complexes

$$B_r(X) \hookrightarrow B_r(X \cup Y) \hookleftarrow B_r(Y)$$



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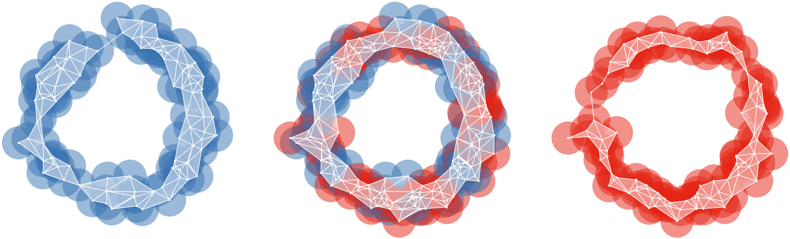

$\mathrm{Del}_r(X)$ $\qquad$ $\mathrm{Del}_r(X \cup Y)$ $\qquad$ $\mathrm{Del}_r(Y)$

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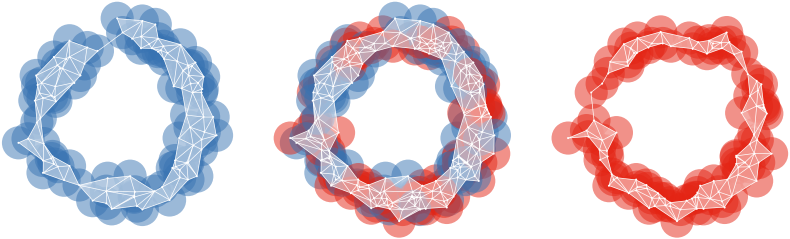

$$\begin{tikzcd}
\mathrm{Del}_r(X) \arrow[dr, hook] & & \mathrm{Del}_r(X \cup Y) \arrow[dl, hook'] \arrow[dr, hook] & & \mathrm{Del}_r(Y) \arrow[dl, hook'] \\
& \mathrm{Del}_r(X \cup Y, X) & & \mathrm{Del}_r(X \cup Y, Y) &
\end{tikzcd}$$

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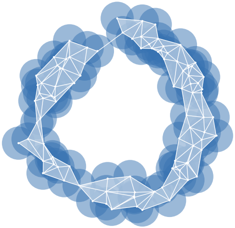
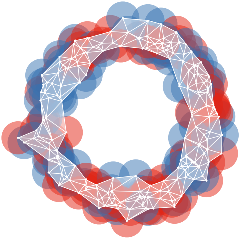
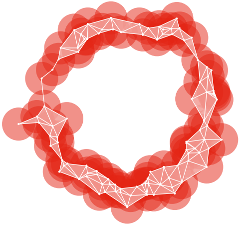

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Wrap complexes

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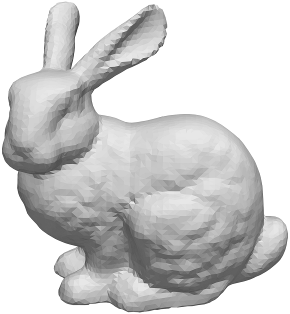
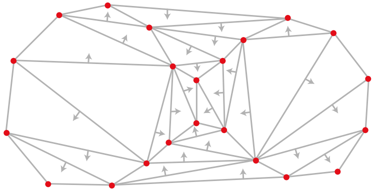
Generalizes and greatly simplifies the surface reconstruction algorithm *Wrap* (Edelsbrunner 1995, Geomagic)



Wrap complexes

Consider the Delaunay function f_D of X.



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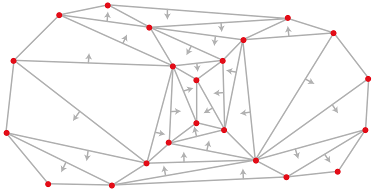

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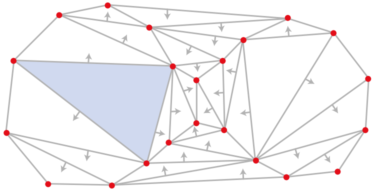

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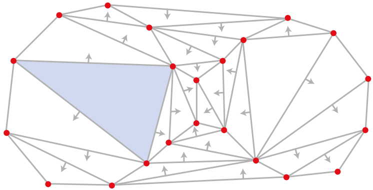

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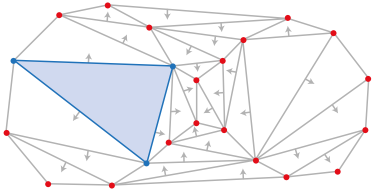

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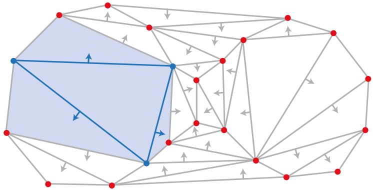

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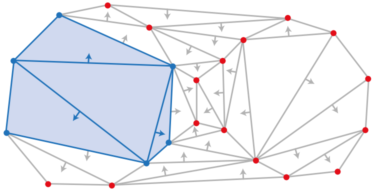

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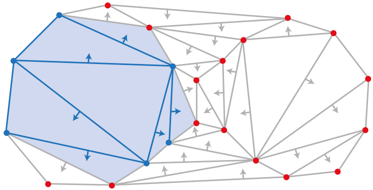

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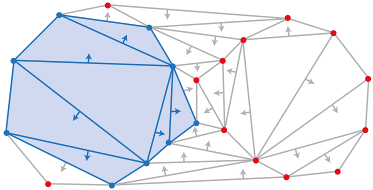

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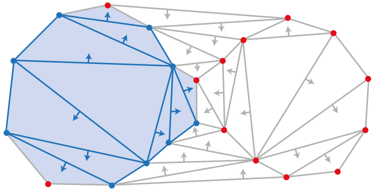

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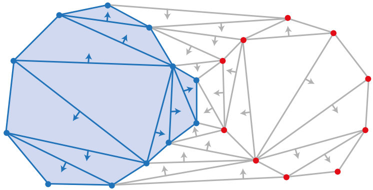

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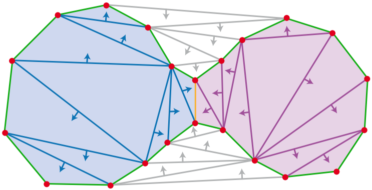

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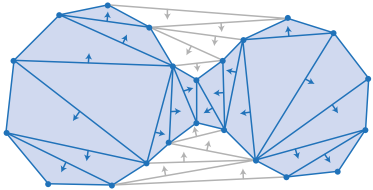

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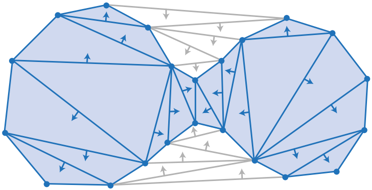

The Delaunay intervals induce a collapse $\mathrm{Del}_r \searrow \mathrm{Wrap}_r$.

Wrapping up

- Čech and Delaunay complexes from Morse functions
- Explicit homotopy equivalence by simplicial collapses
- Simple definition and generalization of *Wrap* complexes