Ripser

Efficient Computation of Vietoris–Rips Persistence Barcodes

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TUM

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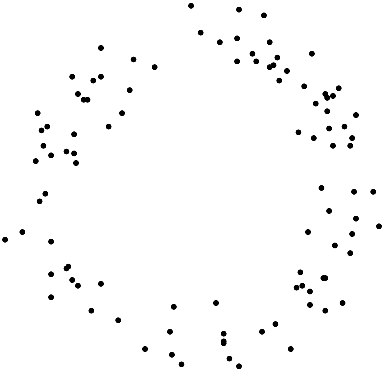
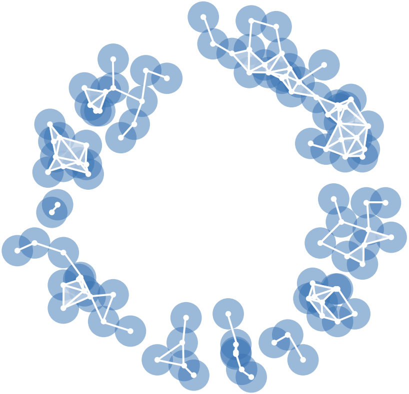
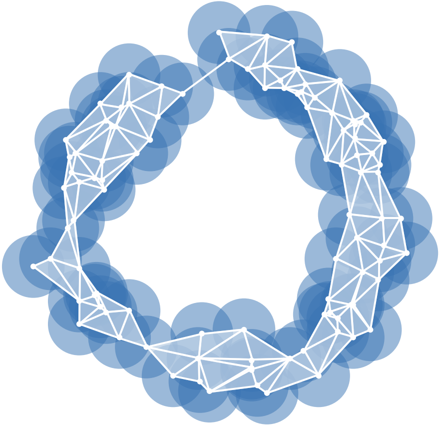
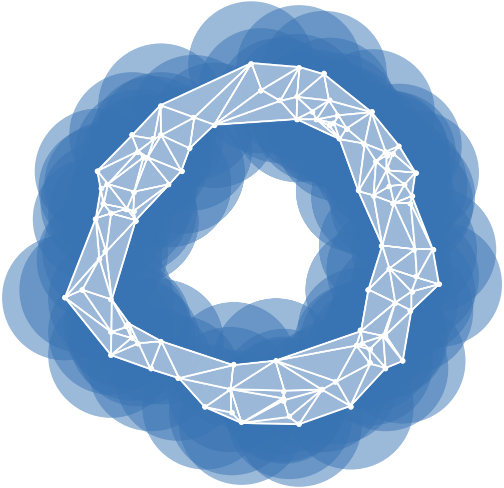
Special Hausdorff Program *Applied and Computational Algebraic Topology*
Hausdorff Research Institute for Mathematics, Bonn

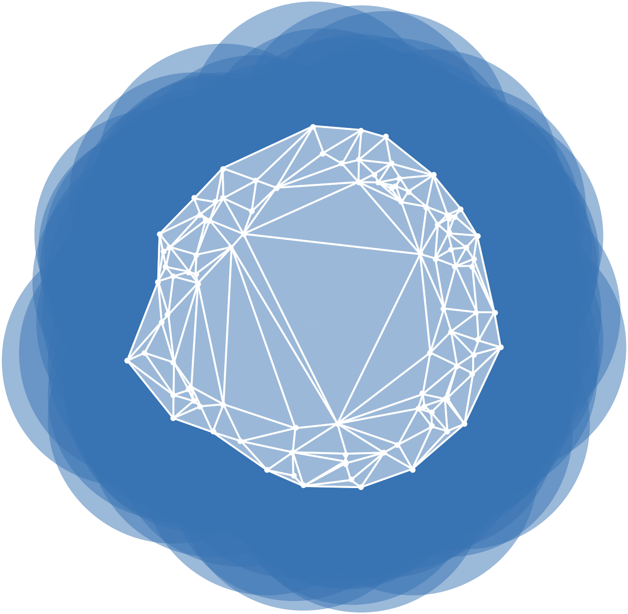
Persistent homology

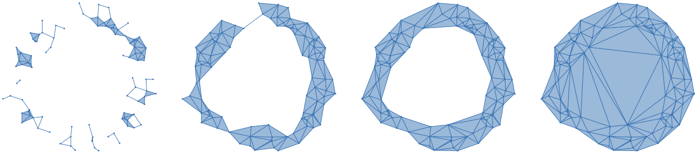


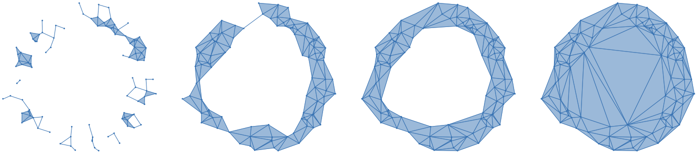


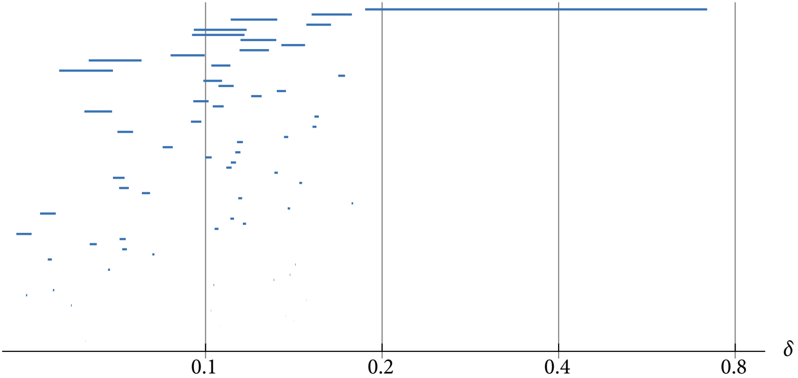












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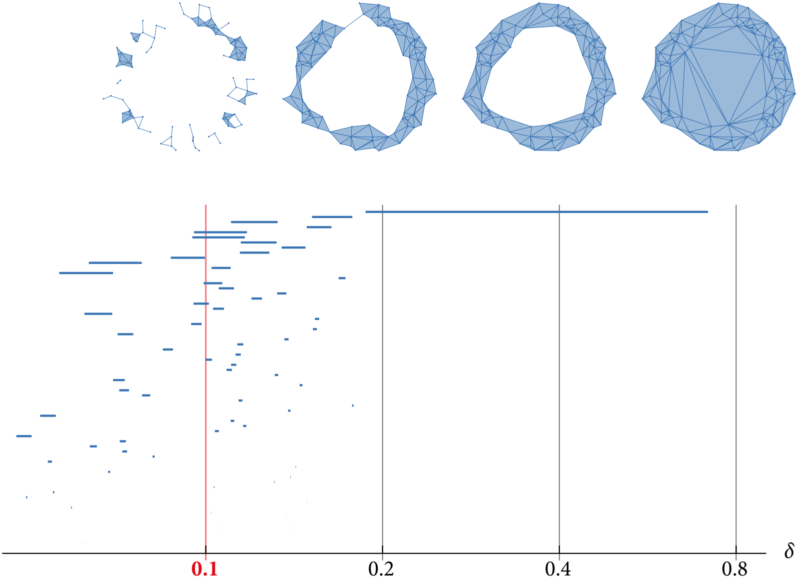

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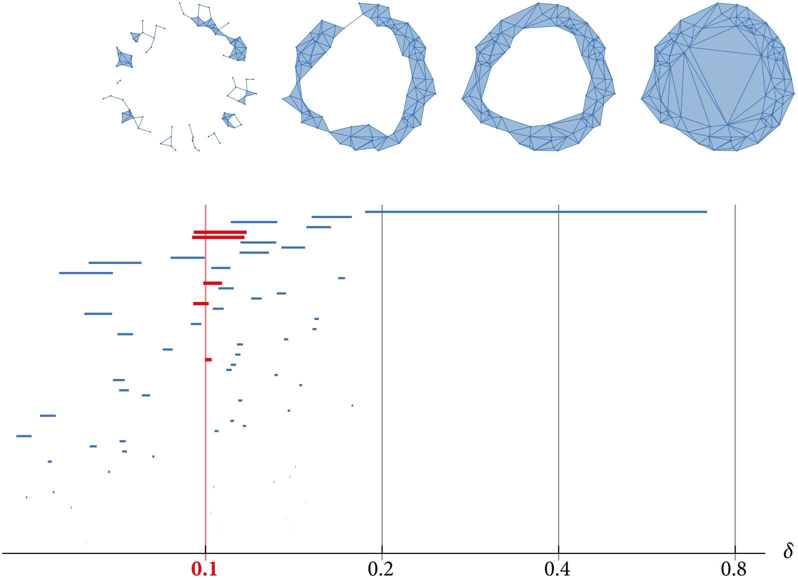
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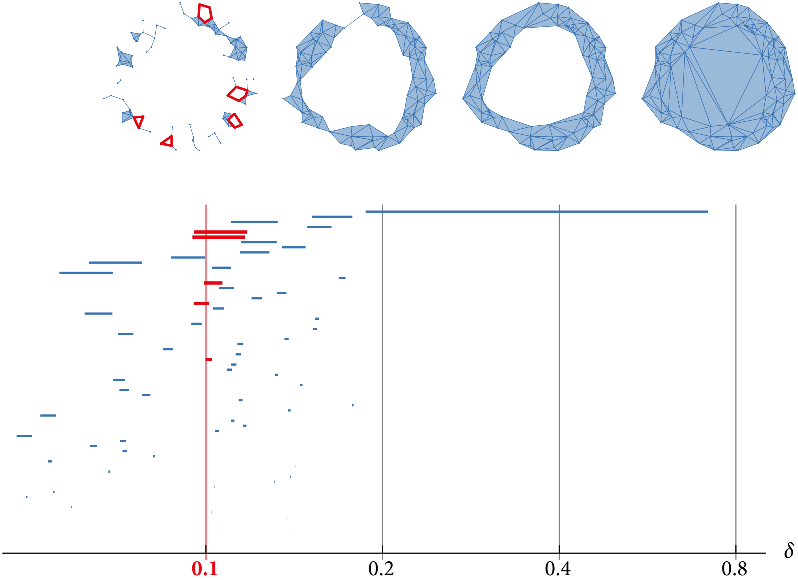
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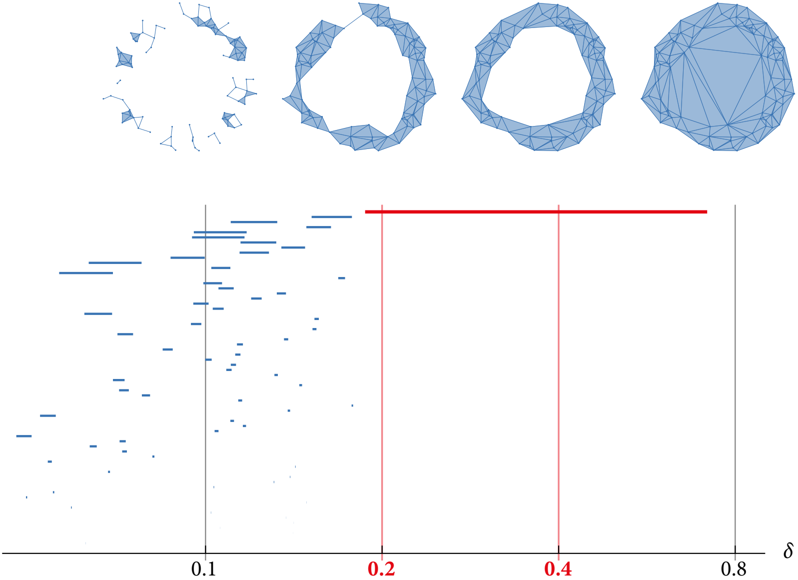

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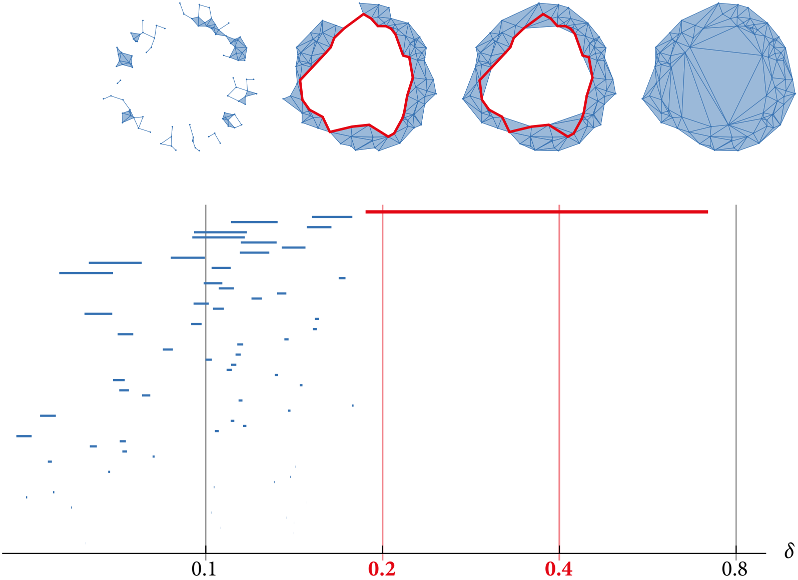

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Vietoris–Rips persistence

Vietoris–Rips filtrations

Consider a finite metric space (X, d).
The *Vietoris–Rips complex* is the simplicial complex

$$\mathrm{Rips}_t(X) = \{S \subseteq X \mid \operatorname{diam} S \leq t\}$$

- 1-skeleton: all edges with pairwise distance $\leq t$
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Goal:

- compute persistence barcodes for $H_d(\mathrm{Rips}_t(X))$ (in dimensions $0 \leq d \leq k$)

Demo: Ripser

Example data set:

- 192 points on $\mathbb{S}^2$
- persistent homology barcodes up to dimension 2
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Comparison with other software:

- javaplex: 3200 seconds, 12 GB
- Dionysus: 533 seconds, 3.4 GB
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- 2016 ATMCS Best New Software Award (jointly with RIVET)

Design goals

Goals for previous projects:

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  fast persistence computation (matrix reduction only)
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Goals for Ripser:

- Use as little memory as possible
- Be reasonable about computation time

The four special ingredients

The improved performance is based on 4 insights:

- Clearing inessential columns [Chen, Kerber 2011]
- Computing cohomology [de Silva et al. 2011]
- Implicit matrix reduction
- Apparent and emergent pairs

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Lessons from PHAT:

- Clearing and cohomology yield considerable speedup,
- but only when *both* are used in conjuction!

Matrix reduction

Matrix reduction algorithm

Setting:

- finite metric space X, n points
- persistent homology $H_d(\mathrm{Rips}_t(X); \mathbb{F}_2)$ in dimensions $d \leq k$

Notation:

- D: boundary matrix of filtration
- R_i: ith column of R

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Result:

- $R = D \cdot V$ is reduced (unique pivots)
- V is full rank upper triangular

Compatible basis cycles

For a reduced boundary matrix $R = D \cdot V$, call

$$P = \{i : R_i = 0\} \qquad \textit{positive indices,}$$

$$N = \{j : R_j \neq 0\} \qquad \textit{negative indices,}$$

$$E = P \setminus \operatorname{pivots} R \qquad \textit{essential indices.}$$

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- Persistence intervals: $\{[i, j) \mid i = \operatorname{pivot} R_j\} \cup \{[i, \infty) \mid i \in E\}$
- Columns with non-essential positive indices never used!

Clearing

Clearing non-essential positive columns

Idea [Chen, Kerber 2011]:

- Don't reduce at non-essential positive indices
- Reduce boundary matrices of $\partial_d : C_d \to C_{d-1}$ in decreasing dimension $d = k + 1, \ldots, 1$
- Whenever $i = \operatorname{pivot} R_j$ (in matrix for ∂_d)
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Note:

- reducing *positive* columns typically harder than negative
- with clearing: need only reduce *essential* positive columns

Cohomology

Persistent cohomology

We have seen: many columns of $R = D \cdot V$ are not needed

- Skip those inessential columns in matrix reduction

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For persistence barcodes in low dimensions $d \leq k$:

- Number of skipped indices for reducing D^T (cohomology) is much larger than for D (homology)
  - reducing boundary matrix produces basis for $H_{k+1}(K_{k+1})$, which is not needed
- The resulting persistence barcode is the same [de Silva et al. 2011]

Counting homology column reductions

- standard matrix reduction:

$$\sum_{d=1}^{k+1} \underbrace{\binom{n}{d+1}}_{\dim C_d(K)} = \sum_{d=1}^{k+1} \underbrace{\binom{n-1}{d}}_{\dim B_{d-1}(K)} + \sum_{d=1}^{k+1} \underbrace{\binom{n-1}{d+1}}_{\dim Z_d(K)}$$

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$k = 2, n = 192$: $\quad 1\,161\,472 = 1 + 1\,161\,471$

Observations

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Previous example ($k = 2, n = 192$):

- Only 845 out of $1\,161\,471$ columns have to be reduced

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Approach for Ripser:

- Boundary matrix D for lexicographically ordered basis
    - Implicitly defined and recomputed when needed
- Matrix reduction in Ripser: store only coefficient matrix V
    - recompute previous columns of $R = D \cdot V$ when needed
    - Typically, V is much sparser and smaller than R

Oblivious matrix reduction

Algorithm variant:

- $R = D$
- for $j = 1, \ldots, n$
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Corollary

The rank of an $m \times n$ matrix can be computed in $O(n)$ memory.

Apparent and emergent pairs

Natural filtration settings

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• simplexwise filtration	persistence (computation)

Conclusion:

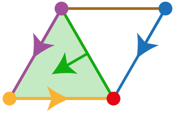
- Discrete Morse theory sits in the middle
  between persistence and persistence (!)

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a cell complex is a partition of the set of cells into

- singleton sets $\{\phi\}$ *(critical cells)*, and
- pairs $\{\sigma, \tau\}$, where σ is a facet of τ.



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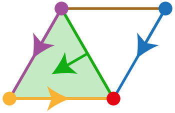

A function $f : K \to \mathbb{R}$ on a cell complex is a *discrete Morse function* if

- sublevel sets are subcomplexes, and

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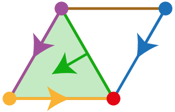

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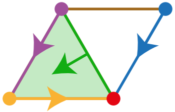

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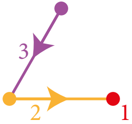

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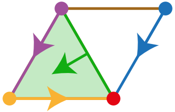

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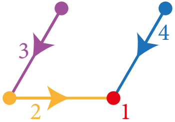

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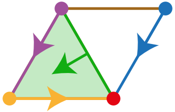

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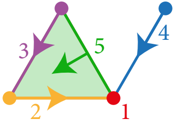

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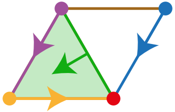

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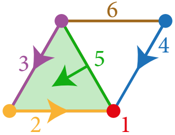

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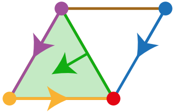

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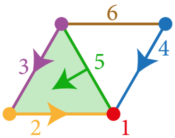

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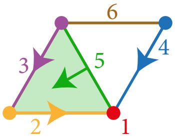

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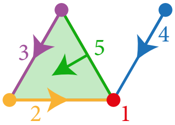

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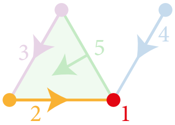

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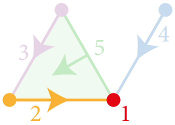

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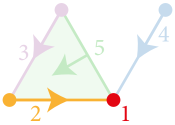

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This homotopy equivalence is compatible with the filtration.

Corollary

K and M have isomorphic persistent homology (with regard to the sublevel sets of f).

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Consider a *Morse filtration* (one or two simplices at a time).

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Consider a *simplexwise filtration* (one simplex at a time).
Persistence pair (σ, τ):

- inserting simplex σ creates a new *homological* feature
- inserting τ destroys that feature again

Apparent pairs

Definition

In a simplexwise filtration, (σ, τ) is an *apparent* pair if

- σ is the youngest face of τ
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Lemma

Any apparent pairs is a persistence pair.

Lemma

The apparent pairs form a discrete gradient.

- Generalizes a construction proposed by [Kahle 2011] for the study of random Rips filtrations

From Morse theory to persistence and back

Proposition (from Morse to persistence)

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From Morse theory to persistence and back

Proposition (from Morse to persistence)

The pairs of a Morse filtration are apparent 0-persistence pairs for the canonical simplexwise refinement of the filtration.

Proposition (from persistence to Morse)

Consider an arbitrary filtration with a simplexwise refinement.
The apparent 0-persistence pairs yield a Morse filtration

- *refining the original one, and*
- *refined by the simplexwise one.*

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Consider the *lexicographically refined Rips filtration:*

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Then (σ, τ) *is an* emergent *persistence pair.*

- Includes all apparent pairs with persistence 0
- Can be identified *without* enumerating all cofaces of σ
  - Provides a shortcut for computation

Ripser Live: users from 156 different cities