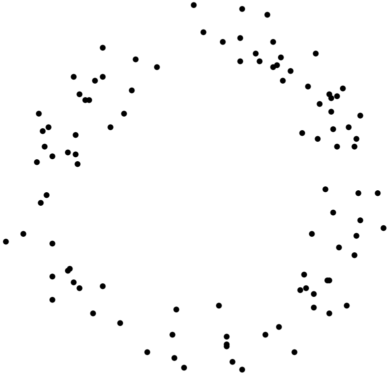
Induced matchings and the algebraic stability of persistence barcodes

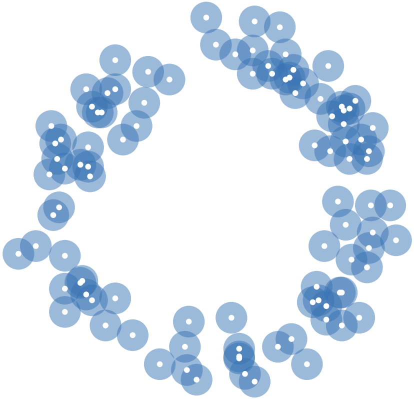
Ulrich Bauer

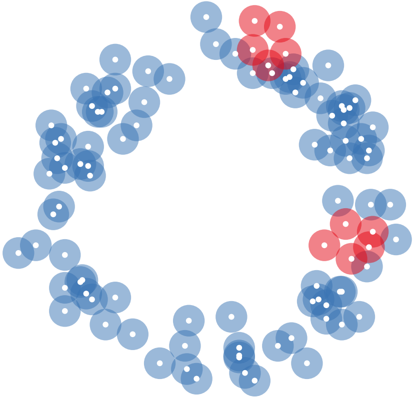
TUM

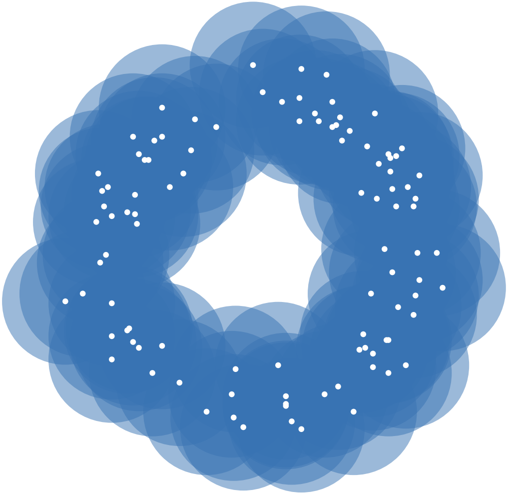
February 25, 2015

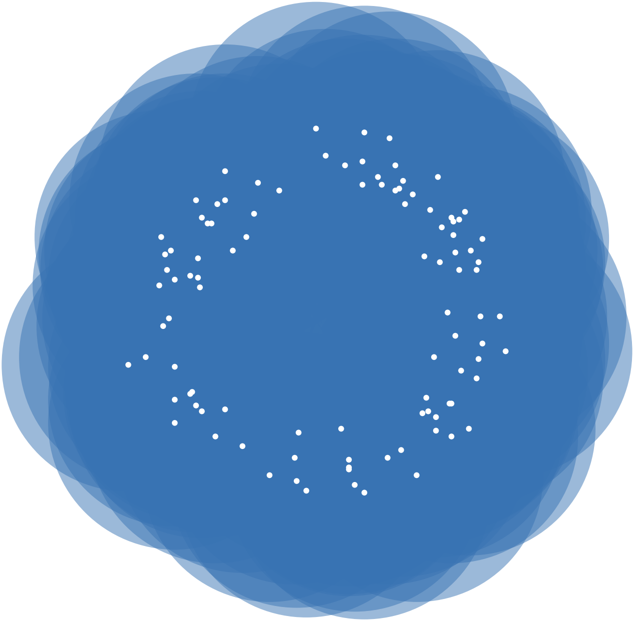
Joint work with Michael Lesnick (IMA)

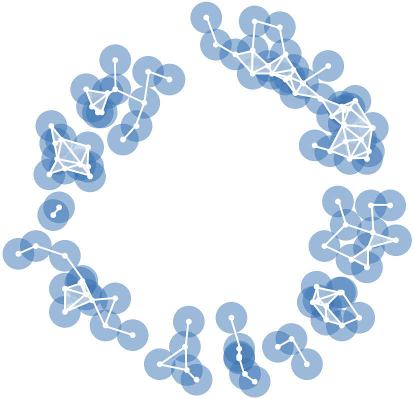


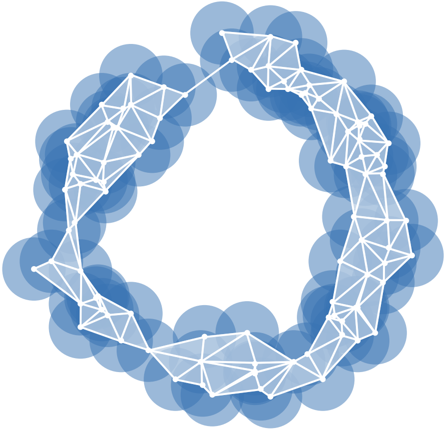


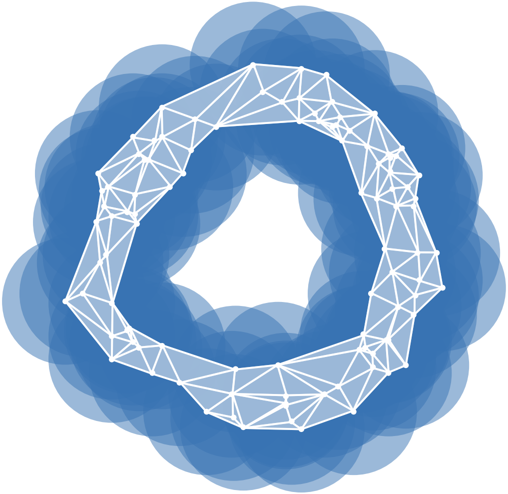


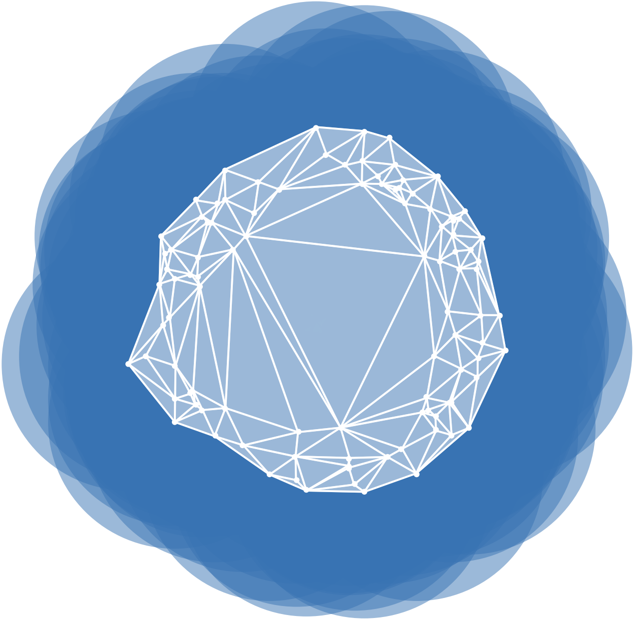




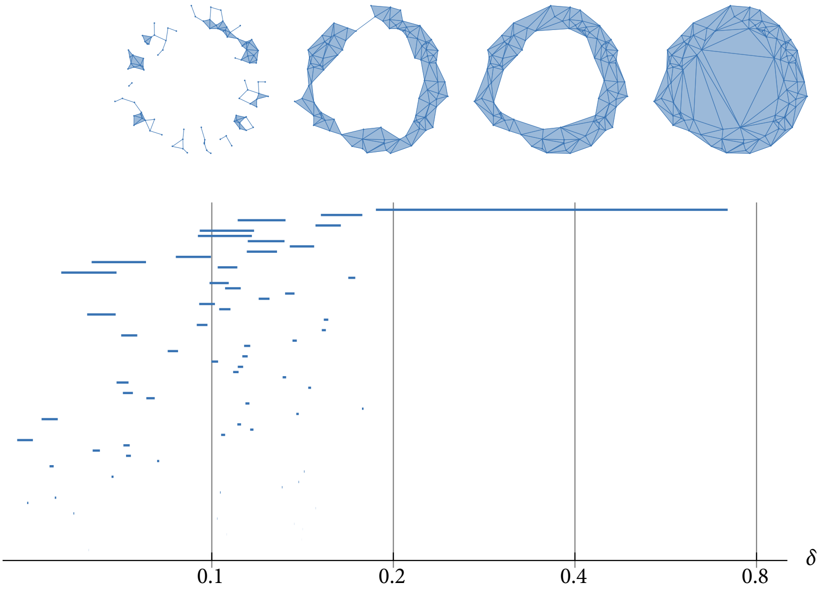








What is persistent homology?



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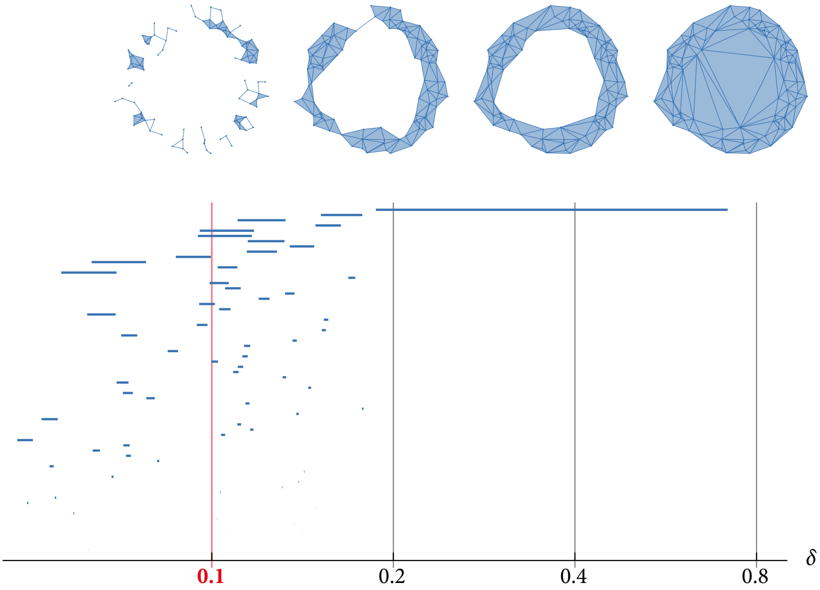

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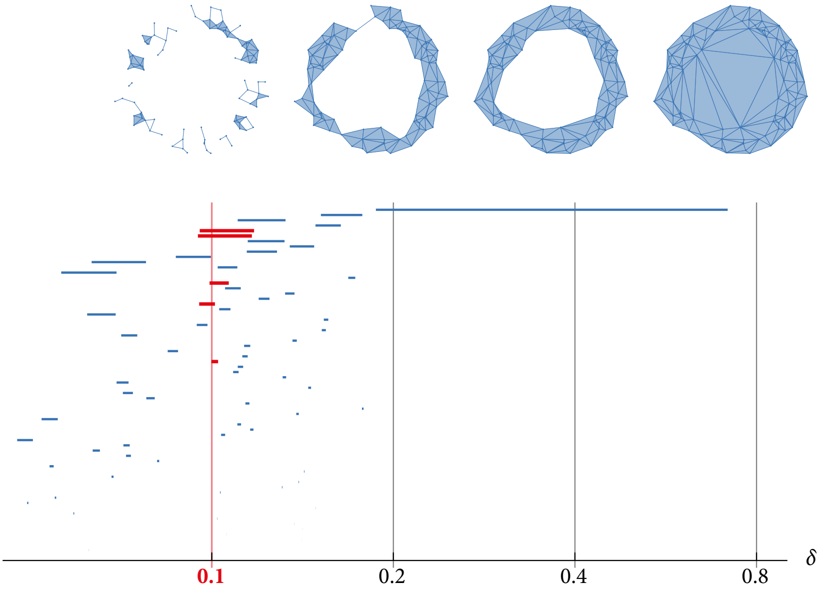

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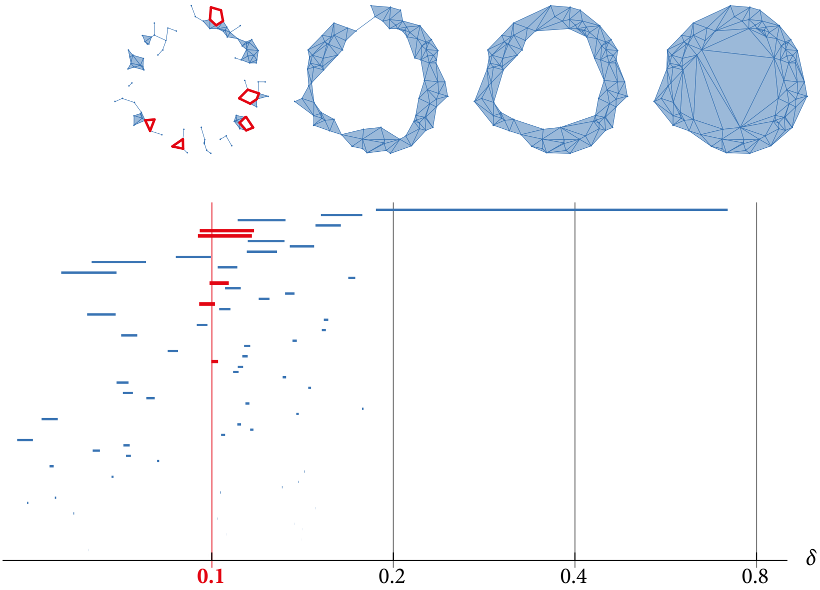

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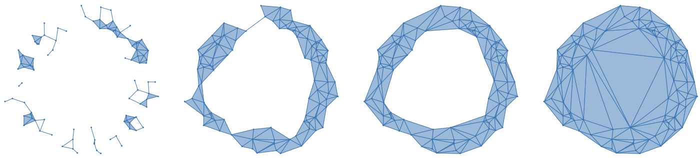

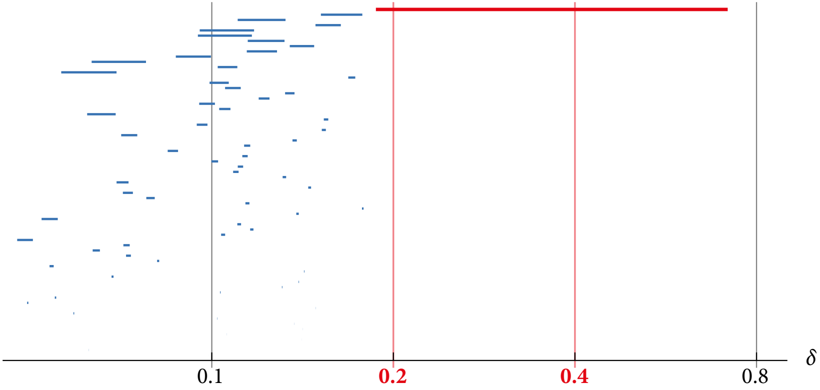

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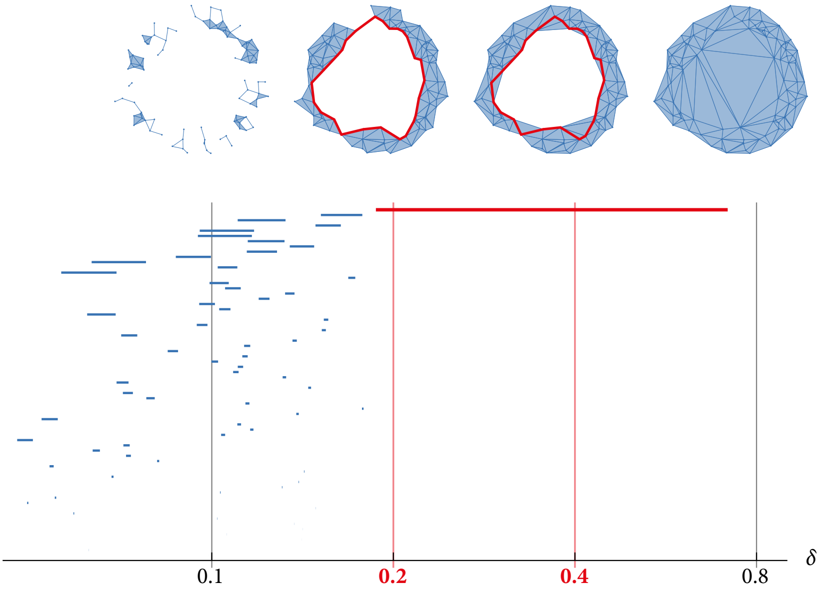

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  - A topological space K_t for each $t \in \mathbb{R}$
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- $\mathbf{R}$ is the poset category of $(\mathbb{R}, \leq)$

Inference

Homology inference

Problem (Homology inference)

Determine the homology $H_*(\Omega)$ of a shape $\Omega \subset \mathbb{R}^d$ from a finite sample $P \subset \Omega$.

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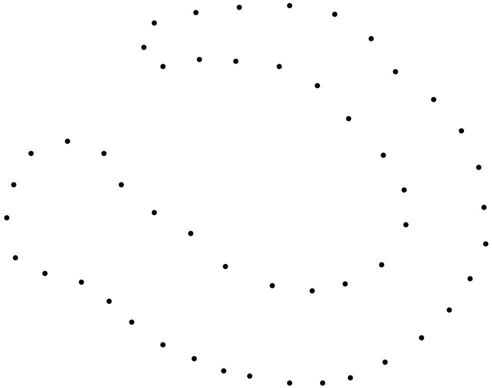
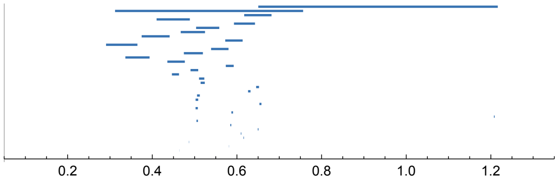
Idea:

- approximate the shape by a thickening $B_\delta(P)$ covering Ω

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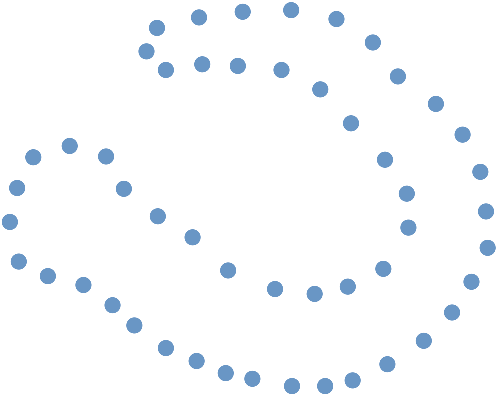

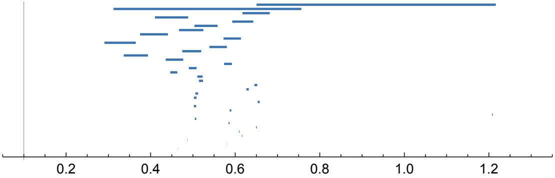
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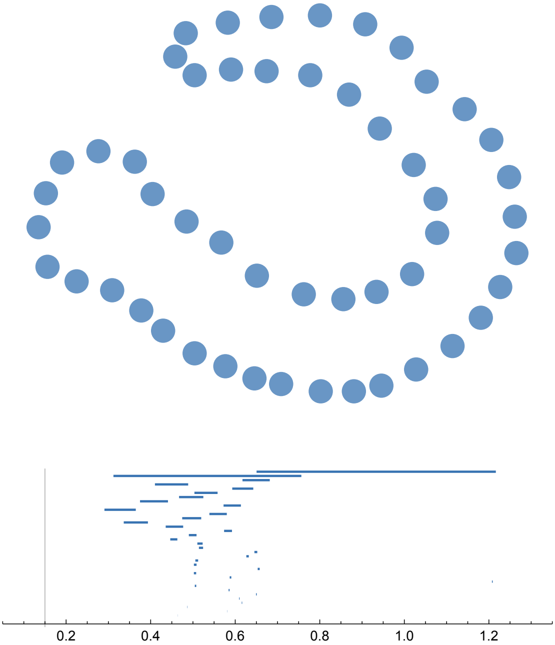
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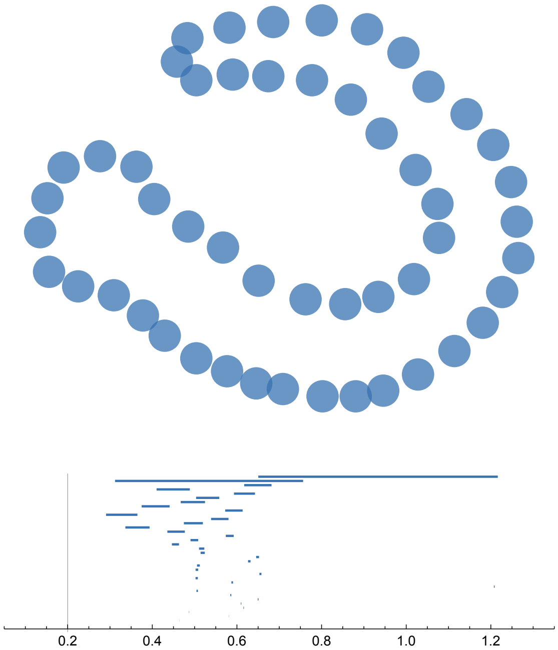
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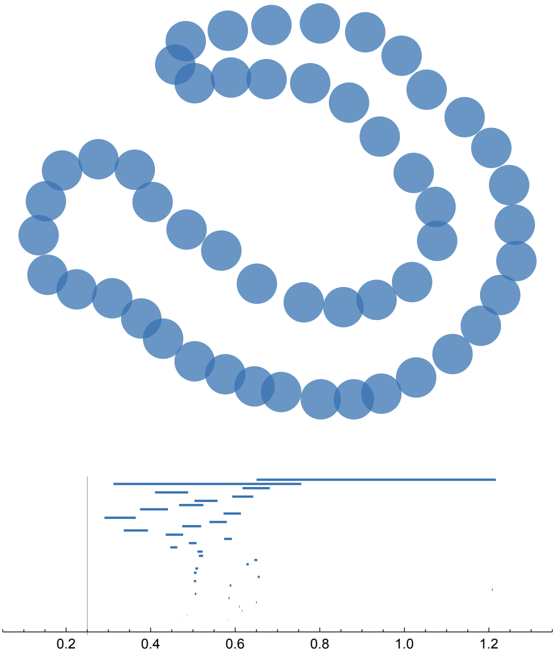
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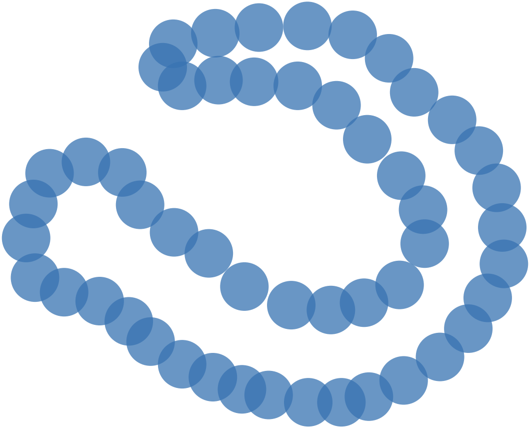

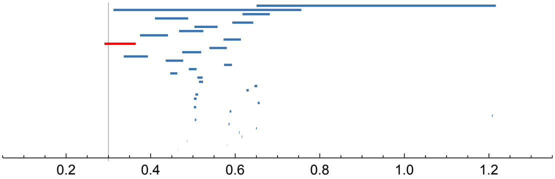
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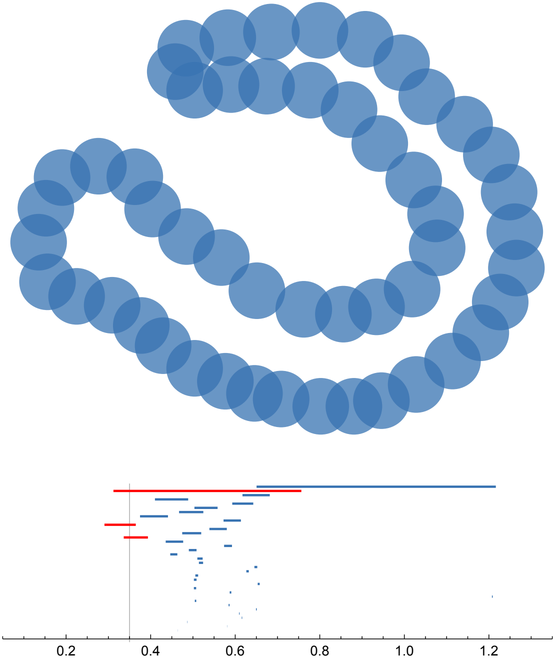
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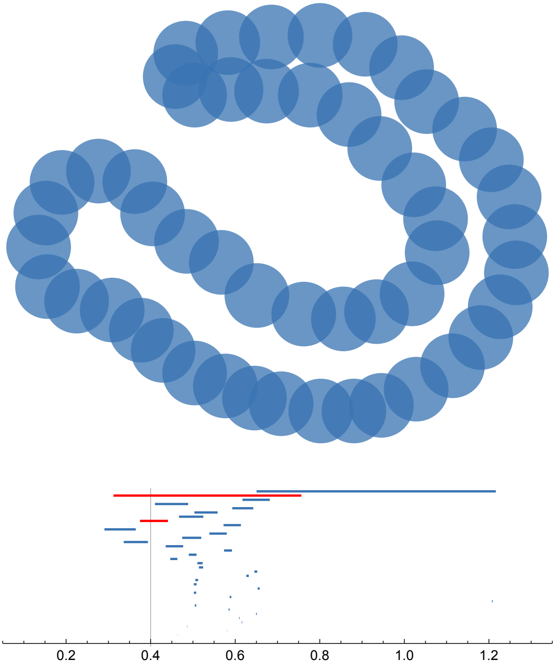
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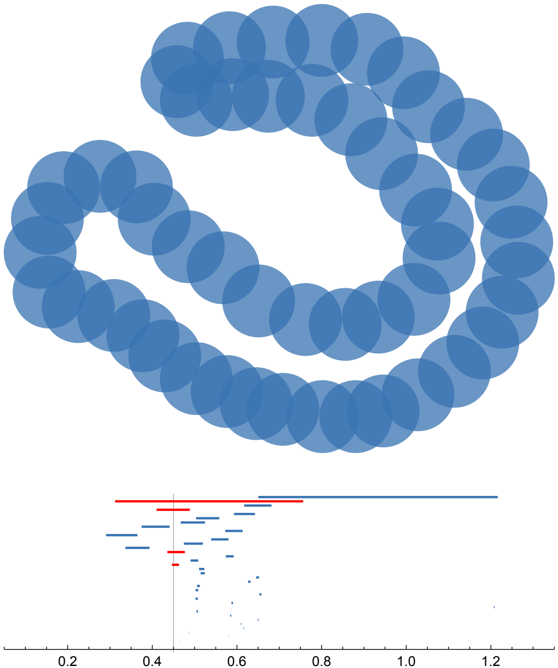
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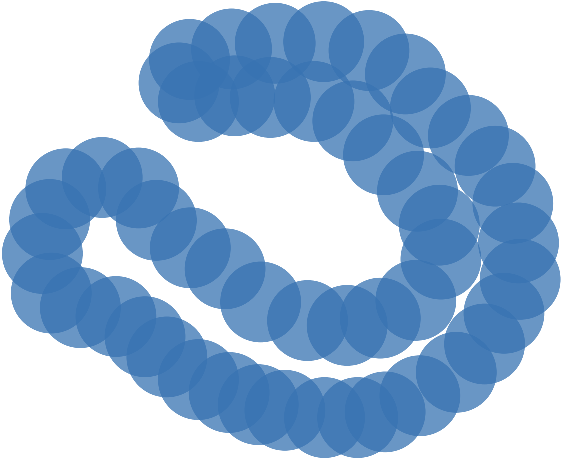

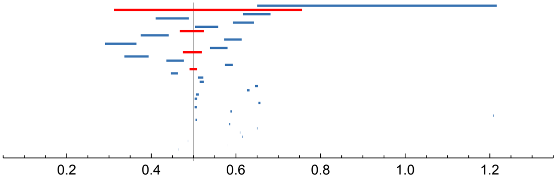
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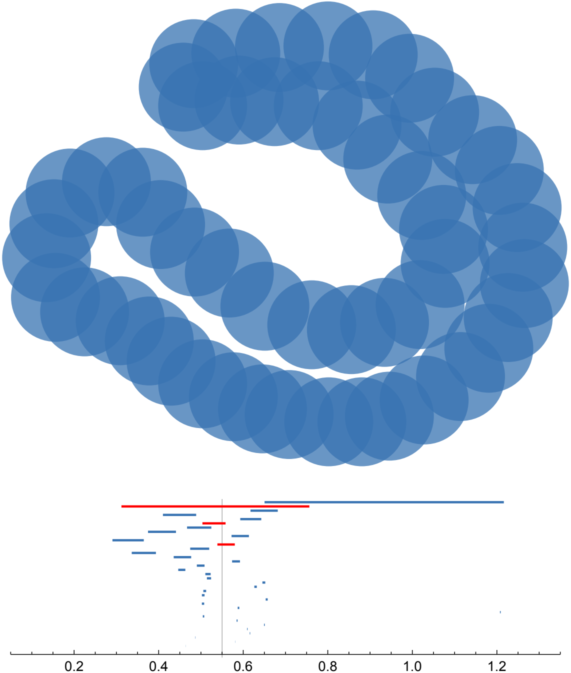

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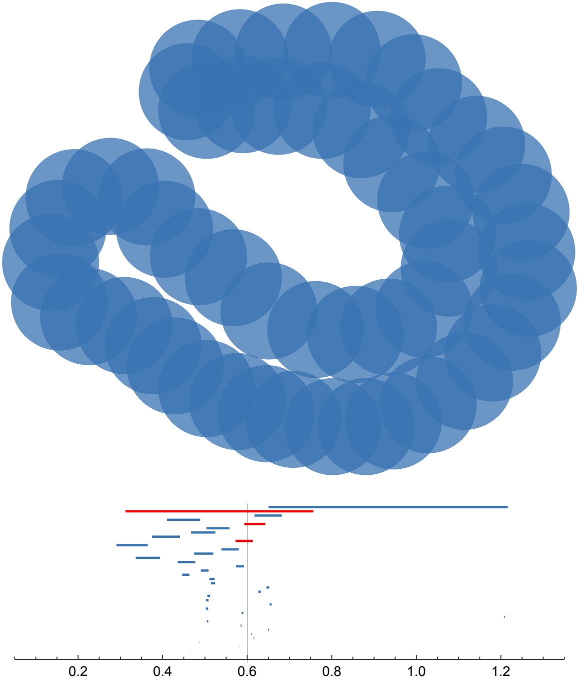

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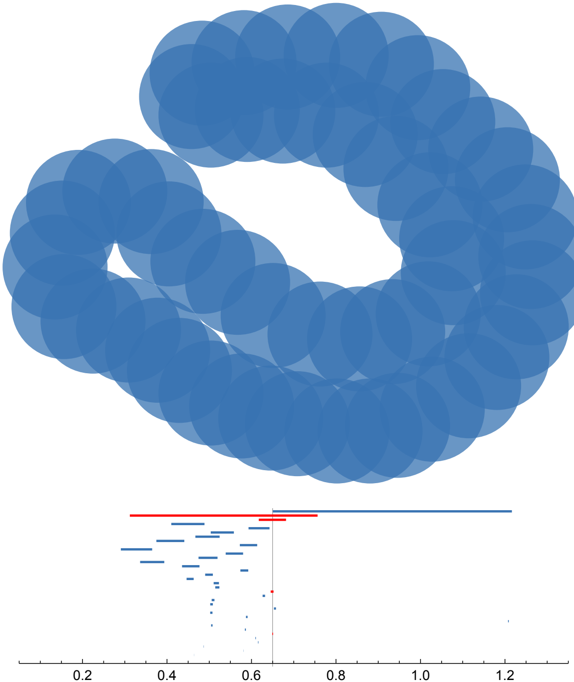
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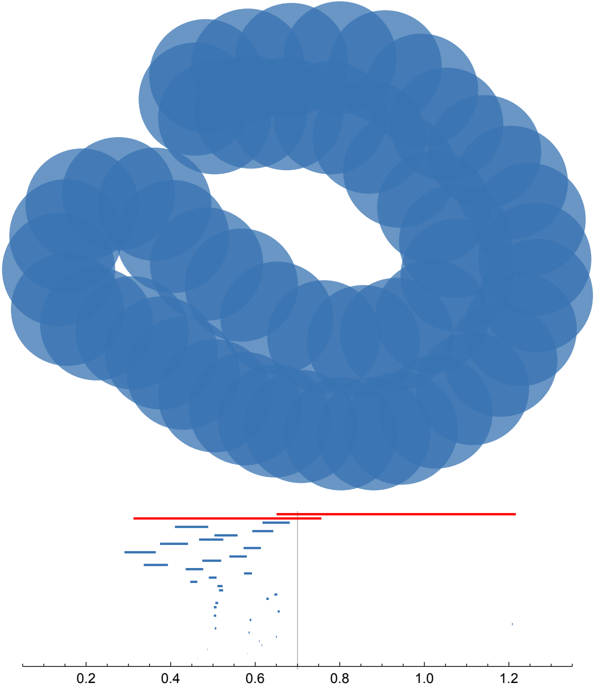
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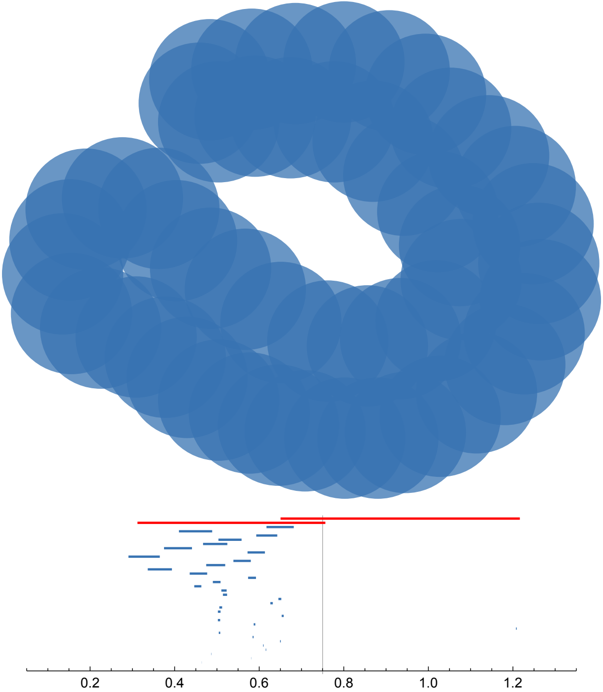
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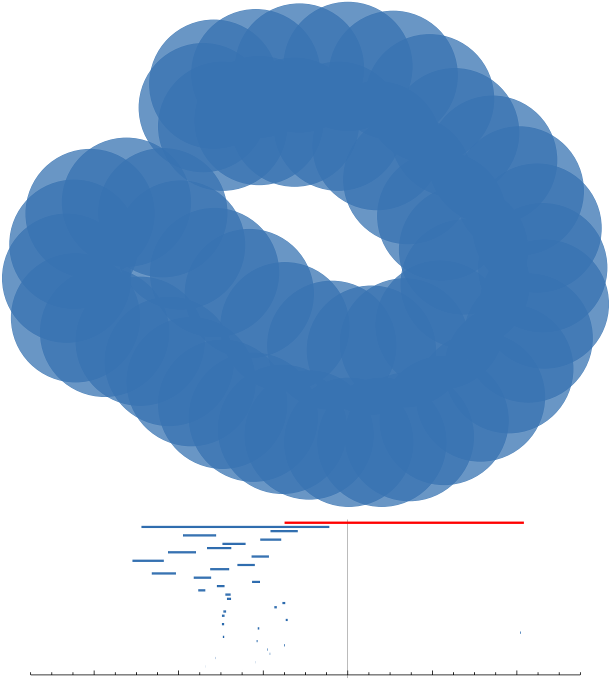

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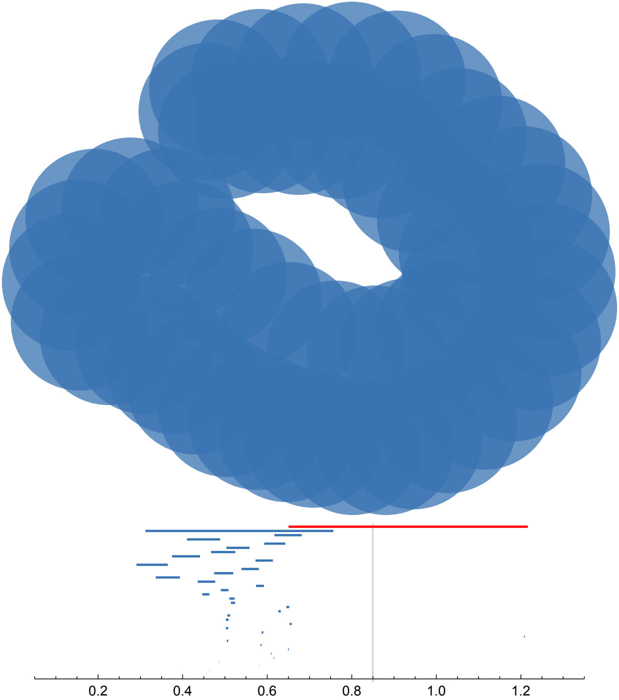
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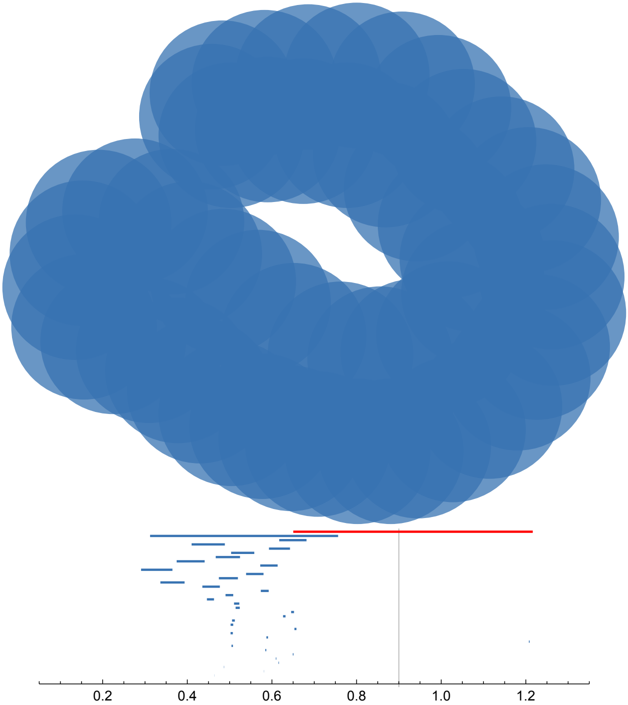
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Homology inference using persistent homology

$P_\delta = B_\delta(P)$: δ-neighborhood (union of balls) around P

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $\Omega \subset \mathbb{R}^d$. Let $P \subset \Omega$ be such that

- $\Omega \subseteq P_\delta$ *for some* $\delta > 0$ *and*
- *both* $H_*(\Omega \hookrightarrow \Omega_\delta)$ *and* $H_*(\Omega_\delta \hookrightarrow \Omega_{2\delta})$ *are isomorphisms.*

Then

$$H_*(\Omega) \cong \operatorname{im} H_*(P_\delta \hookrightarrow P_{2\delta}).$$

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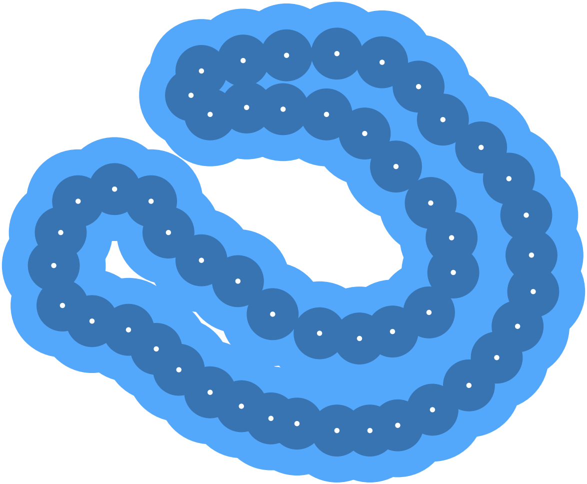

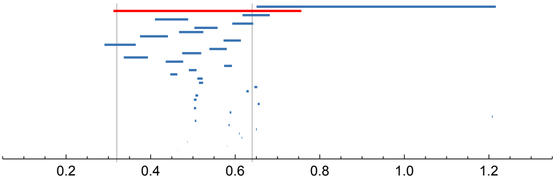
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Proof.

$$
\begin{tikzcd}
\Omega \arrow[dr, hook] & & \Omega_\delta \arrow[dr, hook] & & \Omega_{2\delta} \\
& P_\delta \arrow[ur, hook] & & P_{2\delta} \arrow[ur, hook] &
\end{tikzcd}
$$

Proof.

$$\begin{tikzcd}
H_*(\Omega) \arrow[dr] & & H_*(\Omega_\delta) \arrow[dr] & & H_*(\Omega_{2\delta}) \\
& H_*(P_\delta) \arrow[ur] & & H_*(P_{2\delta}) \arrow[ur] &
\end{tikzcd}$$

Proof.

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\begin{tikzcd}
H_*(\Omega) \arrow[rr, hook, two heads] \arrow[dr] & & H_*(\Omega_\delta) \arrow[rr, hook] \arrow[dr] & & H_*(\Omega_{2\delta}) \\
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\begin{tikzcd}
H_*(\Omega) \arrow[rr, hook, two heads] \arrow[dr] & & H_*(\Omega_\delta) \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, dashed, leftrightarrow, "\cong"] & & H_*(\Omega_{2\delta}) \\
& H_*(P_\delta) \arrow[ur, two heads] \arrow[dr, two heads] & & H_*(P_{2\delta}) \arrow[ur] & \\
& & \operatorname{im} H_*(P_\delta \hookrightarrow P_{2\delta}) \arrow[ur, hook] & &
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& & \operatorname{im} H_*(P_\delta \hookrightarrow P_{2\delta}) \arrow[ur, hook] & &
\end{tikzcd}
$$

$\square$

The pipeline of topological data analysis

$$\begin{tikzcd}
\text{point cloud} \arrow[d, "\text{distance}"] \\
\text{function} \arrow[d, "\text{sublevel sets}"] \\
\text{topological spaces} \arrow[d, "\text{homology}"] \\
\text{vector spaces} \arrow[d, "\text{barcode}"] \\
\text{intervals}
\end{tikzcd}$$

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$$

$P \subset \mathbb{R}^d$

The pipeline of topological data analysis

$$\begin{tikzcd}
\text{point cloud} \arrow[d, "\text{distance}"] & P \subset \mathbb{R}^d \arrow[d, "x \mapsto d(x,P)"] \\
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\text{function} & & f : \mathbb{R}^d \to \mathbb{R} \\
\Big\downarrow \text{sublevel sets} & & \Big\downarrow {\scriptstyle t \mapsto f^{-1}(-\infty, t]} \\
\text{topological spaces} & & \\
\Big\downarrow \text{homology} & & \\
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\text{topological spaces} \arrow[d, "\text{homology}"] & K : \mathbf{R} \to \mathbf{Top} \arrow[d, "t \mapsto H_*(K_t; \mathbb{F})"] \\
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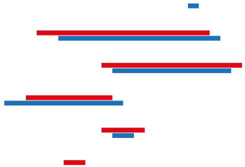
Stability

Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

If two functions $f, g : K \to \mathbb{R}$ have distance $\|f - g\|_\infty \leq \delta$ then there exists a δ-matching of their barcodes.



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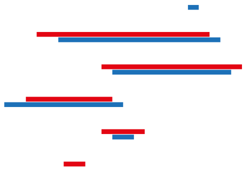

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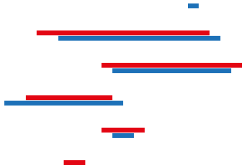

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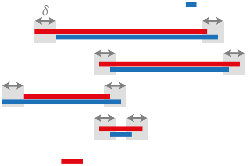

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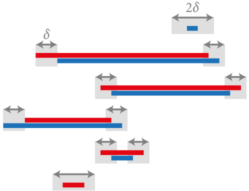

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Homology is a *functor:* homology groups are interleaved too.

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Interval Persistence Modules

Let $\mathbb{K}$ be a field. For an arbitrary interval $I \subseteq \mathbb{R}$, define the *interval persistence module* $C(I)$ by

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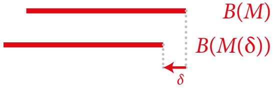
- define $M(\delta)$ by $M(\delta)_t = M_{t+\delta}$ (shift barcode to the left by δ)



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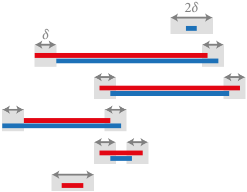

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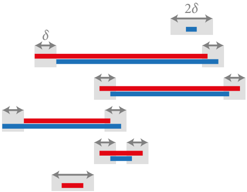

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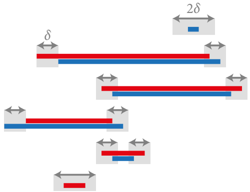

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- indirect proof, 80 page paper (Chazal et al. 2012)

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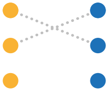
- relies on *partial functoriality* of the induced matching

The matching category

A *matching* $\sigma : S \nrightarrow T$ is a bijection $S' \to T'$, where $S' \subseteq S$, $T' \subseteq T$.



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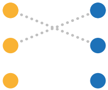

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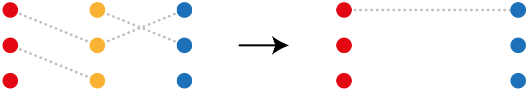

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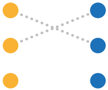

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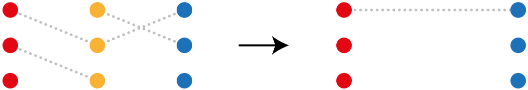
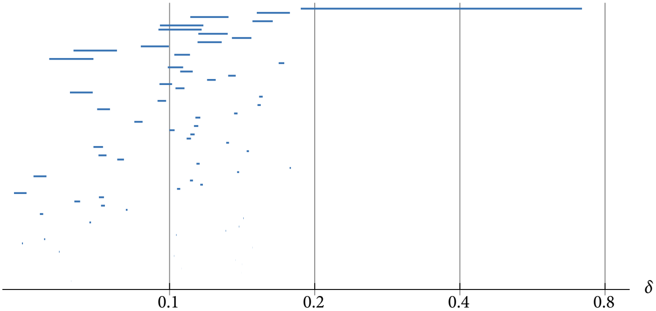


Matchings form a category $\mathbf{Mch}$

- objects: sets
- morphisms: matchings

Barcodes as matching diagrams

We can regard a barcode B as a functor $\mathbf{R} \to \mathbf{Mch}$:



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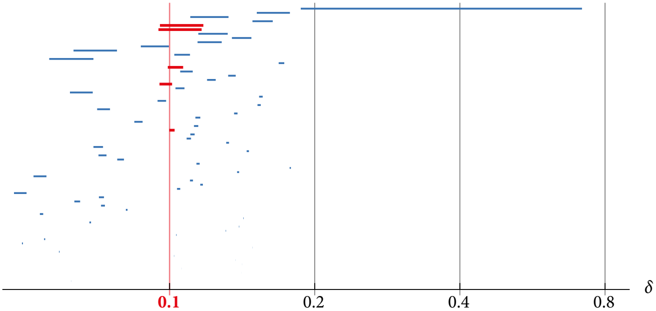

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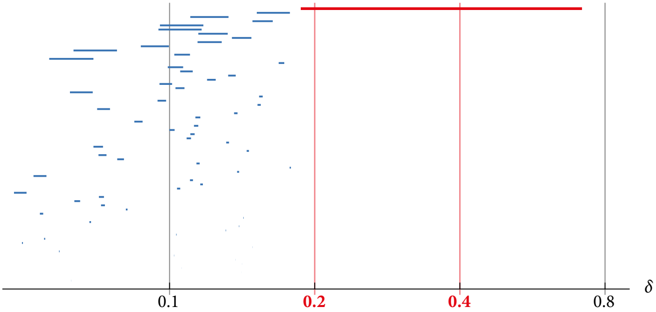
- for each $s \leq t$, define the matching $B_s \nrightarrow B_t$ to be the identity on $B_s \cap B_t$.



Barcode matchings as natural transformations

We can regard certain matchings of barcodes $\sigma : A \nrightarrow B$ as natural transformations of functors $\mathbf{R} \to \mathbf{Mch}$.

- consider restrictions $\sigma_t : A_t \nrightarrow B_t$ of σ to $A_t \times B_t$:

$$\begin{tikzcd} A_s \arrow[r, "|" marking] \arrow[d, "|" marking, "\sigma_s"'] & A_t \arrow[d, "|" marking, "\sigma_t"] \\ B_s \arrow[r, "|" marking] & B_t \end{tikzcd}$$

- requirement on the matching σ:
  if $I \in A$ is matched to $J \in B$, then I overlaps J to the right.

Barcode matchings as interleavings

We can regard a δ-matching of barcodes $\sigma : A \nrightarrow B$
as a δ-interleaving of functors $\mathbf{R} \to \mathbf{Mch}$:

$$
\begin{tikzcd}
A_t \arrow[rr, "|" marking] \arrow[dr, "|" marking] & & A_{t+2\delta} \arrow[dr, "|" marking] & \\
& B_{t+\delta} \arrow[ur, "|" marking] \arrow[rr, "|" marking] & & B_{t+3\delta}
\end{tikzcd}
$$

- each matching $A_t \nrightarrow B_{t+\delta}$ is the restriction of σ

Stability via functoriality?

$$
\begin{tikzcd}
F_t \arrow[rr, hook] \arrow[dr, hook] & & F_{t+2\delta} \arrow[dr, hook] & \\
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$$\begin{tikzcd}
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\end{tikzcd}$$

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$$\begin{tikzcd}
B(H_*(F_t)) \arrow[r] \arrow[dr] & B(H_*(F_{t+2\delta})) \arrow[dr] & \\
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Non-functoriality of the persistence barcode

Theorem (B, Lesnick 2014)

There exists no functor $\mathbf{Vect}^{\mathbf{R}} \to \mathbf{Mch}$ *sending each persistence module to its barcode.*

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- In particular, there is no natural choice of basis for vector spaces

Structure of submodules and quotient modules

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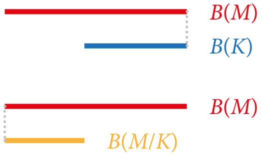

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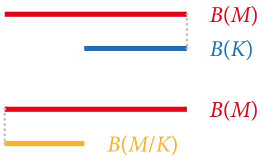

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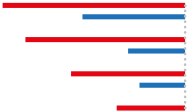
- If multiple bars have same endpoint: match in order of decreasing length



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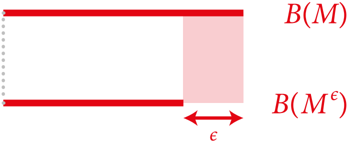
Similar for surjections.

The induced matching theorem

Define M^ϵ by shrinking bars of $B(M)$ from the right by ϵ.



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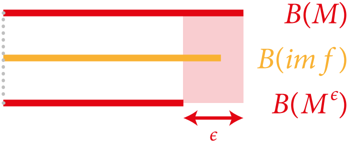
Lemma

Let $f : M \to N$ be a morphism such that $\ker f$ is ϵ-trivial (all bars of $B(\ker f)$ are shorter than ϵ). Then M^ϵ is a quotient module of $\operatorname{im} f$.



$$\begin{tikzcd} & M \arrow[dl, two heads] \arrow[dr, two heads] & \\ M^\epsilon & & \operatorname{im} f \arrow[ll, two heads] \end{tikzcd}$$

The induced matching theorem

Define ${}^{\epsilon}N$ by shrinking bars of $B(N)$ from the left by ϵ.

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Let $f : M \to N$ be a morphism such that $\operatorname{coker} f$ is ϵ-trivial (all bars of $B(\operatorname{coker} f)$ are shorter than ϵ).
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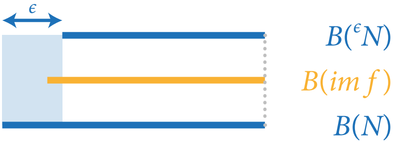

$$\begin{tikzcd} & N & \\ \operatorname{im} f \arrow[ur, hook] & & {}^{\epsilon}N \arrow[ul, hook'] \arrow[ll, hook'] \end{tikzcd}$$

The induced matching theorem

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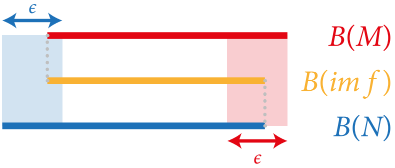

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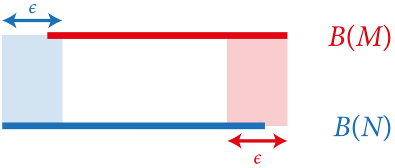

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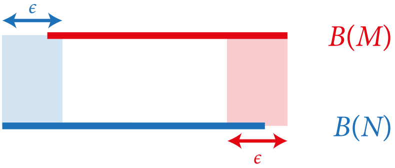
Let $f : M \to N$ *be a morphism with* $\ker f$ *and* $\operatorname{coker} f$ ϵ*-trivial.*



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Let $f : M \to N$ *be a morphism with* $\ker f$ *and* $\operatorname{coker} f$ ϵ*-trivial.*
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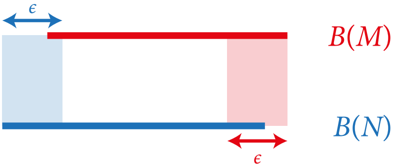

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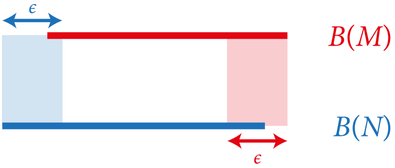
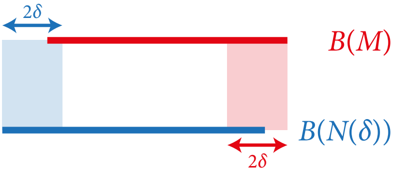
If $B(f)$ matches $[b, d) \in B(M)$ to $[b', d') \in B(N)$, then
$b' \leq b \leq b' + \epsilon$ *and* $d - \epsilon \leq d' \leq d$.



The induced matching theorem

Let $f : M \to N(\delta)$ be an interleaving morphism.

Then $\ker f$ and $\operatorname{coker} f$ are 2δ-trivial.



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Let $f : M \to N(\delta)$ be an interleaving morphism.

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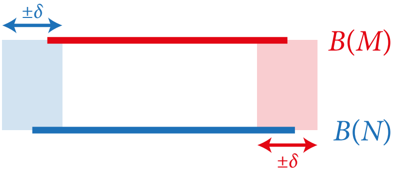
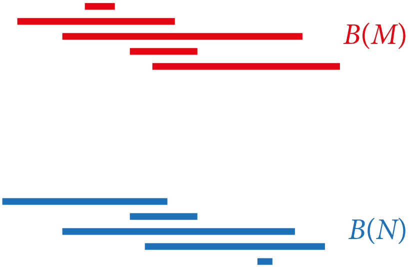
Corollary (Algebraic stability via induced matchings)

A δ-interleaving between persistence modules induces a δ-matching of their persistence barcodes.



Stability via induced matchings



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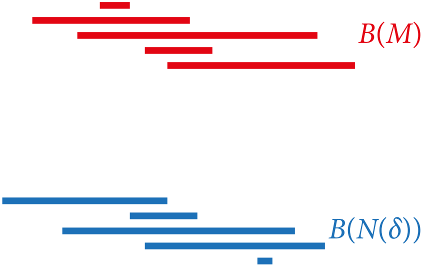

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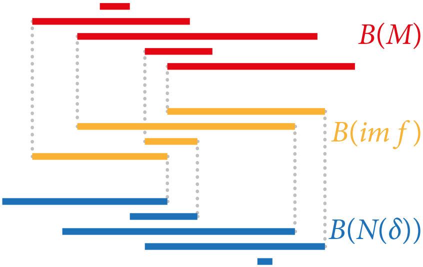

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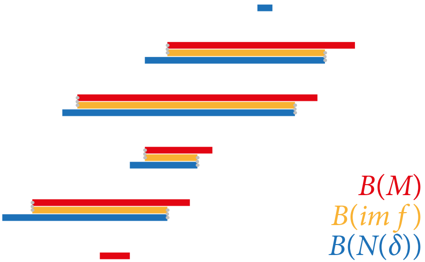

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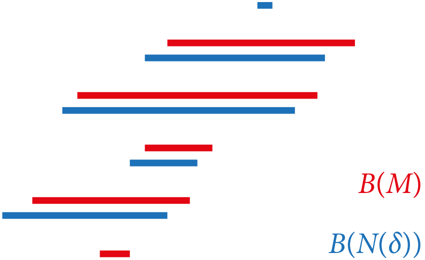

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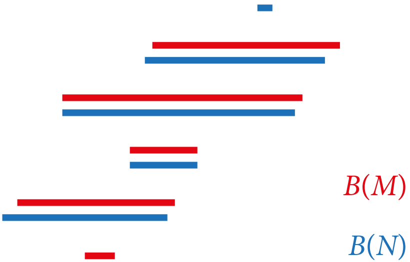

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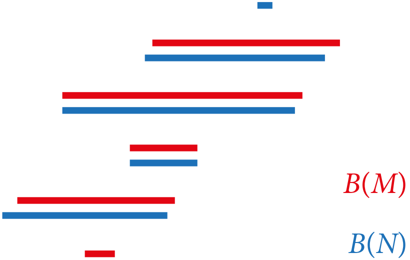

Thanks for your attention!