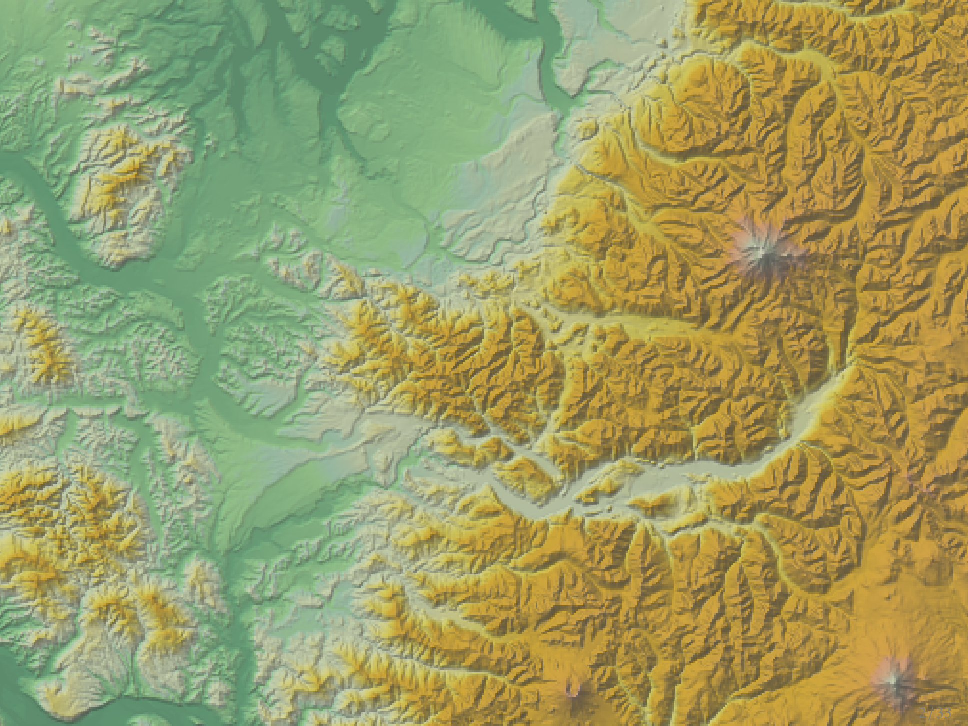
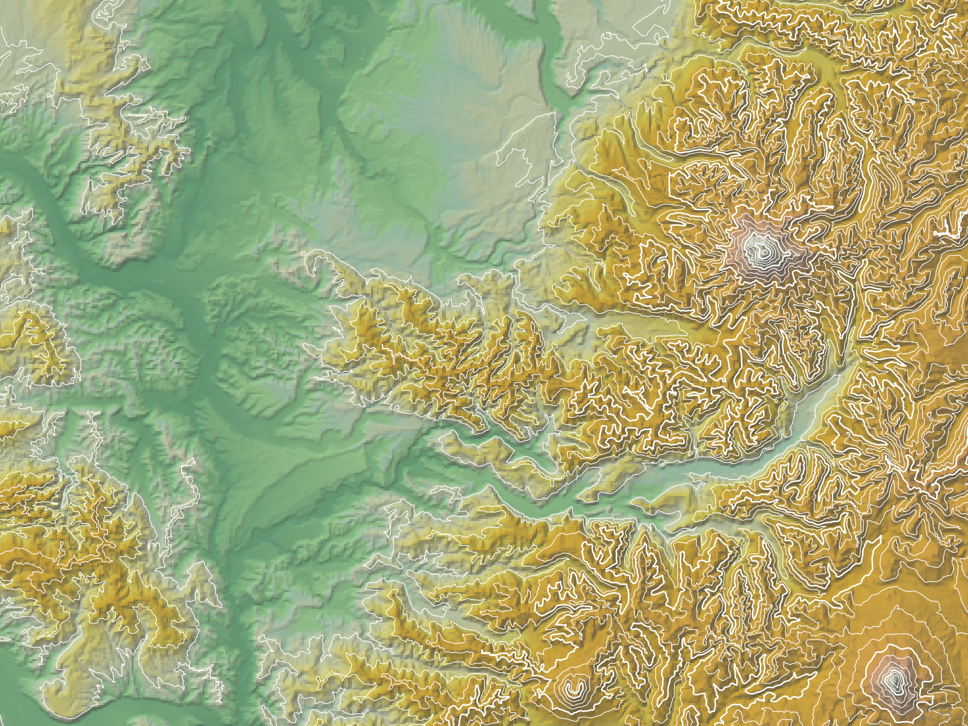
Topological simplification of functions on surfaces

Ulrich Bauer

Georg-August-Universität Göttingen

June 28, 2011





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Given a function f on a surface and $\delta > 0$, find a function f_δ that:

- minimizes number of critical points
- stays close to input function: $\|f_\delta - f\|_\infty \leq \delta$

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Using:

- Discrete Morse theory [Forman 1998]
  - provides notion of critical point in the discrete setting
- Homological persistence [Edelsbrunner et al. 2002]
  - quantifies significance of critical points

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Think of generalized simplicial complexes

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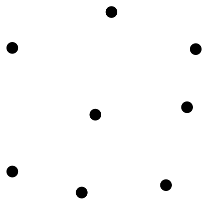

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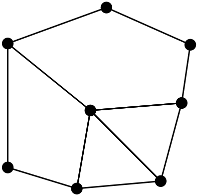

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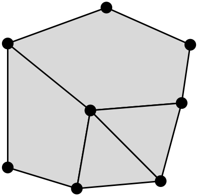

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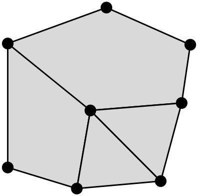
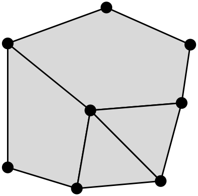
- *regular* CW complex: all attaching maps are topological embeddings



Discrete Morse theory [Forman, 1998]

Finite regular CW complex $\mathcal{K}$



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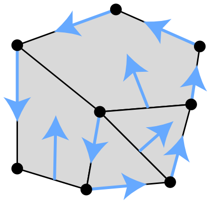

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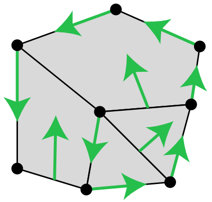

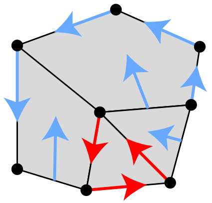

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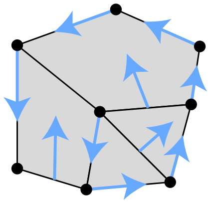

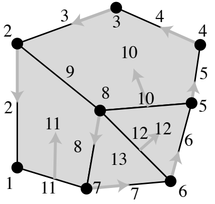

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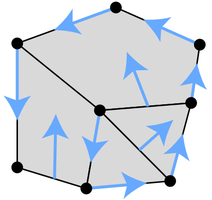

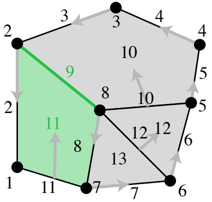

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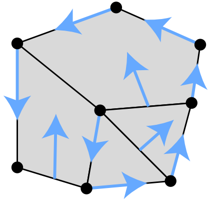

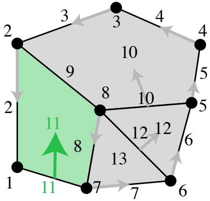

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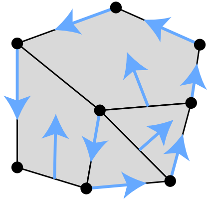

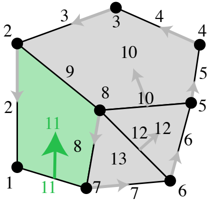

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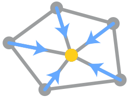

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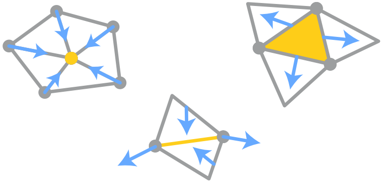

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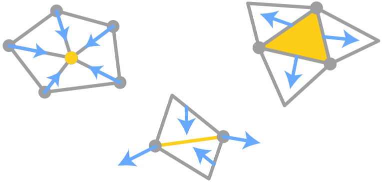
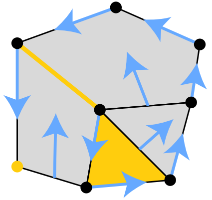
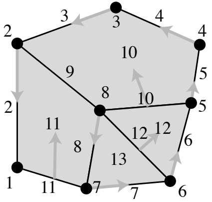


  - No multi-saddle by definition!

Discrete Morse Theory [Forman, 1998]

- *Level subcomplex*:
  - union of all closed cells below a certain value





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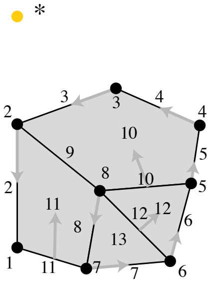

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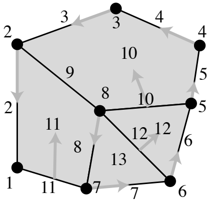

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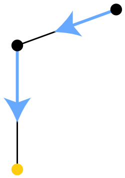

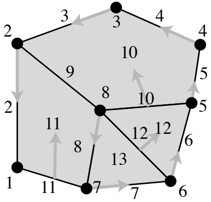

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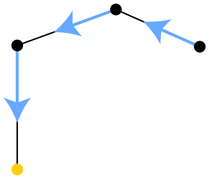

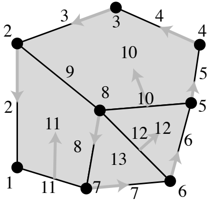

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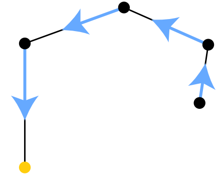

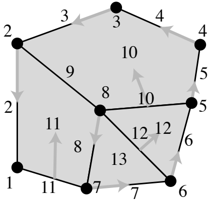

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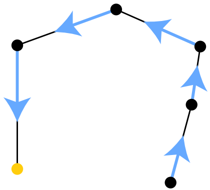

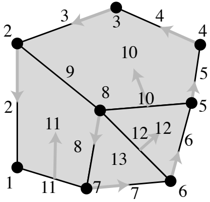

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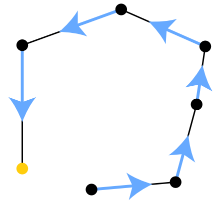

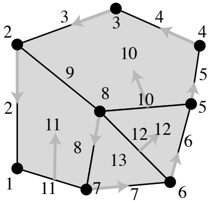

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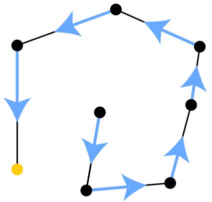

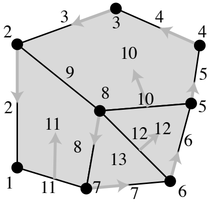

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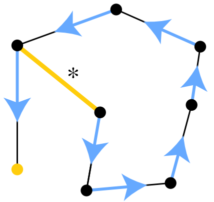

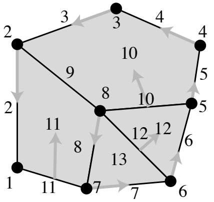

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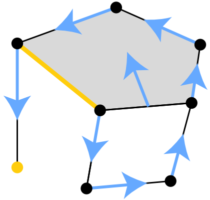

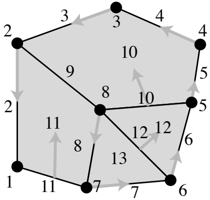

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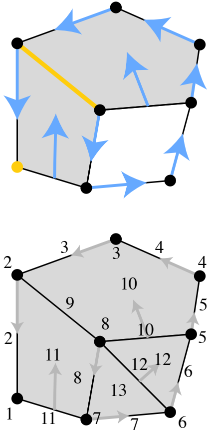

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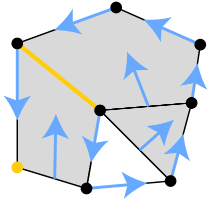

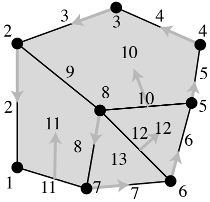

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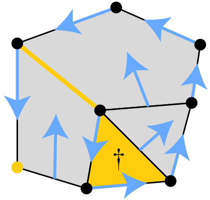

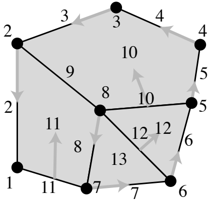

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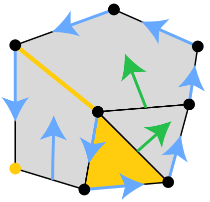

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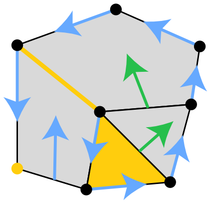

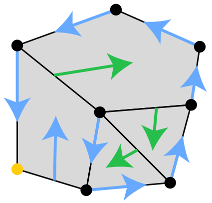

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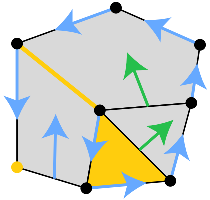

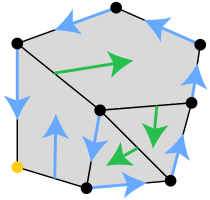

Level subcomplexes

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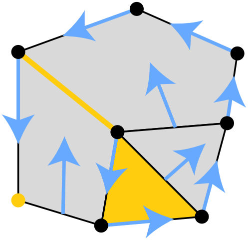

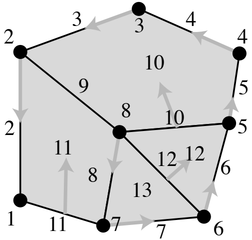

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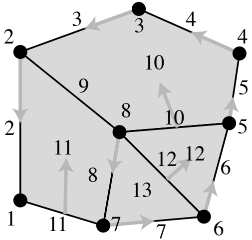

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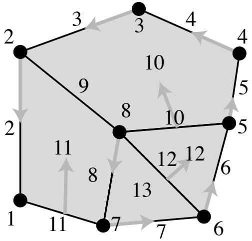

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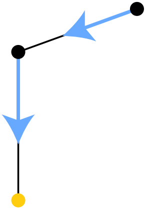

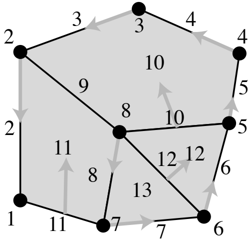

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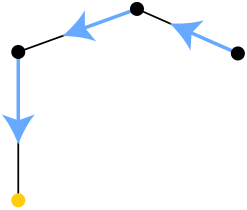

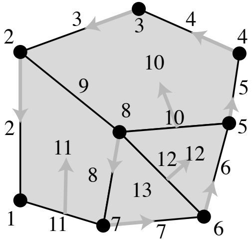

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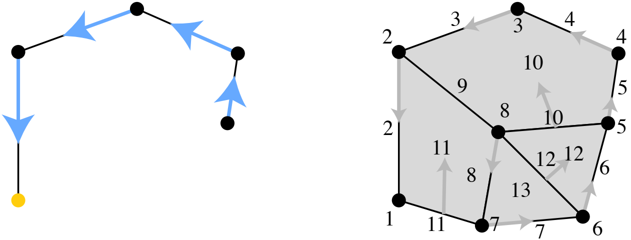

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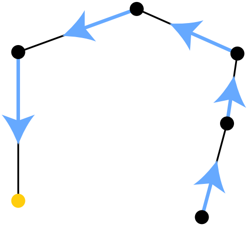

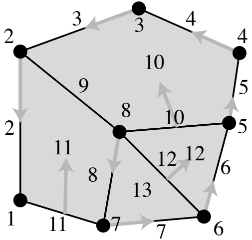

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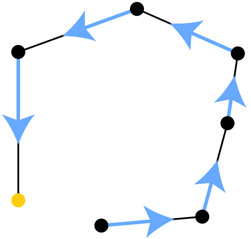

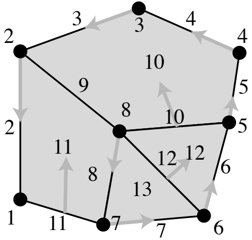

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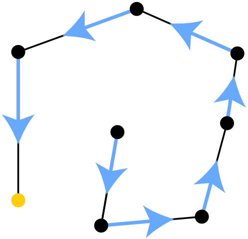

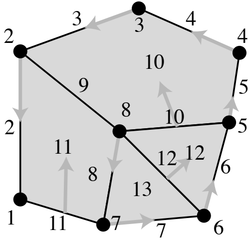

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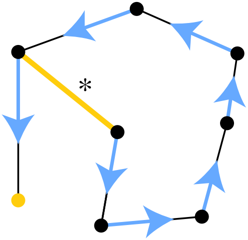

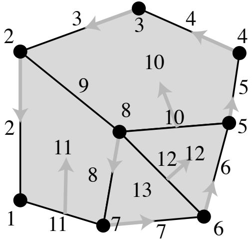

Homotopy type (and hence homology) changes only at critical cells

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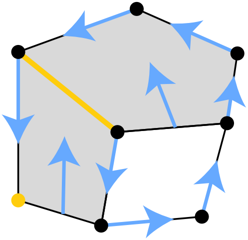

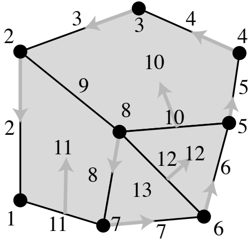

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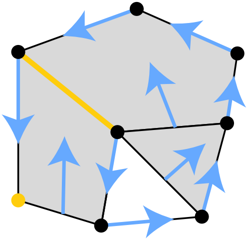

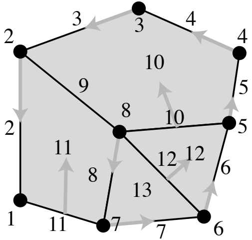

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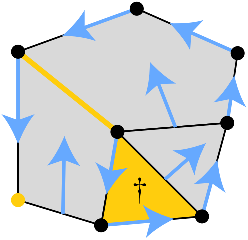

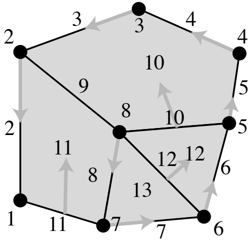

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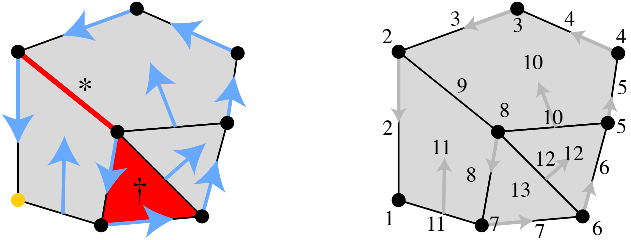

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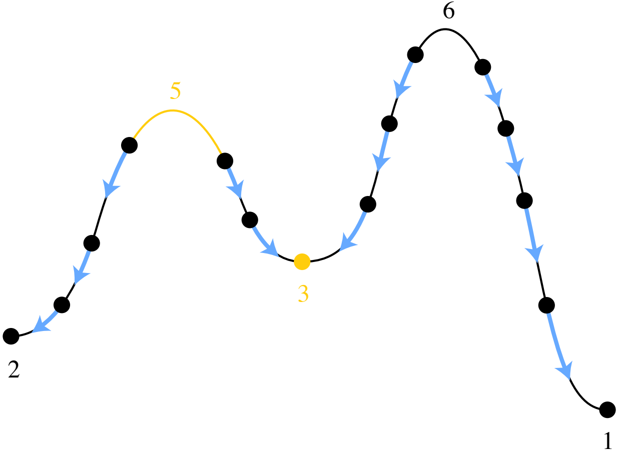
Canceling critical points of a gradient vector field



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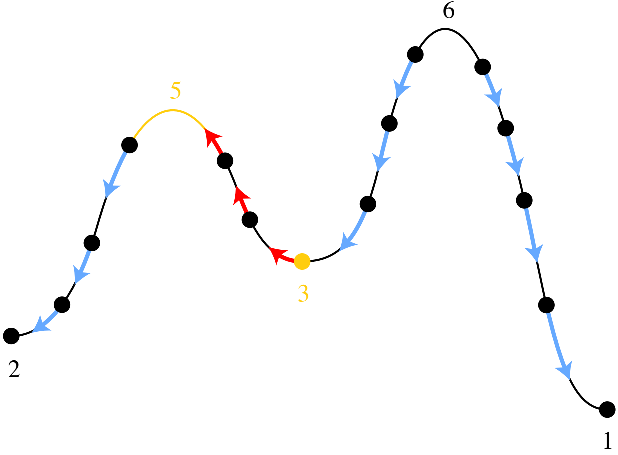
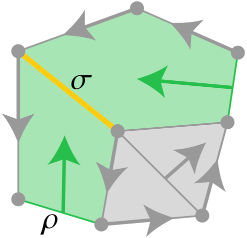


Ascending and descending sets

A gradient vector field V imposes inequalities on values of a consistent function

- *Ascending set* of a cell σ:
  $\{\rho : g(\rho) \geq g(\sigma)$ for *any* g consistent with $V\}$



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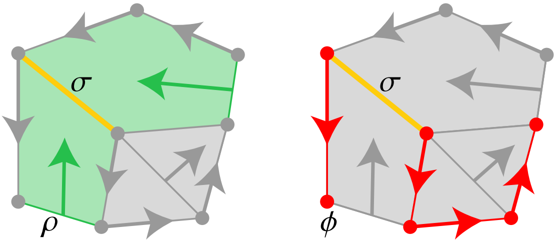

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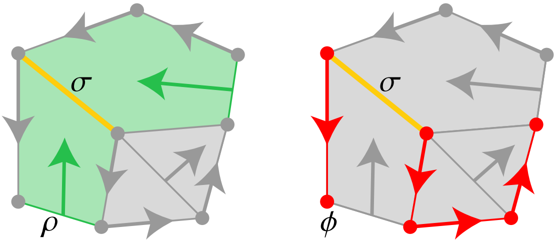
V induces partial order $\prec_V$ on cells: write $\sigma \prec_V \rho$ and $\phi \prec_V \sigma$



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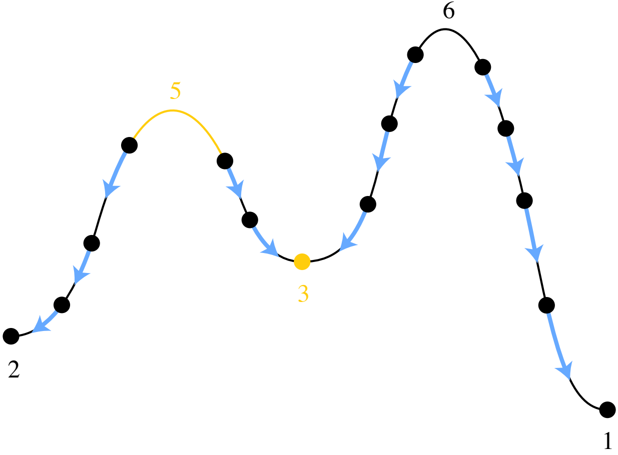

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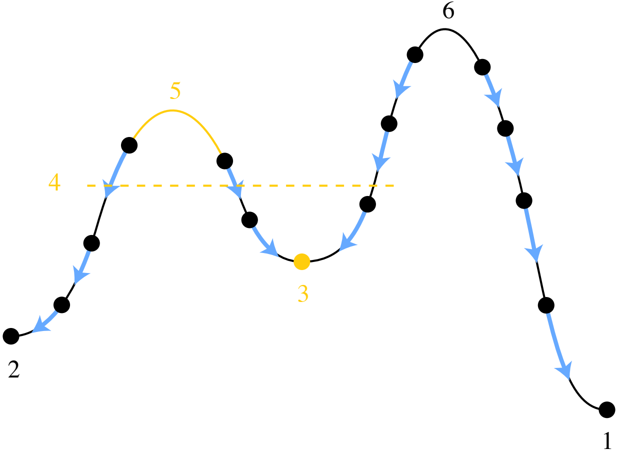

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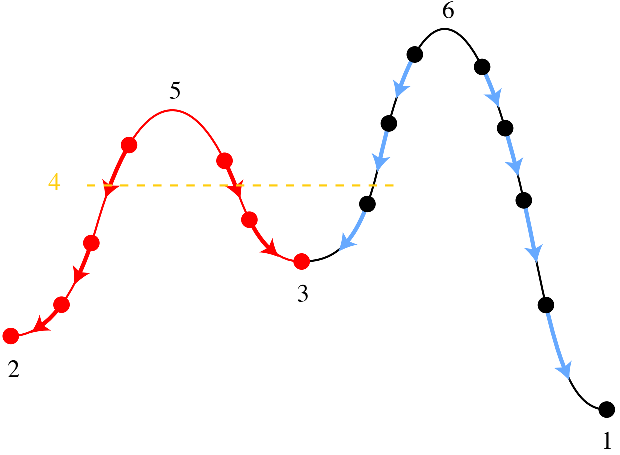

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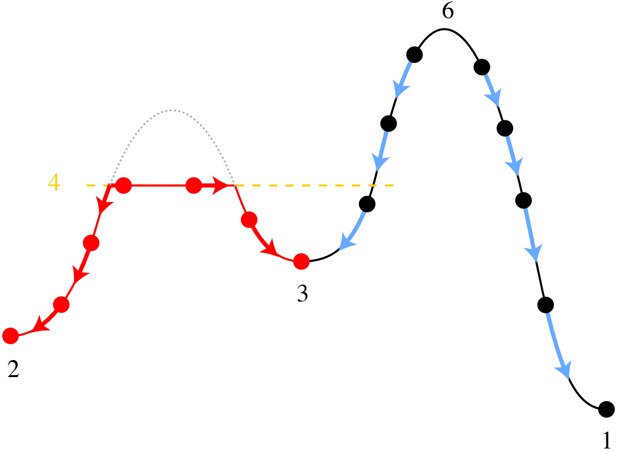

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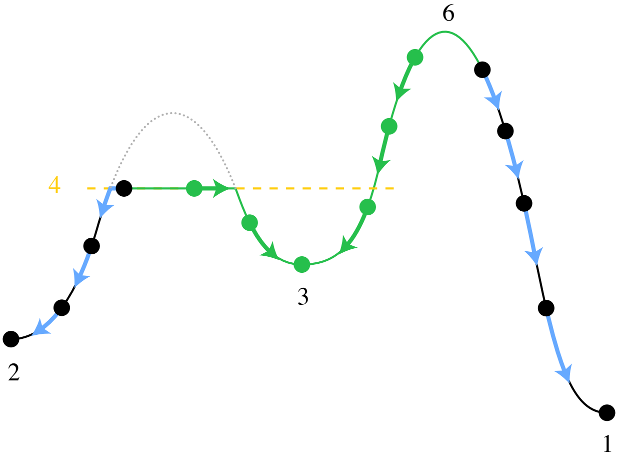

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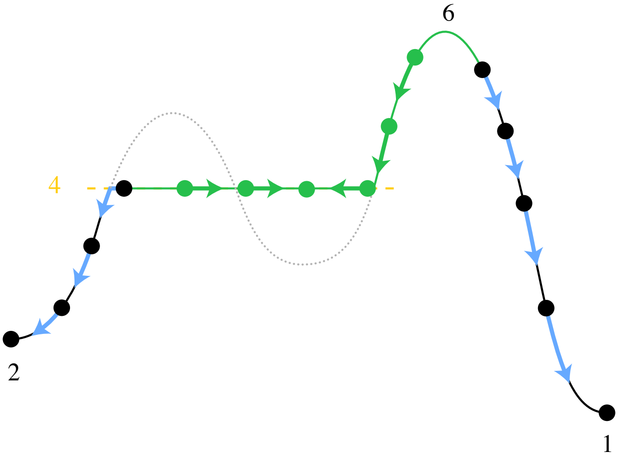

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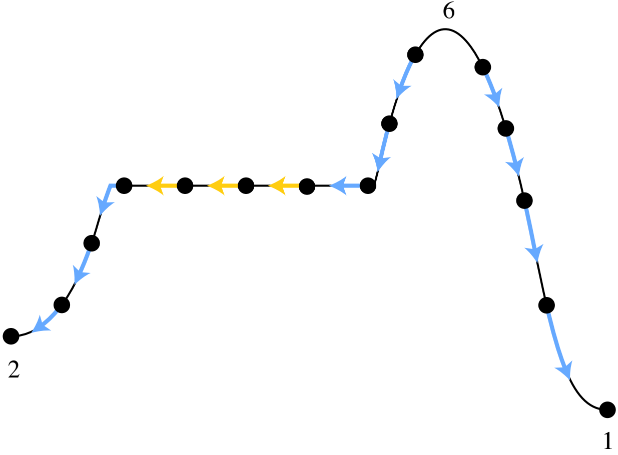
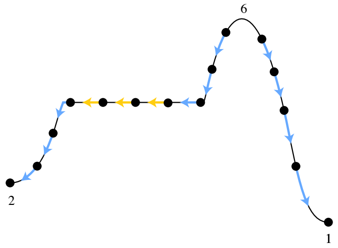


Pseudo-Morse functions



After cancelation, function is no longer Morse

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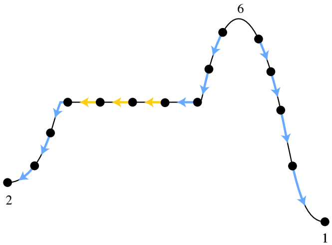

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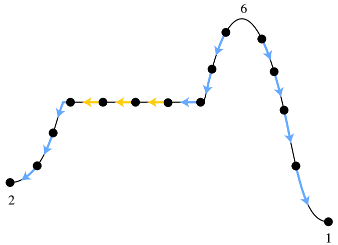

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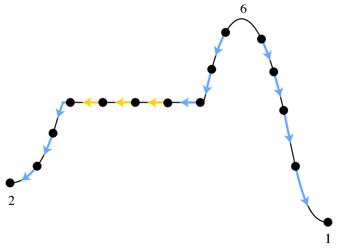

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A symbolic perturbation scheme

Lemma

A function f is a pseudo-Morse function consistent with V if and only if for every $\epsilon > 0$ there is a discrete Morse function f_ϵ consistent with V and $\|f_\epsilon - f\|_\infty \leq \epsilon$.

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- treat the pair (f, V) like a Morse function
- can be extended to situations where *unique* critical values are required

Comparing Discrete and PL Morse theory

Why not just use PL functions?

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We can easily convert one type of function into the other (PL $\to$ pseudo-Morse $\to$ PL: barycentric subdivision)

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Theorem (Gray et al., 2010; B., 2011)

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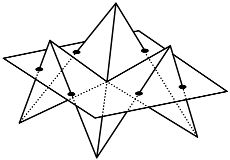

Difficulty comes from multi-saddles

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We have seen how to cancel 2 critical points from a function

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- How many critical points can be removed by cancelation?

Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for sublevel sets

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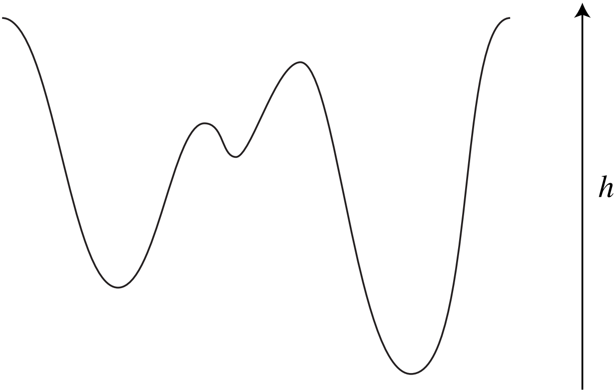
Investigate change of homology for sublevel sets

Example: connected components in 1D



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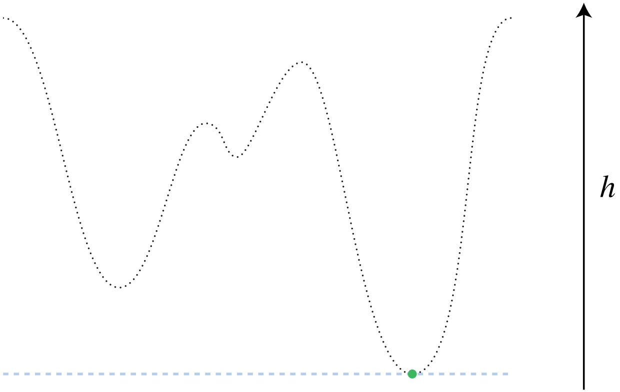

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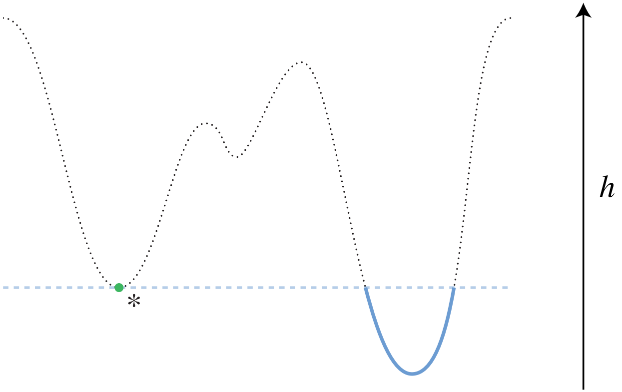

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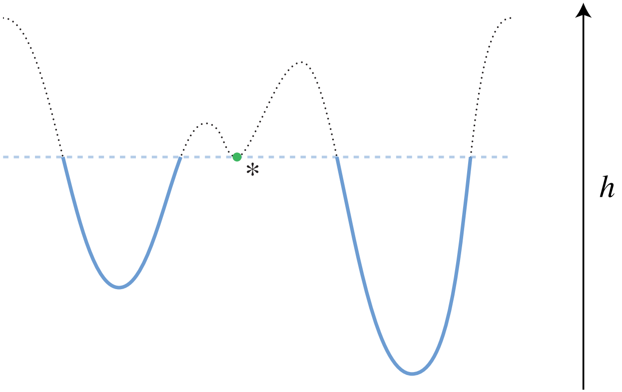

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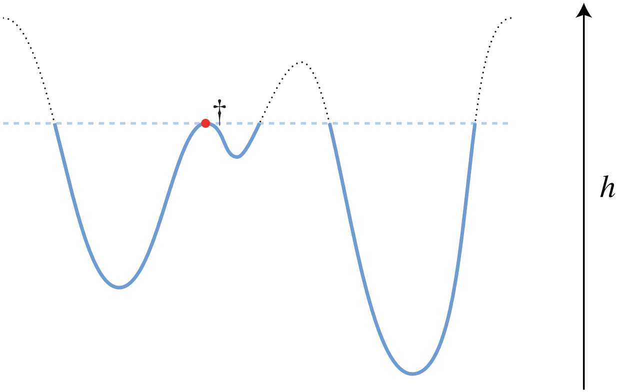

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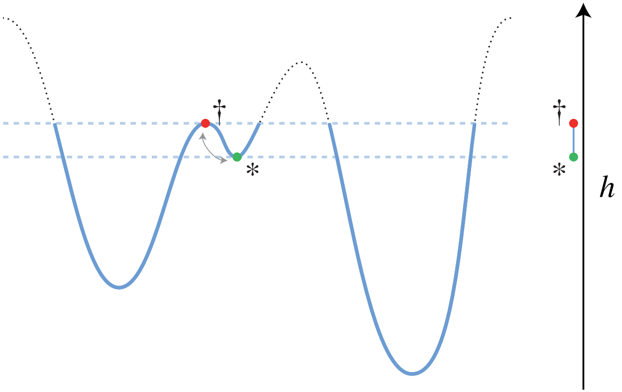

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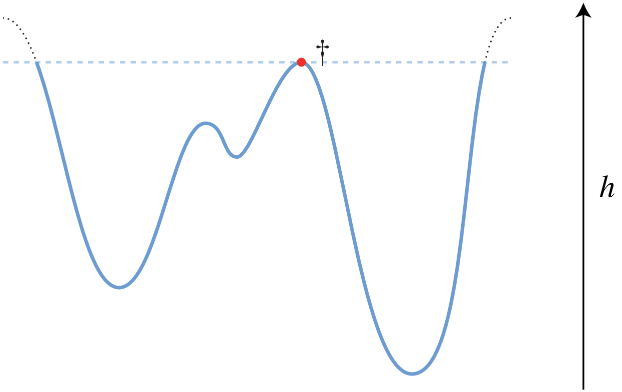

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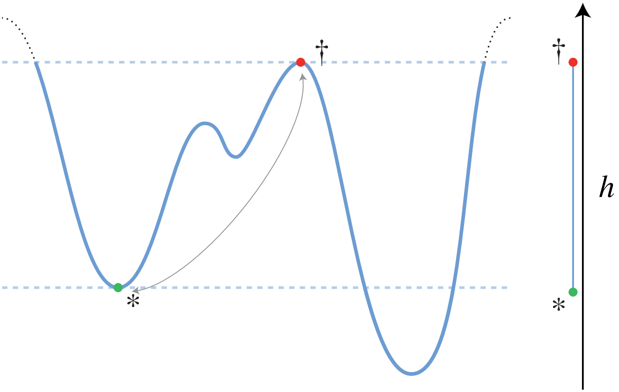
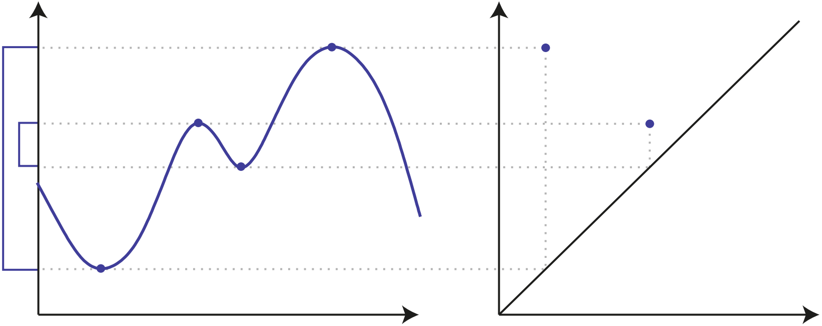


Persistence diagrams [Cohen-Steiner et al., 2005]



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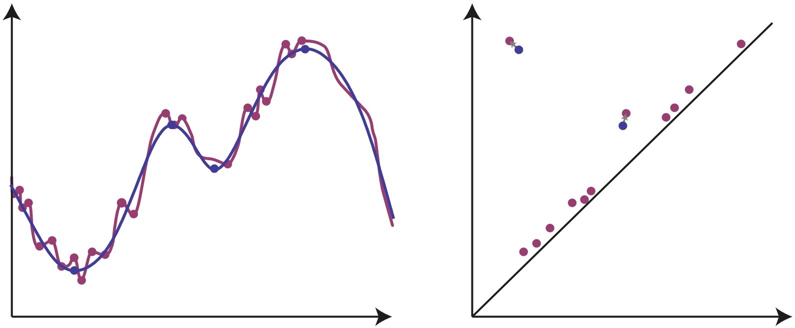

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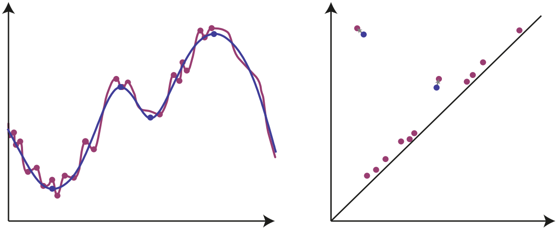

A bound on number of critical points

Corollary

Let f be a discrete Morse function on a surface and let $\delta > 0$.

Then for every function f_δ with $\|f_\delta - f\|_\infty \leq \delta$ we have:

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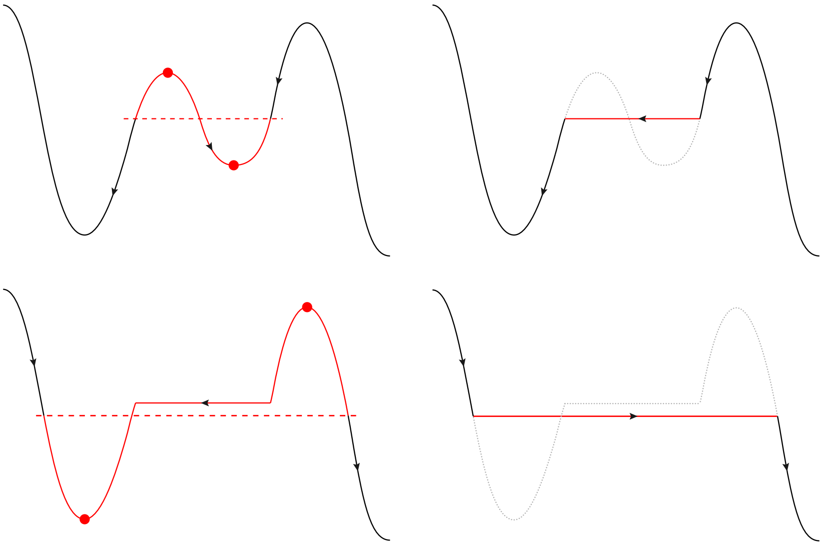
- Is cancelation always allowed?
- Order of cancelation?

Canceling two nested persistence pairs



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Corollary

On a surface, it is possible to cancel just the persistence pairs with persistence $\leq 2\delta$ *(without canceling the other pairs).*

Connecting persistence and Morse theory

Theorem (B., Lange, Wardetzky, 2010)

Consider a persistence pair (σ, τ) *of a discrete Morse function on a surface. Then* (σ, τ) *can be canceled after all persistence pairs* $(\tilde{\sigma}, \tilde{\tau})$ *with* $f(\sigma) < f(\tilde{\sigma}) < f(\tilde{\tau}) < f(\tau)$ *have been canceled:*
(i) there exists a gradient path from τ *to* σ*, and*
(ii) this path is unique.

Corollary

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What about the tolerance? Does the error accumulate?

Tightness of the stability bound

Theorem (B., Lange, Wardetzky, 2010)

Let f be a pseudo-Morse function on a surface and let $\delta > 0$.

Then a nested cancelation sequence yields a function f_δ with:

- $\|f_\delta - f\|_\infty \leq \delta$.
- *All persistence pairs of f with persistence $\leq 2\delta$ are canceled.*

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This function achieves the minimal number of critical points.

Proof of the tightness theorem

Lemma

Let V_i be a vector field obtained from nested persistence cancelation and let (σ, τ) be a remaining persistence pair. Then for each cell ρ,

- $\rho \succ_{V_i} \sigma$ *implies* $f(\rho) \geq f(\sigma)$, *and*
- $\rho \prec_{V_i} \tau$ *implies* $f(\rho) \leq f(\tau)$.

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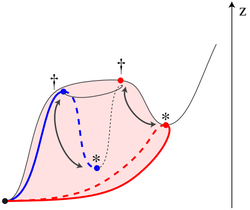
Need to consider interplay between dimensions!

An example in 2D



input

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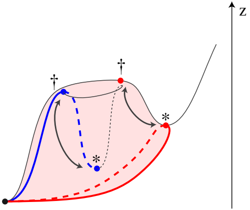
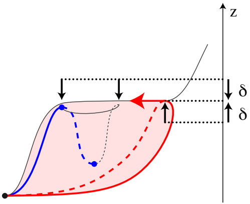
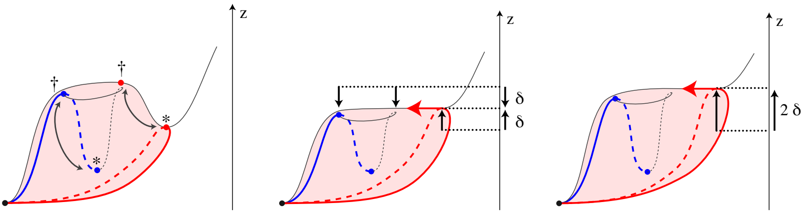


input



optimal

An example in 2D



input optimal not optimal

On surfaces, cancelation may affect other critical values!

An efficient algorithm

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- Computing persistence pairs: $O(\mathrm{sort}(n))$
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- Computing simplified gradient vector field: $O(n)$
  (DFS graph traversal)
- Computing simplified function: $O(n)$
  (topological sort by DFS graph traversal)

Combination with energy methods

Recall: simplified vector field V_δ imposes inequalities on simplified function

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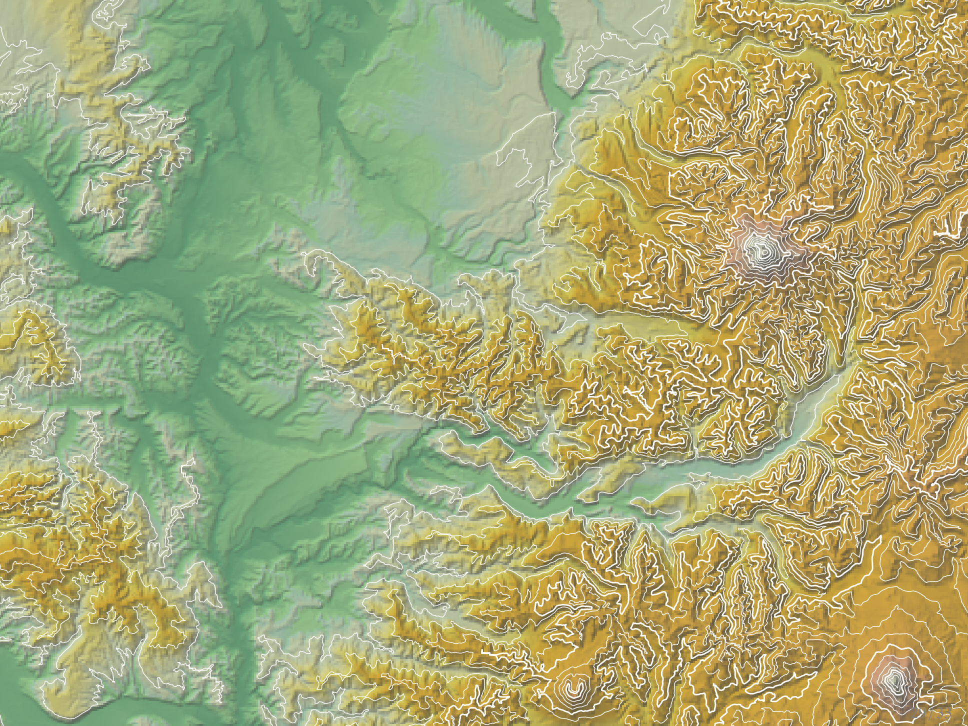
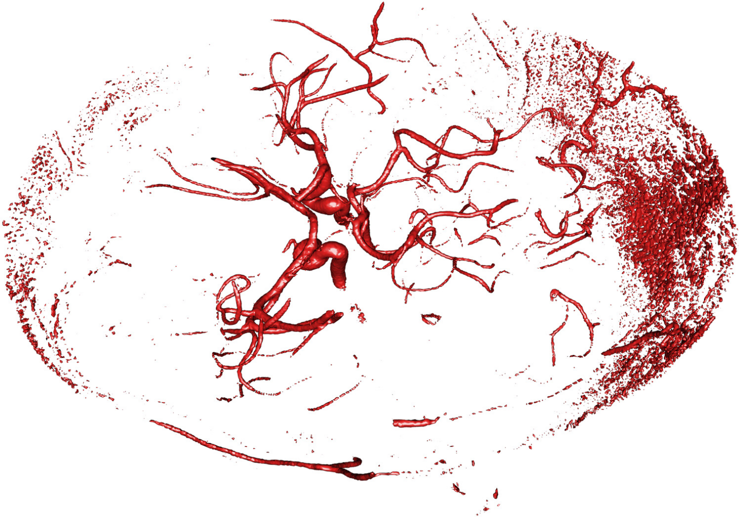
- defines convex set of solutions
- find the "best" solution using your favorite energy functional



Removing local extrema from 3D data



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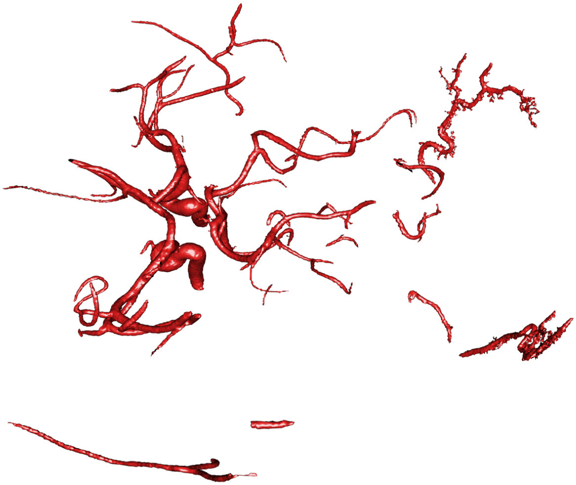

Future work

- Connection to total variation denoising
- Extension to higher dimensions (weaker assumptions on critical points)
- Connection to singularity theory?

Present work

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Persistence in Discrete Morse Theory.
PhD thesis, University of Göttingen, 2011.

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Proceedings of ICNAAM 2010, 1022–1026.
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Past work

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Thanks for your attention!