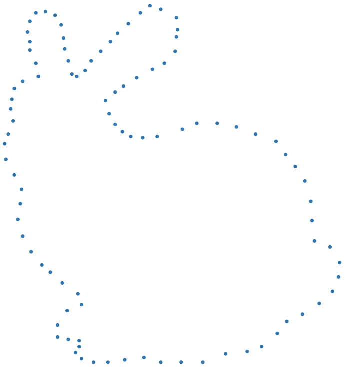
The Morse theory of Čech and Delaunay filtrations

Ulrich Bauer Herbert Edelsbrunner

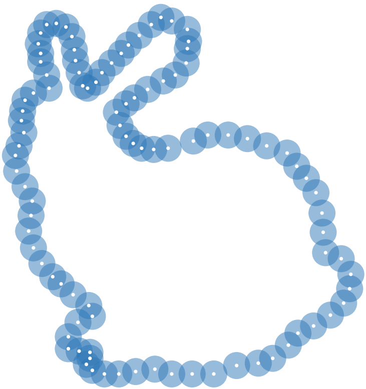
IST Austria

SoCG 2014

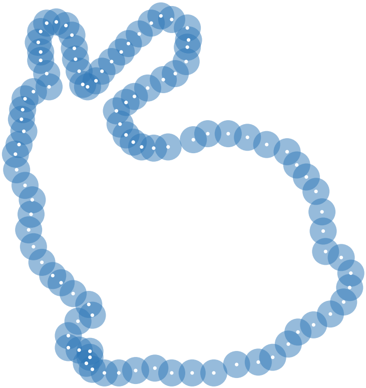
Connect the dots: topology from geometry



Connect the dots: topology from geometry

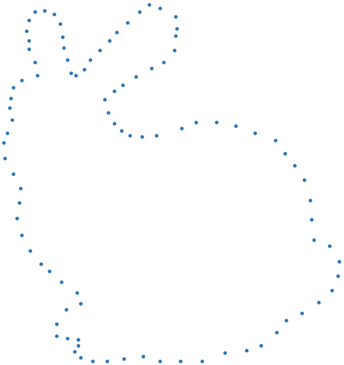


Connect the dots: topology from geometry



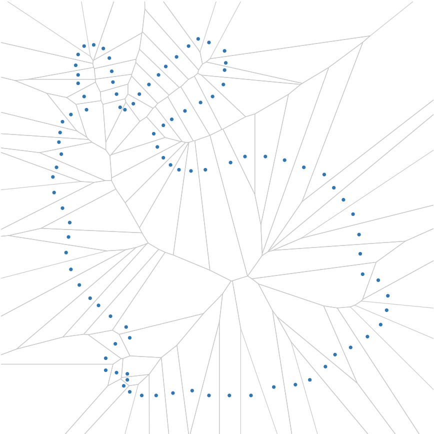
$$\mathrm{Cech}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} B_r(p) \neq \emptyset \right\}$$

Connect the dots: topology from geometry



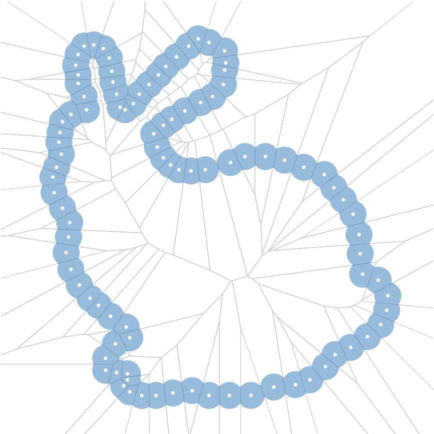
$$\mathrm{Cech}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} B_r(p) \neq \emptyset \right\}$$

Connect the dots: topology from geometry



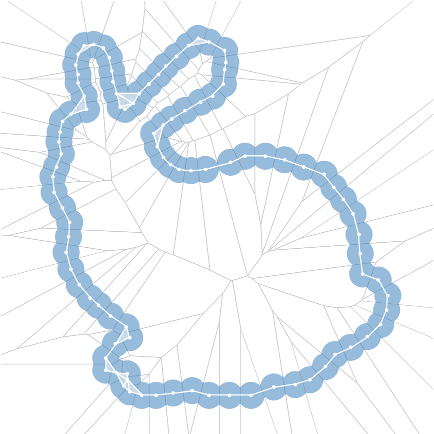
$$\mathrm{Cech}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} B_r(p) \neq \emptyset \right\}$$

Connect the dots: topology from geometry



$$\mathrm{Cech}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} B_r(p) \neq \emptyset \right\}$$

Connect the dots: topology from geometry



$$\mathrm{Cech}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} B_r(p) \neq \emptyset \right\}$$

$$\mathrm{Del}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{p \in Q} \left(B_r(p) \cap \mathrm{Vor}_X(p)\right) \neq \emptyset \right\}$$

Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)

Čech and Delaunay functions

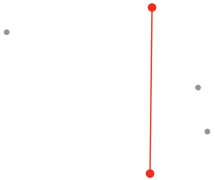
$X \subset \mathbb{R}^d$: finite point set (in general position)

Simplices $\Delta(X)$: nonempty subsets of X

Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)

Simplices $\Delta(X)$: nonempty subsets of X



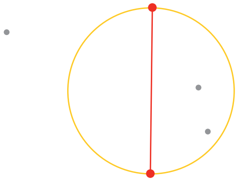
Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)

Simplices $\Delta(X)$: nonempty subsets of X

Two functions on simplices $Q \subseteq X$:

Čech function $f_C(Q)$:
radius of smallest enclosing sphere of Q



Čech and Delaunay functions

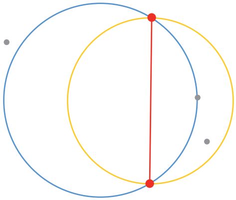
$X \subset \mathbb{R}^d$: finite point set (in general position)

Simplices $\Delta(X)$: nonempty subsets of X

Two functions on simplices $Q \subseteq X$:

Čech function $f_C(Q)$:
radius of smallest enclosing sphere of Q

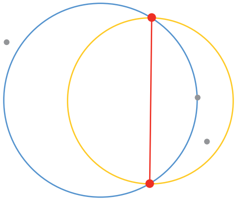
Delaunay function $f_D(Q)$:
radius of smallest empty circumsphere



Čech and Delaunay functions

$X \subset \mathbb{R}^d$: finite point set (in general position)

Simplices $\Delta(X)$: nonempty subsets of X



Two functions on simplices $Q \subseteq X$:

Čech function $f_C(Q)$:
radius of smallest enclosing sphere of Q

Delaunay function $f_D(Q)$:
radius of smallest empty circumsphere

- defined only if Q has an empty circumsphere: $Q \in \mathrm{Del}(X)$

Čech and Delaunay complexes from functions

Define for any radius r:

Čech and Delaunay complexes from functions

Define for any radius r:

- *Čech complex* $\mathrm{Cech}_r = f_C^{-1}(-\infty, r]$

Čech and Delaunay complexes from functions

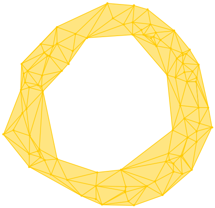
Define for any radius r:

- *Čech complex* $\mathrm{Cech}_r = f_C^{-1}(-\infty, r]$
  - all simplices having an enclosing sphere of radius $\leq r$

Čech and Delaunay complexes from functions

Define for any radius r:

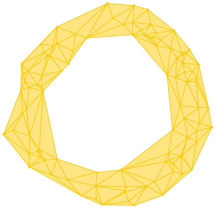
- *Čech complex* $\mathrm{Cech}_r = f_C^{-1}(-\infty, r]$
  - all simplices having an enclosing sphere of radius $\leq r$
- *Delaunay–Čech complex* $\mathrm{DelCech}_r = \mathrm{Cech}_r \cap \mathrm{Del}$



Čech and Delaunay complexes from functions

Define for any radius r:

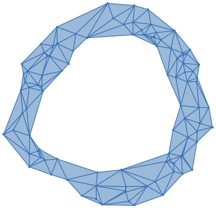
- *Čech complex* $\mathrm{Cech}_r = f_C^{-1}(-\infty, r]$
  - all simplices having an enclosing sphere of radius $\leq r$
- *Delaunay–Čech complex* $\mathrm{DelCech}_r = \mathrm{Cech}_r \cap \mathrm{Del}$
  - restriction of Čech complex to Delaunay simplices



Čech and Delaunay complexes from functions

Define for any radius r:

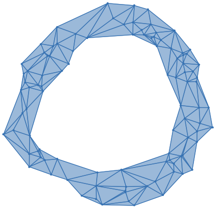
- *Čech complex* $\mathrm{Cech}_r = f_C^{-1}(-\infty, r]$
  - all simplices having an enclosing sphere of radius $\leq r$
- *Delaunay–Čech complex* $\mathrm{DelCech}_r = \mathrm{Cech}_r \cap \mathrm{Del}$
  - restriction of Čech complex to Delaunay simplices
- *Delaunay complex* (α-*complex*, for $\alpha = r$) $\mathrm{Del}_r = f_D^{-1}(-\infty, r]$



Čech and Delaunay complexes from functions

Define for any radius r:

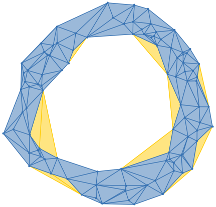
- *Čech complex* $\mathrm{Cech}_r = f_C^{-1}(-\infty, r]$
  - all simplices having an enclosing sphere of radius $\leq r$
- *Delaunay–Čech complex* $\mathrm{DelCech}_r = \mathrm{Cech}_r \cap \mathrm{Del}$
  - restriction of Čech complex to Delaunay simplices
- *Delaunay complex* (*α-complex*, for $\alpha = r$) $\mathrm{Del}_r = f_D^{-1}(-\infty, r]$
  - all simplices having an empty circumsphere of radius $\leq r$



Čech and Delaunay complexes from functions

Define for any radius r:

- *Čech complex* $\mathrm{Cech}_r = f_C^{-1}(-\infty, r]$
  - all simplices having an enclosing sphere of radius $\leq r$
- *Delaunay–Čech complex* $\mathrm{DelCech}_r = \mathrm{Cech}_r \cap \mathrm{Del}$
  - restriction of Čech complex to Delaunay simplices
- *Delaunay complex* (*α-complex*, for $\alpha = r$) $\mathrm{Del}_r = f_D^{-1}(-\infty, r]$
  - all simplices having an empty circumsphere of radius $\leq r$



Properties of Čech and Delaunay complexes

By the *Nerve theorem* (Borsuk 1947):

$$\mathrm{Del}_r(X) \simeq \mathrm{Cech}_r(X) \simeq B_r(X).$$

Properties of Čech and Delaunay complexes

By the *Nerve theorem* (Borsuk 1947):

$$\mathrm{Del}_r(X) \simeq \mathrm{Cech}_r(X) \simeq B_r(X).$$

But we also have

$$\mathrm{Del}_r(X) \subseteq \mathrm{DelCech}_r(X) \subseteq \mathrm{Cech}_r(X).$$

Properties of Čech and Delaunay complexes

By the *Nerve theorem* (Borsuk 1947):

$$\mathrm{Del}_r(X) \simeq \mathrm{Cech}_r(X) \simeq B_r(X).$$

But we also have

$$\mathrm{Del}_r(X) \subseteq \mathrm{DelCech}_r(X) \subseteq \mathrm{Cech}_r(X).$$

- Are all three complexes homotopy equivalent?

Properties of Čech and Delaunay complexes

By the *Nerve theorem* (Borsuk 1947):

$$\mathrm{Del}_r(X) \simeq \mathrm{Cech}_r(X) \simeq B_r(X).$$

But we also have

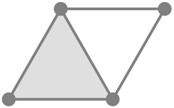
$$\mathrm{Del}_r(X) \subseteq \mathrm{DelCech}_r(X) \subseteq \mathrm{Cech}_r(X).$$

- Are all three complexes homotopy equivalent?
- Are they related by a sequence of simplicial collapses?

Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

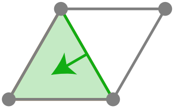


Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

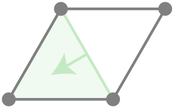
- A *free face* of K is a simplex which is contained in a unique proper coface.



Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

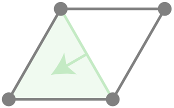


- A *free face* of K is a simplex which is contained in a unique proper coface.
- We can *collapse* K to a subcomplex L by removing a free face F, along with its proper coface.

Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.

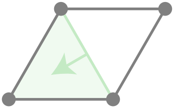


- A *free face* of K is a simplex which is contained in a unique proper coface.
- We can *collapse* K to a subcomplex L by removing a free face F, along with its proper coface.
- L is homotopy equivalent to K. In particular, they have isomorphic homology.

Simplicial collapses

Definition (Whitehead 1938)

Let K be a simplicial complex.



- A *free face* of K is a simplex which is contained in a unique proper coface.
- We can *collapse* K to a subcomplex L by removing a free face F, along with its proper coface.
- L is homotopy equivalent to K. In particular, they have isomorphic homology.

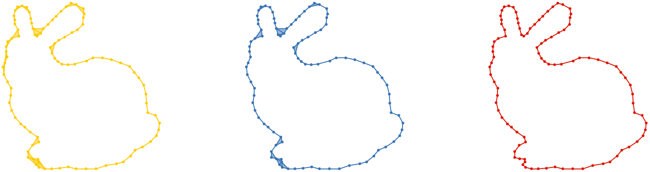
If there is a sequence of these elementary collapses from K to L, we say that K *collapses* to L (written as $K \searrow L$).

Main result: a sequence of collapses

Theorem (B, Edelsbrunner)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent. In particular,

$$\mathrm{Cech}_r \searrow \mathrm{DelCech}_r \searrow \mathrm{Del}_r \searrow \mathrm{Wrap}_r \,.$$

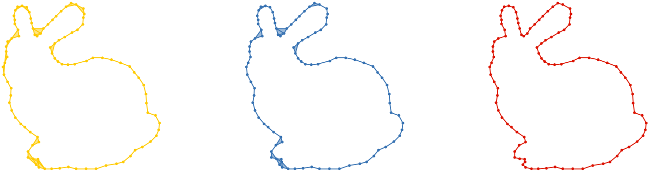


Main result: a sequence of collapses

Theorem (B, Edelsbrunner)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent. In particular,

$$\mathrm{Cech}_r \searrow \mathrm{DelCech}_r \searrow \mathrm{Del}_r \searrow \mathrm{Wrap}_r \,.$$



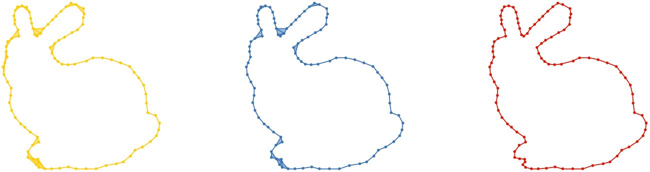
- All collapses are induced by a single *discrete gradient field*

Main result: a sequence of collapses

Theorem (B, Edelsbrunner)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent. In particular,

$$\mathrm{Cech}_r \searrow \mathrm{DelCech}_r \searrow \mathrm{Del}_r \searrow \mathrm{Wrap}_r \,.$$



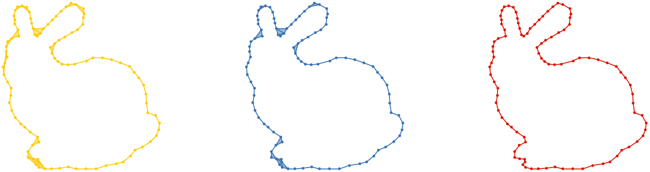
- All collapses are induced by a single *discrete gradient field*
- The filtrations have isomorphic persistent homology

Main result: a sequence of collapses

Theorem (B, Edelsbrunner)

Čech, Delaunay–Čech, Delaunay, and Wrap complexes are homotopy equivalent. In particular,

$$\mathrm{Cech}_r \searrow \mathrm{DelCech}_r \searrow \mathrm{Del}_r \searrow \mathrm{Wrap}_r\,.$$

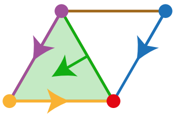


- All collapses are induced by a single *discrete gradient field*
- The filtrations have isomorphic persistent homology
- Also works for weighted point sets

Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a simplicial complex
is a partition of the simplices
into singletons and *facet pairs* (Q, R)
(Q is a face of R with codimension 1).

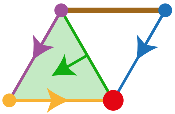


Discrete Morse theory

Definition (Forman 1998)

A *discrete vector field* on a simplicial complex
is a partition of the simplices
into singletons and *facet pairs* (Q, R)
(Q is a face of R with codimension 1).

The singletons are called *critical simplices.*

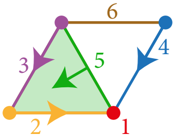


Discrete Morse theory

Definition (Forman 1998)

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *discrete Morse function* if for all $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes



Discrete Morse theory

Definition (Forman 1998)

A function $f : K \to \mathbb{R}$ on a simplicial complex
is a *discrete Morse function* if for all $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$
  are subcomplexes

Discrete Morse theory

Definition (Forman 1998)

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *discrete Morse function* if for all $t \in \mathbb{R}$:

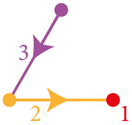
- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes

Discrete Morse theory

Definition (Forman 1998)

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *discrete Morse function* if for all $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes

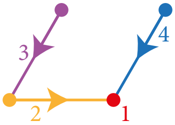


Discrete Morse theory

Definition (Forman 1998)

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *discrete Morse function* if for all $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes

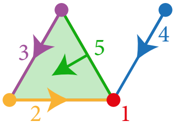


Discrete Morse theory

Definition (Forman 1998)

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *discrete Morse function* if for all $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes

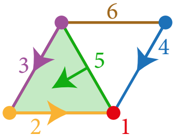


Discrete Morse theory

Definition (Forman 1998)

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *discrete Morse function* if for all $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes

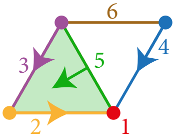


Discrete Morse theory

Definition (Forman 1998)

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *discrete Morse function* if for all $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes
- the level sets $f^{-1}(t)$ form a discrete vector field (the *discrete gradient* of f)

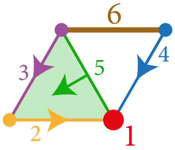


Discrete Morse theory

Definition (Forman 1998)

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *discrete Morse function* if for all $t \in \mathbb{R}$:

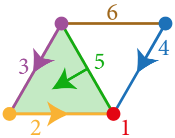
- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes
- the level sets $f^{-1}(t)$ form a discrete vector field (the *discrete gradient* of f)



If $f^{-1}(t) = \{Q\}$ then t is a *critical value*.

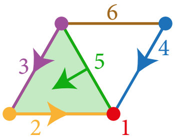
Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.



Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.



Theorem (Forman 1998)

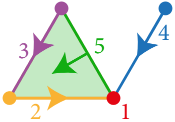
If $(s, t]$ contains no critical value of f,
then K_t collapses to *K_s (written as $K_t \searrow K_s$).*

Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.

Theorem (Forman 1998)

If $(s, t]$ contains no critical value of f,
then K_t collapses to K_s *(written as $K_t \searrow K_s$).*

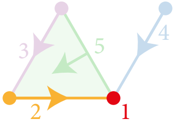


Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.

Theorem (Forman 1998)

If $(s,t]$ contains no critical value of f,
then K_t collapses to K_s *(written as $K_t \searrow K_s$).*

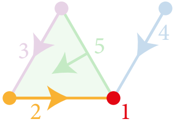


Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.

Theorem (Forman 1998)

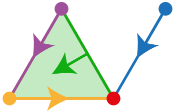
If $(s, t]$ contains no critical value of f, then K_t collapses to K_s (written as $K_t \searrow K_s$).



Let V be a discrete gradient field on a simplicial complex K, and let L be a subcomplex of K.

Corollary

If $K \smallsetminus L$ is the union of facet pairs of V, then $K \searrow L$.

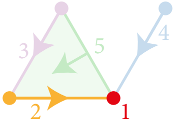


Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.

Theorem (Forman 1998)

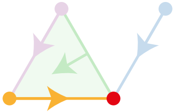
If $(s, t]$ contains no critical value of f, then K_t collapses to K_s (written as $K_t \searrow K_s$).



Let V be a discrete gradient field on a simplicial complex K, and let L be a subcomplex of K.

Corollary

If $K \setminus L$ is the union of facet pairs of V, then $K \searrow L$.

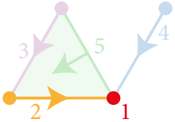


Collapses from Morse functions and gradients

Let f be a discrete Morse function on a simplicial complex K.

Theorem (Forman 1998)

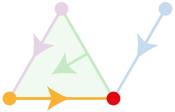
If $(s,t]$ contains no critical value of f,
then K_t collapses to K_s *(written as $K_t \searrow K_s$).*



Let V be a discrete gradient field on a simplicial complex K, and let L be a subcomplex of K.

Corollary

If $K \smallsetminus L$ is the union of facet pairs of V,
then $K \searrow L$.



We say that V *induces the collapse $K \searrow L$.*

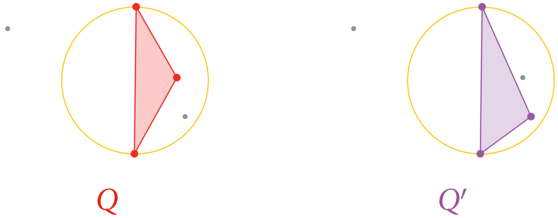
Unfortunately…

Neither the Čech nor the Delaunay functions are discrete Morse functions!

Unfortunately…

Neither the Čech nor the Delaunay functions are discrete Morse functions!

- Example: two simplices Q, Q' with $f_C(Q) = f_C(Q')$ that do not form a facet pair:

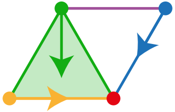


Generalized discrete Morse theory

Definition (Freij 2009)

A *generalized discrete vector field* on a simplicial complex is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}.$$

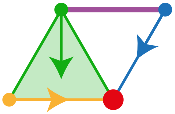


Generalized discrete Morse theory

Definition (Freij 2009)

A *generalized discrete vector field* on a simplicial complex is a partition of the simplices into intervals of the face poset:

$$[L, U] = \{Q : L \subseteq Q \subseteq U\}.$$



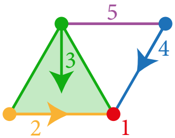
The singletons are called *critical simplices.*

Generalized discrete Morse theory

Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes



Generalized discrete Morse theory

Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes

Generalized discrete Morse theory

Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex
is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

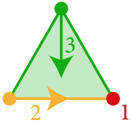
- the sublevel sets $K_t = f^{-1}(-\infty, t]$
  are subcomplexes

Generalized discrete Morse theory

Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex
is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$
  are subcomplexes

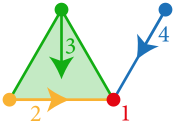


Generalized discrete Morse theory

Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes

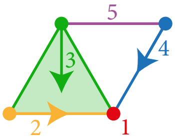


Generalized discrete Morse theory

Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex
is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes

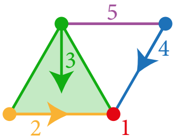


Generalized discrete Morse theory

Definition

A function $f : K \to \mathbb{R}$ on a simplicial complex
is a *generalized discrete Morse function* if for $t \in \mathbb{R}$:

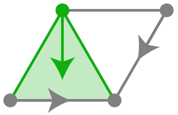
- the sublevel sets $K_t = f^{-1}(-\infty, t]$ are subcomplexes
- the level sets $f^{-1}(t)$ form a generalized vector field (the *discrete gradient* of f)



Refining generalized vector fields

A generalized vector field V can be refined to a vector field.

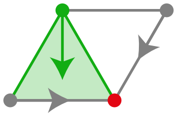
For each non-critical interval $[L, U] \in V$:



Refining generalized vector fields

A generalized vector field V can be refined to a vector field.
For each non-critical interval $[L, U] \in V$:

- choose an arbitrary vertex $x \in U \setminus L$

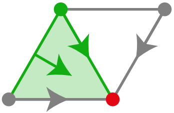


Refining generalized vector fields

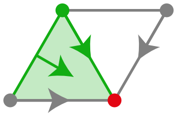
A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:

- choose an arbitrary vertex $x \in U \setminus L$
- partition $[L, U]$ into facet pairs $(Q \setminus \{x\}, Q \cup \{x\})$ for all $Q \in [L, U]$.



Refining generalized vector fields



A generalized vector field V can be refined to a vector field.

For each non-critical interval $[L, U] \in V$:

- choose an arbitrary vertex $x \in U \setminus L$
- partition $[L, U]$ into facet pairs $(Q \setminus \{x\}, Q \cup \{x\})$ for all $Q \in [L, U]$.

Therefore the collapsing theorems also hold for generalized discrete Morse functions.

Morse theory of Čech and Delaunay complexes

Proposition

The Čech function and the Delaunay function are generalized discrete Morse functions.

Morse theory of Čech and Delaunay complexes

Proposition

The Čech function and the Delaunay function are generalized discrete Morse functions.

The following are equivalent for each simplex $Q \subseteq X$:

- $f_D(Q) = f_C(Q)$

Morse theory of Čech and Delaunay complexes

Proposition

The Čech function and the Delaunay function are generalized discrete Morse functions.

The following are equivalent for each simplex $Q \subseteq X$:

- $f_D(Q) = f_C(Q)$
- *Q is a critical simplex of f_C*

Morse theory of Čech and Delaunay complexes

Proposition

The Čech function and the Delaunay function are generalized discrete Morse functions.

The following are equivalent for each simplex $Q \subseteq X$*:*

- $f_D(Q) = f_C(Q)$
- Q *is a critical simplex of* f_C
- Q *is a critical simplex of* f_D

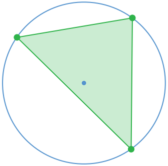
Morse theory of Čech and Delaunay complexes

Proposition

The Čech function and the Delaunay function are generalized discrete Morse functions.

The following are equivalent for each simplex $Q \subseteq X$:

- $f_D(Q) = f_C(Q)$
- *Q is a critical simplex of f_C*
- *Q is a critical simplex of f_D*
- *Q is a* centered *Delaunay simplex (containing the circumcenter in the interior)*



Čech intervals

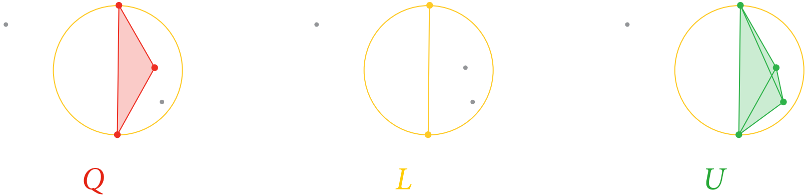
Lemma

Let $Q \subseteq X$ be a simplex with smallest enclosing sphere S.
Then $Q' \subseteq X$ has the same smallest enclosing sphere S iff

$$Q' \in [L, U],$$

where

- *L: points of X on S,*
- *U: points of X on or inside of S.*



The front face of a simplex

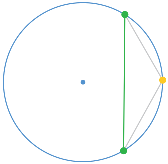
Let $T \subseteq X$ be a simplex with smallest circumsphere S.
Write the center z of S as an affine combination

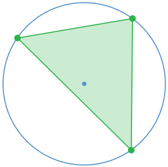
$$z = \sum_{x \in T} \mu_x x, \qquad 1 = \sum_{x \in T} \mu_x.$$

We call

$$\operatorname{front} T = \{x \in T \mid \mu_x > 0\},$$
$$\operatorname{back} T = \{x \in T \mid \mu_x < 0\}$$

the *front face* and the *back face* of T, respectively.





Delaunay intervals

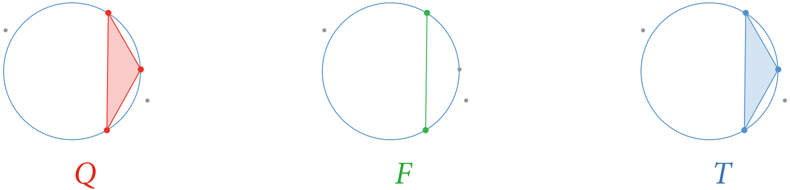
Lemma

Let $Q \subseteq X$ be a simplex with smallest empty circumsphere S.
Then $Q' \subseteq X$ has the same smallest empty circumsphere S iff

$$Q' \in [F, T],$$

where

- *T: points of X on S,*
- *F: front face of T.*



Sphere optimization problems

Both Čech and Delaunay function are defined using the smallest sphere S satisfying certain constraints:

$$\begin{aligned} \underset{r,z}{\text{minimize}} \quad & r \\ \text{subject to} \quad & \|z - q\| \le r, \quad q \in Q, \\ & \|z - e\| \ge r, \quad e \in E. \end{aligned}$$

Here r is the radius and z is the center of the sphere S.

Sphere optimization problems

Both Čech and Delaunay function are defined using the smallest sphere S satisfying certain constraints:

$$\begin{array}{ll} \underset{r,z}{\text{minimize}} & r \\ \text{subject to} & \|z - q\| \leq r, \quad q \in Q, \\ & \|z - e\| \geq r, \quad e \in E. \end{array}$$

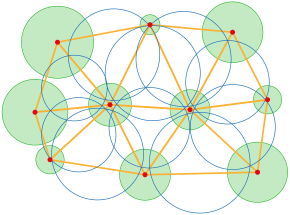
Here r is the radius and z is the center of the sphere S.

- Čech function: choose $E = \emptyset$
- Delaunay function: choose $E = X$

Weighted point sets

All constructions can be generalized to weighted point sets

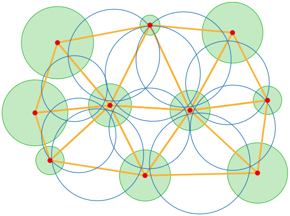
- Intuition: spheres with given squared radius
- Instead of circumspheres, consider *orthogonal spheres*



Weighted point sets

All constructions can be generalized to weighted point sets

- Intuition: spheres with given squared radius
- Instead of circumspheres, consider *orthogonal spheres*



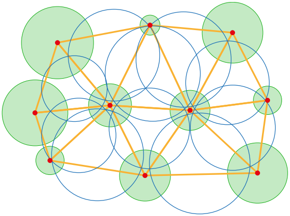
The optimization problem is:

$$\begin{aligned} \underset{r,z}{\text{minimize}} \quad & r^2 \\ \text{subject to} \quad & \|z - q\|^2 \le r^2 + w_q, \quad q \in Q, \\ & \|z - e\|^2 \ge r^2 + w_e, \quad e \in E. \end{aligned}$$

Weighted point sets

All constructions can be generalized to weighted point sets

- Intuition: spheres with given squared radius
- Instead of circumspheres, consider *orthogonal spheres*



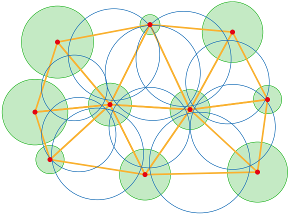
The optimization problem (substituting $a = \|z\|^2 - r^2$) is:

$$\begin{aligned} \underset{a,z}{\text{minimize}} \quad & \|z\|^2 - a \\ \text{subject to} \quad & \|z - q\|^2 \le \|z\|^2 - a + w_q, \quad q \in Q, \\ & \|z - e\|^2 \ge \|z\|^2 - a + w_e, \quad e \in E. \end{aligned}$$

Weighted point sets

All constructions can be generalized to weighted point sets

- Intuition: spheres with given squared radius
- Instead of circumspheres, consider *orthogonal spheres*



The optimization problem (substituting $a = \|z\|^2 - r^2$) is:

$$\begin{aligned} \underset{a,z}{\text{minimize}} \quad & \|z\|^2 - a \\ \text{subject to} \quad & 2\langle z, q\rangle - a \geq \|q\|^2 - w_q, \quad q \in Q, \\ & 2\langle z, e\rangle - a \leq \|e\|^2 - w_e, \quad e \in E. \end{aligned}$$

The Karush–Kuhn–Tucker optimality conditions

Consider an optimization problem

$$\begin{aligned}
\underset{x}{\text{minimize}} \quad & f(x) \\
\text{subject to} \quad & g_i(x) \le 0, \quad i \in I, \\
& h_j(x) = 0, \quad j \in J.
\end{aligned}$$

The Karush–Kuhn–Tucker optimality conditions

Consider an optimization problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0, \quad i \in I, \\ & h_j(x) = 0, \quad j \in J. \end{aligned}$$

Assume that the functions g_i, h_j are affine and f is convex. Then a feasible point x is an optimal solution iff there exist *Lagrange multipliers* $(v_i)_{i \in I}$ and $(\lambda_j)_{j \in J}$ such that

$$\begin{aligned} \nabla f(x) + \sum_{i \in I} v_i \nabla g_i(x) + \sum_{j \in J} \lambda_j \nabla h_j(x) &= 0, \\ v_i &\ge 0, \quad i \in I, \\ v_i\, g_i(x) &= 0, \quad i \in I. \end{aligned}$$

KKT conditions for smallest spheres

The KKT conditions for our sphere optimization problem are:

Proposition

A sphere S enclosing Q and empty of E is minimal iff its center z can be written as an affine combination

$$z = \sum_{x \in Q \cup E} \mu_x x, \qquad 1 = \sum_{x \in Q \cup E} \mu_x$$

such that

- $\mu_x \geq 0$ *for* $x \notin E$,
- $\mu_x \leq 0$ *for* $x \notin Q$, *and*
- $\mu_x = 0$ *for* $x \notin S$.

Čech and Delaunay intervals from KKT

Corollary

A sphere S enclosing Q and empty of E is minimal iff

$$z \in \operatorname{aff} T,$$
$$Q \in [F, U], \text{ and}$$
$$E \in [B, W],$$

where

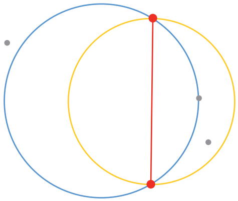
- *z: center of S,*
- *T, U, W: points of X on S, on or inside S, on or outside of S,*
- *F, B: front, back face of T.*

Intersections of Čech and Delaunay intervals

Lemma (Excluded Singularity)

The intersection of non-singular Čech and Delaunay intervals is a non-singular interval.

- Consider a non-critical Delaunay simplex Q

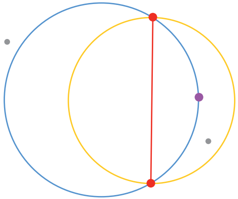


Intersections of Čech and Delaunay intervals

Lemma (Excluded Singularity)

The intersection of non-singular Čech and Delaunay intervals is a non-singular interval.

- Consider a non-critical Delaunay simplex Q
- There is a point p inside the Čech sphere and on the Delaunay sphere

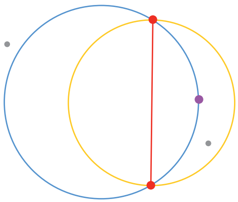


Intersections of Čech and Delaunay intervals

Lemma (Excluded Singularity)

The intersection of non-singular Čech and Delaunay intervals is a non-singular interval.

- Consider a non-critical Delaunay simplex Q
- There is a point p inside the Čech sphere and on the Delaunay sphere
- $Q' = Q \setminus \{p\}$ and $Q'' = Q \cup \{p\}$ have the same Čech and Delaunay sphere

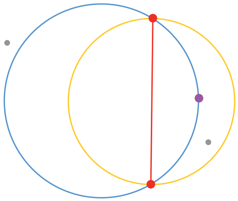


Intersections of Čech and Delaunay intervals

Lemma (Excluded Singularity)

The intersection of non-singular Čech and Delaunay intervals is a non-singular interval.

- Consider a non-critical Delaunay simplex Q
- There is a point p inside the Čech sphere and on the Delaunay sphere
- $Q' = Q \setminus \{p\}$ and $Q'' = Q \cup \{p\}$ have the same Čech and Delaunay sphere

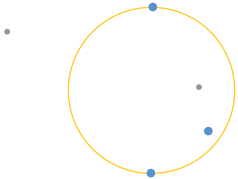


Corollary

The pairs (Q', Q'') yield a vector field that induces a collapse $\mathrm{DelCech}_r \searrow \mathrm{Del}_r$.

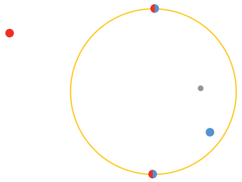
Collapsing non-Delaunay simplices

- Consider a non-Delaunay simplex Q together with its Čech sphere S



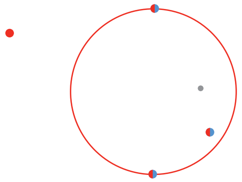
Collapsing non-Delaunay simplices

- Consider a non-Delaunay simplex Q together with its Čech sphere S
- Start with the excluded set $E = X \setminus \operatorname{int} S$



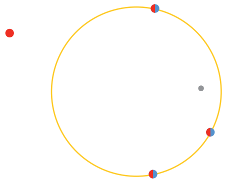
Collapsing non-Delaunay simplices

- Consider a non-Delaunay simplex Q together with its Čech sphere S
- Start with the excluded set $E = X \smallsetminus \operatorname{int} S$
- Exclude more and more points until no feasible sphere exists



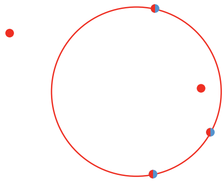
Collapsing non-Delaunay simplices

- Consider a non-Delaunay simplex Q together with its Čech sphere S
- Start with the excluded set $E = X \setminus \operatorname{int} S$
- Exclude more and more points until no feasible sphere exists



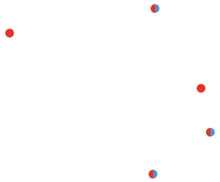
Collapsing non-Delaunay simplices

- Consider a non-Delaunay simplex Q together with its Čech sphere S
- Start with the excluded set $E = X \setminus \operatorname{int} S$
- Exclude more and more points until no feasible sphere exists



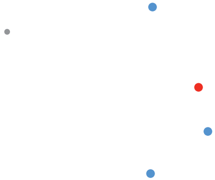
Collapsing non-Delaunay simplices

- Consider a non-Delaunay simplex Q together with its Čech sphere S
- Start with the excluded set $E = X \setminus \operatorname{int} S$
- Exclude more and more points until no feasible sphere exists



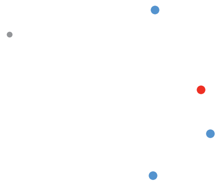
Collapsing non-Delaunay simplices

- Consider a non-Delaunay simplex Q together with its Čech sphere S
- Start with the excluded set $E = X \setminus \operatorname{int} S$
- Exclude more and more points until no feasible sphere exists
- Let p be the last excluded point



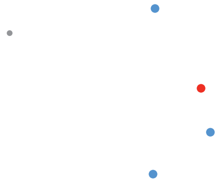
Collapsing non-Delaunay simplices

- Consider a non-Delaunay simplex Q together with its Čech sphere S
- Start with the excluded set $E = X \setminus \operatorname{int} S$
- Exclude more and more points until no feasible sphere exists
- Let p be the last excluded point
- $Q' = Q \setminus \{p\}$ and $Q'' = Q \cup \{p\}$ have the same Čech sphere and no Delaunay sphere



Collapsing non-Delaunay simplices

- Consider a non-Delaunay simplex Q together with its Čech sphere S
- Start with the excluded set $E = X \setminus \operatorname{int} S$
- Exclude more and more points until no feasible sphere exists
- Let p be the last excluded point
- $Q' = Q \setminus \{p\}$ and $Q'' = Q \cup \{p\}$ have the same Čech sphere and no Delaunay sphere

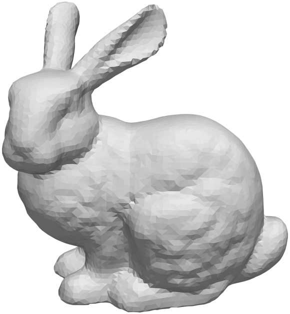


Corollary

The pairs (Q', Q'') *yield a vector field that induces a collapse* $\mathrm{Cech}_r \searrow \mathrm{DelCech}_r$.

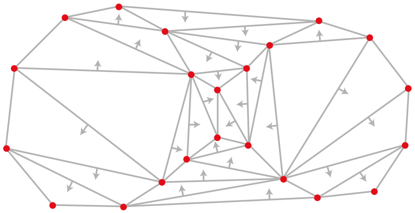
Wrap complexes

Generalizes and greatly simplifies the surface reconstruction algorithm *Wrap* (Edelsbrunner 1995, Geomagic)



Wrap complexes

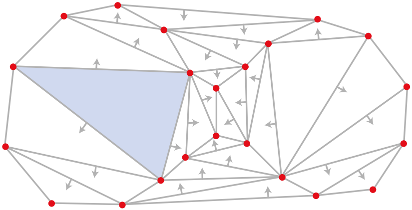
Consider the Delaunay function f_D of X.



Wrap complexes

Consider the Delaunay function f_D of X.

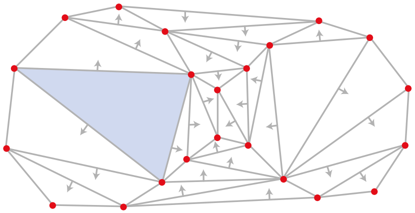
- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$



Wrap complexes

Consider the Delaunay function f_D of X.

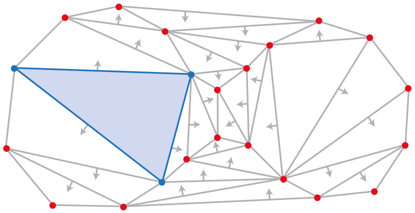
- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$
- Let $\downarrow C$ denote the descending set of Delaunay intervals (with the partial order induced by the face relation)



Wrap complexes

Consider the Delaunay function f_D of X.

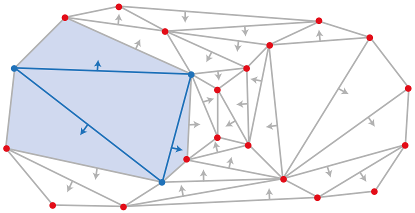
- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$
- Let $\downarrow C$ denote the descending set of Delaunay intervals (with the partial order induced by the face relation)



Wrap complexes

Consider the Delaunay function f_D of X.

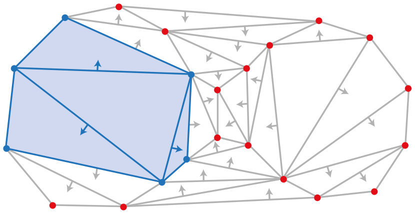
- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$
- Let $\downarrow C$ denote the descending set of Delaunay intervals (with the partial order induced by the face relation)



Wrap complexes

Consider the Delaunay function f_D of X.

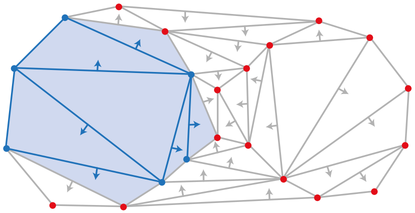
- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$
- Let $\downarrow C$ denote the descending set of Delaunay intervals (with the partial order induced by the face relation)



Wrap complexes

Consider the Delaunay function f_D of X.

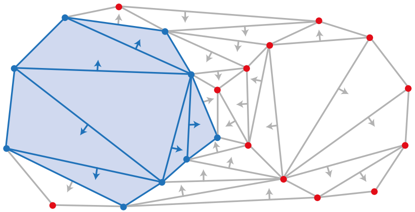
- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$
- Let $\downarrow C$ denote the descending set of Delaunay intervals (with the partial order induced by the face relation)



Wrap complexes

Consider the Delaunay function f_D of X.

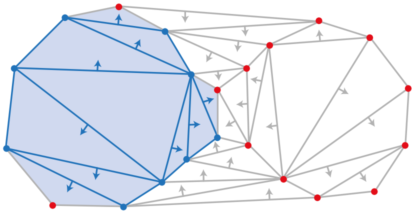
- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$
- Let $\downarrow C$ denote the descending set of Delaunay intervals (with the partial order induced by the face relation)



Wrap complexes

Consider the Delaunay function f_D of X.

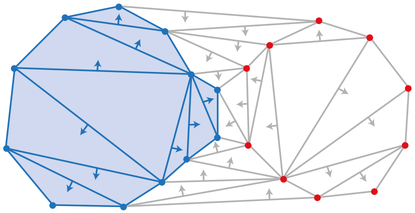
- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$
- Let $\downarrow C$ denote the descending set of Delaunay intervals (with the partial order induced by the face relation)



Wrap complexes

Consider the Delaunay function f_D of X.

- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$
- Let $\downarrow C$ denote the descending set of Delaunay intervals (with the partial order induced by the face relation)



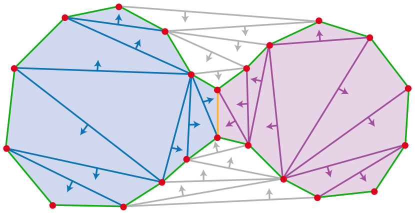
Wrap complexes

Consider the Delaunay function f_D of X.

- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$
- Let $\downarrow C$ denote the descending set of Delaunay intervals (with the partial order induced by the face relation)

Define

$$\mathrm{Wrap}_r = \bigcup_{C \in \mathrm{Crit}_r} \downarrow C$$



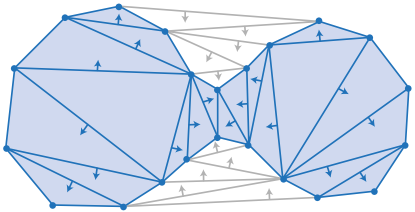
Wrap complexes

Consider the Delaunay function f_D of X.

- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$
- Let $\downarrow C$ denote the descending set of Delaunay intervals (with the partial order induced by the face relation)

Define

$$\mathrm{Wrap}_r = \bigcup_{C \in \mathrm{Crit}_r} \downarrow C$$



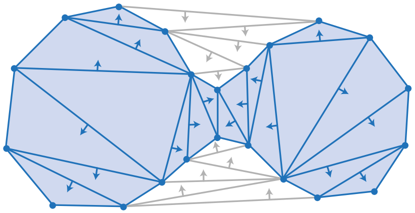
Wrap complexes

Consider the Delaunay function f_D of X.

- Let $C \in \mathrm{Crit}_r$ be a critical simplex with value $f_D(C) \leq r$
- Let $\downarrow C$ denote the descending set of Delaunay intervals (with the partial order induced by the face relation)

Define

$$\mathrm{Wrap}_r = \bigcup_{C \in \mathrm{Crit}_r} \downarrow C$$



By construction: $\mathrm{Del}_r \searrow \mathrm{Wrap}_r$.

Wrapping up

- Čech and Delaunay complexes from Morse functions
- Explicit construction of simplicial collapses
- Simple definition and generalization of *Wrap* complexes