Persistent diagrams as diagrams

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TUM

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The many faces of persistence

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- Matching diagrams: sequence of partial bijections
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Inerval decompositions and persistence modules

Theorem (Crawley-Boewey 2015)

Any pointwise finite-dimensional (pfd) persistence module (a diagam $M : \mathbb{R} \to \mathbf{vect}$) has an essentially unique decomposition as a direct sum of indecomposable interval modules, *isomorphic to*

$$0 \to \cdots \to 0 \to \underbrace{\mathbb{K} \to \cdots \to \mathbb{K}}_{\textit{supported by an interval } I \subseteq \mathbb{R}} \to 0 \to \cdots$$

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- The points in the *persistence diagram* are the endpoints of the intervals in the barcode.
- This is not a diagram in the sense of category theory (functor)!

Persistence and stability: the big picture

point cloud	$P \subset \mathbb{R}^d$	Hausdorff distance

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$\downarrow$ distance		
function	$f : \mathbb{R}^d \to \mathbb{R}$	L_∞ distance

Persistence and stability: the big picture

$$\begin{tikzcd}
\text{point cloud} \arrow[d, "\text{distance}"] \\
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\text{topological spaces (filtration)}
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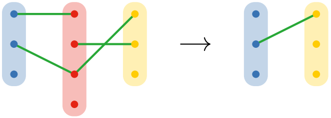
$\downarrow$ decomposition		
intervals (persistence barcode)	$B : \mathbf{R} \to \mathbf{Mch}$	Bottleneck/interleaving distance

The category of matchings

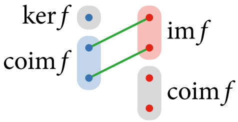
Consider the category $\mathbf{Mch}$ (a subcategory of the category $\mathbf{Rel}$ of sets and relations) with

- objects: sets,
- morphisms: matchings (partial bijections).

Composition:



(Co)kernel/image:



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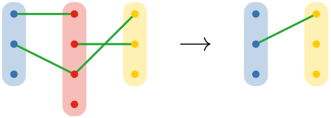

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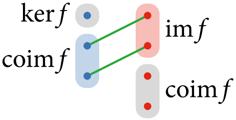


$\mathbf{Mch}$ is *Puppe-exact (p-exact)*:

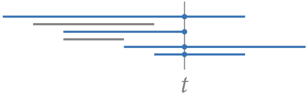
- it has a zero object ($\varnothing$)
- it has all (co)kernels
- every mono (epi) is (co)kernel
- every morphism $f : A \to B$ has an epi-mono factorization $A \twoheadrightarrow \operatorname{im} f \hookrightarrow B$

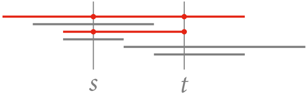
but not additive:

- it does not have all (co)products

From barcodes to matching diagrams (and back)

- A barcode (collection of intervals) can be read as a diagram $\mathbb{R} \to \mathbf{Mch}$:





$t \mapsto \{$intervals in barcode containing $t\}$ $\quad (s \leq t) \mapsto \{$intervals containing both $s, t\}$

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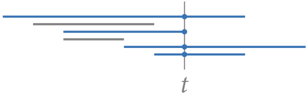

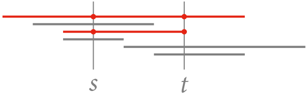

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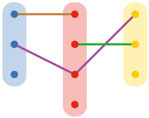

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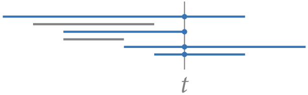

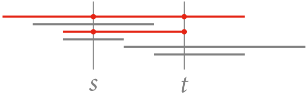

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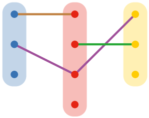

  - equivalence classes $\mathcal{E}(D) := \left(\bigcup_{t \in \mathbb{R}} \{t\} \times D_t \right) / \sim$, where $(s, x) \sim (t, y)$ for all $s \leq t,\ x \in D_s,\ y \in D_t$

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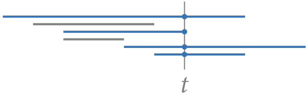

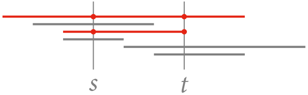

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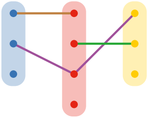

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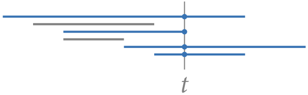

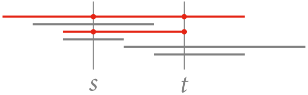

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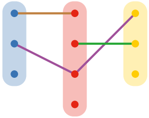

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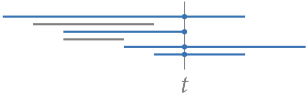

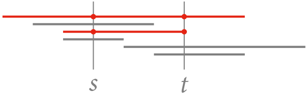

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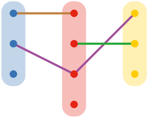

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Turn this into an equivalence of categories $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbb{R}}$

A category of barcodes

Proposition

The functor category is equivalent to $\mathbf{Barc}$*, the category with*

- *objects: barcodes (as a disjoint union of intervals),*
- *morphisms:* overlap matchings $X \nrightarrow Y$*:*
  if $I \in U$ *is matched to* $J \in V$*, then* I overlaps J to the right*:*
  - I *bounds* J *above (every* $s \in J$ *is bounded above by some* $t \in I$*),*
  - J *bounds* I *below,*
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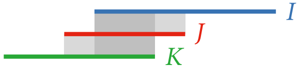
- *composition:* $\tau \bullet \sigma = \{(I, K) \in \tau \circ \sigma \mid I \text{ overlaps } K \text{ above}\}$.



$(I, K) \in \tau \bullet \sigma$ *(overlap)*

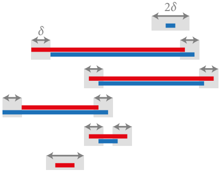
$(I, K) \notin \tau \bullet \sigma$ *(no overlap)*

Bottleneck distance as an interleaving distance

δ-*matching* between barcodes U, V:

- if I is matched to J, then endpoints are δ-close
- unmatched intervals are 2δ-trivial (shorter than 2δ)

Bottleneck distance: $d_B(U, V) = \inf\{\delta \mid \exists\delta\text{-matching } U \nrightarrow V\}$



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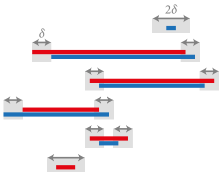

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δ-interleaving between diagrams X, Y indexed over $\mathbb{R}$ (in any category):
natural transformations $f_t : X_t \to Y_{t+\delta}, g_t : Y_t \to X_{t+\delta}$ yielding commutative diagrams

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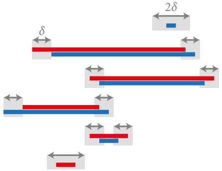

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Proposition

$d_I = d_B$ (using the equivalence $\mathbf{Barc} \simeq \mathbf{Mch}^{\mathbb{R}}$).

Non-functoriality of persistence barcodes

Can a pfd persistence module $M : \mathbf{vect}^{\mathbb{R}}$ be turned into its barcode $B(M) : \mathbf{Mch}^{\mathbb{R}}$ by a functor $B : \mathbf{vect} \to \mathbf{Mch}$ (or $\mathbf{vect}^{\mathbb{R}} \to \mathbf{Mch}^{\mathbb{R}}$)?

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Theorem

There is no functor $\mathbf{vect} \to \mathbf{Mch}$ *sending every vector space* V *to a set of cardinality* $\dim V$ *(equivalently, a linear map* f *to a matching of cardinality* $\operatorname{rank} f$*).*

But: there is a barcode functor for subcategories of monos/epis of persistence modules $\mathbf{vect}^{\mathbb{R}}$:

Structure of persistence sub-/quotient modules

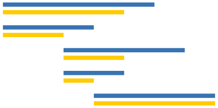
Proposition

Let N be a quotient module of a persistence module M (for $M \twoheadrightarrow N$ an epimorphism).

Then there is an injective map between the barcodes $B(N) \hookrightarrow B(M)$.

If J is mapped to I, then

- *I and J are aligned below, and*
- *I bounds J above.*



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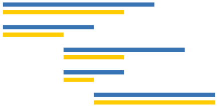

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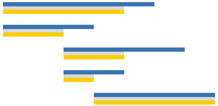
Rephrased for $\mathbf{Mch}^{\mathbb{R}}$:

Proposition

There is a functor from epimorphisms of persistence modules
to epimorphisms of matching diagrams.
(Dually, there is a functor from monos to monos.)



Induced matchings

Theorem

For $f : M \to N$ a morphism of pfd persistence modules, the epi-mono factorization $M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$ gives an induced matching $\chi(f)$ *between their barcodes. If I is matched to J, then*

(i) *I overlaps J above.*

(ii) *If $\ker f$ is δ-trivial, then*

  (a) *J bounds $I(\delta)$ above, and*

  (b) *any unmatched interval of $B(M)$ is δ-trivial.*

(iii) *If $\operatorname{coker} f$ is δ-trivial, then*

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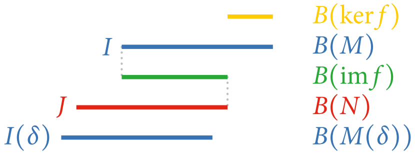

Induced matchings

Theorem

For $f : M \to N$ a morphism of pfd persistence modules, the epi-mono factorization $M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$ gives an induced matching *$\chi(f)$ between their barcodes. If I is matched to J, then*

(i) *I overlaps J above.*

(ii) *If $\ker f$ is δ-trivial, then*

  (a) *J bounds $I(\delta)$ above, and*

  (b) *any unmatched interval of $B(M)$ is δ-trivial.*

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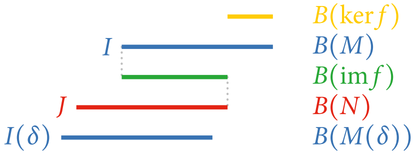
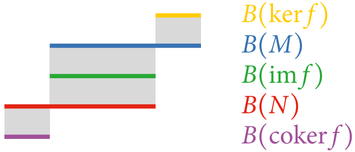


Rephrased in $\mathbf{Mch}^{\mathbb{R}}$:

Theorem

If $f : M \to N$ has δ-trivial (co)kernel, then so does $\chi(f)$.



Algebraic stability via induced matchings

Consider interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(\forall t \in \mathbb{R})$:

$$\begin{tikzcd}
& M_t \arrow[dr, "f_t"] & \\
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- $\operatorname{im} N_{t-\delta,t+\delta} \hookrightarrow \operatorname{im} f_t \hookrightarrow N_{t+\delta}$,
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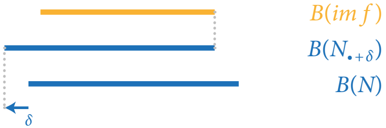

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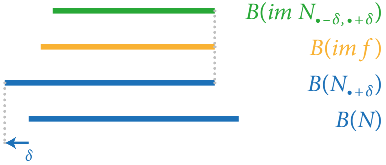

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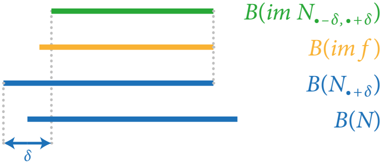

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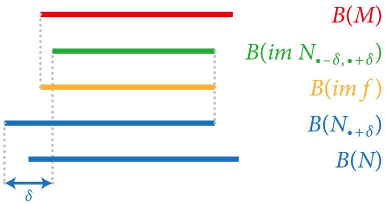

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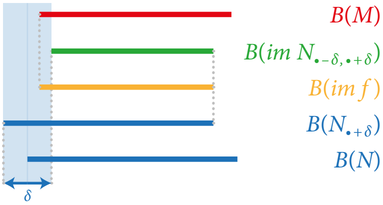
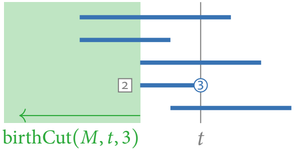


Matching diagrams from persistence modules

Let $M : \mathbf{vect}^{\mathbb{R}}$. For $t \in \mathbb{R}$, $i \in \mathbb{N}$, define

$$\begin{aligned}
\operatorname{birthCut}(M, t, i) &= \{s < t \mid \operatorname{rank} M_{s,t} < i\}, \\
\operatorname{birthOrd}(M, t, i) &= \min\{i - \operatorname{rank} M_{s,t} > 0 \mid s < t\}, \\
\operatorname{birthId}(M, t, i) &= (\operatorname{birthCut}(M, t, i), \operatorname{birthOrd}(M, t, i)).
\end{aligned}$$



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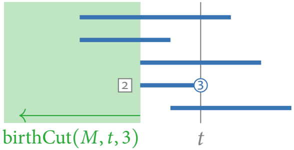

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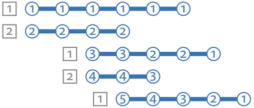
Construct a matching diagram $B(M) : \mathbb{R} \to \mathbf{Mch}$: for all $t \leq u \in \mathbb{R}$, define

$$\begin{aligned} B(M)_t &= \{i \in \mathbb{N} \mid i \leq \dim M_t\} \\ B(M)_{t,u} &= \{(i,j) \mid \mathrm{birthId}(M,t,i) = \mathrm{birthId}(M,u,i)\}. \end{aligned}$$

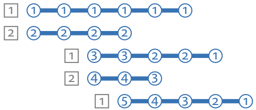
- Yields a barcode without using interval decomposition!



More about matching diagrams

Proposition

- $\operatorname{im} B(M)_{t,u} = \{j \in \mathbb{N} \mid j \leq \operatorname{rank} M_{t,u}\}$.



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- *If* $i \leq j$, *then* $\operatorname{birthCut}(M, t, i) \subseteq \operatorname{birthCut}(M, t, j)$
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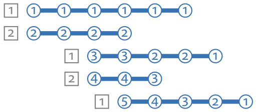

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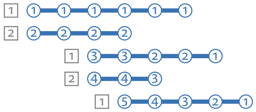

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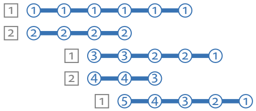

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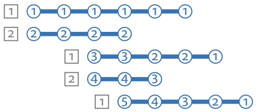
- *If $(i,j),(k,l) \in B(M)_{t,u}$, then $i \geq k \Leftrightarrow j \geq l$*
- *Thus, bars are partially ordered; extends to lexicographic order by*
  - *earlier birth, and (for same birth)*
  - *later death*



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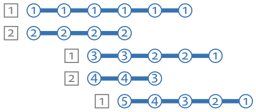

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Applies even to q-tame persistence modules ($\operatorname{rank} M_{t,u} < \infty$ for all $t < u$)!

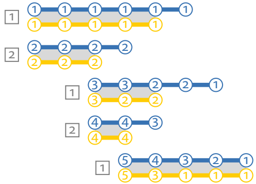
Induced matchings for matching diagrams

Let N be a quotient module of a persistence module M ($M \twoheadrightarrow N$ an epimorphism). Define

$$\chi(M \twoheadrightarrow N)_t = \{(i,j) \mid \mathrm{birthId}(M,t,i) = \mathrm{birthId}(N,t,j)\}.$$

Theorem

B and χ form a functor from epimorphisms of persistence modules to epimorphisms of matching diagrams.
(Dually, there is a functor from monos to monos.)



- This is the structure theorem for sub-/quotient modules, in terms of matching diagrams.
- Using an epi-mono factorization, this yields induced matchings and algebraic stability for q-tame persistence modules.
- Can be used to guide the construction of a decomposition for pdf modules.

Thanks for your attention!