Persistence simplification of discrete Morse functions on surfaces

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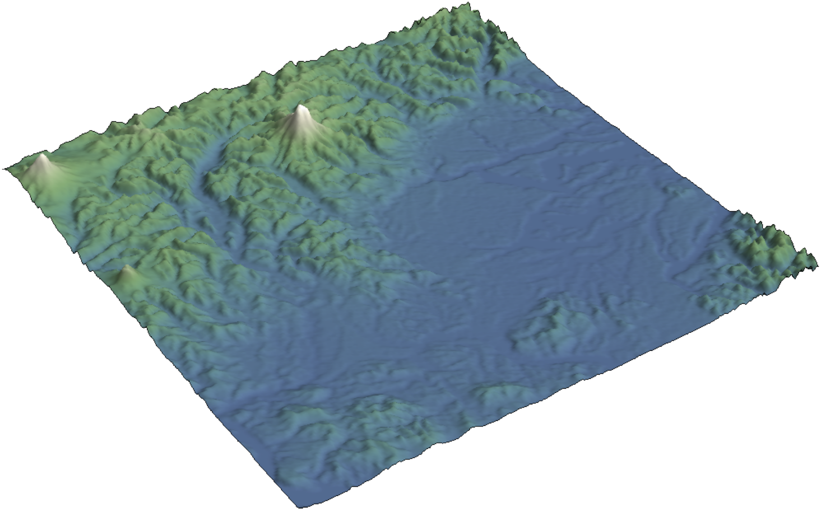

Goal

Topological denoising of functions on surfaces

- minimize number of critical points
- stay close to input function

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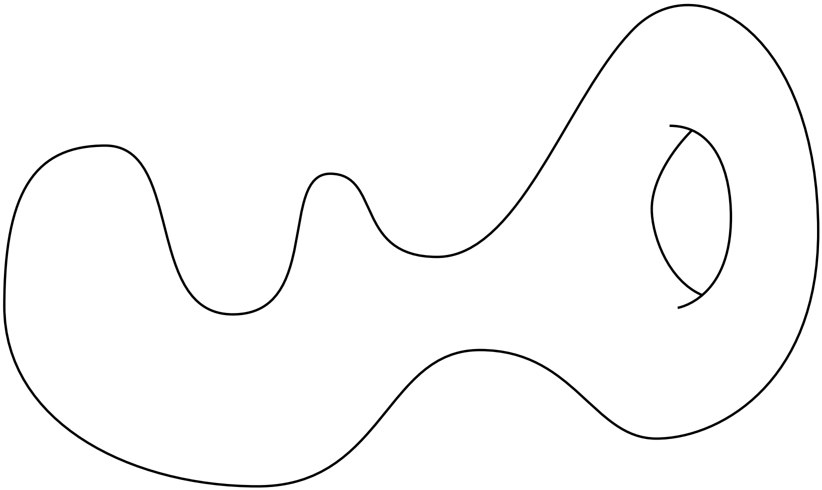
Using:

- Discrete Morse theory [Forman 1998]
  - provides notion of critical point in the discrete setting
- Homological persistence [Edelsbrunner et al. 2002]
  - quantifies significance of critical points

Morse theory at a glance



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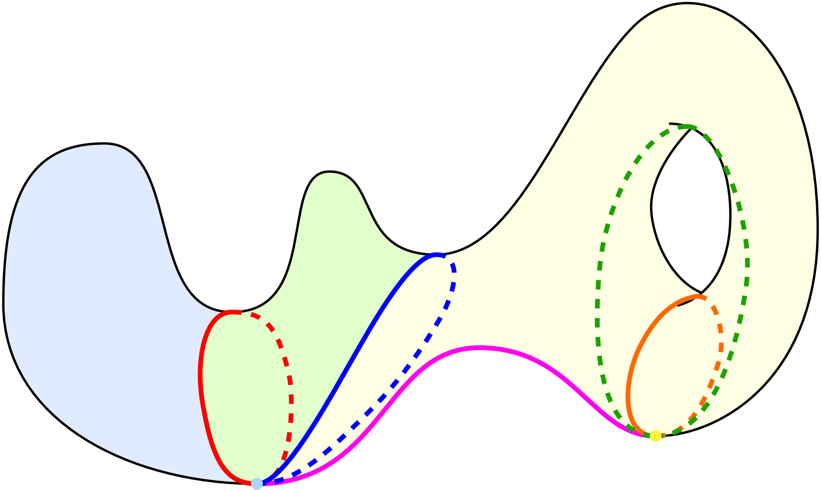

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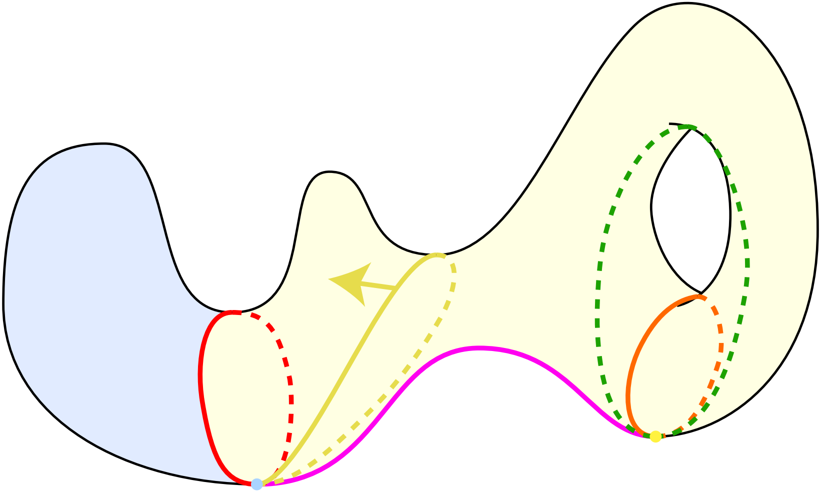

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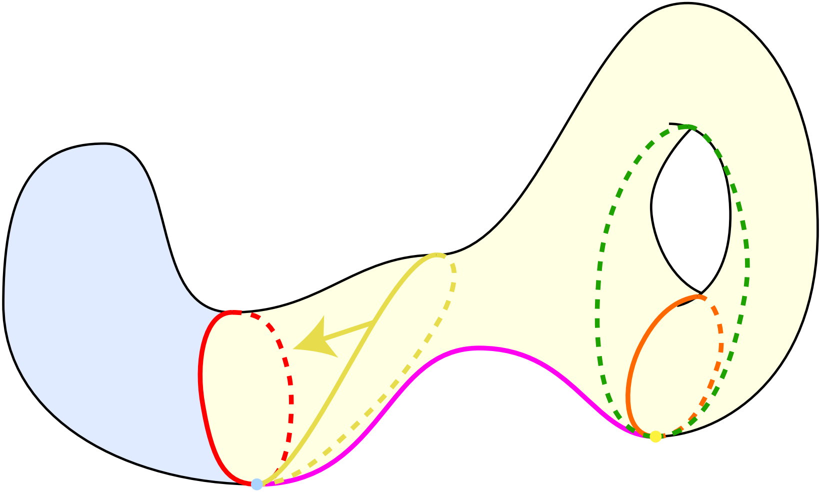

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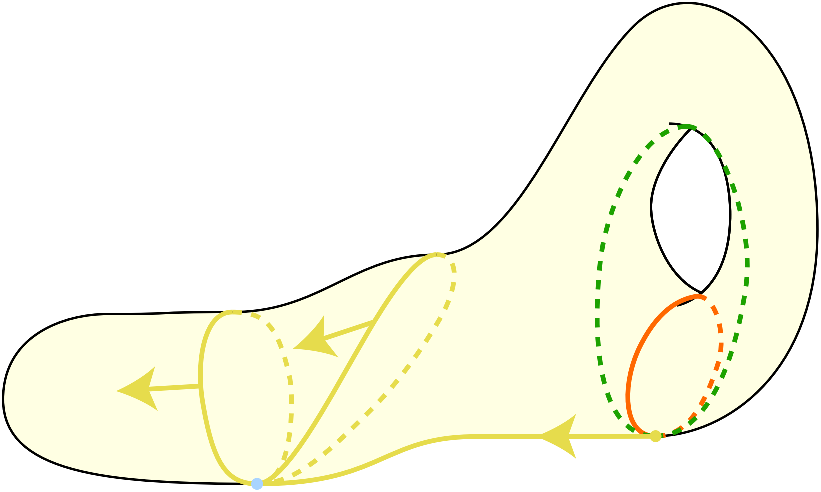
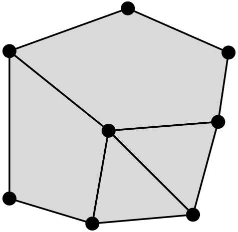


Discrete Morse theory [Forman, 1998]

Consider finite CW complex $\mathcal{K}$.



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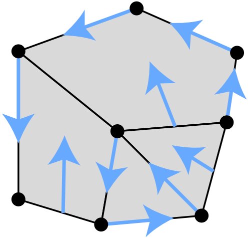
- Discrete vector field:
  - a set of *pairs* of cells (σ, τ), where σ is a regular facet of τ (arrow from σ to τ)
  - each cell in at most one pair



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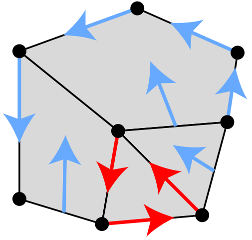
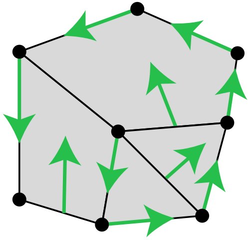
- Discrete *gradient* vector field:
  - no closed paths





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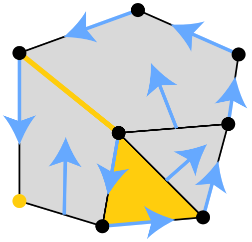

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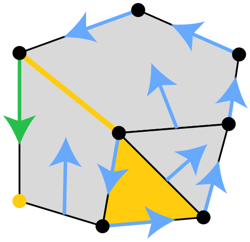

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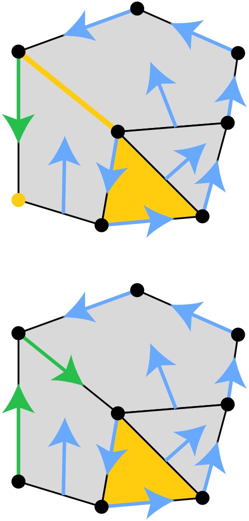

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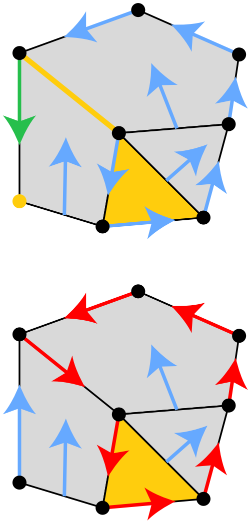

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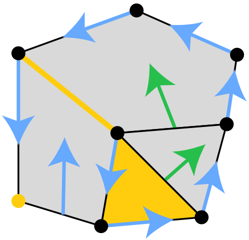

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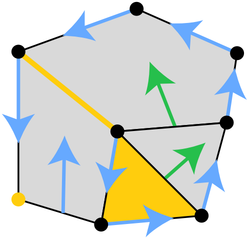

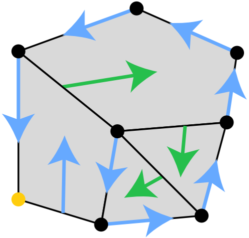

Discrete Morse function [Forman, 1998]

A function f : {cells of $\mathcal{K}$} $\to \mathbb{R}$ and a gradient vector field V_f such that:

- For all σ facet of τ:
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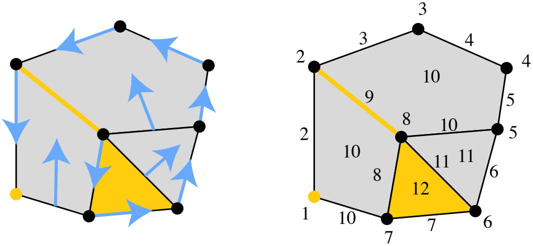
f is *consistent* with V_f.



Discrete Morse function (Pseudo-Morse function)

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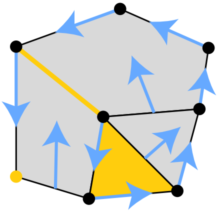
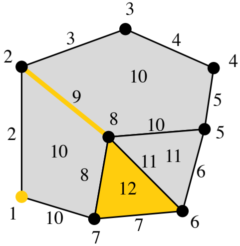
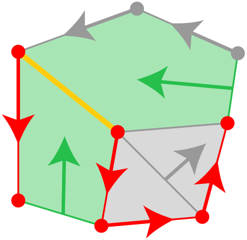




Attracting and repelling sets

A gradient vector field V enforces inequalities on cells

- Attracting set of a (critical) cell σ:
  all cells ρ with $g(\rho) \geq g(\sigma)$ for *any* g consistent with V
- Repelling set: analogously for $g(\rho) \leq g(\sigma)$



Back to our problem

Aim: Cancel critical points from pseudo-Morse function (g, V)

To do: Cancelation requires two steps:

- Reverse gradient vector field (which pairs?)
- Make function consistent to new vector field (how?)

Persistent homology [Edelsbrunner et al., 2002]

Investigate change of homology for growing spaces

Given:

- CW complex $\mathcal{K}$
- A (Pseudo-)Morse function (f, V)
- Critical cells $\{\rho_1, \dots, \rho_N\}$ of V such that $f(\rho_i) < f(\rho_{i+1})$

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- Level Subcomplex $\mathcal{K}(\rho)$:
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- Level Subcomplex $\mathcal{K}(\rho)$:
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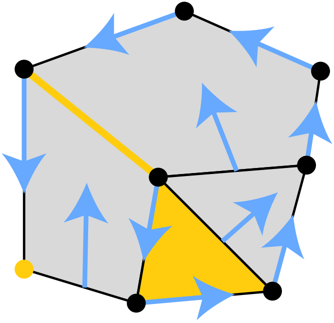
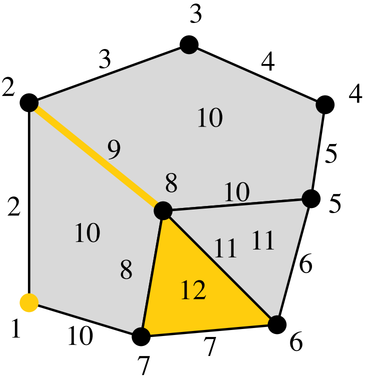
- Investigate change of homology of $\mathcal{K}(\rho_i)$ as i increases

Example: level subcomplexes





Example: Persistent homology

$\mathcal{K}(\rho_1)$

$\beta_0 = 1$

$\beta_1 = 0$

ρ_1 *positive* cell

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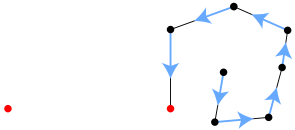

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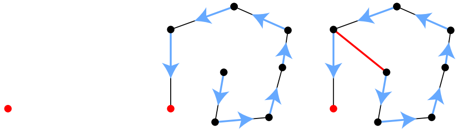

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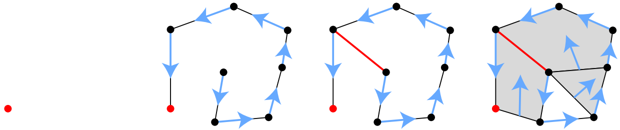

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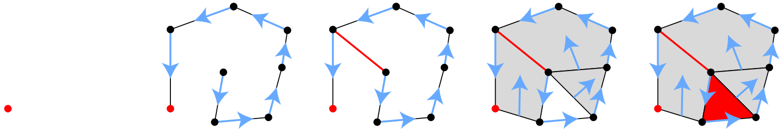

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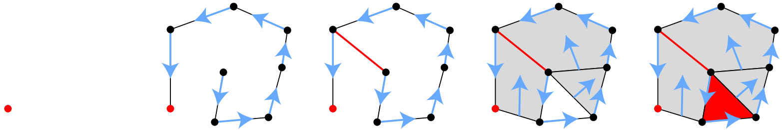

$\mathcal{K}(\rho_1)$

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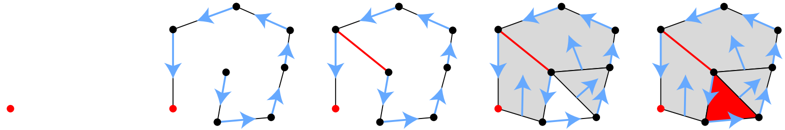

$\mathcal{K}(\rho_1)$	$\mathcal{K}(\rho_2)$	$\mathcal{K}(\rho_3)$
$\beta_0 = 1$	$\beta_0 = 1$	$\beta_0 = 1$
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ρ_1 *positive* cell	ρ_2 *positive* cell	ρ_3 *negative* cell

- (ρ_2, ρ_3) is a *persistence pair*:
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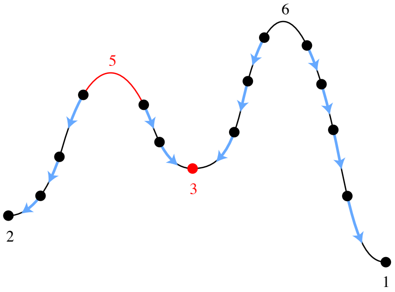
- $f(\rho_3) - f(\rho_2)$ is the *persistence* of (ρ_2, ρ_3)

Canceling a persistence pair



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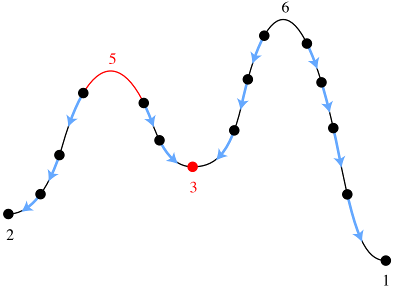

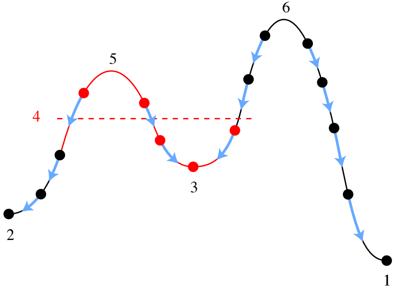

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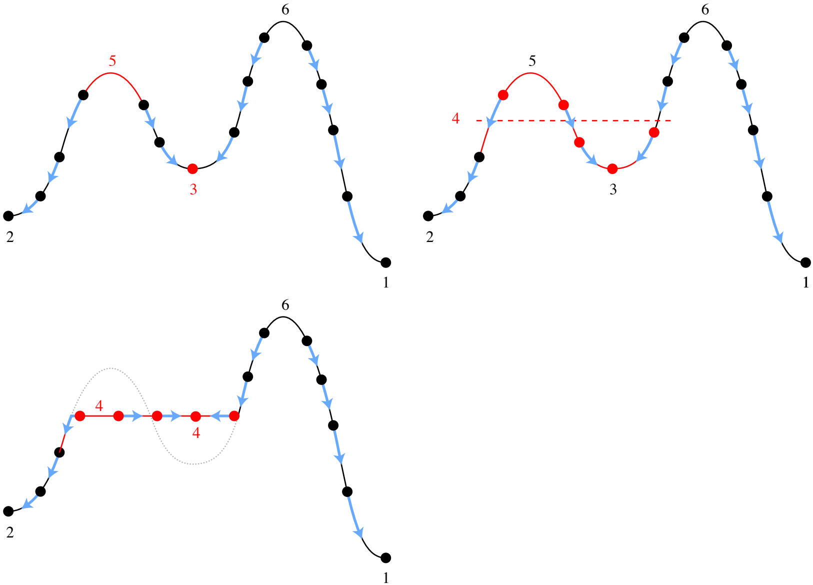

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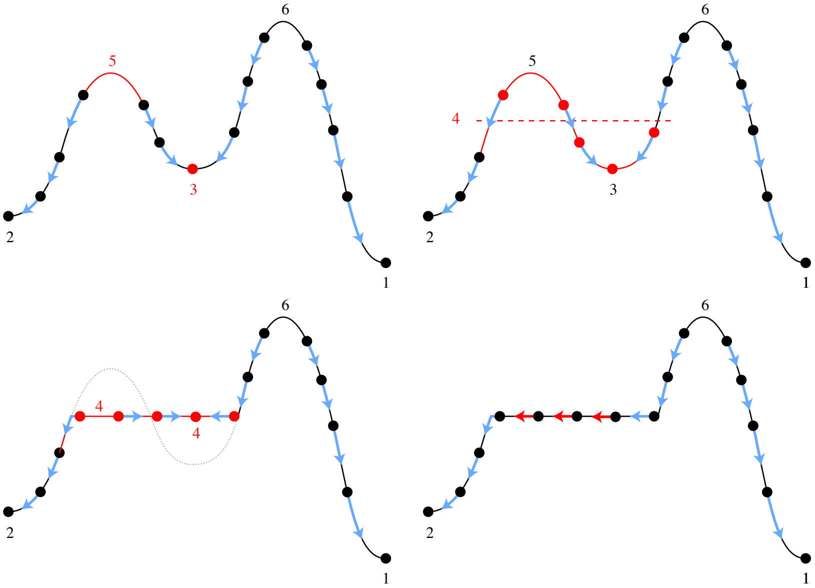
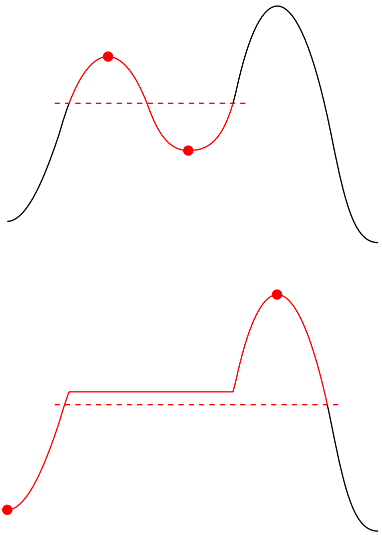


Canceling two nested persistence pairs



The algorithm

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  - Obtain V_{i+1} by reversing V_i along the path from τ to σ
- The function f_n is the desired function

Natural questions

- Can we cancel all low-persistence pairs?
  - Are the assumptions for canceling satisfied?
- If yes, how close can we stay to the original function?

Main result

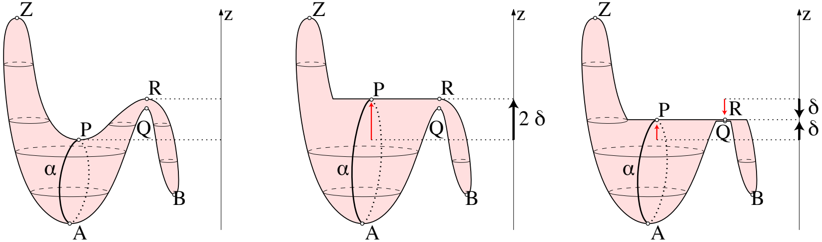
Let (g, V) be a discrete pseudo-Morse function on a combinatorial surface and $\delta > 0$.

Then there exists a pseudo-Morse function (g_δ, V_δ) with:

- $||g - g_\delta||_\infty < \delta$
- All persistence pairs of (g, V) with persistence $< 2\delta$ are canceled

This function achieves the minimal number of critical points.

Comparison with other methods

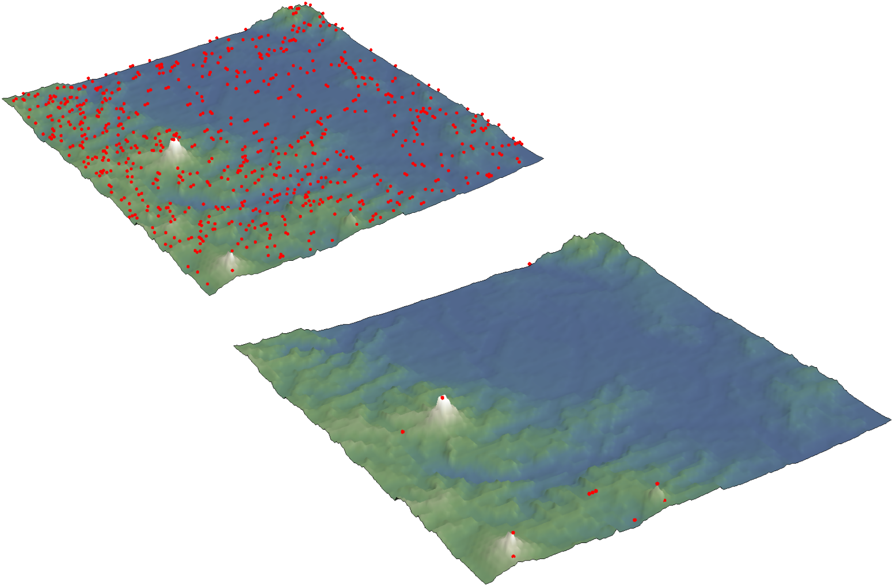


Original function

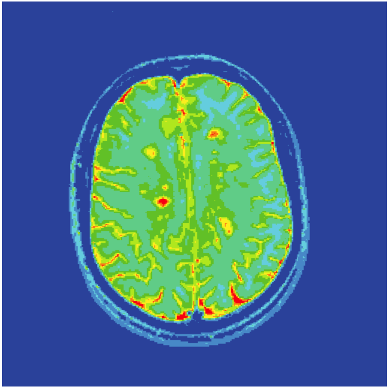
after [Edelsbrunner et al. 2006, Attali et al. 2009]

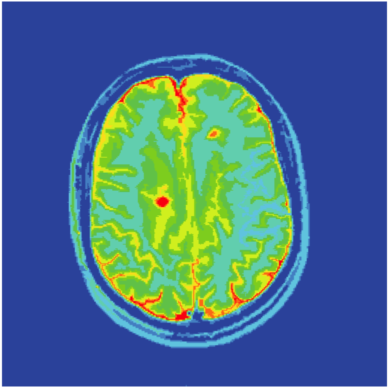
our method

Example: Simplification of terrain



Example: Medical images





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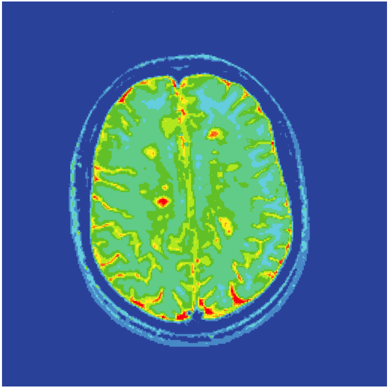 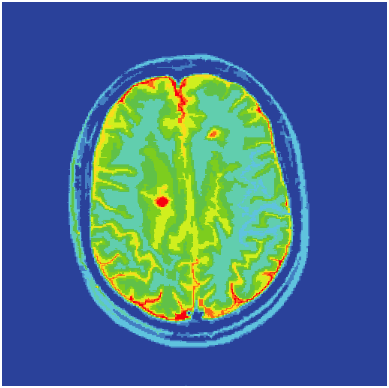

...thanks for your attention!