We have: From lecture/tutorial 3, we have that the amplitude at δt of a plane wave is

$$f(\delta t, x) = \exp\left(ikx/\delta x\right)\left[1 + C\exp(-ik) - C\right]$$

As we take time steps, we pick up a factor of

$$\left[1 + C\exp(-ik) - C\right]$$

at each timestep, so after n steps, we have that

$$f(t, x) = \exp\left(ikx/\delta x\right)\left[1 + C\exp(-ik) - C\right]^n$$

where $n = t/\delta t$. Unlike with fluid mechanics, the evolution here is linear, so we can find the full solution by taking the Fourier transform of the initial conditions, and multiplying the amplitude of each k-mode by the factor in brackets.

To find when the amplitude has been reduced by a factor of 2, we take the absolute value of the factor in brackets, set it equal to 1/2, and solve for k. Note that when solving for $x^n = a$ we can also solve for $x = a^{1/n}$. Also note that when solving $|x| = a$, we can equivalently solve $|x^2| = a^2$, which is easier when working with complex numbers.

$$\left|\,\left[1 + C\exp(-ik) - C\right]^n\,\right| = 1/2$$

$$|1 + C\cos(k) - C - iC\sin(k)| = 2^{-1/n}$$

$$(1 + C\cos(k) - C)^2 + C^2\sin(k)^2 = 2^{-1/2n}$$

Define $\alpha \equiv 2^{-1/2n}$ and replace $\sin^2$ by $1 - \cos^2$:

$$1 + C^2\cos^2(k) + C^2 + 2C\cos(k) - 2C^2\cos(k) - 2C + C^2 - C^2\cos^2(k) = \alpha$$

As in tutorial problem 3, the $\cos^2$ terms cancel and we're left with a first-order expression for $\cos(k)$:

$$\cos(k)(2C - 2C^2) + 1 + 2C^2 - 2C = \alpha$$

or

$$\cos(k) = \frac{\alpha - 1 + 2C - 2C^2}{2C(1 - C)}$$