Mathematics Project

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ISC

CMS Gomti Nagar Extension

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While I was preparing this project file, various information that I found helped me and I am glad that I was able to complete this project and understand many things. Though the preparation of project was an immense learning experience, I inculcated many personal qualities during this process like responsibility, punctuality, confidence and others.

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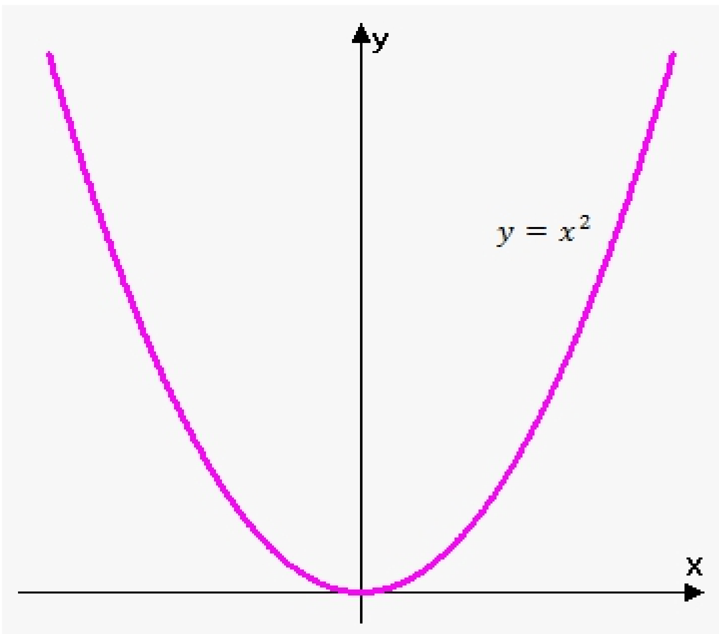
Introduction to Increasing and Decreasing Functions

Increasing and Decreasing function is one of the applications of derivatives. Derivatives are used to identify that the function is increasing or decreasing in a particular interval.

Generally, we know that if something is increasing, it is going upward and if something is decreasing it is going downwards. Hence if we talk graphically, if the graph of function is going upward then it is an increasing function and if the graph is going downward then it is a decreasing function.

Increasing and Decreasing Function



A function is said to be an increasing function if the value of y increases with the increase in x. As we can see from the above figure that at the right of the origin, the curve is going upward as we are going to the right so it is called Increasing Function.

A function is said to be a decreasing function if the value of y decreases with the increase in x. As above, in the left of the origin, the curve is going downward if we are moving from left to right.

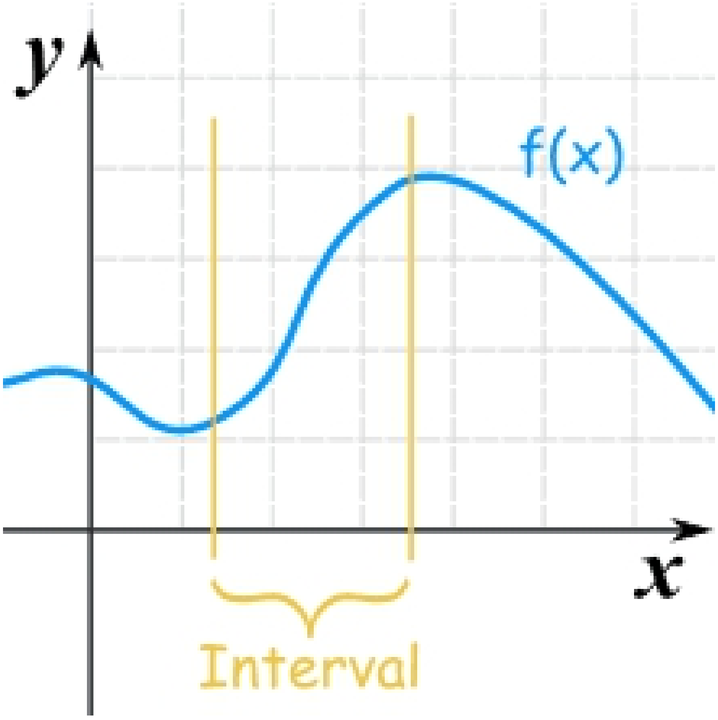
Definition of Increasing Function

This is the definition of a function which is increasing on an interval.

If there is a function $y = f(x)$

A function is increasing over an interval, if for every x1 and x2 in the interval,

$x1 < x2$, $f(x1) \leq f(x$



A function is strictly increasing over an interval, if for every x1 and x2 in the interval,

$x1 < x2$, $f(x1) < f(x2)$

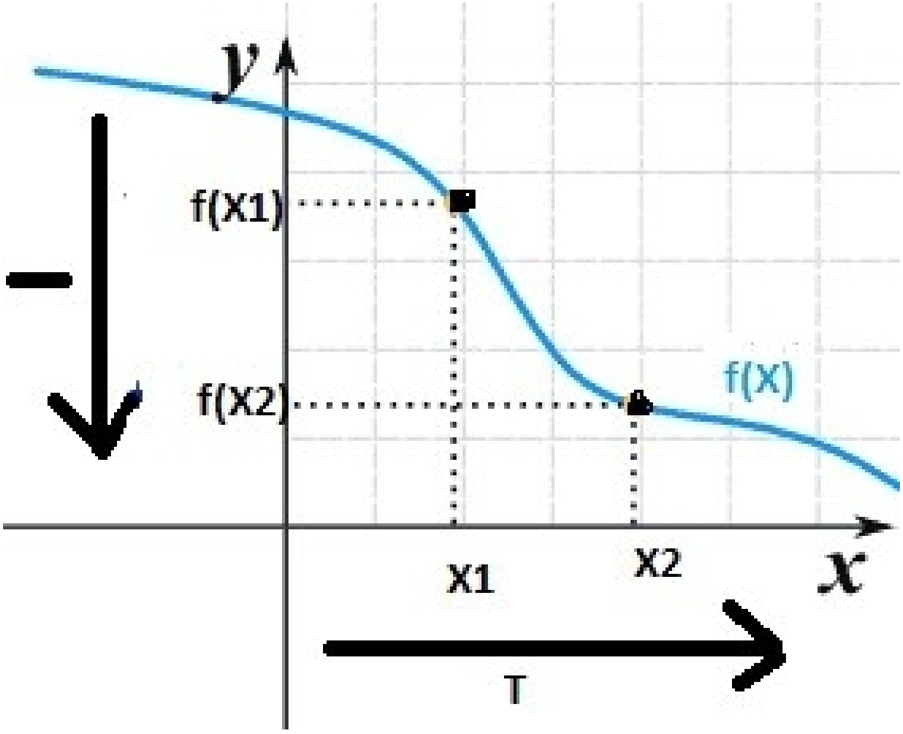
There is a difference of symbol in both the above increasing functions

Definition of Decreasing Function

If there is a function $y = f(x)$

A function is decreasing over an interval, if for every x1 and x2 in the interval,

$x1 < x2, f(x1) \geq f(x2)$



A function is strictly decreasing over an interval, if for every x1 and x2 in the interval,

$$x1 < x2, f(x1) > f(x2)$$

There is a difference of symbol in both the above decreasing functions.

Definition of Increasing and Decreasing function at a point

Let x0 be a point on the curve of a real valued function f.

Then f is said to be increasing, strictly increasing, decreasing or strictly decreasing at x0, if there exists an open interval I containing x0 such that f is increasing, strictly increasing, decreasing or strictly decreasing, respectively in I.

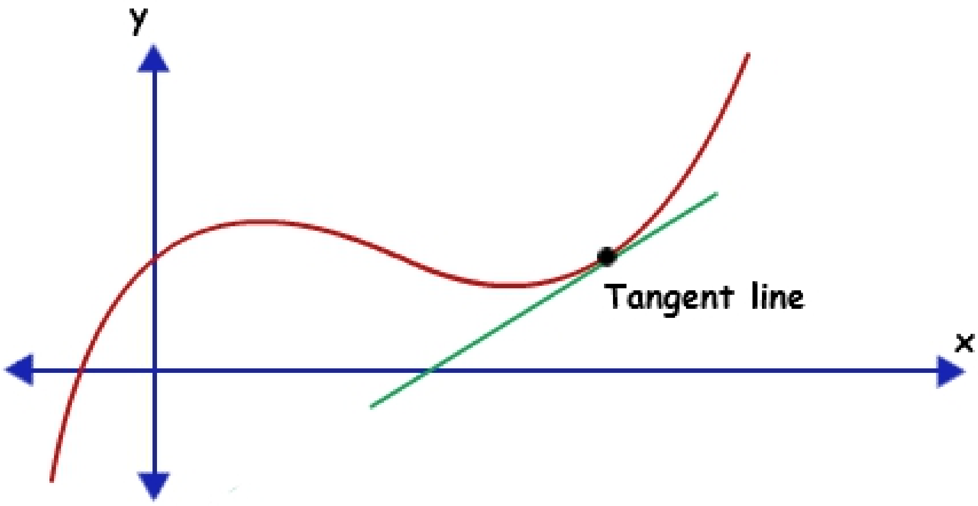
If there is a function f and interval I = (x0 – h, x0 + h), h > 0

It is said to be increasing at x0 if f is increasing in (xo - h, x0 + h)

$$x1 < x2 \text{ in } I \Rightarrow f(x1) \leq f(x2)$$

It is said to be strictly increasing at x0 if f is strictly increasing in (x0 - h, x0 + h)

$$x1 < x2 \text{ in } I \Rightarrow f(x1) < f(x2)$$



How derivatives are used to find whether the function is Increasing or Decreasing Function?

We can use the first derivative test to check whether the function is increasing or decreasing.

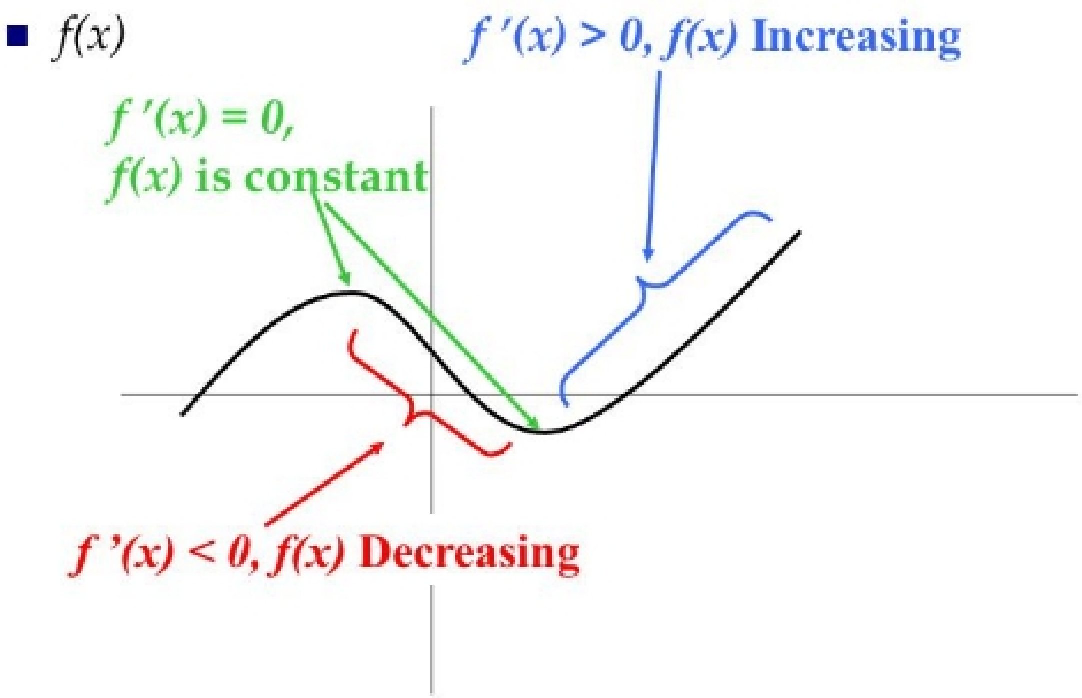
Theorem

Let f be continuous on [a, b] and differentiable on the open interval (a, b). Then

(a) If $f'(x) > 0$ for each $x \in (a, b)$ then f is increasing in interval [a, b]

(b) If $f'(x) < 0$ for each $x \in (a, b)$ then f is decreasing in interval [a, b]

(c) If $f'(x) = 0$ for each $x \in (a, b)$ then f is a constant function in [a, b]



This can be proved with the help of mean value theorem.

Proof:

Let x1, x2 ∈ [a, b] such that x1 < x2

Now we can prove it with the help of Mean value theorem, which says that there is a point c between x1 and x2 so that

Let x1, x2 ∈ [a, b] such that x1 < x2

Now we can prove it with the help of Mean value theorem, which says that there is a point c between x1 and x2 so that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

First Derivative Test for Local Extrema

If the derivative of a function changes sign around a critical point, the function is said to have a local (relative) extremum at that point. If the derivative changes from positive (increasing function) to negative (decreasing function), the function has a local (relative) maximum at the critical point. If, however, the derivative changes from negative (decreasing function) to positive (increasing function), the function has a local (relative) minimum at the critical point.

Example: If f(x) = x 4 − 8 x 2, determine all local extrema for the function.

f(x) has critical points at x = −2, 0, 2. Because f'(x) changes from negative to positive around −2 and 2, f has a local minimum at (−2,−16) and (2,−16). Also, f'(x) changes from positive to negative around 0, and hence, f has a local maximum at (0,0).

When this technique is used to determine local maximum or minimum function values, it is called the First Derivative Test for Local Extrema. Note that there is no guarantee that the derivative will change signs, and therefore, it is essential to test each interval around a critical point.

Example 1

If f(x) = x 4 − 8 x 2, determine all local extrema for the function.

f(x) has critical points at x = −2, 0, 2. Because f'(x) changes from negative to positive around −2 and 2, f has a local minimum at (−2,−16) and (2,−16). Also, f'(x) changes from positive to negative around 0, and hence, f has a local maximum at (0,0).

Example 2

If f(x) = sin x + cos x on [0, 2π], determine all local extrema for the function.

f(x) has critical points at x = π/4 and 5π/4. Because f′(x) changes from positive to negative around π/4, f has a local maximum at $(\pi/4, \sqrt{2})$.

Also f′(x) changes from negative to positive around 5π/4, and hence, f has a local minimum at $(5\pi/4, -\sqrt{2})$

Second Derivative Test for Local Extrema

The second derivative may be used to determine local extrema of a function under certain conditions. If a function has a critical point for which f′(x) = 0 and the second derivative is positive at this point, then f has a local minimum here. If, however, the function has a critical point for which f′(x) = 0 and the second derivative is negative at this point, then f has local maximum here. This technique is called Second Derivative Test for Local Extrema.
Three possible situations could occur that would rule out the use of the Second Derivative Test for Local Extrema:

(1) $f'(x) = 0$ and $f''(x) = 0$

(2) $f'(x) = 0$ and $f''(x)$ does not exist

(3) $f'(x)$ does not exist

Under any of these conditions, the First Derivative Test would have to be used to determine any local extrema.

Another drawback to the Second Derivative Test is that for some functions, the second derivative is difficult or tedious to find.

As with the previous situations, revert back to the First Derivative Test to determine any local extrema.

Example 1:

Find any local extrema of f(x) = x 4 − 8 x 2 using the Second Derivative Test.

f′(x) = 0 at x = −2, 0, and 2. Because f″(x) = 12 x 2 −16, you find that f″(−2) = 32 > 0, and f has a local minimum at (−2,−16); f″(2) = 32 > 0, and f has local maximum at (0,0); and f″(2) = 32 > 0, and f has a local minimum (2,−16).

Example 2: Find any local extrema of f(x) = sin x + cos x on [0,2π] using the Second Derivative Test.

f′(x) = 0 at x = π/4 and 5π/4. Because f″(x) = −sin x −cos x, you find that $f''(\pi/4) = -\sqrt{2}$ and f has a local maximum at $(\pi/4, \sqrt{2})$. Also, $(5\pi/4) = \sqrt{2}$ and f has a local minimum at $(5\pi/4, -\sqrt{2})$

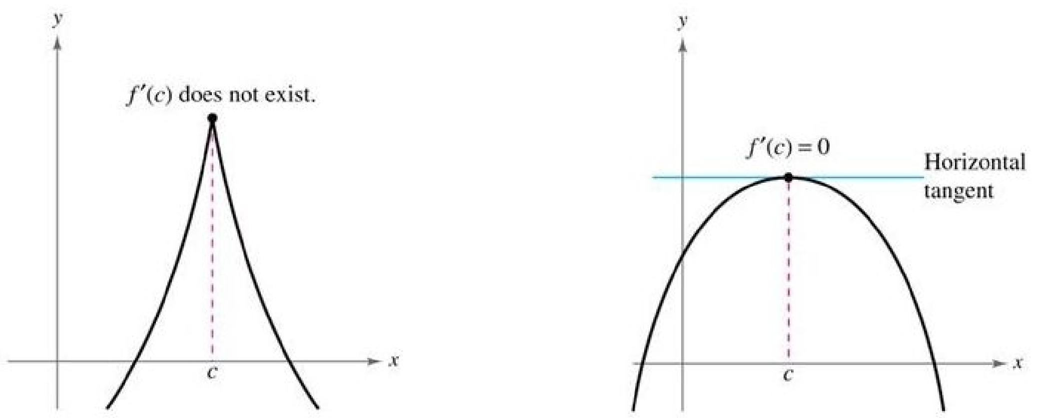
Important:

1. f is strictly increasing in (a, b) if f ' (x) > 0, ∀ x ∈ [a, b]
2. f is strictly decreasing in (a, b) if f ' (x) < 0, ∀ x ∈ [a, b]
3. f is increasing or decreasing on R if it is increasing or decreasing in every interval of R

Definition of Critical Numbers

The Critical Numbers for a function f are those numbers c in the domain of f for which f ' (c) = 0 or does not exists.

A critical point is a point whose x coordinate is the critical number c and the y coordinate is the f(c).



c is a critical number of *f*.

What are intervals of increase and decrease?

Interval is basically all the numbers between given two numbers.

If we talk about curve, we can say the portion of curve which is coming in between the two given numbers on the x-axis is the required interval.

Calculation of intervals of increase or decrease

To calculate the intervals of increase or decrease function, we need to follow some steps:

First of all, we have to differentiate the given function.

Then solve the first derivative as equation to find the value of x.

The first derivative: f'(x) = 0.

Form open intervals with the values of the x which we got after solving the first derivative and the points of discontinuity.

Take a value from every interval and find the sign they have in the first derivative.

If f'(x) > 0 is increasing.

If f'(x) < 0 is decreasing.

Examples of Increasing and Decreasing Functions

Example 1:

Find the intervals of increase and decrease of the function $f(x) = x3 - 3x + 2$.

Solution:

$f(x) = x3 - 3x + 2$.

Find the derivative of f(x)

$f'(x) = 3x2 -3$

Solve the derivative as $f'(x) = 0$

$3x2 -3 = 0$

$3x2 = 3$

$x2 = 1$

$x = -1$ and $x = 1$

Now we have to check the sign of first derivative in every interval to find the function is increasing or decreasing.

On the interval $(-\infty, -1)$,

Let $x = -2$

$f'(-2) = 3(-2)2 -3$

f’(−2) = 12 −3

f’(−2) = 9 > 0

On the interval (−1, 1),

Let x = 0

f'(0) = 3(0)2 −3

f'(0) = −3 < 0

On the interval (1, ∞),
Let x = 2

f'(2) = 3(2)2 −3

f'(2) = 12 −3

f'(2) = 9 > 0

Hence,

The function is Increasing in the intervals: (−∞, −1) and
(1, ∞)

The function is decreasing in the interval: (−1, 1)

Example 2:

Find the intervals in which the function f(x) = 2x3 – 3x2 -36x +7 is

1) Strictly Increasing

2) Strictly Decreasing.

Solution:

A function is strictly increasing or decreasing on an open interval where its derivative is positive or negative.

Given:

$f(x) = 2x3 – 3x2 -36x +7$

The derivative of f(x) will be

$f' (x) = 6x2 – 6x -36$

$= 6(x2 – x -6)$

$= 6(x2 – 3x + 2x – 6)$

$= 6[x(x – 3) + 2(x – 3)]$

$= 6[(x + 2) (x – 3)]$

Now we know that $f' (x) = 0$

So $x = -2, 3$

Now -2 and 3 are dividing the number line in the three disjoint intervals

$(-\infty,-2)$

$(-2, 3)$

$(3, \infty)$

Now we have to check the sign of first derivative in every interval to find the function is increasing or decreasing.

On the interval $(-\infty, -2)$,
Let $x = -3$

$f'(-3) = 6[(x + 2)(x - 3)]$

$f'(-3) = 6[(-3 + 2)(-3 - 3)]$

$f'(-3) = 6[(-1)(-6)]$

$= 6(6)$
$= 36 > 0$
On the interval $(-2, 3)$,
Let $x = 0$
$f'(0) = 6[(0 + 2)(0 - 3)]$
$f'(0) = 6(-6)$
$= -36 < 0$
On the interval $(3, \infty)$,
Let $x = 4$
$f'(4) = 6[(4 + 2)(4 - 3)]$
$f'(4) = 6[6.1]$
$= 36 > 0$

Hence,

The function is strictly Increasing in the intervals:
$(-\infty, -2)$ and $(3, \infty)$

The function is strictly decreasing in the interval:
$(-2, 3)$

Using Vectors to find the area of a trianle and parallelogram

The vectors cannot be added algebraically, like scalar quantities. The angle between the two vectors plays an important role in addition of vectors. Similarly, angle between two vectors plays an important role when the two vectors are multiplied together.

Consider two quantities having magnitudes ‘3’ and ‘2’ respectively. If the quantities are scalars then then the result of their multiplication cannot be anything other than 6. We are at liberty to express these product as 3.2 = 6 or 32 = 6. If the two quantities are vector quantities, the result of their multiplication can have any value lying in between -6 and +6. The result depends upon the angle between them. Moreover we are not free to put a dot (.) or a cross (X) in between.

Multiplication of Vector by a Scalar

Let vector a is multiplied by a scalar m. If m is a positive quantity, only magnitude of the vector will change by a factor ‘m’ and its direction will remain same. If m is a negative quantity the direction of the vector will be reversed.

Multiplication of a Vector by a Vector

There are two ways in which two vectors can be multiplied together.

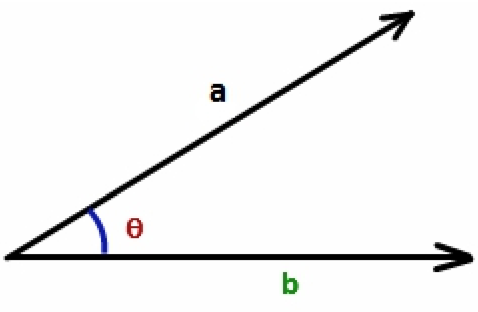
Dot Product or Scalar Product

The dot product of two vectors a and b is defined as the product of their magnitudes and the cosine of the smaller angle between the two.

It is written by putting a dot (.) between two vectors. The result of this product does not possess any direction. So, it is a scalar quantity. Hence it is also called a scalar product.

$$\vec{a}\vec{b} = |\vec{a}||\vec{b}|\cos\theta = ab\cos\theta$$

where a and b are the magnitudes of the respective vectors and θ is the angle between them. The final product is a scalar quantity. If two vectors are mutually perpendicular then θ = 900 and cos 90 = 0, Hence, their dot product is zero.



a.b = |a| |b| Cosθ

Cross Product or Vector Product

Cross product of two vectors $\vec{A}$ and $\vec{B}$ is defined as a single vector $\vec{C}$ whose magnitude is equal to the product of their individual magnitudes and the sine of the smaller angle between them and is directed along tyhe normal to the plane containing $\vec{A}$ and $\vec{B}$.

$$\vec{a} \times \vec{b} = \vec{c} = ab\sin\theta\hat{n}$$

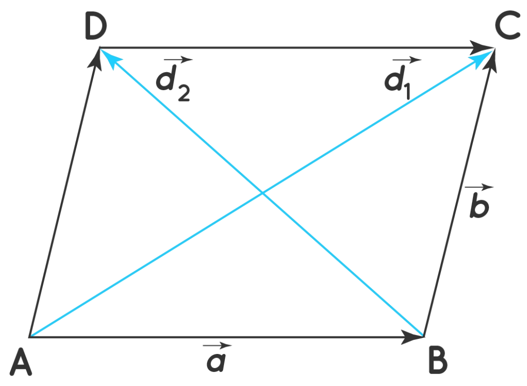
Here $\hat{n}$ is the unit vector in a direction perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. Cross product of two vectors being a vector quantity is also known as vector product.

To specify the sense of the vector c, refer to the figure given below.

Imagine rotating a right hand screw whose axis is perpendicular to the plane formed by vectors a and b so as to turn it from vectors a to b trough the angle between them. Then the direction of advancement of the screw gives the direction of the vector product vectors $\vec{A}$ and $\vec{B}$.

Area of Parallelogram in Vector Form

The area of the parallelogram can be calculated using different formulas even when either the sides or the diagonals are given in the vector form. Consider a parallelogram ABCD as shown in the figure below,



Area of parallelogram in vector form using the adjacent sides is,

$|\vec{a} \times \vec{b}|$

where, $\vec{a}$ and $\vec{b}$ are vectors representing two adjacent sides.

Here,

$\vec{a} + \vec{b} = \vec{d_1} \rightarrow$ i) and,

$\vec{b} + (-\vec{a}) = \vec{d_2}$

or, $\vec{b} - \vec{a} = \vec{d_2} \rightarrow$ ii)

$\Rightarrow \vec{d_1} \times \vec{d_2} = (\vec{a} + \vec{b})(\vec{b} - \vec{a})$

$= \vec{a} \times (\vec{b} - \vec{a}) + \vec{b} \times (\vec{b} - \vec{a})$

$= \vec{a} \times \vec{b} - \vec{a} \times \vec{a} + \vec{b} \times \vec{b} - \vec{b} \times \vec{a}$

Since $\vec{a} \times \vec{a} = 0$, and $\vec{b} \times \vec{b} = 0$

$\Rightarrow \vec{a} \times \vec{b} - 0 + 0 - \vec{b} \times \vec{a}$

Since $\vec{a} \times \vec{b} = - \vec{b} \times \vec{a}$,

$\vec{d_1} \times \vec{d_2} = \vec{a} \times \vec{b} - (-(\vec{a} \times \vec{b}))$

$= 2(\vec{a} \times \vec{b})$

Therefore, area of parallelogram when diagonals are given in vector form

$= 1/2 \, |(\vec{d_1} \times \vec{d_2})|$

where, $\vec{d_1}$ and $\vec{d_2}$ are diagonals.

Here, we know that area of parallelogram is equal to half of the product of diagonals into the sine of the angle between the two sides, which is the same as given in the above derivation, to further prove it:

We take an example:

The angle between any two sides of a parallelogram is 30 degrees. If the length of the two parallel sides is 4 units and 6 units respectively, then find the area.

Using Cross Product:
= 4 . 6 . sin (30^0)
= 12 sq units

Using regular mensuration:
Area of parallelogram = Base x Height

Let base be the side with 4 units

Using trignometry, we can calculate the height by : 6.sin(30^0)

Hence, area = 4.6.sin(30^0) = 12 sq units (Proved)
Similarly, the area of a triangle can also be calculated since it is just half of the area of the parallelogram formed by the two vectors.

Bibliography

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