Vlasov equation with normalization derivation

Jeff Ye

April 2024

1 Introduction

Start from the Vlasov-Poisson equation:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \dot{v}\frac{\partial f}{\partial v} = 0 \tag{1}$$

- f represents particle distribution function in 1D.
- t represents time.
- x represents spatial position.
- v represents velocity, indicating the rate at which f changes with respect to x.
- $\dot{v}$ represents the time derivative of velocity, which is the acceleration.

The acceleration $\dot{v}$ of charged particles (such as electrons) in an electric field can also be expressed as:

$$\dot{v} = \frac{\delta E q}{m_e} \tag{2}$$

Consider the particle distribution function f can be expanded as:

$$f = f_M + \delta f \tag{3}$$

where:

- f_M is the Maxwellian distribution.
- δf represents small perturbations to the distribution.

Substituting the expanded distribution back into the motion equation, we get:

$$\frac{\partial(f_M + \delta f)}{\partial t} + v\frac{\partial(f_M + \delta f)}{\partial x} + \dot{v}\frac{\partial(f_M + \delta f)}{\partial v} = 0$$

$$\frac{f_M}{\partial t} + v\frac{f_M}{\partial x} + \frac{q}{m_e}E_0\frac{f_M}{\partial_v} + \frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial x} + \frac{q}{m_e}(\delta E\frac{f_M}{\partial_v} + E_0\frac{\partial \delta f}{\partial_v} + \delta E\frac{\partial \delta f}{\partial v}) = 0$$

with zero net charge and consider linear only $\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial x} + \frac{\delta E q}{m_e}\frac{\partial f_M}{\partial_v} = 0$

(4)

$$\text{where } \delta f = g(v,t)e^{ikx}, \quad g(v,t) = \sum_{n=0}^{\infty} g_n \psi_n \left(\frac{v-u}{\alpha}\right) \tag{5}$$

$$E = E_k e^{ikx} + E_0 \tag{6}$$

The goal is to derive the perturbations particle distribution function $\partial \delta f$over time t using Hermite polynomials by applying $\int_{-\infty}^{\infty} H_m(v-ve)$ to all terms, where v_e is the initial speed set for electron that different from main flow.

2 Initial simplification

2.1 For $\frac{\delta E q}{m_e}\frac{\partial f_M}{\partial_v}$

$$\begin{aligned}
\nabla \cdot \delta E &= \frac{\rho}{\epsilon_0} = -q\frac{\int \delta f \, dv}{\epsilon_0} \\
&= \frac{\partial \delta E}{\partial x} = ik\delta E (\text{1D Fourier Space}) \\
ik\delta E &= -q\frac{\int \delta f \, dv}{\epsilon_0} \\
\delta E &= \frac{\int -q\delta f \, dv}{\epsilon_0 ik}
\end{aligned} \tag{7}$$

Second part:

$$\frac{\partial f_M}{\partial v} = \frac{\partial e^{-v^2}}{\partial v} \tag{8}$$

Combine the two parts:

$$\begin{aligned}
\frac{\delta E q}{m_e}\frac{\partial f_M}{\partial_v} &= -\frac{q\int q\delta f \, dv}{\epsilon_0 ik m_e}\frac{\partial e^{-v^2}}{\partial v} \\
&= (2v)e^{-v^2}\frac{q\int q\delta f dv}{m_e \epsilon_0 ik}
\end{aligned} \tag{9}$$

2.2 For $v\frac{\partial \delta f}{\partial x}$

$$\begin{aligned}
\text{Fourier Space } \Rightarrow \frac{\partial \delta f}{\partial x} &= ik\delta f \text{ where } \delta f \sim e^{ikx} \\
v\frac{\partial \delta f}{\partial x} &= ikv\delta f
\end{aligned} \tag{10}$$

3 Hermite polynomials

3.1 Definition and Properties

The Hermite polynomials can be defined through the Rodrigues' formula:

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$$

3.2 Orthogonality

They are orthogonal with respect to the weight function e^{-x^2}, which is expressed as:

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = \sqrt{\pi} 2^n n! \delta_{mn} \tag{11}$$

where δ_{mn} is the Kronecker delta.

3.3 Recurrence relations

Hermite polynomials satisfy recurrence relations that are instrumental for computational and analytical purposes. These relations are expressed as follows:

For the Hermite polynomials $H_n(v)$:

$$\begin{aligned} H_{m+1}(v) &= 2vH_m(v) - 2mH_{m-1}(v) \\ 2vH_m(v) &= H_{m+1}(v) + 2mH_{m-1}(v) \\ vH_m(v) &= \frac{1}{2}H_{m+1}(v) + mH_{m-1}(v) \end{aligned} \tag{12}$$

3.4 Apply Hermite polynomials

Let δf be defined and set u = v-v_e as:

$$\delta f = \sum_{n=0}^{\infty} H_n(u) \frac{\delta f_n(t)}{\sqrt{\pi 2^n n!}} e^{-u^2} \tag{13}$$

also apply $\int_{-\infty}^{\infty} H_m(u) du$ to all terms

where $H_n(u)$ and $H_m(u)$ are Hermite polynomials, $f_n(t)$ and $f_m(t)$ are time-dependent perturbations = $\partial\delta f_{n;m}(t)$.

3.4.1 For $\frac{\partial \delta f}{\partial t}$

$$\begin{aligned} \int_{-\infty}^{\infty} H_m(u) \frac{\partial \delta f}{\partial t} du &= \int_{-\infty}^{\infty} H_m(u) \sum_{n=0}^{\infty} H_n(u) \frac{f_n(t)}{\partial t \sqrt{\pi 2^n n!}} e^{-u^2} du \\ &= \sum_{n=0}^{\infty} \frac{f_n(t)}{\partial t \sqrt{\pi 2^n n!}} \int_{-\infty}^{\infty} H_n(u) H_m(u) e^{-u^2} du \\ &= \sum_{n=0}^{\infty} \frac{f_n(t)}{\partial t \sqrt{\pi 2^n n!}} \sqrt{\pi} 2^n n! \delta_{mn} \\ &= \frac{f_m(t)}{\partial t \sqrt{\pi 2^m m!}} \sqrt{\pi} 2^m m! \\ &= \frac{f_m(t)}{\partial t} \sqrt{2^m m!} \end{aligned} \tag{14}$$

3.4.2 For $v\frac{\partial \delta f}{\partial x}$

$$
\begin{aligned}
\int_{-\infty}^{\infty} H_m(u) v \frac{\partial \delta f}{\partial x} du &= \int_{-\infty}^{\infty} H_m(u) ikv \sum_{n=0}^{\infty} H_n(u) \frac{f_n(t)}{\sqrt{\pi 2^n n!}} e^{-u^2} du \\
&= ik \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} H_m(u) u H_n(u) \frac{f_n(t)}{\sqrt{\pi 2^n n!}} \ e^{-u^2} du \\
&+ ik \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} H_m(u) v_e H_n(u) \frac{\delta f_n(t)}{\sqrt{\pi 2^n n!}} \ e^{-u^2} du \\
&= ik \sum_{n=0}^{\infty} \frac{f_n(t)}{\sqrt{\pi 2^n n!}} \int_{-\infty}^{\infty} H_n \left(\frac{1}{2} H_{m+1}(u) + m H_{m-1}(u) \right) e^{-u^2} du \\
&+ v_e ik \sum_{n=0}^{\infty} \frac{f_n(t)}{\sqrt{\pi 2^n n!}} \int_{-\infty}^{\infty} H_m(u) H_n(u) e^{-u^2} du \\
&= ik \sum_{n=0}^{\infty} \frac{f_n(t)}{\sqrt{\pi 2^n n!}} \left(\frac{1}{2} \right) \int_{-\infty}^{\infty} H_n H_{m+1} e^{-u^2} du + m \int_{-\infty}^{\infty} H_n H_{m-1} e^{-u^2} du) \\
&+ v_e ik \sum_{n=0}^{\infty} \frac{f_n(t)}{\sqrt{\pi 2^n n!}} \sqrt{\pi} 2^n n! \delta_{mn} \\
&= ik \left(\frac{f_{m+1}(t)}{\sqrt{\pi 2^{m+1}(m+1)!}} \frac{1}{2} \sqrt{\pi} 2^{m+1} (m+1)! + \frac{f_{m-1}(t)}{\sqrt{\pi 2^{m-1}(m-1)!}} m \sqrt{\pi} 2^{m-1} (m-1)! \right) \\
&+ v_e ik \frac{f_m(t)}{\sqrt{\pi 2^m m!}} \sqrt{\pi} 2^m m! \\
&= ik \left(f_{m+1}(t) \sqrt{2^{m-1}(m+1)!} + f_{m-1}(t) m \sqrt{2^{m-1}(m-1)!} + v_e f_m(t) \sqrt{2^m m!} \right)
\end{aligned}
\tag{15}
$$

3.4.3 Calculation of the special term Integral

Consider the integral involving Hermite polynomials $H_0(v)$ under the condition $n = m = 0$. The Hermite polynomial $H_0(v)$ is defined as:

$$H_0(v) = 1$$

Therefore, the integral we need to evaluate simplifies to:

$$\int_{-\infty}^{\infty} H_0(v) H_0(v) v e^{-2v^2} \, dv = \int_{-\infty}^{\infty} v e^{-2v^2} \, dv$$

The function v is an odd function and e^{-2v^2} is an even function. The product of an odd function and an even function is an odd function, and the integral of an odd function over a symmetric interval from $-\infty$ to ∞ is zero. Therefore:

$$\int_{-\infty}^{\infty} v e^{-2v^2} \, dv = 0$$

3.4.4 For $\dot{v}\frac{\partial f_M}{\partial v}$

$$\begin{aligned}
\int q\partial\delta f dv &= \int q\sum_{n=0}^{\infty} H_n(v)\frac{f_n(t)}{\sqrt{\pi}2^n n!}e^{-u^2}dv \\
&= q\int\sum_{n=0}^{\infty} H_0 H_n(v)\frac{f_n(t)}{\sqrt{\pi}2^n n!}e^{-u^2}dv \\
&= q\int\sum_{n=0}^{\infty}\sqrt{\pi}2^n n!\frac{f_n(t)}{\sqrt{\pi}2^n n!}\delta_{mn}(m=0) \\
&= qf_0
\end{aligned} \tag{16}$$

$$\begin{aligned}
\frac{\delta E q}{m_e}\frac{\partial f_M}{\partial_v} &= (2v)e^{-v^2}\frac{q\int q\delta f dv}{\epsilon_0 m_e ik} \\
\int_{-\infty}^{\infty} H_m(v)\dot{v}\frac{\partial f_M}{\partial v} &= \int_{-\infty}^{\infty} H_m(v)(-(2v)e^{-v^2}\frac{q^2 f_0}{\epsilon_0 m_e ik}) \\
&= \frac{q^2 f_0}{\epsilon_0 m_e ik}\int_{-\infty}^{\infty} H_m(u)(2v)e^{-v^2} \\
&= \frac{q^2 f_0}{\epsilon_0 m_e ik}\int_{-\infty}^{\infty} H_m(u)H_1 e^{-v^2} \\
&= \frac{q^2 f_0}{\epsilon_0 m_e ik}\sqrt{\pi}2^n n!\delta_{1,m}(n=1) \\
&= \frac{2q^2 f_0\sqrt{\pi}}{\epsilon_0 m_e ik}\delta_{1,m}
\end{aligned} \tag{17}$$

4 Normalizing

This section details the specific process of normalizing a typical physical equation that involves charge, velocity, and time:

$$\begin{aligned}
\frac{\partial f_m(t)}{\partial t}\sqrt{2^m m!} = &-ik(\delta f_{m+1}(t)\sqrt{2^{m-1}(m+1)!} - \delta f_{m-1}(t)m\sqrt{2^{m-1}(m-1)!} \\
&- v_e f_m(t)\sqrt{2^m m!}) + \frac{2q^2\sqrt{\pi}f_0}{\epsilon_0 m_e ik}\delta_{1,m}
\end{aligned} \tag{18}$$

4.1 Normalization Strategy

- Normalize time t using a characteristic time scale T such that $T = \frac{1}{\omega_{\text{pe}} v_{th}}$, where ω_{pe} is the plasma frequency $\omega_{\text{pe}} = \sqrt{\frac{n_e e^2}{m^* \varepsilon_0}}$, [rad/s]; v_{th} is the thermal velocity of the electron set as 1 in this case then $n_e = \sqrt{\pi}$.
- Define dimensionless time τ as $\tau = \omega_{\text{pe}} v_{th} t$.

- Adjust the charge terms to match the new time scale and simplify the expression.

After applying the normalization:

$$\frac{\partial f_m(t)}{\partial \tau} = -\frac{ik}{\omega_{\mathrm{pe}}}\left(f_{m+1}(t)\sqrt{\frac{m+1}{2}} + \sqrt{\frac{m}{2}}f_{m-1}(t) + v_e f_m(t)\right) + \frac{q^2\sqrt{2\pi}f_0}{\epsilon_0 m_e ik\omega_{\mathrm{pe}}}\delta_{1,m}$$
$$\frac{\partial f_m(t)}{\partial \tau} = -i\sqrt{2}C\left(f_{m+1}(t)\sqrt{\frac{m+1}{2}} + \sqrt{\frac{m}{2}}f_{m-1}(t) + v_e f_m(t)\right) + \frac{1}{iC}f_0\delta_{1,m} \quad (19)$$

where $C = \frac{kv_{th}}{\omega_{\mathrm{pe}}}$,dimensionless.

5 Final result

$$\frac{\partial f_m(t)}{\partial t} = -if_{m+1}(t)\sqrt{\frac{m+1}{2}} - i\sqrt{\frac{m}{2}}f_{m-1}(t) - iv_e f_m(t) - Cf_0\delta_{1,m} \quad (20)$$