Stokes' theorem

In vector calculus, and more generally differential geometry, **Stokes' theorem** (sometimes spelled **Stokes's theorem**, and also called the **generalized Stokes theorem** or the **Stokes–Cartan theorem**[1]) is a statement about the integration of differential forms on manifolds, which both simplifies and generalizes several theorems from vector calculus. Stokes' theorem says that the integral of a differential form ω over the boundary of some orientable manifold Ω is equal to the integral of its exterior derivative $d\omega$ over the whole of Ω, i.e.,

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega.$$

Stokes' theorem was formulated in its modern form by Élie Cartan in 1945,[2] following earlier work on the generalization of the theorems of vector calculus by Vito Volterra, Édouard Goursat, and Henri Poincaré.[3][4]

This modern form of Stokes' theorem is a vast generalization of a classical result that Lord Kelvin communicated to George Stokes in a letter dated July 2, 1850.[5][6][7] Stokes set the theorem as a question on the 1854 Smith's Prize exam, which led to the result bearing his name. It was first published by Hermann Hankel in 1861.[7][8] This classical Kelvin–Stokes theorem relates the surface integral of the curl of a vector field $\mathbf{F}$ over a surface in Euclidean three-space to the line integral of the vector field over its boundary: Let $\gamma: [a, b] \to \mathbf{R}^2$ be a piecewise smooth Jordan plane curve. The Jordan curve theorem implies that γ divides $\mathbf{R}^2$ into two components, a compact one and another that is non-compact. Let D denote the compact part that is bounded by γ and suppose $\psi: D \to \mathbf{R}^3$ is smooth, with $S := \psi(D)$. If Γ is the space curve defined by $\Gamma(t) = \psi(\gamma(t))$[note 1] and $\mathbf{F}$ is a smooth vector field on $\mathbf{R}^3$, then:[9][10][11]

$$\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{\Gamma} = \iint_{S} \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

This classical statement, along with the classical divergence theorem, the fundamental theorem of calculus, and Green's theorem are simply special cases of the general formulation stated above.

Contents

Introduction

The fundamental theorem of calculus states that the integral of a function f over the interval $[a, b]$ can be calculated by finding an antiderivative F of f:

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Stokes' theorem is a vast generalization of this theorem in the following sense.

- By the choice of F, $\frac{dF}{dx} = f(x)$. In the parlance of differential forms, this is saying that $f(x)\, dx$ is the exterior derivative of the 0-form, i.e. function, F: in other words, that $dF = f\, dx$. The general Stokes theorem applies to higher differential forms ω instead of just 0-forms such as F.
- A closed interval $[a, b]$ is a simple example of a one-dimensional manifold with boundary. Its boundary is the set consisting of the two points a and b. Integrating f over the interval may be generalized to integrating forms on a higher-dimensional manifold. Two technical conditions are needed: the manifold has to be orientable, and the form has to be compactly supported in order to give a well-defined integral.
- The two points a and b form the boundary of the closed interval. More generally, Stokes' theorem applies to oriented manifolds M with boundary. The boundary ∂M of M is itself a manifold and inherits a natural orientation from that of M. For example, the natural orientation of the interval gives an orientation of the two boundary points. Intuitively, a inherits the opposite orientation as b, as they are at opposite ends of the interval. So, "integrating" F over two boundary points a, b is taking the difference $F(b) - F(a)$.

In even simpler terms, one can consider the points as boundaries of curves, that is as 0-dimensional boundaries of 1-dimensional manifolds. So, just as one can find the value of an integral ($f\, dx = dF$) over a 1-dimensional manifold ($[a, b]$) by considering the anti-derivative (F) at the 0-dimensional boundaries ($\{a, b\}$), one can generalize the fundamental theorem of calculus, with a few additional caveats, to deal with the value of integrals ($d\omega$) over n-dimensional manifolds (Ω) by considering the antiderivative (ω) at the $(n-1)$-dimensional boundaries ($\partial\Omega$) of the manifold.

So the fundamental theorem reads:

$$\int_{[a,b]} f(x)\, dx = \int_{[a,b]} dF = \int_{\{a\}^- \cup \{b\}^+} F = F(b) - F(a).$$

Formulation for smooth manifolds with boundary

Let Ω be an oriented smooth manifold with boundary of dimension n and let α be a smooth n-differential form that is compactly supported on Ω. First, suppose that α is compactly supported in the domain of a single, oriented coordinate chart $\{U, \varphi\}$. In this case, we define the integral of α over Ω as

$$\int_\Omega \alpha = \int_{\varphi(U)} \left(\varphi^{-1}\right)^* \alpha,$$

i.e., via the pullback of α to $\mathbf{R}^n$.

More generally, the integral of α over Ω is defined as follows: Let $\{\psi_i\}$ be a partition of unity associated with a locally finite cover $\{U_i, \varphi_i\}$ of (consistently oriented) coordinate charts, then define the integral

$$\int_\Omega \alpha \equiv \sum_i \int_{U_i} \psi_i \alpha \, ,$$

where each term in the sum is evaluated by pulling back to $\mathbf{R}^n$ as described above. This quantity is well-defined; that is, it does not depend on the choice of the coordinate charts, nor the partition of unity.

The generalized Stokes theorem reads:

> **Theorem.** *(Stokes–Cartan) If ω is a smooth $(n-1)$-form with compact support on smooth n-dimensional manifold-with-boundary Ω, $\partial\Omega$ denotes the boundary of Ω given the induced orientation, and $i : \partial\Omega \hookrightarrow \Omega$ is the inclusion map, then*
>
> $$\int_\Omega d\omega = \int_{\partial\Omega} i^*\omega.$$

Conventionally, $\int_{\partial\Omega} i^*\omega$ is abbreviated as $\int_{\partial\Omega} \omega$, since the pullback of a differential form by the inclusion map is simply its restriction to its domain: $i^*\omega = \omega|_{\partial\Omega}$. Here d is the exterior derivative, which is defined using the manifold structure only. The right-hand side is sometimes written as $\oint_{\partial\Omega} \omega$ to stress the fact that the $(n-1)$-manifold $\partial\Omega$ has no boundary.[note 2] (This fact is also an implication of Stokes' theorem, since for a given smooth n-dimensional manifold Ω, application of the theorem twice gives $\int_{\partial(\partial\Omega)} \omega = \int_\Omega d(d\omega) = 0$ for any $(n-2)$-form ω, which implies that $\partial(\partial\Omega) = \emptyset$.) The right-hand side of the equation is often used to formulate *integral* laws; the left-hand side then leads to equivalent *differential* formulations (see below).

The theorem is often used in situations where Ω is an embedded oriented submanifold of some bigger manifold, often $\mathbf{R}^k$, on which the form ω is defined.

Topological preliminaries; integration over chains

Let M be a smooth manifold. A (smooth) singular k-simplex in M is defined as a smooth map from the standard simplex in $\mathbf{R}^k$ to M. The group $C_k(M, \mathbf{Z})$ of singular k-chains on M is defined to be the free abelian group on the set of singular k-simplices in M. These groups, together with the boundary map, ∂, define a chain complex. The corresponding homology (resp. cohomology) group is isomorphic to the usual singular homology group $H_k(M, \mathbf{Z})$ (resp. the singular cohomology group $H^k(M, \mathbf{Z})$), defined using continuous rather than smooth simplices in M.

On the other hand, the differential forms, with exterior derivative, d, as the connecting map, form a cochain complex, which defines the de Rham cohomology groups $H^k_{dR}(M, \mathbf{R})$.

Differential k-forms can be integrated over a k-simplex in a natural way, by pulling back to $\mathbf{R}^k$. Extending by linearity allows one to integrate over chains. This gives a linear map from the space of k-forms to the kth group of singular cochains, $C^k(M, \mathbf{Z})$, the linear functionals on $C_k(M, \mathbf{Z})$. In other words, a k-form ω defines a functional

$$I(\omega)(c) = \oint_c \omega.$$

on the k-chains. Stokes' theorem says that this is a chain map from de Rham cohomology to singular cohomology with real coefficients; the exterior derivative, d, behaves like the *dual* of ∂ on forms. This gives a homomorphism from de Rham cohomology to singular cohomology. On the level of forms, this means:

1. closed forms, i.e., $d\omega = 0$, have zero integral over *boundaries*, i.e. over manifolds that can be written as $\partial\sum_c M_c$, and

2. exact forms, i.e., $\omega = d\sigma$, have zero integral over *cycles*, i.e. if the boundaries sum up to the empty set:
$\sum_c M_c = \varnothing$.

De Rham's theorem shows that this homomorphism is in fact an isomorphism. So the converse to 1 and 2 above hold true. In other words, if $\{c_i\}$ are cycles generating the kth homology group, then for any corresponding real numbers, $\{a_i\}$, there exist a closed form, ω, such that
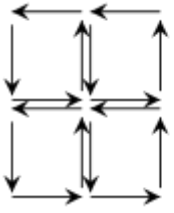
$$\oint_{c_i} \omega = a_i \,,$$

and this form is unique up to exact forms.

Stokes' theorem on smooth manifolds can be derived from Stokes' theorem for chains in smooth manifolds, and vice versa.[12] Formally stated, the latter reads:[13]

Theorem. *(Stokes' theorem for chains) If c is a smooth k-chain in a smooth manifold M, and ω is a smooth $(k-1)$-form on M, then*

$$\int_{\partial c} \omega = \int_c d\omega.$$


Underlying principle

To simplify these topological arguments, it is worthwhile to examine the underlying principle by considering an example for $d = 2$ dimensions. The essential idea can be understood by the diagram on the left, which shows that, in an oriented tiling of a manifold, the interior paths are traversed in opposite directions; their contributions to the path integral thus cancel each other pairwise. As a consequence, only the contribution from the boundary remains. It thus suffices to prove Stokes' theorem for sufficiently fine tilings (or, equivalently, simplices), which usually is not difficult.

Generalization to rough sets

The formulation above, in which Ω is a smooth manifold with boundary, does not suffice in many applications. For example, if the domain of integration is defined as the plane region between two x-coordinates and the graphs of two functions, it will often happen that the domain has corners. In such a case, the corner points mean that Ω is not a smooth manifold with boundary, and so the statement of Stokes' theorem given above does not apply. Nevertheless, it is possible to check that the conclusion of Stokes' theorem is still true. This is because Ω and its boundary are well-behaved away from a small set of points (a measure zero set).

A version of Stokes' theorem that extends to rough domains was proved by Whitney.[14] Assume that D is a connected bounded open subset of $\mathbf{R}^n$. Call D a *standard domain* if it satisfies the following property: There exists a subset P of ∂D, open in ∂D, whose complement in ∂D has Hausdorff $(n-1)$-measure zero; and such that every point of P has a *generalized normal vector*. This is a vector $\mathbf{v}(x)$ such that, if a coordinate system is chosen so that $\mathbf{v}(x)$ is the first basis vector, then, in an open neighborhood around x, there exists a smooth function $f(x_2, \ldots, x_n)$ such that P is the graph
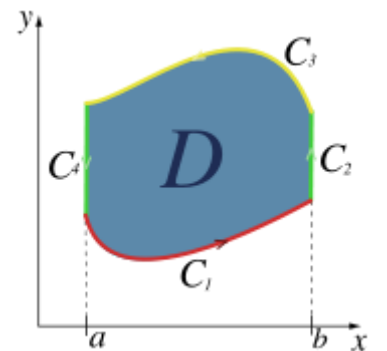
A region (here called D instead of Ω) with piecewise smooth boundary. This is a manifold with corners, so its boundary is not a smooth manifold.

$\{\, x_1 = f(x_2, \ldots, x_n)\, \}$ and D is the region $\{\, x_1 < f(x_2, \ldots, x_n)\, \}$. Whitney remarks that the boundary of a standard domain is the union of a set of zero Hausdorff $(n-1)$-measure and a finite or countable union of smooth $(n-1)$-manifolds, each of which has the domain on only one side. He then proves that if D is a standard domain in $\mathbf{R}^n$, ω is an $(n-1)$-form which is defined, continuous, and bounded on $D \cup P$, smooth on D, integrable on P, and such that $d\omega$ is integrable on D, then Stokes' theorem holds, that is,

$$\int_P \omega = \int_D d\omega \,.$$

The study of measure-theoretic properties of rough sets leads to geometric measure theory. Even more general versions of Stokes' theorem have been proved by Federer and by Harrison.[15]
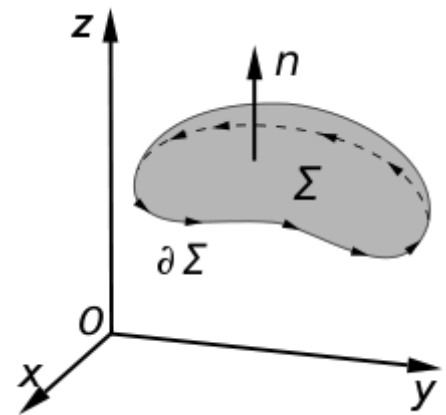
Special cases

The general form of the Stokes theorem using differential forms is more powerful and easier to use than the special cases. The traditional versions can be formulated using Cartesian coordinates without the machinery of differential geometry, and thus are more accessible. Further, they are older and their names are more familiar as a result. The traditional forms are often considered more convenient by practicing scientists and engineers but the non-naturalness of the traditional formulation becomes apparent when using other coordinate systems, even familiar ones like spherical or cylindrical coordinates. There is potential for confusion in the way names are applied, and the use of dual formulations.

Kelvin–Stokes theorem

This is a (dualized) $(1+1)$-dimensional case, for a 1-form (dualized because it is a statement about vector fields). This special case is often just referred to as *Stokes' theorem* in many introductory university vector calculus courses and is used in physics and engineering. It is also sometimes known as the **curl** theorem.

The classical Kelvin–Stokes theorem relates the surface integral of the curl of a vector field over a surface Σ in Euclidean three-space to the line integral of the vector field over its boundary. It is a special case of the general Stokes theorem (with $n = 2$) once we identify a vector field with a 1-form using the metric on Euclidean 3-space. The curve of the line integral, $\partial\Sigma$, must have positive orientation, meaning that $\partial\Sigma$ points counterclockwise when the surface normal, $\mathbf{n}$, points toward the viewer.

An illustration of the Kelvin–Stokes theorem, with surface Σ, its boundary $\partial\Sigma$ and the "normal" vector n.

One consequence of the Kelvin–Stokes theorem is that the field lines of a vector field with zero curl cannot be closed contours. The formula can be rewritten as:

Suppose $\mathbf{F}=(P(x,y,z),Q(x,y,z),R(x,y,z))$ is defined in region with smooth surface Σ and has first-order continuous partial derivatives. Then

$$\iint_\Sigma \left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \right)$$
$$= \oint_{\partial\Sigma} \left(P\,dx + Q\,dy + R\,dz \right),$$

where P, Q and R are the components of $\mathbf{F}$, and $\partial\Sigma$ is boundary of region with smooth surface Σ.

Green's theorem

Green's theorem is immediately recognizable as the third integrand of both sides in the integral in terms of P, Q, and R cited above.

In electromagnetism

Two of the four Maxwell equations involve curls of 3-D vector fields, and their differential and integral forms are related by the Kelvin–Stokes theorem. Caution must be taken to avoid cases with moving boundaries: the partial time derivatives are intended to exclude such cases. If moving boundaries are included, interchange of integration and differentiation introduces terms related to boundary motion not included in the results below (see Differentiation under the integral sign):

Name	Differential form	Integral form (using Kelvin–Stokes theorem plus relativistic invariance, $\int \frac{\partial}{\partial t} \ldots \rightarrow \frac{d}{dt} \int \ldots$)
Maxwell–Faraday equation Faraday's law of induction:	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = \iint_S \nabla \times \mathbf{E} \cdot d\mathbf{A}$ $= -\iint_S \dfrac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}$ (with C and S not necessarily stationary)
Ampère's law (with Maxwell's extension):	$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \nabla \times \mathbf{H} \cdot d\mathbf{A}$ $= \iint_S \mathbf{J} \cdot d\mathbf{A} + \iint_S \dfrac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{A}$ (with C and S not necessarily stationary)

The above listed subset of Maxwell's equations are valid for electromagnetic fields expressed in SI units. In other systems of units, such as CGS or Gaussian units, the scaling factors for the terms differ. For example, in Gaussian units, Faraday's law of induction and Ampère's law take the forms:[16][17]

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\,,$$
$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}\,,$$

respectively, where c is the speed of light in vacuum.

Divergence theorem

Likewise, the Ostrogradsky–Gauss theorem (also known as the divergence theorem or Gauss's theorem)

$$\int_{\text{Vol}} \nabla \cdot \mathbf{F} \, d_{\text{Vol}} = \oint_{\partial\text{Vol}} \mathbf{F} \cdot d\mathbf{\Sigma}$$

is a special case if we identify a vector field with the $(n-1)$-form obtained by contracting the vector field with the Euclidean volume form. An application of this is the case $\mathbf{F} = f\mathbf{c}$ where $\mathbf{c}$ is an arbitrary constant vector. Working out the divergence of the product gives

$$\mathbf{c} \cdot \int_{\mathrm{Vol}} \nabla f \, d_{\mathrm{Vol}} = \mathbf{c} \cdot \oint_{\partial \mathrm{Vol}} f \, d\mathbf{\Sigma} \,.$$

Since this holds for all $\mathbf{c}$ we find

$$\int_{\mathrm{Vol}} \nabla f \, d_{\mathrm{Vol}} = \oint_{\partial \mathrm{Vol}} f \, d\mathbf{\Sigma} \,.$$

See also

- Chandrasekhar–Wentzel lemma

Footnotes

1. γ and Γ are both loops, however, Γ is not necessarily a Jordan curve

2. For mathematicians this fact is known, therefore the circle is redundant and often omitted. However, one should keep in mind here that in thermodynamics, where frequently expressions as $\oint_W \{d_{\mathrm{total}} U\}$ appear (wherein the total derivative, see below, should not be confused with the exterior one), the integration path W is a one-dimensional closed line on a much higher-dimensional manifold. That is, in a thermodynamic application, where U is a function of the temperature $\alpha_1 := T$, the volume $\alpha_2 := V$, and the electrical polarization $\alpha_3 := P$ of the sample, one has

   $$\{d_{\mathrm{total}} U\} = \sum_{i=1}^{3} \frac{\partial U}{\partial \alpha_i} \, d\alpha_i \,,$$

   and the circle is really necessary, e.g. if one considers the *differential* consequences of the *integral* postulate

   $$\oint_W \{d_{\mathrm{total}} U\} \overset{!}{=} 0 \,.$$

References

1. *Physics of Collisional Plasmas – Introduction to | Michel Moisan | Springer* (https://www.springer.com/gp/book/9789400745575).

2. Cartan, Élie (1945). *Les Systèmes Différentiels Extérieurs et leurs Applications Géométriques*. Paris: Hermann.

3. Katz, Victor J. (1979-01-01). "The History of Stokes' Theorem". *Mathematics Magazine*. **52** (3): 146–156. doi:10.2307/2690275 (https://doi.org/10.2307%2F2690275). JSTOR 2690275 (https://www.jstor.org/stable/2690275).

4. Katz, Victor J. (1999). "5. Differential Forms". In James, I. M. (ed.). *History of Topology*. Amsterdam: Elsevier. pp. 111–122. ISBN 9780444823755.

5. See:

- Katz, Victor J. (May 1979). "The history of Stokes' theorem". *Mathematics Magazine*. **52** (3): 146–156. doi:10.1080/0025570x.1979.11976770 (https://doi.org/10.1080%2F0025570x.1979.11976770).
- The letter from Thomson to Stokes appears in: Thomson, William; Stokes, George Gabriel (1990). Wilson, David B. (ed.). *The Correspondence between Sir George Gabriel Stokes and Sir William Thomson, Baron Kelvin of Largs, Volume 1: 1846–1869* (https://books.google.com/books?id=YrjkOEdC83gC&pg=PA97#v=one page&q&f=false). Cambridge, England: Cambridge University Press. pp. 96–97.
- Neither Thomson nor Stokes published a proof of the theorem. The first published proof appeared in 1861 in: Hankel, Hermann (1861). *Zur allgemeinen Theorie der Bewegung der Flüssigkeiten* (http://babel.hathitrust.or g/cgi/pt?id=mdp.39015035826760#page/34/mode/1up) [*On the general theory of the movement of fluids*]. Göttingen, Germany: Dieterische University Buchdruckerei. pp. 34–37. Hankel doesn't mention the author of the theorem.
- In a footnote, Larmor mentions earlier researchers who had integrated, over a surface, the curl of a vector field. See: Stokes, George Gabriel (1905). Larmor, Joseph; Strutt, John William, Baron Rayleigh (eds.). *Mathematical and Physical Papers by the late Sir George Gabriel Stokes* (https://books.google.com/books?id =O28ssiqLT9AC&pg=PA320#v=onepage&q&f=false). **5**. Cambridge, England: University of Cambridge Press. pp. 320–321.

6. Darrigol, Olivier (2000). *Electrodynamics from Ampère to Einstein*. Oxford, England. p. 146. ISBN 0198505930.

7. Spivak (1965), p. vii, Preface.

8. See:

- The 1854 Smith's Prize Examination is available online at: Clerk Maxwell Foundation (http://www.clerkmaxwel lfoundation.org/SmithsPrizeExam_Stokes.pdf). Maxwell took this examination and tied for first place with Edward John Routh. See: Clerk Maxwell, James (1990). Harman, P. M. (ed.). *The Scientific Letters and Papers of James Clerk Maxwell, Volume I: 1846–1862* (https://books.google.com/books?id=zfM8AAAAIAAJ& pg=PA237#v=onepage&q&f=false). Cambridge, England: Cambridge University Press. p. 237, footnote 2. See also Smith's prize or the Clerk Maxwell Foundation (http://www.clerkmaxwellfoundation.org/SmithsPrizeS olutions2008_2_14.pdf).
- Clerk Maxwell, James (1873). *A Treatise on Electricity and Magnetism* (https://books.google.com/books?id=9 2QSAAAAIAAJ&pg=PA27#v=onepage&q&f=false). **1**. Oxford, England: Clarendon Press. pp. 25–27. In a footnote on page 27, Maxwell mentions that Stokes used the theorem as question 8 in the Smith's Prize Examination of 1854. This footnote appears to have been the cause of the theorem's being known as "Stokes' theorem".

9. Stewart, James (2010). *Essential Calculus: Early Transcendentals* (https://books.google.com/books?id=btIhvKZC kTsC&pg=PA786). Cole.

10. This proof is based on the Lecture Notes given by Prof. Robert Scheichl (University of Bath, U.K) [1] (http://www. maths.bath.ac.uk/~masrs/ma20010/), please refer the [2] (http://www.maths.bath.ac.uk/~masrs/ma20010/stokesp roofs.pdf)

11. This proof is also same to the proof shown in (https://www.proofwiki.org/wiki/Classical_Stokes'_Theorem)

12. Renteln, Paul (2014). *Manifolds, Tensors, and Forms*. Cambridge, UK: Cambridge University Press. pp. 158–175. ISBN 9781107324893.

13. Lee, John M. (2013). *Introduction to Smooth Manifolds*. New York: Springer. p. 481. ISBN 9781441999818.

14. Whitney, *Geometric Integration Theory,* III.14.

15. Harrison, J. (October 1993). "Stokes' theorem for nonsmooth chains". *Bulletin of the American Mathematical Society*. New Series. **29** (2). arXiv:math/9310231 (https://arxiv.org/abs/math/9310231).

16. Jackson, J. D. (1975). *Classical Electrodynamics* (https://archive.org/details/classicalelectro00jack_0) (2nd ed.). New York, NY: Wiley.

17. Born, M.; Wolf, E. (1980). *Principles of Optics* (6th ed.). Cambridge, England: Cambridge University Press.

Further reading

- Grunsky, Helmut (1983). *The General Stokes' Theorem* (https://archive.org/details/generalstokesthe0000grun). Boston: Pitman. ISBN 0-273-08510-7.
- Katz, Victor J. (May 1979). "The History of Stokes' Theorem". *Mathematics Magazine*. **52** (3): 146–156. doi:10.2307/2690275 (https://doi.org/10.2307%2F2690275). JSTOR 2690275 (https://www.jstor.org/stable/2690275).
- Loomis, Lynn Harold; Sternberg, Shlomo (2014). *Advanced Calculus* (https://archive.org/details/LoomisL.H.SternbergS.AdvancedCalculusRevisedEditionJonesAndBartlett). Hackensack, New Jersey: World Scientific. ISBN 978-981-4583-93-0.
- Madsen, Ib; Tornehave, Jørgen (1997). *From Calculus to Cohomology: De Rham cohomology and characteristic classes* (https://archive.org/details/MadsenI.H.TornehaveJ.FromCalculusToCohomologyDeRhamCohomologyAndCharacteristicClasses1996). Cambridge, UK: Cambridge University Press. ISBN 0-521-58956-8.
- Marsden, Jerrold E.; Anthony, Tromba (2003). *Vector Calculus* (5th ed.). W. H. Freeman.
- Lee, John (2003). *Introduction to Smooth Manifolds* (https://archive.org/details/GraduateTextsInMathematics218LeeJ.M.IntroductionToSmoothManifoldsSpringer2012). Springer-Verlag. ISBN 978-0-387-95448-6.
- Rudin, Walter (1976). *Principles of Mathematical Analysis* (https://archive.org/details/1979RudinW). New York, NY: McGraw–Hill. ISBN 0-07-054235-X.
- Spivak, Michael (1965). *Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus*. San Francisco: Benjamin Cummings. ISBN 0-8053-9021-9.
- Stewart, James (2009). *Calculus: Concepts and Contexts* (https://books.google.com/books?id=Vou3MZu_7tcC&pg=PA960). Cengage Learning. pp. 960–967. ISBN 0-495-55742-0.
- Stewart, James (2003). *Calculus: Early Transcendental Functions* (5th ed.). Brooks/Cole.
- Tu, Loring W. (2011). *An Introduction to Manifolds* (2nd ed.). New York: Springer. ISBN 978-1-4419-7399-3.

External links

- Hazewinkel, Michiel, ed. (2001) [1994], "Stokes formula" (https://www.encyclopediaofmath.org/index.php?title=p/s090310), *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4
- Proof of the Divergence Theorem and Stokes' Theorem (http://higheredbcs.wiley.com/legacy/college/hugheshallett/0471484822/theory/hh_focusontheory_sectionm.pdf)
- Calculus 3 – Stokes Theorem from lamar.edu (http://tutorial.math.lamar.edu/classes/calcIII/stokestheorem.aspx) – an expository explanation