

1 Electrons in a metal as an ideal Fermi gas (7 points)

A metal contains conduction electrons in a volume V . The temperature is $T = 0$ so that the chemical potential is $\mu = \varepsilon_F$. The magnetic moment of the electrons is μ_m . A magnetic field B is switched on, hence the energy of particles with spin parallel (+) and anti-parallel (-) to the magnetic field is

$$\varepsilon_{\pm}(p) = \frac{p^2}{2m} \mp \mu_m B \quad (1)$$

respectively. In the following the magnetic field is assumed so weak that only its first order effect should be considered.

a) Calculate the mean number $\langle N_{\pm} \rangle$ of electrons with spin parallel and anti-parallel to the magnetic field, respectively. (3 points)

Total number of particles:

$$\langle N \rangle = 2 \cdot \frac{V}{h^3} \cdot \int_0^{p_F} d^3p \quad (2)$$

$$= 2 \cdot \frac{V}{h^3} \cdot 4\pi \cdot \int_0^{p_F} p^2 dp \quad (3)$$

$$= 2 \cdot \frac{4\pi V}{3h^3} \cdot p_F^3 \quad (4)$$

$$\Rightarrow \langle N_{\pm} \rangle = \frac{4\pi V}{3h^3} p_{F\pm}^3 \quad (5)$$

with

$$p_{F+} = \sqrt{2m(\mu + \mu_m B)} \quad \text{and} \quad p_{F-} = \sqrt{2m(\mu - \mu_m B)} \quad (6)$$

b) Calculate the mean magnetization M and the magnetic susceptibility χ . (2 points)

$$M = \mu_m \cdot (\langle N_+ \rangle - \langle N_- \rangle) \quad (7)$$

$$= \mu_m \cdot \left(\frac{4\pi V}{3h^3} p_{F+}^3 - \frac{4\pi V}{3h^3} p_{F-}^3 \right) \quad (8)$$

$$= \mu_m \cdot \frac{4\pi V}{3h^3} \cdot \left([2m \cdot (\mu + \mu_m B)]^{3/2} - [2m \cdot (\mu - \mu_m B)]^{3/2} \right) \quad (9)$$

$$(10)$$

$$\Rightarrow \chi = \left(\frac{\partial M}{\partial B} \right) \quad (11)$$

$$= \mu_m \cdot \partial_B \left(\langle N_+ \rangle - \langle N_- \rangle \right) \quad (12)$$

$$= \mu_m \cdot \left(\frac{3}{2} \cdot \sqrt{2m \cdot (\mu + \mu_m B)} - \frac{3}{2} \cdot \sqrt{2m \cdot (\mu - \mu_m B)} \right) \quad (13)$$

$$= 3\sqrt{\frac{m}{2}} \cdot \mu_m \cdot \left(\sqrt{\mu + \mu_m B} - \sqrt{\mu - \mu_m B} \right) \quad (14)$$

Comment: Taylor expand these expressions!

c) Express the chemical potential μ in terms of the mean total number of electrons $\langle N \rangle$ and eliminate the chemical potential from the formula for the susceptibility. (2 points)

$$\mu = \frac{1}{2m} \left(\frac{3h^3 N}{8\pi V} \right)^{\frac{2}{3}}. \quad (15)$$

2 Particle-hole symmetry for fermions (4 points)

Consider a system of spin 1/2 fermions at temperature T , that occupy a finite number M of energy levels ε_i with $i = 1, 2, \dots, M$.

a) Determine the grand canonical partition sum Z_G . (1 point)

HINT: Pay attention to the degeneracy.

We can have the following configurations for one energy level ε_i :

1. The energy level is not occupied ($E = 0$).
2. The energy level is occupied with one spin-up electron ($E = \varepsilon_i$).
3. The energy level is occupied with one spin-down electron ($E = \varepsilon_i$).
4. The energy level is occupied with two electron ($E = 2\varepsilon_i$).

Since spin-up and spin-down electrons are distinguishable, we have two microstates with energy ε_i . Thus, the partition sum for one energy level is:

$$z_i = 1 + 2e^{-\beta(\varepsilon_i - \mu)} + e^{-2\beta(\varepsilon_i - \mu)} = \left(1 + e^{-\beta(\varepsilon_i - \mu)}\right)^2. \quad (16)$$

Remark: In general, for degeneracy g , the partition sum would be given by

$$z_i = \left(1 + e^{-\beta(\varepsilon_i - \mu)}\right)^g. \quad (17)$$

For M independent energy levels, the grand canonical partition sum is

$$Z_G = \prod_{i=1}^M \left(1 + e^{-\beta(\varepsilon_i - \mu)}\right)^2. \quad (18)$$

b) Use the relation $F = -k_B T \ln Z_G + \mu \langle N \rangle$ between the free energy, the grand canonical partition sum, the chemical potential and the mean particle number to show: the thermodynamic properties of the system stay the same if one, instead of considering the $\langle N \rangle$ particles, distributes $2M - \langle N \rangle$ "holes" with chemical potential $-\mu$ on the energy levels $-\varepsilon_i$. (3 points)

You need to show that the free energy (from which all the physical properties of the system are determined by derivatives) of the "holes" differs at most by a constant term: $F' = F + C$. The free energy of the particles is

$$F = \mu \langle N \rangle - 2k_B T \sum_{i=1}^M \ln \left(1 + e^{-\beta(\varepsilon_i - \mu)}\right). \quad (19)$$

The free energy of the "holes" is

$$F' = -\mu(2M - \langle N \rangle) - 2k_B T \sum_{i=1}^M \ln \left(1 + e^{\beta(\varepsilon_i - \mu)}\right). \quad (20)$$

Indeed one can show that

$$F' = F - 2 \sum_{i=1}^M \varepsilon_i. \quad (21)$$

3 Ultra-relativistic ideal Fermi fluid (*4 points*)

An ultra-relativistic ideal Fermi fluid is contained in a volume V . The chemical potential is μ . Consider only the special case $T = 0$ in the following. Calculate the mean particle number $\langle N \rangle$ and the mean total energy $\langle E \rangle$. Express $\langle E \rangle$ in terms of $\langle N \rangle$ and μ .

HINT: Ultra-relativistic means that you can neglect the mass term in the relativistic formula for the energy of the particles:

$$\varepsilon(\vec{p}) = \sqrt{m^2 c^4 + \vec{p}^2 c^2} \approx pc. \quad (22)$$

We use the following identity from the lecture script:

$$\sum_{\vec{k}, m_S} (...) = 2 \cdot \frac{V}{h^3} \int d\vec{p} (...) = \frac{2V}{h^3} \int_0^\infty dp 4\pi p^2 (...). \quad (23)$$

Thus, the mean particle number is given by

$$\langle N \rangle = \sum_{\vec{k}, m_S} n_{\vec{k}, m_S} = \frac{8\pi V}{c^3 h^3} \int_0^\infty d\varepsilon \varepsilon^2 n(\varepsilon), \quad (24)$$

where we used the linear dispersion relation $\varepsilon(p) = pc$ for ultra-relativistic particles. In the limit $T = 0$, the Fermi function becomes $n(\varepsilon) = \theta(\mu - \varepsilon)$:

$$\langle N \rangle = \frac{8\pi V}{c^3 h^3} \int_0^\mu d\varepsilon \varepsilon^2 = \frac{8\pi V}{3c^3 h^3} \mu^3. \quad (25)$$

A similar calculation is done for the mean energy:

$$\langle E \rangle = \frac{8\pi V}{c^3 h^3} \int_0^\infty d\varepsilon \varepsilon^2 n(\varepsilon) \varepsilon = \frac{8\pi V}{c^3 h^3} \int_0^\mu d\varepsilon \varepsilon^3 = \frac{2\pi V}{c^3 h^3} \mu^4 = \frac{3}{4} \langle N \rangle \mu. \quad (26)$$