# 1 Electrons in a metal as an ideal Fermi gas *(7 points)*

A metal contains conduction electrons in a volume $V$. The temperature is $T = 0$ so that the chemical potential is $\mu = \varepsilon_F$. The magnetic moment of the electrons is $\mu_m$. A magnetic field B is switched on, hence the energy of particles with spin parallel (+) and anti-parallel (-) to the magnetic field is

$$\varepsilon_{\pm}(p) = \frac{p^2}{2m} \mp \mu_m B \tag{1}$$

respectively. In the following the magnetic field is assumed so weak that only its first order effect should be considered.

**a) Calculate the mean number $\langle N_{\pm} \rangle$ of electrons with spin parallel and anti-parallel to the magnetic field, respectively.** ***(3 points)***

Total number of particles:

$$\begin{aligned} \langle N \rangle &= 2 \cdot \frac{V}{h^3} \cdot \int_0^{p_F} d^3p &(2)\\ &= 2 \cdot \frac{V}{h^3} \cdot 4\pi \cdot \int_0^{p_F} p^2 dp &(3)\\ &= 2 \cdot \frac{4\pi V}{3h^3} \cdot p_F^3 &(4)\\ \Rightarrow \langle N_{\pm} \rangle &= \frac{4\pi V}{3h^3} p_{F\pm}^3 &(5) \end{aligned}$$

with

$$p_{F+} = \sqrt{2m(\mu - \mu_m B)} \quad \text{and} \quad p_{F-} = \sqrt{2m(\mu + \mu_m B)} \tag{6}$$

**b) Calculate the mean magnetization $M$ and the magnetic susceptibility $\chi$.** ***(2 points)***

$$\begin{aligned} M &= \mu_m \cdot (\langle N_+ \rangle - \langle N_- \rangle) &(7)\\ &= \mu_B \cdot \left( \frac{4\pi V}{3h^3} p_{F+}^3 - \frac{4\pi V}{3h^3} p_{F-}^3 \right) &(8)\\ &= \mu_B \cdot \frac{4\pi V}{3h^3} \cdot \left( [2m \cdot (\mu - \mu_m B)]^{3/2} - [2m \cdot (\mu + \mu_m B)]^{3/2} \right) &(9)\\ \Rightarrow \chi &= \left( \frac{\partial M}{\partial B} \right) &(10)\\ &= \mu_m \cdot \partial_B \Big( \langle N_+ \rangle - \langle N_- \rangle \Big) &(11)\\ &= \mu_m \cdot \left( \frac{3}{2} \cdot \sqrt{2m \cdot (\mu + \mu_m B)} - \frac{3}{2} \cdot \sqrt{2m \cdot (\mu - \mu_m B)} \right) &(12)\\ &= 3\sqrt{\frac{m}{2}} \cdot \mu_m \cdot \left( \sqrt{\mu + \mu_m B} - \sqrt{\mu - \mu_m B} \right) &(13) \end{aligned}$$

**c) Express the chemical potential $\mu$ in terms of the mean total number of electrons $\langle N \rangle$ and eliminate the chemical potential from the formula for the susceptibility.** ***(2 points)***

$$\langle N \rangle = \langle N_+ \rangle + \langle N_- \rangle \tag{14}$$

## 2 Particle-hole symmetry for fermions *(4 points)*

Consider a system of spin 1/2 fermions at temperature $T$, that occupy a finite number $M$ of energy levels $\varepsilon_i$ with $i = 1, 2, ..., M$.

**a) Determine the grand canonical partition sum $Z_G$. *(1 point)***

HINT: Pay attention to the degeneracy.

$$Z_G = \sum_{i=1}^{M} \left[ 1 + 2e^{-\beta(\varepsilon_i - 2\mu)} + e^{-\beta(\varepsilon_i - \mu)} \right] \tag{15}$$

**b) Use the relation $F = -k_B T \ln Z_G + \mu \langle N \rangle$ between the free energy, the grand canonical partition sum, the chemical potential and the mean particle number to show: the thermodynamic properties of the system stay the same if one, instead of considering the ⟨N⟩ particles, distributes $2M - \langle N \rangle$ "holes" with chemical potential $\mu$ on the energy levels $-\varepsilon_i$.**

## 3 Ultra-relativistic ideal Fermi fluid *(4 points)*

An ultra-relativistic ideal Fermi fluid is contained in a volume $V$. The chemical potential is $\mu$. Consider only the special case $T = 0$ in the following. Calculate the mean particle number $\langle N \rangle$ and the mean total energy $\langle E \rangle$. Express $\langle E \rangle$ in terms of $\langle N \rangle$ and $\mu$.

HINT: Ultra-relativistic means that you can neglect the mass term in the relativistic formula for the energy of the particles:

$$\varepsilon(\vec{p}) = \sqrt{m^2c^4 + \vec{p}^2c^2} \approx pc\,. \tag{16}$$

We use the following identity from the lecture script:

$$\sum_{\vec{k},m_S} (...) = 2 \cdot \frac{V}{h^3} \int \mathrm{d}\vec{p}\,(...) = \frac{2V}{h^3} \int_0^\infty \mathrm{d}p\, 4\pi p^2 (...)\,. \tag{17}$$

Thus, the mean particle number is given by

$$\langle N \rangle = \sum_{\vec{k},m_S} n_{\vec{k},m_S} = \frac{8\pi V}{c^3h^3} \int_0^\infty \mathrm{d}\varepsilon\, \varepsilon^2 n(\varepsilon)\,, \tag{18}$$

where we used the linear dispersion relation $\varepsilon(p) = pc$ for ultra-relativistic particles. In the limit $T = 0$, the Fermi function becomes $n(\varepsilon) = \theta(\mu - \varepsilon)$:

$$\langle N \rangle = \frac{8\pi V}{c^3h^3} \int_0^\mu \mathrm{d}\varepsilon\, \varepsilon^2 = \frac{8\pi V}{3c^3h^3}\mu^3\,. \tag{19}$$

A similar calculation is done for the mean energy:

$$\langle E \rangle = \frac{8\pi V}{c^3h^3} \int_0^\infty \mathrm{d}\varepsilon\, \varepsilon^2 n(\varepsilon)\varepsilon = \frac{8\pi V}{c^3h^3} \int_0^\mu \mathrm{d}\varepsilon\, \varepsilon^3 = \frac{2\pi V}{c^3h^3}\mu^4 = \frac{3}{4}\langle N \rangle \mu\,. \tag{20}$$