1 Electrons in a metal as an ideal Fermi gas *(7 points)*

A metal contains conduction electrons in a volume V. The temperature is $T = 0$ so that the chemical potential is $\mu = \varepsilon_F$. The magnetic moment of the electrons is μ_m. A magnetic field B is switched on, hence the energy of particles with spin parallel (+) and anti-parallel (-) to the magnetic field is

$$\varepsilon_\pm(p) = \frac{p^2}{2m} \mp \mu_m B \tag{1}$$

respectively. In the following the magnetic field is assumed so weak that only its first order effect should be considered.

a) Calculate the mean number $\langle N_\pm \rangle$ of electrons with spin parallel and anti-parallel to the magnetic field, respectively. ***(3 points)***

Total number of particles:

$$\begin{aligned}
\langle N \rangle &= 2 \cdot \frac{V}{h^3} \cdot \int_0^{p_F} d^3p &(2)\\
&= 2 \cdot \frac{V}{h^3} \cdot 4\pi \cdot \int_0^{p_F} p^2 dp &(3)\\
&= 2 \cdot \frac{4\pi V}{3h^3} \cdot p_F^3 &(4)\\
\Rightarrow \langle N_\pm \rangle &= \frac{4\pi V}{3h^3} p_{F\pm}^3 &(5)
\end{aligned}$$

with

$$p_{F+} = \sqrt{2m(\mu + \mu_m B)} \quad \text{and} \quad p_{F-} = \sqrt{2m(\mu - \mu_m B)} \tag{6}$$

b) Calculate the mean magnetization M and the magnetic susceptibility χ. ***(2 points)***

$$\begin{aligned}
M &= \mu_m \cdot (\langle N_+ \rangle - \langle N_- \rangle) &(7)\\
&= \mu_B \cdot \left(\frac{4\pi V}{3h^3} p_{F+}^3 - \frac{4\pi V}{3h^3} p_{F-}^3 \right) &(8)\\
&= \mu_B \cdot \frac{4\pi V}{3h^3} \cdot \left([2m \cdot (\mu + \mu_m B)]^{3/2} - [2m \cdot (\mu - \mu_m B)]^{3/2} \right) &(9)\\
& &(10)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \chi &= \left(\frac{\partial M}{\partial B} \right) &(11)\\
&= \mu_m \cdot \partial_B \Big(\langle N_+ \rangle - \langle N_- \rangle \Big) &(12)\\
&= \mu_m \cdot \left(\frac{3}{2} \cdot \sqrt{2m \cdot (\mu + \mu_m B)} - \frac{3}{2} \cdot \sqrt{2m \cdot (\mu - \mu_m B)} \right) &(13)\\
&= 3\sqrt{\frac{m}{2}} \cdot \mu_m \cdot \left(\sqrt{\mu + \mu_m B} - \sqrt{\mu - \mu_m B} \right) &(14)
\end{aligned}$$

Comment: Taylor expand these expressions!

c) Express the chemical potential μ in terms of the mean total number of electrons $\langle N \rangle$ and eliminate the chemical potential from the formula for the susceptibility. ***(2 points)***

$$\mu = \frac{1}{2m} \left(\frac{3h^3 N}{8\pi V} \right)^{\frac{2}{3}} . \tag{15}$$

2 Particle-hole symmetry for fermions *(4 points)*

Consider a system of spin 1/2 fermions at temperature T, that occupy a finite number M of energy levels ε_i with $i = 1, 2, ..., M$.

a) Determine the grand canonical partition sum Z_G. *(1 point)*

HINT: Pay attention to the degeneracy.

We can have the following configurations for one energy level ε_i:

1. The energy level is not occupied ($E = 0$).
2. The energy level is occupied with one spin-up electron ($E = \varepsilon_i$).
3. The energy level is occupied with one spin-down electron ($E = \varepsilon_i$).
4. The energy level is occupied with two electron ($E = 2\varepsilon_i$).

Since spin-up and spin-down electrons are distinguishable, we have two microstates with energy ε_i. Thus, the partition sum for one energy level is:

$$z_i = 1 + 2e^{-\beta(\varepsilon_i-\mu)} + e^{-2\beta(\varepsilon_i-\mu)} = \left(1 + e^{-\beta(\varepsilon_i-\mu)}\right)^2 . \tag{16}$$

Remark: In general, for degeneracy g, the partition sum would be given by

$$z_i = \left(1 + e^{-\beta(\varepsilon_i-\mu)}\right)^g . \tag{17}$$

For M independent energy levels, the grand canonical partition sum is

$$Z_G = \prod_{i=1}^{M} \left(1 + e^{-\beta(\varepsilon_i-\mu)}\right)^2 . \tag{18}$$

b) Use the relation $F = -k_BT \ln Z_G + \mu\langle N\rangle$ between the free energy, the grand canonical partition sum, the chemical potential and the mean particle number to show: the thermodynamic properties of the system stay the same if one, instead of considering the ⟨N⟩ particles, distributes $2M - \langle N\rangle$ "holes" with chemical potential $-\mu$ on the energy levels $-\varepsilon_i$. *(3 points)*

You need to show that the free energy (from which all the physical properties of the system are determined by derivatives) of the "holes" differs at most by a constant term: $F' = F + C$. The free energy of the particles is

$$F = \mu\langle N\rangle - 2k_BT \sum_{i=1}^{M} \ln\left(1 + e^{-\beta(\varepsilon_i-\mu)}\right) . \tag{19}$$

The free energy of the "holes" is

$$F' = -\mu(2M - \langle N\rangle) - 2k_BT \sum_{i=1}^{M} \ln\left(1 + e^{\beta(\varepsilon_i-\mu)}\right) . \tag{20}$$

Indeed one can show that

$$F' = F - 2\sum_{i=1}^{M} \varepsilon_i . \tag{21}$$

3 Ultra-relativistic ideal Fermi fluid *(4 points)*

An ultra-relativistic ideal Fermi fluid is contained in a volume V. The chemical potential is μ. Consider only the special case $T = 0$ in the following. Calculate the mean particle number $\langle N\rangle$ and the mean total energy $\langle E\rangle$. Express $\langle E\rangle$ in terms of $\langle N\rangle$ and μ.

HINT: Ultra-relativistic means that you can neglect the mass term in the relativistic formula for the energy of the particles:

$$\varepsilon(\vec{p}) = \sqrt{m^2c^4 + \vec{p}^2c^2} \approx pc\,. \tag{22}$$

We use the following identity from the lecture script:

$$\sum_{\vec{k},m_S} (...) = 2 \cdot \frac{V}{h^3} \int \mathrm{d}\vec{p}\,(...) = \frac{2V}{h^3} \int_0^\infty \mathrm{d}p\, 4\pi p^2 (...)\,. \tag{23}$$

Thus, the mean particle number is given by

$$\langle N\rangle = \sum_{\vec{k},m_S} n_{\vec{k},m_S} = \frac{8\pi V}{c^3h^3} \int_0^\infty \mathrm{d}\varepsilon\, \varepsilon^2 n(\varepsilon)\,, \tag{24}$$

where we used the linear dispersion relation $\varepsilon(p) = pc$ for ultra-relativistic particles. In the limit $T = 0$, the Fermi function becomes $n(\varepsilon) = \theta(\mu - \varepsilon)$:

$$\langle N\rangle = \frac{8\pi V}{c^3h^3} \int_0^\mu \mathrm{d}\varepsilon\, \varepsilon^2 = \frac{8\pi V}{3c^3h^3}\mu^3\,. \tag{25}$$

A similar calculation is done for the mean energy:

$$\langle E\rangle = \frac{8\pi V}{c^3h^3} \int_0^\infty \mathrm{d}\varepsilon\, \varepsilon^2 n(\varepsilon)\varepsilon = \frac{8\pi V}{c^3h^3} \int_0^\mu \mathrm{d}\varepsilon\, \varepsilon^3 = \frac{2\pi V}{c^3h^3}\mu^4 = \frac{3}{4}\langle N\rangle\mu\,. \tag{26}$$