1 Electrons in a metal as an ideal Fermi gas *(7 points)*

A metal contains conduction electrons in a volume V. The temperature is $T = 0$ so that the chemical potential is $\mu = \varepsilon_F$. The magnetic moment of the electrons is μ_m. A magnetic field B is switched on, hence the energy of particles with spin parallel (+) and anti-parallel (-) to the magnetic field is

$$\varepsilon_\pm(p) = \frac{p^2}{2m} \mp \mu_m B \tag{1}$$

respectively. In the following the magnetic field is assumed so weak that only its first order effect should be considered.

a) Calculate the mean number $\langle N_\pm \rangle$ of electrons with spin parallel and anti-parallel to the magnetic field, respectively. ***(3 points)***

$$Z = Z_+ + Z_- \text{ ???} \tag{2}$$

$$Z_\pm = \int_0^\infty \mathrm{d}\varepsilon \cdot \Big(1 + \exp(-\beta\varepsilon)\Big) \cdot n_\pm(\varepsilon) \tag{3}$$

$$\langle N \rangle = -\frac{1}{\beta}\partial_\mu \ln(Z) \tag{4}$$

b) Calculate the mean magnetization M and the magnetic susceptibility χ. ***(2 points)***

$$M = \mu_m \cdot (\langle N_+ \rangle - \langle N_- \rangle) \tag{5}$$

$$\chi = \left(\frac{\partial M}{\partial B}\right) \tag{6}$$

c) Express the chemical potential μ in terms of the mean total number of electrons $\langle N \rangle$ and eliminate the chemical potential from the formula for the susceptibility. ***(2 points)***

$$\langle N \rangle = \langle N_+ \rangle + \langle N_- \rangle \tag{7}$$

0/7

Q
1 0/7
2 0/4
3 4/4
Σ 4/15

2 Particle-hole symmetry for fermions *(4 points)*

Consider a system of spin 1/2 fermions at temperature T, that occupy a finite number M of energy levels ε_i with $i = 1, 2, ..., M$.

a) Determine the grand canonical partition sum Z_G. *(1 point)*

HINT: Pay attention to the degeneracy.

X

$$Z_G = \sum_{i=1}^{M} \left[1 + 2e^{-\beta(\varepsilon_i - 2\mu)} + e^{-\beta(\varepsilon_i - \mu)} \right] ??? \quad (8)$$

b) Use the relation $F = -k_B T \ln Z_G + \mu\langle N\rangle$ between the free energy, the grand canonical partition sum, the chemical potential and the mean particle number to show: the thermodynamic properties of the system stay the same if one, instead of considering the ⟨N⟩ particles, distributes $2M - \langle N\rangle$ "holes" with chemical potential μ on the energy levels $-\varepsilon_i$.

→ for each state ε_i, have

spin states $\emptyset$ (no particle)

$\uparrow$, $\downarrow$ (2x single particle state)

$\uparrow\downarrow$ (2 particle state)

$$z_i = 1 + 2e^{-\beta(\varepsilon_i - \mu)} + e^{-\beta(2\varepsilon_i - 2\mu)}$$
$$= \left(1 + e^{-\beta(\varepsilon_i - \mu)}\right)^2$$

0/4

3 Ultra-relativistic ideal Fermi fluid *(4 points)*

An ultra-relativistic ideal Fermi fluid is contained in a volume V. The chemical potential is μ. Consider only the special case $T = 0$ in the following. Calculate the mean particle number $\langle N\rangle$ and the mean total energy $\langle E\rangle$. Express $\langle E\rangle$ in terms of $\langle N\rangle$ and μ.

HINT: Ultra-relativistic means that you can neglect the mass term in the relativistic formula for the energy of the particles:

$$\varepsilon(\vec{p}) = \sqrt{m^2c^4 + \vec{p}^2c^2} \approx pc\,. \tag{9}$$

We use the following identity from the lecture script:

$$\sum_{\vec{k},m_S}(\ldots) = 2\cdot\frac{V}{h^3}\int \mathrm{d}\vec{p}\,(\ldots) = \frac{2V}{h^3}\int_0^\infty \mathrm{d}p\,4\pi p^2(\ldots)\,. \tag{10}$$

Thus, the mean particle number is given by

$$\langle N\rangle = \sum_{\vec{k},m_S} n_{\vec{k},m_S} = \frac{8\pi V}{c^3h^3}\int_0^\infty \mathrm{d}\varepsilon\,\varepsilon^2 n(\varepsilon)\,, \tag{11}$$

where we used the linear dispersion relation $\varepsilon(p) = pc$ for ultra-relativistic particles. In the limit $T = 0$, the Fermi function becomes $n(\varepsilon) = \theta(\mu - \varepsilon)$:

$$\langle N\rangle = \frac{8\pi V}{c^3h^3}\int_0^\mu \mathrm{d}\varepsilon\,\varepsilon^2 = \frac{8\pi V}{3c^3h^3}\mu^3\,. \tag{12}$$

A similar calculation is done for the mean energy:

$$\langle E\rangle = \frac{8\pi V}{c^3h^3}\int_0^\infty \mathrm{d}\varepsilon\,\varepsilon^2 n(\varepsilon)\varepsilon = \frac{8\pi V}{c^3h^3}\int_0^\mu \mathrm{d}\varepsilon\,\varepsilon^3 = \frac{2\pi V}{c^3h^3}\mu^4 = \frac{3}{4}\langle N\rangle\mu\,. \tag{13}$$

$\propto \mu^4$

4/4