Assignment 10

Handout 29.01.2021 – Return 05.02.2021 – Discussion 11./12.02.2021

Exercise 10.1 [6 points]: Canonical treatment of paramagnetism

As a classical model for paramagnetism one can consider a system of N particles with the Hamiltonian

$$\mathcal{H} = -hM \text{ with } M = \mu \sum_{i=1}^{N} \cos\theta_i \,, \tag{1}$$

where h is an external homogeneous magnetic field, μ the magnetic moment of a single particle and θ_i the angle between the magnetic field $\vec{h}$ and the magnetic moment $\vec{\mu}$ of particle i.

1. Use the canonical distribution to calculate the average magnetization, $\langle M \rangle$, as a function of h and temperature T. *(3 points)*
2. The ratio of which quantities determines the average magnetization? Sketch the functional dependence of the average magnetization on this ratio. *(1 point)*
3. Discuss the two limiting cases: high temperature/weak field vs. low temperature/strong field. *(2 points)*

Exercise 10.2 [6 points]: One-dimensional lattice gas

Consider a one-dimensional lattice model for a non-ideal gas with N lattice sites and periodic boundary conditions. Each lattice site i is either empty (occupancy $n_i = 0$) or occupied by at most one atom (occupancy $n_i = 1$). There is an attractive energy J between atoms occupying neighbouring sites. The chemical potential of the atoms is μ. The Hamiltonian of this lattice gas is

$$H = -J \sum_{\langle ij \rangle} n_i n_j - \mu \sum_i n_i \tag{2}$$

where $\sum_{\langle ij \rangle}$ is the sum over all pairs of neighbouring sites.

1. Express the partition sum of the one-dimensional lattice gas in terms of the transfer matrix T. Calculate the transfer matrix T and its eigenvalues. *(2 points)*
2. Find a transformation of the occupancies n_i to map the lattice gas model to the Ising model with spins S_i. *(2 points)*
3. Derive an expression for the average $\langle n_i \rangle$ in the limit of $N \to \infty$ in terms of the eigenvalues of the transfer matrix. *(2 points)*

Exercise 10.3 [3 points]: Renormalization of the Ising chain

In the lecture we have derived the following RG flow equation for the coupling constant K of the Ising chain without magnetic field: the new value K' is given by

$$K'(K) = \frac{1}{2} \ln \cosh(2K). \tag{3}$$

In addition we have derived the absolute increase in free energy per spin arising in each iteration:

$$g(K) = \frac{1}{2} \ln 2 + \frac{1}{4} \ln(\cosh(2K)). \tag{4}$$

1. Write a short computer program (e.g. in Mathematica or Python) that defines the flow equation $K'(K)$ and the free energy increase $g(K)$ as functions. Start with a coupling constant $K_0 = 1$ and iterate through K_1, K_2, K_3 up to K_4. Also calculated the corresponding values $g_0 = g(K_0)$ to $g_4 = g(K_4)$. What are the limits for these two series ? *(1.5 points)*

2. Use these results to estimate the dimensionless free energy per spin $f = -\beta F/N$ in fourth order (simply cut the appropriate sum after the term with g_4; you can also include the next order term, but now by simply using the first term in $g(K)$). Compare to the known exact result for the Ising chain. How good is the numerical agreement ? *(1.5 points)*