# 1 Problematic moments of a probability distribution

Consider the probability density function

$$f(x) = c\left[\frac{\gamma}{(x-x_0)^2+\gamma^2}\right] \tag{1}$$

with $\gamma > 0$ and $-\infty < x_0 < \infty$, which is known as the Cauchy distribution in mathematics and as the Lorentz distribution in physics.

**a) Calculate the normalization constant $c > 0$ and the median of the distribution for $\gamma = 1$ and $x_0 = 0$.**

Any probability density function $f(x)$ must fulfill the normalization condition

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)\cdot dx && (2)\\
&= \int_{-\infty}^{\infty} c\cdot\left[\frac{\gamma}{(x-x_0)^2+\gamma^2}\right]\cdot dx && (3)\\
&= \frac{c}{\gamma}\cdot\int_{-\infty}^{\infty}\left[\frac{1}{(\frac{x-x_0}{\gamma})^2+1}\right]\cdot dx && (4)\\
&= \frac{c}{\gamma}\cdot\int_{-\infty}^{\infty}\left[\frac{1}{z^2+1}\right]\cdot dz \qquad \text{with } z := \frac{x-x_0}{\gamma} && (5)\\
&= \frac{c}{\gamma}\cdot\Big[\arctan(z)\Big]_{-\infty}^{\infty} && (6)\\
&= \frac{c}{\gamma}\cdot(\frac{\pi}{2}-(-\frac{\pi}{2})) && (7)\\
&= \frac{c}{\gamma}\cdot\pi && (8)\\
\Rightarrow c &= \frac{\gamma}{\pi} && (9)
\end{aligned}
$$

For the median $x_m$, we have to solve

$$
\begin{aligned}
\frac{1}{2} &= \int_{-\infty}^{x_m} f(x)\cdot dx && (10)\\
&= \int_{-\infty}^{x_m} c\cdot\left[\frac{\gamma}{(x-x_0)^2+\gamma^2}\right]\cdot dx && (11)\\
&= \frac{c}{\gamma}\cdot\Big[\arctan(z)\Big]_{-\infty}^{x_m} && (12)\\
&= \frac{c}{\gamma}\cdot\left(\arctan(x_m)+\frac{\pi}{2}\right) && (13)\\
\Rightarrow x_m &= \tan\left(\frac{\gamma}{2c}-\frac{\pi}{2}\right) && (14)
\end{aligned}
$$

**b) Try to evaluate the first two moments of the distribution. What problem arises? Compare the situation to the Gaussian distribution and comment on the implications for large derivations from the median.**

First moment:

$$\int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int_{-\infty}^{\infty} c\left[\frac{\gamma \cdot x}{(x-x_0)^2+\gamma^2}\right] \cdot dx \tag{15}$$
$$= \dots \tag{16}$$

Second moment:

$$\int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx = \int_{-\infty}^{\infty} c\left[\frac{\gamma \cdot x^2}{(x-x_0)^2+\gamma^2}\right] \cdot dx \tag{17}$$
$$= \dots \tag{18}$$

## 2 Stationary limit of the overdamped Fokker-Planck equation

In the lecture, the Fokker-Planck equation was derived for constant drift velocity $v$ and constant diffusion constant $D$. In the case that they are not constant, the Fokker-Planck equation reads

$$\dot{p}(x,t) = -\partial_x\Big(v(x)p(x,t)\Big) + \partial_x^2\Big(D(x)p(x,t)\Big) \tag{19}$$

To find the stationary limit, the left hand side $\dot{p}(x,t)$ is set to zero.

**a) Consider an overdamped particle in one dimension with a harmonic potential $U(x) = \frac{1}{2}kx^2$ ($k$ is the spring constant). This could be e.g. a colloid in an optical trap. Use the balance between friction force $\xi v$ ($\xi$ is the friction coefficient) and potential force $-\partial_x U(x)$ to replace $v(x)$ in this equation. $D$ is assumed to be constant. Solve for the stationary distribution $p_s(x)$. Note that both $p_s(x)$ and its derivative should vanish at $x = \pm\infty$.**

From the balance between friction force $\xi v$ and potential force $-\partial_x U(x)$ we can deduce

$$\begin{aligned} \xi v(x) &= -\partial_x U(x) \\ &= -kx \\ \Rightarrow v(x) &= -\frac{kx}{\xi}. \end{aligned} \tag{20}$$

Assuming $D$ to be constant, we get

$$\begin{aligned} \dot{p}(x,t) &= -\partial_x\Big(v(x)p(x,t)\Big) + \partial_x^2\Big(D(x)p(x,t)\Big) \\ \dot{p}(x,t) &= \frac{k}{\xi}p(x,t) + \frac{kx}{\xi}\partial_x p(x,t) + D\partial_x^2 p(x,t). \end{aligned} \tag{21}$$

The stationary solution is given where $\dot{p}(x,t) = 0$. Thus

$$0 = \frac{k}{\xi D}p(x,t) + \frac{kx}{\xi D}p'(x,t) + p''(x,t). \tag{22}$$

This is the equation for the damped harmonic oscillator...

**b) Compare to the Boltzmann distribution $p_s(x) \sim \exp\left(-\frac{U(x)}{k_B T}\right)$ and from this derive a relation between diffusion constant $D$ and friction coefficient $\xi$.**

...

**c) Calculate the first and second moments of the stationary distribution. Interpret your results in terms of physics.**

...

# 3 Information entropy

**a) Reservoirs for energy and particle number: Consider a system in a fixed volume that not only exchanges energy but also particles with the surrounding. To calculate the probability distribution $p_i$, maximize the information (Shannon) entropy $S = -\sum_i p_i \log p_i$ under the joint conditions of normalization, of $U = \sum_i p_i E_i$ (mean energy) and $N = \sum_i p_i N_i$ (mean particle number). What physical meaning has the new Lagrangian multiplier associated to the condition in $N$?**

Consider the function

$$f(p_i) = -\sum_i p_i \ln p_i - \lambda_1 \left(\sum_i p_i E_i - U\right) - \lambda_2 \left(\sum_i p_i N_i - N\right). \tag{23}$$

The variation with respect to the probability distribution yields

$$\delta f(p_i) = -\sum_i p_i \left(\ln p_i + 1 + \lambda_1 E_i + \lambda_2 N_i\right) \delta p_i \overset{!}{=} 0. \tag{24}$$

Which in fact yields:

$$p_i = e^{-(1+\lambda_1 E_i + \lambda_2 N_i)}. \tag{25}$$

**b) Rational probabilities: if one does not know the probabilities of events, one can define so-called rational probabilities $\bar{p}$ such that entropy is maximized subject to the constraints imposed by the available information. Assume that in a certain game, a player can score any integer $n = 0, 1...$ and it is known that the mean score is $\mu$. Use again the entropy and the method of Lagrange multipliers to show that when imposing the relevant constraints the rational choice is $\bar{p}_n = \frac{\mu^n}{(1+\mu)^{n+1}}$**

Now the function would look like this:

$$f(p_i) = -\sum_i p_i \ln p_i - \lambda_1 \left(\sum_n p_n n - \mu\right). \tag{26}$$

Variation with respect to $p_n$ yields

$$p_n = e^{-(1+\lambda_1 n)}. \tag{27}$$

Constraint for $\mu$:

$$\mu = \sum_n n \exp(-\lambda n + 1) = \frac{e^{\lambda - 1}}{(e^\lambda - 1)^2} \tag{28}$$

$$\Rightarrow 0 = \mu(e^\lambda - 1)^2 - e^{\lambda-1} \tag{29}$$

$$= \mu e^{2\lambda} - 2\mu e^\lambda + \mu - e^{\lambda - 1} \tag{30}$$

$$= z^2 - \left(2 + \frac{1}{\mu e}\right) z + 1 \qquad \text{with } z = e^\lambda \tag{31}$$

This should now be solvable using the quadratic formula, which unfortunately does not seem to lead to the expected result.