

1 Stirling's formula

Start with the definition of the continuous factorial:

$$n! = \int_0^\infty x^n \cdot e^{-x} \cdot dx \quad (1)$$

$$= \int_0^\infty e^{-x+n \log x} \cdot dx \quad (2)$$

$$= n \cdot \int_0^\infty e^{-ny+n \log(ny)} \cdot dy \quad (3)$$

With $f(y) = -y + \log(ny)$ this can be written as

$$= n \cdot \int_0^\infty e^{nf(y)} \cdot dy. \quad (4)$$

The maximum of $f(y)$ is at $y_0 = 1$. Taylor expansion around y_0 :

$$f(y) \approx \log(n) - 1 - \frac{(y-1)^2}{2}. \quad (5)$$

Thus

$$\begin{aligned} n! &\approx ne^{n \log(n)-n} \cdot \int_0^\infty e^{-n(y-1)^2/2} \cdot dy \\ &= ne^{n \log(n)-n} \sqrt{\frac{\pi}{2n}} \left(1 + \text{Erf} \left(\sqrt{\frac{n}{2}} \right) \right) \\ &\approx \sqrt{2\pi n} e^{n \log(n)-n}, \end{aligned} \quad (6)$$

where we have used $\text{Erf}(x) \approx 1$ for large x . Taking the logarithm yields the Stirling formula:

$$\log(n!) = n \log(n) - n + \frac{1}{2} \log(2\pi n). \quad (7)$$

2 Adding two Gaussian distributions

a) Without characteristic function

Let p_x and p_y be Gaussian distributions with parameters μ_x , μ_y and σ .

$$p_x(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \mu_x)^2}{2\sigma^2}\right), \quad (8)$$

$$p_y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y - \mu_y)^2}{2\sigma^2}\right). \quad (9)$$

$$p_z(z) = \int dx \int dy \cdot \delta(z - (x + y)) \cdot p(x, y) \quad (10)$$

$$= \int dx \int dy \cdot \delta(z - (x + y)) \cdot p(x) \cdot p(y) \quad (11)$$

$$= \frac{1}{2\pi\sigma^2} \int dx \int dy \cdot \delta(z - (x + y)) \cdot \exp\left(-\frac{(y - \mu_y)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{(x - \mu_x)^2}{2\sigma^2}\right) \quad (12)$$

$$= \dots \quad (13)$$

$$= \frac{1}{\sqrt{4\pi}\sigma^2} \cdot \exp\left(-\frac{(z - \mu_x - \mu_y)^2}{4\sigma^2}\right). \quad (14)$$

With $\mu_z = \mu_x + \mu_y$ and $\sigma_z = \sqrt{2}\sigma$, this can be rewritten as

$$p_z(z) = \frac{1}{\sqrt{2\pi}\sigma_z^2} \cdot \exp\left(-\frac{(z - \mu_z)^2}{2\sigma_z^2}\right). \quad (15)$$

b) With characteristic function

The Fourier transform of a normal density with mean μ and standard deviation σ is

$$\hat{p}(t) = \int_{-\infty}^{\infty} p(x) e^{-itx} dx = e^{-i\mu t} e^{-\frac{(\sigma t)^2}{2}}. \quad (16)$$

The product of two distributions with the same σ is

$$\hat{p}_z(t) = \hat{p}_x(t) \hat{p}_y(t) = e^{-i(\mu_x + \mu_y)t} e^{-(\sigma t)^2}, \quad (17)$$

which leads also to the definitions $\mu_z = \mu_x + \mu_y$ and $\sigma_z = \sqrt{2}\sigma$ and a Gaussian distribution.

3 Computer exercise on random numbers

Write a little computer program that sums up N random numbers drawn from $[-1, 1]$. Divide by N and compare the obtained distribution to the Gaussian distribution, for $N = 10, 100, 1000$, by plotting the histogram and the appropriate analytical function together.

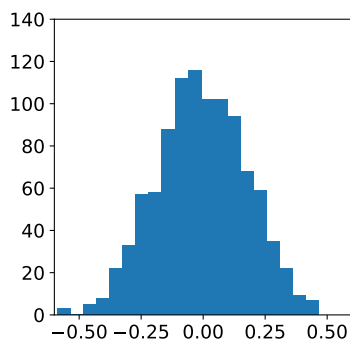
To create a distribution we run a loop for 1000 iterations, at each iteration we create a random number by summing up N number between -1 and 1. The computer program can be implemented using the following python snippet:

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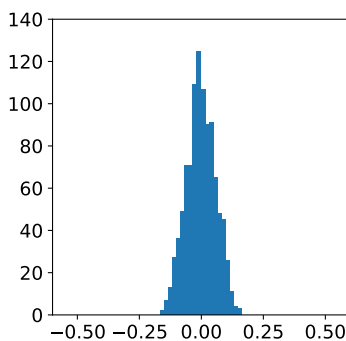
1 import random
2
3 import matplotlib.pyplot as plt
4 import numpy as np
5 from scipy.optimize import curve_fit
6
7 M = 1000
8 for N in [10, 100, 1000]:
9     rs = []
10    for _ in range(M):
11        r = 0
12        for i in range(N):
13            r += random.uniform(-1, 1)
14        rs.append(r / N)
15
16    plt.figure(figsize=(3, 3))
17    plt.xlim(-0.6, 0.6)
18    plt.ylim(0, 130)
19    plt.hist(rs, bins=20)
20    plt.savefig(f'../figures/{N}.pdf')

```

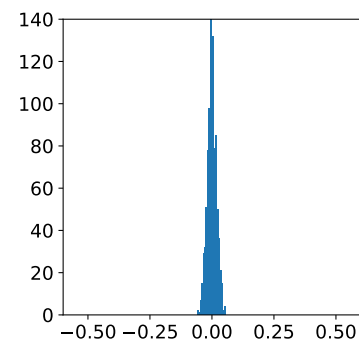
This leads to these plots:



(a) $N = 10$



(b) $N = 100$



(c) $N = 1000$

As is to be expected, the expectation value of the distribution does not change with increasing N , but stays constant at ≈ 0 . The width of the curve decreases for a large number of summands. This is due to the fact that we divide the sum by N .