Institute for Theoretical Physics | Heidelberg University
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Theoretical Statistical Physics (MKTP1) | Winter term 2020/21

Assignment 8

Handout 15.01.2020 – Return 21./22.01.2020 – Discussion 28./29.01.2020

Exercise 8.1 [7 points]: Electrons in a metal as an ideal Fermi gas

A metal contains conduction electrons in a volume V. The temperature is $T = 0$ so that the chemical potential is $\mu = \epsilon_{\mathrm{F}}$. The magnetic moment of the electrons is μ_{m}. A magnetic field $\boldsymbol{B}$ is switched on, hence the energy of particles with spin parallel $(+)$ and anti-parallel $(-)$ to the magnetic field is

$$\epsilon_{\pm}(\boldsymbol{p}) = \frac{p^2}{2m} \mp \mu_{\mathrm{m}} B\,, \tag{1}$$

respectively. In the following the magnetic field is assumed so weak that only its first order effect should be considered.

1. Calculate the mean number $\langle N_{\pm} \rangle$ of electrons with spin parallel and anti-parallel to the magnetic field, respectively. *(3 points)*
2. Calculate the mean magnetization $M = \mu_{\mathrm{m}} \left(\langle N_+ \rangle - \langle N_- \rangle \right)$ and the magnetic susceptibility $\chi = \left(\frac{\partial M}{\partial B} \right)_{N,V,T=0}$. *(2 points)*
3. Express the chemical potential μ in terms of the mean total number of electrons $\langle N \rangle = \langle N_+ \rangle + \langle N_- \rangle$ and eliminate the chemical potential from the formula for the susceptibility. *(2 points)*

Exercise 8.2 [4 points]: Particle-hole symmetry for fermions

Consider a system of spin 1/2 fermions at temperature T, that occupy a finite number M of energy levels ϵ_i, $i = 1, 2, ..., M$.

1. Determine the grand canonical partition sum Z_G *(1 point)*
   HINT: Pay attention to the degeneracy.
2. Use the relation
   $$F = -k_B T \ln Z_G + \mu \langle N \rangle \tag{2}$$
   between the free energy, the grand canonical partition sum, the chemical potential and the mean particle number to show: the thermodynamic properties of the system stay the same if one, instead of considering the $\langle N \rangle$ particles, distributes $2M - \langle N \rangle$ "holes" with chemical potential $-\mu$ on the energy levels $-\epsilon_i$. *(3 points)*

Exercise 8.3 [4 points]: Ultra-relativistic ideal Fermi fluid

An ultra-relativistic ideal Fermi fluid is contained in a volume V. The chemical potential is μ. Consider only the special case $T = 0$ in the following. Calculate the mean particle number $\langle N \rangle$ and the mean total energy $\langle E \rangle$. Express $\langle E \rangle$ in terms of $\langle N \rangle$ and μ.
HINT: Ultra-relativistic means that you can neglect the mass term in the relativistic formula for the energy of the particles: $\epsilon(\boldsymbol{p}) = \sqrt{m^2 c^4 + \boldsymbol{p}^2 c^2} \approx pc$.