1 Quantum corrections to the classical limit *(8 points)*

The classical limit of quantum fluids arises as the linear term in a Taylor expansion in fugacity $z = e^{\beta\mu}$ around $z = 0$ (corresponding to $\mu = -\infty$). This also corresponds to an expansion in density $\rho = N/V$ (the classical gas is a dilute gas, because then wavefunctions do not overlap). The quadratic terms in these expansions correspond to the first quantum corrections to the classical limit. As always, all quantities of interest can be calculated from the grandcanonical potential, which for the ideal quantum fluids reads

$$\Psi(T,V,\mu) = \mp g_S k_B T \frac{V}{h^3} \cdot \int \mathrm{d}\vec{p} \cdot \ln\left[1 \pm\ z e^{-\beta\frac{\vec{p}^2}{2m}}\right] \tag{1}$$

with the different signs corresponding to fermions and bosons, respectively. From thermodynamics, we also know $\Psi = -pV$.

1. Expand the integrand in the formula for the grandcanonical potential Ψ to second order in z and perform the two integrals. ***(2 points)***

Taylor expansion of the natural logarithm:

$$\ln(1 \pm\ x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n} = \pm\ x - \frac{x^2}{2} + O(x^3) \tag{2}$$

We define $C = g_S k_B T \frac{V}{h^3}$ and plug the definition of the Taylor expansion into Equation 1.

$$\Psi(T,V,\mu) \approx \mp C \cdot \int \mathrm{d}\vec{p} \cdot \left[\pm\ z e^{-\beta\frac{\vec{p}^2}{2m}} - \frac{z^2}{2} e^{-2\beta\frac{\vec{p}^2}{2m}}\right] \tag{3}$$

$$= \mp 4\pi C \cdot \left[\pm\ z \int p^2 \mathrm{d}p \cdot e^{-\beta\frac{p^2}{2m}} - \frac{z^2}{2} \cdot \int p^2 \mathrm{d}p \cdot e^{-\beta\frac{p^2}{m}}\right] \tag{4}$$

We now identify $p^2 e^{-\beta\frac{p^2}{2m}} = -2m \cdot \partial_\beta e^{-\beta\frac{p^2}{2m}}$ and $p^2 e^{-\beta\frac{p^2}{m}} = -m \cdot \partial_\beta e^{-\beta\frac{p^2}{m}}$:

$$\Psi(T,V,\mu) = \mp 4\pi C \cdot \left[\mp 2mz \int \mathrm{d}p \cdot \partial_\beta e^{-\beta\frac{p^2}{2m}} + \frac{mz^2}{2} \cdot \int \mathrm{d}p \cdot \partial_\beta e^{-\beta\frac{p^2}{m}}\right] \tag{5}$$

$$= \mp 4\pi \cdot Cm \cdot \left[\mp 2z \cdot \partial_\beta \int \mathrm{d}p \cdot e^{-\beta\frac{p^2}{2m}} + \frac{z^2}{2} \cdot \partial_\beta \int \mathrm{d}p \cdot e^{-\beta\frac{p^2}{m}}\right] \tag{6}$$

Substituting $q = \sqrt{\frac{\beta}{2m}} \cdot p$ and $r = \sqrt{\frac{\beta}{m}} \cdot p$

$$\Psi(T,V,\mu) = \mp 4\pi \cdot Cm \cdot \left[\mp 2z \cdot \partial_\beta \sqrt{\frac{2m}{\beta}} \int e^{-q^2} \mathrm{d}q + \frac{z^2}{2} \cdot \partial_\beta \sqrt{\frac{m}{\beta}} \int e^{-r^2} \mathrm{d}r\right] \tag{7}$$

$$= \mp 4\pi \cdot Cm \cdot \sqrt{m\pi} \cdot \left[\mp 2\sqrt{2} z \cdot \partial_\beta \frac{1}{\sqrt{\beta}} + \frac{z^2}{2} \cdot \partial_\beta \frac{1}{\sqrt{\beta}}\right] \tag{8}$$

$$= \pm\ C \cdot \left(\frac{\pi m}{\beta}\right)^{3/2} \cdot \left[\mp 2\sqrt{2} z + \frac{z^2}{2}\right] \tag{9}$$

$$= \pm\ g_s V \cdot k_B T \cdot \left(\frac{\pi m}{\beta h^2}\right)^{3/2} \cdot \left[\mp 2\sqrt{2} z + \frac{z^2}{2}\right] \tag{10}$$

$$= \pm\ g_s V \cdot k_B T \cdot \left(\frac{\pi m}{\beta h^2}\right)^{3/2} \cdot \left[\mp \sqrt{8} e^{\beta\mu} + \frac{1}{2} e^{2\beta\mu}\right] \tag{11}$$

2. Calculate the mean particle number $N = -\partial_\mu \Psi$ in the same order. Invert this relation to get $u = \rho\lambda^3/g_s$ as a function of z. (2.5 points)

Particle number $N = -\partial_\mu \Psi$:

$$N = \mp\, g_s V \cdot k_B T \cdot \beta \left(\frac{\pi m}{\beta h^2}\right)^{3/2} \cdot \left[\mp \sqrt{8} e^{\beta\mu} + e^{2\beta\mu}\right] \tag{12}$$

Density $\rho = N/V$:

$$\rho = \mp\, g_s \cdot k_B T \cdot \beta \left(\frac{\pi m}{\beta h^2}\right)^{3/2} \cdot \left[\mp \sqrt{8} e^{\beta\mu} + e^{2\beta\mu}\right] \tag{13}$$

With $u = \rho\lambda^3/g_s$ and $z = e^{\beta\mu}$:

$$\begin{aligned} u &= \mp\, k_B T \cdot \beta \left(\frac{\pi m \lambda^2}{\beta h^2}\right)^{3/2} \cdot \left[\mp \sqrt{8} e^{\beta\mu} + e^{2\beta\mu}\right] && (14)\\ &= \mp\, k_B T \cdot \beta \left(\frac{\pi m \lambda^2}{\beta h^2}\right)^{3/2} \cdot \left[\mp \sqrt{8} z + z^2\right] && (15) \end{aligned}$$

3. Combine your results for Ψ and u to obtain the first two terms for pressure p in an expansion in ρ to second order. *(2.5 points)*

$$\begin{aligned} p &= -\Psi/V && (16)\\ &= \mp\, g_s \cdot k_B T \cdot \left(\frac{\pi m}{\beta h^2}\right)^{3/2} \cdot \left[\mp \sqrt{8} e^{\beta\mu} + \frac{1}{2} e^{2\beta\mu}\right] && (17)\\ &= \ldots? && (18) \end{aligned}$$

4. Discuss your results for the classical limit and the first quantum correction. Where does degeneracy g_s show up? What do the differences in sign mean? *(1 point)*

The grandcanonical potential with the minus sign Ψ_- describes the ideal bosonic gas, while Ψ_+ describes the ideal fermionic gas.

2 Two-dimensional ideal Bose fluid *(7 points)*

Consider a *two*-dimensional, ideal Bose fluid.

a) What is the expression for the mean particle number for a 2D system? *(1 point)*

Recall that the particle number N for spin-0 bosons (degeneracy g = 1) for a 3D system is given by

$$N = \sum_{\vec{k}} n_{\vec{k}} = \frac{V}{h^3} \int \mathrm{d}\vec{p} \, \frac{1}{e^{\beta(\epsilon(p)-\mu)} - 1} \,, \tag{19}$$

where we encounter the sum over all modes and the Bose distribution. The corresponding expression for a 2D system is given by

$$N = \frac{A}{h^2} \int \mathrm{d}\vec{p} \, \frac{1}{e^{\beta(\epsilon(p)-\mu)} - 1} = \frac{2\pi A}{h^2} \int_0^\infty \mathrm{d}p \, \frac{p}{e^{\beta(\epsilon(p)-\mu)} - 1} \,. \tag{20}$$

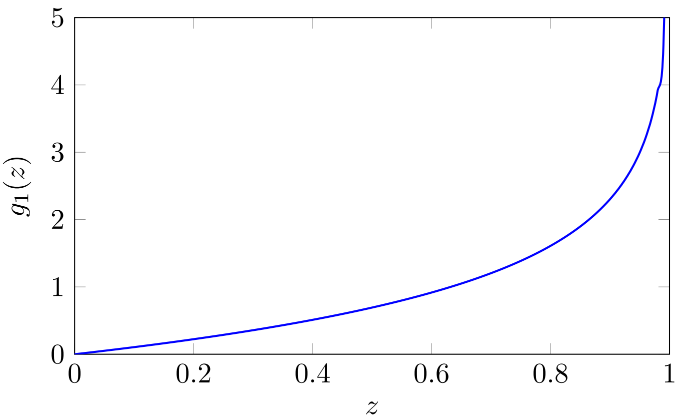
b) First consider particles with energy $\epsilon(p) = \frac{p^2}{2m}$. Evaluate the mean particle number and investigate whether there is a Bose-Einstein condensation (BEC) at finite temperature. *(2.5 points)*

With the formula from above, we get $\mathrm{d}p = m\mathrm{d}\epsilon/p$ and

$$\begin{aligned} N &= \frac{2\pi A}{h^2} \int_0^\infty \mathrm{d}p \, \frac{p}{e^{\beta(\epsilon(p)-\mu)} - 1} = \frac{2\pi m A}{h^2} \int_0^\infty \mathrm{d}\epsilon \, \frac{1}{e^{\beta(\epsilon-\mu)} - 1} \\ &= \frac{2\pi m A}{h^2} \sum_{l=1}^{\infty} e^{\beta\mu l} \int_0^\infty \mathrm{d}\epsilon \, e^{-\beta\epsilon l} = \frac{2\pi m A}{\beta h^2} \sum_{l=1}^{\infty} \frac{z^l}{l} \\ &= \frac{A}{\lambda^2} \, g_1(z) \,, \end{aligned} \tag{21}$$

where we introduced the fugacity $z = e^{\beta\mu}$, the thermal wavelength $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$ and the generalized Riemann Zeta function $g_\nu(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^\nu}$.

Plot $g_1(z) = -\ln(1-z)$ from $z = 0$ to $z = 1$:



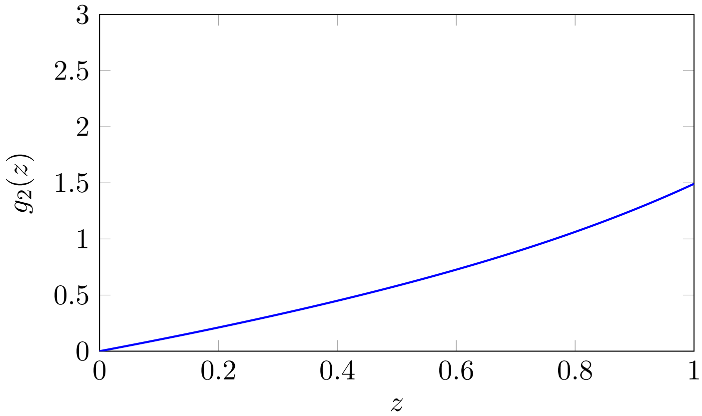
See that it diverges for $z \longrightarrow 1$. Thus, no BEC!

c) Now consider massless bosons with energy $\epsilon(p) = cp$. Again evaluate the mean particle number and investigate whether there is a BEC at finite temperature. ***(2.5 points)***

With the formula from above, we get $\mathrm{d}p = \mathrm{d}\epsilon/c$ and

$$\begin{aligned} N &= \frac{2\pi A}{h^2} \int_0^\infty \mathrm{d}p \, \frac{p}{e^{\beta(\epsilon(p)-\mu)} - 1} = \frac{2\pi A}{c^2 h^2} \int_0^\infty \mathrm{d}\epsilon \, \frac{\epsilon}{e^{\beta(\epsilon-\mu)} - 1} \\ &= \frac{2\pi A}{c^2 h^2} \sum_{l=1}^{\infty} e^{\beta\mu l} \int_0^\infty \mathrm{d}\epsilon \, \epsilon \, e^{-\beta\epsilon l} = \frac{2\pi A}{\beta^2 c^2 h^2} \sum_{l=1}^{\infty} \frac{z^l}{l^2} \\ &= \frac{2\pi A}{\beta^2 c^2 h^2} \, g_2(z) \, . \end{aligned} \tag{22}$$

Plot $g_2(z) = \mathrm{Li}_2(z)$ from $z = 0$ to $z = 1$:



See that it converges for $z \longrightarrow 1$. Thus, BEC!

d) In case you find a BEC, give the respective expression for the critical temperature. ***(1 point)***

We have found a BEC for the case of massless particles.
When the system reaches the critical temperature, the chemical potential is zero ($\mu = 0$).
Thus, by defining the particle density $\rho = N/A$, we get

$$\rho = \frac{2\pi}{\beta^2 c^2 h^2} \, g_2(1) \quad \longleftrightarrow \quad T_c = \frac{hc}{k_B} \sqrt{\frac{\rho}{2\pi g_2(1)}} = \frac{hc}{k_B} \sqrt{\frac{3\rho}{\pi^3}} \, . \tag{23}$$