Assignment 5

Handout 04.12.2020 – Return 10./11.12.2020 – Discussion 17./18.12.2020

Exercise 5.1 [8 points]: Mixture of ideal gases

Enclosed in a box of volume V are N_1 molecules of species 1 with mass m_1 and N_2 molecules of species 2 with mass m_2. The system can be considered as an ideal gas with the total energy E.

1. Calculate the phase space volume $\Omega(E, V, N_1, N_2)$ for the case $N_1 \gg 1$ and $N_2 \gg 1$. *(2 points)* HINT: As in the single species case, use a suitable transformation to obtain an integral over a $3N$ dimensional sphere. In the result, it is very helpful to introduce the reduced mass, $\mu := \frac{m_1 m_2}{m_1+m_2}$, and to identify a part that has the form of an ideal gas with $N = N_1 + N_2$ particles of mass μ.

2. Calculate the entropy of the system as function of the ratio N_1/N_2 for constant $N = N_1 + N_2$. Discuss the dependence of the entropy on N_1/N_2 and determine the value of N_1/N_2 at which the entropy is maximal. *(3 points)*

3. Calculate the pressure, $p = T\left(\frac{\partial S}{\partial V}\right)_{E,N_1,N_2}$ of the system. Determine how the different molecular species contribute to the total pressure. *(1 point)*

4. Consider a situation in which the container is separated into two volumes V_1 and V_2 containing only molecules of species 1 and 2, respectively. The total energy of the system is E and the two compartments are in thermal and mechanical equilibrium. Calculate the entropy of mixing, that is, compare the entropy of the demixed state with that of the mixture. *(2 points)*

Exercise 5.2 [4 points]: Thermodynamics of a black hole

A black hole is the end state of massive stars which are too heavy to support themselves under gravity and thus collapse. From general relativity it is known that a black hole of mass M has a radius

$$R = G\frac{2M}{c^2} \tag{1}$$

with G the gravitational constant and c the speed of light. Hawking calculated the emission of radiation from a black hole and from this concluded that its effective temperature is

$$T = \frac{\hbar c^3}{8\pi G M k_B}. \tag{2}$$

According to Einstein, the energy of a black hole of mass M is $E = Mc^2$.

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1. Calculate the specific heat of a black hole, which should come out to be negative. *(1 point)*
2. Calculate the entropy of the black hole using the definition of temperature and assuming the entropy is zero at mass $M = 0$. Show that the result is proportional to the surface area of the black hole, measured in units of the Planck length $L_P = \sqrt{\hbar G/c^3}$. *(2 points)*
3. Use the entropy of a black hole to estimate the maximum number of bits that can be stored in a black hole of small radius (e.g. one centimeter). The entropy of a black hole is huge; it absorbs a lot of information from the environment without giving it back. *(1 point)*

Exercise 5.3 [3 points]: Maxwell relations

Assume that we know the fundamental equation as $E = E(S, V, N)$ for some system of interest and that it is well-behaved (i.e. at least $\mathcal{C}^2$). Then it should not matter in which sequence we take the partial derivatives for a second derivative like $\frac{\partial^2 E}{\partial S \partial V}$. This is the basis of the so-called "Maxwell relations".

1. Use this idea to derive

$$\frac{\partial T}{\partial V}_{|S,N} = -\frac{\partial p}{\partial S}_{|V,N} . \tag{3}$$

   How many such Maxwell relations exist ? *(1.5 points)*

2. Calculate explicitly that the Maxwell relation Eq. (3) is true for the ideal gas. *(1.5 points)*
   HINT: Invert the known entropy $S(E, V, N)$ to get $E(S, V, N)$.