Three Body Problem

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Abstract

In this project, we have discussed the restricted Three-Body problem and investigated how the motion of a body with negligible mass varies with different initial conditions. We have used Maple to plot the graphs for non-linear differential equations. We assume that the motion takes place in two dimensions. We observe that the body with negligible mass revolves around the centre of mass of the three body system only when its relative angular velocity is non-zero. The number of revolutions and the time taken for the body to reach singularity depends on the initial conditions. We also note the existence of five Lagrange (equilibrium) points in the system.

1 Introduction

We consider two bodies that are in motion around their common centre of mass. The third body has negligible mass and is constrained by the motion of the two massive bodies. We set the origin to be the centre of mass of the two massive bodies. We define the three bodies to be O_1, O_2, and O_3 with respective masses M_1, M_2, and negligible mass m, with position vectors $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, and $\boldsymbol{r}$ from the origin, respectively.

Let the distance between the two massive bodies O_1 and O_2 be:

$$R = |\boldsymbol{r_1} - \boldsymbol{r_2}|$$

Let the combined mass of the two massive bodies O_1 and O_2 be:

$$M = M_1 + M_2$$

This implies that the potential energy of O_3 is:

$$U_3 = -\frac{GMm}{r}, \tag{1}$$

O_3 is of negligible mass and does not affect the motion of the two massive bodies. We are basically considering a restricted 3 body problem. To analyse the motion of O_3, we consider a rotating frame of reference with respect to the centre of mass of the system. The radius vector $\boldsymbol{R} = \boldsymbol{r_1} - \boldsymbol{r_2}$ is rotating with an angular velocity of ω in the inertial frame of reference [2]. In this frame the two massive bodies are at rest. Let us use cylindrical coordinates to analyze the motion:

ρ denotes the vector from the origin to O_3 in the x-y plane.

θ denotes the counterclockwise angle from the origin to O_3.

z denotes the vector from the origin to O_3, perpendicular to the x-y plane.

We know that $\mathfrak{L}$ is difference of the kinetic energy and the potential energy of the system. Since we are considering a rotating frame of reference, O_1 and O_2 are at rest. The kinetic energy is:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Where, $\boldsymbol{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is the vector from the origin to O_3. Let us convert this to cylindrical coordinates:

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta,$$
$$\implies \dot{x} = \dot{\rho}\cos\theta - \rho\sin\theta\dot{\theta}, \quad \dot{y} = \dot{\rho}\sin\theta + \rho\cos\theta\dot{\theta}, \quad \dot{z} = \dot{z}$$

Thus,

$$\begin{aligned}\dot{x}^2 + \dot{y}^2 &= (\dot{\rho}\cos\theta)^2 - 2(\dot{\rho}\cos\theta\rho\sin\theta\dot{\theta}) + (\rho\sin\theta\dot{\theta})^2 \\ &\quad + (\dot{\rho}\sin\theta)^2 + 2\dot{\rho}\sin\theta\rho\cos\theta\dot{\theta} + (\rho\cos\theta\dot{\theta})^2 \\ \implies \dot{x}^2 + \dot{y}^2 &= \dot{\rho}^2 + \rho^2\dot{\theta}^2\end{aligned}$$

Therefore, the kinetic energy is:

$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right)$$

Note that, since we have transformed to a rotating frame, we substitute $\theta - \omega t$ instead of θ. So, the Kinetic energy is:

$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2(\dot{\theta} - \omega)^2 + \dot{z}^2\right)$$

Let us compute the potential energy of the system. Consider the two body system O_1 and O_2. We know that the two massive bodies are at rest in our rotational frame of reference. Hence, the distance between them remains constant and we previously defined the distance to be R.

$$U = U_{12} + U_{21}$$

$$U = -\frac{GM_1M_2}{R} - \frac{GM_1M_2}{R}$$

We know that:

$$r^2 = x^2 + y^2 + z^2 = \rho^2 + z^2$$

Substituting for r in (1),

$$U_3 = -\frac{GMm}{(\rho^2 + z^2)^{1/2}}$$

Our Lagrangian is then:

$$\mathfrak{L} = \frac{1}{2}m(\dot{\rho}^2 + \rho^2(\dot{\theta} - \omega)^2 + \dot{z}^2) + \frac{GMm}{(\rho^2 + z^2)^{1/2}} - \frac{GM_1M_2}{R} - \frac{GM_1M_2}{R}$$

Now we notice that the last two terms do not depend upon ρ, θ or z. Hence, we ignore them when finding the Lagrange equations. Therefore, we get [2] :

$$\mathfrak{L} = \frac{1}{2}m(\dot{\rho}^2 + \rho^2(\dot{\theta} - \omega)^2 + \dot{z}^2) + \frac{GMm}{(\rho^2 + z^2)^{1/2}} \tag{2}$$

Let us analyze the equilibrium points of the system. A body placed at an equilibrium point remains at rest. Such points in space are called Lagrange Points. First, using Kepler's Laws, let us calculate ω:

$$\omega^2 = \frac{G(M_1 + M_2)}{R^3} \implies \omega = \sqrt{\frac{GM}{R^3}}$$

Since we are considering a rotating frame of reference, we obtain the forces on O_3 as [4] :

$$\boldsymbol{F_{O_3}} = -\frac{GM_1m}{|\boldsymbol{r} - \boldsymbol{r_1}|^3}(\boldsymbol{r} - \boldsymbol{r_1}) - \frac{GM_2m}{|\boldsymbol{r} - \boldsymbol{r_2}|^3}(\boldsymbol{r} - \boldsymbol{r_2}) + m\omega^2\boldsymbol{r} - 2m\omega \times \dot{\boldsymbol{r}} \tag{3}$$

2 Equations

First, let us compute the Lagrange equation for ρ using (2):

$$\begin{aligned}
\frac{\partial \mathfrak{L}}{\partial \rho} &= m\rho(\dot{\theta}-\omega)^2 - \frac{GMm\rho}{(\rho^2+z^2)^{3/2}}, \\
\frac{\partial \mathfrak{L}}{\partial \dot{\rho}} &= m\dot{\rho} \implies \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{\rho}} = m\ddot{\rho}, \\
\frac{\partial \mathfrak{L}}{\partial \rho} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{\rho}} &\implies m\rho(\dot{\theta}-\omega)^2 - \frac{GMm\rho}{(\rho^2+z^2)^{3/2}} = m\ddot{\rho} \\
&\implies \rho(\dot{\theta}-\omega)^2 - \frac{GM\rho}{(\rho^2+z^2)^{3/2}} = \ddot{\rho}, \qquad (\because m \neq 0)
\end{aligned}$$

Therefore we obtain the Lagrange equation for ρ as:

$$\rho(\dot{\theta}-\omega)^2 - \frac{GM\rho}{(\rho^2+z^2)^{3/2}} = \ddot{\rho}, \tag{4}$$

Let us compute the Lagrange equation for θ using (2):

$$\begin{aligned}
\frac{\partial \mathfrak{L}}{\partial \theta} &= 0, \\
\frac{\partial \mathfrak{L}}{\partial \dot{\theta}} &= m\rho^2(\dot{\theta}-\omega) \implies \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{\theta}} = 2m\rho\dot{\rho}(\dot{\theta}-\omega) + m\rho^2\ddot{\theta}, \\
\frac{\partial \mathfrak{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{\theta}} &\implies 2m\rho\dot{\rho}(\dot{\theta}-\omega) + m\rho^2\ddot{\theta} = 0, \\
&\implies 2\dot{\rho}(\dot{\theta}-\omega) + \rho\ddot{\theta} = 0, \qquad (\because m \neq 0, \rho \neq 0)
\end{aligned}$$

Therefore we obtain the Lagrange equation for θ as:

$$2\dot{\rho}(\dot{\theta}-\omega) + \rho\ddot{\theta} = 0 \tag{5}$$

Let us compute the Lagrange equation for z using (2):

$$\begin{aligned}
\frac{\partial \mathfrak{L}}{\partial z} &= -\frac{GMmz}{(\rho^2+z^2)^{3/2}}, \\
\frac{\partial \mathfrak{L}}{\partial \dot{z}} &= m\dot{z} \implies \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{z}} = m\ddot{z}, \\
\frac{\partial \mathfrak{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{z}} &\implies -\frac{GMmz}{(\rho^2+z^2)^{3/2}} = m\ddot{z}, \\
&\implies -\frac{GMz}{(\rho^2+z^2)^{3/2}} = \ddot{z}, \qquad\qquad (\because m \neq 0)
\end{aligned}$$

Therefore we obtain the Lagrange equation for z as:

$$-\frac{GMz}{(\rho^2+z^2)^{3/2}} = \ddot{z} \tag{6}$$

We obtain the Lagrange points of the system [1]:

$$L_1 : \left(R\left[1-\left(\frac{\alpha}{3}\right)^{1/3}\right], 0\right) \tag{7}$$

$$L_2 : \left(R\left[1+\left(\frac{\alpha}{3}\right)^{1/3}\right], 0\right) \tag{8}$$

$$L_3 : \left(-R\left[1+\frac{5}{12}\alpha\right], 0\right) \tag{9}$$

$$L_4 : \left(\frac{R}{2}\left(\frac{M_1-M_2}{M_1+M_2}\right), \frac{\sqrt{3}}{2}R\right) \tag{10}$$

$$L_5 : \left(\frac{R}{2}\left(\frac{M_1-M_2}{M_1+M_2}\right), \frac{-\sqrt{3}}{2}R\right) \tag{11}$$

3 Analysis

Let us apply dimensional analysis to the Lagrange equations we obtained in the previous section. Consider (4):

$$\rho(\dot{\theta}-\omega)^2 - \frac{GM\rho}{(\rho^2+z^2)^{3/2}} = \ddot{\rho}$$
$$\left[\rho(\dot{\theta}-\omega)^2\right] = \left[LT^{-2}\right]$$
$$\left[\frac{GM\rho}{(\rho^2+z^2)^{3/2}}\right] = \frac{[L^3M^{-1}T^{-2}][ML]}{[L^3]} = \left[LT^{-2}\right]$$
$$[\ddot{\rho}] = \left[LT^{-2}\right]$$

Notice that each term evaluates to the same dimension. Consider (5):

$$2\dot{\rho}(\dot{\theta}-\omega) + \rho\ddot{\theta} = 0$$
$$\left[2\dot{\rho}(\dot{\theta}-\omega)\right] = \left[LT^{-1}\right]\left[T^{-1}\right] = \left[LT^{-2}\right]$$
$$\left[\rho\ddot{\theta}\right] = [L][T^{-2}]$$

Notice that each term evaluates to the same dimension. Consider (6):

$$-\frac{GMz}{(\rho^2+z^2)^{3/2}} = \ddot{z}$$
$$\left[-\frac{GMz}{(\rho^2+z^2)^{3/2}}\right] = \frac{[L^3M^{-1}T^{-2}][ML]}{[L^3]} = [LT^{-2}]$$
$$[\ddot{z}] = \left[LT^{-2}\right]$$

Notice that each term evaluates to the same dimension. Therefore, the three Lagrange equations are dimensionally correct. For the analysis of Lagrange points, we define the constants [1]:

$$\alpha = \frac{M_2}{M_1 + M_2}, \beta = \frac{M_1}{M_1 + M_2}$$

To compute the first three Lagrange points, we set $\dot{\boldsymbol{r}}(t) = 0$ and look for solutions to $\boldsymbol{F_{O_3}} = 0$ ((3)). We are unable to compute solutions for general values of α. So, we obtain the approximate points ((7), (8), (9)) in the limit $\alpha \ll 1$ [1]. We find the other two Lagrange points ((10), (11)) by balancing the centrifugal force and the gravitational forces.

4 Nonlinear solutions

Case 1: Zero initial angular velocity:

Let us define the initial conditions:

$$G = 1,\ M = 10^6,\ \theta(0) = 0,\ z(0) = 0,\ \rho(0) = 10^3,\ \dot{\rho}(0) = 0,\ \dot{z}(0) = 0,$$

We also define $\omega = 1$. We wish to analyse the motion of O_3 such that its angular velocity in the rotating frame is zero. So, we define:

$$\frac{d\theta}{dt} = \omega = 1$$

In Figure 1, we notice that ρ decreases with time. This makes physical sense as gravitational force is attractive and O_3 moves towards the centre of mass of the system. Note that O_3 travels in a straight line to the origin. Also, notice that the slope of the graph becomes steeper as ρ decreases. This suggests that the speed of O_3 increases as the distance between the centre of mass and O_3 decreases. This also makes physical sense, as the total energy ($E = T + U$) of the three bodies is constant. As O_3 moves closer to the origin, ρ decreases, causing the decrease in total potential energy of the system:

$$U = -\frac{Gm}{(\rho^2 + z^2)^{1/2}} - \frac{GM_1M_2}{R} - \frac{GM_1M_2}{R}$$

This implies that the kinetic energy of the system must increase. This explains why the speed of the body increases as time increases. If we increase the initial value of ρ to 10^5 as shown in Figure 2, we observe that the motion is similar to the original case.

Case 2: Non-zero initial angular velocity:

We consider the same initial conditions as Case 1, except for $\frac{d\theta}{dt}$.

$$\frac{d\theta}{dt} = \omega + 0.00001 = 1.00001$$

We chose the above value so that the initial change in angular velocity in the rotating frame is non-zero but small. In Figure 3, we can see that initially ρ decreases. As explained in Case 1, this is due to gravitational attraction. Now we consider the portions of the graph where ρ increases with time. As gravitational force is attractive, the speed of the body decreases as time increases. We observe that ρ does not actually reach its initial magnitude.

Let us now, discuss the most interesting case, when the body is actually near the centre of mass. As we can see in Figure 3, there is a very large change in ρ and the sign of $\frac{d\rho}{dt}$ changes at some points. This suggests that the body approaches very close to the centre of mass where the gravitational attraction is very high. Then, the body revolves around the centre of mass, and moves away. However, to be sure that this is actually the case, we look at Figure 4. $\frac{d\theta}{dt}$ gives the angular velocity of O_3 in the rotating frame of reference. Let us define $\theta_1 = \theta - \omega t$ to be the angle in the inertial frame of reference. In Figure 4, we observe that θ_1 changes very steadily whenever the body is moving away or towards the centre of mass. We look at the regions where there is a large change in θ_1. Initially, $\delta\theta_1 = 6.165$. As $6.165/2Pi = 0.98$, we can say that the body nearly completes a full revolution around the centre of mass of the system. Note that $\delta\theta_1$ becomes larger with time. This means that the body revolves around the centre of mass more number of times. The interesting point to note here is that the large and steep changes in ρ and θ_1 occur at the same points in time. This verifies that the body revolves around the centre of mass of the system. Furthermore, in Figure 5, note that Maple cannot the graph after 45.408243 seconds, and reports that O_3 may have reached a singularity beyond that point. A possible explanation for this is that the body reaches the centre of mass at that point in time. As $\rho = 0$, and $z = 0$, the denominator becomes 0 in the Lagrange equations (4) and (6). Now, we modify ρ to be 10^5 in Figure 6. O_3 cycles many more times around the centre of mass before hitting the singularity. Furthermore, each time O_3 revolves, the value of ρ nearly reaches the initial value of ρ, which is in contrast to the previous case in Figure 3.

Case 3: Lagrange Points:

We consider the Initial Conditions as:

$$G = 1 \ M_1 = 10^{10} \ M_2 = 10^6 \ R = 10^8$$

The initial conditions are chosen such that we obtain $\alpha \ll 1$. In Figure 7, we plot the five Lagrange points. Observe that L_1 and L_2 are situated on either side of O_2. L_3 lies beyond O_1 in the direction opposite to O_2. L_4 and L_5 form equilateral triangles with O_1 and O_2.

5 Conclusions

In this project, we discussed specific cases of the restricted three body problem. Notably, we analyzed the case when the object with negligible mass has a non-zero relative angular velocity. For Case 1 in our non-linear solutions, we considered the body with negligible mass to arise at an arbitrary point in the two body system. For Case 2, we considered the body with negligible mass approached the two body system from infinity. This explains how it has an initial relative angular velocity. We note that the two cases contrast each other in such a way that it helps us understand more about the physics of the restricted three body problem.

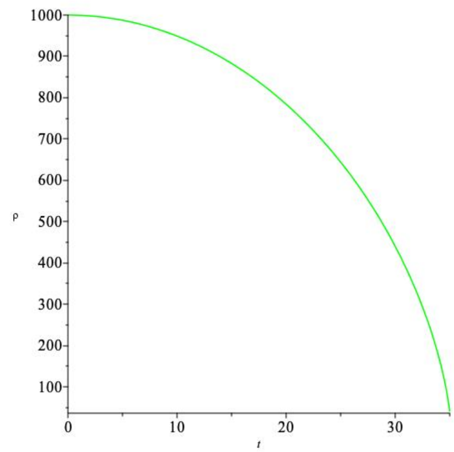


Figure 1: The graphs show how $\rho(t)$ varies with time when $\rho(0) = 10^3$

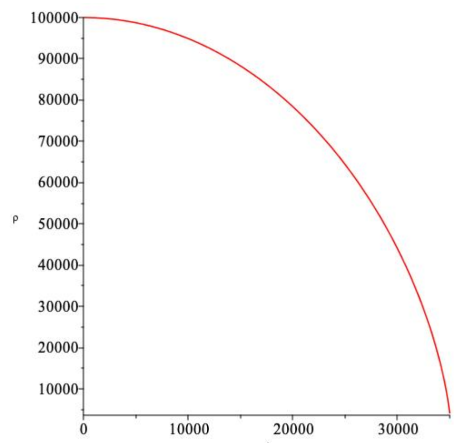


Figure 2: The graphs show how $\rho(t)$ varies with time when $\rho(0) = 10^5$

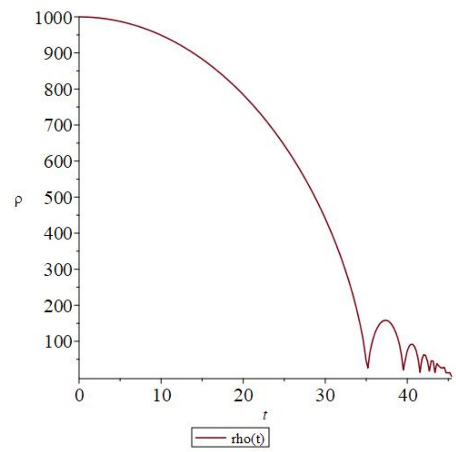


Figure 3: The graphs show how $\rho(t)$ varies with time for $0 \leq t \leq 45$.

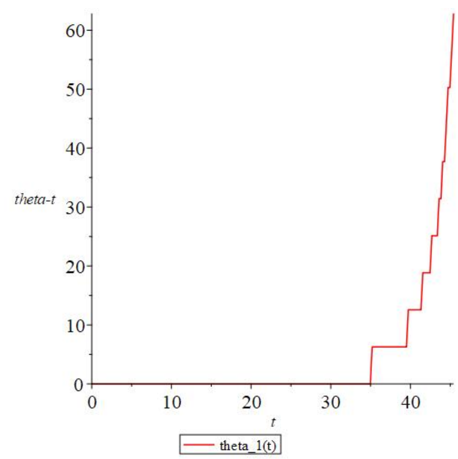


Figure 4: The graphs show how $\theta(t)$ varies with time for $0 \leq t \leq 45$.

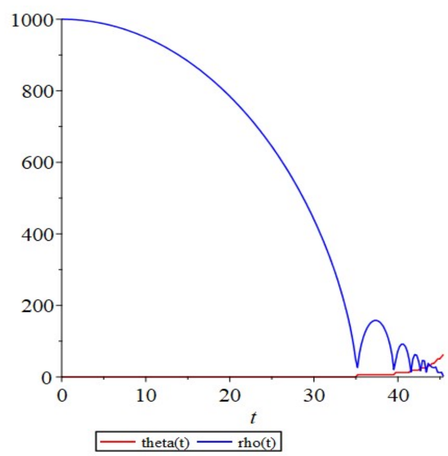


Figure 5: The graphs show how $\rho(t)$ and $\theta(t)$ vary with time for $0 \leq t \leq 45$.

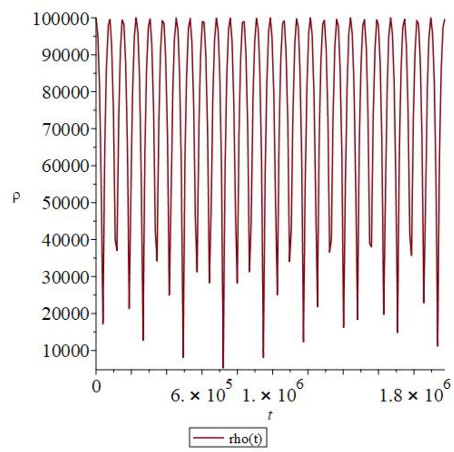


Figure 6: The graphs show how $\rho(t)$ varies with time in the case when $rho(0) = 10^5$.

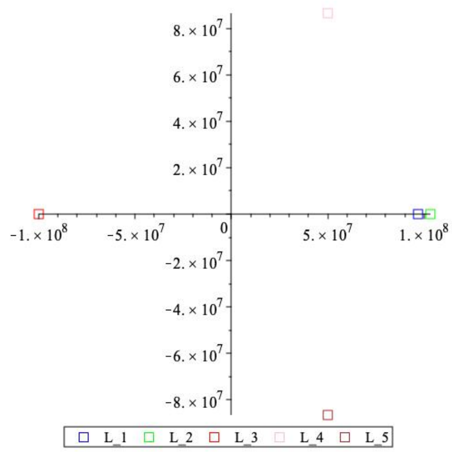


Figure 7: The graph shows the Lagrange points of the system where $M_1 = 10^{10}$, $M_2 = 10^6$, and $R = 10^8$.

References

[1] Cornish, N. J. (1998). *The Lagrange Points.* Retrieved from https://map.gsfc.nasa.gov/ContentMedia/lagrange.pdf

[2] Goldstein et al., (2002) *Classical Mechanics, Third edition* San Francisco.

[3] Taylor, J. R. *Classical Mechanics.* University Science Books, 2005.

[4] Westra, D. (2017). *Lagrangian Points.* Retrieved from https://www.mat.univie.ac.at/ westra/lagrangepoints.pdf

[5] Widnall, S. (2008). Lecture L18 - Exploring the Neighborhood: The Restricted Three-Body Problem. Retrieved November 06, 2020

$restart;$
$with(plots):$
$with(DEtools):$
#We define the Ordinary Differential Equations for the x, y, and z components respectively.

$ODEx := \text{rho}(t)\cdot(diff(\text{theta}(t), t) - \text{omega})^2 - \frac{G\cdot M\cdot\text{rho}(t)}{(\rho(t)^2 + z(t)^2)^{\frac{3}{2}}} = diff(\text{rho}(t), t, t);$

$$ODEx := \rho(t)\left(\frac{d}{dt}\theta(t) - \omega\right)^2 - \frac{G\,M\,\rho(t)}{\left(\rho(t)^2 + z(t)^2\right)^{3/2}} = \frac{d^2}{dt^2}\rho(t) \quad \textbf{(1)}$$

$ODEy := 2\cdot diff(\text{theta}(t) - \text{omega}\cdot t, t)\cdot diff(\text{rho}(t), t) + \text{rho}(t)\cdot diff(\text{theta}(t) - \text{omega}\cdot t, t, t) = 0;$

$$ODEy := 2\left(\frac{d}{dt}\theta(t) - \omega\right)\left(\frac{d}{dt}\rho(t)\right) + \rho(t)\left(\frac{d^2}{dt^2}\theta(t)\right) = 0 \quad \textbf{(2)}$$

$ODEz := \frac{-G\cdot M\cdot z(t)}{(\text{rho}(t)^2 + z(t)^2)^{\frac{3}{2}}} = diff(z(t), t, t);$

$$ODEz := -\frac{G\,M\,z(t)}{\left(\rho(t)^2 + z(t)^2\right)^{3/2}} = \frac{d^2}{dt^2}z(t) \quad \textbf{(3)}$$

$ICs := \text{rho}(0) = 10^3, \text{D}(\text{rho})(0) = 0, \text{theta}(0) = 0, \text{D}(\text{theta})(0) = \text{omega}, z(0) = 0, \text{D}(z)(0) = 0;$

$$ICs := \rho(0) = 1000, \text{D}(\rho)(0) = 0, \theta(0) = 0, \text{D}(\theta)(0) = \omega, z(0) = 0, \text{D}(z)(0) = 0 \quad \textbf{(4)}$$

$ICs2 := \text{rho}(0) = 10^5, \text{D}(\text{rho})(0) = 0, \text{theta}(0) = 0, \text{D}(\text{theta})(0) = \text{omega}, z(0) = 0, \text{D}(z)(0) = 0;$

$$ICs2 := \rho(0) = 100000, \text{D}(\rho)(0) = 0, \theta(0) = 0, \text{D}(\theta)(0) = \omega, z(0) = 0, \text{D}(z)(0) = 0 \quad \textbf{(5)}$$

$G := 1; M := 10^6; \text{omega} := 1;$

$$G := 1$$
$$M := 1000000$$
$$\omega := 1 \quad \textbf{(6)}$$

$sol\_list := dsolve(\{ODEx, ODEy, ODEz, ICs\}, numeric, output = listprocedure)$

$$sol\_list := \left[t = \textbf{proc}(t)\ \dots\ \textbf{end proc}, \rho(t) = \textbf{proc}(t)\ \dots\ \textbf{end proc}, \frac{d}{dt}\rho(t) = \textbf{proc}(t) \quad \textbf{(7)}\right.$$

...

$$\textbf{end proc}, \theta(t) = \textbf{proc}(t)\ \dots\ \textbf{end proc}, \frac{d}{dt}\theta(t) = \textbf{proc}(t)\ \dots\ \textbf{end proc}, z(t) = \textbf{proc}(t)$$

...

$$\left.\textbf{end proc}, \frac{d}{dt}z(t) = \textbf{proc}(t)\ \dots\ \textbf{end proc}\right]$$

$sol\_list2 := dsolve(\{ODEx, ODEy, ODEz, ICs2\}, numeric, output = listprocedure)$

$$sol\_list2 := \left[t = \mathbf{proc}(t) \;\ldots\; \mathbf{end\ proc}, \rho(t) = \mathbf{proc}(t) \;\ldots\; \mathbf{end\ proc}, \frac{d}{dt}\rho(t) = \mathbf{proc}(t) \right. \tag{8}$$

...

$$\mathbf{end\ proc}, \theta(t) = \mathbf{proc}(t) \;\ldots\; \mathbf{end\ proc}, \frac{d}{dt}\theta(t) = \mathbf{proc}(t) \;\ldots\; \mathbf{end\ proc}, z(t) = \mathbf{proc}(t)$$

...

$$\left. \mathbf{end\ proc}, \frac{d}{dt} z(t) = \mathbf{proc}(t) \;\ldots\; \mathbf{end\ proc} \right]$$

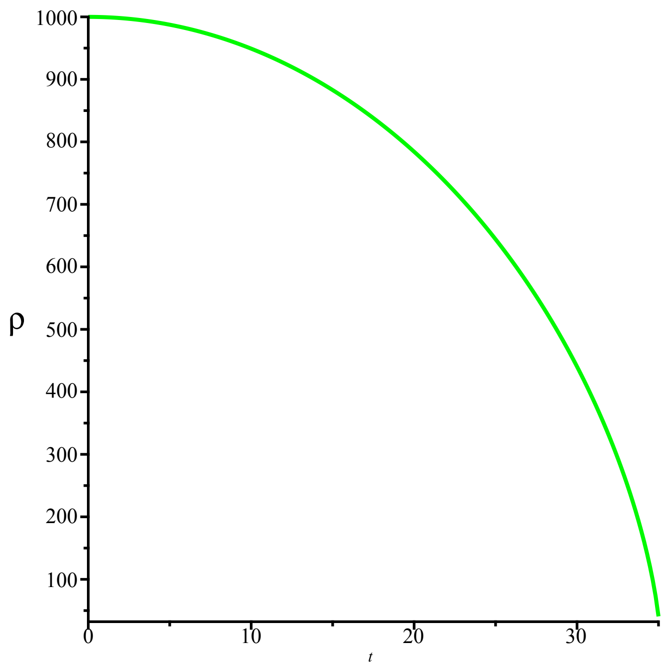
$rho\_sol := rhs(sol\_list[2])$

$$rho\_sol := \mathbf{proc}(t) \;\ldots\; \mathbf{end\ proc} \tag{9}$$

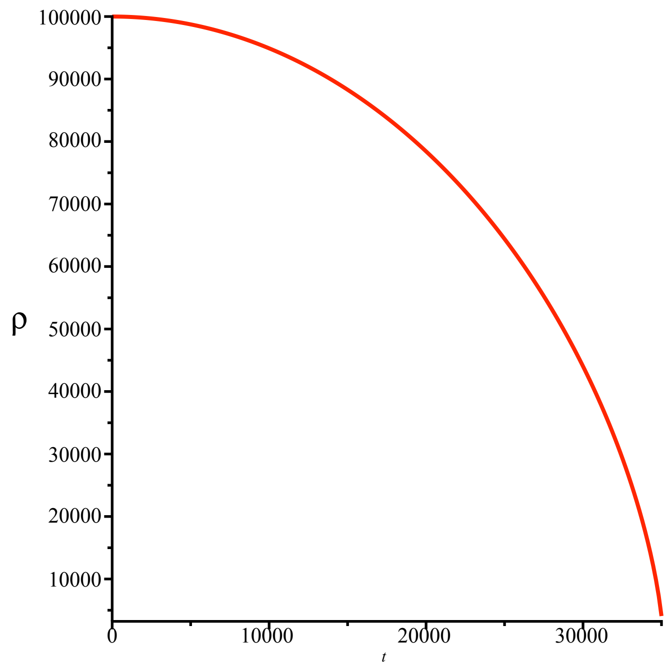
$rho\_sol2 := rhs(sol\_list2[2])$

$$rho\_sol2 := \mathbf{proc}(t) \;\ldots\; \mathbf{end\ proc} \tag{10}$$

$plot1 := odeplot(rho\_sol, [t, \mathrm{rho}(t)], t = 0..35, color = green)$



$plot2 := odeplot(rho\_sol2, [t, \mathrm{rho}(t)], t = 0..35000, color = red)$



100000
90000
80000
70000
60000
ρ
50000
40000
30000
20000
10000
0
10000
20000
30000
t

$restart;$
$with(plots):$

$$ODEx := \mathrm{rho}(t)\cdot(diff(\mathrm{theta}(t),t) - omega\_1)^2 - \frac{G\cdot M\cdot \mathrm{rho}(t)}{(\rho(t)^2 + z(t)^2)^{\frac{3}{2}}} = diff(\mathrm{rho}(t),t,t);$$

$$ODEx := \rho(t)\left(\frac{d}{dt}\,\theta(t) - 1\right)^2 - \frac{1000000\,\rho(t)}{\left(\rho(t)^2 + z(t)^2\right)^{3/2}} = \frac{d^2}{dt^2}\,\rho(t) \tag{1}$$

$$ODEy := 2\cdot(diff(\mathrm{theta}(t),t) - omega\_1)\cdot diff(\mathrm{rho}(t),t) + \mathrm{rho}(t)\cdot diff(\mathrm{theta}(t),t,t) = 0;$$

$$ODEy := 2\left(\frac{d}{dt}\,\theta(t) - 1\right)\left(\frac{d}{dt}\,\rho(t)\right) + \rho(t)\left(\frac{d^2}{dt^2}\,\theta(t)\right) = 0 \tag{2}$$

$$ODEz := \frac{-G\cdot M\cdot z(t)}{(\mathrm{rho}(t)^2 + z(t)^2)^{\frac{3}{2}}} = diff(z(t),t,t);$$

$$ODEz := -\frac{1000000\,z(t)}{\left(\rho(t)^2 + z(t)^2\right)^{3/2}} = \frac{d^2}{dt^2}\,z(t) \tag{3}$$

$$ICs := \mathrm{rho}(0) = 1000, \mathrm{D}(\mathrm{rho})(0) = 0, \mathrm{theta}(0) = 0, \mathrm{D}(\mathrm{theta})(0) = 1.00001, z(0) = 0, \mathrm{D}(z)(0) = 0;$$

$$ICs := \rho(0) = 1000, \mathrm{D}(\rho)(0) = 0, \theta(0) = 0, \mathrm{D}(\theta)(0) = 1.00001, z(0) = 0, \mathrm{D}(z)(0) = 0 \tag{4}$$

#Now we define the constants:

$G := 1; M := 10^6; omega\_1 := 1;$

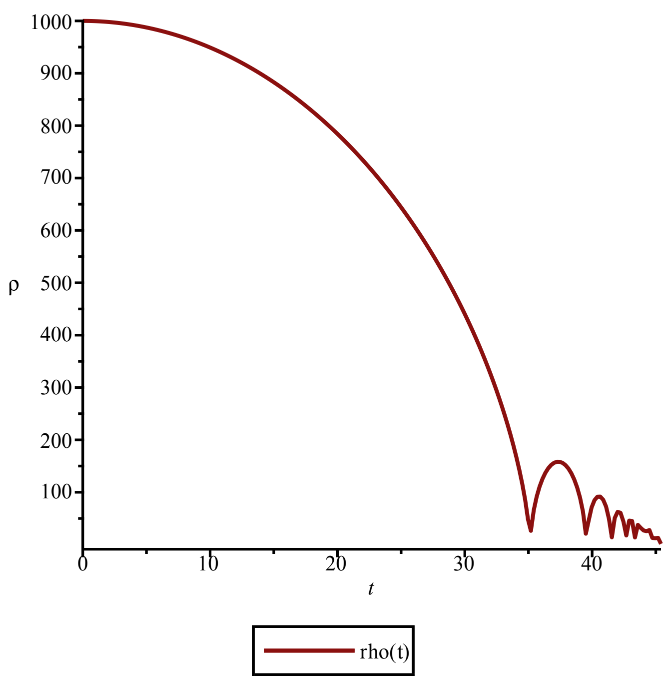
$$G := 1$$
$$M := 1000000$$
$$omega\_1 := 1 \tag{5}$$

$sol := dsolve(\{ODEx, ODEy, ODEz, ICs\}, numeric)$

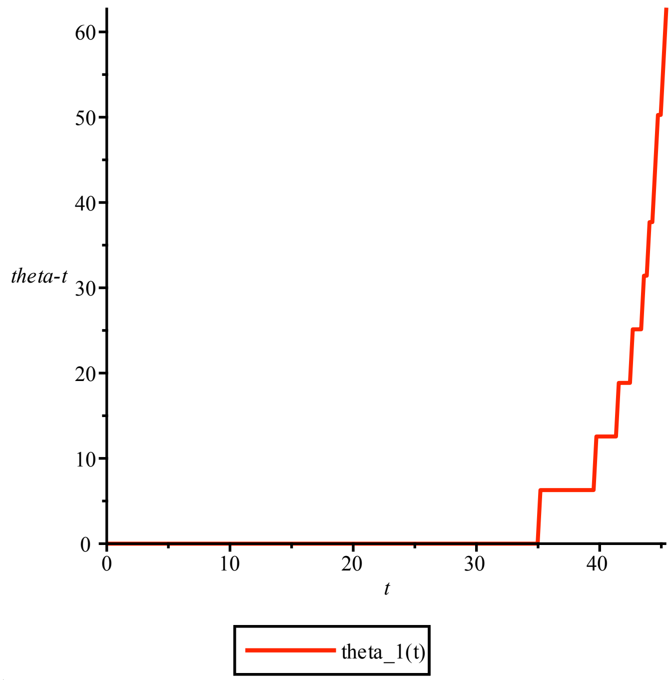
$$sol := \mathbf{proc}(x\_rkf45) \ \ldots \ \mathbf{end\ proc} \tag{6}$$

$plot1 := odeplot(sol, [t, \mathrm{rho}(t)], t = 0..45.408243, \ legend = \text{"rho(t)"}) :$
$plot2 := odeplot(sol, [t, \mathrm{theta}(t) - t\cdot omega\_1], t = 0..45.408243, color = red, legend = \text{"theta\_1(t)"}) :$

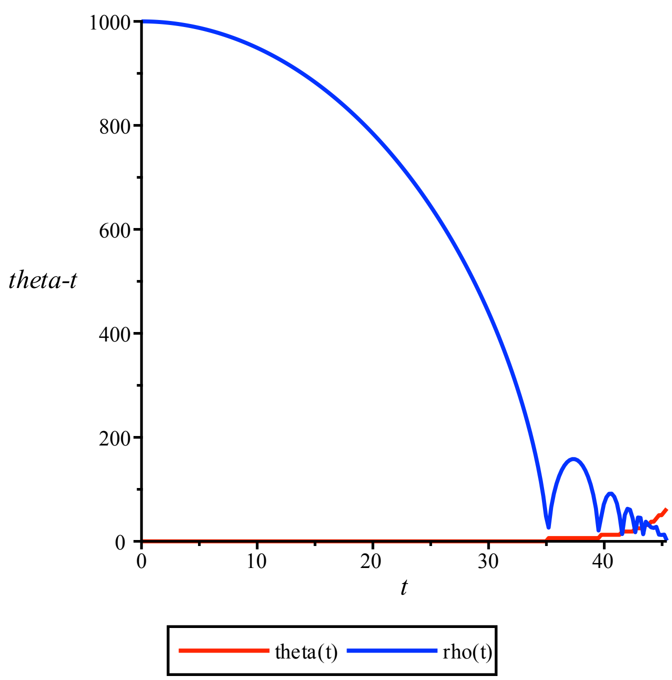
$display([plot1]);$



$display([plot2]);$



display([*plot2*, *plot1*]);



#Now we solve for different initial conditions:

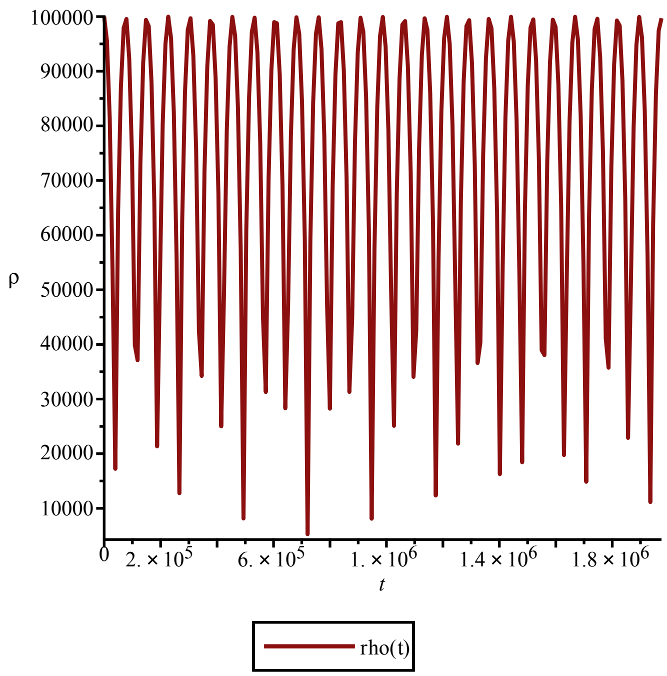
$ICs1 := \text{rho}(0) = 100000, \text{D}(\text{rho})(0) = 0, \text{theta}(0) = 0, \text{D}(\text{theta})(0) = 1.00001, z(0) = 0, \text{D}(z)(0) = 0;$

$$ICs1 := \rho(0) = 100000, \mathrm{D}(\rho)(0) = 0, \theta(0) = 0, \mathrm{D}(\theta)(0) = 1.00001, z(0) = 0, \mathrm{D}(z)(0) = 0 \quad \textbf{(7)}$$

sol := *dsolve*({*ODEx*, *ODEy*, *ODEz*, *ICs1*}, *numeric*)

sol := **proc**(*x_rkf45*) ... **end proc** **(8)**

plot3 := *odeplot*(*sol*, [*t*, rho(*t*)], *t* = 0 ..1974390, *legend* = "rho(t)") :
display([*plot3*])



100000
90000
80000
70000
60000
50000
40000
30000
20000
10000
ρ
0
2. × 10^5
6. × 10^5
1. × 10^6
1.4 × 10^6
1.8 × 10^6
t
rho(t)

```
restart;
with(plots):
with(DEtools):
with(plottools):
```

Lagrange Points Approximation.

$M\_1 := 10^{10}$

$$M\_1 := 10000000000 \tag{1}$$

$M\_2 := 10^{6}$

$$M\_2 := 1000000 \tag{2}$$

$R := 10^{8}$

$$R := 100000000 \tag{3}$$

$\text{alpha} := \dfrac{M\_2}{M\_1 + M\_2}$

$$\alpha := \frac{1}{10001} \tag{4}$$

$p1 := point\left(\left[R\cdot\left(1-\left(\dfrac{\text{alpha}}{3}\right)^{\left(\frac{1}{3}\right)}\right), 0\right], color = blue, symbol = box, symbolsize = 15, legend = \text{"L\_1"}\right)$

$$p1 := POINTS\left(\left[\begin{array}{cc} 9.67818093200000\ 10^{7} & 0. \end{array}\right], COLOUR(RGB, 0., 0., 1.00000000), LEGEND(\text{"L\_1"}), SYMBOL(BOX, 15)\right) \tag{5}$$

$p2 := point\left(\left[R\cdot\left(1+\left(\dfrac{\text{alpha}}{3}\right)^{\left(\frac{1}{3}\right)}\right), 0\right], color = green, symbol = box, symbolsize = 15, legend = \text{"L\_2"}\right)$

$$p2 := POINTS\left(\left[\begin{array}{cc} 1.03218190700000\ 10^{8} & 0. \end{array}\right], COLOUR(RGB, 0., 1.00000000, 0.), LEGEND(\text{"L\_2"}), SYMBOL(BOX, 15)\right) \tag{6}$$

$p3 := point\left(\left[-R\cdot\left(1+\left(\dfrac{5\cdot\text{alpha}}{12}\right)\right), 0\right], color = red, symbol = box, symbolsize = 15, legend = \text{"L\_3"}\right)$

$$p3 := POINTS\left(\left[\begin{array}{cc} -1.00004166200000\ 10^{8} & 0. \end{array}\right], COLOUR(RGB, 1.00000000, 0., 0.), \right. \tag{7}$$

$LEGEND("L\_3"), SYMBOL(BOX, 15))$

$$p4 := point\left(\left[\frac{R}{2}\cdot\left(\frac{(M\_1 - M\_2)}{M\_1 + M\_2}\right), \frac{\mathrm{sqrt}(3)}{2}\cdot R\right], color = pink, symbol = box, symbolsize = 15, legend = \text{"L\_4"}\right)$$

$$p4 := POINTS\left(\left[\begin{array}{cc} 4.9990001\ 10^{7} & 8.66025404000000\ 10^{7}\end{array}\right], COLOUR(RGB, 1.00000000, 0.75294118, 0.79607843), LEGEND(\text{"L\_4"}), SYMBOL(BOX, 15)\right) \quad \textbf{(8)}$$

$$p5 := point\left(\left[\frac{R}{2}\cdot\left(\frac{(M\_1 - M\_2)}{M\_1 + M\_2}\right), \frac{-\mathrm{sqrt}(3)}{2}\cdot R\right], color = brown, symbol = box, symbolsize = 15, legend = \text{"L\_5"}\right)$$

$$p5 := POINTS\left(\left[\begin{array}{cc} 4.9990001\ 10^{7} & -8.66025404000000\ 10^{7}\end{array}\right], COLOUR(RGB, 0.64705882, 0.16470588, 0.16470588), LEGEND(\text{"L\_5"}), SYMBOL(BOX, 15)\right) \quad \textbf{(9)}$$

$display(p1, p2, p3, p4, p5)$