Amrita School of Engineering, Bengaluru-35
23MAT106
Mathematics for Intelligent Systems – 1
Lab Practice Sheet-5
(Span of vectors, Fundamental Subspaces of a matrix)

- **3D Scatterplot of space spanned by given vectors / 3D Scatterplot of a Vector Space given basis vectors**
  - **Span of one vector:**

```
b1=[1;1;1] % enter basis vector as a column vector
pts=[]
for i=1:1000
  k1=-1+2*rand(1) %generates 1000 random numbers(scalars) between -1 and 1
  pts=[pts,k1*b1];
                  %generates 1000 points in the span set in the form of 3×1000 matrix
end
scatter3(pts(1,:),pts(2,:),pts(3,:),1);
        %gives the scatterplot of generated 1000 points. Each point size is given as 1.
```

(Check 'help scatter3' to find the variety of scatter3 options w.r.t. size and colour.)

  - **Span of two vectors:**

```
b1=[1;0;0]  % enter one basis vector as a column vector
b2=[0;1;0]  % enter second basis vector as a column vector
pts=[]
for i=1:10000
     k1=-1+2*rand(1) %generates 10000 random numbers(scalars) between -1 and 1
     k2=-1+2*rand(1) %generates 10000 random numbers(scalars) between -1 and 1
     pts=[pts,k1*b1+k2*b2];
                  %generates 10000 points in the span set in the form of 3×10000 matrix
end
scatter3(pts(1,:),pts(2,:),pts(3,:),1);
        %gives the scatterplot of generated 10000 points
```

- **Obtaining the basis for rowspace, columnspace, nullspace and the nullity of a given matrix**

```
A=randi([0,9],5,2)*randi([0,9],2,5)
[RR, ic]=rref(A);
        % RR is the row-reduced echelon form of A,
             ic is a row vector that tells the column position of the leading pivotal element.
r = length(ic);
R=RR(1:r,:)
        % gives a matrix, whose row vectors forms the basis of row-space of A
C=A(:,ic)
        % gives a matrix, whose column vectors forms the basis of column-space of A
N=null(A)
        % gives a matrix, whose column vectors forms the basis of null-space of A
S=size(N);
Nullity=S(:,2)
        % gives the nullity of the matrix
```

- ✓ Another way to find nullity: S=size(A);NC=S(:,2);Nulity=NC-rank(A)
- ✓ Another way to find basis of column space is to find the basis of rowspace of A^T
- ✓ Basis for left null space can be found similarly using the command null(A^T)

- **3D scatterplot of Row space and column space of any random 3×3 matrix with rank 2**
    - ```
      A= randi([-3,3],3,2)*randi([-3,3],2,3)
      [RR, ic]=rref(A);
      r = length(ic);
      R=RR(1:r,:)
      RSB1=(R(1,:))’; RSB2=(R(2,:))’;
                % generating basis vectors of row space and writing it as column vectors
      [RR, ic]=rref(A’);
      r = length(ic);
      C=RR(1:r,:)
      CSB1=(C(1,:))’; CSB2=(C(2,:))’;
                % generating basis vectors of column space as column vectors
      RSpts=[];
      CSpts=[];
      for i=1:10000
          k1=-1+2*rand(1);
          k2=-1+2*rand(1);
          a1=-1+2*rand(1);
          a2=-1+2*rand(1) ;
          RSpts=[RSpts,k1*RSB1+k2*RSB2];
          CSpts=[CSpts,a1*CSB1+a2*CSB2];
      end
      scatter3(RSpts(1,:),RSpts(2,:),RSpts(3,:),1);
      hold on
      scatter3(CSpts(1,:),CSpts(2,:),CSpts(3,:),1);
      ```

- **How to check if a given vector y is in any of the subspaces generated by a matrix A ?**
    - If rank of $[\mathrm{A} \mid \mathbf{y}] = \mathrm{rank}(\mathrm{A})$, then y in column space of A
      where $[\mathrm{A} \mid \mathbf{y}]$ is matrix with y vector appended as last column of A
    - If rank of $\begin{bmatrix} \mathrm{A} \\ \mathbf{y}^{\mathrm{T}} \end{bmatrix} = \mathrm{rank}(\mathrm{A})$, then y is in row space of A
      where $\begin{bmatrix} \mathrm{A} \\ \mathbf{y}^{\mathrm{T}} \end{bmatrix}$ is matrix with y^{T} vector appended as last row of A.
    - If $A\boldsymbol{y} = \mathbf{0}$**,** then **y** is in the null space of A.
    - If $\boldsymbol{y}^T A = \mathbf{0}$**,** then **y** is in the left null space of A.

- **Generation of two orthogonal vectors to a given 3D vector**
    - ```
      A=[1,2,3];
      N=null(A);
                % gives a matrix with the two columns that are orthogonal to vector A
                  gives the matrix that has each column vector as orthogonal basis of nullspace of A
      OrthVec1=N(:,1);
      OrthVec2=N(:,2);
      ```

Practise Questions

1. Find a 3D scatter plot of the span of the vector (1,0,2).
2. Find a 3D scatter plot of the span of the vectors (1,1,0) and (0,1,0). Is the span set same as XY plane?
3. Find a scatter plot of the span of the vectors (1,2) and (-2,1).

4. Find the null space and row space of $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Explain what each of them represents geometrically and plot them in MATLAB using scatter plot.
5. Find the null space of the following matrices manually. Also find the scatterplot of these null spaces

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

6. Generate a random 3×3 matrix of rank 2. Provide the scatter plot of the row-space and the column space of the matrix in the same figure.
7. Generate a random 3×3 matrix of rank 1. Provide the scatter plot of the column-space and the left null space of the matrix in the same figure.
8. Given, a matrix, $A = \begin{pmatrix} 1 & 3 & 4 & 7 \\ 2 & 4 & 6 & 10 \\ 3 & 5 & 8 & 13 \\ 4 & 6 & 10 & 16 \end{pmatrix}$. Find out the subspace associated with A in which each of the following vectors lie ?

(i) $u = \begin{pmatrix} -2 \\ -3 \\ 1 \\ 1 \end{pmatrix}$, (ii)$v = \begin{pmatrix} 5 \\ 8 \\ 11 \\ 14 \end{pmatrix}$ (iii) $w = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ (iv) $y = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}$ (v) $m = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$

9. Given, a matrix, $A = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & 1 & 4 \\ 1 & 1 & 3 & 1 \\ 2 & 0 & 5 & 4 \end{pmatrix}$. Find out the subspace associated with A in which each of the following vectors lie ?

(i) $v = \begin{pmatrix} 5 \\ 1 \\ -2 \\ 0 \end{pmatrix}$, (ii) $w = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 2 \end{pmatrix}$, (iii) $u = \begin{pmatrix} -1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$ (iv) $m = \begin{pmatrix} 3 \\ -1 \\ 7 \\ 7 \end{pmatrix}$

10. Using MATLAB generate a 9×9 matrix A of rank 2.
    (a) Obtain a symmetric matrix B=A+A' and find rank of B.
    (b) Find the eigenvalues and eigenvectors of B using MATLAB.
    (c) Are their complex eigenvalues?(Check the same for more symmetric matrices). Explain why or why not?
    (d) How many eigenvalues of B are zero ? Is there a relation between the number of zero eigenvalues of B and the rank of B? Explain.
    (e) Verify if the eigenvectors of B are orthogonal.
    (f) Find a basis for row space, column space and null space of B using MATLAB.
    (g) Find the nullity of A and B.

11. Using MATLAB generate a 10×5 matrix A of rank 3.
    (a) Obtain a symmetric matrix $S1=A*A^T$ and $S2=A^T*A$
    (b) Find the rank of S1 and S2.
    (c) Find the eigenvalues of S1 and S2. What do you notice about them? Explain.
    (d) Find the nullity of A, S1 and S2.