23MAT112
Mathematics for Intelligent Systems – 2
Practice Sheet – 2
Fundamental subspaces, Left Inverse, Right Inverse, Projection, Least Squares

1. Row vectors of matrix product AB is a linear combination of row vectors of B. Thus row space of AB and B are same (rref(AB) will be same as rref(B)). Using this concept, we can obtain infinitely many matrices that has the same rref as B, by pre-multiplying it by any suitable matrix A.

   a. Create 5 different 3×4 matrices that have the rref matrix as $R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   Answer: A = randi([0,9],3,3)*R. Can be verified by checking rref(A).

   b. Create a 5×4 matrix that has non-zero rows in rref same as in $R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   Answer: B = randi([0,9],5,3)*R. Can be verified by checking rref(B).

   c. Create a 4×4 matrix that has non-zero rows in rref same as in $R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   Answer: B = randi([0,9],4,3)*R. Can be verified by checking rref(B).

2. Every matrix $A_{m\times n}$ matrix of rank r can be decomposed into A=CR, where C is an m×r matrix with column vectors as the independent columns of A and R is an r×n matrix with row vectors as non-zero rows in rref(A).

   C and R can be obtained in MATLAB using these lines of code:

```
%Enter matrix A
[RR, ic]=rref(A);
r = length(ic);
R=RR(1:r, :);
C=A(:, ic);
%Verify by checking A=CR
```

   Decompose the following matrices in A=CR form and verify the result.

   a. $A = \begin{pmatrix} 1 & 3 & 4 & 7 \\ 2 & 4 & 6 & 10 \\ 3 & 5 & 8 & 13 \\ 4 & 6 & 10 & 16 \end{pmatrix}$ b. $B = \begin{pmatrix} 35 & 28 & 35 & 14 \\ 40 & 32 & 32 & 14 \\ 40 & 32 & 24 & 12 \\ 75 & 60 & 67 & 28 \end{pmatrix}$

   c. Any random 7×9 integer matrix with rank 3.

   d. Any random 9×4 integer matrix with rank 3.

3. **Left Inverse of a matrix:** If all columns of matrix A of size mxn are independent (or rank(A) equals n), then the matrix has a left inverse such that

$$A_{LI} \times A = I_{n\times n}; \quad A_{LI} \equiv \textit{Left} \text{ inverse of A}$$

$$A_{LI} = \left(A^T A\right)^{-1} A^T$$

If all columns are independent, you may notice that 'm' should be greater than or equal to 'n'. If m = n , then it is a square matrix and left inverse is same as original inverse.

Perform the following experiments:

a. Create a random 4 by 3 matrix

b. Check whether rank (A) = n =3

c. Find Left inverse of A

Answer: ALI= inv(A'*A)*A'

d. Check whether ALI*A produces an identity matrix.

4. **Right Inverse of a matrix:** If all rows of matrix A of size mxn are independent (or rank(A) equals m) , then the matrix has a right inverse such that

$$A \times A_{RI} = I_{m\times m}; \quad A_{RI} \equiv \textit{Right} \text{ inverse of A}$$

$$A_{RI} = A^T \left(AA^T\right)^{-1}$$

Create such matrices and verify the result.

If all rows are independent, you may notice that 'm' should be less than or equal to 'n'. If m = n, then it is a square matrix and right inverse is same as original inverse.

Perform the following experiments:

a. Create a random 5 by 7 matrix

b. Check whether rank (A) = m = 5

c. Find right inverse of A

Answer: ARI= A'*inv(A*A')

d. Check whether A* ARI produces an identity matrix.

5. **Pseudoinverse of an m×n matrix, A**

Every matrix $A_{m\times n}$ has a pseudoinverse $\mathbf{A^+}$ or **pinv(A)**

- If m = n = rank of A, then A is invertible and $\mathbf{A^+ = A^{-1}}$
- If rank of A = n, (all columns of A are independent), then $\mathbf{A^+ = (A^TA)^{-1}A^T}$ = **Left Inverse of A,** and $\mathbf{A^+A = I_n}$.
- If rank of A = m, (all rows of A are independent), then $\mathbf{A^+ = A^T (AA^T)^{-1}}$ = **Right Inverse of A,** and $\mathbf{AA^+ = I_m}$.

- Pseudoinverse $\mathbf{A}^{+}$ of any matrix A can be obtained in MATLAB using the command 'pinv(A)'.

  a. Generate a 9×9 random integer matrix and verify that inv(A) = pinv(A).
  b. Generate a 5×4 random integer matrix and verify that pinv(A) is same as the left inverse of A.
  c. Generate a 3×7 random integer matrix and verify that pinv(A) is same as the right inverse of A.

6. Projection of a vector $\boldsymbol{v}$ on a subspace S, which is a column space of matrix A(with all independent columns) is $\mathbf{p} = A(A^TA)^{-1}A^T\mathbf{v}$.

   If some columns in A are dependent (i.e., if $\text{Rank}(A_{m\times n}) < n$), then the projection of a vector $\boldsymbol{v}$ on a subspace S, which is a column space of matrix A is $\mathbf{p}$ = A*pinv(A)*$\mathbf{v}$, where pinv(A) is the pseudo inverse of A.

   a. Find the projection of $v = \begin{pmatrix} -3 \\ -3 \\ 8 \\ 9 \end{pmatrix}$ on the space spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$.

   b. Find the projection of $u = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ on the space spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

   c. Find the projection of $u = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ on the column space of $= \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \end{pmatrix}$.