Amrita School of Engineering, Bengaluru-35
23MAT106
Mathematics for Intelligent Systems – 1
Lab Practice Sheet-6
(Eigenvalues, Eigenvectors)

- A=[1,2;5,4]; E = eig(A)

  % produces a column vector E containing the eigenvalues of a square matrix A.

- A=[1,2;5,4]; [V,D] = eig(A)

  % produces a diagonal matrix D of eigenvalues and a full matrix V whose columns are the corresponding eigenvectors so that A*V = V*D.

- **Finding eigenvalues and eigenvectors of a random square matrix of order 3**

```
A=randi([-5,5],3)
[V, D] = eig(A);
Lambda1=D(1,1);
Lambda2=D(2,2);
Lambda3=D(3,3);
Eigenvector1=V(:,1);
Eigenvector2=V(:,2);
Eigenvector3=V(:,3);
```

- A=randi([-5,5],3); [R,D,L] = eig(A)

  % produces a diagonal matrix D of eigenvalues and a full matrix R whose columns are the corresponding right eigenvectors (eigenvectors) and also produces a full matrix L whose columns are the corresponding left eigenvectors so that L'*A = D*L'

- **Finding Characteristic polynomial of a matrix**

```
A=  [ 1 2 3 ; 2 3 4; 3 5 8];
syms   x;
charpoly(A,x)
```

  % Command for obtaining characteristic polynomial of a matrix A.

- M=magic(9);eig(M)

  % Finds eigenvalues of a magic square matrix of order 9.
  magic(N) is an N-by-N matrix constructed from the integers 1 through N^2 with equal row, column, and diagonal sums. Produces valid magic squares for all N > 0 except N = 2.

- **Generation of a matrix with given eigenvalues and eigenvectors**

```
Ld1=1;
Ld2=2;
Ld3=3;
Ev1=[1;1;1];
Ev2=[2;1;-1];
Ev3=[1;2;-1];
V=[Ev1,Ev2,Ev3]
D=diag([Ld1,Ld2,Ld3])
A=V*D*inv(V)
```

  % Generates a matrix A with given eigenvalues Ld1,Ld2,Ld3 and independent eigenvectors Ev1,Ev2 and Ev3.

Practise Questions

1. Find the eigenvalues and eigenvectors of A, B and C:

$$A = \begin{bmatrix} -45 & 153 & 3 \\ -18 & 60 & 1 \\ 8 & -26 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -13 & -42 & 98 \\ 73 & 199 & -469 \\ 29 & 78 & -184 \end{bmatrix} \quad C= \begin{bmatrix} -1 & 2 & 20 & -22 & 2 & 2 \\ -5 & 6 & 20 & -22 & 2 & 2 \\ -4 & 0 & 25 & -22 & 2 & 2 \\ -3 & 0 & 18 & -16 & 2 & 2 \\ -2 & 0 & 12 & -12 & 3 & 2 \\ -1 & 0 & 6 & -6 & 0 & 4 \end{bmatrix}$$

2. For the matrices in the previous question, verify the following properties:
   (a) Trace of a matrix = sum of eigenvalues of the matrix
   (b) Det of a matrix = Product of eigenvalues of the matrix

3. Enter an upper triangular matrix of order 7 and find its eigenvalues.
4. Enter an anti-diagonal matrix of order 6 and find it’s eigenvalues.

   Hint: Examples of anti-diagonal matrices are A=$\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$; B=$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

5. Enter an anti-diagonal matrix of order 5 and find it’s eigenvalues.

6. Generate a magic matrix of order 5 and find it’s eigenvalues. Do you notice anything about it’s eigenvalue?
   Hint: Compare the row sum/ column sum of the magic matrix and eigenvalues.

7. Generate random integer square matrices A of order 4, B of order 10 and C of order 15. Find their eigenvalues and eigenvectors.
   (a) Verify the following properties of eigenvalues for A, B and C:
      i. Trace of a matrix = sum of eigenvalues of the matrix
      ii. Det of a matrix = Product of eigenvalues of the matrix
      iii. Complex eigenvalues of a matrix appear in pairs
   (b) Find eigenvalues of 5A, B/3 and C^{-1}. Compare them with eigenvalues of A, B and C respectively. What do you observe?
   (c) Find eigenvectors of 5A, B/3 and C^{-1}. Compare them with eigenvectors of A, B and C respectively. What do you observe?
   (d) From A, using S=A+A’, generate a symmetric matrix. Find the eigenvalues and eigenvectors of S.

8. Generate a random integer square matrix A of order 3.
   (a) Find the eigenvalues, right eigenvectors and left eigenvectors of this matrix.
   (b) Find the characteristic polynomial for this matrix.
   (c) Replace x in the characteristic polynomial by the matrix A and evaluate the value. What is the value obtained? Try this with other square matrices. The result obtained verifies the **Cayley-Hamilton theorem.**
   Cayley-Hamilton theorem – Every square matrix satisfies it’s own characteristic equation.

9. Generate an integer matrix A of order 5×3. Find AA^T and A^TA. Evaluate the eigenvalues of AA^T and A^TA. Verify the result that “ **The non-zero eigenvalues of AA^T and A^TA are same**”. This result will be the base for many applications in future.

10. Generate a matrix with eigenvalues as 3,5,7,11 and eigenvectors as [1,1,2,2], [1,0,1,0],[2,1,2,0] and [0,1,1,0].

11. Using MATLAB generate a 5 × 5 matrix A of rank 2.
    (a) Obtain a symmetric matrix B=A+A’ and find rank of B.
    (b) Find the eigenvalues and eigenvectors of B using MATLAB.
    (c) Are their complex eigenvalues? (Check the same for more symmetric matrices). Explain why or why not?

(d) How many eigenvalues of B are zero? Is there a relation between the number of zero eigenvalues of B and the rank of B?
(e) Verify if the eigenvectors of B are orthogonal.

12. Using MATLAB generate a 10×5 matrix A of rank 3.
    (a) Obtain a symmetric matrix $S1=A*A^T$ and $S2=A^T*A$
    (b) Find the rank of S1 and S2.
    (c) Find the eigenvalues of S1 and S2. What do you notice about them? Explain.

13. Using MATLAB generate a 7×7 matrix A of rank 2.
    (a) Obtain a skew-symmetric matrix B=A-A' and find rank of B.
    (b) Find the eigenvalues and eigenvectors of B using MATLAB.
    (c) Are their real eigenvalues? (Check the same for more skew-symmetric matrices). Explain why or why not?
    (d) Do you have one eigenvalue as zero? Can you explain why?
    (e) In a similar manner generate a skew-symmetric matrix C from an 8×8 matrix. Does C have a zero eigenvalue? Why or Why not?

14. Generate a random integer row vector V of dimension 10.
    (a) How many independent vectors can you produce that are orthogonal to V. Explain. Also find these orthogonal vectors.
    (b) Find $W=V^T V$.
    (c) Find the rank and eigenvalues of W.
    (d) Find $U=VV^T$.
    (e) Verify the result $\mathrm{Trace}(VV^T) = \mathrm{Trace}(V^TV)= \|V\|^2$, where $\|V\|$ is Euclidean/Frobenius norm of vector/matrix V

    Euclidean norm of a vector $\boldsymbol{v} = (v_1, v_2, \dots, v_n)$ is $\|v\| = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2}$

    Frobenius norm of a matrix $M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $\|M\| = \sqrt{{a_{11}}^2 + {a_{12}}^2 + {a_{21}}^2 + {a_{22}}^2}$

15. Generate a matrix that will transform the points on a circular membrane to an elliptic membrane with a stretch factor of 5 and 9 along the directions [1,1] and [1,-1] respectively.