

23MAT112
Mathematics for Intelligent Systems – 2
Practice Sheet – 4
LU, QR, SVD, Polar Decompositions

- LU Decomposition using MATLAB

```
>>[L,U] = lu(A)
```

% returns an upper triangular matrix in U and a permuted lower triangular matrix in L, such that $A = L \cdot U$. The input matrix A can be full or sparse.

- Using MATLAB we can find the QR decomposition of a matrix A by:

```
>> [Q,R]=qr(A,0)
```

- QR Decomposition for finding eigenvalues of matrices numerically :

```
A=randi(10,2,2);
```

```
eig(A)
```

```
B=A;
```

```
for i= 1:10
```

```
[Q R]=qr(B,0); % B=QR.
```

```
B=R*Q;
```

% Creating a new B=RQ and repeat the computation. B=RQ retains eigenvalues of A (as RQ is similar to A) but approaches to an upper triangular matrix. That is, diagonal elements of B approaches to eigenvalues of A.

```
end
```

```
diag(B)
```

% Note that eig(A) and diag(B) are almost the same values.

- Singular Value Decomposition using MATLAB

```
>>[U,Z,V] = svd(A)
```

% produces a diagonal matrix Z, of the same dimension as A and with nonnegative diagonal elements in decreasing order, and unitary matrices U and V so that $A = U \cdot Z \cdot V'$.

```
>>Z = svd(A)
```

% returns a vector containing the singular values.

- Understanding of SVD through computational experiments using MATLAB

1. First find the SVD of a matrix, $A = \begin{bmatrix} 9 & 10 \\ 10 & 7 \\ 2 & 1 \end{bmatrix}$.

```
>> format bank % display results in 2 decimal places
```

```
>>A=[9 10; 10 7; 2 1];
```

```
>>[U Z V]=svd(A); % full SVD
```

2. Verify that U and V are orthogonal by evaluating the following:

```
>> U'*U
```

```
>>U*U'
```

```
>> V'*V
```

```
>>V*V'
```

3. Compute eigenvectors of $A^T A$ and compare with the columns of V.

```
>> [V1, L1]=eig(A'*A)
```

```
>>V
```

% Notice that the column vectors of V1 and V are the same, that is the eigenvectors of $A^T A$.

4. Compute eigenvectors of AA^T and compare with the columns of U.

```
>> [U1, L]=eig(A*A')
```

```
>>U
```

% Notice that the column vectors of U1 and U are the same, that is the eigenvectors of AA^T .

5. Compute the eigenvalues of $A^T A$ and AA^T and compare with the singular values of A.

```
>> eig(A'*A)
```

```
>>eig(A*A')
```

```
>>S=diag(Z) % singular values of A are diagonal elements of Z
```

```
>>Ssquare=S.^2 % square of singular values
```

% Notice that the square of singular values of A are same as the non-zero eigenvalues of $A^T A$ and AA^T .

6. Find the Frobenious norm of A and compare with the sum of squares of singular values of A (sum of non-zero eigenvalues of $A^T A$ and AA^T).

```
>> norm(A,'fro')^2 % square of Frobenious norm,  $\sum_i \sum_j a_{ij}^2$  for a matrix  $[a_{ij}]$ 
```

```
>> S=diag(Z); Sum=sum(S.^2) % sum of squares of singular values of A
```

% Notice that sum of squares of singular values of A is equal to square of the Frobenius norm of the matrix A and this value is the total variation of data given in form of matrix A.

7. Find the rank and nullity of the matrix A.

```
>> r=rank(A); m=size(A); nullity=m(2)-r
```

8. Verify that the third column in U forms a basis for the left nullspace of A.

```
>> A'*U(:,3).
```

% Notice that this is a zero vector, hence column vectors of A are orthogonal to third column in U and hence it is a basis for left nullspace

- Given the SVD of a matrix is $A=U\Sigma V^T$, the pseudo-inverse of A can be decomposed as $A^+ = V \Sigma^+ U^T$

```
>> A=[1,1,0;0,1,1];[ u,z,v]=svd(A)
```

```
>> pseudoA=pinv(A);
```

```
>> v*pinv(z)*u' % Verify if this is same as pseudoA
```

- Polar Decomposition of a matrix $A=QS$, where $Q=UV^T$ and $S=V\Sigma V^T$, U and V are matrices in SVD of A

```
>> A=[3,0;4,5];
```

```
>> [u,z,v]=svd(A);
```

```
>> q=u*v';
```

```
>> s=v*z*v'
```

```
>> q*s % Verify if it is A
```

Practice questions

1. Using QR decomposition find the eigenvalues of a random integer symmetric matrix of order 15 and verify the result using direct evaluation of eigenvalues.
2. Generate a random integer 7 by 9 matrix A of rank 6.
 - (a) Verify the rank of A is 6
 - (b) Find the SVD of the matrix
 - (c) Using SVD find an orthonormal basis for all the fundamental subspaces of the matrix A
 - (d) Verify the results obtained in (c) by checking the orthogonality of RS & NS and CS & LNS.
 - (e) Find the singular values of A
 - (f) Find the sum of square of the singular values of A using Frobenious norm.
 - (g) Write a MATLAB code to obtain the singular values of A without using the command ‘svd’.
 - (h) Find the polar decomposition of the matrix
 - (i) Find A^+ , the pseudo-inverse of A and show the decomposition of it
3. Find the polar decomposition ($A=QS$) of the matrix $A = \begin{bmatrix} 9 & 3 \\ 4 & 5 \end{bmatrix}$. Verify that Q is an orthogonal matrix and using the graph of the surface $f(\mathbf{x}) = \mathbf{x}^T \mathbf{S} \mathbf{x}$, verify that S is a positive definite matrix.
4. Using QR decomposition find the eigenvalues of the following matrices using 50 iterations in MATLAB.
 - (a) 4th order Pascal’s matrix
 - (b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 5 & 7 & 0 \end{bmatrix}$
 - (c) $B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$
 - (d) a random 50 by 50 matrix
5. Find the polar decomposition of the matrix: $A = \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}$.
6. Generate a square matrix A of order 9 and rank 9. Find the polar decomposition $A=QS$ of this matrix. Verify that the matrix S is positive definite.
7. Generate a square matrix A of order 9 and rank 7. Find the polar decomposition $A=QS$ of this matrix. Verify that the matrix S is not positive definite but is positive semi-definite.
8. Find the pseudo inverse of A (in question 5, 6 and 7) and SVD of A. Then verify that:
9. Generate a 5 by 3 matrix of rank 3. Using SVD find a basis for the left null space of the matrix and the null space of the matrix.
10. Generate a 6 by 5 matrix of rank 4. Using SVD find a basis for all the fundamental subspaces of matrix A
11. Solve $9x-2y+3z=1$, $4x+9y-5z=3$, $6x-y-z=3$; using LU decomposition.
12. Express the Pascal matrix of order 7 as LU using elementary matrices, by reducing it to upper triangular form manually.