

Volatility modeling

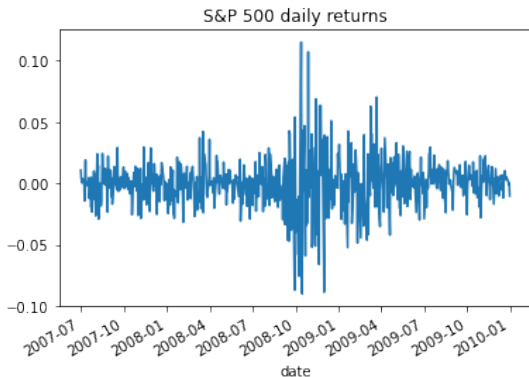
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Motivation: Time-varying, persistent volatility



- Many datasets, especially financial data, exhibit time-varying and persistent volatility. We'd like to model this for many reasons, especially risk management.
- Note that it is really the *persistence* that is important to model.

Outline

- Engle's ARCH-LM test for persistence in volatility.
- Modeling persistence in volatility with GARCH.
- Application: Value-at-risk calculations.

A general theme is to use r_t^2 as an estimate of σ_t^2 , and adapt the tools that we already know from prior weeks around this idea.

Engle's ARCH-LM test for persistence in volatility

When σ_t^2 is high, is σ_{t+1}^2 likely to also be high?

- This is like checking for time-series effects in any dataset.
- Could we just apply one of our existing tests to data on σ_t^2 ?
- The major problem is, you can't directly measure σ_t^2 .
- Instead, focus on r_t^2 as an indirect measure.

The standard test is known as *Engle's test* or the *ARCH-LM* test.

- Run the following regression: $r_t^2 = \alpha + \beta r_{t-1}^2 + \varepsilon_t$
- Under the null of no persistence in σ_t^2 , we have $N \times R^2 \sim \chi^2$.
- Implemented in statsmodels with the *het_arch* command.
- You can also extend the regression to include p lags of r_t^2 .
Then the test statistic is asymptotically $\chi^2(p)$.
By default, *het_arch* will choose an “optimal” number of lags.

Modeling persistent volatility: Moving average

One popular way to measure volatility over time is through a moving average. This works much like the moving average from week 1.

But, a complication is that you cannot actually measure volatility at any point in time. Instead, the moving-average estimate of volatility is based on an average of r_t^2 terms:

$$\sigma_{t,CMA}^2 = \frac{1}{m} \left(\frac{1}{2} r_{t-m}^2 + r_{t-m+1}^2 + \cdots + r_{t+m-1}^2 + \frac{1}{2} r_{t+m}^2 \right)$$

This is a fast and easy way to describe and plot volatility over time.

Modeling persistent volatility: EWMA

- EWMA stands for Exponentially Weighted Moving Average.
- The basic idea is like exponential smoothing from week 2.
- You choose a smoothing parameter α , and an initial σ_0^2 , which is often the sample value from the first few observations.
- Then update the values of σ_t following the formula:

$$\sigma_{t+1}^2 = \alpha \times r_t^2 + (1 - \alpha) \times \sigma_t^2$$

- Higher α moves you close to just estimating tomorrow's volatility with today's r^2 . Lower α gives smoother results that react more slowly to changes, for better and for worse.

Smoothing vs model-based approaches

- We can make some of the same comments here that we did after we covered Holt-Winters forecasting:
- Smoothing-based approaches give an easy and effective way to describe the data and build accurate forecasts.
If that is our only goal, they are likely to be good enough.
- But these approaches will break down as we move our attention to other goals like building confidence intervals, or understanding relationships between different series.
- At some point we have to adopt the *model-based* approach that is more standard in statistics.
- Before, this led us to ARMA. Here it leads us to a set of volatility models that will be our main focus.

Modeling persistence in volatility: ARCH

The simplest and oldest volatility model is called ARCH
(which stands for “**A**uto**R**egressive **C**onditional **H**eteroskedasticity”)

$$y_t = \mu + u_t \quad , \quad u_t \sim \mathcal{N}(0, \sigma_t^2)$$
$$\sigma_t^2 = \omega + \alpha u_{t-1}^2$$

- The first line could be replaced with any ARMA model for y_t . However, in practice volatility modeling is most important in situations where there probably is no predictability in y_t .
- The important thing is the second line: When today's (squared) error is large, tomorrow's error has larger variance, through α .
- More generally, an ARCH(q) model includes q lags of u_t^2 . The above model is an ARCH(1).
- ARCH models are typically estimated by MLE.

Modeling persistence in volatility: GARCH

In practice, ARCH models have generally been replaced by GARCH (“**G**eneralized **A**uto**R**egressive **C**onditional **H**eteroskedasticity”)

$$y_t = \mu + u_t \quad , \quad u_t \sim \mathcal{N}(0, \sigma_t^2) \\ \sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2$$

- The new thing compared to ARCH is the $\beta \sigma_{t-1}^2$ term.
This extra flexibility greatly improves the model performance.
- This general category of models is called GARCH(p,q), where
 - p is the number of lags of σ_t^2 ,
 - q is the number of lags of u_t^2 ,

The above model is a GARCH(1,1).

In this class we will only look at ARCH(1) and GARCH(1,1).

- Again, GARCH models are typically estimated by MLE.

Finding a good model for volatility in the data

- Decide on an ARMA model for y (often just $y_t = \mu + u_t$). In the first step, don't yet model any volatility effects.
- Estimate the model, apply Engle's ARCH-LM test to the sample residuals \hat{u}_t , and examine ACF and PACF plots of \hat{u}_t^2 .
- If the test rejects, or you see important patterns in the plots, add a simple ARCH or GARCH specification to your model.
- Apply Engle's test to the *standardized* residuals $\hat{u}_t/\hat{\sigma}_t$, where $\hat{\sigma}_t$ are the model's estimates of volatility over time (labeled as `conditional_volatility` in Python's ARCH results). And plot the ACF and PACF of the squared values $(\hat{u}_t/\hat{\sigma}_t)^2$.
- Add just enough ARCH and GARCH terms, so that the test fails to reject and you don't see important patterns in the plots.

Application: Value-at-risk (VAR)

- The **value-at-risk** (VAR) of a portfolio is a percentile of its return distribution over some specified amount of time.
- For example, the 10-day, 5% value-at-risk is the 5th percentile of the distribution of returns over the next 10 days.
 - To be more precise, it is the *absolute value* of this number, because we are usually thinking about low percentiles (losses).
- Example: If the 10-day, 5% VAR is 8%, we would say you have a 5% chance of *losing more than 8%* over the next 10 days.
 - It is also common to express VAR in dollar terms:
If your portfolio is worth \$1m, and the 10-day 5% VAR is 8%, then you have a 5% chance of losing more than \$80k.

Calculating VAR from a normal distribution

- Suppose you have daily return data ending at time T , and you want to know the α % VAR over the next K days.
- Let's use $r_{T \rightarrow T+K}$ to label the return over the next K days. The calculation is easy, if we know the *distribution* of $r_{T \rightarrow T+K}$.
- We will use log returns, so $r_{T \rightarrow T+K} = r_{T+1} + r_{T+2} + \dots + r_{T+K}$.
- Assume that each day's log return follows a normal distribution.
(This is not critical, but it also is not a bad approximation.)
- Then $r_{T \rightarrow T+K}$ is also normal (a sum of normal variables).
If we can find its mean $\mu_{T \rightarrow T+K}$ and its volatility $\sigma_{T \rightarrow T+K}$, then the VAR is just $\mu_{T \rightarrow T+K}$ minus a multiple of $\sigma_{T \rightarrow T+K}$.
 - A 5% VAR is approximately $\mu_{T \rightarrow T+K} - 1.645 \times \sigma_{T \rightarrow T+K}$.

Calculating $\mu_{T \rightarrow T+K}$ from daily data

- We typically assume there is no predictability in stock returns.
- Then $\mu_{T \rightarrow T+K} = K \times \mu$, where μ is the average *one-day* return.

This follows from:

$$\mathbb{E}[r_{T \rightarrow T+K}] = \mathbb{E}[r_{T+1}] + \mathbb{E}[r_{T+2}] + \dots + \mathbb{E}[r_{T+K}] = \underbrace{\mu + \mu + \dots + \mu}_{K \text{ times}}$$

- If you *do* think there is predictability in returns, you can just fit a model such as ARMA and forecast in the standard way.
But in practice, this is not likely to give you meaningful results.

Calculating $\sigma_{T \rightarrow T+K}$, ignoring predictability in σ

- As in the previous slide, let's start with the simplest case, where we just assume that σ is completely unpredictable.
 - (But to be clear, the whole point of this topic is that this is *not* the appropriate approach, and will give us misleading results.)
- In this case, $\sigma_{T \rightarrow T+K} = \sqrt{K} \times \sigma$, where σ is one-day volatility. This follows from

$$\begin{aligned}\text{Var}(r_{T \rightarrow T+K}) &= \text{Var}(r_{T+1} + r_{T+2} + \dots + r_{T+K}) \\ &= \text{Var}(r_{T+1}) + \dots + \text{Var}(r_{T+K}) = \underbrace{\sigma^2 + \dots + \sigma^2}_{K \text{ times}} = K \times \sigma^2\end{aligned}$$

Then the K -day, 5% VAR would be $K \times \mu - 1.645 \times \sqrt{K} \times \sigma$.

Calculating $\sigma_{T \rightarrow T+K}$, modeling predictability in σ

- Revisit the previous calculation. Now suppose σ is predictable.
- The first step from before is still correct:

$$\begin{aligned}\text{Var}(r_{T \rightarrow T+K}) &= \text{Var}(r_{T+1} + r_{T+2} + \dots + r_{T+K}) \\ &= \text{Var}(r_{T+1}) + \text{Var}(r_{T+2}) + \dots + \text{Var}(r_{T+K})\end{aligned}$$

This is still correct because daily returns are uncorrelated.

- Now replace each $\text{Var}(r)$ with a *forecast* of that day's variance.
 - Before, we assumed that variance/volatility were unpredictable, so our forecast for each day was just the one-day variance σ^2 .
 - Now, we instead use forecasts from a GARCH model.

How to build volatility forecasts from a GARCH model

- Suppose you have chosen and estimated a GARCH model, as described earlier. Then, you can forecast future values of σ^2 following a similar approach as with ARMA models.
- In statsmodels, the `forecast()` function will do this for you. Let's understand how it works.
- For example, suppose your model was `GARCH(1,1)`. This means you specified that $\sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2$, and you now have estimates of all these things up to time T .
- Your first forecast σ_{T+1}^2 is just $\hat{\omega} + \hat{\alpha} \hat{u}_T^2 + \hat{\beta} \hat{\sigma}_T^2$.
 - In statsmodels, $\hat{\omega}$, $\hat{\alpha}$, and $\hat{\beta}$ are reported in "params"; \hat{u} is in "resid," and $\hat{\sigma}$ is in "conditional_volatility".
- What about the later forecasts? See next slide...

Forecasting volatility with a GARCH model (2)

What is your forecast of σ_{T+2}^2 ?

- The model says $\sigma_{T+2}^2 = \omega + \alpha u_{T+1}^2 + \beta \sigma_{T+1}^2$.
- We can use the same estimates of ω , α , and β as we did before, and for σ_{T+1}^2 we substitute the forecast from the previous slide.
- But what about u_{T+1}^2 ? Although our best forecast of u_{T+1} is zero, our best forecast of its *squared* value is its variance σ_{T+1}^2 . So we substitute the forecast from the previous slide there too!
- This means our forecast of σ_{T+2}^2 is $\hat{\omega} + (\hat{\alpha} + \hat{\beta}) \times \hat{\sigma}_{T+1}^2$, where $\hat{\sigma}_{T+1}^2$ is the forecast calculated on the previous slide.
- All later forecasts follow the same logic.
For example we forecast σ_{T+3}^2 as $\hat{\omega} + (\hat{\alpha} + \hat{\beta}) \times \hat{\sigma}_{T+2}^2$, and so on.

Back to VAR calculations

We can finally complete the VAR calculation from earlier:

$$\begin{aligned}\text{Var}(r_{T \rightarrow T+K}) &= \text{Var}(r_{T+1} + r_{T+2} + \dots + r_{T+K}) \\ &= \text{Var}(r_{T+1}) + \text{Var}(r_{T+2}) + \dots + \text{Var}(r_{T+K}) \\ &= \hat{\sigma}_{T+1}^2 + \hat{\sigma}_{T+2}^2 + \dots + \hat{\sigma}_{T+K}^2\end{aligned}$$

and just substitute in the forecasts from the previous two slides. The square root of this sum is $\sigma_{T \rightarrow T+K}$, the volatility of $r_{T \rightarrow T+K}$.

Finally, we can calculate the VAR:

Start with $\mu_{T \rightarrow T+K}$, which is just $K \times \mu$, and subtract an appropriate multiple of $\sigma_{T \rightarrow T+K}$. For a 5% VAR, this multiple is 1.645.

Illustration: VAR during a crisis

- The notebook on Canvas illustrates today's concepts: You imagine doing a VAR calculation in April 2020.
- This basically comes down to estimating $\sigma_{T \rightarrow T+K}$.
- If you ignore predictability in σ , you will report a VAR that is much too optimistic. That is, you will understate your risk.
 - This approach implicitly assumes that the next K days will look like the “typical” day in the data.
 - This is not appropriate, because high volatility tends to persist, and volatility in Apr 2020 was at an all-time high!
- A GARCH model takes into account that (1) recent volatility is high, and (2) this means future volatility will also be high.
- When you use GARCH to forecast $\sigma_{T \rightarrow T+K}$, you report a VAR that is more realistic, and much larger.