

# Elaboration on the kinetic approach of Derbenev and Kondratenko to spin-polarized beams in electron storage rings \*

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## Abstract

We present a detailed account of the kinetic approach for describing the effect of synchrotron radiation on electron and positron spin polarization in storage rings. This approach was introduced in 1974 by Derbenev and Kondratenko and was extended by us in 2019. The kinetic approach is much less frequently utilized but it is more general than the original non-kinetic approach of Derbenev and Kondratenko from 1972 since the kinetic approach is not centered on the invariant spin field. As with the non-kinetic approach the kinetic approach covers the radiative depolarization effect, the Sokolov-Ternov effect and its Baier-Katkov correction as well as the kinetic polarization effect but it enables the calculation of corrections to the original Derbenev-Kondratenko formulas and thereby provides estimates of the reliability of the latter. It is applicable to storage rings with energies from a few GeV up to the energies of the FCC-ee and CEPC and beyond. The kinetic approach is based on the spin-orbit Wigner functions which lead to the so-called Bloch equation for the polarization density which is a generalization of Fokker-Planck equations to spin motion. In turn, as discovered in 2019, the Bloch equation is based on stochastic ordinary differential equations which can be used to develop Monte-Carlo spin-tracking codes covering the key effects beyond just radiative depolarization.

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# 1 Introduction

## 1.1 Generalities

We first make some general remarks on the dynamics of spin-polarized beams. Spin dynamics for electrons and positrons in the electric and magnetic fields in particle accelerators and storage rings is influenced by two main effects, namely spin precession following the Thomas-BMT (T-BMT) equation (see the second equation in (144)) and polarization build-up during spin flipping by the emission of synchrotron radiation (the Sokolov-Ternov effect) [BR23]. Moreover, the stochastic emission of the photons comprising the synchrotron radiation causes particle recoil and thereby injects noise into the otherwise deterministic motion. This injects noise via the particle momentum and the electric and magnetic fields in the T-BMT precession vector field (see (36)) and thereby noise into the spin motion. We call this spin diffusion or radiative depolarization which, if left unchecked, causes the beam to be depolarized. However, the Sokolov-Ternov effect counteracts the radiative depolarization effect and may even lead, in a practically feasible time, to a nonzero polarization close to the equilibrium polarization. Among the accelerators we have in mind in this work are the electron resp. positron storage rings in the future machines, the FCC-ee at CERN, the EIC at Brookhaven National Lab and the CEPC.

Analytical calculations of the equilibrium polarization usually rely on the formalism of Derbenev and Kondratenko exhibited in [DK72, DK73, Man87-1]. That approach exploits use of parameters applicable to electron (positron) storage rings that have been in use so far and it provides a framework for understanding how to maximize the attainable polarization [BR23]. However with the pioneering, [DK72], it was already clear that this approach has the potential drawback of relying on the assumption that on average the spin vectors of the particles in a bunch are aligned along the so-called invariant spin field (see related remarks at the end of Section 4 where also the concept of radiative invariant spin field is outlined).

## 1.2 Outline of the kinetic approach

We now turn to the kinetic approach. This approach does not rely on the use of the invariant spin field which opens the possibility of exploring new parameter regimes. The kinetic approach has been suggested by Derbenev and Kondratenko in 1974, in Russian, (the English translation is [DK75]) where they consider the orbital density (= density in orbital phase space),  $\rho[W]$ , and the polarization density,  $\vec{\mathcal{P}}[W]$ , with their time evolution in the presence of synchrotron radiation. The present work, which is an outgrowth of discussions between the authors, is an exposition of the kinetic approach with the aim of making this much less practiced but more general approach better known and more transparent.

The dynamics of the kinetic approach is described in [DK75] by a linear PDE system consisting of the orbital Fokker-Planck equation for the  $\mathbb{R}$ -valued orbital density  $\rho[W]$  (see (6)) and an equation of the Fokker-Planck type for the  $\mathbb{R}^3$ -valued polarization density  $\vec{\mathcal{P}}[W]$ , namely the so-called full Bloch equation (see (35)). Thus Sections 3 and 4 contain all the dynamical information given to us from [DK75]. The above  $[W]$ -notation is justified by the fact that  $\rho[W]$ ,  $\vec{\mathcal{P}}[W]$  are the building blocks of the so-called spin-1/2 Wigner function (sometimes called the spin-orbit Wigner function),  $W$ , see (1), whereby the description of a

bunch in terms of  $\rho[W], \vec{\mathcal{P}}[W]$  is equivalent to a description in terms of  $W$ .

As is shown in Section 5, the PDE system for  $\rho[W], \vec{\mathcal{P}}[W]$  is equivalent to a linear PDE system (see (65)) for the  $\mathbb{C}^{2 \times 2}$ -valued function  $W$  reflecting the fact that the description in terms of  $\rho[W], \vec{\mathcal{P}}[W]$  is equivalent to the perhaps more elegant (as evidenced by Section 5) description in terms of  $W$ . Both descriptions will be used in the present work. Note that, (6), (35) (but not (65)) were presented in [DK75] and that neither (6), (35) nor (65) were derived in [DK75] (and they are not derived here as well). However (6), (35), (65) were derived (from QED) in Kondratenko's thesis [Kon82] which underlies [DK75] (for more remarks on [Kon82], see Section 5).

Following [DK75] we use in this work cartesian coordinates, i.e., the so-called laboratory frame. Moreover we use the same units as in [DK75] which one may interpret as SI units with  $c = 1$ . By dealing with a kinetic approach we have  $W = W(t, q)$  and  $\rho[W] = \rho[W](t, q), \vec{\mathcal{P}}[W] = \vec{\mathcal{P}}[W](t, q)$  where  $t \in \mathbb{R}$  is time and  $q = (\vec{r}, \vec{\pi}) \in \mathbb{R}^6$  with  $\vec{r} \in \mathbb{R}^3$  being the position vector and  $\vec{\pi} \in \mathbb{R}^3$  being the momentum vector of the particle (in the present work,  $\vec{r}, \vec{\pi}$  and  $q$ , as all vectors, are column vectors). Thus the polarization vector of the whole bunch (at time,  $t$ ) is given by

$$\vec{P}[W](t) = \int_{\mathbb{R}^6} \vec{\mathcal{P}}[W](t, q) d^6 q ,$$

and the polarization by  $|\vec{P}[W](t)|$ . See (83) too. Note that in contrast to the kinetic approach, in a fluid approach the focus would be on the position space,  $\mathbb{R}^3$ , not on the orbital phase space,  $\mathbb{R}^6$ .

Note that in practical computations one generally does not work in the cartesian coordinates but in machine coordinates., i.e., in a so-called beam frame. Thus the equations of the present work are not directly used when it comes to numerical work. However there is a straightforward and time-honored method to transform every equation from cartesian coordinates to machine coordinates. We will come back to this point in Section 10.

The kinetic approach encapsulates, as the aforementioned non-kinetic approach, the radiative depolarization effect due to spin diffusion, the Sokolov-Ternov effect (plus the Baier-Katkov correction) as well as the so-called kinetic polarization effect. Note that the terminology 'kinetic' has different meanings in the phrases 'kinetic polarization effect' and 'kinetic approach' (in 'kinetic approach' it alludes to the fact that  $q$  is a phase space vector, not a position vector). For details on where the aforementioned effects are located in the evolution equations of the kinetic approach, see Section 16. By design, in the kinetic formalism also a wide range of other effects can be included, e.g., resonant depolarization via oscillating magnetic fields or the weak-strong beam-beam effect.

Sometimes it is convenient to write the polarization density as  $\rho[W]\vec{\mathcal{P}}_{loc}[W]$  where  $\vec{\mathcal{P}}_{loc}[W]$  is the so-called local polarization vector field. From the definition of  $\vec{\mathcal{P}}_{loc}[W]$ , it is straightforward to show that the polarization density is proportional to the density of spin angular momentum in phase space. This, in turn, is what permits the polarization density to obey an equation of the Fokker-Planck type, namely the aforementioned full Bloch equation. For more details on the local polarization vector field, see Section 6.

Whenever it is illuminating, e.g., in Sections 11 and 14, we will comment on the so-called reduced setup (see (60)). The reduced setup is important since it is the special case

that suffices for studying the radiative depolarization. For the evolution equation of  $W$  in the reduced setup, see (86). As an aside we note that the description in terms of (86) is equivalent to the description in terms of the orbital Fokker-Planck equation, (6), and the so-called reduced Bloch equation (see (61)). In fact (6), (61) and (86) are built around just the orbital Fokker-Planck operator,  $l_{orb}$ , (see (28)) and the T-BMT precession vector field,  $\vec{\Omega}_{TBMT}$ , reflecting the aforementioned fact that the radiative depolarization effect results from the interplay between the orbital noise and the T-BMT precession.

### 1.3 Outline of the extension of the kinetic approach

We now outline an extension which was developed since 2001. While in [DK75] the bunch is described in terms of  $\rho[W]$  and  $\vec{\mathcal{P}}[W]$  (and thus implicitly in terms of  $W$ ) we have developed, starting in [BH01] and completed in [HABBE19], an extension of the kinetic approach. In this extension, which is an implication of (6), (35), the bunch is described in terms of the so-called spin-orbit density,  $f = f(t, q, \vec{s})$ , whose time evolution is determined by a single Fokker-Planck equation on the spin-orbit phase space,  $\mathbb{R}^9$ , called the full spin-orbit Fokker-Planck equation, (132) (for the precise definition of spin-orbit densities, see Section 12). The search for a Fokker-Planck equation of  $f$  was inspired by  $\rho[W]$  satisfying a Fokker-Planck equation and by the full Bloch equation for  $\vec{\mathcal{P}}[W]$  being of Fokker-Planck type.

The description in terms of  $f$ , which is the subject matter of Sections 7-16, is just an extension, not a modification, of the kinetic approach since the dynamical content of (132) is the same as the dynamical content of the time evolution of  $\rho[W]$ ,  $\vec{\mathcal{P}}[W]$  resp.  $W$ . Although (132) adds no new dynamical information to the kinetic approach the description of the bunch in terms of  $f$  allows us, in Sections, 7-16, to share insights into the statistics and dynamics of a spin-polarized bunch which go beyond [DK75, BH01, HABBE19]. In particular (132) is presented and derived from (6), (35) in Sections 7-10. In fact to our knowledge no derivation of (132) was previously published so that the importance of Sections 7-10 is underscored. Moreover the extension of the kinetic approach leads to a derivation of the so-called Baier-Katkov-Strakhovenko equation (see Section 15) which is very different from the original derivation in [Bai69] and in [BKS70] and thus leads to new insights into the Baier-Katkov-Strakhovenko equation. In fact, as shown in Section 15, the extension of the kinetic approach can be used to tie the Baier-Katkov-Strakhovenko equation with a Fokker-Planck equation (see (178)).

The description in terms of  $f$  provides new insights not only via the full spin-orbit Fokker-Planck equation. In fact since (132) is a Fokker-Planck equation it has an associated Ito SDE system (see (133)) which has the same dynamical content as (132) and which we call the full spin-orbit SDE system (SDE=stochastic ODE). The interesting aspect of (133) is being neither a PDE nor a system of PDEs and so is potentially of great practical interest since (133) may be numerically solved, e.g., by a standard SDE solver or by the time-honored technique of particle tracking. In fact this aspect was our key reason to become interested in the extension of the kinetic approach via the description in terms of  $f$ . In other words it was the search for the full spin-orbit SDE system, and not so much the search for the full spin-orbit Fokker-Planck equation, which captured our imagination since 2001.

One possible approach of numerically solving (133) is to extend an existing Monte-Carlo

spin tracking code (developed for computing the so-called radiative depolarization time, e.g., a code from Bmad) to (133) (Bmad is a software library at Cornell University). The existing Monte-Carlo spin tracking codes we are aware of are not modeled after (133) but after the, simpler, reduced spin-orbit SDE system (see (143)). Note that (143) is obtained by applying the reduced setup to (133) and that (143) is the Ito SDE system associated to the reduced spin-orbit Fokker-Planck equation (see (142)) the latter being obtained by applying the reduced setup to (132). Note also that (143) (as well as (142)) contains all the information of the radiative depolarization effect (see Section 11). For more comments on numerically solving (133) see the discussions after (137).

This completes the outline of the kinetic approach to the dynamics of spin-polarized beams in electron or positron storage rings. We hope that the present work will lead to progress in (and better understanding of) this field of Accelerator Physics.

## 1.4 The motivation of the kinetic approach

Let us finally mention the motivation behind the kinetic approach. One obvious motivation underlying [DK75] was to shed light on the accuracy of the Derbenev-Kondratenko formulas, e.g., a formula for the so-called equilibrium polarization (these formulas are the central results of [DK72, DK73, Man87-1]). For example, using the evolution equation of  $W$ , terms are introduced in [DK75] which are somehow to be merged with the Derbenev-Kondratenko formulas, aimed at accounting for so-called uncorrelated resonance crossings, see, e.g., [Kon74, DKS79]. Such terms are of special interest for circular colliders with very high electron or positron energies like the future machines, the FCC-ee and the CEPC (or even the EIC) but their validity and use in the form presented requires clarification. For a recent critical assessment of these terms in the context of the CEPC, see [XDBWWG23]. While the Derbenev-Kondratenko formulas and any correction terms are not our main concern here, we hope that the present work will strengthen understanding of these two notions.

Finally, the extension of the kinetic approach had its own motivation namely to obtain an SDE system which carries the dynamical content of the kinetic approach (as mentioned above, this system turned out to be the full spin-orbit SDE system).

## 2 The spin-1/2 Wigner function: Basic facts

In this section we give a basic outline of  $W$  needed for Sections 3 and 4 (in Section 5 we take a closer look at  $W$ ).

A bunch can be described by a so-called spin-1/2 Wigner function,  $W$ , which is a  $\mathbb{C}^{2 \times 2}$ -valued function that we write, for cartesian coordinates, in the form

$$W(t, q) = \frac{1}{2} \left( I_{2 \times 2} \rho[W](t, q) + \sigma_i \mathcal{P}_i[W](t, q) \right), \quad (1)$$

where  $I_{2 \times 2}$  is the unit  $2 \times 2$ -matrix and

$$\vec{\mathcal{P}}[W] \equiv \begin{pmatrix} \mathcal{P}_1[W] \\ \mathcal{P}_2[W] \\ \mathcal{P}_3[W] \end{pmatrix}, \quad \vec{\sigma} \equiv \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}, \quad (2)$$

with the so-called Pauli matrices,  $\sigma_1, \sigma_2, \sigma_3$  defined by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3)$$

We use in the present work the summation convention for repeated lower indices, e.g.,  $\sigma_i \mathcal{P}_i[W] \equiv \sigma_1 \mathcal{P}_1[W] + \sigma_2 \mathcal{P}_2[W] + \sigma_3 \mathcal{P}_3[W]$  and  $q_i q_i \equiv q_1 q_1 + \dots + q_6 q_6$  (note that (1) is our first, but not last use of this convention). Note also that (1) is eq. 1 in [HABBE19], albeit in different notation.

For later reference we note, by (3) and for  $i, j = 1, 2, 3$ , that

$$Tr_{2 \times 2}[\sigma_i] = 0, \quad Tr_{2 \times 2}[\sigma_i \sigma_j] = 2\delta_{i,j}, \quad (4)$$

where  $\delta_{i,j}$  is the Kronecker symbol and  $Tr_{2 \times 2}$  symbolizes the trace operation over  $2 \times 2$ -matrices. Note, by (1), (4) and for  $t \in \mathbb{R}, q \in \mathbb{R}^6$ ,

$$\rho[W](t, q) = Tr_{2 \times 2}[W(t, q)], \quad \vec{\mathcal{P}}[W](t, q) = \begin{pmatrix} Tr_{2 \times 2}[\sigma_1 W(t, q)] \\ Tr_{2 \times 2}[\sigma_2 W(t, q)] \\ Tr_{2 \times 2}[\sigma_3 W(t, q)] \end{pmatrix}. \quad (5)$$

We will use (1), (5) in Section 5 to show that the description of the bunch in terms of  $W$  is equivalent to a description in terms of  $\rho[W]$  and  $\vec{\mathcal{P}}[W]$  (the latter description is utilized in [DK75]).

The proper description of the bunch in terms of  $W$ , which is the subject matter of Section 5 below, involves so-called statistical conditions on  $W$  (equivalent to statistical conditions on  $\rho[W], \vec{\mathcal{P}}[W]$ ) and an evolution equation for  $W$ . The statistical conditions are used in the present work to characterize physically meaningful  $\rho[W], \vec{\mathcal{P}}[W]$  and  $W$ . To obtain the evolution equation for  $W$  in Section 5 we first have to provide, in Sections 3,4, the evolution equations for  $\rho[W]$  and  $\vec{\mathcal{P}}[W]$  as they are given to us from [DK75].

### 3 The orbital Fokker-Planck equation

In this section we outline the dynamics of  $\rho[W]$ . Recall from Section 1 that  $\rho[W]$  is  $\mathbb{R}$ -valued.

The evolution equation for  $\rho[W]$ , as given to us from [DK75], is the following linear PDE:

$$\frac{\partial \rho[W]}{\partial t} + \frac{\partial}{\partial r_j}(v_j \rho[W]) + \frac{\partial}{\partial \pi_j}(\mathcal{F}_j \rho[W]) = St \rho[W], \quad (6)$$

where the velocity vector field,  $\vec{v}$ , the Lorentz force field,  $\vec{\mathcal{F}}$ , and the linear operator  $St$  will be defined below in this section.

Before we define  $\vec{v}, \vec{\mathcal{F}}$  and  $St$  let us make some remarks on (6). First of all, the rhs of (6) is its radiative part. Secondly, for some details on the theoretical origins of (6), see the discussions after (18). Thirdly, we call (6) (and every equation which only differs from (6) in terms of notation) the orbital Fokker-Planck equation (see also the discussions after (28)). Fourthly, the orbital Fokker-Planck equation was presented, but not derived, in [DK75] (except for some remarks). Instead it was derived in [Kon82] which is a work which

underlies [DK75] (for more remarks on [Kon82] see Section 5). Fifthly, in Section 5 we will determine which solutions  $\rho[W]$  of (6) we will call orbital densities.

We now define  $\vec{v}$ ,  $\vec{\mathcal{F}}$  and  $St$ . The  $\mathbb{R}^3$ -valued function,  $\vec{v}$ , is the velocity vector field, i.e.,

$$\begin{pmatrix} v_1(q) \\ v_2(q) \\ v_3(q) \end{pmatrix} \equiv \vec{v}(q) := \frac{\vec{\pi}}{m\gamma(q)}, \quad \gamma(q) := \sqrt{\frac{1}{1 - |\vec{v}(q)|^2}}, \quad (7)$$

reflecting the fact that  $\vec{\pi}$  is the particle's kinetic momentum vector. Note that  $m$  is the mass of the particle and  $|\cdot|$  is the Euclidean norm. Note also that  $\vec{v}$  determines the time derivative of the position vector, see (31) (which is the same in the radiationless case, namely the first equality of (33)). The  $\mathbb{R}^3$ -valued function,  $\vec{\mathcal{F}}$ , is the Lorentz force field for the particle in the external electromagnetic field, i.e.,

$$\begin{pmatrix} \mathcal{F}_1(t, q) \\ \mathcal{F}_2(t, q) \\ \mathcal{F}_3(t, q) \end{pmatrix} \equiv \vec{\mathcal{F}}(t, q) := e \left( \vec{E}(t, \vec{r}) + \vec{v}(q) \times \vec{B}(t, \vec{r}) \right), \quad (8)$$

where  $e$  is the charge of the particle and where  $\vec{E}$  and  $\vec{B}$  are the external electric and magnetic fields. The linear operator  $St$  is defined by

$$St := -\frac{\partial}{\partial \pi_j} \mathcal{C}_j + \frac{1}{2} \frac{\partial}{\partial \pi_i} \mathcal{E}_{i,j} \frac{\partial}{\partial \pi_j}, \quad (9)$$

where the  $\mathbb{R}^3$ -valued function,  $\vec{\mathcal{C}}$ , and the  $\mathbb{R}^{3 \times 3}$ -valued function  $\mathcal{E}$  are defined by

$$\begin{pmatrix} \mathcal{C}_1(t, q) \\ \mathcal{C}_2(t, q) \\ \mathcal{C}_3(t, q) \end{pmatrix} \equiv \vec{\mathcal{C}}(t, q) := -\frac{2e^2}{3m} \gamma^3(q) |\vec{a}_{\mathcal{F}}(t, q)|^2 \vec{\pi}, \quad (10)$$

$$\begin{pmatrix} \mathcal{E}_{1,1}(t, q) & \mathcal{E}_{1,2}(t, q) & \mathcal{E}_{1,3}(t, q) \\ \mathcal{E}_{2,1}(t, q) & \mathcal{E}_{2,2}(t, q) & \mathcal{E}_{2,3}(t, q) \\ \mathcal{E}_{3,1}(t, q) & \mathcal{E}_{3,2}(t, q) & \mathcal{E}_{3,3}(t, q) \end{pmatrix} \equiv \mathcal{E}(t, q) \\ := \frac{55}{24\sqrt{3}} \lambda(t, q) \vec{\pi} \vec{\pi}^T, \quad \lambda(t, q) := \hbar \frac{e^2}{m^2} \gamma^5(q) |\vec{a}_{\mathcal{F}}(t, q)|^3. \quad (11)$$

Here  $\vec{a}_{\mathcal{F}}$  is the acceleration field of the nonradiative particle, i.e.,

$$\vec{a}_{\mathcal{F}}(t, q) = \frac{e}{m\gamma(q)} \left( \vec{v}(q) \times \vec{B}(t, \vec{r}) + \vec{E}(t, \vec{r}) - \vec{v}(q) v_i(q) E_i(t, \vec{r}) \right). \quad (12)$$

Note that  $\vec{a}_{\mathcal{F}}$  is denoted in [DK75] by  $\dot{\mathbf{v}}$ . Note also that  $\vec{\mathcal{C}}$  is the classical radiation-force field, see the discussion after (29). As is common and also (tacitly) practiced in [DK75] we neglect electric bremsstrahlung effects, i.e., we only account for magnetic bremsstrahlung, that is, synchrotron radiation and so (12) simplifies to

$$\vec{a}_{\mathcal{F}}(t, q) := \frac{e}{m\gamma(q)} \left( \vec{v}(q) \times \vec{B}(t, \vec{r}) \right), \quad (13)$$

which, in the present work, is the definition of  $\vec{a}_{\mathcal{F}}$ . This completes our definition of  $\vec{v}$ ,  $\vec{\mathcal{F}}$  and  $St$ .

The expression, (6), of the Fokker-Planck equation for  $\rho[W]$  is not always the most convenient one and so we will present in this section three more expressions, namely (18), (26) and (27). To derive (18) from (6) we rewrite the rhs of (6). We thus compute, for  $i = 1, 2, 3$ ,

$$\mathcal{E}_{i,j} \frac{\partial}{\partial \pi_j} = \frac{\partial}{\partial \pi_j} \mathcal{E}_{i,j} - 2\mathcal{Q}_i , \quad (14)$$

where the  $\mathbb{R}^3$ -valued function  $\vec{\mathcal{Q}}$  is the quantum radiation-force field defined, for  $i = 1, 2, 3$ , by

$$\mathcal{Q}_i := \frac{1}{2} \frac{\partial \mathcal{E}_{i,j}}{\partial \pi_j} . \quad (15)$$

For later reference we note, by using (7), (11), (13), (11), (15), that one can show:

$$\vec{\mathcal{Q}}(t, q) = \frac{55}{48\sqrt{3}} \left( 6 + \frac{1}{\gamma^2(q)} \right) \lambda(t, q) \vec{\pi} . \quad (16)$$

For the terminology used for  $\vec{\mathcal{Q}}$ , see the discussion after (29). From (14) we get

$$\frac{\partial}{\partial \pi_i} \mathcal{E}_{i,j} \frac{\partial}{\partial \pi_j} = \frac{\partial}{\partial \pi_i} \frac{\partial}{\partial \pi_j} \mathcal{E}_{i,j} - 2 \frac{\partial}{\partial \pi_i} \mathcal{Q}_i ,$$

resulting, by (9), in:

$$St = - \frac{\partial}{\partial \pi_j} \left( \mathcal{C}_j + \mathcal{Q}_j \right) + \frac{1}{2} \frac{\partial}{\partial \pi_i} \frac{\partial}{\partial \pi_j} \mathcal{E}_{i,j} . \quad (17)$$

With (17) one can write (6) as

$$\begin{aligned} & \frac{\partial \rho[W]}{\partial t} + \frac{\partial}{\partial r_j} (v_j \rho[W]) + \frac{\partial}{\partial \pi_j} (\mathcal{F}_j \rho[W]) \\ &= - \frac{\partial}{\partial \pi_j} \left( (\mathcal{C}_j + \mathcal{Q}_j) \rho[W] \right) + \frac{1}{2} \frac{\partial}{\partial \pi_i} \frac{\partial}{\partial \pi_j} (\mathcal{E}_{i,j} \rho[W]) . \end{aligned} \quad (18)$$

It is clear by its derivation that (18) is the same as (6).

Before we will write the orbital Fokker-Planck equation in the forms, (26), (27) we make some comments on (18). We first comment on the theoretical origins of (18). According to the discussion after eq. 1 in [DK75],  $-\frac{\partial}{\partial \pi_j} \mathcal{C}_j$ , accounts for the radiation friction whereas  $(1/2) \frac{\partial}{\partial \pi_i} \mathcal{E}_{i,j} \frac{\partial}{\partial \pi_j}$  and thus:  $(1/2) \frac{\partial}{\partial \pi_i} \frac{\partial}{\partial \pi_j} \mathcal{E}_{i,j} - \frac{\partial}{\partial \pi_i} \mathcal{Q}_i$  accounts for the quantum fluctuations of the radiation. Thus, by (17),  $St$  is responsible for all radiative terms of the orbital Fokker-Planck equation, (18), i.e., the rhs is the radiative part of (18) (and thus the rhs of (6) is the radiative part of (6)). We now take a look at the lhs of (18). Neglecting the radiative effects, i.e., neglecting the rhs of (18) we get

$$\frac{\partial \rho[W]}{\partial t} + \frac{\partial}{\partial r_j} (v_j \rho[W]) + \frac{\partial}{\partial \pi_j} (\mathcal{F}_j \rho[W]) = 0 , \quad (19)$$

which, by the above, is the Liouville equation [wikiLT] describing the orbital motion of a nonradiating particle in the external electromagnetic field. Thus the orbital Fokker-Planck equation is conditioned by the combination of Hamiltonian and radiative dynamics. We finally remark that the orbital Fokker-Planck equation has a prehistory dating back at least as far as 1954. In fact it would be possible to rederive it by using [Dir38, Sch54] and some simple theoretical assumptions.

After these comments on (18) we will next write the orbital Fokker-Planck equation in the forms, (26), (27). We thus define the  $\mathbb{R}^6$ -valued functions,  $\mathcal{D}_{orb,nrad}$ ,  $\mathcal{D}_{orb,rad}$ ,  $\mathcal{D}_{orb}$ ,  $\mathcal{B}_{orb}$ , by

$$\mathcal{D}_{orb,nrad}(t, q) := \begin{pmatrix} \vec{v}(q) \\ \vec{\mathcal{F}}(t, q) \end{pmatrix}, \quad (20)$$

$$\mathcal{D}_{orb,rad}(t, q) := \begin{pmatrix} \vec{0} \\ \vec{\mathcal{C}}(t, q) + \vec{\mathcal{Q}}(t, q) \end{pmatrix}, \quad (21)$$

$$\mathcal{D}_{orb} := \mathcal{D}_{orb,nrad} + \mathcal{D}_{orb,rad}, \quad (22)$$

$$\mathcal{B}_{orb}(t, q) := \sqrt{\frac{55}{24\sqrt{3}}} \sqrt{\lambda(t, q)} \begin{pmatrix} \vec{0} \\ \vec{\pi} \end{pmatrix}. \quad (23)$$

It follows from the above that  $\mathcal{D}_{orb,rad}$  is the radiative and  $\mathcal{D}_{orb,nrad}$  is the nonradiative part of  $\mathcal{D}_{orb}$ . Because of (18) and by the discussion after (18) we call  $\vec{\mathcal{C}} + \vec{\mathcal{Q}}$  the radiation-force field. It follows from (11), (20), (21), (22), (23) that

$$\frac{\partial}{\partial q_k} \mathcal{D}_{orb,k} = \frac{\partial}{\partial r_j} v_j + \frac{\partial}{\partial \pi_j} (\mathcal{F}_j + \mathcal{C}_j + \mathcal{Q}_j), \quad (24)$$

$$\frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \mathcal{B}_{orb,k} \mathcal{B}_{orb,l} = \frac{\partial}{\partial \pi_j} \frac{\partial}{\partial \pi_i} \mathcal{E}_{i,j}, \quad (25)$$

and so we can write (18) as

$$\frac{\partial \rho[W]}{\partial t} = -\frac{\partial}{\partial q_k} (\mathcal{D}_{orb,k} \rho[W]) + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} (\mathcal{B}_{orb,k} \mathcal{B}_{orb,l} \rho[W]). \quad (26)$$

We also write (26) more compactly as

$$\frac{\partial \rho[W]}{\partial t} = l_{orb} \rho[W], \quad (27)$$

where the linear operator  $l_{orb}$  is defined by

$$\begin{aligned} l_{orb} &:= -\frac{\partial}{\partial r_j} v_j - \frac{\partial}{\partial \pi_j} (\mathcal{F}_j + \mathcal{C}_j + \mathcal{Q}_j) + \frac{1}{2} \frac{\partial}{\partial \pi_i} \frac{\partial}{\partial \pi_j} \mathcal{E}_{i,j} \\ &= -\frac{\partial}{\partial q_k} \mathcal{D}_{orb,k} + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \mathcal{B}_{orb,k} \mathcal{B}_{orb,l} \\ &= -\frac{\partial}{\partial q_k} \mathcal{D}_{orb,k} + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} (\mathcal{B}_{orb} \mathcal{B}_{orb}^T)_{k,l} = -\frac{\partial}{\partial r_j} v_j - \frac{\partial}{\partial \pi_j} \mathcal{F}_j + St, \end{aligned} \quad (28)$$

and where in the fourth equation of (28) we used (17). It is clear by their derivations that (26), (27) are the same as (18) (and thus are the same as (6)). Note that, (27), is eq. 2 in [HABBE19].

The key point about (26) is being an expression in terms of the so-called drift vector field and of the so-called diffusion matrix field. Thus (26) is that expression of the orbital Fokker-Planck equation by which it is immediately clear that it is a Fokker-Planck equation. In fact  $\mathcal{D}_{orb}$  is the so-called drift vector field and  $\mathcal{B}_{orb}\mathcal{B}_{orb}^T$  is the so-called diffusion matrix field of (26). In particular  $\mathcal{D}_{orb}$  encapsulates the spatial drift via the velocity field,  $\vec{v}$ , and the momentum drift via the Lorentz force field,  $\vec{\mathcal{F}}$ , and via the radiation-force field,  $\vec{\mathcal{C}} + \vec{\mathcal{Q}}$ . Also,  $\mathcal{B}_{orb}\mathcal{B}_{orb}^T$  encapsulates the diffusion effect due to the synchrotron radiation. Since  $\mathcal{B}_{orb}$  is a vector field, every value of  $\mathcal{B}_{orb}\mathcal{B}_{orb}^T$  is an outer product matrix. The so-called noise vector field,  $\mathcal{B}_{orb}$ , is not uniquely determined by (26) since each of the values of  $\mathcal{B}_{orb}$  can be multiplied by either 1 or  $-1$  without affecting  $\mathcal{B}_{orb}\mathcal{B}_{orb}^T$  (our choice of  $\mathcal{B}_{orb}$  is given by (23)). For the notions of Fokker-Planck equation, drift vector field and diffusion matrix field see, e.g., [Gar04].

In the remaining parts of this section we make further comments on the orbital Fokker-Planck equation. First of all, one can show that  $1/\lambda$  is of the order of the Sokolov-Ternov polarization time and thus much larger than the orbital damping time (the latter being associated with  $\vec{\mathcal{C}}$ ). Therefore, and because of (16) it is common in applications to neglect  $\vec{\mathcal{Q}}$ , i.e., to approximate  $\vec{\mathcal{C}} + \vec{\mathcal{Q}}$  by  $\vec{\mathcal{C}}$ . Nevertheless, in the present work, we do not neglect  $\vec{\mathcal{Q}}$  except when explicitly mentioned. In particular this allows us to display the close relation between  $\vec{\mathcal{Q}}$  and the orbital noise vector field. In fact, by (16), (23),  $\mathcal{B}_{orb}$  is related to  $\vec{\mathcal{Q}}$  via

$$\sqrt{|\vec{\mathcal{Q}}(t, q)|} \mathcal{B}_{orb}(t, q) = \gamma(q) \sqrt{\frac{2m\sqrt{\gamma^2(q) - 1}}{6\gamma^2(q) + 1}} \begin{pmatrix} \vec{0} \\ \vec{\mathcal{Q}}(t, q) \end{pmatrix}. \quad (29)$$

Secondly,  $l_{orb}$  is the sum of a zeroth order part  $\hbar$  and a first order part in  $\hbar$ , i.e., the sum of a classical and a quantum term. In fact, by (11), (23),  $\mathcal{B}_{orb}\mathcal{B}_{orb}^T$ , is of first order  $\hbar$ . Moreover, by (7), (8), (10), (16),  $\vec{v}, \vec{\mathcal{F}}, \vec{\mathcal{C}}$  are of zeroth order  $\hbar$  and  $\vec{\mathcal{Q}}$  is of first order  $\hbar$ . Thus, by (20), (21), (22),  $\mathcal{D}_{orb}$ , has a part of zeroth order  $\hbar$ , namely  $\begin{pmatrix} \vec{v} \\ \vec{\mathcal{F}} + \vec{\mathcal{C}} \end{pmatrix}$  and a part of first order  $\hbar$ , namely  $\begin{pmatrix} \vec{0} \\ \vec{\mathcal{Q}} \end{pmatrix}$ . This justifies the terminologies classical resp. quantum radiation-force field (recall that  $\vec{\mathcal{C}} + \vec{\mathcal{Q}}$  is the radiation-force field).

We conclude, by (28), that  $l_{orb}$  has a part of zeroth order  $\hbar$ , namely  $-\frac{\partial}{\partial r_j} v_j - \frac{\partial}{\partial \pi_j} (\mathcal{F}_j + \mathcal{C}_j)$  and a part of first order  $\hbar$ , namely  $-\frac{\partial}{\partial \pi_j} \mathcal{Q}_j + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \mathcal{B}_{orb,k} \mathcal{B}_{orb,l}$ . Thirdly, the quantum part of  $l_{orb}$  is of a radiative nature (the converse does not hold because of the term,  $-\frac{\partial}{\partial \pi_j} \mathcal{C}_j$ ). Fourthly, the Ito SDE system associated with the orbital Fokker-Planck equation reads, by (26), as

$$Q' = \mathcal{D}_{orb}(t, Q) + \mathcal{B}_{orb}(t, Q)\nu(t), \quad (30)$$

where  $\nu$  is the one-dimensional (=scalar) white noise process (recall that SDE=stochastic ODE). Note that we denote dependent variables (like  $Q$  in (30)) by roman capital letters.

For the notion of associated Ito SDE system see, e.g., [Gar04]. By (20), (21), (22), (23) the Ito SDE system, (30), can be written more explicitly as

$$\vec{R}' = \vec{v}(Q) , \quad (31)$$

$$\vec{\Pi}' = \vec{\mathcal{F}}(t, Q) + \vec{\mathcal{C}}(t, Q) + \vec{\mathcal{Q}}(t, Q) + \sqrt{\frac{55}{24\sqrt{3}}} \sqrt{\lambda(t, Q)} \vec{\Pi} \nu(t) . \quad (32)$$

Fifthly, neglecting all radiative effects, the SDE system, (31), (32), simplifies to

$$\vec{R}' = \vec{v}(Q) , \quad \vec{\Pi}' = \vec{\mathcal{F}}(t, Q) . \quad (33)$$

Note that the solutions  $\begin{pmatrix} \vec{R} \\ \vec{\Pi} \end{pmatrix}$  of (33) are characteristics of the Liouville equation, (19), in the sense that they satisfy:

$$\frac{d}{dt} \left( \rho[W] \left( t, \vec{R}(t), \vec{\Pi}(t) \right) \right) = 0 , \quad (34)$$

where  $\rho[W]$  is any solution of (19).

## 4 The full Bloch equation for the polarization density

In this section we outline the dynamics of  $\vec{\mathcal{P}}[W]$ . Note, by Section 1, that  $\vec{\mathcal{P}}[W]$  is  $\mathbb{R}^3$ -valued.

The evolution equation for  $\vec{\mathcal{P}}[W]$  is given to us from [DK75] as the following linear inhomogeneous PDE system:

$$\begin{aligned} \frac{\partial \vec{\mathcal{P}}[W]}{\partial t} + \frac{\partial}{\partial r_j} (v_j \vec{\mathcal{P}}[W]) + \frac{\partial}{\partial \pi_j} (\mathcal{F}_j \vec{\mathcal{P}}[W]) - \vec{\Omega}_{TBMT} \times \vec{\mathcal{P}}[W] \\ = St \vec{\mathcal{P}}[W] + \Delta l_{hom} \vec{\mathcal{P}}[W] + l_{inhom} \rho[W] , \end{aligned} \quad (35)$$

where  $\rho[W]$  is a solution of the orbital Fokker-Planck equation. The function  $\vec{\Omega}_{TBMT}$  and the linear operators  $\Delta l_{hom}$  and  $l_{inhom}$  will be defined below in this section (recall that  $\vec{v}$ ,  $\vec{\mathcal{F}}$  and  $St$  are defined in Section 3 above).

Before we define  $\vec{\Omega}_{TBMT}$ ,  $\Delta l_{hom}$  and  $l_{inhom}$  let us make some remarks on (35). First of all, the rhs of (35) is its radiative part (see the discussion after (44)). Secondly, for some details on the theoretical origins of (35), see the discussion after (44). Thirdly we call (35) (and every equation which only differs from (35) in terms of notation) the full Bloch equation (see also the discussion after (49)). Fourthly, in Section 5 we will determine which solutions  $\vec{\mathcal{P}}[W]$  of (35) we will call polarization densities.

We now define  $\vec{\Omega}_{TBMT}$ ,  $\Delta l_{hom}$  and  $l_{inhom}$ . The  $\mathbb{R}^3$ -valued function,  $\vec{\Omega}_{TBMT}$ , is the T-BMT

precession vector field, i.e.,

$$\begin{aligned} \vec{\Omega}_{TBMT}(t, q) := & \frac{e}{m\gamma(q)} \left( - \left( 1 + \frac{g-2}{2} \gamma(q) \right) \vec{B}(t, \vec{r}) \right. \\ & + \frac{g-2}{2} \frac{\gamma^2(q)}{1+\gamma(q)} \vec{v}(q) v_i(q) B_i(t, \vec{r}) \\ & \left. + \left( \frac{g-2}{2} \gamma(q) + \frac{\gamma(q)}{1+\gamma(q)} \right) \left( \vec{v}(q) \times \vec{E}(t, \vec{r}) \right) \right), \end{aligned} \quad (36)$$

where  $g$  is the particle's  $g$ -factor. The linear operator  $\Delta l_{hom}$  is a multiplication operator defined by

$$\left( \Delta l_{hom} \vec{j} \right)(t, q) := -\frac{5\sqrt{3}}{8} \lambda(t, q) \left( I_{3 \times 3} - \frac{2}{9m^2\gamma^2(q)} \vec{\pi} \vec{\pi}^T \right) \vec{j}(t, q), \quad (37)$$

where  $\vec{j}$  is any function:  $\mathbb{R}^7 \rightarrow \mathbb{R}^3$ . The linear operator  $l_{inhom}$  is defined by

$$\left( l_{inhom} k \right)(t, q) := - \left( 1 + \frac{\partial}{\partial \pi_j} \pi_j \right) \lambda(t, q) \frac{1}{m\gamma(\vec{\pi})} \frac{\vec{\pi} \times \vec{a}_{\mathcal{F}}(t, q)}{|\vec{a}_{\mathcal{F}}(t, q)|} k(t, q), \quad (38)$$

where  $k$  is any differentiable function:  $\mathbb{R}^7 \rightarrow \mathbb{R}$ . To rewrite (38) into a more convenient form we define the  $\mathbb{R}^3$ -valued function

$$\vec{\mathcal{B}}_{spin}(t, q) := \sqrt{\frac{24\sqrt{3}}{55}} \sqrt{\lambda(t, q)} \frac{1}{m\gamma(q)} \frac{\vec{\pi} \times \vec{a}_{\mathcal{F}}(t, q)}{|\vec{a}_{\mathcal{F}}(t, q)|}. \quad (39)$$

Thus we obtain from (23):

$$\begin{aligned} \lambda(t, q) \frac{1}{m\gamma(\vec{\pi})} \frac{\vec{\pi} \times \vec{a}_{\mathcal{F}}(t, q)}{|\vec{a}_{\mathcal{F}}(t, q)|} &= \sqrt{\frac{55}{24\sqrt{3}}} \sqrt{\lambda(t, q)} \vec{\mathcal{B}}_{spin}(t, q), \\ \frac{\partial}{\partial \pi_j} \pi_j \lambda(t, q) \frac{1}{m\gamma(\vec{\pi})} \frac{\vec{\pi} \times \vec{a}_{\mathcal{F}}(t, q)}{|\vec{a}_{\mathcal{F}}(t, q)|} &= \frac{\partial}{\partial \pi_j} \pi_j \sqrt{\frac{55}{24\sqrt{3}}} \sqrt{\lambda(t, q)} \vec{\mathcal{B}}_{spin}(t, q) \\ &= \frac{\partial}{\partial \pi_j} \mathcal{B}_{orb,j+3}(t, q) \vec{\mathcal{B}}_{spin}(t, q) = \frac{\partial}{\partial q_j} \mathcal{B}_{orb,j}(t, q) \vec{\mathcal{B}}_{spin}(t, q), \end{aligned}$$

which implies, by (38),

$$l_{inhom} k = - \sqrt{\frac{55}{24\sqrt{3}}} \sqrt{\lambda} k - \frac{\partial}{\partial q_j} (\mathcal{B}_{orb,j} \vec{\mathcal{B}}_{spin} k). \quad (40)$$

It follows from (40) that

$$l_{inhom} = l_{inhom,1} + l_{inhom,2}, \quad (41)$$

where the linear operators  $l_{inhom,1}, l_{inhom,2}$  are defined by

$$\left( l_{inhom,1} k \right)(t, q) := -\sqrt{\frac{55}{24\sqrt{3}}} \sqrt{\lambda(t, q)} \vec{\mathcal{B}}_{spin}(t, q) k(t, q) , \quad (42)$$

$$l_{inhom,2} k := -\frac{\partial}{\partial q_j} (\mathcal{B}_{orb,j} \vec{\mathcal{B}}_{spin} k) , \quad (43)$$

where  $k$  is any differentiable function:  $\mathbb{R}^7 \rightarrow \mathbb{R}$ . Note that  $l_{inhom,1}$  is a multiplication operator. This completes the definition of  $\vec{\Omega}_{TBMT}, \Delta l_{hom}$  and  $l_{inhom}$ .

The expression, (35), of the full Bloch equation is not always the most convenient one and so we will present in this section three more expressions, namely (44), (49), (59). It follows from (40) that (35) can be written as

$$\begin{aligned} & \frac{\partial \vec{\mathcal{P}}[W]}{\partial t} + \frac{\partial}{\partial r_j} (v_j \vec{\mathcal{P}}[W]) + \frac{\partial}{\partial \pi_j} (\mathcal{F}_j \vec{\mathcal{P}}[W]) - \vec{\Omega}_{TBMT} \times \vec{\mathcal{P}}[W] \\ &= St \vec{\mathcal{P}}[W] + \Delta l_{hom} \vec{\mathcal{P}}[W] + l_{inhom,1} \rho[W] + l_{inhom,2} \rho[W] . \end{aligned} \quad (44)$$

It is clear by its derivation that (44) is the same as (35).

Before we write the full Bloch equation in the forms of (49), (59) we make some comments on (44). We first comment on the theoretical origins of (44). According to the discussion after (18),  $St$  accounts for the radiation friction and the quantum fluctuations of the radiation. Moreover, by (37), (42), (43) and according to the discussion after eq. 2 in [DK75],  $\Delta l_{hom} \vec{\mathcal{P}}[W] + l_{inhom,1} \rho[W]$  comprises the so-called direct action of the radiation on the polarization whereas  $l_{inhom,2} \rho[W]$  comprises the spin-orbit interaction with the classical radiation (the latter effect is not to be confused with the radiation friction effect!). We conclude that the rhs is the radiative part of (44) (and thus the rhs of (35)). We now take a look at the lhs of (44). Neglecting the rhs of (44) we get

$$\frac{\partial \vec{\mathcal{P}}[W]}{\partial t} + \frac{\partial}{\partial r_j} (v_j \vec{\mathcal{P}}[W]) + \frac{\partial}{\partial \pi_j} (\mathcal{F}_j \vec{\mathcal{P}}[W]) - \vec{\Omega}_{TBMT} \times \vec{\mathcal{P}}[W] = \vec{0} , \quad (45)$$

which describes, by the above, a nonradiative spinning particle moving in the external electromagnetic field. We finally remark that the full Bloch equation was presented but not derived in [DK75] (except for some remarks). For the origin of the full Bloch equation, see the corresponding remarks in Section 1.

After these comments on (44) we now rewrite (44) into the form, (49). We thus define the linear operators  $l_{TBMT}, l_{hom}$  by

$$\left( l_{TBMT} \vec{j} \right)(t, q) := \vec{\Omega}_{TBMT}(t, q) \times \vec{j}(t, q) , \quad (46)$$

$$l_{hom} := l_{TBMT} + \Delta l_{hom} , \quad (47)$$

where  $\vec{j}$  is any function:  $\mathbb{R}^7 \rightarrow \mathbb{R}^3$ . Note that  $l_{TBMT}$  and  $l_{hom}$  are multiplication operators. From (28) we get:

$$-\frac{\partial}{\partial r_j} v_j - \frac{\partial}{\partial \pi_j} \mathcal{F}_j + St + \Delta l_{hom} = l_{orb} + \Delta l_{hom} , \quad (48)$$

so that, by (46), (47), we can write (44) as

$$\frac{\partial \vec{\mathcal{P}}[W]}{\partial t} = (l_{orb} + l_{hom})\vec{\mathcal{P}}[W] + l_{inhom}\rho[W], \quad (49)$$

which will be needed in Section 5. It is clear by its derivation that (49) is the same as (44) (and thus is the same as (35)).

Before we write the full Bloch equation in the form of (59) we make some comments on (49). First of all, (49) is eq. 8 in [HABBE19] and so, following [HABBE19], we call (49) the full Bloch equation (and thus, (35), (44), are the full Bloch equation). Secondly, we now comment on how  $\hbar$  appears in the full Bloch equation, (49). We identified after (29) the parts of  $l_{orb}$  which are of zeroth order resp. first order  $\hbar$ . Moreover, by (36), (46),  $l_{TBMT}$  is of zeroth order  $\hbar$  whereas, by (11), (37),  $\Delta l_{hom}$  is of first order  $\hbar$ . We thus conclude, by (47), that  $l_{TBMT}$  is the part of  $l_{hom}$  which is of zeroth order  $\hbar$  and that  $\Delta l_{hom}$  is the part of  $l_{hom}$  which is of first order  $\hbar$ . Thus, by the discussion after (44), the quantum part of  $l_{hom}$  is of a radiative nature and, conversely, the radiative part of  $l_{hom}$  is of quantum nature. We conclude, by the discussion after (29), that the quantum part of  $l_{orb} + l_{hom}$  is of a radiative nature (the converse does not hold because of the term,  $-\frac{\partial}{\partial \pi_j} \mathcal{C}_j$ , in  $l_{orb}$ ). By (11), (23), (39),  $l_{inhom}$  is of first order  $\hbar$  and is of a radiative nature. In particular  $l_{inhom}$  is of first order  $\hbar$  (although it is tied via its term,  $l_{inhom,2}$ , to the classical radiation!).

After these comments on (49) we now rewrite the full Bloch equation into the form, (59). We first define the  $\mathbb{R}^3$ -valued function  $\vec{\mathcal{D}}_{spin,0,ST}$  and the  $\mathbb{R}^{3 \times 3}$ -valued functions  $\mathcal{D}_{spin,+}, \mathcal{D}_{spin,+}, \mathcal{D}_{spin,+}$ :

$$\vec{\mathcal{D}}_{spin,0,ST}(t, q) := -\sqrt{\frac{55}{24\sqrt{3}}} \sqrt{\lambda(t, q)} \vec{\mathcal{B}}_{spin}(t, q), \quad (50)$$

$$\mathcal{D}_{spin,+}, TBMT(t, q) \vec{s} := \vec{\Omega}_{TBMT}(t, q) \times \vec{s}, \quad (51)$$

$$\mathcal{D}_{spin,+}, ST(t, q) \vec{s} := -\frac{5\sqrt{3}}{8} \lambda(t, q) \vec{s}, \quad (52)$$

$$\mathcal{D}_{spin,+}, BK(t, q) := \frac{5\sqrt{3}}{36m^2\gamma^2(q)} \lambda(t, q) \vec{\pi} \vec{\pi}^T, \quad (53)$$

where  $t \in \mathbb{R}, q \in \mathbb{R}^6, \vec{s} \in \mathbb{R}^3$ . We also define the  $\mathbb{R}^{3 \times 3}$ -valued function  $\mathcal{D}_{spin,+}$  and the  $\mathbb{R}^3$ -valued function  $\vec{\mathcal{D}}_{spin}$  by

$$\mathcal{D}_{spin,+} := \mathcal{D}_{spin,+}, TBMT + \mathcal{D}_{spin,+}, ST + \mathcal{D}_{spin,+}, BK, \quad (54)$$

$$\vec{\mathcal{D}}_{spin}(t, q, \vec{s}) := \vec{\mathcal{D}}_{spin,0,ST}(t, q) + \mathcal{D}_{spin,+}(t, q) \vec{s}, \quad (55)$$

where  $t \in \mathbb{R}, q \in \mathbb{R}^6, \vec{s} \in \mathbb{R}^3$ . In Section 15 the notations,  $\mathcal{D}_{spin,0,ST}, \mathcal{D}_{spin,+}, ST$ , will be justified in terms of the Sokolov-Ternov effect and the notation,  $\mathcal{D}_{spin,+}, BK$ , in terms of the Baier-Katkov correction while the notation,  $\mathcal{D}_{spin,+}, TBMT$ , will be justified in terms of the Thomas-BMT precession effect. It follows from (41), (42), (43), (46), (47), (50), (54), that

for any function  $\vec{j} : \mathbb{R}^7 \rightarrow \mathbb{R}^3$  and any differentiable function  $k : \mathbb{R}^7 \rightarrow \mathbb{R}$ ,

$$\left( l_{hom} \vec{j} \right)(t, q) = \mathcal{D}_{spin,+}(t, q) \vec{j}(t, q) , \quad (56)$$

$$\left( l_{inhom,1} k \right)(t, q) = \vec{\mathcal{D}}_{spin,0,ST}(t, q) k(t, q) , \quad (57)$$

$$\left( l_{inhom} k \right)(t, q) = \vec{\mathcal{D}}_{spin,0,ST}(t, q) k(t, q) - \frac{\partial}{\partial q_k} \left( \mathcal{B}_{orb,k}(t, q) \vec{\mathcal{B}}_{spin}(t, q) k(t, q) \right) . \quad (58)$$

With (56), (58) at hand we can write the full Bloch equation, (49), as

$$\frac{\partial \vec{\mathcal{P}}[W]}{\partial t} = l_{orb} \vec{\mathcal{P}}[W] + \mathcal{D}_{spin,+} \vec{\mathcal{P}}[W] + \vec{\mathcal{D}}_{spin,0,ST} \rho[W] - \frac{\partial}{\partial q_k} (\mathcal{B}_{orb,k} \vec{\mathcal{B}}_{spin} \rho[W]) . \quad (59)$$

It is clear by the derivation that (59) is the same as (49) (and thus is the same as (35), (44)). Note that the expression, (59), of the full Bloch equation will be used in Section 9 below.

The fact that  $\mathcal{B}_{orb}$  occurs twice on the rhs of (59) (namely in the first and fourth terms) may get unnoticed at a first read but it is crucial for obtaining the extension of the kinetic approach (recall that this extension is the topic of Sections 7-16). A reader who is interested in this fine point should carefully read the discussion after (132).

We now make some more remarks on the non-kinetic approach mentioned at the beginning of Section 1. The non-kinetic approach is the subject matter of [DK72, DK73, Man87-1] and it relies on the assumption that on average the spin vectors in a bunch are aligned along the invariant spin field. In fact this assumption corresponds in the kinetic approach to the condition that the polarization density points along the invariant spin field. However, as already hinted at in [DK75], this condition is inconsistent with the full Bloch equation (and even inconsistent with the reduced Bloch equation). This inconsistency led to the concept of the radiative invariant spin field, starting with [Hei97] (see also [BH15]) and still under investigation by us (the radiative invariant spin field is denoted in [BH15] by  $\hat{p}$ ). The invariant spin field is an approximation of the radiative invariant spin field which we believe diminishes the usefulness of the Derbenev-Kondratenko formulas with growing beam energies like in FCC-ee and CEPC. In contrast it is believed that the Derbenev-Kondratenko formulas are useful (even at those high energies) when the invariant spin field is replaced by the radiative invariant spin field. The aforementioned inconsistency is also studied, for the reduced setup, in Beznosov's PhD thesis [Bez20] (for remarks on this thesis, see Section 16).

We finally introduce the reduced Bloch equation. For that purpose we consider the so-called reduced setup, defined by

$$l_{hom} = l_{TBMT} , \quad l_{inhom} = 0 . \quad (60)$$

In the reduced setup the full Bloch equation simplifies to

$$\frac{\partial \vec{\mathcal{P}}[W]}{\partial t} = l_{orb} \vec{\mathcal{P}}[W] + \vec{\Omega}_{TBMT} \times \vec{\mathcal{P}}[W] . \quad (61)$$

Note that (61) is obtained from (46), (49), (60). Following [HABBE19] we call (61) the reduced Bloch equation. This justifies the terminology of reduced setup.

We recall from Section 1 that the orbital Fokker-Planck equation, in combination with (61), contains all the information needed to study the radiative depolarization effect (by the discussion before, (29), this is even true if one neglects  $\vec{Q}$ ). Note also, by (46), (55), (56), (58), that (60) can be written as

$$\vec{\mathcal{D}}_{\text{spin}}(t, q, \vec{s}) = \vec{\Omega}_{TBMT} \times \vec{s}, \quad (62)$$

$$\vec{\mathcal{B}}_{\text{spin}}(t, q) = \vec{0}, \quad (63)$$

where  $t \in \mathbb{R}, q \in \mathbb{R}^6, \vec{s} \in \mathbb{R}^3$ .

For later reference we call the general situation, where the full Bloch equation holds, i.e., where we ignore (60) resp. (62), (63), the full setup.

## 5 The spin-1/2 Wigner function: The statistical conditions and the evolution equation

In this section we first derive, from Sections 3 and 4, the evolution equation, (65), for  $W$ . We even show equivalence, i.e., that (65) is equivalent to the PDE system, (27), (49), for  $\rho[W], \vec{\mathcal{P}}[W]$ . Secondly, we define the so-called statistical conditions on  $W$  leading us to the useful notion of the spin-1/2 Wigner function of a bunch which in turn is equivalent to the notions of the orbital density and polarization density of a bunch. We thus see that describing the bunch in terms of physically meaningful  $W$  is equivalent to a description in terms of physically meaningful  $\rho[W]$  and  $\vec{\mathcal{P}}[W]$ . Thirdly, the remaining parts of this section are devoted to making comments on  $W$ , e.g., on its relation to  $\rho[W]$  and  $\vec{\mathcal{P}}[W]$  and on the theoretical origins of  $W$ .

To derive an evolution equation for  $W$  we first note, by (1), that

$$\frac{\partial W}{\partial t} = \frac{1}{2} \left( I_{2 \times 2} \frac{\partial \rho[W]}{\partial t} + \sigma_i \frac{\partial \mathcal{P}_i[W]}{\partial t} \right). \quad (64)$$

Using (1), (5) and inserting the orbital Fokker-Planck equation (27) and the full Bloch equation (49) into the rhs of (64) we get

$$\begin{aligned} \frac{\partial W}{\partial t} &= \frac{1}{2} \left( I_{2 \times 2} l_{orb} \rho[W] + \sigma_i \left( (l_{orb} + l_{hom}) \vec{\mathcal{P}}[W] + l_{inhom} \rho[W] \right)_i \right) \\ &= l_{orb} \frac{1}{2} \left( I_{2 \times 2} \rho[W] + \sigma_i \mathcal{P}_i[W] \right) + \frac{1}{2} \sigma_i \left( l_{hom} \vec{\mathcal{P}}[W] + l_{inhom} \rho[W] \right)_i \\ &= l_{orb} W + \frac{1}{2} \sigma_i \left( l_{hom} \begin{pmatrix} Tr_{2 \times 2}[\sigma_1 W] \\ Tr_{2 \times 2}[\sigma_2 W] \\ Tr_{2 \times 2}[\sigma_3 W] \end{pmatrix} + l_{inhom} Tr_{2 \times 2}[W] \right)_i, \end{aligned}$$

in short, we got the linear PDE system:

$$\frac{\partial W}{\partial t} = l_{orb} W + \frac{1}{2} \sigma_i \left( l_{hom} \begin{pmatrix} Tr_{2 \times 2}[\sigma_1 W] \\ Tr_{2 \times 2}[\sigma_2 W] \\ Tr_{2 \times 2}[\sigma_3 W] \end{pmatrix} + l_{inhom} Tr_{2 \times 2}[W] \right)_i, \quad (65)$$

which is the evolution equation for  $W$ .

Having thus shown that (65) follows from (27), (49) we now show that the converse is also true. We first note, by (5), that (65) implies

$$\frac{\partial W}{\partial t} = l_{orb}W + \frac{1}{2}\sigma_i \left( l_{hom}\vec{\mathcal{P}}[W] + l_{inhom}\rho[W] \right)_i . \quad (66)$$

We compute, by (4), (5), (66),

$$\begin{aligned} \frac{\partial \rho[W]}{\partial t} &= \frac{\partial}{\partial t} Tr_{2\times 2}[W] = Tr_{2\times 2}\left[\frac{\partial W}{\partial t}\right] \\ &= Tr_{2\times 2}\left[l_{orb}W + \frac{1}{2}\sigma_i \left( l_{hom}\vec{\mathcal{P}}[W] + l_{inhom}\rho[W] \right)_i\right] \\ &= Tr_{2\times 2}[l_{orb}W] = l_{orb}Tr_{2\times 2}[W] = l_{orb}\rho[W] , \end{aligned} \quad (67)$$

so that (27) holds. Thus (27) follows from (65). To show that (49) follows from (65) we multiply (66) from the left by  $\sigma_j$  resulting in

$$\sigma_j \frac{\partial W}{\partial t} = \sigma_j l_{orb}W + \frac{1}{2}\sigma_j \left( \sigma_i \left( l_{hom}\vec{\mathcal{P}}[W] + l_{inhom}\rho[W] \right)_i \right) , \quad (68)$$

where  $j = 1, 2, 3$ . Note that (68) follows from (65) because (66) follows from (65). It follows from (4), (5), (68) that, for  $j = 1, 2, 3$ ,

$$\begin{aligned} \frac{\partial \mathcal{P}_j[W]}{\partial t} &= \frac{\partial}{\partial t} Tr_{2\times 2}[\sigma_j W] = Tr_{2\times 2}[\sigma_j \frac{\partial W}{\partial t}] \\ &= Tr_{2\times 2}\left[\sigma_j l_{orb}W + \frac{1}{2}\sigma_j \left( \sigma_i \left( l_{hom}\vec{\mathcal{P}}[W] + l_{inhom}\rho[W] \right)_i \right)\right] \\ &= l_{orb}Tr_{2\times 2}[\sigma_j W] + \frac{1}{2}Tr_{2\times 2}\left[\sigma_j \sigma_i \left( l_{hom}\vec{\mathcal{P}} + l_{inhom}\rho \right)_i\right] \\ &= l_{orb}\mathcal{P}_j[W] + \left( l_{hom}\vec{\mathcal{P}}[W] + l_{inhom}\rho[W] \right)_j , \end{aligned}$$

so that  $\frac{\partial \vec{\mathcal{P}}[W]}{\partial t} = (l_{orb} + l_{hom})\vec{\mathcal{P}}[W] + l_{inhom}\rho[W]$ , i.e., (49) holds. Thus (49) follows from (65).

Therefore (65) implies (27), (49) which completes the proof that the linear PDE system, (65), is equivalent to the linear PDE system, (27), (49). While (27), (49) are the central piece of [DK75] we note that (65) does not occur in [DK75] (where  $W$  is merely mentioned). However  $W$  plays a key role in [Kon82] (recall from Section 1 that Kondratenko's thesis underlies [DK75]).

To arrive at the notion of the spin-1/2 Wigner function of a bunch we need statistical conditions on  $W$  (which will lead us to statistical conditions on  $\rho[W]$  and  $\vec{\mathcal{P}}[W]$  as well). We require the following three conditions on  $W$  (which for lack of a better word we call

statistical conditions) to be valid for  $t \in \mathbb{R}, q \in \mathbb{R}^6$ ,

$$\left( W(t, q) \right)^\dagger = W(t, q) , \quad (69)$$

$$\int_{\mathbb{R}^6} Tr_{2 \times 2} \left[ W(t, \tilde{q}) \right] d^6 \tilde{q} = 1 , \quad (70)$$

$$\left| \int_{\mathbb{R}^6} \begin{pmatrix} Tr_{2 \times 2}[\sigma_1 W(t, \tilde{q})] \\ Tr_{2 \times 2}[\sigma_2 W(t, \tilde{q})] \\ Tr_{2 \times 2}[\sigma_3 W(t, \tilde{q})] \end{pmatrix} d^6 \tilde{q} \right| \leq 1 , \quad (71)$$

where  $\dagger$  denotes the complex conjugate of the transpose of a matrix and where (69) is the common hermiticity condition on Wigner functions while (70) reflects the well known statistical interpretation of Wigner functions and (71) is the normalization condition of the polarization. For more details on (69), (70), (71) see the discussion after (84).

The statistical conditions on  $W$  can be expressed in terms of its building blocks,  $\rho[W]$  and  $\vec{\mathcal{P}}[W]$  as follows. First of all, since  $I_{2 \times 2}$  and  $\sigma_1, \sigma_2, \sigma_3$  are Hermitian  $2 \times 2$ -matrices, it follows from (5) that (69) is equivalent to

$$\rho[W](t, q) \in \mathbb{R} , \quad \vec{\mathcal{P}}[W](t, q) \in \mathbb{R}^3 , \quad (72)$$

where  $t \in \mathbb{R}, q \in \mathbb{R}^6$ . Secondly, it follows from (5) that (70) is equivalent to

$$\int_{\mathbb{R}^6} \rho[W](t, q) d^6 q = 1 , \quad (73)$$

where  $t \in \mathbb{R}$ . Thirdly, we note, by (5) and for  $t \in \mathbb{R}$ , that

$$\int_{\mathbb{R}^6} \begin{pmatrix} Tr_{2 \times 2}[\sigma_1 W(t, q)] \\ Tr_{2 \times 2}[\sigma_2 W(t, q)] \\ Tr_{2 \times 2}[\sigma_3 W(t, q)] \end{pmatrix} d^6 q = \int_{\mathbb{R}^6} \vec{\mathcal{P}}[W](t, q) d^6 q , \quad (74)$$

which entails that (71) is equivalent to:

$$\left| \int_{\mathbb{R}^6} \vec{\mathcal{P}}[W](t, q) d^6 q \right| \leq 1 . \quad (75)$$

Fourthly, we conclude that (69), (70) and (71) are equivalent to (72), (73) and (75).

We thus arrive at the notion of the spin-1/2 Wigner function of a bunch as follows. We call  $W$  the spin-1/2 Wigner function of a bunch iff  $W$  satisfies (65), (69), (70), (71). By the above this is equivalent to  $\rho[W], \vec{\mathcal{P}}[W]$  satisfying (27), (49), (72), (73), (75). We thus call  $\rho[W]$  the orbital density and  $\vec{\mathcal{P}}[W]$  the polarization density of a bunch iff  $W$  is the spin-1/2 Wigner function of a bunch.

As an aside we make a remark on (71). In fact we briefly discuss the option of strengthening (71) to

$$|\vec{\mathcal{P}}[W]| \leq \rho[W] , \quad (76)$$

while sticking to (69) and (70). For later reference we note that (76) implies

$$\rho[W] \geq 0 . \quad (77)$$

To show that (76) is stronger than (71) we first compute, by (73), (74) and (76) and for  $t \in \mathbb{R}$ ,

$$\begin{aligned} & \left| \int_{\mathbb{R}^6} \begin{pmatrix} Tr_{2 \times 2}[\sigma_1 W(t, q)] \\ Tr_{2 \times 2}[\sigma_2 W(t, q)] \\ Tr_{2 \times 2}[\sigma_3 W(t, q)] \end{pmatrix} d^6 q \right| = \left| \int_{\mathbb{R}^6} \vec{\mathcal{P}}[W](t, q) d^6 q \right| \\ & \leq \int_{\mathbb{R}^6} |\vec{\mathcal{P}}[W](t, q)| d^6 q \leq \int_{\mathbb{R}^6} \rho[W](t, q) d^6 q = 1 , \end{aligned} \quad (78)$$

so that (76) is at least as strong as (71). Secondly, it is easy to choose  $W(0, \cdot)$  such that, at  $t = 0$ , (71) holds and (76) does not hold. The ‘trick’ here is to modify, if necessary,  $\rho[W](0, \cdot)$ , e.g., by a scaling transformation such that  $\rho[W](0, q)$  is transformed to  $\eta^6 \rho[W](0, \eta q)$  where  $\eta$  is a positive constant while  $\vec{\mathcal{P}}[W](0, \cdot)$  is kept unchanged (we leave the details to the reader). Thus indeed (76) is stronger than (71). The strengthening, (76), of (71) can be used to interpret certain features of the kinetic approach (see the discussion after (93)). However in the present work we do not use (76) except when explicitly mentioned. Even the condition, (77) (which is an implication of (76)), is not assumed in the present work except when explicitly mentioned. Nevertheless (77) is expected to hold for a real bunch.

After this aside we now give, in the remaining parts of this section, further comments on  $W$ . We first consider the notions of observable and expectation value. An arbitrary (scalar) observable is, in the spin-1/2 Wigner function formalism, a function  $A : \mathbb{R}^7 \rightarrow \mathbb{C}^{2 \times 2}$  whose values are Hermitian matrices, i.e., for  $t \in \mathbb{R}, q \in \mathbb{R}^6$ ,

$$A(t, q) = I_{2 \times 2} A_{orb}(t, q) + \sigma_i A_{spin,i}(t, q) , \quad (79)$$

where  $A_{orb}(t, q) \in \mathbb{R}$  and  $\vec{A}_{spin}(t, q) \in \mathbb{R}^3$ . If  $W$  is the spin-1/2 Wigner function of a bunch, i.e., if (65), (69), (70) and (71) holds we define the function  $\langle A \rangle_W : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\begin{aligned} \langle A \rangle_W(t) &:= \int_{\mathbb{R}^6} Tr_{2 \times 2} \left[ A(t, q) W(t, q) \right] d^6 q \\ &= \int_{\mathbb{R}^6} \left( A_{orb}(t, q) \rho[W](t, q) + A_{spin,i}(t, q) \mathcal{P}_i[W](t, q) \right) d^6 q , \end{aligned} \quad (80)$$

where we also used (1), (5), and (79). We call  $\langle A \rangle_W(t)$  the expectation value of  $A$  w.r.t.  $W$  at time  $t$ . The expectation values of vector observables like  $\vec{\sigma}$  are defined by using (80) componentwise. For example

$$\langle \vec{\sigma} \rangle_W(t) := \int_{\mathbb{R}^6} \begin{pmatrix} Tr_{2 \times 2}[\sigma_1 W(t, q)] \\ Tr_{2 \times 2}[\sigma_2 W(t, q)] \\ Tr_{2 \times 2}[\sigma_3 W(t, q)] \end{pmatrix} d^6 q = \int_{\mathbb{R}^6} \vec{\mathcal{P}}[W](t, q) d^6 q , \quad (81)$$

where in the second equation of (81) we used (74). For later reference we note, by (81), that the statistical condition, (71), is equivalent to

$$|\langle \vec{\sigma} \rangle_W(t)| \leq 1 . \quad (82)$$

If  $W$  is the spin-1/2 Wigner function of a bunch then, motivated by (81), we define the function  $\vec{P}[W] : \mathbb{R} \rightarrow \mathbb{R}^3$ , by

$$\vec{P}[W](t) := \langle \vec{\sigma} \rangle_W(t) = \int_{\mathbb{R}^6} \vec{\mathcal{P}}[W](t, q) d^6q , \quad (83)$$

where we also used (81).

By (83),  $\vec{P}[W](t)$  is the expectation value of  $\vec{\sigma}$  w.r.t.  $W$  at time,  $t$ , and this underscores the importance of expectation values. In fact if  $W$  is the spin-1/2 Wigner function of a bunch then by the above  $\vec{\mathcal{P}}[W]$  is the polarization density of a bunch and thus, by (83),  $\vec{P}[W](t)$  is the so-called polarization vector of a bunch (at time,  $t$ ) and therefore  $|\vec{P}[W](t)|$  is the so-called polarization of a bunch (at time,  $t$ ). Note, by (82) and (83), that the statistical condition, (71), is equivalent to

$$|\vec{P}[W](t)| \leq 1 . \quad (84)$$

With the above we can now justify our statistical conditions, (69), (70) and (71), in more detail. First of all, orbital densities are  $\mathbb{R}$ -valued and polarization densities are  $\mathbb{R}^3$ -valued, i.e., satisfy (72). Also, from the discussion before (72) we know that (72) is equivalent to (69). Secondly, orbital densities are normalized by (73). Also, from the discussion before (73) we know that (73) is equivalent to (70). For another justification of (69), (70), see the discussion before (86). Thirdly, polarization vectors satisfy (84) which, by the discussion before (84), is equivalent to (71). This completes the justification of (69), (70) and (71).

We now discuss how the statistical conditions, (69), (70) and (71), on  $W$  fit to (65), i.e., to the dynamics of  $W$ . First of all, we recall that (69) is equivalent to (72). Moreover the linear operators  $l_{orb}$ ,  $l_{hom}$  and  $l_{inhom}$  turn real functions into real functions which guarantees that (72) holds for all  $t$  if it holds for  $t = 0$  and if the coefficient functions of (65) are sufficiently regular. Thus, by the equivalence of (69) and (72), (69) indeed holds for all  $t$  if it holds for  $t = 0$  (nevertheless the present work does not rely on this criterion, except when explicitly mentioned like in Section 15). Secondly, it follows from (67) that

$$\frac{\partial}{\partial t} Tr_{2 \times 2}[W] = l_{orb} Tr_{2 \times 2}[W] ,$$

so that, for  $t \in \mathbb{R}$  and by (28),

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^6} Tr_{2 \times 2}[W(t, q)] d^6q &= \int_{\mathbb{R}^6} \frac{\partial}{\partial t} Tr_{2 \times 2}[W(t, q)] d^6q \\ &= \int_{\mathbb{R}^6} l_{orb} Tr_{2 \times 2}[W(t, q)] d^6q = \int_{\mathbb{R}^6} \left( \left( -\frac{\partial}{\partial q_k} \mathcal{D}_{orb,k}(t, q) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \mathcal{B}_{orb,k}(t, q) \mathcal{B}_{orb,l}(t, q) \right) Tr_{2 \times 2}[W(t, q)] \right) d^6q = 0 . \end{aligned} \quad (85)$$

It follows from (85) that the normalization condition, (70), obeys the following conservation law: (70) holds for all  $t$  if it holds for  $t = 0$  and if the coefficient functions of (65) are

sufficiently regular (nevertheless the present work does not rely on this criterion, except when explicitly mentioned). Thirdly, in contrast to the constraints of (69) and (70) we are not aware if (71) fits to the dynamics of  $W$  (thus we are not aware if (75), (82) or (84) fit to the dynamics of  $W$  either). Therefore we leave the following question open: Does (71) hold for all  $t$  if it holds for  $t = 0$ ? Note that for the unknown ‘true’ dynamics of the bunch the answer would be: yes. Also we believe that (65) is a good approximation of the ‘true’ dynamics of the bunch and thus we believe that in general the answer to the above question is: yes (if the coefficient functions of (65) are sufficiently regular). Nevertheless the present work does not rely on the positive answer to this question, except when explicitly mentioned. In summary, we believe that if  $W$  satisfies (65) then in general (69), (70), (71) hold if they hold at  $t = 0$ . Fourthly, we will reformulate, in Section 12, the statistical conditions on  $W$ , as statistical conditions on the spin-orbit density  $f$ .

We now wrap up this section with further comments on  $W$ . First of all, the evolution equation for  $W$  was derived from the QED of the Dirac equation in [Kon82] (recall from Section 1 that Kondratenko’s thesis underlies [DK75]). Kondratenko’s derivation rests on the fact that  $W$  is the so-called Wigner-Weyl transform of the density operator of the bunch and it relies on various approximations, including a Markov approximation (necessitated by having traced out unwelcome degrees of freedom, e.g., photon degrees of freedom). In other words, the spin-1/2 Wigner function of a bunch is a Wigner-Weyl transform of an approximation of the density operator of the true bunch (this fact can also be used as another justification of (69), (70)). By [Kon82] the time evolution takes into account the effect of the external electromagnetic field and the associated (incoherent) synchrotron radiation on the bunch. Other effects like coherent synchrotron radiation effects, intrabeam scattering, the weak-strong beam-beam effect and radiative  $g - 2$  terms [Man87-2] may be included as well (but this is not pursued in the present work). Secondly, in the reduced setup, (60), the evolution equation, (65), simplifies to:

$$\frac{\partial W}{\partial t} = l_{orb}W + \frac{1}{2}\sigma_i \left( \vec{\Omega} \times \begin{pmatrix} Tr_{2 \times 2}[\sigma_1 W] \\ Tr_{2 \times 2}[\sigma_2 W] \\ Tr_{2 \times 2}[\sigma_3 W] \end{pmatrix} \right)_i, \quad (86)$$

where we also used (46), (47). We recall from Section 1 that (86), contains all the information needed to study the radiative depolarization effect (by the discussion before (29) this is even true if one neglects  $\vec{Q}$  in (86)). Even in the reduced setup we stick to our language used in this section. In particular in analogy to the discussion after (75) we call, in the reduced setup,  $W$  the spin-1/2 Wigner function of a bunch iff  $W$  satisfies (69), (70), (71) and (86). Moreover in analogy to the discussion after (75) we call, in the reduced setup,  $\rho[W]$  the orbital density and  $\vec{P}[W]$  the polarization density of a bunch iff  $W$  is the spin-1/2 Wigner function of a bunch, i.e., iff  $\rho[W], \vec{P}[W]$  satisfy (72), (73), (75) and the orbital Fokker-Planck equation as well as the reduced Bloch equation. Furthermore we stick to the definition of  $\vec{P}[W](t)$ : If in the reduced setup  $W$  is the spin-1/2 Wigner function of a bunch then by the above  $\vec{P}[W]$  is the polarization density of a bunch and thus, by (83),  $\vec{P}[W](t)$  is the so-called polarization vector of a bunch (at time,  $t$ ) and therefore  $|\vec{P}[W](t)|$  is the so-called polarization of a bunch (at time,  $t$ ). Thirdly, we note that there is an extensive literature on spin-1/2 Wigner functions. For example the reader may consult [OCW84]. Note also that

spin-1/2 Wigner functions are sometimes called Stratonovich functions.

## 6 The local polarization vector field

To keep the discussion concise we assume in this section that

$$\rho[W] > 0 . \quad (87)$$

Following [BH01], we define

$$\vec{\mathcal{P}}_{loc}[W](t, q) := \frac{\vec{\mathcal{P}}[W](t, q)}{\rho[W](t, q)} , \quad (88)$$

so that

$$\vec{\mathcal{P}}[W](t, q) = \rho[W](t, q) \vec{\mathcal{P}}_{loc}[W](t, q) , \quad (89)$$

which entails, by (83),

$$\vec{\mathcal{P}}[W](t) = \int_{\mathbb{R}^6} \rho[W](t, q) \vec{\mathcal{P}}_{loc}[W](t, q) d^6q . \quad (90)$$

If  $W$  is the spin-1/2 Wigner function of a bunch, i.e., if it satisfies (65), (69), (70) and (71) that is, if  $\rho[W]$  is the orbital density and  $\vec{\mathcal{P}}[W]$  is the polarization density of a bunch (recall the discussion after (75)) then, by following [BH01], we call  $\vec{\mathcal{P}}_{loc}[W]$  the local polarization vector field. The fact that we define the local polarization vector field only in a special case, namely (87), indicates that this field only plays a side role in this work.

In the remaining parts of this section we comment on the relation between (76) and  $\vec{\mathcal{P}}_{loc}[W]$ . First of all, by (87),

$$|\vec{\mathcal{P}}[W]| = |\rho[W]| |\vec{\mathcal{P}}_{loc}[W]| = \rho[W] |\vec{\mathcal{P}}_{loc}[W]| . \quad (91)$$

Secondly, if

$$|\vec{\mathcal{P}}_{loc}[W]| \leq 1 , \quad (92)$$

then, due to (91), we get (76), i.e.,  $|\vec{\mathcal{P}}[W]| \leq \rho[W]$ . Thirdly, it follows from (76), (91) that

$$\rho[W] |\vec{\mathcal{P}}_{loc}[W]| \leq \rho[W] , \quad (93)$$

so that, by (87), we get (92). Thus (92) is equivalent to the strengthening, (76), of (71).

Fourthly, if (76) holds at a point,  $(t, q)$ , i.e., if (92) holds at  $(t, q)$  then, because of (90), one may interpret,  $\vec{\mathcal{P}}_{loc}[W](t, q)$ , as the spin polarization vector at  $(t, q)$  namely the ensemble average of normalized single-particle spin vectors at that point. In the opposite case, where at  $(t, q)$  (71) holds but where (76) does not hold, one may interpret,  $\vec{\mathcal{P}}_{loc}[W](t, q)$ , as the weighted spin polarization vector at  $(t, q)$  where the possibly  $t$ -dependent weight may be defined in the vein of the discussion after (78), i.e., via a scaling transformation of  $\rho[W]$ . These interpretations of  $\vec{\mathcal{P}}_{loc}[W]$  will be used in Section 10 (see the discussion after (137)).

## 7 The dynamical condition used to derive the full spin-orbit Fokker-Planck equation

Sections 7-10 are focused on the following task: Translate, if possible, the dynamical information of the kinetic approach, i.e., the information contained in the evolution equation, (65), of  $W$  into a Fokker-Planck equation on the spin-orbit phase space,  $\mathbb{R}^9$ ! Such an equation we call spin-orbit Fokker-Planck equation (moreover we call it the full resp. reduced spin-orbit Fokker-Planck equation whether we are in the full or reduced setup). Note that we use the terminology of spin-orbit Fokker-Planck equation whenever we leave open if we talk about the full or the reduced setup. In the full setup, i.e., when (65) holds the aforementioned task was fulfilled in [HABBE19] resulting in the full spin-orbit Fokker-Planck equation, (130), which is presented in Section 10. Being a Fokker-Planck equation, (130) has the same dynamical content as its associated Ito SDE system, the latter also being presented in Section 10.

Since (86) is a special case of (65) it is no surprise that the reduced spin-orbit Fokker-Planck equation, (141), is a special case of (130). Thus in our work of Sections 7-10 we focus on the full setup and do not have to care about the reduced setup (the reduced setup is covered in Section 11). In [BH01] a derivation of the reduced spin-orbit Fokker-Planck equation was given for the special case where  $\vec{Q} = \vec{0}$  and this derivation was partly based on physical intuition, whereas so far we did neither publish a derivation of the full nor of the reduced spin-orbit Fokker-Planck equation (not even in [HABBE19]). Thus the derivation of the full spin-orbit Fokker-Planck equation is a major piece of the present work. The key element of our derivation is the so-called dynamical condition which is the topic of this section.

Before we define the dynamical condition we need to make some preparations and we first note that the full spin-orbit Fokker-Planck equation, being a Fokker-Planck equation, can be formally written, for  $f = f(t, q, \vec{s})$ , as

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial y_j}(d_{orb,spin,j}f) + \frac{1}{2}\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_k}\left((b_{orb,spin}b_{orb,spin}^T)_{k,j}f\right), \quad (94)$$

where  $d_{orb,spin}$  is  $\mathbb{R}^9$ -valued and  $b_{orb,spin}$  is  $\mathbb{R}^{9 \times K}$ -valued with  $K$  being a positive integer and

$$y \equiv \begin{pmatrix} q \\ \vec{s} \end{pmatrix}. \quad (95)$$

Thus the aforementioned task of Sections 7-10 boils down to find  $d_{orb,spin}$  and  $b_{orb,spin}$ . One calls the  $\mathbb{R}^9$ -valued function,  $d_{orb,spin}$ , the drift vector field and the  $\mathbb{R}^{9 \times 9}$ -valued function,  $b_{orb,spin}b_{orb,spin}^T$ , the diffusion matrix field of (94). We call the  $\mathbb{R}^{9 \times K}$ -valued function,  $b_{orb,spin}$ , the noise matrix field of (94).

The Ito SDE system associated with the formal Fokker-Planck equation, (94), reads, for  $Q = Q(t)$ ,  $\vec{S} = \vec{S}(t)$ , as

$$\begin{pmatrix} Q' \\ \vec{S}' \end{pmatrix} = d_{orb,spin}(t, Q, \vec{S}) + b_{orb,spin}(t, Q, \vec{S})\omega(t), \quad (96)$$

where  $\omega$  is the  $K$ -dimensional white noise process.

How do we arrive at a dynamical condition which will allow us to translate the information in (65) into, (94)? Answer: Following [BH01] there is a natural way to translate  $f$  into a spin-1/2 Wigner function (which we denote by  $W_f$ ). In fact it is natural to mimick the trace operation in (5) by the operation of  $\vec{s}$ -integration over  $\mathbb{R}^3$  giving us the following ansatz for  $\rho[W_f], \vec{\mathcal{P}}[W_f]$ :

$$\begin{aligned}\rho[W_f](t, q) &= \int_{\mathbb{R}^3} f(t, q, \vec{s}) d^3 s, \quad \vec{\mathcal{P}}[W_f](t, q) = \int_{\mathbb{R}^3} \begin{pmatrix} s_1 f(t, q, \vec{s}) \\ s_2 f(t, q, \vec{s}) \\ s_3 f(t, q, \vec{s}) \end{pmatrix} d^3 s \\ &= \int_{\mathbb{R}^3} \vec{s} f(t, q, \vec{s}) d^3 s,\end{aligned}$$

and thus, by using (1), we arrive at the following ansatz for  $W_f$ :

$$W_f(t, q) := \frac{1}{2} \left( I_{2 \times 2} \int_{\mathbb{R}^3} f(t, q, \vec{s}) d^3 s + \sigma_i \int_{\mathbb{R}^3} s_i f(t, q, \vec{s}) d^3 s \right). \quad (97)$$

For later reference we note, by (5), (97), that

$$\rho[W_f](t, q) = Tr_{2 \times 2}[W_f(t, q)] = \int_{\mathbb{R}^3} f(t, q, \vec{s}) d^3 s, \quad (98)$$

$$\vec{\mathcal{P}}[W_f](t, q) = \begin{pmatrix} Tr_{2 \times 2}[\sigma_1 W_f(t, q)] \\ Tr_{2 \times 2}[\sigma_2 W_f(t, q)] \\ Tr_{2 \times 2}[\sigma_3 W_f(t, q)] \end{pmatrix} = \int_{\mathbb{R}^3} \vec{s} f(t, q, \vec{s}) d^3 s, \quad (99)$$

so that, by (97),

$$W_f(t, q) = \frac{1}{2} \left( I_{2 \times 2} \rho[W_f](t, q) + \sigma_i \mathcal{P}_i[W_f](t, q) \right). \quad (100)$$

Note that (98), (99) explicitly display the mimicking of  $Tr_{2 \times 2}[\dots]$  by  $\int_{\mathbb{R}^3} d^3 s \dots$ .

With the correspondence, (97), between  $f$  and  $W_f$  we can now define the dynamical condition which reads as follows: If  $f$  satisfies, (94), and if  $W_f$  in (97) exists and is sufficiently regular then  $W_f$  satisfies, (65), i.e.,

$$\frac{\partial W_f}{\partial t} = l_{orb} W_f + \frac{1}{2} \sigma_i \left( l_{hom} \begin{pmatrix} Tr_{2 \times 2}[\sigma_1 W_f] \\ Tr_{2 \times 2}[\sigma_2 W_f] \\ Tr_{2 \times 2}[\sigma_3 W_f] \end{pmatrix} + l_{inhom} Tr_{2 \times 2}[W_f] \right)_i. \quad (101)$$

Recalling from Section 5 that (65) is equivalent to the system (26), (49) we note that (101) is equivalent to

$$\frac{\partial \rho[W_f]}{\partial t} = -\frac{\partial}{\partial q_j} (\mathcal{D}_{orb,j} \rho[W_f]) + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} (\mathcal{B}_{orb,k} \mathcal{B}_{orb,l} \rho[W_f]), \quad (102)$$

$$\frac{\partial \vec{\mathcal{P}}[W_f]}{\partial t} = (l_{orb} + l_{hom}) \vec{\mathcal{P}}[W_f] + l_{inhom} \rho[W_f]. \quad (103)$$

Thus the dynamical condition is equivalent to the following criterion: If  $f$  satisfies (94) and if  $W_f$  exists and is sufficiently regular then (102), (103) have to hold. This criterion will give us the full spin-orbit Fokker-Planck equation, (130).

In the remaining parts of this section we make some comments on the dynamical condition. First of all, the correspondence, (97), between  $f$  and  $W_f$  is independent of the dynamics of  $W_f$ , i.e., it would also be used if one would modify, (101), e.g., by adding further physical effects like the weak-strong beam-beam effect. Secondly, for an arbitrary solution  $f$  of (94),  $W_f$  does not necessarily exist (pick for example  $f(0, \cdot, \cdot)$  as a constant function!). Thirdly, if  $W_f$  exists then it is uniquely determined by  $f$ , but  $f$  is not uniquely determined by  $W_f$ . Thus  $W_f$  and  $f$  play similar roles as the fields and their potentials in classical electrodynamics. In other words it can happen that, under the dynamical condition, solutions  $f, g$  of (94) exist such that  $W_f = W_g$  and  $f \neq g$ . Explicit examples of  $f, g$  are presented, for the reduced setup, in Section 14 below, see (171), (175). Fourthly, conditions on  $f$  which guarantee that  $W_f$  is the spin-1/2 Wigner function of a bunch will be stated in Section 12 (if these conditions hold then we will call  $f$ , from Section 12 onwards, a spin-orbit density). Of course, by the dynamical condition, one of these conditions is the validity of the full spin-orbit Fokker-Planck equation. Fifthly, it follows from (83), (99) that if  $W_f$  is the spin-1/2 Wigner function of a bunch, e.g., if  $f$  is a spin-orbit density then  $\vec{P}[W_f]$  is well defined and reads as

$$\vec{P}[W_f](t) = \langle \vec{\sigma} \rangle_{W_f}(t) = \int_{\mathbb{R}^6} \vec{\mathcal{P}}[W_f](t, q) d^6 q = \int_{\mathbb{R}^9} \vec{s} f(t, q, \vec{s}) d^3 s d^6 q. \quad (104)$$

## 8 The kinematical conditions used to derive the full spin-orbit Fokker-Planck equation

We mentioned in Section 7 the nonuniqueness of  $f$  in  $W_f$  and there is even more to say about nonuniqueness as follows. Every choice of the coefficient functions of the formal Fokker-Planck equation, (94), which satisfies the dynamical condition, reproduces the evolution equation, (101), of  $W_f$ . Thus the uniqueness of the full spin-orbit Fokker-Planck equation is not required (nor is it known to us). In other words we are not interested in the question, whether (130) is the only possible choice of (94) which satisfies the dynamical condition. Even this can be understood in terms of our analogy (from Section 7) with classical electrodynamics: While the electrodynamical fields obey the same PDE system, the potentials obey PDE systems which vary (since they may belong to different gauges).

Nevertheless in order to facilitate the specification of the coefficient functions of (94) (not in order to make the coefficient functions unique!) we add two kinematical conditions to the dynamical condition (kinematical means non-dynamical). These two kinematical conditions read as follows:

- (i) Choose the positive integer,  $K$ , in  $b_{orb,spin}$  as small as possible (recall that  $b_{orb,spin}$  is  $\mathbb{R}^{9 \times K}$ -valued!). In other words, first try  $K = 1$  then try  $K = 2$  and so on.
- (ii) Let all coefficient functions of the formal Fokker-Planck equation, (94), depend on  $t, q, \vec{s}$  such that the dependence on  $\vec{s}$  is of at most first order.

In the remaining parts of this work we make some comments on (i),(ii). First of all, (i) is Occam's razor while (ii) is motivated by dealing with a spin-1/2 particle (in fact (ii) would be modified in the case of a particle of spin different from 1/2). Secondly, (i) facilitates our work in an obvious way. In fact we will see in Section 9 that,  $K = 1$ , will suffice. Thirdly, (ii) facilitates our work since it will ensure that only few nonphysical entries occur in (116), (117) (which are then eliminated via the dynamical condition). Nevertheless without (ii) we would arrive at the same full spin-orbit Fokker-Planck equation, (130). In contrast, we leave open if we would arrive at (130) if we would drop (i). In other words we leave unanswered the question if a non-scalar white noise process  $\omega$  in (96) is consistent with the dynamical condition. However this is not a drawback since, as mentioned above, the uniqueness of the coefficient functions of (94) is not required. Nevertheless, thanks to the naturalness of the two kinematical conditions, (i), (ii), we believe that the explicit form of the full spin-orbit Fokker-Planck equation, (130), is the simplest possible one. Fourthly, as mentioned above, in [HABBE19] we introduced (130) without derivation, i.e., we did not explain in [HABBE19] how we got (130). In fact unpublished derivation relied on the dynamical condition and parts of the two kinematical conditions as well as some educated guesses. Fifthly, in contrast to [HABBE19] in this work we explicitly derive (130) and our derivation here is completely based on the dynamical condition and the two kinematical conditions. In particular we show that (130) satisfies the dynamical condition and thereby put (130) on a firm ground. It is however not surprising that we get the same equation full spin-orbit Fokker-Planck equation as in [HABBE19], namely (130), since our conditions here are very similar to the ones underlying [HABBE19]. Sixthly, the following question arises: In which order should the conditions be enforced? Answer: First enforce the kinematical conditions, (i), (ii), and then enforce the dynamical condition!

## 9 Deriving the full spin-orbit Fokker-Planck equation

As announced in Section 8 we impose here, on the formal Fokker-Planck equation (94), the two kinematical conditions and then the dynamical condition (thus Section 9 is divided into Sections 9.1 and 9.2). In particular we will show that the simplest case,  $K = 1$ , of the kinematical conditions is sufficient.

### 9.1 Enforcing the two kinematical conditions

We begin with the kinematical condition, (i), whereby in the case,  $K = 1$ ,  $b_{orb,spin}$  is  $\mathbb{R}^9$ -valued. Because  $d_{orb,spin}(t, q, \vec{s}) \in \mathbb{R}^9$  and  $b_{orb,spin}(t, q, \vec{s}) \in \mathbb{R}^9$  we can write

$$d_{orb,spin}(t, q, \vec{s}) = \begin{pmatrix} d_{orb}(t, q, \vec{s}) \\ \vec{d}_{spin}(t, q, \vec{s}) \end{pmatrix}, \quad (105)$$

$$b_{orb,spin}(t, q, \vec{s}) = \begin{pmatrix} b_{orb}(t, q, \vec{s}) \\ \vec{b}_{spin}(t, q, \vec{s}) \end{pmatrix}, \quad (106)$$

where  $d_{orb}(t, q, \vec{s}) \in \mathbb{R}^6$ ,  $b_{orb}(t, q, \vec{s}) \in \mathbb{R}^6$  and  $\vec{d}_{spin}(t, q, \vec{s}) \in \mathbb{R}^3$ ,  $\vec{b}_{spin}(t, q, \vec{s}) \in \mathbb{R}^3$ .

For any solution,  $f$ , of (94) we now compute, by (95), (105), (106),

$$\begin{aligned} -\frac{\partial}{\partial y_j}(b_{orb,spin,j}f) &= -\frac{\partial}{\partial q_j}(d_{orb,j}f) - \frac{\partial}{\partial s_j}(d_{spin,j}f) , \\ \frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l}(b_{orb,spin,l}b_{orb,spin,k}f) &= \frac{\partial}{\partial y_k}\left(\frac{\partial}{\partial q_l}(b_{orb,l}b_{orb,spin,k}f) + \frac{\partial}{\partial s_l}(b_{spin,l}b_{orb,spin,k}f)\right) \\ &= \frac{\partial}{\partial q_k}\frac{\partial}{\partial q_l}(b_{orb,l}b_{orb,k}f) + \frac{\partial}{\partial s_k}\frac{\partial}{\partial q_l}(b_{orb,l}b_{spin,k}f) \\ &\quad + \frac{\partial}{\partial q_k}\frac{\partial}{\partial s_l}(b_{spin,l}b_{orb,k}f) + \frac{\partial}{\partial s_k}\frac{\partial}{\partial s_l}(b_{spin,l}b_{spin,k}f) \\ &= \frac{\partial}{\partial q_k}\frac{\partial}{\partial q_l}(b_{orb,l}b_{orb,k}f) + 2\frac{\partial}{\partial s_k}\frac{\partial}{\partial q_l}(b_{orb,l}b_{spin,k}f) + \frac{\partial}{\partial s_k}\frac{\partial}{\partial s_l}(b_{spin,l}b_{spin,k}f) , \end{aligned}$$

so that, for the case,  $K = 1$ , the formal Fokker-Planck equation, (94), can be written under the kinematical condition, (i), as

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\frac{\partial}{\partial q_j}(d_{orb,j}f) - \frac{\partial}{\partial s_j}(d_{spin,j}f) + \frac{1}{2}\frac{\partial}{\partial q_k}\frac{\partial}{\partial q_l}(b_{orb,l}b_{orb,k}f) \\ &\quad + \frac{\partial}{\partial s_k}\frac{\partial}{\partial q_l}(b_{orb,l}b_{spin,k}f) + \frac{1}{2}\frac{\partial}{\partial s_k}\frac{\partial}{\partial s_l}(b_{spin,l}b_{spin,k}f) . \end{aligned} \quad (107)$$

Thus in the case,  $K = 1$ , the Ito SDE system associated with the formal Fokker-Planck equation can be written, under the kinematical condition, (i), as

$$Q' = d_{orb}(t, Q, \vec{S}) + b_{orb}(t, Q, \vec{S})\nu(t) , \quad (108)$$

$$\vec{S}' = \vec{d}_{spin}(t, Q, \vec{S}) + \vec{b}_{spin}(t, Q, \vec{S})\nu(t) , \quad (109)$$

where  $\nu$  is the one-dimensional white noise process. Note that (108), (109) is obtained by inserting, (105), (106), into (96).

Having enforced the kinematical condition, (i), for the case,  $K = 1$ , we arrived at (107). To enforce both kinematical conditions we now impose the kinematical condition, (ii), onto (107). Recalling the definition of (ii) in Section 8 we get from (105), (106)

$$\begin{aligned} d_{orb}(t, q, \vec{s}) &= d_{orb,0}(t, q) + d_{orb,+}(t, q)\vec{s} , \\ \vec{d}_{spin}(t, q, \vec{s}) &= \vec{d}_{spin,0}(t, q) + d_{spin,+}(t, q)\vec{s} , \\ b_{orb}(t, q, \vec{s}) &= b_{orb,0}(t, q) + b_{orb,+}(t, q)\vec{s} , \\ \vec{b}_{spin}(t, q, \vec{s}) &= \vec{b}_{spin,0}(t, q) + b_{spin,+}(t, q)\vec{s} , \end{aligned} \quad (110)$$

where  $d_{spin,0}(t, q), \vec{b}_{spin,0}(t, q) \in \mathbb{R}^3$ ;  $d_{orb,0}(t, q), b_{orb,0}(t, q) \in \mathbb{R}^6$  and  $d_{spin,+}(t, q), b_{spin,+}(t, q) \in \mathbb{R}^{3 \times 3}$  as well as  $d_{orb,+}(t, q), b_{orb,+}(t, q) \in \mathbb{R}^{6 \times 3}$ . Note, by (105), (106), (110), that

$$d_{orb,spin}(t, q, \vec{s}) = \begin{pmatrix} d_{orb,0}(t, q) + d_{orb,+}(t, q)\vec{s} \\ \vec{d}_{spin,0}(t, q) + d_{spin,+}(t, q)\vec{s} \end{pmatrix} , \quad (111)$$

$$b_{orb,spin}(t, q, \vec{s}) = \begin{pmatrix} b_{orb,0}(t, q) + b_{orb,+}(t, q)\vec{s} \\ \vec{b}_{spin,0}(t, q) + b_{spin,+}(t, q)\vec{s} \end{pmatrix} . \quad (112)$$

With (111), (112) our task, mentioned after (95) and to be fulfilled in this section, which is to specify  $d_{orb,spin}$ ,  $b_{orb,spin}$ , boils down to the specification of the four vector fields,  $d_{orb,0}$ ,  $\vec{d}_{spin,0}$ ,  $b_{orb,0}$ ,  $\vec{b}_{spin,0}$ , and of the four matrix fields,  $d_{orb,+}$ ,  $d_{spin,+}$ ,  $b_{orb,+}$ ,  $b_{spin,+}$ . Note that under the kinematical conditions and for the case,  $K = 1$ , the formal Fokker-Planck equation can be written in terms of these eight functions as

$$\begin{aligned} \frac{\partial f}{\partial t} = & -\frac{\partial}{\partial q_j} \left( (d_{orb,0,j} + d_{orb,+,j,k} s_k) f \right) - \frac{\partial}{\partial s_j} \left( (d_{spin,0,j} + d_{spin,+,j,k} s_k) f \right) \\ & + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( (b_{orb,0,l} + b_{orb,+,l,i} s_i) (b_{orb,0,k} + b_{orb,+,k,j} s_j) f \right) \\ & + \frac{\partial}{\partial s_k} \frac{\partial}{\partial q_l} \left( (b_{orb,0,l} + b_{orb,+,l,i} s_i) (b_{spin,0,k} + b_{spin,+,k,j} s_j) f \right) \\ & + \frac{1}{2} \frac{\partial}{\partial s_k} \frac{\partial}{\partial s_l} \left( (b_{spin,0,l} + b_{spin,+,l,i} s_i) (b_{spin,0,k} + b_{spin,+,k,j} s_j) f \right). \end{aligned} \quad (113)$$

Note that (113) is obtained by inserting, (110), into (107).

As an aside we note that under the kinematical conditions and for the case,  $K = 1$ , the Ito SDE system associated with the formal Fokker-Planck equation can be written as:

$$Q' = d_{orb,0}(t, Q) + d_{orb,+}(t, Q) \vec{S} + \left( b_{orb,0}(t, Q) + b_{orb,+}(t, Q) \vec{S} \right) \nu(t), \quad (114)$$

$$\vec{S}' = \vec{d}_{spin,0}(t, Q) + d_{spin,+}(t, Q) \vec{S} + \left( \vec{b}_{spin,0}(t, Q) + b_{spin,+}(t, Q) \vec{S} \right) \nu(t), \quad (115)$$

where  $\nu$  is the one-dimensional white noise process. Note that (114), (115) are obtained by inserting, (110), into (108), (109). This completes our enforcement of the kinematical conditions for the case,  $K = 1$  (as we will see at the end of Section 9.2, we will not have to go beyond,  $K = 1$ ).

Having enforced both kinematical conditions (for the case,  $K = 1$ ) we have arrived at (113). Thus we will now prepare ourselves for enforcing the dynamical condition by computing  $\frac{\partial \rho[W_f]}{\partial t}$  and  $\frac{\partial \vec{\rho}[W_f]}{\partial t}$  where  $f$  is an arbitrary solution of the formal Fokker-Planck equation, (113).

We begin with  $\frac{\partial \rho[W_f]}{\partial t}$  and compute, by (98), (99), (107),

$$\begin{aligned} \frac{\partial \rho[W_f]}{\partial t}(t, q) &= \int_{\mathbb{R}^3} \frac{\partial f}{\partial t}(t, q, \vec{s}) d^3 s \\ &= -\frac{\partial}{\partial q_j} \int_{\mathbb{R}^3} \left( d_{orb,0,j}(t, q) + d_{orb,+,j,k}(t, q) s_k \right) f(t, q, \vec{s}) d^3 s \\ &\quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \int_{\mathbb{R}^3} \left( (b_{orb,0,l}(t, q) + b_{orb,+,l,i}(t, q) s_i) \right. \\ &\quad \left. \cdot (b_{orb,0,k}(t, q) + b_{orb,+,k,j}(t, q) s_j) \right) f(t, q, \vec{s}) d^3 s \end{aligned}$$

$$\begin{aligned}
&= -\frac{\partial}{\partial q_j} \left( d_{orb,0,j}(t, q) \int_{\mathbb{R}^3} f(t, q, \vec{s}) d^3 s \right) - \frac{\partial}{\partial q_j} \left( d_{orb,+,j,k}(t, q) \int_{\mathbb{R}^3} s_k f(t, q, \vec{s}) d^3 s \right) \\
&+ \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \int_{\mathbb{R}^3} \left( b_{orb,0,l}(t, q) b_{orb,0,k}(t, q) \right. \\
&\quad + b_{orb,+,l,i}(t, q) s_i b_{orb,0,k}(t, q) + b_{orb,0,l}(t, q) b_{orb,+,k,j}(t, q) s_j \\
&\quad \left. + b_{orb,+,l,i}(t, q) s_i b_{orb,+,k,j}(t, q) s_j \right) f(t, q, \vec{s}) d^3 s \\
&= -\frac{\partial}{\partial q_j} \left( d_{orb,0,j}(t, q) \rho[W_f](t, q) \right) - \frac{\partial}{\partial q_j} \left( d_{orb,+,j,k}(t, q) \mathcal{P}_k[W_f](t, q) \right) \\
&+ \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,0,l}(t, q) b_{orb,0,k}(t, q) \rho[W_f](t, q) \right) \\
&+ \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,+,l,i}(t, q) \mathcal{P}_i[W_f](t, q) b_{orb,0,k}(t, q) \right) \\
&+ \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,0,l}(t, q) b_{orb,+,k,j}(t, q) \mathcal{P}_j[W_f](t, q) \right) \\
&+ \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( \int_{\mathbb{R}^3} b_{orb,+,l,i}(t, q) b_{orb,+,k,j}(t, q) s_i s_j f(t, q, \vec{s}) d^3 s \right) \\
&= -\frac{\partial}{\partial q_j} \left( d_{orb,0,j}(t, q) \rho[W_f](t, q) \right) - \frac{\partial}{\partial q_j} \left( d_{orb,+,j,k}(t, q) \mathcal{P}_k[W_f](t, q) \right) \\
&+ \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,0,l}(t, q) b_{orb,0,k}(t, q) \rho[W_f](t, q) \right) \\
&+ \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,0,l}(t, q) b_{orb,+,k,j}(t, q) \mathcal{P}_j[W_f](t, q) \right) \\
&+ \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,+,l,i}(t, q) b_{orb,+,k,j}(t, q) \int_{\mathbb{R}^3} s_i s_j f(t, q, \vec{s}) d^3 s \right),
\end{aligned}$$

in short

$$\begin{aligned}
\frac{\partial \rho[W_f]}{\partial t}(t, q) &= -\frac{\partial}{\partial q_j} \left( d_{orb,0,j}(t, q) \rho[W_f](t, q) \right) \\
&- \frac{\partial}{\partial q_j} \left( d_{orb,+,j,k}(t, q) \mathcal{P}_k[W_f](t, q) \right) \\
&+ \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,0,l}(t, q) b_{orb,0,k}(t, q) \rho[W_f](t, q) \right) \\
&+ \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,0,l}(t, q) b_{orb,+,k,j}(t, q) \mathcal{P}_j[W_f](t, q) \right) \\
&+ \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,+,l,i}(t, q) b_{orb,+,k,j}(t, q) \int_{\mathbb{R}^3} s_i s_j f(t, q, \vec{s}) d^3 s \right), \tag{116}
\end{aligned}$$

where we also used that (107) implies (113). The second and fourth entries on the rhs of (116) are Stern-Gerlach-like terms and also the fifth entry is nonphysical (in the sense that

$\int_{\mathbb{R}^3} s_i s_j f(t, q, \vec{s}) d^3 s$  cannot be expressed in terms of  $\rho[W_f], \vec{\mathcal{P}}[W_f]$ . In fact these nonphysical entries will be eliminated in Section 9.2 when we enforce the first part of the dynamical condition.

Having computed  $\frac{\partial \rho[W_f]}{\partial t}$  we now compute  $\frac{\partial \vec{\mathcal{P}}[W_f]}{\partial t}$  where again  $f$  is an arbitrary solution of the formal Fokker-Planck equation, (113). We first compute, by (99), (107) and via integration by parts,

$$\begin{aligned}
\frac{\partial \mathcal{P}_m[W_f]}{\partial t}(t, q) &= \int_{\mathbb{R}^3} s_m \frac{\partial f}{\partial t}(t, q, \vec{s}) d^3 s = \int_{\mathbb{R}^3} s_m \left( -\frac{\partial}{\partial q_j} d_{orb,j}(t, q, \vec{s}) - \frac{\partial}{\partial s_j} d_{spin,j}(t, q, \vec{s}) \right. \\
&\quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} b_{orb,l}(t, q, \vec{s}) b_{orb,k}(t, q, \vec{s}) + \frac{\partial}{\partial s_k} \frac{\partial}{\partial q_l} b_{orb,l}(t, q, \vec{s}) b_{spin,k}(t, q, \vec{s}) \\
&\quad \left. + \frac{1}{2} \frac{\partial}{\partial s_k} \frac{\partial}{\partial s_l} b_{spin,l}(t, q, \vec{s}) b_{spin,k}(t, q, \vec{s}) \right) f(t, q, \vec{s}) d^3 s \\
&= -\frac{\partial}{\partial q_j} \int_{\mathbb{R}^3} d_{orb,j}(t, q, \vec{s}) s_m f(t, q, \vec{s}) d^3 s \\
&\quad - \int_{\mathbb{R}^3} s_m \frac{\partial}{\partial s_j} d_{spin,j}(t, q, \vec{s}) f(t, q, \vec{s}) d^3 s \\
&\quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \int_{\mathbb{R}^3} b_{orb,l}(t, q, \vec{s}) b_{orb,k}(t, q, \vec{s}) s_m f(t, q, \vec{s}) d^3 s \\
&\quad + \int_{\mathbb{R}^3} s_m \frac{\partial}{\partial s_k} \frac{\partial}{\partial q_l} b_{orb,l}(t, q, \vec{s}) b_{spin,k}(t, q, \vec{s}) f(t, q, \vec{s}) d^3 s \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} s_m \frac{\partial}{\partial s_k} \frac{\partial}{\partial s_l} b_{spin,l}(t, q, \vec{s}) b_{spin,k}(t, q, \vec{s}) f(t, q, \vec{s}) d^3 s \\
&= -\frac{\partial}{\partial q_j} \int_{\mathbb{R}^3} d_{orb,j}(t, q, \vec{s}) s_m f(t, q, \vec{s}) d^3 s \\
&\quad + \int_{\mathbb{R}^3} d_{spin,m}(t, q, \vec{s}) f(t, q, \vec{s}) d^3 s \\
&\quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \int_{\mathbb{R}^3} b_{orb,l}(t, q, \vec{s}) b_{orb,k}(t, q, \vec{s}) s_m f(t, q, \vec{s}) d^3 s \\
&\quad - \frac{\partial}{\partial q_l} \int_{\mathbb{R}^3} b_{orb,l}(t, q, \vec{s}) b_{spin,m}(t, q, \vec{s}) f(t, q, \vec{s}) d^3 s ,
\end{aligned}$$

where  $m = 1, 2, 3$  which implies, by (98), (99), (110),

$$\begin{aligned}
\frac{\partial \mathcal{P}_m[W_f]}{\partial t}(t, q) &= -\frac{\partial}{\partial q_j} \int_{\mathbb{R}^3} \left( d_{orb,0,j}(t, q) + d_{orb,+j,i}(t, q) s_i \right) s_m f(t, q, \vec{s}) d^3 s \\
&\quad + \int_{\mathbb{R}^3} \left( d_{spin,0,m}(t, q) + d_{spin,+m,i}(t, q) s_i \right) f(t, q, \vec{s}) d^3 s \\
&\quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \int_{\mathbb{R}^3} \left( b_{orb,0,l}(t, q) + b_{orb,+l,i}(t, q) s_i \right) \\
&\quad \cdot \left( b_{orb,0,k}(t, q) + b_{orb,+k,j}(t, q) s_j \right) s_m f(t, q, \vec{s}) d^3 s
\end{aligned}$$

$$\begin{aligned}
& -\frac{\partial}{\partial q_l} \int_{\mathbb{R}^3} \left( b_{orb,0,l}(t, q) + b_{orb,+l,i}(t, q) s_i \right) \left( b_{spin,0,m}(t, q) + b_{spin,+m,j}(t, q) s_j \right) f(t, q, \vec{s}) d^3 s \\
& = -\frac{\partial}{\partial q_j} d_{orb,0,j}(t, q) \int_{\mathbb{R}^3} s_m f(t, q, \vec{s}) d^3 s - \frac{\partial}{\partial q_j} d_{orb,+j,i}(t, q) \int_{\mathbb{R}^3} s_i s_m f(t, q, \vec{s}) d^3 s \\
& \quad + d_{spin,0,m}(t, q) \int_{\mathbb{R}^3} f(t, q, \vec{s}) d^3 s + d_{spin,+m,i}(t, q) \int_{\mathbb{R}^3} s_i f(t, q, \vec{s}) d^3 s \\
& \quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} b_{orb,0,l}(t, q) b_{orb,0,k}(t, q) \int_{\mathbb{R}^3} s_m f(t, q, \vec{s}) d^3 s \\
& \quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} b_{orb,0,l}(t, q) b_{orb,+k,j}(t, q) \int_{\mathbb{R}^3} s_j s_m f(t, q, \vec{s}) d^3 s \\
& \quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} b_{orb,+l,i}(t, q) b_{orb,0,k}(t, q) \int_{\mathbb{R}^3} s_i s_m f(t, q, \vec{s}) d^3 s \\
& \quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} b_{orb,+l,i}(t, q) b_{orb,+k,j}(t, q) \int_{\mathbb{R}^3} s_i s_j s_m f(t, q, \vec{s}) d^3 s \\
& \quad - \frac{\partial}{\partial q_l} b_{orb,0,l}(t, q) b_{spin,0,m}(t, q) \int_{\mathbb{R}^3} f(t, q, \vec{s}) d^3 s \\
& \quad - \frac{\partial}{\partial q_l} b_{orb,0,l}(t, q) b_{spin,+m,j}(t, q) \int_{\mathbb{R}^3} s_j f(t, q, \vec{s}) d^3 s \\
& \quad - \frac{\partial}{\partial q_l} b_{orb,+l,i}(t, q) b_{spin,0,m}(t, q) \int_{\mathbb{R}^3} s_i f(t, q, \vec{s}) d^3 s \\
& \quad - \frac{\partial}{\partial q_l} b_{orb,+l,i}(t, q) b_{spin,+m,j}(t, q) \int_{\mathbb{R}^3} s_i s_j f(t, q, \vec{s}) d^3 s \\
& = -\frac{\partial}{\partial q_j} d_{orb,0,j}(t, q) \mathcal{P}_m[W_f](t, q) - \frac{\partial}{\partial q_j} d_{orb,+j,i}(t, q) \int_{\mathbb{R}^3} s_i s_m f(t, q, \vec{s}) d^3 s \\
& \quad + d_{spin,0,m}(t, q) \rho[W_f](t, q) + d_{spin,+m,i}(t, q) \mathcal{P}_i[W_f](t, q) \\
& \quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} b_{orb,0,l}(t, q) b_{orb,0,k}(t, q) \mathcal{P}_m[W_f](t, q) \\
& \quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} b_{orb,0,l}(t, q) b_{orb,+k,j}(t, q) \int_{\mathbb{R}^3} s_j s_m f(t, q, \vec{s}) d^3 s \\
& \quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} b_{orb,+l,i}(t, q) b_{orb,0,k}(t, q) \int_{\mathbb{R}^3} s_i s_m f(t, q, \vec{s}) d^3 s \\
& \quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} b_{orb,+l,i}(t, q) b_{orb,+k,j}(t, q) \int_{\mathbb{R}^3} s_i s_j s_m f(t, q, \vec{s}) d^3 s \\
& \quad - \frac{\partial}{\partial q_l} b_{orb,0,l}(t, q) b_{spin,0,m}(t, q) \rho[W_f](t, q) - \frac{\partial}{\partial q_l} b_{orb,0,l}(t, q) b_{spin,+m,j}(t, q) \mathcal{P}_j[W_f](t, q) \\
& \quad - \frac{\partial}{\partial q_l} b_{orb,+l,i}(t, q) b_{spin,0,m}(t, q) \mathcal{P}_i[W_f](t, q) \\
& \quad - \frac{\partial}{\partial q_l} b_{orb,+l,i}(t, q) b_{spin,+m,j}(t, q) \int_{\mathbb{R}^3} s_i s_j f(t, q, \vec{s}) d^3 s ,
\end{aligned}$$

where  $m = 1, 2, 3$  and thus we got

$$\begin{aligned}
\frac{\partial \vec{\mathcal{P}}[W_f]}{\partial t}(t, q) = & -\frac{\partial}{\partial q_j} \left( d_{orb,0,j}(t, q) \vec{\mathcal{P}}[W_f](t, q) \right) \\
& -\frac{\partial}{\partial q_j} \left( d_{orb,+j,i}(t, q) \int_{\mathbb{R}^3} \vec{s} s_i f(t, q, \vec{s}) d^3 s \right) \\
& + \vec{d}_{spin,0}(t, q) \rho[W_f](t, q) + d_{spin,+}(t, q) \vec{\mathcal{P}}[W_f](t, q) \\
& + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,0,l}(t, q) b_{orb,0,k}(t, q) \vec{\mathcal{P}}[W_f](t, q) \right) \\
& + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,0,l}(t, q) b_{orb,+k,j}(t, q) \int_{\mathbb{R}^3} \vec{s} s_j f(t, q, \vec{s}) d^3 s \right) \\
& + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,+l,i}(t, q) b_{orb,0,k}(t, q) \int_{\mathbb{R}^3} \vec{s} s_i f(t, q, \vec{s}) d^3 s \right) \\
& + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,+l,i}(t, q) b_{orb,+k,j}(t, q) \int_{\mathbb{R}^3} \vec{s} s_i s_j f(t, q, \vec{s}) d^3 s \right) \\
& - \frac{\partial}{\partial q_l} \left( b_{orb,0,l}(t, q) \vec{b}_{spin,0}(t, q) \rho[W_f](t, q) \right) \\
& - \frac{\partial}{\partial q_l} \left( b_{orb,0,l}(t, q) b_{spin,+}(t, q) \vec{\mathcal{P}}[W_f](t, q) \right) \\
& - \frac{\partial}{\partial q_l} \left( b_{orb,+l,i}(t, q) \mathcal{P}_i[W_f](t, q) \vec{b}_{spin,0}(t, q) \right) \\
& - \frac{\partial}{\partial q_l} \left( b_{orb,+l,i}(t, q) b_{spin,+}(t, q) \int_{\mathbb{R}^3} s_i \vec{s} f(t, q, \vec{s}) d^3 s \right). \tag{117}
\end{aligned}$$

Note that the second, sixth, seventh, eighth and twelfth entries on the rhs of (117) are nonphysical in the sense that terms of the form:  $\int_{\mathbb{R}^3} \vec{s} s_n f(t, q, \vec{s}) d^3 s$  and of the form:  $\int_{\mathbb{R}^3} \vec{s} s_n s_p f(t, q, \vec{s}) d^3 s$  cannot be expressed via  $\rho[W_f]$ ,  $\vec{\mathcal{P}}[W_f]$ . We will see below that also the tenth entry on the rhs of (117) is nonphysical. All these nonphysical entries will be eliminated in Section 9.2 when we enforce the second part of the dynamical condition.

## 9.2 Enforcing the dynamical condition

With (116), (117) we have computed  $\frac{\partial \rho[W_f]}{\partial t}$  and  $\frac{\partial \vec{\mathcal{P}}[W_f]}{\partial t}$  where  $f$  is a solution of the formal Fokker-Planck equation, (107) (and thus is a solution of the formal Fokker-Planck equation, (113)). Therefore (116), (117) hold under the kinematical conditions for the case,  $K = 1$ ). Thus we are ready to enforce the dynamical condition, i.e., we will try (and in fact we will succeed) to choose the eight coefficient functions,  $d_{orb,0}, \dots, b_{spin,+}$ , in (113) such that the dynamical condition holds, i.e., such that (116), (117) are the same as (102), (103).

We begin with the first part of the dynamical condition whereby we have to ensure that (116) is the same as (102). Thus in particular we have to get rid of the nonphysical entries in (116) which were specified in the discussion after (116). This we accomplish by choosing

$d_{orb,+}, b_{spin,+}$  as zero matrix fields:

$$d_{orb,+} := 0_{6 \times 3}, \quad b_{orb,+} := 0_{6 \times 3}, \quad (118)$$

whereby (116) simplifies to

$$\frac{\partial \rho[W_f]}{\partial t} = -\frac{\partial}{\partial q_j} \left( d_{orb,0,j} \rho[W_f] \right) + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( b_{orb,0,l} b_{orb,0,k} \rho[W_f] \right). \quad (119)$$

Thus the task of enforcing the first part of the dynamical condition has been boiled down to the task of ensuring that (119) is the same as (102). This we accomplish by choosing:

$$d_{orb,0} := \mathcal{D}_{orb}, \quad b_{orb,0} := \mathcal{B}_{orb}. \quad (120)$$

With the choices, (118), (120), of  $d_{orb,0}, b_{orb,0}, d_{orb,+}, b_{orb,+}$ , we observe that (119) is the same as (102) and so the first part of the dynamical condition has been successfully enforced.

Thus we now enforce the second part of the dynamical condition. We will accomplish this by using (118), (120), and by using the freedom of properly choosing the still arbitrary coefficient functions,  $\vec{d}_{spin,0}, \vec{b}_{spin,0}, d_{spin,+}, b_{spin,+}$ . Enforcing the second part of the dynamical condition therefore means that we have to ensure that (117) is the same as (103). We recall from Section 4 that (49) is the same as (59) and thus (103) is the same as

$$\begin{aligned} \frac{\partial \vec{\mathcal{P}}[W_f]}{\partial t} &= l_{orb} \vec{\mathcal{P}}[W_f] + \mathcal{D}_{spin,+} \vec{\mathcal{P}}[W_f] + \vec{\mathcal{D}}_{spin,0,ST} \rho[W_f] \\ &\quad - \frac{\partial}{\partial q_k} (\mathcal{B}_{orb,k} \vec{\mathcal{B}}_{spin} \rho[W_f]), \end{aligned} \quad (121)$$

which will be used in this section and in Section 16.

As an aside we note, by (54), that (121) can be written as

$$\begin{aligned} \frac{\partial \vec{\mathcal{P}}[W_f]}{\partial t} &= l_{orb} \vec{\mathcal{P}}[W_f] + \mathcal{D}_{spin,+} \vec{\mathcal{P}}[W_f] + \mathcal{D}_{spin,+} \vec{\mathcal{P}}[W_f] \\ &\quad + \mathcal{D}_{spin,+} \vec{\mathcal{P}}[W_f] + \vec{\mathcal{D}}_{spin,0,ST} \rho[W_f] - \frac{\partial}{\partial q_k} (\mathcal{B}_{orb,k} \vec{\mathcal{B}}_{spin} \rho[W_f]), \end{aligned} \quad (122)$$

which will be used in Section 16.

Enforcing the second part of the dynamical condition now means that we have to ensure that (117) is the same as (121). In particular we have to get rid of the nonphysical entries in (117) which are specified in the discussion after (117). Thanks to our choice, (118), (120), of  $d_{orb,0}, b_{orb,0}, d_{orb,+}, b_{orb,+}$ , (117) simplifies to

$$\begin{aligned} \frac{\partial \vec{\mathcal{P}}[W_f]}{\partial t} &= -\frac{\partial}{\partial q_j} (\mathcal{D}_{orb,j} \vec{\mathcal{P}}[W_f]) + \vec{d}_{spin,0} \rho[W_f] + d_{spin,+} \vec{\mathcal{P}}[W_f] \\ &\quad + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} \left( \mathcal{B}_{orb,l} \mathcal{B}_{orb,k} \vec{\mathcal{P}}[W_f] \right) \\ &\quad - \frac{\partial}{\partial q_l} \left( \mathcal{B}_{orb,l} \vec{b}_{spin,0} \rho[W_f] \right) - \frac{\partial}{\partial q_l} \left( \mathcal{B}_{orb,l} b_{spin,+} \vec{\mathcal{P}}[W_f] \right), \end{aligned} \quad (123)$$

so that most of the nonphysical entries of (117) have disappeared without having yet enforced the second part of the dynamical condition. We next note, by (28), that (123) can be written as

$$\begin{aligned} \frac{\partial \vec{\mathcal{P}}[W_f]}{\partial t} &= l_{orb} \vec{\mathcal{P}}[W_f] + \vec{d}_{spin,0} \rho[W_f] + d_{spin,+} \vec{\mathcal{P}}[W_f] \\ &\quad - \frac{\partial}{\partial q_l} \left( \mathcal{B}_{orb,l} \vec{b}_{spin,0} \rho[W_f] \right) - \frac{\partial}{\partial q_l} \left( \mathcal{B}_{orb,l} b_{spin,+} \vec{\mathcal{P}}[W_f] \right). \end{aligned} \quad (124)$$

Thus the task of enforcing the second part of the dynamical condition has been boiled down to the task of ensuring that (124) is the same as (121). In other words we have to choose  $\vec{d}_{spin,0}, \vec{b}_{spin,0}, d_{spin,+}, b_{spin,+}$  such that (124) is the same as (121). To accomplish this we next note that phase space derivatives of  $\vec{\mathcal{P}}[W_f]$  occur in (121) only via the entry,  $l_{orb} \vec{\mathcal{P}}[W_f]$ , whereas phase space derivatives of  $\vec{\mathcal{P}}[W_f]$  occur in (124) not only via the entry,  $l_{orb} \vec{\mathcal{P}}[W_f]$ , but also via the fifth entry on the rhs of (124). Thus the fifth entry on the rhs of (124) is nonphysical (and therefore the tenth entry on the rhs of (117) is nonphysical!) and so we have to eliminate it. This we accomplish by choosing  $b_{spin,+}$  as the zero matrix field:

$$b_{spin,+} := 0_{3 \times 3}, \quad (125)$$

whereby (124) simplifies to

$$\frac{\partial \vec{\mathcal{P}}[W_f]}{\partial t} = l_{orb} \vec{\mathcal{P}}[W_f] + \vec{d}_{spin,0} \rho[W_f] + d_{spin,+} \vec{\mathcal{P}}[W_f] - \frac{\partial}{\partial q_l} \left( \mathcal{B}_{orb,l} \vec{b}_{spin,0} \rho[W_f] \right). \quad (126)$$

Thus the task of enforcing the second part of the dynamical condition has been simplified to the task of ensuring that (126) is the same as (121). This we accomplish by choosing:

$$\vec{d}_{spin,0} := \vec{\mathcal{D}}_{spin,0,ST}, \quad (127)$$

$$d_{spin,+} := \mathcal{D}_{spin,+}, \quad (128)$$

$$\vec{b}_{spin,0} := \vec{\mathcal{B}}_{spin}, \quad (129)$$

where  $\vec{\mathcal{B}}_{spin}, \vec{\mathcal{D}}_{spin,0,ST}, \mathcal{D}_{spin,+}$  are defined by (39), (50), (54).

With the choice, (127), (128), (129), we note that (126) is the same as (121) and so the second part of the dynamical condition has been successfully enforced as well. Thus both parts of the dynamical condition have been successfully enforced and so the aim of this section is fulfilled. In particular we see that the case,  $K = 1$ , is consistent with the kinematical and dynamical conditions.

## 10 The full spin-orbit Fokker-Planck equation and the full spin-orbit stochastic ODE system

With Section 9 at hand we can now write down the full spin-orbit Fokker-Planck equation and its associated Ito SDE system.

Choosing the coefficient functions,  $d_{orb,0}, \vec{d}_{spin,0}, b_{orb,0}, \vec{b}_{spin,0}$ , and  $d_{orb,+}, d_{spin,+}, b_{orb,+}, b_{spin,+}$  via (118), (120), (125), (127), (128), (129), the formal Fokker-Planck equation, (113), becomes

$$\begin{aligned} \frac{\partial f}{\partial t} = & -\frac{\partial}{\partial q_j}(\mathcal{D}_{orb,j}f) - \frac{\partial}{\partial s_j}\left((\mathcal{D}_{spin,0,ST,j} + \mathcal{D}_{spin,+,j,k}s_k)f\right) \\ & + \frac{1}{2}\frac{\partial}{\partial q_k}\frac{\partial}{\partial q_l}(\mathcal{B}_{orb,l}\mathcal{B}_{orb,k}f) + \frac{\partial}{\partial s_k}\frac{\partial}{\partial q_l}(\mathcal{B}_{orb,l}\mathcal{B}_{spin,k}f) \\ & + \frac{1}{2}\frac{\partial}{\partial s_k}\frac{\partial}{\partial s_l}(\mathcal{B}_{spin,l}\mathcal{B}_{spin,k}f). \end{aligned} \quad (130)$$

By its derivation in Section 9 it is clear that (130) satisfies the dynamical condition. Thus, by Section 7, (130) is a Fokker-Planck equation and it is called the full spin-orbit Fokker-Planck equation. Also, by Section 7, for every solution  $f$  of (130),  $W_f$  satisfies (101) (if  $W_f$  is sufficiently regular).

Using (55) we can rewrite (130) into the following simpler form

$$\begin{aligned} \frac{\partial f}{\partial t} = & -\frac{\partial}{\partial q_j}(\mathcal{D}_{orb,j}f) - \frac{\partial}{\partial s_j}(\mathcal{D}_{spin,j}f) + \frac{1}{2}\frac{\partial}{\partial q_k}\frac{\partial}{\partial q_l}(\mathcal{B}_{orb,l}\mathcal{B}_{orb,k}f) \\ & + \frac{\partial}{\partial s_k}\frac{\partial}{\partial q_l}(\mathcal{B}_{orb,l}\mathcal{B}_{spin,k}f) + \frac{1}{2}\frac{\partial}{\partial s_k}\frac{\partial}{\partial s_l}(\mathcal{B}_{spin,l}\mathcal{B}_{spin,k}f), \end{aligned} \quad (131)$$

where  $\mathcal{D}_{orb}, \mathcal{B}_{orb}, \vec{\mathcal{B}}_{spin}, \vec{\mathcal{D}}_{spin}$  are defined by (22), (23), (39), (55). By its derivation it is clear that (131) is the same as (130) and so it is the full spin-orbit Fokker-Planck equation.

For the casual reader, neither (130) nor (131) may look like a Fokker-Planck equation. In order to make this property clearly visible we now write (131) as

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial y_j}\left(\left(\begin{array}{c} \mathcal{D}_{orb} \\ \vec{\mathcal{D}}_{spin} \end{array}\right)_j f\right) + \frac{1}{2}\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_l}\left(\left(\begin{array}{c} \mathcal{B}_{orb} \\ \vec{\mathcal{B}}_{spin} \end{array}\right)_l \left(\begin{array}{c} \mathcal{B}_{orb} \\ \vec{\mathcal{B}}_{spin} \end{array}\right)_k f\right). \quad (132)$$

Note, by (95), that  $y = \begin{pmatrix} q \\ s \end{pmatrix}$ . Clearly (132) is the same as (131) (and so the same as (130)).

The key point about (132) is being an expression in terms of the so-called drift vector field and of the so-called diffusion matrix field. Thus (132) is that expression of the full spin-orbit Fokker-Planck equation by which it is immediately clear that it is a Fokker-Planck equation.

In fact  $\begin{pmatrix} \mathcal{D}_{orb} \\ \vec{\mathcal{D}}_{spin} \end{pmatrix}$  is the so-called drift vector field and  $\begin{pmatrix} \mathcal{B}_{orb} \\ \vec{\mathcal{B}}_{spin} \end{pmatrix} \begin{pmatrix} \mathcal{B}_{orb} \\ \vec{\mathcal{B}}_{spin} \end{pmatrix}^T$  is the so-called diffusion matrix field of (132) (for the physical interpretation of both fields, see Section 16 below). Since  $\begin{pmatrix} \mathcal{B}_{orb} \\ \vec{\mathcal{B}}_{spin} \end{pmatrix}$  is a vector field, which we call the noise vector field, every value of the diffusion matrix field is an outer product matrix. The noise vector field is not uniquely determined by (132) since each of its values can be multiplied by either 1 or  $-1$  without affecting the diffusion matrix field (our choice of the noise vector field is given by (23) and (39)). For the notions of Fokker-Planck equation, drift vector field and diffusion matrix field see, e.g., [Gar04].

Let us briefly continue the discussion of a fine point which began after (59) and which is based on the observation that  $\mathcal{B}_{orb}$  occurs twice on the rhs of (59). In fact it follows from (132) that  $\mathcal{B}_{orb}$  occurs twice on the rhs of (131) (and thus twice on the rhs of (130)). The point to be made here is the following: The fact that  $\mathcal{B}_{orb}$  occurs twice on the rhs of (130) we owe, by Section 9, to the fact that  $\mathcal{B}_{orb}$  occurs twice on the rhs of (121). Thus the fact that  $\mathcal{B}_{orb}$  occurs twice on the rhs of the full Bloch equation (121) is crucial for obtaining the extension of the kinetic approach. Note also that  $\mathcal{B}_{orb}$  occurs twice on the rhs of (121) because it occurs twice on the rhs (59). In summary, the extension of the kinetic approach is made possible partly because  $\mathcal{B}_{orb}$  occurs twice on the rhs of (59).

The Ito SDE system associated with the full spin-orbit Fokker-Planck equation, (132), reads as

$$\begin{aligned} Q' &= \mathcal{D}_{orb}(t, Q) + \mathcal{B}_{orb}(t, Q)\nu(t), \\ \vec{S}' &= \vec{\mathcal{D}}_{spin}(t, Q, \vec{S}) + \vec{\mathcal{B}}_{spin}(t, Q)\nu(t), \end{aligned} \quad (133)$$

where  $\nu$  is the one-dimensional white noise process. We call, (133), the full spin-orbit SDE system. Note that the first equality in (133) is the same as (30). For the notion of associated Ito SDE system we recommend again, [Gar04].

It is sometimes convenient to express the full spin-orbit Fokker-Planck equation in terms of  $l_{orb}$ . Thus using (28) we write for later reference the full spin-orbit Fokker-Planck equation, (131), as:

$$\frac{\partial f}{\partial t} = l_{orb}f - \frac{\partial}{\partial s_j}(\mathcal{D}_{spin,j}f) + \frac{\partial}{\partial s_k}\frac{\partial}{\partial q_l}(\mathcal{B}_{orb,l}\mathcal{B}_{spin,k}f) + \frac{1}{2}\frac{\partial}{\partial s_k}\frac{\partial}{\partial s_l}(\mathcal{B}_{spin,l}\mathcal{B}_{spin,k}f). \quad (134)$$

This expression of the full spin-orbit Fokker-Planck equation will be used in Section 11.

It is sometimes also convenient to express the full spin-orbit Fokker-Planck equation in a more explicit form than in the expressions above. Thus using (54), (55) we write the full spin-orbit Fokker-Planck equation, (131), as

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\frac{\partial}{\partial q_j}(\mathcal{D}_{orb,j}f) - \frac{\partial}{\partial s_j}\left(\left(\vec{\mathcal{D}}_{spin,0,ST}(t, q) + \mathcal{D}_{spin,+TBMT}(t, q)\vec{s}\right.\right. \\ &\quad \left.\left. + \mathcal{D}_{spin,+,ST}(t, q)\vec{s} + \mathcal{D}_{spin,+,BK}(t, q)\vec{s}\right)_j\right)f \\ &\quad + \frac{1}{2}\frac{\partial}{\partial q_k}\frac{\partial}{\partial q_l}(\mathcal{B}_{orb,l}\mathcal{B}_{orb,k}f) + \frac{\partial}{\partial s_k}\frac{\partial}{\partial q_l}(\mathcal{B}_{orb,l}(t, q)\mathcal{B}_{spin,k}(t, q)f) \\ &\quad + \frac{1}{2}\frac{\partial}{\partial s_k}\frac{\partial}{\partial s_l}(\mathcal{B}_{spin,l}(t, q)\mathcal{B}_{spin,k}(t, q)f). \end{aligned} \quad (135)$$

Note that (135) will be used in Section 16. By its derivation it is clear that (135) is the same as (131) and thus (135) is the same as (130), (132), (134) and so it is the full spin-orbit Fokker-Planck equation.

The Ito SDE system associated with the full spin-orbit Fokker-Planck equation, (135), is given by

$$\begin{aligned} Q' &= \mathcal{D}_{orb}(t, Q) + \mathcal{B}_{orb}(t, Q)\nu(t) , \\ \vec{S}' &= \vec{\mathcal{D}}_{spin,0,ST}(t, Q) + \mathcal{D}_{spin,+,TBMT}(t, Q)\vec{S} \\ &\quad + \mathcal{D}_{spin,+ ST}(t, Q)\vec{S} + \mathcal{D}_{spin,+ BK}(t, Q)\vec{S} + \vec{\mathcal{B}}_{spin}(t, Q)\nu(t) . \end{aligned} \tag{136}$$

Note that (136) is obtained by inserting (54), (55) into (133). Note also that (136) will be used in Section 16. Since (135) is equal to (132) both have the same associated Ito SDE system and so (136) is the same as (133). In particular (136) is the full spin-orbit SDE system. Most importantly, the full spin-orbit SDE system is identical to eq. 19 in [HABBE19]. Thus having derived the SDE system of [HABBE19] we have achieved an important aim of the present work (see Section 1 for the aims of the present work).

As an aside we use (20), (21), (22), (50), (51), (52), (53) to write (136) more explicitly as

$$\begin{aligned} Q' &= \left( \vec{\mathcal{F}}(t, Q) + \vec{\mathcal{C}}(t, Q) + \vec{\mathcal{Q}}(t, Q) \right) + \vec{\mathcal{B}}_{orb}(t, Q)\nu(t) , \\ \vec{S}' &= -\sqrt{\frac{55}{24\sqrt{3}}}\sqrt{\lambda(t, Q)}\vec{\mathcal{B}}_{spin}(t, Q) + \vec{\Omega}_{TBMT}(t, Q) \times \vec{S} \\ &\quad - \frac{5\sqrt{3}}{8}\lambda(t, Q)\vec{S} + \frac{5\sqrt{3}}{36m^2\gamma^2(Q)}\lambda(t, Q)\vec{\Pi}\vec{\Pi}^T\vec{S} + \vec{\mathcal{B}}_{spin}(t, Q)\nu(t) . \end{aligned} \tag{137}$$

By its derivation (137) is equal to (136) and thus equal to the full spin-orbit SDE system.

Let us now give some practical instructions on numerical solution methods of the full spin-orbit SDE system, e.g., standard SDE solvers or particle tracking codes (the latter perhaps obtained by extending an existing Monte-Carlo spin tracking code developed for computing the radiative depolarization time). As pointed out in Section 1 in order to perform such computations one transforms, from cartesian coordinates to machine coordinates, all equations involved, that is, the full spin-orbit SDE system, the definition, (104), of the polarization vector etc. To be specific let us sketch the typical computational procedure to be performed by a numerical solution method: Compute the initial polarization vector and the polarization vector at a later time (here ‘time’ in the sense of machine coordinates)! To fulfill this task one may pick  $N$  spin-orbit vectors in machine coordinates,  $(Q_{0,n}, \vec{S}_{0,n})$ , where  $n = 1, \dots, N$  and  $Q_{0,n} \in \mathbb{R}^6, \vec{S}_{0,n} \in \mathbb{R}^3$  with  $|\vec{S}_{0,n}| \leq 1$ . One now defines  $p_1, \dots, p_N$  as the values of the initial orbital density at the initial points  $(0, Q_{0,1}), \dots, (0, Q_{0,N})$ . The initial polarization vector is now defined as  $\frac{p_1\vec{S}_{0,1} + \dots + p_N\vec{S}_{0,N}}{p_1 + \dots + p_N}$ . Thus  $|\frac{p_1\vec{S}_{0,1} + \dots + p_N\vec{S}_{0,N}}{p_1 + \dots + p_N}| \leq 1$ . One then uses the numerical solution method to evolve the  $N$  spin-orbit vectors to a final ‘time’ resulting in  $(Q_n, \vec{S}_n)$  where  $n = 1, \dots, N$ . The final polarization vector is thus defined as  $\frac{p_1\vec{S}_1 + \dots + p_N\vec{S}_N}{p_1 + \dots + p_N}$ . Note that this way of computing the initial and final polarization vector is schematic and may even be modified but it reflects the main idea which is familiar from Monte-Carlo spin tracking codes developed for computing the radiative depolarization time.

As an aside we note that, in contrast to the reduced spin-orbit SDE system, in the full spin-orbit SDE system  $|\vec{S}(t)|$  is not conserved in time. Therefore we cannot interprete  $\vec{S}(t)$  in the full spin-orbit SDE system as the spin of a single particle. To find an interpretation we assume, for simplicity, in this paragraph, that (87) holds, i.e.,  $\rho[W_f] > 0$  so that we can apply Section 6 and in particular the remarks after (93). If (76) holds at  $(t, Q(t))$ , i.e.,  $|\vec{\mathcal{P}}[W_f](t, Q(t))| \leq \rho[W_f](t, Q(t))$ , then, by the remarks after (93), we can interprete  $\vec{S}(t)$  in the full spin-orbit SDE system as the spin polarization vector at  $(t, Q(t))$ . In the case where (76) does not hold at  $(t, Q(t))$  then, by the remarks after (93), we can interprete  $\vec{S}(t)$  in the full spin-orbit SDE system as the weighted spin polarization vector at  $(t, Q(t))$ . Most importantly the distinction between the two cases where (76) does or does not hold at a point does not have a negative impact on practical computations. In fact in the above sketch of computing the initial and final polarization vector the two cases are treated in the same way. This comes as a relief since determining, which case holds at a point, would be computationally expensive. Note also that from now on in this work the condition, (87), will not be used.

We finally make some remarks on the full spin-orbit SDE system and the full spin-orbit Fokker-Planck equation. We recall from Sections 3 and 4 that when neglecting all radiative effects, the functions  $\vec{\mathcal{C}}, \vec{\mathcal{Q}}, \lambda, \mathcal{B}_{orb}, \vec{\mathcal{D}}_{spin,0,ST}, \mathcal{D}_{spin,+ST}, \mathcal{D}_{spin,+BK}, \vec{\mathcal{B}}_{spin}$  vanish and  $\mathcal{D}_{orb}$  becomes  $\mathcal{D}_{orb,nrad}$ . Thus we note, for later reference, that when neglecting all radiative effects in (137) we get

$$\vec{R}' = \vec{v}(Q) , \quad \vec{\Pi}' = \vec{\mathcal{F}}(t, Q) , \quad \vec{S}' = \vec{\Omega}_{TBMT}(t, Q) \times \vec{S} , \quad (138)$$

and, when neglecting all radiative effects in (135), we get

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial r_j}(v_j f) - \frac{\partial}{\partial \pi_j}(\mathcal{F}_j f) - \frac{\partial}{\partial s_j}\left(\left(\vec{\Omega}_{TBMT} \times \vec{s}\right)_j f\right) , \quad (139)$$

where we also used (20), (51). Note that the first two equations in (138) are the same as (33). Note also that the solutions  $\begin{pmatrix} \vec{R} \\ \vec{\Pi} \\ \vec{S} \end{pmatrix}$  of (138) are characteristics of (139) in the sense

that they satisfy:

$$\frac{d}{dt}\left(f\left(t, \vec{R}(t), \vec{\Pi}(t), \vec{S}(t)\right)\right) = 0 , \quad (140)$$

where  $f$  is any solution of (139).

## 11 The reduced spin-orbit Fokker-Planck equation and the reduced spin-orbit stochastic ODE system

In this section we revisit Section 10 by focusing on the reduced setup.

In the reduced setup, defined by (60) resp. (62), (63), the full spin-orbit Fokker-Planck equation, (134), simplifies to the so-called reduced spin-orbit Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = l_{orb}f - \frac{\partial}{\partial s_j} \left( \left( \vec{\Omega}_{TBMT} \times \vec{s} \right)_j f \right). \quad (141)$$

Note that (141) is obtained by inserting (62), (63) into (134). By (28) we can write (141) also in the form:

$$\begin{aligned} \frac{\partial f}{\partial t} = & -\frac{\partial}{\partial q_j} (\mathcal{D}_{orb,j} f) + \frac{1}{2} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_l} (\mathcal{B}_{orb,l} \mathcal{B}_{orb,k} f) \\ & - \frac{\partial}{\partial s_j} \left( \left( \vec{\Omega}_{TBMT} \times \vec{s} \right)_j f \right). \end{aligned} \quad (142)$$

The Ito SDE system associated with the reduced spin-orbit Fokker-Planck equation, (142), reads as

$$\begin{aligned} Q' &= \mathcal{D}_{orb}(t, Q) + \mathcal{B}_{orb}(t, Q)\nu(t), \\ \vec{S}' &= \vec{\Omega}_{TBMT}(t, Q) \times \vec{S}, \end{aligned} \quad (143)$$

where  $\nu$  is the one-dimensional white noise process. We call, (143), the reduced spin-orbit SDE system. With (20), (21), (22) we can write (143) more explicitly as

$$\begin{aligned} Q' &= \left( \begin{array}{c} \vec{v}(Q) \\ \vec{\mathcal{F}}(t, Q) + \vec{\mathcal{C}}(t, Q) + \vec{\mathcal{Q}}(t, Q) \end{array} \right) + \vec{\mathcal{B}}_{spin}(t, Q)\nu(t), \\ \vec{S}' &= \vec{\Omega}_{TBMT}(t, Q) \times \vec{S}. \end{aligned} \quad (144)$$

Clearly (143) is the same as (144) and so (144) is the reduced spin-orbit SDE system. If one neglects  $\vec{\mathcal{Q}}$  then (144) becomes the SDE system which was already derived in [BH01] and (141) becomes the Fokker-Planck equation which was already derived in [BH01] (recall from Section 3 that  $\vec{\mathcal{Q}}$  is negligible).

Note that if  $f$  satisfies the reduced spin-orbit Fokker-Planck equation, (141), then (if  $W_f$  is sufficiently regular), by the dynamical condition stated in Section 7,  $W_f$  satisfies (86), i.e.,

$$\frac{\partial W_f}{\partial t} = l_{orb}W_f + \frac{1}{2}\sigma_i \left( \vec{\Omega} \times \left( \begin{array}{c} Tr_{2 \times 2}[\sigma_1 W_f] \\ Tr_{2 \times 2}[\sigma_2 W_f] \\ Tr_{2 \times 2}[\sigma_3 W_f] \end{array} \right) \right)_i. \quad (145)$$

By our remarks made after (86) we also note that (145) contains all the information to study the radiative depolarization effect (same for the reduced spin-orbit Fokker-Planck equation and for the reduced spin-orbit SDE system). Thus by the dynamical condition the reduced spin-orbit Fokker-Planck equation (as well as the reduced spin-orbit SDE system) contains all the information to study the radiative depolarization effect.

## 12 Spin-orbit densities: The physically meaningful solutions of the full and reduced spin-orbit Fokker-Planck equations

In this section we define and discuss the physically meaningful solutions of the spin-orbit Fokker-Planck equations and we will call them spin-orbit densities (recall from Section 7 the convention that an equation is called spin-orbit Fokker-Planck equation iff it is either the full or the reduced spin-orbit Fokker-Planck equation).

Viewing every spin-1/2 Wigner function of a bunch as physically meaningful we thus require for every spin-orbit density,  $f$ , that  $W_f$  is the spin-1/2 Wigner function of a bunch. In other words we want  $W_f$  to satisfy the statistical conditions, (69), (70), (71), that is, for  $t \in \mathbb{R}, q \in \mathbb{R}^6$ ,

$$\left( W_f(t, q) \right)^\dagger = W_f(t, q) , \quad (146)$$

$$\int_{\mathbb{R}^6} \text{Tr}_{2 \times 2} \left[ W_f(t, \tilde{q}) \right] d^6 \tilde{q} = 1 , \quad (147)$$

$$\left| \int_{\mathbb{R}^6} \text{Tr}_{2 \times 2} \left[ \vec{\sigma} W_f(t, \tilde{q}) \right] d^6 \tilde{q} \right| \leq 1 , \quad (148)$$

and we want  $W_f$  to satisfy the evolution equation (which in the full setup is (101) and which in the reduced setup is (145)).

We know from Section 5 that the statistical conditions, (69), (70), (71), are equivalent to the conditions, (72), (73), (75), on  $\rho[W_f], \vec{\mathcal{P}}[W_f]$  which read for  $W = W_f$  as

$$\rho[W_f](t, q) \in \mathbb{R} , \quad \vec{\mathcal{P}}[W_f](t, q) \in \mathbb{R}^3 , \quad (149)$$

$$\int_{\mathbb{R}^6} \rho[W_f](t, \tilde{q}) d^6 \tilde{q} = 1 , \quad (150)$$

$$\left| \int_{\mathbb{R}^6} \vec{\mathcal{P}}[W_f](t, \tilde{q}) d^6 \tilde{q} \right| \leq 1 , \quad (151)$$

where  $t \in \mathbb{R}, q \in \mathbb{R}^6$ .

Motivated by (149), (150), (151) and the dynamical condition stated in Section 7 we call  $f$  a spin-orbit density iff it is a solution of the spin-orbit Fokker-Planck equation and satisfies, for  $t \in \mathbb{R}, q \in \mathbb{R}^6, \vec{s} \in \mathbb{R}^3$ ,

$$f(t, q, \vec{s}) \in \mathbb{R} , \quad (152)$$

$$\int_{\mathbb{R}^9} f(t, \tilde{q}, \vec{\tau}) d^3 \tau d^6 \tilde{q} = 1 , \quad (153)$$

$$\left| \int_{\mathbb{R}^9} \vec{\tau} f(t, \tilde{q}, \vec{\tau}) d^3 \tau d^6 \tilde{q} \right| \leq 1 . \quad (154)$$

Note, by (98), that (149), (150) are equivalent to (152), (153) except that the condition, (152), is, for the sake of convenience, stronger than (149). Note also that (154) is equivalent to (151) (because of (99)).

Thus if  $f$  is a spin-orbit density then, by the discussion before (149),  $W_f$  satisfies the statistical conditions, (146), (147), (148), and, by the dynamical condition,  $W_f$  satisfies the evolution equation (101) (in the full setup) and (145) (in the reduced setup). Thus, by Section 5, if  $f$  is a spin-orbit density then  $W_f$  is the spin-1/2 Wigner function of a bunch.

We expect, in analogy to the discussions after (85), that the three conditions, (152), (153), (154), on the spin-orbit density,  $f$ , fit to the dynamics of  $f$ . In fact, since the spin-orbit Fokker-Planck equation is a Fokker-Planck equation, (152), (153) hold for all  $t$  if they hold for  $t = 0$  (and if the coefficient functions of the spin-orbit Fokker-Planck equation are sufficiently regular). In analogy to the discussion after (85) we leave the following question unanswered: Does (154) hold for all  $t$  if it holds for  $t = 0$ ? Nevertheless since (154) is equivalent to:

$$\left| \int_{\mathbb{R}^6} \begin{pmatrix} Tr_{2 \times 2}[\sigma_1 W_f(t, q)] \\ Tr_{2 \times 2}[\sigma_2 W_f(t, q)] \\ Tr_{2 \times 2}[\sigma_3 W_f(t, q)] \end{pmatrix} d^6 q \right| \leq 1 , \quad (155)$$

and by the discussion after (85) we believe that in general the answer to this question is: yes (if the coefficient functions of the spin-orbit Fokker-Planck equation are sufficiently regular). In summary, we believe that if  $f$  satisfies the spin-orbit Fokker-Planck equation and if, for  $q \in \mathbb{R}^6, \vec{s} \in \mathbb{R}^3$ ,

$$f(0, q, \vec{s}) \in \mathbb{R} , \quad (156)$$

$$\int_{\mathbb{R}^9} f(0, \tilde{q}, \vec{\tau}) d^3 \tau d^6 \tilde{q} = 1 , \quad (157)$$

$$\left| \int_{\mathbb{R}^9} \vec{\tau} f(0, \tilde{q}, \vec{\tau}) d^3 \tau d^6 \tilde{q} \right| \leq 1 , \quad (158)$$

then  $f$  is a spin-orbit density (and thus  $W_f$  is the spin-1/2 Wigner function of a bunch and  $\rho[W_f]$  is the orbital density and  $\vec{\mathcal{P}}[W_f]$  is the polarization density of that bunch). Nevertheless the present work does not rely on this belief, except when explicitly mentioned (e.g., in Section 15).

For later reference we now discuss expectation values w.r.t.  $W_f$ . We first note by the above that we do not require that a spin-orbit density,  $f$ , is a probability density, i.e., negative values of  $f$  are not problematic (for explicit examples with negative values, see Section 14). To consider these expectation values let  $f$  be a spin-orbit density (either of the full or reduced spin-orbit Fokker-Planck equation). Thus  $W_f$  is the spin-1/2 Wigner function of a bunch and so the expectation values  $\langle A \rangle_{W_f}$  are well defined and can be directly expressed in terms of  $f$ . In fact, by (80), (98), (99), we have

$$\begin{aligned} \langle A \rangle_{W_f}(t) &= \int_{\mathbb{R}^6} \left( A_{orb}(t, q) \rho[W_f](t, q) + A_{spin,j}(t, q) \mathcal{P}_j[W_f](t, q) \right) d^6 q \\ &= \int_{\mathbb{R}^6} \left( A_{orb}(t, q) \rho[W_f](t, q) + A_{spin,j}(t, q) \mathcal{P}_j[W_f](t, q) \right) d^6 q \\ &= \int_{\mathbb{R}^9} \left( A_{orb}(t, q) + s_j A_{spin,j}(t, q) \right) f(t, q, \vec{s}) d^3 s d^6 q , \end{aligned}$$

in short

$$\langle A \rangle_{W_f}(t) = \int_{\mathbb{R}^9} \left( A_{orb}(t, q) + s_j A_{spin,j}(t, q) \right) f(t, q, \vec{s}) d^3 s d^6 q . \quad (159)$$

In the special case,  $\vec{A}_{spin} = \vec{0}$ , (159) results in

$$\begin{aligned} \langle I_{2 \times 2} A_{orb} \rangle_{W_f}(t) &= \int_{\mathbb{R}^6} A_{orb}(t, q) \rho[W_f](t, q) d^6 q \\ &= \int_{\mathbb{R}^9} A_{orb}(t, q) f(t, q, \vec{s}) d^3 s d^6 q . \end{aligned} \quad (160)$$

In the special case,  $A_{orb} = 0$ , (159) results in

$$\begin{aligned} \langle \sigma_j A_{spin,j} \rangle_{W_f}(t) &= \int_{\mathbb{R}^6} A_{spin,j}(t, q) \mathcal{P}_j[W_f](t, q) d^6 q \\ &= \int_{\mathbb{R}^9} s_j A_{spin,j}(t, q) f(t, q, \vec{s}) d^3 s d^6 q , \end{aligned} \quad (161)$$

in particular we get for the polarization vector

$$\langle \vec{\sigma} \rangle_{W_f}(t) = \int_{\mathbb{R}^9} \vec{s} f(t, q, \vec{s}) d^3 s d^6 q = \vec{P}[W_f](t) , \quad (162)$$

where in the second equation of (162) we used (104).

Note finally that in (159)-(162)  $f$  may have negative values.

### 13 Some properties of the full and the reduced spin-orbit Fokker-Planck equations

We here discuss some features of the spin-orbit densities,  $f$ , of the spin-orbit Fokker-Planck equation (full or reduced). These features are related to (163).

A simple observation is the following: If  $f = f(t, q, \vec{s})$  then we can write:

$$f(t, q, \vec{s}) = f_{even}(t, q, \vec{s}) + f_{odd}(t, q, \vec{s}) , \quad (163)$$

where

$$f_{even}(t, q, \vec{s}) := \frac{f(t, q, \vec{s}) + f(t, q, -\vec{s})}{2} , \quad (164)$$

$$f_{odd}(t, q, \vec{s}) := \frac{f(t, q, \vec{s}) - f(t, q, -\vec{s})}{2} . \quad (165)$$

Note, by (164), (165), that  $f_{even}$  is even in  $\vec{s}$  and  $f_{odd}$  is odd in  $\vec{s}$ .

If  $f$  is a spin-orbit density of the Fokker-Planck equation (either full or reduced) we observe, by (98), (99), (164), (165), that  $\rho[W_{f_{even}}]$ ,  $\rho[W_{f_{odd}}]$  and  $\vec{\mathcal{P}}[W_{f_{even}}]$ ,  $\vec{\mathcal{P}}[W_{f_{odd}}]$  exist and satisfy

$$\rho[W_{f_{even}}] = \rho[W_f] , \rho[W_{f_{odd}}] = 0 , \vec{\mathcal{P}}[W_{f_{even}}] = \vec{0} , \vec{\mathcal{P}}[W_{f_{odd}}] = \vec{\mathcal{P}}[W_f] . \quad (166)$$

With (166) we arrive at the observation that the information about  $\rho[W_f]$  resides entirely in  $f_{even}$  and that the information about  $\vec{\mathcal{P}}[W_f]$  resides entirely in  $f_{odd}$ . Note also by the second equality of (166) that  $f_{odd}$  is never a spin-orbit density.

Because of (163), (164), (165), the spin-orbit Fokker-Planck equation (full or reduced) is equivalent to a system of two linear PDEs for  $f_{even}, f_{odd}$  (to be presented elsewhere). In case of the full spin-orbit Fokker-Planck equation these two equations couple  $f_{even}$  and  $f_{odd}$  because of the following two terms of (135):

$$-\frac{\partial}{\partial s_j}(\mathcal{D}_{spin,0,ST,j}f) , \quad (167)$$

$$\frac{\partial}{\partial s_k}\frac{\partial}{\partial q_l}(\mathcal{B}_{orb,l}\mathcal{B}_{spin,k}f) . \quad (168)$$

In Section 16 we will see that (167) is a Sokolov-Ternov term of (135) and that (168) carries the kinetic polarization effect in (135). Both (167) and (168) are of order  $\hbar$  and so the combination of (163) and the two PDEs for  $f_{even}, f_{odd}$  opens up an avenue of studying  $f$  (which should be pursued in future work), e.g., via a perturbative treatment of  $f_{even}, f_{odd}$  w.r.t. the perturbation parameter,  $\hbar$ .

If  $f$  satisfies the full spin-orbit Fokker-Planck equation then in general neither  $f_{even}$  nor  $f_{odd}$  do. In contrast, if  $f$  satisfies the reduced spin-orbit Fokker-Planck equation, then both terms in (167) and (168) vanish and  $f_{even}, f_{odd}$  satisfy the reduced spin-orbit Fokker-Planck equation (this case will be discussed in Section 14). Moreover if  $f$  is a spin-orbit density of the reduced spin-orbit Fokker-Planck equation then  $f_{even}$  is, too (while, by the above,  $f_{odd}$  is not a spin-orbit density).

## 14 Further properties of the reduced spin-orbit Fokker-Planck equation

The properties discussed Section 13 hold in particular in the reduced setup. In this section we focus entirely on the reduced setup by studying further properties valid for this setup (the extension of this study to the full setup is left to future work). Thus the spin-orbit Fokker-Planck equation underlying this section is the reduced spin-orbit Fokker-Planck equation.

We know from Section 12 that if  $f$  is a spin-orbit density then  $W_f$  is the spin-1/2 Wigner function of a bunch so one may wonder if the converse is also true. In other words, the following question, which is the key question underlying this section, will now be addressed: If  $W$  is the spin-1/2 Wigner function of a bunch, does a spin-orbit density,  $f$ , exist such that:

$$W = W_f . \quad (169)$$

Perhaps surprisingly, the answer is: yes! In fact, motivated by (163), we define the function,  $f$ , by

$$f(t, q, \vec{s}) := (2\pi)^{-3/2} \exp\left(-\frac{1}{2}s_i s_i\right) \left( \rho[W](t, q) + s_j \mathcal{P}_j[W](t, q) \right) , \quad (170)$$

and observe that if  $W$  is the spin-1/2 Wigner function of a bunch then  $f$  in (170) satisfies (169) and is a spin-orbit density (in particular  $f$  satisfies the reduced spin-orbit Fokker-Planck equation, (142)). To prove that  $f$  in (170) is a spin-orbit density one may use, from Section 5, the statistical conditions for  $W$  and the fact that  $\rho[W]$  satisfies the orbital Fokker-Planck equation and that  $\vec{\mathcal{P}}[W]$  satisfies the reduced Bloch equation. To prove that  $f$  in (170) satisfies (169) one may insert (170) into (97) and then show, by (1), that  $W = W_f$ . The details we leave to the reader.

Having settled with (170) the existence problem of  $f$  we now discuss the nonuniqueness of  $f$  which was already alleged in Section 7. In other words the following question will now be addressed: If  $W$  is the spin-1/2 Wigner function of a bunch, does more than one spin-orbit density,  $f$ , exists which satisfies (169)? Unsurprisingly, the answer is: yes! In fact, we define for an arbitrary positive number,  $\eta$ , the function,  $f$ , by

$$f(t, q, \vec{s}) := \frac{1}{\eta^5} (2\pi)^{-3/2} \exp\left(-\frac{1}{2\eta^2} s_i s_i\right) \left( \eta^2 \rho[W](t, q) + s_j \mathcal{P}_j[W](t, q) \right), \quad (171)$$

and observe that if  $W$  is the spin-1/2 Wigner function of a bunch then each  $f$  in (171) satisfies, (169), and is a spin-orbit density. We leave the proof, which is almost the same as for (170), to the reader. Of course, the functions in (171) are infinitely many and pairwise distinct and so the nonuniqueness problem has been solved.

We now make some remarks on (169), (170), (171). First of all,  $f$  in (171) is equal to  $f$  in (170) iff  $\eta = 1$ . Secondly, if  $W$  is the spin-1/2 Wigner function of a bunch then, since every spin-orbit density  $f$  in (171) satisfies (169), we can write every such function as

$$f(t, q, \vec{s}) = \frac{1}{\eta^5} (2\pi)^{-3/2} \exp\left(-\frac{1}{2\eta^2} s_i s_i\right) \left( \eta^2 \rho[W_f](t, q) + s_j \mathcal{P}_j[W_f](t, q) \right), \quad (172)$$

where, as in (171),  $\eta$  is an arbitrary positive number. Thirdly, by (163), (164), (165), (169) and for  $f$  in (171),

$$\begin{aligned} f_{even}(t, q, \vec{s}) &= \frac{1}{\eta^3} (2\pi)^{-3/2} \exp\left(-\frac{1}{2\eta^2} s_i s_i\right) \rho[W](t, q) \\ &= \frac{1}{\eta^3} (2\pi)^{-3/2} \exp\left(-\frac{1}{2\eta^2} s_i s_i\right) \rho[W_f](t, q), \end{aligned} \quad (173)$$

$$\begin{aligned} f_{odd}(t, q, \vec{s}) &= \frac{1}{\eta^5} (2\pi)^{-3/2} \exp\left(-\frac{1}{2\eta^2} s_i s_i\right) s_j \mathcal{P}_j[W](t, q) \\ &= (2\pi)^{-3/2} \exp\left(-\frac{1}{2} s_i s_i\right) s_j \mathcal{P}_j[W_f](t, q). \end{aligned} \quad (174)$$

Fourthly, it follows from the discussions after (168) that  $f_{odd}, f_{odd}$  in (173), (174) satisfy the reduced spin-orbit Fokker-Planck equation, (142), and that  $f_{even}$  is a spin-orbit density while  $f_{odd}$  is not. In fact all this can be checked directly by using (173), (174) (the details we leave to the reader). Fifthly, let  $W$  be the spin-1/2 Wigner function of a bunch. We here note that  $f$  in (171) has some negative values whenever  $\vec{\mathcal{P}}[W]$  has some nonzero values. In other words, all values of  $f$  in (171) are nonnegative only in the trivial case where  $\vec{\mathcal{P}}[W] = \vec{0}$ . Sixthly, we point out that the spin-orbit densities of the form (171) are not the only spin-orbit densities

which satisfy (169). In fact, we define for an arbitrary positive number,  $\eta$ , the function,  $f$ , by

$$f(t, q, \vec{s}) := \frac{1}{4\pi\eta^4} \delta(|\vec{s}| - \eta) \left( \eta^2 \rho[W](t, q) + 3s_j \mathcal{P}_j[W](t, q) \right), \quad (175)$$

where  $\delta$  is Dirac's delta function and we observe that if  $W$  is the spin-1/2 Wigner function of a bunch then each  $f$  in (175) satisfies (169) and is a spin-orbit density. We leave the proof, which is almost the same as for (170) to the reader. Note that the factor  $\delta(|\vec{s}| - \eta)$  in (175) reflects the fact that, in the reduced setup,  $|\vec{S}(t)|$  is conserved in time (this follows from (144)). Seventhly, we point out, without proof, that even the spin-orbit densities of the forms, (171), and, (175), are not the only spin-orbit densities which satisfy (169).

## 15 Deriving the Baier-Katkov-Strakhovenko equation from the full spin-orbit Fokker-Planck equation

In this section we derive from the full spin-orbit Fokker-Planck equation the BKS equation (BKS=Baier-Katkov-Strakhovenko), the latter having been introduced in [Bai69] and [BKS70].

In fact we will accomplish even a bit more, see the discussion around (201). Deriving the BKS equation will help us, in Section 16, to identify the physical meaning of the terms of the full spin-orbit Fokker-Planck equation, of the full spin-orbit SDE system and of the full Bloch equation.

To derive the BKS equation from the full spin-orbit Fokker-Planck equation we first have to simplify, motivated by [Bai69] and [BKS70], the full spin-orbit Fokker-Planck equation to (178) since, as we will see, the BKS equation does not hold in the full setup. Thus in this section we simplify the full setup to what we call the BKS setup which is characterized by neglecting all orbital radiative terms, i.e., the BKS setup is defined by

$$\mathcal{D}_{orb} = \mathcal{D}_{orb,nrad}, \quad \mathcal{B}_{orb} = 0, \quad (176)$$

where  $\mathcal{D}_{orb,nrad}$  is defined by (20). We can now phrase the task of this section more precisely: Derive the BKS equation from the full spin-orbit Fokker-Planck equation in the BKS setup!

Before we begin with the derivation of the BKS equation we make some remarks on the BKS setup. First of all, in the BKS setup the orbital Fokker-Planck equation simplifies to (19) in particular we have  $\vec{\mathcal{C}} = \vec{\mathcal{Q}} = \vec{0}$ . Secondly, in the BKS setup, the full spin-orbit SDE system, (133), simplifies to:

$$\begin{pmatrix} Q \\ \vec{S} \end{pmatrix}' = \begin{pmatrix} \mathcal{D}_{orb,nrad}(t, Q) \\ \vec{\mathcal{D}}_{spin}(t, Q, \vec{S}) \end{pmatrix} + \begin{pmatrix} 0 \\ \vec{\mathcal{B}}_{spin}(t, Q) \end{pmatrix} \nu(t). \quad (177)$$

Thirdly, in the BKS setup the radiative spin terms, unlike the radiative orbital terms, are not neglected. In fact, by Section 4,  $\vec{\mathcal{B}}_{spin}$  and  $\vec{\mathcal{D}}_{spin}$ , which occur in (177), both carry radiative terms. Fourthly, in the BKS setup the full spin-orbit Fokker-Planck equation,

(130) simplifies, by (176), to

$$\begin{aligned}\frac{\partial f}{\partial t} = & -\frac{\partial}{\partial q_j}(\mathcal{D}_{orb,nonrad,j}f) - \frac{\partial}{\partial s_j}\left((\mathcal{D}_{spin,0,ST,j} + \mathcal{D}_{spin,+j,k}s_k)f\right) \\ & + \frac{1}{2}\frac{\partial}{\partial s_k}\frac{\partial}{\partial s_l}(\mathcal{B}_{spin,l}\mathcal{B}_{spin,k}f).\end{aligned}\quad (178)$$

Unsurprisingly, (177) is the Ito SDE system associated with (178). Fifthly, no additional approximation beyond the BKS setup is required in this section, i.e., we here deal with exact solutions of (178). We call  $f$  a spin-orbit density of the BKS setup iff it satisfies, (178), and the statistical conditions, (152), (153), (154). Sixthly, to keep things simple we stick in this section to the belief, stated after (158), that the three statistical conditions on  $f$  fit to the dynamics. In the BKS setup this belief means the following: If  $f$  satisfies (156), (157), (158), (178) then  $f$  is a spin-orbit density of the BKS setup. Seventhly, the orbital Fokker-Planck equation, (102), and the full Bloch equation, (121), read, by (22), (28), (176), in the BKS setup as

$$\frac{\partial \rho[W_f]}{\partial t} = -\frac{\partial}{\partial q_j}(\mathcal{D}_{orb,nrad,j}\rho[W_f]), \quad (179)$$

$$\frac{\partial \vec{\mathcal{P}}[W_f]}{\partial t} = -\frac{\partial}{\partial q_j}(\mathcal{D}_{orb,nrad,j}\vec{\mathcal{P}}[W_f]) + \mathcal{D}_{spin,+}\vec{\mathcal{P}}[W_f] + \vec{\mathcal{D}}_{spin,0,ST}\rho[W_f]. \quad (180)$$

After these remarks on the BKS setup we now start the derivation of the BKS equation. The key point of this section is to show that the BKS equation, (200), is an ODE for the polarization vector,  $\vec{P}[W_f]$ , if one properly restricts the generality of  $f$  (in fact,  $f$  will eventually be restricted via (192)). To come to this we first compute, by (104), (180) and without restriction on  $f$ , i.e., for an arbitrary spin-orbit density  $f$  of the BKS setup,

$$\begin{aligned}\frac{d}{dt}\vec{P}[W_f](t) &= \frac{d}{dt}\int_{\mathbb{R}^6}\vec{\mathcal{P}}[W_f](t,q)d^6q = \int_{\mathbb{R}^6}\frac{\partial \vec{\mathcal{P}}[W_f]}{\partial t}(t,q)d^6q \\ &= \int_{\mathbb{R}^6}\left(-\frac{\partial}{\partial q_j}\left(\mathcal{D}_{orb,nrad,j}(t,q)\vec{\mathcal{P}}[W_f]\right)\right. \\ &\quad \left.+\mathcal{D}_{spin,+}(t,q)\vec{\mathcal{P}}[W_f](t,q)+\vec{\mathcal{D}}_{spin,0,ST}(t,q)\rho[W_f](t,q)\right)d^6q \\ &= \int_{\mathbb{R}^6}\left(\mathcal{D}_{spin,+}(t,q)\vec{\mathcal{P}}[W_f](t,q)+\vec{\mathcal{D}}_{spin,0,ST}(t,q)\rho[W_f](t,q)\right)d^6q,\end{aligned}$$

in short,

$$\frac{d}{dt}\vec{P}[W_f](t) = \int_{\mathbb{R}^6}\left(\mathcal{D}_{spin,+}(t,q)\vec{\mathcal{P}}[W_f](t,q)+\vec{\mathcal{D}}_{spin,0,ST}(t,q)\rho[W_f](t,q)\right)d^6q. \quad (181)$$

Although  $f$  in (181) is a spin-orbit density of the BKS setup (i.e.,  $f$  carries the ‘right’ dynamics and the ‘right’ statistics) it is too general. In fact for such a general  $f$  the evolution equation (181) is not an ODE for  $\vec{P}[W_f]$ , let alone the BKS equation for  $\vec{P}[W_f]$ .

To properly restrict the generality of  $f$  in (181) we have to recall from [Bai69] or [BKS70] that the BKS equation is defined w.r.t. (arbitrary) deterministic orbital motion. Thus in order for (181) to become an ODE for  $\vec{P}[W_f]$ ,  $f$  not only has to be a spin-orbit density of the BKS setup but also has to render the orbital motion to be deterministic. To get insight into this matter we first note that the orbital part,

$$Q' = \mathcal{D}_{orb,nrad}(t, Q) , \quad (182)$$

of the Ito SDE system, (177), is an ODE system and so an arbitrary solution,  $Q_0 \equiv \begin{pmatrix} \vec{R}_0 \\ \vec{\Pi}_0 \end{pmatrix}$ , of (177) satisfies

$$Q'_0 = \mathcal{D}_{orb,nrad}(t, Q_0) , \quad (183)$$

$$Q_0(t) = \varphi(t, Q_0(0)) , \quad (184)$$

with  $Q_0(0) \in \mathbb{R}^6$  being arbitrary and  $\varphi$  being the solution function of the ODE system, (182).

We can now state what it means for a spin-orbit density,  $f$ , of the BKS setup to render the orbital motion to be given by  $Q_0$ , i.e., to render the orbital motion to be deterministic. Roughly speaking it means that  $\rho[W_f](t, q)$  is nonzero only if  $q = Q_0(t)$ . Mathematically speaking it means that

$$\rho[W_f](t, q) = \delta(q - Q_0(t)) , \quad (185)$$

where we also used (150). Note, by (185), that

$$\rho[W_f](0, q) = \delta(q - Q_0(0)) . \quad (186)$$

For later reference we note that (160) simplifies by (185), to

$$\langle I_{2 \times 2} A_{orb} \rangle_{W_f}(t) = \int_{\mathbb{R}^9} A_{orb}(t, q) \rho[W_f](t, q) d^6 q = A_{orb}(t, Q_0(t)) , \quad (187)$$

where the orbital observable,  $I_{2 \times 2} A_{orb}$ , is arbitrary. Note that  $\rho[W_f]$  in (185), solves the orbital Fokker-Planck equation, (179). This can be shown either directly or by using (186) in combination with the method of characteristics (note that (179) is a first-order PDE!). Thus an orbital density,  $f$ , of the BKS setup renders the orbital motion to be deterministic iff (186) holds (for arbitrary  $Q_0(0) \in \mathbb{R}^6$ ).

Therefore our task of deriving the BKS equation narrows down to finding spin-orbit densities,  $f$ , of the BKS setup which satisfy (186) and to showing that (181) has become, for such  $f$ , the BKS equation. Recalling the discussion before (179) we assume that  $f$  satisfies (156), (157), (158),(178), (186). Note that (186) implies (157) and so the task reduces to showing that (181) is the BKS equation if one imposes (156), (158), (178),(186) on  $f$ . In order to construct  $f$  we need  $f(0, \cdot, \cdot)$  to satisfy (156), (158), (186). In fact (98), (186) imply

$$f(0, q, \vec{s}) = \delta(q - Q_0(0)) h_0(\vec{s}) , \quad (188)$$

where, in order to satisfy, (156),  $h_0$  is  $\mathbb{R}$ -valued. Note, by (98), (188), that

$$\rho[W_f](0, q) = \delta(q - Q_0(0)) \int_{\mathbb{R}^3} h_0(\vec{s}) d^3 s , \quad (189)$$

so that, in order to obey, (186), we require

$$\int_{\mathbb{R}^3} h_0(\vec{s}) d^3 s = 1 . \quad (190)$$

By (188) and in order to obey, (158), we demand

$$\left| \int_{\mathbb{R}^3} \vec{s} h_0(\vec{s}) d^3 s \right| \leq 1 . \quad (191)$$

Therefore  $f$  satisfies (156), (158), (178),(186) iff it satisfies (178) and  $f(0, \cdot, \cdot)$  satisfies (188), (190),(191) where  $Q_0(0) \in \mathbb{R}^6$  is arbitrary and  $h_0$  is  $\mathbb{R}$ -valued.

To show that (181) is for such an  $f$  the BKS equation, we take a closer look at  $f$ . In fact it follows from (178), (184), (188) that

$$f(t, q, \vec{s}) = \delta\left(q - Q_0(t)\right) h(t, \vec{s}) , \quad (192)$$

where

$$h(0, \vec{s}) = h_0(\vec{s}) , \quad (193)$$

and where  $h$  is determined by  $h_0$  as follows. In fact, using (187), inserting (192) into (178) and integrating the result over  $q$  leads to

$$\begin{aligned} \frac{\partial h}{\partial t} &= -\frac{\partial}{\partial s_j} \left( \left( \mathcal{D}_{spin,+j,k}(t, Q_0(t)) s_k + \mathcal{D}_{spin,0,ST,j}(t, Q_0(t)) \right) h \right) \\ &\quad + \frac{1}{2} \frac{\partial}{\partial s_k} \frac{\partial}{\partial s_l} \left( \mathcal{B}_{spin,l}(t, Q_0(t)) \mathcal{B}_{spin,k}(t, Q_0(t)) h \right) . \end{aligned} \quad (194)$$

Since (194) is a Fokker-Planck equation for  $h$  we note that  $h$  exists and is uniquely determined by  $h_0$  via (193), (194) (if the coefficient functions of (194) are sufficiently regular). As an aside we note that (194) has Gaussian solutions (we leave the proof to the reader).

As promised we now show that (181) is an ODE for  $\vec{P}[W_f]$  if  $f$  satisfies (178) and if  $f(0, \cdot, \cdot)$  satisfies (188), (190),(191) where  $Q_0(0) \in \mathbb{R}^6$  is arbitrary and  $h_0$  is  $\mathbb{R}$ -valued. In fact it follows from (99), (192), that

$$\vec{P}[W_f](t, q) = \delta\left(q - Q_0(t)\right) \int_{\mathbb{R}^3} \vec{s} h(t, \vec{s}) d^3 s , \quad (195)$$

and so, by (104),

$$\vec{P}[W_f](t) = \int_{\mathbb{R}^3} \vec{s} h(t, \vec{s}) d^3 s . \quad (196)$$

Note, by (195), (196), that

$$\vec{\mathcal{P}}[W_f](t, q) = \delta\left(q - Q_0(t)\right) \vec{P}[W_f](t). \quad (197)$$

Inserting (197) into (181) and using (185) results in

$$\begin{aligned} \frac{d}{dt} \vec{P}[W_f](t) &= \vec{P}[W_f](t) \int_{\mathbb{R}^6} \mathcal{D}_{spin,+}(t, q) \delta\left(q - Q_0(t)\right) d^6q \\ &\quad + \int_{\mathbb{R}^6} \vec{\mathcal{D}}_{spin,0,ST}(t, q) \delta\left(q - Q_0(t)\right) d^6q \\ &= \mathcal{D}_{spin,+}(t, Q_0(t)) \vec{P}[W_f](t) + \vec{\mathcal{D}}_{spin,0,ST}(t, Q_0(t)), \end{aligned}$$

in short,

$$\frac{d}{dt} \vec{P}[W_f](t) = \mathcal{D}_{spin,+}(t, Q_0(t)) \vec{P}[W_f](t) + \vec{\mathcal{D}}_{spin,0,ST}(t, Q_0(t)), \quad (198)$$

which indeed is an ODE for  $\vec{P}[W_f]$ . By the above derivation of (198) it is clear that  $f$  in (198) is a spin-orbit density of the BKS setup which renders the orbital motion to be deterministic.

Using (54) we can write (198) more explicitly as

$$\begin{aligned} \frac{d}{dt} \vec{P}[W_f](t) &= \mathcal{D}_{spin,+TBMT}\left(t, Q_0(t)\right) \vec{P}[W_f](t) \\ &\quad + \mathcal{D}_{spin,+ST}\left(t, Q_0(t)\right) \vec{P}[W_f](t) + \mathcal{D}_{spin,+BK}\left(t, Q_0(t)\right) \vec{P}[W_f](t) \\ &\quad + \vec{\mathcal{D}}_{spin,0,ST}\left(t, Q_0(t)\right), \end{aligned} \quad (199)$$

and using (50), (51), (52), (53) we can write (199) more explicitly as

$$\begin{aligned} \frac{d}{dt} \vec{P}[W_f](t) &= \vec{\Omega}_{TBMT}\left(t, Q_0(t)\right) \times \vec{P}[W_f](t) \\ &\quad - \frac{5\sqrt{3}}{8} \lambda\left(t, Q_0(t)\right) \vec{P}[W_f](t) + \frac{5\sqrt{3}}{36} \lambda\left(t, Q_0(t)\right) \vec{\Pi}_0(t) \vec{\Pi}_0^T(t) \vec{P}[W_f](t) \\ &\quad - \sqrt{\frac{55}{24\sqrt{3}}} \sqrt{\lambda(t, Q_0(t))} \vec{\mathcal{B}}_{spin}\left(t, Q_0(t)\right). \end{aligned} \quad (200)$$

Comparing with [BKS70] or with [Bai69] we observe that (200) (and thus: (198), (199)) is the BKS equation. It follows from [Bai69, BKS70] that the first term on the rhs of (200) and of (199) encapsulates the Thomas-BMT precession effect and that the second and fourth terms on the rhs of (200) and of (199) encapsulate the Sokolov-Ternov effect while the third term on the rhs of (200) and of (199) encapsulates the Baier-Katkov correction. These remarks, in combination with (50), (51), (52), (53), justify the notations

$\mathcal{D}_{spin,+TBMT}, \mathcal{D}_{spin,+ST}, \mathcal{D}_{spin,+BK}, \vec{\mathcal{D}}_{spin,0,ST}$ . This completes our derivation of the BKS equation.

Inspection of our above derivation of the BKS equation reveals that the full spin-orbit Fokker-Planck equation not only implies the BKS equation but also the so-called generalized BKS equation which is the same as (200) except that  $Q_0$  is now a solution of

$$Q'_0 = \mathcal{D}_{orb}(t, Q_0). \quad (201)$$

Thus for the generalized BKS equation, the BKS setup, (176), is replaced by the generalized BKS setup, defined by:  $\mathcal{B}_{orb} = 0$ . Therefore the radiative force fields  $\vec{\mathcal{C}}, \vec{\mathcal{Q}}$  are allowed to be nonzero in case of the generalized BKS setup, i.e., the generalized BKS equation takes into account the radiative force fields.

## 16 Interpreting the full spin-orbit Fokker-Planck equation, the full spin-orbit stochastic ODE system and the full Bloch equation

This section wraps up our presentation of the kinetic approach. In fact in this section we show how each of the key effects on the polarization, mentioned in Section 1, are tied to specific terms of the full spin-orbit SDE system, (136), of the full spin-orbit Fokker-Planck equation, (135), and of the full Bloch equation, (121).

To accomplish this task we take advantage of the fact that (136) is the Ito SDE system associated to (135) whereby the terms in (136) correspond in a well-defined way to the terms in (135) (this correspondence is also clearly visible in the pair, (132), (133)). Moreover we take advantage of how (121) was used in Section 9 to derive (130) (and thus to derive (135)).

We first recall from the discussion after (145) that all terms on the rhs of the reduced SDE system, (144), encapsulate the radiative depolarization effect. Moreover the terms on the rhs of (144) correspond in the full spin-orbit SDE system, (136), to the terms on the rhs of the first equation of (136), and to the second term on the rhs of the second equation of (136). Thus these terms in (136) encapsulate the radiative depolarization effect. Therefore and since (136) is the Ito SDE system associated to the Fokker-Planck equation, (135), the terms on the rhs of the first equation in (136) in combination with the second term on the rhs of the second equation in (136) correspond to the first, third and sixth terms on the rhs of (135). Thus these terms in (135) encapsulate the radiative depolarization effect. This completes the account of how the full spin-orbit SDE system and the full spin-orbit Fokker-Planck equation encapsulate the radiative depolarization effect.

Note that the above comments are related to Beznosov's PhD thesis [Bez20] where numerical approximations of the radiative depolarization time are studied in some detail. Moreover, [Bez20] addresses for the reduced setup the fact, mentioned in Section 4, that the polarization density does not exactly point along the invariant spin field.

We next recall from Section 15 that the second and fourth terms on the rhs of the BKS equation, (199), encapsulate the Sokolov-Ternov effect. Clearly these two terms correspond to the first and third terms on the rhs of the second equation of (136). Thus these two terms in (136) encapsulate the Sokolov-Ternov effect. Also since (136) is the Ito SDE system

associated to the Fokker-Planck equation, (135), the first and third terms on the rhs of the second equation of (136) correspond to the second and fourth terms on the rhs of (135). Thus these two terms in (135) encapsulate the Sokolov-Ternov effect. This completes the account of how the full spin-orbit SDE system and the full spin-orbit Fokker-Planck equation encapsulate the Sokolov-Ternov effect.

We now recall from Section 15 that the third term on the rhs of the BKS equation, (199), encapsulates the Baier-Katkov correction to the Sokolov-Ternov effect. Clearly this term corresponds to the fourth term on the rhs of the second equation of (136). Thus this term in (136) encapsulates the Baier-Katkov correction to the Sokolov-Ternov effect. Also since (136) is the Ito SDE system associated to the Fokker-Planck equation, (135), the fourth term on the rhs of the second equation of (136) corresponds to the fifth term on the rhs of (135). Thus this term in (135) encapsulates the Baier-Katkov correction to the Sokolov-Ternov effect. This completes the account of how the full spin-orbit SDE system and the full spin-orbit Fokker-Planck equation encapsulate the Baier-Katkov correction to the Sokolov-Ternov effect.

We now address the kinetic polarization effect. To begin with, we note from [DK75] that the fourth term on the rhs of the full Bloch equation, (121), is that term of (121) which is responsible for the kinetic polarization part in the Derbenev-Kondratenko formula for the equilibrium polarization. Also, we know from our derivation, in Section 9, of the full spin-orbit Fokker-Planck equation that the fifth term on the rhs of the full Bloch equation, (130) (and thus the seventh term on the rhs of the full spin-orbit Fokker-Planck equation, (135)), comes from the fourth term on the rhs of (121). Also since (136) is the Ito SDE system associated to the Fokker-Planck equation, (135), the seventh term on the rhs of (135) corresponds to the noise terms in (136), i.e., corresponds to  $\mathcal{B}_{orb}(t, Q)\nu(t)$  and  $\vec{\mathcal{B}}_{spin}(t, Q)\nu(t)$ . Thus the two noise terms in (136) encapsulate the kinetic polarization effect. This completes the account of how the full spin-orbit SDE system and the full spin-orbit Fokker-Planck equation encapsulate the kinetic polarization effect.

Before we continue we make some remarks on the above discussion of the kinetic polarization effect. First of all, the aforementioned cooperation between  $\mathcal{B}_{orb}(t, Q)\nu(t)$  and  $\vec{\mathcal{B}}_{spin}(t, Q)\nu(t)$  can be interpreted as an expression of the interference effect which was credited in [MSY05] as the QED mechanism underlying the kinetic polarization effect. For the QED origin of the kinetic polarization effect, see also [Mon84]. Secondly, if one would neglect at least one of the two noise terms in (136) then the kinetic polarization effect would disappear. This happens for example in the reduced setup considered in Section 11 where  $\vec{\mathcal{B}}_{spin}(t, Q)\nu(t)$  is absent. It also happens, in the BKS setup considered in Section 15, where  $\mathcal{B}_{orb}(t, Q)\nu(t)$  is absent. Thirdly, the noise term,  $\mathcal{B}_{orb}(t, Q)\nu(t)$ , in (136) plays a double role: In fact it follows from the above that it contributes to the kinetic polarization effect and to the radiative depolarization effect. Fourthly, having just identified how the full spin-orbit SDE system and the full spin-orbit Fokker-Planck equation encapsulate the various physical effects, we can imagine new roads of studying these effects, in particular the kinetic polarization effect. We already mentioned Section 1 one avenue namely numerically solving the full spin-orbit SDE system. Here we mention an analytical avenue whereby one obtains, via an appropriate approximation of the full spin-orbit Fokker-Planck equation, an ODE system for the polarization vector  $\langle \vec{S} \rangle_{W_f}(t)$  and for  $\langle Q_j \rangle_{W_f}(t), \langle Q_j Q_k \rangle_{W_f}(t)$

where  $j, k = 1, \dots, 6$ . For the definition of  $\langle \dots \rangle_{W_f}$ , see Section 12. The dependence of this ODE system on the kinetic polarization term,  $\frac{\partial}{\partial s_k} \frac{\partial}{\partial q_l} (\mathcal{B}_{orb,l} \mathcal{B}_{spin,k} f)$ , in the full spin-orbit Fokker-Planck equation, (135), leads to a dependence of its solutions on this term which can be used to quantitatively studying the kinetic polarization effect. Many other analytical approximations of (135) (including even higher moments of  $\vec{S}$  and  $Q$ ) can be pursued as well. This completes our remarks on the kinetic polarization effect.

Finally, by the above interpretation of the full spin-orbit Fokker-Planck equation, (135), we can interprete the terms of the full Bloch equation, (121), as well. In fact using that (121) is equal to (122) and using the derivation of the full spin-orbit Fokker-Planck equation, in Section 9, we see by the above interpretation of (135) that the first and second terms on the rhs of (122) encapsulate the radiative depolarization effect, that the third and fifth terms on the rhs of (122) encapsulate the Sokolov-Ternov effect and that the fourth term on the rhs of (122) encapsulates the Baier-Katkov correction whereas the sixth term on the rhs of (122) encapsulates, by an earlier remark, the kinetic polarization effect.

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