MAT235

Calculus II

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Contents

[0] With references from our textbook [1] and lecture notes.

List of Definitions

References

[1] James Stewart. *Multivariable Calculus*. Brooks Cole; 8 edition, 2015.

10 Parametric Equations and Polar Coordinates

10.1 Curves Defined by Parametric Equations

Definition 10.1 – Parametric equations. Let x and y be give as functions of third variable t with equations

$$x = f(t)\ y = g(t)$$

each values of x and y determines a point and as t varies we have $(x, y) = (f(t), g(t))$. The curve traced out by plotting these points are called **parametric curve**.

Note that same parametric equations can represent the same curve. We therefore distinguish between a curve and a parametric curve (the points are traced in a particular way).

10.2 Calculus with Parametric Curves

Suppose y is a differentiable function of x. Then the chain rule gives us

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Theorem 10.1 – Arc length. If a curve C is described by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Theorem 10.2 – Surface Area. If the curve given by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, is rotated about the x-axis, where f', g' are continuous and $g(t) \geq 0$, then the area of the resulting surface is given by

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

10.3 Polar Coordinates

Definition 10.2 – Polar coordinate system. Let the origin **pole** be O. The **polar axis** is a half line starting at O. Let P be any point in this system, then r is the distance from O to P and θ is the angle between line OP and the polar axis. The point P is then represented by (r, θ) and r, θ are called polar coordinates of P. If $r < 0$, it lies in the quadrant on the opposite side of the pole from positive r.

Note every point represent a unique point in the Cartesian coordinate system, but each point can have many representations in the polar coordinate system.
The following expressions denote the relationship between polar and Cartesian coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta$$

This can be used to find the Cartesian coordinates when the polar coordinates are known. The following relationship can be used to find the polar coordinates when the Cartesian coordinates are known.

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

Definition 10.3 – Polar Curves. The graph of a polar equation $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Consider the following symmetry properties:

1. If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.
2. If the equation is unchanged when r is replaced by $-r$ or when θ is replaced by $\theta + \pi$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)
3. If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \pi/2$.

10.4 Areas and Lengths in Polar Coordinates

The formula for the area A of polar region is

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta = \int_a^b \frac{1}{2} r^2 d\theta$$

Note that it is sometimes difficult to find all intersections of two polar curves since a single point has many representations. Thus, to find all points of intersection of two polar curves, it is recommended to draw the graphs of both curves.

The formula for the length of a curve with polar equation $r = f(\theta), a \leq \theta \leq b$ is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

10.5 Conic Sections

Definition 10.4 – Parabola. A parabola is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**. The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

Similarly, an parabola equation with focus $(p, 0)$ and directrix $y = -p$ is

$$y^2 = 4px$$

Definition 10.5 – Ellipse. An ellipse is the set of points in a plane the sum of whose distances from two fixed points F1 and F2 is a constant. These two fixed points are called the **foci** (plural of focus).

The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a \geq b > 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

Definition 10.6 – Hyperbola. A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points F1 and F2 (the foci) is a constant.

The equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$. The x-intercept is $\pm a$ and there is no y-intercept.
To sketch a hyperbola, it is useful to identify the asymptotes $y = (b/a)x$ and $y = -(b/a)x$
To consider questions of shifted Conics, we determine a, b, c by obtaining the distance and foci with given points or information. For instance, a shifted ellipse should have the form

$$\frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1$$

12 Vectors and the Geometry of Space

12.1 Three-Dimensional Coordinate Systems

Definition 12.1 – Three-dimensional rectangular coordinate system. The one-to-one correspondence between points and ordered triple in $\mathbb{R}^3$. $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\}$

Theorem 12.1 – Distance Formula in Three Dimensions.
The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Consider a application of the above formula: equation of sphere.
By definition, a sphere is the set of all points $P(x, y, z)$ whose distance from C is r. An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere would be

$$x^2 + y^2 + z^2 = r^2$$

12.2 Vectors

Definition 12.2 – Vector addition. If u and v are vectors positioned so the initial point of v is at the terminal point of u , then the sum $u + v$ is the vector from the initial point of u to the terminal point of v.

Definition 12.3 – Scalar multiplication. If c is a scalar and v is a vector, then the scalar multiple cv is the vector whose length is $|c|$ times the length of v and whose direction is the same as v if $c > 0$ and is opposite to v if $c < 0$. If $c = 0$ or $v = 0$, then $cv = 0$.

Definition 12.4 – Magnitude/length. The length of the two-dimensional vector $a = \langle a_1, a_2 \rangle$ is

$$|a| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector is computed in a similar manner.

Definition 12.5 – Standard basis vectors.
The following vectors i, j, k with length 1 and point in the directions of the positive $x-, y-$, and $z-$ axes $i = \langle 1, 0, 0 \rangle \quad j = \langle 0, 1, 0 \rangle \quad k = \langle 0, 0, 1 \rangle$ are called the standard basis vectors.

Note that any vectors in V_3 can be expressed in terms of i, j, k.

12.3 The Dot Product

Definition 12.6 – Dot product. If $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\boldsymbol{a}$ and $\boldsymbol{b}$ is the number $\boldsymbol{a} \cdot \boldsymbol{b}$ given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Theorem 12.2. If θ is the angle between the vector $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

This theorem can be proved using Law of Cosines. Note that the corollary below follows from this theorem. If θ is the angle between the vector $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Theorem 12.3. Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Definition 12.7 – Direction angles**.** The direction angles of a nonzero vector $\boldsymbol{a}$ are the angles α,β, and γ that $\boldsymbol{a}$ makes with the positive x, y and z axes.

We then obtain the following properties

$$\cos\alpha = \frac{\boldsymbol{a} \cdot \boldsymbol{i}}{|\boldsymbol{a}||\boldsymbol{i}|} = \frac{a_1}{|\boldsymbol{a}|}$$

Similarly we also have

$$\cos\beta = \frac{a_2}{|\boldsymbol{a}|} \qquad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

Equivalently, we have

$$\boldsymbol{a} = |\boldsymbol{a}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$$

This shows that the direction cosines of $\boldsymbol{a}$ are the components of the unit vector in the direction of $\boldsymbol{a}$.

Definition 12.8 – Vector projection**.** The vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is denoted by $\text{proj}_{\mathbf{a}}\mathbf{b}$

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||^2}\mathbf{b}$$

12.4 The Cross Product

Definition 12.9 – Cross product**.** If $\boldsymbol{a} = \langle a_1, a_2, a_3\rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3\rangle$, then the **cross product** of $\boldsymbol{a}$ and $\boldsymbol{b}$ is the vector

$$\boldsymbol{a} \times \boldsymbol{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

Note that the result of the cross product is a vector. To make it easier to remember, we find the cross product to be equivalent to the second-order determinant calculation:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Theorem 12.4. The vector $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$

Theorem 12.5. If θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ (for $0 \leq \theta \leq \pi$), then

$$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta$$

An corollary of this theorem is that two nonzero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if

$$\boldsymbol{a} \times \boldsymbol{b} = 0$$

which follows from the fact that $\sin 0 = \sin \pi = 0$.
Note: the cross product is not commutative and not associative, that is

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} \neq \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$$

However, some of the usual laws of algebra do hold. (e.g., $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$)

> **Definition 12.10** – Scalar triple product. The product $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ is called the scalar triple product of vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. In addition,
>
> $$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Consider the the volume of the parallelepiped determined by the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, it is the magnitude of their scalar triple product:

$$V = |\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})|$$

12.5 Equations of Lines and Planes

12.5.1 Lines

A line L in 3-D space is determined by a point $P_0(x_0, y_0, z_0)$ on L and the direction of L. The direction of L can be described by a vector $\mathbf{v}$ parallel to L. Let $P(x, y, z)$ be an arbitrary point on L and let r_0, r be the position vector of P_0, P. Then we have

$$r = r_0 + \overrightarrow{P_0P} = r_0 + t\mathbf{v}$$

for some scalar t. This is the **vector equation** of L.
If we write $\mathbf{v}$ as $\langle a, b, c \rangle$, then we have $t\mathbf{v} = \langle ta, tb, tc \rangle$. Therefore we can also represent $\mathbf{r} = \langle x, y, z \rangle$ as

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

The following three equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$. Each parameter t gives a point (x, y, z) on L

$$x = x_0 + ta \qquad y = y_0 + tb \qquad z = z_0 + tc$$

If we eliminate the parameter t and if none of a, b, c is 0 we can solve each of these equations for t and obtain the **symmetric equations** of L

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

> **Definition 12.11** – Skew lines. lines that do not intersect and are not parallel (and therefore do not lie in the same plane).

12.5.2 Planes

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ and a vector $\mathbf{n}$ that is orthogonal to the plane. $\mathbf{n}$ is referred to as the **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane and let r_0, r be the position vector of P_0, P. $\overrightarrow{P_0P}$ is therefore a vector on the plane and can be represented by $\mathbf{r} - \mathbf{r_0}$. We then obtain

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$$

Equivalently,

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$$

This is called the **vector equation of the plane**. Now consider $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$. It follows from the vector equation that

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Equivalently

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is called the scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$.
We can also rewrite it into the linear equation in x, y and z.

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$. If a, b, c are not all 0, the linear equation represents a plane with normal vector $\langle a, b, c \rangle$.
Two planes are **parallel** if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.
The distance between two planes can be computed by

$$D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

for $\mathbf{n}$ representing the normal vector $\langle a, b, c \rangle$ and $\mathbf{b} = \overrightarrow{P_0P_1}$ for some $P_0(x_0, y_0, z_0), P_1(x_1, y_1, z_1)$ on the planes. Since P_0 is on the plane, we can substitute in and get

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

In summary, to compute the distance between two parallel planes, we choose any point on one plane and calculate its distance to the other plane by applying this formula.

12.6 Cylinders and Quadric Surfaces

> **Definition 12.12** – Cylinder. A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes.
For instance, when sketching the surface of $y^2 + z^2 = 1$, we realize x is missing. We therefore sketch the cross-sections, which is circles of $y^2 + z^2 = 1, x = 0$, and move parallel on the x-axis.

> **Definition 12.13** – Quadric Surfaces. A quadric surface is the graph of a second-degree equation in three variables x, y and z. By translation and rotation it can be brought into one of the two standard forms
>
> $$Ax^2 + By^2 + Cz^2 + J = 0 \qquad Ax^2 + By^2 + Iz = 0$$

The following table shows the main quadric surfaces. The main strategy when sketching these surfaces is to identify the cross section in two of the other planes.

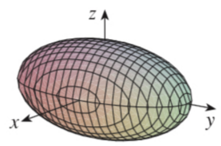
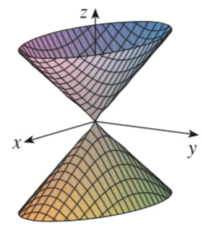
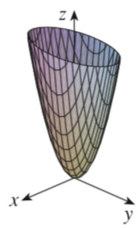
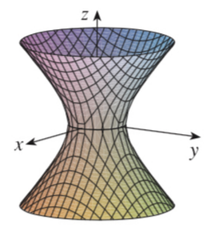
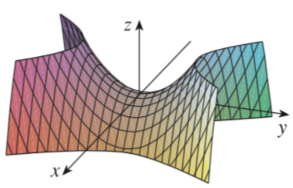
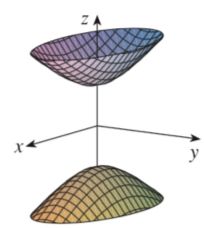
Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Figure 1: Graphs of quadric surfaces

In addition, if possible, identify the

- Type of the surface
- Intercepts
- For hyperboloid - 1 or 2 sheets, or paraboloid - axis of symmetry
- Opening (upwards/downwards)

13 Vector Functions

13.1 Vector Functions and Space Curves

Definition 13.1 – Vector function. A function whose domain is a set of real numbers and whose range is a set of vectors.

Consider the unique vector $\mathbf{r}(t)$ in V_3 for some vector function $\mathbf{r}$ and for every number t in its domain. If $f(t), g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f, g, and h are real-valued functions called the **component functions** of $\mathbf{r}$ and we write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Theorem 13.1 – Limit of vector functions. If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then

$$\lim_{t\to a}\mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$$

if the limits of the component functions exist.

Theorem 13.2 – Continuity of vector functions. A vector function $\mathbf{r}$ is continuous at a if

$$\lim_{t\to a}\mathbf{r}(t) = \mathbf{r}(a)$$

Definition 13.2 – Space curve. The set C of all points (x, y, z) in space where

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and t varies throughout the interval I is called a **space curve**. These equations are called **parametric equations of** C and t is called a parameter.

Any continuous vector function $\mathbf{r}$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$. To describe the curve of a vector function, we identify a vector it is parallel to and a point on the line.
If we are given two point and asked to find a vector equation for the line segment, we see $\mathbf{r}(t)$ can be described as a $\mathbf{r}_0$ on the line and the direction vector $\mathbf{r_1}$. We therefore have

$$\mathbf{r}(t) = \langle x_0 + x_1 t, y_0 + y_1 t, z_0 + z_1 t\rangle$$

for t in the domain of $\mathbf{r}$.
If we wish to find the **parametrization** of the curve C, we should make use of the following properties

$$\cos^2 t + \sin^2 t = 1 \qquad \cosh^2 t - \sinh^2 t = 1$$

where $\cosh t = \dfrac{e^t + e^{-t}}{2}$, and $\sinh t = \dfrac{e^t - e^{-t}}{2}$.

13.2 Derivatives and Integrals of Vector Functions

Definition 13.3 – Derivative of vector function. The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined as

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists.

Definition 13.4 – Tangent vector. Let point P have position vector $\mathbf{r}(t)$ then the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, if $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq 0$. We also define the **unit tangent vector** to be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Theorem 13.3. If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where $f(t), g(t)$ and $h(t)$ are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

The **second derivative** of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, namely $\mathbf{r}''$.

Theorem 13.4 – Differentiation Rules. Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Definition 13.5 – Definite integral of vector function. The integral of $\mathbf{r}$ can be evaluated by evaluating each of the component function f, g, and h

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}$$

note that the result is a vector.

We can extend the Fundamental Theorem of Calculus to continuous vector functions $\mathbf{r}$

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

13.3 Arc Length and Curvature

Definition 13.6 – Length of a space curve. Suppose the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where $a \leq t \leq b$. If the curve is traversed once as t goes from a to b, then the length

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt = \int_a^b |\mathbf{r}'(t)|dt$$

13.3.1 Reparametrization With Respect to Arc Length

Consider the arc length function

$$L = \int_a^b |\mathbf{r}'(t)|dt$$

If we wish to measure the distance travelled from $\mathbf{r}(a)$ at time t we may replace b with t, namely let

$$s(t) = \int_a^t |\mathbf{r}'(u)| du$$

If we reparametrize $\mathbf{r}(t)$ with the above expression as input, then we may find the position after travelling distance s from $\mathbf{r}(a)$.
For example, consider the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ measured from $(1, 0, 0)$ in the direction of increasing t.
After calculations we get

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{2}$$

The initial point $(1, 0, 0)$ implies $a = 0$.

$$s = s(t) = \int_0^t |\mathbf{r}'(u)| du = \int_0^t \sqrt{2} du = \sqrt{2} t$$

We see the relationship $t = s/\sqrt{2}$ and if we substitute it into $\mathbf{r}$ we get a function of the position vector in terms of the units travelled. Namely,

$$\mathbf{r}(t(s)) = \langle \cos s/\sqrt{2}, \sin s/\sqrt{2}, s/\sqrt{2} \rangle$$

Thus if $s = 3$, then the point 3 units of length along the curve from the starting point is $\langle \cos 3/\sqrt{2}, \sin 3/\sqrt{2}, 3/\sqrt{2} \rangle$.

13.3.2 Curvature

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq 0$ on I. If C is a smooth curve defined by the vector function $\mathbf{r}$, the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and indicate the direction of the curve.
The **curvature** of C at a given point is a measure of how quickly the curve changes direction at that point. We therefore define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length.

Definition 13.7 – Curvature. The curvature of a curvature is

$$k(t) = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

where $\mathbf{T}$ is the unit tangent vector.

Sometimes it is easier to deal with $\mathbf{r}$ than $\mathbf{T}$, so the following equivalent formula may be useful.

Theorem 13.5. The curvature of the curve given by the vector function $\mathbf{r}$ is

$$k(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

If we have $y = f(x)$, we can choose x as the parameter and write $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$. Then $|\mathbf{r}'(x0| = \sqrt{1 + [f'(x)]^2}$, therefore the following relationship holds

$$k(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

Definition 13.8 – Unit normal vector. At any point where $k \neq 0$ we can define the **principal unit normal vector** $\mathbf{N}(t)$ as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

We also define the vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ to be the **binormal vector**. It is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and is also a unit vector.

13.4 Motion in Space: Velocity and Acceleration

Let $\mathbf{r}(t)$ be the **position** vector at time t of an objection moving in a plane.
The **velocity** at time t is

$$\mathbf{v}(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)$$

Because the **speed** is the rate of change of distance with respect to time, we have from 13.3 that

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = |v(t)|$$

which is the magnitude of the velocity vector. In addition, we find the **acceleration** of the object

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

Let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Some other questions we may be interested include:

- When will the object hit the ground?
  This is when the position vector $\mathbf{r}(t) = 0$
- How far will the object move?
  This can be found by evaluating $x(t)$ where t is the duration of the object's travel.
- What is the maximum height of the object?
  This maximum height of the object is when $y(t)$ attains its maximum, which can be found by taking $y'(t) = 0$ and substituting the time t into $y(t)$.

14 Partial Derivatives

14.1 Functions of Several Variables

Definition 14.1 – Functions of two variables. A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, i.e., $\{f(x, y) \mid (x, y) \in D\}$.

For example, to find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$, we proceed with

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geq 0\} = \{(x, y) \mid x^2 + y^2 \leq 9\}$$

The range of g is therefore

$$\{z \mid z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\}$$

Since $0 \leq z \leq 3$, as $9 - x^2 - y^2 \leq 9$,

$$R = \{z \mid 0 \leq z \leq 3\} = [0, 3]$$

Definition 14.2 – Level curves. The level curves of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant in the range of f.

We may think of level curves as height of f. Drawing level curves is a technique for graphing 3D surfaces.

14.2 Limits and Continuity

Definition 14.3 – Limit of functions of two variables. Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$(x, y) \in D \text{ and } 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \Rightarrow |f(x, y) - L| < \epsilon$$

Abstractly, similar with Calculus I, if the limit **exists**, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Thus, to show $\lim_{(x,y)\to(a,b)} f(x, y)$ **does not exist**, we can proceed by showing $f(x, y)$ has different limits when we approach along two different paths.

Theorem 14.1. If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

To show that the limit exists, we can use several techniques. Consider the following theorems:

Theorem 14.2.

$$\lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b \qquad \lim_{(x,y)\to(a,b)} c = c$$

Theorem 14.3 – Squeeze Theorem. Suppose $f_1(x, y) \leq f(x, y) \leq f_2(x, y)$.
If $\lim_{(x,y)\to(x_0,y_0)} f_1(x, y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} f_2(x, y) = L$. Then $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$.

THEOREM 14.4. Let k be an **even number**. Consider functions in the form of

$$f(x,y) = \frac{x^n y^m}{x^k + y^k}$$

Then to compute the limit as $(x,y) \to (0,0)$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \begin{cases} 0 & \text{if } n+m > k \\ \text{DNE} & \text{if } n+m \le k \end{cases}$$

If we can refactor a multivariable function f to have only a single variable, then we may apply techniques from single variable calculus, such as the L'hopital's Rule, to compute the limit. Consider

$$\lim_{(x,y)\to(0,0)} (x^2+y^2) \ln \sqrt{x^2+y^2}$$

We know that $\lim_{x\to 0} x \ln x = 0$ so we should try to manipulate the function. Let $(x,y) = (r\cos\theta, r\sin\theta)$. Note that as $(x,y) \to (0,0)$, $r \to 0^+$.

$$\begin{aligned} \lim_{(x,y)\to(0,0)} (x^2+y^2) \ln \sqrt{x^2+y^2} &= \lim_{r\to 0^+} r^2 \ln \sqrt{r^2} \\ &= \lim_{r\to 0^+} r^2 \ln r \\ &= \lim_{r\to 0^+} -\frac{r^2}{2} = 0 \end{aligned}$$

(By L'hopital's Rule)

DEFINITION 14.4 – CONTINUITY. We say $z = f(x,y)$ is continuous at (x_0, y_0) if

1. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
2. $f(x_0, y_0)$ is well-defined
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

14.3 Partial Derivatives

If f is a function of two variables x and y, suppose we let x vary while keeping y fixed, say $y = b$ where b is a constant. Then if we were to take the derivative at a, then we call it the **partial derivative of f with respect to x at (a,b)** and denote it by $f_x(a,b)$. In other words,

$$f_x(a,b) = g'(a) \qquad \text{where} \qquad g(x) = f(x,b)$$

DEFINITION 14.5 – PARTIAL DERIVATIVES. If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

An alternative **notation** for $f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y)$.

THEOREM 14.5 – RULES FOR FINDING PARTIAL DERIVATIVES OF $z = f(x, y)$.
1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

14.3.1 Second partial derivatives

Let $z = f(x, y)$
Since the partial derivatives f_x and f_y are also functions of two variables, we can consider their partial derivatives to be the **second partial derivatives** of f. Consider the following second partial derivatives of f at (a, b). They are denoted and defined as follows:

$$f_{xx}(a,b) = \frac{\partial^2 f}{\partial x \partial x}(a,b) = \frac{d}{dx}[f_x(x,b)]_{x=a}$$
$$f_{xy}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b) = \frac{d}{dy}[f_x(a,y)]_{y=b}$$
$$f_{yx}(a,b) = \frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{d}{dx}[f_y(x,b)]_{x=a}$$
$$f_{yy}(a,b) = \frac{\partial^2 f}{\partial y \partial y}(a,b) = \frac{d}{dy}[f_y(a,y)]_{y=b}$$

THEOREM 14.6. If f_{xy} and f_{yx} are continuous near (a, b) then $f_{xy}(a, b) = f_{yx}(a, b)$.

14.4 Tangent Planes and Linear Approximations

14.4.1 Tangent planes

Suppose $z = f(x, y)$ has $f_x(a, b)$ and $f_y(a, b)$ then

$$\vec{a} = \langle 1, 0, f_x(a,b) \rangle \qquad \vec{b} = \langle 0, 1, f_y(a,b) \rangle$$

Then $\vec{a} \times \vec{b}$ is a normal vector to the plane generated by $\vec{a}$ and $\vec{b}$. Also note that

$$\vec{a} \times \vec{b} = \langle -f_x(a,b), -f_y(a,b), 1 \rangle$$

We then get the planes

$$d = -f_x(a,b)x + -f_y(a,b)y + z$$
$$d = -f_x(a,b)a + -f_y(a,b)b + f(a,b)$$

and equivalently

$$z = (x - a)f_x(a,b) + (y - b)f_y(a,b) + f(a,b) = L(x,y)$$

DEFINITION 14.6 – EQUATION OF TANGENT PLANES. Suppose f has continuous partial derivatives.
An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Note that in multivariable calculus, existence of tangent plane does **not** imply differentiability, since the tangent plane will exist if the partial derivatives are defined.

14.4.2 Linear Approximations

Definition 14.7 – Differentiability. If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed as

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

We can interpret differentiability as: the tangent plane approximates the graph of f well near the point of tangency.

Theorem 14.7. If f is differentiable at (a, b) then f is continuous at (a, b).

Theorem 14.8. If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b) then f is differentiable at (a, b).

Recall from Calculus I if f is differentiable then the linear approximation L at point a and f' are very close at a, i.e., $L(a + \Delta x) \approx f(a + \Delta x)$. In multivariable, if f is differentiable at (a, b) then $L(a + \Delta x, b + \Delta y) \approx f(a + \Delta x, b + \Delta y)$. And $L(a + \Delta x, b + \Delta y)$ is the linear approximation of f.

Definition 14.8 – Linear Approximation. Let L be the linear function whose graph is a tangent plane where $L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$. Then the approximation

$$f(x, y) \approx f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

is called the linear approximation or tangent plane approximation of f at (a, b).

14.4.3 Differentials

For a differentiable function with two variables, $z = f(x, y)$, we let dx and dy be independent variables. Then the differential dz is defined as

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

If we take $dx = \Delta x = x - a, dy = \Delta y = y - b$, then we get

$$dz = f_x(x, y)(x - a) + f_y(x, y)(y - b)$$

Note we can also write $f(x, y) \approx f(a, b) + dz$.

14.5 The Chain Rule

For multivariable functions, the Chain Rule has several versions for different types of composite function.

Theorem 14.9 – Chain Rule - Case 1. Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

The following case consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t: $x = g(s, t), y = h(s, t)$.

Theorem 14.10 – Chain Rule - Case 2. Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(s,t)$ and $y = h(s,t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Theorem 14.11 – Chain Rule - General Version. Suppose that u is a differentiable functions of the n variables $x_1, x_2, ..., x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, ..., t_m$. Then u is a function of $t_1, t_2, ..., t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + ... + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

14.5.1 Implicit Differentiation

Consider an equation of the form $F(x,y) = 0$ that defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f. Since both x and y are functions of x,

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \Longleftrightarrow \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Now we consider $z = f(x,y)$ where z is implicitly defined by an equation of the form $F(x,y,z) = 0$. If F and f are differentiable, we can obtain $\frac{\partial z}{\partial x}$ with

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Since $\frac{\partial}{\partial x}x = 1$ and $\frac{\partial}{\partial x}y = 0$, we get $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$, which leads to the following theorem.

Theorem 14.12 – Implicit Function Theorem.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

14.6 Directional Derivatives and the Gradient Vector

14.6.1 Directional Derivatives

To find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b\rangle$, we consider the directional derivative which calculates the change of z with a very small change in the direction of $\mathbf{u}$.

Definition 14.9 – Directional Derivatives. The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_u f(x_0, y_0) = \lim_{h\to 0}\frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Theorem 14.13. if f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$$

Theorem 14.14. If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta\rangle$ and we get

$$D_u f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

14.6.2 The Gradient Vector

We notice from the definition of directional derivative that $D_u f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle \cdot \mathbf{u}$. The first vector is called the gradient of f.

> **Definition 14.10** – Gradient. If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by
> $$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

14.6.3 Maximizing the Directional Derivative

We use the following theorem to find the directions that f change the fastest and the maximum rate of change.

> **Theorem 14.15.** Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_u f(x)$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

14.6.4 Tangent Planes from the Gradient Vector

The gradient vector at point P, which is $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on the surface S with equation $F(x, y, z) = k$ that passes through P.
If $\nabla F(x_0, y_0, z_0) \neq 0$, we can find the **tangent plane to the level surface** $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P with the normal vector $\nabla F(x_0, y_0, z_0)$. In other words, we can find the tangent plane using the gradient of f with
$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

14.7 Maximum and Minimum Values

> **Definition 14.11** – Local and Absolute extrema. A function of two variables has a **local maximum** at (a, b) if $f(x, y) \le f(a, b)$ when (x, y) is near (a, b). The number $f(a, b)$ is called a local maximum value. If $f(x, y) \ge f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a local minimum value.
> If the inequalities above hold for all points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

> **Theorem 14.16.** If f has a local maximum or minimum at (a, b) and the first-order derivatives of f exist there, then $f_x(a, b) = f_y(a, b) = 0$.

> **Definition 14.12** – Critical Point. A point (a, b) is called a critical point of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

To fully determine whether a function have a extreme value at a critical point, we can incorporate the Second Derivative Test that works similarly with the one from single variable calculus.

> **Theorem 14.17** – Second Derivatives Test. Suppose the second partial derivatives of f are continuous on a disk with center (a, b) and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (i.e., (a, b) is a critical point). Let
> $$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$
>
> 1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
> 2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
> 3. If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.
>
> If $D = 0$, this test gives no information.

14.7.1 Absolute Maximum and Minimum Values

A **closed set** is one that contains all its boundary points. A **bounded set** is contained within some disk, i.e., finite in extent.

> **Theorem 14.18** – Extreme Value Theorem for Functions of Two Variables. If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

14.8 Lagrange Multipliers

If we wish to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$, we can use Lagrange Multipliers.

> **Definition 14.13** – Lagrange multiplier. If $\nabla g(x_0, y_0, z_0) \neq 0$, there is a number λ such that
>
> $$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

The procedure to find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ are as following:

1. Find all values of x, y, z and λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k$, namely, solve the system of four equations in the four unknowns x, y, z and λ:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x, y, z) = k$$

2. Evaluate f at all the possible points (x, y, z) that result from solving the above system. The largest of these values is the maximum value of f; and the smallest is the minimum value of f.

15 Multiple Integrals

15.1 Double Integrals over Rectangles

Recall definite integral of a function f:

$$\int_b^a f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

In a similar manner we consider a function f of two variables on a closed rectangle

$$R = [a,b]\times[c,d] = \{(x,y)\in\mathbb{R}^2 \mid a\le x\le b, c\le y\le d\}$$

Definition 15.1 – Double integral. The double integral of f over the rectangle R is

$$\iint\limits_R f(x,y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij},y_{ij})\Delta A$$

if this limit exists.

Theorem 15.1 – Midpoint Rule for Double Integral.

$$\iint\limits_R f(x,y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i,\overline{y}_j)\Delta A$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1},x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1},y_j]$.

15.1.1 Properties of Double Integrals

$$\iint\limits_R [f(x,y)+g(x,y)]dA = \iint\limits_R f(x,y)dA + \iint\limits_R g(x,y)dA$$

$$\iint\limits_R cf(x,y)dA = c\iint\limits_R f(x,y)dA \text{ where } c \text{ is a constant}$$

If $f(x,y)\ge g(x,y)$ for all (x,y) in R, then

$$\iint\limits_R f(x,y)dA \ge \iint\limits_R g(x,y)dA$$

15.2 Iterated Integrals

To evaluate an iterated integral, we calculate two single integrals. Let $A(x)=\int_c^d f(x,y)\,dy$. If we now integrate A with respect to x from a to b we get

$$\int_a^b A(x)\,dx = \int_a^b \Big[\int_c^d f(x,y)\,dx\Big]dy$$

the integral on the right is called an **iterated integral**.

Theorem 15.2 – Fubini's Theorem.
If f is continuous on the rectangle $R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b, c \le y \le d\}$, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

15.3 Double Integrals over General Regions

To integrate a function f over regions D of a general shape, we suppose D can be enclosed in a rectangular region R. Then we can define a new function F with domain R by

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

If F is integrable over R, then we define the **double integral of f over D** by

$$\iint\limits_D f(x, y)\, dA = \iint\limits_R F(x, y)\, dA$$

Specifically, the following theorem enables us to evaluate a double integral over a region as an iterated integral.

Theorem 15.3. If f is continuous on a region D that lies between the graphs of two continuous functions of x such that

$$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\iint\limits_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

15.4 Double Integrals in Polar Coordinates

Recall that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

To change the (x, y) coordinates in a Double Integral, we use the following theorem.

Theorem 15.4. If f is continuous on a polar rectangle R given by $0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint\limits_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

15.5 Applications of Double Integrals

15.5.1 Moments and Centers of Mass

Definition 15.2 – Moment of lamina. We define the moment about the x-axis by

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} y_{ij}\rho(x_{ij}, y_{ij})\Delta A = \iint_D y\rho(x, y)\, dA$$

And similarly, the moment about the y-axis is

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} x_{ij}\rho(x_{ij}, y_{ij})\Delta A = \iint_D x\rho(x, y)\, dA$$

Definition 15.3 – Center of Mass. The center of mass $(\overline{x}, \overline{y})$ is defined so that $m\overline{x} = M_y$ and $m\overline{y} = M_x$.
So the coordinates $(\overline{x}, \overline{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)\, dA \qquad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)\, dA$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\, dA$$

15.5.2 Moment of Inertia

Definition 15.4 – Moment of Inertia. The moment of inertia of the lamina about the x, y-axis is defined as

$$I_x = \iint_D y^2\rho(x, y)\, dA \qquad I_y = \iint_D x^2\rho(x, y)\, dA$$

The moment of inertia about the origin, or the polar moment of inertia is defined as

$$I_0 = \iint_D (x^2 + y^2)\rho(x, y)\, dA$$

15.5.3 Probability

Definition 15.5 – Joint density function. The joint density function of a continuous random variables X and Y is a function f of two variables such that (X, Y) lies in a region D

$$P((X, Y) \in D) = \iint_D f(x, y)\, dA$$

An important properties for **independent random variables** is that their joint denstiy functions is the product of their indivudual density functions:

$$f(x, y) = f_1(x)f_2(y)$$

Definition 15.6 – Expected Values. If X and Y are random variables with joint density function f, we define the expected values of X and Y to be

$$\mu_1 = \iint_{\mathbb{R}^2} xf(x, y)\, dA \qquad \mu_2 = \iint_{\mathbb{R}^2} yf(x, y)\, dA$$

15.6 Surface Area

Definition 15.7 – Surface Area. The area of the surface with equation $z = f(x, y), (x, y) \in D$, where f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA$$

15.7 Triple Integrals

Definition 15.8 – Triple Integral. The triple integral of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta V$$

if this limit exists.

Theorem 15.5 – Fubini's Theorem.
If f is continuous on the rectangle $R = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

The triple integral over a general bounded region E in three dimensional space is similar with that defined in two dimension.

$$\iiint_E f(x, y, z)\, dV = \iint_D \Big[\int_{u_1(x,y)}^{u_2(x,y)} dz\Big]\, dA$$

15.8 Triple Integrals in Cylindrical Coordinates

Definition 15.9 – Cylindrical coordinate system. A point P in three-dimensional space is represented by the-ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-place to P.
To convert from cylindrical to rectangular cooridnates, we use the equations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

To convert from rectangular to cylindrical coordinates

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

15.9 Triple Integrals in Spherical Coordinate

Definition 15.10 – Spherical Coordinate. Consider the spherical coordinates (ρ, θ, ϕ) of a point P. ρ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates and ϕ is the angle between the positive z-axis and the line segment OP. To convert from spherical coordinate to rectangular coordinates:

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

15.10 Change of variables in Multiple Integrals

Change of variables can be useful from the uv-plane to the xy-plane where x and y are related to u and v by the equations

$$x = g(u, v) \qquad y = h(u, v)$$

Definition 15.11 – Jacobian. The Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Definition 15.12 – Change of Variables in a Double Integral. Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint\limits_R f(x, y)\, dA = \iint\limits_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

The triple integral change of variables formula is similar.

16 Vector Calculus

16.1 Vector Calculus

Definition 16.1 – Vector Field. Let D be a set in $\mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$$

Vector field in three dimension is defined similarly.

Definition 16.2 – Gradient Field. A vector field $\mathbf{F}$ is called a **conservative** vector field if it is the gradient of some scalar function, i.e., if there exists a function f such that $\mathbf{F} = \nabla f$.In this case, f is called a potential function for $\mathbf{F}$.

$$\nabla f(x, y, z) = \mathbf{f}(x, y, z)$$

16.2 Line Integrals

Definition 16.3 – Line Integral. If f is defined on a smooth curve C given by

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b$$

then the line integral of f along C is

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

Theorem 16.1. If f is a continuous function, then the limit always exists and the following formula can be used to evaluate the line integral

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

The line integrals with respect to x and y can also be evaluated by expressing everything in terms of

$$x = x(t) \qquad y = y(t) \qquad dx = x'(t)\, dt \qquad dy = y'(t)\, dt$$

Note that it is useful to recall the formula to parametrize a line segment $\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$.
For line integrals in the plane, we evaluate integrals of the form

$$\int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

by expressing $x, y, z,\ dx,\ dy,\ dz$ in terms of the parameter t.

Definition 16.4 – Line Integral of F along C. Let $\mathbf{F}$ be a continuous vector field defined on a smooth cure C given by a vector function $\mathbf{r}(t), a \leq t \leq b$. Then the line integral of F along C is

$$\int_C \mathbf{F} \cdot\, d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C P\, dx + Q\, dy + R\, dz = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

16.3 The Fundamental Theorem for Line Integral

Theorem 16.2. Let C be a smooth curve given by the vector function $\mathbf{r}(t), a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

This theorem implies that the line integral of ∇f is the net change in f i.e., we can evaluate the line integral of a conservative vector field by knowing the value of f at the endpoints of C.

16.3.1 Independence of Path

From the above theorem, we know that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

whenever ∇f is continuous. If $\mathbf{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if $\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial and terminal points.

Definition 16.5 – Closed curve. A curve is called closed if its terminal point coincides with its initial point, i.e., $\mathbf{r}(b) = \mathbf{r}(a)$.

Theorem 16.3. The line integral of any conservative vector field F is independent of path, so $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path.

Theorem 16.4. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

The following theorem says that the only vector fields that are independent of path are conservative, assuming that D is open and connected. (D doesn't have any boundary points and that any two points in D can be joined by a path that lies in D).

Theorem 16.5. Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D.

Theorem 16.6. If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Theorem 16.7. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

16.4 Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

Theorem 16.8 – Green's Theorem. Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

<u>**Notation**</u> to denote the line integral is calculated using the positive orientation of the closed curve C:

$$\oint_C P\,dx + Q\,dy \qquad \oint_C P\,dx + Q\,dy$$

16.5 Curl and Divergence

16.5.1 Curl

Definition 16.6 – Curl **F**. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, R all exist, then the **curl** of **F** is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Theorem 16.9. If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl } (\nabla f) = 0$$

Since terms in $\nabla \times (\nabla f)$ cancel out.

Here is an important theorem that can be used to check whether **F** is a conservative field.

Theorem 16.10. If **F** is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = 0$, then **F** is a conservative vector field.

16.5.2 Divergence

Definition 16.7 – Divergence. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, R all exist, then the **divergence** of **F** is the function of three variables defined by

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Theorem 16.11. If **F** is a vector field defined on all of $\mathbb{R}^3$ whose component functions P, Q, R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

16.5.3 Vector Forms of Green's Theorem

Recall that

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C P\, dy + Q\, dx$$
$$= \iint_D \Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\Big)\, dA$$

But we can rewrite the integrand in the double integral as the divergence of $\mathbf{F}$, which gives

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

16.6 Parametric Surfaces and Their Areas

We describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v. We suppose that

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

is a vector-valued function defined on a region for the uv-plane.

> **Definition 16.8** – Parametric surface. The set of all points (x, y, z) in $\mathbb{R}^3$ such that
>
> $$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$
>
> and (u, v) varies throughout D is called a **parametric surface** S and the three equations above are called the **parametric equations** of S.

16.6.1 Surfaces of Revolution

Consider the surface S obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$ about the x-axis with θ being the angle of rotation. Then we take x and θ as the parameters and obtain the parametric equations

$$x = x \qquad y = \sin x \cos\theta \qquad z = \sin x \sin\theta$$

16.6.2 Tangent Planes

Recall that the tangent plane contains the tangent vectors and the normal vector. If $\mathbf{r}_u \times \mathbf{r}_v \neq 0$, the surface is called smooth. The tangent plane to a smooth surface contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ and the normal vector $\mathbf{r}_u \times \mathbf{r}_v$.

> **Definition 16.9** – Surface Area of smooth parametric surface. Let $\mathbf{r}(u, v)$ be the parametric equation of a smooth surface S and S is covered just once as (u, v) ranges throughout the parameter domain D, then the surface area of S is
>
> $$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

Theorem 16.12. Consider a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives. Then if we take x and y as parameters, we have

$$x = x \qquad y = y \qquad z = f(x, y)$$

This gives us the shortcut

$$\mathbf{r}_u \times \mathbf{r}_v = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

and the surface area formula

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

16.7 Surface Integrals

Definition 16.10 – Surface Integral**.** The **surface integral of** f **over the surface** S is

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

If a surface is projected onto the xy-plane with equation $z = g(x, y)$, the formula becomes

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

16.7.1 Oriented Surfaces

If it is possible to choose a unity normal vector $\mathbf{n}$ at every such point (x, y, z) so that $\mathbf{n}$ varies continuously over S, then S is called an **oriented surface** and the given choice of $\mathbf{n}$ provides S with an outward or inward **orientation**. If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then its unit noraml vector gives

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

Note that this implies the $\mathbf{k}$ component is positive and gives the upward orientation of the surface. The opposite orientation is given by $-\mathbf{n}$.

16.7.2 Surface Integrals of Vector Fields

Definition 16.11 – Flux Integral. If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

This integral is also called the **flux** of $\mathbf{F}$ across S. Note that the third double integral follows from the definition of surface integral.
In the case of a surface S given by graph $z = g(x, y)$, we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

This assumes upward orientation of S, so we would multiply by -1 for a downward orientation.

16.8 Stokes' Theorem

Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S. A **positive orientation of the boundary curve** C means that if you walk around C with your head pointing in the direction of $\mathbf{n}$, then the surface will always be on your left.

Theorem 16.13 – Stokes' Theorem. Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot dr = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Note that this theorem allows us to evaluate either a surface S with curve C or the boundary curve C, whichever one is more convenient.

Theorem 16.14. The vector field curl $\mathbf{F}$ has an independence of surface property. Namely, $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, ds$ is only dependent on the boundary of S, rather than S itself.
In other words, if S_1 and S_2 have the same boundary C,

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

This can be used to simplify calculation when integrating one surface is easier than the other.

16.9 The Divergence Theorem

Theorem 16.15 – The Divergence Theorem. Let E be a simple solide region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot \, d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

Which can make calculations a lot simpler if $\mathbf{F}$ is complicated.

With some examples, the given surface is not closed. For instance, consider the case where we take S_0 to be the part of S without base (S_1).
Then

$$\iint_{S_0} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_S \mathbf{F} \cdot \mathbf{n} \, ds - \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, ds$$