Functional Analysis

Tutorial 2

Contents

1 Assignments

Problem 1.1. Show that in the space s, we have $x_n \to x$ if and only if $\xi_j^{(n)} \to \xi_j$ for all $j = 1, 2, \cdots$, where $x_n = (\xi_j^{(n)})$ and $x = (\xi_j)$. Prove that s is complete.

Solution. The space s is the space of all real (or complex) sequences $x = (\xi_1, \xi_2, \dots)$ equipped with the metric

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}.$$

Let $x_n = (\xi_j^{(n)})_{j=1}^{\infty}$ and $x = (\xi_j)_{j=1}^{\infty}$.

Part 1: $x_n \to x$ in s iff $\xi_j^{(n)} \to \xi_j$ for each j.

($\Rightarrow$) Suppose $d(x_n, x) \to 0$. For a fixed j,

$$\frac{1}{2^j} \frac{|\xi_j^{(n)} - \xi_j|}{1 + |\xi_j^{(n)} - \xi_j|} \le d(x_n, x) \to 0,$$

hence $\frac{|\xi_j^{(n)} - \xi_j|}{1 + |\xi_j^{(n)} - \xi_j|} \to 0$. Since the map $t \mapsto \frac{t}{1+t}$ is increasing for $t \ge 0$, this implies $|\xi_j^{(n)} - \xi_j| \to 0$, so $\xi_j^{(n)} \to \xi_j$.

($\Leftarrow$) Assume $\xi_j^{(n)} \to \xi_j$ for each j. Let $\varepsilon > 0$. Choose N such that $\sum_{j=N+1}^{\infty} \frac{1}{2^j} < \frac{\varepsilon}{2}$. For $j = 1, \dots, N$, pick M_j such that for all $n \ge M_j$,

$$|\xi_j^{(n)} - \xi_j| < \frac{\varepsilon}{2}.$$

Set $M = \max_{1 \le j \le N} M_j$. For $n \ge M$ and $j \le N$,

$$\frac{|\xi_j^{(n)} - \xi_j|}{1 + |\xi_j^{(n)} - \xi_j|} \le |\xi_j^{(n)} - \xi_j| < \frac{\varepsilon}{2},$$

hence

$$\sum_{j=1}^{N} \frac{1}{2^j} \frac{|\xi_j^{(n)} - \xi_j|}{1 + |\xi_j^{(n)} - \xi_j|} < \frac{\varepsilon}{2} \sum_{j=1}^{N} \frac{1}{2^j} < \frac{\varepsilon}{2}.$$

For $j > N$,

$$\frac{1}{2^j} \frac{|\xi_j^{(n)} - \xi_j|}{1 + |\xi_j^{(n)} - \xi_j|} \leq \frac{1}{2^j},$$

so

$$\sum_{j=N+1}^{\infty} \frac{1}{2^j} \frac{|\xi_j^{(n)} - \xi_j|}{1 + |\xi_j^{(n)} - \xi_j|} \leq \sum_{j=N+1}^{\infty} \frac{1}{2^j} < \frac{\varepsilon}{2}.$$

Adding the two parts, for all $n \geq M$, $d(x_n, x) < \varepsilon$. Hence $x_n \to x$ in s.

Part 2: s is complete.

Let $(x_n)_{n=1}^{\infty}$ be Cauchy in s. For each fixed j, the inequality

$$\frac{1}{2^j} \frac{|\xi_j^{(n)} - \xi_j^{(m)}|}{1 + |\xi_j^{(n)} - \xi_j^{(m)}|} \leq d(x_n, x_m)$$

shows that $(\xi_j^{(n)})_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$ (or $\mathbb{C}$), hence convergent. Therefore s is complete. □

Problem 1.2. The distance from a point x to a non-empty subset B of (X, d) is defined to be

$$D(x, B) = \inf\{d(x, b) : b \in B\}.$$

Show that for any $x, y \in X$,

$$D(x, B) - D(y, B) \leq d(x, y).$$

Proof. For any $b \in B$, by the triangle inequality,

$$d(x, b) \leq d(x, y) + d(y, b).$$

Taking the infimum over all $b \in B$ on the right side, we get

$$d(x, b) \leq d(x, y) + D(y, B).$$

Taking the infimum over all $b \in B$ on the left side, we have

$$D(x, B) \leq d(x, y) + D(y, B).$$

Rearranging gives

$$D(x, B) - D(y, B) \leq d(x, y).$$

□

Problem 1.3. The distance between two non-empty subsets A, B of (X, d) is defined to be
$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$
Show that D does not define a metric on the power set of X (the set of all subsets of X).

Proof. Consider $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Let $A = [0, 1]$ and $B = [1, 2]$. Then
$$D(A, B) = \inf\{|a - b| : a \in [0, 1], b \in [1, 2]\} = 0,$$
since we can take $a = 1$ and $b = 1$. However, $A \neq B$. Thus, $D(A, B) = 0$ does not imply $A = B$, violating the identity of indiscernibles property of a metric. Therefore, D does not define a metric on the power set of X. $\square$

Problem 1.4. Show that the image of an open set under a continuous function is not necessarily open.

Proof. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. The set $U = (-1, 1)$ is open in $\mathbb{R}$. However, the image of U under f is
$$f(U) = \{x^2 : x \in (-1, 1)\} = [0, 1),$$
which is not open in $\mathbb{R}$. Thus, the image of an open set under a continuous function need not be open. $\square$

Problem 1.5. Show that any nonempty set $A \subset (X, d)$ is open if and only if it is a union of open balls.

Proof. ($\Rightarrow$) Suppose A is open. For each $x \in A$, there exists $\varepsilon_x > 0$ such that the open ball $B(x, \varepsilon_x) \subset A$. Thus,
$$A = \bigcup_{x \in A} B(x, \varepsilon_x).$$

($\Leftarrow$) Suppose A is a union of open balls. Each open ball is open, and a union of open sets is open. Therefore, A is open. $\square$

Problem 1.6. Show that a compact subset of a metric space is closed and bounded. Is the converse true?

Proof. Consider the space $X = \ell^\infty$, and a subset $\{e_n\}$ where e_n is the sequence with 1 in the n-th position and 0 elsewhere. This set is closed and bounded but not compact, as it has no convergent subsequence. Thus, the converse is not true. $\square$

Problem 1.7. If (x_n) is a Cauchy sequence in a metric space (X, d) and has a convergent subsequence (x_{n_k}) converging to $x \in X$, show that the original sequence (x_n) converges to x.

Proof. Let $\varepsilon > 0$. Since (x_n) is Cauchy, there exists N_1 such that for all $m, n \geq N_1$, $d(x_n, x_m) < \frac{\varepsilon}{2}$. Since (x_{n_k}) converges to x, there exists N_2 such that for all $k \geq N_2$, $d(x_{n_k}, x) < \frac{\varepsilon}{2}$. Let $N = \max(N_1, n_{N_2})$. For any $n \geq N$, choose k such that $n_k \geq N$. Then,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $x_n \to x$. □

2 Q&A

Proposition 2.1. In a metric space, a subset is compact if and only if it is sequentially compact.

Proof. **(Compact $\implies$ Sequentially Compact)**: Let $K \subset X$ be a compact subset of a metric space (X, d). Consider any sequence (x_n) in K. We will show that there exists a subsequence of (x_n) that converges to a point in K.

Since K is compact, for each integer $m \geq 1$, we can cover K with finitely many open balls of radius $\frac{1}{m}$. By the pigeonhole principle, there exists at least one ball that contains infinitely many terms of the sequence (x_n). We can select a subsequence (x_{n_k}) that lies entirely within this ball. Repeating this process for each m, we obtain a nested sequence of balls with radii tending to zero. The intersection of these balls contains exactly one point, say $x \in K$. The subsequence (x_{n_k}) converges to x.

(Sequentially Compact $\implies$ Compact): Let $K \subset X$ be a sequentially compact subset of a metric space (X, d). We will show that every open cover of K has a finite subcover.

Proof. The implication "compact $\implies$ sequentially compact" holds in any metric space (and indeed in any first-countable space). We prove the nontrivial direction:

Sequentially compact $\implies$ compact.

Lemma 2.1 (Sequentially compact $\implies$ totally bounded)**.** If K is sequentially compact, then K is totally bounded.

Proof. Suppose K is not totally bounded. Then there exists $\varepsilon > 0$ such that no finite collection of ε-balls covers K. Construct a sequence (x_n) inductively: pick $x_1 \in K$; having chosen $x_1, \ldots, x_n$, pick

$$x_{n+1} \in K \setminus \bigcup_{i=1}^{n} B(x_i, \varepsilon).$$

By construction, $d(x_i, x_j) \geq \varepsilon$ for all $i \neq j$. This sequence has no Cauchy subsequence, hence no convergent subsequence, contradicting sequential compactness. Therefore K is totally bounded. $\square$

Lemma 2.2 (Sequentially compact $\implies$ complete)**.** If K is sequentially compact, then K is complete (as a metric subspace).

Proof. Let (x_n) be a Cauchy sequence in K. By sequential compactness, it has a convergent subsequence $x_{n_k} \to x \in K$. A Cauchy sequence with a convergent subsequence converges to the same limit, so $x_n \to x \in K$. Hence K is complete. $\square$

Lemma 2.3 (Totally bounded and complete $\implies$ compact)**.** In a metric space, if K is totally bounded and complete, then K is compact.

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of K. First we show the existence of a Lebesgue number: there exists $\delta > 0$ such that every subset of K with diameter less than δ is contained in some U_α.

If not, then for each n there exists $C_n \subset K$ with $\operatorname{diam}(C_n) < \frac{1}{n}$ not contained in any U_α. Pick $x_n \in C_n$. By sequential compactness (which follows from total boundedness and completeness, but here we already have it from hypothesis), some subsequence $x_{n_k} \to x \in K$. Choose α with $x \in U_\alpha$, and choose $r > 0$ such that $B(x, r) \subset U_\alpha$. For large k, $\operatorname{diam}(C_{n_k}) < \frac{1}{n_k} < r/2$ and $d(x_{n_k}, x) < r/2$, hence $C_{n_k} \subset B(x, r) \subset U_\alpha$, contradiction. So a Lebesgue number $\delta > 0$ exists.

Now, total boundedness gives a finite $\frac{\delta}{3}$-net $\{y_1, \ldots, y_m\} \subset K$. Each ball $B(y_i, \delta/3)$ has diameter $< \delta$, hence lies in some U_{α_i}. Since the balls cover K, the sets $U_{\alpha_1}, \ldots, U_{\alpha_m}$ form a finite subcover. Hence K is compact. $\square$

By the three lemmas, if K is sequentially compact, then it is totally bounded and complete, hence compact. $\square$

Therefore, every open cover of K must have a finite subcover, and thus K is compact. $\square$