Functional Analysis

Tutorial 1

1 Course Information

1.1 Grading

Grades will be a weighted combination of the scores of homework assignments (20%), a couple of bi-weekly quizzes (15%), a midterm exam (25%) and a final exam (40%). The exam schedule is:

- Midterm Exam: 25th March, in class.
- Final Exam: 6th May, in class.

1.2 Homework

There will be weekly homework assignments. The homework will be posted on the bb, and will be due at the beginning of tutorial (TBD) on the due date. Late homework will not be accepted.

1.3 Contact Information

- TA Name: 辛起 (Qi Xin)
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  - Office Hours: 5:15-6:15pm, Wednesday, ZR322 (or by appointment).
  - Appointment in advance is highly recommended.

1.4 Textbook

- **Main Textbook:** *Introductory Functional Analysis with Applications*, Erwin Kreyszig, Wiley, 1978.
- **Reference Books (My personal recommendation):**
  - 泛函分析讲义, 张恭庆，林源渠编著，高等教育出版社， 2004.
  - *Functional Analysis*, Yosida, Springer, 6th edition, 1980.

2 Quiz Problem

Proposition 2.1. *The space $C[a,b]$ with the metric*

$$d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|$$

is complete, but is incomplete with the metric

$$d_1(f,g) = \int_a^b |f(x) - g(x)|\, dx.$$

Proof. Let (f_n) be a Cauchy sequence in $C[a,b]$ with respect to the metric d. For each $x \in [a,b]$, the sequence $(f_n(x))$ is a Cauchy sequence in $\mathbb{C}$, and hence converges to some limit $f(x)$. Define the function $f : [a,b] \to \mathbb{C}$ by $f(x) = \lim_{n\to\infty} f_n(x)$. We will show that f is continuous and that $f_n \to f$ in $C[a,b]$. Since (f_n) is Cauchy, for every $\epsilon > 0$, there exists an integer N such that for all $m, n \geq N$, we have

$$d(f_n, f_m) = \max_{x\in[a,b]} |f_n(x) - f_m(x)| < \epsilon.$$

Taking the limit as $m \to \infty$, we get

$$\max_{x\in[a,b]} |f_n(x) - f(x)| \leq \epsilon.$$

Thus f_n converges uniformly to f, and hence f is continuous. Therefore, $C[a,b]$ is complete with respect to the metric d.

To show that $C[a,b]$ is incomplete with respect to the metric d_1, consider the sequence of functions (f_n) defined by

$$f_n(x) = \begin{cases} nx, & 0 \leq x \leq \frac{1}{n}, \\ 1, & \frac{1}{n} < x \leq 1 \end{cases}.$$

Each f_n is continuous on $[0,1]$. We will show that (f_n) is a Cauchy sequence in $C[0,1]$ with respect to the metric d_1, but does not converge to any function in $C[0,1]$. For $m, n \geq 1$, we have

$$d_1(f_n, f_m) = \int_0^1 |f_n(x) - f_m(x)|\, dx \leq \int_0^{\max(\frac{1}{n},\frac{1}{m})} |f_n(x) - f_m(x)|\, dx \leq \max(\frac{1}{n}, \frac{1}{m}).$$

Thus, for every $\epsilon > 0$, there exists an integer N such that for all $m, n \geq N$, we have

$$d_1(f_n, f_m) < \epsilon.$$

Therefore, (f_n) is a Cauchy sequence in $C[0,1]$ with respect to the metric d_1. However, the pointwise limit of (f_n) is the function

$$f(x) = \begin{cases} 0, & x = 0, \\ 1, & 0 < x \leq 1 \end{cases},$$

which is not continuous on $[0,1]$. Hence, (f_n) does not converge to any function in $C[0,1]$ with respect to the metric d_1. This shows that $C[a,b]$ is incomplete with respect to the metric d_1. $\square$

3 Tutorial Problems

Example 3.1. *Minkowski's inequality states that for any $p \geq 1$ and any sequences $x = (\xi_j)$ and $y = (\eta_j)$ in ℓ^p, we have*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Proof of Minkowski's Inequality. We prove the Minkowski's inequality in three steps.

- **Step 1:** Prove Young's inequality: for any $a, b \geq 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$
Hint: Use the convexity of the exponential function.

- **Step 2:** Prove Hölder's inequality: for any sequences of non-negative numbers (a_j) and (b_j), we have
$$\sum_{j=1}^{\infty} a_j b_j \leq \left(\sum_{j=1}^{\infty} a_j^p\right)^{1/p} \left(\sum_{j=1}^{\infty} b_j^q\right)^{1/q}.$$

- **Step 3:** Use Hölder's inequality to prove Minkowski's inequality: for any sequences $x = (\xi_j)$ and $y = (\eta_j)$ in ℓ^p, we have
$$\left(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{1/p} + \left(\sum_{j=1}^{\infty} |\eta_j|^p\right)^{1/p}.$$

For step 1, we can use the convexity of the exponential function to show that for any $t \in [0, 1]$,

$$e^{t \ln a + (1-t) \ln b} \leq t e^{\ln a} + (1-t) e^{\ln b} = ta + (1-t)b.$$

Setting $t = \frac{1}{p}$ and rearranging the terms gives Young's inequality.

For step 2, let $A = \left(\sum_{j=1}^{\infty} a_j^p\right)^{1/p}$ and $B = \left(\sum_{j=1}^{\infty} b_j^q\right)^{1/q}$, and define normalized sequences $\tilde{a}_j = \frac{a_j}{A}$ and $\tilde{b}_j = \frac{b_j}{B}$. Applying Young's inequality to each term $\tilde{a}_j \tilde{b}_j$ and summing over j, we obtain

$$\sum_{j=1}^{\infty} \tilde{a}_j \tilde{b}_j \leq \frac{1}{p} \sum_{j=1}^{\infty} \tilde{a}_j^p + \frac{1}{q} \sum_{j=1}^{\infty} \tilde{b}_j^q = \frac{1}{p} + \frac{1}{q} = 1.$$

Multiplying both sides by AB gives Hölder's inequality.

Finally, for step 3, let $u_j = |\xi_j|$ and $v_j = |\eta_j|$. Then it suffices to show that

$$\left(\sum_{j=1}^{\infty}(u_j + v_j)^p\right)^{1/p} \leq \left(\sum_{j=1}^{\infty} u_j^p\right)^{1/p} + \left(\sum_{j=1}^{\infty} v_j^p\right)^{1/p}.$$

Notice that

$$(u_j + v_j)^p = (u_j + v_j)^{p-1}u_j + (u_j + v_j)^{p-1}v_j.$$

Applying Hölder's inequality to the sequences (u_j) and $((u_j + v_j)^{p-1})$, and similarly to (v_j) and $((u_j + v_j)^{p-1})$, we get

$$\sum_{j=1}^{\infty}(u_j+v_j)^p \leq \left(\sum_{j=1}^{\infty} u_j^p\right)^{1/p} \left(\sum_{j=1}^{\infty}(u_j + v_j)^p\right)^{(p-1)/p} + \left(\sum_{j=1}^{\infty} v_j^p\right)^{1/p} \left(\sum_{j=1}^{\infty}(u_j + v_j)^p\right)^{(p-1)/p}.$$

If $\sum_{j=1}^{\infty}(u_j + v_j)^p = 0$, the inequality is trivial. Otherwise, dividing both sides by $\left(\sum_{j=1}^{\infty}(u_j + v_j)^p\right)^{(p-1)/p}$, we obtain

$$\left(\sum_{j=1}^{\infty}(u_j + v_j)^p\right)^{1/p} \leq \left(\sum_{j=1}^{\infty} u_j^p\right)^{1/p} + \left(\sum_{j=1}^{\infty} v_j^p\right)^{1/p},$$

which completes the proof of Minkowski's inequality. □

Definition 3.1 (Convergence)**.** *Let $(X, \mathcal{T})$ be a topological space. A sequence (x_n) in X is said to converge to a point $x \in X$ if for every open set $U \in \mathcal{T}$ containing x, there exists an integer N such that for all $n \geq N$, we have $x_n \in U$. We denote this by $x_n \to x$.*

One can use sequences to characterize topological notions.

1. A point x_0 is a limit point of M if and only if there exists a sequence (x_n) in M such that $x_n \to x_0$ and $x_n \neq x_0$ for all n.

2. Let $F \subset X$ be closed. A point x is in F if and only if there exists a sequence (x_n) in F such that $x_n \to x$.

3. A function $f : X \to Y$ between two topological spaces is continuous if and only if for every sequence (x_n) in X that converges to a point $x \in X$, the sequence $(f(x_n))$ converges to $f(x)$ in Y.

4. A set M is dense in X if and only if for every point $x \in X$, there exists a sequence (x_n) in M such that $x_n \to x$.

5. A function $f : X \to Y$ between two topological spaces is continuous if and only if

$$x_n \to x \text{ in } X \implies f(x_n) \to f(x) \text{ in } Y.$$

Remark 3.1. *Notice that a single point space has no limit points.*

Definition 3.2 (Compactness)**.** *A subset $K \subset X$ is called sequentially compact if every sequence in K has a subsequence that converges to a point in K.*

A subset $K \subset X$ is called compact if every open cover of K has a finite subcover. That is, for any collection of open sets $\{U_\alpha\}_{\alpha \in A}$ such that

$$K \subset \bigcup_{\alpha \in A} U_\alpha,$$

there exists a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset A$ such that

$$K \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_n}.$$

Proposition 3.1. *In a metric space, a subset is compact if and only if it is sequentially compact.*

The proof is skipped in class but provided here for completeness.

Proof. (Sketch) We will prove both directions of the equivalence.

(Compact $\implies$ Sequentially Compact): Let $K \subset X$ be a compact subset of a metric space (X, d). Consider any sequence (x_n) in K. We will show that there exists a subsequence of (x_n) that converges to a point in K.

Since K is compact, for each integer $m \geq 1$, we can cover K with finitely many open balls of radius $\frac{1}{m}$. By the pigeonhole principle, there exists at least one ball that contains infinitely many terms of the sequence (x_n). We can select a subsequence (x_{n_k}) that lies entirely within this ball. Repeating this process for each m, we obtain a nested sequence of balls with radii tending to zero. The intersection of these balls contains exactly one point, say $x \in K$. The subsequence (x_{n_k}) converges to x.

(Sequentially Compact $\implies$ Compact): Let $K \subset X$ be a sequentially compact subset of a metric space (X, d). We will show that every open cover of K has a finite subcover.

Suppose, for the sake of contradiction, that there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of K that does not have a finite subcover. We can construct a sequence (x_n) in K such that each x_n is not contained in any finite union of the sets in the cover. Since K is sequentially compact, there exists a subsequence (x_{n_k}) that converges to some point $x \in K$. However, since $\{U_\alpha\}_{\alpha \in A}$ is an open cover of K, there exists some $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$. By the definition of convergence, there exists an integer N such that for all $k \geq N$, we have $x_{n_k} \in U_{\alpha_0}$. This contradicts our construction of the sequence (x_n).

Therefore, every open cover of K must have a finite subcover, and thus K is compact. □

Remark 3.2. *The equivalence between compactness and sequential compactness does not hold in general topological spaces. There exist topological spaces that are compact but not sequentially compact, and vice versa.*

Theorem 3.1. *Let (X, d) be a metric space and $K \subset X$ be compact. Then K is closed and bounded.*

Proposition 3.2. *In $\mathbb{R}^n$ or $\mathbb{C}^n$ with the standard Euclidean metric, a subset is compact if and only if it is closed and bounded.*

Proof. This is a direct consequence of the Heine-Borel theorem. □

However, the above proposition does not hold in general metric spaces. For example, in the space ℓ^∞, the closed unit ball

$$B = \{x \in \ell^\infty \mid d(x, 0) \leq 1\}$$

is closed and bounded but not compact.

The following result holds in general topological space.

Theorem 3.2. *Let K be a compact (resp. sequentially compact) subset of a topological space X and F be a closed subset of X. Then F is also compact (resp. sequentially compact).*