FIN512: Problem Set 11

Question 1

Part (a):

The risk-neutral probabilities are:

$$p = \frac{e^{r\Delta t} - d}{u - d}, \quad 1 - p = \frac{u - e^{r\Delta t}}{u - d}$$

$$\Delta t = \frac{T}{N} = 0.5, \quad e^{r\Delta t} = e^{0.05\times 0.5} = 1.02532$$

$$p = \frac{1.02532 - 0.8}{1.25 - 0.8} = 0.5007, \quad 1 - p = 0.4993$$

At maturity:

$$S_u = S_0 \cdot u = 100 \cdot 1.25 = 125, \quad S_d = S_0 \cdot d = 100 \cdot 0.8 = 80$$

$$P_u = \max(K - S_u, 0) = \max(105 - 125, 0) = 0, \quad P_d = \max(K - S_d, 0) = \max(105 - 80, 0) = 25$$

$$P_0 = e^{-r\Delta t} \cdot (p \cdot P_u + (1 - p) \cdot P_d)$$

$$P_0 = e^{-0.05\cdot 0.5} \cdot (0.5 \cdot 0 + 0.5 \cdot 25) = 0.975309 \cdot 12.5 = 12.174$$

$$\Delta = \frac{P_u - P_d}{S_u - S_d} = \frac{0 - 25}{125 - 80} = -0.5556$$

Part (b): Value of a European Put (N = 2)

$$\Delta t = \frac{T}{N} = 0.25, \quad e^{r\Delta t} = e^{0.05\times 0.25} = 1.012$$

$$p = \frac{1.012 - 0.8}{1.25 - 0.8} = 0.4724, \quad 1 - p = 0.5276$$

At $t = 0.25$:

$$S_{uu} = S_0 \cdot u^2 = 100 \cdot (1.25)^2 = 156.25, \quad S_{ud} = S_{du} = S_0 \cdot u \cdot d = 100 \cdot 1.25 \cdot 0.8 = 100$$

$$S_{dd} = S_0 \cdot d^2 = 100 \cdot (0.8)^2 = 64$$

$$P_{uu} = \max(K - S_{uu}, 0) = \max(105 - 156.25, 0) = 0$$

$$P_{ud} = P_{du} = \max(K - S_{ud}, 0) = \max(105 - 100, 0) = 5$$

$$P_{dd} = \max(K - S_{dd}, 0) = \max(105 - 64, 0) = 41$$

At $t = 0.25$:

$$P_u = e^{-r\Delta t} \cdot (p \cdot P_{uu} + (1 - p) \cdot P_{ud}) = 2.6052$$

$$P_d = e^{-r\Delta t} \cdot (p \cdot P_{ud} + (1 - p) \cdot P_{dd}) = 23.6957$$

At $t = 0$:

$$P_0 = e^{-r\Delta t} \cdot (p \cdot P_u + (1 - p) \cdot P_d) = 13.5620$$

$$\Delta_u = \frac{P_{uu} - P_{ud}}{S_{uu} - S_{ud}} = \frac{0 - 5}{156.25 - 100} = -0.089$$

$$\Delta_d = \frac{P_{ud} - P_{dd}}{S_{ud} - S_{dd}} = \frac{5 - 41}{100 - 64} = -1$$

$$\Delta = \frac{P_u - P_d}{S_u - S_d} = \frac{2.6052 - 23.6957}{125 - 80} = -0.4687$$

Part (c):

At $t = 0.25$**:**

$$S_u = S_0 \cdot u = 100 \cdot 1.25 = 125, \quad S_d = S_0 \cdot d = 100 \cdot 0.8 = 80$$

At $t = 0.5$**:**

$$S_{uu} = S_u \cdot u = 125 \cdot 1.25 = 156.25, \quad S_{ud} = S_u \cdot d = 125 \cdot 0.8 = 100$$

$$S_{du} = S_d \cdot u = 80 \cdot 1.25 = 100, \quad S_{dd} = S_d \cdot d = 80 \cdot 0.8 = 64$$

At maturity ($t = 0.5$), the payoff of the American put option is:

$$P_{ij} = \max(K - S_{ij}, 0)$$

$$P_{uu} = \max(105 - 156.25, 0) = 0, \quad P_{ud} = \max(105 - 100, 0) = 5$$

$$P_{du} = \max(105 - 100, 0) = 5, \quad P_{dd} = \max(105 - 64, 0) = 41$$

For the u**-node:**

$$P_u = e^{-r\Delta t} \cdot (p \cdot P_{uu} + (1-p) \cdot P_{ud}) = 0.987577 \cdot (0.472 \cdot 0 + 0.528 \cdot 5)$$

$$P_u = 0.987577 \cdot 2.64 = 2.61$$

Immediate exercise value:

$$P_u^{\text{exercise}} = \max(K - S_u, 0) = \max(105 - 125, 0) = 0$$

Since $P_u > P_u^{\text{exercise}}$, no early exercise occurs at the u-node.

For the d**-node:**

$$P_d = e^{-r\Delta t} \cdot (p \cdot P_{du} + (1-p) \cdot P_{dd}) = 0.987577 \cdot (0.472 \cdot 5 + 0.528 \cdot 41)$$

$$P_d = 0.987577 \cdot 23.41 = 23.11$$

Immediate exercise value:

$$P_d^{\text{exercise}} = \max(K - S_d, 0) = \max(105 - 80, 0) = 25$$

Since $P_d^{\text{exercise}} > P_d$, early exercise occurs at the d-node.

Option Value at $t = 0$

$$P_0 = e^{-r\Delta t} \cdot (p \cdot P_u + (1-p) \cdot P_d) = 0.987577 \cdot (0.472 \cdot 2.61 + 0.528 \cdot 25)$$

$$P_0 = 0.987577 \cdot 14.78 = 14.242$$

- American put option value at $t = 0$: $P_0 = 14.242$
- American put option value at $t = 0.25$:
  - u-node: $P_u = 2.61$ (no early exercise)
  - d-node: $P_d = 25$ (early exercise occurs)

Part (d): Value of an American Put (N = 2)

The risk-neutral probabilities are:

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}$$

$$\Delta t = \frac{T}{N} = 0.5$$

$$p = \frac{1.0 - 0.8}{1.25 - 0.8} = 0.4444, \quad 1 - p = 0.5556$$

At $t = 0.25$

$$S_u = S_0 \cdot u = 100 \cdot 1.25 = 125, \quad S_d = S_0 \cdot d = 100 \cdot 0.8 = 80$$

At $t = 0.5$:

$$S_{uu} = S_u \cdot u = 125 \cdot 1.25 = 156.25, \quad S_{ud} = S_u \cdot d = 125 \cdot 0.8 = 100$$

$$S_{du} = S_d \cdot u = 80 \cdot 1.25 = 100, \quad S_{dd} = S_d \cdot d = 80 \cdot 0.8 = 64$$

At maturity ($t = 0.5$), the payoff of the put option is:

$$P_{ij} = \max(K - S_{ij}, 0)$$

$$P_{uu} = \max(105 - 156.25, 0) = 0, \quad P_{ud} = \max(105 - 100, 0) = 5$$

$$P_{du} = \max(105 - 100, 0) = 5, \quad P_{dd} = \max(105 - 64, 0) = 41$$

For European, at $t = 0.25$:

$$P_u = e^{-r\Delta t} \cdot (p \cdot P_{uu} + (1-p) \cdot P_{ud}) = 0.987577 \cdot (0.4444 \cdot 0 + 0.5556 \cdot 5) = 2.743$$

$$P_d = e^{-r\Delta t} \cdot (p \cdot P_{du} + (1-p) \cdot P_{dd}) = 0.987577 \cdot (0.4444 \cdot 5 + 0.5556 \cdot 41) = 24.691$$

At $t = 0$:

$$P_0 = e^{-r\Delta t} \cdot (p \cdot P_u + (1-p) \cdot P_d) = 0.987577 \cdot (0.4444 \cdot 2.743 + 0.5556 \cdot 24.691) = 14.750$$

For American, at $t = 0.25$, compare the immediate exercise value ($K - S$) to the continuation value (P):

$$P_u = \max(105 - 125, 2.743) = 2.743, \quad P_d = \max(105 - 80, 24.691) = 25$$

At $t = 0$,

$$P_0 = e^{-r\Delta t}(p * P_u + (1-p) * P_d) = 14.920$$

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Question 2

European Put Option Price (N = 100): 13.90, a littble bit higher than question 1 b) : 12.19 American Put Option Price (N = 100): 14.21, a littble bit lower than question 1 c) :14.242 The steps keep increasing, however the error between 2 prices didn't converge and maintains at 0.3. —

Question 3

Part (a): Optimal Strategy

Using dynamic programming:

$$V_t(x) = \max(10 \cdot \text{x}, \mathbb{E}[V_t(x)])$$

$$V_3(x) = 10 * x, x = [1, 2, 3, 4, 5, 6]$$

$$V_2(x) = max(10 * x, 35)$$

At t=2, if x$\geq$ 4, stop.

$$V_1(x) = max(10 * x, E(V_1(x))$$

$$E(V_2(x)) = \frac{1}{6}(35 + 35 + 35 + 40 + 50 + 60) = 42.5$$

At t=1, if $x \geq 5$, stop.

Part (b): Fair Price

$$E(V_1(x)) = \frac{1}{6}(42.5 * 5 + 50 + 60) = 46.67$$

Part (c): Impact of Additional Throws

When number of thorws increases, the optimal stop point for each throw increases, and the expected return of this game would increase due to more times of choice. However, the entrance fee also increases. —