# Midterm exam

Fall 2024

**Financial computing** Monday 10/21

This exam has four questions.

You are expected to pick and solve three of the four questions.

All three questions you pick will be equally weighted.

The exam is **a take-home exam**.

It is due by class time (3:00 pm Central time) on Monday 10/28.

You are **not** allowed to collaborate or communicate with others about the questions in this exam.

You are **perfectly allowed** to discuss and collaborate with me!

Answer every question to the best of your knowledge.

Make sure to show your work for **partial credit**.

Good luck!

### Question 1: Linear algebra and the Simplex method

(a) In class, we wrote a basic version of the Simplex method. In this question, we ask you to add some functionality to it: specifically, we want you to implement one method to identify a starting basic feasible solution for Simplex. The method is called the big-$M$ method. Assume that you are provided with a constraint matrix in the form of $Ax = b$, which includes some row (let it be the $i$-th row) of the form $x_1 - x_2 - x_3 = 3$. Then, it can be converted into the form of $x_1 - x_2 - x_3 + y_i = 3$, where $y_i$ is an artificial variable with a very hight cost (big-$M$ cost). You can now begin Simplex using the basis consisting of $y_1, y_2, \ldots, y_m$. After some iterations, all $y$ variables should be out of the basis, if the problem is feasible. The first such basis is the actual initial basis of Simplex! For example, consider the following linear program:

$$\begin{aligned} \min \quad & 7x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 3 \\ & -2x_1 + x_2 - x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Now, convert it into:

$$\begin{aligned} \min \quad & 7x_1 + 3x_2 + My_1 + My_2 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + y_1 = 3 \\ & -2x_1 + x_2 - x_4 + y_2 = 1 \\ & x_1, x_2, x_3, x_4, y_1, y_2 \geq 0. \end{aligned}$$

Then, the basis consisting of $y_1, y_2$ is feasible, and after iterating twice, we would remove $y_1, y_2$ in favor of a basis consisting of $x_2, x_3$. **Add the method `findInitialBasis()` to the Simplex class we wrote in class.**

(b) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \\ x & x & x & x & 0 \\ 0 & 0 & 0 & 0 & y \end{bmatrix},$$

where $x$ and $y$ are some real numbers (between $-\infty$ and $+\infty$). Answer the following questions:

- What are the values of $x$ and $y$ that minimize the rank of the matrix? What is the minimum rank?
- What are the values of $x$ and $y$ that maximize the rank of the matrix? What is the maximum rank?
- Can $Av = b$ have a solution, if $b = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$ have a solution? For which values of $x$ and $y$ does it have a solution?
- Can $Av = b$ have a solution, if $b = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^T$ have a solution? For which values of $x$ and $y$ does it have a solution?
- Can $Av = b$ have a solution, if $b = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ have a solution? For which values of $x$ and $y$ does it have a solution?

### Question 2: Portfolio optimization and side constraints

In class, we discussed the traditional portfolio optimization model and provided simpler and more advanced formulations for solving this. In this exam question, we ask you to build your own portfolio optimization model that also takes into consideration some other side constraints. You can continue using the `portfolio.txt` we also used in class. That said, you will also need access to a `extra_information.txt` file, which contains some additional details and requirements for each of the stocks available to you.

(a) Our first extension works as follows. Assume you have a traditional portfolio optimization model, where you would like to minimize the total risk (covariance) given that you achieve a minimum (desired) expected return. However, you can assume that we are interested in minimizing our total exposure to specific fields (like transportation and logistics, software, pharmaceutical companies, etc.). To achieve that, we ask that each of the fields $f \in F$ only has at most $N_f$ stocks that we invest in. For example, assuming the existence of three fields and $N_1 = 1, N_2 = 3, N_3 = 0$, this means that we can have at most 1 stock from field 1, 3 stocks from field 2, and none of the stocks from field 3. You may assume that each stock is categorized as being in exactly one field. The field is marked with a character ('A', 'B', 'C', ...) in the `extra_information.txt` file. **Create the mathematical model for the problem. Be sure to mention your decision variables, constraints, and objective function. Then, solve the program for $N_f \in \{3, 5, 10\}$, for all $f \in F$.**

(b) The second extension builds on top of the portfolio optimization model that incorporates transaction fees. In class, we assumed the existence of a single transaction fee for each stock bought. In this extension, we ask you to have transaction fees that vary depending on the volume of the stock bought. We assume that each stock $i \in \mathcal{A}$ has three levels of transaction fees: a higher level of fees for investments (allocations) of $x_i \leq \ell_i$, a lower level of transaction fees for allocations of $x_i \geq u_i$, and an intermediate level of fees for allocations of $\ell_i \leq x_i \leq u_i$, where $x_i$ is the fraction of your budget invested in stock $i$. You may assume that for each stock we have that the levels satisfy $\ell_i < u_i$. This information is also available in the last two numbers in each row of the `extra_information.txt` file. **Once again, create the mathematical model for the problem. Be sure to properly define your decision variables, constraints, and objective function. Then, solve the program for the data in the file.**

### Question 3: Collections of bonds convexity and duration

(a) In class, we discussed bond convexity and duration and wrote some useful `C++` functionality to help us calculate these properties for various bonds. Specifically, consider the information in Table 1. Assume you have a future obligation of $Q = \$100000$ in $N = 10$ years. **What is the optimal allocation** you can do using the bonds in the Table so as to fully immunize yourselves for that future obligation, while maximizing the convexity of the bond portfolio? Be sure to use the two-bond theorem to find the best allocation.

| **Bond** | B1 | B2 | B3 | B4 | B5 | B6 | B7 | B8 | B9 | B10 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Duration** | 6 | 6 | 8 | 9 | 9.5 | 9.7 | 10 | 12 | 12.2 | 14 |
| **Convexity** | 4 | 3 | 5 | 5.2 | 5.2 | 5.5 | 3 | 5 | 2.7 | 2.5 |

Table 1: The collection of bonds you can use to create a portfolio.

(b) Now, assume that you are provided with a series of bonds $I$ and their information (i.e., their maturity ages, their coupon rates, and their value). However, further assume that the bonds are separated in groups of bonds: these groups are supposed to be bought *together* in the given allocation. Assume that each collection includes a portion of $\alpha_i^c$ of each bond. For example, if we had 5 bonds ($i = 1, 2, 3, 4, 5$), then collection $c$ could have an allocation $\alpha_1^c = 0.3, \alpha_2^c = 0.0, \alpha_3^c = 0.2, \alpha_4^c = 0.4, \alpha_5^c = 0.1$ (that is, 30% bond 1, 20% bond 3, 40% bond 4, 10% bond 5). You may use that $\sum_{i \in I} \alpha_i^c = 1$, for all collections $c$.
Based on our discussion in class, we can write a linear program to find the optimal allocation in the case of single bonds. Is that also true for collections of bonds? **Write the linear program explaining your variables, constraints, and objective function. Also: is the two-bond theorem (extended to a two-collection theorem) still valid in this case? Why or why not?**

### Question 4: A "network of stocks" problem

(a) Assume that for any two stocks, we have their covariance $\sigma_{ij}^2$. Build a network of stocks such that two stocks are adjacent (connected by an edge) if their covariance is higher than a threshold ($\sigma_{ij}^2 > t$). We say that a group of stocks is "disparate", if all stocks in the group are not adjacent to each other. For example, consider stocks $A, B, C, D, E, F$ such that $\sigma_{AB}^2 = \sigma_{AE}^2 = \sigma_{BD}^2 = \sigma_{BE}^2 = \sigma_{CE}^2 = \sigma_{DE}^2 = \sigma_{EF}^2 = 0.8$ and all other $\sigma_{ij}^2$ are 0.1. Then, the group of $B$, $C$, and $F$ are "disparate" for $t = 0.5$, since $B, C, F$ are not adjacent in the network. See Figure 1 for the graph that can be created using a threshold $t = 0.5$.

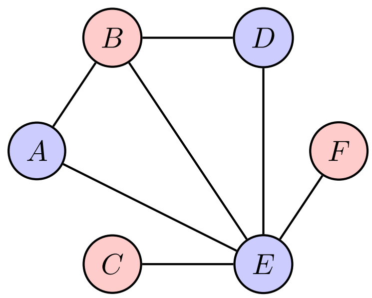


Figure 1: An example of a "disparate" group of stocks for a threshold of $t = 0.5$, shown in red. Note that a larger group of "disparate" stocks can be produced in the form of $A, C, D, F$.

**Create the mathematical model of detecting the "disparate" group of stocks with the largest number of stocks.**

(b) Assume that for any two stocks, we have their covariance $\sigma_{ij}^2$. Build a network of stocks such that two stocks are adjacent (connected by an edge) if their covariance is higher than a threshold ($\sigma_{ij}^2 > t$). We say that a group of stocks is "friendly", if all stocks in the group are adjacent to each other. In Figure 2, we provide an example using the same correlations as in part (a).

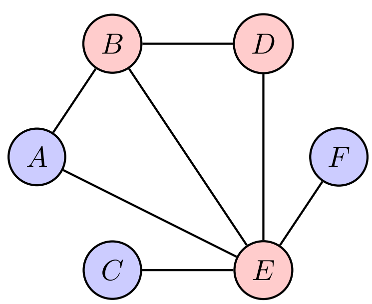


Figure 2: An example of a "friendly" group of stocks for a threshold of $t = 0.5$, shown in red. Note that this is also the largest "friendly" group that can be created in this network.

**Create the mathematical model of detecting the "friendly" group of stocks with the largest number of stocks.**