FIN500: Problem Set 2

Due: Monday September 13th before 11:59pm, please submit electronic copies through the assignment function on Canvas

Question 1: FRA replication (4 points)

Consider the following zero coupon bond prices on a principal of \$1.

Maturity	Price
0.25	0.98704503
0.50	0.97418412
0.75	0.96246391
1.00	0.95238095

You can ignore day count conventions in this question but assume that the rates are SOFR rates (i.e use simple compounding)

a) What are the forward rates, $F_0(0.25,0.5)$, $F_0(0.5,0.75)$, $F_0(0,75,1)$ and $F_0(0,5,1)$?

$$F_0(t_1,t_2) = \frac{1}{t_2-t_1}\left(\frac{1+r(0,t_2)\cdot t_2}{1+r(0,t_1)\cdot t_1} - 1\right) = \frac{1}{t_2-t_1}\left(\frac{P(0,t_1)}{P(0,t_2)} - 1\right)$$

$$F_0(0.25,0.5) = \frac{1}{0.5-0.25}\left(\frac{P(0,0.25)}{P(0,0.5)} - 1\right) = 0.0528$$

$$F_0(0.5,0.75) = \frac{1}{0.75-0.5}\left(\frac{P(0,0.5)}{P(0,0.75)} - 1\right) = 0.0487$$

$$F_0(0.75,1) = \frac{1}{1-0.75}\left(\frac{P(0,0.75)}{P(0,1)} - 1\right) = 0.0423$$

$$F_0(0.5,1) = \frac{1}{1-0.5}\left(\frac{P(0,0.5)}{P(0,1)} - 1\right) = 0.0458$$

Now Consider a (long) forward rate agreement where you will receive $r_{0.25}(0.25,0.5)$ at time 0.5 and pay a known fixed rate K, on a notional principal of \$1,000,000.

b) Following the techniques in class how you can recreate the payoff of the contract by entering into positions with the zero coupon bonds (this is called *replication*.). Determine a formula for the current value of the FRA (or bond portfolio)?

The payoff from the long (buyer) FRA is:

$$Payoff = N \times (r(T, T+\tau)\tau - K\tau) = 1000000(r_{0.25}(0.25,0.5) - K) \times 0.25$$

c) By using your answer to (b) or otherwise, calculate the **current ($t = 0$) value** of an existing FRA (i.e its value is not necessarily zero) on a principal of \$1,000,000, where in 9 months time you will receive $r_{0.5}(0.5,0.75)$ and pay a fixed rate K = 5.0%?

$$Payoff = 1000000(r_{0.5}(0.5,0.75) - 0.05) \times 0.25 = -425$$

d) What value of K will lead the FRA paying $r_{0.75}(0.75,1)$ and receiving K having a current value of 0?

$$Payoff = -N \times (r(T, T+\tau)\tau - K\tau) = 1000000\big(K - r_{0.5}(0.75,1)\big) = 0$$

$$K = r_{0.5}(0.75,1) = 0.0414$$

Question 2 (3 points)
We have the following information about term SOFR rates quoted on October 1, 2024:

Period of rate	End of deposit period (T)	Number of days between settlement and deposit	Spot Rate, r(0,T)
3 months	1 January 2025	90	5.25%
6 months	1 April 2025	180	5.35%
9 months	1 July 2025	270	5.25%
12 months	1 October 2025	360	5.20%

The SOFR day-count basis is actual/360. Assume that these day counts are correct DO NOT recount.

(a) What is the current price of a zero coupon bond maturing on 1 October 2025, i.e what is P(0, 1). (1 point)

$$P(0,1) = \frac{1}{(1+r)^1} = 0.9506$$

(b) What is the forward rate covering the period from the maturity of 9-month SOFR to the maturity of 12-month SOFR i.e. $F_0(0.75,1)$? Express the forward rate using the conventions of SOFR, that is simple interest and the actual/360 day-count convention.

$$F_0(0.75,1) = \frac{1}{1-0.75}\left(\frac{1+r(0,1)\cdot 1}{1+r(0,0.75)\cdot 0.75} - 1\right) = 0.0486$$

On October 1, 2024 you wish to hedge a 90-day \$50m **loan** which you will enter into on July 1, 2025 paying interest equal to three-month SOFR (i.e. r(0.75, 1)), on October 1, 2025.

There are forward rate agreements (FRAs) available on any notional principal with a fixed rate of 5.00% and a floating rate of 3-month SOFR rate set in 9 months (i.e r(0.75, 1))).

(c) If the 3-month SOFR spot rate (= r(0.75,1.0)) on August 1 is 5.75% show that you can use the FRA to lock in an interest payment equivalent to 5.00% on your investment.

$$\text{Interest Amount} = \text{Principal} \times \text{Rate} \times \frac{90}{360} = 718750$$

$$\text{FRA payoff} = (\text{SOFR Rate} - \text{FRA Rate}) \times \frac{90}{360} \times \text{Principal} = 93750$$

$$\text{Total Borrowing Costs} = -\text{Interest} + \text{FRA payoff} = 625000$$

$$\text{Which is the same with Principal} \times \text{FRA Rate} \times \frac{90}{360} = 625000$$

(d) On October 1, 2024, what would be the fair price to pay for the forward rate agreement above (fixed rate of 5.00%) on a principal of \$50m? **You should use your answer to part b) here.**

$$FRA_0 = N \times P(0, T + \tau)(F_0(T, T + \tau)\tau - K\tau)$$
$$= 50m \times P(0,1)(F_0(0.75,1) - K) \times 0.25 = -16635.5$$

(e) Do you expect that this FRA will have a positive or negative payoff at T + τ = 1? Explain your answer.

Assuming no significant market upheavals, it is more likely that the FRA will have a negative payoff for the buyer at T+τ=1. The buyer will probably end up paying the difference since the market rate is expected to be lower.

Question 3 (3 points)

(In these questions, don't worry about the day-count conventions, and ignore the fact that SOFR is actually quoted for settlement in 2 business days.)

It is October 1, 2024. You have the following rates and zero coupon bond prices:

Maturity date (T)	T in years (actual/360)	Zero Coupon Bond Price P(0,T)	Forward Rate, $F_0(T-0.5, T)$
4/1/25	0.5	0.97442	5.25%
10/1/25	1.0	0.95065	5.50%
4/1/26	1.5	0.92341	5.90%
10/1/26	2.0	0.90045	5.10%

(a) Fill in the missing entries in the table (i,ii)

Consider a swap contract making the following payments (in millions of dollars):

Time 0.5	Time 1.0	Time 1.5
0.5(K - r(0,0.5))×10	0.5(K - *r*(0.5,1.0))×10	0.5(K - *r*(1.0,1.5)) ×10

(c) Calculate the (par) swap rate for this swap.

$$PV_{Fixed\ Payment} = \sum 0.5 \times K \times 10 \times P(0,T)$$

$$PV_{Floating\ Payments} = \sum 0.5 \times r(t,t+0.5) \times 10 \times P(0,T)$$

Equating PV(Fixed) to PV(Floating), get K $= 0.0554$

Question 4 (3 points)

Consider a 1-year received-fixed, pay-floating interest rate swap with payments every 3 months based on 3-month SOFR. The fixed rate of the swap is 5.5% percent, the notional principal is USD 100 million, and a floating index value of 5.25 percent is to be used in calculating the first payment. The cash flows of the swap (in millions) are shown in the table below.

Time 0.25	Time 0.5	Time 0.75	Time 1
0.25(5.5%- 5.25%) ×100	0.25(5.5% - $r_{0.25}$(0.25,0.5) ×100	0.25(5.5%- $r_{0.5}$(0.5,0.75) ×100	0.25(5.5% - $r_{0.75}$(0.75,1) ×100

The current date is time 0. The swap was transacted 5 minutes ago, and interest rates have changed in the last 5 minutes, so that the current market value of the swap is no longer zero. In particular, the current forward rates are $F_0(0,0.25) = 5.30\%$, $F_0(0.25,0.5) = 5.35\%$, $F_0(0.5,0.75) = 5.45\%$, and $F_0(0.75,1) = 5.55\%$. These forward rates are expressed on an annual basis, with simple compounding.

What is the current market value of the interest rate swap?

$$F_0(t_1, t_2) = \frac{1}{t_2 - t_1}\left(\frac{1 + r(0,t_2)\cdot t_2}{1 + r(0,t_1)\cdot t_1} - 1\right)$$

$$r(0, 0.25) = 0.053$$

$$r(0, 0.5) = \frac{\left((t_2-t_1)\cdot F_0(t_1,t_2)+1\right)\left(r(0,t_1)\cdot t_1+1\right) - 1}{t_2} = 0.0536$$

$$r(0, 0.75) = \frac{\left((t_2-t_1)\cdot F_0(t_1,t_2)+1\right)\left(r(0,t_1)\cdot t_1+1\right) - 1}{t_2} = 0.0544$$

$$r(0, 1) = \frac{\left((t_2-t_1)\cdot F_0(t_1,t_2)+1\right)\left(r(0,t_1)\cdot t_1+1\right) - 1}{t_2} = 0.0552$$

$$PV_{swap} = \frac{0.25(5.5\% - 5.25\%)\times 100}{(1+0.25\times r(0,0.25))} + \frac{0.25(5.5\% - 5.35\%)\times 100}{(1+0.5\times r(0,0.5))} + \frac{0.25(5.5\% - 5.45\%)\times 100}{(1+0.75\times r(0,0.75))} +$$

$$\frac{0.25(5.5\% - 5.55\%)\times 100}{(1+r(0,1))} = 0.0984\ million$$

Question 5 (2 points)

Consider a 2-year inverse floating rate note (a so-called "inverse floater") with a principal of N = \$100 million that pays interest every six months. The interest payment at time t is given by $0.5[0.04 - r_{t-0.5}(t - 0.5, t)]\, N$. The cash flows (per dollar of principal) of the note are:

Time 0.5	Time 1	Time 1.5	Time 2
$0.5[0.04 - r_0(0,0.5)]$	$0.5[0.04 - r_{0.5}(0.5,1)]$	$0.5[0.04 - r_1(1,1.5)]$	$0.5[0.04 - r_{1.5}(1.5,2)]+1$

The note is called an inverse floating rate note because the interest payments are negatively related to SOFR.

The quoted swap rate on a 2-year swap with semi-annual payments based on 6-month SOFR is 2 percent per year.

<u>What is the current (time 0) value of the inverse floating rate note? Please briefly explain.</u>

Hint: Interpret the inverse floating rate note as a portfolio of a fixed-rate note and a swap. You cannot use the valuation approach you used above, because you do not have enough information to compute the various forward rates.

$$1 = \tau s_0 P(0,T_1) + \tau s_0 P(0,T_2) + \cdots + \tau s_0 P(0,T_n) + P(0,T_n)$$
$$= \tau s_0 (P(0,T_1) + \cdots + P(0,T_n)) + P(0,T_n)$$

we can derive then the quoted swap rate by:

$$s_0 = \frac{1 - P(0,T_n)}{\tau \sum_{i=1}^{n} P(0,T_i)},\ where\ s_0 = 2\%$$

Thus,

$$2\% = \frac{1 - P(0,T_n)}{0.5 \sum_{i=1}^{n} P(0,T_i)} \qquad (*)$$

By (*), we have:

$$0.01 \sum_{i=1}^{n} P(0,T_i) + P(0,T_n) = 1 \qquad (**)$$

Therefore, the present value of the swap is:

$$PV_{swap} = par\ rate\ swap + \frac{0.5[0.02 - r_0(0,0.5)] \times 100}{(1 + 0.5 \times r(0,0.5))} + \frac{0.5[0.02 - r_{0.5}(0.5,1)] \times 100}{(1 + 0.5 \times r(0,1))}$$
$$+ \frac{0.5[0.02 - r_1(1,1.5)] \times 100}{(1 + 0.5 \times r(0,1.5))} + \frac{0.5[0.02 - r_{1.5}(1.5,2)] \times 100}{(1 + 0.5 \times r(0,2))}$$
$$= par\ rate\ swap + 100\left(0.02 \times 0.5 \sum_{i=1}^{n} P(0,T_i) + P(0,T_n)\right) = 100m\ , by\ (**)$$