FIN500 HW10

Junru Wang

Question 1: Buffered PLUS Securities

(a) Payoff Diagram

The payoff is calculated as:

$$\text{Payoff} = \begin{cases} 200 + 1000 \times \left(\frac{S_T}{S_0}\right), & \text{if } S_T < 0.8 \times S_0, \\ 1000, & \text{if } 0.8 \times S_0 \leq S_T \leq S_0, \\ 1000 + 2000 \times \frac{S_T - S_0}{S_0}, & \text{if } S_0 < S_T \leq 1.076 \times S_0, \\ 1152, & \text{if } S_T > 1.076 \times S_0. \end{cases}$$

(b) Replicating Portfolio

Let:

$$\begin{aligned} S_0 &= 5460.48 \quad \text{(initial index level)}, \\ K_1 &= 0.8 \times S_0 = 4368.38 \quad \text{(strike for put)}, \\ K_2 &= S_0 = 5460.48 \quad \text{(strike for the first call)}, \\ K_3 &= 1.076 \times S_0 = 5873.69 \quad \text{(strike for the second call)}. \end{aligned}$$

The portfolio weights are:

$$\begin{aligned} w_1 &= \frac{1000}{S_0} = 0.1831 \quad \text{(weight for the put option)}, \\ w_2 &= \frac{2000}{S_0} = 0.3662 \quad \text{(weight for the first call option)}, \\ w_3 &= \frac{2000}{S_0} = 0.3662 \quad \text{(weight for the second call option)}. \end{aligned}$$

The net payoff is given by:

$$\text{Net Payoff} = \text{Bond Payoff} + w_1 \times \text{Put Payoff} + w_2 \times \text{Call Payoff (K2)} + w_3 \times \text{Call Payoff (K3)}.$$

Each component is defined as:

$$\begin{aligned} \text{Put Payoff} &= -\max(K_1 - S_T, 0), \\ \text{Call Payoff (K2)} &= \max(S_T - K_2, 0), \\ \text{Call Payoff (K3)} &= -\max(S_T - K_3, 0). \end{aligned}$$

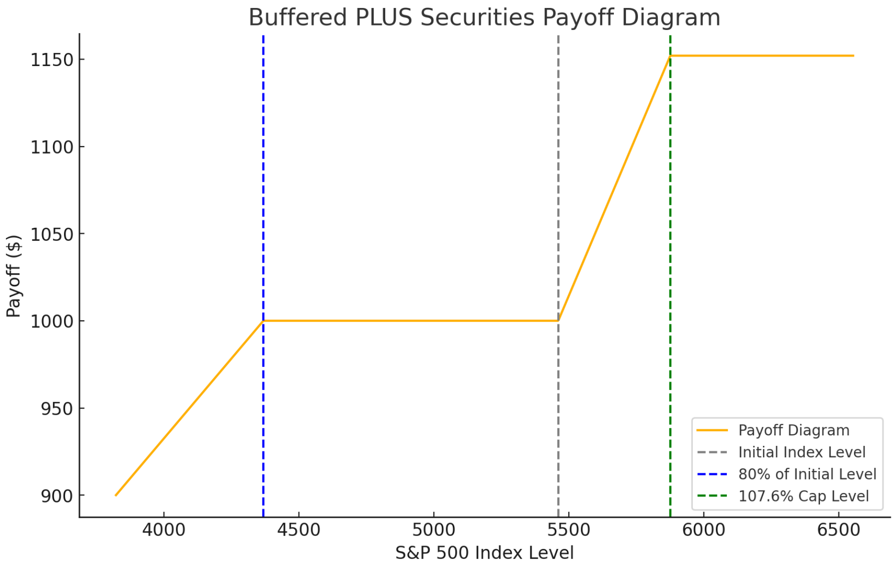

Figure 1: Payment

Conclusion

The replicating portfolio consists of:

- One zero-coupon bond maturing to $1,000.
- Sell 0.1831 European put option, with K = 4368.38.
- Buy 0.3662 European call option with K = 5460.48.
- Sell 0.3662 European put option with K = 5873.69.

Part (c): Payoff and Return Calculation

Scenario 1: $S_T = 5733.504$

- **Index percentage increase**:

$$\text{Index Percent Increase} = \frac{S_T - S_0}{S_0} = \frac{5733.504 - 5460.48}{5460.48} = 0.05\,(5\%).$$

- **Payoff**:

$$\text{Payoff} = 1000 + 2000 \times 0.05 = 1000 + 100 = 1100.$$

- **Return**:

$$\text{Return} = \frac{\text{Payoff} - \text{Investment}}{\text{Investment}} = \frac{1100 - 1000}{1000} = 0.1\,(10\%).$$

Scenario 2: $S_T = 6006.528$

- **Index percentage increase**:

$$\text{Index Percent Increase} = \frac{S_T - S_0}{S_0} = \frac{6006.528 - 5460.48}{5460.48} = 0.1\,(10\%).$$

- **Payoff (Capped)**:

$$\text{Payoff} = 1152 \quad (\text{Capped at } S_{\text{cap}}).$$

- **Return**:

$$\text{Return} = \frac{\text{Payoff} - \text{Investment}}{\text{Investment}} = \frac{1152 - 1000}{1000} = 0.152\,(15.2\%).$$

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Question 2: Early Exercise of American Call Option

Part (a): Cash Flows for Strategy A

At $t = 1$:

$$\text{Cash flow} = S_1 - K.$$

At $t = 2$:

$$\text{Cash flow} = 0.$$

Part (b): Cash Flows for Strategy B ($S_2 > K$)

If option executes:

At $t = 1$:

$$\text{Net cash flow} = S_1 - K.$$

At $t = 2$:

- **Payoff from option:**

$$\text{Payoff from option} = S_2 - K.$$

- **Forward Contract:** The forward price is given by:

$$F_{1,2} = S_1(1 + r).$$

- **Loan Repayment:**

$$\text{Loan repayment} = (S_1 - K)(1 + r).$$

- **Net Cash Flow:**

$$\text{Net cash flow} = (S_2 - K) + (S_1(1 + r) - S_2) - (S_1 - K)(1 + r).$$

Simplifying:

$$\text{Net cash flow} = Kr.$$

If the option expires:

At $t = 1$:

$$\text{Net cash flow} = S_1 - K.$$

At $t = 2$:

$$\text{Option payoff} = 0.$$

$$\text{Payoff from forward} = F_{1,2} - S_2.$$

- Loan repayment:

$$\text{Loan repayment} = (S_1 - K)(1 + r).$$

- Net Cash Flow:

$$\text{Net cash flow} = S_1(1 + r) - S_2 - (S_1 - K)(1 + r) = K(1 + r) - S_2 < Kr.$$

Therefore, Net cash flow should be Kr, in which we don't execute the option.

Part (c): Cash Flows for Strategy B ($S_2 \leq K$)

$$\text{Net cash flow} = S_1(1 + r) - S_2 - (S_1 - K)(1 + r) = K(1 + r) - S_2 \geq Kr.$$

Net cash flow should be $K(1 + r) - S_2$, in which we execute.

Part (d): Dividend Impact

At $t = 1$:

$$\text{Net cash flow} = S_1 - K.$$

At $t = 2$:

$$\text{Net cash flow B} = max(K(1 + r) - S_2, Kr) \geq Kr > 0.$$

If the stock pays a large dividend D at $t = 2$, the forward price becomes:

$$F_{1,2} = S_1(1 + r) - D.$$

At $t = 2$, S_2 is the price after dividend:

$$\text{Net cash flow B} = max(K(1 + r) - S_2 - D, Kr - D) \geq Kr - D.$$

If $D < Kr$, our conclusion still works.

Part (e): Early Exercise of American Put Option

Strategy A: At $t = 1$:

$$\text{Cash flow} = K - S_1.$$

At $t = 2$:

$$\text{Cash flow} = 0.$$

Strategy B: At $t = 1$, borrow $(K - S_1)$:

$$\text{Net cash flow} = K - S_1.$$

At $t = 2$

- **Payoff from option:**

$$\text{Payoff from option} = K - S_2.$$

- **Forward Contract:** Buy stocks at $t = 2$:

$$F_{1,2} = S_1(1 + r),$$

- **Loan Repayment:**

$$\text{Loan Repayment} = (K - S_1)(1 + r).$$

- **Net Cash Flow (if execute):**

$$\text{Net Cash Flow} = (K - S_2) + (S_2 - S_1(1 + r)) - (K - S_1)(1 + r).$$

  Simplifying:

$$\text{Net Cash Flow} = -Kr.$$

- **Net Cash Flow (if not execute):**

$$\text{Net Cash Flow} = (S_2 - S_1(1 + r)) - (K - S_1)(1 + r).$$

  Simplifying:

$$\text{Net Cash Flow} = S_2 - K(1 + r).$$

Therefore,

$$\text{Net cash flow} = max(S_2 - K(1 + r), -Kr).$$

If $S_2 < K(1 + r)$, we should exercise early, else we should delay it till t=2. —

Question 3: Expected Returns of Options

Part (a): Option Pricing

Using the parameters:

$$\begin{aligned}
S_0 &= 345 \quad \text{(Initial stock price)},\\
K &= 350 \quad \text{(Strike price)},\\
u &= e^{0.6\sqrt{0.5}} \approx 1.329 \quad \text{(Up factor)},\\
d &= e^{-0.6\sqrt{0.5}} \approx 0.752 \quad \text{(Down factor)},\\
r &= 0.05 \quad \text{(3-month interest rate)},\\
T &= 0.5 \quad \text{(Time to maturity in years)}.
\end{aligned}$$

The risk-neutral probability is:

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05 \cdot 0.5} - 0.752}{1.329 - 0.752} \approx 0.4245.$$

The stock prices at maturity are:

$$S_u = S_0 \cdot u = 345 \cdot 1.329 \approx 458.51,$$
$$S_d = S_0 \cdot d = 345 \cdot 0.752 \approx 259.44.$$

The option prices under risk-neutral probabilities are:

$$\text{Call Price} = e^{-rT} \left[p \cdot \max(S_u - K, 0) + (1-p) \cdot \max(S_d - K, 0) \right],$$

$$\text{Put Price} = e^{-rT} \left[p \cdot \max(K - S_u, 0) + (1-p) \cdot \max(K - S_d, 0) \right].$$

After substituting values, we get:

$$\text{Call Price} \approx 73.41, \quad \text{Put Price} \approx 69.76.$$

Part (b): Real-World Probabilities

The real-world probabilities are calculated using the true expected stock price:

$$\mathbb{E}[S_T] = S_0 \cdot e^{\mu T}, \quad \mathbb{E}[S_T] = 345 \cdot e^{0.09 \cdot 0.5} \approx 361.7.$$

$$p_{\text{real}}^{\text{up}} = \frac{\mathbb{E}[S_T] - S_0 \cdot d}{S_0 \cdot (u-d)}, \quad p_{\text{real}}^{\text{down}} = 1 - p_{\text{real}}^{\text{up}}.$$

Substituting the values:

$$p_{\text{real}}^{\text{up}} = \frac{361.7 - 345 \cdot 0.752}{345 \cdot (1.329 - 0.752)} \approx 0.448, \quad p_{\text{real}}^{\text{down}} \approx 0.552.$$

Part (c): Expected Returns

The expected payoff under real-world probabilities is:

$$E[\text{Call Payoff}] = e^{-rT} \left[p^* \cdot \max(S_u - K, 0) + q^* \cdot \max(S_d - K, 0) \right],$$

$$E[\text{Put Payoff}] = e^{-rT} \left[p^* \cdot \max(K - S_u, 0) + q^* \cdot \max(K - S_d, 0) \right].$$

Substituting values, we get:

$$E[\text{Call Payoff}] \approx 77.50, \quad E[\text{Put Payoff}] \approx 66.89.$$

It is not surprising. The real world probability of down case here is smaller than risk-neutral probability, therefore the put option price is smaller.

The price difference between the real-world and risk-neutral pricing methods for a put option is given by:

$$\Delta P = \frac{e^{(r_{\text{real}} - r)T} - 1}{u - d} \cdot \left[\max(K - S_u, 0) - \max(K - S_d, 0) \right],$$

Given $S_u < K < S_d, r_{real} > r$, we have $\Delta P < 0$, that is $P_{real} < P_{neutral}$.

Figure 2: Price Analysis

Part (d): Strike Price Analysis

The expected return varies with the strike price:

- For low strike prices ($K \to 0$), the call option becomes deep in-the-money, and the return converges to the real-world stock return.
- For high strike prices ($K \to \infty$), the call option becomes out-of-the-money, and the expected return decreases significantly due to low probabilities of payoff.
- Put option returns follow a similar trend in reverse: higher expected returns for lower strike prices and diminishing returns for higher strike prices.