Back to bonds: ① Convexity and Duration

② Swap Payoff $=\sum_{i=1}^{n}\tau N(r(T_{i-1},T_i)-S_0)\,p(0,T_i)$

$P_{Swap}=$

$S_0=\frac{1-p(0,T_n)}{\tau\sum_{i=1}^{n}p(0,T_i)} \to p(0,T_n)+\sum_{i=1}^{n}\tau_{T_i}\,p(0,T_i)=1$

浮动的现值=面值=1

$P_{Swap}=N(1-S_0\tau\sum_{i=1}^{n}p(0,T_i)-p(0,T_n))$

③ CMT: $S_{t_0}=\frac{\sum P(t_0,t_i)\,f_{t_0}(t_{i-1},t_i)}{\sum_{i=1}^{n}P(t_0,t_i)}$，远期利率加权

↓

"$r(0,t_n)$"

$$D=\frac{\sum_{t=1}^{nm}t\cdot\frac{c/m}{(1+\frac{y}{m})^{tm}}+n\cdot\frac{F}{(1+\frac{y}{m})^{n\cdot m}}}{\sum_{t=1}^{nm}\frac{c/m}{(1+\frac{y}{m})^{tm}}+\frac{F}{(1+\frac{y}{m})^{nm}}}=\frac{\sum t x_t+ny}{\sum x_t+y}$$

↓

$$\frac{1+\frac{y}{m}}{y}-\frac{1+\frac{y}{m}+n(\frac{c}{m}-\frac{y}{m})}{\frac{c}{m}((1+\frac{y}{m})^{n}-1)+y}-t/T$$

n：剩余期限数

m：每次间隔

① $P=\sum Ce^{-rt}+Fe^{-rT}$

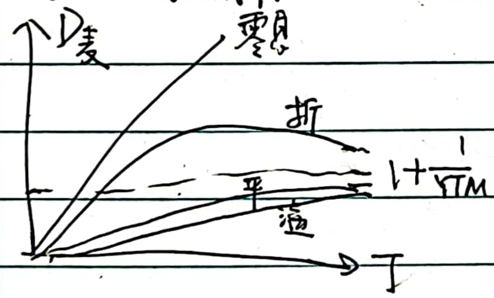
$D=\frac{\sum_{t=1}^{T}tCe^{-rt}+TFe^{-rT}}{P}$，加权回本时间

$D_{mod}=-\frac{1}{P}\cdot\frac{dP}{dr}$ (弹性)，Convexity $=\frac{1}{P}\cdot\frac{d^2P}{dr^2}$ 曲

$=-\frac{1}{P}\cdot(\sum Ce^{-rt}\cdot(-t)+(-T)\cdot Fe^{-rT})$

$=D$，如果是离散 $D_{mod}=\frac{D}{1+r}$ → 付息区间，不一定为年

修正久期没有单位



Convexity $=\frac{1}{P}\cdot(\sum Ce^{-rt}t^2+T^2Fe^{-rT})$

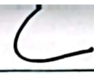
$\Delta P=-D_{mod}\cdot P\cdot\Delta r+\frac{1}{2}\,\text{convexity}\cdot P\cdot(\Delta r)^2$

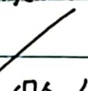
N：半年一次，总付次数　CPT → PV　CF → 2nd+CE/C 清除 ☆

I/Y：半年利率　[500] [+/-] [enter]

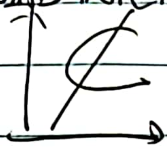
PMT：每次付息　先DATA后STAT　[↓] … → [CPT

FV：终值/面值　[NPV]

回到最初的问题，切点是什么？ 如果风险资产组合，得到 

加入无风险，得到 ，在无风险为0权重时，二者相交，

同时，加入无风险的组合优化的结果"更优"，不然无风险权重取0，则为"C"上点

所以不可能出现 ，只可能  相切，切点处 $W_{r_f}=0$

代入该点得到二个线的斜率相同(麻烦)

该点就是有无风险的情况下，风险资产的最优解 → 市场组合

反过来，一基金定理简过 r_f 和 r_M 可以画出 ""。

↳ 权重固定 → ~~一般假设市值加权~~

六、Beta of Portfolio

$$\beta^P = \frac{cov(r_p, r_M)}{Var(r_M)} = \frac{cov(\Sigma W_i r_i, r_M)}{Var(r_M)}$$

$$= \frac{\Sigma W_i\, cov(r_i, r_M)}{Var(r_M)} = \Sigma W_i \beta_i$$

SML: $r_i = r_f + \beta_i(r_m - r_f)$

七、Risk: $\sigma_i^2 = \beta_i^2\sigma_M^2 + Var(\varepsilon_i)$, $r_i - r_f = \alpha + \beta_i(r_m - r_f) + \varepsilon_i$

$$\downarrow$$

$$\frac{\Sigma\, cov^2(r_i, r_M)}{\sigma_m^2\,(Var(r_M))}$$

$$\sigma_{ij}^2 = \beta_i\beta_j\cdot\sigma_m^2$$

$$cov(r_i - r_f, r_j - r_f) = [\beta_i, s_i, h_i]\begin{bmatrix}\sigma^2_{r_m-r_f} & \sigma_{r_m-r_f,SMB} & \sigma_{r_m-r_f,HML}\end{bmatrix}\begin{bmatrix}\beta_j\\ s_j\\ h_j\end{bmatrix} + cov(\varepsilon_i,$$

（CML与有效前沿的切点、？）就是市场组合

五、CAPM ☆

Think again, Minimize $\frac{1}{2}w'\Sigma w$

Such that $r_f + w'(\bar{r}-r_f I) - \bar{r}_p = 0$

$w_0 + w'I = 1$

Then we got $\Sigma w + \lambda(\bar{r}-r_f I) = 0$ ~~$\bar{r}_p - r_f$~~ 0

$$w'(\bar{r}-r_f I) = \bar{r}_p - r_f$$

$$-\frac{1}{\lambda} = \frac{w'(\bar{r}-r_f I)}{w'\Sigma w} \qquad \bar{r}-r_f I = -\frac{\Sigma w}{\lambda}$$

$$= -\Sigma w \cdot \frac{w'(\bar{r}-r_f I)}{w'\Sigma w}$$

$$\mathrm{cov}(r_i, w'r) = \mathrm{cov}\left(r_i, \sum_{j=1}^{n} w_j r_j\right) = \sum_{j=1}^{n} w_j \sigma_{ij}$$

$$\Sigma w = \begin{bmatrix} \sum_{j=1}^{n} w_j \sigma_{1j} \\ \vdots \\ \sum_{j=1}^{n} w_j \sigma_{nj} \end{bmatrix} = \begin{bmatrix} \mathrm{cov}(r_1, w'r) \\ \vdots \\ \mathrm{cov}(r_n, w'r) \end{bmatrix}$$

$$\bar{r} - r_f I = -\begin{bmatrix} \mathrm{cov}(r_1, w'r) \\ \vdots \\ \mathrm{cov}(r_n, w'r) \end{bmatrix} \cdot \frac{w'(\bar{r}-r_f I)}{w'\Sigma w}$$

$$\begin{bmatrix} \bar{r}_1 - r_f \\ \bar{r}_2 - r_f \\ \vdots \end{bmatrix}$$

for each row: $\bar{r}_i - r_f = \frac{\mathrm{cov}(r_i, w'r)}{w'\Sigma w} \cdot w'(\bar{r}-r_f I)$

$$= \frac{\mathrm{cov}(r_i, w'r)}{\mathrm{Var}(w'r)} \cdot w'(\bar{r}-r_f I)$$

$$w'\Sigma w = \sum\sum w_i w_j \cdot \sigma_{ij}^2$$
$$= \sum\sum w_i w_j \, \overset{\mathrm{cov}}{\cancel{\mathrm{Var}}}(r_i, r_j)$$

$$\mathrm{Var}(w'r) = \mathrm{Var}\left(\sum w_i r_i\right) = \sum w_i^2 \mathrm{Var}(r_i) + \sum w_i w_j \mathrm{cov}(w_i r_i, w_j r_j)$$
$$= w'\Sigma w$$

↓

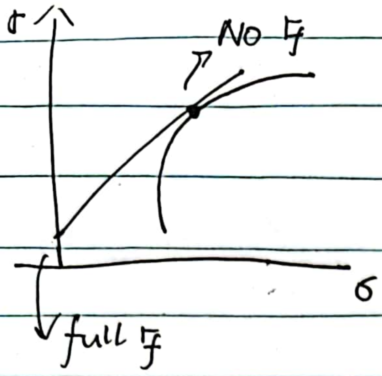
拆开有协方差

取市场组合，$r_i - r_f = \frac{\mathrm{cov}(r_i, r_M)}{\mathrm{Var}(r_M)} \cdot (\bar{r}_M - r_f)$，此处 $w'I + w_0 = 1$

↓

β_i

四、Capital Market Line $r_p = r_f + \frac{E(r_M)-r_f}{\sigma_M}\cdot\sigma_p$



Tangency point is "Market portfolio"

直觉上，$r_p = E(r_M)$时，$\sigma_M = \sigma_p$，

引入无风险资产(此时我们完全不知其形态、公式)，

~~$\bar{r}_p = W_M E(r_M)$~~

Minimize $\frac{1}{2} w'\Sigma w$ (W_0不影响方差)

Such that $(1-w'I) r_f + w'(\bar{r}) = \bar{r}_p \to r_f + (w'\bar{r} - I'w r_f) = \bar{r}_p$

$W_0 + I'W = 1$

$$L(w,\lambda) = \frac{1}{2} w'\Sigma w + \lambda\underbrace{(r_f + w'\bar{r} - I'w r_f - \bar{r}_p)}_{w'(\bar{r} - r_f I) - \bar{r}_p}$$

$$\frac{\partial L}{\partial w} = \Sigma w + \lambda(\bar{r} - r_f I)' = 0$$

$$w = \lambda \Sigma^{-1}(\bar{r} - r_f I)$$

$$\frac{\partial L}{\partial \lambda} = r_f + w'(\bar{r} - r_f I) = \alpha\bar{r}_p \Rightarrow (\bar{r} - r_f I)'\cdot\lambda\Sigma^{-1}(\bar{r} - r_f I) = \alpha\bar{r}_p$$

$$\lambda = (\bar{r} - r_f I)'^{-1}\cdot(\bar{r}_p - r_f)(\bar{r} - r_f I)^{-1}\cdot\Sigma$$

$$= (\bar{r} - r_f I)'^{-1}(\bar{r}_p - r_f)\frac{1}{\Sigma^{-1}(\bar{r} - r_f I)}$$

$$= \frac{\bar{r}_p - r_f}{\cancel{\bar{r} - r_f}(\bar{r} - r_f I)'\Sigma^{-1}(\bar{r} - r_f I)}$$

注意没有逆矩阵

$$W_0 = 1 - I'w, \quad w = \frac{(\bar{r}_p - r_f)\Sigma^{-1}(\bar{r} - r_f I)}{(\bar{r} - r_f I)'\Sigma^{-1}(\bar{r} - r_f I)} = \cancel{(\bar{r}_p - r_f)(\bar{r} - r_f I)'^{-1}}$$

二基金 $\xrightarrow{\text{无风险}}$ 一基金 ~~$W_0 = 1 - I'$~~ $\sigma_p^2 = w'\Sigma w = (\bar{r}_p - r_f)^2\cdot(K)$ 常数K

$\sigma_p = (\bar{r}_p - r_f)\cdot\sqrt{K}$，斜率$\sqrt{K}$，可知是一条直线

此时风险组合的权重比例不变，同比变动

Market Model: $r_i = \alpha + \beta r_m + \varepsilon_i \xrightarrow{\alpha = r_f(1-\beta)}$ CAPM

Part I: effective portfolio (from mean-variance)

① 系统性风险

$r_p = \sum W_i r_i$, $\cancel{Var}\ \sigma_p^2 = \sum W_i^2 \sigma_i^2 + \sum_{i\neq j} W_i W_j \sigma_{i,j}$

Suppose $W_i = \frac{1}{N}$, $\sigma_p^2 = \frac{\sum \sigma_i^2}{N^2} + \frac{1}{N^2}\sum_{i\neq j}\sigma_{i,j} = \frac{\sum \sigma_i^2}{N^2} + \frac{1}{N^2}\cdot C_N^2 \times 2 \times \overline{\sigma_{i,j}}$

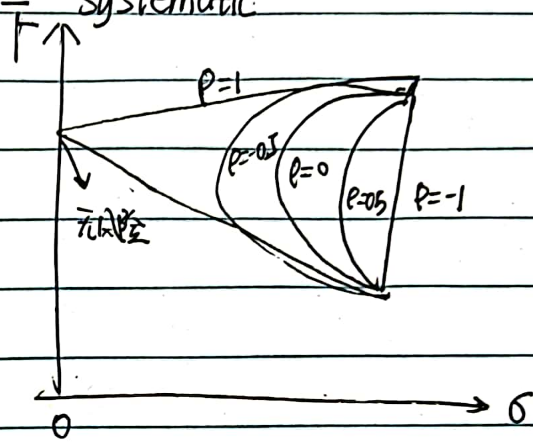
$\xlongequal{N\to\infty} 0 + \frac{N-1}{N}\overline{\sigma_{i,j}} \Rightarrow \overline{\sigma_{i,j}}$

systematic

② $\rho = 1$, $\sigma_p = W_A\sigma_A + W_B\sigma_B$，无法分散

$0<\rho<1$, $\rho\downarrow$, 风险分散越好

$\rho<0$ ，可以构建无风险



③ 如果不考虑 r_p 给定

$L(w,\lambda) = w'\Sigma w + \lambda(1 - I'w)$

$\frac{\partial L}{\partial w} = 2\Sigma w - \lambda I = 0$, $\Sigma w = \frac{\lambda}{2} I$, $\cancel{\lambda = 2\Sigma w \cdot I^{-1} = 2\Sigma \cdot (I'^{-1} \cdot I \cdot I^{-1}) = 2\cdot(I'\Sigma^{-1}I)^{-1} = \frac{2}{I'\Sigma^{-1}I}}$

$W = \cancel{I^{-1}\varnothing}$

I 没有逆矩阵

$\begin{cases} I'w = 1 \\ \lambda I = 2\Sigma w \cancel{I^{-1}} \end{cases} \Rightarrow \cancel{I' \frac{\lambda I}{2} \Sigma^{-1} = I'w = 1}$

$I'\Sigma^{-1}\frac{\lambda}{2}I \cdot \cancel{1} = I'w\cancel{1} = 1$ $\cancel{\lambda =}$

$\frac{\lambda}{\cancel{2}} = \frac{2}{I'\Sigma^{-1}I}$, $w = \frac{1}{2}\Sigma^{-1}\cdot\lambda I$

$= \frac{1}{2}\Sigma^{-1}\cdot\frac{2}{I'\Sigma^{-1}I}\cdot I$

$= \frac{\Sigma^{-1}I}{I'\Sigma^{-1}I}$ $\left(\Sigma w = \frac{I}{I'\Sigma^{-1}I}\right)$ 协方差归一