Problem Set 4 – Question 1 Solution

JunruWang

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Question 1: Using Itô's Lemma and the Itô Integral

Let $X(t)$ be a standard Brownian motion. We use Itô's lemma to prove the following identities.

(a) Show that

$$\int_0^t X(\tau)\,dX(\tau) = \frac{1}{2}X^2(t) - \frac{1}{2}t$$

Solution: Let $f(X(t)) = X^2(t)$. Applying Itô's lemma:

$$df = \frac{d}{dt}X^2(t) = 2X(t)dX(t) + \frac{1}{2}\cdot 2 \cdot 1 \cdot dt = 2X(t)dX(t) + dt$$

Integrate both sides from 0 to t:

$$\int_0^t d(X^2(\tau)) = \int_0^t 2X(\tau)dX(\tau) + \int_0^t d\tau$$

$$X^2(t) = 2\int_0^t X(\tau)dX(\tau) + t$$

Rearranging gives:

$$\int_0^t X(\tau)dX(\tau) = \frac{1}{2}X^2(t) - \frac{1}{2}t$$

(b) Show that

$$\int_0^t \tau\,dX(\tau) = tX(t) - \int_0^t X(\tau)\,d\tau$$

Solution: Let $f(\tau, X(\tau)) = \tau X(\tau)$. Apply Itô's lemma:

$$df = \frac{\partial f}{\partial \tau}d\tau + \frac{\partial f}{\partial X}dX(\tau) + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial X^2}d\tau$$

$$\frac{\partial f}{\partial \tau} = X(\tau), \quad \frac{\partial f}{\partial X} = \tau, \quad \frac{\partial^2 f}{\partial X^2} = 0$$

So,

$$df = X(\tau)d\tau + \tau dX(\tau)$$

Integrating both sides:

$$\int_0^t d(\tau X(\tau)) = \int_0^t X(\tau)d\tau + \int_0^t \tau dX(\tau)$$

Since $\tau X(\tau)\big|_0^t = tX(t)$, we get:

$$tX(t) = \int_0^t X(\tau)d\tau + \int_0^t \tau dX(\tau)$$

Thus:

$$\int_0^t \tau dX(\tau) = tX(t) - \int_0^t X(\tau)d\tau$$

(c) Show that

$$\int_0^t X^2(\tau)\, dX(\tau) = \frac{1}{3}X^3(t) - \int_0^t X(\tau)\, d\tau$$

Solution: Let $f(X(t)) = X^3(t)$. Applying Itô's lemma:

$$df = 3X^2(t)dX(t) + \frac{1}{2} \cdot 6X(t)dt = 3X^2(t)dX(t) + 3X(t)dt$$

Integrate both sides:

$$X^3(t) = 3\int_0^t X^2(\tau)dX(\tau) + 3\int_0^t X(\tau)d\tau$$

Divide both sides by 3:

$$\int_0^t X^2(\tau)dX(\tau) = \frac{1}{3}X^3(t) - \int_0^t X(\tau)d\tau$$

Question 2: Process Followed by the Futures Price

Let the stock price follow the stochastic differential equation (SDE):

$$dS(t) = (\mu - \delta)S(t)dt + \sigma S(t)dX(t)$$

We are asked to find the SDE for the futures price:

$$F(S(t), t) = S(t)e^{(r-\delta)(T-t)}$$

Step 1: Apply Itô's Lemma to $F(S,t)$F(S,t)

Note that $F(S,t)$ is a function of both S and t. Define:

$$f(S,t) = S \cdot e^{(r-\delta)(T-t)}$$

Compute the partial derivatives:

$$\frac{\partial F}{\partial S} = e^{(r-\delta)(T-t)}$$

$$\frac{\partial F}{\partial t} = S \cdot e^{(r-\delta)(T-t)} \cdot (-(r-\delta)) = -(r-\delta)F$$

$$\frac{\partial^2 F}{\partial S^2} = 0$$

Now apply Itô's Lemma:

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2} \cdot \frac{\partial^2 F}{\partial S^2} \cdot (dS)^2$$

Since $\frac{\partial^2 F}{\partial S^2} = 0$, the last term vanishes:

$$dF = -(r-\delta)Fdt + e^{(r-\delta)(T-t)} \cdot dS$$

Substitute $dS = (\mu - \delta)Sdt + \sigma S dX(t)$:

$$dF = -(r-\delta)Fdt + e^{(r-\delta)(T-t)} \left[(\mu-\delta)Sdt + \sigma S dX(t)\right]$$

Now note that:

$$e^{(r-\delta)(T-t)} \cdot S = F(S,t)$$

So:

$$dF = -(r-\delta)Fdt + (\mu-\delta)Fdt + \sigma F dX(t)$$

Group the drift terms:

$$dF = \left[(\mu-\delta) - (r-\delta)\right]Fdt + \sigma F dX(t) = (\mu - r)Fdt + \sigma F dX(t)$$

Final Answer:

$$\boxed{dF = (\mu - r)Fdt + \sigma F dX(t)}$$

This is the stochastic process followed by the futures price.

Question 3: A Practical Example of Using Itô's Lemma

Let the price processes be:

$$dS_{1t} = \mu_1 S_{1t} dt + \sigma_1 S_{1t} dX_{1t}$$
$$dS_{2t} = \mu_2 S_{2t} dt + \sigma_2 S_{2t} dX_{2t}$$
$$dB_t = r B_t dt$$

with $\mathbb{E}[dX_{1t} dX_{2t}] = \rho dt$, where $\rho \in [-1, 1]$ is the correlation between the two Brownian motions.

(a) Find the process for $Y_t = S_{1t} S_{2t}$ Yt = S1t * S2t

We apply the product rule for Itô processes:

$$dY_t = d(S_{1t} S_{2t}) = S_{1t} dS_{2t} + S_{2t} dS_{1t} + dS_{1t} dS_{2t}$$

Substitute the dynamics:

$$\begin{aligned} dY_t &= S_{1t}(\mu_2 S_{2t} dt + \sigma_2 S_{2t} dX_{2t}) + S_{2t}(\mu_1 S_{1t} dt + \sigma_1 S_{1t} dX_{1t}) + \sigma_1 \sigma_2 S_{1t} S_{2t} \rho dt \\ &= \mu_1 Y_t dt + \mu_2 Y_t dt + \sigma_1 Y_t dX_{1t} + \sigma_2 Y_t dX_{2t} + \rho \sigma_1 \sigma_2 Y_t dt \\ &= Y_t \left[(\mu_1 + \mu_2 + \rho \sigma_1 \sigma_2) dt + \sigma_1 dX_{1t} + \sigma_2 dX_{2t} \right] \end{aligned}$$

(b) Write the geometric average option payoff

The payoff is:

$$V(S_{1T}, S_{2T}, T) = \max\left(\sqrt{S_{1T} S_{2T}} - K,\, 0 \right)$$

Let $Y_T = S_{1T} S_{2T}$, so:

$$V(Y_T) = \max\left(\sqrt{Y_T} - K,\, 0 \right)$$

(c) Find the process for $Z_t = \sqrt{Y_t}$ Zt = sqrt(Yt)

Let $Z_t = Y_t^{1/2}$, apply Itô's lemma:

$$dZ_t = \frac{1}{2} Y_t^{-1/2} dY_t - \frac{1}{8} Y_t^{-3/2} (dY_t)^2$$

From part (a), recall:

$$dY_t = Y_t \left[\mu_Y dt + \sigma_1 dX_{1t} + \sigma_2 dX_{2t} \right]$$

where $\mu_Y = \mu_1 + \mu_2 + \rho \sigma_1 \sigma_2$.
Compute the square of the diffusion term:

$$(dY_t)^2 = Y_t^2(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)dt = Y_t^2\sigma_3^2 dt$$

where:

$$\sigma_3^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$$

Therefore:

$$\begin{aligned} dZ_t &= \frac{1}{2}Y_t^{-1/2} \cdot Y_t \left[\mu_Y dt + \sigma_1 dX_{1t} + \sigma_2 dX_{2t}\right] - \frac{1}{8}Y_t^{-3/2} \cdot Y_t^2 \sigma_3^2 dt \\ &= \frac{1}{2}\sqrt{Y_t}\left[\mu_Y dt + \sigma_1 dX_{1t} + \sigma_2 dX_{2t}\right] - \frac{1}{8}\sqrt{Y_t} \cdot \sigma_3^2 dt \\ &= \sqrt{Y_t}\left[\left(\frac{1}{2}\mu_Y - \frac{1}{8}\sigma_3^2\right) dt + \frac{1}{2}\sigma_1 dX_{1t} + \frac{1}{2}\sigma_2 dX_{2t}\right] \end{aligned}$$

Define a new Brownian motion X_{3t} such that:

$$\sigma_3 dX_{3t} = \sigma_1 dX_{1t} + \sigma_2 dX_{2t}, \quad \sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}$$

Then:

$$dZ_t = \sqrt{Y_t}\left[\left(\frac{1}{2}\mu_Y - \frac{1}{8}\sigma_3^2\right) dt + \frac{1}{2}\sigma_3 dX_{3t}\right]$$

(d) Real-world expected return of the geometric average

The expected return is the drift term in dZ_t, divided by $Z_t = \sqrt{Y_t}$. Therefore, the expected rate of return is:

$$\frac{1}{Z_t} \cdot \mathbb{E}[dZ_t] = \frac{1}{2}\mu_Y - \frac{1}{8}\sigma_3^2 = \frac{1}{2}(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2) - \frac{1}{8}(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)$$

$$= \frac{1}{2}(\mu_1 + \mu_2) + \frac{1}{2}\rho\sigma_1\sigma_2 - \frac{1}{8}(\sigma_1^2 + \sigma_2^2) - \frac{1}{4}\rho\sigma_1\sigma_2$$

$$= \frac{1}{2}(\mu_1 + \mu_2) - \frac{1}{8}(\sigma_1^2 + \sigma_2^2) + \left(\frac{1}{2} - \frac{1}{4}\right)\rho\sigma_1\sigma_2$$

$$= \boxed{\frac{1}{2}(\mu_1 + \mu_2) - \frac{1}{8}(\sigma_1^2 + \sigma_2^2) + \frac{1}{4}\rho\sigma_1\sigma_2}$$

This is the expected rate of return of the geometric average of the two stocks.

Question 4: One with Exchange Rates

We are given the following stochastic differential equations:

$$\begin{aligned} dB_e &= r_e B_e dt \\ dB_p &= r_p B_p dt \\ dp &= \mu_p p dt + \sigma_p p dX_{1t} \\ de &= \mu_e e dt + \sigma_e e dX_{2t} \end{aligned}$$

with $\mathbb{E}[dX_{1t}dX_{2t}] = \rho dt$. These represent money market accounts and spot exchange rates for GBP and EUR, with prices in USD.

(a) Determine the processes followed by $Z_{pt} = p_t B_{pt}$ and $Z_{et} = e_t B_{et}$

Use the product rule for Itô processes.

For the British Pound bond:

$$\begin{aligned} dZ_{pt} = d(p_t B_{pt}) &= p_t dB_{pt} + B_{pt} dp_t + dB_{pt} dp_t \\ &= p_t r_p B_{pt} dt + B_{pt}(\mu_p p_t dt + \sigma_p p_t dX_{1t}) + 0 \quad (\text{since } dB_{pt}dp_t \text{ is order } dt^2) \\ &= Z_{pt}(r_p + \mu_p)dt + \sigma_p Z_{pt} dX_{1t} \end{aligned}$$

So:

$$\boxed{dZ_{pt} = Z_{pt}\left[(r_p + \mu_p)dt + \sigma_p dX_{1t}\right]}$$

Similarly, for the Euro bond:

$$\begin{aligned} dZ_{et} = d(e_t B_{et}) &= e_t dB_{et} + B_{et} de_t + 0 \\ &= e_t r_e B_{et} dt + B_{et}(\mu_e e_t dt + \sigma_e e_t dX_{2t}) \\ &= Z_{et}(r_e + \mu_e)dt + \sigma_e Z_{et} dX_{2t} \end{aligned}$$

Thus:

$$\boxed{dZ_{et} = Z_{et}\left[(r_e + \mu_e)dt + \sigma_e dX_{2t}\right]}$$

(b) Define $Y_t = \frac{e_t}{p_t}$. What is dY_t?

Let:

$$Y_t = \frac{e_t}{p_t}$$

Use Itô's lemma for the quotient of two stochastic processes:

$$dY_t = \frac{1}{p_t} de_t - \frac{e_t}{p_t^2} dp_t + \frac{1}{p_t^2} d\langle e_t, p_t \rangle$$

Compute:
- $de_t = \mu_e e_t dt + \sigma_e e_t dX_{2t}$ - $dp_t = \mu_p p_t dt + \sigma_p p_t dX_{1t}$ - $d\langle e_t, p_t \rangle = \sigma_e \sigma_p e_t p_t \rho dt$
Then:

$$\begin{aligned} dY_t &= \frac{1}{p_t}(\mu_e e_t dt + \sigma_e e_t dX_{2t}) - \frac{e_t}{p_t^2}(\mu_p p_t dt + \sigma_p p_t dX_{1t}) + \frac{1}{p_t^2}\sigma_e \sigma_p e_t p_t \rho dt \\ &= Y_t \left[(\mu_e - \mu_p + \rho \sigma_e \sigma_p) dt + \sigma_e dX_{2t} - \sigma_p dX_{1t} \right] \end{aligned}$$

Now define a new Brownian motion X_{3t} such that:

$$\sigma_3 dX_{3t} = \sigma_e dX_{2t} - \sigma_p dX_{1t}$$

Then the variance of dX_{3t} is:

$$\sigma_3^2 = \sigma_e^2 + \sigma_p^2 - 2\rho \sigma_e \sigma_p$$

So the final process for Y_t is:

$$\boxed{dY_t = Y_t \left[(\mu_e - \mu_p + \rho \sigma_e \sigma_p) dt + \sigma_3 dX_{3t} \right]}$$

Question 5: Black-Scholes-Merton PDE and Expected Returns

Let the stock price process be:

$$dS(t) = (\mu - \delta) S(t) dt + \sigma S(t) dX(t)$$

Let the risk-free bond process be:

$$dB(t) = r B(t) dt$$

Let $V(S(t), t)$ be the value of an option on the stock.
The Black-Scholes-Merton PDE is:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} = rV$$

(a) Use Itô's Lemma to find dV

Apply Itô's lemma to $V(S(t), t)$:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2$$

Substitute $dS = (\mu - \delta)Sdt + \sigma SdX$, and $(dS)^2 = \sigma^2 S^2 dt$:

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\left[(\mu - \delta)Sdt + \sigma SdX\right] + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} \cdot \sigma^2 S^2 dt \\ &= \left(\frac{\partial V}{\partial t} + (\mu - \delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dX \end{aligned}$$

(b) Compare to BSM PDE and determine required drift

From part (a), the drift of V is:

$$\mu_V = \frac{\partial V}{\partial t} + (\mu - \delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

From the Black-Scholes PDE, the left-hand side is:

$$\mathcal{L}(V) = \frac{\partial V}{\partial t} + (r - \delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

So, to match the PDE, we must choose the expected return of the underlying to be:

$$\boxed{\mu = r}$$

This ensures that the drift of V under the real-world measure equals the LHS of the BSM PDE under the risk-neutral measure.

(c) What assumption is made about the expected return of the option?

Using the real-world SDE for V:

$$\mathbb{E}[dV] = \mu_V dt = \left(\frac{\partial V}{\partial t} + (\mu - \delta)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

The expected rate of return on the option is:

$$\frac{\mathbb{E}[dV]}{V} \quad \Rightarrow \quad \text{To satisfy the PDE, we assume this equals } r$$

So the implicit assumption is:

$$\boxed{\text{The expected rate of return on the option is } r}$$

This is consistent with risk-neutral pricing.

(d) Return to Question 2 – what is the process followed by $F(S(t), t)$ under $\mu = r$?

From Question 2, the process for:

$$F(S(t), t) = S(t)e^{(r-\delta)(T-t)}$$

was shown to be:

$$dF = (\mu - r)Fdt + \sigma F dX(t)$$

If we now set $\mu = r$, then:

$$dF = \sigma F dX(t)$$

$$\boxed{dF = \sigma F dX(t)}$$

So the futures price is a **martingale** under the real-world measure when $\mu = r$, i.e., it has zero drift.