FIN500 Review

1 Forwards

1.1 Notation

- Zero Coupon Bond: pay 1 at maturity. Disount factor

$$P(t,T) = e^{-r(t,T)(T-t)}$$

- Coupon Bond: $B(t,T,c,n)$, c is annual coupon rate. remember pay face value at time T.
- Par Bond: $P = F(r = YTM)$; Discount Bond: $P < F$; PremiumBond:$P > F$;
- YTM is annual return of coupon bond, zero-coupon rate is annual return of zero-cb.
- Forward Rates:

$$F_0(t_1,t_2) = \frac{1}{t2 - t1}[(1 + r(0,t_2) * t_2) - (1 + r(0,t_1) * t_1)]$$

$$F_0(t_1,t_2) = \frac{1}{t2 - t1}[\frac{P(0,t_1)}{P(0,t_2)} - 1]$$

- Par Coupon Rate: coupon rate that makes $B(0,T,c,n) = 1$

$$c = \frac{1 - P(0,T)}{\sum_{i=1}^{n} P(0,t_i)}$$

1.2 FRA

$$Payoff_{long} = N * [r(T,T+t) * t - Kt]$$

$$Payoff_{short} = -Payoff_{long}$$

$$FRA_0 = P(0,T+t)[F_0(T,T+t) * t - Kt]$$

$$FRA_0 = P(0,T) - (1 + Kt)P(0,T+t)$$

1.3 Swap

$$Swap(0) = N(1 - S_0 * t\sum_{i=1}^{n} P(0,T_i)) - P(0,T_n)]$$

$$V(Swap) = Float - Fix$$

$$N * S_0 * t\sum_{i=1}^{n} P(0,T_i) = N * [1 - P(0,T_n)]$$

Use forward rate to replace future float rate $tN * r(T_{i-1},T_i) -> tN * F_0(T_{i-1},T_i)$

2 Term Structure

- Expectations Hypo

$$E_0[r_1(1,2)] = F_0(1,2)$$

- Risk-Adjusted Expectation: Long term includes a risk premium

$$r(0,n) = \frac{1}{n} E_0[r(0,1) + r(1,2) + ... + r(n-1,n)] + rp(n)$$

- Forward and Risk netrual, Q is RN measure.

$$F_0(t,t+1) \geq E_0^Q[r(t,t+1)]$$

3 Duration

- Macaulay

$$D = \frac{\sum_{i=1}^{n} PV(t_i) * t_i}{PV_{total}}$$

- Modifed

$$D_m = -\frac{1}{P}\frac{dP}{dy} = \frac{D}{1+y/m}$$
$$\Delta P = -D_m * p * \Delta y$$

- basis is 0.01%

Mean-Variance Optimization Problem

$$Var[r_p] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

diversity reduces idiosyncratic risk(non system)

Problem Setup

The mean-variance optimization problem is defined as follows:

$$\begin{aligned}
&\text{Minimize} \quad \frac{1}{2} w' \Sigma w \quad \text{(Portfolio variance)} \\
&\text{Subject to:} \\
&w' r = r_p \quad \text{(Constraint on the mean return)} \\
&w' 1 = 1 \quad \text{(Weights sum to 1)}
\end{aligned}$$

Lagrangian Formulation

The Lagrangian function is given by:

$$L(w, \lambda, \mu) = \frac{1}{2} w' \Sigma w + \lambda (w' r - r_p) + \mu (w' 1 - 1)$$

First Order Conditions (FOCs)

Taking the partial derivatives with respect to w, λ, and μ, and setting them to zero, we obtain:

$$\begin{aligned}
\frac{\partial L}{\partial w} &= \Sigma w + \lambda r + \mu 1 = 0 \quad (1) \\
\frac{\partial L}{\partial \lambda} &= w' r - r_p = 0 \quad (2) \\
\frac{\partial L}{\partial \mu} &= w' 1 - 1 = 0 \quad (3)
\end{aligned}$$

Solution

From equation (1), we solve for w:

$$w = -\lambda\Sigma^{-1}r - \mu\Sigma^{-1}1$$

Substitute into the constraints (2) and (3). Define the following constants:

$$A = r'\Sigma^{-1}r, \quad B = r'\Sigma^{-1}1 = 1'\Sigma^{-1}r, \quad D = 1'\Sigma^{-1}1$$

Solving for λ and μ, we get:

$$\mu = \frac{r_pB - A}{AD - B^2}, \quad \lambda = \frac{-\mu B + r_p}{A}$$

Mean-Variance Optimization with a Risk-Free Asset

Problem Setup

$$\text{Minimize} \quad \frac{1}{2}w'\Sigma w \quad \text{(Portfolio variance)}$$
$$\text{Subject to:} \quad r_f + w'(r - r_f) = r_p \quad \text{(Target portfolio return)}$$
$$L(w,\lambda) = \frac{1}{2}w'\Sigma w + \lambda\left(r_f + w'(r - r_f) - r_p\right)$$

First-Order Conditions (FOCs)

$$\frac{\partial L}{\partial w} = \Sigma w + \lambda(r - r_f) = 0 \quad (1)$$
$$\frac{\partial L}{\partial \lambda} = r_f + w'(r - r_f) - r_p = 0 \quad (2)$$

— From equation (1), solve for w:

$$w = -\lambda\Sigma^{-1}(r - r_f)$$

Rearranging gives:

$$\lambda = \frac{r_f - r_p}{(r - r_f)'\Sigma^{-1}(r - r_f)}$$

If we don't set r_p, min variance result should be $w = \frac{\sum.^{-1}1}{1'\sum.^{-1}1}$, which makes variance matrix into (0,1).

4 Twofund Theory

- Allow Shortselling
- No riskfree
- All investors have same estimation
- 2 minVariance portfolios can repulicate all.

5 Onefund Theory

- allow shortselling
- one riskfree
- estimate same

$$\sigma_p = \frac{r_p - r_f}{\sqrt{(r - r_f1)'\sum.^{-1}(r - r_f1)}}$$
$$r_p = r_f + \sigma_p * ()$$

Here we got CML.

6 CAPM

$$\beta_i = \frac{cov(r_i, r_M)}{var(r_M)}$$

$$r_i - r_f = \beta_i w'(r - r_f 1)$$

Suppose all investors choose the same, so w is market portfolio, $w'r = r_M$ SML: for all assets

$$r_i = r_f + \beta_i(r_M - r_f)$$

CML: for Mean-Variance portfolios

$$r_p = r_f + \sigma_p * \frac{r_m - r_f}{\sigma_m}$$

$$\sigma_i^2 = \beta_i^2 * \sigma_m^2 + var(e_i)$$

7 Fama-French

$$r_i - r_f = \beta_i(r_M - r_f) + s_i(SMB) + h_i(HML) + e_i$$

Divide by B/M into High, Medium and Low, then divide by market into Small, Big.

$$HML = \frac{1}{2}(r_{SH} + r_{BH}) - \frac{1}{2}(r_{SL} + r_{BL})$$

$$SMB = \frac{1}{3}(r_{SH} + r_{SM} + r_{SL}) - \frac{1}{3}(r_{BH} + r_{BM} + r_{BL})$$

8 Fundamentals of Efficient Markets

8.1 Definition

- Prices reflect all info
- Prices react to new info
- instantaneously and unbiasedly

Problem: investors didnt believe this theory, they try to achieve and analyze info, which is irrational behavior.

Fix: We should take info cost into the model. Extro cost to reach the info offsets the return from it.

8.2 Random Walk Hypothesis

- $E[r_t] = \mu, corr(r_t, r_{t+i}) = 0$
- Prediction by past data is invalid
- Prove Efficient Markets theory. Rejection of the RWH is not sufficient to reject the weak form of the efficient market hypothesis.

8.3 Trading Rules

- Informative: $E[r_{t+1}|q_{t+1}]$ is not same for all possible q_{t+1}, function of info history I_t
- Example: BLL, double moving average, $R_t = \frac{a_{t,S} - a_{t,L}}{a_{t,L}}, R_t > B, Buy, |R_t| \leq B, keep$

9 Derivatives

9.1 (1)Forwards

Must perform; Initial cost is 0.

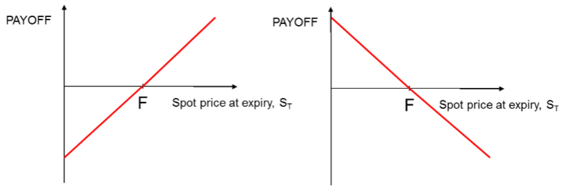
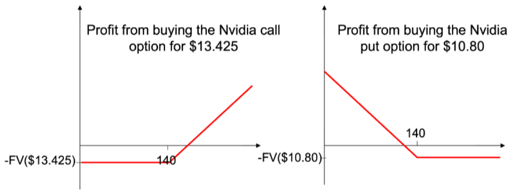
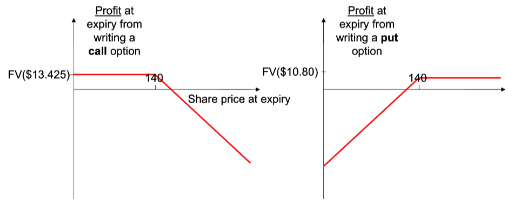


Figure 1: Forward Payoff



(a) Buy option



(b) **Pay option**

9.2 (2)Option

9.3 (3)Futures

$$F_{0,T} = FP_{0,t}(1 + rT)$$
$$F_{0,T} = S_0(1 + rT)$$
$$F_{0,T} = S_0(1 + rT) - FV(Div) = S_0 e^{(r-d)T}$$

9.4 Abitrage

No risk, no net invest, strictly positive profits. (Today and future)

- Stock = futures+zero-coupon bond
- Under this assumption, portfolio value at any time should be the same after discounted. If not, there is an abitrage.

9.4.1 (1)Cash&Carry Arbitrage

- Buy asset: Pay S_0 at time 0.
- Sell futures: Receive $F_{0,T}$ at time T.
- Borrow S_0 at time 0, pay $S_0(1 + rT)$ at time T.
- Receive Div.
- $F_{0,T} - (S_0(1 + rT) - FV(Div))$

9.4.2 (2)Reverse Cash&Carry

- Sell asset, buy futures, invest money we sell.

9.4.3 (3) Forward Contract Pricing

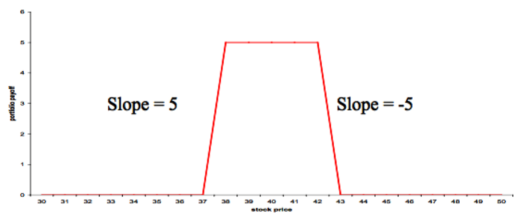
- Borrow EUR
- Lend USD
- Buy spot USD by EUR

$$F_{0,t} = \frac{1 + r_{usd}(0, t) * t}{1 + r_{eur}(0, t) * t} e_0$$

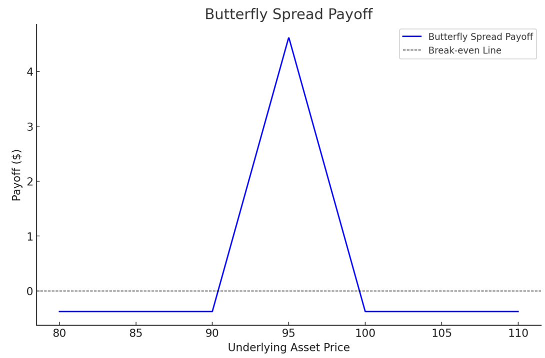
$F_{0,t}$: foward exchange rate, e_0: spot exchange rate(EUR/USD=EUR for USD)

9.4.4 Butterfly Spread

- 4 options, same maturity, different K: Buy 1 call, K1; Sell 2 call, K2 ; Buy 1 call, K3; $K1 < K2 < K3$



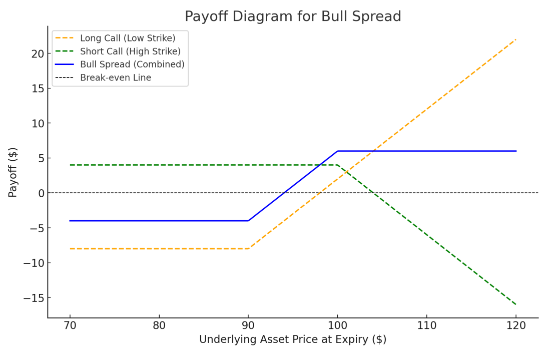
(a) Example



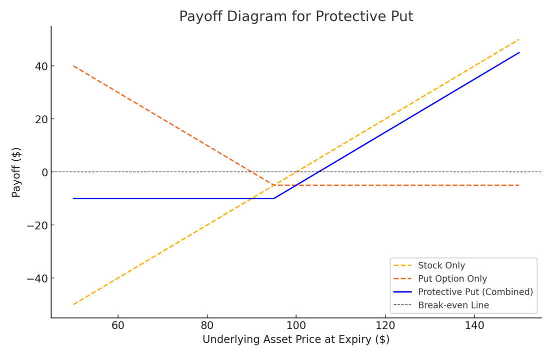
(b) **Butterfly Payoff**

9.4.5 Bull Spread

- Buy call+ sell call($K1 < K2$), low invest, low profit.



(a) Bull Spread



(b) Protective Put

9.4.6 Protective Put

- Buy asset at S0 and put.(mind putcall parity)

9.4.7 Example

Buy 5 call options K1, sell 5 call options K2, sell 5 call options K3, buy 5 call options K4, $K1 < K2 < K3 < K4$. At t=0, sell $(5C1 - 5C2 - 5C3 + 5C4)$ riskfree bonds.

9.5 Put-Call Parity

$$C_t - P_t = S_t - Ke^{-r(T-t)} - De^{-r(\tau-t)}$$

$$C_t - P_t = S_t e^{-d(T-t)} - Ke^{-r(T-t)}$$

$$C_t(A) \geq C_t(E), P_t(A) \geq P_t(E)$$

Right side is a forward, if dividend, $S_t = S_t - PV(Div)$ A: Buy euro put + buy asset B: Buy euro call + bond$(Ke^{-r(T-t)} + De^{-r\tau})$, τ is dividend day.

9.5.1 Intrinsic Value

Profit you can get from exercisin now. price = intrin value + time value

9.5.2 Price Bounds

$$max(S_t - Ke^{-r(T-t)}, 0) \leq c_t \leq S_t$$
$$max(Ke^{-r(T-t)} - S_t, 0) \leq p_t \leq Ke^{-r(T-t)}$$

For american put espeically:

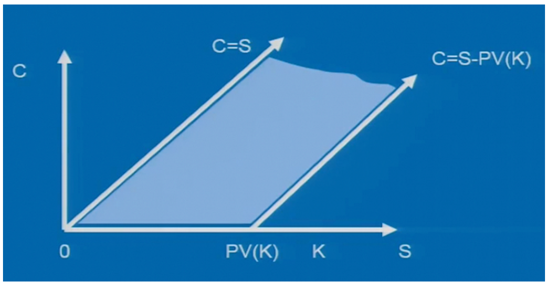
$$max(0, K - S_t) \leq p_t \leq K$$

Prove: Right: sell call, buy stock/ sell put, lend money(buy bond). Left: buy call and bond, compare stock/buy put and stock, compare bond.

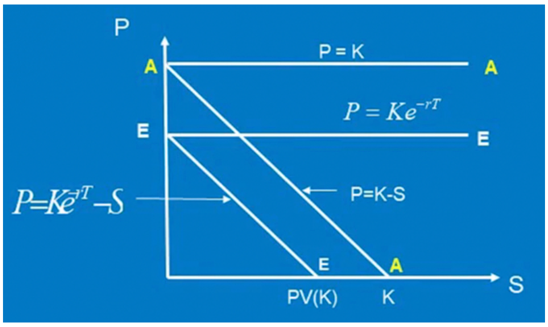
Why not early exercise?: Buy 1 call + Bond, we pay $Ke^{-rT} + c \geq S_0$, we receive $S_t - K + Ke^{-r(T-t)} \leq S_t, S_T \leq max(S_T, K)$. So if we early exericse, it's better to hold a stock.

Or we can use:

$$C_t(A) \geq C_t(E) = P_t(E) + S_t - Ke^{-r(T-t)} \geq S_t - K$$
$$P_t(A) \geq C_t(E) = C_t(E) - S + Ke^{-r(T-t)}$$



(a) Call Bound



(b) Put Bound

9.6 Binomial Tree

$$V_0 = E^{RN}[\textbf{payoff}]/(1 + r_f) = E[\textbf{payoff}]/(1 + r_{true})$$

9.6.1 Continuous Dividends

$$\pi = \frac{e^{(r-\delta)\Delta t} - d}{u - d}$$

9.6.2 Proportional Dividends

$$S- > S(1 - D)$$

9.6.3 American Option Pricing

$$V_{i,j} = max(V_{hold}, V_{exercise})$$