Probelm1: Derivation of Black-Scholes PDE from the Binomial Model

Step 1: Binomial Model Setup

Consider a binomial model with a small time step dt, where the up and down factors are given by:

$$u = 1 + \sigma\sqrt{dt}, \quad d = 1 - \sigma\sqrt{dt}$$

The option value at time t is denoted as $V = V(S, t)$, with:

$$V^+ = V(uS, t + dt), \quad V^- = V(dS, t + dt)$$

Step 2: Taylor Expansion

We expand V^+ and V^- in a Taylor series around (S, t), keeping terms up to first order in t and second order in S:

$$\begin{aligned}
V^+ &= V + (uS - S)\frac{\partial V}{\partial S} + dt\frac{\partial V}{\partial t} + \frac{(uS - S)^2}{2}\frac{\partial^2 V}{\partial S^2} \\
V^- &= V + (dS - S)\frac{\partial V}{\partial S} + dt\frac{\partial V}{\partial t} + \frac{(dS - S)^2}{2}\frac{\partial^2 V}{\partial S^2}
\end{aligned}$$

Since $uS - S = \sigma S\sqrt{dt}$ and $dS - S = -\sigma S\sqrt{dt}$, we substitute:

$$\begin{aligned}
V^+ &= V + \sigma S\sqrt{dt}\frac{\partial V}{\partial S} + dt\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2 dt}{2}\frac{\partial^2 V}{\partial S^2} \\
V^- &= V - \sigma S\sqrt{dt}\frac{\partial V}{\partial S} + dt\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2 dt}{2}\frac{\partial^2 V}{\partial S^2}
\end{aligned}$$

Step 3: Binomial Pricing Formula

The standard binomial pricing equation is:

$$V(S, t) = e^{-rdt}\left(qV^+ + (1 - q)V^-\right)$$

where q is the risk-neutral probability:

$$q = \frac{e^{rdt} - d}{u - d}$$

For small dt, we approximate:

$$e^{rdt} \approx 1 + rdt, \quad u - d = 2\sigma\sqrt{dt}$$

Substituting u and d:

$$\begin{aligned} q &= \frac{1 + rdt - (1 - \sigma\sqrt{dt})}{(1 + \sigma\sqrt{dt}) - (1 - \sigma\sqrt{dt})} \\ &= \frac{rdt + \sigma\sqrt{dt}}{2\sigma\sqrt{dt}} \\ &= \frac{rdt}{2\sigma\sqrt{dt}} + \frac{1}{2} \end{aligned}$$

Step 4: Expanding the Pricing Formula

Substituting q, V^+, and V^- into the binomial pricing equation:

$$\begin{aligned} V &= e^{-rdt}\left(qV^+ + (1-q)V^-\right) \\ &\approx (1 - rdt)\left[\left(\frac{1}{2} + \frac{rdt}{2\sigma\sqrt{dt}}\right)V^+ + \left(\frac{1}{2} - \frac{rdt}{2\sigma\sqrt{dt}}\right)V^-\right] \end{aligned}$$

Expanding:

$$V \approx (1 - rdt)\left[\frac{1}{2}(V^+ + V^-) + \frac{rdt}{2\sigma\sqrt{dt}}(V^+ - V^-)\right]$$

Using the Taylor expansions of V^+ and V^-:

$$\begin{aligned} V^+ + V^- &= 2V + 2dt\frac{\partial V}{\partial t} + \sigma^2 S^2 dt \frac{\partial^2 V}{\partial S^2} \\ V^+ - V^- &= 2\sigma S\sqrt{dt}\frac{\partial V}{\partial S} \end{aligned}$$

Substituting these:

$$\begin{aligned} V &\approx (1 - rdt)\left[V + dt\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2 dt}{2}\frac{\partial^2 V}{\partial S^2} + \frac{rdt}{\sigma\sqrt{dt}}\sigma S\sqrt{dt}\frac{\partial V}{\partial S}\right] \\ &= (1 - rdt)\left[V + dt\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2 dt}{2}\frac{\partial^2 V}{\partial S^2} + rSdt\frac{\partial V}{\partial S}\right] \end{aligned}$$

Expanding $(1 - rdt)V \approx V - rdtV$:

$$V - rdtV = V + dt\frac{\partial V}{\partial t} + rSdt\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2 dt}{2}\frac{\partial^2 V}{\partial S^2}$$

Canceling V and dividing by dt:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV = 0$$