FIN512 Financial Derivatives

Static Replication of Dual Directional Trigger Jump Securities

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1 Introduction

This report provides a comprehensive valuation analysis of the Dual Directional Trigger Jump Securities (hereinafter referred to as 'DDTJ Securities'), a structured financial product issued by Morgan Stanley Finance LLC, with its performance tied to the EURO STOXX 50 Index. A distinguishing feature of this security is that the underlying index is denominated in Euros (€), while the final payment is settled in U.S. dollars ($). This introduces an additional layer of complexity, as the valuation approach must incorporate the impact of cross-currency dynamics, including foreign exchange (FX) risk, interest rate differentials, and quanto adjustments.

To accurately price this security, we employ static replication techniques using options, leveraging the Black-Scholes-Merton (BSM) framework. However, due to the cross-currency nature of the product, the standard BSM model must be adjusted to ensure proper valuation. A detailed discussion of these adjustments is presented in Section 2.

Section 3 outlines the data sources and parameter estimation methodology used in our valuation. In Section 4, we construct a static replication strategy, demonstrating how the payoff structure of the DDTJ Securities can be decomposed into multiple quanto options for pricing. Section 5 presents a sensitivity analysis, evaluating the impact of key model parameters on valuation accuracy and evaluating the robustness of our pricing methodology.

2 Pricing Methodology

A quanto option is a type of derivative that allows investors to gain exposure to an asset denominated in a foreign currency while receiving the payoff in their domestic currency, without being affected by exchange rate fluctuations. This is achieved through a quanto adjustment, which incorporates the correlation between the asset price and the exchange rate into the pricing model.

In a later section, we will use quanto European options and quanto digital options to construct the replication portfolio and determine the pricing. Consequently, in this section, we will first explain how to price these two types of options using the Black-Scholes-Merton (BSM) formula.

2.1 Quanto European Options

The BSM closed-form solution for the quanto European call option is given by:

$$C = S_0 \cdot e^{(-r_f - r_d - d - \rho \cdot \sigma_S \cdot \sigma_E)T} N(d_1) - K \cdot e^{-r_d T} N(d_2) \tag{1}$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r_f - d - \rho \cdot \sigma_S \cdot \sigma_E + \frac{\sigma_S^2}{2}\right) T}{\sigma_S \sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r_f - d - \rho \cdot \sigma_S \cdot \sigma_E - \frac{\sigma_S^2}{2}\right) T}{\sigma_S \sqrt{T}} = d_1 - \sigma_S \sqrt{T}$$

where:

- S_0 = underlying asset price in a foreign currency at time 0
- K = strike price in foreign currency
- r_d = domestic risk-free interest rate
- r_f = foreign risk-free interest rate
- T = time to maturity
- d = dividend yield of the foreign asset

- σ_S = volatility of the foreign asset
- σ_E = volatility of the domestic exchange rate
- ρ = correlation between the foreign asset and domestic exchange rate

It has the following adjustments compared to the standard BSM formula:

1. The term $\rho\sigma_S\sigma_E$ accounts for the correlation between the asset price and the exchange rate.
2. The foreign risk-free rate r_f and dividend yield d impact the discounted value of the foreign asset.
3. The domestic risk-free rate r_d is used to discount the strike price.

For quanto European put options, we have:

$$P = Ke^{-r_d T}N(-d_2) - S_0 e^{(-r_f - r_d - d - \rho\sigma_S\sigma_E)T}N(-d_1) \tag{2}$$

where the parameters are defined as above.

2.2 Quanto Digital Options

A digital option, also known as a binary option, is a type of financial derivative that offers a fixed payout if the price of the underlying asset meets a specific condition at expiration. Digital options can introduce jumps in the portfolio's payoff structure.

The BSM closed-form solution for the quanto digital call option is given by:

$$DC = Ae^{-r_d T}N(d_2) \tag{3}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r_f - d - \rho\sigma_S\sigma_E - \frac{\sigma_S^2}{2}\right)T}{\sigma_S\sqrt{T}}$$

where:

- A = fixed payoff in the domestic currency (USD)
- The remaining parameters are identical to those in Section 2.1 and will not be reiterated here.

For quanto-digital put options, we have:

$$DP = Ae^{-r_d T}N(-d_2) \tag{4}$$

where the parameters are defined as above.

2.3 Implied Volatility Interpolation–SVI Quasi-Parametric

For most of the parameters in the BSM model, readily available data can be directly used. However, since we do not have direct access to the implied volatility data for the corresponding moneyness, we use interpolation to estimate it.

Here we apply the SVI (Stochastic Volatility Inspired) model, and it fits the implied volatility surface using a two-step optimization process.

2.3.1 SVI Quasi-Parametric Model

The implied variance $w(y)$ is defined as:

$$w(y) = a + dy + c\sqrt{y^2 + 1}$$

where:

- $y = \frac{x-m}{\sigma}$ is the normalized moneyness,
- $x = \ln\left(\frac{K}{S_0}\right)$ is the log-moneyness,
- a, d, c, m, σ are the SVI model parameters.

2.3.2 Optimization Process

To estimate m and σ, we minimize the sum of squared errors:

$$\min_{m,\sigma} \sum \left(a + dy + c\sqrt{y^2 + 1} - IV\right)^2$$

Given m and σ, we solve for a, d, c using the least squares method:

$$\begin{bmatrix} 1 & \frac{\sqrt{2}}{2}(y + \sqrt{y^2 + 1}) & \frac{\sqrt{2}}{2}(-y + \sqrt{y^2 + 1}) \end{bmatrix} \begin{bmatrix} a \\ d \\ c \end{bmatrix} = \begin{bmatrix} IV \end{bmatrix}$$

2.3.3 Interpolation

Once the SVI parameters (a, d, c, m, σ) are determined, the implied volatility for any moneyness x can be interpolated using:

$$IV(x) = \sqrt{w(x)}$$

3 Data and Parameters

3.1 Data

All data used in this report are obtained from Bloomberg Terminal and cover underlying asset prices, option market data, risk-free rates, and foreign exchange (FX) rates. These datasets are essential for valuation, static replication, discounting, and sensitivity analysis.

3.1.1 Underlying Asset Data

The underlying asset is the EURO STOXX 50 Index (SX5E). The current market price and historical data are retrieved as follows:

- **Spot Price (S_0)**: Obtained via `SX5E Index` (Date: 2024-11-14).
- **Historical Price Data**: Retrieved via `SX5E Index HP` for a 1-year daily time series.
- **Dividend Yield (q)**: Extracted from `SX5E Index DVD`.

3.1.2 Option Market Data

To conduct static replication and valuation, we use EUR-denominated options:

- **Implied Volatility (σ)**: Retrieved via `OVME`.
- **Option Prices (EUR)**: Obtained from `SX5E Index OMON`.

3.1.3 Risk-Free Rates

For discounting:

- **EUR OIS (5-Year)**: Retrieved via `ESTR OIS`.
- **USD OIS (5-Year)**: Retrieved via `SOFR OIS`.

3.1.4 Exchange Rate Data

- **Spot FX Rate**: `EURUSD Curncy`.
- **FX Implied Volatility**: Retrieved from `EURUSDV1M Curncy` (1M) and `EURUSDV1Y Curncy` (1Y).

3.1.5 Correlation Data

$$\rho = \mathrm{Corr}(\mathrm{Returns}_{\mathrm{SX5E}}, \mathrm{Returns}_{\mathrm{EUR/USD}})$$

3.2 Data Resource

Data Item	Bloomberg Command	Description
Foreign Asset Price (S_0)	SX5E Index	EURO STOXX 50 Index price
Strike Price (K)	Exchange contract	Defined by option contract
Domestic Risk-Free Rate (r_d)	USSWAP5	5-year USD Swap Rate
Foreign Risk-Free Rate (r_f)	EUSWAP5	5-year EUR Swap Rate
Time to Maturity (T)	Defined by contract	Example: 5 years
Dividend Yield (q)	DVD → SX5E Index	12-month rolling dividend yield
Foreign Asset Volatility (σ_S)	SX5E Index → IMPV	ATM implied volatility for SX5E
FX Volatility (σ_E)	EURUSD Curncy → IMPV	ATM implied volatility for EUR/USD
Asset-FX Correlation (ρ)	CRNC → SX5E, EURUSD	Historical correlation of SX5E and EUR/USD

Table 1: Bloomberg

4 Static Replication of DDTJ Securities

4.1 Overview of the DDTJ securities

Key Terms:

Issuer:	Morgan Stanley Finance LLC
Pricing date:	November 14, 2024
Original issue date:	November 19, 2024
Valuation date:	November 14, 2029
Maturity date:	November 19, 2029
Underlying index:	EURO STOXX 50® Index (SX5E)
Initial index value:	4,833.53
Estimated value on the pricing date:	$953.10 per security

Payoff Structure:

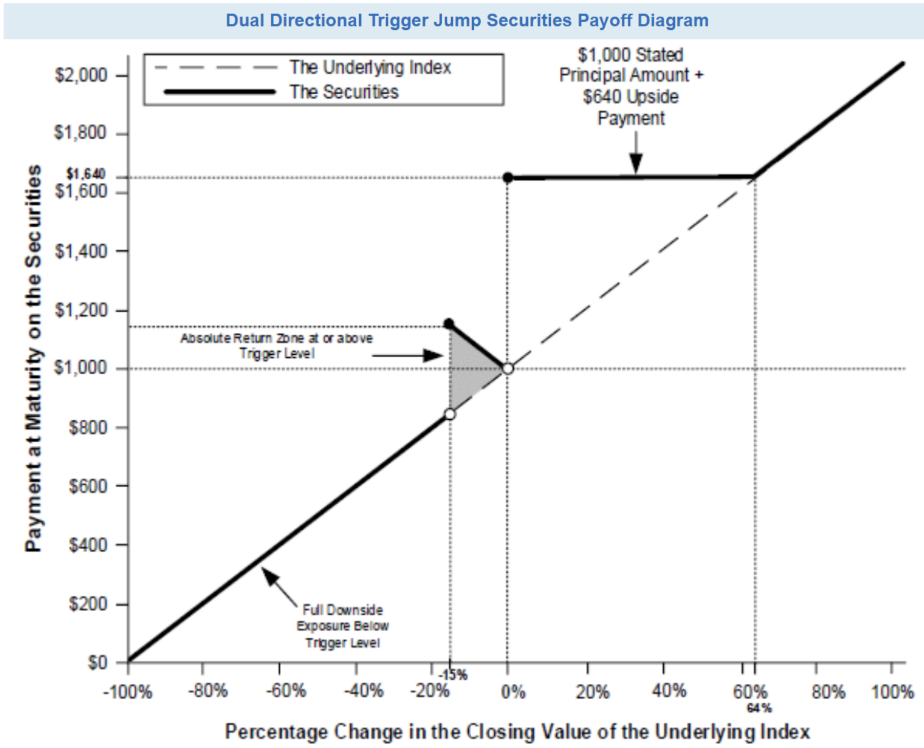

Figure 1: The Payoff Diagram from Pricing Supplement

4.2 Replication and Pricing

A Dual Directional Trigger Jump Securitie priced on November 14, 2024 is equivalent to a portfolio of:

1. A zero-coupon bond with a face value of $1000 maturing on November 19, 2029.
2. $(\frac{1000}{4833.53})$ units of written European SX5E put options expiring on November 14, 2029, with a strike price of $4,108.50$ $(4833.53 \times 85\%)$.
3. A written digital SX5E put options expiring on November 14, 2029, with a strike price of $4,108.50$ $(4833.53 \times 85\%)$ and a fixed payment of $150.
4. $(\frac{1000}{4833.53})$ units of written European SX5E call options expiring on November 14, 2029, with a strike price of $4,108.50$ $(4833.53 \times 85\%)$.
5. A purchased digital SX5E call options expiring on November 14, 2029, with a strike price of $4,108.50$ $(4833.53 \times 85\%)$ and a fixed payment of $150.
6. $(\frac{1000}{4833.53})$ units of purchased European SX5E call options expiring on November 14, 2029, with a strike price of $4,833.53$.
7. A purchased digital SX5E call options expiring on November 14, 2029, with a strike price of $4,833.53$ and a fixed payment of $640.
8. $(\frac{1000}{4833.53})$ units of purchased European SX5E call options expiring on November 14, 2029, with a strike price of $7,926.9892$ $(4833.53 \times 164\%)$.

Next, we only need to price them separately and sum them up to obtain the value of the DDTJ securities.

1. A zero-coupon bond is always easy to price:

$$Bond = \$1000 \times \exp(-rT)$$

where $r = r_d$= 0.03906675, T =5 . The result is \$822.56.

For the remaining options, we can use the BSM formulas from Section 2 to price them. Furthermore, they share the following parameters:

$$S_0 = 4833.53, \quad r_d = 0.03906675, \quad r_f = 0.020673, \quad T = 5,$$
$$d = 0.031413, \quad \sigma_E = 0.082225, \quad \rho = 0.39$$

2. The written European SX5E put option can be valued using Equation (2), where K= 4,108.50 and $\sigma_S = 0.15301129$. The result is $\frac{1000}{4833.53} \times \$344.95 = \$71.36$.

3. The written digital SX5E put options can be valued using Equation (4), where K= 4,108.50, σ_S = 0.15301129 and A = 150. The result is \$57.99.

4. The written European SX5E call options can be valued using Equation (1), where K= 4,108.50 and $\sigma_S = 0.15301129$. The result is $\frac{1000}{4833.53} \times \$642.14 = \$132.85$.

5. The purchased digital SX5E call options can be valued using Equation (3), where K= 4,108.50, σ_S = 0.15301129 and A = 150. The result is \$65.39.

6. The purchased European SX5E call options can be valued using Equation (1), where K= 4,833.53 and $\sigma_S = 0.16635691$. The result is $\frac{1000}{4833.53} \times \$423.41 = \$87.60$.

7. A purchased digital SX5E call options can be valued using Equation (3), where K= 4,833.53, σ_S = 0.16635691 and A = 640. The result is \$181.00.

8. The purchased European SX5E call options can be valued using Equation (1), where K= 7,926.9892 and $\sigma_S = 0.23731903$. The result is $\frac{1000}{4833.53} \times \$171.31 = \$35.44$.

Finally, the price of a DDTJ security is equal to the price of the replication portfolio as follows:

$$\begin{aligned}\text{Price of DDTJ Security} &= 822.56 - 71.36 - 57.99 - 132.85 + 65.39 + 87.60 + 81 + 35.44 \\ &= 929.79(\$).\end{aligned}$$

5 Sensitivity Analysis and Accuracy Evaluation

5.1 Sensitivity Analysis

5.1.1 Introduction

Sensitivity analysis is conducted to assess the impact of key market factors on the pricing of the Quanto Option. The primary objective is to quantify how changes in FX volatility (σ_{fx}), dividend yield (q) affect the option's valuation.
The underlying asset, EURO STOXX 50, has a market price of **953.10**. The option pricing is based on the Black-Scholes-Merton (BSM) model with Quanto adjustment.

5.1.2 Results

Variable	Impact on Option Price
FX Volatility (σ_{fx})	Higher σ_{fx} decreases option value (Quanto Adjustment effect)
Dividend Yield (q)	Higher q reduces option price (Lower growth expectation)

Table 2: Key Sensitivity Factors in Quanto Option Pricing

	FX Volatility									
Dividend Yield	0.0500	0.0611	0.0722	0.0833	0.0944	0.1056	0.1167	0.1278	0.1389	0.1500
0.0200	999.00	993.52	988.06	982.63	977.22	971.84	966.48	961.15	955.84	950.56
0.0233	980.68	975.24	969.84	964.45	959.09	953.76	948.45	943.17	937.91	932.68
0.0267	962.41	957.03	951.67	946.34	941.04	935.76	930.50	925.27	920.07	914.89
0.0300	944.22	938.89	933.59	928.31	923.06	917.84	912.64	907.47	902.32	897.20
0.0333	926.11	920.84	915.60	910.38	905.19	900.02	894.89	889.77	884.69	879.63
0.0367	908.11	902.90	897.72	892.56	887.44	882.33	877.26	872.21	867.19	862.19
0.0400	890.23	885.09	879.97	874.88	869.81	864.78	859.77	854.79	849.83	844.90
0.0433	872.49	867.41	862.36	857.34	852.34	847.38	842.44	837.52	832.64	827.78
0.0467	854.90	849.89	844.91	839.96	835.04	830.14	825.28	820.44	815.63	810.84
0.0500	837.48	832.55	827.64	822.77	817.92	813.10	808.30	803.54	798.81	794.10

Table 3: Quanto Portfolio Price Table: Effect of FX Volatility and Dividend Yield

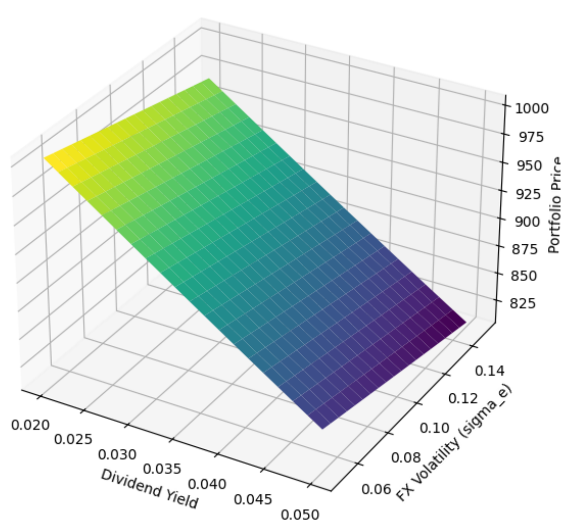

Figure 2: Price Surface

5.1.3 Discussion

FX Volatility (σ_{fx})	Option Price ($)
5.00%	999.00
6.11%	993.52
7.22%	988.06
8.33%	982.63
9.44%	977.22
10.56%	971.84
11.67%	966.48
12.78%	961.15
13.89%	955.84
15.00%	950.56

Table 4: Quanto Option Sensitivity to FX Volatility (σ_{fx}) at $q = 2.00\%$

Impact of FX Volatility (σ_{fx}) Observations:

- As FX volatility increases, the option price decreases due to a lower Quanto Adjustment.
- **Vega Calculation:**

$$\text{Vega} = \frac{\partial C}{\partial \sigma_{fx}} \approx -4.1$$

Meaning that each 1% increase in FX volatility decreases the option price by approximately 4.1 USD.

Dividend Yield (q)	Option Price ($)
2.00%	999.00
2.33%	980.68
2.67%	962.41
3.00%	944.22
3.33%	926.11
3.67%	908.11
4.00%	890.23
4.33%	872.49
4.67%	854.90
5.00%	837.48

Table 5: Quanto Option Sensitivity to Dividend Yield (q) at $\sigma_{fx} = 5.00\%$

Impact of Dividend Yield (q) Observations:

- Higher dividend yields reduce option prices due to lower expected asset growth.
- **Dividend Rho Calculation:**

$$\text{Dividend Rho} = \frac{\partial C}{\partial q} \approx -53.8$$

Meaning that each 1% increase in q decreases the option price by approximately 53.8 USD.

5.2 Accuracy Evaluation

5.2.1 Error Quantification

The absolute and relative errors are calculated as:

$$\text{Absolute Error} = |C_{\text{model}} - C_{\text{true}}| = |929.79 - 953.10| = 23.31$$

$$\text{Relative Error} = \frac{|C_{\text{model}} - C_{\text{true}}|}{C_{\text{true}}} \times 100\% = \frac{23.31}{953.10} \times 100\% \approx 2.45\%$$

5.2.2 Potential Sources of Discrepancy

Several factors may contribute to this pricing deviation:

Parameter Mismatch

- FX Volatility (σ_{fx}) Misestimation: Small changes in σ_{fx} can significantly impact Quanto adjustments.
- Dividend Yield (q) Uncertainty: An incorrect estimate of q affects the forward price of the underlying asset.

Numerical Approximation Errors

- Implied volatility surfaces may use different interpolation methods (e.g., Cubic Spline), affecting the final computed option price.
- Quanto discounting approximations may not fully capture correlation effects.

Model Assumptions

- The model may assume constant volatility, whereas market data may follow a stochastic volatility process.
- Incorrect discounting rates: Ensuring EUR OIS is used for EUR cash flows and USD SOFR rates for Quanto adjustments is crucial.

5.3 Conclusion

- Quanto Option pricing is highly sensitive to dividend yield and FX volatility.
- FX volatility has a negative impact on price, while higher dividend yields reduce option value.
- Differences in model assumptions, market data, and interpolation techniques can lead to pricing variations.

Future improvements could include:

- Using the Heston Model to introduce stochastic volatility.
- Applying Local Volatility Surface adjustments for more accurate Quanto discounting.
- Performing regression analysis with market data to refine pricing accuracy.