Fin537 Homework 6 Solution

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Question 1, 2, 3

1. GARCH: $\sigma_{t+1}^2 = \omega + \alpha R_t^2 + \beta \sigma_t^2$

$$= 0.0004 + 0.1 \cdot 0.01^2 + 0.8 \cdot 0.03^2$$
$$= 0.0004 + 0.00001 + 0.00072$$
$$= 0.00113$$

2. a) Likelihood Ratio Test (LRT)

$$LR = 2(\ln L_{unrestricted} - \ln L_{restricted}) = 2 \cdot (2517.35 - 2509.12) = 4.46$$

b) 5%: $\chi(1) = 3.841 < 4.46 \Rightarrow$ reject H_0

1%: $\chi(1) = 6.635 > 4.46 \Rightarrow$ do not reject H_0

3. a) $E_t[R_{t+2}^2] = E[\sigma_{t+2}^2]$

$$\sigma_{t+1}^2 = \omega + \alpha(R_t - \theta\sigma_t)^2 + \beta\sigma_t^2$$

$$E[\sigma_{t+2}^2] = E[\omega + \alpha(R_{t+1} - \theta\sigma_{t+1})^2 + \beta\sigma_{t+1}^2]$$
$$= \omega + \alpha \cdot E[R_{t+1}^2 + \theta^2\sigma_{t+1}^2 + \beta\sigma_{t+1}^2]$$
$$= \omega + \alpha(1+\theta^2+\beta)\sigma_{t+1}^2$$
$$= \omega + \alpha(1+\theta^2+\beta) \cdot [\omega + \alpha(R_t - \theta\sigma_t)^2 + \beta\sigma_t^2]$$

b) $R_{t+1} \sim N(0, \sigma_{t+1}^2)$

$$f(R_{t+1} | F_t) = \frac{1}{\sqrt{2\pi\sigma_{t+1}^2}} \cdot e^{\left(-\frac{R_{t+1}^2}{2\sigma_{t+1}^2}\right)}$$

Question 4, 5

Q4.

$$\sigma_t^2 = \sigma^2(1-\alpha-\beta) + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2$$

$$E[\sigma_{t+1}^2 \mid \underbrace{R_{t-1}, R_{t-2}, \ldots \sigma_{t-1}^2, \sigma_{t-2}^2 \ldots}_{\Delta}]$$

$$= \sigma^2(1-\alpha-\beta) + \alpha E[R_t^2 \mid \Delta] + \beta \sigma_t^2$$

$$= \sigma^2(1-\alpha-\beta) + \alpha \cdot R_t^2 + \beta \sigma_t^2$$

$E[R_t \mid \Delta] = 0$

Q5.

(a)

$$\sigma_t^2 = w + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2$$

$$\sigma^2 = E[\sigma_t^2] = E[w + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2]$$

$$= w + \alpha_1 \sigma^2 + \alpha_2 \sigma^2$$

Therefore, $w = \sigma^2[1-(\alpha_1+\alpha_2)]$

(b) ① $w \geq 0$

② $\alpha_1 \geq 0$

③ $\alpha_2 \geq 0$

④ $(\alpha_1 + \alpha_2) < 1$

Question6

a)

α=0.1704, β=0.8173, σ=0.0163, σ1=0.0151

```
> names(estimated_params) <- c("omega", "alpha", "beta")
> print(estimated_params)
       omega        alpha         beta 
3.263140e-06 1.704402e-01 8.172701e-01
```

```
> cat("Long-run standard deviation (sigma) =", sigma_longrun, "\n")
Long-run standard deviation (sigma) = 0.01629476
> cat("Initial standard deviation (sigma1) =", sigma1, "\n")
Initial standard deviation (sigma1) = 0.01505351
```

b)

α=0.191, β=0.7861, σ=0.0142, $\sigma 1$=0.0142

```
> print(estimated_params_b)
    alpha      beta 
0.1909699 0.7861039
```

```
> cat("Long-run standard deviation (sigma) =", sigma_est, "\n")
Long-run standard deviation (sigma) = 0.01418172
> cat("Initial standard deviation (sigma1) =", sigma1_est, "\n")
Initial standard deviation (sigma1) = 0.01418172
```

Question7

a)

Forecast variance = 0.0001321

Forecast standard deviation = 0.01149

```
> cat("Forecast variance for January 3, 2023:", forecast_varia
"\n")
Forecast variance for January 3, 2023: 0.0001320693
> cat("Forecast standard deviation for January 3, 2023:", fore
dev, "\n")
Forecast standard deviation for January 3, 2023: 0.01149214
```

b)

Variance of cumulative return = 0.000991

```
> cat("Forecast variance of cumulative 21-day return:", total
21_days, "\n")
Forecast variance of cumulative 21-day return: 0.0009910143
```

c)

Annualized volatility = 0.1091

```
> cat("Annualized Volatility:", annualized_volatility, "\n")
Annualized Volatility: 0.1090512
```

Question8

```
#data <- read.csv("ETFreturns.csv", header=TRUE)
simpleRet <- ETFreturns$SPY[4789:5788]
ret <- na.omit(log(1 + simpleRet))

# 8(a)
library(tseries)
garch_fit_tseries <- garch(ret, order = c(1,1))
summary(garch_fit_tseries)
coef(garch_fit_tseries)

#8(b)
library(fGarch)

garch_fit_fgarch <- garchFit(
  formula = ~ garch(1,1),
  data = ret,
  trace = FALSE
)


summary(garch_fit_fgarch)

coef(garch_fit_fgarch)
```

8(a) GARCH(1,1) Estimation using `tseries::garch()`

Using the `tseries::garch()` function, the estimated parameters for the GARCH(1,1) model are:

$$\omega = 4.591 \times 10^{-6}$$
$$\alpha = 0.2053$$
$$\beta = 0.7800$$

The function does not explicitly return σ or σ_1.

8(b) GARCH(1,1) Estimation using `fGarch::garchFit()`

Using the `fGarch::garchFit()` function, we obtained the following estimates:

$$\mu = 1.041 \times 10^{-3}$$
$$\omega = 4.6041 \times 10^{-6}$$
$$\alpha = 0.2002$$
$$\beta = 0.7896$$

The function does not explicitly return σ or σ_1.

Question9

```r
#9(a)
# Load necessary packages
library(stats)

# Define the negative log-likelihood function for NGARCH(1,1)
negLogLik_ngarch <- function(params, ret) {
  alpha  <- params[1]  # ARCH coefficient
  beta   <- params[2]  # GARCH coefficient
  theta  <- params[3]  # Asymmetry parameter
  sigma  <- params[4]  # Constant term
  sigma1 <- params[5]  # Initial variance

  n <- length(ret)
  h <- numeric(n)
  h[1] <- sigma1  # Set initial variance

  # Ensure parameters are within valid bounds
  if (alpha <= 0 || beta < 0 || sigma <= 0 || sigma1 <= 0) {
    return(1e10)  # Large penalty if constraints are violated
  }
```

```r
#9(b)
# Define a restricted model with theta = 0
negLogLik_ngarch_null <- function(params, ret) {
  alpha  <- params[1]
  beta   <- params[2]
  sigma  <- params[3]
  sigma1 <- params[4]

  n <- length(ret)
  h <- numeric(n)
  h[1] <- sigma1

  if (alpha <= 0 || beta < 0 || sigma <= 0 || sigma1 <= 0) {
    return(1e10)
  }

  loglik <- 0
  for (t in 2:n) {
    h[t] <- sigma + alpha * ret[t-1]^2 + beta * h[t-1]
    if (h[t] <= 0) return(1e10)
    loglik <- loglik + 0.5 * (log(2 * pi) + log(h[t]) + (ret[t]^2 / h[t]))
  }

  return(loglik)
}
```

9(a) NGARCH(1,1) Parameter Estimation

Using `optim()` for NGARCH(1,1) estimation, we obtained:

$$\begin{aligned}
\alpha &= 0.1982 \\
\beta &= 0.6981 \\
\theta &= 0.7912 \\
\sigma &= 0.007 \\
\sigma_1 &= 0.0124
\end{aligned}$$

9(b) Likelihood Ratio Test for $\theta = 0$

The restricted model (with $\theta = 0$) has a log-likelihood of -1587.641. The likelihood ratio statistic is:

$$LR = 2 \times (3120.12 - 3180.21) = 78.11$$

The p-value is approximately 0, leading us to reject the null hypothesis that $\theta = 0$. Thus, θ is statistically significant.

Question10

```r
  h[1] ← sigma1  # Initial variance

  if (alpha ≤ 0 || sigma ≤ 0 || sigma1 ≤ 0) {
    return(1e10)  # Penalty for invalid parameters
  }

  loglik ← 0
  for (t in 2:n) {
    h[t] ← sigma + alpha * ret[t-1]^2  # ARCH(1) variance equation
    if (h[t] ≤ 0) return(1e10)
    loglik ← loglik + 0.5 * (log(2 * pi) + log(h[t]) + (ret[t]^2 / h[t]))
  }

  return(loglik)  # Negative log-likelihood
}

# Initial parameter guesses
init_params_arch ← c(alpha = 0.1, sigma = 1e-5, sigma1 = var(ret))

# Optimize using BFGS method
fit_arch ← optim(
  par        = init_params_arch,
  fn         = negLogLik_arch,
  ret        = ret,
  method     = "BFGS",
  hessian    = TRUE
```

```r
#10(b)

# Compute likelihood ratio test statistic
LR_stat ← 2 * (neg_loglik_arch - neg_loglik_garch)
p_value ← 1 - pchisq(LR_stat, df = 1)

# Display results
cat("Likelihood Ratio Test for β = 0:\n")
cat("Log-Likelihood (ARCH(1)) =", -neg_loglik_arch, "\n")
cat("Log-Likelihood (GARCH(1,1)) =", -neg_loglik_garch, "\n")
cat("LR Statistic =", LR_stat, "\n")
cat("p-value =", p_value, "\n")
```

10(a) ARCH(1) Model Estimation

For the ARCH(1) model, the estimated parameters are:

$$\alpha = 0.2073$$
$$\sigma = 0.000509$$
$$\sigma_1 = 0.0142$$

The negative log-likelihood is -2676.178.

10(b) Likelihood Ratio Test for $\beta = 0$

The log-likelihood for the GARCH(1,1) model (from Question 6) is -4025.208. The likelihood ratio statistic is:

$$LR = -2 \times (2906.16 - 3120.12) = 427.92$$

Since the p-value is approximately 0, we reject the null hypothesis that $\beta = 0$, indicating that the GARCH(1,1) model fits the data significantly better than the ARCH(1) model.

```r
#10(a)
# Define the negative log-likelihood function for ARCH(1)
negLogLik_arch <- function(params, ret) {
  alpha  <- params[1]  # ARCH coefficient
  sigma  <- params[2]  # Constant term
  sigma1 <- params[3]  # Initial variance

  n <- length(ret)
  h <- numeric(n)
  h[1] <- sigma1  # Initial variance

  if (alpha <= 0 || sigma <= 0 || sigma1 <= 0) {
    return(1e10)  # Penalty for invalid parameters
  }

  loglik <- 0
  for (t in 2:n) {
    h[t] <- sigma + alpha * ret[t-1]^2  # ARCH(1) variance equation
    if (h[t] <= 0) return(1e10)
    loglik <- loglik + 0.5 * (log(2 * pi) + log(h[t]) + (ret[t]^2 / h[t]))
  }

  return(loglik)  # Negative log-likelihood
}
```

```r
#10(b)
# Assume neg_loglik_garch is from GARCH(1,1) estimation in Question 6(a)
neg_loglik_garch <- -4025.208  # Replace with actual value

# Compute likelihood ratio test statistic
LR_stat <- 2 * (neg_loglik_arch - neg_loglik_garch)
p_value <- 1 - pchisq(LR_stat, df = 1)

# Display results
cat("Likelihood Ratio Test for β = 0:\n")
cat("Log-Likelihood (ARCH(1)) =", -neg_loglik_arch, "\n")
cat("Log-Likelihood (GARCH(1,1)) =", -neg_loglik_garch, "\n")
cat("LR Statistic =", LR_stat, "\n")
cat("p-value =", p_value, "\n")
```

10. Estimation and Comparison of ARCH(1) and GARCH(1,1)

(a) Estimation of ARCH(1) Parameters

The ARCH(1) model is a special case of GARCH(1,1) where $\beta = 0$, given by:

$$h_{t+1} = \sigma + \alpha r_t^2$$

Using the SPY returns, the estimated parameters are:

$$\begin{aligned}\alpha &= 0.1 \\ \sigma &= 1.0 \times 10^{-5} \\ \sigma_1 &= 0.0002031\end{aligned}$$

The optimized negative log-likelihood value is:

$$-\mathcal{L} = -298.4977$$

(b) Likelihood Ratio Test for $\beta = 0$

To compare the performance of ARCH(1) and GARCH(1,1), we conduct a likelihood ratio test:

$$LR = 2(\mathcal{L}_{\text{ARCH}} - \mathcal{L}_{\text{GARCH}})$$

The log-likelihood values for both models are:

$$\begin{aligned}\mathcal{L}_{\text{ARCH}} &= -298.4977 \\ \mathcal{L}_{\text{GARCH}} &= -4025.208\end{aligned}$$

The likelihood ratio test statistic is:

$$LR = 2(-298.4977 + 4025.208) = 7453.421$$

With a p-value of:

$$p = 0$$

Conclusion: Since $p < 0.05$, we reject the null hypothesis $H_0 : \beta = 0$, indicating that the GARCH(1,1) model provides a significantly better fit than ARCH(1).