Question 3: Poisson Process and Coin Tosses

Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be two independent Poisson processes with rates λ_1 and λ_2, respectively. With probability p, we observe $N_1(t)$, and with probability $1-p$, we observe $N_2(t)$. Define a stochastic process $\{N(t), t \geq 0\}$ as:

$$N(t) = \begin{cases} N_1(t), & \text{with probability } p, \\ N_2(t), & \text{with probability } 1-p. \end{cases}$$

(a) Does $\{N(t), t \geq 0\}$ possess stationary increments?

For $\{N(t)\}$ to have stationary increments, the probability of an increment, $P(N(t+s) - N(s) = k)$, should depend only on t and not on s.

Given:

$$P(N(t+s)-N(s) = k) = p \cdot P(N_1(t+s)-N_1(s) = k) + (1-p) \cdot P(N_2(t+s)-N_2(s) = k),$$

where:

$$P(N_1(t+s) - N_1(s) = k) = \frac{(\lambda_1 t)^k e^{-\lambda_1 t}}{k!},$$
$$P(N_2(t+s) - N_2(s) = k) = \frac{(\lambda_2 t)^k e^{-\lambda_2 t}}{k!}.$$

Thus:

$$P(N(t+s) - N(s) = k) = p \cdot \frac{(\lambda_1 t)^k e^{-\lambda_1 t}}{k!} + (1-p) \cdot \frac{(\lambda_2 t)^k e^{-\lambda_2 t}}{k!}.$$

The probabilities depend only on t, not on s. Therefore, $\{N(t)\}$ has **stationary increments**.

(b) Does $\{N(t), t \geq 0\}$ possess independent increments?

For $\{N(t)\}$ to have independent increments, the increments $N(t+s) - N(s)$ and $N(s)$ should be independent.

Since $N_1(t)$ and $N_2(t)$ are independent Poisson processes, their increments are independent:

$$P(N(t+s) - N(s) = i, N(s) = j) = P(N(t+s) - N(s) = i) \cdot P(N(s) = j),$$

where:

$$P(N(t+s)-N(s)=i)=p\cdot\frac{(\lambda_1 t)^i e^{-\lambda_1 t}}{i!}+(1-p)\cdot\frac{(\lambda_2 t)^i e^{-\lambda_2 t}}{i!},$$

$$P(N(s)=j)=p\cdot\frac{(\lambda_1 s)^j e^{-\lambda_1 s}}{j!}+(1-p)\cdot\frac{(\lambda_2 s)^j e^{-\lambda_2 s}}{j!}.$$

Thus, the increments $N(t+s)-N(s)$ and $N(s)$ are not independent. Except when p=0 or 1.

(c) Is $\{N(t), t \geq 0\}$ a Poisson process?

A process is a Poisson process if it satisfies the following: 1. $N(0) = 0$. 2. The process has stationary increments. 3. The process has independent increments.

$\{N(t)\}$ is not a **Poisson process** except p=0 or 1.