Problems and Solutions to
Chapter 2: The Klein-Gordan Field

嘤呜

(Dated: 2021.4.4-2021.4.)

I. PROBLEM 2.1

Classical electromagnetism (with no sources) follow from the action

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \quad \text{where} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

(a) Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components of $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$.

Solution:

The basic idea is to use Euler-Lagrange equation directly with $\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = 0 \tag{1}$$

but we should start with

$$\frac{\partial F_{\rho\sigma}}{\partial \partial_\mu A_\nu} = \frac{\partial}{\partial \partial_\mu A_\nu}\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right) = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho$$

$$\frac{\partial F^{\rho\sigma}}{\partial \partial_\mu A_\nu} = g^{\rho\lambda}g^{\sigma\eta}\frac{\partial}{\partial \partial_\mu A_\nu}\left(\partial_\lambda A_\eta - \partial_\eta A_\lambda\right) = \delta^{\rho\mu}\delta^{\sigma\nu} - \delta^{\sigma\mu}\delta^{\rho\nu}$$

so that

$$\frac{\partial \mathcal{L}_{EM}}{\partial \partial_\mu A_\nu} = -\frac{1}{4}(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)F^{\rho\sigma} - \frac{1}{4}F_{\rho\sigma}(\delta^{\rho\mu}\delta^{\sigma\nu} - \delta^{\sigma\mu}\delta^{\rho\nu}) = -F^{\mu\nu}$$

notice that there is no A_μ term in $\mathcal{L}_{\text{EM}}$, so we have from Eq. (1) that

$$\partial_\mu F^{\mu\nu} = 0$$

Identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$, we have explicitly for $\nu = 0$

$$\partial_i E^i = 0, \ i.e. \quad \nabla \cdot \vec{E} = 0$$

and for $\nu = i$

$$\partial_0 E^i - \partial_j \epsilon^{ijk} B^k = 0, \ i.e. \quad \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B}$$

Another two equations follows from Bianchi Identity: $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$. Specific derivations can be found in this question on Physics Stack Exchange(Equivalent form of Bianchi identity in electromagnetism).

(b) Construct the energy-momentum tensor for this theory. Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is anti-symmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu,$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum indices:

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2); \quad \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}$$

Solution:

The conservation of energy-momentum tensor comes from the invariance of EOM under the space-time translation: $x^\mu \to x^\mu - a^\mu$, the corresponding transformations of the electromagnetic field and the Lagragian are

$$A_\mu(x) \to A_\mu(x+a) = A_\mu(x) + a^\nu \partial_\nu A_\mu(x), \mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} + a^\nu \partial_\mu(\delta^\mu_\nu \mathcal{L})$$

make a substitution $\alpha\Delta\phi(x) \to a^\nu \partial_\nu A_\mu(x)$ in Eq(2.9)-(2.12) in the book, we have immediately

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial_\nu A_\rho - \delta^\mu_\nu \mathcal{L} = -F^{\mu\rho}\partial_\nu A_\rho - \delta^\mu_\nu \mathcal{L}$$

obviously, this is not a symmetric tensor, and the conservation of this currents still holds for $\hat{T}^{\mu\nu}$ since $\partial_\mu \partial_\lambda K^{\lambda\mu\nu} = \partial_\lambda \partial_\mu K^{\mu\lambda\nu} = -\partial_\lambda \partial_\mu K^{\lambda\mu\nu} = 0$.

$$\begin{aligned}
\hat{T}^{\mu\nu} &= -F^{\mu\rho}\partial^\nu A_\rho - \delta^{\mu\nu}\mathcal{L} + \partial_\lambda(F^{\mu\lambda}A^\nu) \\
&= -F^{\mu\rho}\partial^\nu A_\rho + F^{\mu\lambda}\partial_\lambda A^\nu - \delta^{\mu\nu}\mathcal{L} \\
&= -F^\mu_\rho \partial^\nu A^\rho + F^\mu_\rho \partial^\rho A^\nu - \delta^{\mu\nu}\mathcal{L} \\
&= F^\mu_\rho F^{\rho\nu} - \delta^{\mu\nu}\mathcal{L}
\end{aligned}$$

this symmetric tensor that is equivalent to the canonical energy-momentum tensor is not unique, such a tensor is called Belinfante energy-momentum tensor, first discovered by F. J. Belinfante [1]. With such a energy-momentum tensor, we can have the familiar energy and momentum densities for the electromagnetic field:

$$\begin{aligned}
\hat{T}^{00} &= F^0_i F^{i0} + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} = \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2) \\
\hat{T}^{0i} &= F^0_j F^{ji} = (\vec{E} \times \vec{B})_i
\end{aligned}$$

II. PROBLEM 2.2 THE COMPLEX SCALAR FIELD

Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi)$$

It is easiest to analyze the theory by considering $\phi(x)$ and $\phi^*(x)$, rather thant the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

(a) Find the conjugate momenta of $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi)$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

[1] F. J. Belinfante, Physica **7**, 449 (1940).