Inverse and Direct Dynamic Modeling of Gough-Stewart Robots

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Inverse and Direct Dynamic Modeling of Gough-Stewart Robots

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***Abstract*--This paper presents closed form solutions for the inverse and direct dynamic models of the Gough-Stewart parallel robot. The models are obtained in terms of the Cartesian dynamic model elements of the legs and of the Newton-Euler equation of the platform. The final form has an interesting and intuitive physical interpretation. The base inertial parameters of the robot, which constitute the minimum number of inertial parameters, are explicitly determined. The number of operations to compute the inverse and direct dynamic models are given.**

***Index Terms*--Parallel robots, inverse dynamic model, direct dynamic model, base parameters, computational cost.**

I. INTRODUCTION

The inverse dynamic modeling is important for high performance control algorithms of robots, and the direct dynamic model is required for their simulation. The dynamic modeling of parallel robots presents an inherent complexity due to their closed-loop structure and kinematic constraints. To obtain the dynamics of parallel robots, many methods have used the classical procedure of computing the dynamic model of an equivalent tree structure, then the system constraints are considered by the use of the Lagrange multipliers [1]-[5]. The principle of virtual work has been used in [6]-[8]. The Newton-Euler formulation has been applied as well, for instance:

- Reboulet et al. [9] have proposed a matrix formulation for the dynamics of a simplified Stewart parallel robot. They neglected the piston rod mass and the rotation of the legs around their main axes.
- Gosselin [10] has proposed an inverse dynamic model for a general Stewart parallel robot. This method is difficult to generalize to other structures and the direct dynamic problem has not been treated.
- Dasgupta et al. [11][12] have proposed closed form dynamic equations of the general Stewart platform. They applied their algorithm to several planar and spatial parallel robots [13]. The computational cost of this method is not optimized.
- Ji [14] has discussed the influence of the leg inertia on the dynamic model.

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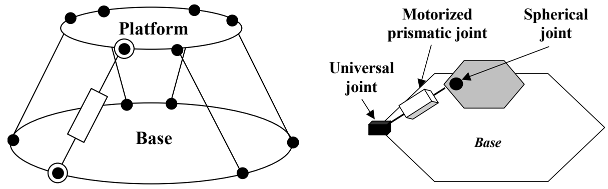

Fig. 1. Description of the 6 degree of freedom Gough-Stewart robot.

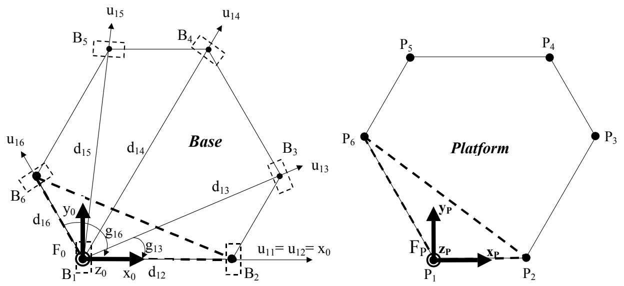

Fig. 2. Frame F_0 fixed with the base and frame F_p fixed with the mobile platform.

This paper presents closed form solutions for the complete inverse and direct dynamic models. The models are obtained in terms of the dynamic models of the legs. Thus, one can use the method with which he is familiar to obtain this model (Lagrange, Newton-Euler, Kane, …). A closed form solution to determine the base inertial parameters, which represent the minimum number of parameters to compute the dynamic model, is also presented.

This paper is organized as follows:

Section II describes the structure of the robot and recalls its geometric modeling. Section III reviews the kinematic modeling of the robot. Then, the inverse and the direct dynamic models of the robot are presented in section IV and section V respectively. Section VI determines the base inertial parameters of the robot. Finally, section VII gives the computational cost of the proposed models.

II. DESCRIPTION OF THE ROBOT

The 6 d.o.f. Gough-Stewart robot is composed of a moving platform connected to a fixed base by six extendable legs. The extremities of each leg are fitted with a 2 degree of freedom (d.o.f.) universal joint at the base and a 3 d.o.f. spherical joint at the platform (fig. 1). The universal joint center and the spherical joint center are denoted by B_i and P_i (i = 1 to 6)

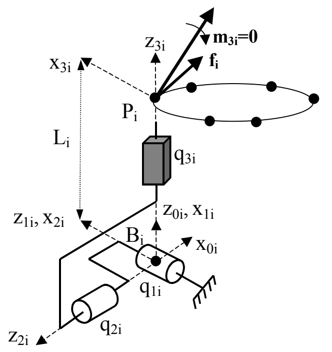

Fig. 3. Link frames of a leg and reaction force $\mathbf{f_i}$ between leg i and the platform

TABLE I
GEOMETRIC PARAMETERS OF THE LEG i FRAMES

j_i	$a(j_i)$	μ_{ji}	σ_{ji}	γ_{ji}	b_{ji}	α_{ji}	d_{ji}	θ_{ji}	r_{ji}
1i	0	0	0	γ_{1i}	b_{1i}	$-\pi/2$	d_{1i}	q_{1i}	0
2i	1i	0	0	0	0	$\pi/2$	0	q_{2i}	0
3i	2i	1	1	0	0	$\pi/2$	0	0	q_{3i}

for i = 1, …, 6 and $\gamma_{11} = b_{11} = d_{11} = \gamma_{12} = b_{12} = b_{16} = 0$

respectively. The length of each leg is actuated using an active prismatic joint.

We define the frame F_0 fixed with the base and the frame F_p fixed with the mobile platform. To minimize the number of geometric parameters that are not equal to zero, we place these frames as follows (fig. 2) [15]:

- The origin of frame F_0 is B_1, the x_0 axis is along B_1B_2 and the (x_0,y_0) plane is defined by (B_1,B_2,B_6).
- The origin of frame F_p is P_1, the x_p axis is along P_1P_2 and the (x_p,y_p) plane is defined by (P_1,P_2,P_6).

The axes of the first revolute joint of each leg are placed as shown in figure 2. The Khalil and Kleinfinger notations [16][17] are used to describe the geometry of the tree structure, which is composed of the base and the six legs. Each leg is composed of three moving links and three joints (2 passive revolute joints and 1 active prismatic joint).

Let j_i denotes the link j of leg i, the local link frames are defined as seen in (fig. 3). The geometric parameters defining these frames are given in table I, where a(j) denotes the antecedent of link j, the parameter μ_j is one for a motorized joint and zero for a passive joint, $\sigma_j = 0$ for a revolute joint, and $\sigma_j = 1$ for a prismatic joint. The parameters (γ_j, b_j, α_j, d_j, θ_j, r_j) are used to determine the location of frame F_j with respect to its antecedent F_i. It is denoted by the transformation matrix [17]:

$$ {}^{i}\mathbf{T}_j = \begin{bmatrix} {}^{i}\mathbf{R}_j & {}^{i}\mathbf{P}_j \\ \mathbf{0}_{(1x3)} & 1 \end{bmatrix} \qquad (1)$$

Where:

${}^{i}\mathbf{R}_j$ is the (3×3) rotation matrix :

$$ {}^{i}\mathbf{R}_j = \begin{bmatrix} {}^{i}\mathbf{s}_j & {}^{i}\mathbf{n}_j & {}^{i}\mathbf{a}_j \end{bmatrix} \qquad (2)$$

${}^{i}\mathbf{P}_j$ is the (3×1) position vector.

III. KINEMATIC MODELING

The following notations are used:

$\mathbf{a_{3i}}$ unit vector along the z_{3i} axis,

${}^{0}\mathbf{L_i}$ represents the position vector $\mathbf{P_1 P_i}$ referred to frame F_0, in the following the upper left exponent indicates the projection frame.

${}^{0}\hat{\mathbf{L}}_i$ denotes the (3×3) skew symmetric matrix associated with the vector ${}^{0}\mathbf{L_i}$.

${}^{0}\mathbf{V_p}$ linear velocity of the origin of the platform P_1.

${}^{0}\boldsymbol{\omega}_\mathbf{p}$ angular velocity of the platform,

${}^{0}\dot{\mathbf{V}}_\mathbf{p}$ linear acceleration of the origin of the platform,

${}^{0}\dot{\boldsymbol{\omega}}_\mathbf{p}$ angular acceleration of the platform,

${}^{0}\mathbb{V}_\mathbf{P}$ (6x1) spatial velocity vector. It is composed of the linear and the angular velocities of the platform:

$$ {}^{0}\mathbb{V}_\mathbf{P} = \begin{bmatrix} {}^{0}\mathbf{V_P} \\ {}^{0}\boldsymbol{\omega}_\mathbf{P} \end{bmatrix} \qquad (3)$$

${}^{0}\dot{\mathbb{V}}_\mathbf{P}$ The spatial acceleration vector:

$$ {}^{0}\dot{\mathbb{V}}_\mathbf{P} = \begin{bmatrix} {}^{0}\dot{\mathbf{V}}_\mathbf{P} \\ {}^{0}\dot{\boldsymbol{\omega}}_\mathbf{P} \end{bmatrix} \qquad (4)$$

${}^{0}\mathbf{V_{pi}}$ linear velocity of point P_i of the platform:

$$ {}^{0}\mathbf{V_{Pi}} = {}^{0}\mathbf{V_P} + {}^{0}\boldsymbol{\omega}_\mathbf{P} \times {}^{0}\mathbf{L_i} \qquad (5)$$

${}^{0}\dot{\mathbf{V}}_\mathbf{pi}$ linear acceleration of point P_i of the platform:

$$ {}^{0}\dot{\mathbf{V}}_\mathbf{Pi} = {}^{0}\dot{\mathbf{V}}_\mathbf{P} + {}^{0}\dot{\boldsymbol{\omega}}_\mathbf{P} \times {}^{0}\mathbf{L_i} + {}^{0}\boldsymbol{\omega}_\mathbf{P} \times \left({}^{0}\boldsymbol{\omega}_\mathbf{P} \times {}^{0}\mathbf{L_i} \right) \qquad (6)$$

It can also be written as:

$$ {}^{0}\dot{\mathbf{V}}_\mathbf{Pi} = \begin{bmatrix} \mathbf{I_3} & -{}^{0}\hat{\mathbf{L}}_i \end{bmatrix} {}^{0}\dot{\mathbb{V}}_\mathbf{P} + {}^{0}\boldsymbol{\omega}_\mathbf{P} \times \left({}^{0}\boldsymbol{\omega}_\mathbf{P} \times {}^{0}\mathbf{L_i} \right) \qquad (7)$$

Where $\mathbf{I_3}$ is the (3×3) identity matrix.

$\boldsymbol{\Gamma}$ (6×1) vector of the prismatic joint forces:

$$ \boldsymbol{\Gamma} = \begin{bmatrix} \Gamma_{31} & \cdots & \Gamma_{36} \end{bmatrix}^T \qquad (8)$$

$\mathbf{q_a}$ (6×1) vector of the motorized joint variables:

$$ \mathbf{q_a} = \begin{bmatrix} q_{31} & \cdots & q_{36} \end{bmatrix}^T \qquad (9)$$

$\dot{\mathbf{q}}_i$ (3×1) vector of the joint velocities of leg i:

$$ \dot{\mathbf{q}}_i = \begin{bmatrix} \dot{q}_{1i} & \dot{q}_{2i} & \dot{q}_{3i} \end{bmatrix}^T \qquad (10)$$

$\ddot{\mathbf{q}}_i$ (3×1) vector of the joint acceleration of leg i:

$$ \ddot{\mathbf{q}}_i = \begin{bmatrix} \ddot{q}_{1i} & \ddot{q}_{2i} & \ddot{q}_{3i} \end{bmatrix}^T \qquad (11)$$

The following models are well-known [17][18] and will be used in the following:

i) The inverse kinematic model of the robot, which gives the velocities of the active joints $\dot{q}_{3i}$ (i = 1 to 6) as a function of the spatial velocity of the platform. It is defined by:

$$ \dot{\mathbf{q}}_\mathbf{a} = {}^{0}\mathbf{J}_\mathbf{p}^{-1}\, {}^{0}\mathbb{V}_\mathbf{p} \qquad (12)$$

The inverse Jacobian matrix of the robot is given as [18]:

$$ {}^{0}\mathbf{J}_{p}^{-1} = \begin{bmatrix} {}^{0}\mathbf{a}_{31}^{T} & \left({}^{0}\hat{\mathbf{L}}_{1}\,{}^{0}\mathbf{a}_{31}\right)^{T} \\ \vdots & \vdots \\ {}^{0}\mathbf{a}_{36}^{T} & \left({}^{0}\hat{\mathbf{L}}_{6}\,{}^{0}\mathbf{a}_{36}\right)^{T} \end{bmatrix} \quad (13) $$

Where ${}^{0}\mathbf{a}_{3i}$ is the unit vector along the z_{3i} axis.

We recall that for a well designed parallel robot the matrix ${}^{0}\mathbf{J}_{p}^{-1}$ should be regular in the reachable space [18], otherwise the robot risk to be destroyed when approaching singularity.

ii) The inverse kinematic model of a leg, which gives the joint velocities of the leg i ($\dot{q}_{1i}, \dot{q}_{2i}, \dot{q}_{3i}$) as a function of the linear velocity of point P_i:

$$ \dot{\mathbf{q}}_{i} = {}^{0}\mathbf{J}_{3i}^{-1}\,{}^{0}\mathbf{V}_{pi} \quad (14) $$

${}^{0}\mathbf{J}_{3i}^{-1}$ is the inverse Jacobian matrix of leg i.

The Jacobian matrix of the terminal point of leg i projected into frame F_{3i} is obtained as:

$$ {}^{3i}\mathbf{J}_{3i} = \begin{bmatrix} 0 & q_{3i} & 0 \\ -q_{3i}\sin(q_{2i}) & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (15) $$

Its inverse is given as:

$$ {}^{3i}\mathbf{J}_{3i}^{-1} = \begin{bmatrix} 0 & -1/\left(q_{3i}\sin(q_{2i})\right) & 0 \\ 1/q_{3i} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (16) $$

Note that:

$$ {}^{0}\mathbf{J}_{3i}^{-1} = {}^{3i}\mathbf{J}_{3i}^{-1}\,{}^{3i}\mathbf{R}_{0}, \quad {}^{0}\mathbf{J}_{3i}^{-T} = {}^{0}\mathbf{R}_{3i}\,{}^{3i}\mathbf{J}_{3i}^{-T} \quad (17) $$

Where ${}^{0}\mathbf{J}_{3i}^{-T}$ denotes $\left({}^{0}\mathbf{J}_{3i}^{-1}\right)^{T}$. This Jacobian is singular when $\sin(q_{2i}) = 0$ and/or $q_{3i} = 0$ which are physically impossible.

iii) The second order inverse kinematic model of the leg i:

$$ \ddot{\mathbf{q}}_{i} = {}^{0}\mathbf{J}_{3i}^{-1}\left({}^{0}\dot{\mathbf{V}}_{pi} - {}^{0}\dot{\mathbf{J}}_{3i}\,\dot{\mathbf{q}}\right) \quad (18) $$

Substituting equation (7) into equation (18), we obtain:

$$ \ddot{\mathbf{q}}_{i} = {}^{0}\mathbf{J}_{3i}^{-1}\left(\begin{bmatrix}\mathbf{I}_{3} & -{}^{0}\hat{\mathbf{L}}_{i}\end{bmatrix}{}^{0}\dot{\mathbb{V}}_{P} + {}^{0}\boldsymbol{\omega}_{P}\times\left({}^{0}\boldsymbol{\omega}_{P}\times{}^{0}\mathbf{L}_{i}\right) - {}^{0}\dot{\mathbf{J}}_{3i}\,\dot{\mathbf{q}}_{i}\right) \quad (19) $$

IV. INVERSE DYNAMIC MODEL

The inverse dynamic model gives the motorized joint forces as a function of the position, velocity and acceleration of the mobile platform. It is denoted by $\Gamma = \mathbf{f}({}^{0}\mathbf{T}_{P}, {}^{0}\mathbb{V}_{P}, {}^{0}\dot{\mathbb{V}}_{P})$. Using the inverse geometric and kinematic models of the legs, we can compute the joint positions, velocities and accelerations of the legs $(\mathbf{q}_{i}, \dot{\mathbf{q}}_{i}, \ddot{\mathbf{q}}_{i})$ in terms of the platform trajectory. Since the platform and the legs are connected by spherical joints, then only pure reaction force $\mathbf{f}_i$ exists between leg i and the platform (fig. 3) [9][10].

The inverse dynamic model will be obtained by first computing $\mathbf{f}_i$ as a function of Γ_{3i}, by the use of the dynamic model of leg i, then all the motorized joint forces Γ will be obtained from the Newton-Euler equations of the platform.

A. Computation of f_i

The general form of the inverse dynamic model of a leg i, is written as [17]:

$$ \Gamma_{i} = \mathbf{H}_{i}\left(\mathbf{q}_{i}, \dot{\mathbf{q}}_{i}, \ddot{\mathbf{q}}_{i}\right) + {}^{0}\mathbf{J}_{3i}^{T}\,{}^{0}\mathbf{f}_{i} \quad (20) $$

Γ_i is the (3×1) vector of the torques/forces of leg i, where Γ_{1i} and Γ_{2i} are zero, and Γ_{3i} is the prismatic joint force.

Using equation (20) the reaction force can be written as:

$$ {}^{0}\mathbf{f}_{i} = -\mathbf{H}_{xi}\left(\mathbf{q}_{i}, \dot{\mathbf{q}}_{i}, \ddot{\mathbf{q}}_{i}\right) + {}^{0}\mathbf{J}_{3i}^{-T}\,\Gamma_{i} \quad (21) $$

Where:

$$ \mathbf{H}_{xi} = {}^{0}\mathbf{J}_{3i}^{-T}\,\mathbf{H}_{i}\left(\mathbf{q}_{i}, \dot{\mathbf{q}}_{i}, \ddot{\mathbf{q}}_{i}\right) \quad (22) $$

$\mathbf{H}_{xi}$ transforms $\mathbf{H}_{i}\left(\mathbf{q}_{i}, \dot{\mathbf{q}}_{i}, \ddot{\mathbf{q}}_{i}\right)$ from the joint space into the position Cartesian space at point P_i [19][20].

Using equations (16) and (17) we obtain:

$$ {}^{0}\mathbf{f}_{i} = -\mathbf{H}_{xi}\left(\mathbf{q}_{i}, \dot{\mathbf{q}}_{i}, \ddot{\mathbf{q}}_{i}\right) + {}^{0}\mathbf{a}_{3i}\,\Gamma_{3i} \quad (23) $$

B. Computation of the motor forces

The Newton-Euler equation of the platform is written as [17] [21][22]:

$$ {}^{0}\mathbb{F}_{P} = {}^{0}\mathbb{I}_{P}\,{}^{0}\dot{\mathbb{V}}_{P} + \begin{bmatrix} {}^{0}\boldsymbol{\omega}_{P}\times\left({}^{0}\boldsymbol{\omega}_{P}\times{}^{0}\mathbf{MS}_{P}\right) \\ {}^{0}\boldsymbol{\omega}_{P}\times\left({}^{0}\mathbf{I}_{P}\,{}^{0}\boldsymbol{\omega}_{P}\right) \end{bmatrix} - \begin{bmatrix} M_{P}\mathbf{I}_{3} \\ {}^{0}\mathbf{M}\hat{\mathbf{S}}_{P} \end{bmatrix}{}^{0}\mathbf{g} \quad (24) $$

With:

${}^{0}\mathbb{F}_{p}$ total external forces and moments on the platform about the origin P_1, ${}^{0}\mathbf{g}$ acceleration of gravity, $\mathbf{I}_3$ (3×3) identity matrix, M_p mass of the platform,

${}^{0}\mathbf{I}_{p}$ (3×3) inertia tensor of the platform with respect to frame F_0, it is obtained by:

$$ {}^{0}\mathbf{I}_{p} = {}^{0}\mathbf{R}_{p}\,{}^{p}\mathbf{I}_{p}\,{}^{0}\mathbf{R}_{p}^{T} \quad (25) $$

${}^{P}\mathbf{I}_{p}$ (3×3) inertia tensor of the platform with respect to frame F_p. It is written as:

$$ {}^{P}\mathbf{I}_{P} = \begin{bmatrix} XX_{P} & XY_{P} & XZ_{P} \\ XY_{P} & YY_{P} & YZ_{P} \\ XZ_{P} & YZ_{P} & ZZ_{P} \end{bmatrix} \quad (26) $$

${}^{0}\mathbf{MS}_{p}$ first moments of the platform:

$$ {}^{P}\mathbf{MS}_{P} = \left[MX_{P}\ MY_{P}\ MZ_{P}\right]^{T} \quad (27) $$

$$ {}^{0}\mathbf{MS}_{P} = {}^{0}\mathbf{R}_{P}\,{}^{P}\mathbf{MS}_{P} \quad (28) $$

${}^{0}\mathbb{I}_{p}$ (6×6) spatial inertia matrix of the platform [21]:

$$ {}^{0}\mathbb{I}_{p} = \begin{bmatrix} M_{P}\mathbf{I}_{3} & -{}^{0}\mathbf{M}\hat{\mathbf{S}}_{P} \\ {}^{0}\mathbf{M}\hat{\mathbf{S}}_{P} & {}^{0}\mathbf{I}_{P} \end{bmatrix} \quad (29) $$

The external forces and moments on the platform, which are due to the reaction forces of the legs are given by:

$$ {}^{0}\mathbb{F}_{p} = \sum_{i=1}^{6}\begin{bmatrix} \mathbf{I}_{3} \\ {}^{0}\hat{\mathbf{L}}_{i} \end{bmatrix}{}^{0}\mathbf{f}_{i} \quad (30) $$

Substituting equation (23) into equation (30) and using (13) , we obtain [23]:

$$ \Gamma = {}^{0}\mathbf{J}_{p}^{T}\left({}^{0}\mathbb{F}_{p} + {}^{0}\mathbb{F}_{leg}\right) \quad (31) $$

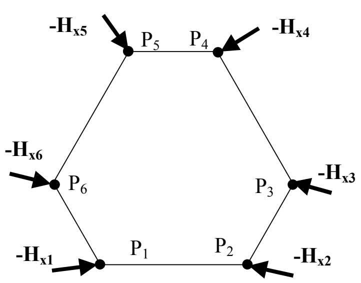

Fig. 4. The equivalent body of Gough-Stewart robot.

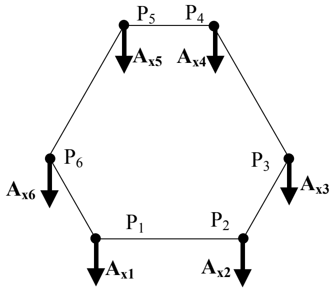

Fig. 5. Equivalent spatial inertia matrix of the legs.

With:

$$\mathbb{F}_{\mathbf{leg}} = \sum_{i=1}^{6}\left[\begin{bmatrix}\mathbf{I_3}\\ {}^{0}\hat{\mathbf{L}}_{i}\end{bmatrix}\mathbf{H}_{xi}\left(\mathbf{q}_i,\dot{\mathbf{q}}_i,\ddot{\mathbf{q}}_i\right)\right] \tag{32}$$

Equation (31) represents the closed form solution of the inverse dynamic model of the parallel robot. To compute the inverse dynamic model, beside the computation of the Jacobian matrices, we need to determine $\mathbf{H}_i\left(\mathbf{q}_i,\dot{\mathbf{q}}_i,\ddot{\mathbf{q}}_i\right)$, which represents the inverse dynamic model of leg i. Different methods can be used to calculate this vector numerically or symbolically [24][25]. To reduce the computational cost, the recursive Newton-Euler method using customized symbolic technique and the base inertial parameters could be used [17][26].

C. *Generality of the algorithm*

From equation (32) we deduce that the effect of each leg i dynamics on the platform is equivalent to the application of an external force $-\mathbf{H}_{xi}\left(\mathbf{q}_i,\dot{\mathbf{q}}_i,\ddot{\mathbf{q}}_i\right)$ at each point P_i (fig. 4).

Thus, the inverse dynamic model of a general parallel robot can be computed as follows:

i) Calculate the dynamic model of each leg $\mathbf{H}_i\left(\mathbf{q}_i,\dot{\mathbf{q}}_i,\ddot{\mathbf{q}}_i\right)$.

ii) Calculate the Cartesian dynamic model of each leg $\mathbf{H}_{xi}$.

iii) Calculate the forces and moments $\mathbb{F}_{leg}$, which represent the forces and/or moments to overcome all $\mathbf{H}_{xi}$.

iv) Find the forces and/or moments required to move the platform, $\mathbb{F}_{\mathbf{P}}$, as given by (24).

v) Compute the motor forces by $\mathbf{\Gamma} = {}^{0}\mathbf{J}_{p}^{T}\left(\mathbb{F}_{P} + \mathbb{F}_{leg}\right)$.

We have to define, on a case by case basis, the components of the forces or moment of $\mathbf{H}_{xi}$ and the degrees of freedom of the Newton-Euler equations of the platform. These two things are easy to determine. For example, if the platform has only translational motion, we just take into account the first 3 equations of Newton-Euler [27]. In [28] we show the application of this method for different parallel robots.

V. DIRECT DYNAMIC MODEL

The direct dynamic model of the robot gives the platform acceleration as a function of the state of the robot (platform position and velocity) and the input forces of the active joints. It is denoted by ${}^{0}\dot{\mathbb{V}}_{p} = \mathbf{f}\left({}^{0}\mathbf{T}_{p}, {}^{0}\mathbb{V}_{p}, \Gamma\right)$.

This model will be obtained by first calculating the joint accelerations of the legs $\ddot{\mathbf{q}}_i$ as a function of ${}^{0}\dot{\mathbb{V}}_{p}$ in the inverse dynamic model of leg i. Then ${}^{0}\dot{\mathbb{V}}_{p}$ will be obtained from the Newton-Euler equations of the platform.

The dynamic model of the leg i can be rewritten as [17][29]:

$$\mathbf{H}_i\left(\mathbf{q}_i,\dot{\mathbf{q}}_i,\ddot{\mathbf{q}}_i\right) = \mathbf{A}_i\ddot{\mathbf{q}}_i + \mathbf{h}_i\left(\mathbf{q}_i,\dot{\mathbf{q}}_i\right) \tag{33}$$

Where :

$\mathbf{A_i}$ is the (3×3) inertia matrix of leg i and $\mathbf{h_i}$ is the (3×1) vector of Coriolis, centrifugal and gravity forces.

Multiplying (33) by ${}^{0}\mathbf{J}_{3i}^{-T}$ and using (19), we obtain:

$$\begin{aligned}\mathbf{H}_{xi} = & -\mathbf{A}_{xi}\left[\mathbf{I}_3 \quad -{}^{0}\hat{\mathbf{L}}_i\right]{}^{0}\dot{\mathbb{V}}_{P} \\ & -\mathbf{A}_{xi}\left({}^{0}\omega_{P}\times\left({}^{0}\omega_{P}\times{}^{0}\mathbf{L}_i\right)-{}^{0}\dot{\mathbf{J}}_{3i}\dot{\mathbf{q}}_i\right)-\mathbf{h}_{xi}\left(\mathbf{q}_i,\dot{\mathbf{q}}_i\right)\end{aligned} \tag{34}$$

Where:

$\mathbf{A}_{xi} = {}^{0}\mathbf{J}_{3i}^{-T}\mathbf{A}_i{}^{0}\mathbf{J}_{3i}^{-1}$ is the (3×3) Cartesian inertia matrix of leg i referred at point P_i, and $\mathbf{h}_{xi} = {}^{0}\mathbf{J}_{3i}^{-T}\mathbf{h}_i$ is the Cartesain, Coriolis, centrifugal and gravity forces of leg i at point P_i [19][20].

Substituting equation (34) into (32), and using (31) we obtain the direct dynamic model by:

$$\mathbf{A}_{\mathbf{robot}}\,{}^{0}\dot{\mathbb{V}}_{\mathbf{P}} = -{}^{0}\mathbf{J}_{\mathbf{p}}^{-\mathbf{T}}\,\mathbf{\Gamma} + \mathbf{h}_{\mathbf{robot}} \tag{35}$$

With:

$$\mathbf{A}_{\mathbf{robot}} = \sum_{i=1}^{6}\left\{\begin{bmatrix}\mathbf{A}_{xi} & -\mathbf{A}_{xi}{}^{0}\hat{\mathbf{L}}_i\\ {}^{0}\hat{\mathbf{L}}_i\mathbf{A}_{xi} & -{}^{0}\hat{\mathbf{L}}_i\mathbf{A}_{xi}{}^{0}\hat{\mathbf{L}}_i\end{bmatrix}\right\} + \begin{bmatrix}M_P\mathbf{I_3} & -{}^{0}\mathbf{M}\hat{\mathbf{S}}_{\mathbf{P}}\\ {}^{0}\mathbf{M}\hat{\mathbf{S}}_{\mathbf{P}} & {}^{0}\mathbf{I_P}\end{bmatrix} \tag{36}$$

$$\begin{aligned}\mathbf{h}_{\mathbf{robot}} = & \sum_{i=1}^{6}\left\{\begin{bmatrix}\mathbf{I_3}\\ {}^{0}\hat{\mathbf{L}}_i\end{bmatrix}\left(\mathbf{A}_{xi}\left({}^{0}\omega_{P}\times\left({}^{0}\omega_{P}\times{}^{0}\mathbf{L}_i\right)-{}^{0}\dot{\mathbf{J}}_{3i}\dot{\mathbf{q}}_i\right)+\mathbf{h}_{xi}\left(\mathbf{q}_i,\dot{\mathbf{q}}_i\right)\right)\right\} \\ & + \begin{bmatrix}{}^{0}\omega_{P}\times\left({}^{0}\omega_{P}\times{}^{0}\mathbf{MS_P}\right)\\ {}^{0}\omega_{P}\times\left({}^{0}\mathbf{I_P}\,{}^{0}\omega_{P}\right)\end{bmatrix} - \begin{bmatrix}M_P\mathbf{I_3}\\ {}^{0}\mathbf{M}\hat{\mathbf{S}}_{\mathbf{P}}\end{bmatrix}\mathbf{g}\end{aligned} \tag{37}$$

The symmetric and definite positive (6x6) matrix $\mathbf{A_{robot}}$ is the total inertia matrix of the platform and the legs. The (6x1) vector $\mathbf{h_{robot}}$ is the total Coriolis, centrifugal and gravity effects. The computation of the direct dynamic model is based, beside the calculation of the Jacobian matrices, on the computation of $\mathbf{A_i}$ and $\mathbf{h_i}$ of each leg. Many methods are available to compute them [17][25][30]. To reduce the computational cost, customized symbolic methods and base inertial parameters can be used to obtain $\mathbf{A_i}$ and $\mathbf{h_i}$ [26][31].

We note that:

i) Equation (35) represents the closed form solution of the direct dynamic model.

ii) The contribution of leg i on the inertia matrix of the robot $\mathbf{A_{robot}}$, is represented by the (3×3) mass matrix $\mathbf{A_{xi}}$ located at P_i (fig. 5). Each mass matrix $\mathbf{A_{xi}}$ leads to the (6x6) symmetric spatial matrix:

$$\mathbb{A}_{\mathbf{xi}} = \begin{bmatrix} \mathbf{A_{xi}} & -\mathbf{A_{xi}}\,{}^{\mathbf{0}}\hat{\mathbf{L}}_{\mathbf{i}} \\ {}^{\mathbf{0}}\hat{\mathbf{L}}_{\mathbf{i}}\,\mathbf{A_{xi}} & -{}^{\mathbf{0}}\hat{\mathbf{L}}_{\mathbf{i}}\mathbf{A_{xi}}\,{}^{\mathbf{0}}\hat{\mathbf{L}}_{\mathbf{i}} \end{bmatrix} \tag{38}$$

ii) The mass matrix $\mathbf{A_{xi}}$ induces a centrifugal force, which is equal to $\mathbf{A_{xi}}\left({}^{\mathbf{0}}\boldsymbol{\omega}_{\mathbf{P}} \times \left({}^{\mathbf{0}}\boldsymbol{\omega}_{\mathbf{P}} \times {}^{\mathbf{0}}\mathbf{L_i}\right) - {}^{\mathbf{0}}\dot{\mathbf{J}}_{\mathbf{3i}}\dot{\mathbf{q}}_{\mathbf{i}}\right)$ and a moment that is given by: ${}^{\mathbf{0}}\hat{\mathbf{L}}_{\mathbf{i}}\mathbf{A_{xi}}\left({}^{\mathbf{0}}\boldsymbol{\omega}_{\mathbf{P}} \times \left({}^{\mathbf{0}}\boldsymbol{\omega}_{\mathbf{P}} \times {}^{\mathbf{0}}\mathbf{L_i}\right) - {}^{\mathbf{0}}\dot{\mathbf{J}}_{\mathbf{3i}}\dot{\mathbf{q}}_{\mathbf{i}}\right)$.

These terms are similar to the second term of the right hand side of the Newton-Euler equation of the platform (24).

iii) The Coriolis, centrifugal and gravity forces of each leg are transformed into the (3×1) Cartesian force $-\mathbf{h_{xi}}$ at point P_i.

VI. BASE INERTIAL PARAMETERS OF THE ROBOT

The dynamic models are obtained in terms of the inertial parameters of the links of the legs and of those of the platform. We use the following ten parameters for each link j:

- XX_j, XY_j, XZ_j, YY_j, YZ_j, ZZ_j: representing the six independent elements of the inertia matrix around the origin of frame j (see equation (26));
- MX_j, MY_j, MZ_j: defining the first moments of link j;
- M_j: is the mass of link j.

The inverse dynamic model and energy model of the robot are linear with respect to these parameters [17][22][32], which we call standard inertial parameters. The base inertial parameters represent the minimum number of parameters from which the dynamic model can be calculated. The dynamic model complexity is reduced when computed by the base inertial parameters. Besides, they constitute the only identifiable parameters [22][26]. They can be obtained from the standard parameters of the links, by eliminating the parameters that have no effect on the dynamic model and by grouping some others.

A. *Basic conditions for computing the base parameters*

Since the kinetic energy and the potential energy are linear in the inertial parameters, thus:

$$H = E + U = \sum_{i=1}^{10m} \frac{\partial H}{\partial K_i} K_i = \sum_{i=1}^{10m} h_i K_i \tag{39}$$

Where:

m is the number of the links.

H is the total energy of the robot.

E is the total kinetic energy of the robot.

U is the total potential energy of the robot.

K_i denotes an inertial parameter, h_i is its coefficient in the energy.

The expressions of the energy function for each inertial parameter are given in appendix A.

To determine the base inertial parameters, the following cases are considered [17][32], which can be deduced from the Lagrange equation:

1) An inertial parameter K_i has no effect on the dynamic model, if its energy function is constant:

$$h_i = \text{constant} \tag{40}$$

In this case the parameter K_i is eliminated, it can be set as zero.

2) An inertial parameter K_i can be grouped with some other parameters $K_{i1},\ldots,K_{ir}$ if the energy function h_i can be expressed linearly in terms of the energy functions $h_{i1}, \ldots, h_{ir}$ [17][33]:

$$h_i = t_{i1}\,h_{i1} + \ldots + t_{ir}\,h_{ir} = \sum_{j=1}^{r} t_{ij}\,h_{ij} \tag{41}$$

With t_{ij} is constant. The grouped parameters will be given as:

$$K_{ijR} = K_{ij} + t_{ij}K_i \qquad j = 1 \text{ to } r \tag{42}$$

The index R indicates that some parameters are grouped with that one.

The base inertial parameters of the Gough-Stewart robot will be calculated by first determining the base inertial parameters of the legs, then the parameters of the platform will be taken into account.

B. *Base inertial parameters of the legs*

It has been shown that, if the Khalil and Kleinfinger notations are used to assign the link frames, then the base inertial parameters of a general tree structure can be completely determined using simple rules without computing the energy functions [17][33]. Concerning each leg of the Gough-Stewart robot, we can make use of the following three rules, which are applied recursively from the terminal link to the base:

Rule 1: If joint j is revolute, the parameters M_j, MZ_j and YY_j can be grouped with the parameters of links j and i, where $i = a(j)$ is the link antecedent to link j. The grouping relations (when b_j and γ_j are zero) are given in appendix B.

Rule 2: If joint j is prismatic, the parameters of the inertia matrix ${}^{\mathbf{j}}\mathbf{I_j}$ can be grouped with the parameters of the inertia matrix ${}^{\mathbf{i}}\mathbf{I_i}$ with i = a(j), using the following relation:

$$\mathbf{{}^{i}I_{iR} = {}^{i}I_i + {}^{i}R_j\,{}^{j}I_j\,{}^{j}R_i} \tag{43}$$

Where ${}^{j}\mathbf{I}_j$ is the inertia matrix of link j, it is the same as (26) after replacing p by j.

This relation can be deduced from the fact that, in this case the rotational velocity of links i and j are the same. Relation (43) is developed in terms of the geometric parameters in appendix B.

Rule 3: If joint j is revolute and if a(j)=0, that is to say that link j is articulated on the base, then the parameters XX_j, XY_j, XZ_j, YY_j, YZ_j, MZ_j ,M_j have no effect on the dynamic model. This rule comes from the fact that, in this case, the x and y components of the rotational velocity of link j are zero and that the velocity of the origin of frame F_j is zero too.

Assuming general inertial parameters, there will be 30 standard inertial parameters for each leg i. Applying rule 2 on link 3i, the parameters XX_{3i}, XY_{3i}, XZ_{3i}, YY_{3i}, YZ_{3i}, ZZ_{3i} can be grouped with the parameters of link 2 using the expressions given in appendix B. Then using rule 1 on link 2i, the

TABLE II
BASE PARAMETERS OF THE GOUGH-STEWART ROBOT

Leg i (i = 1 to 6)									
					ZZ_{1Ri}	MX_{1i}	MY_{1Ri}		
XX_{2Ri}	XY_{2Ri}	XZ_{2Ri}		$YZ2_{Ri}$	$ZZ2_{Ri}$	MX_{2i}	MY_{2i}		
						MX_{3i}	MY_{3i}	MZ_{3i}	
Platform									
XX_{RP}	XY_{RP}	XZ_{P}	YY_{RP}	YZ_{P}	ZZ_{RP}	MX_{RP}	MY_{RP}	MZ_{P}	M_{RP}

TABLE III
ESSENTIAL PARAMETERS OF THE GOUGH-STEWART ROBOT

Leg i (i = 1 to 6)									
					ZZ_{1Ri}				
XX_{2Ri}					$ZZ2_{Ri}$		MY_{2i}		
						MX_{3i}	MY_{3i}	MZ_{3i}	
Platform									
XX_{RP}	XY_{RP}	XZ_{P}	YY_{RP}	YZ_{P}	ZZ_{RP}	MX_{RP}	MY_{RP}	MZ_{P}	M_{RP}

parameters YY_{2i}, MZ_{2i} and M_{2i} can be grouped with the parameter YY_{2i} and with the parameters of link 1. This recursive calculation ends with the application of rule 3 on link 1i to eliminate the parameters XX_{1i}, XY_{1i}, XZ_{1i}, YY_{1i}, YZ_{1i}, MZ_{1i}, M_{1i}. Thus, we have 14 base inertial parameters for each leg. They are given by: ZZ_{1Ri}, MX_{1i}, MY_{1Ri}, XX_{2Ri}, XY_{2Ri}, XZ_{2Ri}, YZ_{2Ri}, ZZ_{2Ri}, MX_{2i}, MY_{2i}, MX_{3i}, MY_{3i}, MZ_{3i}, M_{3i}. The grouping relations are:

$$ZZ_{1Ri} = YY_{2i} + ZZ_{1i} + ZZ_{3i} \tag{44}$$

$$MY_{1Ri} = MY_{1i} - MZ_{2i} \tag{45}$$

$$XX_{2Ri} = XX_{2i} + XX_{3i} - YY_{2i} - ZZ_{3i} \tag{46}$$

$$XY_{2Ri} = XY_{2i} - XZ_{3i} \tag{47}$$

$$XZ_{2Ri} = XY_{3i} + XZ_{2i} \tag{48}$$

$$YZ_{2Ri} = YZ_{2i} - YZ_{3i} \tag{49}$$

$$ZZ_{2Ri} = YY_{3i} + ZZ_{2i} \tag{50}$$

C. *Base inertial parameters of the closed loop structure*

More parameters could be eliminated or grouped when considering the closed structure (legs and platform). For this step we present a solution leading to determine them explicitly for a general Gough-Stewart structure. We note that for the legs, we grouped the inertial parameters of a link with its antecedent. For the platform we choose to group the parameters of the legs with those of the platform, such that the inertial parameters of the different legs remain identical.

Since the origin of the terminal frame of each leg (frames 3i) and the point P_i are the same thus:

$$\mathbf{V_{3i}} = \mathbf{V_P} + \boldsymbol{\omega}_\mathbf{P} \times \mathbf{L_i} \tag{51}$$

Applying relation (73), given in appendix A, the energy function of the mass M_{3i} is obtained as:

$$h_{M3i} = \frac{1}{2}\, {}^{3i}\mathbf{V}_{3i}^{T}\, {}^{3i}\mathbf{V}_{3i} - {}^{0}\mathbf{g}^{T}\, {}^{0}\mathbf{P_i} \tag{52}$$

Since:

$${}^{3i}\mathbf{V}_{3i} = {}^{P}\mathbf{V}_{Pi} = \left({}^{P}\mathbf{V}_{P} + {}^{P}\boldsymbol{\omega}_{p} \times {}^{P}\mathbf{L}_{i}\right) \tag{53}$$

and:

$${}^{0}\mathbf{P}_{i} = \left({}^{0}\mathbf{P}_{p} + {}^{0}\mathbf{R}_{p} \times {}^{P}\mathbf{L}_{i}\right) \tag{54}$$

Equation (52) can be developed as follows:

$$\begin{aligned} h_{M3i} = & \frac{1}{2}\omega_{1,P}\,\omega_{2,P}\left({}^{P}L_{2,i}^{2} + {}^{P}L_{3,i}^{2}\right) - \omega_{1,P}\,\omega_{1,P}\,{}^{P}L_{1,i}\,{}^{P}L_{2,i} \\ & + \frac{1}{2}\omega_{3,P}\,\omega_{3,P}\left({}^{P}L_{1,i}^{2} + {}^{P}L_{2,i}^{2}\right) - \omega_{2,P}\,\omega_{3,P}\,{}^{P}L_{2,i}\,{}^{P}L_{3,i} \\ & + \frac{1}{2}\omega_{2,P}\,\omega_{2,P}\left({}^{P}L_{1,i}^{2} + {}^{P}L_{3,i}^{2}\right) - \omega_{1,P}\,\omega_{3,P}\,{}^{P}L_{1,i}\,{}^{P}L_{3,i} \\ & + V_{2,P}\,\omega_{3,P}\,{}^{P}L_{1,i} - V_{3,P}\,\omega_{2,P}\,{}^{P}L_{1,i} - {}^{0}\mathbf{g}^{T}\,{}^{0}\mathbf{s_P}\,{}^{P}L_{1,i} \\ & + V_{3,P}\,\omega_{1,P}\,{}^{P}L_{2,i} - V_{1,P}\,\omega_{3,P}\,{}^{P}L_{2,i} - {}^{0}\mathbf{g}^{T}\,{}^{0}\mathbf{n_P}\,{}^{P}L_{2,i} \\ & + V_{1,P}\,\omega_{2,P}\,{}^{P}L_{3,i} - V_{2,P}\,\omega_{1,P}\,{}^{P}L_{3,i} - {}^{0}\mathbf{g}^{T}\,{}^{0}\mathbf{a_P}\,{}^{P}L_{3,i} \\ & + \frac{1}{2}\,{}^{P}\mathbf{V}_{P}^{T}\,{}^{P}\mathbf{V_P} - {}^{0}\mathbf{g}^{T}\,{}^{0}\mathbf{P_P} \end{aligned} \tag{55}$$

Where:

$${}^{P}\mathbf{V_P} = \begin{bmatrix} V_{1,P} & V_{2,P} & V_{3,P} \end{bmatrix}^{T},\ {}^{P}\boldsymbol{\omega}_\mathbf{P} = \begin{bmatrix} \omega_{1,P} & \omega_{2,P} & \omega_{3,P} \end{bmatrix}^{T},$$

$${}^{0}\mathbf{R_P} = \begin{bmatrix} {}^{0}\mathbf{s_P} & {}^{0}\mathbf{n_P} & {}^{0}\mathbf{a_P} \end{bmatrix}^{T},\ {}^{P}\mathbf{L_i} = \begin{bmatrix} {}^{P}L_{1,i} & {}^{P}L_{2,i} & {}^{P}L_{3,i} \end{bmatrix}^{T}.$$

From equations (55) and equation (73), given in appendix A, we can deduce that the energy function of the parameters M_{3i} (i = 1 to 6) can be expressed in term of the energy functions of the inertial parameters of the platform, such that:

$$\begin{aligned} h_{M3i} = & \left({}^{P}L_{2,i}^{2} + {}^{P}L_{3,i}^{2}\right)h_{XXP} - {}^{P}L_{1,i}\,{}^{P}L_{2,i}\,h_{XYP} \\ & - {}^{P}L_{1,i}\,{}^{P}L_{3,i}\,h_{XZP} + \left({}^{P}L_{1,i}^{2} + {}^{P}L_{3,i}^{2}\right)h_{YYP} \\ & - {}^{P}L_{2,i}\,{}^{P}L_{3,i}\,h_{YZP} + \left({}^{P}L_{1,i}^{2} + {}^{P}L_{2,i}^{2}\right)h_{ZZP} \\ & + h_{MP} + {}^{P}L_{1,i}\,h_{MXP} + {}^{P}L_{2,i}\,h_{MYP} + {}^{P}L_{3,i}\,h_{MZP} \end{aligned} \tag{56}$$

Thus, using (41) and (42), we can group the parameters M_{3i} with the inertial parameters of the platform. Taking into account that ${}^{P}L_{1,1}$, ${}^{P}L_{2,1}$, ${}^{P}L_{3,1}$, ${}^{P}L_{1,2}$, ${}^{P}L_{3,2}$ and ${}^{P}L_{3,6}$ are zero, the grouped relations are:

$$XX_{RP} = XX_{P} + \sum_{i=2}^{6}\left\{\left({}^{P}L_{2,i}^{2} + {}^{P}L_{3,i}^{2}\right)M_{3i}\right\} \tag{57}$$

$$XY_{RP} = XY_{P} - \sum_{i=3}^{6}\left\{{}^{P}L_{1,i}\,{}^{P}L_{2,i}\,M_{3i}\right\} \tag{58}$$

$$XZ_{RP} = XZ_{P} - \sum_{i=3}^{5}\left\{{}^{P}L_{1,i}\,{}^{P}L_{3,i}\,M_{3i}\right\} \tag{59}$$

$$YY_{RP} = YY_{P} + \sum_{i=3}^{6}\left\{\left({}^{P}L_{1,i}^{2} + {}^{P}L_{3,i}^{2}\right)M_{3i}\right\} \tag{60}$$

$$YZ_{RP} = YZ_{P} - \sum_{i=3}^{5}\left\{{}^{P}L_{2,i}\,{}^{P}L_{3,i}\,M_{3i}\right\} \tag{61}$$

$$ZZ_{RP} = ZZ_{P} + \sum_{i=2}^{6}\left\{\left({}^{P}L_{1,i}^{2} + {}^{P}L_{2,i}^{2}\right)M_{3i}\right\} \tag{62}$$

$$MX_{RP} = MX_{P} + \sum_{i=3}^{6}\left\{{}^{P}L_{1,i}\,M_{3i}\right\} \tag{63}$$

$$MY_{RP} = MY_{P} + \sum_{i=2}^{6}\left\{{}^{P}L_{2,i}\,M_{3i}\right\} \tag{64}$$

$$MZ_{RP} = MZ_{P} + \sum_{i=3}^{5}\left\{{}^{P}L_{3,i}\,M_{3i}\right\} \tag{65}$$

TABLE IV
COMPARISON OF THE NUMBER OF OPERATIONS OF THE INVERSE DYNAMICS

	Kinematic Modelling	Inverse Dynamic Mod.	Total
Our method[1] [35]	659 '*' and 299 '+'	631 '*' and 489 '+'	2078
Our method[2] [35]	659 '*' and 299 '+'	715 '*' and 609 '+'	2282
Gosselin [10]	468 '*' and 282 '+'	834 '*' and 566 '+'	2150
Dasgupta [11]	-	-	4489

[1] using essential inertial parameters.
[2] using base inertial parameters .

$$M_{RP} = M_P + \sum_{i=1}^{6} M_{3i} \tag{66}$$

Physically, these grouped inertial parameters can be seen as the sum of those of the platform and the effect of masses M_{3i} suspended at points P_i for i = 1, …, 6.

Finally, we obtain 88 base inertial parameters for the Gough-Stewart robot (13 for each leg and 10 for the platform), they are summarized in table II, recall that the number of the standard inertial parameters is 190. The application of the QR numerical method to determine the base inertial parameters [17][34] validates this result.

Moreover, if we consider that the links are symmetrical and that the links 1 and 2 constitute one real link, then we can define the set of 52 essential parameters, given in table III.

VII. COMPUTATIONAL COST

The proposed algorithms of the dynamic modeling of the Gough-Stewart robot are implemented using customized symbolic technique and the base inertial parameters [22][31]. The numbers of operations to compute the different steps are given in [35]. The computation of the inverse dynamic model including the kinematic procedures of the legs needs 788 '+' and 1290 '*'. Where as the computation of the direct dynamic model needs 1120 '+' and 1823 '*'.

The comparison with the number of operations obtained by [10] and [11] is given in table IV. From the computational cost point of view, our method is equivalent to that of Gosselin [10], but we remind that our method can provide the inverse and the direct models and can be easily generalized for other structures.

Since the computation of the inverse dynamic model of the legs can be carried out in parallel, thus the proposed dynamic algorithms could be easily distributed on seven processors [11]. In this case, the computation cost of the inverse dynamic model is reduced to 226 '+' and 300 '*', and that of the direct dynamic model will be reduced to 387 '+' and 414 '*'.

VIII. CONCLUSION

In this paper, we highlight an interesting physical interpretation of the final form of the inverse and direct dynamic models of the Gough-Stewart robot. These models take into account all the dynamic parameters of the links. The approach is straightforward and could be applied to most parallel manipulators. These models are computed in terms of the dynamic models of the legs. Consequently, the computation of these models can make use of the techniques, which were developed for the serial robots. To reduce the computation cost, the base inertial parameters of the robot have been determined analytically. Moreover, the computation of these models could be easily distributed on parallel processors. The numbers of operations of the inverse and the direct dynamic models are given.

APPENDIX A: EXPRESSIONS OF THE ENERGY FUNCTION $\mathbf{h_j}$

The total energy of link j is given as a linear relation in terms of the inertial parameters as follows [32]:

$$H_j = E_j + U_j = \mathbf{h_j}\,\mathbf{K^j} \tag{67}$$

Where:

$\mathbf{K^j}$ is the (10×1) vector of the 10 standard inertial parameters;
The kinetic and the potential energy are computed by:

$$E_j = \frac{1}{2}\left({}^j\boldsymbol{\omega}_j^T\, {}^j\mathbf{I_j}\, {}^j\boldsymbol{\omega}_j + M_j\, {}^j\mathbf{V}_j^T\, {}^j\mathbf{V}_j + 2\, {}^j\mathbf{MS}_j^T \left({}^j\mathbf{V}_j \times {}^j\boldsymbol{\omega}_j \right) \right) \tag{68}$$

$$U_j = -\begin{bmatrix} {}^0\mathbf{g}^T & 0 \end{bmatrix} {}^0\mathbf{T_j} \begin{bmatrix} {}^j\mathbf{MS_j} \\ M_j \end{bmatrix} \tag{69}$$

${}^0\mathbf{T_j}$ is the transformation matrix of frame j relative to frame 0:

$${}^0\mathbf{T_j} = \begin{bmatrix} {}^0\mathbf{s_j} & {}^0\mathbf{n_j} & {}^0\mathbf{a_j} & {}^0\mathbf{P_j} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{70}$$

$\mathbf{h_j}$ is the (1×10) vector defined by:

$$\mathbf{h_j} = \begin{bmatrix} \frac{\partial H_j}{\partial XX_j} & \frac{\partial H_j}{\partial XY_j} & \cdots & \frac{\partial H_j}{\partial MZ_j} & \frac{\partial H_j}{\partial M_j} \end{bmatrix}^T \tag{71}$$

Thus:

$$\mathbf{h_j} = \begin{bmatrix} h_{XXj}\, h_{XYj}\, h_{XZj}\, h_{YYj}\, h_{YZ_j}\, h_{ZZj}\, h_{MXj}\, h_{MYj}\, h_{MZj}\, h_{Mj} \end{bmatrix}^T \tag{72}$$

Where:

$$\begin{aligned}
&h_{XXj} = \frac{1}{2}\omega_{1,j}\omega_{1,j}\,,\, h_{XYj} = \omega_{1,j}\omega_{2,j}\,,\, h_{XZj} = \omega_{1,j}\omega_{3,j} \\
&h_{YYj} = \frac{1}{2}\omega_{2,j}\omega_{2,j}\,,\, h_{YZj} = \omega_{2,j}\omega_{3,j}\,,\, h_{ZZj} = \frac{1}{2}\omega_{3,j}\omega_{3,j} \\
&h_{MXj} = \omega_{3,j}V_{2,j} - \omega_{2,j}V_{3,j} - {}^0\mathbf{g}^T\, {}^0\mathbf{s_j} \\
&h_{MYj} = \omega_{1,j}V_{3,j} - \omega_{3,j}V_{1,j} - {}^0\mathbf{g}^T\, {}^0\mathbf{n_j} \\
&h_{MZj} = \omega_{2,j}V_{1,j} - \omega_{1,j}V_{2,j} - {}^0\mathbf{g}^T\, {}^0\mathbf{a_j} \\
&h_{Mj} = \frac{1}{2}\, {}^j\mathbf{V}_j^T\, {}^j\mathbf{V}_j - {}^0\mathbf{g}^T\, {}^0\mathbf{P_j}
\end{aligned} \tag{73}$$

APPENDIX B: GENERAL GROUPING RELATIONS

In the following we consider the special case where b_j and γ_j are zero. Two cases are considered:

1)If joint j is revolute, the parameters YY_j, MZ_j and M_j can be grouped with the parameter XX_j and the parameters of the antecedent link i using the following relations [32][33]:

$$XX_{jR} = XX_j - YY_j$$

$$XX_{iR} = XX_i + YY_j + 2\,r_j\,MZ_j + r_j^2\,M_j$$

$$
\begin{aligned}
XY_{iR} &= XY_i + d_j\,S\alpha_j\,MZ_j + d_j\,r_j\,S\alpha_j\,MZ_j \\
XZ_{iR} &= XZ_i - d_j\,C\alpha_j\,MZ_j - d_j\,r_j\,C\alpha_j\,MZ_j \\
YY_{iR} &= YY_i + CC\alpha_j\,YY_j + 2\,r_j\,CC\alpha_j\,MZ_j + \left(d_j^2 + r_j^2\,CC\alpha_j\right)M_j \\
YZ_{iR} &= YZ_i + CS\alpha_j\,YY_j + 2\,r_j\,CS\alpha_j\,MZ_j + r_j^2\,CS\alpha_j\,M_j \\
ZZ_{iR} &= ZZ_i + SS\alpha_j\,YY_j + 2\,r_j\,SS\alpha_j\,MZ_j + \left(d_j^2 + r_j^2\,SS\alpha_j\right)M_j \\
MX_{iR} &= MX_i + d_j\,M_j \\
MY_{iR} &= MY_i - S\alpha_j\,MZ_j + r_j\,S\alpha_j\,M_j \\
MZ_{iR} &= MZ_i + C\alpha_j\,MZ_j + r_j\,C\alpha_j\,M_j \\
M_{iR} &= M_i + M_j
\end{aligned}
\tag{74}
$$

2) If joint j is prismatic, the inertia matrix of link j can be grouped with that of link i. The grouping relations are [32][33]:

$$
\begin{aligned}
XX_{iR} &= XX_i + CC\theta_j\,XX_j - 2\,CS\theta_j\,XY_j + SS\theta_j\,YY_j \\
XY_{iR} &= XY_i + CS\theta_j\,C\alpha_j\,XX_j + \left(CC\theta_j - SS\theta_j\right)C\alpha_j\,XY_j \\
&\quad - C\theta_j\,S\alpha_j\,XZ_j - CS\theta_j\,C\alpha_j\,YY_j + S\theta_j\,S\alpha_j\,YZ_j \\
XZ_{iR} &= XZ_i + CS\theta_j\,S\alpha_j\,XX_j + \left(CC\theta_j - SS\theta_j\right)S\alpha_j\,XY_j \\
&\quad + C\theta_j\,C\alpha_j\,XZ_j - CS\theta_j\,S\alpha_j\,YY_j - S\theta_j\,C\alpha_j\,YZ_j \\
YY_{iR} &= YY_i + SS\theta_j\,CC\alpha_j\,XX_j + 2\,CS\theta_j\,CC\alpha_j\,XY_j \\
&\quad - 2\,S\theta_j\,CS\alpha_j\,XZ_j + CC\theta_j\,CC\alpha_j\,YY_j \\
&\quad - 2\,C\theta_j\,CS\alpha_j\,YZ_j + SS\alpha_j\,ZZ_j \\
YZ_{iR} &= YZ_i + SS\theta_j\,CS\alpha_j\,XX_j + 2\,CS\theta_j\,CS\alpha_j\,XY_j \\
&\quad + S\theta_j\left(CC\alpha_j - SS\alpha_j\right)XZ_j + CC\theta_j\,CS\alpha_j\,YY_j \\
&\quad + C\theta_j\left(CC\alpha_j - SS\alpha_j\right)YZ_j - CS\alpha_j\,ZZ_j \\
ZZ_{iR} &= ZZ_i + SS\theta_j\,SS\alpha_j\,XX_j + 2\,CS\theta_j\,SS\alpha_j\,XY_j \\
&\quad + 2\,S\theta_j\,CS\alpha_j\,XZ_j + CC\theta_j\,SS\alpha_j\,YY_j \\
&\quad + 2\,C\theta_j\,CS\alpha_j\,YZ_j + CC\alpha_j\,ZZ_j
\end{aligned}
\tag{75}
$$

Where: C(.) = cos(.), S(.) = sin(.), CC(.) = cos²(.), SS(.) = sin²(.) and CS(.) = cos(.) sin(.). The geometric parameters α_j, θ_j, r_j and d_j define frame link j with respect to frame i.

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