Mechanical Systems and Signal Processing (1996) **10**(6), 697–709

IDENTIFICATION OF RIGID BODY INERTIA PROPERTIES FROM EXPERIMENTAL DATA

A. Fregolent and A. Sestieri

Department of Mechanics and Aeronautics, University of Rome "La Sapienza", Rome, Italy

(*Received July* 1995, *accepted January* 1996)

The estimate of mass properties from the massline characteristic of experimental FRFs has attracted the attention of the modal analysis community during the last decade. The problem is typically non-linear and, in the technical literature, iterative and direct methods were proposed for its solution. In this paper a direct procedure is developed. The non-linear problem is divided into a series of simpler linear problems and solved in subsequent steps. A careful analysis is performed on the conditioning of the systems to assess the minimum number of required measurements necessary to solve the problem even in noisy environments. The proposed approach produces accurate results for two real experimentally tested systems: a 2- and a 3-D structure. An accurate measurement of the masslines characteristics is required. An extended discussion on this point is presented with particular emphasis on the bias introduced in the low frequency range when testing free structures.

1. INTRODUCTION

The extensive and successful application of the experimental frequency response function (FRF) in different fields of dynamics and vibrations led several researchers, during the last decade, to propose the use of FRF data for estimating the inertia properties of structures. The inertia of complex system is often derived from trifilar pendulum tests, which are prone to large errors and are difficult to perform, whilst FRF data are now a common and generally reliable framework that can efficiently circumvent other complex methods and provide good estimates.

Although a method to determine the rigid body characteristic using rigid body modes and modes shapes obtained from time domain data [1] was recently developed, the general approach consists of using the massline, i.e. the FRF inertia restraint of a soft suspended structure as the input for a set of kinematic and dynamic equations, from which the rigid body characteristics (position of the centre of mass, values of mass, principal directions and moments of inertia) can be determined.

Several researchers [2–4] have explained how to determine the FRF massline, typical of a free structure, from data obtained from a weakly suspended structure. Others [5–8] outlined different procedures that either utilise masslines values or identify rigid body modes to obtain the required mass properties. Among this group of investigators, Okubo, Furukawa, Bretl and Conti deserve particular mention. Okubo and Furukawa [5] assumed an a priori knowledge of one principal axis and the body's mass. Then they solve a non-linear problem through an iterative procedure, using an initial estimate of the rigid body mass properties. The inertia parameters are computed through minimisation of the difference between the values of the FRF in the low frequency range and the massline

0888–3270/96/060697 + 13 $25.00/0

values obtained by a curve-fitting model based on the unknown mass parameters. The work of Bretl and Conti [6] presents, in the authors' opinion, the most promising approach in this field and the one on which many other approaches are based. In that paper the authors propose two different methods and, between them, the procedure using masslines as input requires the knowledge of the body's mass. Unlike Okubo's method, the procedure is non-iterative: the mass properties are obtained through a direct solution of two linear problems. Wei and Reis [7] develop a method that assembles part of the formulation used in [5] with the one in [6]. Furusawa [8] proposes an approach that exploits FRF data and some knowledge on the body's geometry.

The purpose of this paper is twofold. First a new direct procedure is introduced which is similar to Bretl and Conti's approach but does not require the knowledge of the mass value. A careful analysis is performed on the conditioning of the equations to be solved in order to assess the minimum number of measurements, necessary even in noisy environments. Two real structures are analysed (2- and 3-D, respectively) and comparisons between analytical and experimental results are given. The second goal is aimed to stress what is generally overlooked in most of the papers on the subject: that the identification of inertia parameters is not a direct by-product of already obtained FRFs. It will be shown that a particular set of measurements is required. Moreover, the estimate of masslines characteristics of a free system is quite difficult for 3-D bodies, even when soft suspensions are used.

2. THEORETICAL BACKGROUND: KINEMATIC AND DYNAMIC EQUATIONS

The proposed procedure requires, as input, the estimated values of massline, obtained from measured FRFs (see Appendix A). Let us assume at this stage that these values are known precisely. Let $Oxyz$ be a general reference frame with respect to which excitation and transduction co-ordinates are known, G the centre of mass, and $\dot{\omega}$ the angular acceleration vector. The principal inertia directions as well as the co-ordinates of the centre of mass G, the value of the mass and the moments of inertia are unknown.

Using the acceleration of the centre of mass, the linearised acceleration of any point of a 3-D system can be expressed in the $Oxyz$ reference frame as:

$$\ddot{\mathbf{x}}_T = \ddot{\mathbf{x}}_G + \dot{\boldsymbol{\omega}} \wedge (\mathbf{x}_T - \mathbf{x}_G) \tag{1}$$

On the contrary, when the acceleration of the origin O is considered, it is:

$$\ddot{\mathbf{x}}_T = \ddot{\mathbf{x}}_0 + \dot{\boldsymbol{\omega}} \wedge (\mathbf{x}_T - \mathbf{x}_0) \tag{2}$$

In this case it is obviously $\mathbf{x}_0 = \mathbf{0}$.

For an unrestrained body, the linearised dynamic equation of motion with respect to the centre of gravity is, in matrix form:

$$\begin{bmatrix} \mathbf{f} \\ \mathbf{m} \end{bmatrix} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}}_G \\ \dot{\boldsymbol{\omega}} \end{bmatrix} \tag{3}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{f}$ and $\mathbf{m}$ the force and moment vectors, respectively and $\mathbf{J}$ the tensor of inertia:

$$\mathbf{J} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

If principal directions are assumed, the tensor of inertia becomes diagonal and its elements are the principal moments of inertia.

The moment components can be expressed as a function of the exciting force, centre of mass and excitation co-ordinates, namely:

$$\mathbf{m} = (\mathbf{x}_E - \mathbf{x}_G) \wedge \mathbf{f} \tag{4}$$

whilst the acceleration components of the centre of mass can be expressed by equation (1).

By rearranging these equations, different systems can be obtained, depending on the parameters used in the coefficient matrix and in the vector of known terms. However, independently of the above choice, all these systems share a non-linear character. Note that in equations (1) and (2), masslines never appear explicitly. However, dividing by the applied force, the values of masslines are immediately obtained from curve-fitting the FRF in the low frequency range.

Assuming that the mass is known, Bretl and Conti obtain a linear system, whose solution requires, at least, three excitation points and six transduction points for each direction, that is a total of 18 measurements. However, in Bretl and Conti's work, a conditioning analysis of the solving equations is missing and the use of the proposed triaxial accelerometer causes singular conditions for some systems. This drawback is overlooked in their paper.

2.1. KINEMATICS

Two possible procedures can be followed.

2.1.1. *Method* 1

This method uses equation (2) that, in matrix form, is written as:

$$\begin{bmatrix} (\ddot{x}_T - \ddot{x}_0)/F \\ (\ddot{y}_T - \ddot{y}_0)/F \\ (\ddot{x}_T - \ddot{z}_0)/F \end{bmatrix} = \begin{bmatrix} 0 & z_T & -y_T \\ -z_T & 0 & x_T \\ y_T & -x_T & 0 \end{bmatrix} \begin{bmatrix} \dot{\omega}_x/F \\ \dot{\omega}_y/F \\ \dot{\omega}_z/F \end{bmatrix} \tag{5}$$

where F is a force applied in any of the co-ordinates' directions. In order to solve these equations for the unknown angular acceleration components, six masslines must be determined, three for the transduction points and three for the origin, along the three co-ordinate directions. However, since the matrix is skew-symmetric, it becomes a singular matrix when a single point is used for the three directions (triaxial accelerometer). It follows that a single transduction point cannot be used for more than two directions.

2.1.2. *Method* 2

As before, equation (2) is used, but the acceleration components of the origin O are here considered as unknowns, i.e.:

$$[\ddot{x}_T/F \quad \ddot{y}_T/F \quad \ddot{z}_T/F] = \begin{bmatrix} 1 & 0 & 0 & 0 & z_T & -y_T \\ 0 & 1 & 0 & -z_T & 0 & x_T \\ 0 & 0 & 1 & y_Y & -x_T & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_0/F \\ \ddot{y}_0/F \\ \ddot{z}_0/F \\ \dot{\omega}_x/F \\ \dot{\omega}_y/F \\ \dot{\omega}_z/F \end{bmatrix} \tag{6}$$

Here the acceleration components at the origin O must not be measured and the masslines of these components do not need identification. In order to solve for the unknowns, at least two sets of transduction accelerations must be used to invert a 6×6 system.

Consider six different transduction points. By using equation (2) and appropriately subtracting corresponding rows, the following system is obtained:

$$\begin{bmatrix} (\ddot{x}_1 - \ddot{x}_2)/F \\ (\ddot{y}_3 - \ddot{y}_4)/F \\ (\ddot{z}_5 - \ddot{z}_6)/F \end{bmatrix} = \begin{bmatrix} 0 & (z_1 - z_2) & -(y_1 - y_2) \\ -(z_3 - z_4) & 0 & (x_3 - x_4) \\ (y_5 - y_6) & -(x_5 - x_6) & 0 \end{bmatrix} \begin{bmatrix} \dot{\omega}_x/F \\ \dot{\omega}_y/F \\ \dot{\omega}_z/F \end{bmatrix} \tag{7}$$

In this way a 3×3 problem only must be solved, but the number of measurements required is still six for each direction of excitation.

2.2. DYNAMICS

The dynamic problem is non-linear. However three possible solution methods can be proposed to solve the problem in linear form.

2.2.1. *Method* 1

By substituting the second set of equation (3) into equation (4) and defining A_{ij} as the components of the inverse inertia tensor ($\mathbf{A} = \mathbf{J}^{-1}$), we can write:

$$\begin{aligned}
\dot{\omega}_x/F_x &= A_{12}(z_E - z_G) - A_{13}(y_E - y_G) \\
\dot{\omega}_y/F_x &= A_{22}(z_E - z_G) - A_{23}(y_E - y_G) \\
\dot{\omega}_z/F_x &= A_{32}(z_E - z_G) - A_{33}(y_E - y_G) \\
\dot{\omega}_x/F_y &= A_{13}(x_E - x_G) - A_{11}(z_E - z_G) \\
\dot{\omega}_y/F_y &= A_{23}(x_E - x_G) - A_{21}(z_E - z_G) \\
\dot{\omega}_z/F_y &= A_{33}(x_E - x_G) - A_{31}(z_E - z_G) \\
\dot{\omega}_x/F_z &= A_{11}(y_E - y_G) - A_{21}(x_E - x_G) \\
\dot{\omega}_y/F_z &= A_{21}(y_E - y_G) - A_{22}(x_E - x_G) \\
\dot{\omega}_z/F_z &= A_{31}(y_E - y_G) - A_{32}(x_E - x_G)
\end{aligned} \tag{8}$$

Since the A_{ij} elements and the co-ordinates of the centre of mass are unknown, this is a non-linear system 9×9. However we can transform this set of equations into a new one 9×21 by defining a new vector of 21 unknowns as follows: the first six elements containing the lower triangle elements of the matrix **A** only, the other 15 expressing suitable products between the coordinates of the centre of mass and the A_{ij} terms. This system is defined by a matrix, in which the first part (9×6) is dependent on the excitation co-ordinates whilst the second (9×21) is composed by ones and zeros only. In order to transform this system into a linear one, we can delete the columns and the relative unknowns from 7 to 21 by subtracting from this system a similar one, relative to different excitation points, thus obtaining:

$$
\begin{bmatrix}
(\dot{\omega}_x/F_x)_1-(\dot{\omega}_x/F_x)_4\\
(\dot{\omega}_y/F_x)_1-(\dot{\omega}_y/F_x)_4\\
(\dot{\omega}_z/F_x)_1-(\dot{\omega}_z/F_x)_4\\
(\dot{\omega}_x/F_y)_2-(\dot{\omega}_x/F_y)_5\\
(\dot{\omega}_y/F_y)_2-(\dot{\omega}_y/F_y)_5\\
(\dot{\omega}_z/F_y)_2-(\dot{\omega}_z/F_y)_5\\
(\dot{\omega}_x/F_x)_3-(\dot{\omega}_x/F_z)_6\\
(\dot{\omega}_y/F_z)_3-(\dot{\omega}_y/F_z)_6\\
(\dot{\omega}_z/F_z)_3-(\dot{\omega}_z/F_z)_6
\end{bmatrix}
=
\begin{bmatrix}
0 & z_1-z_4 & 0 & y_4-y_1 & 0 & 0\\
0 & 0 & z_1-z_4 & 0 & y_4-y_1 & 0\\
0 & 0 & 0 & 0 & z_1-z_4 & y_4-y_1\\
z_5-z_2 & 0 & 0 & x_2-x_5 & 0 & 0\\
0 & z_5-z_2 & 0 & 0 & x_2-x_5 & 0\\
0 & 0 & 0 & z_5-z_2 & 0 & x_2-x_5\\
y_3-y_6 & x_6-x_3 & 0 & 0 & 0 & 0\\
0 & y_3-y_6 & x_6-x_3 & 0 & 0 & 0\\
0 & 0 & 0 & y_3-y_6 & x_6-x_3 & 0
\end{bmatrix}
\begin{bmatrix}
A_{11}\\ A_{12}\\ A_{22}\\ A_{31}\\ A_{32}\\ A_{33}
\end{bmatrix}
$$

By solving the above set of equations, the inverse components of the inertia tensor can be determined. Using equation (8) written for two sets of excitation points, a new set of equations 9×3 in the unknowns x_G, y_G and z_G can be written and solved. Since the eigenvalues of the tensor matrix are the principal moments of inertia, they can be easily evaluated. With this approach the mass does not appear either as input nor as unknown.

Since the subtraction operation requires two sets of measurements for any excitation (12 measurements for any direction), the construction of this set of equations leads to a large number of measurements: globally 36 measurements for the three directions. Moreover the conditioning of the system is hardly controllable: in fact subtraction among co-ordinates can lead to singularities or ill-conditioning if some of them are equal or very close.

2.2.2. *Method* 2

Once the kinematic problem is solved, we can rewrite equation (1) with force and response measured in the same direction. Under this condition:

$$\ddot{x}_G/F_x = \ddot{y}_G/F_y = \ddot{z}_G/F_z = 1/M$$

so that we have:

$$
\begin{bmatrix}
\ddot{x}_T/F_x-(\dot{\omega}_y/F_x)z_T+(\dot{\omega}_z/F_x)y_T\\
\ddot{y}_T/F_y-(\dot{\omega}_z/F_y)x_T+(\dot{\omega}_x/F_y)z_T\\
\ddot{z}_T/F_z-(\dot{\omega}_x/F_z)y_T+(\dot{\omega}_y/F_z)x_T
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & -\dot{\omega}_z/F_x & \dot{\omega}_y/F_x\\
1 & \dot{\omega}_z/F_y & 0 & -\dot{\omega}_x/F_y\\
1 & -\dot{\omega}_y/F_z & \dot{\omega}_x/F_z & 0
\end{bmatrix}
\begin{bmatrix}
\dfrac{1}{M}\\ x_G\\ y_G\\ z_G
\end{bmatrix}
\tag{9}
$$

Unlike method 1, here the mass appears among the unknowns, but the solution of the problem requires four excitations and 24 transductions. Once the co-ordinates of the centre of mass have been determined, equations (8) can be used to evaluate the inverse components of the inertia tensor and thus the principal moments of inertia.

With this approach the solution of the dynamic problem is significantly simplified. The minimum number of necessary measurements to solve the problem, including the mass, is only 24. However a difficulty arises when, using experimental data, a least square solution would be used for the dynamic problem. For any new row added to the system of equations, six further measurements are necessary. When the kinematic problem must also be solved in a least square sense, at least one further measurement must be performed for any direction: in total nine new measurements are necessary if a single row is added, and this is a heavy limitation. Equation (8) cannot be solved as a determined system (i.e. the same number of equations and unknowns) by using the first six equations only, because the matrix is singular and, as a result, only two equations can be used for any excitation group.

2.2.3. *Method* 3

Using equations (3) and (4) together, we can write:

$$\begin{bmatrix} F_x - M\ddot{x}_0 \\ F_y - M\ddot{y}_0 \\ F_z - M\ddot{z}_0 \\ \hline y_E F_z - z_E F_y \\ z_E F_x - x_E F_z \\ x_E F_y - y_E F_x \end{bmatrix} =$$

$$\left[\begin{array}{ccc|cccccc} 0 & -M\dot{\omega}_z & M\dot{\omega}_y & 0 & 0 & 0 & 0 & 0 & 0 \\ M\dot{\omega}_z & 0 & -M\dot{\omega}_x & 0 & 0 & 0 & 0 & 0 & 0 \\ -M\dot{\omega}_y & M\dot{\omega}_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & F_z & -F_y & \dot{\omega}_x & 0 & 0 & -\dot{\omega}_y & 0 & -\dot{\omega}_z \\ -F_z & 0 & F_x & 0 & \dot{\omega}_y & 0 & -\dot{\omega}_x & -\dot{\omega}_z & 0 \\ F_y & -F_x & 0 & 0 & 0 & \dot{\omega}_z & 0 & -\dot{\omega}_y & -\dot{\omega}_x \end{array}\right] \begin{bmatrix} x_G \\ y_G \\ z_G \\ \hline J_{xx} \\ J_{yy} \\ J_{zz} \\ J_{xy} \\ J_{yz} \\ J_{xz} \end{bmatrix}$$

The first set of equations is uncoupled from the second. Therefore, moving the mass to the left-hand side of the equations, we have:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \left[\begin{array}{c|ccc} \ddot{x}_0 & 0 & -\dot{\omega}_x & \dot{\omega}_y \\ \ddot{y}_0 & \dot{\omega}_z & 0 & -\dot{\omega}_x \\ \ddot{z}_0 & -\dot{\omega}_y & \dot{\omega}_x & 0 \end{array}\right] \begin{bmatrix} M \\ Mx_G \\ My_G \\ Mz_G \end{bmatrix}$$

Operatively, instead of the previous system of equations, a different one is used in which only a force in a single direction is applied. Therefore, dividing by this force (say F_x) we actually have:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \ddot{x}_0/F_x & | & 0 & -\dot{\omega}_x/F_x & \dot{\omega}_y/F_x \\ \ddot{y}_0/F_x & | & \dot{\omega}_z/F_x & 0 & -\dot{\omega}_x/F_x \\ \ddot{z}_0/F_x & | & -\dot{\omega}_y/F_x & \dot{\omega}_x/F_x & 0 \end{bmatrix} \begin{bmatrix} M \\ Mx_G \\ My_G \\ Mz_G \end{bmatrix}$$

Once the kinematic problem is solved, M, $x_G = Mx_G/M$ etc. can be easily determined. Note, however, that the 3×3 submatrix on the right is singular, so that it is necessary to consider only two equations or a single equation (where the force is applied) for any excitation.

This approach is more flexible than the one shown in the second method. To solve its overdetermined version in a least square sense, other measurements, beyond the three basically used, are not necessary. In fact, even considering the above singularity, two equations are available for any excitation.

The second set of equations can now be solved to obtain the moments of inertia directly. It would be possible to solve equation (8), instead of the previous one, because the matrix elements are written in the excitation co-ordinates, that are more reliable values than the components of the angular acceleration. However, in this case the inverse components of the inertia tensor are determined so that an inversion operation is required.

Using equation (6) for the kinematic problem and the third method for the dynamic one, a determined system of equations is obtained, with three sets of measurements, one for any excitation direction. For each group, six transduction points are necessary: therefore 18 measurements are necessary globally, including the computation of the mass value.

2.3. A PARTICULAR CASE: THE PLATE

For a 2-D system such as a plate, the problem is noticeably simplified. In fact, on the assumption that the plate is infinitesimally thin, one of the co-ordinates of the centre of mass (say z_G) is known because it lies in the plane of the plate. Moreover only the x and y components of the tensor of inertia need to be considered if the reference frame $Oxyz$ is chosen so that the z axis is orthogonal to the plate plane, which implicitly defines the direction of the z principal axis.

The kinematic problem is then simplified into the single equation:

$$\ddot{z}_T/F_z = \ddot{z}_0/F_z + \dot{\omega}_x y_T/F_z - \dot{\omega}_y x_T/F_z \tag{10}$$

in which the unknowns are $\ddot{z}_0$, $\dot{\omega}_x$ and $\dot{\omega}_y$. To solve the problem in determined form only three transductions points are necessary for any excitation.

Following method 3, the dynamic problem can be written as:

$$\begin{bmatrix} F_z - M\ddot{z}_0 \\ y_E F_z \\ -x_E F_z \end{bmatrix} = \begin{bmatrix} -M\dot{\omega}_y & M\dot{\omega}_x & 0 & 0 & 0 \\ 0 & F_z & \dot{\omega}_x & 0 & -\dot{\omega}_y \\ -F_z & 0 & 0 & \dot{\omega}_y & -\dot{\omega}_x \end{bmatrix} \begin{bmatrix} x_G \\ y_G \\ J_{xx} \\ J_{yy} \\ J_{xy} \end{bmatrix} \tag{11}$$

which, as before, can be partitioned into:

$$F_z = [\ddot{z}_0 \quad -\dot{\omega}_y \quad \dot{\omega}_x] \begin{bmatrix} M \\ Mx_G \\ My_G \end{bmatrix} \tag{12}$$

and

$$\begin{bmatrix} (y_E - y_G)F_z \\ -(x_E - x_G)F_z \end{bmatrix} = \begin{bmatrix} \dot{\omega}_x & 0 & -\dot{\omega}_y \\ 0 & \dot{\omega}_y & -\dot{\omega}_x \end{bmatrix} \begin{bmatrix} J_{xx} \\ J_{yy} \\ J_{xy} \end{bmatrix} \tag{13}$$

To solve the first equation three different excitation points are necessary. Therefore the problem requires three sets of measurements with three transduction points, i.e. globally nine measurements are necessary.

3. THEORETICAL AND EXPERIMENTAL RESULTS: COMMENTS

The procedure developed above has been applied to two different structures, namely a plate and a 3-D structure (Fig. 1), suitably designed to permit an easy theoretical computation of the mass characteristics. Both structures were investigated using theoretically computed mass-lines and experimental data.

For the plate (dimensions: 0.3 m length, 0.14 m width, 0.005 m thickness) that was suspended through very soft springs to simulate free boundary conditions, the results are shown in Table 1. Both theoretical and experimental results were obtained through an overdetermined system of equations using 12 measurements (three excitation forces and four transducers). They both agree very well with the expected inertia values: theoretical results present a maximum error of 0.2% for J_y and experimental results scatter from the expected ones by 4.6% maximum. It should cause no surprise that the experimental results agree so well with the theoretical ones, because the system is well-conditioned and the estimated masslines agree quite well with the real ones.

The 3-D system, whose expected mass characteristics were computed by a program that assembles the results of each substructure, presented a more severe test. This structure was analysed first theoretically and then experimentally. The theoretical results were obtained using 18 theoretical masslines (determined system of equations) and are in perfect agreement with the actual values. Also the experimental results, obtained by two different overdetermined systems of equations and reported in Tables 2 and 3, match up

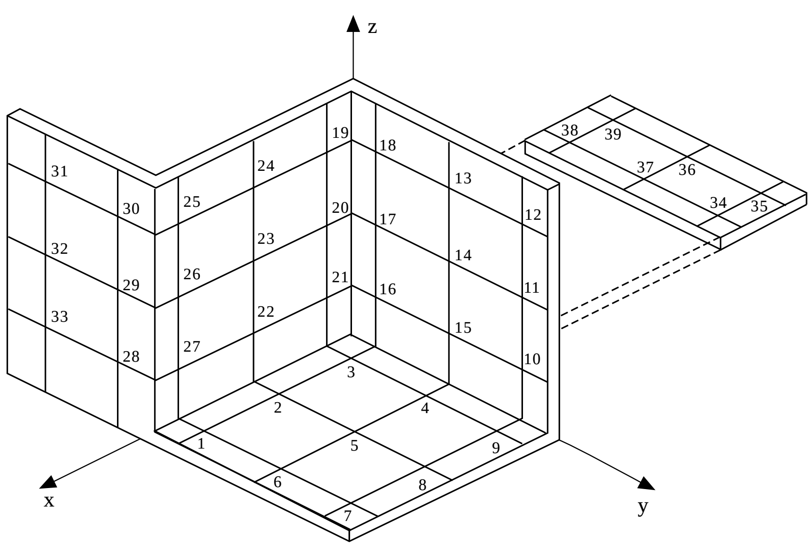

Figure 1. Sketch of the 3-D system.

TABLE 1

Estimate of the mass characteristics of a plate

	Theoretical data		
	Expected	Estimated	Error %
M	5.67×10^{-1}	5.67×10^{-1}	0
x_G	0	0	0
y_G	6.36×10^{-2}	6.36×10^{-2}	0
J_x	4.25×10^{-3}	4.25×10^{-3}	0
J_y	9.26×10^{-4}	9.24×10^{-4}	0.2
	Experimental data		
	Expected	Estimated	Error %
M	5.67×10^{-1}	5.41×10^{-1}	4.6
x_G	0	0	0
y_G	6.36×10^{-2}	6.28×10^{-2}	1.25
Jx	4.25×10^{-3}	4.08×10^{-3}	4
J_y	9.26×10^{-4}	9.28×10^{-4}	0.21

TABLE 2

Estimate of the mass characteristics of the 3-D structure (overdetermined system 16 × 10)

	Experimental data		
	Experimented	Estimated	Error %
M	3.91	3.81	2.6
x_G	4.72×10^{-2}	5.01×10^{-2}	6.1
y_G	3.85×10^{-2}	3.64×10^{-2}	5.4
z_G	5.35×10^{-2}	4.95×10^{-2}	7.5
J_x	1.31×10^{-2}	1.28×10^{-2}	1.6
J_y	3.29×10^{-2}	3.34×10^{-2}	1.5
J_z	2.78×10^{-2}	2.69×10^{-2}	3.2

TABLE 3

Estimate of the characteristics of the 3-D structure (overdetermined system 13 × 10)

	Experimental data		
	Expected	Estimated	Error %
M	3.91	3.81	2.6
x_G	4.72×10^{-2}	4.41×10^{-2}	6.6
y_G	3.85×10^{-2}	3.73×10^{-2}	3.1
z_G	5.35×10^{-2}	4.99×10^{-2}	6.7
J_x	1.31×10^{-2}	1.31×10^{-2}	0
J_y	3.29×10^{-2}	3.04×10^{-2}	7.6
J_z	2.78×10^{-2}	2.78×10^{-2}	0

satisfactorily with the expected ones. Both tests were performed by means of 27 measured masslines: a single excitation along the three co-ordinate directions and nine transduction points for any different excitation were used to solve the kinematic problem [equation (6)]. Then the dynamic problem was solved through method three, using two different combinations of the previously measured data: Table 2 used globally an overdetermined system 16×10, whilst Table 3 used a system 13×10. In Table 2 a maximum error of 7.5% was obtained whilst in Table 3, reporting results of a smaller overdetermined system, a maximum error of 7.6% was obtained, although globally the average error was lower than in the previous case.

It is worth pointing out that such satisfactory results were obtained after a careful design of the test rig, in order to get sufficiently correct values of masslines. In fact, for 3-D structures the measurement of masslines is a delicate operation. Besides the usual computation that must be used to fit the masslines from a someway suspended structure, here different problems arise and the results are quite sensitive to the suspension mechanism. It is worth considering that, for 3-D systems, the masslines must be determined not only for responses along the same direction of the exciting forces, but also for responses measured orthogonally to the applied forces.

Therefore we first checked the influence on the massline values of the non-linear terms of the kinematic equations, usually neglected in these computations, and the effect of the suspension mechanism, by considering theoretically the response of a three-dimensional system suspended as a pendulum through a very flexible spring. Although they both have some small effect, their influence can be neglected.

Subsequently, we analysed the drawback connected with responses measured orthogonally to the exciting forces. Due to the relative direction between force and transduction when the force acts orthogonally to the measured response, the transverse sensitivity of accelerometers (usually about $3 \div 5\%$), in some cases makes the response measured normally to the principal transducer axis of the same order of magnitude as the response along the principal axis. Another problem connected with the previous situation, which makes more critical this set of measurements, arises when, due to small errors in the mounting of the accelerometer or in the direction of the applied force, undesired transverse components of acceleration are measured which amplify the previously described inconvenient. This situation may provoke a significant difference between the expected values of masslines and the measured ones. Although in our tests, a careful analysis of the performed measurements showed which elements would require corrections, we preferred to accept these errors, in order to avoid a complex manipulation of data that would make the procedure particularly complicated. Obviously this results in some scatter of masslines values with respect to the expected ones. The average error on the whole set of measured masslines was about 2.3% whilst the variance of these data was about 12%.

Notwithstanding this scatter of data, the results obtained for the rigid body characteristics were satisfactory. This proves that the problem is well-posed, and the proposed approach leads to a straightforward technique for the determination of the mass characteristics of structures.

4. CONCLUSIONS

Different procedures for determining the rigid body inertia properties have been presented in this paper. All the proposed methods use experimental FRF data in the low frequency range to identify the massline characteristics, which are required as input to the solution algorithms. A careful analysis of the condition of the system of equations shows the minimum set of necessary measurements for each approach and the preferable procedure to adopt, even in presence of noise.

Although apparently one can think that, following these procedures, the mass properties can be derived as a by-product of available FRF data, previously obtained for other purposes, unfortunately this is not usually possible. In fact particular sets of data are required, to avoid singular or ill-conditioned matrices. Moreover, even in the lucky circumstance of a large distance between rigid body and flexible modes, which makes the massline frequency region nearly constant, the massline values identification is particularly troublesome, especially for 3-D structures. An inaccurate mounting of accelerometers and/or an incorrect direction of the applied force may cause relevant errors in the masslines values. Finally it was observed that, for 3-D structures, masslines values are also quite sensitive to the suspension mechanism, and generally a pendulum suspension through a very flexible spring is the most reliable.

Notwithstanding all these limitations, the proposed procedures showed an appreciable agreement between theoretical and experimental data, confirming that they are quite efficient and represent very good alternatives to traditional trifilar pendulum tests.

ACKNOWLEDGEMENTS

The present work has been supported by MURST (Ministero Università e Ricerca Scientifica) through grants 40% and 60%.

REFERENCES

1. S. M. Pandit and Z.-Q. Hu 1994 *Journal of Sound and Vibration* **177**, 31–41. Determination of rigid body characteristics from time domain test data.
2. M. A. Lamontia 1982 *Proceedings of the* 1*st IMAC*, Orlando, FL. On the determination and use of residual flexibilities, inertia restraints and rigid body modes.
3. J. Crowley, T. Rockrin and D. Brown 1986 *Proceedings of the* 4*th IMAC*, Los Angeles, CA. Use of rigid body calculations in tests.
4. M. Furusawa and T. Tominaga 1986 *Proceedings of the* 4*th IMAC*, Los Angeles, CA. Rigid body modes enhancement and RDOF estimation for experimental modal analysis.
5. N. Okubo and T. Furukawa 1983 *Proceedings of the* 2*nd IMAC*, Orlando, FL. Measurement of the rigid body modes for dynamic design.
6. J. Bretl and P. Conti 1987 *Proceedings of the* 5*th IMAC*, London. Rigid body mass properties from test data.
7. Y. S. Wei and J. Reis 1989 *Proceedings of the* 7*th IMAC*, Las Vegas, NV. Experimental determination of rigid body inertia properties.
8. M. Furusawa 1989 *Proceedings of the* 7*th IMAC*, Las Vegas, NV. A method of determining rigid body inertia properties.
9. A. Fregolent and G. Saginario 1992 *Atti AIMETA*. A model to estimate errors at low frequencies when measuring FRF data of free systems. (In Italian)

APPENDIX A: MASSLINE ESTIMATION FROM EXPERIMENTAL DATA

When the FRF of a free structure is experimentally determined, distressing bias are introduced in the low frequency range because of the need of hanging up the structure. The suspension constraints the rigid body degrees of freedom and induces resonance conditions at low frequencies. These resonances can affect the estimate of masslines values when the rigid body modes interact with the elastic modes.

The inertance of an unrestrained structure can be written as the sum of two contributions: the rigid body motion and the elastic motion with respect to the undeformed configuration:

$$\mathbf{H}_{ij}(\omega) = \frac{\ddot{\mathbf{x}}_i(\omega)}{\mathbf{f}_j(\omega)} = R_B^{ij} + \sum_r \frac{-\omega^2 \phi_i^r \phi_j^r}{\omega_r^2 - \omega^2 + j\eta_r\omega_r^2} \tag{14}$$

being R_B the contribution of the rigid body modes, ω_r the rth natural frequency, η_r the damping loss factor of the rth mode, ϕ_i^r and ϕ_j^r the values of mode r at the points i and j. The constant massline contribution is present at any frequency and can be determined using different procedures, depending on the frequency band of analysis. Generally it is possible to identify, in the inertance of a suspended structure, three typical frequency bandwidths (Fig. A1):

- zone A: this is the frequency band where the rigid body motion is predominant. Resonance peaks, due to the suspension, are present;
- zone B: this is a frequency band where the rigid body motion is no longer affected by the suspension, and the elastic modes are not still present;
- zone C: this is the frequency band of the elastic modes.

In zone B the correct estimation of the massline values is relatively simple. In fact the almost constant trend of the inertance yields the estimate of the massline value as a simple average of the real part of the FRF in the considered bandwidth, where the imaginary part of the FRF is minimal. In zone C the contribution of the rigid body motion is constant although it is superimposed to the response of the elastic modes. Therefore the massline can be determined using a curve-fitting procedure, estimating the low frequency residual on the left of the first elastic mode and on the right of the last rigid mode. However this curve fitting is problematical when the rigid body modes interact with the elastic ones or when the modal density is high. Zone A can be dealt as zone C whenever the structure-suspension system presents a linear behaviour. However, in this case too, the curve fitting procedure is troublesome. In fact, the suspension modes are restricted to a narrow band and even using a high frequency resolution it could be very difficult to obtain a correct curve-fitting.

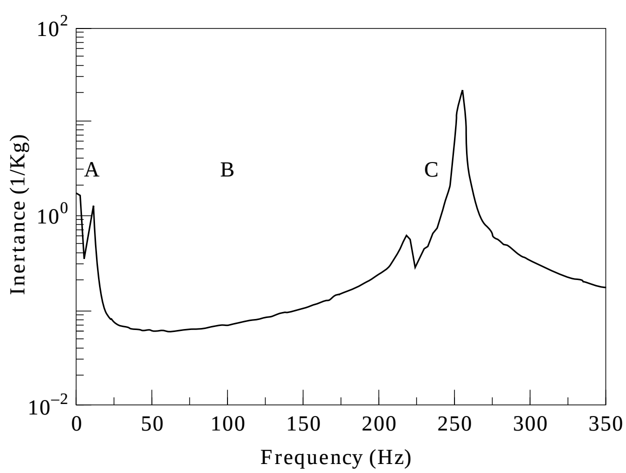

Figure A1. Typical inertance trend of a suspended structure.

Thus the simplest and most reliable action to determine the massline value is using zone B, which would be wide enough to avoid the influence of adjacent modes. When the rigid body and elastic modes are coupled, the previous condition is not verified and also the use of zone B becomes critical. It is then necessary to reduce the suspension influence.

Preliminary simulated tests were performed on the 3-D structure shown in Fig. 1 to determine which parameters mostly affect the FRF values in the low frequency range. Among those carefully considered were: the spring stiffness, its length, the suspension damping, the point linking the suspension to the structure, and the impulse duration. (This last parameter was considered because the masslines were identified from FRFs obtained by impulsive tests). In the model, the torsional stiffness of the spring is neglected so that the suspension-structure system has five restrained degrees of freedom: the motion along the spring axis and four rotations around the two ends of the spring. While the translational mode is always clearly identified, the four rotational modes are highly coupled and cannot often be distinguished. By changing the spring's stiffness it was observed that the higher the stiffness, the higher the amplitude of the translational mode (and the natural frequency of the correspondent mode) so that the corresponding FRF approaches the constant value of the massline at higher frequencies. Therefore a soft spring is convenient to reduce the frequency bandwidth where resonances have influence. With respect to the suspension length, it was observed that this parameter affects the rotational modes around the spring's ends and for an equal stiffness, the longer the spring, the lower the amplitude (and the natural frequencies) of the correspondent modes.

Several tests were performed using different values of damping, observing that this parameter does not affect the rigid body modes at low frequencies. On the contrary, the choice of the point connecting the spring to the structure can present a high incidence in the low frequency range. By comparing the results obtained from a connecting point closer or further from the centre of mass, it is observed that the rotational modes are those more affected. When the connecting point is close to the centre of mass, the rotational modes around this point tend to shift towards the lower frequencies. Thus a connection point close to the centre of mass reduces the amplitude of zone A.

Finally, with respect to the impulse duration, the simulated tests show that when the impulse has a finite duration and triangular form, the massline values in zone B are quite similar to those obtained in the ideal case of a delta function. In fact, whilst the delta acts at $t = 0$ only, the real force acts on successive times, when the internal force of the suspension is present. This superposition affects the massline values. The estimated results are not presented here but can be found in [9].