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THE INERTIAL CHARACTERISTICS OF DYNAMIC ROBOT MODELS

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Abstract—Robot dynamics are embedded in the mathematical foundations of classical mechanics to introduce novel physical interpretations and structural characteristics of the Lagrangian dynamic robot model. Within this framework, the centrality of the inertial matrix emerges. The physical significance of the inertial coefficients is further illuminated by the introduction of the *coefficient of coupling* of robotic manipulators. The properties of the inertial matrix follow directly from the kinematic and dynamic parameters of the robot. These properties translate into the characteristics of the centrifugal, Coriolis and gravitational components of the dynamic robot model. The novel approach reinforces the need to integrate the mechanical and controller designs of robotic manipulators. The conceptual framework leads to design guidelines for simplifying and reducing the nonlinear kinematic and dynamic coupling of robot dynamics. The development of the paper is applied to illustrate the properties and structural characteristics of industrial robots.

1. INTRODUCTION

The pioneering work of Uicker[1] and Kahn and Roth[2] on the dynamic behavior of spatial mechanisms inaugurated extensive research on the dynamics of robotic manipulators. Recursive and nonrecursive algorithms have been implemented to formulate, simulate and control the dynamic behavior of robotic manipulators. (For an excellent review, the reader is referred to Brady *et al.*[3, Chapter 2].) In the past fifteen years, robot dynamics have evolved into a mature discipline.

Technological progress and increased understanding have not altered the complex nature of robot dynamics. The complete dynamic model of a manipulator is characterized by a system of highly coupled and nonlinear, second-order differential equations[4]. The coefficients of the dynamic robot model, which exemplify the structure and depend upon the state (coordinates and velocities) of the manipulator, are prescribed by lengthy formulae, especially as the number, N, of degree-of-freedom (DOF) increases. At Carnegie–Mellon University, the computer program ARM (Algebraic Robot Modeller) has been implemented to generate symbolically the forward solution and complete Lagrangian dynamic robot model for control engineering applications[5]. The symbolic dynamic model of the six DOF PUMA robot, which occupies forty typewritten pages, dramatizes the highly coupled and nonlinear characteristics of robot dynamics.

Upon neglecting mechanical dissipation, the standard matrix-vector formulation of the dynamic robot model is[4]

$$\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \mathbf{F}(t). \qquad (1)$$

Standard robotic terminology is utilized throughout the paper and departures from the symbolic notation in the literature are noted. In (1), $\mathbf{q}$ is the N-vector of generalized coordinates; $\mathbf{D}(q) = [d_{ij}]$ is the inertial ($N \times N$) matrix; $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = [\dot{q}_j c_{jm}(i) \dot{q}_m]$ is the centrifugal and Coriolis N-vector; $\mathbf{G}(\mathbf{q})$ is the gravitational N-vector; and $\mathbf{F}(t)$ is the N-vector of generalized external forces.

The relative significance of the dynamic components in (1), which has been the subject of considerable controversy in the literature, depends upon the manipulator design and trajectory. Hollerbach[3, Chapter 2] shows that there are realistic cases in which all of the dynamic contributions are significant, relative to each other. Simulations at Carnegie–Mellon University illustrate the relative importance of the generalized force components in the dynamic robot model[6].

This paper focuses on the physical interpretation of the structural characteristics of the dynamic robot model in (1) for control engineering applications[7 and 8]. The coupled and nonlinear dynamic robot model presents a formidable problem for control engineers. The continuously increasing demands for enhanced performance and productivity emphasize the need to integrate the mechanical and controller designs of robotic manipulators. Within this control engineering framework, emphasis is placed upon designing a robot which reduces the inertial, centrifugal, Coriolis and gravitational coupling and nonlinearities, thus yielding a model that is simple and accurate. The engineering problem is to determine geometrical configurations that lead naturally to:

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• Diagonal (or diagonally dominant) and almost constant inertial matrix $\mathbf{D}(\mathbf{q})$;

• Close to zero centrifugal and Coriolis vector $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$; and

• Almost constant or zero gravitational vector $\mathbf{G}(\mathbf{q})$.

These conditions cannot be satisfied simultaneously because (except for degenerate cases) they are contradictory. Nevertheless, the design approach developed in this paper can provide the robot engineer with guidelines that can be applied to a multitude of manipulator applications.

Within this framework, the centrality of the inertial matrix $\mathbf{D}(\mathbf{q})$ emerges and dominates the physical interpretation and structural characteristics of the dynamic robot model in (1). This paper thus concentrates on the central role of the inertial matrix $\mathbf{D}(\mathbf{q}) = [d_{ij}]$ and introduces novel physical interpretations and structural characteristics of the inertial coefficients d_{ij}. These properties of the inertial matrix are induced by the link parameters (relative positions and orientations of the links) and the masses and centers-of-gravity of the links. In turn, these properties of $\mathbf{D}(\mathbf{q})$ propagate into the centrifugal and Coriolis components, and impact the gravitational components of the dynamic robot model in (1).

An engineering contribution of this paper are design guidelines, which specify configurations for which:

• The self-inertial coefficients d_{ii} are constant;

• The mutual inertial coefficients $d_{ij}(i \neq j)$ are zero;

• The centrifugal and Coriolis coefficients $c_{jm}(i)$ (for all j, m, i) are zero; and

• The gravitational coefficients G_i are constant or zero.

The search for these guidelines motivated the conceptual framework highlighted in this paper and these guidelines resolve the engineering problem formulated at the outset of this section.

There are two distinct approaches to the dynamic modeling of open-loop kinematic chains[3]: the Lagrangian formulation and the Newton–Euler formulation. These equivalent approaches[9] accomplish different goals. The Lagrangian formulation leads to a compact system of equations-of-motion which is appropriate for modeling and control applications and incorporates considerable redundancy. The Newton–Euler formulation is more suitable for real-time simulation and control but requires an explicit representation of the internal forces in the system. This is the basic distinction between computational methods that produce algebraic formulae and algorithms that are difficult to interpret[7, 8, 10 and 11] and conceptual approaches that illuminate the dynamics but are not appropriate for real-time computation. In this paper we follow the Lagrangian formulation, but we do not introduce the accompanying Uicker–Kahn formalism. Instead, we embed robot dynamics in the mathematical foundations of classical mechanics[12]. We adopt this formalism to unravel the physical properties and structural characteristics of the dynamic robot model in (1).

This paper is organized as follows. The robot dynamics scenario is reviewed (in Section 2) to establish the framework for the formulation of the dynamic robot model. The properties of the inertial, centrifugal and Coriolis, and gravitational coefficients are then interpreted in Sections 3, 4, and 5, respectively. In Section 6, we apply the framework developed in this paper to illustrate the characteristics of practical robots. Finally, in Section 7, we summarize the contributions of this paper.

2. ROBOT DYNAMICS SCENARIO

We consider the articulated chain in Fig. 1 of N rigid bodies (or links), which are serially connected by joints. Our analysis is motivated by the standard Uicker–Kahn Lagrangian formulation of manipu-

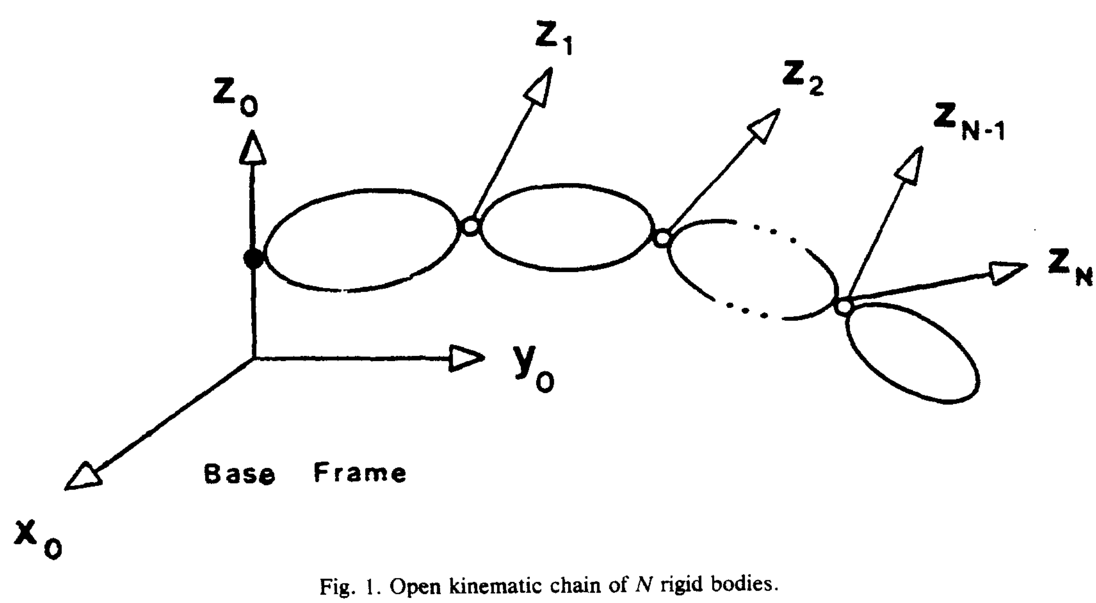

Fig. 1. Open kinematic chain of N rigid bodies.

lator dynamics[3, p. 75]. By convention[2], each link translates along or rotates around the z-axis of a coordinate frame embedded in the previous link. The first link moves with respect to the z-axis of the zero (or base) coordinate frame which is a *Cartesian inertial frame*. In robotic manipulators, the origin of the N-th frame often coincides with that of the (N-1)-th frame, and its orientation is specified by the task executed by the chain.

We proceed to derive the dynamic equations-of-motion for the N DOF mechanism (in Fig. 1) in terms of the generalized coordinates $q_1, \ldots, q_N$. The generalized coordinate q_i is defined as either a rotational angle around $\mathbf{z}_{i-1}$ or a translational displacement along $\mathbf{z}_{i-1}$. Accordingly, link i is characterized as rotational or translational. Throughout the paper, we use geometrical (3×1) vectors, rather than the (4×1) vectors of the Uicker–Kahn formalism.

The N-dimensional space defined by the generalized coordinates $(q_1, \ldots, q_N)$ is called the *configuration space* of the N DOF mechanism. This configuration space is Riemannian and not Euclidean. In the configuration space, there is no coordinate system in which the distance between any two points equals the square-root of the sum-of-the-squares of the coordinate differentials. The derivation and interpretation of the equations-of-motion in the configuration space thus requires the application of (the concepts of) Riemannian geometry to analytical mechanics[13].

We evaluate the kinetic energy (in Section 2.1) and potential energy (in Section 2.2) of the articulated chain in Fig. 1, to derive the Lagrangian equations-of-motion (in Section 2.3).

2.1 *Kinetic energy*

The kinetic energy T of the articulated chain equals the sum of the kinetic energies of the point masses that constitute the bodies of the chain:

$$T = \frac{1}{2} \sum_a m_a \dot{\mathbf{r}}_a^2. \tag{2}$$

In (2), $\mathbf{r}_a = \mathbf{r}_a(q_1, \ldots, q_N)$ is the (coordinate-dependent) position vector of point mass m_a in the Cartesian base frame. Thus,

$$\dot{\mathbf{r}}_a = \sum_{i=1}^{N} \frac{\partial \mathbf{r}_a}{\partial q_i} \dot{q}_i. \tag{3}$$

Upon substituting the velocities of each particle in (3), the kinetic energy of the collection of point masses in (2) becomes

$$T = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \dot{q}_i \, d_{ij} \, \dot{q}_j. \tag{4}$$

The coefficients d_{ij} of the quadratic form in (4) are

$$d_{ij} = \sum_a m_a \left\{ \frac{\partial \mathbf{r}_a}{\partial q_i} \cdot \frac{\partial \mathbf{r}_a}{\partial q_j} \right\}.$$

If the point mass m_a belongs to link k, then $\mathbf{r}_a = \mathbf{r}_a(q_1, \ldots, q_k)$, and

$$d_{ij} = \sum_{k=\max(i,j)}^{N} \left[\sum_{a \in k} m_a \left\{ \frac{\partial \mathbf{r}_a}{\partial q_i} \cdot \frac{\partial \mathbf{r}_a}{\partial q_j} \right\} \right]. \tag{5}$$

The description of the motion of the mechanism (in Fig. 1) by generalized coordinates is one of the principal features of analytical mechanics[13, p. 17]. The generalized coordinates $(q_1, \ldots, q_N)$ are the coordinates of a Riemannian space[12]. Riemannian geometry is based on one single differential quantity, called the line element $\mathrm{d}s$. The significance of the line element is "the distance between two neighboring points of space, expressed in terms of the coordinates and their differentials"[13, p. 18]. The line element (or Riemannian metric) is

$$\mathrm{d}s = \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} \mathrm{d}q_i \, d_{ij}(\mathbf{q}) \, \mathrm{d}q_j \right\}^{1/2}. \tag{6}$$

The line element in (6) is related to the kinetic energy in (4) according to

$$T = \frac{1}{2} \left(\frac{\mathrm{d}s}{\mathrm{d}t} \right)^2 = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{D}(\mathbf{q}) \dot{\mathbf{q}}. \tag{7}$$

The practical implications of the intimate relationship between the kinetic energy and the Riemannian metric are threefold:

1. The inertial matrix $\mathbf{D}(\mathbf{q})$ is symmetric; i.e. $d_{ij} = d_{ji}$ for all i and j;
2. The inertial matrix $\mathbf{D}(\mathbf{q})$ is positive definite[7 and 8]; i.e. the kinetic energy in (7) is positive for all values of the link coordinates and (nonzero) velocities; and
3. The self (d_{ii})-and mutual (d_{ij}) inertial coefficients satisfy the inequality[14, p. 427]

$$d_{ij}^2 < d_{ii} d_{jj} \quad \text{for all } i \neq j.$$

2.2 *Potential energy*

The potential energy V of the kinematic chain (in Fig. 1) equals the sum of the potential energies of the point masses that constitute the links of the chain:

$$V = - \sum_a m_a \, (\mathbf{g} \cdot \mathbf{r}_a) + \text{constant}, \tag{8}$$

where $\mathbf{g}^T = [g_x \; g_y \; g_z]$ is the gravity vector ($|\mathbf{g}| = 9.81 \mathrm{m/s^2}$). The minus sign in (8) indicates that g_z is considered to be a negative number. The constant depends upon the specified reference plane of zero potential energy. We assume that the robot operates in a uniform gravitational field. Hence, we use the terms center-of-mass and center-of-gravity interchangeably. By defining the position vector $\mathbf{r}_k$ from the origin of the base frame to the center-of-gravity of link k according to

$$m_k \mathbf{r}_k = \sum_{a \in k} m_a \mathbf{r}_a,$$

the potential energy in (8) becomes

$$V = -\mathbf{g} \cdot \sum_{k=1}^{N} m_k \mathbf{r}_k + \text{constant}. \tag{9}$$

2.3 *Equations-of-motion*

The Lagrangian equations-of-motion are[15]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i} = F_i(t) \quad \text{for } i = 1, 2, \ldots, N, \tag{10}$$

where F_i is the generalized force that acts on joint i; F_i is a force for a translational joint and a torque for a rotational joint. We note that F_i must be in the direction of q_i; i.e. in the direction of $\mathbf{z}_{i-1}$. Thus, F_i is the projection on $\mathbf{z}_{i-1}$ of the sum of the external generalized forces acting on joint i.

By substituting (4) and (9) into (10), the Lagrangian equations-of-motion for an articulated chain become

$$\sum_{j=1}^{N} d_{ij}\ddot{q}_j + \sum_{j=1}^{N}\sum_{m=1}^{N} \dot{q}_j\, c_{jm}(i)\, \dot{q}_m + G_i = F_i(t) \quad \text{for } i = 1, 2, \ldots, N. \tag{11}$$

In (11), $c_{jm}(i)$ is the centrifugal coefficient for $j = m$ and the Coriolis coefficient for $j \neq m$ of link i. The $c_{jm}(i)$ are specified by the Christoffel symbol[15]:

$$c_{jm}(i) = \frac{1}{2}\left\{\frac{\partial d_{ij}}{\partial q_m} + \frac{\partial d_{im}}{\partial q_j} - \frac{\partial d_{jm}}{\partial q_i}\right\}. \tag{12}$$

The Christoffel symbol in (12) reinforces the centrality of the inertial matrix $\mathbf{D} = [d_{ij}]$ in the dynamic robot model in (1).

The gravitational coefficient G_i of link i is

$$G_i = -\frac{\partial V}{\partial q_i}. \tag{13}$$

In the sequel, we identify the physical properties and structural characteristics of the dynamic robot model in (11). We address the inertial coefficients d_{ij}, centrifugal and Coriolis coefficients $c_{jm}(i)$, and gravitational coefficients G_i in Sections 3, 4, and 5, respectively.

3. INERTIAL COEFFICIENTS

3.1 *Introduction*

In this section, we develop a geometrical framework in which to interpret the physical characteristic and identify the structural properties of the inertial matrix $\mathbf{D}(\mathbf{q})$. Figure 2 serves as the foundation for the position vectors in our analysis. In Fig. 2 and throughout this section, we assume that $i \leq j$. Because of the symmetry of the inertial matrix, we make this assumption without loss of generality.

Vector addition (Fig. 2) leads to

$$\begin{aligned}\mathbf{r}_a &= \mathbf{R}_i(q_1, \ldots, q_{i-1}) + \mathbf{R}_{ij}(q_1, \ldots, q_{j-1}) \\ &\quad + \mathbf{p}_{ja}(q_1, \ldots, q_k) \\ &= \mathbf{R}_i(q_1, \ldots, q_{i-1}) + \mathbf{p}_{ia}(q_1, \ldots, q_k).\end{aligned} \tag{14}$$

We introduce the Boolean variable

$$\sigma_i = \begin{cases} 1 & \text{if joint } i \text{ is translational} \\ 0 & \text{if joint } i \text{ is rotational}\end{cases}$$

and denote the complement $(1 - \sigma_i)$ by σ_i^*. The partial derivatives of $\mathbf{r}_a$ in (14), with respect to the generalized coordinates q_i and q_j, are[16]

$$\frac{\partial \mathbf{r}_a}{\partial q_i} = \frac{\partial \mathbf{p}_{ia}}{\partial q_i} = \sigma_i \mathbf{z}_{i-1} + \sigma_i^* (\mathbf{z}_{i-1} \times \mathbf{p}_{ia}) \tag{15}$$

and

$$\frac{\partial \mathbf{r}_a}{\partial q_j} = \frac{\partial \mathbf{p}_{ja}}{\partial q_j} = \sigma_j \mathbf{z}_{j-1} + \sigma_j^* (\mathbf{z}_{j-1} \times \mathbf{p}_{ja}). \tag{16}$$

Substituting (15) and (16) into (2), and applying the fundamental property of the scalar triple product; i.e. $\mathrm{v}_1 \cdot (\mathrm{v}_2 \times \mathrm{v}_3) = (\mathrm{v}_1 \times \mathrm{v}_2) \cdot \mathrm{v}_3$[14], culminate in the following explicit formula for the inertial coefficients:

$$\begin{aligned} d_{ij} &= \sigma_i\sigma_j(\mathbf{z}_{i-1}\cdot\mathbf{z}_{j-1}) \sum_{k=j}^{N}\sum_{a\in k} m_a \\ &\quad + \sigma_i\sigma_j^* \times (\mathbf{z}_{i-1}\times\mathbf{z}_{j-1})\cdot \sum_{k=j}^{N}\sum_{a\in k} m_a\mathbf{p}_{ja} \\ &\quad + \sigma_i^*\sigma_j(\mathbf{z}_{j-1}\times\mathbf{z}_{i-1})\cdot \sum_{k=j}^{N}\sum_{a\in k} m_a\mathbf{p}_{ia} \\ &\quad + \sigma_i^*\sigma_j^* \sum_{k=j}^{N}\sum_{a\in k} m_a(\mathbf{z}_{i-1}\times\mathbf{p}_{ia})\cdot(\mathbf{z}_{j-1}\times\mathbf{p}_{ja}). \end{aligned} \tag{17}$$

The double sums appearing in (17) are

$$\sum_{k=j}^{N}\sum_{a\in k} m_a = m_j + \cdots + m_N \tag{18}$$

$$\sum_{k=j}^{N}\sum_{a\in k} m_p\mathbf{p}_{ja} = (m_j + \cdots + m_N)\mathbf{p}_j \tag{19}$$

$$\sum_{k=j}^{N}\sum_{a\in k} m_a\mathbf{p}_{ia} = (m_j + \cdots + m_N)\mathbf{p}_i \tag{20}$$

$$\sum_{k=j}^{N}\sum_{a\in k} m_a(\mathbf{z}_{i-1}\times\mathbf{p}_{ia})\cdot(\mathbf{z}_{j-1}\times\mathbf{p}_{ja}) = J_{ij} = (m_j + \cdots + m_N)K_{ij}. \tag{21}$$

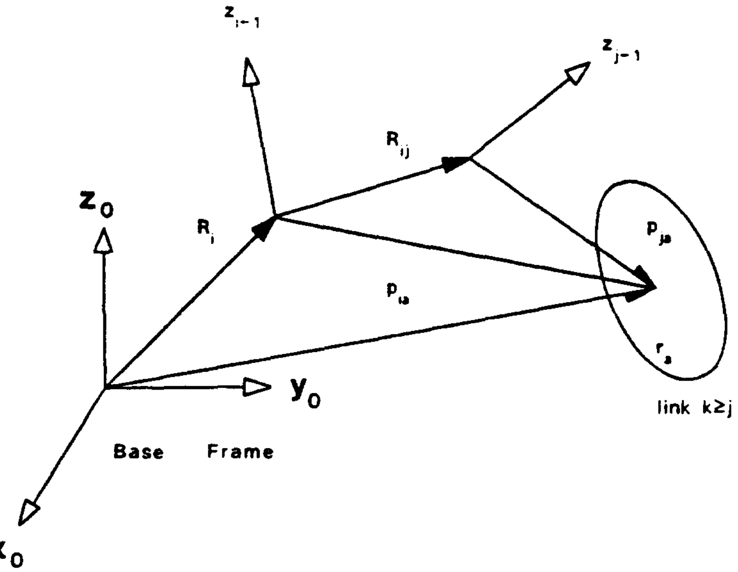

Fig. 2. Position vectors in an articulated kinematic chain. (All vectors are expressed in the base frame.) $\mathbf{z}_{i-1}$ is the unit vector that defines the reference axis for coordinate q_i. $\mathbf{z}_{j-1}$ is the unit vector that defines the reference axis for coordinate q_j. $\mathbf{r}_a$ is the position vector from the origin of the base frame to the point mass m_a in link $k \geq j$; $\mathbf{r}_a$ is a function of $(q_1, \ldots, q_k)$. $\mathbf{R}_i$ is the position vector from the origin of the base frame to the origin of frame $(i - 1)$; $\mathbf{R}_i$ is a function of $(q_1, \ldots, q_{i-1})$. $\mathbf{R}_{ij}$ is the position vector from the origin of frame $(i - 1)$ to the origin of frame $(j - 1)$; $\mathbf{R}_{ij}$ is a function of $(q_1, \ldots, q_{j-1})$. p_{ja} is the position vector from the origin of frame $(j - 1)$ to the point mass m_a in link $k \geq j$; p_{ja} is a function of $(q_1, \ldots, q_k)$. p_{ia} is the position vector from the origin of frame $(i - 1)$ to the point mass m_a in link $k \geq j$; p_{ia} is a function of $(q_1, \ldots, q_k)$.

In (19) and (20), $\mathbf{p}_j$ and $\mathbf{p}_i$ are the position vectors of the centers-of-gravity of links j through N in the $(j - 1)$ and $(i - 1)$ frames, respectively. In (21), J_{ij} is the projected inertia of links j through N with respect to frame $(i - 1)$. We interpret the projected inertia as follows. The product $(\mathbf{z}_{i-1} \times \mathbf{p}_{ia}) \cdot (\mathbf{z}_{j-1} \times p_{ja})$ in (21) is the inner product of the vector distances of point mass m_a from axes $\mathbf{z}_{i-1}$ and $\mathbf{z}_{j-1}$. Thus, $\mathbf{J}_{ij}$ is the inertia of links j through N (which are rotating around $\mathbf{z}_{j-1}$) with respect to $\mathbf{z}_{i-1}$. The projected inertia J_{ij} is positive or negative, depending upon the relative rotation of the two rotational axes. Accordingly, the projected radius-of-gyration K_{ij} is defined as the signed square distance from $\mathbf{z}_{i-1}$ to a point, with mass equal to $(m_j + \cdots + m_N)$, that exhibits the same projected inertia J_{ij} as links j through N. Thus, K_{ij} depends upon the generalized coordinates $q_{i+1}, \ldots, q_N$. When the axes $\mathbf{z}_{i-1}$ and $\mathbf{z}_{j-1}$ coincide (i.e. $\mathbf{z}_{i-1} \equiv \mathbf{z}_{j-1}$), J_{ij} is the inertia J_{ii} of links i through N with respect to $\mathbf{z}_{i-1}$, and K_{ij} is the square of the coordinate-dependent radius-of-gyration K_{ii}[16].

3.2 *Self-inertial and mutual inertial coefficients*

Upon substituting (18) through (21) into (17), the inertial coefficient formula in (17) becomes

$$\begin{aligned} d_{ij} = (m_j + \cdots + m_N)\,\{&\sigma_i\sigma_j(\mathbf{z}_{i-1} \cdot \mathbf{z}_{j-1}) \\ &+ \sigma_i\sigma_j^*(\mathbf{z}_{i-1} \times \mathbf{z}_{j-1}) \cdot \mathbf{p}_j \\ &+ \sigma_i^*\sigma_j(\mathbf{z}_{j-1} \times \mathbf{z}_{i-1}) \cdot \mathbf{p}_i \\ &+ \sigma_i^*\sigma_j^*\, K_{ij}\}. \end{aligned} \tag{22}$$

In Appendix A, we apply the concepts of the Denavit–Hartenberg convention[18] to transform (22) into a computationally attractive algorithm.

When $i = j$, (22) specifies the self-inertial coefficients as

$$d_{ii} = (m_i + \cdots + m_N)\{\sigma_i + \sigma_i^*\, K_{ii}^2\}. \tag{23}$$

The self-inertial coefficients are the sum of a constant positive bias and a function of $q_{i+1}, \ldots, q_N$[7 and 8]. The constant positive bias (of the inertial coefficient) of a translational link is the mass of the link, and the constant positive bias of a rotational link is the inertia of the link with respect to the axis-of-rotation. The square of the radius-of-gyration K_{ii}^2 in (23) is thus the sum of a constant positive bias and a function of $q_{i+1}, \ldots, q_N$.

Since the inertial matrix $\mathbf{D}(\mathbf{q})$ is symmetric (Section 2.1), there are $N(N + 1)/2$ independent inertial coefficients d_{ij}: N self-inertial coefficients d_{ii} in (23) and $N(N - 1)/2$ mutual inertial coefficients d_{ij} in (22). The physical significance of the self-inertial and mutual inertial coefficients emerges naturally from (22) and (23). In the sequel, we distinguish between translational and rotational joints in Sections 3.3 and 3.4, respectively. In Section 3.5, we introduce the coefficient of coupling of robotic manipulators.

3.3 *Translational joint*

When joint i is translational ($\sigma_i = 1$), the Lagrangian dynamic robot model in (11) for joint i becomes a force-balancing equation. The inertial component balances the net force F_i^{NET}; that is,

$$d_{ii}\ddot{q}_i + \sum_{j\neq i} d_{ij}\ddot{q}_j = F_i^{\mathrm{NET}}. \tag{24}$$

From (23), the self-inertial coefficient[4, p. 176]

$$d_{ii} = (m_i + \cdots + m_N) = \text{constant} > 0. \tag{25}$$

Hence, the self-inertial coefficient of a translational joint is constant and equals the total mass translated by the joint.

When joint j is translational ($\sigma_j = 1$), equation (22) becomes

$$d_{ij} = (m_j + \cdots + m_N)\,(\mathbf{z}_{i-1}\cdot\mathbf{z}_{j-1}) \quad \text{for } j > i. \tag{26}$$

The mutual inertial coefficient d_{ij} in (26) between two translational joints signifies the contribution of the translation of joint j to the translation of joint i. For instance, if the translations occur in parallel directions ($\mathbf{z}_{i-1}\cdot\mathbf{z}_{j-1} = 1$), the external force that activates joint j contributes fully to the work effort of F_i^{NET}. If the translations occur in orthogonal directions ($\mathbf{z}_{i-1}\cdot\mathbf{z}_{j-1} = 0$), the joints are uncoupled and $d_{ij} = 0$. For all intermediate orientations, the contribution of one translation to the other depends upon the relative orientation of the translational axes, and d_{ij} is a function of $q_{i+1}, \ldots, q_{j-1}$ (for $j > i$). For consecutive translational axes, $d_{i(i+1)}$ is always constant.

When joint j is rotational ($\sigma_j = 0$), equation (22) becomes

$$d_{ij} = (m_j + \cdots + m_N)\,(\mathbf{z}_{i-1}\times\mathbf{z}_{j-1})\cdot\mathbf{p}_j \quad \text{for } j > i. \tag{27}$$

The mutual inertial coefficient d_{ij} in (27) between a translational and a rotational joint signifies the contribution of the rotation of joint j to the translation of joint i. If the external torque that rotates joint j also translates the center-of-mass of links j through N along q_i, the external torque contributes (through $\ddot{q}_j$) to the work effort of F_i^{NET}. The contribution depends upon the relative orientation of the axes of translation and rotation, and the orientation of the position vector of the centers-of-mass of links j through N with respect to the axis-of-rotation. In general, therefore, d_{ij} is a function of $q_{i+1}, \ldots, q_N$ (for $j > i$). When the axes of rotation and translation are parallel, $d_{ij} = 0$.

3.4 *Rotational joint*

When joint i is rotational ($\sigma_i = 0$), the Lagrangian dynamic robot model in (11) for joint i becomes a torque-balancing equation. The inertial component balances the net torque T_i^{NET}; that is,

$$d_{ii}\ddot{q}_i + \sum_{j\neq i} d_{ij}\ddot{q}_j = T_i^{\mathrm{NET}}. \tag{28}$$

From (23), the self-inertial coefficient

$$d_{ii} = (m_i + \cdots + m_N)\,K_{ii}^2 > 0, \tag{29}$$

where K_{ii} is the radius-of-gyration of links i through N with respect to $\mathbf{z}_{i-1}$. Hence, the self-inertial coefficient for a rotational joint equals the total inertia with respect to the axis of rotation of the links rotated by the joint and is a function of $q_{i+1}, \ldots, q_N$. The constant positive bias of d_{ii} for a rotational link is the inertia of the link with respect to its axis-of-rotation.

When joint j is translational ($\sigma_j = 1$), equation (22) becomes

$$d_{ij} = (m_j + \cdots + m_N)\,(\mathbf{z}_{j-1}\times\mathbf{z}_{i-1})\cdot\mathbf{p}_i \quad \text{for } j > i. \tag{30}$$

The mutual inertial coefficient d_{ij} signifies the contribution of the external force that translates joint j to the rotational effort of T_i^{NET}. The contribution depends upon the relative orientation of the axes of translation and rotation, and the orientation of the position vector of the centers-of-mass of links j through N with respect to the axis-of-rotation. The contribution is independent of the displacement of q_j. Thus, d_{ij} is a function of $q_{i+1}, \ldots, q_{j-1}, q_{j+1}, \ldots, q_N$ (for $j > i$). When the axes of rotation and translation are parallel, $d_{ij} = 0$.

When joint j is rotational ($\sigma_j = 0$), equation (22) becomes

$$d_{ij} = (m_j + \cdots + m_N)\,K_{ij} \quad \text{for } j > i. \tag{31}$$

The mutual inertial coefficient d_{ij} signifies the contribution of the external torque that rotates joint j to the rotational effort of T_i^{NET}. The contribution depends upon the variable (coordinate dependent) distribution of mass of links j through N with respect to axes $\mathbf{z}_{i-1}$ and $\mathbf{z}_{j-1}$. The mass distribution (characterized by the projected radius-of-gyration K_{ij}) and mutual inertial coefficient d_{ij} are functions of $q_{i+1}, \ldots, q_N$.

3.5 *Coefficient of coupling*

The physical significance of the inertial coefficients can be further illuminated by the introduction of the coefficient of coupling k_{ij}. The coefficient of coupling† k_{ij}, between joints i and j, is defined as the ratio of the absolute value of the mutal inertial

† We use the lower-case k_{ij} to specify the coefficient of coupling in (32) and the upper-case K_{ij} to denote the projected radius-of-gyration in (21).

coefficient d_{ij} to the geometric mean of the self-inertial coefficients d_{ii} and d_{jj}:

$$k_{ij} \triangleq \frac{|d_{ij}|}{(d_{ii}d_{jj})^{1/2}} \quad \text{for } i \neq j. \tag{32}$$

The coordinate–dependent coefficient of coupling k_{ij} in (32), which is a function of $q_{s+1}, \ldots, q_N$, where $s = \min(i, j)$, is the robotic manipulator analog of the coefficient of coupling between two inductors[17, p. 381]. Since the inertial matrix is positive definite (Section 2.1), the coefficient of coupling in (32) lies between zero and one, and provides a measure of the degree of structural coupling. As $k_{ij} \to 0$, joints i and j become uncoupled. As $k_{ij} \to 1$, joints i and j become tightly coupled.

From (22), the coefficient of coupling k_{ij} can be interpreted as the product of a constant loading factor, LF, and a coordinate-dependent kinematic factor, KF; that is,

$$k_{ij} = \text{Loading Factor} \times \text{Kinematic Factor}. \tag{33}$$

The loading factor ($0 < \text{LF} < 1$) is the square-root of the ratio of the masses moved by joints j and i, respectively:

$$\text{LF} = \left\{\frac{m_j + \cdots + m_N}{(m_i + \cdots + m_{j-1}) + (m_j + \cdots + m_N)}\right\}^{1/2} \quad \text{for } j > i. \tag{34}$$

The kinematic factor ($0 \leq \text{KF} \leq 1$) reflects the relative orientation of the axes of joints i and j, and the distribution of mass with respect to these axes:

$$\begin{aligned}\text{KF} = {} & \sigma_i\sigma_j |\mathbf{z}_{i-1}\cdot\mathbf{z}_{j-1}| + \sigma_i\sigma_j^* \frac{1}{K_{jj}} |(\mathbf{z}_{i-1}\times\mathbf{z}_{j-1})\cdot\mathbf{p}_j| \\ & + \sigma_i^*\sigma_j \frac{1}{K_{ii}} |(\mathbf{z}_{j-1}\times\mathbf{z}_{i-1})\cdot\mathbf{p}_i| \\ & + \sigma_i^*\sigma_j^* \frac{|K_{ij}|}{K_{ii}K_{jj}} \quad \text{for } j > i.\end{aligned} \tag{35}$$

The engineer confronted with the task of designing a robot (or analyzing an existing manipulator) can attempt to reduce the dynamic coupling by specifying a diagonal inertial matrix. Since such a design is not always achievable (especially for nontrivial manipulators of more than three DOF), the designer should seek configurations for which the inertial matrix is diagonally dominant, or almost diagonally dominant. Diagonal dominance of $\mathbf{D}(\mathbf{q})$ indicates weak coupling between the joints, and thus a practical design objective is to minimize the coefficients of coupling. Such a design can be accomplished by reducing both the loading and kinematic factors. The loading factor in (34) is reduced when the masses $m_1, \ldots, m_N$ of the links constitute a decreasing sequence. The kinematic factor in (35) can be reduced by assigning the coordinate frames so that the resulting mass distribution (of the links) reduces the contribution of the net forces and torques to the links.

Before we proceed (in Section 4) to highlight the centrifugal and Coriolis coefficients, we summarize in Table 1 the salient physical properties and structural characteristics of the inertial coefficients.

Table 1. Properties and characteristics of the inertial coefficients

Inertial ($N \times N$) Matrix $\mathbf{D}(q) = [d_{ij}]$

Symmetric:	$d_{ij} = d_{ji}$	for all i and j
Positive Definitive:	$d_{ii} > 0$	for all i
	$d_{ij}^2 < d_{ii}d_{jj}$	for all $i \neq j$

Self-inertial Coefficients (for $i = j$)

Joint i	d_{ii}	Functional dependence
Translational:	$m_i + \ldots + m_N$	constant
Rotational:	$(m_i + \ldots + m_N)\, K_{ii}^2$	$f(q_{i+1}, \ldots, q_N)$

Mutual Inertial Coefficients (for $j > i$)

Joint i	Joint j	d_{ij}	Functional dependence
Translational	Translational	$(m_j + \ldots + m_N)(\mathbf{z}_{i-1}\cdot\mathbf{z}_{j-1})$	$f(q_{i+1}, \ldots, q_{j-1})$
Translational	Rotational	$(m_j + \ldots + m_N))\mathbf{z}_{i-1}\times\mathbf{z}_{j-1})\cdot\mathbf{p}_j$	$f(q_{i+1}, \ldots, q_N)$
Rotational	Translational	$(m_j + \ldots + m_N)(\mathbf{z}_{j-1}\times\mathbf{z}_{i-1})\cdot\mathbf{p}_i$	$f(q_{i+1}, \ldots, q_{j-1}, q_{j+1}, \ldots, q_N)$
Rotational	Rotational	$(m_j + \ldots + m_N)\, K_{ij}$	$f(q_{i+1}, \ldots, q_N)$

Coefficient of Coupling

$$k_{ij} = \frac{|d_{ij}|}{(d_{ii}d_{jj})^{1/2}} \quad \text{for all } i \neq j$$

4. CENTRIFUGAL AND CORIOLIS COEFFICIENTS

Whereas the inertial coefficients (in Table 1) reflect the geometry of the kinematic chain and characterize the structural coupling between the degrees-of-freedom of the manipulator, the centrifugal and Coriolis coefficients indicate the velocity coupling of the links.

The centrifugal and Coriolis coefficients are introduced into the equations-of-motion in (11) through the Christoffel symbol in (12). The Christoffel symbol is essential for delineating the properties of the centrifugal and Coriolis coefficients. The Christoffel symbol, coupled with the functional dependence of the inertial coefficients in Table 1, i.e.

$$d_{ij} = f(q_{s+1}, \ldots, q_N), \quad \text{where } s = \min(i, j),$$

simplify the formulation of the centrifugal and Coriolis coefficients $c_{jm}(i)$ in (11).

We characterize the centrifugal coefficients $c_{jj}(i)$ in Section 4.1 and the Coriolis coefficients $c_{jm}(i)$, for $j \neq m$, in Section 4.2.

4.1 *Centrifugal coefficients*

Since $\partial d_{ij}/\partial q_m = 0$ for $m \le \min(i, j)$, the centrifugal coefficients $c_{jj}(i)$ are

$$c_{jj}(i) = \begin{cases} -\dfrac{1}{2}\dfrac{\partial d_{jj}}{\partial q_i} & \text{for } i > j \\ 0 & \text{for } i = j \\ \dfrac{\partial d_{ij}}{\partial q_j} & \text{for } i < j. \end{cases} \tag{36}$$

The formulation in (36) indicates that there are $N(N - 1)$ independent centrifugal coefficients. Furthermore, the functional dependence of the centrifugal coefficients is [7 and 8]

$$c_{jj}(i) = f(q_{s+1}, \ldots, q_N), \quad \text{where } s = \min(i, j).$$

The centrifugal coefficient $c_{jj}(i)$ characterizes the effect of the velocity of joint j on link i. Beyond the fact that $c_{jj}(j) = 0$[11], we observe that $c_{jj}(i)$ is introduced by the coordinate dependence of the inertial coefficients d_{ij}. The centrifugal coefficient $c_{jj}(i)$ vanishes when d_{jj} is independent of q_i (for $i > j$), or when d_{ij} is independent of q_j (for $i < j$). For example, when joint j is translational ($\sigma_j = 1$), d_{jj} is constant and d_{ij} is independent of q_j for $i < j$ (Table 1). Consequently, a translational joint does not exert centrifugal forces on any link. The centrifugal force is thus a unique characteristic of a rotational joint.

4.2 *Coriolis coefficients*

Based again upon the functional dependence of the inertial coefficients (in Table 1), the Christoffel symbol in (12) generates the Coriolis coefficients $c_{jm}(i)$ for $j \neq m$ as

$$c_{jm}(i) = \begin{cases} \dfrac{1}{2}\left\{\dfrac{\partial d_{ij}}{\partial q_m} - \dfrac{\partial d_{jm}}{\partial q_i}\right\} & \text{for } j < m \text{ and } i \\ \dfrac{1}{2}\left\{\dfrac{\partial d_{im}}{\partial q_j} - \dfrac{\partial d_{jm}}{\partial q_i}\right\} & \text{for } m < j \text{ and } i \\ \dfrac{1}{2}\left\{\dfrac{\partial d_{ij}}{\partial q_m} + \dfrac{\partial d_{im}}{\partial q_j}\right\} & \text{for } i < j \text{ and } m. \end{cases} \tag{37}$$

The Coriolis coefficient $c_{jm}(i)$ characterizes the effect of the coupling of velocities of links j and m on link i. From (37), the functional dependence of the Coriolis coefficients is [7 and 8]:

$$c_{jm}(i) = f(q_{s+1}, \ldots, q_N), \quad \text{where } s = \min(i, j, m).$$

Furthermore, the symmetry of the inertial coefficients d_{ij} (in Table 1) leads to the symmetry of the Coriolis coefficients[11]; i.e. $c_{jm}(i) = c_{mj}(i)$. Thus, there appears to be $N^2(N - 1)/2$ independent Coriolis coefficients. This number is reduced to $N(N - 1)(N - 2)/3$ by incorporating the non-interacting and reflective coupling properties of the Coriolis coefficients[11], which are immediate consequences of (37). These properties are

$$\text{Non-interacting} \quad c_{ji}(i) = 0 \quad \text{for all } j \le i \tag{38}$$

$$\text{Reflective coupling} \quad c_{jm}(i) = -c_{ji}(m) \quad \text{for all } j \le i, m. \tag{39}$$

The non-interacting property states that there is no Coriolis force on a link due to the coupling of its velocity with the velocity of an inner link. The reflective coupling property states that the Coriolis coefficient for link i, due to the coupling of the velocities of links j and m ($j \le i$, m), is equal and opposite to the Coriolis coefficient for link m due to the coupled velocities of links j and i.

The Coriolis forces in (11) account for the difference beteen forces measured in rotating and non-rotating coordinate frames[16]. The difference results from the observation that the energies of a particle moving in the two frames are different. The rotation of the first frame adds (to the total energy) a component which depends upon the coordinates of the particle and is proportional to the angular velocity of the rotating frame. The practical significance of this fact is that a pair of links, the inner of which is translational, does not exert Coriolis forces on any link. For instance, when $i < j$ and $\sigma_i = 1$, eqn (15) specifies that $\partial^2 \mathbf{r}_a/\partial q_i \partial q_j = 0$ and substitution of this result into (12) leads to $c_{ij}(m) = 0$. Consequently, the Coriolis force is a unique characteristic of the coupling of the velocity of a rotational link to the velocity of an outer link.

Once the inertial coefficients are calculated, (36) and (37) can be implemented to compute the centrifugal and Coriolis coefficients through symbolic differentiation.

5. GRAVITATIONAL COEFFICIENTS

To conclude our conceptualization of the dynamic robot model in (1), we turn in this section to the gravitational coefficient for joint i in (13). Upon applying (15),

$$G_i = (m_i + \cdots + m_N)\,\{\sigma_i(\mathbf{g} \cdot \mathbf{z}_{i-1}) + \sigma_i^* [\mathbf{g} \cdot (\mathbf{z}_{i-1} \times \mathbf{p}_i)]\}. \quad (40)$$

In Appendix B, we apply the concepts of the Denavit–Hartenberg convention[18] to transform (40) into a computationally attractive algorithm.

From (40), the functional dependence of the gravitational coefficients are [7 and 8]:

Translational joint ($\sigma_i = 1$) $\quad G_i = G_i(q_1, \ldots, q_{i-1})$

Rotational joint ($\sigma_i = 0$) $\quad G_i = G_i(q_1, \ldots, q_N)$.

Two configurations of practical importance are identified. If link i translates in a direction perpendicular to the gravity field (as does link 3 of a cylindrical robot[7]), $\mathbf{g} \cdot \mathbf{z}_{i-1}$ and $G_i = 0$. If link i rotates around an axis parallel to the gravity field (as does link 1 of the PUMA robot[5]), $\mathbf{g} \cdot (\mathbf{z}_{i-1} \times \mathbf{p}_i) = 0$ and $G_i = 0$. Hence, the gravitational coefficient of a joint characterizes the work production of that joint in the gravity field. In practice, counterbalances are introduced to reduce the effects of gravity. A counterbalance is added to a link to alter its mass distribution and reduce the corresponding gravitational coefficient.

6. ENGINEERING APPLICATIONS

In Sections 1–5, we have developed a conceptual framework in which to interpret physically the coefficients of the dynamic robot model in (11). The objective of this section is twofold: to transform the physical interpretations into design guidelines; and to utilize (to the extent possible) these guidelines to simplify dynamic robot models. We thus apply our conceptual framework to illustrate the structural characteristics of robotic manipulators, with specific attention to the 3 DOF positioning systems of industrial robots[7, 19 and 20].

We believe that the integrated mechanical and controller design of manipulators and the need for simpler and less coupled robot dynamics introduce the following design guidelines:

1. A robotic manipulator should possess as many translational joints as the application allows. A translational joint possesses a constant self-inertial coefficient, does not produce centrifugal forces, and eliminates velocity coupling (through zero Coriolis coefficients) with outer links.

2. The degrees-of-freedom of the manipulator should be placed in parallel or orthogonal directions. This arrangement of the z-axes eliminates mutual inertial coefficients (and thereby the derivative centrifugal and Coriolis coefficients through the Christofell symbol in (12)), and simplies the remaining coefficients. Furthermore, this arrangement prevents some of the links from producing work in the gravity field, thus eliminating the corresponding gravitational coefficients.

3. The masses of the links should decrease, in progressing from the base to the end-effector. Reduction of the loading factor (and hence of the coupling factor) reduces the coefficients of inertial coupling in (33).

4. To the extent possible, the first DOF should be configured so that it does not produce work in the gravity field. Elimination of the gravitational coefficient in the equation-of-motion of the first link results in a dynamic robot model that is independent of the first coordinate[7 and 8].

Application of these guidelines reduces significantly the number of nonzero independent coefficients in the dynamic robot model. In the sequel, we quantify this reduction for the 3 DOF positioning systems (i.e. the arm) of industrial robots[7].

In Table 2, we enumerate the number of independent coefficients[21] in the dynamic robot models of three and six DOF manipulators. For this range of N DOF manipulators, the number of independent coefficients is $\sim N^{2.55}$.

In Table 3, we compare the number of nonzero independent coefficients in the dynamic robot models of the three DOF positioning system configurations[7, 19 and 20]:

- Cartesian (z-x-y) robot[7];
- Cylindrical (θ-z-r) robot[7];

Table 2. Independent coefficients of N DOF dynamic robot models

Coefficients	Symbol	N DOF	3 DOF	6 DOF
Self-inertial	d_{ii}	N	3	6
Mutual inertial	d_{ij}	$N(N-1)/2$	3	15
Centrifugal	$c_{jj}(i)$	$N(N-1)$	6	30
Coriolis	$c_{jm}(i)$	$N(N-1)(N-2)/3$	2	40
Gravitational	G_i	N	3	6
Total		$N(2N^2+3N+7)/6$	17	97

Table 3. Comparative complexity of N DOF state-of-the-art dynamic robot models

Manipulator	N	N_R	d_{ii}	d_{ij}	Coefficients $c_{ii}(m)$	$c_{ij}(m)$	G_i	Total	Sparsity
Cartesian (z–x–y) $z \perp x \perp y$	3	0	3(3)	0	0	0	1(1)	4	0.235
Cylindrical (θ–z–r) $\theta \parallel z \perp r$	3	1	3(2)	0	1	0	1(1)	5	0.294
Spherical (θ–ϕ–r) $\theta \perp \phi \perp r$	3	2	3(1)	0	3	0	2	8	0.471
Articulated (q_1–q_2–q_3) $q_1 \perp q_2 \parallel q_3$	3	3	3(1)	3	6	2	2	16	0.941
Stanford $q_1 \perp q_2 \perp q_3$ and $q_3 \parallel q_4 \perp q_5 \perp q_6$	6	5	6(1)	12	19	25	4	66	0.680
PUMA $q_1 \perp q_2 \parallel q_3$ and $q_3 \perp q_4 \perp q_5 \perp q_6$	6	6	6(1)	14	30	34	4	88	0.907

- Spherical (θ-ϕ-r) robot[7]; and
- Articulated (q_1-q_2-q_3) robot[5].

We highlight these 3 DOF configurations in the perspective of state-of-the-art designs by including the 6 DOF Stanford manipulator and 6 DOF PUMA robot in Table 3. All of the robots are assumed to operate on a level foundation and consequently $G_1 = 0$. With the exception of the Cartesian robot, all of these configurations are characterized by nonlinear models[7]. Furthermore, the arrangement of the axes in the 3 DOF Cartesian, cylindrical and spherical robots leads to diagonal inertial matrices. The salient link orientation information is included in Table 3. The numbers in parentheses indicate the number of constant coefficients.

To indicate the complexity of the dynamic robot model in (11), we introduce a sparsity measure which is the ratio of the number of actual non-zero independent coefficients (in Table 3) to the number of independent coefficients (in Table 2). For the $N = 3$ DOF configurations, the total number of coefficients (and hence the sparsity) are observed to scale according to

$$\left[1 + \left(\frac{N_R}{N}\right)^2\right]^2,$$

where N_R is the number of rotational joints. The number of actual non-zero independent coefficients and sparsity of the 3 DOF Cartesian robot follow directly from (25), (26), (36) and (37). By adopting the Cartesian sparsity (0.235) as a basis for comparison, all of the sparsities in Table 3 obey the empirical formula

$$(0.235)\left[1 + \left(\frac{N_R}{N}\right)^2\right]^2.$$

In the robotics literature[22], the ratio (N_R/N) is called the degree-of-anthropomorphism and is a characteristic of the robot structure. The foregoing empirical observations of the sparsities in Table 3 (as a function of the degree-of-anthropomorphism) dramatically illustrate the complexity introduced by rotational joints and reinforce the design guidelines outlined at the outset of this section.

7. CONCLUSIONS

The objective of this section is to summarize the contributions of the paper. To interpret the characteristics of the dynamic robot model in the mathematical foundation of classical mechanics, we have developed a robot scenario that leads to novel physical insights and structural characteristics of robot dynamics. The presentation revolves around the centrality of the inertial matrix $\mathbf{D}(\mathbf{q})$.

The following contributions, which have direct application to dynamic robot modeling and control, highlight the development of the paper:

1. Application of Riemannian geometry leads directly to the properties and structure of the inertial matrix $\mathbf{D}(\mathbf{q})$ in Table 1. The properties of the inertial matrix propagate into the centrifugal and Coriolis coefficients through the Christoffel symbol in (12).
2. The robot dynamics scenario (outlined in Section 2) leads to the explicit formulae (22) and (23) for the inertial coefficients which facilitate their physical interpretation.
3. The coefficient of coupling is introduced in (32) to illuminate dynamic link coupling.
4. The effect of translational joints on the simplification of robot dynamics is quantified, and an empirical formula (based upon the degree-of-anthropomorphism) is introduced (in Section 6) for practical manipulators.
5. Formulae for the inertial and gravitational coefficients are transformed (in Appendixes A and B) through the Denavit–Hartenberg convention into computational algorithms.
6. The conceptual framework reinforces the need to integrate the mechanical and controller design, and focuses on a set of design guidelines (in Section 6) for manipulators.

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APPENDIX A

Denavit–Hartenberg representation of the inertial coefficients

The objective of this appendix is to relate the inertial coefficient formula in (17) and (22) to the computational formulae of the Denavit–Hartenberg convention. In the Denavit–Hartenberg convention[18], the links are assigned coordinate frames so that their relative positions and orientations can be described by homogeneous transformations. Homogeneous transformations are represented by (4×4) matrices which incorporate the geometrical properties and configurations of the links[4, p. 42]. For instance, the relative positions and orientations of the coordinate frames of links $(i-1)$, and $(j-1)$, where $j \geq i$, are related by the transformation matrix

$$\mathbf{A}_{i-1}^{j-1} \triangleq \begin{vmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{vmatrix} = [\mathbf{n}\ \mathbf{o}\ \mathbf{a}\ \mathbf{p}]. \qquad \text{(A-1)}$$

The right-handed set of unit vectors $(\mathbf{n}, \mathbf{o}, \mathbf{a})$ in Fig. 3 defines the orientation of the $(x_{j-1}, y_{j-1}, z_{j-1})$ axes with respect to the $(i-1)$ coordinate frame, and the vector $\mathbf{p}$ defines the origin of the $(j-1)$ coordinate frame[4].

The dot and scalar triple products appearing in (17) and (22) are invariant (coordinate-free) quantities. By selecting a convenient coordinate frame, we can express these products in terms of the vector elements in (A–1) as follows:

$$\text{Frame}\,(j-1)\text{:}\ \mathbf{z}_{i-1} \cdot \mathbf{z}_{j-1} = a_z \qquad \text{(A-2)}$$

$$\text{Frame}\,(j-1)\text{:}\ (\mathbf{z}_{i-1} \times \mathbf{z}_{j-1}) \cdot \mathbf{p}_j = o_z x_j - n_z y_j \qquad \text{(A-3)}$$

$$\text{Frame}\,(i-1)\text{:}\ (\mathbf{z}_{j-1} \times \mathbf{z}_{i-1}) \cdot \mathbf{p} = a_y p_x - a_x p_y. \qquad \text{(A-4)}$$

From (A–3) and (A–4):

$$(\mathbf{z}_{j-1} \times \mathbf{z}_{i-1}) \cdot \mathbf{p}_i = (\mathbf{z}_{j-1} \times \mathbf{z}_{i-1}) \cdot \mathbf{p}_j + (\mathbf{z}_{j-1} \times \mathbf{z}_{i-1}) \cdot \mathbf{p}$$
$$= -(o_z x_j - n_z y_j) + (a_y p_x - a_x p_y). \qquad \text{(A-5)}$$

For the dot product that defines the mutual inertial coefficient between two rotational joints:

$$\text{Frame}\,(i-1)\text{:}\quad \mathbf{z}_{i-1} \times \mathbf{p} = [-p_y\, p_x\, 0]^T \qquad \text{(A-6)}$$

$$\text{Frame}\,(j-1)\text{:}\quad \mathbf{z}_{i-1} \times \mathbf{p}_{ja} = [(o_z p_{jaz} - a_z p_{jay})$$
$$-(n_z p_{jaz} - a_z p_{jax})\ (n_z p_{jay} - o_z p_{jax})]^T \qquad \text{(A-7)}$$

$$\text{Frame}\,(j-1)\text{:}\quad \mathbf{z}_{j-1} \times \mathbf{p}_{ja} = [-p_{jay}\, p_{jax}\, 0]^T. \qquad \text{(A-8)}$$

From (A–6), (A–7) and (A–8):

$$(\mathbf{z}_{i-1} \times \mathbf{p}_{ia}) \cdot (\mathbf{z}_{j-1} \times \mathbf{p}_{ja}) = (\mathbf{z}_{i-1} \times \mathbf{p}) \cdot (\mathbf{z}_{j-1} \times \mathbf{p}_{ja})$$
$$+ (\mathbf{z}_{i-1} \times \mathbf{p}_{ja}) \cdot (\mathbf{z}_{j-1} \times \mathbf{p}_{ja}) = (n_x p_y - n_y p_x) p_{jay}$$
$$- (o_x p_y - o_y p_x) p_{jax} - o_z p_{jay} p_{jaz}$$
$$- n_z p_{jax} p_{jaz} + a_z (p_{jax}^2 + p_{jay}^2). \qquad \text{(A-9)}$$

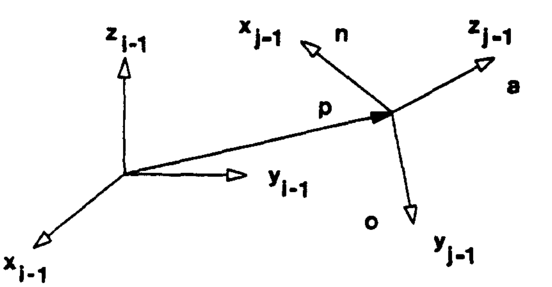

Fig. 3. Geometric illustration of the **n**, **o**, **a** and **p** vectors.

Upon substituting (A–2), (A–3), (A–5) and (A–9) into (17), the inertial coefficient d_{ij} becomes (for $j \geq i$):

$$\begin{aligned} d_{ij} = (m_j + \cdots + m_N)\{\sigma_i\sigma_j a_z + \sigma_i\sigma_j^*(o_z x_j - n_z y_j) \\ + \sigma_i^*\sigma_j[(n_z y_j - o_z x_j) + (p_x a_y - a_x p_y)]\} \\ + \sigma_i^*\sigma_j^*\{a_z I_{jzz} - o_z I_{jyz} - n_z I_{jxz} + (m_j + \cdots + m_N) \\ \cdot [(n_x p_y - n_y p_x)y_j - (o_x p_y - o_y p_x)x_j]\}, \quad \text{(A–10)} \end{aligned}$$

where (x_j, y_j, z_j) are the coordinates of the center-of-gravity and $(I_{jzz}, I_{jyz}, I_{jxz})$ are the moments and products of inertia of links j through N in the $(j-1)$ frame. Equation (A–10) removes the redundancy† in the Denavit–Hartenberg formulation and is computationally attractive for evaluating (numerically and symbolically) the inertial coefficients. In fact, (A–10) is a viable computational alternative to the Q-matrix formulation of the inertial coefficients[7 and 10]:

$$d_{ij} = \sum_{k=j}^{N} Tr\{(\mathbf{A}_i \ldots \mathbf{A}_{j-1}\mathbf{Q}_j\mathbf{A}_j \ldots \mathbf{A}_k)\mathbf{J}_k(\mathbf{Q}_i\mathbf{A}_i \ldots \mathbf{A}_j \ldots \mathbf{A}_k)^T\}, \quad \text{(A–11)}$$

where $\mathbf{A}_s \triangleq \mathbf{A}_{s-1}^s$, for $s = i, \ldots, N$. The recursive formulation of (A–11) is implemented in ARM[5] to calculate symbolically the inertial coefficients.

In the Denavit–Hartenberg convention, it is standard practice to specify the coordinates of the centers-of-mass of the links, and the moments and products of inertia of each link in its own coordinate frame. This information, and the mass of the link, are included in the (4×4) pseudo-inertia matrix $\mathbf{J}_k$[4]. The elements of $\mathbf{J}_k$ are transformed from frame k to frame $(j-1)$ according to the formula

$$\mathbf{J}_k^{(j-1)} = \mathbf{A}_{j-1}^k \mathbf{J}_k (\mathbf{A}_{j-1}^k)^T. \quad \text{(A–12)}$$

Hence, the user-supplied information consists of the homogeneous transformation matrices $\mathbf{A}_{k-1}^k$, the pseudo-inertia matrices $\mathbf{J}_k$, and the type (rotational or translational) of the links; i.e. the Boolean variable σ_k for $k = 1, 2, \ldots, N$. The algorithm for evaluating the inertial coefficient d_{ij} $(j \geq i)$ consists of three steps:

1. Compute the product of the transformation matrices $\mathbf{A}_{i-1}^{j-1} = \mathbf{A}_{i-1}^i \ldots \mathbf{A}_{j-2}^{j-1}$ to generate the elements of the $\mathbf{n}$, $\mathbf{o}$, $\mathbf{a}$ and $\mathbf{p}$ vectors in (A–1).
2. Transform the pseudo-inertia matrices $\mathbf{J}_k$ from frame k to frame $(j-1)$ according to (A–12), and sum the transformed matrices for $k = j, j+1, \ldots, N$, to obtain (x_j, y_j, z_j) and $(I_{jzz}, I_{jxz}, I_{jyz})$.
3. Substitute the quantities computed in steps (1) and (2) into (A–10).

† A companion formula, which also removes the redundancy, appears in [4, p. 174]. That formula, however, is not computationally appealing, but rather is directed toward the simplification of the inertial coefficients by neglecting the elements having a sufficiently small magnitude.

The computational effort of the algorithm is alleviated by the fact that only the nonzero terms in (A–10), which are identified by the Boolean variable σ, have to be computed. Once the inertial coefficients are computed, the centrifugal and Coriolis coefficients are evaluated from (36) and (37) through symbolic differentiation.

APPENDIX B

Denavit–Hartenberg representation of the gravitational coefficients

The objective of this appendix is to relate the gravitational coefficient formula in (40) to the computational formulae of the Denavit–Hartenberg convention. The development parallels that of Appendix A.

In the following analysis, we assume that the gravity vector $\mathbf{g}^T = [g_x\ g_y\ g_z]$, where $|\mathbf{g}| = 9.81$ m/s^2. The dot and scalar triple products appearing in (40) are invariant (coordinate-free) quantities. By selecting a convenient coordinate frame, we can express these products in terms of the vector elements in (A–1) as follows:

$$\text{Frame}\,(i-1):\ \mathbf{g} \cdot \mathbf{z}_{i-1} = g_x a_x + g_y a_y + g_z a_z \quad \text{(B–1)}$$

$$\text{Frame}\,(i-1):\ \mathbf{g} \cdot (\mathbf{z}_{i-1} \times \mathbf{p}_i) = x_i(g_x o_x + g_y o_y + g_z o_z) - y_i(g_x n_x + g_y n_y + g_z n_z), \quad \text{(B–2)}$$

where x_i and y_i are the coordinates of the center-of-gravity of link i in its own frame, and the elements of the $\mathbf{n}$, $\mathbf{o}$, and $\mathbf{a}$ vectors belong to the $\mathbf{A}_0^{i-1} = \mathbf{A}_1^0 \ldots \mathbf{A}_{i-2}^{i-1}$ transformation matrix in (A–1).

Upon substituting (B–1) and (B–2) into (40),

$$\begin{aligned} G_i = (m_i + \cdots + m_N)\{\sigma_i(g_x a_x + g_y a_y + g_z a_z) \\ + \sigma_i^*[x_i(g_x o_x + g_y o_y + g_z o_z) - y_i(g_x n_x + g_y n_y \\ + g_z n_z)]\} \text{ for } i = 1, 2, \ldots, N. \quad \text{(B–3)} \end{aligned}$$

Since x_i, y_i and g_x, g_y, g_z are user-supplied information, the algorithm for evaluating the inertial coefficients consists of two steps:

1. Compute the transformation matrix $\mathbf{A}_0^{i-1}$ for $i = 1, 2, \ldots, N$, to generate the elements of the $\mathbf{n}$, $\mathbf{o}$ and $\mathbf{a}$ vectors in (B–3). (These elements are computed in the first step of the algorithm in Appendix A for the inertial coefficients.)
2. Substitute the quantities computed in step (1) and the user-supplied information into (B–3).

For manipulators which operate on a level foundation, $\mathbf{g}^T = [0\ 0\ -g]$ and the gravitational coefficient G_i in (B–3) reduces to

$$\begin{aligned} G_i = -(m_i + \cdots + m_N)g\{\sigma_i a_z \\ + \sigma_i^* \cdot (x_i o_z - y_i n_z)\} \quad \text{for } i = 1, 2, 3, \ldots, N. \quad \text{(B–4)} \end{aligned}$$

If the first link translates in a direction perpendicular to the gravity field, $\sigma_1 = 1$ and $a_z = 0$. If the first link rotates around an axis parallel to the gravity field, $\sigma_1^* = 1$ and $n_z = o_z = 0$. In both configurations, $G_1 = 0$.

CARACTERISTIQUES INERTIELLES DES MODELES DYNAMIQUES ROBOTIQUES

Résumé—En se basant sur les principes mathématiques de la mécanique classique, on trouve des caractéristiques physiques et des interprétations nouvelles du modèle robotique lagrangien. Comme les propriétés des coefficients inertiaux fixent les caractéristiques des coefficients centrifuges, de Coriolis et de gravité du modèle, on met en évidence dans cette étude l'importance de la matrice inertielle. La signification physique des coefficients inertiaux est explicitée d'avantage par l'introduction du "coefficient de couplage" des manipulateurs robotiques. Cette nouvelle approche du problème souligne la nécessité d'intégrer la construction mécanique au problème du contrôle des robots.