1 Formal Properties

Definition 1. *A closed embedding $i : Z \hookrightarrow X$ is a* ***nilpotent thickening*** *if its associated ideal sheaf $\mathcal{I}$ is nilpotent. In this case, we say that i is an n-**order thickening** if $\mathcal{I}^{n+1} = 0$. We will often refer to 1-order thickenings as* ***first-order thickenings****.*

Fix a base $S \in \mathsf{Space}$. Our interest is in a general S-space $f : X \to S$. Given $i : T \hookrightarrow T'$ an order-n thickening, let $g : T \to X$ be a morphism fitting into a commutative diagram

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T' \arrow[r] & S \end{tikzcd}$$

Consider the set

$$\mathrm{Def}(g; f, i) := \{\widetilde{g} \in \mathrm{Hom}_{\mathsf{Space}}(T', X) : g = \widetilde{g} \circ i\}$$

of **deformations** of g with respect to f and i. Each such deformation $\widetilde{g}$ gives us a commutative diagram

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T' \arrow[r] \arrow[ur, dotted, "\widetilde{g}"] & S \end{tikzcd}$$

We will be especially interested in the case that $i : T \hookrightarrow T'$ is a first-order thickening of affine schemes.

Definition 2. *Let $f : X \to S$ be a map of spaces. We say that f is* ***formally smooth*** *(resp.,* ***formally unramified****,* ***formally étale****) if, given any commutative diagram*

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T' \arrow[r] & S \end{tikzcd}$$

with $i : T \hookrightarrow T'$ a first-order thickening of affine schemes, the set $\mathrm{Def}(g; f, i)$ is nonempty (resp., at most a singleton, a singleton).

Remark 3. *The condition that T' be affine can be dropped when checking if f is formally unramified or formally étale. This is no longer the case when checking if f is formally smooth.*

How do we obtain first-order thickenings of affine S-schemes? Suppose we have $T = \mathrm{Spec}\, C \in \mathsf{Aff}_{/S}$ and assume for simplicity that $S = \mathrm{Spec}\, A$. Then, $\Delta_{T/S} : T \to T \times_S T$ corresponds to the surjective multiplication map

$$C \otimes_A C \to C, \qquad c_1 \otimes c_2 \mapsto c_1 c_2$$

and so $\Delta_{T/S}$ is a closed embedding. Letting I be the kernel of the multiplication map, we obtain a first-order thickening

$$\mathrm{Spec}\, C \xrightarrow{\sim} \mathrm{Spec}(C \otimes_A C)/I \hookrightarrow \mathrm{Spec}(C \otimes_A C)/I^2.$$

If we merely assume that S is a scheme then we can factor $\Delta_{T/S}$ as

$$T \xhookrightarrow{\text{closed}} W \xhookrightarrow{\text{open}} T \times_S T$$

The scheme W need not be affine, though regardless the closed embedding $T \hookrightarrow W$ corresponds to a quasicoherent ideal sheaf $\mathcal{I}$ on W with $T \cong \operatorname{Spec}_W \mathcal{O}_W/\mathcal{I}$. There is then an epimorphism $\mathcal{O}_W/\mathcal{I}^2 \twoheadrightarrow \mathcal{O}_W/\mathcal{I}$ inducing a first-order thickening $T \hookrightarrow T'$.

Let $f : X \to S$ be any map of schemes. As above, $\Delta_{X/S} : X \to X \times_S X$ is an immersion with factorization $X \hookrightarrow W \hookrightarrow X \times_S X$ and $X \cong \operatorname{Spec}_W \mathcal{O}_W/\mathcal{I}$ for $\mathcal{I}$ some quasicoherent ideal sheaf on W. We let $X \hookrightarrow X^{(n)}$ denote the order-n thickening given by $X \xrightarrow{\sim} \operatorname{Spec}_W \mathcal{O}_W/\mathcal{I} \hookrightarrow \operatorname{Spec}_W \mathcal{O}_W/\mathcal{I}^n$. We can also think of $X^{(2)}$ as a **first-order infinitesimal neighborhood** of the diagonal and denote it by $\Delta^{(1)}$. Note that the factorization of $\Delta_{X/S}$ as an immersion need not be unique.[1] However, by the above we can canonically define $X^{(n)}$ when $\Delta_{X/S}$ is a closed embedding (i.e., f is separated).

Switching gears a bit,

Let $A \in \mathsf{CRing}$, $B \in \mathsf{CAlg}_A$, and $M \in \mathsf{Mod}_B$. Viewing both B and M as A-modules, we may define the set $\operatorname{Der}_A(B, M)$ of A-linear **derivations** from B to M to be the set of $\delta \in \operatorname{Hom}_{\mathsf{Mod}_A}(B, M)$ satisfying the *Leibniz rule*:

$$\delta(fg) = g\delta(f) + f\delta(g) \text{ for all } f, g \in B.$$

Exercise 4. *Let $\varphi : A \to B$ be the structure map and $\delta \in \operatorname{Der}_A(B, M)$. Show that δ kills $\varphi(A)$ and that $\delta = 0$ if φ is surjective or a localization map.*

This determines a functor $\operatorname{Der}_A(\cdot, \cdot) : \mathsf{CAlg}_A^{\mathrm{op}} \times \mathsf{Mod}_A \to \mathsf{Set}$ which in fact factors through Mod_A (i.e., we can add derivations and scale them by elements of A). This tells us what happens if we change the inputs B and M, but what happens if we change A itself? Let $A' \to A$ be a ring map. This equips both B and M with the structure of A'-modules. We obtain a map $\operatorname{Der}_A(B, M) \to \operatorname{Der}_{A'}(B, M)$ which is the identity on the level of sets. The latter set $\operatorname{Der}_{A'}(B, M)$ carries no natural A-module structure so there is clearly something more going on here.

Exercise 5. *Let $A \to B \to C$ be a sequence of ring maps and $M \in \mathsf{Mod}_C$. Show that there is a natural exact sequence*

$$0 \longrightarrow \operatorname{Der}_B(C, M) \longrightarrow \operatorname{Der}_A(C, M) \longrightarrow \operatorname{Der}_A(B, M)$$

of A-modules that is functorial in M.

Define the **cotangent module** $\Omega^1_{B/A} \in \mathsf{Mod}_B$ via the universal property that $\operatorname{Hom}_{\mathsf{Mod}_B}(\Omega^1_{B/A}, \cdot) \cong \operatorname{Der}_A(B, \cdot)$.[2] The data of $\Omega^1_{B/A}$ is entirely encoded by $\mathrm{id} \in \operatorname{Hom}_{\mathsf{Mod}_B}(\Omega^1_{B/A}, \Omega^1_{B/A})$, which corresponds to a derivation $d = d_{B/A} \in \operatorname{Der}_A(B, \Omega^1_{B/A})$. This derivation is *universal* in the sense that, given any $M \in \mathsf{Mod}_B$ and $\delta \in \operatorname{Der}_A(B, M)$, there is a unique B-module map $\Omega^1_{A/B} \to M$ such that the diagram

[1] The ambiguity in W comes from choosing suitable affine open coverings of X and S.

[2] This determines $\Omega^1_{B/A}$ up to unique isomorphism by an application of Yoneda's Lemma.

$$\begin{tikzcd} B \arrow[r, "\delta"] \arrow[dr, "d"'] & M \\ & \Omega^1_{B/A} \arrow[u, dotted, "\exists!"'] \end{tikzcd}$$

commutes. Setting aside the matter of existence for the moment, let's deduce some properties of cotangent modules.

Exercise 6. *Fix a ring map $\varphi : A \to B$. Make use of the universal property of localization for the following problems.*

(a) *Let $S \subseteq B$ be a multiplicative set. Show that there is a canonical isomorphism $\Omega^1_{S^{-1}B/A} \cong S^1\Omega^1_{B/A}$ of $S^{-1}B$-modules.*

(b) *Let $S \subseteq A$ be a multiplicative set such that $\varphi(S) \subseteq B^\times$. Show that there is a canonical isomorphism $\Omega^1_{B/S^{-1}A} \cong \Omega^1_{B/A}$ of B-modules.*

Proposition 7. *Let $A \to B \to C$ be a sequence of ring maps and $M \in \mathsf{Mod}_C$. Then, there is a canonical isomorphism*

$$\mathrm{Der}_A(B, M) \cong \mathrm{Hom}_{\mathsf{Mod}_C}(C \otimes_B \Omega^1_{B/A}, M)$$

of A-modules functorial in M.

Proof. We have

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Mod}_C}(C \otimes_B \Omega^1_{B/A}, M) &\cong \mathrm{Hom}_{\mathsf{Mod}_B}(C \otimes_B \Omega^1_{B/A}, M) \\ &\cong \mathrm{Hom}_{\mathsf{Mod}_B}(\Omega^1_{B/A}, \mathrm{Hom}_{\mathsf{Mod}_B}(C, M)) \\ &\cong \mathrm{Der}_A(B, \mathrm{Hom}_{\mathsf{Mod}_B}(C, M)). \end{aligned}$$

The latter module is a subset of $\mathrm{Hom}_{\mathsf{Mod}_A}(B, \mathrm{Hom}_{\mathsf{Mod}_B}(C, M))$, which by tensor-Hom adjunction is isomorphic to $\mathrm{Hom}_{\mathsf{Mod}_A}(C \otimes_B B, M)$. Further identifications give

$$\mathrm{Hom}_{\mathsf{Mod}_A}(C \otimes_B B, M) \cong \mathrm{Hom}_{\mathsf{Mod}_A}(C, M) \cong \mathrm{Hom}_{\mathsf{Mod}_A}(B, M).$$

One can then show that the image of $\mathrm{Der}_A(B, \mathrm{Hom}_{\mathsf{Mod}_B}(C, M))$ in $\mathrm{Hom}_{\mathsf{Mod}_A}(B, M)$ is precisely $\mathrm{Der}_A(B, M)$. This construction is evidently functorial in M. $\square$

Corollary 8. *Let $A \to B \to C$ be a sequence of ring maps and $M \in \mathsf{Mod}_C$. Then, there is a canonical exact sequence*

$$C \otimes_B \Omega_{B/A} \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0$$

of C-modules.

Exercise 9. *Let $B_1, B_2 \in \mathsf{CAlg}_A$ and $C := B_1 \otimes_A B_2$ (which we view as a commutative algebra over both B_1 and B_2 in the usual manner). Show that there is a canonical isomorphism*

$$(C \otimes_{B_1} \Omega^1_{B_1/A}) \oplus (C \otimes_{B_2} \Omega^1_{B_2/A}) \cong \Omega^1_{C/A}$$

of C-modules.

Let $A \in \mathsf{CRing}$ and $M \in \mathsf{Mod}_A$. We equip the abelian group $A \oplus M$ with a commutative A-algebra structure as follows. Define

$$(a, m) \cdot (b, n) := (a + b, an + bm).$$

We call the resulting algebra a **split square-zero extension**.[3] We immediately see that the projection $\mathrm{pr}_1 : A \oplus M \to A$ is a ring map equipping A with an $(A \oplus M)$-module structure and allowing us to view $A \oplus M$ as an $(A \oplus M)$-module.[4] If it floats your boat, $A \oplus M$ is naturally an object of $\mathsf{CAlg}_{A//A}$, the category of commutative rings equipped with a ring map both from and to A.

Exercise 10. *Show that the projection* $\mathrm{pr}_2 : A \oplus M \to M$ *is an A-linear derivation.*

Exercise 11. *Identify M with the subset $\{0\} \times M \subseteq A \oplus M$. Show that M is an ideal of $A \oplus M$ with $M^2 = 0$. This explains half of the name.*

Where does this sort of construction come from?

Example 12. *Let k be a field and consider the k-algebra $k[x]/(x^2)$ called the **dual numbers** over k. This may equivalently be viewed as the k-algebra $k[\epsilon]$ for ϵ a formal element such that $\epsilon^2 = 0$. As a k-vector space we have $k[\epsilon] \cong k \times k\epsilon$, with multiplication $(a + m\epsilon)(b + n\epsilon) = ab + (an + bm)\epsilon$. It follows that $k[\epsilon] \cong k \oplus k\epsilon$ as k-algebras.*

We can repeat the above construction given any $B \in \mathsf{CAlg}_A$ to get $B \oplus M \in \mathsf{CAlg}_A$.

Exercise 13. *Show that elements of* $\mathrm{Der}_A(B, M)$ *correspond canonically to A-algebra sections of* $\mathrm{pr}_1 : B \oplus M \to B$ *– i.e., A-algebra maps $\sigma : B \to B \oplus M$ such that* $\mathrm{pr}_1 \circ \sigma = \mathrm{id}_B$.

Just as before we may view M as an ideal of $B \oplus M$ with $M^2 = 0$.

[3] Half of the name comes from the fact that $A \oplus M$ is a split extension of A-modules.

[4] Note that the identity map on $A \oplus M$ need not equip $A \oplus M$ with the structure of a module over itself since coordinate-wise operations need not make $A \oplus M$ into a ring – indeed, multiplication on M need not even be defined!