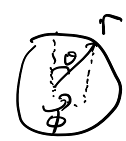
We have coordinate: (r, θ, ϕ)

$r > 0, \quad 0 \le \theta < \pi \quad , \quad 0 \le \phi < 2\pi$



lets define (u, v, w) to be components in r, θ, ϕ.

Each grid (r_i, θ_i, ϕ_i) is the center of the cell.

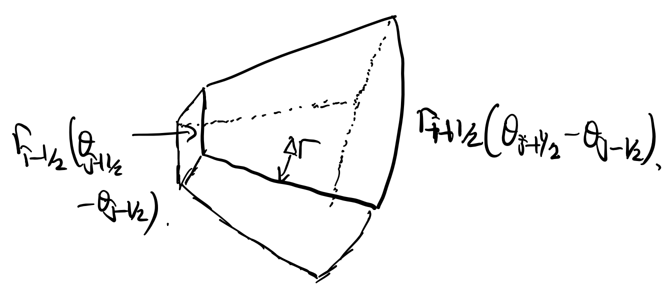
A cell is given by $r_{i-1/2} \le r < r_{i+1/2}$.

$$\theta_{j-1/2} \le \theta < \theta_{j+1/2}$$

$$\phi_{k-1/2} \le \phi < \theta_{k+1/2}$$

The cell walls in r and θ direction are curved, but straight in ϕ direction

Given a scalar quantity: q:



$$\frac{\partial q}{\partial t} + \vec{\nabla}(q\vec{u}) = 0$$

$$\hookrightarrow = \frac{\partial q}{\partial t} + \frac{1}{r^2}\frac{\partial (r^2 q u)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, q v)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(q w)}{\partial \phi} = 0$$

We see there are geometric factors:

Given cell (i, j, k): $r_{i-1/2} \le r < r_{i+1/2}$, $\theta_{j-1/2} \le \theta < \theta_{j+1/2}$

$$\phi_{k-1/2} \le \phi < \phi_{k+1/2}$$

Volume: $V_{ijk} = \frac{1}{3}\left(r^3_{i+1/2} - r^3_{i-1/2}\right)\left(\cos\theta_{j-1/2} - \cos\theta_{j+1/2}\right)\left(\phi_{k+1/2} - \phi_{k-1/2}\right)$

The $i+1/2$ surface of the cell in r-direction (i.e. when r = constant)

$$S_{(r),i+1/2,j,k} = r^2_{i+1/2}\left(\cos\theta_{j-1/2} - \cos\theta_{j+1/2}\right)\left(\phi_{k+1/2} - \phi_{k-1/2}\right)$$

Similarly: The $j+1/2$ surface of the cell in θ-direction (i.e. θ = constant)

$$S_{(\theta),i,j+1/2,k} = \frac{1}{2}\left(r^2_{i+1/2} - r^2_{i-1/2}\right)\sin\theta_{j+1/2}\left(\phi_{k+1/2} - \phi_{k-1/2}\right)$$

For $k+1/2$:

$$S_{(\phi),i,j,k+1/2} = \frac{1}{2}\left(r^2_{i+1/2} - r^2_{i-1/2}\right)\left(\theta_{j+1/2} - \theta_{j-1/2}\right)$$

If we difference equations in conservative form:

$$\frac{\partial(q_{ijk}V_{ijk})}{\partial t} = F_{i-1/2,j,k}S_{(r),i-1/2,j,k} - F_{i+1/2,j,k}S_{(r),i+1/2,j,k}$$
$$+ F_{i,j-1/2,k}S_{(\theta),i,j-1/2,k} - F_{i,j+1/2,k}S_{(\theta),i,j+1/2,k}$$
$$+ F_{i,j,k-1/2}S_{(\phi),i,j,k-1/2} - F_{i,j,k+1/2}S_{(\phi),i,j,k+1/2}$$

$$\frac{\partial(q_{i,j,k}V_{i,j,k})}{\partial t} + \left(\Delta_r F_r S_r\right)_{i,j,k} + \left(\Delta_\theta F_\theta S_\theta\right)_{i,j,k} + \left(\Delta_\phi F_\phi S_\phi\right)_{i,j,k} = 0$$

Above works for scalar quantity like ρ and e_{tot}.

☆ Euler Equation in Polar Coordinates:

Density:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

$$\hookrightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial(r^2\rho u)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,\rho v)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho w)}{\partial\phi} = 0$$

Energy:

$$\frac{\partial(\rho e)}{\partial t} + \vec{\nabla}\cdot(\rho e\vec{u} + p\vec{u}) = 0$$

$$\hookrightarrow = \frac{\partial(\rho e)}{\partial t} + \frac{1}{r^2}\frac{\partial(r^2(\rho e+p)u)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta(\rho e+p)v)}{\partial\theta}$$

$$+ \frac{1}{r\sin\theta}\frac{\partial((\rho e+p)w)}{\partial\phi} = 0$$

Momentum Equation: $\frac{\partial(\rho\vec{u})}{dt} + \vec{\nabla}\cdot(\rho\vec{v}\otimes\vec{v}) + \nabla p = 0$

r - radial:

$$\frac{\partial(\rho u)}{\partial t} + \frac{1}{r^2}\frac{\partial(r^2\rho u^2)}{\partial r} + \frac{\partial p}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,\rho uv)}{\partial\theta}$$

$$+\frac{1}{r\sin\theta}\frac{\partial(\rho u w)}{\partial\phi} - \frac{\rho(v^2+w^2)}{r} = 0$$

In discrete form for Riemann solver:

$$\frac{\partial(\rho u V)}{\partial t} + \Delta_r(F_{rr} S_r) + \Delta_\theta(F_{r\theta} S_\theta) + \Delta_\phi(F_{r\phi} S_\phi)$$
$$= \left[\frac{2P}{r} + \frac{\rho(v^2+w^2)}{r}\right] V$$

where
$$\left.\begin{aligned} F_{rr} &= \rho u^2 + P \\ F_{r\theta} &= \rho u v \\ F_{r\phi} &= \rho u w \end{aligned}\right\} \text{Momentum Fluxes.}$$

θ - Equation:

$$\frac{\partial(\rho v)}{\partial t} + \frac{1}{r^2}\frac{\partial(r^2\rho u v)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,\rho v^2)}{\partial\theta}$$
$$+\frac{1}{r}\frac{\partial P}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho w v)}{\partial\phi} + \rho\frac{uv}{r}$$
$$-\frac{\cos\theta}{\sin\theta}\rho\frac{w^2}{r} = 0$$

In discrete Riemann solver:

$$\frac{\partial(\rho v V)}{\partial t} + \Delta_r(F_{\theta r} S_r) + \Delta_\theta(F_{\theta\theta} S_\theta) + \Delta_\phi(F_{\theta\phi} S_\phi)$$
$$= -\left[\rho \frac{vu}{r} - \frac{\cos\theta}{\sin\theta} \frac{\rho w^2 + P}{r}\right] V$$

for $F_{\theta r} = \rho v u$ $F_{\theta\theta} = \rho v^2 + P$, $F_{\theta\phi} = \rho v w$

ϕ- momentum equation:

$$\frac{\partial(\rho w)}{\partial t} + \frac{1}{r^2}\frac{\partial(r^2 \rho u w)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta \rho v w)}{\partial\theta}$$
$$+ \frac{1}{r\sin\theta}\frac{\partial(\rho w^2 + P)}{\partial\phi} + \frac{\rho u w}{r} + \frac{\cos\theta}{\sin\theta}\frac{\rho v w}{r} = 0$$

In discrete Riemann solver:

$$\frac{\partial(\rho w V)}{\partial t} + \Delta_r(F_{\phi r} S_r) + \Delta_\theta(F_{\phi\theta} S_\theta) + \Delta_\phi(F_{\phi\phi} S_\phi)$$
$$= -\left[\frac{\rho u w}{r} + \frac{\cos\theta}{\sin\theta}\frac{\rho v w}{r}\right] V$$

$F_{\phi r} = \rho w u$ $F_{\phi\theta} = \rho w v$ $F_{\phi\phi} = \rho w^2 + P$

procedure in applying Riemann Solver to polar coordinates:

1) precompute cell volumes, cell surfaces and cosine and sine values.

2) Do strang-splitting.

3) Compute left and right interface states, and feed them to Riemann solvers.

4) Add source terms if needed ?

Replacing ϕ-momentum with angular momentum in r, θ advection

let $l = w \sin\theta r$

If we replace w by l

$$\frac{\partial(\rho l)}{\partial t} + \frac{\partial(\rho u l)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v l)}{\partial \theta} + \frac{\partial(\rho w^2 + P)}{\partial \phi}$$

$$+ 2\sin\theta \rho u w + \cos\theta \rho v w = 0$$

then rewrite:

$$\frac{\partial(\rho l)}{\partial t} + \frac{1}{r^2}\frac{\partial(r^2 \rho u l)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta \rho v l)}{\partial \theta} + \frac{\partial(\rho w^2 + P)}{\partial \phi} = 0$$

then discrete Riemann solver has form:

$$\frac{\partial(\rho l V)}{\partial t} + \Delta_r(r\sin\theta F_{\phi r} S_r) + \Delta_\theta(r\sin\theta F_{\phi\theta} S_\theta) + \Delta_\phi(r\sin\theta F_{\phi\phi} S_\phi) = 0$$

No more

Then

$$\tilde{F}_{\phi r} = r\sin\theta F_{\phi r} = \rho u l$$

$$\tilde{F}_{\phi\theta} = r\sin\theta F_{\phi\theta} = \rho v l$$

$$\tilde{F}_{\phi\phi} = r\sin\theta F_{\phi\phi} = \rho w l + P r\sin\theta$$

Source terms.