Euler Equations:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho \vec{v}) = 0$$

$$\frac{\partial(\rho \vec{v})}{\partial t} + \vec{\nabla}\cdot(\rho \vec{v}\otimes\vec{v}) + \nabla P = 0$$

$$\frac{\partial(\rho E)}{\partial t} + \vec{\nabla}\cdot(\rho E\vec{v} + P\vec{v}) = 0$$

$E = e + \frac{1}{2}|v|^2$ with EOS: $P = P(\rho, e)$

γ-law: $P = \rho e(\gamma - 1)$

For 2D: $\vec{v} = (u, v)$

Density:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho(u\hat{x} + v\hat{y})) = 0$$

$$\hookrightarrow \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\vec{v}\otimes\vec{v} = \begin{pmatrix} u \\ v \end{pmatrix}\begin{pmatrix} u & v \end{pmatrix} = \begin{pmatrix} u^2 & uv \\ vu & v^2 \end{pmatrix}$$

Momentum

$$\frac{\partial(\rho(u\hat{x} + v\hat{y}))}{\partial t} + \vec{\nabla}\cdot\left(\rho\begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}\right) + \frac{\partial P}{\partial x}\hat{x} + \frac{\partial P}{\partial y}\hat{y} = 0$$

X-Momentum:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial P}{\partial x} = 0$$

Y-Momentum

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho u v)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial P}{\partial y} = 0$$

Energy:

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial[(\rho E+P)u]}{\partial x} + \frac{\partial[(\rho E+P)v]}{\partial y} = 0$$

In 2D, 4 equations with conserved quantity: $\rho, \rho u, \rho v, \rho E$

① $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$

② $\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2+P)}{\partial x} + \frac{\partial(\rho u v)}{\partial y} = 0$

③ $\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho u v)}{\partial x} + \frac{\partial(\rho v^2+P)}{\partial y} = 0$

④ $\frac{\partial(\rho E)}{\partial t} + \frac{\partial[(\rho E+P)u]}{\partial x} + \frac{\partial[(\rho E+P)v]}{\partial y} = 0$

With γ-law eos: $P = \rho e(\gamma-1)$

$$\rho = \frac{P}{e(\gamma-1)}$$

$$\rho e = \frac{P}{\gamma-1}$$

The conserved states are:

$$Q = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix} = \begin{pmatrix} q_0 \\ \rho q_1 \\ \rho q_2 \\ \frac{q_3}{\gamma-1} + \frac{1}{2}\rho(u^2+v^2) \end{pmatrix}$$

The Primitive Variable:

$$q = \begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix} = \begin{pmatrix} Q_0 \\ Q_1/\rho \\ Q_2/\rho \\ \rho(\gamma-1)\underbrace{\left[\frac{Q_3 - \frac{1}{2}\rho(u^2+v^2)}{\rho}\right]}_{=e} \end{pmatrix}$$

Flux Construction:

1) Convert conservative quantities to primitive.

$$Q = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{Bmatrix} \Rightarrow q = \begin{Bmatrix} \rho \\ u \\ v \\ p \end{Bmatrix}$$

2) whether or not use flattening coefficient $\xi \in [0, 1]$ to flat the slope.

It is used to add additional dissipation at shock, to ensure they're smeared out over ~ 2 zones.

Procedure:

If x-direction:

checks pressure.

$$t1 = \text{abs}\left(p(i+1/2, j) - p(i-3/2, j) \right)$$

$$t2 = \mathrm{abs}\left(p(i+3/2, j) - p(i-5/2, j)\right)$$

$$z = \frac{t_1}{\max[t_1, 10^{-10}]}$$

$$t2 = \frac{t_1}{\min\left(p(i+1/2, j),\ p(i-3/2, j)\right)}$$

$$t_1 = -\left(p(i+1/2, j) - p(i-3/2, j)\right)$$

If Y-direction: swap $i \to j$

$$\zeta = \min\left[1.0,\ \max\left(0.0,\ 1.0 - \left(\frac{z - z_0}{z_1 - z_0}\right)\right)\right], \quad z_0 \sim 0.75,\ z_1 \sim 0.85$$

$\zeta =$ If ($t1 > 0.0$ and $t2 >$ delta) delta ~ 0.33

check whether we have shock

then $\zeta = \zeta$

else. $\zeta = 1.0$

After finding ζ for X- and Y- direction:

$PX =$ where $\left(p(i+1/2, j) - p(i-3/2, j) > 0,\ \zeta\_X(i-3/2, j)\right)$

else $\zeta\_X(i+1/2)$

$PY =$ where $\left(p(i, j+1/2) - p(i, j-3/2) > 0,\ \zeta\_Y(i, j-3/2)\right)$

else $\zeta\_Y(i, j+1/2)$

then $\xi = \min\left(\min(\xi_x, \rho_x), \min(\xi_y, \rho_y)\right)$

3) Construct $\frac{\partial q}{\partial x}$ and $\frac{\partial q}{\partial y}$ with limiters and multiply with ξ to flatten.

$$ldx = \xi \frac{\partial q}{\partial x} \qquad ldy = \xi \frac{\partial q}{\partial y}$$

4) Construct left and right interface state for x and y.