Hydrodynamics:

$f(x_i, v_i, t)$ $\qquad$ $\int d^3v \, Q f(x_i, v_i, t) = \langle Q \rangle (x_i)$

we go from microscopic (distribution) to macroscopic (continuum) by considering equations for $n\langle \chi \rangle$, i.e. amount of χ per unit volume.

Consider different conserved quantities:

$$\chi = \begin{pmatrix} m \\ m\vec{u} \\ \frac{1}{2} m |\vec{u}|^2 \end{pmatrix} \begin{matrix} \leftarrow \text{mass} \\ \leftarrow \text{momentum} \\ \leftarrow \text{energy} \end{matrix}$$

Start from Boltzmann Equations:

$$\frac{\partial}{\partial t}\left(n\langle \chi \rangle\right) + \frac{\partial}{\partial x_i}\left(n \langle v_i \chi \rangle\right) - \cancel{n\left\langle v_i \frac{\partial \chi}{\partial x_i}\right\rangle} - \frac{n}{m}\left\langle F_i \frac{\partial \chi}{\partial v_i}\right\rangle - \frac{n}{m}\cancel{\left\langle \frac{\partial F_i}{\partial v_i} \chi \right\rangle} = 0$$

1) consider $\vec{F}$ to be independent of velocity $\Rightarrow \frac{\partial F_i}{\partial v_i} = 0$

2) consider $\chi = \chi(v_i)$, so independent of position $x_i \Rightarrow \frac{\partial}{\partial x_i}\chi = 0$

$$\hookrightarrow \boxed{\frac{\partial}{\partial t}\left(n\langle \chi \rangle\right) + \frac{\partial}{\partial x_i}\left(n\langle v_i \chi \rangle\right) = \frac{n}{m}\left\langle F_i \frac{\partial \chi}{\partial v_i}\right\rangle}$$

I) consider the conserved quantity is mass, $\underline{\underline{\chi = m}}$:

$$\frac{\partial}{\partial t}(nm) + \frac{\partial}{\partial x_i}(nm\langle u_i \rangle) = 0$$

Now define $\rho = nm$ and $\vec{v} = u_i$: average velocity

$$\therefore \quad \boxed{\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}(\rho v_i) = 0}$$: continuity equation (Mass conservation)

II) Now consider conserved quantity is momentum : $p = m\vec{u}$

$$\frac{\partial}{\partial t}(n\langle m\vec{u}\rangle) + \frac{\partial}{\partial x_i}(n\langle m u_i u_j \rangle) = \frac{n}{m}\langle F_i \underbrace{\frac{\partial}{\partial u_i} m u_j}_{m\delta_{ij}} \rangle$$

$$\hookrightarrow \quad \frac{\partial}{\partial t}(\rho\vec{v}) + \frac{\partial}{\partial x_i}(\rho\langle u_i u_j \rangle) = n\langle F_j \rangle$$

Note: since $\langle Q \rangle = \dfrac{\int d^3v \, f(x_i, v_i, t)\, Q}{\int d^3v \, f(x_i, v_i, t)}$, so if $Q \neq Q(v_i)$, then $\langle Q \rangle = Q$

We previously assumed $\vec{F} \neq \vec{F}(v_i)$, hence $\langle F_j \rangle = F_j$

Now let's define a symmetric tensor P_{ij}:

$$P_{ij} = \rho\langle (u_i - v_i)(u_j - v_j) \rangle$$

Note: u_i is the velocity of individual particle.

v_i is the average velocity of the fluid: $\langle u_i \rangle = v_i$

so $u_i - v_i$ is the velocity wrt the average flow.

Now let's expand $P_{ij} = \rho \langle (u_i - v_i)(u_j - v_j) \rangle$

↳ $$P_{ij} = \rho \langle (u_i - v_i)(u_j - v_j) \rangle$$

$$= \rho \{ \langle u_i u_j - u_i v_j - v_i u_j + v_i v_j \rangle \}$$

$$= \rho \{ \langle u_i u_j \rangle - \langle u_i v_j \rangle - \langle v_i u_j \rangle + \langle v_i v_j \rangle \}$$

$$= \rho \{ \langle u_i u_j \rangle - v_i v_j - v_i v_j + v_i v_j \}$$

$$P_{ij} = \rho \langle u_i u_j \rangle - \rho v_i v_j$$

Hence $$\boxed{\rho \langle u_i u_j \rangle = P_{ij} + \rho v_i v_j}$$

Now substitute back to momentum equation:

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_i}(\rho \langle u_i u_j \rangle) = n F_j$$

↳ $$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_i}(P_{ij} + \rho v_i v_j) = n F_j$$

$\therefore$ $$\boxed{\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) + \frac{\partial}{\partial x_j} P_{ij} = \frac{\rho}{m} F_i}$$: Momentum eq

(Following Shu)

III) Now consider conserved quantity is energy $\chi = \frac{1}{2} m u_i^2$

Let's define $u_i = v_i + w_i$

where $v_i = \langle u_i \rangle$, w_i = deviation where $\langle w_i \rangle = 0$

then

$$\chi = \frac{1}{2} m |\vec{u}|^2$$
$$= \frac{1}{2} m |v_i + w_i|^2$$
$$\chi = \frac{1}{2} m \{ v_i^2 + 2 v_i w_i + w_i^2 \}$$

Now substitute into conservation equation:

$$\frac{\partial}{\partial t}(n\langle \chi \rangle) + \frac{\partial}{\partial x_i}(n \langle u_i \chi \rangle) = \frac{n}{m} \langle F_i \frac{\partial}{\partial u_i} \chi \rangle$$

$$\hookrightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho \{ v_j^2 + \cancel{2v_j \langle w_j \rangle} + \langle w_j^2 \rangle \}\right)$$
$$+ \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho \langle (v_i + w_i)(v_j^2 + 2v_j w_j + w_j^2) \rangle\right) = \frac{\rho}{m} F_i \langle \frac{\partial}{\partial u_i}\left(\frac{1}{2} u_j^2\right) \rangle$$

Now consider the RHS:

$$\frac{1}{2}\frac{\rho}{m} F_i \langle \frac{\partial}{\partial u_i}(u_j u_j) \rangle = \frac{1}{2} \frac{\rho}{m} F_i \langle \left[u_j \underbrace{\frac{\partial u_j}{\partial u_i}}_{\delta_{ij}} + u_j \underbrace{\frac{\partial u_j}{\partial u_i}}_{\delta_{ij}} \right] \rangle$$
$$= \frac{\rho}{m} F_j \langle u_j \rangle$$

$$\Rightarrow \frac{1}{2}\frac{\rho}{m} F_i \langle \frac{\partial}{\partial u_i}(u_j u_j) \rangle = \frac{\rho}{m} F_j v_j$$

Now consider $\langle (v_i + w_i)\, |\vec{v} + \vec{w}|^2 \rangle$ inside $\frac{\partial}{\partial x_i}$:

$$\langle (v_i + w_i)|\vec{v}+\vec{w}|^2\rangle = \langle (v_i+w_i)(v_j^2 + 2v_j w_j + w_j^2)\rangle$$

$$= v_i v_j^2 + 2 v_i v_j \underbrace{\langle w_j\rangle}_{=0} + v_i \langle w_j^2\rangle$$

$$+ \underbrace{\langle w_i\rangle}_{=0} v_j^2 + 2v_j\langle w_i w_j\rangle + \langle w_i w_j^2\rangle$$

$$\langle (v_i+w_i)(v_j+w_j)^2\rangle = v_i v_j^2 + v_i\langle w_j^2\rangle + 2v_j\langle w_i w_j\rangle + \langle w_i w_j^2\rangle$$

1) Now define (per mass) specific internal energy : $e = \frac{1}{2}\langle |w|^2\rangle$

Then internal energy density : $\rho e = \frac{1}{2}\rho\langle |w|^2\rangle$

2) Define conductive heat flux (energy flux) : $q_i = \frac{1}{2}\rho\langle w_i |w|^2\rangle$

3) Note $P_{ij} = \rho\langle (u_i - v_i)(u_j - v_j)\rangle = \rho\langle w_i w_j\rangle$

so $$\langle (v_i+w_i)(v_j+w_j)^2\rangle = v_i v_j^2 + v_i\underbrace{\langle w_j^2\rangle}_{2e} + 2v_j\underbrace{\langle w_i w_j\rangle}_{\frac{1}{\rho}P_{ij}} + \underbrace{\langle w_i w_j^2\rangle}_{\frac{2}{\rho}q_i}$$

$$= v_i v_j^2 + 2e v_i + 2\frac{1}{\rho} v_j P_{ij} + 2\frac{1}{\rho} q_i$$

Previously : $\frac{\partial}{\partial t}\left(\frac{1}{2}\rho[v_j^2 + \langle w_j^2\rangle]\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho\langle (v_i+w_i)(v_j+w_j)^2\rangle\right) = \frac{\rho}{m}F_j v_j$

Full Energy Equation

$$\therefore \boxed{\frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\rho v_j^2 + \rho e\right)}_{\rho E} + \frac{\partial}{\partial x_i}\left(\underbrace{v_i\left(\frac{1}{2}\rho v_j^2 + \rho e\right)}_{\rho E \vec{v}}\right) + \frac{\partial}{\partial x_i}(v_j P_{ij}) + \frac{\partial}{\partial x_i} q_i = \frac{\rho}{m}F_j v_j}$$

How about <u>internal energy</u>, $e = \frac{1}{2}\langle |w|^2 \rangle$, specifically?

start with momentum equation and multiply it by v_j

$$\Rightarrow v_j \left\{ \underbrace{\frac{\partial}{\partial t}(\rho v_j) + \frac{\partial}{\partial x_i}(\rho v_i v_j) + \frac{\partial}{\partial x_i} P_{ij} = \frac{\rho}{m} F_j}_{\text{Momentum equation.}} \right\}$$

Now subtract the above from the total energy equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v_j^2 + \rho e\right) + \frac{\partial}{\partial x_i}\left(v_i\left[\frac{1}{2}\rho v_j^2 + \rho e\right]\right) + \frac{\partial}{\partial x_i}(v_j P_{ij}) + \frac{\partial}{\partial x_i} q_i = \cancel{\frac{\rho}{m} F_j v_j}$$

$$- \left\{ v_j \frac{\partial}{\partial t}(\rho v_j) + v_j \frac{\partial}{\partial x_i}(\rho v_i v_j) + v_j \frac{\partial}{\partial x_i} P_{ij} = \cancel{\frac{\rho}{m} F_j v_j} \right\}$$

$$\hookrightarrow = \frac{\partial}{\partial t}(\rho e) + \frac{1}{2} v_j^2 \frac{\partial}{\partial t}\rho + \cancel{\rho v_j \frac{\partial}{\partial t} v_j} + \frac{\partial}{\partial x_i}(v_i \rho e) + \frac{\partial}{\partial x_i}\left(v_i \frac{1}{2}\rho v_j^2\right)$$

$$+ \underbrace{\frac{\partial}{\partial x_i}(v_j P_{ij})}_{\cancel{v_j \frac{\partial}{\partial x_i} P_{ij}} + P_{ij}\frac{\partial}{\partial x_i} v_j} + \frac{\partial}{\partial x_i} q_i - \left[v_j^2 \frac{\partial}{\partial t}\rho + \cancel{\rho v_j \frac{\partial}{\partial t} v_j} \right] - v_j \frac{\partial}{\partial x_i}(\rho v_i v_j) - \cancel{v_j \frac{\partial}{\partial x_i} P_{ij}} = 0$$

$$\overbrace{\frac{1}{2} v_j^2 \left[\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}(\rho v_i) \right] = 0}^{\text{continuity eq}}$$

$$\hookrightarrow \frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x_i}(v_i \rho e) + \frac{\partial}{\partial x_i} q_i + P_{ij}\frac{\partial}{\partial x_i} v_j + \frac{1}{2} v_j^2 \frac{\partial}{\partial t}\rho + \frac{1}{2} v_j^2 \frac{\partial}{\partial x_i}(\rho v_i)$$

$$+ \cancel{\rho v_i v_j \frac{\partial}{\partial x_i} v_j} - \cancel{\rho v_i v_j \frac{\partial}{\partial x_i} v_j} \underbrace{- v_j^2 \frac{\partial}{\partial x_i}(\rho v_i) - v_j^2 \frac{\partial}{\partial t}\rho}_{-v_j^2 \left[\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}(\rho v_i) \right] = 0} = 0$$

$$-v_j^2 \Big[\underbrace{\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x_i}(\rho v_i)}_{\text{continuity eq.}} \Big] = 0$$

Note that the <u>Bulk KE terms all cancel</u>, so the momentum equation captures all the Bulk KE contributions.

Now we have:

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x_i}(v_i \rho e) + \frac{\partial}{\partial x_i} q_i + P_{ij}\frac{\partial}{\partial x_i} v_j = 0$$

Rewrite $P_{ij}\frac{\partial}{\partial x_i} v_j$, knowing $P_{ij} = \rho \langle (u_i - v_i)(u_j - v_j)\rangle$

symmetric tensor, i.e. $P_{ij} = P_{ji}$

↳ $$P_{ij}\frac{\partial}{\partial x_i} v_j = \frac{1}{2}\left(P_{ij}\frac{\partial}{\partial x_i} v_j + P_{ji}\frac{\partial}{\partial x_j} v_i\right)$$

$$= P_{ij}\,\frac{1}{2}\left(\frac{\partial}{\partial x_i} v_j + \frac{\partial}{\partial x_j} v_i\right)$$

$$\boxed{P_{ij}\frac{\partial}{\partial x_i} v_j = P_{ij}\Lambda_{ij} \quad \text{where } \Lambda_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} v_j + \frac{\partial}{\partial x_j} v_i\right)}$$

All together: Internal Energy equation follows: $e = \frac{1}{2}\langle |w|^2 \rangle$

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x_i}(v_i \rho e) + P_{ij}\Lambda_{ij} + \frac{\partial}{\partial x_i} q_i = 0$$

Alternatively: define $P_{ij} = p\,\delta_{ij} + \Pi_{ij}$

pressure (p); viscous stress tensor (Π_{ij})

then $P_{ij}\frac{\partial}{\partial x_i} v_j = P_{ij}\Lambda_{ij} = p\,\delta_{ij}\frac{\partial}{\partial x_i} v_j + \Pi_{ij}\frac{\partial}{\partial x_i} v_j$

↳ $$\frac{\partial}{\partial t}(\rho e) + \underbrace{\frac{\partial}{\partial x_i}(v_i \rho e)}_{\vec{\nabla}\cdot(\rho e \vec{v})} + \underbrace{p\frac{\partial}{\partial x_i} v_i}_{p(\vec{\nabla}\cdot\vec{v})} + \Pi_{ij}\frac{\partial}{\partial x_i} v_j + \underbrace{\frac{\partial}{\partial x_i} q_i}_{\vec{\nabla}\cdot\vec{q}} = 0$$

where $\vec{q} = \frac{1}{2}\rho\langle w_i |w|^2\rangle$

energy flux / conductive heat flux

Choudhuri 3.3: How do we know whether the system will behave like a fluid (continuum)?

Answer: If collisions are important enough in a system of particles to produce local Maxwellian distribution within different parts of the system, then we expect system to behave fluid-like.

proof: Assume the distribution function at each point within a system of particles is Maxwellian.

$$f^{(0)}(\vec{x},\vec{u},t)=n(\vec{x},t)\left(\frac{m}{2\pi k_B T(\vec{x},t)}\right)^{3/2}\exp\left\{-\frac{m[\vec{u}-\vec{v}(\vec{x},t)]^2}{2k_B T(\vec{x},t)}\right\}$$

Now let's compute some quantities using above distribution.

$$P_{ij} = \rho \langle (u_i-v_i)(u_j-v_j)\rangle$$

$$= \rho \frac{\int f^{(0)}(u_i-v_i)(u_j-v_j)\,d^3u}{\int f^{(0)}d^3u}$$

let $\omega_i = u_i - v_i$

$$= \frac{\rho}{n}\int d^3w\ \omega_i\omega_j\, n\left(\frac{m}{2\pi k_B T}\right)^{3/2}\exp\left\{-\frac{m w^2}{2k_B T}\right\}$$

If $i\neq j$, then integral is odd so it vanishes, so δ_{ij}

$$\boxed{P_{ij} = \underbrace{n k_B T}_{P}\,\delta_{ij}}$$

$\Leftarrow$ For Maxwellian distribution $P_{ij} = P\delta_{ij}$, so no viscosity.

Similarly, for energy flux or conductive heat flux : q

$$\vec{q} = \frac{1}{2}\rho \langle (\vec{u}-\vec{v}) |\vec{u}-\vec{v}|^2 \rangle$$

$$= \frac{1}{2}\rho \frac{\int f^{(0)} \vec{w} |\vec{w}|^2 d^3w}{\int f^{(0)} d^3w}$$

$$\boxed{\vec{q} = 0}$$

⇐ conductive heat flux vanishes for Maxwellian distribution.

For internal energy: e

$$e = \frac{1}{2} \langle |\vec{u}-\vec{v}|^2 \rangle$$

$$= \frac{1}{2} \frac{\int f^{(0)} |w|^2 d^3w}{\int f^{(0)} d^3w}$$

$$\boxed{e = \frac{3}{2}\frac{k_B T}{m}}$$

⇐ Monoatomic particles assuming Maxwellian distribution.

Finally:

$$P_{ij}\Lambda_{ij} = p\delta_{ij}\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$= p\frac{\partial v_i}{\partial x_i}$$

$$\boxed{P_{ij}\Lambda_{ij} = p\vec{\nabla}\cdot\vec{v}}$$

⇐ Assuming Maxwellian distribution.

Now using above results, plug into momentum / energy equations:

<u>Momentum</u>: $\frac{\partial}{\partial t}(\rho v_j) + \frac{\partial}{\partial x_i}(\rho v_i v_j) + \frac{\partial}{\partial x_i} P_{ij} = \frac{\rho}{m} F_j$

$$\hookrightarrow \boxed{\frac{\partial}{\partial t}(\rho v_j) + \frac{\partial}{\partial x_i}(\rho v_i v_j) + \underbrace{\frac{\partial}{\partial x_j} p}_{\vec{\nabla} p} = \frac{\rho}{m} F_j}$$

<u>Internal energy</u>: $\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x_i}(v_i \rho e) + \frac{\partial}{\partial x_i} q_i + P_{ij} \Lambda_{ij} = 0$

$$\hookrightarrow \boxed{\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x_i}(v_i \rho e) + \underbrace{p \frac{\partial}{\partial x_i} v_i}_{p \vec{\nabla} \cdot \vec{v}} = 0}$$

Along with <u>continuity</u>: $\boxed{\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0}$

Remarks: 1) These 3 equations make up dynamical evolution of 5 individual scalar variables

2) There are a total of 6 variables: $\vec{v}, p, e, \rho$
Note; we neglect $\vec{F}$ since it's an external force, so it has to be prescribed.

3) But for monoatomic gas, $p = n k_B T$, $e = \frac{3}{2} \frac{k_B T}{m}$, and $\rho = nm$
So there are only 2 independent variables: $n(\vec{x}, t)$, $T(\vec{x}, t)$

4) So we actually have 5 independent variables and exactly 5 equations, so we can solve for this dynamical system.

Transport Phenomena (Choudhuri 3.4)

Transport Phenomena (3.4 choudhuri)

→ Previously, we only have diagonal terms in P_{ij}, meaning we neglected viscosity, we also have $\vec{q}=0$, i.e. there is no flow of internal energy, i.e., heat.

→ Therefore, there wasn't transport phenomena.

→ Consider perturbation from Maxwellian distribution to handle transport phenomena.

$$f(\vec{x},\vec{u},t) = \underbrace{f^{(0)}(\vec{x},\vec{u},t)}_{\substack{\text{Maxwellian distribution} \\ \text{(local equilibrium)}}} + \underbrace{g(\vec{x},\vec{u},t)}_{\text{small perturbation.}}$$

Now substitute the above distribution in Boltzmann equation:

$$\frac{\partial f}{\partial t} + \vec{u}\cdot\vec{\nabla} f + \frac{\vec{F}}{m}\cdot\vec{\nabla}_u f = \int d^3u_1 \int d\Omega\, |\vec{u}-\vec{u}_1|\, \sigma(\Omega)\left(f' f_1' - f f_1\right)$$

To first order, i.e. $\mathcal{O}(g)$:

$$\hookrightarrow \frac{Df}{Dt} \simeq \int d^3u_1 \int d\Omega\, |\vec{u}-\vec{u}_1|\, \sigma(\Omega)\left(f^{(0)'} g_1' + g' f_1' - f^{(0)} g - g f_1^{(0)}\right)$$

to consider the order of magnitude, only consider the last term.

$$\hookrightarrow \frac{Df}{Dt} \simeq -\int d^3u_1 \int d\Omega\, |\vec{u}-\vec{u}_1|\, \sigma(\Omega)\, g f_1^{(0)}$$

$$= -g(\vec{x},\vec{u},t)\, \underbrace{n\,\sigma_{tot}\,\vec{u}_{rel}}_{\approx \frac{1}{\tau} \leftarrow \text{collision time.}}$$

then we have:

$$\Rightarrow \left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla} + \frac{\vec{F}}{m}\cdot\vec{\nabla}_u\right) f = -\frac{f-f^{(0)}}{\tau} \quad \text{where } g = f - f^{(0)}$$

known as Bhatnagar, Gross and Krook equation.

Now if a system has a strong gradient, then $\vec{u}\cdot\vec{\nabla}$ is what gives rise to the RHS.

$$\text{suppose} \quad \underbrace{(\vec{u}\cdot\vec{\nabla})f \approx \frac{|u| f^{(0)}}{L}}_{\text{LHS}} \approx \underbrace{-\frac{g}{\tau}}_{\text{RHS}}$$

where $|u|$ is the typical molecular velocity and L is the typical length scale over which the system changes notably.

$$\text{then} \quad \frac{|g|}{f^{(0)}} \approx \frac{|u|\tau}{L} \approx \frac{\lambda}{L} \leftarrow \text{mean-free-path.}$$

so the perturbation is small from pure Maxwellian if $\lambda \ll L$, then say

$$\alpha \equiv \frac{\lambda}{L}$$

then expand in α: (Chapman – Enskog expansion)

$$f = f^{(0)} + \alpha f^{(1)} + \alpha^2 f^{(2)} + \cdots$$

If we only consider the zeroth order approximation, $f \simeq f^{(0)}$

then : $$\left(\frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} + \frac{F_i}{m}\frac{\partial}{\partial u_i}\right) f^{(0)} = -\frac{g}{\tau}$$

but since $f^{(0)}$ depends on t, x_i through $n(x_i, t)$, $T(x_i, t)$ and $\vec{v}(x_i, t)$ then

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial n}{\partial t}\frac{\partial f^{(0)}}{\partial n} + \frac{\partial T}{\partial t}\frac{\partial f^{(0)}}{\partial T} + \frac{\partial v_i}{\partial t}\frac{\partial f^{(0)}}{\partial v_i}$$

similarly: $$\frac{\partial f^{(0)}}{\partial x_i} = \frac{\partial n}{\partial x_i}\frac{\partial f^{(0)}}{\partial n} + \frac{\partial T}{\partial x_i}\frac{\partial f^{(0)}}{\partial T} + \frac{\partial v_i}{\partial x_i}\frac{\partial f^{(0)}}{\partial v_i}$$

then we find

$$\Rightarrow \left[\frac{1}{T}\frac{\partial T}{\partial x_i} U_i \left(\frac{m}{2k_BT}U^2 - \frac{5}{2}\right) + \frac{m}{k_BT}\Lambda_{ij}\left(U_iU_j - \frac{1}{3}\delta_{ij}U^2\right)\right] f^{(0)} = -\frac{g}{\tau}$$

→ We see that g depends linearly on the temperature gradient, flow velocity gradient, $\Lambda_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$, and collision timescale, τ.

→ If τ, collision timescale, is large, the fluid will be able to stream faraway without colliding with other particles. This adds perturbation to the local Maxwellian equillibrium, hence increases, g.

→ If τ, collision time scale, is small, then collisions will be effective, so the local equilibrium is maintained.

Now that we determined g, we can determine the corresponding P_{ij}, $\vec{q}$, and e,

we see $\quad \vec{q} = \frac{1}{2n} \int d^3U \, U^2 g$

$$\vec{q} \overset{!}{=} -K \vec{\nabla} T \quad \leftarrow \text{flux of internal energy (heat flux)}$$

where $K = \frac{\tau m}{6T} \int d^3U \, U^4 \left(\frac{m}{2k_B T} U^2 - \frac{5}{2} \right) f^{(0)} = \frac{5}{2} \tau n \frac{k_B^2 T}{m}$

coefficient of thermal conductivity.

Similarly, $\quad P_{ij} = p \delta_{ij} + \Pi_{ij}$

where $\quad \Pi_{ij} = m \int d^3U \, U_i U_j g$

$$\overset{!}{=} -\frac{\tau m^2}{k_B T} \Lambda_{kl} \int d^3U \, U_i U_j \left(U_k U_l - \frac{1}{3} \delta_{kl} U^2 \right) f^{(0)}$$

→ Note that Π_{ij} is a traceless symmetric tensor, so $\Pi_{ii} = 0$.

→ Since Π_{ij} depends linearly on Λ_{kl}, it has a traceless form:

$$\Pi_{ij} = -2\mu \left(\Lambda_{ij} - \frac{1}{3} \delta_{ij} \vec{\nabla} \cdot \vec{v} \right)$$

→ By comparing the two equations, we see

coefficient of viscosity → $\mu = \frac{\tau m^2}{k_B T} \int d^3U \, U_1^2 U_2^2 f^{(0)} = \tau n k_B T$ ← (require mean-free-path to be smaller than system size)

Summary: By considering small perturbation from local equilibrium solution, i.e. Maxwellian, we can calculate the transport coefficient, K and μ, which are coefficient of thermal conductivity and viscosity.

Surprising Remarks :

we can take $\tau = \frac{\lambda}{\langle u \rangle} = \frac{1}{4na^2}\left(\frac{m}{\pi k_B T}\right)^{1/2}$

then since $\eta = \tau n k_B T$

$\hookrightarrow \quad \eta = \frac{1}{4a^2}\left(\frac{m k_B T}{\pi}\right)^{1/2}$ ← coefficient of viscosity.

Note its for $\boxed{\text{gas}}$ only :

Remark 1: $\boxed{\text{Viscosity is independent of density !}}$

→ Even though a denser gas has more molecules to transport, but since mean-free-path also decreases, each individual molecule is less effecient at transporting.

→ i.e. when density is high, τ, collision time scale drops,

Remark 2: Viscosity increases with temperature for gas !

→ We saw it increases as $\sqrt{T}$, but it increases faster than $\sqrt{T}$ in real-life since gas molecules are not rigid spheres, so as $T \uparrow$, a, effective size of molecule decreases. So overall, it increases faster than $\sqrt{T}$.

→ Note in liquid, viscosity decrease with temperature.