General form of Euler Equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{U}) = 0$$

$$\frac{\partial (\rho \vec{U})}{\partial t} + \vec{\nabla} \cdot (\rho \vec{U} \otimes \vec{U}) + \vec{\nabla} p = 0$$

$$\frac{\partial (\rho E)}{\partial t} + \vec{\nabla} \cdot \left(\rho E \vec{U} + \rho \vec{U} \right) = 0$$

Then energy / mass:

$$E = e + \frac{1}{2} |U|^2$$

With Gamma law eos: $p = \rho e (\gamma - 1)$

In 1D:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} = 0$$

$$\frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u E + u p)}{\partial x} = 0$$

Write in conservative form:

$$\frac{\partial}{\partial t}\mathcal{U} + [F(\mathcal{U})]_x = 0$$

$$\mathcal{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} \qquad F(\mathcal{U}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u E + u p \end{pmatrix}$$

If assume gamma law eos: $p = \rho e(\gamma - 1)$

$$F(\mathcal{U}) = \begin{pmatrix} \rho u \\ \frac{1}{2}\frac{(\rho u)^2}{\rho}(3-\gamma) + \rho E(\gamma - 1) \\ \frac{\rho^2 u E}{\rho}\gamma - \frac{1}{2}\frac{(\rho u)^3}{\rho^2}(\gamma - 1) \end{pmatrix}$$

then

$$\frac{\partial \vec{F}}{\partial \mathcal{U}} = \vec{A}(\mathcal{U}) = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2}u^2(3-\gamma) & u(3-\gamma) & \gamma - 1 \\ \frac{1}{2}(\gamma-2)u^3 - \frac{uc^2}{\gamma-1} & \frac{3-2\gamma}{2}u^2 + \frac{c^2}{\gamma-1} & u\gamma \end{pmatrix}$$

Then $\mathcal{U}_t + [F(\mathcal{U})]_x = 0$

$$\hookrightarrow \mathcal{U}_t + \frac{\partial F}{\partial \mathcal{U}}\frac{\partial \mathcal{U}}{\partial x} = 0$$

$$\mathcal{U}_t + \vec{A}(\mathcal{U})\,\mathcal{U}_x = 0$$

If we change to primitive variables: $q_t + \vec{A}(q)\, q_x = 0$

$$\vec{q} = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} \qquad \vec{A}(q) = \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{pmatrix}$$

then the eigenvalues of $\vec{A}(q)$: $|\vec{A} - \lambda \vec{I}| = 0$ are :

$$\lambda = \begin{pmatrix} u-c \\ u \\ u+c \end{pmatrix}$$

We find the left and right eigenvectors :

$$\vec{L} = \begin{pmatrix} l^{(-)} \\ l^{(0)} \\ l^{(+)} \end{pmatrix} = \begin{pmatrix} 0 & \frac{-\rho}{2c^2} & \frac{1}{2c^2} \\ 1 & 0 & \frac{-1}{c^2} \\ 0 & \frac{\rho}{2c^2} & \frac{1}{2c^2} \end{pmatrix}$$

$$\vec{R} = \begin{pmatrix} r^{(-)} & r^{(0)} & r^{(+)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \frac{-c}{\rho} & 0 & \frac{c}{\rho} \\ c^2 & 0 & c^2 \end{pmatrix}$$

If we let $\quad d\vec{w} = \vec{L}\, d\vec{q} \quad \leftarrow \vec{w}$ are characteristic variables

$$\vec{L}\left(q_t + A q_x\right) = 0$$

$$\vec{L} q_t + \vec{L}\vec{A}\vec{R}\vec{L} q_x = 0$$

$$\vdots$$

$$L q_t + \vec{\lambda} \vec{L} q_x = 0$$

$$\vec{w}_t + \vec{\lambda} \vec{w}_x = 0$$

then it can be shown that:

$$\frac{\partial \vec{w}}{\partial t} + \vec{\lambda} \frac{\partial \vec{w}}{\partial x} = 0$$

writing things out:

$$\left.\begin{aligned} w_t^{(-)} + \lambda^{(-)} w_x^{(-)} &= 0 \\ w_t^{(0)} + \lambda^{(0)} w_x^{(0)} &= 0 \\ w_t^{(+)} + \lambda^{(+)} w_x^{(+)} &= 0 \end{aligned}\right\}$$

3 decoupled nonlinear advection equations.
3 waves traveling at speed u, $u-c$, $u+c$.

We can then recover the primitive variables as:

$$\begin{pmatrix} \rho \\ u \\ p \end{pmatrix} = d\vec{q} = L^{-1} d\vec{w} = \underset{\rightarrow}{R} d\vec{w} = R \begin{pmatrix} w^{(-)} \\ w^{(0)} \\ w^{(+)} \end{pmatrix}$$

Here we note that for $r^{(0)}$, only density proportion is non-zero, so there will only be a density jump over the middle wave.

Entropy Formulation: used to understand the condition of rarefaction since entropy is constant for rarefaction. (For shock, entropy increases)

we can also write: $\tilde{q}_s = (\rho, u, s)^T$

we need to get rid of pressure gradient term in velocity equation:

$$\frac{\partial p(\rho, s)}{\partial x} = \left.\frac{\partial p}{\partial s}\right|_\rho \frac{\partial s}{\partial x} + \left.\frac{\partial p}{\partial \rho}\right|_s \frac{\partial \rho}{\partial x} = \left.\frac{\partial p}{\partial s}\right|_\rho \frac{\partial s}{\partial x} + \frac{p\Gamma_1}{\rho}\frac{\partial \rho}{\partial x}$$

then the velocity equation becomes:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\left[\left.\frac{\partial p}{\partial s}\right|_\rho \frac{\partial s}{\partial x} + \frac{p\Gamma_1}{\rho}\frac{\partial \rho}{\partial x}\right] = 0$$

such that $\quad q_{st} + A_s(q_s)\, q_{sx} = 0$

$$A_s = \begin{pmatrix} u & \rho & 0 \\ \frac{c^2}{\rho} & u & p_s/\rho \\ 0 & 0 & u \end{pmatrix} \quad \text{here } p_s = \left.\frac{\partial p}{\partial s}\right|_\rho$$

then we still have the same eigenvalues:

$$\vec{\lambda} = \begin{pmatrix} u-c & & \\ & u & \\ & & u+c \end{pmatrix}$$

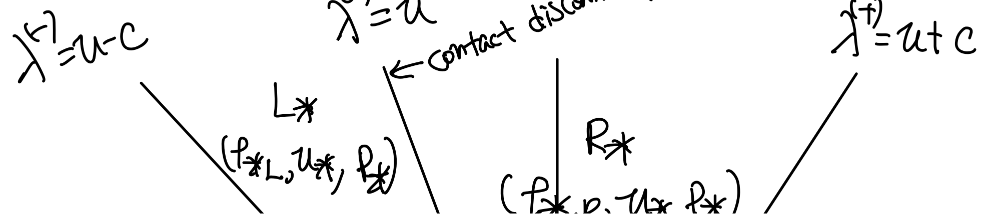
$$R = \begin{pmatrix} r^{(-)} & r^{(0)} & r^{(+)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \frac{-c}{\rho} & 0 & c/\rho \\ 0 & \frac{-c^2}{P_s} & 0 \end{pmatrix}$$

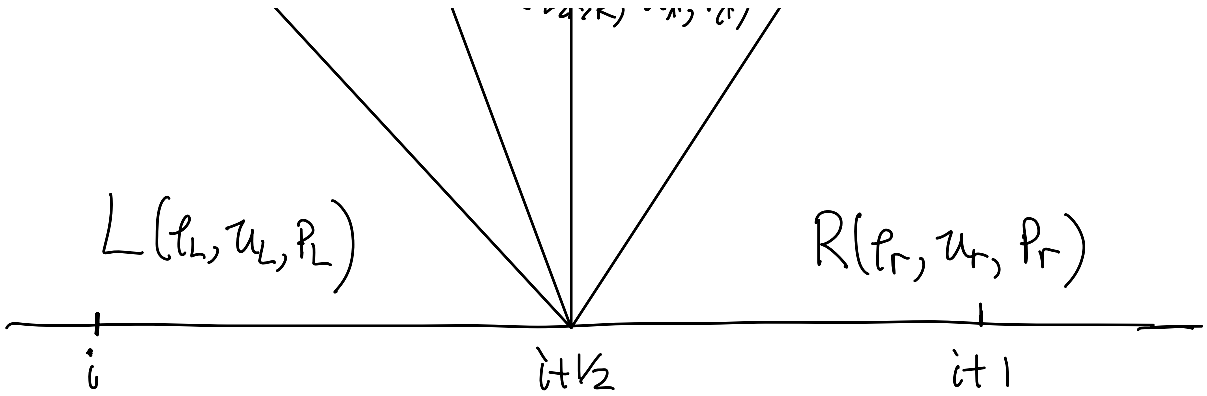
$$L = \begin{pmatrix} l^{(-)} \\ l^{(0)} \\ l^{(+)} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\rho}{c} & \frac{1}{c^2} \\ \frac{P_s}{P_s+1} & 0 & \frac{-P_s}{c^2(P_s+1)} \\ 0 & \frac{\rho}{c} & \frac{1}{c^2} \end{pmatrix}$$

Euler Riemann Solver:

Given left and right states in primitive variable, output an unique interface state:

$$q_{i+1/2} = R(q_{i+1/2,L} \, , \, q_{i+1/2,R})$$





Properties?

1) The middle wave, speed at u, it is neither a shock or rarefaction since velocity remains constant across this wave. Only density changes.

2) The left and right waves can either be a shock or rarefaction.

3) L and R are the original left and right states since no waves have reach ~~to~~ them, they're unchanged

3) Given L and R states, we have 4 unknowns,
$(u_*, P_*, \rho_{*L}, \rho_{*R})$
← A total of 4 unknowns

Rarefaction conditions:

⇒ Since entropy remains constant across a rarefaction, $\frac{Ds}{Dt} = 0$

In the primitive variable, replace pressure with entropy. $\vec{q}_s = (\rho, u, s)^T$.

Then we need to express the pressure gradient, $\frac{\partial p}{\partial x}$ term in momentum Euler equations in terms of $\vec{q}_s$.

$\vec{q}_s = (\rho, u, s)^T$.

$$\frac{\partial p(\rho, s)}{\partial x} = \left.\frac{\partial p}{\partial \rho}\right|_s \frac{\partial \rho}{\partial x} + \left.\frac{\partial p}{\partial s}\right|_\rho \frac{\partial s}{\partial x} = \left.\frac{\partial p}{\partial s}\right|_\rho \frac{\partial s}{\partial x} + \frac{p\Gamma_1}{\rho}\frac{\partial \rho}{\partial x}$$

$$\hookrightarrow \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\left[\left.\frac{\partial p}{\partial s}\right|_\rho \frac{\partial s}{\partial x} + \frac{p\Gamma_1}{\rho}\frac{\partial \rho}{\partial x}\right] = 0$$

Then $\quad q_t + A q_x = 0 \Rightarrow q_{s,t} + A_s q_{s,x} = 0$

where

$$\vec{A}_s = \begin{pmatrix} u & \rho & 0 \\ c^2/\rho & u & \frac{1}{\rho}\left.\frac{\partial p}{\partial s}\right|_\rho \\ 0 & 0 & u \end{pmatrix}$$

$$R = \left(r^{(-)} \; r^{(0)} \; r^{(+)} \right) = \begin{pmatrix} 1 & 1 & 1 \\ \frac{-c}{\rho} & 0 & c/\rho \\ 0 & \frac{-c^2}{p_s} & 0 \end{pmatrix}$$

Since $d\vec{q}_s = \vec{R}\, d\vec{w}$, we see that entropy change over the $(-)$ and $(+)$ waves are zero.

$$L = \begin{pmatrix} l^{(-)} \\ l^{(0)} \\ l^{(+)} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\rho}{c} & \frac{1}{c^2} \\ \frac{p_s}{p_s+1} & 0 & \frac{-p_s}{c^2(p_s+1)} \\ 0 & \frac{\rho}{c} & \frac{1}{c^2} \end{pmatrix}$$

Now, we return to the original primitive variable $\vec{q}$;

Consider the $(+)$ wave, across this wave, $w^{(+)}$ will jump, but constant across the $(-)$ and (0) wave.

Similarly constancy of the middle and right characteristic variables across the left wave $(-)$ gives:

$$l^{(+)} \cdot d\vec{q} = 0 = \begin{pmatrix} 0 & \frac{\rho}{2c} & \frac{1}{2c^2} \end{pmatrix} \begin{pmatrix} d\rho \\ du \\ dp \end{pmatrix} = 0$$

$$l^{(0)} \cdot d\vec{q} = 0 = \begin{pmatrix} 1 & 0 & \frac{-1}{c^2} \end{pmatrix} \begin{pmatrix} d\rho \\ du \\ dp \end{pmatrix} = 0$$

↳ Then we have:

$$du + \frac{1}{\rho c} dp = 0$$

$$d\rho - \frac{1}{c^2} dp = 0$$

→ Define Lagrangian sound speed: $C = \rho c$ and $\tau = 1/\rho$
then:

$$du = \frac{-dp}{C} \qquad d\tau = -\frac{dp}{C^2}$$ across the left wave.

Similarly

$$du = \int_{p_{L,R}}^{p_*} \frac{dp}{C} \qquad d\tau = \int_{p_{L,R}}^{p_*} -\frac{dp}{C^2}$$ across the right wave.

Until now, these are general. Finding the solution across the rarefaction involves integrating the system with EOS to provide $C \equiv C(\tau, P)$. The solutions are called Riemann Invariants.

Now consider gamma law eos: $P = K\rho^{\gamma}$

$$dp = \gamma K \rho^{\gamma - 1} d\rho$$

then only need to integrate a single equation:

$$du = -\int \frac{\gamma k \rho^{\gamma-1} d\rho}{\rho\sqrt{\frac{\gamma p}{\rho}}}$$

$$= -\int \frac{\gamma k \rho^{\gamma-1}\, d\rho}{\sqrt{\gamma \rho k \rho^{\gamma}}}$$

$2\gamma-2-(\gamma+1) = \gamma-3$

$$\stackrel{!}{=} \int \sqrt{\gamma k\, \rho^{\gamma-3}}\, d\rho$$

$$\stackrel{!}{=} -\sqrt{\gamma k} \int \rho^{\frac{\gamma}{2}-\frac{3}{2}}\, d\rho$$

$$\stackrel{!}{=} -\sqrt{\gamma k}\, \frac{1}{\frac{\gamma}{2}-\frac{1}{2}} \sqrt{\rho^{\gamma}}$$

$$u \stackrel{!}{=} -\frac{2c}{\gamma-1} + \text{const}$$

then $u + \frac{2c}{\gamma-1} = \text{constant}$ $\leftarrow$ Riemann Invariant across left wave

Similarly $u - \frac{2c}{\gamma-1} = \text{constant}$ $\leftarrow$ Invariant across right wave.

using the Riemann Invariant to the left wave:

$$u_L + \frac{2c_L}{\gamma-1} = u_* + \frac{2c_*}{\gamma-1}$$

and
$$K = \frac{p_L}{\rho_L^{\gamma}} = \frac{p_*}{\rho_*^{\gamma}}$$

We have:

$$u_* = u_L + \frac{2c_L}{\gamma-1}\left[1 - \left(\frac{p_*}{p_L}\right)^{\frac{\gamma-1}{2\gamma}}\right] \quad \text{for the left rarefaction}$$

$$u_* = u_R - \frac{2c_R}{\gamma-1}\left[1 - \left(\frac{p_*}{p_R}\right)^{\frac{\gamma-1}{2\gamma}}\right] \quad \text{for the right rarefaction.}$$

<u>Shock Jump Condition:</u> (Entropy increases in shock)

When the left or right wave is a shock:

We use Rankine-Hugoniot Condition:

In general:

$$\frac{F(U_*) - F(U_s)}{U_* - U_s} = S$$

where $s \in (L, R)$, left or right interface state.

and S is the shock speed.

We define velocity in the shock frame:

L,R interface velocity $\rightarrow$ $\hat{u}_s = u_s - S$

star state velocity $\rightarrow$ $\hat{u}_* = u_* - S$

Using velocity in shock frame:

$$\frac{F(\hat{u}_*) - F(\hat{u}_s)}{\hat{u}_* - \hat{u}_s} = 0$$

For 1D Euler: where $s \in [L, R]$

① $\rho_* \hat{u}_* = \rho_s \hat{u}_s$

② $\rho_* \hat{u}_*^2 + P_* = \rho_s \hat{u}_s^2 + P_s$

③ $\rho_* \hat{u}_* e_* + \frac{1}{2} \rho_* \hat{u}_*^3 + \hat{u}_* P_* = \rho_s \hat{u}_s e_s + \frac{1}{2} \rho_s \hat{u}_s^3 + \hat{u}_s P_s$

For a general EOS:

from Eq. (1): $\hat{u}_* = \frac{\rho_S}{\rho_*} \hat{u}_S$

Plug above into Eq. (2):

$$\rho_S^2 \left(\frac{1}{\rho_S} - \frac{1}{\rho_*} \right) \hat{u}_S^2 = P_* - P_S$$

let $[P] = P_* - P_S$

then

$$-[\tau] = \frac{1}{\rho_S} - \frac{1}{\rho_*} = \frac{[P]}{\rho_S^2 \hat{u}_S^2}$$

$\tau = \frac{1}{\rho}$

Now introduce mass flux, $\underline{\underline{W_S}}$.

For the shock separating, L, L*, mass will be moving through the shock to the right. Then in the frame of the shock:

$$W_L = \rho_L \hat{u}_L = \rho_* \hat{u}_*$$

Likewise, for the shock separating R/R*, mass will move to the left passing the shock:

$$W_R = -\rho_R \hat{u}_R = -\rho_* \hat{u}_*$$

Then our first jump condition becomes:

$$-[\tau] = \frac{[p]}{W_s^2} \quad 1 \text{ density relation}$$

Now going back to the momentum equation

$$\rho_* \hat{u}_*^2 + p_* = \rho_s \hat{u}_s^2 + p_s$$

For left shock:

$$W_L \hat{u}_* + p_* = W_L \hat{u}_L + p_L$$

$$\hookrightarrow \quad \hat{u}_* - \hat{u}_L = \frac{-p_* + p_L}{W_L}$$

$$\hookrightarrow \quad [u] = \frac{-[p]}{W_L} \quad 2 \text{ velocity relation of left shock}$$

For right shock:

$$-W_R \hat{u}_* + p_* = -W_R \hat{u}_R + p_R$$

$$[u] = \frac{[p]}{W_R} \quad 3 \text{ velocity relation of right shock.}$$

Need condition for energy:

Use energy equation with mass flux: W_s

$$W_s e_* + \frac{1}{2} W_s \hat{u}_*^2 + \frac{W_s}{\rho_*} p_* = W_s e_s + \frac{1}{2} W_s \hat{u}_s^2 + \frac{W_s}{\rho_s} p_s$$

Canceling W_s:

$$[e] + \frac{p_*}{\rho_*} - \frac{p_s}{\rho_s} + \frac{1}{2}\left(\hat{u}_*^2 - \hat{u}_s^2\right) = 0$$

Substitute $\hat{u}_*^2 = W_s^2 / \rho_*^2$ and $\hat{u}_s^2 = W_s^2/\rho_s^2$

$$[e] + \frac{p_*}{\rho_*} - \frac{p_s}{\rho_s} + \frac{1}{2} W_s^2 \left(\frac{1}{\rho_*^2} - \frac{1}{\rho_s^2}\right) = 0$$

Using the first jump condition: $W_s^2 = \frac{-[p]}{[\tau]}$

$$[e] + \frac{p_*}{\rho_*} - \frac{p_s}{\rho_s} + \frac{1}{2}\left(p_s - p_*\right)\left(\frac{1}{\rho_*} - \frac{1}{\rho_s}\right)^{-1}\left(\frac{1}{\rho_*^2} - \frac{1}{\rho_s^2}\right) = 0$$

with algebra:

$$\hookrightarrow \quad [e] = -\frac{p_* + p_s}{2}[\tau]$$

Summary of jump conditions across shock:

① $[\tau] = -\frac{[p]}{W_s^2}$

② $[u] = \mp \frac{[p]}{W_s}$ '$-$' for left, '$+$' for right.

③ $[e] = -\frac{p_s + p_*}{2}[\tau]$

Procedure:

① root find to find ρ_* corresponding to P_*:
- guess a value for ρ_*
- use eos, express $e_* = e(P_*, \rho_*)$
- use Newton or other methods

② Compute:

$$\frac{1}{W_s^2} = -\frac{[\tau]}{[p]}$$

③ Find the star velocity:

$$u_* = u_s \mp \frac{[p]}{W_s} \quad \text{'$-$' for left shock, '$+$' for right shocks}$$

Shock speed then can be calculated by:

$$S = u_s \mp \frac{W_s}{\rho_s} \quad \text{'$-$' for left, '$+$' for right.}$$

Consider a simple case with gamma law eos:

First, find ρ_*:

Introduce:

$$e = \frac{P}{\rho}\frac{1}{\gamma-1}$$

plug into

$$[e] = -\left(\frac{P_* + P_s}{2}\right)[\tau]$$

then: $$\rho_{*,s} = \rho_s \left[\frac{\frac{P_*}{P_s}(\gamma+1) + (\gamma-1)}{(\gamma+1) + \frac{P_*}{P_s}(\gamma-1)} \right]$$

Second: compute $\frac{1}{W_s^2} = -\frac{[\tau]}{[P]}$

$$W_s^2 = \frac{1}{2} P_s \rho_s \left[\left(\frac{P_*}{P_s}\right)(\gamma+1) + (\gamma-1) \right]$$

Third: Find star velocity: u_*:

$$u_* = u_s \pm c_s \left[\frac{2}{\gamma(\gamma-1)} \right]^{1/2} \frac{1 - \frac{P_*}{P_s}}{\left(\frac{P_*}{P_s} \frac{\gamma+1}{\gamma-1} + 1 \right)^{1/2}}$$

'+' for left shock
'−' for right shock

Lastly: Find shock speed:

$$S = u_s \mp c_s \left[\left(\frac{P_*}{P_s}\right) \frac{\gamma+1}{2\gamma} + \frac{\gamma-1}{2\gamma} \right]^{1/2}$$

'−' for left shock
'+' for right shock

Now that we have both star velocity, u_*, from rarefaction and shock condition, we determine which to use based on pressure.

$$u_{*,L}(p) = \begin{cases} u_{*,L}^{\text{shock}} & p > P_L \\ u_{*,L}^{\text{rare}} & p \le P_L \end{cases}$$

$$u_{*,R}(p) = \begin{cases} u_{*,R}^{\text{shock}} & p > p_R \\ u_{*,R}^{\text{rare}} & p \le p_R \end{cases}$$

Therefore, we need to solve for pressure, p_* using condition:

$$u_{*,L}(p_*) - u_{*,R}(p_*) = 0$$

This means that velocity and pressure do not jump across the contact wave.

For gamma law eos:

$$u_{*,s}(p_*) = \begin{cases} u_s \pm \dfrac{2c}{\gamma-1}\left[1 - \left(\dfrac{p_*}{p_s}\right)^{\frac{\gamma-1}{2\gamma}}\right] & \text{for } p_* \le p_s \\ u_s \pm c\left[\dfrac{2}{\gamma(\gamma-1)}\right]^{1/2} \dfrac{1 - \frac{p_*}{p_s}}{\left(\frac{p_*}{p_s}\frac{\gamma+1}{\gamma-1} + 1\right)^{1/2}} & \text{for } p_* > p_s \end{cases}$$

rarefaction

Shock

Solve for p_* such that

$$u_{*,L}(p_*) - u_{*,R}(p_*) = 0$$