ASEN 5519 HW #2 | 10/06/20 | LAOUAR, ZAKARIYA

1) • $H_{head}: x^2+y^2-r_1^2 \leq 0$

• $H_{eye}: -\left((x-x_2)^2+(y-y_2)^2-r_2^2\right) \leq 0$

• $H_{mouth}: \left(\frac{x}{3}-y\right) \leq 0$ upper

$-\frac{x}{3}-y \leq 0$ lower

• $H_{hat}: -\left(\frac{-1}{(y_3/x_3)}(x-x_3)+y_3-y\right) \leq 0$ lower trapezoid

$+\left(\frac{-1}{(y_4/x_4)}(x-x_4)+y_4-y\right) \leq 0$ upper trapezoid

$+\left(\frac{-1}{(y_5/x_5)}(x-x_5)+y_5-y\right) \leq 0$ right trapezoid

$-\left(\frac{-1}{(y_6/x_6)}(x-x_6)+y_6-y\right) \leq 0$ left trapezoid

$\left(-(x+r_1)/2-y-(r_1+0.5)\right) \leq 0$ top pointy part

$-\left(-2(x+(r_1+0.3))-y-(r_1+0.5)\right) \leq 0$ bottom pointy part

Semi-Algebraic Set:

$$\boxed{\begin{aligned}\text{Pacman} = {} & H_{head} \cap H_{eye} \cap H_{mouth} \cap H_{hat_1} \cap H_{hat_2} \cap H_{hat_3} \cap H_{hat_4} \\ & \cap H_{hat_5} \cap H_{hat_6}\end{aligned}}$$

2. a) Determine matrix $R(\alpha, \beta, \gamma)$. $R(\alpha,\beta,\gamma) = R_z(\gamma)R_y(\beta)R_z(\alpha)$

$$R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$\Rightarrow R(\alpha,\beta,\gamma) = R_z(\gamma) \cdot \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= R_z(\gamma) \cdot \begin{bmatrix} \cos\beta\cos\alpha & -\cos\beta\sin\alpha & \sin\beta \\ \sin\alpha & \cos\alpha & 0 \\ -\sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\gamma & \sin\beta\cos\gamma \\ \cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\beta\sin\gamma \\ -\sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{bmatrix}$$

2.cont. b) show that $R(\alpha,\beta,\gamma) = R(\alpha-\pi, -\beta, \gamma-\pi)$

$$\cos(\alpha-\pi) = -\cos(\alpha), \quad \sin(\alpha-\pi) = -\sin(\alpha)$$
$$\cos(\gamma-\pi) = -\cos(\gamma), \quad \sin(\gamma-\pi) = -\sin(\gamma)$$
$$\cos(-\beta) = \cos(\beta), \quad \sin(-\beta) = -\sin(\beta)$$

$\Rightarrow R(\alpha-\pi, -\beta, \gamma-\pi) =$

$$\begin{bmatrix} -\cos\alpha\cdot\cos\beta\cdot -\cos\gamma - (\sin\alpha\cdot -\sin\gamma) & \sin\alpha\cos\beta\cdot -\cos\gamma - (-\cos\alpha\cdot -\sin\gamma) & -\sin\beta\cdot -\cos\gamma \\ (-\cos\alpha)(\cos\beta)(-\sin\gamma) + (-\sin\alpha)(-\cos\gamma) & -(-\sin\alpha)(\cos\beta)(-\sin\gamma) + (-\cos\alpha)(-\cos\gamma) & (-\sin\beta)(-\sin\gamma) \\ -(-\sin\beta)(-\cos\alpha) & (-\sin\beta)(-\sin\alpha) & \cos\beta \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\gamma & \sin\beta\cos\gamma \\ \cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\beta\sin\gamma \\ -\sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{bmatrix} = R(\alpha,\beta,\gamma)}$$

2.cont. c) $R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$ $\quad S_\beta = \sin(\beta)$, $C_\beta = \cos(\beta)$

if $R_{33} = 1$, then $\beta = 0$ and $R = \begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{\alpha+\gamma} & -S_{\alpha+\gamma} & 0 \\ S_{\alpha+\gamma} & C_{\alpha+\gamma} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We can't uniquely solve for α and γ since only the sum, $\alpha+\gamma$ is represented in this case.

If $R_{33} \neq 1$ or -1 $\Rightarrow$ R_{31} and R_{32} cant both be zero and thus α can be defined

↳ Then $\cos(\beta) = R_{33}$ $\Rightarrow$ $\sin^2(\beta) + \cos^2(\beta) = 1$ $\Rightarrow$ $\sin(\beta) = \pm\sqrt{1 - R_{33}^2}$

$\Rightarrow \beta = \tan^{-1}\left(\frac{\sin(\beta)}{\cos(\beta)}\right)$

$\Rightarrow \alpha = \tan^{-1}\left(\frac{R_{32}}{R_{31}}\right)$, $\gamma = \tan^{-1}\left(\frac{R_{23}}{R_{13}}\right)$

however if $R_{33} = \pm 1 \Rightarrow \beta = 0, \pi \Rightarrow$ only the sum $\alpha \pm \gamma$ can be defined

$\alpha + \gamma = \tan^{-1}\left(\frac{R_{21}}{R_{11}}\right)$, there infinitely many solutions

3. a) $(\theta_1, \theta_2, \theta_3) = \left(\frac{\pi}{4}, \frac{\pi}{2}, -\frac{\pi}{6}\right) \Rightarrow (a,b,c) = ?$

Pose of A_1 relative to the global frame:

$$T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pose of A_2 relative to A_1's frame:

$$T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & a_1 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pose of A_3 relative to A_2's frame:

$$T_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & a_2 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Point a:

$$\begin{bmatrix} x_a \\ y_a \\ 1 \end{bmatrix} = T_1 \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 2.8284 \\ 2.8284 \end{matrix}} \\ 1 \end{bmatrix}$$

Point b:

$$\begin{bmatrix} x_b \\ y_b \\ 1 \end{bmatrix} = T_1 T_2 \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} -0.7071 \\ 12.0208 \end{matrix}} \\ 1 \end{bmatrix}$$

Point c:

$$\begin{bmatrix} x_c \\ y_c \\ 1 \end{bmatrix} = T_1 T_2 T_3 \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} -3.2953 \\ 19.7482 \end{matrix}} \\ 1 \end{bmatrix}$$

3. b) $d = (0, 4)$

System of equations: $(a_1, a_2 = 8, a_3 = 9)$

$$x_3 = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) + a_3 \cos(\theta_1 + \theta_2 + \theta_3)$$
$$y_3 = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) + a_3 \sin(\theta_1 + \theta_2 + \theta_3)$$
$$\phi = \theta_1 + \theta_2 + \theta_3$$

* Shorthand Notation: $C_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$, $C_\phi = \cos(\phi)$

$$x_2 = x_3 - a_3 C_\phi$$
$$y_2 = y_3 - a_3 S_\phi$$
$$\cos\theta_2 = \frac{x_2^2 + y_2^2 - a_1^2 - a_2^2}{2 a_1 a_2} \Rightarrow \boxed{\theta_2 = \cos^{-1}\left(\frac{x_2^2 + y_2^2 - a_1^2 - a_2^2}{2 a_1 a_2}\right)}$$

$$x_2 = a_1 C_1 + a_2 C_{12}, \quad y_2 = a_1 S_1 + a_2 S_{12}$$

$$\Rightarrow x_2 = C_1(a_1 + a_2 C_2) - S_1(a_2 S_2)$$
$$y_2 = C_1(a_2 S_2) + S_1(a_1 + a_2 C_2)$$

$$\Rightarrow \cos(\theta_1) = \frac{(a_1 + a_2\cos(\theta_2))x_2 + a_2 \sin\theta_2 \, y_2}{x_2^2 + y_2^2}$$

$$\Rightarrow \sin(\theta_1) = \frac{(a_1 + a_2\cos(\theta_2))y_2 - a_2 \sin(\theta_2)\, x_2}{x_2^2 + y_2^2}$$

$$\Rightarrow \tan(\theta_1) = \frac{\sin(\theta_1)}{\cos(\theta_1)} \Rightarrow \boxed{\theta_1 = \tan^{-1}\left(\frac{\sin(\theta_1)}{\cos(\theta_1)}\right)}$$

$$\Rightarrow \boxed{\theta_3 = \phi - (\theta_1 + \theta_2)}$$

There are infinitely many configurations for this 3-link robot arm.

One possible configuration, setting our constraint of $\boxed{\phi = \frac{3\pi}{2}}$

$\Rightarrow$ $\boxed{\theta_1 = 0.8491 \text{ rads}, \ \theta_2 = 1.4455 \text{ rads}, \ \theta_3 = 2.4184 \text{ rads.}}$

Scanned with CamSc

4.

a) One train on one track can be represented by $\mathbb{R}^1$
The second can also be represented by $\mathbb{R}^1$

$$q = (\underbrace{x_1}_{\mathbb{R}^1}, \underbrace{x_2}_{\mathbb{R}^1})$$

thus the c-space can be represented as

$$\boxed{\mathbb{R}^1 \times \mathbb{R}^1 \quad \text{with dimension} = 2}$$

b) A spacecraft translating in 2D is represented by $\mathbb{R}^2$
A spacecraft rotating in 2D is represented by S

thus $q = (\underbrace{x, y}_{\mathbb{R}^2}, \underbrace{\theta}_{S}) \Rightarrow$ c-space represented as $\boxed{\mathbb{R}^2 \times S \quad \text{with dimension} = 3}$

c) Having two translating and rotating robots each with 2D translation & 1D rotation → would make $q = (x_1, y_1, \theta_1, x_2, y_2, \theta_2)$
we would have 4 translation dimensions to make $\mathbb{R}^4$
and 2 rotation dimensions to make $S^1 \times S^1 = T^2$

thus the c-space is represented as

$$\boxed{\mathbb{R}^4 \times T^2 \quad \text{with dimension} = 6}$$

d) Since the two robots are connected rigidly, they cant rotate or translate individually making it act like one planar, rotating, translating robot.

$\Rightarrow$ thus the c-space is represented by

$q = (x, y, \theta) \Rightarrow \boxed{\mathbb{R}^2 \times S^1 \quad \text{with dimenson} = 3}$

Scanned with CamSc

4.cont. e) The rod can translate in 3D, represented by $\mathbb{R}^3$
The rod can only rotate in 2 dimensions
however, represented by $S_1 \times S_1 = T_2$
thus $q = (\underbrace{x, y, z}_{\mathbb{R}^3}, \underbrace{\theta_1, \theta_2}_{T^2})$

and c-space $\Rightarrow$ $\boxed{\mathbb{R}^3 \times T^2 \text{ with dimension } = 5}$

f) A spacecraft that can translate and rotate in 3D can be represented as a 3D special Euclidean group $SE(3)$. A 3-link robot arm can be represented as $S^1 \times S^1 \times S^1 = T^3$

The overall spacecraft has 9 DOF
with $q = (x_1, x_2, x_3, \theta_1, \theta_2, \theta_3, \alpha_1, \alpha_2, \alpha_3)$
and c-space $\Rightarrow$ $\boxed{SE(3) \times T^3 \text{ with dimension} = 9}$

g) 7 revolute joints =

$\boxed{\text{c-space} \Rightarrow S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1 = T^7 \text{ with dimension} = 7}$

6. $W \subseteq \mathbb{R}^n$

Subset $X \subset \mathbb{R}^n$ is called <u>convex</u> iff for any pair of points in X all points along the line segment that connects them are contained in X, i.e.,

(1) $(\lambda z_1 + (1-\lambda) z_2) \in Z \quad \forall x_1, x_2 \in Z, \quad \forall \lambda \in [0,1]$

We know that the robot represented by X and the obstacle in the workspace represented by Y are both <u>convex</u>. Let $Z = Y \ominus X$

$$\Rightarrow (\lambda(y_1 - x_1) + (1-\lambda)(y_2 - x_2)) \in Y \ominus X = Z$$

Distribute:

$$\Rightarrow \lambda y_1 - \lambda x_1 + y_2 - x_2 - \lambda y_2 + \lambda x_2 \in Y \ominus X$$

Group:

$$\Rightarrow \boxed{(\lambda y_1 + (1-\lambda) y_2) - (\lambda x_1 + (1-\lambda x_2)) \in Y \ominus X = Z}$$

This is in the form of a definitional convex shape subtracting a definitional convex shape

$\blacksquare$

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Housekeeping

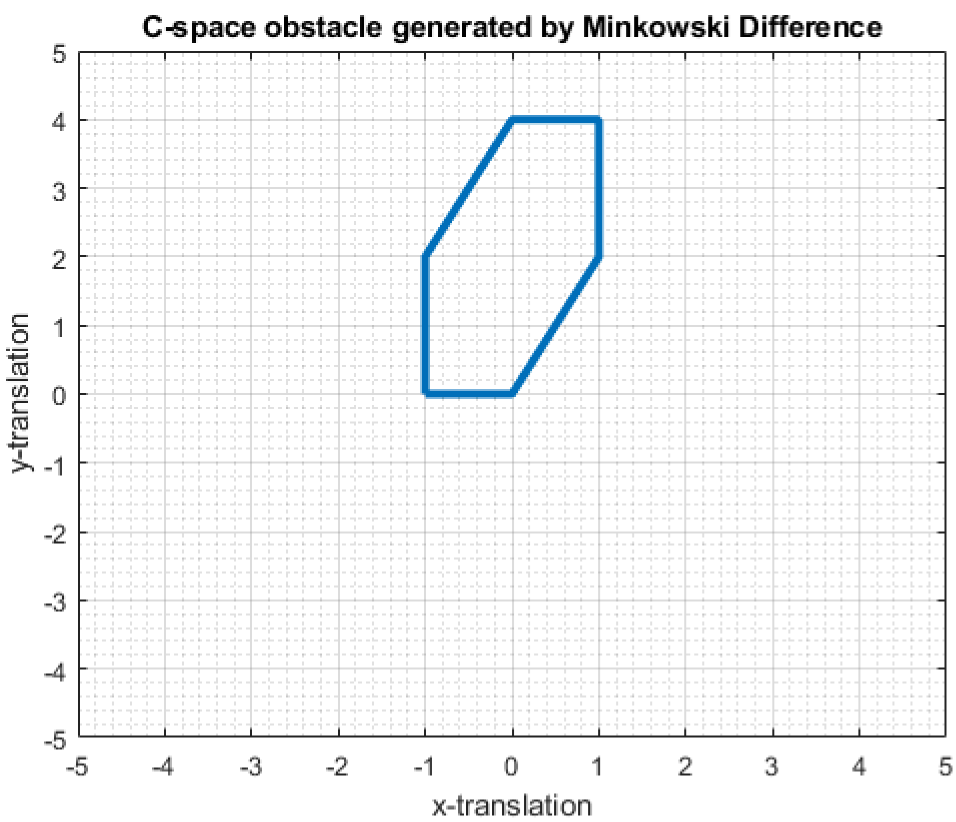
Problem 5a)

Load path from csv

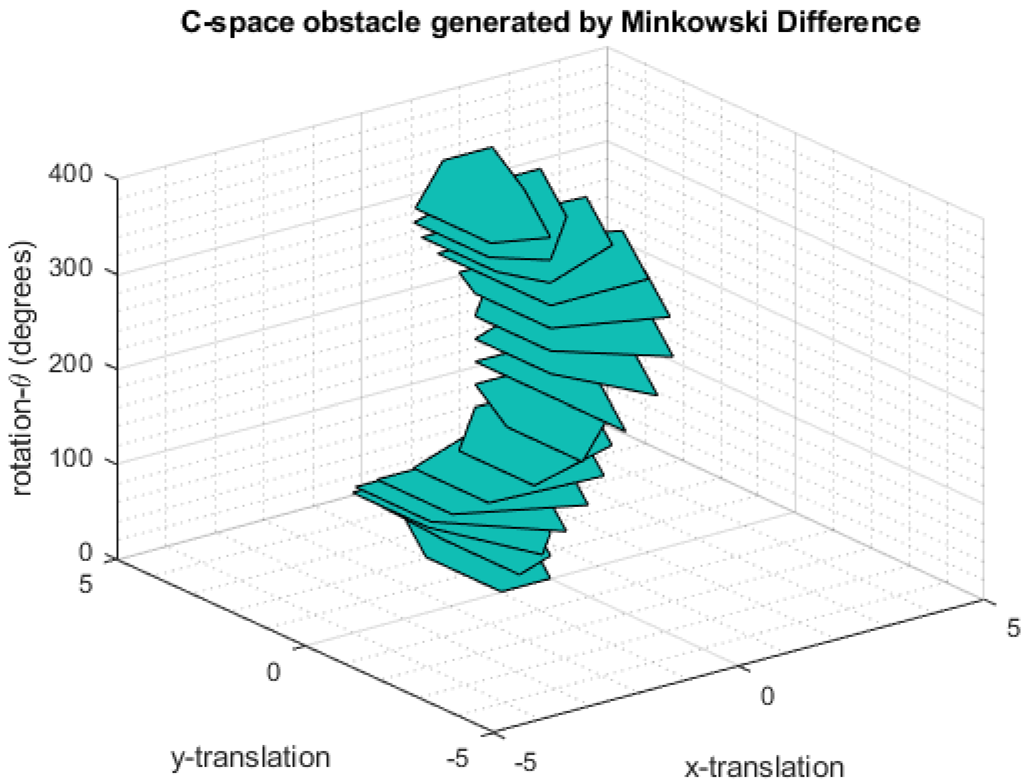
```
The vertices of the cspace obstacle are:

mat =

    -1     0
     0     0
     1     2
     1     4
     0     4
    -1     2
```



Problem 5b)



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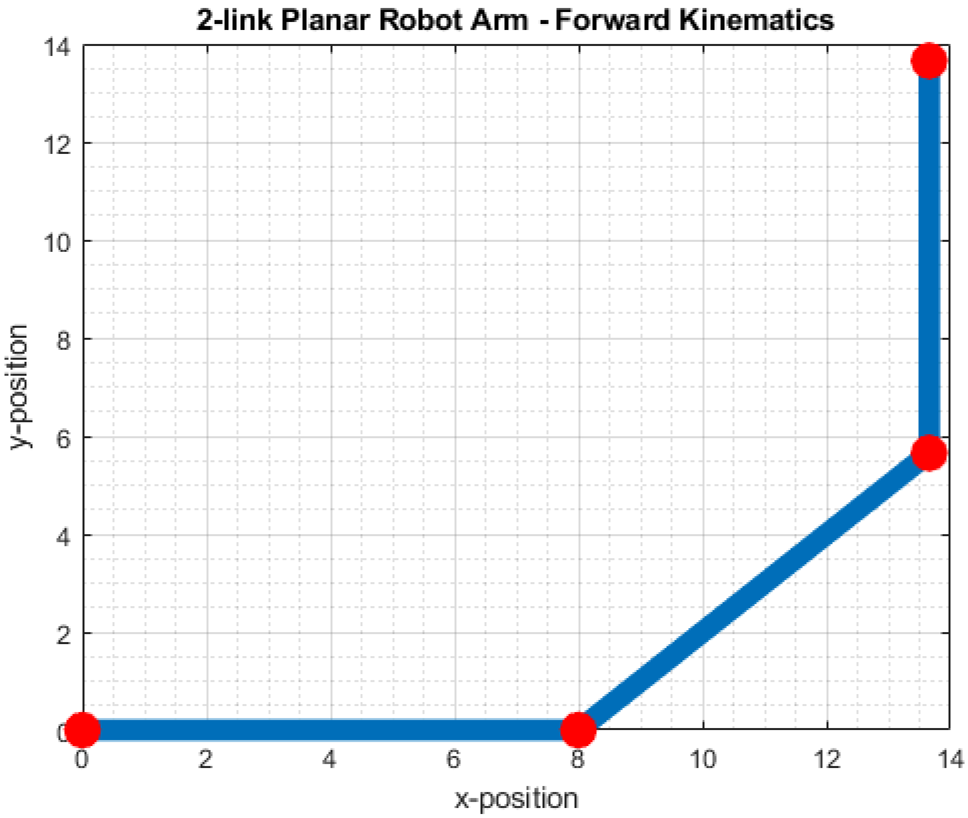
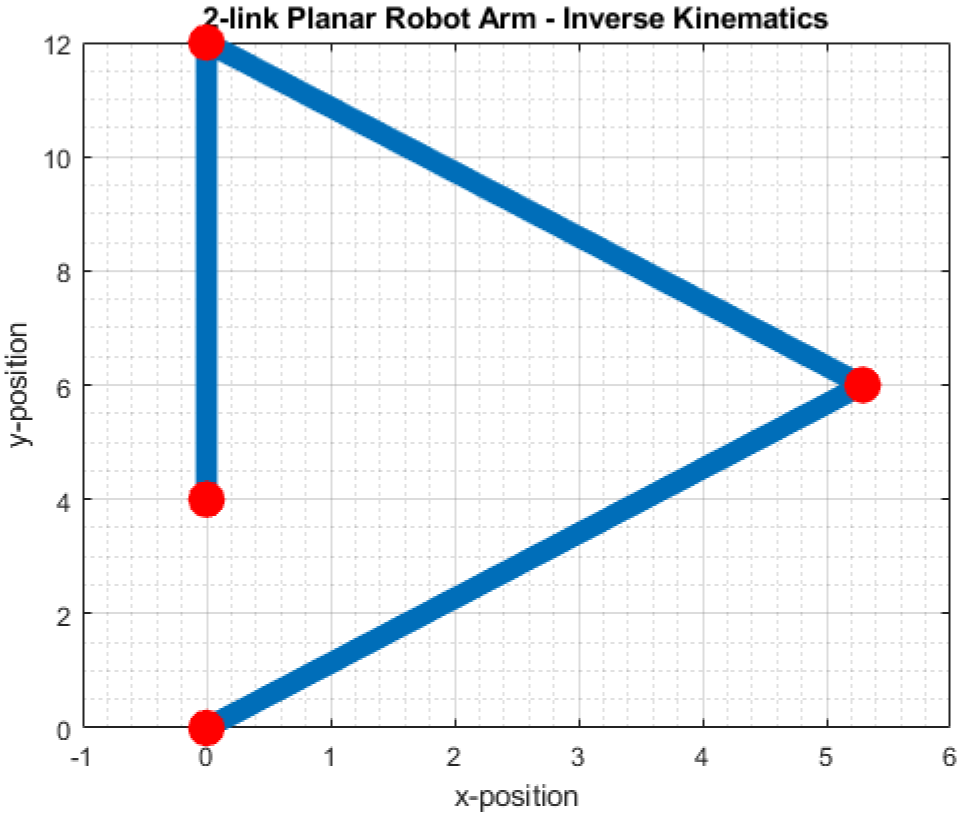
Housekeeping

Get Endpoint



Get Configuration

```
The final configuration of the robot arm is:
Theta1: 0.848062 rad, Theta2: 1.445468 rad, Theta3: 2.418858 rad
```



Functions

The final position of the endpoint is: (13.656854,13.656854)

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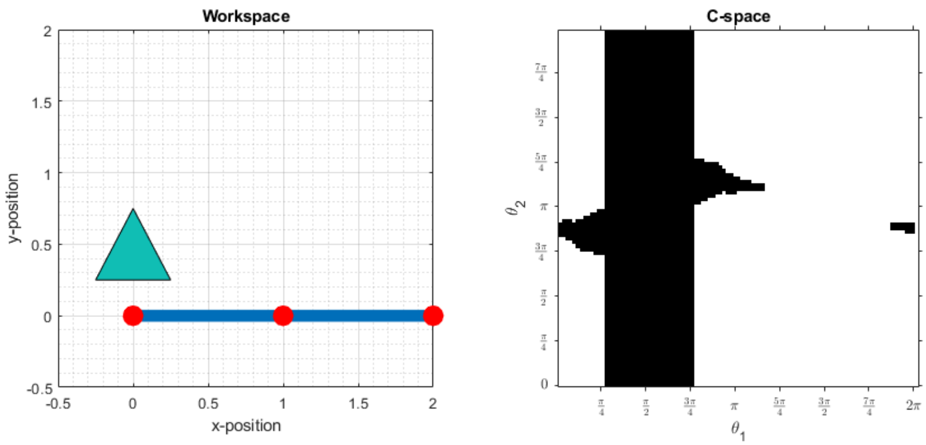
Housekeeping

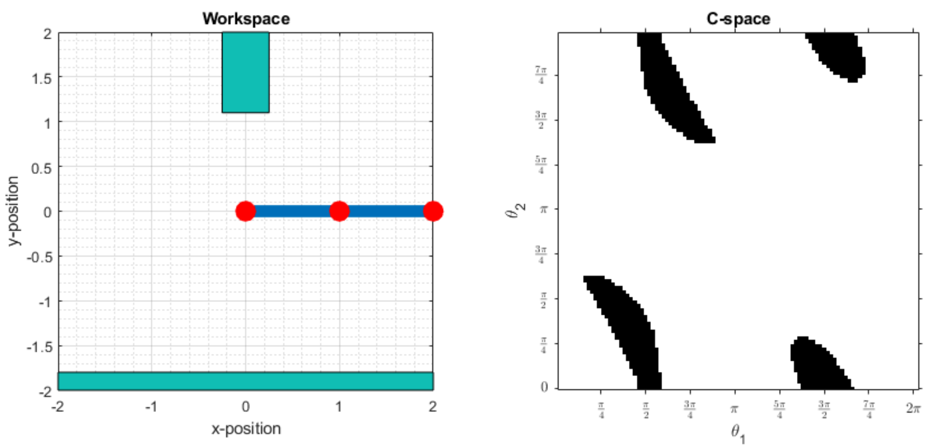
Case (a)

Plot 2-link robot



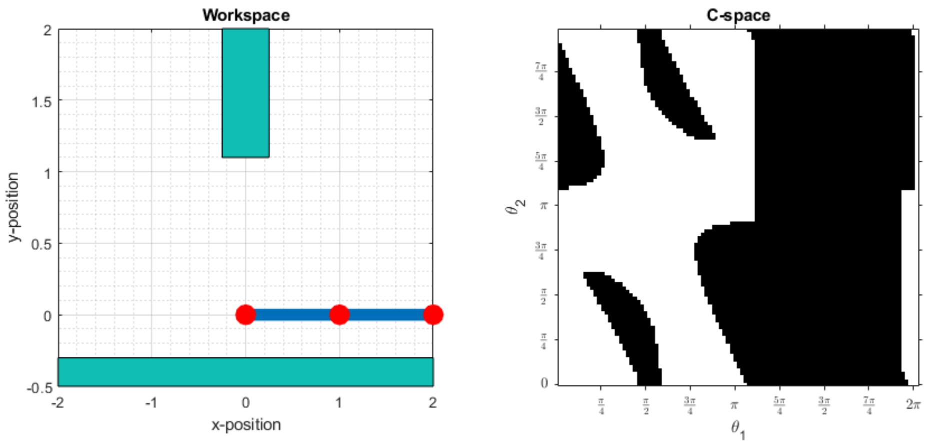
Case (b)

Plot 2-link robot



Case (c)

Plot 2-link robot



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