BM20A9200 Mathematics A – Exercise set 8

To be done by 6.–10.11.2023

Text in blue or red is not part of the problem or its solution. It's there as extra information to help you learn.

Exercise 1. Prove that for any integer $n \geq 2$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n} = \frac{n-1}{n}.$$

Solution.

Proof. Let $V(n)$ denote the equation in the claim.

Base case: We have

$$\frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{2-1}{2}$$

so $V(2)$ is true.

Induction hypothesis: $V(k)$ is true for some integer $k \geq 2$.

Induction step: Assume that the induction hypothesis is true and let's show that $V(k+1)$ is then true. We have

$$\begin{aligned}
&\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{((k+1)-1) \cdot (k+1)} \\
&\quad = \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k-1) \cdot k}\right) + \frac{1}{((k+1)-1) \cdot (k+1)} \\
&\quad = \frac{k-1}{k} + \frac{1}{((k+1)-1) \cdot (k+1)}
\end{aligned}$$

and we have

$$\frac{k-1}{k} + \frac{1}{((k+1)-1) \cdot (k+1)} = \frac{1}{k}\left(k - 1 + \frac{1}{k+1}\right) = \frac{1}{k}\frac{k^2-1+1}{k+1}$$

which is equal to $k/(k+1) = ((k+1)-1)/(k+1)$. Hence if the induction hypothesis holds it is true that $V(k+1)$.

By induction the claim is true. $\square$

Exercise 2. Is the following proof correct? If yes explain why. If not explain where is the mistake.

Claim. In a group of $n \geq 1$ people everyone has the same name.

Proof. Let $V(n) =$ "in a group of n people everybody has the same name.".

Base case: $V(1)$ is true because in a group of one person everybody indeed has the same name.

Induction hypothesis: $V(k)$ is true for some $k \geq 1$.

Induction step: Let the induction hypothesis be true. Consider a group of $k+1$ people. Arrange them in a line. Then the first k people form a group and have the same name by the induction hypothesis. Similarly the last k people have the same name. Because of the overlap they all have the same name.

The base case and the induction step are true. Hence everybody in any group of $n \geq 1$ people have the same name. □

Solution. The proof is incorrect (otherwise we'd all be called Emilia). The base case $k = 1$ is correct, indeed if there's only one person, then everybody (=that one person) has the same name as that one person.

The mistake is in the induction step. Consider what happens for $V(2)$. The induction hypothesis is true for $k = 1$, because everybody in a group of 1 person has the same name. But the "proof" continues:

> Consider a group of $1 + 1$ people. Arrange them in a line. Then the first 1 people form a group and have the same name by the induction hypothesis. Similarly the last 1 people have the same name. Because of the overlap they all have the same name.

There is no overlap! When there are $k+1 = 2$ people, the first $k = 1$ people and last $k = 1$ people don't overlap. So the first person and last person can have different names, and the induction step breaks down.

Exercise 3. The **Fibonacci numbers** F_n, $n \geq 0$ integers, are such that each of them is the sum of the preceding two, starting from 0 and 1. That is, $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The first few Fibonacci numbers F_n are:

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

Prove that for all $n \in \mathbb{N}$

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

Solution.

Proof. Denote by $V(n)$ the claim.

Base case: $V(0)$ says that $F_0 = F_2 - 1$. We have $F_0 = 0$ and $F_2 = 1$, so indeed $V(0)$ is true.

Induction hypothesis: $V(k)$ is true for some $k \in \mathbb{N}$.

Induction step: Assume that the induction hypothesis is true. We will show that $V(k+1)$ is true then. By the induction hypothesis we have

$$\begin{aligned} F_0 + F_1 + F_2 + \cdots + F_{k+1} &= \big(F_0 + F_1 + F_2 + \cdots + F_k\big) + F_{k+1} \\ &= F_{k+2} - 1 + F_{k+1}. \end{aligned}$$

But recall that by the definition of the Fibonacci nubmers we have $F_{k+2} + F_{k+1} = F_{k+3}$. Hence

$$F_0 + F_1 + F_2 + \cdots + F_{k+1} = F_{k+3} - 1 = F_{(k+1)+2} - 1.$$

This is $V(k+1)$.

The base case and induction steps are true. Hence the claim is true for all natural numbers. $\square$

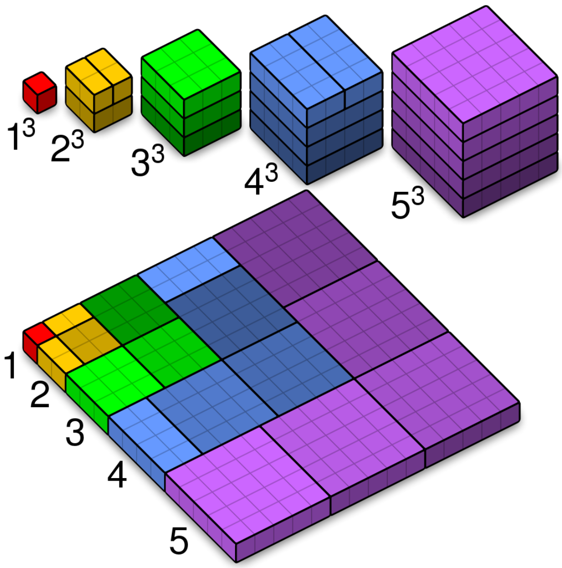
Exercise 4. Prove Nicomachus's theorem:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$$

for all positive integers n. **Hint:** Use the result $1 + 2 + \ldots + n = n(n+1)/2$ for all positive integers n.

Solution.

Proof. There are several proofs of this fact. Some are "proofs without words", like the one below



However not everyone can be convinced by such a picture. For example, what about $n = 6$ or $n = 7$? Or what is the logic of splitting the cubes and why do they fit to form the big square? If you claim this to be a proof, you need to be ready to answer this kind of questions to whoever is asking. If you answer with a picture in the exam, you must write down the answers to any questions I might have... Remember, you need to explain your answers; even if your answer is a picture.

Proof by induction:

Base case $n = 1$: $1^3 = 1^2$ is obviously true.

Induction hypothesis: Assume that $1^3 + 2^3 + \ldots + k^3 = (1 + 2 + \ldots + k)^2$ for some positive integer k.

Induction step: Assume that the induction hypothesis is true. We will show that

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \big(1 + 2 + 3 + \ldots + k + (k+1)\big)^2 \qquad (1)$$

is true. The left-hand side of (1) is equal to

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 &= \left(1^3 + 2^3 + 3^3 + \ldots + k^3\right) + (k+1)^3 \\ &= (1 + 2 + 3 + \ldots + k)^2 + (k+1)^3 \end{aligned} \quad (2)$$

because of the induction hypothesis. By the hint this is equal to

$$\ldots = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = \left(\frac{k^2}{4} + k + 1\right)(k+1)^2. \quad (3)$$

Let's investigate the right-hand side of (1) next. By the hint it is equal to

$$\begin{aligned} \left(1 + 2 + 3 + \ldots + k + (k+1)\right)^2 &= \left(\frac{k(k+1)}{2} + (k+1)\right)^2 \\ &= \frac{k^2(k+1)^2}{4} + 2 \cdot \frac{k(k+1)}{2} \cdot (k+1) + (k+1)^2 \\ &= \frac{k^2}{4} \cdot (k+1)^2 + k \cdot (k+1)^2 + (k+1)^2 \\ &= \left(\frac{k^2}{4} + k + 1\right)(k+1)^2 \end{aligned} \quad (4)$$

which is equal to the left-hand side (3). The induction step shows that if the claim is true for $n = k$ then we can deduce it to be true for $n = k + 1$. We have shown it to be true for $n = 1$. By induction the claim is true for all integers $n \geq 1$. $\square$

Exercise 5. Recall the definition of integer divisibility.

a) Show that $5 \mid 75$, $-3 \mid 51$ and $55 \mid 0$ by definition.

b) Find all $n \in \mathbb{Z}$ such that $n \mid (3n + 2)$. Remember negative solutions too!

Solution. We have $a \mid b$ if and only if there is $k \in \mathbb{Z}$ such that $b = k \cdot a$.

a) $5 \mid 75$ because $75 = 15 \cdot 5$. We have $-3 \mid 51$ because $51 = (-17) \cdot (-3)$. We have $55 \mid 0$ because $0 = 0 \cdot 55$.

b) We have $n \mid (3n + 2)$ if and only if $3n + 2 = k \cdot n$ for some $k \in \mathbb{Z}$. This is equivalent to $2 = (k - 3) \cdot n$ for some integer k, and this is equivalent to $2 = k' \cdot n$ for some $k' \in \mathbb{Z}$. The number 2 can be written as a product of two integers only in the following ways:

$$2 = 1 \cdot 2, \qquad 2 = 2 \cdot 1, \qquad 2 = (-1) \cdot (-2), \qquad 2 = (-2) \cdot (-1).$$

This gives the solution set $\{2, 1, -2, -1\}$.

Exercise 6. Determine the greatest common divisor of 2023 and 1428, and find some integers x and y such that $2023x - 1428y = \gcd(2023, 1428)$.

Solution. Let's use the Euclidean algorithm for finding the greatest common divisor:

$$\begin{aligned} 2023 &= 1 \cdot 1428 + 595 \\ 1428 &= 2 \cdot 595 + 238 \\ 595 &= 2 \cdot 238 + 119 \\ 238 &= 2 \cdot 119 + 0 \end{aligned}$$

This shows that $\gcd(2023, 1418) = 119$. Let's solve $2023x - 1428y = 119$ next.

From the calculation above we get

$$\begin{aligned} 119 &= 595 - 2 \cdot 238 \\ &= 595 - 2 \cdot (1428 - 2 \cdot 595) = 5 \cdot 595 - 2 \cdot 1428 \\ &= 5 \cdot (2023 - 1 \cdot 1428) - 2 \cdot 1428 = 5 \cdot 2023 - 7 \cdot 1428 \end{aligned}$$

Hence a solution is $(x, y) = (5, 7)$. All solutions are given by

$$\begin{aligned} x &= 5 + \frac{1428}{119}k = 5 + 12k \\ y &= 7 + \frac{2023}{119}k = 7 + 17k \end{aligned}$$

but the problem did not ask for all solutions.