BM20A9200 Mathematics A – Exercise set 5

To be done by 9.–13.10.2023

Text in blue or red is not part of the problem or its solution. It's there as extra information to help you learn.

Exercise 1. Are the following statements true for all sets A, B, C and D?

a) $A \times (B - D) \subseteq (A \times B) - (C \times D)$.

b) $A \times (B - D) = (A \times B) - (C \times D)$.

Solution.

a) The claim is true. Assume that $x \in A \times (B - D)$. Then $x = (a, b)$, where $a \in A$ and $b \in B - D$. Because $b \in B - D$, the definition of set difference implies $b \in B$ and $b \notin D$. Hence $a \in A$ and $b \in B$, so the definition of the set product implies $x = (a, b) \in A \times B$. On the other hand $b \notin D$, so $x = (a, b) \notin C \times D$. Now $x \in A \times B$ and $x \notin C \times D$, so the definition of the set difference gives $x \in (A \times B) - (C \times D)$.

   In conclusion $A \times (B - D) \subseteq (A \times B) - (C \times D)$ is true for any sets A, B, C and D.

b) Let's show that the claim is false by providing a counter example. Select $A = B = \{1, 2\}$ and $C = D = \{2, 3\}$. Now $A \times (B - D) = \{(1, 1), (2, 1)\}$ and $(A \times B) - (C \times D) = \{(1, 1), (2, 1), (1, 2)\}$. So in this particular case $A \times (B - D) \neq (A \times B) - (C \times D)$. So the statement cannot be true for all sets A, B, C ja D.

Exercise 2. Prove by contrapositive the following claim for sets A and B:

If $(A \cup B) \setminus B = A$ then $A \cap B = \emptyset$.

Solution. Let's prove the claim by contrapositive. Let's assume that $A \cap B \neq \emptyset$, meaning that there is an element $x \in A \cap B$. Then $x \in A$ and $x \in B$. On the other hand, since $x \in B$ then by the definition of set difference we have $x \notin (A \cup B) \setminus B$.
Let's recall what we know about the element x:

- We have $x \in A$, but
- $x \notin (A \cup B) \setminus B$.

A set is determined by its elements. Since there is an element that's in A but not in $(A \cup B) \setminus B$ that means that these sets are different. This is the negation of the first proposition of the implication in the claim. The claim has thus been proven by contrapositive.

Exercise 3. Denote $X = \{0, 1, 2, 3\}$. Which of the following rules define a function? Why?

a) $f\colon X \to X$, $f(n) = n^2 + n \cdot (-1)^{n+1}$

b) $g\colon X \to X$, $g(x) = 3$

c) $\sigma\colon \mathbb{R} \to \mathbb{R}$, $\sigma(x) = \sqrt{x-3}$

d) $\tau\colon \mathbb{Q} \to \mathbb{Q}$, $\tau(x) = \dfrac{2a-b}{a^2+2b^2}$, when the rational number x is written in the form $x = \dfrac{a}{b}$, where $a, b \in \mathbb{Z}$.

Info: The letters σ and τ are the Greek letters sigma and tau.

Hint: Recall that the definition of a function/mapping (Jouni's notes PDF number 4, page 1) has actually two conditions. We associate to every element in the domain (first set) an element of the codomain (second set). And we must not associate more than one element of the codomain to any element of the domain.

Solution.

a) The rule f is not a mapping, because according to it we would have $f(3) = 3^2 + 3 \cdot (-1)^4 = 12$ but 12 is not an element of the codomain X.

b) The rule g is a mapping, because it associates of every element of the domain a single elements of the codomain. In fact, it associates the number 3 to every element.

c) The rule σ is not a mapping because the square root function is not defined for negative real numbers. In other words, if x is a real number with $x < 3$ then the rule σ does not tell us which element of the codomain $\mathbb{R}$ should we associate to x.

d) The rule τ is not a mapping because for example the fraction $1/3$ is the same rational number as the fraction $2/6$ but

$$\tau\left(\frac{1}{3}\right) = \frac{2-3}{1^2 + 2 \cdot 3^2} = -\frac{1}{19}$$

and

$$\tau\left(\frac{2}{6}\right) = \frac{4-6}{2^2 + 2 \cdot 6^2} = -\frac{1}{38}.$$

Therefor the rule τ associates the two numbers $-1/19$ and $-1/38$ of the codomain to the number $1/3$ of the domain. For this reason, τ is not a function.

Exercise 4. Consider the function $g\colon \mathbb{R} \to \mathbb{R}$ for which $g(x) = 4 - x^2$ for all $x \in \mathbb{R}$.
Is the mapping g an injection? Is it a surjection?
Explain your answer in detail using the definitions of injections and surjections.

Solution. Let's show that g is not an injection. Let's investiage the images of the numbers $1 \in \mathbb{R}$ and $-1 \in \mathbb{R}$:

$$g(1) = 4 - 1^2 = 4 - 1 = 3$$

and

$$g(-1) = 4 - (-1)^2 = 4 - 1 = 3$$

Hence $g(1) = 3 = g(-1)$ but $1 \neq -1$. Therefore g is not an injection.

Let's show that g is not a surjection by showing that no elements of the domain map to some element of the codomain. Let's show that no elements map to $y = 5$. Let's prove that indirectly: assume that some element $x \in \mathbb{R}$ would map to $y = 5$, namely $g(x) = 5$. This means that the equation $4 - x^2 = 5$ has a solution $x \in \mathbb{R}$. This equation is equivalent to $x^2 = -1$ though. The squares of real numbers are always non-negative, we have a contradiction, so the original claim (there is no $x \in \mathbb{R}$ such that $g(x) = 5$) is true. Therefore g is not a surjection.

Info: An alternate way to prove this is the following. Let $x \in \mathbb{R}$. We know that $x^2 \geq 0$. This implies that $4 - x^2 \leq 4$, so $g(x) \leq 4$ for all $x \in \mathbb{R}$. If $y \in \mathbb{R}$ and $y > 4$ then there is no $x \in \mathbb{R}$ such that $g(x) = y$. Therefore g is not a surjection.

Exercise 5. Solve x in the following equations:

(a) $2^{3x-2} = 16$

(b) $3 \log_6 x = 21$

(c) $\log_2(3x - 4) = 5$

(d) $\log_4 x + \log_4(x - 6) = 2$

Solution. **Info:** recall the definition and properties of logarithms. For example $\log_9 x = 2 \Leftrightarrow x = 9^2$. Also $\log_2 x + \log_2 y = \log_2(xy)$.

We will use multiplications by non-zero numbers, divisions, and raising a number to a power. All these are reversible operations under suitable subsets of the real numbers (logarithm has to have a positive input) so we will have equivalences between the equations.

(a) This equation is defined for any $x \in \mathbb{R}$.

$$\begin{aligned}
&x \in \mathbb{R} \wedge 2^{3x-2} = 16 \\
&\Leftrightarrow x \in \mathbb{R} \wedge \log_2(2^{3x-2}) = \log_2 16 \\
&\Leftrightarrow x \in \mathbb{R} \wedge 3x - 2 = 4 \\
&\Leftrightarrow x \in \mathbb{R} \wedge 3x = 6 \\
&\Leftrightarrow x = 2.
\end{aligned}$$

(b) In here x must be positive. It makes no sense to look for solutions when $x \leq 0$ because the logarithm is not defined then.

$$\begin{aligned} & x \in \mathbb{R}_+ \ \wedge \ 3\log_6 x = 21 \\ \Leftrightarrow\ & x \in \mathbb{R}_+ \ \wedge \ \log_6 x = 7 \\ \Leftrightarrow\ & x \in \mathbb{R}_+ \ \wedge \ 6^{\log_6 x} = 6^7 \\ \Leftrightarrow\ & x \in \mathbb{R}_+ \ \wedge \ x = 6^7 \\ \Leftrightarrow\ & x = 279\,936. \end{aligned}$$

(c) In here we must look at the solution in the set where $3x - 4 > 0$, i.e. $x > 4/3$.

$$\begin{aligned} & x \in (4/3, \infty) \ \wedge \ \log_2(3x-4) = 5 \\ \Leftrightarrow\ & x \in (4/3, \infty) \ \wedge \ 2^{\log_2(3x-4)} = 2^5 \\ \Leftrightarrow\ & x \in (4/3, \infty) \ \wedge \ 3x - 4 = 32 \\ \Leftrightarrow\ & x \in (4/3, \infty) \ \wedge \ 3x = 36 \\ \Leftrightarrow\ & x = 12. \end{aligned}$$

(d) Here we are looking for solutions where $x > 0$ and $x - 6 > 0$. Therefore in $(6, \infty)$.

$$\begin{aligned} & x \in (6, \infty) \ \wedge \ \log_4 x + \log_4(x-6) = 2 \\ \Leftrightarrow\ & x \in (6, \infty) \ \wedge \ \log_4(x(x-6)) = 2 \\ \Leftrightarrow\ & x \in (6, \infty) \ \wedge \ \log_4(x^2 - 6x) = 2 \\ \Leftrightarrow\ & x \in (6, \infty) \ \wedge \ 4^{\log_4(x^2-6x)} = 4^2 \\ \Leftrightarrow\ & x \in (6, \infty) \ \wedge \ x^2 - 6x = 16 \\ \Leftrightarrow\ & x \in (6, \infty) \ \wedge \ x^2 - 6x - 16 = 0 \end{aligned}$$

The second degree polynomial has solutions $x = 8$ and $x = -2$ in $\mathbb{R}$, but $x = -2$ is not in the set where we are looking for solutions. Hence $x = 8$ is the one and only solution here.

Exercise 6. Prove:

$\log_6(15)$ is irrational.

Hint: You probably need the fact that *if n is odd, then n^k is odd for any $k \in \mathbb{N}$.*

Solution.

Proof. Let's prove the claim in directly. Therefore assume that $\log_6(15)$ is rational. This means that there are integers a and $b \neq 0$ such that $\log_6(15) = a/b$. Because $\log_6(15) > 0$ we can assume that a and b are both positive (instead of both being negative). Then $a = b\log_6(15)$ and hence $a = \log_6(15^b)$. Raising 6 to that power gives

$$6^a = 6^{\log_6(15^b)} = 15^b.$$

Since $a > 0$ is an integer, we have 6^a even. Hence 15^b must be too. Because 15 is odd and b a positive integer, 15^b is odd. A contradiction. Hence $\log_6(15)$ is irrational. $\square$