BM20A9200 Mathematics A – Exercise set 6

To be done by 16.–20.10.2023

Text in blue or red is not part of the problem or its solution. It's there as extra information to help you learn.

This set (nbr 6) starts having topics from the week after (lesson 7). You can try to solve them already by reading the background material, or wait till lesson nbr 7 which is on 17.10.

Exercise 1. Solve the equation

$$2\sqrt{1-x^2} = x - 1$$

in the set of real numbers.

Solution. If a real number x solves the equation then it will also solve the equation squared:

$$4(1-x^2) = (x-1)^2.$$

Expanding the parentheses, this gives $4 - 4x^2 = x^2 - 2x + 1$ which is equivalent to $5x^2 - 2x - 3 = 0$. Using the formula for solving second degree equations we get

$$x = \frac{2 \pm \sqrt{2^2 - 4 \cdot 5 \cdot (-3)}}{2 \cdot 5} = \frac{2 \pm 8}{10}.$$

Hence the solutions, if there are any, are $x = -3/5$ and/or $x = +1$.

Warning! We are not done yet. We have only shown that "if x is a solution, then $x = -3/5$ or $x = +1$." We don't know if the converse (if $x = \ldots$ then x is a solution) is true.

If $x = 1$ then $2\sqrt{1-x^2} = 0 = x - 1$, so $x = 1$ is a solution. If $x = -3/5$ then $2\sqrt{1-x^2} = 8/5$ and $x - 1 = -8/5$ so $x = -3/5$ is not a solution. Therefore the equation has exactly one solution: $x = 1$.

Exercise 2. Consider the mapping $f\colon \mathbb{R} \setminus \{2\} \to \mathbb{R}$ defined by $f(x) = x/(2-x)$.

a) Is it an injection? Prove it.

b) Is it a surjection? Prove it.

Solution.

a) To see if it's an injection, assume that $f(x) = f(y)$ for some x and y in its domain $\mathbb{R} \setminus \{2\}$. Then

$$x = \frac{y}{2-y}(2-x) = \frac{2y}{2-y} - \frac{y}{2-y}x.$$

This then implies

$$\left(1 + \frac{y}{2-y}\right)x = \frac{2y}{2-y}$$

which simplifies to

$$\frac{2}{2-y}x = \frac{2y}{2-y}.$$

Multiplying by $2-y$ and dividing by 2 gives $x = y$. Hence $f(x) = f(y)$ implies $x = y$, so f is an injection.

b) Let's first investigate what values can f get. Assume that $v \in \mathbb{R}$ is such that $f(x) = v$ for some $x \in \mathbb{R} \setminus \{2\}$. This means $x/(2-x) = v$. This implies the following:

$$\frac{x}{2-x} = v \Rightarrow x = 2v - vx \Rightarrow (1+v)x = 2v$$

and this has a solution x whenever $v \neq -1$. Let's show that f does not get the value $v = -1$.

Claim. $f(x) \neq -1$ for all $x \in \mathbb{R} \setminus \{2\}$.

Proof. To prove the claim suppose that $x/(2-x) = -1$ for some $x \in \mathbb{R} \setminus \{2\}$. This implies that $x = -(2-x) = x-2$. But this implies $0 = -2$, a contradiction. Hence the original claim is true by the indirect proof. □

Since f cannot obtain the value -1, it is not a surjection.

Exercise 3. Define the functions $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 5 - (x-3)^2$ and $h\colon \mathbb{R} \to \mathbb{R}$

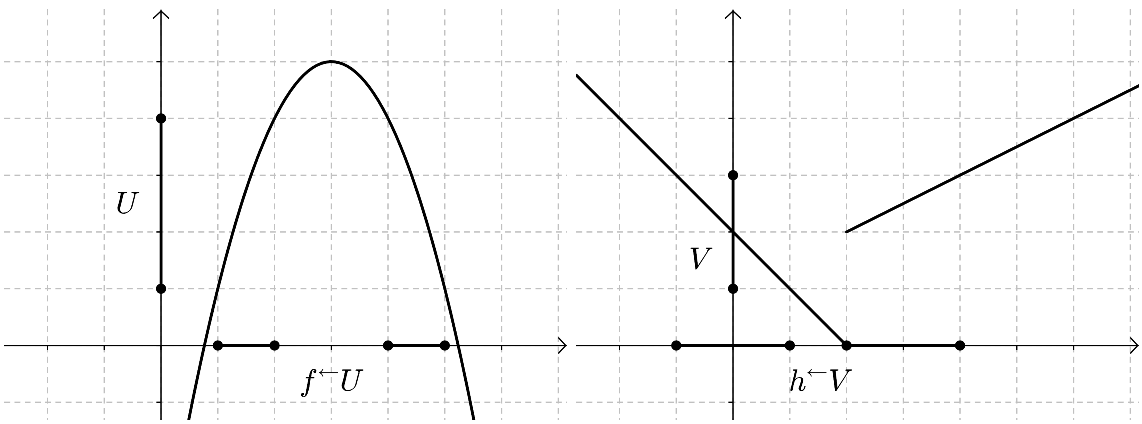
$$h(x) = \begin{cases} 2 - x, & \text{if } x < 2 \\ \frac{1}{2}x + 1, & \text{if } x \geq 2. \end{cases}$$

Draw the graphs of the functions f and h.

a) Denote $U = [1,4]$. Determine the preimage $f^{\leftarrow}(U)$ using the graph of the function f. Mark clearly U and $f^{\leftarrow}(U)$.

b) Denote $V = [1,3]$. Determine the preimage $h^{\leftarrow}(V)$ using the graph of h. Mark clearly V and $h^{\leftarrow}(V)$.

Solution. The graphs of f and h are below.



Graph of f Graph of h

a) By definition $f^{\leftarrow}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$. Looking at points on the graph whose y-coordinate is at least 1 and at most 4 we see that there are two such curves on the parabola. The x-coordinates of all the points on the curves form $f^{\leftarrow}(U)$. The graph tells us that $f^{\leftarrow}(U) = [1,2] \cup [4,5]$.

b) Be definition $h^{\leftarrow}(V) = \{x \in \mathbb{R} \mid h(x) \in V\}$. Looking first at the y-coordinates of points on the graph, then their x-coordinates like in a), we see that $h^{\leftarrow}(V) = [-1, 1] \cup [2, 4]$.

Exercise 4. Find a bijection between the sets $\{1, 2, 3, 4, \ldots\}$ and $\{8, 13, 18, 23, \ldots\}$ where the next number is the previous plus 5.

Solution. Such a bijection will need to increase the value of its output by 5 when the input increases by 1. Let's try something of the form $f(n) = 5n + a$ for some number a. To determine a we can decide to map 1 to 8, so $8 = f(1) = 5 + a$ which gives $a = 3$. Our try is $f(n) = 5n + 3$.

Let's verify that it is of the right form.

- It maps $\{1, 2, 3, 4, \ldots\}$ to the correct set: we have $f(1) = 8$, and $f(n + 1) = 5n + 5 + 3 = f(n) + 5$ so the next value is the previous plus 5.
- It is an injection because $f(n_1) = f(n_2) \Rightarrow 5n_1 + 3 = 5n_2 + 3 \Rightarrow 5n_1 = 5n_2 \Rightarrow n_1 = n_2$.
- It is a surjection. Note that $f(1) = 8$ is the smallest number in the set $\{8, 13, 18, 23, \ldots\}$ and every other number there is gotten by adding a certain number $k \geq 0$ of fives. Then this number is $8 + 5k = 3 + 5(k + 1) = f(k + 1)$ and $k + 1 \in \{1, 2, 3, 4, \ldots\}$. Hence f is a surjection.
- In conclusion $f\colon \{1, 2, 3, 4, \ldots\} \to \{8, 13, 18, 23, \ldots\}$ is a bijection.

Exercise 5. a) Let $X = \{1, 2\}$. Find all possible functions/mappings $f\colon X \to X$. You can use a diagram, or tell how the elements of X map using some other way.

b) Let $Y = \{1, 2, 3\}$. How many different possible mappings $g\colon Y \to Y$ are there?

c) Let $Z = \{1, 2, \ldots, n\}$. How many different possible mappings $h\colon Z \to Z$ do exist?

Solution.

a) By the definition, a mapping $f\colon X \to X$ associates to every element of the domain X exactly one element of the codomain X. We can select either 1 or 2 as the image of the element 1 (two choices). Also, these two options are valid for the image of element 2 independent of which choice we made first. All in all there are 4 mappings, denoted f_1, f_2, f_3 and f_4:

$$\begin{cases} f_1(1) = 1 \\ f_1(2) = 1 \end{cases} \qquad \begin{cases} f_2(1) = 1 \\ f_2(2) = 2 \end{cases} \qquad \begin{cases} f_3(1) = 2 \\ f_3(2) = 1 \end{cases} \qquad \begin{cases} f_4(1) = 2 \\ f_4(2) = 2 \end{cases}$$

b) Mappings $g\colon Y \to Y$ associate exactly one element of the codomain Y to each element of the domain Y. The image of 1 in the domain can be any of 1, 2 or 3 (three choices). The same is true for elements 2 and 3. The choices are independent so we multiply the numbers of choices. Therefore there are $3 \cdot 3 \cdot 3 = 27$ different such mappings.

c) As before, mappings $h\colon Z \to Z$ associate exactly one element of the codomain to each element of the domain. The choices of images are $1, 2, \ldots, n$ (n options). The same number of choices holds for all n elements of the domain. Therefore there are $n \cdot n \cdot \cdots \cdot n = n^n$ different mappings.

Exercise 6. Expand $(1+x)^7$.

Solution. Use Proposition 10 in the notes:

$$(1+x)^7 = \binom{7}{0} + \binom{7}{1}x^1 + \binom{7}{2}x^2 + \binom{7}{3}x^3 + \binom{7}{4}x^4 + \binom{7}{5}x^5 + \binom{7}{6}x^6 + \binom{7}{7}x^7.$$

Let's calculate the binomial coefficients. We can do it one by one like below,

$$\begin{aligned}
\binom{7}{0} &= 1, \\
\binom{7}{1} &= 7, \\
\binom{7}{2} &= \frac{7 \cdot 6}{2} = 21, \\
\binom{7}{3} &= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35, \\
\binom{7}{4} &= \frac{7!}{4! \cdot 3!} = \binom{7}{3} = 35, \\
\binom{7}{5} &= \frac{7!}{5! \cdot 2!} = \binom{7}{2} = 21, \\
\binom{7}{6} &= \frac{7!}{6! \cdot 1!} = \binom{7}{1} = 7, \\
\binom{7}{7} &= \frac{7!}{7! \cdot 0!} = 1,
\end{aligned}$$

or by calculating the row that starts with $1, 7, \ldots$ in Pascal's triangle:

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&2&&1&&&&& \\
&&&&1&&3&&3&&1&&&& \\
&&&1&&4&&6&&4&&1&&& \\
&&1&&5&&10&&10&&5&&1&& \\
&1&&6&&15&&20&&15&&6&&1& \\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}$$

Hence

$$(1+x)^7 = 1 + 7x^1 + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7.$$