BM20A9200 Mathematics A – Exercise set 8

To be done by 6.–10.11.2023

Text in blue or red is not part of the problem or its solution. It's there as extra information to help you learn.

Exercise 1. Prove that for any integer $n \geq 2$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)\cdot n}=\frac{n-1}{n}.$$

Exercise 2. Is the following proof correct? If yes explain why. If not explain where is the mistake.

Claim. In a group of $n \geq 1$ people everyone has the same name.

Proof. Let $V(n)$ = "in a group of n people everybody has the same name.".

Base case: $V(1)$ is true because in a group of one person everybody indeed has the same name.

Induction hypothesis: $V(k)$ is true for some $k \geq 1$.

Induction step: Let the induction hypothesis be true. Consider a group of $k+1$ people. Arrange them in a line. Then the first k people form a group and have the same name by the induction hypothesis. Similarly the last k people have the same name. Because of the overlap they all have the same name.

The base case and the induction step are true. Hence everybody in any group of $n \geq 1$ people have the same name. □

Exercise 3. The **Fibonacci numbers** F_n, $n \geq 0$ integers, are such that each of them is the sum of the preceding two, starting from 0 and 1. That is, $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The first few Fibonacci numbers F_n are:

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

Prove that for all $n \in \mathbb{N}$

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

Exercise 4. Prove Nicomachus's theorem:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$$

for all positive integers n. **Hint:** Use the result $1 + 2 + \ldots + n = n(n+1)/2$ for all positive integers n.

Exercise 5. Recall the definition of integer divisibility.

a) Show that $5|75$, $-3|51$ and $55|0$ by definition.

b) Find all $n \in \mathbb{Z}$ such that $n|(3n+2)$. Remember negative solutions too!

Exercise 6. Determine the greatest common divisor of 2023 and 1428, and find some integers x and y such that $2023x - 1428y = \gcd(2023, 1428)$.