BM20A9200 Mathematics A – Exercise set 7

To be done by 30.10.–3.11.2023

Text in blue or red is not part of the problem or its solution. It's there as extra information to help you learn.

Recall that there is no lesson or exercise sessions during the week of 23.–27.10. We do not have a mid-term exam either; we only have a final exam in January and later.

Exercise 1. Let $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of digits.

(a) How many 4-digit pin codes are there?

(b) How many increasing sequences of 4 digits is there ($x_1 \le x_2 \le x_3 \le x_4$)?

(c) How many subsets of 4 elements of D are there?

Be ready to explain how you arrived at your conclusion.

Solution.

1. This is a selection of $k = 4$ items from a set of $n = 10$, with repetition allowed and the order matters. The answer is $n^k = 10^4 = 10000$. (Alternatively we can see that all the possible pin codes are 0000, 0001, 0002, ..., 9999 of which there are 10000.)

2. Here we select $k = 4$ items from a set of $n = 10$. Repetition is allowed but the order we pick the elements doesn't matter because they will be ordered by increasing size in the end (for example if we pick $4, 3, 2, 1$ this gives the same final sequence than if we picked $1, 2, 3, 4$). Hence the answer is $\binom{n+k-1}{k} = \binom{13}{4} = 13!/(4! \cdot 9!) = 13 \cdot 12 \cdot 11 \cdot 10/(1 \cdot 2 \cdot 3 \cdot 4) = 13 \cdot 12 \cdot 11 \cdot 10/24 = 143 \cdot 5 = 715$.

3. We select $k = 4$ items from a set of $n = 10$, with repetition not allowed and order not mattering (we cannot repeat an element in a set, and the order of elements does not matter in sets). Hence the number is $\binom{n}{k} = \binom{10}{4} = 10!/(4! \cdot 6!) = 10 \cdot 9 \cdot 8 \cdot 7/(1 \cdot 2 \cdot 3 \cdot 4) = 10 \cdot 3 \cdot 7 = 210$.

Exercise 2. Find the values of

$$\binom{70}{5} \quad \text{and} \quad \binom{121}{115}.$$

Explain how you calculated the values.

Solution. We have for example

$$\begin{aligned}\binom{70}{5} &= \frac{70!}{5! \cdot 65!} = \frac{70 \cdot 69 \cdot 68 \cdot 67 \cdot 66}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{70}{5} \cdot \frac{69}{3} \cdot \frac{68}{4} \cdot 67 \cdot \frac{66}{2} \\ &= 14 \cdot 23 \cdot 17 \cdot 67 \cdot 33 = 12\,103\,014.\end{aligned}$$

Remember that in a division $a \cdot b/(c \cdot d)$ we can do the division in any order, for example $a \cdot (b/d)/c$. That's why $70!/(5! \cdot 65!) = (70!/65!)/5! = (70 \cdot 69 \cdot 68 \cdot 67 \cdot 66)/(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$.

For the second case we have

$$\begin{aligned}\binom{121}{115} &= \frac{121!}{115! \cdot 6!} = \frac{121 \cdot 120 \cdot 119 \cdot 118 \cdot 117 \cdot 116}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= 121 \cdot \frac{120}{4 \cdot 5 \cdot 6} \cdot 119 \cdot \frac{118}{2} \cdot \frac{117}{3} \cdot 116 \\ &= 121 \cdot 1 \cdot 119 \cdot 59 \cdot 39 \cdot 116 = 3\,843\,323\,484\end{aligned}$$

Exercise 3. You go to a sushi restaurant for lunch. You will buy 10 pieces of sushi. The restaurant has the following on offer:

- Anago (sea eel)
- Ebi (shrimp)
- Kappa maki (cucumber maki)
- Maguro (blue fin tuna)
- Sake (salmon)

In how many different ways is it possible to make the selection? (Only the final count of every type of sushi matters, not the order)

Solution. Here we have a set of $n = 5$ elements and we pick $k = 10$ with repetition allowed, and the order of picking doesn't matter. Hence the answer is

$$\binom{n+k-1}{k} = \binom{14}{10} = \frac{14!}{10! \cdot 4!} = \frac{14 \cdot 13 \cdot 12 \cdot 11}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 13 \cdot 1 \cdot 11 = 1001.$$

Exercise 4. Prove the following for any natural numbers n:

$$n^2 = \binom{n}{2} + \binom{n+1}{2}.$$

Solution. We have

$$\begin{aligned}\binom{n}{2} + \binom{n+1}{2} &= \frac{n!}{2! \cdot (n-2)!} + \frac{(n+1)!}{2!(n+1-2)!} \\ &= \frac{n(n-1)}{2} + \frac{(n+1)n}{2} = \frac{n^2 - n + n^2 + n}{2} = \frac{2n^2}{2} = n^2.\end{aligned}$$

Exercise 5. Prove Pascal's rule. It states that for positive natural numbers n and k we have

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}.$$

Solution. Let's calculate the left-hand side:

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{(n-k)(n-1)!}{k!(n-k)!} + \frac{k(n-1)!}{k!(n-k)!} \\ &= \frac{(n-k+k)(n-1)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.\end{aligned}$$

Exercise 6. Prove that for any positive integer n

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution.

Proof. Let's prove it by induction. For any positive integer n write the predicate $V(n) =$ "$1^1 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$".

Base case: We have $1 \cdot (1+1) \cdot (2 \cdot 1 + 1) = 1 \cdot 2 \cdot 3 = 6$ and hence $V(1)$ is true.

Induction hypothesis: $V(k)$ is true for some positive integer k.

Indection step: Let's prove that $V(k+1)$ is true when the induction hypothesis is true. By the induction hypothesis

$$\begin{aligned}1^2 + 2^2 + 3^3 + \cdots + (k+1)^2 &= \left(1^2 + 2^2 + 3^3 + \cdots + k^2\right) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)(k \cdot (2k+1) + 6(k+1))}{6} = \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6}\end{aligned}$$

and on the other hand

$$\big((k+1)+1\big)\big(2(k+1)+1\big) = (k+2)(2k+3) = 2k^2 + 7k + 6.$$

Hence

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

so $V(k+1)$ is true.

The base case and the induction step hold. Hence the claim $V(k)$ is true for all positive integers k. $\square$