BM20A9200 Mathematics A – Exercise set 13

To be done by 22.12.2022 23:59

Text in blue or red is not part of the problem or its solution. It's there as extra information to help you learn.

These exercises will not be checked face-to-face. Instead upload a PDF or images of your solution to the bottom of the course's Moodle page. Mark **clearly** your name, student ID and which exercises you have done on the first page.

Exercise 1. Each of the following three set descriptions are wrong (the notation has one of more problems). Figure out what they try to mean and rewrite them using correct mathematical notation.

(a) $\{n \in \mathbb{Z}, 5n\}$

(b) $\{\emptyset, \{\}, 0, 1, 2, 1$

(c) $\{x^2 \geq 2\}$

Solution.

(a) I guess this set tries to denote all numbers of the form $5n$ with $n \in \mathbb{Z}$. A correct notation would be $\{5n \mid n \in \mathbb{Z}\}$. Other notations could be for example: $5\mathbb{Z}$, $\{k \in \mathbb{Z} \mid k = 5n,\ n \in \mathbb{Z}\}$, ...

(b) First of all there's the right bracket missing: $\}$. Then the element 1 is written twice. But also the empty set is written twice because $\emptyset = \{\}$. A correct notation would be: $\{\emptyset, 0, 1, 2\}$.

(c) This seems to try to describe a set of elements x for which $x^2 \geq 2$. It does not say if x must be an integer, a natural number, a rational number, a real number or something else. Any of the following could be a corrected version:

$$\{x \in \mathbb{Z} \mid x^2 \geq 2\}$$
$$\{x \in \mathbb{N} \mid x^2 \geq 2\}$$
$$\{x \in \mathbb{Q} \mid x^2 \geq 2\}$$
$$\{x \in \mathbb{R} \mid x^2 \geq 2\}$$

or others.

Exercise 2. (a) Prove that $\neg A \Rightarrow B$ is logically equivalent to $A \vee B$.

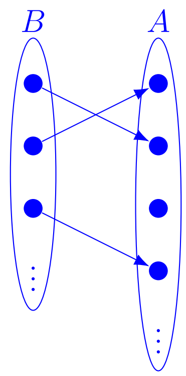
(b) Write a **formal** proof for the claim "Let A, B be sets. If there is no surjection $A \to B$ then there is no injection $B \to A$."

Solution.

(a) We will prove that by calculating the truth tables and seeing that both propositions have the same truth values on the same rows.

A	B	$\neg A$	$\neg A \Rightarrow B$	$A \vee B$
0	0	1	0	0
0	1	1	1	1
1	0	0	1	1
1	1	0	1	1

(b) A direct proof seems difficult, because we have nothing to work with. If we do a proof by contraposition it will give us something. If there is an injection $f : B \to A$, we will show that then there is a surjection $A \to B$. We will do it by reversing the arrow in the following diagram, and mapping elements of A without arrow to some arbitrary element of B.



It's important to note that each $a \in A$ has at most one arrow pointing to it because f is an injection.

Now the formal proof:

Proof. Assume that there is an injection $f : B \to A$. Let us define a function $g : A \to B$: Let $a \in A$ be an arbitrary element. If $f(b) = a$ for some $b \in B$ then there is only one such b because f is an injection. In that case define $g(a) = b$. For the other case, let $b_0 \in B$ be some element. If there is no $b \in B$ such that $f(b) = a$ we just define $g(a) = b_0$. In conclusion:

$$g(a) = \begin{cases} f^{-1}(a), & \text{if } a \in fB, \\ b_0, & \text{otherwise.} \end{cases}$$

The relation g is a function $A \to B$ because each $a \in A$ has a unique $g(a)$ associated to it (since f was an injection). Now we need to show that g is a surjection $A \to B$: let $b \in B$. Then $f(b) \in A$ and $g(f(b)) = b$ because $f(b) \in fB$. Hence any element of B is reached. Thus there is a surjection $A \to B$ and the proof by contraposition is complete. $\square$

Exercise 3. Prove the following claim by induction:

$$3 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \ldots + 2 \cdot 3^n = 3^{n+1}$$

for all $n \in \{1, 2, 3, 4, \ldots\}$.

Solution. Let's prove it by induction on n.

Proof. Initial step: When $n = 1$ the left-hand side is $3 + 2 \cdot 3^1 = 9$. The right-hand side is $3^{1+1} = 9$. Both sides are equal so the claim holds when $n = 1$.
Induction step: Let $k \in \{1, 2, 3, 4, \ldots\}$ and assume that the claim is true when $n = k$. Let's investigate the left-hand side when $n = k + 1$. By the induction assumption

$$\begin{aligned} &3 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \ldots + 2 \cdot 3^k + 2 \cdot 3^{k+1} \\ &\quad = \left(3 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + \ldots + 2 \cdot 3^k\right) + 2 \cdot 3^{k+1} \\ &\quad = 3^{k+1} + 2 \cdot 3^{k+1} = 3 \cdot 3^{k+1} = 3^{k+2} \end{aligned}$$

which is the right-hand side of the claim when $n = k + 1$. The claim is true by induction. □

Exercise 4. Recall the different formulas for calculating the number of ways of selecting k items from a set of n options and in which situations you can use them.
There are 30 students. Each of them can get a grade 0,1,2,3,4 or 5 from the course.

(a) In how many different ways the teacher can assign the grades to the 30 students?

(b) After the semester, the teacher must report how many students got a 0, how many got a 1, etc. How many different such reports is it possible to make?

Solution.

(a) If we order the students on a line with 30 columns, then the possible list of scores are:

$$\begin{array}{ccccccc} 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 2 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 \\ & & & \vdots & & & \\ 5 & 5 & 5 & \ldots & 5 & 5 & 3 \\ 5 & 5 & 5 & \ldots & 5 & 5 & 4 \\ 5 & 5 & 5 & \ldots & 5 & 5 & 5 \end{array}$$

This has 6^{30} combinations, so there is 6^{30} ways of grading the students.

We can reach the same conclusion by noting that we make $k = 30$ choices from $n = 6$ options (one grade for each student), with repetition allowed (the score of a student doesn't restrict the score of anyone else), and with order mattering (if we re-order the scores some students' scores will change). Hence the total is $n^k = 6^{30}$.

(b) If we look at any row in the score list of (a), in the report we don't care about the order of the scores (order does not matter), and we are allowed to have a score appearing more than once (repetition allowed). There is again $n = 6$ options and $k = 30$ selections. Hence the answer is

$$\binom{n-1+k}{n-1} = \binom{35}{5} = \frac{35!}{5!30!} = \frac{35 \cdot 34 \cdot 33 \cdot 32 \cdot 31}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 324632.$$

Alternativetly, we can think of forcing all the zeros first, the ones then, etc. if $\#0$ denotes the number of zeros, the situation $\#0 = 10$, $\#1 = 0$, $\#2 = 5, \#3 = 5$, $\#4 = 8$, $\#5 = 2$ can be represented as

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5$$

or as

$$xxxxxxxxxx||xxxxx|xxxxx|xxxxxxxx|xx$$

where the left-most x's represent 0's, the $|$ represents switching to the next score etc. In this way we see that there are $30 + 6 - 1$ slots for a symbol x or $|$ and we must select where to put the $6 - 1$ symbols $|$ and the rest will be x's. The number of such selections is "choose 5 among 35 slots, i.e. $\binom{35}{5}$.

Exercise 5. Recall the definition of congruence and how to calculate the GCD.

(a) Is there $x \in \mathbb{Z}$ such that $6x \equiv 1 \pmod{70}$? If yes find one. If no explain why not.

(b) Is there $y \in \mathbb{Z}$ such that $6y \equiv 2 \pmod{70}$? If yes find one. If no explain why not.

Solution.

(a) For any $x \in \mathbb{Z}$ we have

$$6x \equiv 1 \pmod{70} \quad \Leftrightarrow \quad (\exists k \in \mathbb{Z})(6x - 1 = 70k).$$

If there is such an x then $1 = 6x - 70k = 2(3x - 35k)$ and $3x - 35k \in \mathbb{Z}$, and this would mean that 1 is even. A contradiction. Hence there is no solution.

(b) We have by definition for any $x \in \mathbb{Z}$

$$6x \equiv 2 \pmod{70} \quad \Leftrightarrow \quad (\exists k \in \mathbb{Z})(6x - 2 = 70k) \quad \Leftrightarrow \quad (\exists k \in \mathbb{Z})(3x - 1 = 35k).$$

Because 3 is prime and does not divide 35 we have $\gcd(3, 35) = 1$. Hence the equation $3x - 35k = 1$ has a solution. Let's find it using the Euclidean algorithm:

$$35 = 11 \cdot 3 + 2$$
$$3 = 1 \cdot 2 + 1.$$

Then $1 = 3 - 2 = 3 - (35 - 11 \cdot 3) = -35 + 12 \cdot 3$ so a solution is $k = 1$ and $x = 12$. Just to check if we made a calculation mistake: $6 \cdot 12 = 72 \equiv 2 \pmod{70}$, so it's all good!

Exercise 6. Consider the following system of equations:

$$\begin{aligned} 3a - 3b + c &= 7 \\ -2a - 3b - 3c &= 9 \\ 2a + 3b + 4c &= -3 \end{aligned}$$

(a) Write the system as an augmented matrix.

(b) Use the Gauss–Jordan method on the augmented matrix.

(c) Read the values of a, b, c from the reduced row-echelon form, and verify that they solve the system above.

Solution.

(a) This is just

$$\left[\begin{array}{rrr|r} 3 & -3 & 1 & 7 \\ -2 & -3 & -3 & 9 \\ 2 & 3 & 4 & -3 \end{array}\right]$$

(b) We have

$$\left[\begin{array}{rrr|r} 3 & -3 & 1 & 7 \\ -2 & -3 & -3 & 9 \\ 2 & 3 & 4 & -3 \end{array}\right] \overset{R_1/3 \to R_1}{\Rightarrow} \left[\begin{array}{rrr|r} 1 & -1 & 1/3 & 7/3 \\ -2 & -3 & -3 & 9 \\ 2 & 3 & 4 & -3 \end{array}\right] \overset{2R_1 + R_2 \to R_2}{\Rightarrow}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 1/3 & 7/3 \\ 0 & -5 & -7/3 & 41/3 \\ 2 & 3 & 4 & -3 \end{array}\right] \overset{-2R_1 + R_3 \to R_3}{\Rightarrow} \left[\begin{array}{rrr|r} 1 & -1 & 1/3 & 7/3 \\ 0 & -5 & -7/3 & 41/3 \\ 0 & 5 & 10/3 & -23/3 \end{array}\right] \overset{-R_2/5 \to R_2}{\Rightarrow}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 1/3 & 7/3 \\ 0 & 1 & 7/15 & -41/15 \\ 0 & 5 & 10/3 & -23/3 \end{array}\right] \overset{R_2 + R_1 \to R_1}{\Rightarrow} \left[\begin{array}{rrr|r} 1 & 0 & 4/5 & -2/5 \\ 0 & 1 & 7/15 & -41/15 \\ 0 & 5 & 10/3 & -23/3 \end{array}\right] \overset{-5R_2 + R_3 \to R_3}{\Rightarrow}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 4/5 & -2/5 \\ 0 & 1 & 7/15 & -41/15 \\ 0 & 0 & 1 & 6 \end{array}\right] \overset{-4R_3/5 + R_1 \to R_1}{\Rightarrow} \left[\begin{array}{rrr|r} 1 & 0 & 0 & -26/5 \\ 0 & 1 & 7/15 & -41/15 \\ 0 & 0 & 1 & 6 \end{array}\right] \overset{-7R_3/15 + R_2 \to R_2}{\Rightarrow}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & -26/5 \\ 0 & 1 & 0 & -83/15 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

(c) The values given by the solution of (b) are $a = -26/5$, $b = -83/15$ and $c = 6$. Then

$$\begin{aligned} &3a - 3b + c = -78/5 + 83/5 + 6 = 5/5 + 6 = 7, \\ &-2a - 3b - 3c = 52/5 + 83/3 - 18 = 135/5 - 18 = 27 - 18 = 9, \\ &2a + 3b + 4c = -52/5 - 83/3 + 24 = -135/5 + 24 = -27 + 24 = -3, \end{aligned}$$

so our calculations indeed gave the correct answer.