BM20A9200 Mathematics A – Exercise set 4

To be done by 2.–6.10.2023

Text in blue or red is not part of the problem or its solution. It's there as extra information to help you learn.

Exercise 1. Let P, Q and R be logical propositions. Write the truth table of $P \Rightarrow (Q \Leftrightarrow R)$.

Solution.

P	Q	R	$Q \Leftrightarrow R$	$P \Rightarrow (Q \Leftrightarrow R)$
0	0	0	1	1
0	0	1	0	1
0	1	0	0	1
0	1	1	1	1
1	0	0	1	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

Exercise 2. Binary numbers:

(a) How do you write the decimal number 69 in binary form? Use the methods described in the lecture notes.

(b) What decimal number corresponds to the binary number 101 0101 0101?

Solution.

(a) Let's do the calculation in a table:

Div. by 2	Quotient	Remainder
69/2	34	1
34/2	17	0
17/2	8	1
8/2	4	0
4/2	2	0
2/2	1	0
1/2	0	1

B = 100 0101.

(b)

Binary	1	0	1	0	1	0	1	0	1	0	1
Position	10	9	8	7	6	5	4	3	2	1	0

Then

$$\begin{aligned} D &= 2^{10} + 2^8 + 2^6 + 2^4 + 2^2 + 2^0 \\ &= 1024 + 256 + 64 + 16 + 4 + 1 = 1365. \end{aligned}$$

Exercise 3. Write the interpretation of the following propositions about integers into English:

1. $\exists n(n = -n)$
2. $\exists n(n^2 = 2)$
3. $\forall n(3n \leq 4n)$
4. $\forall n(n^2 \geq n)$

Which of the statements are true and which false? Explain your answer.

Solution.

1. In English:
   (a) There is some integer n which satisfies $n = -n$. This is true because $0 \in \mathbb{Z}$ satisfies $0 = -0$.
   (b) The square of some integer is 2. It is false because the squares of integers are $0, 1, 4, 9$ and larger.
   (c) The triple of every integer if at most that integer multiplied by four. It's false: for example if $n = -1$ then $3n = -3 > -4 = 4n$.
   (d) The square of every integer is at least the number itself. True: if $n \geq 1$ then $n^2 = n \cdot n \geq 1 \cdot n = n$. If $n = 0$ then $n^2 = 0 \geq 0 = n$. And if $n < 0$ then $n^2 > 0 > n$. In each case $n^2 \geq n$.

Exercise 4 (Source: Advanced Mathematics Matriculation Exam 25.9.2017)**.** Juha[1] tries to prove the following proposition: "If a positive integer is divisible by 3 then it is divisible by 6." He suggests the following proof:

Step 1: Assume that a is divisible by 6.

Step 2: Then there is an integer b such that $a = 6b$.

Step 3: Now $a = 3 \cdot 2b$.

Step 4: Therefore a is divisible by 3.

Show that Juha's proposition is false. What is wrong with Juha's proof? Which statement does Juha's "proof" show, if any?

Solution. (Source: YTL:n hyvän vastauksen piirteet) For example 9 is a positive integer that is divisible by 3. But it's not divisible by 6, so the proposition is false.

The deduction assumes the conclusion and proves the assumption. Therefore it is not valid.

The proof shows that if an integer is divisible by 6 then it is divisible by 3.

[1]A common Finnish man's name

Exercise 5. Let $a, b \in \mathbb{R}$. Show that $a^2 - b^2 = 0$ if and only if[2] $a = b$ or $a + b = 0$.

Solution.

Proof. Let's show the implication in both directions. Thus let's show first that "if $a^2 - b^2 = 0$ then $a = b$ or $a + b = 0$". After that we will show that if $a = b$ or $a + b = 0$ then $a^2 + b^2 = 0$.

$\Rightarrow$ Assume that $a^2 - b^2 = 0$. Because $a^2 - b^2 = (a + b)(a - b)$ the assumption can be written as $(a + b)(a - b) = 0$. By the zero product property[3] if follows that $a + b = 0$ or $a - b = 0$, and the latter implies $a = b$. This direction has been proven.

$\Leftarrow$ Assume that $a = b$ or $a + b = 0$. Let's consider both cases. Assume that $a = b$. Then $a^2 - b^2 = a^2 - a^2 = 0$, so the equation $a^2 - b^2 = 0$ holds. Assume that $a + b = 0$, i.e. $a = -b$. Then $a^2 - b^2 = (-b)^2 - b^2 = b^2 - b^2 = 0$. So the equation $a^2 - b^2 = 0$ is true in this case too. Because $a^2 - b^2 = 0$ in both possible cases, this direction has been proven too.

Because we proved the implication in both directions, the claim has been proven. □

Exercise 6. Consider a right triangle whose legs have lengths a and b and the hypotenuse has length c. The goal is to show $a + b > c$.

1. What is the counter-assumption of the claim?
2. By squaring both sides of the inequality, show that the counter-assumption leads to a contradiction with the Pythagorean theorem.
3. Using the previous, prove the claim using an indirect proof. You can assume the Pythagorean theorem to be true.

Solution.

1. The counter-assumption is the negation of the conclusion of the original claim: $a + b \not> c$, or in other words $a + b \leq c$.
2. Let's raise the counter-assumption to the second power, which gives us $(a + b)^2 \leq c^2$. Expanding this we see that $a^2 + 2ab + b^2 \leq c^2$. Because $2ab > 0$, this contradicts the Pythagorean theorem which says that $a^2 + b^2 = c^2$.
3. *Proof.* Let's make a counter-assumptions: $a + b \leq c$. Squaring both sides we get $(a + b)^2 \leq c^2$ which implies $a^2 + 2ab + b^2 \leq c^2$. The last inequality is contradicting the Pythagorean theorem which says that $a^2 + b^2 = c^2$ because $2ab > 0$. Because the counter-assumption led to a contradiction the original claim must be true. □

[2] Proving "A if and only if B" means proving both $A \Rightarrow B$ and $B \Rightarrow A$.
[3] If a product of two numbers is zero then at least one of the numbers must be zero.