BM20A9200 Mathematics A – Exercise set 3

To be done by 25.–29.9.2023

Text in blue or red is not part of the problem or its solution. It's there as extra information to help you learn.

Exercise 1. Let $X = \{4, 5, 6\}$, $Y = \{a, b, c\}$ and $Z = \{l, m, n\}$. Consider the relations R from X to Y and S from Y to Z defined by:

$$R = \{(4, a), (5, b), (5, c), (6, b), (6, c)\},$$
$$S = \{(a, l), (a, m), (b, l), (b, m), (b, n), (c, m), (c, n)\}.$$

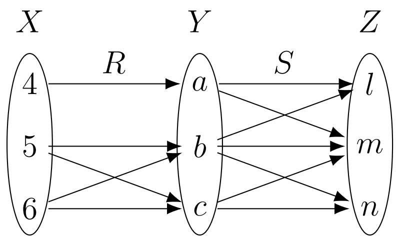
Find the following compositions of relations:

(a) RS

(b) RR^{-1}

Solution.

(a) Let's draw R and S as arrow diagrams side by side:

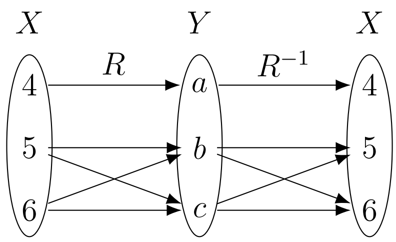


Then $xRSz$ if and only if there is $y \in Y$ such that xRy and ySz. In other words if in the diagram we can follow arrows from x to z.

We see for example that from 4 we can go to a, and from a to l or m. Hence from 4 we get to l and m. Using similar logic we see that

$$RS = \{(4, l), (4, m), (5, l), (5, m), (5, n), (6, l), (6, m), (6, n)\}.$$

(b) For RR^{-1} let's draw R^{-1} instead of S in the above diagram. R^{-1} is just R with arrows and X, Y flipped the other direction.



Using the method in (a) we see that

$$RR^{-1} = \{(4, 4), (5, 5), (5, 6), (6, 5), (6, 6)\}.$$

Exercise 2. Let us denote $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, that is, $\mathbb{R}^+$ is the set of all positive real numbers.

(a) Let us define a binary relation T on $\mathbb{R}^+$ so that T consists of the pairs (x, x^2) where $x \in \mathbb{R}^+$. What is the inverse relation of T?

(b) Let us define a binary relation S on $\mathbb{R}$ so that S consists of the pairs (x, x^2), where $x \in \mathbb{R}$. What is the inverse relation of S?

Solution. The inverse operator is gotten by flipping the first and second component of each pair in the original relation.

(a) Since
$$T = \{(x, x^2) \mid x \in \mathbb{R}_+\}$$
then
$$T^{-1} = \{(x^2, x) \mid x \in \mathbb{R}_+\}.$$

(b) Since
$$S = \{(x, x^2) \mid x \in \mathbb{R}\}$$
then
$$S^{-1} = \{(x^2, x) \mid x \in \mathbb{R}\}.$$

Extra info: Some people want to write this kind of relations as a "graph of a function." That is ok but it is not required here.

To achieve that it means that the pair would have the "free variable" in its first item. Then (a) would become
$$T^{-1} = \{(y, \sqrt{y}) \mid y \in \mathbb{R}_+\}.$$
We have used the symbol y instead of x to show that the new pairs have been gotten from the original ones with the change of variebles $y = \sqrt{x}$ which is allowed when $x \geq 0$.

The same trick cannot be used directly for (b) because to each x^2 there is two values of the second component in S. For example: $(4, 2) \in S$ and $(4, -2) \in S$. To avoid this problem we'll need to split $S^{-1} = S_1^{-1} \cup S_2^{-1}$. For example:
$$S_1^{-1} = \{(x^2, x) \mid x \in \mathbb{R}, x \geq 0\}, \qquad S_2^{-1} = \{(x^2, x) \mid x \in \mathbb{R}, x < 0\}.$$
In S_1^{-1} we can write $y = x^2$ and see that then $x = \sqrt{y}$. Hence
$$S_1^{-1} = \{(y, \sqrt{y}) \mid y \in \mathbb{R}, y \geq 0\}.$$
In S_2^{-1}, if $z = x^2$ then $x = -\sqrt{z}$. But pay attention that the condition "$x \in \mathbb{R}$, $x < 0$" changes to "$z \in \mathbb{R}$, $z > 0$." Using that
$$S_2^{-1} = \{(z, -\sqrt{z}) \mid z \in \mathbb{R}, z > 0\}.$$
Taking $S^{-1} = S_1^{-1} \cup S_2^{-1}$ into account we get
$$S^{-1} = \{(y, \sqrt{y}) \mid y \in \mathbb{R}, y \geq 0\} \cup \{(z, -\sqrt{z}) \mid z \in \mathbb{R}, z > 0\}.$$

Exercise 3. Let A and B be sets. Suppose that R and S are relations from A to B, that is, $R, S \subseteq A \times B$. Prove that

$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}.$$

Solution. We need to prove that two sets are equal. They are equal if and only if they have the same elements. Let's write out the left-hand side using set notation:

Proof.

$$\begin{aligned}(R \cup S)^{-1} &= \{(x, y) \mid (y, x) \in R \cup S\} \\ &= \{(x, y) \mid (y, x) \in R \vee (y, x) \in S\} \\ &= \{(x, y) \mid (y, x) \in R\} \cup \{(x, y) \mid (y, x) \in S\} \\ &= R^{-1} \cup S^{-1}.\end{aligned}$$

The first equality if the definition of the inverse of a relation. The second equality if the definition of the union $R \cup S$. The third equality is the definition of $\cup$ using the set builder notation, for example $\{\ldots \mid P \vee Q\} = \{\ldots \mid P\} \cup \{\ldots \mid Q\}$. The last equality is the definition of the inverse of a relation used twice. □

Info: In the exam it will be a good idea to explain each '=' in the proof like above. This will convince me that you know what you are doing. If I have to guess, it is more likely that you will lose points.

Info: Set notation is very powerful. On a course like Mathematics A it is very useful to write the definitions of objects using set builder notation "$\{x \mid \text{condition on } x\}$." After that use definitions and propositional logic (rules of logic, truth tables) to deduce that two different-looking sets are actually the same.

Exercise 4. Let P be the proposition "I think" and Q the proposition "I am". The following table shows a few propositions and their meaning in natural language:

$P \Rightarrow Q$	If I think, then I am. (I think, therefore I am –Decartes)
$\neg Q \Rightarrow \neg P$	If I am not, then I don't think.
$P \wedge \neg Q$	I think and I am not.
$\neg P \Rightarrow \neg Q$	If I don't think, then I am not.

(a) Write the truth tables for the sentences.

(b) Which of the propositions are logically equivalent?

(c) Using P and Q, invent a few new propositions which are equivalent to some of the propositions above, and prove that they are so using a truth table.

Solution.

(a) The truth tables of the propositions are:

P	Q	$P \Rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

P	Q	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$
1	1	0	0	1
1	0	0	1	0
0	1	1	0	1
0	0	1	1	1

P	Q	$\neg Q$	$P \wedge \neg Q$
1	1	0	0
1	0	1	1
0	1	0	0
0	0	1	0

P	Q	$\neg P$	$\neg Q$	$\neg P \Rightarrow \neg Q$
1	1	0	0	1
1	0	0	1	1
0	1	1	0	0
0	0	1	1	1

(b) By looking at the truth tables we see that propositions $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ have the same truth values. No others do. Therefore these are the only two propositions in the exercise that are logically equivalent.

(c) The proposition $\neg(P \Rightarrow Q)$ is logically equivalent to the proposition $P \wedge \neg Q$. This means that they have the same truth values:

P	P	$\neg Q$	$P \wedge \neg Q$
1	1	0	0
1	0	1	1
0	1	0	0
0	0	1	0

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$
1	1	1	0
1	0	0	1
0	1	1	0
0	0	1	0

Also, the proposition $P \vee \neg Q$ is logically equivalent to the proposition $\neg P \Rightarrow \neg Q$ because they have the same truth tables:

P	Q	$\neg P$	$\neg Q$	$\neg P \Rightarrow \neg Q$
1	1	0	0	1
1	0	0	1	1
0	1	1	0	0
0	0	1	1	1

P	Q	$\neg Q$	$P \vee \neg Q$
1	1	0	1
1	0	1	1
0	1	0	0
0	0	1	1

Exercise 5. In mathematics, it's often useful to figure out what is the negation of a given proposition. For example if we can't prove some theorem, maybe we can prove its negation. Also, we will need the ability to form the negation of a proposition when studying indirect proof methods later.

(a) Connect each proposition on the left to its negation on the right (and prove why).

$$\begin{array}{rl} P_0 \wedge P_1 & \qquad P_0 \wedge \neg P_1 \\ P_0 \vee P_1 & \qquad \neg P_0 \wedge \neg P_1 \\ P_0 \Rightarrow P_1 & \qquad \neg P_0 \vee \neg P_1 \end{array}$$

(b) The following propositions say something about an unknown real number x. Write the negation in each case, but do it in such a way that the negation symbol $\neg$ does not appear there. You can use the symbols $\leq$, $<$, $\geq$, $\notin$ etc.

1. $(x^2 = 1) \wedge (x > 0)$
2. $(x^2 = 2) \vee (x \in \mathbb{Q})$
3. $(x^2 = 1) \Rightarrow (x > 0)$
4. $(x^2 = 2) \Leftrightarrow (x \in \mathbb{Q})$

Hint: You don't need to try to interpret the propositions to find their negation. Try using (a) as a guide.

Solution.

(a) Let's write the truth tables:

P_0	P_1	$P_0 \wedge P_1$
1	1	1
1	0	0
0	1	0
0	0	0

P_0	P_1	$\neg P_1$	$P_0 \wedge \neg P_1$
1	1	0	0
1	0	1	1
0	1	0	0
0	0	1	0

P_0	P_1	$P_0 \vee P_1$
1	1	1
1	0	1
0	1	1
0	0	0

P_0	P_1	$\neg P_0$	$\neg P_1$	$\neg P_0 \wedge \neg P_1$
1	1	0	0	0
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

P_0	P_1	$P_0 \Rightarrow P_1$
1	1	1
1	0	0
0	1	1
0	0	1

P_0	P_1	$\neg P_0$	$\neg P_1$	$\neg P_0 \vee \neg P_1$
1	1	0	0	0
1	0	0	1	1
0	1	1	0	1
0	0	1	1	1

According to the truth tables:

$$\begin{gathered}\neg(P_0 \wedge P_1) \Leftrightarrow \neg P_0 \vee \neg P_1 \\ \neg(P_0 \Rightarrow P_1) \Leftrightarrow P_0 \wedge \neg P_1 \\ \neg(P_0 \vee P_1) \Leftrightarrow \neg P_0 \wedge \neg P_1.\end{gathered}$$

(b) Using (a) for 1.–3., we get

$$\begin{gathered}\neg((x^2 = 1) \wedge (x > 0)) \Leftrightarrow (x^2 \neg 1) \vee (x \leq 0) \\ \neg((x^2 = 2) \vee (x \in \mathbb{Q})) \Leftrightarrow (x^2 \neg 2) \wedge (x \notin \mathbb{Q}) \\ \neg((x^2 = 1) \Rightarrow (x > 0)) \Leftrightarrow (x^2 = 1) \wedge (x \leq 0).\end{gathered}$$

For the last one, we notice that the equivalence $\Leftrightarrow$ is false if and only if one of the sentences is true and the other is false. This way,

$$\neg\big((x^2 = 2) \Leftrightarrow (x \in \mathbb{Q})\big) \quad \Leftrightarrow \quad \big((x^2 = 2) \wedge (x \notin \mathbb{Q})\big) \vee \big((x^2 \neq 2) \wedge (x \in \mathbb{Q})\big).$$

Exercise 6. In lectures it was shown that $P \Rightarrow Q$ is logically equivalent to $\neg P \vee Q$. In other words, the operation $\Rightarrow$ can be defined by using $\vee$ and $\neg$.

(a) Define $\vee$ using $\neg$ and $\Rightarrow$.

(b) Define $\wedge$ using $\neg$ and $\vee$.

Prove the correctness of your answer by using truth tables.

Solution. **Info:** There are two ways for finding these. One is to try things until they almost work, and then make small adjustments. The other is to use the properties of $\wedge$ and $\vee$ to give us a guess, and then check & prove it with truth tables.

For $\vee$ using $\neg$ and $\Rightarrow$:

- $A \vee B$ is false only when both $A = 0$ and $B = 0$.
- $C \Rightarrow D$ is false only when $C = 1$ and $D = 0$.

This suggests trying $C = \neg A$ and $D = B$, which gives the correct guess $\neg A \Rightarrow B$.

For $\wedge$ using $\neg$ and $\vee$:

- $A \wedge B$ is true only when both $A = 1$ and $B = 1$.
- $C \vee D$ is false only when both $C = 0$ and $D = 0$.

This suggests selecting $C = \neg A$, $D = \neg B$ and adding a negation in front: $\neg(\neg A \vee \neg B)$.

(a) The correct answer is that $A \vee B$ is equivalent to $\neg A \Rightarrow B$ (or $\neg B \Rightarrow A$).

Proof. We see that they are logically equivalent from the following truth table:

A	B	$\neg A$	$\neg A \Rightarrow B$	$A \vee B$
0	0	1	0	0
0	1	1	1	1
1	0	0	1	1
1	1	0	1	1

$\square$

(b) The answer is $A \wedge B$ is equivalent to $\neg(\neg A \vee \neg B)$.

Proof. They are logically equivalent by the following truth table

A	B	$\neq A$	$\neg B$	$\neg A \vee \neg B$	$\neg(\neg A \vee \neg B)$	$A \wedge B$
0	0	1	1	1	0	0
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	0	1	1

$\square$

Info: Proving things makes mathematicians convinced. However convincing humans is a different matter. If you want to convince yourself, try these with the example A : "it is raining", B : "it looks pretty outside" and consider the different truth values of A and B and the propositions. For example, "it is not raining and it is pretty outside" has the opposite message than "it is raining or it's an ugly weather outside".