BM20A9200 Mathematics A – Exercise set 11

To be done by 27.11.–1.12.2023

Text in blue or red is not part of the problem or its solution. It's there as extra information to help you learn.

Exercise 1. a) Of the two numbers 3515215 and 1535450, which are divisible by 9 and/or 11? Do not use a calculator. Remember to show your reasoning!

b) Of the following 3 calculations two of them have a mistake. Which two?

$$\begin{aligned}
&485\,845\,416\,843\,583 \cdot 5\,414\,518\,454\,648\,546\,841 \\
&\qquad = 2\,630\,618\,975\,615\,996\,095\,288\,915\,945\,771\,302 \\
&8\,435\,485\,486\,753\,254 \cdot 52\,455\,421\,515 \\
&\qquad = 442\,486\,946\,891\,306\,887\,367\,859\,810 \\
&99\,999\,999\,999\,999\,999\,999 \cdot 999\,999\,999\,989\,999\,999 \\
&\qquad = 99\,999\,999\,998\,999\,999\,999\,000\,000\,000\,010\,000\,001
\end{aligned}$$

Solution.

a) A number

- is divisible by 9 iff its reduced sum (mod 9) is zero,
- is divisible by 11 iff the alternating sum of its digits (every other digit added, every other one subtracted) is a multiple of 11.

As a reminder, the reduced sum (mod 9) of digits of a number is the sum of its digits where at any point in the calculation we can replace any digit 9 by a zero. The reduced sum of 3515215 (mod 9) is

$$3515215 \equiv 1 + 5 + 3 + 5 + 4 + 5 + 0 \equiv 0 \pmod 9$$

so this number is divisible by 9.

The reduced sum (mod 9) of 1535450 is

$$1535450 \equiv 1 + 5 + 3 + 5 + 4 + 5 + 0 \equiv 5 \not\equiv 0 \pmod 9$$

so $9 \nmid 1535450$.

The alternating sum of digits of a number is the first digit added, the second subtracted, the third added etc. For example the alternating sum of digits of 3515215 is

$$3 - 5 + 1 - 5 + 2 - 1 + 5 = 0$$

which is a multiple of 11. Hence $11 \mid 3515215$.

The alternating sum of digits of 1535450 is

$$1 - 5 + 3 - 5 + 4 - 5 + 0 = -7$$

which is not a multiple of 11. Hence $11 \nmid 1535450$.

b) One way to find mistakes in multiplications is to look at the calculation modulo 9. Let's calculate the reduced sum of digits modulo 9, and to make it easier let's do it in groups of 3 digits. For example $485 \equiv 4 + 8 + 5 \equiv 17 \equiv 8$. Let's also name the numbers:

$$\begin{aligned}
a_1 &= 485\,845\,416\,843\,583 \equiv 8 + 8 + 2 + 6 + 7 \equiv 4 \pmod 9 \\
a_2 &= 5\,414\,518\,454\,648\,546\,841 \equiv 5 + 0 + 5 + 4 + 0 + 6 + 4 \equiv 6 \pmod 9 \\
a_3 &= 2\,630\,618\,975\,615\,996\,095\,288\,915\,945\,771\,302 \\
&\equiv 2 + 0 + 6 + 3 + 3 + 6 + 5 + 0 + 6 + 0 + 6 + 5 \equiv 6 \pmod 6
\end{aligned}$$

We have $4 \cdot 6 = 24 \equiv 6 \pmod 9$ so we can't deduce if there's a mistake in this case using congruence modulo 9. Let's investigate the other calculations.

The second calculation:

$$\begin{aligned}
b_1 &= 8\,435\,485\,486\,753\,254 \equiv 8 + 3 + 8 + 0 + 6 + 2 \equiv 0 \pmod 9 \\
b_2 &= 52\,455\,421\,515 \equiv 7 + 5 + 7 + 2 \equiv 3 \pmod 9 \\
b_3 &= 442\,486\,946\,891\,306\,887\,367\,859\,810 \\
&\equiv 1 + 0 + 1 + 0 + 0 + 5 + 7 + 4 + 0 \equiv 0
\end{aligned}$$

Since $0 \cdot 3 \equiv 0 \pmod 9$ we can't deduce if there's a mistake in this one either by looking at congruence modulo 9.

For the last one, each digit 9 will be reduced to 0 in the reduced sum, so we can calculate the reduced sum by summing only the digits that are not 9:

$$\begin{aligned}
c_1 &= 99\,999\,999\,999\,999\,999\,999 \equiv 0 \pmod 9 \\
c_2 &= 999\,999\,999\,989\,999\,999 \equiv 8 \pmod 9 \\
c_3 &= 99\,999\,999\,998\,999\,999\,999\,000\,000\,000\,010\,000\,001 \\
&\equiv 8 + 1 + 1 \equiv 1 \pmod 9
\end{aligned}$$

Since $0 \cdot 8 \not\equiv 1 \pmod 9$ calculation number 3 has a mistake.

We still need to figure our a mitake in the first or second calculation. Congruence modulo 9 did not help up. Then we should look at other moduli. In fact the simplest case if modulo 2. The last digits of a_1 and a_2 are odd, meaning the numbers themselves are odd. However the last digit of a_3 is even, so $2 \mid a_3$. That calculation has a mistake, because odd times odd cannot be even.

Exercise 2.

a) Is there a multiplicative inverse of 7 modulo 12? If yes, calculate it.

b) Is there a multiplicative inverse of 6 modulo 8? If yes, calculate it.

Solution. x is a multiplicative inverse of a modulo m if $ax \equiv 1 \pmod{m}$. There is such an x iff $\gcd(a, m) = 1$.

a) $\gcd(7, 12) = 1$ so there is a multiplicative inverse. The multiples of 7, namely $1 \cdot 7$, $2 \cdot 7$, $\ldots 11 \cdot 7$ are

$$7, \quad 14, \quad 21 \quad 28, \quad 35, \quad 42, \quad 49, \quad 56, \quad 63, \quad 70, \quad 77$$

and the multiples of 12 are

$$12, \quad 24, \quad 36, \quad 48, \quad 60, \quad 72, \quad 84\ldots$$

We see that $7 \cdot 7 = 49 = 48 + 1$. So $7 \cdot 7 \equiv 1 \pmod{12}$, i.e. 7 is its own multiplicative inverse modulo 12.

An alternative solution is to solve the congruence $7x \equiv 1 \pmod{12}$ by writing it as the Diophantine equation $7x = 1 + 12y$. We have

$$\begin{aligned} 12 &= 1 \cdot 7 + 5 \\ 7 &= 1 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + 1 \end{aligned}$$

from which we deduce

$$\begin{aligned} 1 &= 5 - 2 \cdot 2 && \| 2 = 7 - 5 \\ &= 5 - 2 \cdot (7 - 5) = 3 \cdot 5 - 2 \cdot 7 && \| 5 = 12 - 7 \\ &= 3 \cdot (12 - 7) - 2 \cdot 7 = 3 \cdot 12 - 5 \cdot 7 \end{aligned}$$

This gives $(-5) \cdot 7 \equiv 1 \pmod{12}$, so the multiplicative inverse is -5 (or any element of $[-5]_{12}$, which includes the number 7 we found as the inverse in the first method).

b) $\gcd(6, 8) = 2$ so there is no multiplicative inverse.

Exercise 3. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 & 2 & 5 \\ 3 & -1 & 6 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 4 & -2 \\ 6 & -2 \\ 6 & 1 \end{bmatrix}$$

Calculate

a) $\mathbf{A}^\intercal$, $\mathbf{B}^\intercal$ and $\mathbf{C}^\intercal$,

b) $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{C}$, and

c) $\mathbf{AC}$ and $\mathbf{CA}$.

Solution.

a) Transposing flips the rows and columns, such that row number n becomes column number n etc. Hence

$$\mathbf{A}^{\intercal} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 6 & 1 \end{bmatrix}, \qquad \mathbf{B}^{\intercal} = \begin{bmatrix} 2 & 3 \\ 2 & -1 \\ 5 & 6 \end{bmatrix}, \qquad \mathbf{C}^{\intercal} = \begin{bmatrix} 4 & 6 & 6 \\ -2 & -2 & 1 \end{bmatrix}.$$

b) Addition and subtraction are done componentwise, so

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 5 \\ 3 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+2 & 6+5 \\ 2+3 & 0-1 & 1+6 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 11 \\ 5 & -1 & 7 \end{bmatrix}$$

c) The value at row r and columns c of $\mathbf{AC}$ is the doc product of row r of $\mathbf{A}$ and column s of $\mathbf{C}$. So we get

$$\begin{aligned} \mathbf{AC} &= \begin{bmatrix} 1 & 2 & 6 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 6 & -2 \\ 6 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (1,2,6)\cdot(4,6,6) & (1,2,6)\cdot(-2,-2,1) \\ (2,0,1)\cdot(4,6,6) & (2,0,1)\cdot(-2,-2,1) \end{bmatrix} \\ &= \begin{bmatrix} 4+12+36 & -2-4+6 \\ 8+0+6 & -4+0+1 \end{bmatrix} = \begin{bmatrix} 52 & 0 \\ 14 & -3 \end{bmatrix} \end{aligned}$$

The same logic gives

$$\begin{aligned} \mathbf{CA} &= \begin{bmatrix} 4 & -2 \\ 6 & -2 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 2 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (4,-2)\cdot(1,2) & (4,-2)\cdot(2,0) & (4,-2)\cdot(6,1) \\ (6,-2)\cdot(1,2) & (6,-2)\cdot(2,0) & (6,-2)\cdot(6,1) \\ (6,1)\cdot(1,2) & (6,1)\cdot(2,0) & (6,1)\cdot(6,1) \end{bmatrix} \\ &= \begin{bmatrix} 4-4 & 8+0 & 24-2 \\ 6-4 & 12+0 & 36-2 \\ 6+2 & 12+0 & 36+1 \end{bmatrix} = \begin{bmatrix} 0 & 8 & 22 \\ 2 & 12 & 34 \\ 8 & 12 & 37 \end{bmatrix} \end{aligned}$$

Exercise 4. Recall Exercise 1 of Assignment 3.

Let $X = \{4, 5, 6\}$, $Y = \{a, b, c\}$ and $Z = \{l, m, n\}$. Consider the relations R from X to Y and S from Y to Z defined by:

$$\begin{aligned} R &= \{(4,a), (5,b), (5,c), (6,b), (6,c)\}, \\ S &= \{(a,l), (a,m), (b,l), (b,m), (b,n), (c,m), (c,n)\}. \end{aligned}$$

Find the following compositions of relations using **boolean matrices**

a) RS

b) RR^{-1}

Solution. The element (i, j) of a Boolean matrix is 1 if the element number i is in relation with the element number j of the original sets. Thus

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) We have $\mathbf{M}_{RS} = \mathbf{M}_R \circ \mathbf{M}_s$, where $\circ$ is the Boolean product. So

$$\mathbf{M}_{RS} = \mathbf{M}_R \circ \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} (1,0,0) \cdot_B (1,1,0) & (1,0,0) \cdot_B (1,1,1) & (1,0,0) \cdot_B (0,1,1) \\ (0,1,1) \cdot_B (1,1,0) & (0,1,1) \cdot_B (1,1,1) & (0,1,1) \cdot_B (0,1,1) \\ (0,1,1) \cdot_B (1,1,0) & (0,1,1) \cdot_B (1,1,1) & (0,1,1) \cdot_B (0,1,1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Decoding the boolean matrix we see that

$$RS = \{(4, l), (4, m), (5, l), (5, m), (5, n), (6, l), (6, m), (6, n)\}$$

and this is indeed the same answer as in Exercise 1 of Assignment 3.

(b) Let's first calculate the Boolean matrix $\mathbf{M}_{R^{-1}} = \mathbf{M}_R{}^{\intercal}$. It is simply

$$\mathbf{M}_{R^{-1}} = \mathbf{M}_R{}^{\intercal} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

which funnily enough is exactly the same as $\mathbf{M}_R$ in this case. Then $\mathbf{M}_{RR^{-1}} = \mathbf{M}_R \circ \mathbf{M}_{R^{-1}}$, or

$$\mathbf{M}_{RR^{-1}} = \mathbf{M}_R \circ \mathbf{M}_{R^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} (1,0,0) \cdot_B (1,0,0) & (1,0,0) \cdot_B (0,1,1) & (1,0,0) \cdot_B (0,1,1) \\ (0,1,1) \cdot_B (1,0,0) & (0,1,1) \cdot_B (0,1,1) & (0,1,1) \cdot_B (0,1,1) \\ (0,1,1) \cdot_B (1,0,0) & (0,1,1) \cdot_B (0,1,1) & (0,1,1) \cdot_B (0,1,1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Decoding the matrix gives

$$RR^{-1} = \{(4, 4), (5, 5), (5, 6), (6, 5), (6, 6)\}$$

which is again the same answer as in Exercise 1 of Assignment 3.

Exercise 5. For the following system of equations convert the system into an augmented matrix and use the Gaussian elimination method to find a solutions.

$$7x - 8y = -12$$
$$-4x + 2y = 3$$

Solution. This is one way which leads to a reduced row echelon form:

$$\left[\begin{array}{cc|c} 7 & -8 & -12 \\ -4 & 2 & 3 \end{array}\right] \xrightarrow{+2R_2 \to R_1} \left[\begin{array}{cc|c} -1 & -4 & -6 \\ -4 & 2 & 3 \end{array}\right] \xrightarrow{-1 \to R_1} \left[\begin{array}{cc|c} 1 & 4 & 6 \\ -4 & 2 & 3 \end{array}\right]$$
$$\xrightarrow{+4R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 18 & 27 \end{array}\right] \xrightarrow{18^{-1} \to R_2} \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 1 & \frac{3}{2} \end{array}\right] \xrightarrow{-4R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} \end{array}\right]$$

Hence there is a unique solution and it is $x = 0$, $y = \frac{3}{2}$.

Exercise 6. Find the inverse of the 2×2-matrix

$$\begin{bmatrix} 3 & -6 \\ 6 & 23 \end{bmatrix}.$$

Solution. Let's use the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

This gives

$$\begin{bmatrix} 3 & -6 \\ 6 & 23 \end{bmatrix}^{-1} = \frac{1}{3 \cdot 23 - 6 \cdot (-6)} \begin{bmatrix} 23 & 6 \\ -6 & 3 \end{bmatrix} = \frac{1}{105} \begin{bmatrix} 23 & 6 \\ -6 & 3 \end{bmatrix}.$$