BM20A9200 Mathematics A – Exercise set 3

To be done by 25.–29.9.2023

Text in blue or red is not part of the problem or its solution. It's there as extra information to help you learn.

Exercise 1. Let $X = \{4, 5, 6\}$, $Y = \{a, b, c\}$ and $Z = \{l, m, n\}$. Consider the relations R from X to Y and S from Y to Z defined by:

$$R = \{(4, a), (5, b), (5, c), (6, b), (6, c)\},$$
$$S = \{(a, l), (a, m), (b, l), (b, m), (b, n), (c, m), (c, n)\}.$$

Find the following compositions of relations:

(a) RS

(b) RR^{-1}

Exercise 2. Let us denote $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, that is, $\mathbb{R}^+$ is the set of all positive real numbers.

(a) Let us define a binary relation T on $\mathbb{R}^+$ so that T consists of the pairs (x, x^2) where $x \in \mathbb{R}^+$. What is the inverse relation of T?

(b) Let us define a binary relation S on $\mathbb{R}$ so that S consists of the pairs (x, x^2), where $x \in \mathbb{R}$. What is the inverse relation of S?

Exercise 3. Let A and B be sets. Suppose that R and S are relations from A to B, that is, $R, S \subseteq A \times B$. Prove that

$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}.$$

Exercise 4. Let P be the proposition "I think" and Q the proposition "I am". The following table shows a few propositions and their meaning in natural language:

$P \Rightarrow Q$	If I think, then I am. (I think, therefore I am –Decartes)
$\neg Q \Rightarrow \neg P$	If I am not, then I don't think.
$P \wedge \neg Q$	I think and I am not.
$\neg P \Rightarrow \neg Q$	If I don't think, then I am not.

(a) Write the truth tables for the sentences.

(b) Which of the propositions are logically equivalent?

(c) Using P and Q, invent a few new propositions which are equivalent to some of the propositions above, and prove that they are so using a truth table.

Exercise 5. In mathematics, it's often useful to figure out what is the negation of a given proposition. For example if we can't prove some theorem, maybe we can prove its negation. Also, we will need the ability to form the negation of a proposition when studying indirect proof methods later.

(a) Connect each proposition on the left to its negation on the right (and prove why).

$$\begin{array}{rl} P_0 \wedge P_1 & \qquad P_0 \wedge \neg P_1 \\ P_0 \vee P_1 & \qquad \neg P_0 \wedge \neg P_1 \\ P_0 \Rightarrow P_1 & \qquad \neg P_0 \vee \neg P_1 \end{array}$$

(b) The following propositions say something about an unknown real number x. Write the negation in each case, but do it in such a way that the negation symbol $\neg$ does not appear there. You can use the symbols $\leq$, $<$, $\geq$, $\notin$ etc.

1. $(x^2 = 1) \wedge (x > 0)$
2. $(x^2 = 2) \vee (x \in \mathbb{Q})$
3. $(x^2 = 1) \Rightarrow (x > 0)$
4. $(x^2 = 2) \Leftrightarrow (x \in \mathbb{Q})$

Hint: You don't need to try to interpret the propositions to find their negation. Try using (a) as a guide.

Exercise 6. In lectures it was shown that $P \Rightarrow Q$ is logically equivalent to $\neg P \vee Q$. In other words, the operation $\Rightarrow$ can be defined by using $\vee$ and $\neg$.

(a) Define $\vee$ using $\neg$ and $\Rightarrow$.

(b) Define $\wedge$ using $\neg$ and $\vee$.

Prove the correctness of your answer by using truth tables.