# BM20A9200 Mathematics A – Exercise set 2

To be done by 18.–22.9.2022

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**Exercise 1.** Let $U$ be a set and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be nonempty families of subsets of $U$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Show (=prove) that the following inclusions hold:

(a) $\bigcup \mathcal{F}_1 \subseteq \bigcup \mathcal{F}_2$

(b) $\bigcap \mathcal{F}_1 \supseteq \bigcap \mathcal{F}_2$

**Solution.**

(a) **Info:** To prove that a set $S$ is a subset of a set $W$ we need to show that any element $x \in S$ satisfies $x \in W$ too. Let's start the proof with that.

*Proof of (a).* Let $x \in \bigcup \mathcal{F}_1$. By the definition of $\bigcup$ this means that there is $A_1 \in \mathcal{F}_1$ such that $x \in A_1$. But recall that $\mathcal{F}_1 \subseteq \mathcal{F}_2$, so $A_1 \in \mathcal{F}_2$ too. Hence $x \in A_1 \in \mathcal{F}_2$, and so by the definition of $\bigcup \mathcal{F}_2$ we have $x \in \bigcup \mathcal{F}_2$. □

(b) **Info:** As above, we need to show that any element of $\bigcap \mathcal{F}_2$ is also an element of $\bigcap \mathcal{F}_1$. Recall that $x \in \bigcap \mathcal{F}_1$ means that $x$ should be an element of *every* $A_1 \in \mathcal{F}_1$.

*Proof of (b).* Let $x \in \bigcap \mathcal{F}_2$. This means that for any $A_2 \in \mathcal{F}_2$ we have $x \in A_2$. Now, let $A_1 \in \mathcal{F}_1$ be arbitrary. Because $\mathcal{F}_1 \subseteq \mathcal{F}_2$, we have $A_1 \in \mathcal{F}_2$ and so $x \in A_1$. Hence $x \in A_1$ for any $A_1 \in \mathcal{F}_1$, which means that $x \in \bigcap \mathcal{F}_1$. □

**Exercise 2.** Let $U$ be a set and let $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}(U)$ be a nonempty family of subsets of $U$. Prove the following equalities:

(a) $(\bigcap \mathcal{F})^C = \bigcup \{A^C \mid A \in \mathcal{F}\}$

(b) $(\bigcup \mathcal{F})^C = \bigcap \{A^C \mid A \in \mathcal{F}\}$

Recall that the complement of any $X \subseteq U$ is defined by $X^C = U \setminus X$.

**Solution.** **Info:** We will prove the equalities by showing that an element belongs to one side of the equation if and only if it belongs to the other side.

*Proof of (a).* We have

$$\begin{aligned} x \in (\bigcap \mathcal{F})^C &\iff x \notin \bigcap \mathcal{F} \\ &\iff \text{there is } A \in \mathcal{F} \text{ such that } x \notin A \\ &\iff \text{there is } A \in \mathcal{F} \text{ such that } x \in A^C \\ &\iff x \in \bigcup \{A^C \mid A \in \mathcal{F}\}. \end{aligned}$$

The second equivalence is true because the sentence "there is $A \in \mathcal{F}$ such that $x \notin A$" is the opposite of "for all $A \in \mathcal{F}$ we have $x \in A$" which in turn is equivalent to "$x \in \bigcap \mathcal{F}$. □

*Proof of (b).* This time

$$\begin{aligned} x \in \left(\bigcup \mathcal{F}\right)^C &\iff x \notin \bigcup \mathcal{F} \\ &\iff \text{for all } A \in \mathcal{F} \text{ we have } x \notin A \\ &\iff \text{for all } A \in \mathcal{F} \text{ we have } x \in A^C \\ &\iff x \in \bigcap \{A^C \mid A \in \mathcal{F}\}. \end{aligned}$$

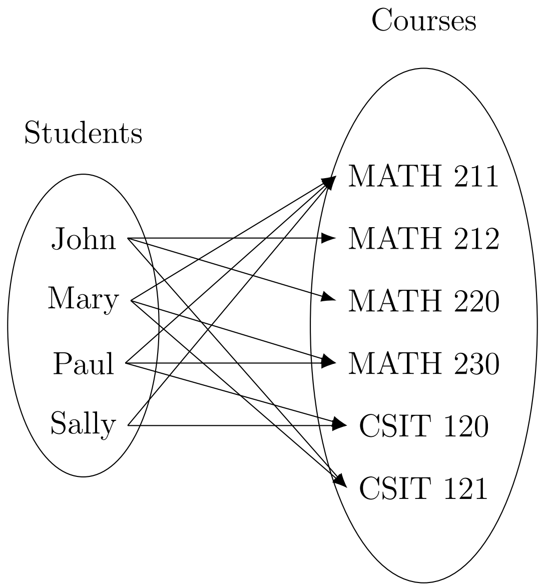
□

**Exercise 3.** The courses taken by John, Mary, Paul and Sally are listed below:

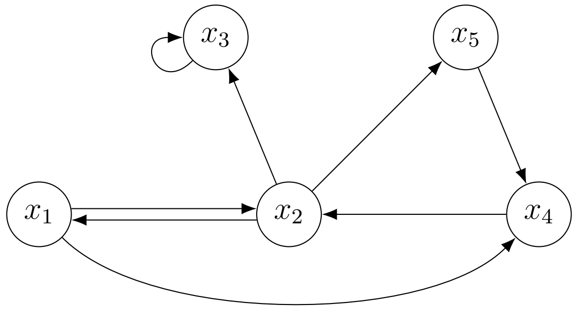
John: MATH 212, CSIT 121, MATH 220
Mary: MATH 230, CSIT 121, MATH 211
Paul: CSIT 120, MATH 211, MATH 230
Sally: MATH 211, CSIT 120

Give a graphical representation of the relation $R$ defined as $aRb$ if student $a$ is taking bourse $b$.

**Solution.**




**Exercise 4.** Write the set of ordered pairs for the relation represented by the following directed graph:



**Solution.** Let $X = \{x_1, x_2, x_3, x_4, x_5\}$. Then the directed graph can be represented by a binary relations $R$ over $X$ where

$$R = \{(x_1, x_2), (x_1, x_4), (x_2, x_1), (x_2, x_3), (x_2, x_5), (x_3, x_3), (x_4, x_2), (x_5, x_4)\}.$$

**Exercise 5.** Let $R$ be a binary relation on the set $\mathcal{P}(\{a, b\})$ with $a \neq b$ defined so that $(A, B) \in R$ holds if $A \,\Delta\, B = \emptyset$. Write out the relation $R$.

**Solution.** Let's calculate $\mathcal{P}(\{a, b\})$ first:

$$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Let's then calculate $A \,\Delta\, B$ for $A, B \in \mathcal{P}(\{a, b\})$:

| $A \,\Delta\, B$ | $\emptyset$ | $\{a\}$ | $\{b\}$ | $\{a, b\}$ |
|---|---|---|---|---|
| $\emptyset$ | $\emptyset$ | $\{a\}$ | $\{b\}$ | $\{a, b\}$ |
| $\{a\}$ | $\{a\}$ | $\emptyset$ | $\{a, b\}$ | $\{b\}$ |
| $\{b\}$ | $\{b\}$ | $\{a, b\}$ | $\emptyset$ | $\{a\}$ |
| $\{a, b\}$ | $\{a, b\}$ | $\{b\}$ | $\{a\}$ | $\emptyset$ |

We see that $(A, B) \in R$ if and only if $A = B$ for $A, B \in \mathcal{P}(\{a, b\})$. In other words,

$$R = \{(\emptyset, \emptyset), (\{a\}, \{a\}), (\{b\}, \{b\}), (\{a, b\}, \{a, b\})\}.$$

**Exercise 6.** Let $A, B, C$ be sets. Prove the following equalities:

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

**Solution.** Let us prove this with the same technique as Exercise 2: by writing out the definitions of the symbols until we are dealing with elementary operators such as $\in$ and logic.

*Proof of (a).* We have

$$\begin{aligned}
(a, b) \in A \times (B \cap C) &\iff a \in A \text{ and } b \in B \cap C \\
&\iff a \in A,\ b \in B \text{ and } b \in C \\
&\iff (a, b) \in A \times B \text{ and } (a, b) \in A \times C \\
&\iff (a, b) \in (A \times B) \cap (A \times C).
\end{aligned}$$

$\square$

*Proof of (b).* This time

$$\begin{aligned}
(a, b) \in A \times (B \cup C) &\iff a \in A \text{ and } b \in B \cup C \\
&\iff a \in A \text{ and } (b \in B \text{ or } b \in C) \\
&\iff (a \in A \text{ and } b \in B) \text{ or } (a \in A \text{ and } b \in C) \\
&\iff (a, b) \in A \times B \text{ or } (a, b) \in A \times C \\
&\iff (a, b) \in (A \times B) \cup (A \times C).
\end{aligned}$$

$\square$