BM20A9200 Mathematics A – Exercise set 12

To be done by 4.–8.12.2023

Text in blue or red is not part of the problem or its solution. It's there as extra information to help you learn.

The exercise sessions of Independence Day 6.12. have been moved to other days. Please check your schedule.

Exercise 1. Let $\mathbf{A}$ be the following augmented matrix

$$\mathbf{A} = \left[\begin{array}{ccc|c} -1 & -2 & 5 & 1 \\ -1 & 7 & 3 & 3 \\ 5 & 1 & 6 & 4 \end{array}\right]$$

Perform the following elementary row operations one after the other starting from $\mathbf{A}$.

a) Multiply the elements of row 2 by 6,

b) then interchange rows 1 and 3,

c) then multiply row 1 by two and add the result to row 2.

Solution.

a)

$$\left[\begin{array}{ccc|c} -1 & -2 & 5 & 1 \\ -6 & 42 & 18 & 18 \\ 5 & 1 & 6 & 4 \end{array}\right]$$

b)

$$\left[\begin{array}{ccc|c} 5 & 1 & 6 & 4 \\ -6 & 42 & 18 & 18 \\ -1 & -2 & 5 & 1 \end{array}\right]$$

c)

$$\left[\begin{array}{ccc|c} 5 & 1 & 6 & 4 \\ 4 & 44 & 30 & 26 \\ -1 & -2 & 5 & 1 \end{array}\right]$$

Exercise 2. Let

$$\mathbf{A} = \begin{bmatrix} 6 & -1 & 2 & 4 \\ 2 & -1 & 6 & 2 \\ -2 & 7 & -3 & 9 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 4 & 0 \\ 1 & -2 & 10 \\ 2 & 2 & -3 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 7 & 5 & 0 \\ 6 & 5 & -3 \\ 9 & 5 & -1 \end{bmatrix}$$

a) Calculate the matrix product $\mathbf{ABC}$.

b) Find some other legal product where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ each appear once and calculate it.

Solution.

a) Let's calculate $\mathbf{AB}$ first using the matrix product rule:

$$\mathbf{AB} = \begin{bmatrix} 6 & -1 & 2 & 4 \\ 2 & -1 & 6 & 2 \\ -2 & 7 & -3 & 9 \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 \\ 2 & 4 & 0 \\ 1 & -2 & 10 \\ 2 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} (6,-1,2,4)\cdot(0,2,1,2) & (6,-1,2,4)\cdot(-2,4,-2,2) & (6,-1,2,4)\cdot(1,0,10,-3) \\ (2,-1,6,2)\cdot(0,2,1,2) & (2,-1,6,2)\cdot(-2,4,-2,2) & (2,-1,6,2)\cdot(1,0,10,-3) \\ (-2,7,-3,9)\cdot(0,2,1,2) & (-2,7,-3,9)\cdot(-2,4,-2,2) & (-2,7,-3,9)\cdot(1,0,10,-3) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -12 & 14 \\ 8 & -16 & 56 \\ 29 & 56 & -59 \end{bmatrix}$$

Then $\mathbf{ABC} = (\mathbf{AB})\mathbf{C}$,

$$(\mathbf{AB})\mathbf{C} = \begin{bmatrix} 8 & -12 & 14 \\ 8 & -16 & 56 \\ 29 & 56 & -59 \end{bmatrix} \begin{bmatrix} 7 & 5 & 0 \\ 6 & 5 & -3 \\ 9 & 5 & -1 \end{bmatrix}$$

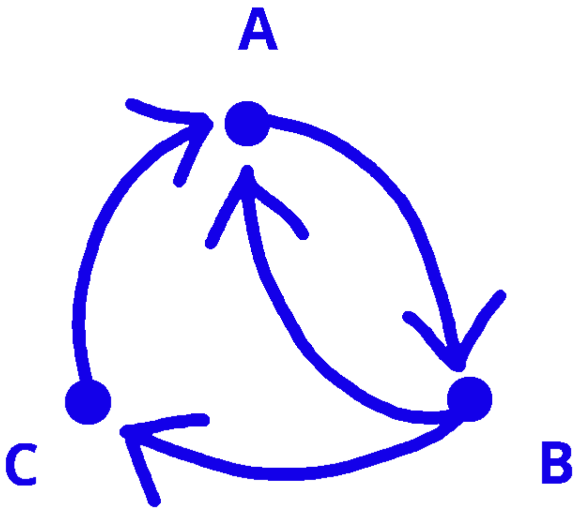
$$= \begin{bmatrix} (8,-12,14)\cdot(7,6,9) & (8,-12,14)\cdot(5,5,5) & (8,-12,14)\cdot(0,-3,-1) \\ (-12,-16,56)\cdot(7,6,9) & (8,-16,56)\cdot(5,5,5) & (8,-16,56)\cdot(0,-3,-1) \\ (29,56,-59)\cdot(7,6,9) & (29,56,-59)\cdot(5,5,5) & (29,56,-59)\cdot(0,-3,-1) \end{bmatrix}$$

$$= \begin{bmatrix} 110 & 50 & 22 \\ 464 & 240 & -8 \\ 8 & 130 & -109 \end{bmatrix}$$

b) Here we list all possible products. This is not needed for solving the problem. It's enough to calculate just one of the options. We have $\mathbf{A} \in \mathbb{R}^{3\times 4}$, $\mathbf{B} \in \mathbb{R}^{4\times 3}$ and $\mathbf{C} \in \mathbb{R}^{3\times 3}$. The product $\mathbf{MN}$ of matrices $\mathbf{M}$ and $\mathbf{N}$ is possible only if $\mathbf{M}$ has as many columns as $\mathbf{N}$ has rows. The only allowed products between $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are thus:

$$\mathbf{AB}, \quad \mathbf{BA}, \quad \mathbf{CA}, \quad \mathbf{BC}$$

If we think of "there being a legal product from matrix 1 to matrix 2" as a binary relation on the set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$, we can display it as a graph:



Looking for paths which go through all the matrices exactly once in the diagram, we see that there are only three options:

$$\mathbf{ABC}, \qquad \mathbf{BCA}, \qquad \mathbf{CAB}$$

The possible choices of matrix products are calculated as:

$$\mathbf{BCA} = \begin{bmatrix} -38 & 43 & -51 & 23 \\ 312 & -152 & 292 & 104 \\ 608 & -158 & 452 & 394 \\ 10 & -25 & 37 & -21 \end{bmatrix}$$

$$\mathbf{CAB} = \begin{bmatrix} 96 & -164 & 378 \\ 1 & 320 & 541 \\ 83 & -244 & 465 \end{bmatrix}$$

and **ABC** is in a).

Exercise 3. Prove that if $\mathbf{A} = [a_{ij}]_{m\times n}$ and $\mathbf{B} = [b_{ij}]_{n\times p}$ then

$$(\mathbf{AB})^\intercal = \mathbf{B}^\intercal\mathbf{A}^\intercal.$$

Hint: It suffices to consider an element in general position (i, j) and show that this element is the same on both sides of the equation above. You will also need to use the general formula for the element of a matrix product.

Solution.

Proof. Let's first make sure that the left (L) and right (R) hand sides have matrices of equal dimensions.

$$\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{B} \in \mathbb{R}^{n\times p} \implies \mathbf{AB} \in \mathbb{R}^{m\times p} \implies L = (\mathbf{AB})^\intercal \in \mathbb{R}^{p\times m}$$

and

$$\mathbf{B}^\intercal \in \mathbb{R}^{p\times n}, \mathbf{A}^\intercal \in \mathbb{R}^{n\times m} \implies R = \mathbf{B}^\intercal\mathbf{A}^\intercal \in \mathbb{R}^{p\times m}$$

so both sides have the same size.

Let's next check an arbitrary element from L. Let ℓ_{ij} be element on row i column j of $L = (\mathbf{AB})^\intercal$. Because of the transpose this is equal to element (j, i) of $\mathbf{AB}$, so it is the dot product of row j of $\mathbf{A}$ and column i of $\mathbf{B}$:

$$\ell_{ij} = \text{row}(\mathbf{A}, j) \cdot \text{col}(\mathbf{B}, i).$$

Now let r_{ij} be element (i, j) of $R = \mathbf{B}^\intercal \mathbf{A}^\intercal$. Then

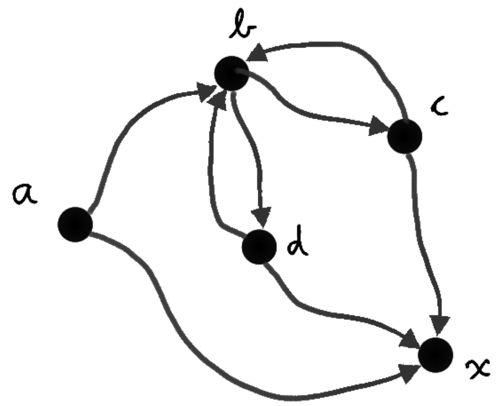
$$r_{ij} = \text{row}(\mathbf{B}^\intercal, i) \cdot \text{col}(\mathbf{A}^\intercal, j).$$

But because of the transpose we have

$$\begin{aligned} \text{row}(\mathbf{B}^\intercal, i) &= \text{col}(\mathbf{B}, i), \\ \text{col}(\mathbf{A}^\intercal, j) &= \text{row}(\mathbf{A}, j). \end{aligned}$$

Because for any vectors $\mathbf{v}, \mathbf{w}$ we have $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, the equations above show that $r_{ij} = \ell_{ij}$. Because this is true for any i, j this means that $L = R$. $\square$

Exercise 4. Consider the following directed graph:



(a) Write its adjacency matrix corresponding to the ordering of its elements (a, b, c, d, x).

(b) In how many different ways can one go from a to x in exactly 5 steps.

Solution.

(a) The adjacency matrix $\mathbf{M} = [m_{ij}]$ has $m_{ij} = 1$ if from vertex number i there is an arrow to vertex number j. Hence

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) The number of walks of length 5 from a to x is element $(1,5)$ of $\mathbf{M}^5$, so let's calculate that.

$$\mathbf{M}^2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}^2 = \mathbf{M}\mathbf{M}^2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}^4 = \mathbf{M}\mathbf{M}^3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and finally,

$$\mathbf{M}^5 = \mathbf{M}\mathbf{M}^4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 4 & 0 & 0 & 4 \\ 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The element $(1,5)$ of $\mathbf{M}^5$ is 4. Hence there are 4 different ways of getting from a to x in exactly 5 steps.

Exercise 5. Find the inverse $\mathbf{A}^{-1}$ of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & -4 \end{bmatrix}.$$

Verify your result.

Solution. Let's reduce $\mathbf{A}$ to a reduced row echelon form, and then do the same row operations to the identity matrix $\mathbf{I}_3$ to get $\mathbf{A}^{-1}$. Let's then verify the resul by calculating $\mathbf{A}\mathbf{A}^{-1}$. There is no need to calculate $\mathbf{A}^{-1}\mathbf{A}$ because $\mathbf{A}$ is a square matrix (Proposition 18 in Jouni's notes. It says that a right inverse or a left inverse of a square matrix is the inverse).

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 2 & 3 & -4 \end{bmatrix} \xrightarrow{-R_1 \to R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 3 & -4 \end{bmatrix} \xrightarrow{-2R_1 \to R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 5 & -4 \end{bmatrix}$$

$$\xrightarrow{-5R_2 \to R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{+R_3 \to R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{+R_2 \to R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's apply the same operations to $\mathbf{I}_3$ in the same order next:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{-R_1 \to R_2} \begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix} \xrightarrow{-2R_1 \to R_3} \begin{bmatrix}1&0&0\\-1&1&0\\-2&0&1\end{bmatrix}$$

$$\xrightarrow{-5R_2 \to R_3} \begin{bmatrix}1&0&0\\-1&1&0\\3&-5&1\end{bmatrix} \xrightarrow{+R_3 \to R_2} \begin{bmatrix}1&0&0\\2&-4&1\\3&-5&1\end{bmatrix} \xrightarrow{+R_2 \to R_1} \begin{bmatrix}3&-4&1\\2&-4&1\\3&-5&1\end{bmatrix}$$

The final verification:

$$\begin{bmatrix}1&-1&0\\1&0&-1\\2&3&-4\end{bmatrix}\begin{bmatrix}3&-4&1\\2&-4&1\\3&-5&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Exercise 6. Matrices are useful in computer graphics. For instance, let a point (x, y) be represented by a column vector $\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]$. Given an angle θ, the product

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

rotates the point (x, y) counterclockwise by θ with respect to the origin and the x-axis. Here sin and cos are trigonometric functions. See the following link for a reminder: https://en.wikipedia.org/wiki/Trigonometric_functions.

Let us consider a rectangle whose corners are at xy-coordinates $(0, 0), (0, 1), (3, 0), (3, 1)$. Describe what happens to these points in the rotation described above when $\theta = 60°$.

Solution. The values of sine and cosine evaluated at $\theta = 60°$ are

$$\sin(60°) = \frac{\sqrt{3}}{2}, \qquad \cos(60°) = \frac{1}{2}.$$

The rotation matrix becomes

$$\mathbf{R} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix}.$$

We can calculate where the points are mapped by calculating

$$\mathbf{R}\begin{bmatrix}0\\0\end{bmatrix}, \quad \mathbf{R}\begin{bmatrix}0\\1\end{bmatrix}, \quad \mathbf{R}\begin{bmatrix}3\\0\end{bmatrix}, \quad \mathbf{R}\begin{bmatrix}3\\1\end{bmatrix}$$

individually, or all in one go

$$\mathbf{R}\begin{bmatrix}0&0&3&3\\0&1&0&1\end{bmatrix}.$$

Let's do the latter:

$$\begin{aligned}
&\begin{bmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{bmatrix}\begin{bmatrix}0&0&3&3\\0&1&0&1\end{bmatrix}\\
&\quad= \begin{bmatrix}1/2\cdot 0 - \sqrt{3}/2\cdot 0 & 1/2\cdot 0 - \sqrt{3}/2\cdot 1 & 1/2\cdot 3 - \sqrt{3}/2\cdot 0 & 1/2\cdot 3 - \sqrt{3}/2\cdot 1\\ \sqrt{3}/2\cdot 0 + 1/2\cdot 0 & \sqrt{3}/2\cdot 0 + 1/2\cdot 1 & \sqrt{3}/2\cdot 3 + 1/2\cdot 0 & \sqrt{3}/2\cdot 3 + 1/2\cdot 1\end{bmatrix}\\
&\quad= \begin{bmatrix}0 & -\sqrt{3}/2 & 3/2 & (3-\sqrt{3})/2\\ 0 & 1/2 & 3\sqrt{3}/2 & (3\sqrt{3}+1)/2\end{bmatrix}
\end{aligned}$$

We can then read column-wise where each point mapped to. In conclusion

$$\begin{bmatrix}0\\0\end{bmatrix} \mapsto \begin{bmatrix}0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix} \mapsto \begin{bmatrix}-\sqrt{3}/2\\1/2\end{bmatrix}, \quad \begin{bmatrix}3\\0\end{bmatrix} \mapsto \begin{bmatrix}3/2\\3\sqrt{3}/2\end{bmatrix}, \quad \begin{bmatrix}3\\1\end{bmatrix} \mapsto \begin{bmatrix}(3-\sqrt{3})/2\\(3\sqrt{3}+1)/2)\end{bmatrix}$$

This picture shows how the points have moved: