Let $f$ be a map on $\mathbb{R}$ and $(x_1, x_2, x_3)$ be a periodic 3 orbit of $f$. Prove that the Lyapunov exponent of the orbit is given by

$$h(x_1) = \frac{\log|f'(x_1)| + \log|f'(x_2)| + \log|f'(x_3)|}{3}$$

$k=3 \quad \{x_1, x_2, x_3\}$ , $a_i \equiv \log|f'(x)|$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} a_i / n$$

list all $b_1 \ldots b_n$ looks like.

$b_n \equiv \frac{\sum_{i=1}^{n} a_i}{n}$ WTS.

$$b_1 = \frac{a_1}{1}$$

$$b_2 = \frac{a_1+a_2}{2}$$

$$b_3 = \frac{a_1+a_2+a_3}{3}$$

$$b_4 = \frac{2a_1+a_2+a_3}{4}$$

$$b_5 = \frac{2a_2+2a_2+a_3}{5}$$

$$b_6 = \frac{2a_2+2a_2+2a_3}{6} = \frac{a_1+a_2+a_3}{3}$$

$$b_7 = \frac{3a_1+2a_2+2a_3}{7}$$

$$b_8 = \frac{3a_1+3a_2+2a_3}{8}$$

$$b_9 = \frac{3a_1+3a_2+3a_3}{9} = \frac{a_1+a_2+a_3}{3}$$

Analyze.

$$b_1 = \frac{a_1}{1}$$

$$b_4 = \frac{2a_1+a_2+a_3}{4}$$

$$b_7 = \frac{3a_1+2a_2+2a_3}{7}$$

$$b_{10} = \frac{4a_1+3a_2+3a_3}{10}$$

$$b_2 = \frac{a_1+a_2}{2}$$

$$b_5 = \frac{2a_1+2a_2+a_3}{5} \qquad \frac{\left(\frac{n-1}{2}\right)}{n}$$

$$b_8 = \frac{3a_1+3a_2+2a_3}{8} \qquad \frac{\left(\frac{n-2}{2}\right)}{n}$$

$$b_{11} = \frac{4a_1+4a_2+3a_3}{11} \qquad \frac{\left(\frac{n-3}{2}\right)}{n}$$

$b_n = \{$ have three case! proof Continoue on back page

$b_n$ have three case, which $k=3$, we have 3 case

$b_n = 3n + l$ for $l \Rightarrow 0 \le l \le n$

when $l=0$, which we have $\dfrac{a_1+a_2+a_3}{3}$

like $b_3, b_6, b_9, b_{12} \ldots$

when $l=1$, which we have $\dfrac{\left(\frac{n+2}{3}\right)a_1 + \left(\frac{n-1}{3}\right)a_2 + \left(\frac{n-1}{3}\right)a_3}{n}$

like $b_1, b_4, b_7, b_{10} \ldots$

when $l=2$, which we have $\dfrac{\left(\frac{n+1}{3}\right)a_1 + \left(\frac{n+1}{3}\right)a_2 + \left(\frac{n-2}{3}\right)a_3}{n}$

like $b_2, b_5, b_8, b_{11} \ldots$

idea from professor luna.

Remember

$$\lim_{n\to\infty}\left(\frac{a_1+a_2+a_3}{3}\right) = hf(x_1)$$

$$\lim_{n\to\infty}\left(\left(\frac{n+2}{3n}\right)a_1 + \left(\frac{n-1}{3n}\right)a_2 + \left(\frac{n-1}{3n}\right)a_3\right) = \lim_{n\to\infty}\left(\frac{a_1}{3} + \frac{a_2}{3} + \frac{a_3}{3}\right) = hf(x_1)$$

$$\lim_{n\to\infty}\left(\left(\frac{n+1}{3n}\right)a_1 + \left(\frac{n+1}{3n}\right)a_2 + \left(\frac{n-2}{3n}\right)a_3\right) = \lim_{n\to\infty}\left(\frac{a_1}{3} + \frac{a_2}{3} + \frac{a_3}{3}\right) = hf(x_1$$

so $h(x_1) = \dfrac{\log|f'(x_1)| + \log|f'(x_2)| + \log|f'(x_3)|}{3}$ Prove that

② Let $f$ be a map on $\mathbb{R}$ and $x_1 \in \mathbb{R}$. Prove that $L$ Lyapunov number of the orbit of $x_1$ under $f$ is $L$ then the lyapunov number of the orbit of $x_1$ under $f^3$ is $L^3$

$k=3$. I want to prove that $Lf^3(x_1) = [Lf(x_1)]^3$

$$Lf^3(x_1) = \lim_{n\to\infty} \left( |f^{3'}(y_1)| \cdot |f^{3'}(y_2)| \cdot |f^{3'}(y_3)| \cdots |f^{3'}(y_n)| \right)^{\frac{1}{n}}$$

$$(f^3)'(y_1) = (f \circ f \circ f)'(y_1)$$
$$= f'(f^2(y_1)) f'(f(y_1)) f'(y_1)$$
$$= f'(x_3) f'(x_2) f'(x_1)$$

$$(f^3)'(y_2) = (f \circ f \circ f)'(y_2)$$
$$= f'(f^2(y_2)) f'(f(y_2)) f'(y_2)$$
$$= f'(x_6) f'(x_5) f'(x_4)$$

$$(f3)'(y_n) = (f \circ f \circ f)'(y_n)$$
$$= f'(f^2(y_n)) f'(f(y_n)) f'(y_n)$$
$$= f'(x_{3n}) f'(x_{3n-2}) f'(x_{3n-1})$$

All of them are step to prove.

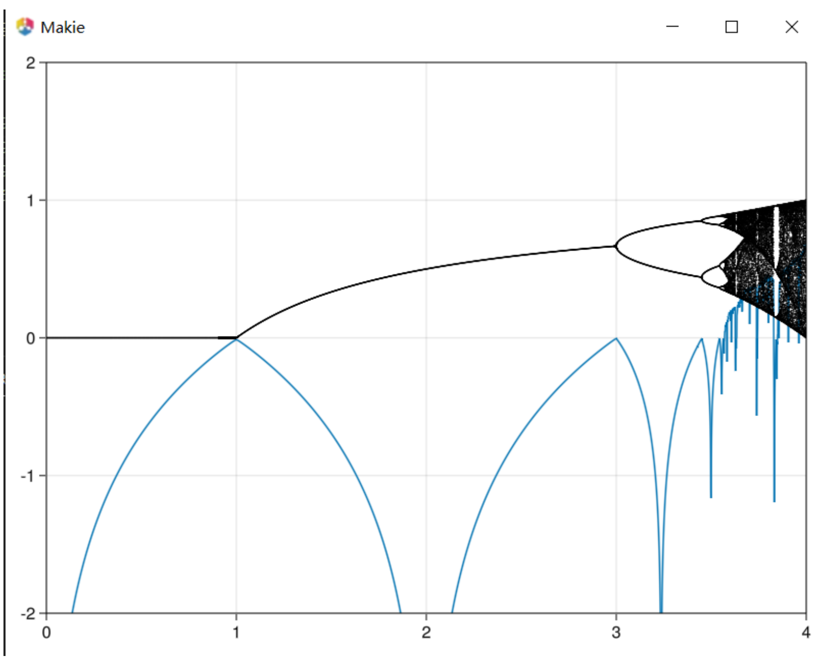
$\hat{n} \equiv 3n$

Taking $\sqrt[3]{\ }$ in $Lf^3(x_1)$

$$\sqrt[3]{Lf^3(x_1)} = (Lf^3(x_1))^{\frac{1}{3}} = \lim_{n\to\infty} \left( |f'(x_1)| \ldots |f'(x_{3n})| \right)^{\frac{1}{3n}}$$
$$= \lim_{\tilde{n}\to\infty} \left( |f'(x_1)| \ldots |f'(x_{\tilde{n}})| \right)^{\frac{1}{\tilde{n}}}$$
$$= Lf^3(x_1) = [Lf(x_1)]^3$$

```
include("chaos_toolsv3.jl") | lyapunovBifurcation (generic function with 3 methods)
GLMakie.activate!() | ✓


g(x) = 3.4x*(1 - x) | g (generic function with 1 method)

# Define a function which takes the parameter a of ax(1-x) and returns the non-zero fixed point.
fixedPointLogistic(a) = if (a != 0) 1 - 1/a else 0 end | fixedPointLogistic (generic function with 1 method)
fixedPointLogistic(3.4) | 0.7058823529411764

# Compute the absolute value of the derivative of g at the fixed point.
gPrime(x) = 3.4(1-2x) | gPrime (generic function with 1 method)
abs(gPrime(0.7)) | 0.9999999999999998
log(1.36) | 0.3074846997479607
#Therefore, the Lyapunov number of 0.6 is 0.5, and its Lyapunov exponent is -0.693147.

# Check that most bounded orbits tend to the fixed point 0.6. Therefore, most bounded orbits are asymptotically periodic.
itg(g, rand(), 20) | 20-element Vector{Real}:
cobwebPlotStartBar(g, 300, 0:0.001:1) | Figure()
```

```
# Let's consider the logistic map in general. Try some Lyapunov exponent approximations.
logistic(a) = x -> a*x*(1-x) | logistic (generic function with 1 method)
logisticPrime(a) = x -> a*(1-2x) | logisticPrime (generic function with 1 method)

# Try a = 3.86 with a random starting point.
lyapunovPoint(x) = log(abs(logisticPrime(3.86)(x))) | lyapunovPoint (generic function with 1 method)

# Notice that the convergence rate can be very slow.
x = rand() | 0.8931397138192239
mean(lyapunovPoint.(itg(logistic(3.86), x, 1000))) | 0.40776647750426653
mean(lyapunovPoint.(itg(logistic(3.86), x, 10000))) | 0.38878676066103884
mean(lyapunovPoint.(itg(logistic(3.86), x, 100000))) | 0.3819061872339732

# Let's plot the Lyapunov exponent against the possible values of a!
lyapunovGraph(0:0.001:4, logistic, 0.4, 5000) | Figure()
lyapunovBifurcation((a, t) -> logistic(a)(t), 1000, 0:0.001:4, 0.4, 0.1:0.2:0.9) | Figure()
```



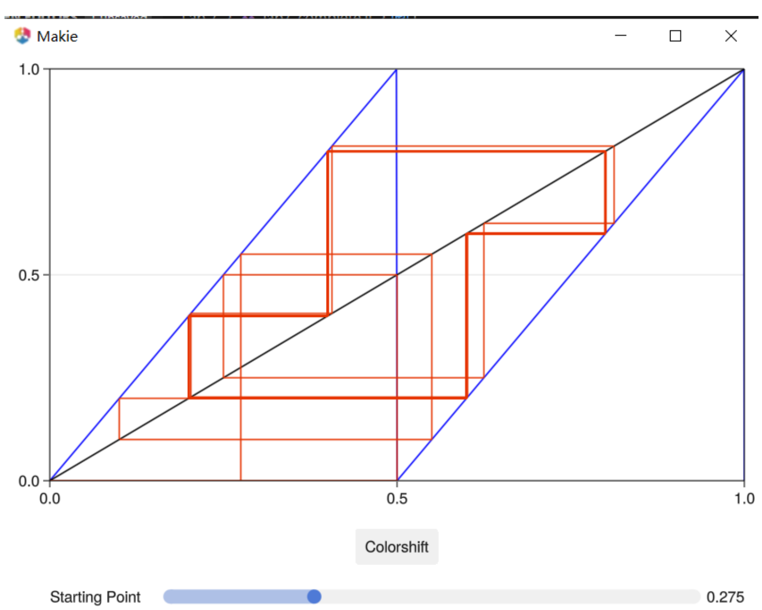

As TA Sylvia explains a different example in lab7, the question for 3.4x(1-x)is almost the same as the example we did on lab7. As you can see in the image, it's interesting that when the blue map point stops at 1, 3 and etc point, we will see that the black map will start to spread out and start to extend in different directions. The larger the number (or infinity) the more the numbers spread out and make the image getting darker than a small number.

```
# Discuss how the Lyapunov exponent corresponds to the bifurcation graph.
# Let's re-evaluate the Lyapunov graph just before chaos breaks at a = 3.56995
lyapunovGraph(3.54:0.00001:3.56995, logistic, 0.4, 5000) Figure()
lyapunovGraph(3.55:0.00001:3.56995, logistic, 0.4, 5000) Figure()
lyapunovGraph(3.565:0.00001:3.56995, logistic, 0.4, 5000) Figure()
lyapunovGraph(3.569:0.000001:3.56995, logistic, 0.4, 50000) Figure()
```

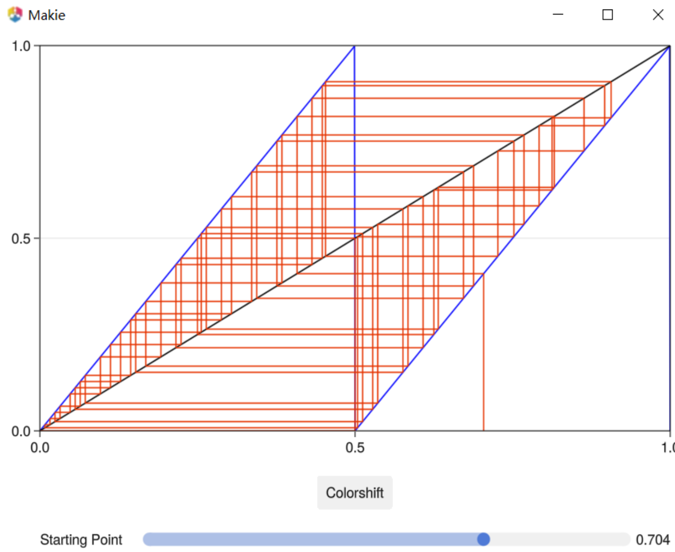
I tested a couple examples for the map and found out that the points are getting more!

# Question 4

```
# Let's try another familiar map.
h(x) = 2x % 1
#domain is 0 to 1
lines(0:0.001:1, h)

# Remember that due to numerical stability issues, all floating point starting points are practically sensitive.
cobwebPlotStartBar(h, 100, 0:0.001:1)

# Let's check Lyapunov numbers!
l = lyapunovNumber(h, rand(),5000)
exp(l)

# Let's try a rational input!
# These two have odd denominator, so they are periodic points
cobwebPlot(h, 1//3, 100, 0:0.001:1)
cobwebPlot(h, 24//59, 100, 0:0.001:1)

# These ones have an even denominator, so they are only "eventually" periodic.
cobwebPlot(h, 1//288, 100, 0:0.001:1)
cobwebPlot(h, 101//288, 100, 0:0.001:1)
cobwebPlot(h, 287//288, 100, 0:0.001:1)
```



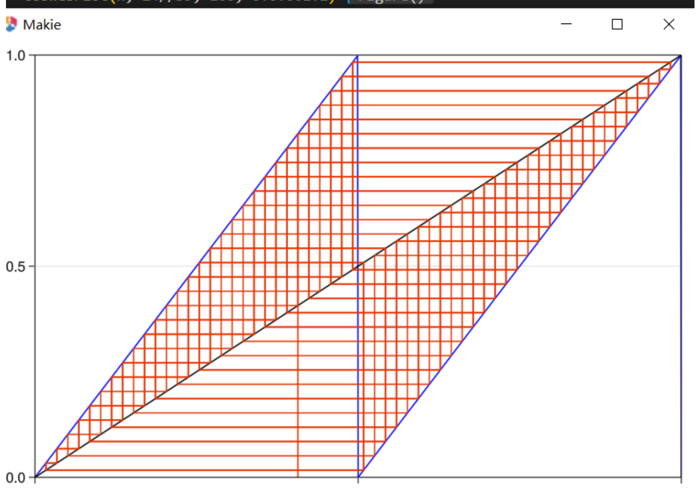

Example for 0.275




Example for 0.704

As TA Sylvia explains the same example in lab7, the question for 2x mod 1 and domain is 0 and 1, which we set a graph that could test different couple number, as you can see on the example on top, the number of the leftest side and rightest are all floating point starting points are practically sensitive.

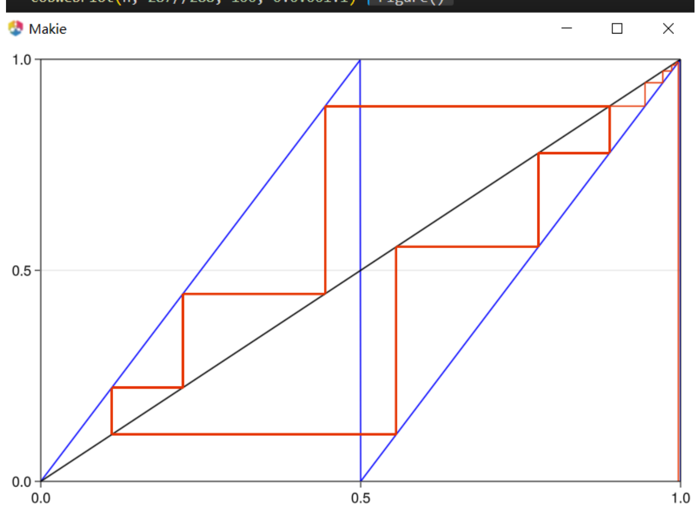
```
cobwebPlot(h, 1//3, 100, 0:0.001:1) | Figure()
cobwebPlot(h, 24//59, 100, 0:0.001:1) | Figure()
```




I also do the following case, in which we can see the difference.

**These two have odd denominator, so they are periodic points**

```
cobwebPlot(h, 1//288, 100, 0:0.001:1) | Figure()
cobwebPlot(h, 101//288, 100, 0:0.001:1) | Figure()
cobwebPlot(h, 287//288, 100, 0:0.001:1) | Figure()
```




**These ones have an even denominator, so they are only "eventually" periodic.**