① Consider the Hénon map for $a = -0.56$ and $b = -0.5$.

$f(x,y) = (-0.56 - x^2 - 0.5y, x)$.

Find and clasifty all its periodic 2 points.

periodic 2 points : $x^2 - (1-b)x + (-a) + (1-b)^2$

$$x^2 - (1+0.5)x + (+0.56) + (1+0.5)^2 = 0$$

$$\Rightarrow x^2 - 1.5x + 2.81 = 0$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{1.5 \pm \sqrt{(1.5)^2 - 4(2.81)(1)}}{2} = \frac{1.5 \pm \sqrt{2.25 - 11.2}}{2} = \frac{1.5 \pm \sqrt{8.95}\, i}{2}$$

There is no period two orbit in this map

Because we got complex number, there is no real solution.

$4a > 3(1-b)^2$ Henon map has period two orbit if this $4a > (3(1-b)^2)$

$\Rightarrow$ $a = -0.56$ $b = -0.5$

$$4(-0.56) > 3(1-(-0.5))^2$$

$$-2.24 > 6.75$$

It's not true. So there is no Period two oribit in this map

proof with professor in office hour.

② Let f be a linear map on $\mathbb{R}^2$ given by.

$f(x,y) = (x + 0.2y, 0.5x + 2y)$.

① Solve: Rectify that $(0,0)$ is saddle point.

$$Df(x,y) = \begin{pmatrix} 1 & 0.2 \\ 0.5 & 2 \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} 1 & 0.2 \\ 0.5 & 2 \end{pmatrix}$$

$Df(x,y) - \lambda$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 0.2 \\ 0.5 & 2-\lambda \end{pmatrix} \Rightarrow (1-\lambda)(2-\lambda) - (0.2)(0.5)$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 - 0.1$$

$$\Rightarrow \lambda^2 - 3\lambda + 1.9$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{3 \pm \sqrt{9 - 4(1)(1.9)}}{2}$$

$$x = 1.5 \pm 0.59$$

$x_1 = 2.09$ $x_2 = 0.91$

$$|1.5 + 0.59| > 1 \Rightarrow |2.09| > 1$$

$$|1.5 - 0.59| < 1 \Rightarrow |0.91| < 1$$

Both point are 2.09 and 0.91.
One direction shows attracting and one direction show repulling. Therefore, it's a saddle point.

② : Find the stable and unstable mainfold of (0,0)

$$\boxed{\begin{array}{l} \text{stable}\, f(0,0) = \text{unstable}\, f^{-1}(0,0) \\ \text{Unstable}\, f(0,0) = \text{Stable}\, f^{-1}(0,0) \end{array}}$$ Quick ideas.

$f(x,y) = (x+0.2y, 0.5x+2y)$

$$f(v) = Av = \begin{pmatrix} 1 & 0.2 \\ 0.5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & 0.2 \\ 0.5 & 2-\lambda \end{pmatrix} \Rightarrow \lambda^2 - 3\lambda + 2 - 0.1 = \lambda^2 - 3\lambda + 1.9$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9 - 4(1)(1.9)}}{2} = 1.5 \pm \frac{1.18}{2}$$

$$= 1.5 \pm 0.59$$

$x_1 = 2.09$. $x_2 = 0.91$

Stable: $\lim_{n\to\infty} |f^n(v) - f^n(p)| = 0$

unstable: $\lim_{n\to\infty} |f^{-n}(v) - f^{-n}(p)| = 0$.

Eigenvector for Stable and unstable

$\text{Stable}\, f(0,0) = \{\alpha x_1 : x \in \mathbb{R}\}$ eigenvector $\lambda_1 < 1$

$\text{unstable}\, f(0,0) = \{\alpha x_2 : x \in \mathbb{R}\}$ eigenvector $\lambda_2 > 1$

eigenvalue 0.91, eigenvalue <1

$\lambda_2 = 0.91$

$\exists\, x_2 = (x, y)$ such that $f(x_2) = \lambda_2 x_2 = 0.91\, x_2$

$$f(x_2) = \begin{bmatrix} 1 & 0.2 \\ 0.5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.91 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{cases} x + 0.2y = 0.91x \implies \dfrac{0.91x - x}{0.2} = y \\ 0.5x + 2y = 0.91y \implies \text{check } 0.5x + 2\left(\dfrac{-0.09x}{0.2}\right) = 0.91\left(\dfrac{-0.09x}{0.2}\right) \end{cases}$$

$0.5x + (-0.9x) \approx -0.40$

Talk and proof with Professor on office hour.

$$\boxed{\text{Stable } f(0,0) = \left\{ \alpha \begin{pmatrix} 1 \\ \frac{(0.91-1)}{0.2} \end{pmatrix} : \alpha \in \mathbb{R} \right\}}$$

eigenvector

$$\begin{pmatrix} x \\ \frac{0.91x - x}{0.2} \end{pmatrix} \xRightarrow{\text{if } x=1} \begin{pmatrix} 1 \\ \frac{(0.91-1)}{0.2} \end{pmatrix}$$

$\lambda_1 = 2.09$, eigenvalue >1.

$\exists\, x_1 = (x, y)$ st $f(x_1) = \lambda_1 x_1 = 2.09 x_1$

$$f(x_1) = \begin{bmatrix} 1 & 0.2 \\ 0.5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2.09 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{cases} x + 0.2y = 2.09x \implies \dfrac{2.09x - x}{0.2} = y \\ 0.5x + 2y = 2.09y \end{cases}$$

eigenvector

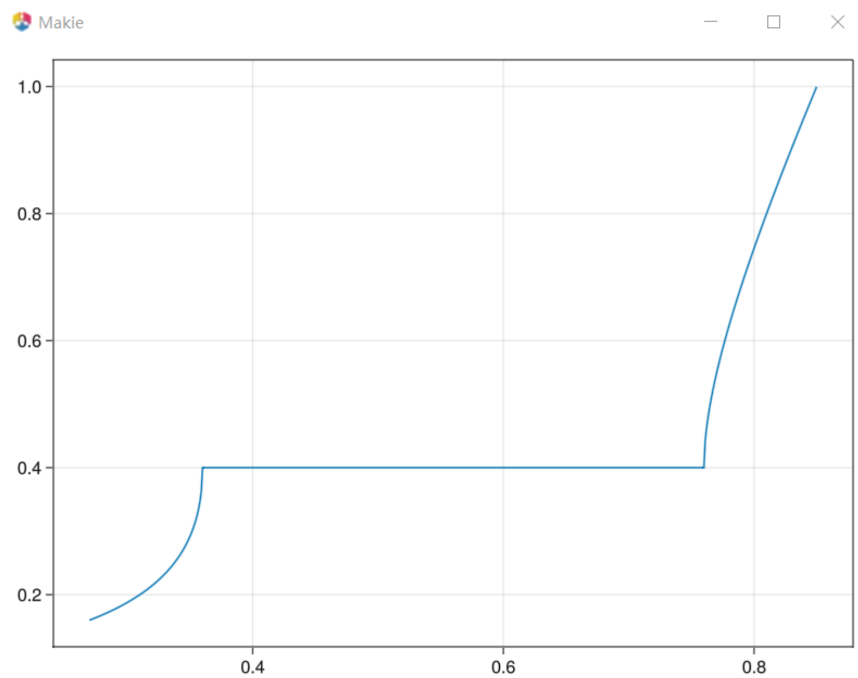
$$\begin{pmatrix} x \\ \frac{1.09x}{0.2} \end{pmatrix} \xRightarrow{x=1} \begin{pmatrix} 1 \\ \frac{1.09}{0.2} \end{pmatrix}$$

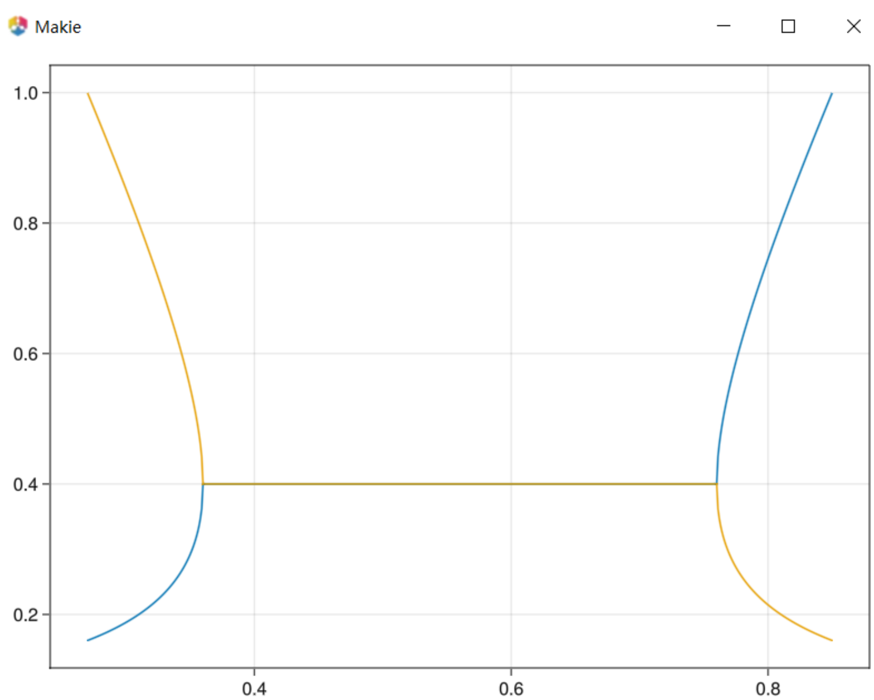
$$\text{unstable } f(0,0) = \left\{ \alpha \begin{pmatrix} 1 \\ \frac{1.09}{0.2} \end{pmatrix} , \alpha \in \mathbb{R} \right\}$$

```julia
include("chaos_toolsv2_2.jl") | parametric_2d
GLMakie.activate!() | ✓
using LinearAlgebra | ✓

# This is the Henon family of maps for varying a and b = 0.4
HenonFamily(a, x, y) = [a - x^2 + 0.4y, x] | HenonFamily (generic function with 1 method)

#x_2(a) = x^2 -0.6x -a +0.36
# Using the quadratic formula, we know that the two fixed points are given by:
x1(a) = (0.6 + sqrt(0.36 - 4*(-a+0.36)))/2 | x1 (generic function with 1 method)
x2(a) = (0.6 - sqrt(0.36 - 4*(-a+0.36)))/2 | x2 (generic function with 1 method)

# We compute the Jacobian of the Henon map with b=0.4. Note that since a changes it by a constant, the Jacobian does not depend on a.

J(a) = [-2*x1(a) 0.4; 1 0] * [-2*x2(a) 0.4; 1 0] | J (generic function with 2 methods)

# Computing the eigenvalues of the Jacobian at the first fixed point (x1, x1)

# Here, λ12 is the *1st* fixed point's *2nd* eigenvalue, hence the notation.
λ21(a) = abs((eigen(J(a)).values)[1]) | λ21 (generic function with 1 method)
λ22(a) = abs((eigen(J(a)).values)[2]) | λ22 (generic function with 1 method)
```

```julia
fig = Figure(); ax = Axis(fig[1,1]) | Axis with 1 plots:
aRange = 0.27:0.001:0.85 | 0.27:0.001:0.85
lines!(ax, aRange, λ21.(aRange)) | Lines{Tuple{Vector{Point{2, Float32}}}}
lines!(ax, aRange, λ22.(aRange)) | Lines{Tuple{Vector{Point{2, Float32}}}}
fig | Figure()


#idea for during the office hour
x1(a) = (0.6 + sqrt(0.36 - 4*(-a+0.36)))/2 | x1 (generic function with 1 method)
x2(a) = (0.6 - sqrt(0.36 - 4*(-a+0.36)))/2 | x2 (generic function with 1 method)

Henon(x,y) = [0.6 - x^2 + 0.4y,x] | Henon (generic function with 1 method)

x = Henon(x1(0.6),x2(0.6)) | 2-element Vector{Float64}:
Henon(x[1], x[2]) | 2-element Vector{Float64}:
[x1(0.5), x2(0.5)] | 2-element Vector{Float64}:
J(x, y) = [-2x 0.4; 1 0] | J (generic function with 2 methods)
```

```julia
Henon(x,y) = [0.6 - x^2 + 0.4y,x] | Henon (generic function with 1 method)

x = Henon(x1(0.6),x2(0.6)) | 2-element Vector{Float64}:
Henon(x[1], x[2]) | 2-element Vector{Float64}:
[x1(0.5), x2(0.5)] | 2-element Vector{Float64}:
J(x, y) = [-2x 0.4; 1 0] | J (generic function with 2 methods)

m(a) = J(x1(a),x1(a))*J(x2(a),x2(a)) | m (generic function with 1 method)
eign1(a)= abs.(eigen(m(a)).values[1]) | eign1 (generic function with 1 method)
lines(0.27:0.001:0.85,eign1) | FigureAxisPlot()
```





Question 3 I do it on both computers and use the class way to solve the problem. As you can see the picture above, the yellow and blue are the henon map we try to solve, when I zoom in, the point between 0.27 and 0.85 are slowly get close to the point 0.6, so which this, because both left and right side are getting close to 0.6, which it is a attracting point.

I also hand proof the point, In the following picture below is all my handwriting and follows what I learn in the class! Finally prove both of the fixed points are sink!

③ why f has a periodic 2 - sink for $0.27 < a < 0.85$

periodic 2 point $\left(x, \frac{a-x^2}{1-b}\right)$

This case

periodic 2 point $\left(x, \left(\frac{a-x^2}{1-0.4}\right)\right) = \left(x, \frac{a-x^2}{0.6}\right)$

one case — two case

$\left(x, \frac{0.27-x^2}{0.6}\right)$ — $\left(x, \frac{0.85-x^2}{0.6}\right)$

my own idea's

on the graph

Since $b=0.4$, when $0.27 < a < 0.85$.

Between this distance, the graph are attracting to eachother. Also I do the hand proof.

periodic 2 point: $x^2 - (1-b)x - a + (1-b)^2 = 0$

If $a=0.27$ and $a=0.85$

Fixed poit $(0.3\ 0.3)$

Fixed point $(0.27) = x^2 - 0.6x + 0.09$ $\quad x = 0.3$

Fixed point $(0.85) = x^2 - 0.6x - 0.49$ $\quad x =$ Fixed point $(0.94, -0.58)$ $(-0.58, 0.94)$

The fixed point on the bottom page is the fixed point for periodic 2.

$$\begin{bmatrix} -0.6-\lambda & 0.4 \\ 1 & -\lambda \end{bmatrix} = \lambda^2+0.6\lambda-0.4$$

$$\lambda = \frac{-0.6 \pm \sqrt{(0.36)-4(1)(-0.4)}}{2}$$

$$\lambda = \frac{-0.6 \pm 1.4}{2}$$

which $\left|\frac{-0.6-1.4}{2}\right| > 1$ and $\left|\frac{-0.6+1.4}{2}\right| < 1$

Fixed point $(0.3, 0.3)$ is a saddle point.

$$\begin{bmatrix} +1.16-\lambda & 0.4 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1.16\lambda - 0.4$$

$$\lambda = \frac{+1.16 \pm \sqrt{(1.16)^2-4(1)(-0.4)}}{2}$$

$$\lambda = \frac{1.16 \pm 1.22}{2}$$

which $\left|\frac{1.16+1.22}{2}\right| > 1$ and $\left|\frac{1.16-1.22}{2}\right| < 1$

Fixed point $(0.94, -0.58)$ and $(-0.58, 0.94)$ are saddle point.

with this, we can check periodic 2 point. stability.

Fixed point

$P_1(0.3, 0.3)$ $\quad P_2 = (0.3, 0.3)$

$$Df^2(P_1) = Df(P_2) \cdot Df(P_1)$$

$$Df(x,y) = \begin{bmatrix} -2x & 0.4 \\ 1 & 0 \end{bmatrix}$$

$$Df(0.3, 0.3) = \underbrace{\begin{bmatrix} -0.6 & 0.4 \\ 1 & 0 \end{bmatrix}}_{(0.3,0.3)} \underbrace{\begin{bmatrix} -0.6 & 0.4 \\ 1 & 0 \end{bmatrix}}_{(0.3,0.3)}$$

$$= \begin{bmatrix} 0.76 & -0.24 \\ -0.6 & 0.4 \end{bmatrix} = \det(A - \lambda I) = 0$$

$$\begin{bmatrix} 0.76-\lambda & -0.24 \\ -0.6 & 0.4-\lambda \end{bmatrix} = (0.76-\lambda)(0.4-\lambda) - 0.144$$

$$= \lambda^2 - 1.16\lambda + 0.304 - 0.144$$

$$= \lambda^2 - 1.16\lambda + 0.16$$

$$\frac{1.16 \pm \sqrt{(1.16)^2 - 4(1)(0.16)}}{2} = \frac{1.16 \pm 0.81}{2}$$

$$\Rightarrow \frac{1.16 + 0.81}{2} = \frac{1.97}{2} = 0.985 < 1$$

$$\Rightarrow \frac{1.16 - 0.81}{2} = \frac{0.36}{2} = 0.18 < 1$$

With both number is less than 1. It's a sink point.

Fixed point for 0.27 is attracting.

Fixed point $(0.94, -0.58)$ $(-0.58, 0.94)$

$$Df(x,y) = \begin{bmatrix} -2x & 0.4 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1.88 & 0.4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1.16 & 0.4 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1.78 & -0.75 \\ 1.16 & 0.4 \end{bmatrix} \Rightarrow \begin{bmatrix} -1.78-\lambda & -0.75 \\ 1.16 & 0.4-\lambda \end{bmatrix}$$

$$\Rightarrow (-\lambda-1.78)(-\lambda+0.4) - (-0.75)(1.16)$$

$$= \lambda^2 + 1.38\lambda - 2.18 + 0.87$$

$$= \lambda^2 + 1.38\lambda - 1.34$$

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1.38 \pm \sqrt{(1.38)^2 - 4(1)(-1.34)}}{2}$$

$$= \frac{-1.38 \pm 3.02}{2}$$

$$\boxed{\begin{matrix} \dfrac{-1.38+3.02}{2} = \dfrac{1.64}{2} < 1 \\ \\ \dfrac{-1.38-3.02}{2} = \dfrac{4.4}{2} < 1 \end{matrix}}$$

Fixed point 0.85 is attracting BC Both point are less than 1. It's Sink.

4. Consider the Saddle fixed point $(0,0)$ of the non linear map $f(x,y) = (\frac{x}{2}, 2y-7x^2)$

$S: \{(x, 4x^2) : x \in \mathbb{R}\}$.

Show that each point in S converges to $(0,0)$ under f.

$\boxed{\text{WTS: } unstable_{f}(0,0) \notin S. \text{ Given } f(x,y) = (\frac{x}{2}, 2y-7x^2) \quad S=\{(x,4x^2): x \in \mathbb{R}\} \quad (0,0) \text{ is saddle}}$

$(x,y) \in S \implies f(x,y) \in stable_{f}(0,0)$

$(x,y) \in (x,4x^2) \qquad f(x,4x^2) \in Stable_{f}(0,0)$

$\lim_{k\to\infty} |f^k(x,4x^2) - f^k(0,0)| = 0$

Inclass we prove $\lim_{k\to\infty} |f^k(0,0)| = 0$, Whichnow, I need to show $\lim_{k\to\infty} |f^k(x,4x^2)| = 0$, So $Stable_{f}(0,0) \subseteq S$, If $q \in S \implies$ $q \in Stable_{f}(0,0)$, $q = (x, 4x^2)$!

In class we did $q = (x, 4x^2+\alpha)$ but $q \in S$, which α is 0 in this case.

Case 1: $f^1(q)$

$f(q) = (\frac{x}{2}, 2(4x^2) - 7(x^2))$

$= (\frac{x}{2}, 8x^2 - 7x^2) = (\frac{x}{2}, x^2)$.

Case $f^2(q)$

$= f^2(\frac{x}{2}, x^2)$

$= (\frac{x}{2^2}, 2(x^2) - 7(\frac{x}{2})^2$

$= (\frac{x}{2^2}, 2x^2 - \frac{7}{4}x^2)$

$= (\frac{x}{2^2}, \frac{x^2}{4})$

now, we see that

$f^n(q) = (\frac{x}{2^n}, \frac{x^2}{2^n})$

$\lim_{n\to\infty} |f^n(q)| = \sqrt{|\frac{x}{2^n}| + |\frac{x^2}{2^n}|} = 0$

go to 0 go to 0.

I prove that $\lim_{n\to\infty} |f^n(x, 4x^2)| = 0$, which each point in S Converges to $(0,0)$ under f.