1). Let A be a following matrix:

$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$. Show that the fixed point $(0,0)$ is a source if $|a| > 1$.

Given following information in class and textbook

$$A^n = a^{n-1}\begin{pmatrix} a & n \\ 0 & a \end{pmatrix} \qquad A^n\begin{pmatrix} x \\ y \end{pmatrix} = a^{n-1}\begin{pmatrix} ax+ny \\ ay \end{pmatrix}$$

Case $|a|>1$.

$$\left|A^n\begin{pmatrix} x \\ y \end{pmatrix}\right| = |a^{n-1}(ax+ny, ay)|$$
$$= |a^{n-1}(ax+ny), a^n y|$$
$$\geq |a^{n-1}(ax+ny), a^n y|$$

two case; if $y=0$ or $y \neq 0$.

$y=0$, then $\lim_{n\to\infty} |(a^n x, 0)| = |(a^n x, 0)| = \infty$

TA sylvia 's idea. Both case with point to ∞.

$$\left|a^{n-1}\begin{pmatrix} 0 \\ ay \end{pmatrix}\right| \leq \left|a^{n-1}\begin{pmatrix} ax+ny \\ ay \end{pmatrix}\right|$$

$y \neq 0$, then $\Longrightarrow \lim_{n\to\infty} [a^n y] \Rightarrow \infty$

Both case $y=0, y\neq 0$ go to ∞, so $\infty > 1$. it's a Source.

② $\begin{pmatrix} 4 & 30 \\ 1 & 3 \end{pmatrix}$

$A = \begin{bmatrix} 4 & 30 \\ 1 & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4 & 30 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 4-\lambda & 30 \\ 1 & 3-\lambda \end{bmatrix}$

$= (4-\lambda)(3-\lambda) - 30$

$= \lambda^2 - 3\lambda - 4\lambda + 12 - 30 = \lambda^2 - 7\lambda - 18$

$(\lambda+2)(\lambda-9) = 0$

$\lambda_1 = -2$ and $\lambda_2 = 9$

$|\lambda_1| = |-2| = 2 > 1$

$|\lambda_2| = |9| = 9 > 1.$

Theorem 2.8 from book 2.4.

The origin is a source if all eigenvalues of A are larger than one in absolute value

therefore; $\begin{bmatrix} 4 & 30 \\ 1 & 3 \end{bmatrix}$ have eigenvalue -2 and 9.

Both $|-2|$ and $|9|$ are greater than 1

It's a Source!

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0.5 \\ 0.25 & 0.75 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0.5 \\ 0.25 & 0.75 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 0.5 \\ 0.25 & 0.75-\lambda \end{pmatrix}$$

$$= (1-\lambda)(0.75-\lambda) - 0.125$$

$$= \lambda^2 - 1.75\lambda + 0.75 - 0.125$$

$$= \lambda^2 - 1.75\lambda + 0.625$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1.75 \pm \sqrt{(1.75)^2 - 4(1)(0.625)}}{2}$$

$$= \frac{1.75 \pm \sqrt{3.0625 - 2.5}}{2} = \frac{1.75 \pm \sqrt{0.5625}}{2}$$

$$= 0.875 \pm 0.375$$

So we have

$|0.875 + 0.375| = |1.25| > 1$ greater than 1

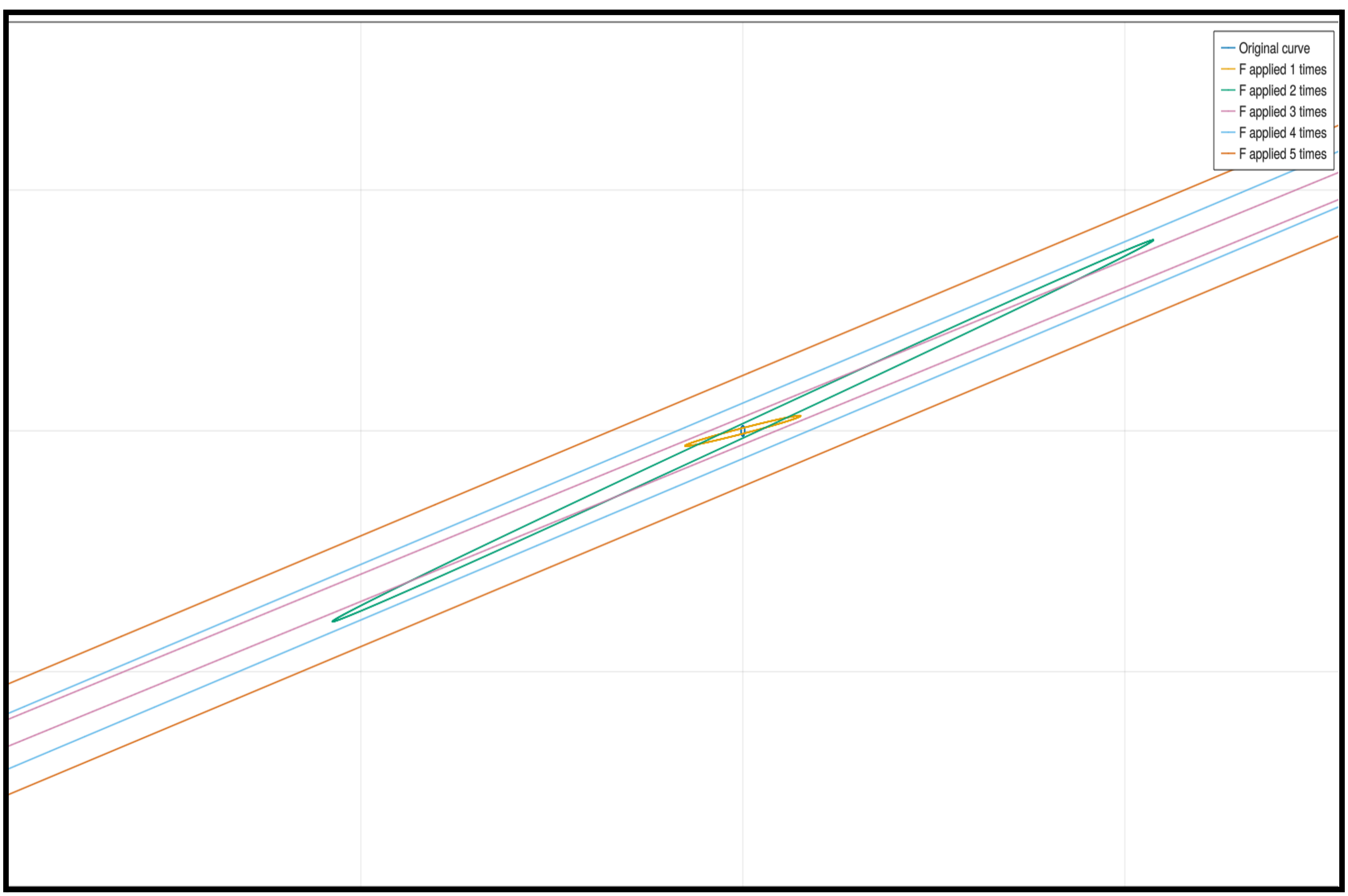
$|0.875 - 0.375| = |0.5| < 1$ less than 1.

We have one eigenvalue of absolute value greater than 1.
We also have one eigenvalue of absolute value less than 1.
It's a saddle

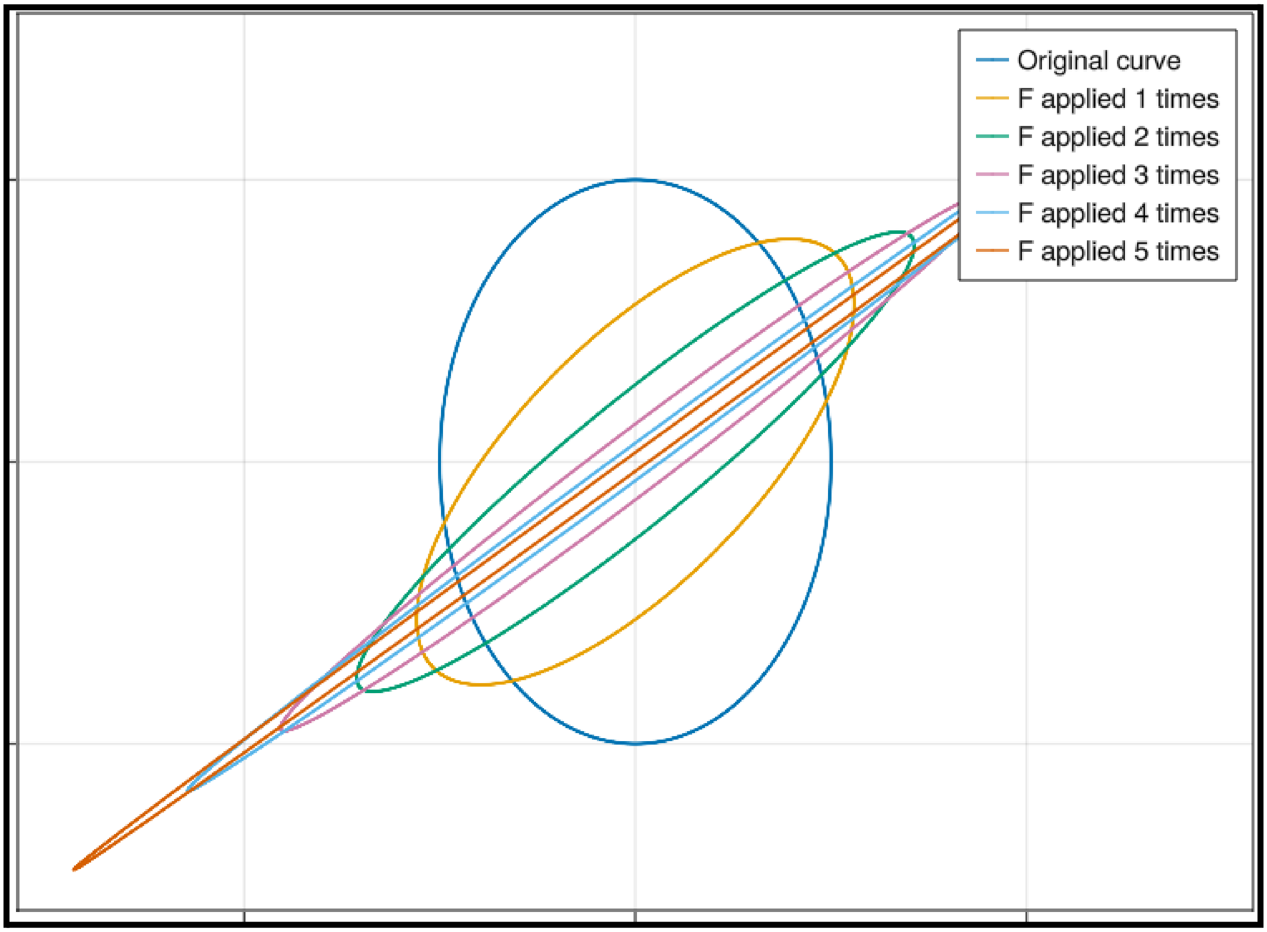
The Code for question 2

```
est > hw4.jl > mat2
using LinearAlgebra  ✓
include("chaos_toolsv2_1.jl")  parametric_2d

mat1 = [4 30;
        1 3]  2×2 Matrix{Int64}:

F1(x, y) = mat1*[x ; y]  F1 (generic function with 1 method)

λs1 = eigen(mat1).values  2-element Vector{Float64}:
γ(t) = [cos(t), sin(t)]  γ (generic function with 1 method)

parametric_2d(F1, 5, γ, 0:0.001:(2π))

#question 2 part 2
mat2 = [1 0.5;
        0.25 0.75]

F2(x, y) = mat2*[x ; y]
λs2 = eigen(mat2).values
γ1(t) = [cos(t), sin(t)]

parametric_2d(F2, 5, γ1, 0:0.001:(2π))
```

Part A answer and graph



Part B answer and graph



Part A shows after F applied 5 times, all of them are repulling, therefore. It's a source!

Part B shows after F applied 5 times, there is one direction attracting and one direction repulling, therefore. It's a saddle!

3. classify the fixed point of $g(x,y) = (x^2-5x+y, x^2)$ as sink, source, or saddles.

$g(x,y) = (x^2-5x+y, x^2)$.

$\begin{cases} x = x^2-5x+y \\ y = x^2 \end{cases}$

$\Rightarrow x = x^2-5x+x^2$

$0 = 2x^2-5x-x$

$0 = 2x^2-6x$

$0 = (2x)(x-3)$

$x=0 \quad x=3$

one fixed point $= (0,0)$

one fixed point $= (3,9)$.

Fixed point $(0,0)$.

$D(f(x,y)) = \begin{bmatrix} 2x-5 & 1 \\ 2x & 0 \end{bmatrix}$ $\quad Df(0,0) = \begin{bmatrix} -5 & 1 \\ 0 & 0 \end{bmatrix}$

solve

$\Rightarrow \det\begin{bmatrix} -5-\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = (-\lambda)(-5-\lambda) - 1 = \lambda^2+5\lambda-1$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-5 \pm \sqrt{25-4(1)(-1)}}{2} = \dfrac{-5\pm\sqrt{29}}{2} = \dfrac{-5\pm\sqrt{29}}{2}$

$x_1 = \dfrac{-5+\sqrt{29}}{2} = \left|\dfrac{-0.38}{2}\right| < 1$ attracting

$x_2 = \dfrac{-5-\sqrt{29}}{2} = \left|\dfrac{-10.38}{2}\right| > 1$ repelling

It's a saddle because one direction is repelling and one direction is attracting. Therefore. It's a saddle

Fixed point $(3, 9)$

$$Df(x,y) = \begin{bmatrix} 2x-5 & 1 \\ 2x & 0 \end{bmatrix} \qquad Df(3,9) = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 6 & -\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - 6 = \lambda^2 - \lambda - 6$$
$$= (\lambda - 3)(\lambda + 2)$$
$$\lambda = 3, \ \lambda = -2.$$

$\lambda_1 = |3| > 1$ Both are greater than 1.

$\lambda_2 = |-2| > 1$ It's a repelling

The direction for Both greater than 1. It's repelling.

4) Consider the Hénon map for $a=-0.56$ and $b=-0.5$.

$f(x,y) = (-0.56 - x^2 - 0.5y, x)$.

Fixed point.

$$\begin{cases} -x^2-0.5y-0.56 = x \\ x=y \end{cases}$$

$-x^2-0.5x-0.56=x$

$0 = x^2+1.5x+0.56$

$x=-0.7$

$x=-0.8$

one fixed point $(-0.7,-0.7)$

one fixed point $(-0.8,-0.8)$

Fix Point $(-0.7,-0.7)$.

$$Df(x,y) = \begin{bmatrix} -2x & -0.5 \\ 1 & 0 \end{bmatrix} \qquad Df(-0.7,-0.7) = \begin{bmatrix} 1.4 & -0.5 \\ 1 & 0 \end{bmatrix}$$

Solve the $\det(Df(-0.7,-0.7))$

$$\det\left(\begin{bmatrix} 1.4 & -0.5 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 1.4-\lambda & -0.5 \\ 1 & -\lambda \end{bmatrix}$$

$|x| = \sqrt{a^2+b^2}$

$= 0.7 \pm 0.1i$

$\frac{1.4}{2} \pm \frac{\sqrt{0.04}}{2} i$

$\Uparrow$

$$\Rightarrow (1.4-\lambda)(-\lambda) + 0.5 = +\lambda^2 - 1.4\lambda + 0.5$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{+1.4 \pm \sqrt{(1.4)^2 - 4(+1)(0.5)}}{+2} = \frac{1.4 \pm \sqrt{0.04}\,i}{2}$$

$x_1 = \sqrt{(0.7)^2+(0.1)^2} < 1$ one point is less than 1

$x_2 = \sqrt{(0.7)^2+(-0.1)^2} < 1$ one point is less than 1

It has two attaracting direction and it's good enough to show

Therefore; $(-0.7,-0.7)$ is a attracting sink

Point $(-0.8, -0.8)$

$$Df(x,y) = \begin{bmatrix} -2x & -0.5 \\ 1 & 0 \end{bmatrix} \qquad Df(-0.8, -0.8) = \begin{bmatrix} 1.6 & -0.5 \\ 1 & 0 \end{bmatrix}$$

Solve det $Df(-0.8, -0.8)$

$$\det\left(\begin{bmatrix} 1.6 & -0.5 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 1.6-\lambda & -0.5 \\ 1 & -\lambda \end{bmatrix}$$

$$\Rightarrow (1.6-\lambda)(-\lambda) + 0.5 = \lambda^2 - 1.6\lambda + 0.5$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1.6 \pm \sqrt{(1.6)^2 - 4(1)(0.5)}}{2} = \frac{1.6 \pm 0.74}{2}$$

$$x_1 = \left|\frac{2.34}{2}\right| = 1.17 > 1. \text{ repelling}$$

$$x_2 = \left|\frac{0.86}{2}\right| = 0.43 < 1. \text{ attracting}$$

one direction is attracting and one is repelling.
It's a saddle for point $(-0.8, -0.8)$.