№4

Дан

$$p(x,\theta) = \theta\{(\overset{-1}{\cancel{\theta}},1)/0\} + f(\theta)\cdot\{\cancel{\ }0,2\}$$

$$\int_{-1}^{1} \theta\, dx + f(\theta)\cdot 2 = 1$$

$$2 f(\theta) = \frac{1}{2} - \theta$$

$$p(x,\theta) = \theta\{(-1,1)/0\} + \left(\frac{1}{2}-\theta\right)\{0,2\}$$

$\vec{X}_n$ - выборка

$$\alpha_1 = M[\xi] = \int_{-1}^{1} \theta x\, dx + 2\left(\frac{1}{2}-\theta\right) = 1 - 2\theta$$

$$\alpha_2 = M[\xi^2] = \int_{-1}^{1} \theta x^2 dx + 4\left(\frac{1}{2}-\theta\right) = \theta\frac{1}{3}\cdot 2 + 2 - 4\theta =$$

$$= \frac{-10}{3}\theta + 2$$

$$D[\xi] = \alpha_2 - \alpha_1^2 = \frac{-10}{3}\theta + 2 - 1 - 4\theta - 4\theta^2 =$$

$$= \frac{2}{3}\theta + 1 - 4\theta^2$$

Метод моментов.

$\alpha_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}$

$\hat{\cancel{1}} - 2\theta = \overline{x} \Rightarrow$

$\Rightarrow \tilde{\theta}_1 = \frac{\hat{\cancel{1}} - \overline{x}}{2}$

~~Несмещенность~~:

~~......~~ ММП:

m – число "г"
k – число "о"
$\Big\}\, u$

$L(\theta) = \theta^{n-u} \cdot \left(\frac{1}{2} - \theta\right)^u$

$\ln L = (\ln\theta)\cdot(n-u) + \left(\ln\left(\frac{1}{2}-\theta\right)\right)\cdot u \to max$

$0 = (\quad)' = (n-u)\frac{1}{\theta} - \frac{u}{\frac{1}{2}-\theta}$

$0 = (n-u)\left(\frac{1}{2}-\theta\right) - u\theta$

$0 = \frac{n}{2} - n\theta - \frac{u}{2} + \cancel{u\theta} - \cancel{u\theta}$

$n\theta = \frac{n}{2} - \frac{u}{2}$

$\tilde{\theta}_2 = \frac{1}{2} - \frac{u}{2n} = \frac{1}{2} - \frac{1}{2}\,?$

b) Несм:

$$M[\tilde{\theta}_1] = M\left[\frac{1-\bar{x}}{2}\right] = \frac{1}{2} - \frac{1}{2} M[\bar{x}] =$$

$$= \frac{1}{2} - \frac{1}{2} M[\xi] = \frac{1}{2} - \frac{1}{2}(1-2\theta) = \theta$$

$$M[\tilde{\theta}_2] = M\left[\frac{1}{2} - \frac{1}{2}\nu\right] = \frac{1}{2} - \frac{1}{2} M[\nu] =$$

$$= \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} - \theta\right)\cdot 2 = \theta$$

Сост:

$$D[\tilde{\theta}_1] = D\left[\frac{2-\bar{x}}{2}\right] = \frac{1}{4} D[\bar{x}] = \frac{1}{4n} D[\xi] =$$

$$= \cancel{\frac{1}{4n}\left(\frac{14}{3}\theta - 2 - 4\theta^2\right)} =$$

$$= \frac{1}{4n}\left(\frac{2}{3}\theta + 1 - 4\theta^2\right) \xrightarrow[n\to\infty]{} 0 \Rightarrow \text{сост}$$

$$D[\tilde{\theta}_2] = D\left[\frac{1}{2} - \frac{1}{2}\nu\right] = \frac{1}{4} D[\nu] =$$

$$\frac{1}{4} \cdot \frac{(1-2\theta)2\theta}{n} \xrightarrow[n\to\infty]{} 0 \Rightarrow \text{сост.}$$

c) $D[\tilde{\theta}_1]$ и $D[\theta_2]$ огр. по $\forall [a, b] \in \left(0, \frac{1}{2}\right)$

$\Rightarrow \tilde{\theta}_1$ и $\tilde{\theta}_2$ — регул. по дост. ум

Регулярность модели

1) $p(x, \theta)$ непр. по θ на Θ

2) $\int\limits_{\mathbb{R}} \frac{\partial}{\partial \theta} p(x, \theta) dx = \int\limits_{-1}^{1} dx + (-1) \cdot 2 = 0$ //

$$\frac{\partial}{\partial \theta} \int\limits_{\mathbb{R}} p(x, \theta) dx = 0 \text{ //}$$

3) $I(\theta) = M\left[\frac{\partial \ln p(x, \theta)}{\partial \theta}\right] =$

$$= 2\left(\frac{\partial \ln\left(\frac{1}{2} - \theta\right)}{\partial \theta}\right)^2 \left(\frac{1}{2} - \theta\right) + \int\limits_{-1}^{1} \left(\frac{\partial \ln \theta}{\partial \theta}\right)^2 \theta dx =$$

↑ дискр.

$= \cancel{1 - 2\theta + 2\theta = 1}$ — ~~непр. дет и~~ > 0

$$= +2\left(\frac{1}{\frac{1}{2} - \theta}\right)^2 \left(\frac{1}{2} - \theta\right) + \frac{2}{\theta} = \frac{[illegible]}{\theta\left(\frac{1}{2} - \theta\right)} \text{ — непр. фун}$$

$a > 0$

Имеем право прим. нер. во.

$$\frac{\left(\frac{1}{2}-\theta\right)\theta}{2\cancel{(3+\Delta\Delta)}n} = D[\tilde{\theta}_2]$$

$\tilde{\theta}_2$ – ~~эфф~~ самая эфф оценка.

№T5

$\vec{X}_n$ – выборк

$$\alpha_1 = M[S] = \int_{\theta}^{2\theta} \frac{1}{\theta} x\,dx = \cancel{\frac{3}{4}\theta^2} \frac{3}{2}\theta$$

$$\alpha_2 = M[S^2] = \int_{\theta}^{2\theta} \frac{1}{\theta} x^2 dx = \frac{7}{3}\theta^2$$

$$D[S] = \alpha_2 - \alpha_1^2 = \frac{7}{6}\theta^4 - \cancel{\frac{3}{4}\theta^3} \cancel{\frac{9}{16}} \left(\frac{7}{3} - \frac{1}{4}\right)\theta^2$$

Метод моментов

$$\frac{3}{2}\theta_1^* = \overline{X}$$

$$\tilde{\theta}_1 = \cancel{\sqrt[3]{\frac{4}{3}\overline{X}}} \frac{2}{3}\overline{X}$$

~~КМ~~ ММП:

$$L(\theta) = \frac{1}{\theta^n}$$

$$\cancel{\ln L} = \cancel{\ldots} \frac{n}{\theta}$$

~~max~~ $\tilde{\theta}_2 = \max \cancel{(\vec{X}_n)} = x_{max}$

$$= \frac{x_{max}}{2}$$

b) Несмещ.

$$M[\tilde{\theta}_1] = M[\tfrac{2}{3}\overline{X}] = \tfrac{2}{3} \overset{M}{\cancel{D}}[S] =$$

$$= \frac{2}{3}\frac{3}{2}\theta = \theta$$

$$\cancel{M[\tilde{\theta}_2] = M[X_{max}] = \int_{\theta}^{2\theta} x \, d\left(\frac{x}{\theta}\right)^n =}$$

$$\cancel{= \int_{\theta}^{2\theta} \frac{1}{\theta} x\, n \left(\frac{x}{\theta}\right)^{n-1} dx = \frac{n(2^{n+1}-1)}{n+1}\theta}$$

$$\cancel{\tilde{\theta}_2' = \frac{n+1}{n(2^{n+1}-1)} x_{max}}$$

Сост.:

$$D[\tilde{\theta}_1] = D[\tfrac{3}{2}\overline{X}] = \frac{9}{4n} \cdot DS = \frac{9}{4n}\left(\frac{7}{3} - \frac{9}{4}\right)\theta^2$$

$$\cancel{D[\tilde{\theta}_2] = D[x_{max}] =}$$

Несмещ.

$$\cancel{M[\tilde{\theta}_2] = M[x_{max} - x_{min}]}$$

$$\varphi(y) = (F(y))^n \cancel{(1 - F(\theta - y))^n} =$$

$$= \left(\frac{1}{\theta}(y-\theta)\right)^n \cancel{\left(1 - \frac{1}{\theta}(\theta - (y-\theta))\right)^n} = \left(\frac{1}{\theta}\right)^n (y-\theta)^n$$

$$\varphi'(y) = n\left(\frac{1}{\theta}\right)^n y^{n-1}$$

$$M[\tilde{\theta}_2] = \int_\theta^{2\theta} n\left(\frac{1}{\theta}\right)^n (y-\theta)^{n-1} y\,dy = \cancel{2n\left(\frac{1}{\theta}\right)} =$$

$$= \int_0^\theta n\left(\frac{1}{\theta}\right)^n x^n\,dx = \frac{(n+1)\theta}{2n+1}$$

$$\tilde{\theta}_2' = \left(\frac{x_{max}}{2}\right)\frac{2n+1}{n+1}$$

Сост.

$$M[\tilde{\theta}_2^2] = \int_{\theta}^{2\theta} 2n\left(\frac{1}{\theta}\right)^{n} (y-\theta)^{n-1} y^2 dy =$$

$$= \cancel{\frac{(8n^2+8n+1)\theta^2}{2n^2+3n+1}} \quad \frac{(4n^2+8n+2)\theta^2}{n^2+3n+2}$$

$$D[\tilde{\theta}_2] = \left(\underbrace{\frac{4n^2+8n+2}{n^2+3n+2}}_{\xrightarrow[n\to\infty]{} 4} - \underbrace{\frac{(2n+1)^2}{(n+1)^2}}_{\xrightarrow[n\to\infty]{} 4}\right)\theta^2 \xrightarrow[n\to\infty]{} 0$$

~~реш~~

$D[\tilde{\theta}_2] \xrightarrow[n\to\infty]{} 0$ сост. //

c) $\cancel{D[\tilde{\theta}_1] = \theta^2 \frac{18+5}{n}}$ $\forall \theta>0 \; \forall n>0 \; D[\tilde{\theta}_1] > D[\tilde{\theta}_2]$

$\cancel{D[\tilde{\theta}_2] =}$ $\Rightarrow \tilde{\theta}_2$ - эффектив

d) $\exists \; f(\theta, \overline{X_n}) = \frac{x_{max}}{\theta}$

$$\frac{x_{max}}{\theta} \sim \varphi(y) = P\left(y > \frac{x_{max}}{\theta}\right) = P(\theta y > x_{max})$$

$$= \left(\frac{\theta y - \theta}{\theta}\right)^n = (y-1)^n$$

$$\varphi(y) = \Phi' = n(y-1)^{n-1}$$

$$t_1 = \eta_{\frac{1-B}{2}} : \int_1^{t_1} \varphi(y)\,dy = \frac{1-B}{2} \Rightarrow t_1 \geq 1$$

$$(t_1-1)^n = \frac{1-B}{2}$$

$$t_1 = 1 + \sqrt[n]{\frac{1-B}{2}}$$

аналогично

$$t_2 = 1 + \sqrt[n]{\frac{1+B}{2}}$$

$$1 + \sqrt[n]{\frac{1-B}{2}} < \frac{X_{max}}{\theta} < 1 + \sqrt[n]{\frac{1+B}{2}}$$

$$\frac{X_{max}}{1+\sqrt[n]{\frac{1-B}{2}}} < \theta < \frac{X_{max}}{1+\sqrt[3]{\frac{1+B}{2}}}$$

e) Асимпт. ОММ

$$\tilde{\theta} = \frac{2}{3}\bar{X} \Rightarrow g(\bar{\alpha}) = \frac{2}{3}\tilde{\alpha}$$

$$g(\alpha) = \frac{2}{3}\alpha_1$$

$$\sigma(\alpha) = \sqrt{\nabla^T g(\alpha) K \cdot \nabla g(\alpha)} = \sqrt{\frac{2}{3}(\alpha_2 - \alpha_1^2)\frac{2}{3}}$$

$$\sqrt{n}\,\frac{g(\bar{\alpha}) - g(\alpha)}{\sigma(\alpha)} \rightsquigarrow N(0,1)$$

$$\sqrt{n}\,\frac{\frac{2}{3}\tilde{\alpha}_1 - \frac{2}{3}\alpha_1}{\sqrt{\frac{2}{3}(\tilde{\alpha}_2 - \tilde{\alpha}_1^2)\frac{2}{3}}} = \sqrt{n}\,\frac{\tilde{\alpha}_1 - \alpha_1}{\sqrt{\alpha_2 - \alpha_1^2}} \rightsquigarrow N(0,1)$$

$$-1{,}96 < \sqrt{n}\,\frac{\tilde{\alpha}_1 - \frac{3}{2}\theta}{\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1}} < 1{,}96$$

$$\frac{2}{3}\left(+1{,}96\sqrt{\frac{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}{n}} + \tilde{\alpha}_1\right) < \theta < \frac{2}{3}\left(-1{,}96\sqrt{\frac{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}{n}} + \tilde{\alpha}_1\right)$$

N T 6

a) $p(x) = \begin{cases} \frac{\theta - 1}{x^{\theta}} & , x \geq 1 ; \quad \theta > 1 \\ 0 & , x < 1 ; \end{cases}$

$$L(\theta) = \frac{(\theta - 1)^n}{(\prod x_i)^{\theta}}$$

$$\ln L(\theta) = n(\theta - 1) - \theta \sum x_i$$

$$\frac{\partial}{\partial \theta}(\ln L) = n - \theta \sum x_i = 0$$

$$\frac{1}{\theta} = \overline{X} \qquad \tilde{\theta} = \frac{1}{\overline{X}}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-n}{(\theta - 1)^2} < 0 \Rightarrow \max$$

~~б) ОМП~~

~~Регулярность модели.~~

~~1) P(x)~~

b) $f(\vec{X}_n, \theta) - ?$

для мед

$$\int_{-\infty}^{\tilde{x}} p(x)\,dx = \frac{1}{2}$$

$$\int_{1}^{\tilde{x}} \frac{\theta - 1}{x^{\theta}}\,dx = -\tilde{x} + 1$$

$$\tilde{x} = \left(\frac{1}{2}\right)^{\frac{1}{1-\theta}}$$

$$med(\theta) = 2^{\frac{1}{\theta - 1}}$$

Регулярность (для ОМП)

1) $p(x, \theta)$ - непр. дифф. и $\exists$ по θ

2) $$\int_{1}^{\infty} \frac{\partial}{\partial \theta} \frac{\theta - 1}{x^{\theta}}\,dx = \int_{1}^{\infty} \frac{x^{\theta} - (\theta - 1)\theta x^{\theta - 1}}{x^{2\theta}} = 0$$

$$\frac{\partial}{\partial \theta} \int_{1}^{\infty} \frac{\theta - 1}{x^{\theta}}\,dx = 0$$

3) $I(\theta) \ominus$

$$\ln(p(x,\theta)) = \ln(\theta-1) - \theta \ln x$$

$$\frac{\partial \ln(p(x;\theta))}{\partial\theta} = \frac{1}{\theta-1} - \ln x$$

$$\theta \int_1^\infty \left(\frac{1}{\theta-1} - \ln x\right)\frac{\theta-1}{x^\theta}\,dx = \frac{1}{(\theta-1)^2}$$ - непр. на Θ, > 0 на Θ

$\Rightarrow$ регулярна,

ОМП:

$$\sqrt{n}\,\frac{f(\hat\theta) - f(\theta)}{\sigma(\theta)} \rightsquigarrow N(0,1)$$

$$\sigma = \sqrt{\nabla^T f(\theta)\, I^{-1}\, \nabla f(\theta)}$$

возьмем $f = \text{med}\,\theta$

$$\nabla f(\theta) = 2^{\frac{1}{\theta-1}} \ln 2 \left(-\frac{1}{(\theta-1)^2}\right)$$

$$\sigma = \frac{2^{\left(\frac{1}{\theta-1}\right)} \ln 2}{\theta-1}$$

$$g(\theta) = 2^{\left(\frac{1}{\theta-1}\right)}$$

$$\sigma = \frac{g(\theta)\ln 2}{\theta-1}$$

$$\sqrt{n}\ \frac{g(\tilde{\theta}) - g(\theta)}{\left(\frac{g(\tilde{\theta})\ln 2}{\tilde{\theta}-1}\right)} \rightsquigarrow N(0,1)$$

$$g(\tilde{\theta}) - \frac{g(\tilde{\theta})\cdot\ln 2}{(\tilde{\theta}-1)\sqrt{n}} < g(\theta) < g(\tilde{\theta}) + \frac{g(\tilde{\theta})\cdot\ln 2}{(\tilde{\theta}-1)\sqrt{n}}$$

c) $$\sqrt{n}\ \frac{\tilde{\theta}-\theta}{\sigma(\theta)} \rightsquigarrow N(0,1)$$

$$\sigma(\tilde{\theta}) = \theta - 1$$

$$\sqrt{n}\ \frac{\tilde{\theta}-\theta}{\theta-1} \rightsquigarrow N(0,1)$$

$$\frac{-1{,}96(\theta-1)}{\sqrt{n}} + \frac{n}{\sum \ln x_i} < \theta < \frac{1{,}96(\theta-1)}{\sqrt{n}} + \frac{n}{\sum \ln x_i}$$