№1

$\xi \sim R(0, \theta) \quad \theta > 0$ 14

Вероятностная модель

θ: $\tilde{\theta}_1 = 2\bar{X}$

$\tilde{\theta}_2 = X_{min}$

$\tilde{\theta}_3 = X_{max}$

$\theta_4 = X_1 + \frac{\sum_{k=2}^{n} x_k}{n-1}$

$p(x) = \frac{1}{\theta}\{(0, \theta)\}$; $M[\xi] = \int_{-\infty}^{\infty} x\, p(x)\, dx = \frac{\theta}{2}$

$M[\xi^2] = \int_{-\infty}^{\infty} x^2 p(x)\, dx = \frac{\theta^2}{3}$

$D[\xi] = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$

$\tilde{\theta}_1$: Несм.

$\forall \theta > 0 \quad M[\tilde{\theta}_1] = M\left[2 \cdot \frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{2}{n}\sum M[x_i] =$

$= 2M[\xi] = \theta$ $\Rightarrow$ несмещ.

(тк случ. вел.)

Сост.

$D[\tilde{\theta}_1] = D\left[\frac{2}{n}\sum x_i\right] = \frac{4}{n^2}\sum D[x_i] = \frac{4}{n} D[x_i] =$

$\ominus \frac{4}{n} D[S] = \frac{\theta}{3n} \xrightarrow[n\to\infty]{} 0 \Rightarrow$ состоят.

дост. усл.

$\tilde{\theta}_2$: Несм:

$\underline{M}[\tilde{\theta}_2] = \int\limits_{-\infty}^{\infty} y\,\varphi(y)dy \ominus$

Φ - функция распред $\quad \Phi(y) = 1 - (1 - F(y))^n$

φ - ~~...~~ плотности $\quad \varphi(y) = \Phi'(y) = n(1 - F(y))^{n-1} \cdot p(y)$

$\frac{\theta}{y}$

$\ominus \int\limits_0^{\theta} yn\left(1 - \frac{\theta}{y}\right)^{n-1} \frac{1}{\theta}\, dy = -\int\limits_1^{\theta} n t^{n-1}(1-e)\theta\, dt = \frac{\theta}{n+1} \Rightarrow$

$t = 1 - \frac{y}{\theta}$

$\Rightarrow$ смещенная

$\tilde{\theta}_2' = (n+1)x_{min}$ - несмещ

сост:

$\tilde{\theta}_2'$ по опред $\quad \forall \theta > 0 \ \forall \varepsilon > 0 \hookrightarrow P(|\tilde{\theta}_2' - \theta| > \varepsilon) \xrightarrow[n\to\infty]{} 0$

$P(|\tilde{\theta}_2' - \theta| < \varepsilon) \geqslant P(\tilde{\theta}_2' \geq \theta + \varepsilon) =$

$= P\left(x_{min} > \frac{\theta + \varepsilon}{n+1}\right) = 1 - P\left(x_{min} < \frac{\theta+\varepsilon}{n+1}\right) =$

$= 1 - \left(1 - \left(1 - F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n\right) = \left(1 - \left(\frac{\theta+\varepsilon}{\theta(n+1)}\right)\right)^n$

$\xrightarrow[n\to\infty]{} \exp\left(-\frac{\theta+\varepsilon}{\theta}\right) \neq 0$ не сост

$\tilde{\theta}_2$ $P(|\tilde{\theta}_2 - \theta| \geq \varepsilon) \underset{\min}{=} P(\tilde{\theta}_2 < \theta - \varepsilon) =$

$= \Phi(\theta-\varepsilon) = 1 - \left(1 - \frac{\theta-\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 1 \Rightarrow$ не сост

$\tilde{\theta}_3$ Несм

$M[\tilde{\theta}_3] = \int_{-\infty}^{\infty} y\varphi(y)\,dy \ominus$

$\Phi(y) = (F(y))^n$

$\varphi(y) = \Phi'(y) = \ldots =$

$= n\left(\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}\{(0,\theta)\}$

$\ominus \int_0^\theta n\frac{y^n}{\theta^n}\,dy = \frac{n}{n+1}$ - смещ

$\tilde{\theta}_3' = \frac{n+1}{n} X_{max}$

$D[\tilde{\theta}_3'] = \frac{(n+1)^2}{n^2} D[\tilde{\theta}_3] \ominus$

$D[\tilde{\theta}_3] = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \frac{n\,\theta^2}{(n+2)(n+1)^2}$

$= \frac{\theta^2}{(n+2)(n+1)^2} \xrightarrow[n\to\infty]{} 0$ сост

По опред состоят.

θ_3:

$$P(|\tilde{\theta}_3 - \theta| \geqslant \varepsilon) = P(|x_{max} - \theta| \geqslant \varepsilon) =$$

$x_{max} \leq \theta$

$$= P(x_{max} \leqslant \theta - \varepsilon) = 0 + (F(\theta-\varepsilon))^n =$$

$$= \left(\frac{\theta-\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 0, \quad \text{При } \varepsilon > \theta \quad F(\theta-\varepsilon) \equiv 0$$

$0 < \varepsilon \leq \theta$

ПОДАРОК

Подарок!

для $\tilde{\theta}_3'$ аналогично:

$$P(|\tilde{\theta}_3' - \theta| \geqslant \varepsilon) = P\left(\left|\frac{n+1}{n} x_{max} - \theta\right| \geqslant \varepsilon\right) =$$

$$= P\left(\frac{n+1}{n} x_{max} \geqslant \varepsilon + \theta\right) = P\left(x_{max} \geqslant \frac{n}{n+1}(\varepsilon+\theta)\right) =$$

$\frac{n+1}{n} x_{max} > \theta$

$$= 1 - P\left(x < \frac{n}{n+1}(\varepsilon+\theta)\right) \xrightarrow[n\to\infty]{} 0$$

$\tilde{\theta}'$ - состоят.

$\tilde{\theta}_{4(5)}$: Несмещ:

$$M[\tilde{\theta}_4] = M\left[x_1 + \frac{1}{n-1}\sum_{k=2}^{n} x_k\right] = M[x_1] + \frac{1}{n-1} \cdot$$

$$\cdot \sum_{k=2}^{n} M[x_k] = \frac{\theta}{2} + \frac{\theta}{2} = \theta$$

несмещ

Сост:

$$D[\tilde{\theta}_4] = D\left[x_1 + \frac{1}{n-1}\sum_{k=2}^{n} x_k\right] = D[\xi] + \frac{1}{(n-1)^2}(n-1)D[\xi] =$$

$$= \frac{\theta^2}{12} + \frac{\theta^2}{12(n-1)} = \frac{\theta^2}{12} \cdot \frac{n}{n-1} \xrightarrow[n\to\infty]{} \not\to 0$$

Несостоят

Доказ: $\tilde{\theta}_4 \xrightarrow{P} \theta$

$$\tilde{\theta}_4 = x_1 + \frac{1}{n-1}\sum_{k=2}^{n} x_k$$

$$x_1 \xrightarrow{P} \xi$$

Применяем ЗБЧ Хинчина для $\left(\frac{1}{n-1}\sum x_k\right)$

$$\frac{1}{n-1}\sum_{k=2}^{n} x_k \xrightarrow{P} M[\xi] = \frac{\theta}{2}$$

$$\tilde{\theta}_4 \to \xi + \frac{\theta}{2} \neq \theta$$

не состоятельна

Эффективность оценок:

$\tilde{\theta}_1$ и $\tilde{\theta}_3'$ – сост. и несм.

$$D[\tilde{\theta}_1] = \frac{\theta^2}{3n}$$

$$\hat{D}[\tilde{\theta}_3'] = \frac{\theta^2}{n(n+2)}$$

$$\frac{1}{n(n+2)} < \frac{1}{3n} \Rightarrow \tilde{\theta}_3' \text{ „эффективнее"}$$

№3

$$p(x) = \begin{cases} e^{-\frac{x}{\theta}}/\theta, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad n=3$$

$$M[S] = \int_0^\infty x e^{-\frac{x}{\theta}} \frac{1}{\theta} dx = \theta$$

$$M[S^2] = \int_0^\infty x^2 e^{-\frac{x}{\theta}} \frac{1}{\theta} dx = \frac{1}{\theta}\left(-\theta x^2 e^{-\frac{x}{\theta}}\Big|_0^\infty + 2\theta \int_0^\infty x e^{-\frac{x}{\theta}} dx\right) =$$

$$= 2\theta^2$$

$$D[S] = M[S^2] - M^2[S] = \theta^2$$

a) $\tilde{\theta}_1$:

$$M[\tilde{\theta}_1] = M\left[\left(\sum x_i\right) \cdot \frac{1}{3}\right] = \frac{1}{3} \cdot 3 \cdot M[S] = \theta$$

несмещ

$\tilde{\theta}_2$:

$$M[\tilde{\theta}_2] = M[x_{(2)}] = \int_0^\infty x \varphi(x) dx \ominus$$

$$F(x) = \int_0^x e^{-\frac{t}{\theta}} \cdot \frac{1}{\theta} dt = 1 - e^{-\frac{x}{\theta}} \quad (x \geq 0)$$

$$\varphi(x) = n \cdot p(x) \cdot C_{n-1}^{k-1} (1 - F(x))^{n-k} (F(x))^{k-1}$$

$$\ominus \int_0^\infty \frac{6}{\theta}\left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right) x \, dx = \frac{5}{6}\theta$$

не смещен

$$\tilde{\theta}_2' = \frac{6}{5}\tilde{\theta}_2$$

b) Эффективность:

$$D[\tilde{\theta}_1] = D\left[\frac{1}{3}\sum_{k=1}^{3} X_k\right] = \frac{1}{9}\cdot 3\, D[S] = \frac{\theta^2}{3}$$

$$D[\tilde{\theta}_2'] = M[\tilde{\theta}_3'^2] - M^2[\tilde{\theta}_3] =$$

$$M[\tilde{\theta}_3^2] = M[X_{(2)}^2] = \frac{6}{\theta}\int_0^{\infty} x^2\left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right)dx =$$

$$= \frac{6}{\theta}\left(0 + 0 + \frac{2\theta}{2}\int_{0}^{+\infty} xe^{-\frac{x\cdot 2}{\theta}} + \frac{2\theta}{3}\int_0^{\infty} xe^{-\frac{3x}{\theta}}dx\right) =$$

$$= \frac{6}{\theta}\left(\int_0^{+\infty} xe^{-\frac{2x}{\theta}} + \frac{2}{3}\int_0^{\infty} xe^{-\frac{3x}{\theta}}\right) = \frac{19}{18}\theta^2$$

$$= \left(\frac{19}{18}\theta^2 - \theta^2\cdot\frac{25}{36}\right)\cdot\frac{6^2}{5^2} = \frac{13}{25}\theta^2$$

$\tilde{\theta}_1$ — "более" эффективна

c) регулярность оценок:

$D[\tilde{\theta}_1]$ и $D[\tilde{\theta}_2']$ огр. на $\forall [a,b] \in (0,\infty)$ $\Rightarrow$

$\Rightarrow$ $\tilde{\theta}_1$ и $\tilde{\theta}_2'$ - регул. по дост. усл

регулярность модели:

1) $p(x,\theta)$ - непр. дифф по θ на Θ

2) $$\frac{\partial}{\partial\theta} P(x,\theta) = \frac{x e^{-\frac{x}{\theta}}}{\theta^3} - \frac{e^{-\frac{x}{\theta}}}{\theta^2}$$

$$\int_0^\infty \frac{\partial}{\partial\theta} P(x,\theta)\,dx = \int_0^\infty \left(\frac{x e^{-\frac{x}{\theta}}}{\theta^3} - \frac{e^{-\frac{x}{\theta}}}{\theta^2}\right) dx = 0$$

$$\int_0^\infty P(x,\theta)\,dx = 1 \qquad \frac{\partial}{\partial\theta}\int_0^\infty P\,dx = 0$$

$$\Rightarrow \frac{\partial}{\partial\theta}\int_0^\infty P\,dx = \int_0^\infty \frac{\partial}{\partial\theta} P(x,\theta)\,dx$$

3) $$I(\theta) = M\left[\left(\frac{\partial \ln p(x,\theta)}{\partial\theta}\right)^2\right] = M\left[\left(\frac{\partial}{\partial\theta}\left(\ln\left(\frac{e^{-\frac{x}{\theta}}}{\theta}\right)\right)\right)^2\right] =$$

$$= M\left[\left(\frac{\partial}{\partial\theta}\left(-\frac{x}{\theta} - \ln\theta\right)\right)^2\right] = M\left[\left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2\right] =$$

$= \frac{1}{\theta^4} M[S^2] + \frac{2}{\theta^3} M[S] + \frac{1}{\theta^2} = \theta^2$ - непр. дифф по θ на Θ

и $I(\theta) > 0$ на Θ

$\Rightarrow$ выполн. усл. для неравенства

Кр.-Рао:

~~$\forall \theta \in \Theta \quad D[\tilde{g}] \geq \frac{g'^2(\theta)}{nI(\theta)}$~~

из следствия:

$\tilde{\theta}_1 : \quad D[\tilde{\theta}_1] = \frac{\theta^2}{3}$

$\frac{1}{nI(\theta)} = \frac{\theta^3}{3}$

$\Rightarrow \tilde{\theta}_1$ - ~~лучшая~~ самая эфф. оценка

$\tilde{\theta}'_3 \neq \tilde{\theta}_1 \Rightarrow \tilde{\theta}_1$ - эффективнее

№ 2

$M[S]=\int_{-\infty}^{\infty} x\, p(x)dx=1$

$M[S^2]=\int_{-\infty}^{\infty} x^2\, p(x)dx=2$

$D[S^2]=2-1^2=1$

По ЦПТ об одинаково распред вел.

$$\frac{\sum_{k=1}^{n} S_k - M[\sum_{k=1}^{n} S_k]}{\sqrt{D[\sum S_k]}} = \frac{\sum S_k - n\, M[S]}{\sqrt{n D[S]}} =$$

$$= \frac{n}{\sqrt{n}} \cdot \frac{\overline{X} - M[S]}{\sqrt{DS}} \rightsquigarrow N(0,1)$$

$$\|$$

$$5\overline{X}-5$$

По теор. о пол. нормальност:

$$5\overline{X}-5 \rightsquigarrow N(0,1) \Rightarrow \overline{X} \rightsquigarrow N\left(1, \frac{1}{25}\right)$$