Multi-Key $\boxed{QS}$

S = {aba, ape, ratto, bar, amo, cane}

i = 1

→ aba
ape
ratto
bar
amo
cane

⇒

$R_<$: ∅

$R_=$: → aba, ape, amo

$R_>$: → ratto, bar, cane

i = 2

$R_<$: ∅
$R_=$: aba ✓
$R_>$: → ape, amo

i = 2

$R_<$: ∅
$R_=$: ape ✓
$R_>$: amo ✓

→ BAR, $R_<$: CANE ⇒

$R_<$:
$R_=$: BAR ✓
$R_>$: CANE ✓

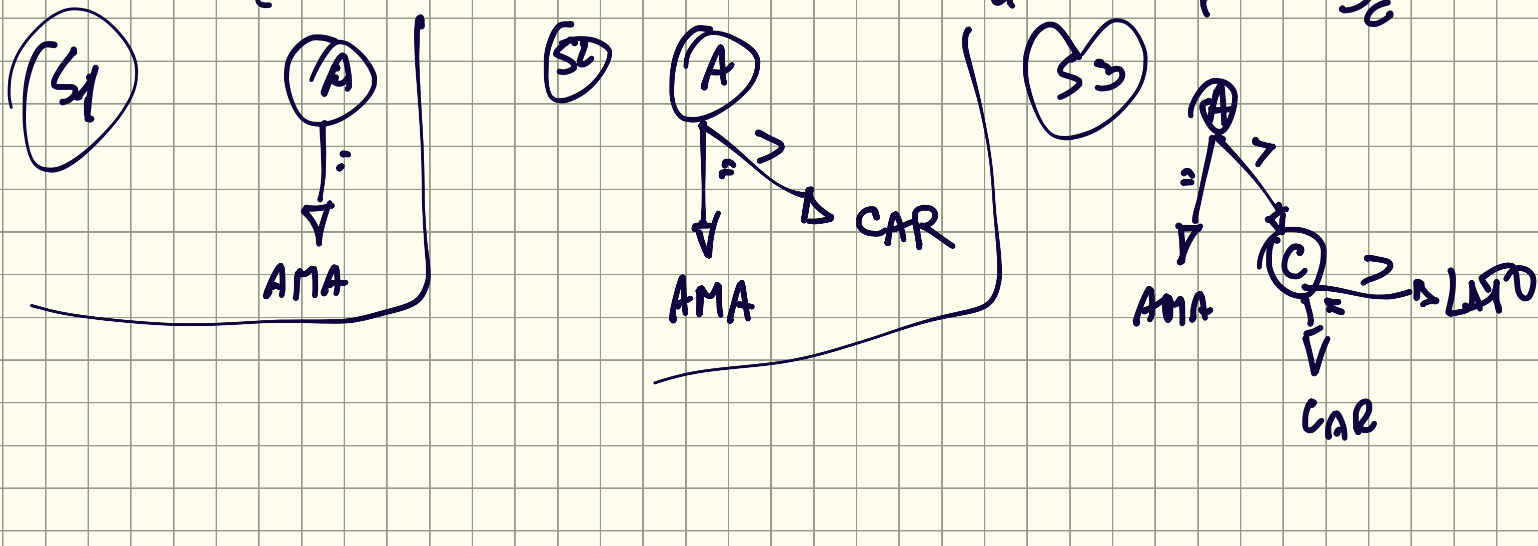
⇒ $R_=$: RATTO ✓

$R_>$:

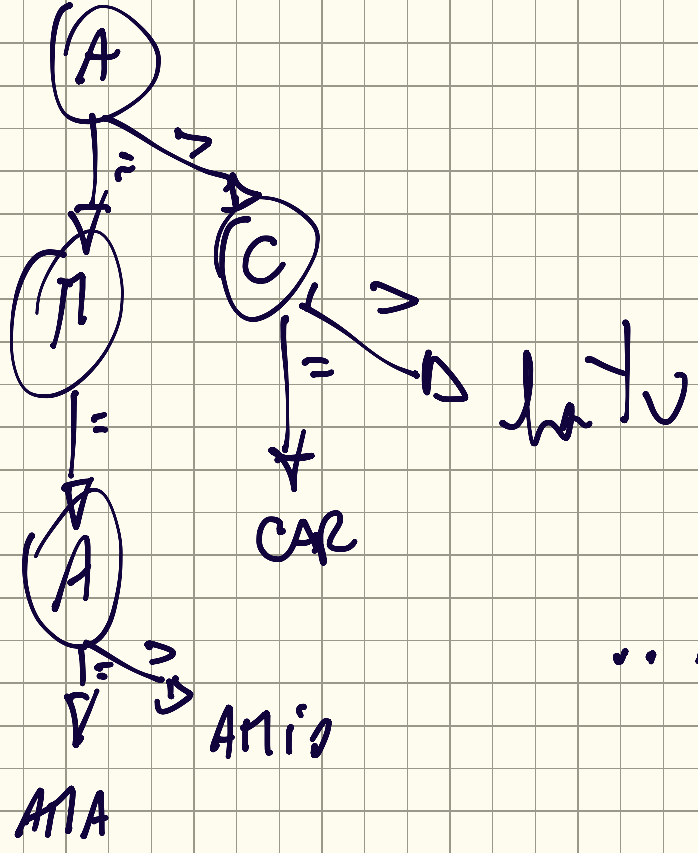
✓

Ternary search TREE

S = {ama, car, lato, amio, figa, figano}
(S_1, S_2, S_3, S_4, S_5, S_6)



S4



...

TREAP

- Each node $x \in T$ has:
  - Key(x) (binary search)
  - priority(x) (MIN/MAX heap)
- $\forall x, v \in T$ if v is a left child:
  $\Rightarrow key(x) > key(v)$
- $\forall x, v \in T$ if v is a right child:
  $\Rightarrow key(x) < key(v)$
- $\forall x, v \in T$ if v is a child of x:
  $\Rightarrow priority(x) > priority(v)$ (if MIN-HEAP)

  else

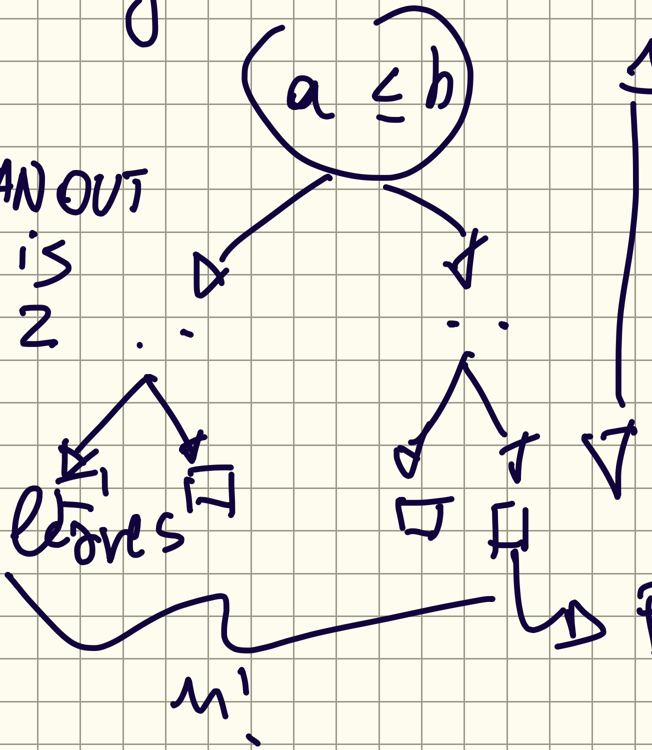
  $\Rightarrow priority(x) > priority(v)$

LOWER BOUND FOR SORTING

In the RAM model:

- every node of a tree is a comparison

=> FANOUT is 2



↳ permutation

height h is

$$2^h \geq \ell$$

$$h \geq \log_2 \ell$$

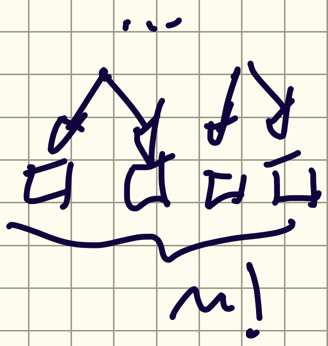
⇓

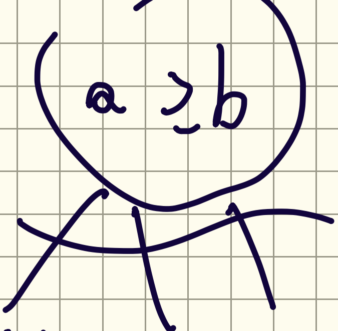
In sorting I've $\ell = n!$

=> lower bound (classic) is $h = \Omega(n \log_2 n)$

In the two-level memory model:

- Account I/Os: every node is an I/O operation
  leaves are still $n!$
- Exploit stuff in M are free





FANOUT here (of each internal node)

is the NUMBER OF COMPARISONS-RESULTS the a single I/O can generate (each node) among items read from disk (B) and items that are in internal Memory (M-B)

=> these B items can be distributed in at most $\binom{M}{B}$ ways among the others (M-B) items in Memory => 1 I/O cannot generate at most $\binom{M}{B}$ different results (not sufficient)

=> Have to consider the PERMUTATIONS among these items

[N.B.: some permutations are already accounted by some previous account]

=> 2 cases: new item / old item (already seen)

New items Old items Permutations

$$\left(\binom{M}{B}\cdot B!\right)^{\frac{n}{B}}\cdot\left(\binom{M}{B}\right)^{t-\frac{n}{B}} \geq n!$$

$$\cancel{\binom{M}{B}^{\frac{n}{B}}}\cdot (B!)^{\frac{n}{B}} \cdot \frac{\binom{M}{B}^{t}}{\cancel{\binom{M}{B}^{\frac{n}{B}}}} \geq n!$$

$$\log_2 (B!)^{\frac{n}{B}} + \log_2\binom{M}{B}^{t} \geq \log_2 n!$$

$$\frac{n}{B}\cdot\log_2(B!) + t\cdot\log_2\binom{M}{B} \geq n\log_2 n$$

$$\frac{n}{\cancel{B}}\cdot\cancel{B}\log_2(B) + t\cdot\log_2\binom{M}{B} \geq n\log_2 n$$

$$t\cdot\log_2\binom{M}{B} \geq n\log_2 n - n\log_2 B$$

$$\frac{t\cdot B\cdot\cancel{\log_2\left(\frac{M}{B}\right)}}{\cancel{B\log_2\frac{M}{B}}} \geq \frac{n\log_2 n - n\log_2 B}{B\log\frac{M}{B}}$$

$$t \geq \frac{n \log_2 n - n \log_2 B}{B \log_{\frac{M}{B}}}$$

$$t \geq \frac{n \log_2 \left(\frac{n}{B}\right)}{B \log_2 \frac{M}{B}} \qquad t \geq \frac{n}{B} \cdot \log_{\frac{M}{B}} \left(\frac{n}{B}\right)$$

Elias - Phano

$S = (1, 2, 1, 1, 4, 7, 7, 4, 1, 3, 9)$

is repeated => want increasing sequence

$S' = (1, 3, 4, 5, 9,$