Equality and Isomorphisms in Type Theory

HoTT from scratch

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Motivation

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- **Higher Observational Type Theory**: is one attempt at making HoTT practical (on a computer) using ideas from logical relations and parametricity.

Outline

1. Substitution Calculus

- **Deductive System**: a collection of rules

1.1 Deductive Systems

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- **Rule**: takes hypotheses and gives a conclusion

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- **Judgement**: hypotheses or conclusions

Substitution Calculus Judgements

Context	Type	Term	Substitution
$\Gamma\ \mathbf{cx}$			

Substitution Calculus Judgements

Context	Type	Term	Substitution
$\Gamma\ \mathbf{cx}$	$\Gamma \vdash A\ \mathbf{Ty}$		

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- **Context**: lists of dependent terms

$$a_1 : A_1$$

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$$a_1 : A_1, a_2 : A_2(a_1)$$

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$$a_1 : A_1, a_2 : A_2(a_1), a_3 : A_3(a_1, a_2), ...a_n : A_n(a_1, a_2, ..., a_{n-1})$$

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- **Context**: lists of dependent terms

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- **Substitutions**: shifts judgements from one context to another

$$\frac{\Gamma \vdash A\ \mathbf{Ty} \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash A[\gamma]\ \mathbf{Ty}}$$

Substitution Calculus Equality Judgements

Type	Term	Substitution
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Substitution Calculus Equality Judgements

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definitionally equal symbols can be replaced by each other

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definitionally equal symbols can be replaced by each other

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash A = B\ \mathbf{Ty}}{\Gamma \vdash a : B}$$

Substitution Calculus Rules

γ Application	**term**		
Morphism			
Pullback			

$$\frac{\Gamma \vdash a : A \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash a[\gamma] : A[\gamma]}\text{substitution-term}$$

Substitution Calculus Rules

γ Application	term	**type**	
Morphism			
Pullback			

$$\frac{\Gamma \vdash A\ \mathbf{Ty} \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash A[\gamma]\ \mathbf{Ty}}\text{substitution-type}$$

SUBSTITUTION CALCULUS RULES

γ Application	term	type	
Morphism	**composition**		
Pullback			

$$\frac{\Gamma_1 \vdash \gamma : \Gamma_0 \quad \Gamma_2 \vdash \gamma' : \Gamma_1}{\Gamma_2 \vdash \gamma \circ \gamma' : \Gamma_0}\text{composition}$$

Substitution Calculus Rules

γ Application	term	type	
Morphism	composition	**identity**	
Pullback			

$$\frac{\Gamma\ \mathbf{cx}}{\Gamma \vdash \mathbf{id} : \Gamma}\,\text{identity}$$

Substitution Calculus Rules

γ Application	term	type	
Morphism	composition	**identity**	
Pullback			

$$\frac{\Delta \vdash \gamma : \Gamma}{\Delta \vdash \gamma \circ \mathbf{id} = \mathbf{id} \circ \gamma = \gamma : \Gamma}\text{unital}$$

SUBSTITUTION CALCULUS RULES

γ Application	term	type	
Morphism	composition	identity	**associativity**
Pullback			

$$\frac{\Gamma_1 \vdash \gamma_0 : \Gamma_0 \quad \Gamma_2 \vdash \gamma_1 : \Gamma_1 \quad \Gamma_3 \vdash \gamma_2 : \Gamma_2}{\Gamma_3 \vdash \gamma_0 \circ (\gamma_1 \circ \gamma_2) = (\gamma_0 \circ \gamma_1) \circ \gamma_2 : \Gamma_0}\text{associativity}$$

Substitution Calculus Rules

γ Application	term	type	
Morphism	composition	identity	associativity
Pullback	**weakening**		

$$\frac{\Gamma \vdash A\ \mathbf{Ty}}{\Gamma.A \vdash \mathbf{p} : \Gamma}\text{weakening}$$

think as adding terms to context

Substitution Calculus Rules

γ Application	term	type	
Morphism	composition	identity	associativity
Pullback	weakening	**variable**	

$$\frac{\Gamma \vdash A \ \mathbf{Ty}}{\Gamma . A \vdash \mathbf{q} : A[\mathbf{p}]}\text{variable}$$

think as debruijn indices at index number of $\mathbf{p}$

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Pullback	weakening	**variable**	

$$\frac{\Gamma \vdash A\ \mathbf{Ty} \quad \Gamma.A \vdash B_1\ \mathbf{Ty}}{\Gamma.A.B_1 \vdash \mathbf{q}[\mathbf{p}] : A[\mathbf{p}^2]}\text{variable}$$

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$$\frac{\Gamma \vdash A \ \mathbf{Ty} \quad \Gamma.A \vdash B_1 \ \mathbf{Ty} \quad \Gamma.A.B_1 \vdash B_2 \ \mathbf{Ty}}{\Gamma.A.B_1.B_2 \vdash \mathbf{q}[\mathbf{p}^2] : A[\mathbf{p}^3]}$$

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Morphism	composition	identity	associativity
Pullback	weakening	**variable**	

$$\frac{\Gamma \vdash A\ \mathbf{Ty} \quad \Gamma.A \vdash B_1\ \mathbf{Ty} \quad \ldots \quad \Gamma.A.B_1 \ldots B_{n-1} \vdash B_n\ \mathbf{Ty}}{\Gamma.A.B_1 \ldots B_n \vdash \mathbf{q}[\mathbf{p}^n] : A[\mathbf{p}^{n+1}]}$$

notationally informally we still write $a : A$

Substitution Calculus Rules

γ Application	term	type	
Morphism	composition	identity	associativity
Pullback	weakening	variable	**substitution extension**

$$\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash A\ \mathbf{Ty} \quad \Delta \vdash a : A[\gamma]}{\Delta \vdash \gamma.a : \Gamma.A}\text{substitution-extension}$$

think removing / dispensing terms from context *the rules are justified / motivated by the natural model for dependent types*

$$\begin{tikzcd}
\Delta \arrow[r, dashed, "\Delta\vdash\gamma.a:\Gamma.A"] \arrow[rr, bend left, "\Delta\vdash a:A[\gamma]"] \arrow[dr, bend right, "\Delta\vdash\gamma:\Gamma"'] & \Gamma.A \arrow[r, "\Gamma.A\vdash\mathbf{q}:A[\mathbf{p}]"] \arrow[d, "\Gamma.A\vdash\mathbf{p}:\Gamma"'] & \mathbf{Tm} \arrow[d, "p"'] \\
& \Gamma \arrow[r, "\Gamma\vdash A\ \mathbf{Ty}"] & \mathbf{Ty}
\end{tikzcd}$$

we notate $\mathbf{Tm}(\Gamma, A)$ *for set of terms and* $\mathbf{Ty}(\Gamma)$ *for set of types*

2. Mapping In Types

2.1 Generally

$$\iota_\Gamma : \mathbf{Tm}(\Gamma, \Upsilon(X)) \cong Y$$

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Introduction	ι_Γ^{-1}

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Formation	Υ
Introduction	ι_Γ^{-1}
Elimination	ι_Γ
Computation / β	$\iota_\Gamma \circ \iota_\Gamma^{-1} = \mathrm{id}$
Uniqueness / η	$\iota_\Gamma^{-1} \circ \iota_\Gamma = \mathrm{id}$

$$\iota_\Gamma : \mathbf{Tm}(\Gamma, \Pi(A, B)) \cong \mathbf{Tm}(\Gamma.A, B)$$

Formation	$\dfrac{\Gamma \vdash A\ \mathbf{Ty} \quad \Gamma.A \vdash B\ \mathbf{Ty}}{\Gamma \vdash \Pi(A, B)\ \mathbf{Ty}}$ Π-formation

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Introduction	$\dfrac{\Gamma.A \vdash b : B}{\Gamma \vdash \lambda(b) : \Pi(A, B)}\Pi\text{-intro}$

2.2 Π

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Introduction	$\dfrac{\Gamma.A \vdash b : B}{\Gamma \vdash \lambda(b) : \Pi(A, B)}\ \Pi\text{-intro}$
Elimination	$\dfrac{\Gamma \vdash a : A \quad \Gamma \vdash f : \Pi(A, B)}{\Gamma \vdash \mathbf{app}(f, a) : B[\mathrm{id}\,.a]}\ \Pi\text{-elim}$

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Elimination	$\dfrac{\Gamma \vdash a : A \quad \Gamma \vdash f : \Pi(A, B)}{\Gamma \vdash \mathbf{app}(f, a) : B[\mathrm{id}\,.a]}\Pi\text{-elim}$
Computation / β	$\dfrac{\Gamma \vdash a : A \qquad \Gamma.A \vdash b : B}{\mathbf{app}(\lambda(b), a) = b[\mathrm{id}\,.a] : B[\mathrm{id}\,.a]}\Pi - \beta$

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Introduction	$\dfrac{\Gamma.A \vdash b : B}{\Gamma \vdash \lambda(b) : \Pi(A, B)}\,\Pi\text{-intro}$
Elimination	$\dfrac{\Gamma \vdash a : A \quad \Gamma \vdash f : \Pi(A, B)}{\Gamma \vdash \mathbf{app}(f, a) : B[\mathrm{id}\,.a]}\,\Pi\text{-elim}$
Computation / β	$\dfrac{\Gamma \vdash a : A \qquad \Gamma.A \vdash b : B}{\mathbf{app}(\lambda(b), a) = b[\mathrm{id}\,.a] : B[\mathrm{id}\,.a]}\,\Pi - \beta$
Uniqueness / η	$\dfrac{\Gamma \vdash f : \Pi(A, B)}{\lambda(\mathbf{app}(f[\mathbf{p}], \mathbf{q})) = f : \Pi(A, B)}\,\Pi - \eta$

$$\iota_\Gamma : \mathbf{Tm}(\Gamma, \Sigma(A, B)) \cong \mathbf{\Sigma}_{a:\mathbf{Tm}(\Gamma, A)} \ \mathbf{Tm}(\Gamma, B[\mathrm{id}\,.a])$$

think of $\mathbf{\Sigma}$ *as metatheory* Σ*, thus set of dependent pairs*

Formation	$\dfrac{\Gamma \vdash A \ \mathbf{Ty} \quad \Gamma.A \vdash B \ \mathbf{Ty}}{\Gamma \vdash \Sigma(A, B) \ \mathbf{Ty}}\ \Sigma\text{-formation}$

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Introduction	$\dfrac{\Gamma \vdash a : A \quad \Gamma \vdash b : B[\mathrm{id}\,.a]}{\Gamma \vdash \mathbf{pair}(a, b) : \Sigma(A, B)}\ \Sigma\text{-intro}$

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Introduction	$\dfrac{\Gamma \vdash a : A \quad \Gamma \vdash b : B[\mathrm{id}\,.a]}{\Gamma \vdash \mathbf{pair}(a, b) : \Sigma(A, B)}\ \Sigma\text{-intro}$
Elimination 1	$\dfrac{\Gamma \vdash p : \Sigma(A, B)}{\Gamma \vdash \mathbf{fst}(p) : A}\ \Sigma\text{-elim}_1$

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Formation	$\dfrac{\Gamma \vdash A \ \mathbf{Ty} \quad \Gamma.A \vdash B \ \mathbf{Ty}}{\Gamma \vdash \Sigma(A, B) \ \mathbf{Ty}}\,\Sigma\text{-formation}$
Introduction	$\dfrac{\Gamma \vdash a : A \quad \Gamma \vdash b : B[\mathrm{id} \, .a]}{\Gamma \vdash \mathbf{pair}(a, b) : \Sigma(A, B)}\,\Sigma\text{-intro}$
Elimination 1	$\dfrac{\Gamma \vdash p : \Sigma(A, B)}{\Gamma \vdash \mathbf{fst}(p) : A}\,\Sigma\text{-elim}_1$
Elimination 2	$\dfrac{\Gamma \vdash p : \Sigma(A, B)}{\Gamma \vdash \mathbf{snd}(p) : B[\mathrm{id} \, . \, \mathbf{fst}(p)]}\,\Sigma\text{-elim}_2$

2.4 Unit

$$\iota_\Gamma : \mathbf{Tm}(\Gamma, \mathbf{Unit}) \cong \{\star\}$$

Formation	$\dfrac{}{\Gamma \vdash \mathbf{Unit}\ \mathbf{Ty}}$ Unit-formation

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Formation	$$\dfrac{}{\Gamma \vdash \mathbf{Unit}\ \mathbf{Ty}}\,\text{Unit-formation}$$
Introduction	$$\dfrac{\Gamma \vdash \mathbf{Unit}\ \mathbf{Ty}}{\Gamma \vdash \mathbf{tt} : \mathbf{Unit}}\,\text{Unit-intro}$$

$$\iota_\Gamma : \mathbf{Tm}(\Gamma, \mathbf{Unit}) \cong \{\star\}$$

Formation	$\dfrac{}{\Gamma \vdash \mathbf{Unit}\ \mathbf{Ty}}$ Unit-formation
Introduction	$\dfrac{\Gamma \vdash \mathbf{Unit}\ \mathbf{Ty}}{\Gamma \vdash \mathbf{tt} : \mathbf{Unit}}$ Unit-intro
Elimination	*what about the elimination rule?*

we will see in mapping out types

these isomorphisms respect substitution

$$\begin{tikzcd}
\mathbf{Tm}(\Gamma, \Upsilon(X)) \arrow[r, leftrightarrow, "\iota_\Gamma"] \arrow[d, "\gamma^*"'] & Y \arrow[d, "\psi_\gamma^*"'] \\
\mathbf{Tm}(\Delta, \Upsilon(X[\gamma])) \arrow[r, leftrightarrow, "\iota_\Delta"] & Y[\gamma]
\end{tikzcd}$$

we have to define rules for these as well for the types and terms of Υ

but we omit them in this presentation

2.6 Extensional Equality

$$\iota_\Gamma : \mathbf{Tm}(\Gamma, \mathbf{Eq}(A, a, b)) \cong \{\star \mid a = b\}$$

Formation	$\dfrac{\Gamma \vdash A\ \mathbf{Ty} \quad \Gamma \vdash a : A \quad \Gamma \vdash b : A}{\Gamma \vdash \mathbf{Eq}(A, a, b)\ \mathbf{Ty}}$ Eq-formation

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Introduction	$\dfrac{\Gamma \vdash a : A}{\Gamma \vdash \mathbf{refl} : \mathbf{Eq}(A, a, a)}$ Eq-intro

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Introduction	$\dfrac{\Gamma \vdash a : A}{\Gamma \vdash \mathbf{refl} : \mathbf{Eq}(A, a, a)}$ Eq-intro
Elimination	$\dfrac{\Gamma \vdash p : \mathbf{Eq}(A, a, b)}{\Gamma \vdash a = b : A}$ Eq-reflection

$$\frac{\Gamma \vdash p : \mathbf{Eq}(A, a, b)}{\Gamma \vdash a = b : A}\text{Eq-reflection}$$

Notice how elimination concludes a definitional equality judgement of terms rather than a term judgement. This departs from the usual elimination rules we have seen before.

$$\frac{\Gamma \vdash p : \mathbf{Eq}(A, a, b)}{\Gamma \vdash a = b : A}\text{Eq-reflection}$$

Notice how elimination concludes a definitional equality judgement of terms rather than a term judgement. This departs from the usual elimination rules we have seen before.

- **Propositional Equality**: equality is **internalized** within the theory e.g.

$$a = b : A \downarrow p : \mathbf{Eq}(A, a, b) \uparrow a = b : A$$

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- e.g. with $p : \mathbf{Eq}(\mathbb{N}, a + b, b + a)$ we can swap $a + b$ for $b + a$ in lets say an argument for length of a vector without having to evaluate a and b

- **Uniqueness of Identity Proofs (UIP)**: there is only one proof of equality between two terms due to equality reflection, i.e. $p, q : \mathbf{Eq}(A, a, b)$ means $p = q$

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- **Normalization**: with UIP our types aren't injective, without injective types we can't have normalization (we won't explore why here, but lets take it for granted for now)

Without normalization we can't have decidable type checking!

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Without normalization we can't have decidable type checking!

We must do propositional equality differently

3. Mapping Out Types

$$\{c \in \mathbf{Tm}(\Gamma.\Upsilon, C) \mid \mathbf{rec}\} \cong \{\star\}$$

we define Υ by the terms it maps out to in C, hence the name

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Void	$X \mapsto 0$	absurd

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$\mathbf{Void}$	$X \mapsto 0$	absurd
$\mathbf{Unit}$	$X \mapsto 1$	$\mathbf{tt}$

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$\mathbb{N}$	$X \mapsto 1 + X$	zero, succ

- **Formation Rule**: $\Upsilon(X)$
- **Intro**: initial algebra constructors
- **Elim**: **rec**

3.2 Unit

the **Unit** *type now defined as a mapping out type*

$$\{c \in \mathbf{Tm}(\Gamma.\ \mathbf{Unit}, C) \mid c = c[\mathrm{id} \,.\, \mathbf{tt}]\} \cong \{\star\}$$

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the **Unit** *type now defined as a mapping out type*

$$\{c \in \mathbf{Tm}(\Gamma.\ \mathbf{Unit}, C) \mid c = c[\mathrm{id}\,.\ \mathbf{tt}]\} \cong \{\star\}$$

$$\mathbf{rec}(\mathbf{tt}, c) = c : C[\mathrm{id}\,.\ \mathbf{tt}]$$

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$$\frac{\Gamma \vdash u : \mathbf{Unit} \quad \Gamma \vdash c : C[\mathrm{id}\,.\ \mathbf{tt}]}{\Gamma \vdash \mathbf{rec}(u, c) : C[\mathrm{id}\,.u]}\text{Unit-elim}$$

the **Bool** *type is a mapping out type*

$$\{c \in \mathbf{Tm}(\Gamma.\,\mathbf{Bool}, C) \mid c_{\mathbf{true}} = c[\mathrm{id}\,.\,\mathbf{true}] \wedge c_{\mathbf{false}} = c[\mathrm{id}\,.\,\mathbf{false}]\} \cong \{\star\}$$

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$$\mathbf{rec}(\mathbf{true}, c_{\mathbf{true}}, c_{\mathbf{false}}) = c_{\mathbf{true}} : C[\mathrm{id}\ .\ \mathbf{true}]$$

$$\mathbf{rec}(\mathbf{false}, c_{\mathbf{true}}, c_{\mathbf{false}}) = c_{\mathbf{false}} : C[\mathrm{id}\ .\ \mathbf{false}]$$

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$$\frac{\Gamma \vdash b : \mathbf{Bool} \quad \Gamma \vdash c_{\mathbf{true}} : C[\mathrm{id} .\ \mathbf{true}] \quad \Gamma \vdash c_{\mathbf{false}} : C[\mathrm{id} .\ \mathbf{false}]}{\Gamma \vdash \mathbf{rec}(b, c_{\mathbf{true}}, c_{\mathbf{false}}) : C[\mathrm{id} . b]}\ \text{Bool-elim}$$

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for **Bool** *often* **rec** *is also written as* **if**

3.4 $A + B$

the disjoint sum type; $A + B$*, is like a* $\mathbf{Bool}$ *with arguments*

$$\{c \in \mathbf{Tm}(\Gamma.A + B, C) \mid c_A = c[\mathrm{id}\,.\ \mathrm{inl}\ a] \wedge c_B = c[\mathrm{id}\,.\ \mathrm{inr}\ b]\} \cong \{\star\}$$

$$\mathbf{rec}(\mathrm{inl}\ a, c_A, c_B) = c_A : C[\mathrm{id}\,.\ \mathrm{inl}\ a]$$
$$\mathbf{rec}(\mathrm{inr}\ b, c_A, c_B) = c_B : C[\mathrm{id}\,.\ \mathrm{inr}\ b]$$

$$\frac{\Gamma \vdash o : A + B \quad \Gamma \vdash c_A : C[\mathrm{id}\,.\ \mathrm{inl}\ a] \quad \Gamma \vdash c_B : C[\mathrm{id}\,.\ \mathrm{inr}\ b]}{\Gamma \vdash \mathbf{rec}(o, c_A, c_B) : C[\mathrm{id}\,.o]}\ A + B\text{-elim}$$

$\mathbb{N}$ *motivates why we call the elimination rule a recursor* **rec**

$$\{c \in \mathbf{Tm}(\Gamma.\mathbb{N}, C) \mid c_z = c[\mathrm{id}\,.\,\mathrm{zero}] \wedge c_s[\mathbf{p}.\mathbf{q}.c] = c[\mathbf{p}.\,\mathrm{succ}(\mathbf{q})]\} \cong \{\star\}$$

$$\mathbf{rec}(\mathrm{zero}, c_z, c_s) = c_z : C[\mathrm{id}\,.\,\mathrm{zero}]$$
$$\mathbf{rec}(\mathrm{succ}(n), c_z, c_s) = c_s[\mathrm{id}\,.n.\,\mathbf{rec}(n, c_z, c_s)] : C[\mathrm{id}\,.\,\mathrm{succ}(n)]$$

$$\frac{\Gamma \vdash n : \mathbb{N} \quad \Gamma.\mathbb{N} \vdash C\ \mathbf{Ty} \quad \Gamma \vdash c_z : C[\mathrm{id}\,.\,\mathrm{zero}] \quad \Gamma.\mathbb{N}.C \vdash c_s : C[\mathbf{p}^2.\,\mathrm{succ}(\mathbf{q}[\mathbf{p}])]}{\Gamma \vdash \mathbf{rec}(n, c_z, c_s) : C[\mathrm{id}\,.n]}\ \mathbb{N}\text{-elim}$$

3.6 **Void**

$\mathbf{Void}$ *has no recursor arguments in* C

$$\mathbf{Tm}(\Gamma.\ \mathbf{Void}, C) \cong \{\star\}$$

$\mathbf{rec}(v) : C[\mathrm{id}\,.v]$ *often written as* $\mathrm{absurd}(v) : C[\mathrm{id}\,.v]$

$$\frac{\Gamma \vdash v : \mathbf{Void} \quad \Gamma.\ \mathbf{Void} \vdash C\ \mathbf{Ty}}{\Gamma \vdash \mathrm{absurd}(v) : C[\mathrm{id}\,.v]}\ \text{Void-elim}$$

notice how ***ANY*** *term in* C *that depends on* $\mathbf{Void}$ *is uniquely* $\mathrm{absurd}(v)$

4. Intensional Equality

We now try to define Propositional Equality as a mapping out type

$$\{c \in \mathbf{Tm}(\Gamma.\ \mathbf{Id}(A, a, b), C) \mid \mathbf{J}\} \cong \{\star\}$$

- **Introduction Rule**: still the same $\mathbf{refl} : \mathbf{Id}(A, a, a)$
- **Elimination Rule**: $\mathbf{J}$ when presented as a "functional program" is as follows:

$$\begin{aligned} &\mathbf{J} : \{A : \mathbf{U}\} \\ &\quad (C : (a, b : A) \to \mathbf{Id}(A, a, b) \to \mathbf{Ty}(\Gamma)) \end{aligned}$$

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&\quad (C : (a, b : A) \to \mathbf{Id}(A, a, b) \to \mathbf{Ty}(\Gamma)) \\
&\to (a : A \to C(a, a, \mathbf{refl}_a))
\end{aligned}$$

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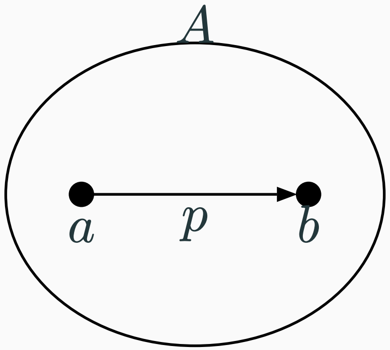
&\to a, b : A \to p : \mathbf{Id}(A, a, b) \\
&\to C(a, b, p)
\end{aligned}$$

we will see why $\mathbf{J}$ *is justified to model equality as follows* *to aid visualization, we introduce the homotopy interpretation of types*

- types are spaces
- terms are points
- identifications are paths



$$\mathbf{J} \Rightarrow \text{subst}$$

$$\begin{aligned}
\mathbf{J} &: \{A : \mathbf{U}\} \\
&\quad (C : (a, b : A) \to \mathbf{Id}(A, a, b) \to \mathbf{Ty}(\Gamma)) \\
&\to (a : A \to C\ a\ a\ \mathbf{refl}_a) \\
&\to a, b : A \to p : \mathbf{Id}(A, a, b) \\
&\to C\ a\ b\ p
\end{aligned}$$

$$\text{subst}\ \{a, b : A\} B\ p = \mathbf{J}$$

4.3 subst

$$\mathbf{J} \Rightarrow \text{subst}$$

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&\to (a : A \to C\ a\ a\ \mathbf{refl}_a) \\
&\to a, b : A \to p : \mathbf{Id}(A, a, b) \\
&\to C\ a\ b\ p
\end{aligned}$$

$$\begin{aligned}
\text{subst}\ \{a, b : A\} B\ p = \mathbf{J} \\
\lambda a\ b\ _.(B\ a \to B\ b)
\end{aligned}$$

$$\mathbf{J} \Rightarrow \text{subst}$$

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&\to C\ a\ b\ p
\end{aligned}$$

$$\begin{aligned}
&\text{subst}\ \{a, b : A\} B\ p = \mathbf{J} \\
&\quad \lambda a\ b\ _.(B\ a \to B\ b) \\
&\quad \lambda_.\lambda b.b
\end{aligned}$$

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$$\begin{aligned}
\mathbf{J} : & \{A : \mathbf{U}\} \\
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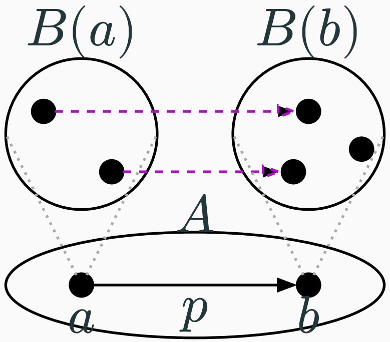
\to & (a : A \to C\ a\ a\ \mathbf{refl}_a) \\
\to & a, b : A \to p : \mathbf{Id}(A, a, b) \\
\to & C\ a\ b\ p
\end{aligned}$$

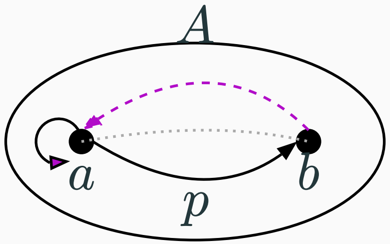
$$\begin{aligned}
\text{subst}\ & \{a, b : A\} B\ p = \mathbf{J} \\
& \lambda a\ b\ _.(B\ a \to B\ b) \\
& \lambda _.\lambda b.b \\
& a\ b\ p
\end{aligned}$$



think subst *gives a function that "transports" terms in Ba to Bb*

$$\mathbf{J} \Rightarrow \mathrm{subst} \Rightarrow \mathrm{sym}$$

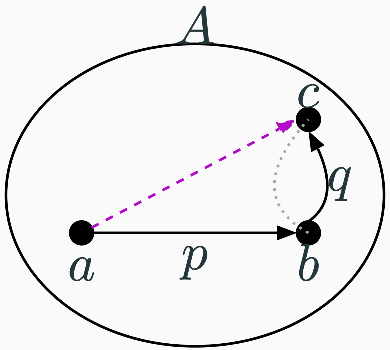
$$\begin{aligned} \mathrm{subst} : \{a, b : A\} \quad & \quad \mathrm{sym}\ p = \mathrm{subst} \\ B : A \to \mathbf{U} \quad & \quad \lambda x.\ \mathbf{Id}(A, x, a) \\ \to \mathbf{Id}(A, a, b) \quad & \quad p \\ \to B\ a \quad & \quad \mathbf{refl}_a \\ \to B\ b \quad & \end{aligned}$$



think sym *"drags" the origin of* $\mathbf{refl}_a$ *along* p *to get its "inverse"*

$$\mathbf{J} \Rightarrow \text{subst} \Rightarrow \text{trans}$$

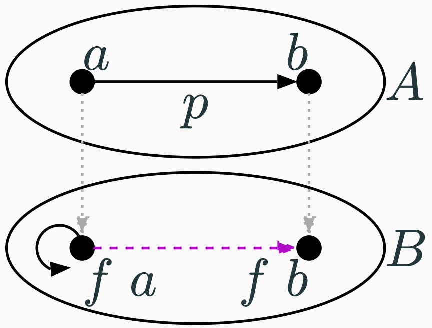
$$\begin{aligned}
\text{subst} : \{a, b : A\} \quad & \quad \text{trans } p\ q = \text{subst} \\
B : A \to \mathbf{U} \quad & \quad \lambda x.\ \mathbf{Id}(A, a, x) \\
\to \mathbf{Id}(A, a, b) \quad & \quad p \\
\to B\ a \quad & \quad q \\
\to B\ b \quad &
\end{aligned}$$



think trans *"drags" end of p along q to get the transitive*

$$\mathbf{J} \Rightarrow \text{subst} \Rightarrow \text{cong}$$

$$\begin{aligned} \text{subst} : \{a, b : A\} \\ B : A \to \mathbf{U} \\ \to \mathbf{Id}(A, a, b) \\ \to B\ a \\ \to B\ b \end{aligned} \qquad \begin{aligned} \text{cong}\ f\ p = \text{subst} \\ \lambda x.\ \mathbf{Id}(B, f\ a, f\ x) \\ p \\ \mathbf{refl}_{f\ a} \end{aligned}$$



think cong *"drags" end of* $\mathbf{refl}_{f\ a}$ *along a path in* B *"parallel" to* p

$$\mathbf{J} \Rightarrow \text{uniq}$$

$$\textit{let } [z] = \Sigma(z : A, \mathbf{Id}(A, x, z)); \textit{ points idetnfied with } z$$

$$\begin{aligned}
\mathbf{J} &: \{A : \mathbf{U}\} \\
&\quad (C : (a, b : A) \to \mathbf{Id}(A, a, b) \to \mathbf{Ty}(\Gamma)) \\
&\to (a : A \to C\ a\ a\ \mathbf{refl}_a) \\
&\to a, b : A \to p : \mathbf{Id}(A, a, b) \\
&\to C\ a\ b\ p
\end{aligned}$$

$$\text{uniq } \{a : A\} b\ p = \mathbf{J}$$

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$$\begin{aligned} &\text{uniq}\ \{a : A\} b\ p = \mathbf{J} \\ &\quad \lambda x, y, p'.\ \mathbf{Id}([z], (x, \mathbf{refl}_x), (y, p')) \end{aligned}$$

$$\mathbf{J} \Rightarrow \mathrm{uniq}$$

$$\textit{let } [z] = \Sigma(z : A, \mathbf{Id}(A, x, z)); \textit{points idetnfied with } z$$

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$\mathbf{J} : \{A : \mathbf{U}\}$

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$\to (a : A \to C\ a\ a\ \mathbf{refl}_a)$

$\to a, b : A \to p : \mathbf{Id}(A, a, b)$

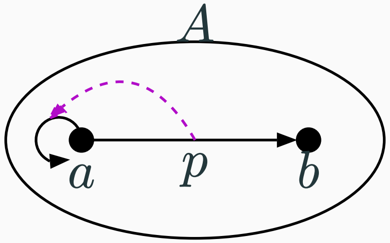
$\to C\ a\ b\ p$

$\text{uniq}\ \{a : A\} b\ p = \mathbf{J}$

$\quad \lambda x, y, p'.\ \mathbf{Id}([z], (x, \mathbf{refl}_x), (y, p'))$

$\quad \lambda x.\mathbf{refl}_{x, \mathbf{refl}_x}$

$\quad a\ b\ p$



think uniq *identifies all identities from* a *to* b *with* $\mathbf{refl}_a$

4.8 Metatheory of $\mathbf{Id}$

- **J is Independent**: $\mathbf{J}$ can still be defined in an $\mathbf{Eq}$ with reflection, but it will be a trivial structure due to UIP

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- **Function Extensionality (funext)**: We can't construct a term for such a type, assuming it as an axiom breaks canonicity (i.e. a bool term is judgementally neither true or false)

$$\text{Funext} = (A : \mathbf{U}) \to (B : A \to \mathbf{U}) \to (f, g : (a : A) \to B\ a)$$
$$\to ((a : A) \to \mathbf{Id}(B\ a, f\ a, g\ a)) \to \mathbf{Id}((a : A) \to B\ a, f, g)$$

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- **No UIP**: We can't construct a term for such a type

$$\mathrm{UIP} = (A : \mathbf{U}) \to (a, b : A) \to (p, q : \mathbf{Id}(A, a, b)) \to \mathbf{Id}(\mathbf{Id}(A, a, b), p, q)$$

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- **Axiom K**: but we can add a second eliminator that identifies identifications of a term to itself with refl to recover UIP whilst still having canonicity and normalization

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$$\begin{aligned}\text{Funext} = (A : \mathbf{U}) \to (B : A \to \mathbf{U}) \to (f, g : (a : A) \to B\, a) \\ \to ((a : A) \to \mathbf{Id}(B\, a, f\, a, g\, a)) \to \mathbf{Id}((a : A) \to B\, a, f, g)\end{aligned}$$

- **No UIP**: We can't construct a term for such a type

$$\text{UIP} = (A : \mathbf{U}) \to (a, b : A) \to (p, q : \mathbf{Id}(A, a, b)) \to \mathbf{Id}(\mathbf{Id}(A, a, b), p, q)$$

- **Axiom K**: but we can add a second eliminator that identifies identifications of a term to itself with refl to recover UIP whilst still having canonicity and normalization
- **Hoffman Conservativity Theorem**: all propositions that are provable (construct terms for a type) in ETT but not in ITT boils down to funext and UIP

- **Full Spectrum**: To use types in our theory, we need a notion of a universe whose terms are types, thus we can omit the formation rules of Υ for introduction rules of $\mathbf{U}$

$$\begin{tikzcd} \mathbf{Tm}(\Gamma, \mathbf{U}) \arrow[rr, bend left, "\mathbf{El}"] & & \mathbf{Ty}(\Gamma) \arrow[ll, bend left, "\mathbf{code}"] \end{tikzcd}$$

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- thus we have dependent version of **rec** called **ind** for inductive types now
- **Recursive Types**: i.e. $\Pi(\mathbf{U}, -)$ can't be made without having a code for $\mathbf{U} : \mathbf{U}$, but doing so causes impredicative paradoxes making the theory inconsistent (we won't prove it here), thus we make a infinite hierarchy of cumulative universes

$$\frac{\Gamma \vdash c : \mathbf{U}_i}{\Gamma \vdash \mathbf{lift}_{i(c)} : \mathbf{U}_{i+1}}\text{Lifting} \qquad \frac{j < i \quad \Gamma \vdash \mathbf{U}_i \text{ type} \quad \Gamma \vdash \mathbf{U}_j \text{ type}}{\Gamma \vdash \mathbf{uni}_{i,j} : \mathbf{U}_i}\text{Universe}$$

5. Univalence

$$\{p \in \mathbf{Tm}(\Gamma.\ \mathbf{Id}(U, A, B), C) \mid \downarrow\} \cong \{\star\}$$

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- the universe is said to be univalent if the map from the identification to C is C as well
- naively we might consider the isomorphisms or bimaps, but this is too strong a condition that it only works on a subset of $\mathbf{U}$ called $\mathbf{HProp}$ which are homotopy propositions, types with only one term, but we will explore them to motivate the full definition

$$\{p \in \mathbf{Tm}(\Gamma.\ \mathbf{Id}(\mathbf{HProp}, A, B), \cong) \mid \mathrm{idtoiso}\} \cong \{\star\}$$

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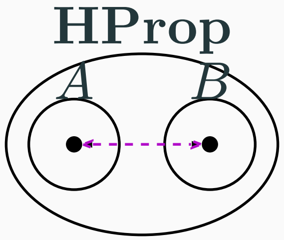
think both types need to have the same amount of terms; this is too restrictive



idtoiso as $p \mapsto (\mathrm{subst}\,\mathrm{id}\,p, \mathrm{subst}\,\mathrm{id}(\mathrm{sym}\ p))$ *and intro an identification given an iso*

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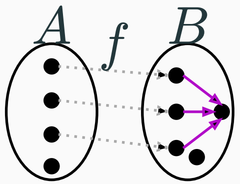	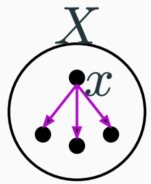	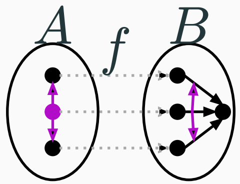	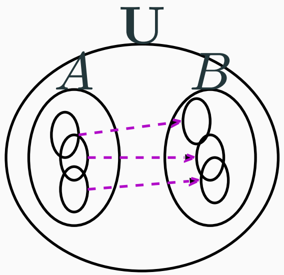

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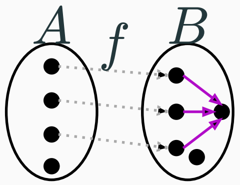	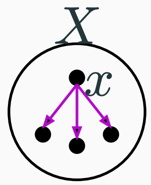	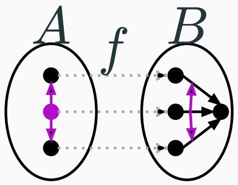	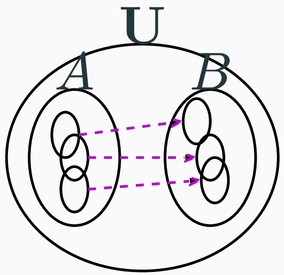

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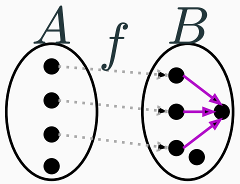	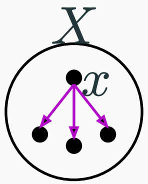	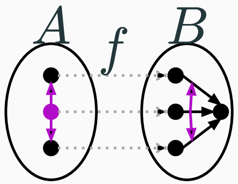	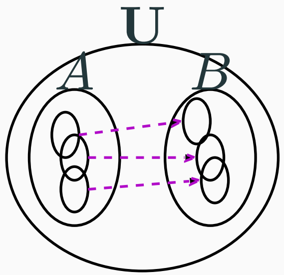

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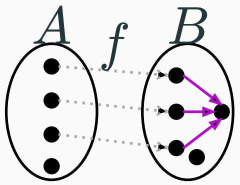	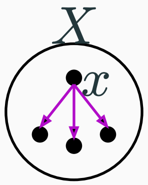	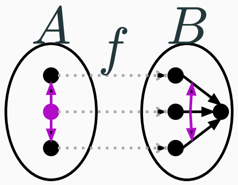	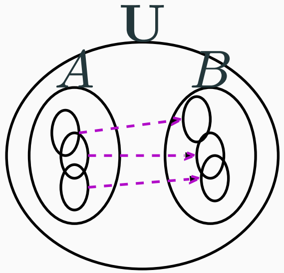

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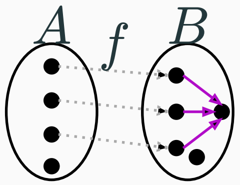	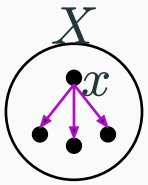	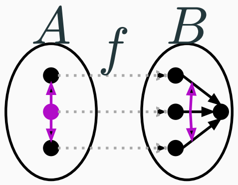	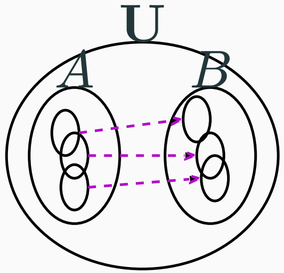

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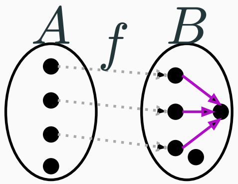

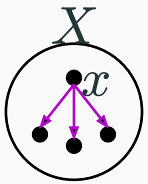

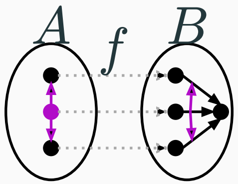

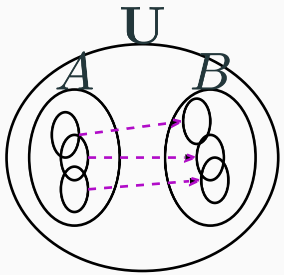

6. Conclusion

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- however ua is an axiom without a computational value, thus we can't implement it in a theorem prover
- an attempt at resolving this is using ideas from logical relations and parametricity to get univalent parametricity

6.2 Summary

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- univalence allows us to do proof transfer ; theoretically, thus we need more