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PAPER VII

LAGRANGE AND MAYER PROBLEMS IN OPTIMAL CONTROL

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1. INTRODUCTION

MODERN control theory uses the most sophisticated mathematical techniques and the engineer, who has surely the greatest stake in the field of control, finds himself increasingly debarred from his own territory by the awesome appearance of today's mathematical language. This is not an unfortunate fact of life which must be uncomplainingly borne; it is a quite unnecessary state of affairs due only to the fact that mathematical rigour demands scholasticism, and scholasticism is difficult and dull. For applications, it is the bare ideas which are required, and these are often very simple. An idea is not to be identified with the language in which it is expressed, and much of modern mathematics, if approached poetically rather than pedantically, can be immediately grasped and harnessed by the engineer. The soul of mathematics has always resided in geometry, and in this paper I turn to geometry to develop the principles of optimal processes, and perhaps to reclaim control for the non-mathematician.

The well-known developments based upon the simplest concepts of the calculus of variations or Pontryagin's "maximum principle" depend more upon analytic ingenuity than direct perception, and require a good deal of justification by hindsight to make them intuitively convincing. More satisfactory from this point of view is Bellman's "dynamic programming", but theoretical studies have scarcely advanced beyond the mere statement of the "principle of optimality" and the immediate derivation of the Jacobi equation for the simplest case—by no means sufficient for general application [1]. The approach taken in this work is perhaps more closely allied to dynamic programming than to other techniques, and may be regarded as a geometric interpretation of it.

The most serious shortcomings of the viewpoint of the calculus of variations are: (a) the Lagrange multipliers have no obvious physical meaning; (b) the system to be controlled enters in the guise of a "side constraint" on the problem of minimization, which is a topsy-turvy way of regarding the control problem. Pontryagin's method may be criticised for introducing adjoint variables "out of the hat" with no obvious motivation. These observations are not mathematical—on the contrary, they are anti-mathematical—they indicate crucial points at which the demands of mathematics override the plea for naturalness, a plea which it is the main purpose of this work to satisfy.

2. NOTATION

Superscripts (e.g. x^j) denote components of a vector. Subscripts denote particular values of a variable (e.g. x_0) or particular members of a family of terms ($B_j(x)$). Partial derivatives are denoted by subscripts,

[1] See PONTRYAGIN's criticism in *The Mathematical Theory of Optimal Processes*. PONTRYAGIN *et. al.* Interscience, New York (1962).

thus

$$\frac{\partial J}{\partial x^i} = J_{x^i}, \qquad \text{and} \qquad \frac{\partial}{\partial x}(\ ) = (\ )_x$$

J_x is the vector $(J_{x1}, J_{x2}, \ldots, J_{xn})$

f_x is the matrix $\begin{bmatrix} f^1_{x1} f^1_{x2}, \ldots, f^1_{xn} & \\ f^2_{x1} & \\ \vdots & \\ f^n_{x1} & f^n_{xn} \end{bmatrix}$

and similarly for u_x.
(x, y) is a scalar product.
Transposed vectors and matrices are not specially indicated.
"m-dimensional" is abbreviated to "m-dim" throughout.

3. THE SYSTEM

We shall be concerned with the behaviour of systems described by

$$\dot{x} = f(x, u(t), t) \tag{1}$$

where x is an n-dim. vector, continuous in the interval

$$I = [t_0, t_f]. \tag{2}$$

The solution $x(x_0, t_0;\ t)$ of (1) (where $x(x_0, t_0;\ t_0) = x_0$) for all t in I is a "trajectory" of the system.

f is an n-dim. vector, continuous, and with continuous derivatives with respect to its arguments.

$u(t)$ is m-dim., piecewise continuous in I.

x is restricted to a given region X of Euclidean n-space, where X is defined by a set of scalar inequalities, continuous, with continuous derivatives:

$$C_i(x) \leqslant 0 \qquad i = 1, 2, \ldots . \tag{3}$$

u is similarly restricted to a region U of m-space, defined by continuously differentiable inequalities:

$$B_j(u) \leqslant 0 \qquad j = 1, 2, \ldots . \tag{4}$$

There may be in addition "mixed constraints" limiting the arguments of f to a region R of $X \times U \times I$ defined by

$$R_k(x, u, t) \leqslant 0 \quad k = 1, 2, \ldots . \tag{5}$$

4. THE OPTIMAL CONTROL PROBLEM

The problem of optimal control is to transfer the state of the system from some point in an initial set S to a terminal set T by choosing $u(t)$ in such a way that a functional

$$\int_{t_0}^{t_f} L(x, u(t), t)\mathrm{d}t \tag{6a}$$

or a function

$$g(x(t_f),\, t_f) \tag{6b}$$

takes a minimum value compared with their values for all other controls and trajectories in R. We have, therefore,

$$x_0 \varepsilon S \subset X; \quad x(x_0,\, t_0; \quad t_f) \varepsilon T \subset X. \tag{7}$$

L and g are continuously differentiable, and

$$L \geqslant 0. \tag{8}$$

S and T will be described by functions of the form

$$S(x_0,\, t_0) = 0 \tag{9a}$$

$$T(x,\, t_f) = 0 \tag{9b}$$

If t, t_0, or t_f enters explicitly as an argument of any of the functions (1–9), the problem is a non-autonomous problem, otherwise it is autonomous. In the latter case t is simply a parameter whose initial value t_0 we can set arbitrarily to zero, and whose final value t_f is defined by (7), i.e. the value of t when the trajectory reaches T. When the problem is non-autonomous, we may add a component x^0 to the state vector x such that

$$\dot{x}^0 = 1 \qquad x_0^0 = 0 \qquad x^0(x_0^0,\, 0;\; t_f) = t_f \tag{10}$$

thereby augmenting the dimension of the state space by one. This is a purely notational manoeuvre which makes the augmented problem autonomous, and shall be assumed to have been done in the sequel, so that t_0 is always zero, and t is never an explicit variable.

When we speak of a "problem" we shall mean the following collection: a system (1) together with its permissible regions X, U, R, (which, if unspecified by constraints (3, 4, 5) shall be taken as the entire open n- or m-dim space, and the whole of $X \times U$ respectively), a cost function (6a or b) and a terminal set T. The only relevant factor which in a practical case would be given, but is not included here, is the initial set S; the omission implies that we are considering the set of optimal control problems for all possible S, and the solution to this general problem is not *one* optimal trajectory, but a *family* of optimal trajectories.

If the cost function is of the type (6a), we have a "Lagrange problem"; if of type (6b), it is a "Mayer problem". Although the two forms are equivalent in the sense that a Lagrange problem can be converted into a Mayer problem and vice versa [2] without any mathematical embarrassment, this can involve the introduction of an artificial variable or an unnatural interpretation of the problem. A terminal miss distance, for example, is not immediately recognizable in the form of a path integral; conversely an integrated error is not convincingly expressed as a terminal point quantity. Add to this the fact that the two formulations lead to quite differently structured developments of the theory of optimal systems, and we have a full justification for treating the two problems separately.

The hybrid "Bolza problem" whose cost function is a sum of the form

$$g(x(t_f)) + \int_0^{t_f} L(x,u)\mathrm{d}t$$

is a combination of the other two formulations, and does not seem to introduce any new principle.

[2] G. A. Bliss: *Lectures on the calculus of variations.* University of Chicago (1946).

5. A BASIC ASSUMPTION

We make at the outset a very far-reaching assumption upon which the whole of the sequel depends:

Given a problem (i.e. the collection described above without a specified S), there exists, from every point in X, one and only one optimal trajectory.

This is a very comprehensive assumption and includes the following by implication:

for all $x_0 \varepsilon X$ there exists at least one control $u(t) \subset U$, for all $t \varepsilon I$, such that, for the corresponding solution $x(x_0, 0;\ t) \subset X$* we have $x(x_0;\ t_f) \varepsilon T$ and $(u(t), x(t)) \varepsilon R$. (11)

In many cases these conditions will not hold: the allowable control region may be too restrictive, so that T is reachable from some points and not others, or there may be admissible trajectories but none truly optimal. The assumption (11) then, is merely a safeguard that the problem has been sensibly posed (i,e. has a solution), and may also be regarded as a further restriction on X: if the region defined by (3) does not satisfy (11), we take as X the subset of that region which does satisfy it. This subset could, without great discomfort, be a disjoint set of trajectories, but this would not allow the simple geometric structure we are seeking, and we shall assume X to be a connected region.

The further assumption of uniqueness is perhaps taking liberty to the brink of licence, and one would hate to be compelled to enumerate the conditions this would impose on the nature of the various functions involved, but—it is an enormously helpful assumption, and from a practical point of view not unreasonable.

6. ISOTIMS†

A. *The problem of Lagrange*

Since there is associated with any initial point x_0 one and only one trajectory, x_0 determines a unique function $x(x_0;\ t)$ and *a fortiori* a unique control $u(x_0;\ t)$. Therefore there is also associated with each point a value of cost

$$J(x_0) = \int_0^{t_f} L(x(x_0;\ t),\quad u(x_0;\ t))\mathrm{d}t \tag{12}$$

Any point $x_1 = x(x_0, 0;\ t_1)$ is itself the initial point for the trajectory $x(x_1, t_1;\ t)$, $t \geqslant t_1$; and the two trajectories $x(x_0, 0;\ t)$, $x(x_1, t_1;\ t)$ coincide for all $t_1 \leqslant t \leqslant t_f$—this is a result of the uniqueness assumption. Indeed every point $x \varepsilon X$ has its associated unique optimal trajectory $x(t)$, optimal control $u(t)$, and optimal cost $J(x)$, and if we reset the value of the parameter t to be zero at any point which is being considered as an "initial point" we have

$$J(x) = \int_0^{t_f} L(x(t), u(t))\mathrm{d}t \tag{13}$$

$J(x)$, then, is a scalar field defined over the region X.

As x moves along a trajectory, the function $J(t)$ is described: it is a continuous, non-negative, non-increasing (see (8)) function, from which it follows that for arbitrary $a > 0$ sufficiently small, there is a point on every optimal trajectory such that

* Some of the arguments of x may be omitted for brevity.

† Greek; $\tau\iota\mu\eta$ = cost.

$$J(x)=a. \tag{14}$$

The set of all points in x for which (14) holds is called "the isotim with value a". An isotim is a set of all points from which T can be reached with the same optimal cost.

The assumption (11) states that there exist single-valued mappings

$$x(x_0, t_0;\ t): \qquad X\times I\times I\to X \tag{15a}$$

$$u(x_0, t_0;\ t): \qquad X\times I\times I\to U \tag{15b}$$

so we may construct the function

$$u(x): \qquad X\to U \tag{16}$$

which is really an abbreviated form for

$$u(x_1, t_1;\ t_1) \text{ (where } x_1=x(x_0, 0;\ t_1)) \text{ for all } x_0 \text{ and all } t_1. \tag{17}$$

Therefore we write (13) as

$$J(x)=\int_0^{t_f} L(x(t),u(x))\mathrm{d}t. \tag{18}$$

The continuity, differentiability etc., properties of $J(x)$ will depend on those of u. We can show that if $u(x)$ is of class C′ then $J(x)$ is of class C′. For, writing (18) less ambiguously as

$$J(x_0)=\int_0^{t_f} L(x(x_0;\ t), \quad u(x(x_0;\ t))\mathrm{d}t \tag{19}$$

we have, when the derivatives exist,

$$J_{x_0}=L(t_f).(t_f)_{x_0}+\int_0^{t_f}(L_x+L_u u_x)x_{x_0}\mathrm{d}t. \tag{20}$$

Consider first the derivative $(t_f)_{xk}$. For δx_0 we must have $x(t_f)+\delta x(t_f)\varepsilon T$. From (9b), $\delta T=0$

$$=(T_x\cdot\delta x) \qquad \text{to first order.} \tag{21}$$
$$=(T_x\cdot x_{x_0}\delta x_0+\dot{x}(t_f)\delta t_f)$$

Let, for any i and all $j\neq i$ in the index set 1, 2, . . . ,n $x^i=\delta$, $x^j=0$; then, letting $\delta\to 0$,

$$(t_f)_{x_0^i}=-(T_x.\ x_{x_0^i})/(T_x\cdot\dot{x}(t_f)) \tag{22}$$

Therefore J_{x_0} exists whenever (i) x_{x_0} exists; (ii) $(T_x\cdot\dot{x})\neq 0$ i.e. the optimal trajectory is not tangent to the terminal set [3]; (iii) u_x exists. Applying a theorem [4] on the differentiability of solutions of differential equations with respect to initial conditions, we find that if the right hand side of

$$\dot{x}=f(x, u(x)) \tag{23}$$

[3] The non-tangency assumption is often made in the calculus of variations. See: [2].
[4] S. Lepschetz: *Differential Equations: Geometric Theory*. Interscience, New York (1963).

is differentiable with respect to x, (which in this case means: if u_x exists) then x_{x_0} exists. The existence of J_{x_0}, then, is entirely dependent upon u_x.

Following our declared intention to investigate practical cases rather than those of great mathematical generality, we might be tempted to assume the existence of u_x. This would be unrealistic, for we could then construct a matrix equation

$d/dt\ x_{x_0} = f_x x_{x_0}$, $x_{x_0}(0) = $ unit matrix, whose solution x_{x_0} cannot be singular [4]. (24)

$\therefore$ for any $\delta x_0 \neq 0$ we have

$$\delta x(t) = x_{x_0}(t)\delta x_0 \neq 0 \qquad t > 0$$

so that trajectories, if anywhere distinct, remain distinct thereafter. The assumption (11), however, demands uniqueness to the right only, not to the left; i.e. trajectories originally distinct may coincide, but must remain coincident thereafter. We cannot assume u_x to exist everywhere, nor can $u(x)$ be everywhere continuous since $u(t)$ is only piecewise continuous, but we assume the following:

> X is composed of separate regions throughout each of which u_x exists. The regions abut one on the other at boundaries which are smooth manifolds on which $u(x)$ is not necessarily continuous, and a fortiori not all components of u_x are defined. (25)

Figure 1 illustrates such a situation involving 3 separate regions. The assumption (25) may be taken as a hypothesis; many systems exhibit such a property and I have not met one which does not. It may well be possible to prove (25) as a consequence of the other properties, but for the present it suffices to accept it as a further assumption.

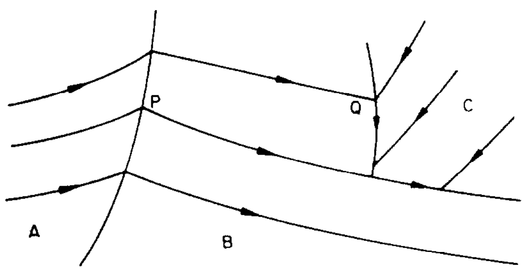

FIG. 1

The isotims (not shown) are smooth $(n-1)$-dim manifolds with gradients J_x within the smooth regions but in general will not have unique normals on the boundaries.

7. NECESSARY CONDITIONS FOR OPTIMALITY

7.1. *A general condition.* We easily derive a very general criterion for the optimality of a trajectory.

Let there be given any point $x_0 \varepsilon X$ and any control function $u(t) \subset U$, $0 \leqslant t \leqslant t_1 \leqslant t_f$, such that $(x(t), u(t)) \subset R$. Let $x_1 = x(x_0;\ t_1)$. The isotims corresponding to x_0, x_1 are

$$J(x) = a_0, \qquad J(x) = a_1 \tag{26}$$

respectively.

In travelling from x_0 to x_1 the (non-optimal) trajectory crosses isotims at the rate $\mathrm{d}J/\mathrm{d}t$.

$$\therefore a_1 - a_0 = \int_0^{t_1} \frac{\mathrm{d}J}{\mathrm{d}t}\mathrm{d}t \tag{27}$$

The value of the integral depends only upon the end points, not upon the path taken, but the actual cost incurred by this trajectory does depend on the path and is given by

$$\text{cost} = \int_0^{t_1} L(x(t), u(t))\mathrm{d}t. \tag{28}$$

If an optimal path is taken from x_1 to T, the total cost of the trajectory from x_0 to T is

$$\int_0^{t_1} L(x(t), u(t))\mathrm{d}t + a_1,$$

but the minimum cost possible for a path from x_0 to T is a_0.

$$\therefore a_0 - a_1 \leq \int_0^{t_1} L\,\mathrm{d}t.$$

Using (27) and the fact that t_1 is arbitrary we have

$$L(x, u) + \frac{\mathrm{d}J}{\mathrm{d}t} \geq 0 \text{ or } \min_u \left[L(x, u) + \frac{\mathrm{d}J}{\mathrm{d}t} \right] = 0 \tag{29}$$

(29) depends only upon the definition of optimality, and the assumption of uniqueness. It does not rely on any particular property of **X**, which may consist merely of isolated trajectories. Because it is very general, it is also not very useful for application. In order to derive something more serviceable we must make simplifying assumptions.

7.2. *The simplest case*. First consider the situation in the interior of a region in which u_x exists, and suppose too that the strict inequalities in (4, 5) hold. We may write

$$\frac{\mathrm{d}J}{\mathrm{d}t} = (J_x \cdot \dot{x}) = (J_x \cdot f(x, u)) \tag{30}$$

and (29) becomes

$$\min_u [L(x, u) + (J_x . f(x, u))] = 0. \tag{31}$$

Define: $H(x, u) = L + (J_x . f)$, then

$$\min_u H = 0, \tag{32}$$

and immediately

$$H_u(x, u) = 0 \tag{33}$$

gives the correct control as long as

$$H_{uu} \neq 0 \tag{34}$$

which we shall assume.

In the region we are considering, every point lies on an optimal trajectory, so that H is zero throughout the region.

$$\therefore \frac{dH}{dx} = 0 = \frac{dL}{dx} + \frac{d}{dx}(J_x . f) \tag{35}$$

(35) ensures the existence of the derivatives $d/dx^i(J_x . f)$, which we write

$$\frac{d}{dx^i}(J_x . f) = \underset{\delta x^i \to 0}{\text{lt}} \frac{(J_x(x+\delta x^i) . f(x+\delta x^i, u(x+\delta x^i))) - (J_x(x) . f(x, u(x)))}{\delta x^i}$$

$$= \text{lt} \frac{(J_x(x+\delta x^i) . f(x, u(x)) + [f_{x^i} + f_u u_{x^i}]\delta x^i + . .) - (J_x(x) . f(x, u(x)))}{\delta x_i}$$

$$= \text{lt}\left(\frac{J_x(x+\delta x^i) - J_x(x)}{\delta x^i} . f(x, u(x))\right) + (J_x(x) . f_{x^i} + f_u u_{x^i}) \tag{36}$$

We see that the second derivatives of J exist, and because they are symmetric,

$$J_{xx^i} f = \frac{d}{dt} J_{x^i} .$$

$$\therefore \frac{d}{dt} J_{x^i} = -L_{x^i} - J_x f_{x^i} - (L_u + J_x f_u) u_{x^i} . \tag{37}$$

$$\text{or} \frac{d}{dt} J_{x^i} = -L_{x^i} - J_x f_{x^i} . \tag{38}$$

applying (33).

The crucial relations (33, 38) may be neatly expressed

$$H_u = 0; \qquad H_x = 0. \tag{39}$$

7.3. *Constraints.* We may now relax the restriction that the equalities in (4, 5) should not hold, and suppose that equality may apply over some parts of the trajectories which still must remain in regions where u_x is defined. For simplicity consider only constraints of the type $R(x, u) \leqslant 0$, for this includes those of type (4).

Using an extension of the technique of undetermined multipliers [5] we introduce a set of multipliers ω_k, one for each of the set of constraints (5), and such that $\omega_k = 0$ whenever $R_k \neq 0$.

Define: $\bar{H} = H(x, u) + \omega(x) R(x, u)$, and since $\omega R = 0$ always, we can immediately extend (32) to min $\bar{H} = 0$ and in place of (39) we shall have

$$\begin{aligned} H_u + \omega(x) R_u &= 0 \\ H_x + \omega(x) R_x &= 0. \end{aligned} \tag{40}$$

(Here we need to assume also that $R_u \neq 0$ when $R = 0$. The expression $\omega_x R$ does not appear in (40) since, if $R \neq 0$, i.e. throughout the interior of the allowable region R, $\omega = 0. \therefore \omega_x = 0$).

[5] This argument is a modification of that in L. BERKOVITZ: Variational methods in problems of control and programming. *J. Math. Anal. and Applications* 3, No. 1 (1961).

We can also deal readily with the case in which (34) does not apply, viz. when H is linear in the variable u. The optimal control is then on the boundary of U or of R. Both cases are covered by setting $R(x, u)=0$. As before, (40) holds, although the actual optimal control must be found by direct evaluation of the possibilities.

7.4. *Boundaries.* A trajectory which is not wholly (with the possible exception of its endpoints) within one smooth region of X will be regarded as consisting of a chain of trajectories. To investigate properties of the junctions we consider first the problem of perturbation of end points of trajectories. Referring to (27), the cost of an optimal trajectory between x_0, x_1 is $J=a_1-a_0$. A perturbation δx_0, δx_1 will give a cost perturbation $\delta J=\delta a_1-\delta a_0$, and if J_x exists at x_0, x_1 then

$$\delta J=(J_{x_1}\cdot\delta x_1)-(J_{x_0}\cdot\delta x_0) \tag{41}$$

If x_0, x_1 are end points, J_x will have at best right and left limiting values respectively, and we are only concerned with components in the directions of δx_0, δx_1.

If x_0, x_1 belong to some sets S, T respectively, and turn out to be the optimal end points for a trajectory from S to T, then the first order cost perturbation is zero. Since δx_0, δx_1 are independent, $J_{x_0}=J_{x_1}=0$.

In Fig. 1, consider the trajectory through P. P is both the terminal point of one arc and the initial point of the succeeding arc. Let P be perturbed, but remain on the boundary (denoted $M=\bigcap_{i=1}^{r} M^i(x)=0$). The superscripts $-$, $+$ will refer to limiting values approaching P from left and right respectively.

$$\therefore \delta J=(J_x^-.\delta x)-(J_x^+.\delta x)=(J_x^--J_x^+.\delta x)=0. \tag{42}$$

Since the only restriction on δx is that it remains in M, we have the result that the component of J_x lying in the intersection of the tangent planes of the M_i at P, is continuous, but the component normal to M need not be. This may be expressed

$$J_x^+-J_x^-+\lambda_i M_x^i=0 \tag{43}$$

λ_i are scalar multipliers—one for each M^i.

If the trajectory crosses M, as at P in Fig. 1, the λ_i are defined, since both J_x^+ and J_x^- are defined; if it remains on the boundary, as at Q, then J_x^+ is not defined, but we have instead of (42)

$$\delta J=(J_x^-.\delta x)-J^+(x+\delta x)-J^+(x)=0$$

$$\therefore \frac{(J_x^-.\delta x)}{|\delta x|}=\frac{J^+(x+\delta x)-J^+(x)}{|\delta x|}$$

The left hand side exists in the limit and is the tangential component of J_x^-. The right hand must exist also, and is the corresponding component of J_x^+, so that (43) still holds, but λ is arbitrary, and the normal component of J_x is not defined.

The reason for this arbitrariness is that the trajectory is now in an (n-r)-space, and the isotims are (n-r-1)-dim. manifolds embedded in it. Clearly there is no unique normal to the isotim in a direction not in the (n-r)-space, just as there is no unique normal to a curve embedded in a 2-surface in 3-space. Indeed, within M, n-r coordinates suffice to describe the behaviour of the trajectories.

We may introduce a variable coordinate transformation $x \to y$ such that the r redundant coordinates are

$$y^1 = y^2 = \dots = y^r \equiv 0 \tag{44}$$

and the $y^{r+1} \dots y^n$ are embedded in M. The tangential components of J_x^+ become $J_y j$, $j = r+1, \dots, n$; the normal components, symbolically written $J_y k$, $k = 1, 2, \dots, r$, are undefined. $(J_x \cdot \dot{x})$ is invariant with respect to this transformation, so (31) becomes

$$\min[L(x, u) + (J_y \cdot \dot{y})] = 0 \tag{45}$$

and (44) shows that the coefficients of the undefined $J_y k$ are identically zero.

Modification of the arguments of section 7.2 result, after transforming back to x coordinates, in equations (33, 38) with J_x replaced by its tangential component. In practice (33, 38) can be used without modification, and with J_x continuous, since the tangential component is continuous, and the normal component is arbitrary, and so can be made continuous.

7.5. *State constraints.* The boundary of X is no different in essence from the boundary of a component region of X, except that here an allowable region exists on only one side of the boundary, so that trajectories must of necessity remain on it, and cannot cross. The arguments of the previous section can be applied almost unchanged, and we shall not go into details here. In practice there is a distinction to be observed, in that the separate regions previously discussed emerged as the natural structure of the optimal fields, whereas in the case of state space constraints the boundary is externally imposed and indeed prevents the trajectories from achieving "free" optimality. This, however, is not a distinction of great profundity and could be removed by mapping the entire n-space into the region defined by the inequality constraints.

B. The Problem of Mayer

The geometric structure of the solution of the Mayer problem is different from the Lagrange problem, but the detailed analysis is in the main quite similar, so we may content ourselves with a mere outline of the properties of the problem.

8. THE REACHABLE SET

Given, as before, a system (1), spaces X, U, $R \subset X \times U$, defined by (3, 4, 5), a terminal set T, and a cost function, this time of the form $g(x(t_f))$, we make the existence and uniqueness assumption (11).

This problem is more subtle than the Lagrange form, for it might be taken to mean: "transfer the system from some initial point to that point in T for which the value of g is least". Now, g can be evaluated over T without reference to the dynamic system, and the point at which it takes its least value is usually immediately obvious; (it may be at infinity). In that case the problem could be restated: "transfer the system to the given terminal point", which is no longer a problem of optimization, and indeed might have many solutions. Furthermore, all trajectories would terminate either at the same "absolutely optimal" point, if g has a minimum only at one point, or at any arbitrary point on a minimum surface.

Obviously this is a misreading of the problem, which can be correctly posed only if it is not possible to reach *any* given point in T, from the initial point, but only a certain subset of T. In that case we may separate the state space into a "reachable set", W and an "unreachable set", and the interpretation of the problem becomes: "move to the point in $W \cap T$ for which g is least".

(Note that if a system is completely controllable with respect to an initial point, i.e. can be transferred to any other point, the Mayer problem cannot be posed without some constraining condition).

Defining an isotim, as before, as the set of all points in X from which T is reachable with the same optimal cost, it is clear that since the cost is determined by the end point of the trajectory, all points on a trajectory lie in the same isotim, and the isotims are composed of sets of trajectories.

The equation of an isotim is

$$J(x) = g(x(t_f)) = a \tag{46}$$

and since $g_x(t_f)$ is presumed to exist, $J(x)$ will have a unique normal wherever $x(t_f)_x$ exists. If we suppose X to be composed of smooth regions with u_x, separated by smooth boundaries, it turns out that, as in Section 6, $J(x)$ has a unique normal interior to these regions, and a unique normal component tangential to the boundaries.

9. NECESSARY CONDITIONS FOR OPTIMALITY

Given some initial point on an isotim, value a, the correct control action will take the state to points from which T can be reached with a cost as low as possible. Points on isotims with value less than a are clearly not reachable, so that x can only move in such a way that $\mathrm{d}J/\mathrm{d}t \geqslant 0$. The best action will be that which makes $\mathrm{d}J/\mathrm{d}t = 0$, so our condition is

$$\min_u \frac{\mathrm{d}J}{\mathrm{d}t} = 0. \tag{47}$$

This condition, like its analogue (29), is quite general, but of little practical use as it stands. Making similar assumptions to those in section 7.2, we write

$$\frac{\mathrm{d}J}{\mathrm{d}t} = (J_x . f)$$

and $\min_u (J_x \cdot f(x,u)) = 0.$ (48)

The implication of this is that the trajectory lies along the boundary of the reachable set, which contains only points for which the cost is greater than, or equal to, a; the isotim a and the reachable set meet along the curve of the optimal trajectory.

The differential equations for J_x, deduced by arguments comparable to those in section 7.2 are

$$\frac{\mathrm{d}}{\mathrm{d}t} J_x = -J_x f_x .$$

The extension of these ideas to take into account constraints of various types is straightforward, and raises no fundamental issue.

10. CONCLUSION

Although in practical cases when the aim is to obtain a numerical solution to a given problem it is of no consequence whether the Lagrange or Mayer formulation is chosen, since the actual equations evolved are identical, nevertheless the underlying concepts involved are radically different and it is worth while bearing the two possibilities in mind when one wants more than just a set of numbers.

The two basic geometric concepts involved here viz., the "isotim", and the "reachable set", are of fundamental importance, and although used here in a very elementary way, have a great role to play in a complete theory of geometric dynamics.

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