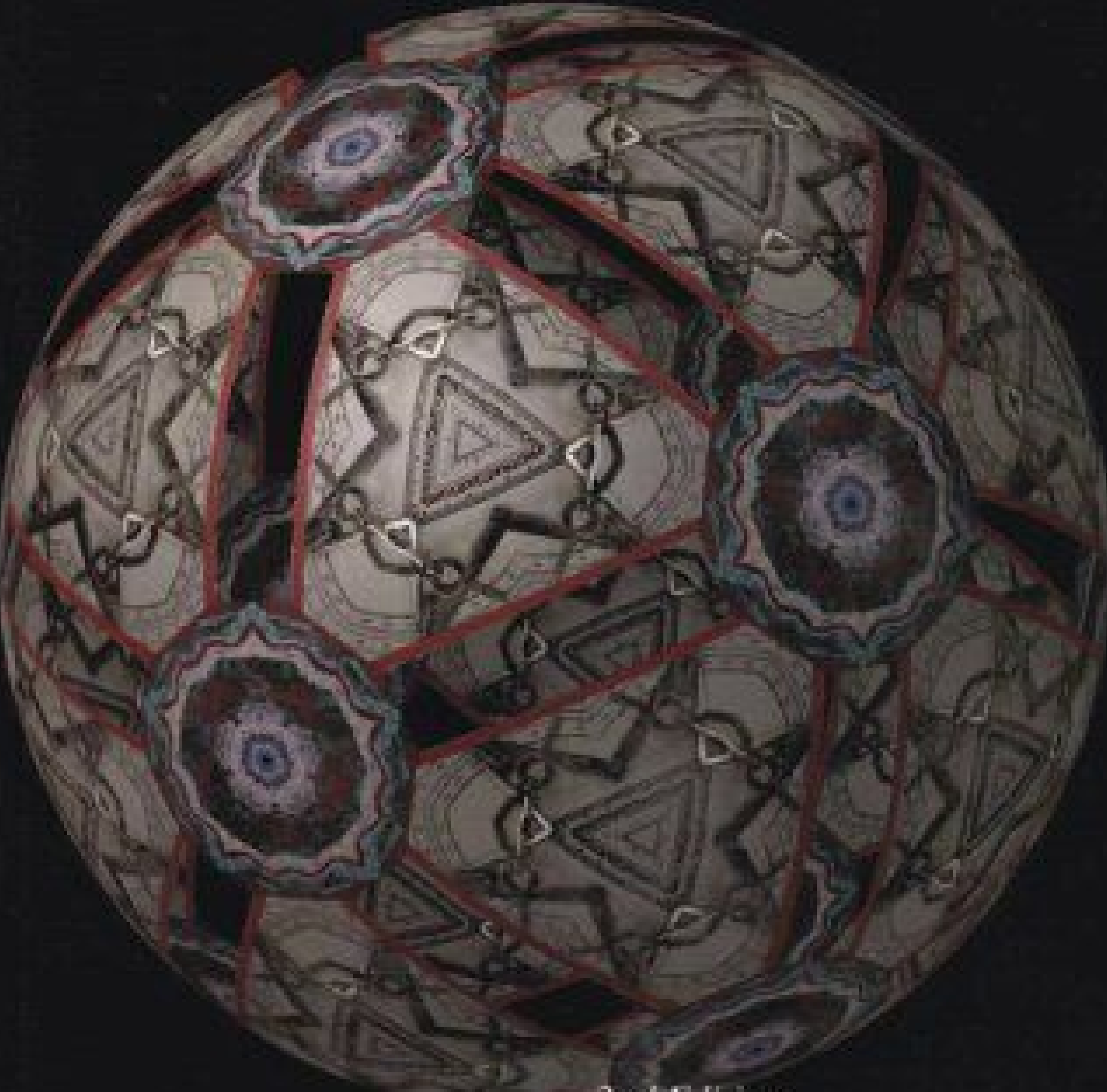
the Art of Problem Solving

Introduction to Geometry

Richard Rusczyk



2nd Edition

the Art of Problem Sol

Introduction to Geometry

The Art of Problem Solving Introduction Series constitutes a complete curriculum for outstanding math students in grades 6-10. The books in the series are:

Introduction to Algebra by Richard Rusczyk
Introduction to Counting & Probability by David Patrick
Introduction to Geometry by Richard Rusczyk
Introduction to Number Theory by Mathew Crawford

Together these books give students a solid background in basic problem-solving mathematics and prepare them for prestigious competitions such as MATHCOUNTS and the American Mathematics Competitions.

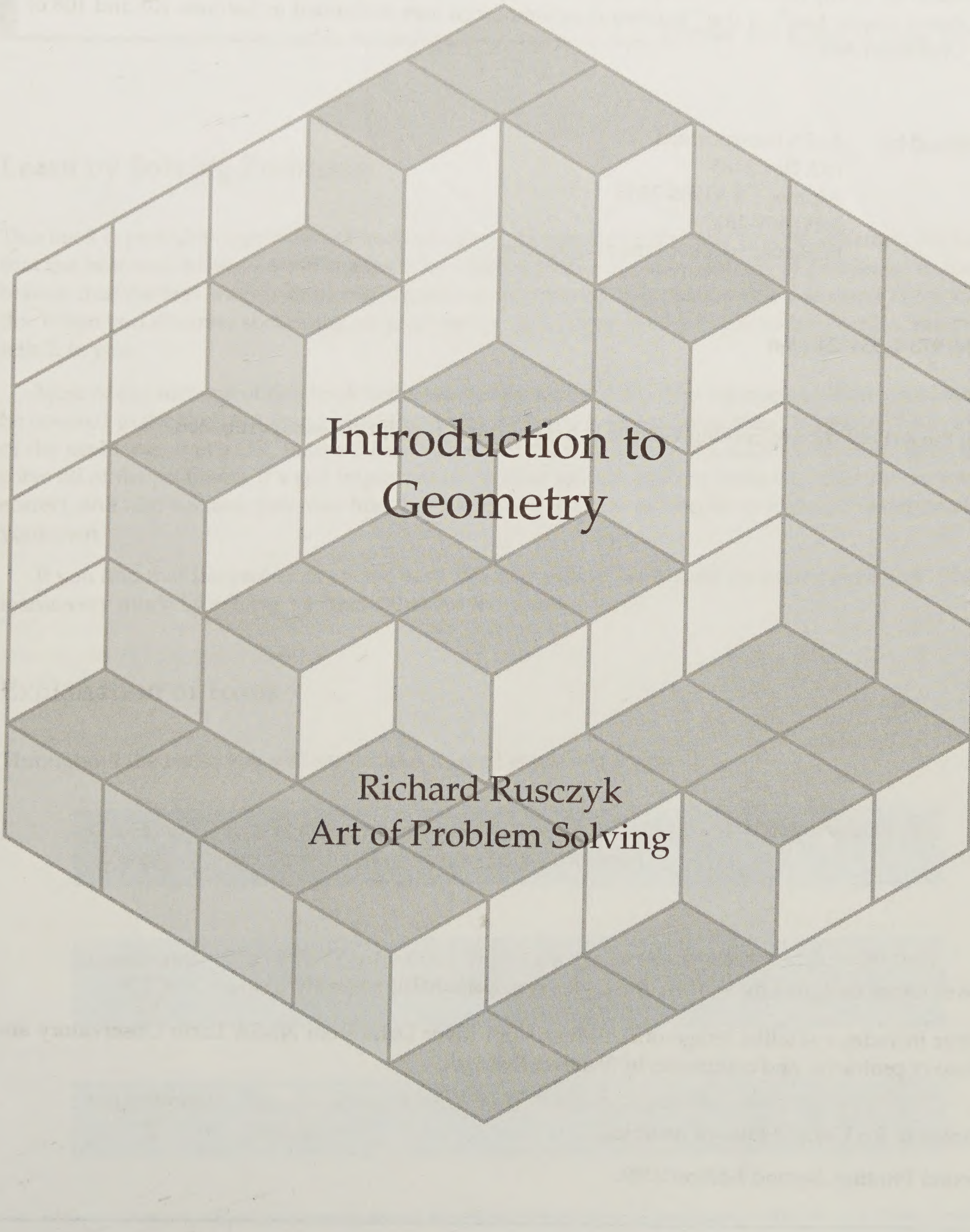
Praise for **Introduction to Geometry**

"[My son] loves your textbook! As a former teacher I know how difficult it is to find a fantastic textbook but yours nailed it! [His] words were 'I love that book.' In my twenty years of teaching, I've never heard a student say that about a textbook before. When asked why he loved it he said because it was easy to follow, and had lots of examples."

Diana Cobbe, parent and teacher

Richard Rusczyk is the founder of www.artofproblemsolving.com. He is co-author of *the Art of Problem Solving, Volumes 1 and 2*, and author of *Introduction to Algebra*. He was a national MATHCOUNTS participant in 1985, a three-time participant in the Math Olympiad Summer Program, and a USA Math Olympiad Winner in 1989.

ISBN 978-1-934124-08-6
90000 >
9 781934 124086



Introduction to Geometry

Richard Rusczyk
Art of Problem Solving

Published by: AoPS Incorporated
P.O. Box 2185
Alpine, CA 91903-2185
(619) 659-1612
books@artofproblemsolving.com

ISBN: 978-1-934124-08-6

Visit the Art of Problem Solving website at http://www.artofproblemsolving.com

Cover image designed by Vanessa Rusczyk using KaleidoTile software.

Cover includes a satellite image of the Mississippi River Delta from NASA Earth Observatory and a photo of protractor and compasses by Vanessa Rusczyk.

Printed in the United States of America.

Second Printing, Second Edition 2009.

How to Use This Book

Learn by Solving Problems

This book is probably very different from most of the math books that you have read before. We believe that the best way to learn mathematics is by solving problems. Lots and lots of problems. In fact, we believe that the best way to learn mathematics is to try to solve problems that you don't know how to do. When you discover something on your own, you'll understand it much better than if someone just tells it to you.

Most of the sections of this book begin with several problems. The solutions to these problems will be covered in the text, but try to solve the problems *before* reading the section. If you can't solve some of the problems, that's OK, because they will all be fully solved as you read the section. Even if you solve all of the problems, it's still important to read the section, both to make sure that your solution is correct, and also because you may find that the book's solution is simpler or easier to understand than your own.

If you find that the problems are too easy, this means that you should try harder problems. Nobody learns very much by solving problems that are too easy for them.

Explanation of Icons

Throughout the book, you will see various shaded boxes and icons.

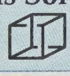
Concept: This will be a general problem-solving technique or strategy. These are the "keys" to becoming a better problem solver!

Important: This will be something important that you should learn. It might be a formula, a solution technique, or a caution.

WARNING!! Beware if you see this box! This will point out a common mistake or pitfall.

Game: Remember, math is fun! This box will contain a game to think about.

Sidenote: This box will contain material which, although interesting, is not part of the main material of the text. It's OK to skip over these boxes, but if you read them, you might learn something interesting!

Bogus Solution:  Just like the impossible cube shown to the left, there's something wrong with any "solution" that appears in this box.

Exercises, Review Problems, and Challenge Problems

Most sections end with several **Exercises**. These will test your understanding of the material that was covered in the section that you just finished. You should try to solve *all* of the exercises. Exercises marked with a ★ are more difficult.

Most chapters have several **Review Problems**. These are problems which test your basic understanding of the material covered in the chapter. Your goal should be to solve most or all of the Review Problems for every chapter – if you're unable to do this, it means that you haven't yet mastered the material, and you should probably go back and read the chapter again.

All of the chapters end with **Challenge Problems**. These problems are generally more difficult than the other problems in the book, and will really test your mastery of the material. Some of them are very, very hard – the hardest ones are marked with a ★. Don't necessarily expect to be able to solve all of the Challenge Problems on your first try – these are difficult problems even for experienced problem solvers. If you are able to solve a large number of Challenge Problems, then congratulations, you are on your way to becoming an expert problem solver!

Hints

Many problems come with one or more hints. You can look up the hints in the Hints section in the back of the book. The hints are numbered in random order, so that when you're looking up a hint to a problem you don't accidentally glance at the hint to the next problem at the same time.

It is very important that you first try to solve the problem without resorting to the hints. Only after you've seriously thought about a problem and are stuck should you seek a hint. Also, for problems which have multiple hints, use the hints one at a time; don't go to the second hint until you've thought about the first one.

Solutions

The solutions to all of the Exercises, Review Problems, and Challenge Problems are in the separate solution book. If you are using this textbook in a regular school class, then your teacher may decide not to make this solution book available to you, and instead present the solutions him/herself. However, if you are using this book on your own to learn independently, then you probably have a copy of the solution book, in which case there are some very important things to keep in mind:

1. Make sure that you make a serious attempt at the problem before looking at the solution. Don't use the solution book as a crutch to avoid really thinking about a problem first. You should think *hard* about a problem before deciding to give up and look at the solution.

2. After you solve a problem, it's usually a good idea to read the solution, even if you think you know how to solve the problem. The solution that's in the solution book might show you a quicker or more concise way to solve the problem, or it might have a completely different solution method that you might not have thought of.

3. If you have to look at the solution in order to solve a problem, make sure that you make a note of that problem. Come back to it in a week or two to make sure that you are able to solve it on your own, without resorting to the solution.

Resources

Here are some other good resources for you to further pursue your study of mathematics:

- *The Art of Problem Solving* books, by Sandor Lehoczky and Richard Rusczyk. Whereas the book that you're reading right now will go into great detail of one specific subject area – geometry – *the Art of Problem Solving* books cover a wide range of problem solving topics across many different areas of mathematics.

- The `www.artofproblemsolving.com` website. The authors of this book are also the webmasters of the Art of Problem Solving website, which contains many resources for students:
  - a discussion forum
  - online classes
  - resource lists of books, contests, and other websites
  - a LaTeX tutorial
  - and much more!

- You can hone your problem solving skills (and perhaps win prizes!) by participating in various math contests. For middle school students in the United States, the major contests are MATHCOUNTS, MOEMS, and the AMC 8. For U.S. high school students, some of the best-known contests are the AMC/AIME/USAMO series of contests (which choose the U.S. team for the International

Mathematics Olympiad), the American Regions Math League (ARML), the Mandelbrot Competition, the Harvard-MIT Mathematics Tournament, and the USA Mathematical Talent Search. More details about these contests are on page vii, and links to these and many other contests are available on the Art of Problem Solving website.

A Note to Teachers

We believe that students learn best when they are challenged with hard problems that at first they may not know how to do. This is the motivating philosophy behind this book.

Rather than first introducing new material and then giving students exercises, we present problems at the start of each section that students should try to solve *before* the new material is presented. The goal is to get students to discover the new material on their own. Often, complicated problems are broken into smaller parts, so that students can discover new techniques one piece at a time. Then the new material is formally presented in the text, and full solutions to each problem are explained, along with problem-solving strategies.

We hope that teachers will find that their stronger students will discover most of the material in this book on their own by working through the problems. Other students may learn better from a more traditional approach of first seeing the new material, then working the problems. Teachers have the flexibility to use either approach when teaching from this book.

The book is linear in coverage. Generally, students and teachers should progress straight through the book in order, without skipping chapters. Sections denoted with a ★ contain supplementary material that may be safely skipped. In general, chapters are not equal in length, so different chapters may take different amounts of classroom time.

Extra! Occasionally, you'll see a box like this at the bottom of a page. This is an "Extra!" and might be a quote, some biographical or historical background, or perhaps an interesting idea to think about.

Acknowledgements

Contests

We would like to thank the following contests for allowing us to use a selection of their problems in this book:

- The **American Mathematics Competitions**, a series of contests for U.S. middle and high school students. The **AMC 8**, **AMC 10**, and **AMC 12** contests are multiple-choice tests which are taken by over 400,000 students every year. Top scorers on the AMC 10 and AMC 12 are invited to take the **American Invitational Mathematics Examination (AIME)**, which is a more difficult, short-answer contest. Approximately 10,000 students every year participate in the AIME. Then, based on the results of the AMC and AIME contests, about 250 students are invited to participate in the **USA Mathematical Olympiad (USAMO)**, a 2-day, 9-hour examination in which each student must show all of his or her work. Results from the USAMO are used to invite a number of students to the Math Olympiad Summer Program, at which the U.S. team for the International Mathematical Olympiad (IMO) is chosen. More information about the AMC contests can be found on the AMC website at `http://www.unl.edu/amc`.

- **MATHCOUNTS®**, the premier contest for U.S. middle school students. MATHCOUNTS is a national enrichment, coaching, and competition program that promotes middle school mathematics achievement through grassroots involvement in every U.S. state and territory. President George W. Bush and former Presidents Clinton, Bush and Reagan have all recognized MATHCOUNTS in White House ceremonies. The MATHCOUNTS program has also received two White House citations as an outstanding private sector initiative. Particularly exciting for MATHCOUNTS Mathletes® were the hour-long ESPN programs on each of the past three National Competitions. More information is available at `www.mathcounts.org`.

- The **Mandelbrot Competition**, which was founded in 1990 by Sandor Lehoczky, Richard Rusczyk, and Sam Vandervelde. The aim of the Mandelbrot Competition is to provide a challenging, engaging mathematical experience which is both competitive and educational. Students compete both as individuals and in teams. The Mandelbrot Competition is offered at the national level for more advanced students and the regional level for less experienced problem solvers. More information can be found at `www.mandelbrot.org`.

- The **Harvard-MIT Mathematics Tournament (HMMT)**, which is an annual math tournament for high school students, held at MIT and at Harvard in alternating years. It is run exclusively by MIT and Harvard students, most of whom themselves participated in math contests in high school. More information is available at `web.mit.edu/hmmt/`.

- The **USA Mathematical Talent Search (USAMTS)**, which was founded in 1989 by Professor George Berzsenyi. The USAMTS is a free mathematics competition open to all United States middle and high school students. As opposed to most mathematics competitions, the USAMTS allows students a full month to work out their solutions. Carefully written justifications are required for each problem. More information is available at `www.usamts.org`.

- The **American Regions Math League (ARML)**, which was founded in 1976. The annual ARML competition brings together nearly 2,000 of the nation's finest students. They meet, compete against, and socialize with one another, forming friendships and sharpening their mathematical skills. The contest is written for high school students, although some exceptional junior high students attend each year. The competition consists of several events, which include a team round, a power question (in which a team solves proof-oriented questions), an individual round, and two relay rounds. More information is available at `www.arml.com`.

How We Wrote This Book

This book was written using the LaTeX document processing system. Specifically, this book was prepared using the MiKTeX installation of pdflatex on a PC running Microsoft Windows XP. We must thank the authors of the various LaTeX packages that we used while preparing this book, and also the brilliant authors of *The LaTeX Companion* for writing a reference book that is not only thorough but also very readable. The diagrams were prepared using METAPOST, a powerful graphics language which is based on Knuth's METAFONT.

About Us

This book is a collaborative effort of the staff of the Art of Problem Solving. Richard Rusczyk was the lead author for this book, and wrote most of the text. Some of the Sidebar and Extra sections were prepared by Ashley Reiter Ahlin, Vanessa Rusczyk, and Naoki Sato. The solutions were written by Ruozhou Jia, Brian Rice, Richard Rusczyk, and Naoki Sato. Extensive proofreading of the manuscript was done by Mathew Crawford, Lisa Davis, Amanda Jones, David Patrick, Tim Lambert, Naoki Sato, and Jake Wildstrom. Vanessa Rusczyk designed the cover and also contributed greatly to the interior design of the book. David Patrick, Naoki Sato, Ravi Boppana, Meena Boppana, Valentin Vornicu, Greg Brockman, Larry Evans, and Joseph Laurendi contributed problems and proofreading to this second edition.

The author would also like to thank Josh Zucker, whose comments about how he learned mathematics inspired the questions-before-the-lessons approach of the text.

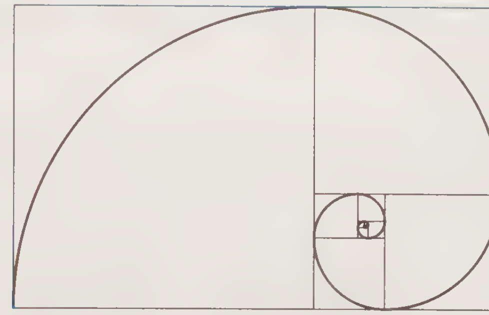
Websites & Errata

We used several websites as source material for the text. Links to these sites, as well as an errata list for the text and solutions, are provided at

`http://www.artofproblemsolving.com/BookLinks/IntroGeometry/links.php`

Contents

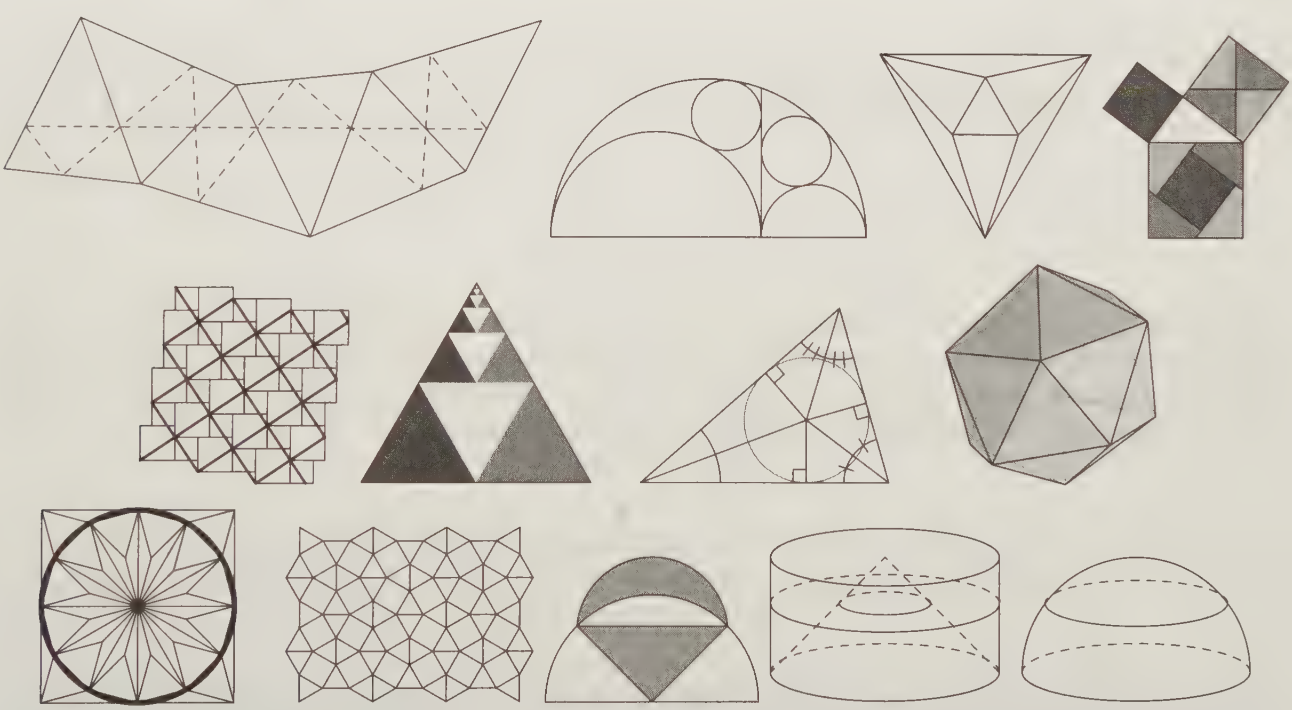
For **Professor Harold Reiter**, who brought me back to education, and for **Vanessa Rusczyk**, whose confidence in me and love of the desert has kept me here ever since.



The Golden Ratio Spiral

Do not laugh at notations; invent them, they are powerful. In fact, mathematics is, to a large extent, invention of better notations. – Richard Feynman

CHAPTER 1 What's in a Name?



Each of these images helps tell a story. Throughout this book we'll share these stories with you, but before we tell these stories, we have to name our characters.

1.1 Why Names and Symbols?

To convince you that names and symbols are useful, we'll start at the end of the book instead of the beginning. Here's the final example problem in this book, written without any special symbols or names.

> Draw three points and connect each to the other two with straight paths. Also, draw the circle that passes through all three of these points. Then, draw a line through one of those three points such that the line goes inside the region you just formed and is equally close to the two other straight paths you formed initially through this point. Draw the circle that goes through the one of your three first points you just drew a line through, through the point where this line hits the straight path that connects the other two of your first three points, and through the point that is half-way between these two other points.
>
> Consider the two paths from the point we drew the extra line through to the other two of our first three points. These paths hit our second circle before they hit these other two points. Show that the distance from where the circle hits these paths to the points where these paths end is the same for both paths.

If you can make much sense of this problem, you're a much more careful reader than I am! We need some special names and symbols so we can communicate mathematical ideas more simply.

1.2 Points, Lines, and Planes

Figure 1.1: A Point

A dot. A speck. In geometry, it's a **point**. If you lived on a point, you'd be awfully bored. There would be no up and down, no right and left. You couldn't move any amount in any direction. Since you can't move on your point in any direction, we say a point has 0 **dimensions**. In order to tell one point from another, we usually label them with capital letters, such as point P above.

Figure 1.2: A Segment

Now, say you got so bored on one point that you just had to go to another point. If there were a straight path from one point to another, that path would be called a **line segment**, or just a **segment**. The two points at the ends of a segment are cleverly called the **endpoints** of the segment. We use these endpoints to label the segment. For example, $\overline{AB}$ is the segment from A to B. To denote the length of the segment, we omit the bar. For example, AB equals 1.5 inches in Figure 1.2.

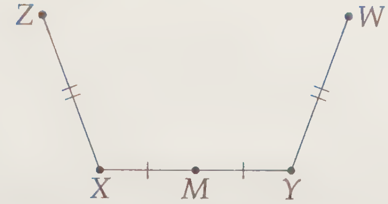


Figure 1.3: A Midpoint and Marking Segments of Equal Length

The endpoints aren't the only points on a segment. There are infinitely many points, since between any two points on the segment, we can find another point. One special point on a segment is the segment's **midpoint**, which is the point halfway between the endpoints. Because the midpoint is the same distance from both endpoints, we say it is **equidistant** from the endpoints. In Figure 1.3, M is the midpoint of $\overline{XY}$. We show that $XM = MY$ in the diagram with the little tick marks along $\overline{XM}$ and $\overline{MY}$. If we have multiple sets of equally long segments, we use a different number of tick marks for each. For example, our diagram above indicates that $ZX = WY$, and that these lengths need not be the same as XM and MY.

Figure 1.4: A Ray

If you're not happy just going from A to B, you can keep going past point B. If you keep going forever, you will make a **ray**. We refer to the ray in Figure 1.4 as $\overrightarrow{AB}$, where the starting point, or **origin**, of the ray comes first. In the diagram, the little arrow indicates that the ray continues forever in that direction.

Figure 1.5: A Line

As you might guess, we could continue forever in both directions. The result is a **line**. Line $\overleftrightarrow{AB}$ is shown in Figure 1.5. We sometimes use a lowercase letter to describe a line, such as line k in the figure. We often leave off the little arrows in the diagrams.

Extra! *[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word.*

–Galileo Galilei

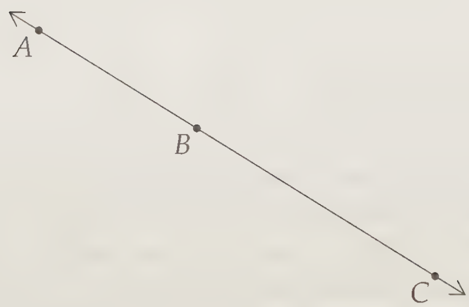


Figure 1.6: Three Collinear Points

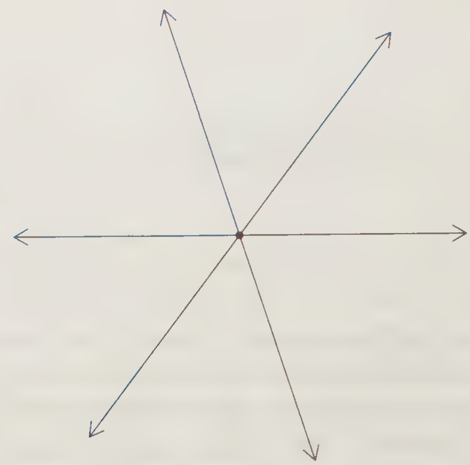


Figure 1.7: Three Concurrent Lines

If three or more points are all on the same line, we say the points are **collinear**, and if three or more lines all pass through the same point, we say the lines are **concurrent**.

Segments, rays, and lines are all one-dimensional figures, since you have only one way you can move along them. Roughly speaking, any path you can draw with a pencil is one-dimensional, meaning you can either move 'forward' on the path or 'backward' on the path.

Once we have the freedom to go off our path and move around on a surface, we're up to two dimensions. On a surface like this page, we might call our dimensions left-right and up-down. If the page extended forever in every direction, we'd call it a **plane**. Most of this book discusses **planar** figures, which are figures that exist in planes.

However, in Chapter 14, we wander off the page and add a third dimension you might think of as 'above-below.' The physical **space** we live in is effectively three-dimensional, and most of what we experience is three-dimensional.

Although it's much harder to think about, there's a great deal of math in higher dimensions. But that's a story for another day.

Exercises

1.2.1 Alice is thinking of a line. How many points on that line does she need to show Bob in order for Bob to know exactly which line she is thinking about?

1.2.2 M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{BM}$. If $BN = 4$, then what is AB?

1.2.3 P, Q, R, S, and T are on line k such that Q is the midpoint of $\overline{PT}$, R is the midpoint of $\overline{QT}$, and S is the midpoint of $\overline{RT}$. If $PS = 9$, then what is PT?

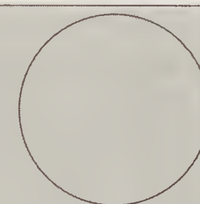
1.2.4★ Points A, B, C, D, and E are five points on a line segment with endpoints A and E. The points are in the order listed above from left to right such that $CD = AB/2$, $BC = CD/2$, $AB = AE/2$, and $AE = 12$. What is the length of $\overline{AD}$? *(Source: MATHCOUNTS)* **Hints:** 203

1.3 Round and Round

Problems

Problem 1.1: Mark a point on a piece of a paper and label it O. Use a ruler to find points on your paper that are 1 inch away from the point O. If you draw all of these points, what figure would you create?

Problem 1.2: The figure shown at right is called a **circle**.



(a) Is it possible to draw a line that does not hit the circle in any points?

(b) Is it possible to draw a line that hits the circle in exactly one point?

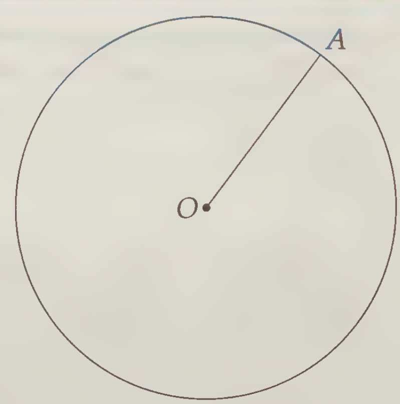
(c) Two points?

(d) Three or more points?

We have many fancy names for things in mathematics. The fancy name we have for a group of points that satisfy certain conditions is a **locus**. While you may never have heard of that word, you've certainly heard of the first locus we'll investigate. (And it's no big deal if you forget the word 'locus' until Intermediate Geometry!)

Problem 1.1: Mark a point on a piece of a paper and label it O. Use a ruler to find points on your paper that are 1 inch away from the point O. If you draw all of these points, what figure would you create?

Solution for Problem 1.1: When we draw all the points that are 1 inch away from O, we form a figure called a **circle**. The point O is called the **center** of the circle. We often refer to a circle by its center, writing 'circle O' or '$\odot O$', where the $\odot$ symbol tells us that we're dealing with a circle. We say that $\overline{OA}$ is a **radius** of the circle because it is a segment connecting the center to a point on the circle. We know that all points on the circle must be 1 inch, from the center, so $OA = 1$ inch. The term 'radius' is also used to mean the length of a radius, so we could write: 'The radius of $\odot O$ is 1 inch.'

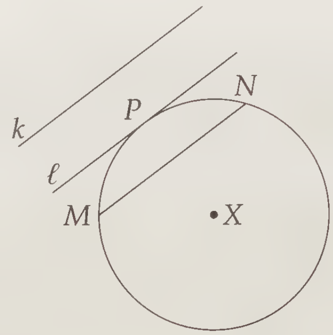


You'll notice that we didn't use a big dot to mark point A. When there's a label near where two figures meet, the label refers to the point where they meet. Therefore, A is the point where our radius hits the circle. □

Much of our work in this book involves both lines and circles.

Problem 1.2: Can a line and a circle intersect in 0 points? 1 point? 2 points? 3 points? More?

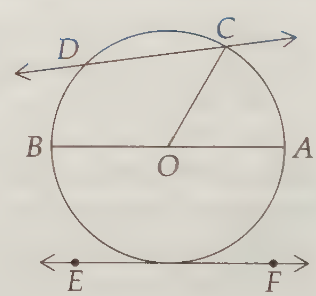
Solution for Problem 1.2: Given $\odot X$, we can clearly find a line that doesn't hit X anywhere. Line k shown below is such a line. Imagine sliding line k closer and closer to $\odot X$ until it touches the circle at exactly one point, such as line ℓ touches $\odot X$ at point P. We say that line ℓ is a **tangent line** to the circle. We can also use 'tangent' as a adjective, and write, 'Line ℓ is tangent to $\odot X$.'



Lines even closer to the center intersect the circle at two points, such as $\overleftrightarrow{MN}$ does. A line that hits a circle at two points is a **secant line**. A segment that connects two points on a circle is a **chord**. $\overline{MN}$ is a chord, while $\overleftrightarrow{MN}$ is a secant line. A chord that passes through the center of a circle is a **diameter**.

Finally, the portion of a circle that connects two points on a circle is called an **arc** of that circle. Of course, we have a symbol for that too: $\overset{\frown}{MN}$ is the shorter of the two arcs that connect M and N. We call the shorter of the two arcs that connect two points on a circle a **minor arc** of the circle. The longer arc that connects the two points is a **major arc** of the circle. We usually use three points to denote a major arc: $\overset{\frown}{PNM}$ is the longer arc connecting P to M, while $\overset{\frown}{PM}$ is the smaller arc connecting them. □

Exercises



1.3.1 In the figure at right, identify whether each of the following is a secant line, a chord, a radius, a diameter, or a tangent line of $\odot O$. (If multiple terms are accurate, list all of the accurate terms.)

(a) $\overline{CO}$

(b) $\overleftrightarrow{EF}$

(c) $\overline{CD}$

(d) $\overline{AB}$

(e) $\overleftrightarrow{CD}$

1.3.2 Suppose point P is outside a given circle. Is it always possible to draw a line through P that is tangent to the circle? (No proof is necessary now; you'll have the tools to prove your answer later in the text.)

1.3.3 What is the maximum number of possible points of intersection of a circle and a triangle? *(Source: AMC 10)* (A triangle is formed by connecting three points with line segments.)

1.3.4★ Two circles and three straight lines lie in the same plane. If neither the circles nor the lines are coincident (meaning the two circles are different and the three lines are all different lines), what is the maximum possible number of points at which at least two of the five figures intersect? *(Source: MATHCOUNTS)* **Hints:** 374

1.4 Construction: Copy a Segment

Classical construction problems are sprinkled throughout the book because a deep understanding of constructions usually leads to a deep understanding of geometry. Construction problems ask us to create precise geometric diagrams with two simple tools. These tools are a **compass**, to make circles, and a **straightedge**, to make straight line segments. Notice that we don't say 'ruler' to make line segments. You don't get to use your straightedge to measure lengths of segments – you can only draw lines. Similarly, you aren't allowed to use your protractor to measure or create angles.

So, what can you do?

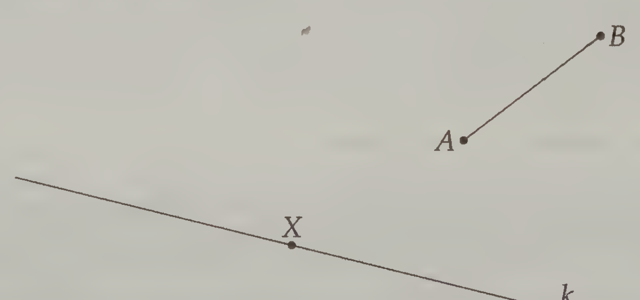
That's the goal of these construction sections: to start learning what you can do with only compass and straightedge. The only operations you can perform with your compass and straightedge are the following:

1. Given a point, you can draw any line through the point.
2. Given two points, you can draw the line that passes through them both.
3. Given a point, you can draw any circle centered at that point.
4. Given a point and a segment, you can draw the circle with its center at that point and with radius equal in length to the length of the segment.
5. Given two points, you can draw the circle through one point such that the other point is the center of the circle.

That's not much, but with these simple operations we can construct an enormous range of diagrams.

Problems

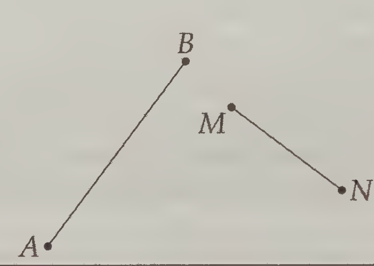
Problem 1.3: Use your compass to find a point Y on k such that $AB = XY$. **You cannot simply use a ruler to measure $\overline{AB}$, then use that measurement to find Y!**



Problem 1.4: Shown below are segments $\overline{AB}$ and $\overline{MN}$.

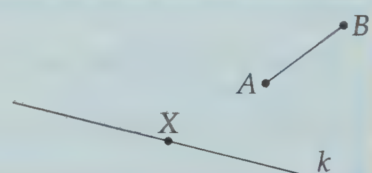
(a) Use straightedge and compass to construct a line segment that has length $AB + MN$. (Reminder: You can't just measure with a ruler!!)

(b) Construct a line segment that has length $AB - MN$. (Even though we don't say 'with a straightedge and compass,' you still can't measure with a ruler! 'Construct' implies 'straightedge and compass' construction.)

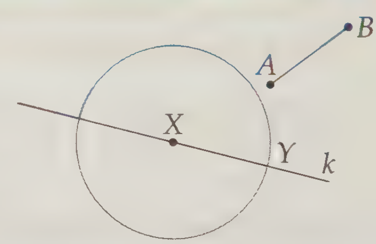


We start our exploration of construction by learning how to copy a segment.

Problem 1.3: Use your compass to find a point Y on k such that $AB = XY$. **You cannot simply use a ruler to measure $\overline{AB}$, then use that measurement to find Y!**



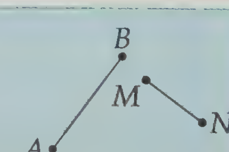
Solution for Problem 1.3: All we can do with a compass is draw circles or parts of circles. To find a point that is AB from X, we first open our compass to a width of AB by putting the point of the compass at A and the compass pencil at B (or vice versa). Then we make a circle with center X and this opening as the radius. Since this $\odot X$ has a radius equal to AB, the two points where it hits k are AB away from X. We can take either one of these as our point Y. □



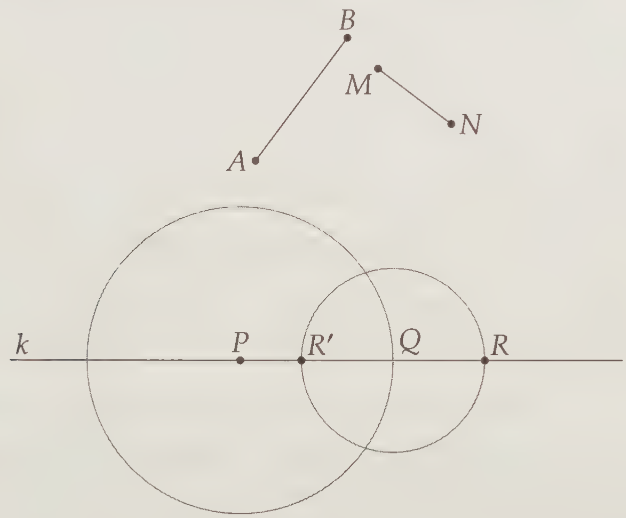
Concept: In nearly all construction problems in which we must make a point, we find that point by constructing two figures that the point must be on. The point we seek is then at the intersection of these two figures. For example, in Problem 1.3, we have line k and construct $\odot X$ that Y must be on. Their intersection gives us the point Y we seek.

Let's try a slightly more challenging construction.

Problem 1.4: Shown are segments $\overline{AB}$ and $\overline{MN}$. Use straightedge and compass to construct a line segment that has length $AB + MN$, and a line segment that has length $AB - MN$.

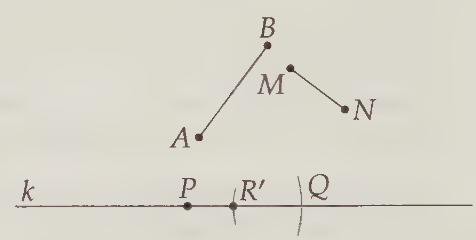


Solution for Problem 1.4: We start by drawing a line k and choosing a point P on line k. We find a point that is $AB + MN$ from P in two steps. First, we find a point a distance of AB from P using our construction technique in Problem 1.3. We do so by opening our compass to width AB and using this radius to draw a circle centered at P. We take one of the points where this circle hits k to be point Q.



Then we find a point that is MN from Q by opening our compass to width MN and using this radius to draw a circle centered at Q. As shown, this circle hits m at two points, R and R'. To get to point R from P, we go a distance of AB to get to Q, then MN more to reach R. Therefore, $PR = AB + MN$. Similarly, to get to R' from P, we first go a distance equal to AB to get to Q, then head *back towards* P a distance of MN to get to R'. So, $PR' = AB - MN$. □

You might have noticed that we didn't need the entire circles we drew in our constructions. We only needed enough of the circle to tell where the circle would hit the line. Typically, these little arcs are all we draw in our constructions. Therefore, our paper when constructing $AB - MN$ in Problem 1.4 might look as shown at right.



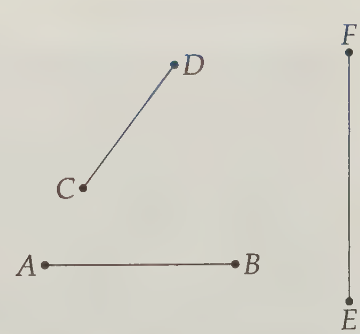
Exercises

1.4.1 Given the segments shown, construct segments with the following lengths:

(a) $AB + CD - EF$.

(b) $2AB$.

(c) $AB - 2EF + 3CD$.



1.5 The Burden of Proof

Earlier we defined a line segment as the direct path that connects two points. It seems obvious that any two points can be connected by a segment. In fact, it seems so obvious that it should be easy to prove. However, it isn't just hard to prove – it's impossible. The statement that any two points can be connected by a straight line segment must be simply accepted as a fact. We call such a statement that must be regarded as fact without proof an **axiom**. Axioms are also sometimes called **postulates**.

When the world's most famous geometer, Euclid, wrote his famous *Elements*, he stated five axioms:

1. Any two points can be connected by a straight line segment.
2. Any line segment can be extended forever in both directions, forming a line.
3. Given any line segment, we can draw a circle with the segment as a radius and one of the segment's endpoints as center.
4. All right angles are congruent. *(We'll talk about right angles and what we mean by 'congruent' shortly!)*
5. Given any straight line and a point not on the line, there is exactly one straight line that passes through the point and never meets the first line.

In Euclid's *Elements*, he combined these axioms to prove ever more complicated mathematical statements. We call such proven mathematical statements **theorems**. A mathematical statement that is not an axiom but hasn't been proved false or true is called a **conjecture**.

In this book, we don't start from Euclid's axioms and prove everything that follows step-by-step. It's a good thing, too! It turns out that even Euclid missed a few axioms. Mathematicians since have shown that Euclid's arguments, in order to be completely valid, would need many more axioms added to these five. In other words, there are some things that even the great Euclid didn't realize are so 'obvious' that they could not be proved. Often when we reach these items in this text, we give a 'common sense' explanation of why we accept these statements as facts. We note when these really are axioms, as opposed to statements that we can prove using previous axioms or theorems.

You can use the proofs we present both as guides for writing your own proofs and as stepping stones to prove interesting theorems of your own.

1.6 Summary

Definitions:

- A **point** is, well, a point. Euclid called a point 'that which has no part.' We can't do much better than that vague description. We typically denote points with capital letters.
- A straight path connecting two points is called a **segment**, and our original two points are the **endpoints** of the segment. We refer to a segment by its endpoints, such as $\overline{AB}$. We remove the bar to denote the length of the segment: AB.

Definitions:

- The point on a segment that is halfway between the endpoints is the **midpoint** of the segment. We also say that this point is **equidistant** from the endpoints.
- If we start at a point, then head in one direction forever, we form a **ray**. Our starting point is the **vertex** of the ray, and we denote a ray as $\overrightarrow{AB}$, where the first point is the vertex of the ray.
- If we continue a line segment past its endpoints forever in both directions, we form a **line**, which we write as $\overleftrightarrow{AB}$.
- If this page were continued forever in every direction, the result would be a **plane**. Since we can move in two general directions, such as right-left and up-down, on a plane, we say the plane has two **dimensions**.
- If we add a third dimension, we are in three-dimensional **space**.

The set of all points that satisfy specific conditions is called a **locus**.

Definitions:

- The set of all points that are the same distance from a given point is a **circle**. The given point is the **center** of the circle, and the fixed distance is the **radius**. We often refer to a circle by its center using the symbol $\odot$, so $\odot O$ refers to a circle centered at O.
- A line that touches a circle at a single point is **tangent** to the circle, while a line that hits a circle at two points is a **secant line**. A segment connecting two points on a circle is a **chord**, and a chord that passes through the center of its circle is a **diameter**. The portion of a circle that connects two points on the circle is an **arc**, which we denote with the endpoints of the arc: $\widehat{MN}$ is the shorter arc that connects M and N.

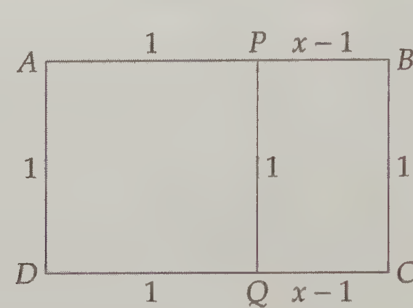
When performing constructions with a straightedge and compass, you can only draw line segments and circular arcs. *You cannot use a ruler to measure segments.* The operations you can perform are:

1. Given a point, you can draw any line through the point.
2. Given two points, you can draw the line that passes through them both.
3. Given a point, you can draw any circle centered at that point.
4. Given a point and a segment, you can draw the circle with its center at that point and with radius equal in length to the length of the segment.
5. Given two points, you can draw the circle through one point such that the other point is the center of the circle.

Extra! *Logic is the art of going wrong with confidence.*

–Morris Kline

Extra! At the top of the first page of each chapter in this book is an image illustrating an interesting geometric fact. The image at the start of this chapter is of the **Golden Ratio Spiral**. A Golden Ratio Spiral is inside a **golden rectangle**, which is a rectangle that can be divided into a square and another rectangle such that the ratio between the dimensions of the new rectangle equals that of the original rectangle.



Shown above is golden rectangle $ABCD$ with dimensions 1 and x. $\overline{PQ}$ divides the rectangle into a square of side 1 and a rectangle with dimensions $x-1$ and 1. Since the ratio of the dimensions of $ABCD$ equals the ratio of the dimensions of $BCQP$, we have

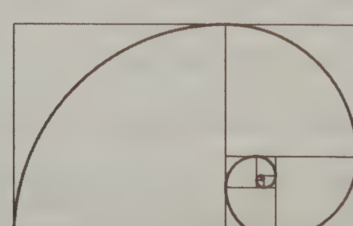
$$\frac{1}{x} = \frac{x-1}{1}.$$

The positive value of x that satisfies this equation is

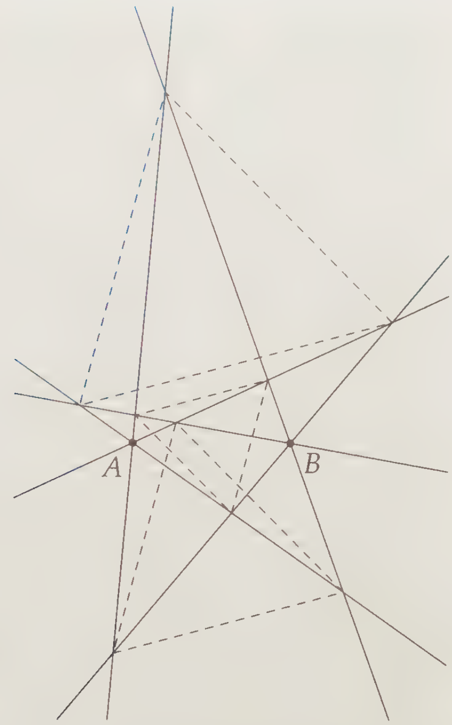
$$x = \frac{1+\sqrt{5}}{2} \approx 1.618034.$$

This number is the **golden ratio** (also sometimes called the **golden mean**), and is often referred to by the Greek letter ϕ ('phi').

When we divide a golden rectangle into a square and a rectangle, the ratio of the dimensions of the smaller rectangle is the same as that of the original rectangle. Therefore, the smaller rectangle is a golden rectangle too, so we can split it into a square and another smaller golden rectangle. We can do this over and over indefinitely, forming the figure shown below.



All of the squares in the diagram together make up our largest golden rectangle. When we omit the largest square, we get our next golden rectangle. Then we omit the next largest square to find the next golden rectangle, and so on. If we then draw a quarter-circle in each of the squares, as shown above, we get the Golden Ratio Spiral.



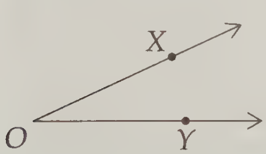
The Lighthouse Theorem

We're going to turn this team around 360 degrees. – Jason Kidd

CHAPTER 2 Angles

2.1 What is an Angle?

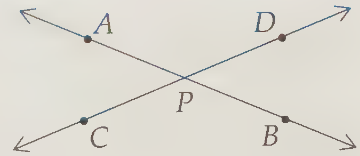
When two rays share an origin, they form an **angle**.



In the diagram at left, rays $\overrightarrow{OX}$ and $\overrightarrow{OY}$ share origin O. We can refer to the angle they form as $\angle XOY$. The $\angle$ symbol tells us we're referring to an angle. The common origin is called the **vertex** of the angle, and the rays $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are called the **sides** of the angle. Notice that when we write the angle as $\angle XOY$, we put the vertex in the middle. We could also refer to the angle as $\angle YOX$, but not as $\angle XYO$. When it's very clear what angle we're talking about, we can just name it with the vertex: $\angle O$.

Of course, two intersecting lines also make angles.

Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ at right intersect at P. Here, we can't just write $\angle P$, since there are many different possible angles this could mean, such as $\angle APC$, $\angle APD$, $\angle DPB$, or $\angle BPC$. We might even be referring to $\angle APB$.



Now that we know what angles are, we need a way to measure them so we can compare one angle to another.

2.2 Measuring Angles

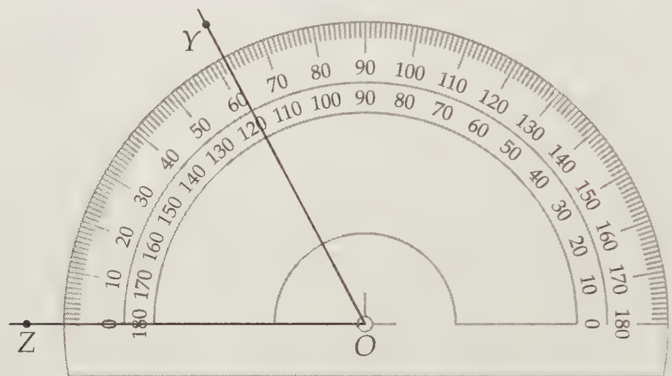


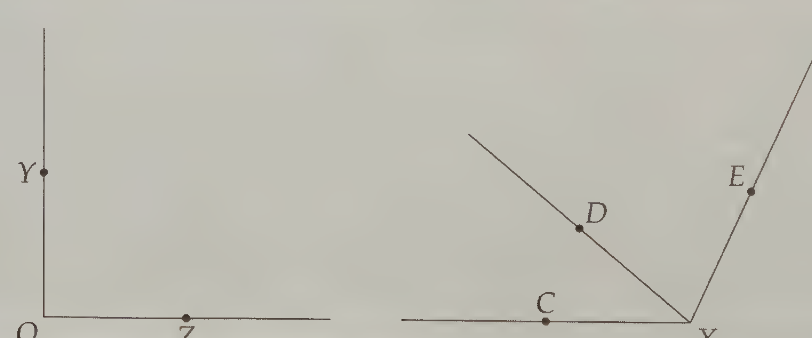
Figure 2.1: A Protractor

Just as we use a ruler to measure the lengths of segments, we can use a **protractor** to measure angles. Roughly speaking, an angle's measure is how 'open' the angle is. Our protractor above shows half a circle (which we call a **semicircle**) divided into 180 equal pieces. Each of these little pieces is considered one **degree** of the semicircle, so that an entire circle is 360 degrees. We use the symbol $^\circ$ to denote degrees, so that a whole circle is 360°.

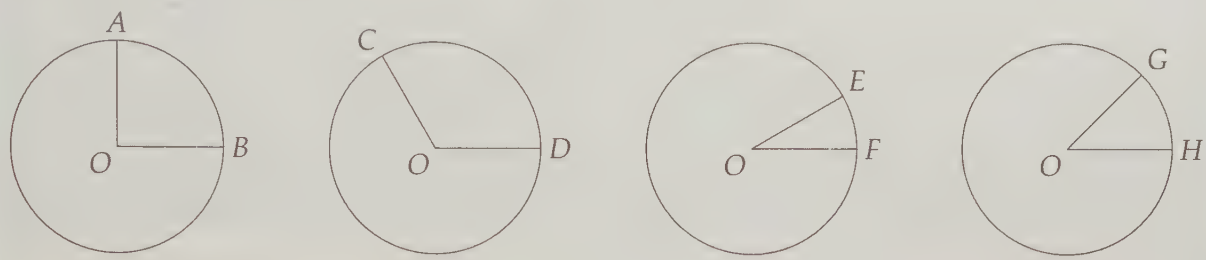
We use a protractor to measure the number of degrees of a circle between the two sides of an angle whose vertex is the center of the circle. For example, in Figure 2.1, the vertex of $\angle YOZ$ is placed at the center of the semicircle. There are 62 degrees between sides $\overrightarrow{OZ}$ and $\overrightarrow{OY}$ of $\angle YOZ$, so we say that $\angle YOZ = 62^\circ$. Sometimes angles are written with an m before $\angle$ to indicate measure: $m\angle YOZ = 62^\circ$.

Problems

Problem 2.1: Use your protractor to find $\angle YOZ$, $\angle CXD$, $\angle DXE$, and $\angle CXE$.



Problem 2.2: The diagram below shows four common angles. In each case, point O is the center of the circle. $\angle AOB$ cuts off 1/4 of a circle, $\angle COD$ cuts off 1/3 of a circle, $\angle EOF$ cuts off 1/12 of a circle, and $\angle GOH$ cuts off 1/8 of a circle.



(a) What is the measure in degrees of $\angle AOB$?

(b) What is the measure in degrees of $\angle COD$?

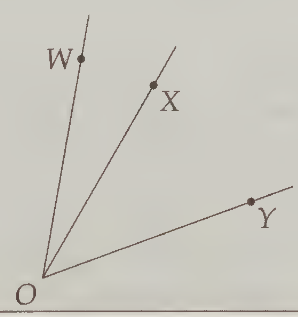
(c) What is the measure in degrees of $\angle EOF$?

(d) What is the measure in degrees of $\angle GOH$?

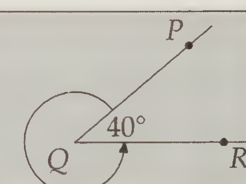
(e) What's so special about 360; why do we use 360 for the number of degrees in a whole circle?

Do not use a protractor; use what you are told about the angles in the text.

Problem 2.3: Given that $\angle WOY = 60°$ and $\angle WOX = 20°$ below, find $\angle XOY$.



Problem 2.4: Suppose instead of measuring an angle the 'regular' way, we go the 'long' way around, as shown in the diagram. The 'regular' angle PQR has measure 40°. What is the measure of the 'long' way around angle?

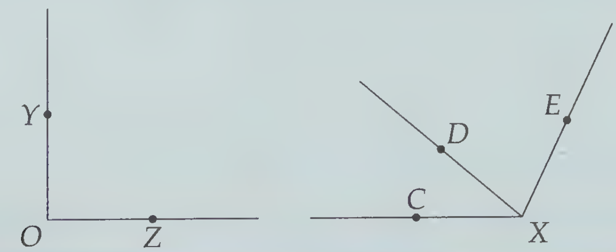


Problem 2.5: Use your protractor to create a 37° angle and a 143° angle.

Extra! *We could use two Eternities in learning all that is to be learned about our own world and the thousands of nations that have arisen and flourished and vanished from it. Mathematics alone would occupy me eight million years.*

–Mark Twain

Problem 2.1: Use your protractor to find $\angle YOZ$, $\angle CXD$, $\angle DXE$, and $\angle CXE$.



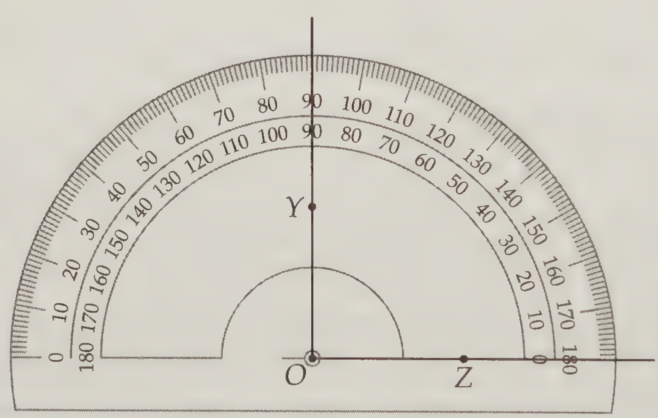
Solution for Problem 2.1: The protractor itself is half a circle (which we call a **semicircle**); we use it to measure the number of degrees of a circle the angle cuts off. Here are the steps we follow to use our protractor to measure angles:

1. Place the protractor on the angle so that the vertex of the angle is exactly where the center of the circle would be if the protractor were a whole circle. Your protractor should clearly show this center point; it's near the middle of the straight side.

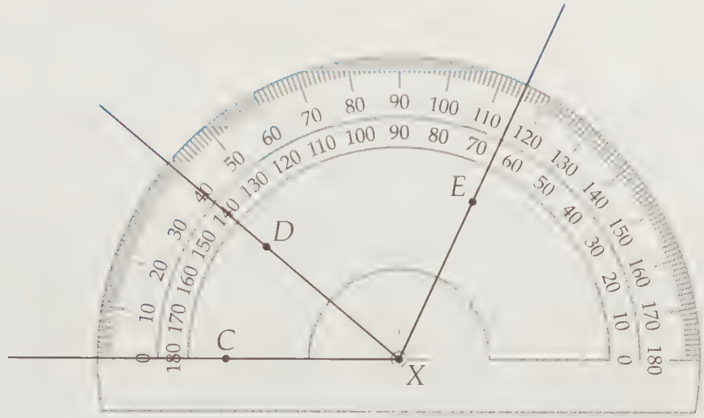
2. Turn the protractor so that one side of the angle is along the 'zero line'; i.e., the line through the center point along the straight edge of the protractor.

3. Find where the other side of the angle meets the curved side of the protractor. The number there tells you the measure of the angle.

For $\angle YOZ$, we put our protractor on the page as shown below. We line up side $\overrightarrow{OZ}$ of the angle with the zero line of the protractor, placing the center point of the protractor over O. We find that side $\overrightarrow{OY}$ hits the curved edge at 90°.

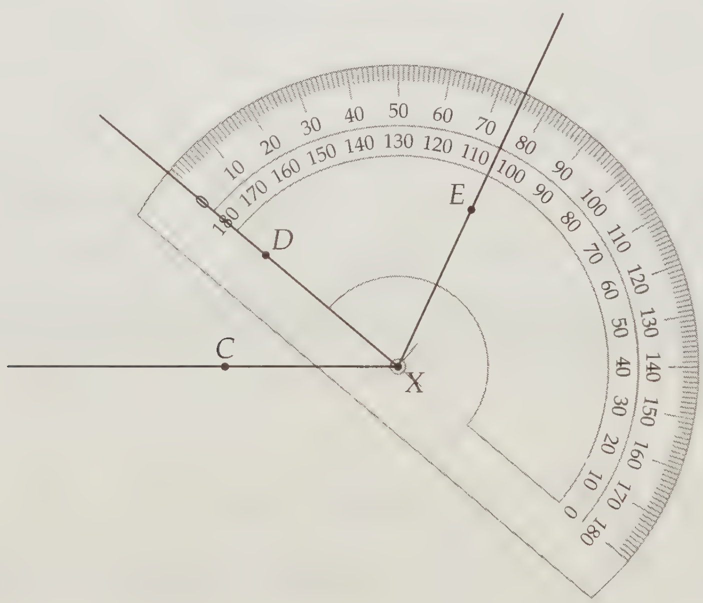


When we follow this procedure with $\angle CXD$, we find that there are two numbers where $\overrightarrow{XD}$ meets the curved edge in the following diagram. We know to take the smaller of these numbers – clearly there are 40 degrees, not 140 degrees, between $\overrightarrow{XC}$ and $\overrightarrow{XD}$. We can also note that $\angle CXD$ is less than half the entire semicircle, so its measure must be the smaller of the two numbers where $\overrightarrow{XD}$ meets the curved edge of the protractor.

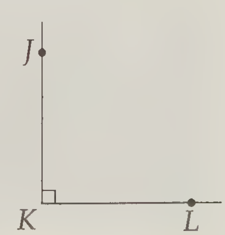


We can also use the above diagram to find the measure of $\angle CXE$. Once again, our angle hits a point on the curved edge with two numbers, but this time we know the angle is greater than 90° (since the angle is more than half the semicircle). Thus, we know that $\angle CXE = 115°$.

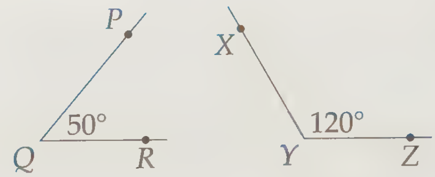
Finally, we can place the protractor as in the diagram below to find that $\angle DXE = 75°$.



Notice that $\angle CXD + \angle DXE = \angle CXE$. This isn't an accident! Since $\angle CXD$ and $\angle DXE$ share a side and a vertex, putting them together gives $\angle CXE$. □

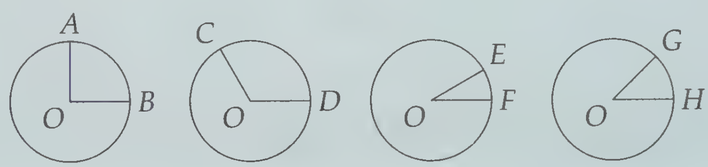


We saw in Problem 2.1 that knowing whether an angle is greater than or less than 90° is necessary for finding its measure using a protractor. This 90° is such an important measure that angles that are 90° have a special name, **right angles**. We usually mark right angles with a little box as shown in $\angle JKL$ at right. Two lines, rays, or line segments that form a right angle are said to be **perpendicular**. $\overline{JK}$ and $\overline{KL}$ are perpendicular; we can use the symbol $\perp$ to write this briefly: $\overline{JK} \perp \overline{KL}$.



Angles that are less than 90° are called **acute**, and those that are greater than 90° but less than 180° are called **obtuse**. Sometimes we write the measure of an angle inside the angle as shown above.

Problem 2.2: The diagram below shows four common angles. In each case, point O is the center of the circle. $\angle AOB$ cuts off 1/4 of a circle, $\angle COD$ cuts off 1/3 of a circle, $\angle EOF$ cuts off 1/12 of a circle, and $\angle GOH$ cuts off 1/8 of a circle.



(a) What is the measure in degrees of $\angle AOB$?

(b) What is the measure in degrees of $\angle COD$?

(c) What is the measure in degrees of $\angle EOF$?

(d) What is the measure in degrees of $\angle GOH$?

(e) Why in the world do we use such a weird number, 360, for the number of degrees in a whole circle?

Do not use a protractor; use what you are told about the angles in the text.

Solution for Problem 2.2: Since a whole circle is 360°, and $\angle AOB$ is 1/4 of a circle, we have

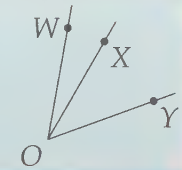
$$\angle AOB = \left(\tfrac{1}{4}\right)(360^\circ) = 90^\circ.$$

We can tackle the other three angles in exactly the same way:

$$\begin{aligned} \angle COD &= \left(\tfrac{1}{3}\right)(360^\circ) &= 120^\circ \\ \angle EOF &= \left(\tfrac{1}{12}\right)(360^\circ) &= 30^\circ \\ \angle GOH &= \left(\tfrac{1}{8}\right)(360^\circ) &= 45^\circ \end{aligned}$$

The number 360 comes from the ancient Babylonians. The Babylonians used a number system with 60 digits, instead of our decimal system, which only has 10 digits. When choosing a number of degrees for a whole circle, they were likely influenced by their number system and possibly by astronomy (a year has around 360 days). However, a look at our answers above points to what might have been the largest factor in choosing 360. The Babylonians, like most people today, probably hated fractions. Since 360 is divisible by lots of different numbers, many common angles in geometry have integer measures. Had we used 100 degrees instead, we'd have to deal with measures such as $12\frac{1}{2}$ degrees, $8\frac{1}{3}$ degrees, and so on. □

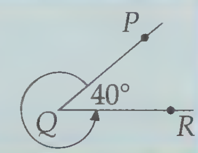
Problem 2.3: Given that $\angle WOY = 60^\circ$ and $\angle WOX = 20^\circ$, find $\angle XOY$.



Solution for Problem 2.3: As we just saw in Problem 2.1, we can combine two angles that share a side and a vertex to make a third whose measure equals the sum of the measures of the first two. Applying this to the diagram in the problem, we see that $\angle WOX + \angle XOY = \angle WOY$. Therefore, $20^\circ + \angle XOY = 60^\circ$, so $\angle XOY = 40^\circ$. □

We call angles that share a side, like $\angle WOX$ and $\angle XOY$ in Problem 2.3, **adjacent angles**.

Problem 2.4: Suppose instead of measuring an angle the 'regular' way, we go the 'long' way around, as shown in the diagram. The 'regular' angle PQR has measure 40°. What is the measure of the 'long' way around angle?



Solution for Problem 2.4: If we imagine our 'regular' angle PQR cutting off a circle, we know it cuts off 40° of the circle. The 'long' way around then must be the rest of the circle. Since a whole circle is 360°, the remainder of our circle is $360^\circ - 40^\circ = 320^\circ$. □

Angles that are greater than 180° are called **reflex angles**. They are rarely important in problems.

We finish our discussion of measuring angles by learning how to draw them given angle measurements.

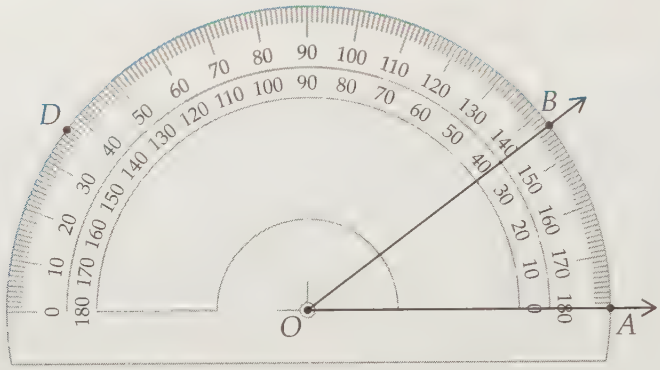
Problem 2.5: Use your protractor to create a 37° angle and a 143° angle.

Solution for Problem 2.5: We start with one side, $\overrightarrow{OA}$, of the 37° angle, which we can draw anywhere. To create the other side, we use our protractor to figure out where the other side would have to go in order to make a 37° angle. We place our protractor over $\overrightarrow{OA}$ as if we are measuring an angle with $\overrightarrow{OA}$ as a side. We then find the 37° point on the curved side, since a 37° angle would have to go through this point. As we've noticed before, our protractor has two 37's, one on each side. Since a 37° angle is clearly acute, we know to choose point B in the figure on the next page, thus making an angle that is less than 90°.

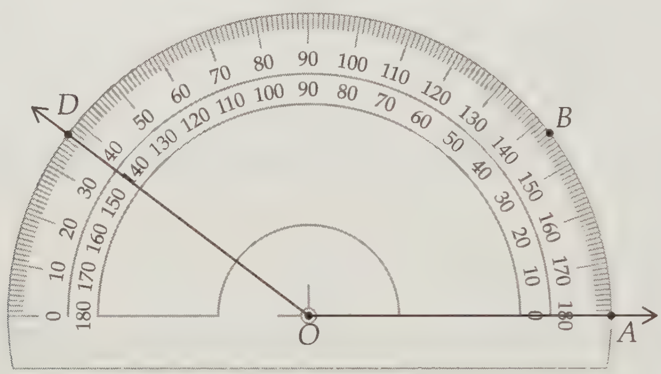
Extra! The golden ratio that we discussed on page 12 does not only appear in geometry! For example, consider the **Fibonacci sequence** shown below.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

The first two terms of the sequence are both 1, and each subsequent term is the sum of the previous two terms. Calculate the ratio between each term and the term before it, such as $34/21 \approx 1.619$. See anything interesting?



Similarly, we have two choices when we build our 143° angle. This time we choose the one that creates the obtuse angle, as shown in the diagram below.



□

Exercises

2.2.1 Use your protractor to make angles with the following measures:

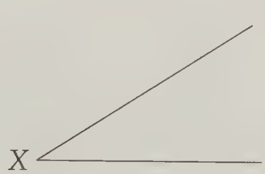
(a) 90° (b) 45° (c) 135° (d) 220°

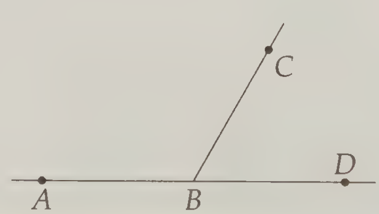
2.2.2 Use your protractor to measure the angles shown. Classify each angle as right, acute, or obtuse.

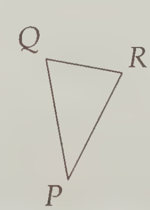
(a) $\angle X$.

(b) $\angle ABC$ and $\angle DBC$.

(c) $\angle PQR$, $\angle PRQ$, and $\angle RPQ$.





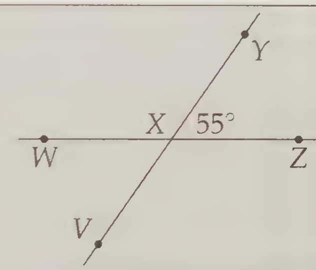


2.3 Straight and Vertical Angles

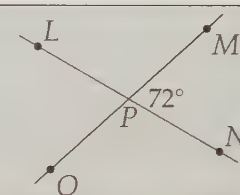
Problems

Problem 2.6: In the figure below, AOB is a straight line. What is the measure of $\angle AOB$?

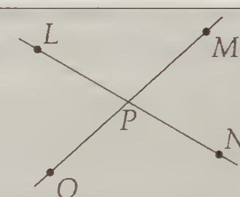
Problem 2.7: In the figure, lines $\overleftrightarrow{VY}$ and $\overleftrightarrow{WZ}$ meet at X, and $\angle YXZ = 55^\circ$. What is the measure of $\angle WXY$?



Problem 2.8: Lines $\overleftrightarrow{LN}$ and $\overleftrightarrow{MO}$ intersect at P such that $\angle MPN = 72^\circ$. Find $\angle LPO$.



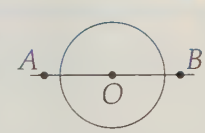
Problem 2.9: Lines $\overleftrightarrow{LN}$ and $\overleftrightarrow{MO}$ intersect at P. Prove that $\angle MPN = \angle LPO$.



As we have noted, two intersecting lines make angles. Can we make an angle with just one line?

Problem 2.6: In the figure, AOB is a straight line. What is the measure of $\angle AOB$?

Solution for Problem 2.6: If we don't see the answer right away, we can try to figure out what portion of a circle the angle cuts off. So, we draw a circle with center O as in the diagram to the right. Now we can see that the angle cuts off half a circle (whichever side of the line we pick). So, $\angle AOB = (1/2)(360^\circ) = 180^\circ$.

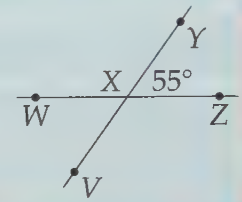


This one's easy to remember: we call an angle that is really a straight line a **straight angle**. □

Extra! *What science can there be more noble, more excellent, more useful for people, more admirably high and demonstrative, than this of mathematics?* –Benjamin Franklin

Straight angles appear too simple to be useful, but often the simplest tools are the best.

Problem 2.7: In the figure, lines $\overleftrightarrow{VY}$ and $\overleftrightarrow{WZ}$ meet at X and $\angle YXZ = 55^\circ$. What is the measure of $\angle WXY$?

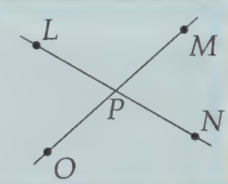


Solution for Problem 2.7: Since $\angle WXY$ and $\angle YXZ$ together make $\angle WXZ$, which is a straight angle, we know that $\angle WXY + \angle YXZ = 180^\circ$. Hence, $\angle WXY = 180^\circ - \angle YXZ = 125^\circ$. □

We call two angles that add to 180° **supplementary angles**. As we have seen, when two lines intersect like $\overleftrightarrow{VY}$ and $\overleftrightarrow{WZ}$ in Problem 2.7, any two adjacent angles thus formed are supplementary because together they make a straight line.

Similarly, we call angles that add to 90° **complementary angles**.

Problem 2.8: Lines $\overleftrightarrow{LN}$ and $\overleftrightarrow{MO}$ intersect at P such that $\angle MPN = 72^\circ$. Find $\angle LPO$.



Solution for Problem 2.8: Angle LPO sure looks equal to $\angle MPN$, and it 'makes sense' that the two are equal, but 'makes sense' isn't good enough in mathematics. We need proof, which we'll tackle in the next problem. For now, we'll try to compute $\angle LPO$.

Since it's not obvious how to compute $\angle LPO$, we start by finding angles we can measure. Since $\angle MPN$ and $\angle NPO$ together make a straight angle, we have $\angle NPO + \angle MPN = 180^\circ$. Thus, $\angle NPO = 180^\circ - 72^\circ = 108^\circ$.

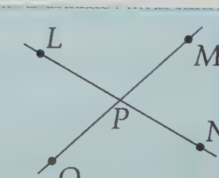
Similarly, since $\angle LPO$ and $\angle NPO$ are supplementary, we have

$$\angle LPO = 180^\circ - \angle NPO = 180^\circ - 108^\circ = 72^\circ.$$

□

Concept: When you can't find the answer right away, try finding whatever you can – you might discover something that leads to the answer! Better yet, you might learn something even more interesting than the answer. The best problem solvers are explorers.

Problem 2.9: Lines $\overleftrightarrow{LN}$ and $\overleftrightarrow{MO}$ intersect at P. Prove that $\angle MPN = \angle LPO$.



Solution for Problem 2.9: What's wrong with this proof:

Bogus Solution: Suppose $\angle MPN = 72^\circ$. Since $\angle MPO$ is a straight angle, we know that $\angle NPO = 180^\circ - 72^\circ = 108^\circ$. Similarly, we have $\angle LPO = 180^\circ - \angle NPO = 72^\circ$. Therefore, $\angle MPN = \angle LPO$.

Every statement in that 'proof' is true. However, it is not a complete proof because it only addresses one case; it only shows that $\angle MPN = \angle LPO$ when $\angle MPN = 72^\circ$.

WARNING!! An example is not a proof!

While examples aren't proofs, they can be useful as guides. Looking at our example, we can quickly construct our proof.

Since $\overleftrightarrow{MPO}$ is a line, we have

$$\angle MPN = 180^\circ - \angle NPO.$$

Since $\overleftrightarrow{LPN}$ is a line, we have

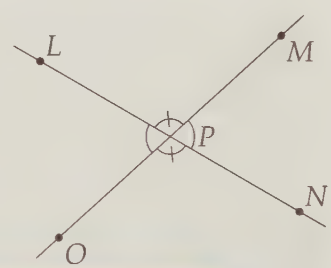
$$\angle LPO = 180^\circ - \angle NPO.$$

Combining these two equations gives $\angle MPN = 180^\circ - \angle NPO = \angle LPO$. □

Notice that our proof does not depend at all on the measure of $\angle MPN$. The proof works no matter how the lines intersect.

Pairs of angles such as $\angle MPN$ and $\angle LPO$ in the diagram below are called **vertical angles**. As we proved in Problem 2.9, vertical angles are always equal to each other.

We often use little arcs to mark equal angles. In the diagram to the right, $\angle MPN$ and $\angle LPO$ each have a single little arc in them to show that they are equal. Angles $\angle LPM$ and $\angle NPO$ also are vertical angles, so they are equal. We put a little hash-mark on the arcs at these angles to show that these two angles are equal to each other, but not necessarily equal to our first pair of equal angles (which have arcs without hash-marks).



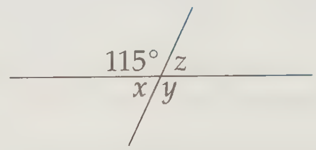
Important: Supplementary, right, obtuse, vertical, acute... by now the number of new names must be driving you nuts. Don't memorize what all these names mean now! The names are not that important. Besides, as you work your way through this book, you'll eventually see them so much you'll just know them anyway.

The concepts are more important than the words for solving problems. 'Angles like $\angle MPN$ and $\angle LPO$ in Problem 2.9 are equal' means something without any more information. 'Vertical angles are equal' doesn't tell you anything until you reach for your math dictionary to look up vertical angles.

The words, however, are important for communicating the concepts. For now, though, focus on the ideas. The words will come naturally.

Exercises

2.3.1 Find x, y, and z in the diagram below.



2.3.2 Find the measure of an angle that is supplementary to each of the following angles:

(a) $\angle AOB = 120^\circ$

(b) $\angle COD = 45^\circ$

(c) $\angle EOF = 90^\circ$

2.3.3 Find the measure of an angle that is complementary to each of the following angles:

(a) $\angle GOM = 30^\circ$

(b) $\angle IOJ = 45^\circ$

(c) $\angle KOL = 75^\circ$

2.4 Parallel Lines

Having dissected what happens when two lines meet, we should wonder about what happens if they don't. If two lines do not meet, we say that they are **parallel**. If lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel, we write $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

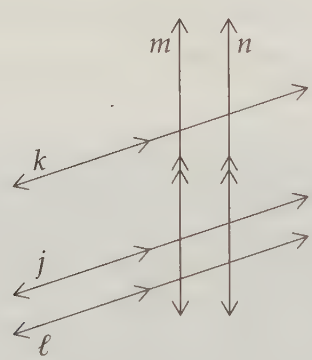


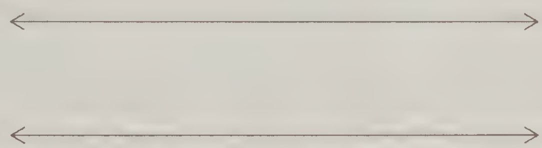
Figure 2.2: Two Sets of Parallel Lines

Just as we use little arcs to mark angles that are equal, we can use little arrows to mark lines that are parallel. In the diagram above, lines j, k, and ℓ are marked parallel, as are lines m and n. You won't see us use this notation all the time, though. Those little arrows can really clutter up a diagram.

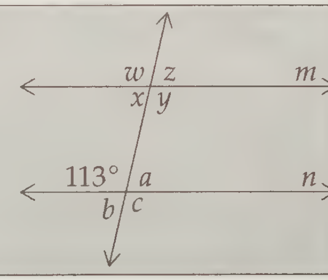
Problems

Problem 2.10: Draw a pair of parallel lines like those shown below. Then draw a line that crosses both of the parallel lines. Measure all the angles formed between your line and both of the parallel lines. Write the angle measures in the angles you form. Try it again with a different crossing line.

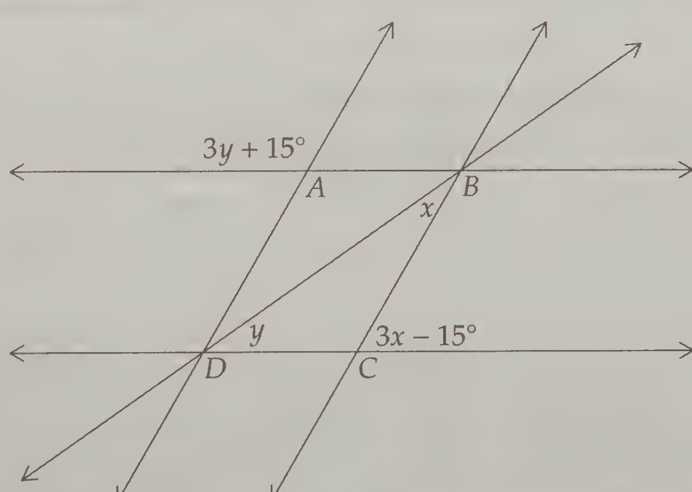
Do you notice anything interesting?



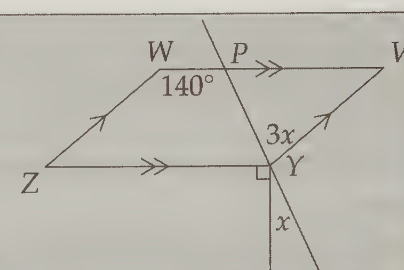
Problem 2.11: Lines m and n are parallel, and we are given the measure of one angle in the diagram as shown. Find the values of a, b, c, w, x, y, and z.



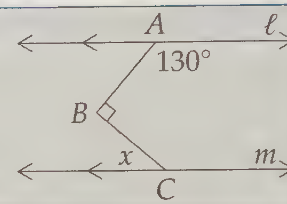
Problem 2.12: In the figure, we have $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. We are also given the measures of four angles as shown in terms of x and y. Find x and y.



Problem 2.13: Given that $\overline{WV} \parallel \overline{YZ}$ and $\overline{WZ} \parallel \overline{VY}$ in the diagram, find x.



Problem 2.14: In the diagram, $\ell \parallel m$ and the angles are as marked. Find x.

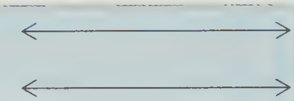


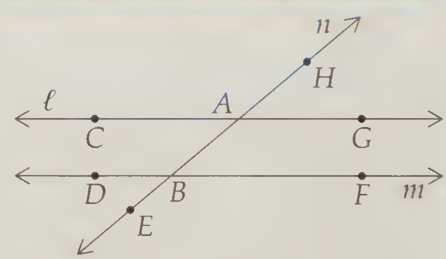
Back on page 10, we noted that one of our axioms (statements we must accept without proof) is

> Given any straight line and a point not on the line, there is exactly one straight line that passes through the point and never meets the first line.

This common-sense statement is sometimes called the **Parallel Postulate**. We'll start our exploration of parallel lines by taking a look at the angles formed when a line intersects a pair of parallel lines.

Problem 2.10: Draw a pair of parallel lines, then draw a line that crosses both of the parallel lines. Measure all the angles formed between your line and both of the parallel lines. Write the angle measures in the angles you form. Try it for a second line (don't worry about the angles where your two lines meet – just focus on the angles between the lines you draw and the parallel lines). Do you notice anything interesting?





Solution for Problem 2.10: In the diagram to the right, we have parallel lines ℓ and m, and we have added line n, which meets ℓ and m at A and B, respectively. We call a line that cuts across parallel lines a **transversal**. We measure $\angle HAG$ and find that it equals 40°. Since $\angle HAG$ and $\angle CAB$ are vertical angles, we don't even have to measure $\angle CAB$. We know that $\angle CAB = \angle HAG = 40^\circ$.

Since $\angle HAG$ and $\angle BAG$ together make up a straight angle, we don't have to measure $\angle BAG$. We know that $\angle BAG = 180^\circ - \angle HAG = 180^\circ - 40^\circ = 140^\circ$. Similarly, $\angle HAC = 140^\circ$.

We might wonder if we need our protractor at all, but then we think about those angles around B. They sure look equal to those around A, and common sense tells us that they are, but we measure to make sure. We find that indeed $\angle ABF = 40^\circ$, from which we deduce that $\angle DBE = 40^\circ$ as well. We can also quickly determine that $\angle ABD = 180^\circ - \angle ABF = 140^\circ$ and $\angle EBF = 140^\circ$.

Seeing that $\angle HAG = \angle ABF$, we wonder if it's always true that a transversal will cut parallel lines at equal angles like $\angle HAG$ and $\angle ABF$. Like the Parallel Postulate, this turns out to be one of those 'obvious' facts that cannot be proved. It must be assumed. As we have seen while finding the angles above, once we know that these two are equal, we can quickly use lines and vertical angles to find the rest of the angles.

Extra! *There is far more imagination in the head of Archimedes than in that of Homer.*

–Voltaire

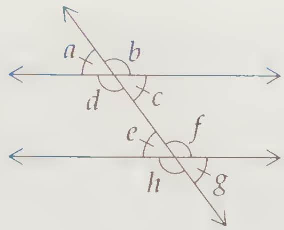


Figure 2.3: Angles Between a Transversal and Two Parallel Lines

Thus, we see that when a parallel line is cut by a transversal, we have two groups of four equal angles. Specifically, in Figure 2.3, we have

$$\begin{aligned} a &= c = e = g \\ b &= d = f = h \end{aligned}$$

Furthermore, the angles in the first group are supplementary to those in the second.

Pairs of these angles have special names to describe their relationships. These names are not terribly important, but you'll see them elsewhere.

a and e are **corresponding angles.**

d and f are **alternate interior angles.**

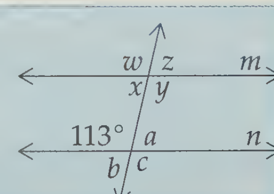
a and g are **alternate exterior angles.**

c and f are **same-side interior angles.**

b and g are **same-side exterior angles.**

Again, the names are not such a big deal. After doing enough geometry, you'll probably know them anyway. Don't bother memorizing them now. Just understand which angles are equal and which are supplementary. □

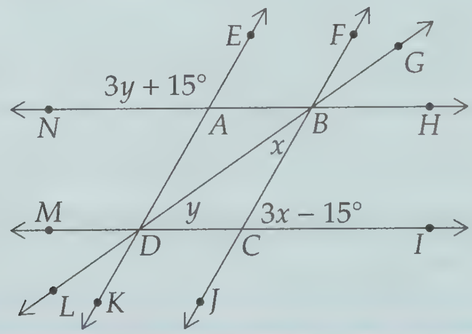
Problem 2.11: Lines m and n are parallel, and we are given the measure of one angle in the diagram as shown. Find the values of a, b, c, w, x, y, and z.



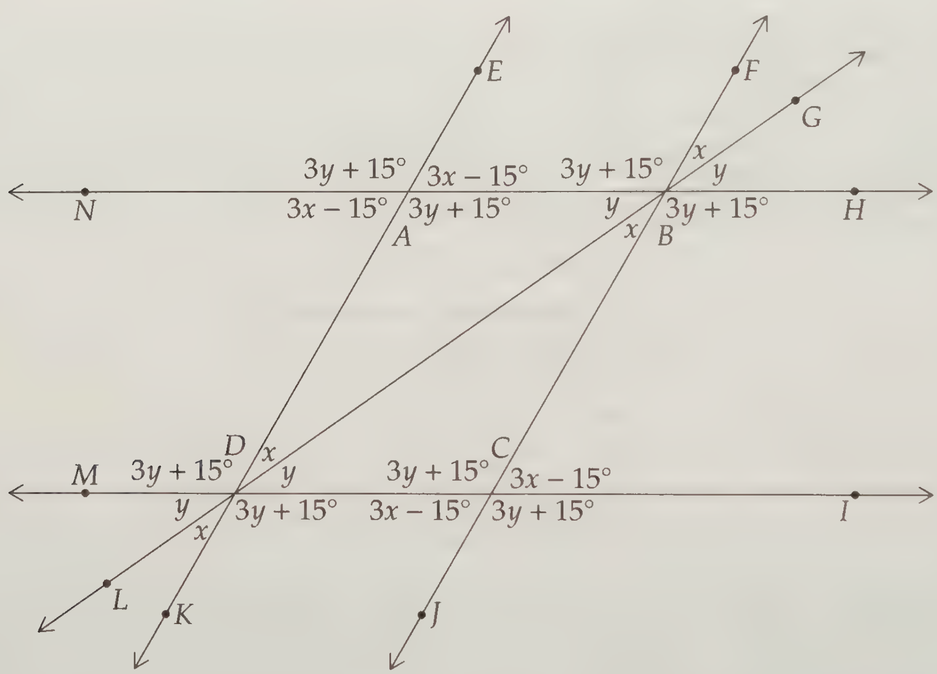
Solution for Problem 2.11: We know that when a transversal cuts parallel lines, equal angles come in groups of four as we saw in Problem 2.10. Therefore, we know that $w = y = c = 113°$. We also know that each angle in the other 'group of four' has a measure that is supplementary to 113°: $x = z = a = b = 180° - 113° = 67°$. □

Now that we understand the relationships between angles when a parallel line is cut by a transversal, let's try a more challenging problem.

Problem 2.12: In the figure, we have $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. We are also given the measures of four angles as shown in terms of x and y. Find x and y.



Solution for Problem 2.12: There's no obvious way to make an equation for x or y, so we start off by using our parallel lines and vertical angles to write the measures of all the angles we know in terms of x and y.



After labeling the angles we know in terms of x and y, we look for ways to build equations. We can use angles that together form straight angles at A and B:

$$\angle EAN + \angle EAB = (3y + 15°) + (3x - 15°) = 180°$$

$$\angle FBA + \angle FBG + \angle GBH = (3y + 15°) + x + y = 180°.$$

Rearranging these gives

$$\begin{aligned} x + y &= 60° \\ x + 4y &= 165° \end{aligned}$$

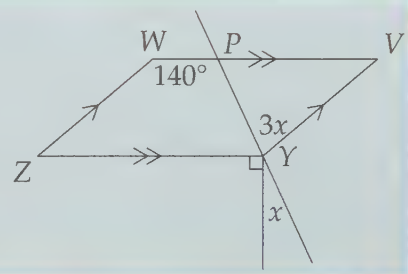
Subtracting the first from the second gives us $3y = 105°$, so $y = 35°$. We can then use substitution to find $x = 25°$. Of course, we didn't have to label every angle above – we could have stopped when we had enough information to set up a pair of equations to solve for x and y.

Note that we could have used parallel line relationships to set up the equations, too. Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, we have $\angle DAN = \angle KDM$, so $3x - 15° = x + y$. Also, $\angle HBC$ and $\angle BCI$ are supplementary, so $(3y + 15°) + (3x - 15°) = 180°$. Solving these two equations gives us the same answer as before. (It better!) □

> **Concept:** Solving a problem with two different methods is an excellent way to check your answer.

We'll finish with two more challenging problems that illustrate how useful parallel lines can be when seeking angle measures.

Problem 2.13: Given that $\overline{WV} \parallel \overline{YZ}$ and $\overline{WZ} \parallel \overline{VY}$ in the diagram, find x.



Solution for Problem 2.13: We start by using what we know about parallel lines to find as many angles as we can. We find $\angle V = 180° - 140° = 40°$ since $\overline{WZ} \parallel \overline{VY}$. Similarly, $\angle Z = 40°$, and $\angle ZYV = 180° - \angle Z = 140°$.

Since $\angle PYV = 3x$, we have $\angle PYZ = \angle VYZ - \angle PYV = 140° - 3x$. Since this angle, the $90°$ angle, and the angle with measure x together give us straight line $\overleftrightarrow{PY}$, we have

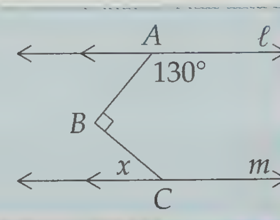
$$\angle PYZ + 90° + x = 180°.$$

We then substitute $\angle PYZ = 140° - 3x$ into this equation, and we have $140° - 3x + 90° + x = 180°$. We solve this equation for x to find that $x = 25°$. □

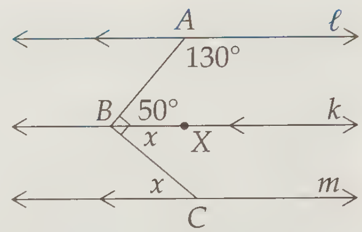
Using information about angles to find information about other angles is often called **angle-chasing**. We've already learned three important tools in angle-chasing: straight angles, vertical angles, and parallel lines. Stay tuned. We'll see plenty more!

> **Concept:** Often when we're angle-chasing, our goal is to build an equation to solve for one of the variables in our problem.

Problem 2.14: In the diagram, $\ell \parallel m$ and the angles are as marked. Find x.



Solution for Problem 2.14: We need to relate our desired angle to angles we know, but neither $\overline{AB}$ nor $\overline{BC}$ cuts both parallel lines. However, if we add a line k through B parallel to ℓ and m, we can do some angle-chasing. Since $k \parallel \ell$, we have $\angle ABX = 180^\circ - 130^\circ = 50^\circ$. Since $k \parallel m$, we have $\angle XBC = x$. Since $\angle XBC + \angle ABX = \angle ABC$, we can now write an equation for x:



$$x + 50^\circ = 90^\circ.$$

Therefore, $x = 40^\circ$. □

Concept: Parallel lines are so useful in problems involving angles that sometimes we'll add new ones to a diagram to help us.

Exercises

2.4.1 Find x and y in the diagram at left below.

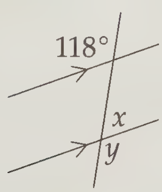


Figure 2.4: Diagram for Problem 2.4.1

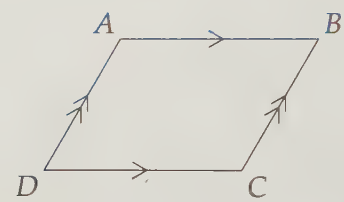


Figure 2.5: Diagram for Problem 2.4.2

2.4.2 Show that $\angle A = \angle C$ and $\angle B = \angle D$ in the diagram at right above.

2.4.3 Find x in the diagram at left below.

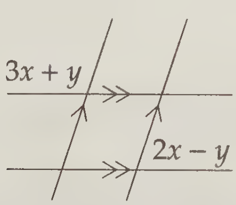


Figure 2.6: Diagram for Problem 2.4.3

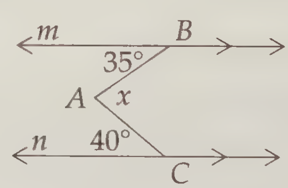
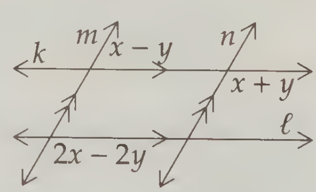


Figure 2.7: Diagram for Problem 2.4.4

2.4.4 In the diagram at right above, $m \parallel n$, and the angles are as marked. Find x. **Hints:** 224

2.4.5 In the diagram at right, $k \parallel \ell$ and $m \parallel n$. If the angles are as marked, find x and y. **Hints:** 481



Extra! *Everything should be made as simple as possible, but not simpler.*

–Albert Einstein

2.5 Angles in a Triangle

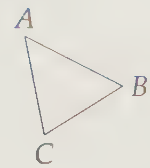


Figure 2.8: A Triangle

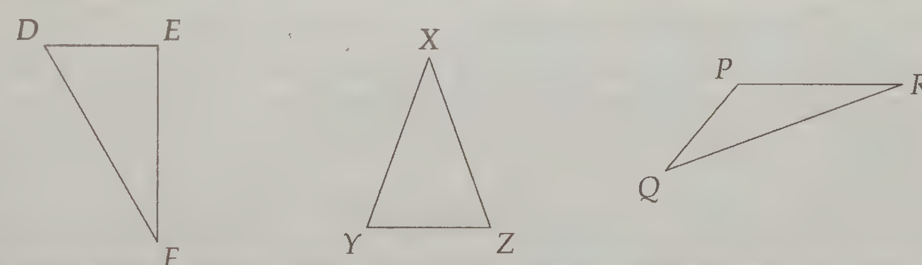
When we connect three points with line segments, we form a **triangle**, as shown in Figure 2.8 above. We will often refer to the triangle as $\triangle ABC$, or sometimes just ABC. The points A, B, and C are called **vertices** of the triangle, and the segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are called **sides**. In this section, however, we will investigate the angles of a triangle.

Problems

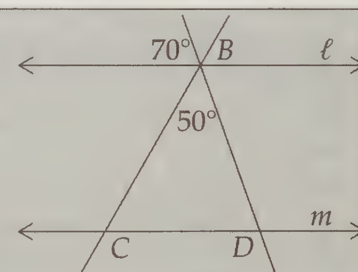
Problem 2.15:

(a) Find the measure of the three angles in each of the triangles below.

(b) Can you guess a statement that is always true about the sum of the angles in a triangle?



Problem 2.16: Given that $\ell \parallel m$, find $\angle BCD$ and $\angle BDC$.



Problem 2.17: Prove that the sum of the measures of the angles in a triangle is always 180°.

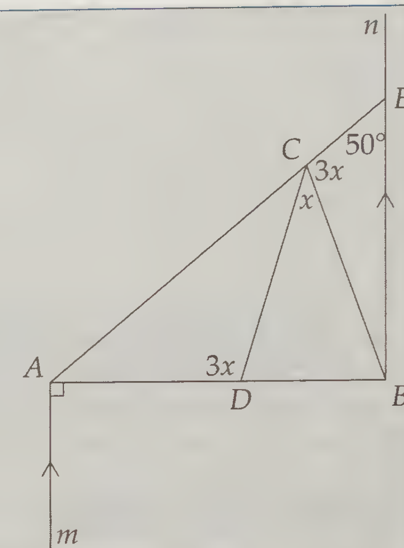
(a) Draw a triangle $\triangle ABC$, and a line k through point A such that $k \parallel \overline{BC}$.

(b) Find angles in your diagram that are equal to $\angle B$ and $\angle C$. Use little arcs as described on page 23 to mark the angles equal.

(c) Prove that $\angle CAB + \angle ABC + \angle BCA = 180°$.

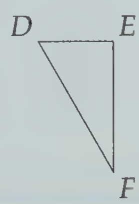
Problem 2.18: One angle in a triangle is twice another angle, and the third angle is 54°. What is the measure of the smallest angle in the triangle?

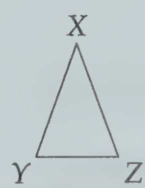
Problem 2.19: In the diagram, $m \parallel n$, $\overline{AB} \perp m$, $\angle ADC = \angle BCE = 3x$, $\angle CEB = 50^\circ$, and $\angle BCD = x$. Find x.

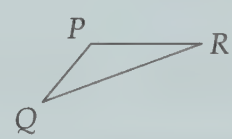


We start our investigation of angles in a triangle by measuring angles in a few triangles.

Problem 2.15:
(a) Find the measure of the three angles in each of the triangles below.
(b) Can you guess a statement that is always true about the sum of the angles in a triangle?







Solution for Problem 2.15: Using our protractor, we can find the measures of the angles in each triangle as below:

$$\begin{array}{llll} \triangle DEF: & \angle D = 60^\circ & \angle E = 90^\circ & \angle F = 30^\circ \\ \triangle XYZ: & \angle X = 40^\circ & \angle Y = 70^\circ & \angle Z = 70^\circ \\ \triangle PQR: & \angle P = 130^\circ & \angle Q = 30^\circ & \angle R = 20^\circ \end{array}$$

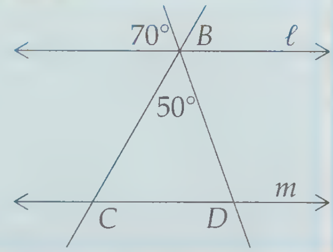
In each case, we see that the sum of the angles is 180°. Hmmm... Is that always true? □

Before tackling the question of whether or not the angles of a triangle always add to 180°, let's try a parallel line problem that includes a triangle.

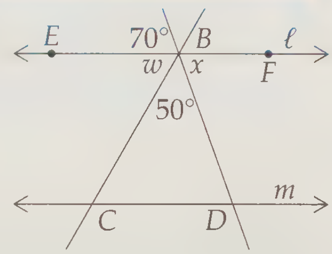
Extra! *The cowboys have a way of trussing up a steer or a pugnacious bronco which fixes the brute so that it can neither move nor think. This is the hog-tie, and it is what Euclid did to geometry.*

–Eric Temple Bell

Problem 2.16: Given that $\ell \parallel m$, find $\angle BCD$ and $\angle BDC$.



Solution for Problem 2.16: We start as we usually do when the answer isn't immediately obvious – we find what we can. We can use vertical angles to find $x = 70^\circ$ in the figure to the right. We can also use either line ℓ or $\overleftrightarrow{BD}$ to find w. $\overleftrightarrow{BD}$ gives $70^\circ + w + 50^\circ = 180^\circ$, so $w = 180^\circ - 70^\circ - 50^\circ = 60^\circ$.



Now we can use our parallel lines to find the angles we want. Since $\ell \parallel m$, we have $\angle BCD = w = 60^\circ$ and $\angle BDC = x = 70^\circ$. □

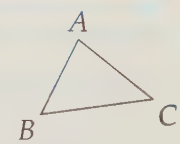
Notice that we don't have to call $\angle BCD$ and $\angle CBE$ 'alternate interior angles' in order to say they are equal. Once we state that $\ell \parallel m$, we can note that $\angle BCD = w$ without writing 'alternate interior angles.' However, if your teacher tells you that you have to include 'alternate interior angles,' you better do it!

Inspired by Problem 2.16, we can now prove that the sum of the angles in a triangle is always 180°.

Problem 2.17: Prove that the sum of the measures of the angles in a triangle is always 180°.

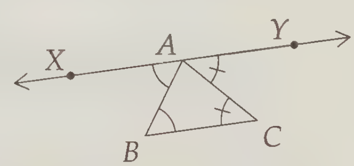
Solution for Problem 2.17: We start by drawing a triangle and by writing what we want to prove:

$$\angle ABC + \angle CAB + \angle BCA = 180^\circ.$$



We don't know a whole lot about angles yet, but we can start by wondering '*Where have we seen* 180° *before?*' Answer: A straight angle. So, we'd like to find a straight line that has all three of our angles, just like $\angle EBC$, $\angle CBD$, and $\angle DBF$ make up line ℓ in our solution to Problem 2.16. However, we don't have any such line yet, so we'll have to add a line somewhere.

We need a line that allows us to use what we know about angles. We don't know much about angles, so we don't have too many options to investigate. So far, probably the most useful angle information we have learned comes from parallel lines. So, we add a line through A parallel to $\overline{BC}$ to create the diagram to the right, which looks curiously like the figure from Problem 2.16.



Since $\overleftrightarrow{XY} \parallel \overline{BC}$, we have $\angle XAB = \angle ABC$ and $\angle BCA = \angle CAY$. Since $\angle XAY$ is a straight angle, we have

$$\angle XAB + \angle CAB + \angle CAY = 180^\circ.$$

Now we're home! We can now use the equalities we found with the parallel lines to get:

$$\angle ABC + \angle CAB + \angle BCA = 180^\circ.$$

□

Important: The sum of the angles in a triangle is always 180°.

There are few geometric relationships you will use more than this one!

Important: Don't view the proof for Problem 2.17 as magic. We see 180°, and that makes us think of finding useful lines. We need a line that has angles equal to the three in $\triangle ABC$. Equal angles make us think of parallel lines, which always give lots of equal angles when cut by a transversal line.
Close the book and try to re-create this proof on your own!

Now let's try solving a problem using the fact that the sum of the angles of a triangle is 180°.

Problem 2.18: One angle in a triangle is twice another angle, and the third angle is 54°. What is the measure of the smallest angle in the triangle?

Solution for Problem 2.18: We know one angle is 54°, but all we know about the other two is that one is twice the other. Therefore, we let the smaller angle be x, so the larger is $2x$. Since the sum of the angles of a triangle is 180°, we have

$$54^\circ + x + 2x = 180^\circ.$$

Solving this, we find $x = 42^\circ$. The angles of our triangle are 42°, 54°, and 84°. (Note we can make a quick check by adding the three and making sure we get 180°.)

Our smallest angle is 42°. □

Concept: The key to tackling word problems in geometry is the same as any other kind of word problem – turn the words into math. Usually this means defining variables and using the words to write equations to solve for the variables.

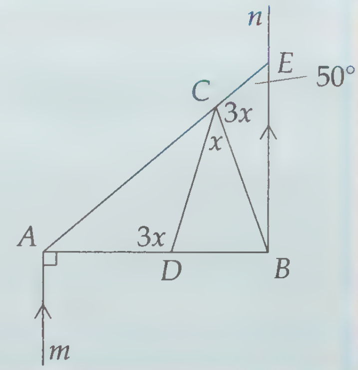
WARNING!! Your last step should be to make sure you've answered the question that is asked.

One of the most common uses of the fact that the angles in a triangle add up to 180° is as an angle-chasing tool. Let's give it a try.

Extra! *In the space of one hundred and seventy-six years the Mississippi has shortened itself two hundred and forty-two miles. Therefore, in the Old Silurian Period the Mississippi River was upward of one million three hundred thousand miles long. Seven hundred and forty-two years from now the Mississippi will be only a mile and three-quarters long. There is something fascinating about science. One gets such wholesome returns of conjecture out of such a trifling investment of fact.*

–Mark Twain

Problem 2.19: In the diagram, $m \parallel n$, $\overline{AB} \perp m$, $\angle ADC = \angle BCE = 3x$, $\angle CEB = 50^\circ$, and $\angle BCD = x$. Find x.



Solution for Problem 2.19: We start with our parallel lines, and see that $\angle ABE = 90^\circ$ because $m \parallel n$ and $m \perp \overline{AB}$. Then, from $\triangle ABE$ we have $\angle EAB + \angle ABE + \angle AEB = 180^\circ$, so

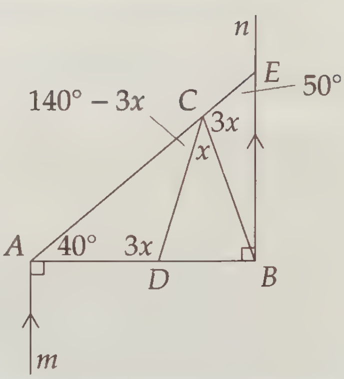
$$\angle EAB = 180^\circ - \angle ABE - \angle AEB = 40^\circ.$$

We can then use $\triangle ACD$ to find

$$\angle ACD = 180^\circ - \angle DAC - \angle ADC = 180^\circ - 40^\circ - 3x = 140^\circ - 3x.$$

Concept: When angle-chasing, it's best to write the values you find for angles on your diagram as you find them, even when these values include variables.

Our diagram now looks like the figure to the right. This picture suggests a way to finish the problem. We have three angles with vertices at C that together make a straight line, so we have



$$\angle ACD + \angle DCB + \angle BCE = 180^\circ.$$

Substitution gives

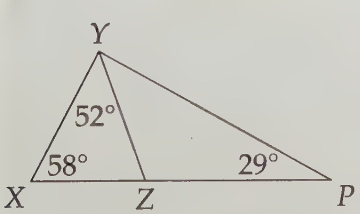
$$140^\circ - 3x + x + 3x = 180^\circ,$$

so $x = 40^\circ$. □

Exercises

2.5.1 Two angles in a triangle have measures 30° and 57°. What is the measure of the third angle?

2.5.2 The angles in a triangle are in the ratio $1 : 2 : 3$. What are the measures of the angles?



2.5.3 Find $\angle ZYP$ in the diagram at the left.

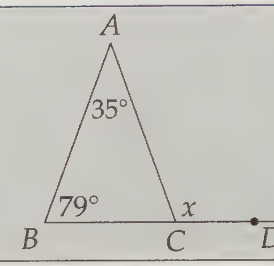
2.5.4 One of the angles in a triangle is a right angle. Show that the other two angles are complementary.

2.5.5★ Using what you know about triangles, find a different solution to Problem 2.14. **Hints:** 109

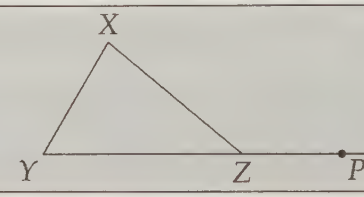
2.6 Exterior Angles

Problems

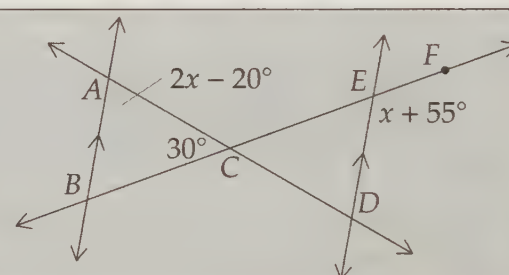
Problem 2.20: Find x given the angle measures shown.



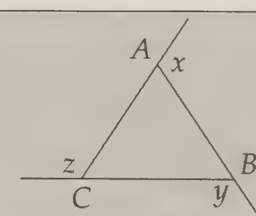
Problem 2.21: Prove that $\angle X + \angle Y = \angle XZP$ in the diagram shown.



Problem 2.22: In the diagram, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$, $\angle BAC = 2x - 20°$, $\angle ACB = 30°$, and $\angle DEF = x + 55°$ as shown. Find $\angle CED$.

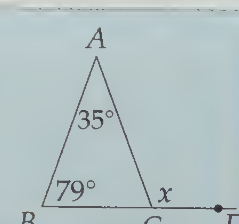


Problem 2.23: Find $x + y + z$.



In the previous section, we examined the **interior angles** of a triangle. In this section, we will take a look at the angles formed when we extend a side of a triangle past a vertex. We cleverly call these the **exterior angles** of the triangle. For example, in the problem below, $\angle ACD$ is an exterior angle of $\triangle ABC$.

Problem 2.20: Find x given the angle measures shown.



Solution for Problem 2.20: From $\triangle ABC$, we have

$$35^\circ + 79^\circ + \angle ACB = 180^\circ,$$

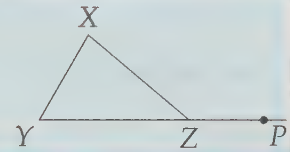
so $\angle ACB = 66^\circ$. From line $\overleftrightarrow{BD}$ we have

$$\angle ACB + x = 180^\circ,$$

so $x = 180^\circ - 66^\circ = 114^\circ$. □

That straightforward solution suggests we can prove something general about an exterior angle of a triangle.

Problem 2.21: Prove that $\angle X + \angle Y = \angle XZP$ in the diagram shown.



Solution for Problem 2.21: Our solution to Problem 2.20 guides the way. From $\triangle XYZ$, we have

$$\angle X + \angle Y + \angle XZY = 180^\circ.$$

From line $\overleftrightarrow{YZP}$, we have

$$\angle XZY + \angle XZP = 180^\circ.$$

Subtracting the second equation from the first gives us

$$\angle X + \angle Y - \angle XZP = 0,$$

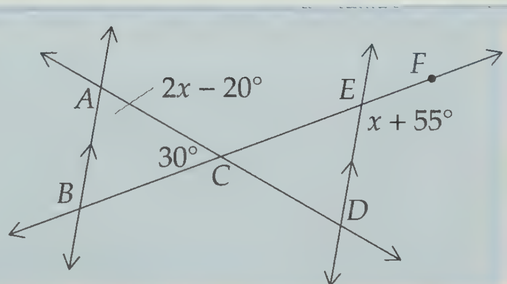
which we can easily rearrange to the desired $\angle X + \angle Y = \angle XZP$.

Notice that our proof does not at all depend on the values of the angles! □

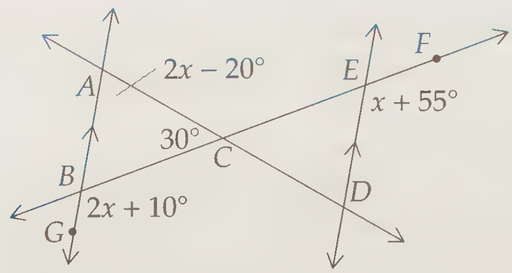
And now, we have more things to name. $\angle X$ and $\angle Y$ are called the **remote interior angles** of exterior angle $\angle XZP$ of $\triangle XYZ$. This name is *really* not important. We mostly give it a name so we can briefly write what we just proved:

Important: Any exterior angle of a triangle is equal to the sum of its remote interior angles.

Problem 2.22: In the diagram, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DE}$, $\angle BAC = 2x - 20^\circ$, $\angle ACB = 30^\circ$, and $\angle DEF = x + 55^\circ$ as shown. Find $\angle CED$.



Solution for Problem 2.22: We know we'll probably need to find x to answer the problem. We could label every angle we know in terms of x, but first we take a minute to look for a faster way to get x.



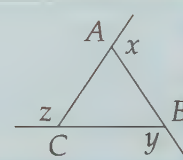
We have $\angle BAC$ and $\angle ACB$ of $\triangle ABC$, so we know the exterior angle $\angle GBC = \angle BAC + \angle ACB = 2x + 10°$. From $\overline{AB} \parallel \overline{DE}$, we know that $\angle GBC = \angle DEF = x + 55°$. Hence, we have $2x + 10° = x + 55°$, so $x = 45°$.

Our desired angle is the supplement of $\angle DEF$, so our answer is

$$\angle CED = 180° - \angle DEF = 180° - (x + 55°) = 80°.$$

There are many other ways we could have approached this problem. This is almost always true when we have problems involving exterior angles. Using exterior angles of a triangle is really just a shortcut for using a line and what we learned in Section 2.5 about the angles of a triangle. □

Problem 2.23: Find $x + y + z$.



Solution for Problem 2.23: We'll use $\angle A$, $\angle B$, and $\angle C$ to refer to the interior angles of $\triangle ABC$. Using what we just learned about exterior angles, we have

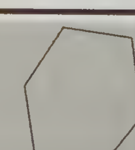
$$\begin{aligned} x &= \angle B + \angle C \\ y &= \angle A + \angle C \\ z &= \angle A + \angle B. \end{aligned}$$

Adding these, and noting that $\angle A + \angle B + \angle C = 180°$, gives:

$$\begin{aligned} x + y + z &= (\angle B + \angle C) + (\angle A + \angle C) + (\angle A + \angle B) \\ &= 2(\angle A + \angle B + \angle C) \\ &= 2(180°) \\ &= 360°. \end{aligned}$$

You'll be seeing this again. Only, next time, we'll be dealing with a figure that has more sides than a simple triangle! □

Extra! For an extra challenge, try to figure out the sum of the interior angles, and the sum of the exterior angles, in a figure with more sides than a triangle, such as the figure at right.



Exercises

2.6.1 Find y in the figure at left below.

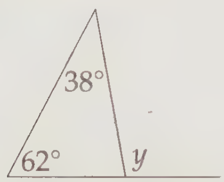


Figure 2.9: Diagram for Problem 2.6.1

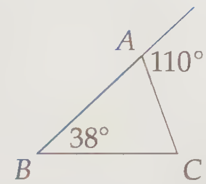
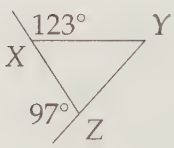


Figure 2.10: Diagram for Problem 2.6.2

2.6.2 Find $\angle C$ in the figure at right above.



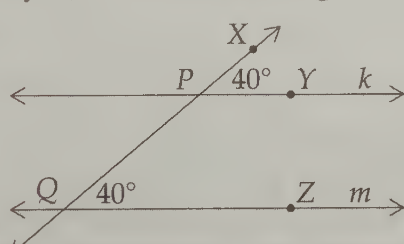
2.6.3 Find $\angle Y$ in the figure at left.

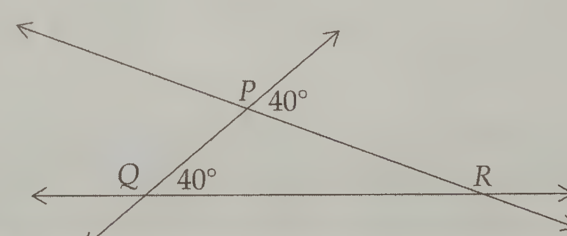
2.6.4 Must an exterior angle of a triangle always be greater than 90°?

2.7 Parallel Lines Revisited

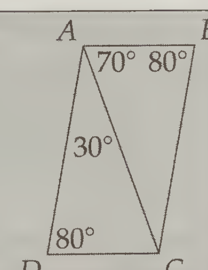
Problems

Problem 2.24:
(a) In the diagram at left below, what can we say about lines k and m?
(b) Why? (Hint: What's wrong with $\triangle PQR$ in the second figure?)



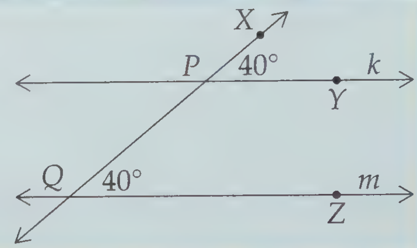


Problem 2.25: Given that the angles have the measures indicated in the diagram, prove that $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$.



In Section 2.4, we learned some useful relationships about angles when we have parallel lines. In this section, we investigate whether or not we can 'go backwards' by using angle relationships to prove lines are parallel.

Problem 2.24: In the diagram to the right, what can we say about lines k and m? Why can we say it?



Solution for Problem 2.24: What's wrong with this solution:

Bogus Solution: If lines k and m were parallel, then $\angle XPY = \angle XQZ$. Since we know that $\angle XPY = \angle XQZ$, lines k and m must be parallel.

This 'solution' is exactly the same as saying the following:

> If an animal is a cat, the animal must have four legs. My pet Spot has four legs. Spot must be a cat.

Clearly, this is bogus, because Spot could be a dog. All we know about Spot is that she has four legs. Our Bogus Solution to the problem has the same flaw!

We know for sure that if lines k and m were parallel, then $\angle XPY = \angle XQZ$. It is **not** logically valid to just run that backwards and say 'If $\angle XPY = \angle XQZ$, then lines k and m are parallel.' We have to prove this second statement separately. In Spot's case, we saw that the 'backwards' version of 'If an animal is a cat, then the animal must have four legs' is not even true! We call this 'backwards' version the **converse** of the original statement.

WARNING!! Suppose we have a true statement of the form:

If this, then that.

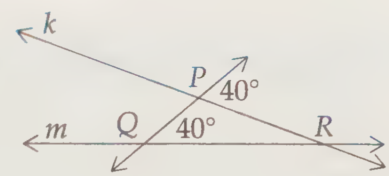
The converse of this statement is:

If that, then this.

Even if our original statement is true, the converse doesn't have to be true. We have to prove the converse separately.

And now, back to our story. We'd like to prove that lines k and m are parallel, and we can't use our angle relationships because we don't know they are parallel. The only other thing we know about parallel lines is that they never meet. So, we wonder if it is possible for the lines to meet if $\angle XPY = \angle XQZ$.

Suppose k and m meet at R as shown in the figure at right. We have a triangle, so we try to use what we know about triangles. First, we note that



$$\angle RPQ = 180^\circ - 40^\circ = 140^\circ.$$

Now that we have two angles of $\triangle PQR$, we can find the third:

$$\angle PRQ = 180^\circ - \angle RPQ - \angle RQP = 180^\circ - 140^\circ - 40^\circ = 0^\circ.$$

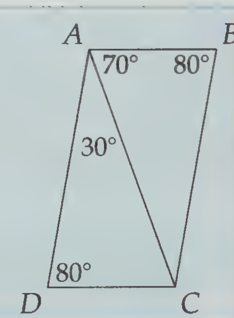
Uh-oh. If k and m meet at R, then $\angle R$ must be 0°, which doesn't make any sense. Therefore, we have shown that it is impossible for k and m to meet to the right of $\overleftrightarrow{PQ}$. (We'll leave the case of k and m meeting to the left of $\overleftrightarrow{PQ}$ as an Exercise.)

Since it is impossible for k and m to meet, they must be parallel. □

Important: Each of the angle relationships regarding parallel lines that we found in Section 2.4 can be used to prove that two lines are parallel. Using these angle relationships is the most common method of proving that two lines are parallel.

Concept: Our solution to the previous problem is an example of **proof by contradiction**. To prove a statement by contradiction, we start by assuming the statement is false. Then we show that this assumption leads us to an impossible statement (such as $\angle PRQ = 0^\circ$ above), which tells us that the assumption itself is false. Having proved the statement cannot be false, we have shown it must be true.

Problem 2.25: Given that the angles are as marked in the diagram, prove that $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$.



Solution for Problem 2.25: We start by finding the angles we can find. Triangles $\triangle ACD$ and $\triangle ABC$ tell us that

$$\angle ACD = 180^\circ - 30^\circ - 80^\circ = 70^\circ \qquad \text{and} \qquad \angle ACB = 180^\circ - 70^\circ - 80^\circ = 30^\circ.$$

Therefore, we have $\angle DAC = \angle ACB$, which means that $\overline{AD} \parallel \overline{BC}$. We also have $\angle BAC = \angle DCA$, which tells us $\overline{AB} \parallel \overline{CD}$.

We also could have noted that $\angle D + \angle DAB = 180^\circ$, so $\overline{AB} \parallel \overline{CD}$. Likewise, $\angle B + \angle BAD = 180^\circ$ gives us $\overline{AD} \parallel \overline{BC}$. □

Exercises

2.7.1 We only did half of the proof in Problem 2.24. Complete the proof by showing that the two lines cannot meet to the left of $\overleftrightarrow{PQ}$.

2.7.2 For the diagram at left below, we have proved that if n cuts k and m such that $x = y$ (that is, corresponding angles are equal), then $k \parallel m$. Use this fact to show that if $y = z$ (that is, alternate interior angles are equal), then $k \parallel m$.

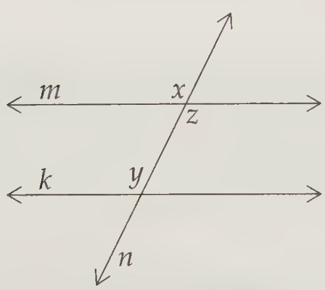


Figure 2.11: Diagram for Problem 2.7.2

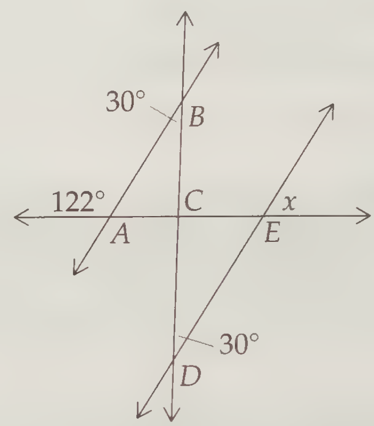


Figure 2.12: Diagram for Problem 2.7.3

2.7.3 Find x in the diagram to the right above. **Hints:** 427, 519

2.7.4 Write the converse of each of the following statements, then identify whether or not that converse is true.

(a) If two teams are playing in the World Cup Finals, then the teams must be playing soccer.

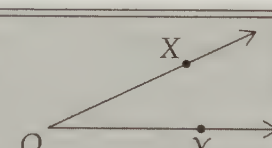
(b) If two of the angles of a triangle add to 80°, then one angle of the triangle must be 100°.

(c) If a river is the longest river in the world, then it must be the Nile.

(d) If an animal is a duck, then it must be a bird.

2.8 Summary

Definitions: Two rays that share an origin form an **angle**. The common origin of the rays is the **vertex** of the angle. We use the symbol $\angle$ to denote an angle, and we use a point on each side and the vertex, or just the vertex, to identify the angle, such as $\angle XOY$ at the right.



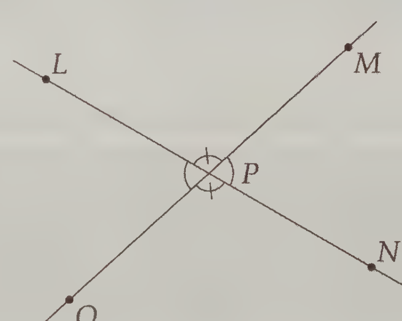
We can use a **protractor** to measure angles (see page 14). The **semicircular** arc of the protractor is divided into 180 degrees, so that a whole circle is 360 degrees.

Definitions: Angles can be classified by their measures.

- A 90° angle is a **right angle**. Lines, segments, or rays that form a right angle are said to be **perpendicular**.
- An angle smaller than 90° is an **acute angle**.
- An angle between 90° and 180° is an **obtuse angle**.
- An angle that measures 180° is a **straight angle**.
- An angle of more than 180° is a **reflex angle**.

Definitions:

- Two angles whose measures add to 180° are **supplementary angles**. Angles that together make up a straight angle form a particularly useful example of supplementary angles.
- Two angles whose measures add to 90° are **complementary angles**.
- When two lines intersect, they form two pairs of **vertical angles**, such as $\angle MPN$ and $\angle LPO$ below. Vertical angles are equal.



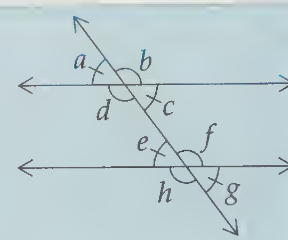
Angles $\angle LPM$ and $\angle NPM$ together form a straight angle, so they are supplementary (i.e. add to 180°).

Important: The concepts are more important than the words for solving problems. 'Angles like $\angle MPN$ and $\angle LPO$ above are equal' means something without any more information. 'Vertical angles are equal' doesn't tell you anything until you reach for your dictionary to look up vertical angles.

The words, however, are important for communicating the concepts. For now, though, focus on the ideas. The words will come naturally.

Definitions: Two lines that do not intersect are **parallel**. A line that cuts across multiple parallel lines is called a **transversal line**.

Important: The angles formed when a transversal cuts across two parallel lines come in two groups of four equal angles as shown:

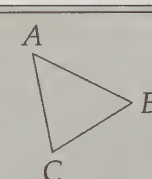


$$a = c = e = g$$
$$b = d = f = h$$

Each of the first set of angles is supplementary to each of the second set of angles. See page 27 for all the special names for pairs of these angles.

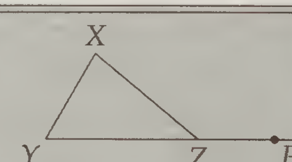
Important: The relationships described above when a transversal cuts two lines can also be used to show that two lines are parallel.

Definitions: When we connect three points with line segments, we form a **triangle**. The points are the **vertices** of the triangle, and the segments are the **sides** of the triangle. The angles inside the triangle formed by the sides are the **interior angles** of a triangle. When we refer to the angles of a triangle, we mean the interior angles.



Important: The sum of the angles in a triangle is always 180°.

Definitions: When we extend a side past a vertex of a triangle, we form an **exterior angle** of the triangle, such as $\angle XZP$ shown. We call $\angle X$ and $\angle Y$ the **remote interior angles** of exterior angle $\angle XZP$.



Important: Any exterior angle of a triangle is equal to the sum of its remote interior angles.

Problem Solving Strategies

Concepts:

- When you can't find the answer right away, try finding whatever you can – you might find something that leads to the answer! Better yet, you might find something even more interesting than the answer. The best problem solvers are explorers.
- Doing a problem two different ways is an excellent way to check your answer.

Continued on the next page. . .

Concepts: *. . . continued from the previous page*

- Often when we're angle-chasing our goal is to build an equation to solve for one of the variables in our problem.
- Parallel lines are so useful in problems involving angles that sometimes we'll add new ones to a diagram to help us.
- When angle-chasing, it's best to write the values you find for angles on your diagram as you find them, even when these values include variables.
- The key to tackling word problems in geometry is the same as any other kind of word problem – turn the words into math. Usually this means defining variables and using the words to write equations to solve for the variables.
- Sometimes using a **proof by contradiction** is much easier than proving a statement directly. To prove a statement by contradiction, we start by assuming the statement is false. Then we show that this assumption leads us to an impossible statement, which tells us that the assumption itself is false. Having proved the statement cannot be false, we have shown it must be true.

Things To Watch Out For!

WARNING!!

- An example is not a proof!
- Your last step should be to make sure you've answered the question that is asked.

Proof is at the heart of mathematics. On page 40, we saw one of the most common logical errors beginners make.

WARNING!! Suppose we have a true statement of the form

If this, then that.

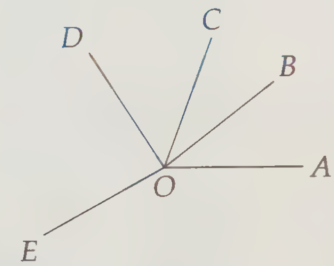
The **converse** of this statement is

If that, then this.

Even if our original statement is true, the converse doesn't have to be true. We have to prove the converse separately.

REVIEW PROBLEMS

2.26 Using your protractor, determine the following angles in the diagram to the right:



(a) $\angle AOB$

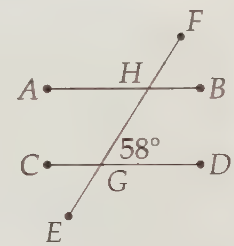
(b) $\angle AOC$

(c) $\angle BOC$

(d) $\angle DOE$

2.27 How many seconds does it take the second hand of a clock to rotate through an angle of 72°?

2.28 Given $\overline{AB} \parallel \overline{CD}$ in the diagram to the right, find:



(a) $\angle CGE$

(b) $\angle HGC$

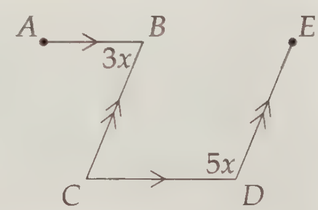
(c) $\angle FHB$

(d) $\angle BHG$

2.29 Two angles of a triangle are 30° and 70°. What is the third angle?

2.30 The angle $\angle B$ in $\triangle ABC$ is 60°. If an exterior angle at A is 170°, what is $\angle C$?

2.31 Find x in the diagram to the right.



2.32 Lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ are parallel, and $\overleftrightarrow{TV} \perp \overleftrightarrow{PQ}$. If $\overleftrightarrow{TV}$ intersects $\overleftrightarrow{PQ}$ at X and $\overleftrightarrow{RS}$ at Y, find $\angle RYX$.

2.33 In the diagram to the left below, the angles are as marked. Find x.

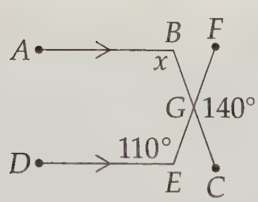


Figure 2.13: Diagram for Problem 2.33

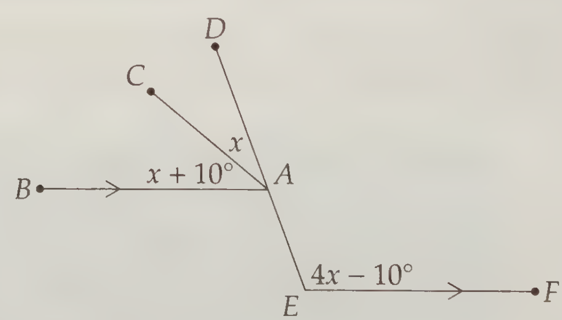


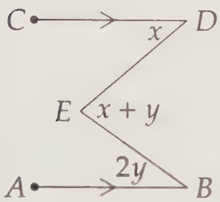
Figure 2.14: Diagram for Problem 2.34

2.34 The measures of the angles are as marked in the diagram to the right above. Find x.

2.35 The three angles $\angle A$, $\angle B$, and $\angle C$ have the property that $\angle A$ is complementary to $\angle B$, $\angle B$ is complementary to $\angle C$, and $\angle C$ is complementary to $\angle A$. Prove that $\angle A = \angle B = \angle C$.

2.36 Let $\triangle ABC$ have (interior) angles in the ratio $3 : 4 : 5$. What is the measure of its smallest exterior angle? **Hints:** 135

2.37 The exterior angles of a triangle are in the ratio $2 : 3 : 4$. What are the (interior) angles of the triangle?



2.38 Is it possible for the angles in the diagram to the left to have the measures indicated? Why or why not?

2.39 Is it possible for two exterior angles of a triangle to be supplementary? Why or why not? **Hints:** 491

2.40 Three straight lines intersect at O and $\angle COD = \angle DOE$ in the diagram at left below. The ratio of $\angle COB$ to $\angle BOF$ is $7 : 2$. What is the number of degrees in $\angle COD$? *(Source: MATHCOUNTS)*

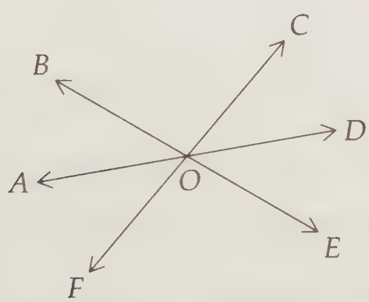


Figure 2.15: Diagram for Problem 2.40

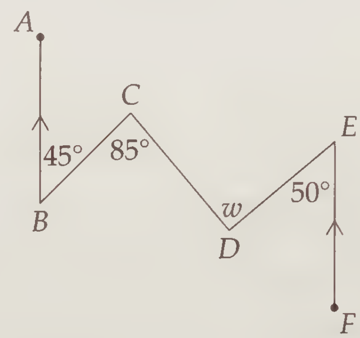
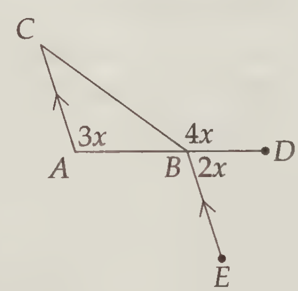


Figure 2.16: Diagram for Problem 2.41

2.41 Find w in the diagram to the right above. **Hints:** 248

2.42 One angle of a triangle is 20°. If the largest angle of the triangle has six times the measure of the smallest, what are the angles of the triangle?

2.43 The measures of the angles in the diagram to the right are as marked. Find $\angle C$.



2.44 The three angles of a triangle have measures $\angle A = x - 2y$, $\angle B = 3x + 5y$, and $\angle C = 5x - 3y$. Find x.

2.45 It is not possible to find the value of y in the previous problem from the given information. What if you are also told that one angle of the triangle is 10°? Is it now possible to compute y? What are the possible positive value(s) of y?

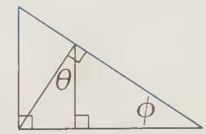
2.46 Show that if a transversal cuts two lines such that the same-side interior angles are supplementary, then the two lines are parallel.

Extra! *Music is the pleasure the human mind experiences from counting without being aware that it is counting.*

–Gottfried Wilhelm von Leibniz

Challenge Problems

2.47 Three angles in the diagram to the right are marked as right angles. If $\phi = 27°$, what is the value of θ? *(Source: Mandelbrot)*



2.48 What is the number of degrees of the angle formed by the minute and hour hands of a clock at 11:10 PM? *(Source: MATHCOUNTS)* **Hints:** 52, 356, 282

2.49 Find x in the diagram at left below. **Hints:** 355

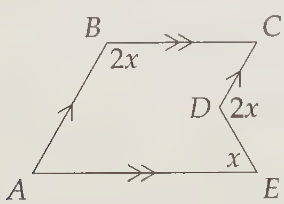


Figure 2.17: Diagram for Problem 2.49

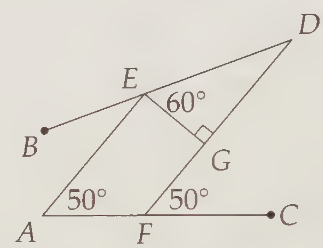


Figure 2.18: Diagram for Problem 2.50

2.50 The angles in the diagram to the right above are as marked. Find $\angle BEA$. **Hints:** 8

2.51 One angle of a triangle is equal to the sum of the other two. Show that the sum of two exterior angles of the triangle is 180° greater than the third. **Hints:** 151, 200

2.52 One of the angles of the triangle in the previous problem is equal to 40°. What are the measures of the other two angles?

2.53 The angles of a triangle are in arithmetic progression. If one of the angles is 100°, what are the measures of the other two angles? (An arithmetic progression is a sequence of numbers in which the difference between each term and its preceding term is always the same.) **Hints:** 581

2.54 It is possible for the interior angles of a triangle to be in the ratio 1 : 2 : 6, but is it possible for the *exterior* angles of a triangle to be in the ratio 1 : 2 : 6? Prove your answer. **Hints:** 14, 444

2.55 Find $\angle A + \angle B + \angle C + \angle D$ in the figure at left below. **Hints:** 358

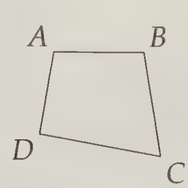


Figure 2.19: Diagram for Problem 2.55

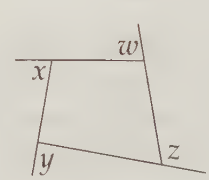
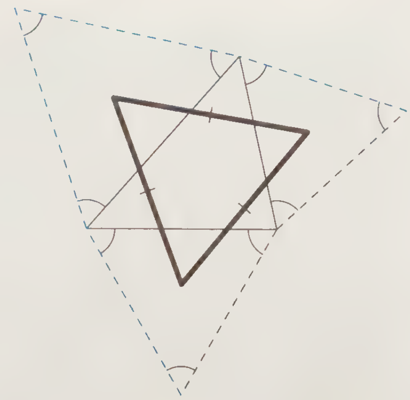


Figure 2.20: Diagram for Problem 2.56

2.56 Find $w + x + y + z$ in the figure at right above. **Hints:** 17, 400

2.57★ Point Z is on side $\overline{PR}$ of $\triangle PQR$ such that $\angle PZQ = \angle PQZ$, and $\angle PQR - \angle PRQ = 42°$. Find $\angle RQZ$. **Hints:** 54, 441, 156



Outer Napoleon Triangle

Equality may perhaps be a right, but no power on earth can ever turn it into a fact. – Honore de Balzac

CHAPTER 3 Congruent Triangles

3.1 Introduction

Two figures are **congruent** if they are exactly the same – in other words, we can slide, spin, and/or flip one figure so that it is exactly on top of the other figure.

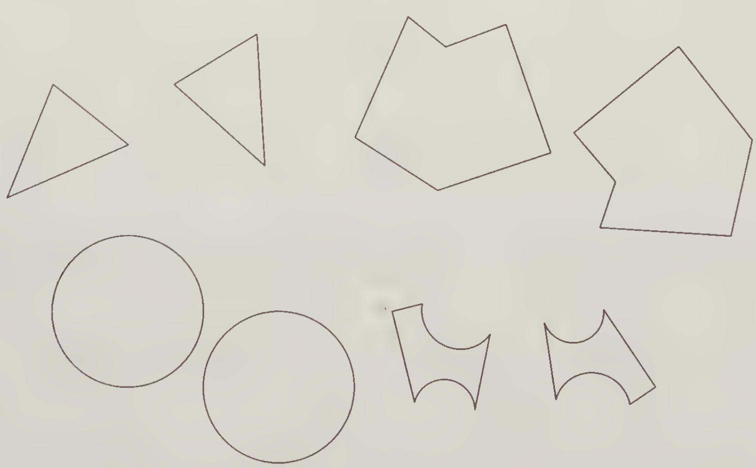


Figure 3.1: Congruent Figures

Figure 3.1 shows four pairs of congruent figures. In each case, we can take one of the figures and slide it, spin it, and/or flip it so that it exactly coincides with the other.

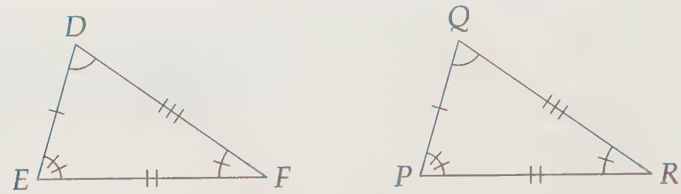


Figure 3.2: Congruent Triangles

We use the symbol $\cong$ to denote that two figures are congruent. For example, to describe the congruent triangles in Figure 3.2, we write $\triangle DEF \cong \triangle QPR$. Notice that we are careful to put the vertices in the same order: D corresponds to Q, E to P, and F to R. We would not, for example, write $\triangle DEF \cong \triangle PQR$ for the triangles in Figure 3.2.

If we have two congruent triangles, all the corresponding pairs of sides are equal, as are the corresponding angles. Conversely, if all pairs of corresponding sides of two triangles have equal lengths, and all the corresponding angles of the two triangles are equal, then the triangles are congruent.

Fortunately, to show that two triangles are congruent, we don't have to go through the hassle of proving each pair of sides and each pair of corresponding angles are congruent. In this chapter, we'll cover various ways requiring considerably less information to show that two triangles are congruent.

3.2 SSS Congruence

Problems

Problem 3.1: In each of the three triangles below, the side lengths are 5, 7, and 8 units. Are the measures of the angles the same in all three triangles? (Yes, you should use your protractor.)

Problem 3.2: In this problem we investigate how many different non-congruent triangles have sides with lengths 2.5 cm, 3.5 cm, and 4 cm by building triangle ABC with these side lengths. Start by drawing segment $\overline{AB}$ with length 4 cm.

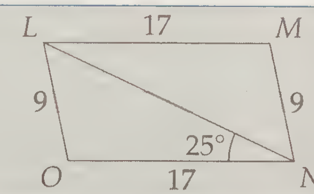
(a) Suppose C is 3.5 cm away from A. What points on your paper are 3.5 cm away from A? Draw the figure that consists of all these points.

(b) We have AB = 4 cm, and our answer to (a) tells us where C must be so that AC = 3.5 cm. We also want C to be 2.5 cm away from B. What points on your paper are 2.5 cm from B? Draw the figure that consists of all these points.

(c) Where in your resulting diagram can C possibly be?

(d) How many different non-congruent triangles have sides with lengths 2.5 cm, 3.5 cm, and 4 cm?

Problem 3.3: In the figure, the sides and angle have measures as shown. Find $\angle MLN$.



Problem 3.4: In this problem we will show that if a radius of a circle bisects (cuts in half) a chord of the circle that is not a diameter, then the radius is perpendicular to the chord.

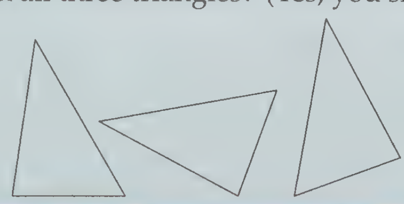
(a) Draw a circle with center O and a chord connecting points A and B on the circle.

(b) Draw a radius of the circle that passes through the midpoint, M, of $\overline{AB}$, thereby bisecting $\overline{AB}$.

(c) Find a pair of congruent triangles and use these triangles to show that $\angle OMA = \angle OMB = 90°$.

Often our best way to get started with a new geometric principle is to experiment. So, we start off by examining some triangles that have the same side lengths and checking if the triangles are indeed congruent.

Problem 3.1: In each of the three triangles below, the side lengths are 5, 7, and 8 units. Are the measures of the angles the same in all three triangles? (Yes, you should use your protractor.)



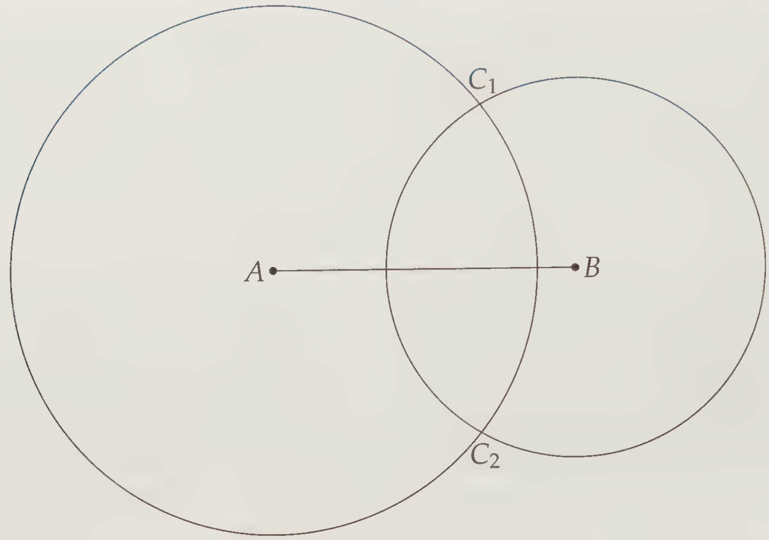
Solution for Problem 3.1: Using our protractor, we find that the angles of each triangle are approximately 82°, 60°, and 38°. Each triangle has the largest angle opposite the largest side and the smallest angle opposite the smallest side. Hence, the sides and the angles appear to be the same in all three triangles, so it looks like they're congruent. □

Let's see if we can get a common-sense explanation for why the triangles of Problem 3.1 are congruent.

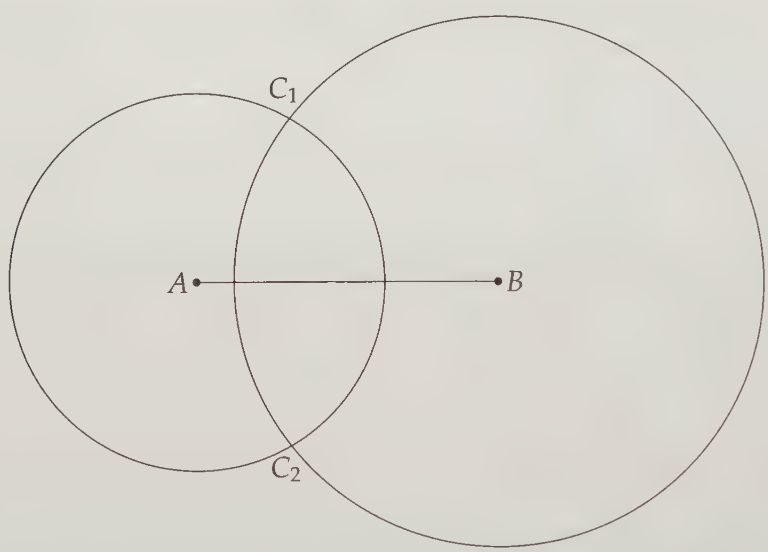
Problem 3.2: Investigate how many different non-congruent triangles have sides with lengths 2.5 cm, 3.5 cm, and 4 cm by building triangle ABC with these side lengths.

Solution for Problem 3.2: It doesn't matter which side we set to 4 cm, so we just pick $\overline{AB}$ to be the long side. We start building $\triangle ABC$ by drawing a 4 cm long segment.

Now we have to use our other side lengths to figure out where C can possibly be. Suppose $AC = 3.5$ cm. Then, C must be 3.5 cm away from A. So, C must be on a circle with center A and radius 3.5 cm. Similarly, since $BC = 2.5$ cm, C must lie on a circle with center B and radius 2.5 cm. We draw both circles and end up with the diagram below. Since C must be on both of the circles, C must be at one of the intersection points. Our two options for C (labeled C_1 and C_2) are merely mirror images of each other.



If we had instead assumed $AC = 2.5$ cm and $BC = 3.5$ cm, we would end up with essentially the same picture, as shown below.



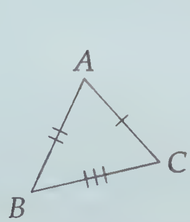
Notice that if we 'flip' our first diagram horizontally, we get the second one. In fact, we don't really even need this second diagram. We could have simply swapped the labels A and B in our first diagram to investigate the possibility that $AC = 2.5$ cm and $BC = 3.5$ cm. □

Concept: Recognizing two seemingly different situations as the same can be a powerful simplifying tool in math problems. In geometry, this often involves recognizing **symmetry**, as we did in noting that C_1 and C_2 are mirror images of each other in each diagram above, and in noting that the two diagrams themselves are mirror images of each other.

Important: We call the principle illustrated in Problems 3.1 and 3.2 the **Side-Side-Side Congruence Theorem**, or **SSS Congruence** for short. SSS states:

> If the lengths of the sides of one triangle equal the lengths of the corresponding sides of another triangle, then the triangles are congruent.

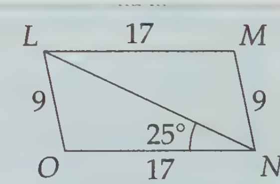
For example, in the diagram below, we have $\triangle ABC \cong \triangle XZY$ by SSS Congruence.



Our 'proof' that all triangles with side lengths 2.5 cm, 3 cm, and 4 cm are congruent skips over a few very important points. First, we didn't prove that triangles $\triangle ABC_1$ and $\triangle ABC_2$ are congruent in each of our cases. Nor did we prove that the triangles in the first case are congruent to the triangles in the second. Intuitively, it seems clear that these triangles are all congruent, but we haven't proved it. In fact, with the tools we have at this point, we cannot prove SSS Congruence at all!

We must accept as an axiom one of the Congruence Theorems we will study in this chapter. We can then use that one theorem, and some other tools, to prove the others. Since some of those other tools are considerably more advanced than those we have now, we'll stick to our intuitive explanations for all of the Congruence Theorems.

Problem 3.3: In the figure, the sides and angle have measures as shown. Find $\angle MLN$.



Solution for Problem 3.3: What's wrong with this:

Bogus Solution: Since $\overline{LM}$ and $\overline{NO}$ are parallel, $\angle MLN = \angle LNO = 25°$.

This Bogus Solution assumes that $\overline{LM} \parallel \overline{NO}$. This might be true, but we have to prove it to use it.

Since $LO = MN$, $ON = LM$, and $LN = LN$, we have $\triangle LNO \cong \triangle NLM$ by SSS. Since angles $\angle MLN$ and $\angle ONL$ are corresponding parts of these two triangles, they must be equal. Therefore,

$$\angle MLN = \angle ONL = 25°.$$

Notice that we can now use $\angle MLN = \angle ONL$ to prove that $\overline{LM} \parallel \overline{NO}$. □

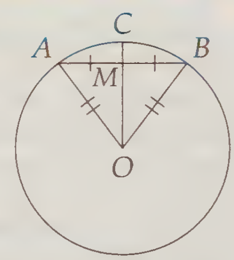
Important: Problem 3.3 shows us a very important general use of congruent triangles. Once we determine that two triangles are congruent, we can apply whatever we know about the sides and angles of one triangle to the other triangle. This obvious principle often goes by the fancy name '**Corresponding Parts of Congruent Triangles are Congruent**,' or **CPCTC**.

Unless your teacher tells you to, you don't have to clutter your paper writing CPCTC all over. Once you have proved that $\triangle LNO \cong \triangle NLM$, you can simply write, 'Since $\triangle LNO \cong \triangle NLM$, we have $\angle MLN = \angle ONL$.'

Let's try using congruent triangles in a proof.

Problem 3.4: Prove that if a radius of a circle bisects (cuts in half) a chord of the circle that is not a diameter, then the radius must be perpendicular to the chord.

Solution for Problem 3.4: We start with a diagram, drawing circle O, chord $\overline{AB}$ with midpoint M, and the radius through M. We then mark the equal pieces of the chord and the equal radii of the circle, and see that we have congruent triangles. (Radii is the plural of radius.) Specifically, $\triangle AMO \cong \triangle BMO$ by SSS. So, $\angle AMO = \angle BMO$. Since $\angle AMO + \angle BMO = 180^\circ$, we must have $\angle AMO = \angle BMO = 90^\circ$. □

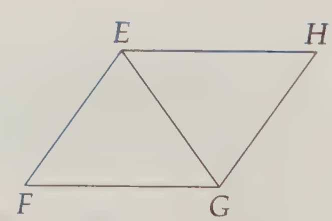


Important: If a radius of a circle bisects a chord of the circle that is not a diameter, then the radius must be perpendicular to the chord.

As we'll see in Chapter 6, this works in reverse, too! A radius perpendicular to a chord must bisect the chord.

Exercises

3.2.1 $EF = GH$ and $FG = EH$ in the diagram at right.



(a) Prove that $\triangle EFG \cong \triangle GHE$.

(b) Show that $\angle EGF = \angle GEH$.

(c) Show that $\overline{HE} \parallel \overline{FG}$.

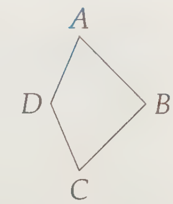
(d) Show that $\overline{HG} \parallel \overline{EF}$.

3.2.2 In triangle ABC, $AB = AC$. Let M be the midpoint of side $\overline{BC}$.

(a) A segment, line, or ray **bisects** an angle if it divides it into two angles with equal measure. Show that $\overline{AM}$ bisects $\angle BAC$ by proving that $\angle BAM = \angle CAM$.

(b) Show that $\overline{AM} \perp \overline{BC}$.

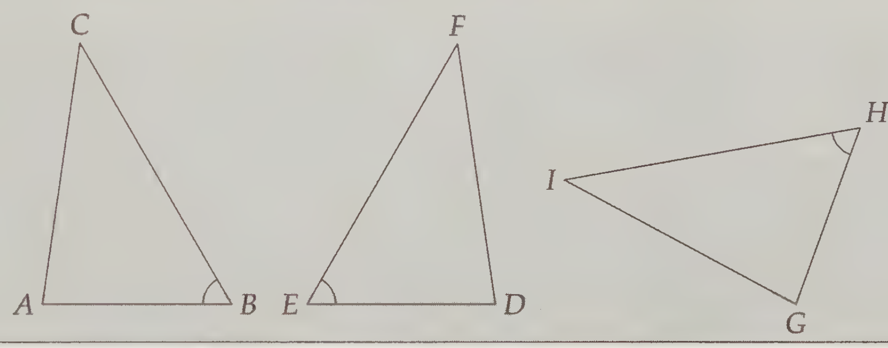
3.2.3 In the diagram at right, $AB = 7$, $AD = 4$, $CD = 4$, and $BC = 7$. Prove that $\angle ABD = \angle CBD$. **Hints:** 165



3.3 SAS Congruence

Problems

Problem 3.5: In each of the triangles below, two sides have lengths 2.5 cm and 4 cm, and the angle between these sides is 60°. Measure the third side and the other two angles of each triangle.



Problem 3.6: Investigate why the triangles in the previous problem appear to be congruent by constructing a triangle with side lengths 2.5 cm and 4 cm such that the angle between these two sides is 60°.

(a) Draw segment $\overline{AB}$ such that $AB = 4$ cm.

(b) Let $\overline{BC}$ be the side that is 2.5 cm long; thus, C is 2.5 cm from B. Construct the figure that consists of all points that are 2.5 cm from B.

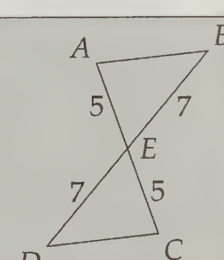
(c) The angle between $\overline{AB}$ and $\overline{BC}$ must be 60°. Where can C be to make $\angle ABC = 60°$? (Forget about $BC = 2.5$ cm for this part.)

(d) Draw your answer to (c).

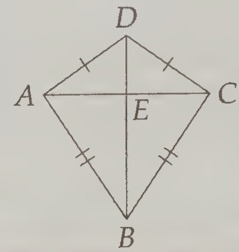
(e) Where can C possibly be located?

(f) Are all triangles with two sides of length 2.5 cm and 4 cm with an angle of 60° between them congruent?

Problem 3.7: Point E is the midpoint of both $\overline{AC}$ and $\overline{BD}$ as shown. Prove that $\overline{AB} \parallel \overline{CD}$.

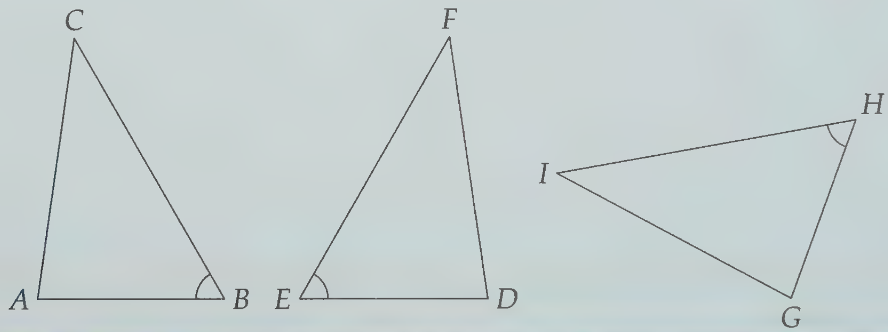


Problem 3.8: In the figure below, $AD = CD$ and $AB = BC$. Prove that $\overline{DB} \perp \overline{AC}$. **Hints:** 461



Showing that all the sides of one triangle equal those of another is not the only way to quickly prove two triangles are congruent. In this section we investigate one way to use angles along with sides.

Problem 3.5: In each of the triangles below, two sides have lengths 2.5 cm and 4 cm, and the angle between these sides is 60°. Measure the third side and the other two angles of each triangle.



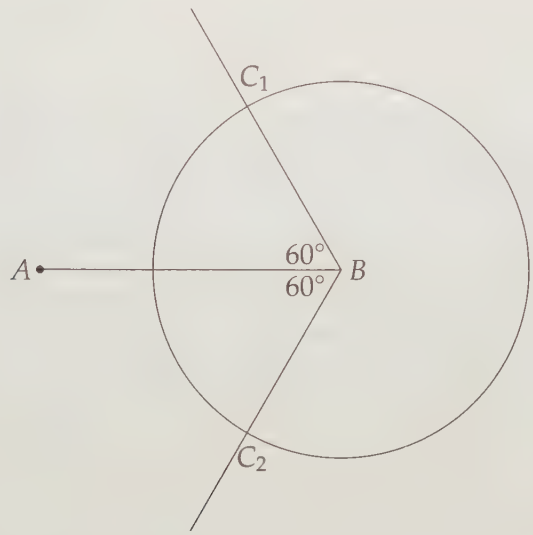
Solution for Problem 3.5: In each case, the third side is 3.5 cm and the other two angles are approximately 82° and 38°, with the 82° angle opposite the 4 cm side. The triangles sure look congruent. □

Sidenote: Perhaps only the ancient Greeks have contributed as much to geometry as the German mathematician **David Hilbert**. Among his wide range of work was *Grundlagen der Geometrie*, which placed geometry in a formal axiomatic framework. Instead of Euclid's 5 axioms, Hilbert used 21 axioms, since Euclid's work included many hidden assumptions that must be accepted as axioms. Among those axioms is SAS Congruence, which, in combination with other axioms, he used to prove the other Congruence Theorems presented in this chapter.

Not only did Hilbert produce much great mathematics, but he also inspired and challenged others. In 1900 he gave an address to the Second International Congress in Paris on 23 unsolved problems of great importance. Since then, some of the problems have been solved, but many still puzzle mathematicians to this day. He also succinctly phrased the passion that guides many great mathematicians and scientists when he closed an address with the words, *"We must know, we shall know."*

Problem 3.6: Investigate why the triangles in the previous problem appear to be congruent by constructing a triangle with side lengths 2.5 cm and 4 cm such that the angle between these two sides is 60°.

Solution for Problem 3.6: We start off just like we did in Problem 3.2, by letting let $AB = 4$ cm. We want $BC = 2.5$ cm and $\angle ABC = 60^\circ$. Since C is 2.5 cm from B, it lies on a circle with center B and radius 2.5 cm. Since $\angle ABC = 60^\circ$, we know that C lies on a ray from B that forms a 60° angle with $\overline{AB}$. We can use our compass and protractor to build the circle and our two possible rays as shown below.



In our diagram there are only two options for C: the two points where one of the rays meets the circle. Just like we saw in Problem 3.2, these C's are mirror images. $\triangle ABC_1$ and $\triangle ABC_2$ have the same side lengths and angle measures.

Therefore, we find that no matter how we build the triangle given two sides and the angle between them, we always get a triangle with the same sides and angles. □

Important: We call the principle illustrated in Problem 3.6 the **Side-Angle-Side Congruence Theorem**, or **SAS** for short. SAS states:

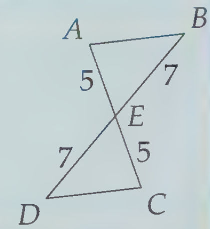
> If two sides of one triangle and the angle between them are equal to the corresponding sides and angle of another triangle, then the two triangles are congruent.

As with our illustration of SSS on page 51, our solution to Problem 3.6 is not a complete proof of the SAS Congruence Theorem. It merely provides an intuitive explanation. Let's see SAS in action.

Extra! *It is impossible to be a mathematician without being a poet in soul.*

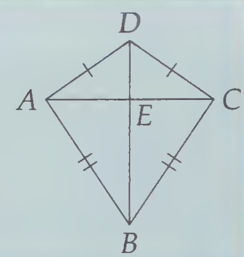
–Sofia Kovalevskaya

Problem 3.7: Point E is the midpoint of both $\overline{AC}$ and $\overline{BD}$ as shown. Prove that $\overline{AB} \parallel \overline{CD}$.



Solution for Problem 3.7: We saw in Section 2.7 that we can use angle equality to prove that lines are parallel. We can get the angle equality we need by using congruent triangles. Since $AE = EC$, $BE = ED$, and $\angle AEB = \angle CED$ (because they are vertical angles), we have $\triangle AEB \cong \triangle CED$ by SAS. Therefore, $\angle ABE = \angle CDE$, so $\angle ABD = \angle CDB$ and we have $\overline{AB} \parallel \overline{CD}$. □

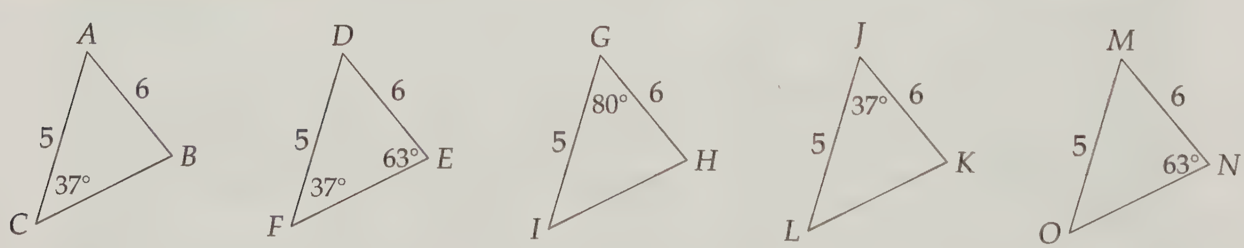
Problem 3.8: In the figure, $AD = CD$ and $AB = BC$. Prove that $\overline{DB} \perp \overline{AC}$.



Solution for Problem 3.8: We have to prove that two lines are perpendicular, but we don't have any information about angles. We look for congruent triangles since that will give us some angle information. The given side equalities along with the obvious $BD = BD$ give us $\triangle DAB \cong \triangle DCB$ by SSS. This doesn't give us any information about angles at point E, but it does give us some angle equalities like $\angle ADE = \angle CDE$. This, together with $AD = DC$ and $DE = DE$, gives us $\triangle ADE \cong \triangle CDE$ by SAS. Therefore, $\angle AED = \angle CED$. Since these two angles are also supplementary, we have $\angle AED = \angle CED = 90°$, so $\overline{BD} \perp \overline{AC}$. □

Exercises

3.3.1 Which two of the triangles below must be congruent and why must they be congruent?

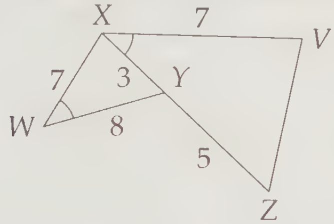


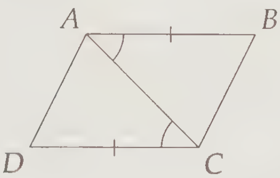
3.3.2 In triangle ABC, $AB = AC$. Let M be the point on $\overline{BC}$ such that $\overline{AM}$ bisects $\angle BAC$ (so that $\angle CAM = \angle BAM$).

(a) Show that M is the midpoint of $\overline{BC}$.

(b) Show that $\overline{AM} \perp \overline{BC}$.

3.3.3 Find VZ in the diagram at the left below. (Note: The diagram is not to scale.) **Hints:** 467



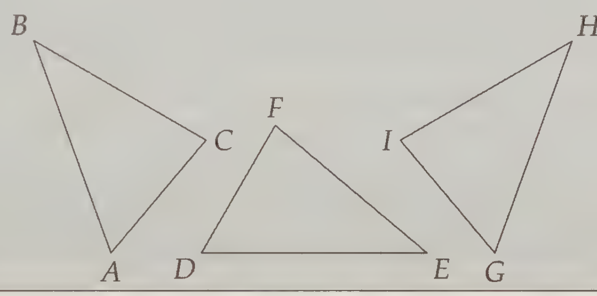


3.3.4 In the figure to the right above, $AB = CD$ and $\angle BAC = \angle DCA$. Prove that $\overline{AD} \parallel \overline{BC}$.

3.4 ASA and AAS Congruence

Problems

Problem 3.9: Shown are three triangles with two angles measuring 60° and 40°, and each triangle has a side of length 3 cm between these two angles. Measure the other two sides of each triangle. Make a guess about triangle congruence given two angles and an included side.



Problem 3.10: Investigate why the triangles in the previous problem are congruent by constructing a triangle with angles 60° and 40° such that the side between the two angles has length 3 cm. Start by drawing a segment $\overline{AB}$ such that $AB = 3$ cm.

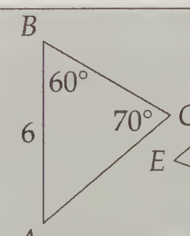
(a) Let the angle between $\overline{AB}$ and $\overline{BC}$ be 60°. Where can C be to make $\angle ABC = 60°$?

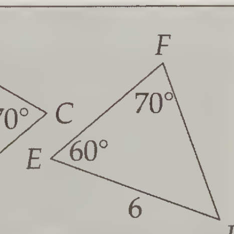
(b) Draw your answer to (a). The angle between $\overline{AB}$ and $\overline{AC}$ must be 40°. Where can C be to make $\angle BAC = 40°$? (Forget about $\angle ABC = 60°$ for this part.)

(c) Draw your answer to (b). Where can C possibly be located?

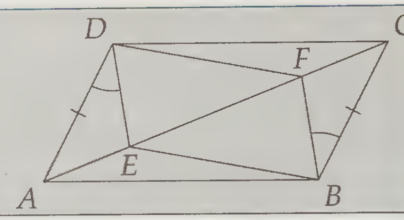
(d) Are all triangles with a side of length 3 cm between angles with measures 60° and 40° congruent?

Problem 3.11: What if the equal sides are not between the equal angles? For example, what if $AB = DE = 6$, $\angle B = \angle E = 60°$, and $\angle C = \angle F = 70°$ as shown? Can we conclude that $\triangle ABC \cong \triangle DEF$? Why or why not?

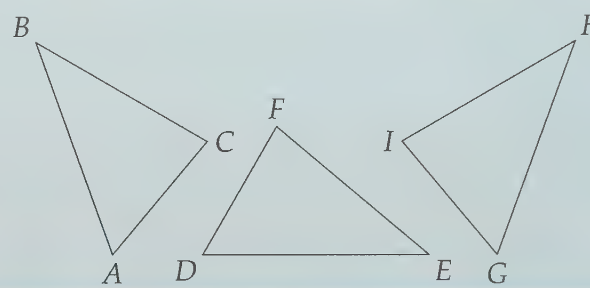




Problem 3.12: In the diagram, $AD = BC$, $\overline{AD} \parallel \overline{BC}$, and E and F are on $\overline{AC}$ so that $\angle ADE = \angle CBF$. Prove that $\overline{AB} \parallel \overline{CD}$ and $DF = EB$.



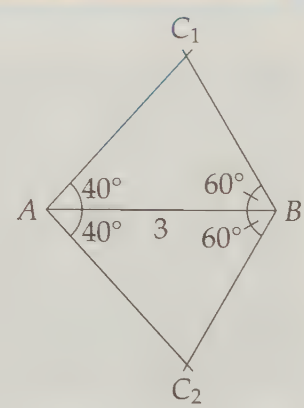
Problem 3.9: Shown are three triangles with two angles measuring 60° and 40°, and each triangle has a side of length 3 cm between these two angles. Measure the other two sides of each triangle. Make a guess about triangle congruence given two angles and an included side.



Solution for Problem 3.9: In each triangle, the side opposite the 40° angle has length about 2 cm, and the side opposite the 60° angle has length around 2.6 cm. The triangles appear to be congruent. □

Problem 3.10: Investigate why the triangles in the previous problem are congruent by constructing a triangle with angles 60° and 40° such that the side between the two angles has length 3 cm.

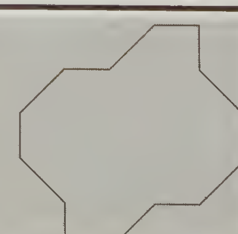
Solution for Problem 3.10: We start with $\overline{AB}$ of length 3 cm. We let $\angle ABC = 60°$. As in Problem 3.6, this means that point C is on one of two rays from B that make an angle of 60° with $\overline{AB}$. Similarly, we have $\angle BAC = 40°$, which means that C is on one of the two rays from A that make an angle of 40° with $\overline{AB}$. Point C must be at one of the two intersections of rays. The two potential C's, marked C_1 and C_2, are mirror images of each other.



Therefore, we find that all the triangles we can build given two angles and the side between them are congruent. □

Extra! Divide the figure at right into four congruent pieces that can be rearranged to form a square.

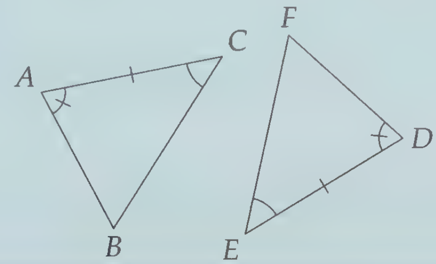
Source: ThinkFun Inc.'s Puzzles webpage. See the links page mentioned in the Acknowledgements for a link to the site.



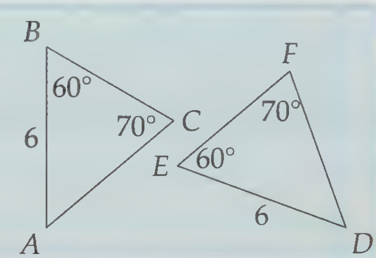
Important: We call the principle illustrated in Problem 3.10 the **Angle-Side-Angle Congruence Theorem**, or **ASA** for short. ASA states that

> If two angles of one triangle and the side between them are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.

For example, in the diagram below, we have $\triangle ABC \cong \triangle DFE$.



Problem 3.11: What if the equal sides are not between the equal angles? For example, what if $AB = DE = 6$, $\angle B = \angle E = 60°$, and $\angle C = \angle F = 70°$? Can we conclude that $\triangle ABC \cong \triangle DEF$? Why or why not?

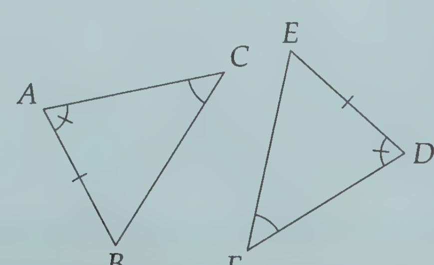


Solution for Problem 3.11: If we have two angles of a triangle, we have the third. In the diagram above we can quickly see that $\angle D = \angle A = 50°$. Hence, we can conclude that $\triangle ABC \cong \triangle DEF$ by ASA. □

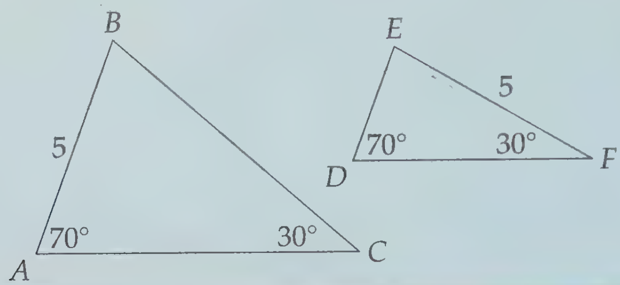
When we have the situation illustrated above, we don't have to find the third angle and invoke ASA. This procedure always works, so it gets an unsurprising theorem name of its own.

Important: We call the principle illustrated in Problem 3.11 the **Angle-Angle-Side Congruence Theorem**, or **AAS** for short. AAS states:

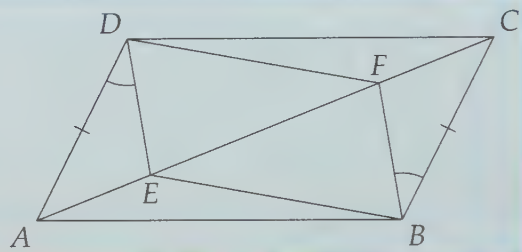
> If two angles and a side of one triangle equal the corresponding angles and side in another triangle as shown below, then the triangles are congruent.

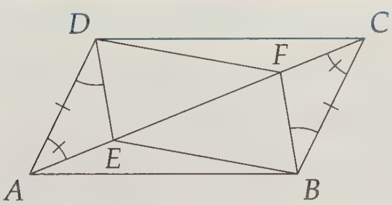


WARNING!! When using AAS, the equal sides must be adjacent to corresponding equal angles. For example, the triangles shown below are **not** congruent because the equal sides are not adjacent to corresponding equal angles!



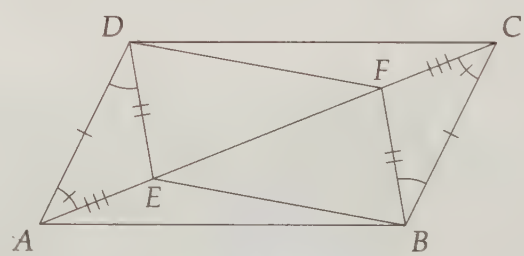
Problem 3.12: In the diagram, $AD = BC$, $\overline{AD} \parallel \overline{BC}$, and E and F are on $\overline{AC}$ so that $\angle ADE = \angle CBF$. Prove that $\overline{AB} \parallel \overline{CD}$ and $DF = EB$.





Solution for Problem 3.12: We go hunting for equal angles to use to prove that $\overline{AB}$ and $\overline{CD}$ are parallel. We start by marking the angles we know are equal. We have a pair in our original diagram, and can add $\angle DAC = \angle ACB$ due to $\overline{AD} \parallel \overline{BC}$, as shown to the left.

Now we can find some congruent triangles. Since $AD = BC$, $\angle DAC = \angle BCA$, and $AC = AC$, we have $\triangle CAD \cong \triangle ACB$ by SAS. Therefore, $\angle DCA = \angle BAC$, so $\overline{AB} \parallel \overline{CD}$.



We also have another pair of congruent triangles: $\triangle DAE \cong \triangle BCF$ by ASA. It's not immediately clear how this will help us prove $BE = DF$, so we go ahead and mark the new equal lengths we know from our new triangle congruence. Focusing on segments $\overline{BE}$ and $\overline{DF}$, we look for congruent triangles that have these as sides. We start by looking for triangles we know something about, which leads us to $\triangle ADF$ and $\triangle CBE$. We already have a side and an angle, and $AF = AE + EF = CF + EF = CE$ gives us a second side. So, $\triangle ADF \cong \triangle CBE$ by SAS. Finally, we conclude that $DF = BE$.

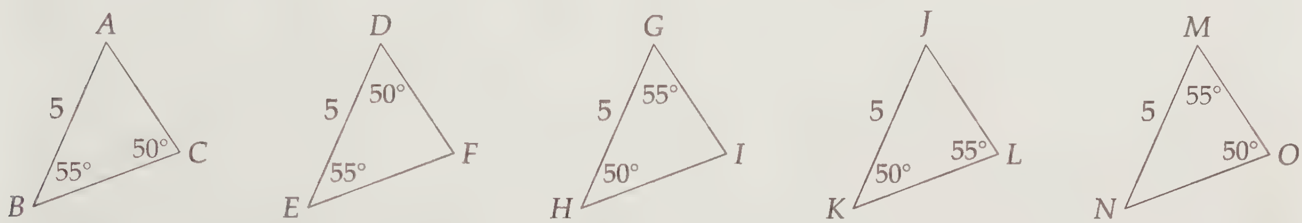
We could also have used SAS to show $\triangle DFE \cong \triangle BEF$, so $DF = BE$. □

Concept: In complicated geometry problems, mark side and angle equalities as you find them (particularly when you find non-obvious ones!)

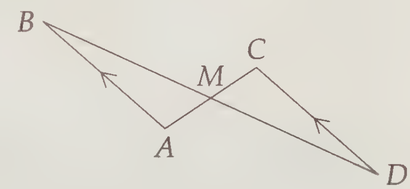
Extra! *The real voyage of discovery consists not in seeking new landscapes, but in having new eyes.*
–Marcel Proust

Exercises

3.4.1 Find all pairs of triangles below that must be congruent. Write out the appropriate congruence (make sure you have the vertices in the right order!), and explain why the triangles must be congruent.



3.4.2 In the diagram at right, $\overline{AB} \parallel \overline{DC}$. If M is the midpoint of $\overline{AC}$, must it also be the midpoint of $\overline{BD}$? Why or why not? **Hints:** 183

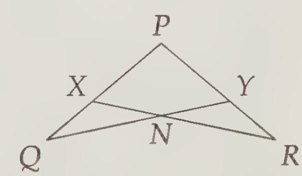


3.4.3 Use ASA Congruence to prove that AAS Congruence is a valid Congruence Theorem. (Do not assume AAS Congruence is a valid theorem for this part – you are asked here to show that any two triangles that satisfy the AAS criteria are indeed congruent *without using AAS*.) **Hints:** 463

3.4.4 In the figure, $PQ = PR$ and $\angle PQY = \angle PRX$.

(a) Prove that $QY = RX$.

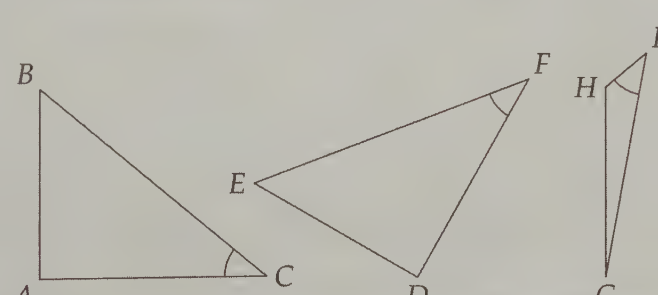
(b)★ Prove that $XN = YN$. **Hints:** 158, 343



3.5 SSA Not-Necessarily Congruence

Problems

Problem 3.13: In the diagram below, we have three triangles with two sides of length 2.5 cm and 3 cm, and an angle of 40°. The 40° angle has the 3 cm segment as a side, but does not have the 2.5 cm segment as a side. Are the triangles congruent?



Problem 3.14: In this problem we explore why there's no 'Side-Side-Angle Congruence Theorem' by building a triangle with $AB = 1.8$ cm, $BC = 1.5$ cm, and $\angle BAC = 40°$.

(a) Draw $AB = 1.8$ cm.

(b) We know that $BC = 1.5$ cm. Draw the points in the diagram that are 1.5 cm from B.

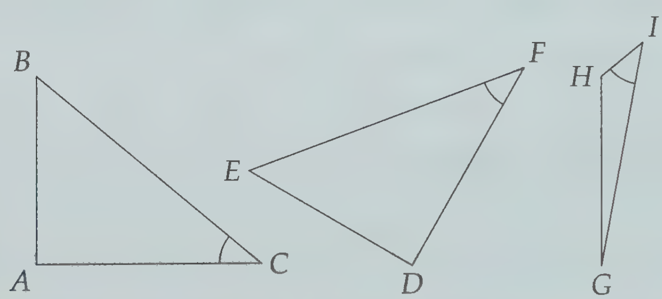
(c) Draw the points in the diagram that can be C such that $\angle BAC = 40°$. (Forget $BC = 1.5$ cm for now.)

(d) Use the previous two parts to determine where C can possibly be.

(e) Is SSA a valid congruence theorem?

(f) Do we ever have a case in which given the lengths of sides $\overline{AB}$ and $\overline{BC}$, and $\angle BAC$, there's only one possible length of $\overline{AC}$?

Problem 3.13: In the diagram, we have three triangles with two sides of length 2.5 cm and 3 cm, and an angle of 40°. The 40° angle has the 3 cm segment as a side, but does not have the 2.5 cm segment as a side. Are the triangles congruent?

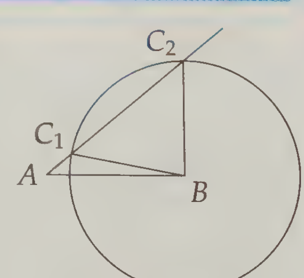


Solution for Problem 3.13: Um, no. Clearly HI is way smaller than EF and BC, so the triangles aren't congruent. □

SSS works. SAS works. ASA, and AAS, too. But SSA, not so much. Why?

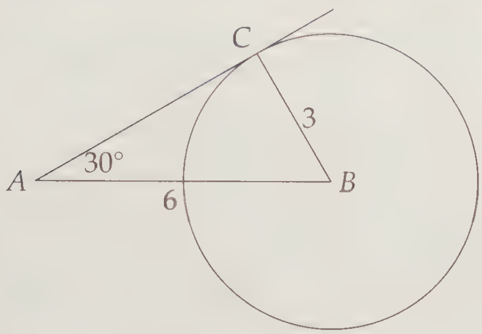
Problem 3.14: Figure out why there's no 'Side-Side-Angle Congruence Theorem' by building a triangle with $AB = 1.8$ cm, $BC = 1.5$ cm, and $\angle BAC = 40°$.

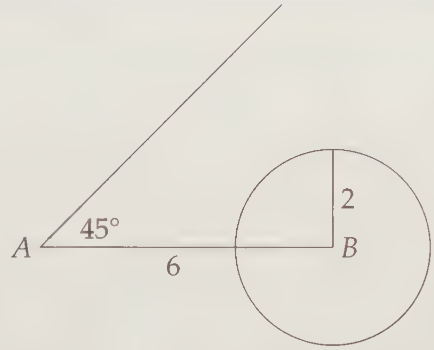
Solution for Problem 3.14: We start as we have before, drawing $\overline{AB}$ with length 1.8 cm. Since $BC = 1.5$ cm, C must be on a circle with center B and radius 1.5 cm. Also, since $\angle BAC = 40°$, C must be on a ray from A that makes a 40° angle with $\overline{AB}$. The ray and the circle are shown to the right.



Uh-oh. The ray hits the circle in two points, C_1 and C_2. Both $\triangle ABC_1$ and $\triangle ABC_2$ match the information we are given about $\triangle ABC$, but they are very obviously not congruent. □

WARNING!! Side-Side-Angle (SSA) is **not** a valid congruence theorem. You cannot use it to prove that two triangles are congruent.





You might have noticed that if we try to build a triangle given two sides and a non-included angle as in Problem 3.14, we don't always get two possible points. The left diagram above shows a case in which we do get exactly one possible C, and the right figure above shows a case in which no triangles are possible. We'll be revisiting the former case later when we have more tools to talk about this special case.

3.6 Isosceles and Equilateral Triangles

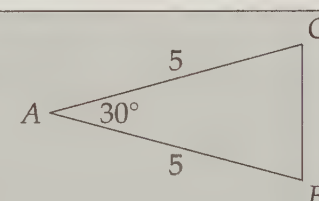
Problems

Problem 3.15: In the diagram, $AB = AC = 5$ and $\angle CAB = 30^\circ$.

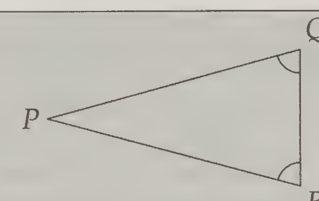
(a) Let M be the midpoint of $\overline{BC}$. Draw $\overline{AM}$. Prove that $\triangle ACM \cong \triangle ABM$.

(b) Find $\angle AMB$.

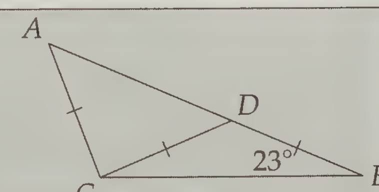
(c) Find $\angle ACB$.



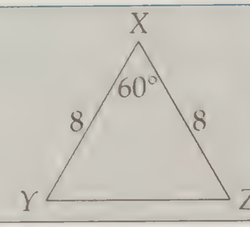
Problem 3.16: Prove that if $\angle PQR = \angle PRQ$, then $PR = PQ$.



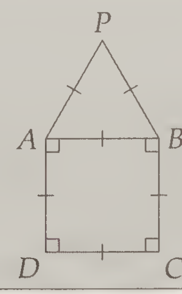
Problem 3.17: In the diagram, $AC = CD = DB$, and $\angle B = 23^\circ$. Find $\angle A$.



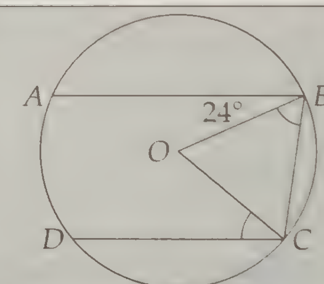
Problem 3.18: In the diagram, $XY = XZ = 8$ and $\angle X = 60^\circ$. Find YZ.



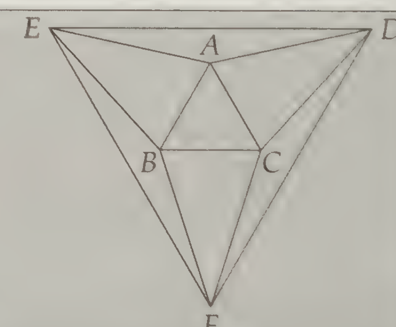
Problem 3.19: Find $\angle PBD$.



Problem 3.20: In the diagram, O is the center of the circle, $\overline{AB} \parallel \overline{CD}$, $\angle ABO = 24^\circ$, and $\angle OBC = \angle OCD$. Find $\angle BOC$.

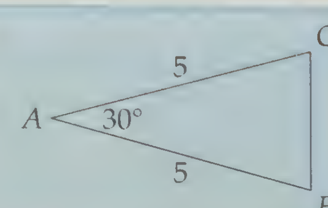


Problem 3.21: In the diagram at right, $AB = BC = AC$ and $AE = EB = BF = FC = CD = DA$ such that $\triangle ABC$ is completely inside $\triangle DEF$. Prove that $DE = EF = DF$.

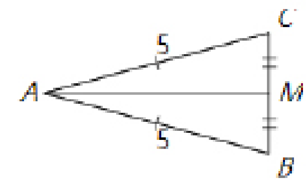


In this section, we investigate triangles that have two or three sides equal in length.

Problem 3.15: In the diagram, $AB = AC = 5$ and $\angle CAB = 30^\circ$. Find $\angle ACB$.



Solution for Problem 3.15: We're a little thin on information. We have a side equality that seems important. But we only have one triangle, so there's no obvious way to use the side equality. We might look for congruent triangles, but we'll have to somehow make two triangles. Splitting $\triangle ABC$ by connecting A to the midpoint, M, of $\overline{BC}$ gives us our congruent triangles.



Since $AB = AC$, $BM = CM$, and $AM = AM$, we have $\triangle AMB \cong \triangle AMC$ by SSS. This tells us that $\angle B = \angle C$, which is just enough to finish the problem. Since the angles of $\triangle ABC$ add up to $180°$, we have

$$180^\circ = \angle A + \angle B + \angle C = 30^\circ + 2(\angle C).$$

Solving for $\angle C$, we have $\angle C = (180^\circ - 30^\circ)/2 = 75^\circ$. □

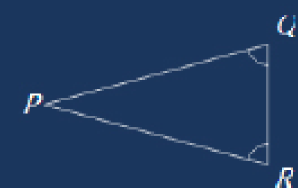
A triangle in which two sides are equal is called an **isosceles triangle**. The equal sides are sometimes called the **legs** of the triangle, and the other side the **base**.

Our general approach in Problem 3.15 can be used to show that if two sides of a triangle are equal, then the angles opposite those sides are equal. These two equal angles are often called the **base angles** of the triangle, and the other angle the **vertex angle**. As we saw in the last problem, if $\triangle ABC$ is isosceles with $\angle B = \angle C$, we have $\angle B = \angle C = (180^\circ - \angle A)/2$.

We might now wonder if this runs the other way: do equal angles imply equal sides?

Problem 3.16 t V

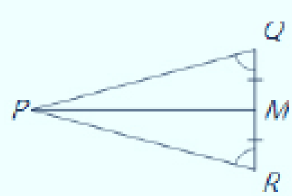
Prove that if $\angle PQR = \angle PRQ$, then $PR = PQ$.



Solution for Problem 3.16: What's wrong with this solution?

Bogus Solution: y

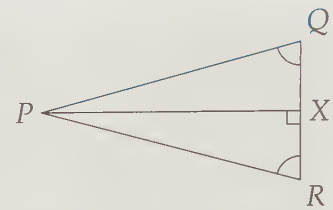
We proceed as we did in Problem 3.15, by connecting P to M, the midpoint of $\overline{QR}$, as shown below. Since $MR = MQ$, $\angle R = \angle Q$, and $PM = PM$, we have $\triangle PMR \cong \triangle PMQ$. Therefore, $PR = PQ$.



This failed solution uses 'SSA' Congruence, which we have shown in Section 3.5 doesn't work! We have to construct our triangles in a way that lets us use a valid congruence theorem.

Extra! *The essence of mathematics is not to make simple things complicated, but to make complicated things simple.*

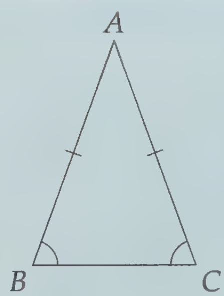
–S. Gudder



Instead of picking the point on $\overline{QR}$ that cuts it into two equal lengths, we make equal angles by connecting P to point X on $\overline{QR}$ such that $\overline{PX} \perp \overline{QR}$. Now we have $\triangle PXR \cong \triangle PXQ$ by AAS. Therefore, $PQ = PR$, as desired. (Note also that $QX = XR$, so that X is the same as point M in our Bogus Solution. This doesn't make that Bogus Solution any more correct, though!) □

Putting our last two problems together gives us a pair of powerful tools.

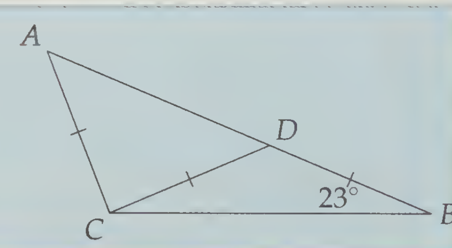
Important:



1. If $AB = AC$ in $\triangle ABC$, then $\angle B = \angle C$.
2. If $\angle B = \angle C$ in $\triangle ABC$, then $AB = AC$

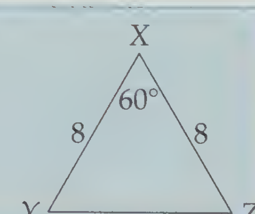
We'll try these tools on a couple more problems, and discover another important type of triangle.

Problem 3.17: In the diagram, $AC = CD = DB$, and $\angle B = 23^\circ$. Find $\angle A$.



Solution for Problem 3.17: Since $DC = DB$, we have $\angle DCB = \angle B = 23^\circ$. We could find $\angle CDB$ to get $\angle ADC$, or we can note that $\angle ADC$ is an exterior angle of $\triangle BCD$, so $\angle ADC = \angle DCB + \angle B = 46^\circ$. Since $AC = DC$, we have $\angle A = \angle ADC = 46^\circ$. □

Problem 3.18: In the diagram, $XY = XZ = 8$ and $\angle X = 60^\circ$. Find YZ.



Solution for Problem 3.18: We can't do much about YZ right away, but we can find $\angle Y$ and $\angle Z$. Since $\angle X = 60^\circ$, $\angle Y + \angle Z = 180^\circ - 60^\circ = 120^\circ$. Since $XY = XZ$, we have $\angle Y = \angle Z$, so $\angle Y = \angle Z = 60^\circ$. Hence, all the angles of $\triangle XYZ$ are equal. Specifically, $\angle X = \angle Y$ means that $YZ = XZ$, so $YZ = 8$. □

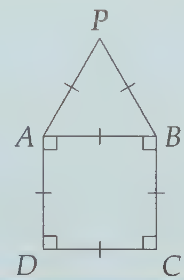
We call the triangle in Problem 3.18 an **equilateral triangle** because all of its sides are equal. As Exercises, you will use an approach a lot like the one above to show:

Important: If all three angles of a triangle are equal, then so are all three sides. Conversely, if all three sides are equal, then all three angles are $60°$.

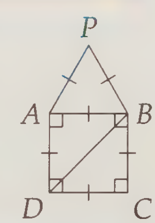
Therefore, in order to prove a triangle is equilateral, we can either prove all the sides are equal, or prove that all the angles are equal.

Let's put our isosceles and equilateral triangles to work.

Problem 3.19: Find $\angle PBD$.



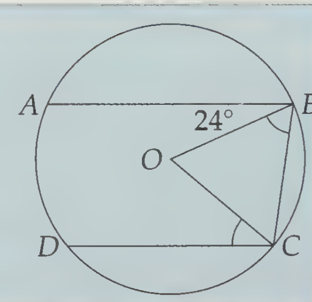
Solution for Problem 3.19: We start by drawing $\overline{BD}$ so we can see the angle we are after. Since $\triangle PAB$ is equilateral, $\angle PBA = 60°$. $\triangle BAD$ is isosceles, since $AB = AD$. Since $\angle BAD = 90°$, the other two angles of $\triangle BAD$ equal $(180° - 90°)/2 = 45°$. Therefore, we have



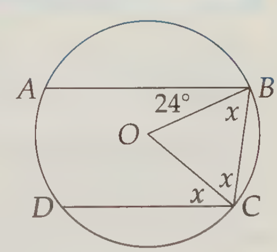
$$\angle PBD = \angle PBA + \angle ABD = 105°.$$

□

Problem 3.20: In the diagram, O is the center of the circle, $\overline{AB} \parallel \overline{CD}$, $\angle ABO = 24°$, and $\angle OBC = \angle OCD$. Find $\angle BOC$.



Solution for Problem 3.20: We have parallel lines and an isosceles triangle ($OB = OC$ because they are radii of the same circle). So, we can do some angle-chasing. We let $\angle OBC = x$, so that we also have $\angle OCD = x$ from the given information and $\angle OCB = x$ from isosceles $\triangle OCB$. Since $\overline{AB} \parallel \overline{CD}$, we have $\angle ABC + \angle BCD = 180°$. Therefore,



$$\angle ABO + \angle OBC + \angle BCO + \angle OCD = 180°.$$

Substitution gives $24° + 3x = 180°$, so $x = 52°$. Therefore, $\angle BOC = 180° - 2x = 76°$. □

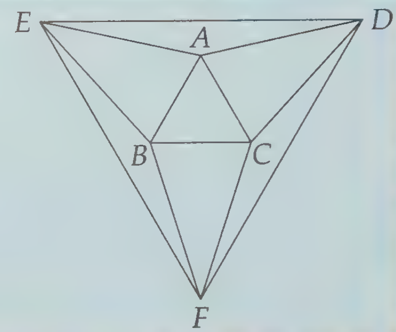
Extra! *Mathematics is the gate and key to the sciences.*

–Roger Bacon

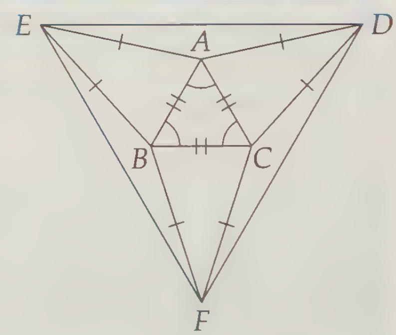
Concept: Assigning a variable to an angle will often help you with your angle-chasing. If you're stuck on an angle problem, assign one of the angle measures a variable and find other angles in terms of that variable. Hopefully, you'll eventually be able to build an equation you can use to solve for the variable.

We'll end this chapter with a proof involving equilateral and isosceles triangles.

Problem 3.21: In the diagram at right, $\triangle ABC$ is equilateral and $AE = EB = BF = CF = AD = CD$ such that $\triangle ABC$ is completely inside $\triangle DEF$. Prove that $\triangle DEF$ is equilateral.



Solution for Problem 3.21: In order to prove that $\triangle DEF$ is equilateral, we must show that either all its sides have the same length, or all its angles have the same measure. We start by marking all the given information. Our most useful tool in showing side lengths are equal is congruent triangles. Therefore, we look for triangles with $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$ as corresponding sides. The triangles that stand out are $\triangle DEA$, $\triangle EFB$, and $\triangle FDC$. If we can show these are congruent, we'll have the desired $DE = EF = FD$.



In order to show that $\triangle DEA \cong \triangle EFB \cong \triangle FDC$, we need only show that $\angle DAE = \angle EBF = \angle FCD$, since we already know that the sides adjacent to these angles in our three triangles are all equal. To learn more about these angles, we look at the other angles at A, B, and C.

We already have $\angle CAB = \angle ABC = \angle BCA$. Furthermore, isosceles triangles $\triangle EAB$, $\triangle FBC$, and $\triangle DCA$ are all congruent by SSS, so their base angles are all equal. Now that we know three of the angles with vertex A equal the corresponding angles with vertex B, we can show $\angle DAE = \angle EBF$:

$$\angle DAE = 360^\circ - \angle EAB - \angle BAC - \angle DAC = 360^\circ - \angle FBC - \angle CBA - \angle EBA = \angle EBF.$$

Similarly, we have $\angle DAE = \angle FCD$. Therefore, by SAS we have $\triangle DEA \cong \triangle EFB \cong \triangle FDC$, which gives us the desired $DE = DF = EF$. So, $\triangle DEF$ is equilateral.

Try finding an alternate solution by showing that all the angles of $\triangle DEF$ are equal. □

We typically prove a triangle equilateral by either proving all its side lengths are equal, or all its angles are equal. This often involves finding congruent triangles.

Concept: Mark the information you have in a problem on your diagram, particularly equal sides and equal angles. This will make congruent triangles easier to find.

We've explored isosceles and equilateral triangles, but we're not finished naming triangles yet! We have a name for two sides equal, and one for all three sides equal. If no two sides of a triangle are equal, the triangle is **scalene**.

Exercises

3.6.1 In $\triangle PQR$, $PQ = PR$ and $\angle P = 43°$. Find $\angle Q$.

3.6.2 Prove that if $AB = AC$ in $\triangle ABC$, then $\angle ABC = \angle ACB$. (Note: You cannot simply state that the triangle is isosceles, so the base angles are equal. You are asked here to prove this fact.)

3.6.3

(a) Prove that if the three sides of a triangle are equal in length, then all three angles of the triangle have measure 60°.

(b) Prove that if the three angles of a triangle are equal, then all three sides of the triangle have the same length.

Note that for this problem, you are not allowed to state that the triangle is equilateral as your proof. You must prove the facts about equilateral triangles that you learned in the text.

3.6.4 Two angles of an equilateral triangle have measures $3x + 27°$ and $2y - 4°$. Find $x + y$.

3.6.5 Point O is the center of the circle in the diagram on the left below. Find $\angle AOB$ if $\angle OAB = 70°$. **Hints:** 79

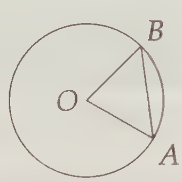


Figure 3.3: Diagram for Problem 3.6.5

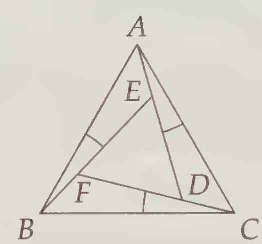
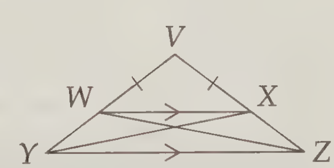


Figure 3.4: Diagram for Problem 3.6.6

3.6.6 Triangle ABC is equilateral in the diagram on the right above. Points D, E, and F are in triangle ABC such that $\angle CAD = \angle ABE = \angle BCF$. D lies on $\overline{CF}$, E lies on $\overline{AD}$, and F lies on $\overline{BE}$. Prove that triangle DEF is equilateral. **Hints:** 192, 574

3.6.7 In the diagram at right, $VW = VX$ and $\overline{WX} \parallel \overline{YZ}$.



(a) Prove that $WY = XZ$. **Hints:** 178, 518

(b) Prove that $YX = WZ$. **Hints:** 315

Extra! *I am interested in mathematics only as a creative art.*

–G. H. Hardy

3.7 Construction: Equilateral Triangle and Perpendicular Bisector

As a reminder, we once more repeat the rules of the game for constructions. The tools of classical construction problems are a compass and a straightedge. Notice that we don't say 'ruler.' You don't get to use your straightedge to measure – you can only draw lines. Similarly, you aren't allowed to use your protractor to measure or create angles.

The only operations you can perform with your compass and straightedge are:

1. Given a point, you can draw any line through the point.
2. Given two points, you can draw the line that passes through them both.
3. Given a point, you can draw any circle centered at that point.
4. Given a point and a segment, you can draw the circle with its center at that point and with radius equal in length to the length of the segment.
5. Given two points, you can draw the circle through one point such that the other point is the center of the circle.

Problems

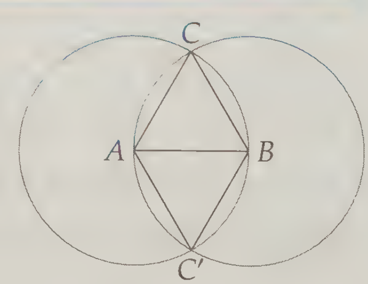
Problem 3.22: Construct an equilateral triangle with a compass and straightedge.

Problem 3.23: Draw a segment $\overline{AB}$. We call the line that is perpendicular to $\overline{AB}$ and passes through the midpoint of $\overline{AB}$ the **perpendicular bisector** of $\overline{AB}$. Construct the perpendicular bisector of $\overline{AB}$.

If you used your protractor to make a 60° angle to construct an equilateral triangle, go back to the beginning of this section and **READ THE RULES**!

Problem 3.22: Construct an equilateral triangle.

Solution for Problem 3.22: We start off by drawing a segment $\overline{AB}$. We want to find point C such that $AC = BC = AB$. Now we have a situation a lot like our SSS 'proof' of Problem 3.2. Since AC must equal AB, we know that C must be on the circle with center A and radius AB. We can construct this circle with our compass. Similarly, C is on the circle with center B and radius AB. Drawing this circle as well gives us the diagram to the right. Our two circles meet at C and C'. Both triangles ABC and ABC' are equilateral.

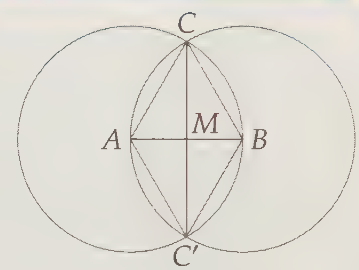


To prove that this construction works, we only have to prove that $AC = AB = BC$. Since each of these segments is a radius of a circle that is defined to have radius AB, clearly all three segments are equal. □

Important: Construction problems are two-part problems. Find the construction, then prove that it works. The second part is usually pretty easy after you've found the construction, but don't forget to do it!

You'll see that in this book we often leave the proof part for the reader. Sometimes we'll ask you to supply the proof that our construction works as an Exercise.

Problem 3.23: Given a segment $\overline{AB}$, construct a line through the midpoint of $\overline{AB}$ that is perpendicular to $\overline{AB}$. We call this line the **perpendicular bisector** of $\overline{AB}$.



Solution for Problem 3.23: In our last construction, we found two points that appear to be directly above and directly below the midpoint of $\overline{AB}$. Perhaps connecting these points will give us the line we want. We start with the same construction we used to make an equilateral triangle. Then we draw $\overline{CC'}$, which meets $\overline{AB}$ at M as shown. By SSS, $\triangle CAC' \cong \triangle CBC'$, so $\angle ACC' = \angle BCC'$. Therefore, $\angle ACM = \angle BCM$. Since we also have $AC = BC$ and $\angle CAM = \angle CBM$ as we found in the previous problem, we have $\triangle ACM \cong \triangle BCM$ by ASA. So, $AM = BM$ and $\angle CMA = \angle CMB$. Since $\angle CMA + \angle CMB = 180^\circ$, we must have $\angle CMA = \angle CMB = 90^\circ$. Therefore, $\overleftrightarrow{CC'}$ is the perpendicular bisector of $\overline{AB}$. □

Concept: Always be thinking about what you already know how to do when trying to do something new!

Exercises

3.7.1 Given a segment $\overline{AB}$, divide the segment into 4 equal pieces with a straightedge and compass.

3.7.2 Is it necessary when constructing the perpendicular bisector of $\overline{AB}$ to use circles with radius AB? Suppose we instead draw two intersecting circles with centers A and B and the same radius. Is the segment connecting the two points where these circles meet the perpendicular bisector of $\overline{AB}$?

3.7.3 Construct a 90° angle. **Hints:** 94

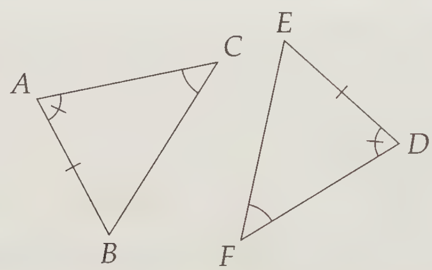
3.7.4 Construct a 30° angle. **Hints:** 504

3.8 Summary

Definition: Two figures are **congruent** if they are exactly the same – in other words, we can slide, spin, and/or flip one figure so that it is exactly on top of the other figure.

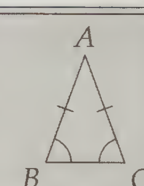
In this chapter we investigated the following theorems for proving that two triangles are congruent:

- **SSS Congruence**: If the sides of one triangle equal the sides of another triangle, then the triangles are congruent. (Section 3.2)
- **SAS Congruence**: If two sides of one triangle and the angle between them are equal to those of another triangle, then the two triangles are congruent. (Section 3.3)
- **ASA Congruence**: If two angles of one triangle and the side between them are equal to those of another triangle, then the two triangles are congruent. (Section 3.4)
- **AAS Congruence**: If two angles and a side of one triangle equal the corresponding angles and side in another triangle as shown below, then the triangles are congruent. (Section 3.4)



Important: Once we determine that two triangles are congruent, we can apply whatever we know about the sides and angles of one triangle to the other triangle. This obvious principle often goes by the fancy name '**Corresponding Parts of Congruent Triangles are Congruent**,' or **CPCTC**.

Definitions: A triangle in which two sides are equal is an **isosceles** triangle. The equal sides are the **legs** of the triangle and the other side is the **base**. The angle between the two equal sides is often called the **vertex angle** of the triangle, and the other two angles are the **base angles**, which are equal to each other.



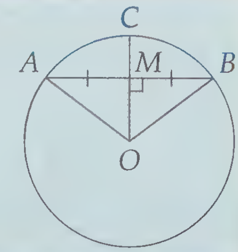
Important:
1. If $AB = AC$ in $\triangle ABC$, then $\angle B = \angle C$.
2. If $\angle B = \angle C$ in $\triangle ABC$, then $AB = AC$

Definition: A triangle in which all three sides are equal is an **equilateral triangle**. All three angles of an equilateral triangle are 60°.

Important: If all three angles of a triangle are equal, then so are all three sides. Conversely, if all three sides of a triangle are equal, then all its angles equal 60°.

Throughout the book we will use congruent triangles to prove many important results. One of the first is:

Important: If a radius of a circle bisects a chord of the circle that is not a diameter, then the radius must be perpendicular to the chord.



Problem Solving Strategies

Concepts:

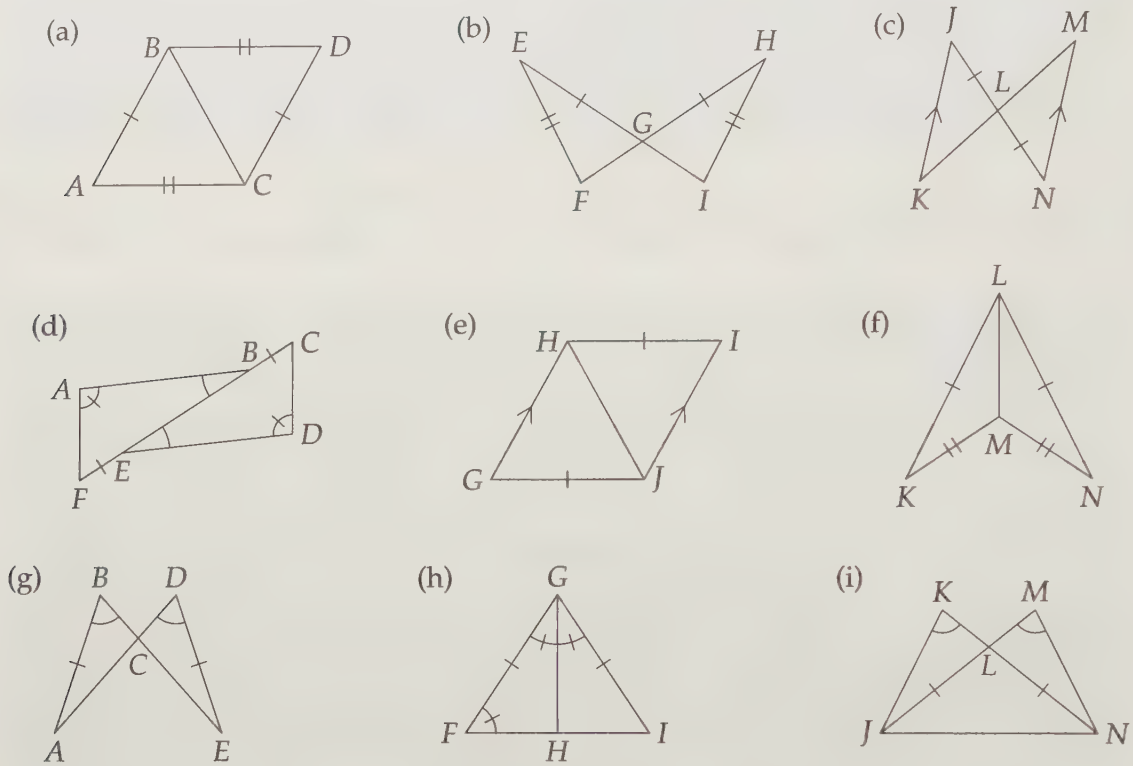
- In more complicated geometry problems, mark side and angle equalities as you find them (particularly when you find non-obvious ones!)
- Dividing isosceles triangles in half by drawing a segment from the vertex between the equal sides to the midpoint of the base can be very effective.
- If you're stuck on an angle problem, assign one of the angle measures a variable and find other angles in terms of that variable. Hopefully, you'll eventually be able to build an equation you can use to solve for the variable.
- Mark the information you have in a problem on your diagram, particularly equal sides and equal angles. This will make congruent triangles particularly easy to find.
- Always be thinking about what you already know how to do when trying something new!

Things To Watch Out For!

WARNING!! SSA (Side-Side-Angle) is **not** a valid congruence theorem. If two sides of one triangle are equal to two sides of another, and the two triangles have equal corresponding angles *that are not the angles between the equal corresponding sides*, then the two triangles are **not necessarily congruent**!

REVIEW PROBLEMS

3.24 In each of the diagrams below, name all pairs of congruent triangles you can identify (without drawing more segments or naming more points). Write the relevant triangle congruences and explain why the triangles are congruent. If there are not any pairs of triangles in a given diagram that must be congruent, state so.



3.25 Find the vertex angle of an isosceles triangle if one base angle is 6° less than half the vertex angle.

3.26 Find x in the diagram shown at left below.

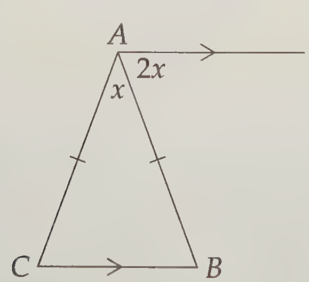


Figure 3.5: Diagram for Problem 3.26

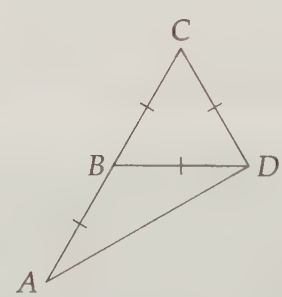


Figure 3.6: Diagram for Problem 3.27

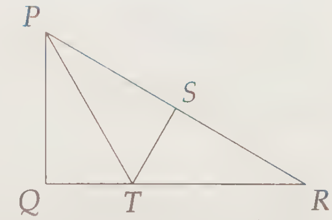
3.27 Find $\angle A$ in the diagram at right above.

3.28 Segments $\overline{AB}$ and $\overline{CD}$ meet at point P such that $AP = CP$ and $BP = DP$.

(a) Is it always true that $\triangle APD \cong \triangle CPB$?

(b) Is it always true that $\overline{AD} \parallel \overline{BC}$? **Hints:** 547

3.29 Let PQR be a triangle as shown at right, and let S and T be points on sides $\overline{PR}$ and $\overline{QR}$, respectively. Triangles PTS, PTQ, and RTS are congruent.



(a) Show that $\angle PST = 90°$.

(b) Find the measures of the angles of the triangle PQR.

3.30 In triangle NOK at left below, $KO = KN$. Let I be the point on $\overline{KN}$ such that $OI = ON$. Let A be the point on $\overline{NO}$ such that $IA = IN$.

(a) Find three angles in the diagram equal in measure to $\angle N$.

(b) Prove that $\overline{IA} \parallel \overline{KO}$.

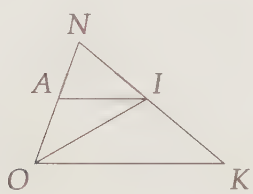


Figure 3.7: Diagram for Problem 3.30

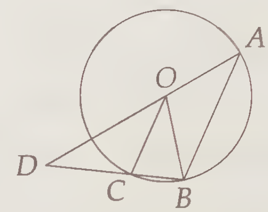


Figure 3.8: Diagram for Problem 3.31

3.31 In the diagram at right above, O is the center of the circle, $OA = CD$, and $\angle AOB = \angle OCD$.

(a) Show that $\triangle BOA \cong \triangle OCD$.

(b) Prove that $\overline{OC} \parallel \overline{AB}$.

(c) Show that $\angle OCB = 2\angle OBA$.

(d) Find $\angle OBA$.

3.32 In the diagram at left below, $AD = BD = AE$ and $DE = EC$. Prove that $AC = BE$. **Hints:** 71

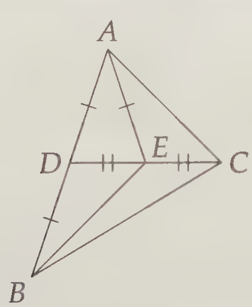


Figure 3.9: Diagram for Problem 3.32

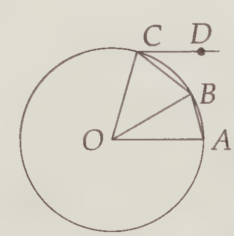


Figure 3.10: Diagram for Problem 3.33

3.33 In the figure at right above, $\angle BOC = 42°$, O is the center of the circle, $\overline{CD} \parallel \overline{OA}$, and $\angle BAO = 2\angle BCD$.

(a) Find $\angle CBO$.

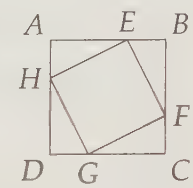
(b) Show that $\angle BOA = 180° - 4\angle BCD$.

(c) Find $\angle BOA$. **Hints:** 145

3.34 $\overline{AB}$ and $\overline{XY}$ are diameters of $\odot O$. Prove that $AX = BY$.

3.35 In triangle ABC, let M be the midpoint of $\overline{BC}$. Prove that if $MA = MB = MC$, then $\angle BAC = 90^\circ$. **Hints:** 546

3.36 All four angles of a square are 90° and all four sides have the same length. Square $EFGH$ is inscribed in square $ABCD$ as shown at right, meaning that each vertex of $EFGH$ is on a side of $ABCD$.



(a) Show that $\angle AHE = \angle BEF$.

(b) Prove that triangles AEH, BFE, CGF, and DHG are congruent.

3.37 In the figure at left below, $\triangle ABC$ is equilateral, $AE = DC = BF$, and $EB = CF = AD$, such that $\triangle ABC$ is completely inside $\triangle DEF$.

(a) Prove that $\triangle DEF$ is equilateral. **Hints:** 73

(b) Would $\triangle DEF$ still be equilateral if $\triangle ABC$ were not fully inside $\triangle DEF$?

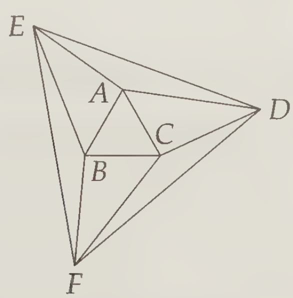


Figure 3.11: Diagram for Problem 3.37

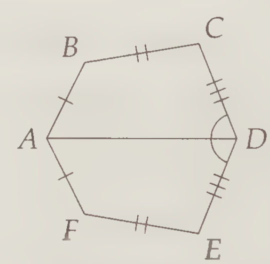
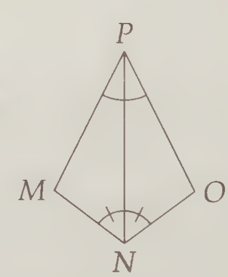


Figure 3.12: Diagram for Problem 3.38

3.38 In $ABCDEF$ at right above, $AB = FA$, $BC = EF$, $CD = DE$, and $\angle CDA = \angle ADE$. Prove that $\angle ABC = \angle EFA$. **Hints:** 33, 500

3.39 $\overline{NP}$ in the diagram bisects both $\angle MNO$ and $\angle MPO$. Which of the following statements must be true? For each statement that must be true, prove that it must be true. For each statement that may not be true, explain why it may not be true.



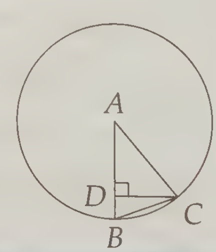
(a) $NO = MN$.

(b) $NO = OP$.

(c) $\overline{MP} \parallel \overline{NO}$.

(d)★ $\overline{MO} \perp \overline{NP}$.

3.40 In the diagram at right, A is the center of the circle and $\overline{CD} \perp \overline{AB}$. We are given that $\angle ACD = 2\angle DCB$.



(a) Let $\angle DCB = x$. Find expressions for $\angle ABC$ and $\angle BAC$ in terms of x.

(b) Find $\angle ABC$.

Challenge Problems

3.41 The measures of two angles of an isosceles triangle are $3x + 4°$ and $x + 17°$. Find all possible values of x. **Hints:** 148

3.42 Triangle ABC is equilateral in the diagram at left below, and $ABDE$, $BCFG$, and $CAHI$ are squares. Prove that triangle DFH is equilateral. **Hints:** 274, 534

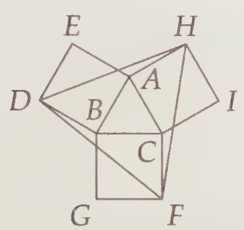


Figure 3.13: Diagram for Problem 3.42

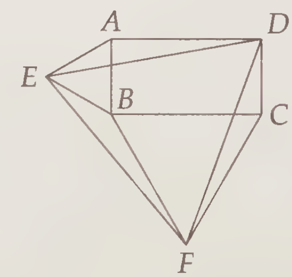


Figure 3.14: Diagram for Problem 3.43

3.43 Let $ABCD$ be a rectangle, meaning that all four of its angles are right angles. Construct equilateral triangles ABE and BCF externally on sides $\overline{AB}$ and $\overline{BC}$ as shown at right above. Prove that triangle DEF is equilateral. **Hints:** 88, 482, 582

3.44 $\triangle ABC$ has a right angle at $\angle C$. Points D and E are on $\overline{AB}$ as shown at left below such that $AD = AC$ and $BE = BC$. Find $\angle DCE$. **Hints:** 401, 232

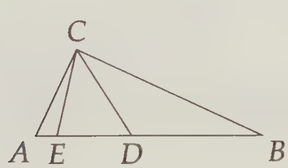


Figure 3.15: Diagram for Problem 3.44

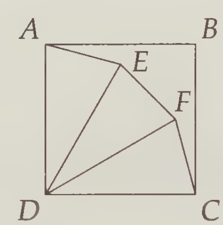
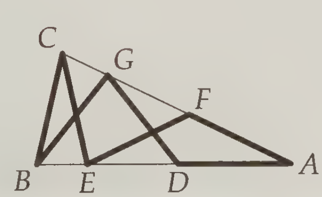
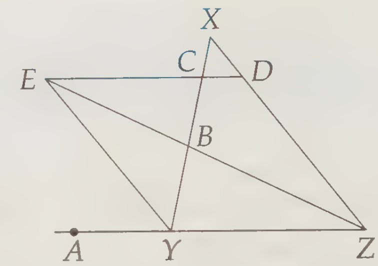


Figure 3.16: Diagram for Problem 3.45

3.45★ Let $ABCD$ be a square, and let E, F be points such that $DA = DE = DF = DC$ and $\angle ADE = \angle EDF = \angle FDC$. Prove that triangle BEF is equilateral. **Hints:** 280, 86, 575

3.46 In triangle ABC in the diagram below, D and E are points on side $\overline{AB}$, and F and G are points on side $\overline{AC}$, such that $AD = DG = GB = BC = CE = EF = FA$. Find $\angle BAC$. **Hints:** 436





3.47 In the diagram at right, $\angle DZE = \angle EZY$, $\angle CYE = \angle AYE$, and $\overline{ED} \parallel \overline{YZ}$. Prove $CD = DZ - CY$. **Hints:** 399

3.48 Describe under what conditions SSA Congruence 'works'. In other words, when is it true that if $AB = XY$ and $BC = YZ$ and $\angle BAC = \angle YXZ$, that we can immediately conclude that $\triangle ABC \cong \triangle XYZ$? (No proof is necessary; give a clear description of when SSA works and an intuitive explanation why it works in these cases.) **Hints:** 142

3.49 In the diagram at left below, $\angle XYZ = \angle AXZ = 90°$, $AX = XY$, and $\angle XZD = \angle YZD$. Prove that $\overline{ZD} \perp \overline{AY}$. **Hints:** 197, 325

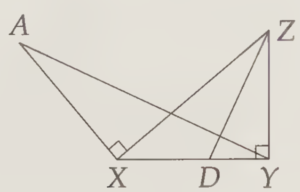


Figure 3.17: Diagram for Problem 3.49

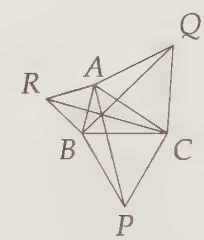
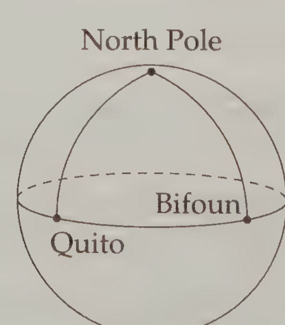


Figure 3.18: Diagram for Problem 3.50

3.50 Draw equilateral triangles BCP, CAQ, and ABR outside $\triangle ABC$ as shown in the diagram at right above. Prove that $AP = BQ = CR$. **Hints:** 455

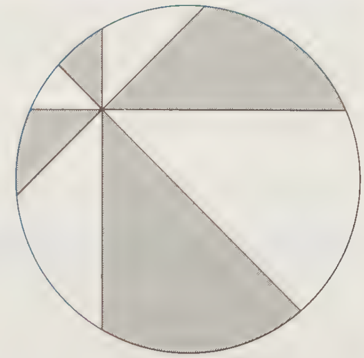
3.51★ There are two possible triangles with $AB = 13$, $BC = 10$, and $\angle A = 40°$. What is the sum of the two possible values of $\angle B$? *(Source: Mandelbrot)* **Hints:** 9, 357, 163

Extra! Take a look at the nearest globe (or imagine one). A woman at the North Pole, looking for the shortest path to Quito, Ecuador, just barely south of the equator, would fly straight down the 78° West line of longitude. Then, to fly to Bifoun, Gabon (which is a quarter of the way around the world to the East), she turns 90° to face East, and fly straight along the Equator. She then returns to the North Pole by turning 90° North and flying along another line of longitude, thus completing a triangular trip. She could begin repeating her trip by turning 90° to the left and then flying to Quito again.



At each stop, you can see that she must turn 90° to head towards the next stop. But that's a total of 270° in this triangle on the surface of the globe! How can that be?

Research **spherical geometry** to answer this mystery.



The Pizza Theorem

There is no transfer into another kind, like the transfer from length to area and from area to a solid. – Aristotle

CHAPTER 4 Perimeter and Area

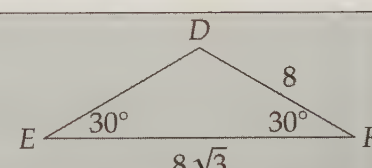
4.1 Perimeter

The **perimeter** of a closed figure is how far you travel if you walk along its boundary all the way around it once. So, the perimeter of a triangle is simply the sum of the lengths of the sides of the triangle.

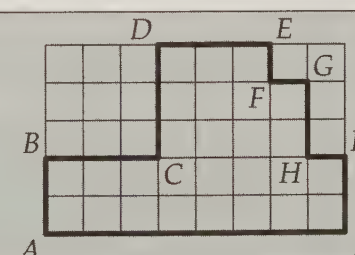
Problems

Problem 4.1: $\triangle ABC$ is equilateral. Find AB given that the perimeter of $\triangle ABC$ is 36.

Problem 4.2: Find the perimeter of $\triangle DEF$.



Problem 4.3: Given that each little square in the grid is a 1×1 square, find the perimeter of $ABCDEFGHIJ$.



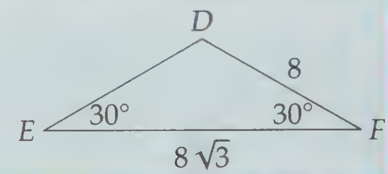
Problem 4.4: The length of each leg of an isosceles triangle is three times the length of the base of the triangle. The perimeter of the triangle is 91. What is the length of the base of the triangle?

Questions about perimeter are usually just questions about segment lengths. We'll try a few.

Problem 4.1: $\triangle ABC$ is equilateral. Find AB given that the perimeter of $\triangle ABC$ is 36.

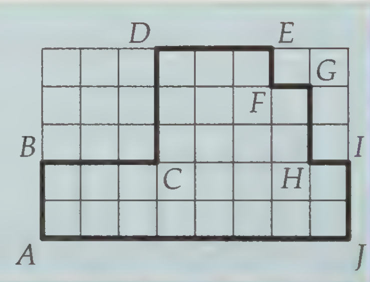
Solution for Problem 4.1: Since $\triangle ABC$ is equilateral, all of its sides are equal. Let each have length s. Then the perimeter is $3s$. Therefore, we have $3s = 36$, so $s = 12$. □

Problem 4.2: Find the perimeter of $\triangle DEF$.



Solution for Problem 4.2: Since $\angle E = \angle F$, we have $DE = DF = 8$. Therefore, our perimeter is $DE + DF + EF = 16 + 8\sqrt{3}$. □

Problem 4.3: Given that each little square in the grid is 1×1 square, find the perimeter of $ABCDEFGHIJ$.



Solution for Problem 4.3: We could simply find the lengths of the 10 sides and add up all the numbers, but there's a slicker approach. All of our sides are either vertical or horizontal. The horizontal side on the bottom has length 8; therefore, the sum of the horizontal sides on top, which together cover the same horizontal distance without backtracking, must be 8. Similarly, $AB + CD = 5$, so the vertical sides on the left have total length 5. The ones on the right have the same total length, so they contribute 5 to the perimeter as well. So, our perimeter is $2(8 + 5) = 26$. □

Problem 4.4: The length of each leg of an isosceles triangle is three times the length of the base of the triangle. The perimeter of the triangle is 91. What is the length of the base of the triangle?

Solution for Problem 4.4: Let the base of the triangle have length x. Then the length of each leg is $3x$. Since the perimeter of the triangle is 91, we must have $x + 3x + 3x = 91$. Solving this equation for x, we find $x = 13$. The length of the base of the triangle is 13. □

Concept: The key to solving most geometric word problems is the same as for most non-geometric word problems: assign variables to unknown quantities and use the words to make equations you can solve.

Exercises

4.1.1 In the diagram at left below, $XY = 5\sqrt{2}$, $YZ = 10$, and $\angle Y = \angle Z = 45^\circ$. Find the perimeter of $\triangle XYZ$.

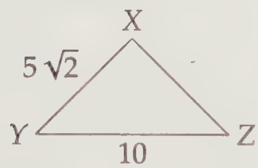


Figure 4.1: Diagram for Problem 4.1.1

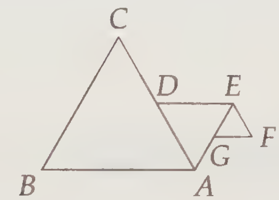


Figure 4.2: Diagram for Problem 4.1.2

4.1.2 Triangles ABC, ADE, and EFG at right above are all equilateral. Points D and G are midpoints of $\overline{AC}$ and $\overline{AE}$, respectively. If $AB = 4$, what is the perimeter of figure $ABCDEFG$? *(Source: AMC 8)*

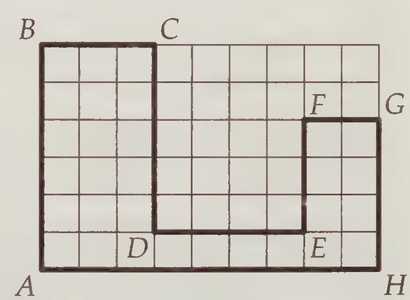
4.1.3

(a) Must two congruent triangles have the same perimeter? Why or why not?

(b) Must two triangles with the same perimeter be congruent? Why or why not?

(c)★ Must two triangles with the same perimeter be congruent if the triangles have one angle measure in common?

4.1.4 Each little square in the grid at right is a 1×1 square. Find the perimeter of $ABCDEFGH$.



4.1.5 A triangle with perimeter 45 has one side that is twice as long as the shortest side and another side that is 50% longer than the shortest side. Find the length of the shortest side of the triangle. **Hints:** 480

4.2 Area

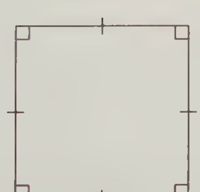


Figure 4.3: A Square

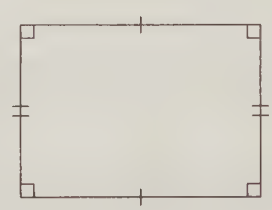
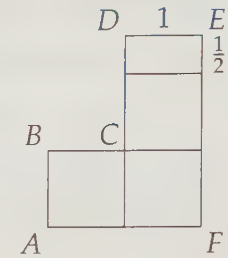


Figure 4.4: A Rectangle

A **square** is a figure that has four sides equal in length and four right angles, as shown in Figure 4.3. We call a four-sided figure with four right angles a **rectangle**, an example of which is shown in Figure 4.4. (A square is a special type of rectangle.) The opposite sides of a rectangle have equal length, as shown.

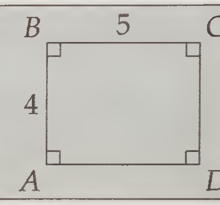
We'll learn more about squares and rectangles in Chapter 8.

Roughly speaking, the **area** of a figure is the total number of 1×1 squares needed to cover the figure, where we add fractional squares when we only need a piece of a square to do the job. For example, we need $3\frac{1}{2}$ squares to cover the region $ABCDEF$ at right, so we say the area of $ABCDEF$ is $3\frac{1}{2}$. We will often use brackets to denote area, like this: $[ABCDEF] = 3\frac{1}{2}$.



Problems

Problem 4.5: Consider rectangle $ABCD$, which has $AB = CD = 4$, $BC = AD = 5$, and right angles at A, B, C, and D. What is the area of $ABCD$?

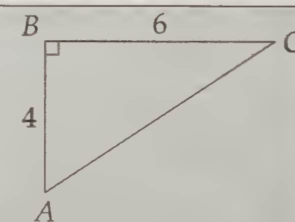


Problem 4.6: The length of one side of a rectangle is 4 less than 3 times an adjacent side. The perimeter of the rectangle is 64. Find the area of the rectangle.

Problem 4.7: A triangle that has a right angle as one of its angles is called a **right triangle**.

(a) Find the area of right triangle $\triangle ABC$, where $AB = 4$, $BC = 6$, and $\angle B = 90^\circ$.

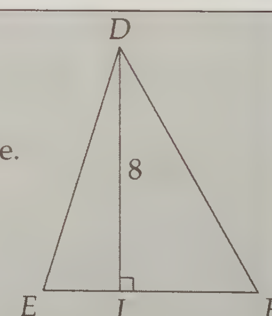
(b) Find a formula for the area of a right triangle given the lengths of its sides.



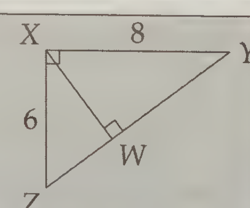
Problem 4.8:

(a) Find the area of $\triangle DEF$ given that J is on $\overline{EF}$, $DJ = 8$, $\overline{DJ} \perp \overline{EF}$, and $EF = 7$.

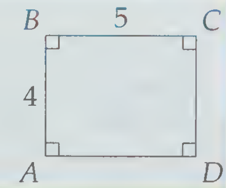
(b) Use the previous part to come up with a way to find the area of any triangle. Does your approach work for obtuse triangles?



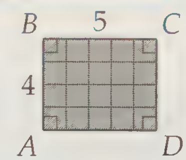
Problem 4.9: Given that $\angle ZXY = \angle XWY = 90^\circ$, $XY = 8$, $XZ = 6$, and $ZY = 10$, find XW.



Problem 4.5: Consider rectangle $ABCD$, which has $AB = CD = 4$, $BC = AD = 5$, and right angles at A, B, C, and D. What is the area of $ABCD$?



Solution for Problem 4.5: We can fill $ABCD$ with four rows of five 1×1 squares each, as shown to the right. So, the area is $(4)(5) = 20$. □



Finding the area of a rectangle by taking the product of consecutive sides works even when the sides aren't integers.

Important: A rectangle with two adjacent sides of lengths ℓ and w has area ℓw. Since a square is just a rectangle in which these lengths are the same, the area of a square equals the square of the length of one of its sides.

Let's apply our knowledge of rectangles to a problem.

Problem 4.6: The length of one side of a rectangle is 4 less than 3 times an adjacent side. The perimeter of the rectangle is 64. Find the area of the rectangle.

Solution for Problem 4.6: Let the lengths of the two sides be x and y. We are given that $x = 3y-4$. Since the opposite sides of a rectangle are equal, the perimeter of the rectangle is $2x + 2y = 2(3y - 4) + 2y = 8y - 8$. We are given that the perimeter is 64, so we have $8y - 8 = 64$. Therefore, $y = 9$ and $x = 3(9) - 4 = 23$, so the area of the rectangle is $9(23) = 207$. □

Now let's use our rectangle formula to find the area of a triangle. We'll start with a special kind of triangle.

Problem 4.7: A triangle that has a right angle as one of its angles is called a **right triangle**.

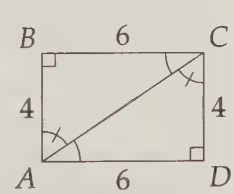
(a) Find the area of right triangle $\triangle ABC$, where $AB = 4$, $BC = 6$, and $\angle B = 90°$.

(b) Find a formula for the area of a right triangle given the lengths of its sides.

Solution for Problem 4.7:

(a) We don't know anything about finding the area of a triangle yet; all we know about are rectangles. However, we know that if we draw a diagonal of a rectangle, we cut the rectangle into two congruent right triangles. In this problem we have a triangle – perhaps we can make a rectangle by just adding another right triangle as shown below.

In the diagram, $\triangle ACD$ is a copy of $\triangle CAB$. Since $\angle BAC + \angle B + \angle BCA = 180°$ and $\angle B = 90°$, we have $\angle BAC + \angle BCA = 90°$. Since $\triangle CAD \cong \triangle ACB$, we have $\angle DAC = \angle BCA$. Therefore, $\angle BAC + \angle DAC = 90°$, so $\angle BAD$ is a right angle. Similarly, $\angle BCD$ is a right angle, so $ABCD$ is a rectangle.



Since $ABCD$ is a rectangle with sides 4 and 6, we know that $[ABCD] = 24$. Our two right triangles are congruent and the sum of their areas is $[ABCD]$. So, each right triangle has area $24/2 = 12$.

(b) Let a and b be the lengths of the two sides adjacent to the right angle of $\triangle ABC$. We call the sides adjacent to a right angle in a right triangle the **legs** of the right triangle. As shown in the previous part, we can build a rectangle with consecutive sides of lengths a and b out of two congruent copies of $\triangle ABC$. The rectangle then has area ab, so each of the two triangles has area $ab/2$. So, the area of any right triangle is half the product of the lengths of the legs of the triangle.

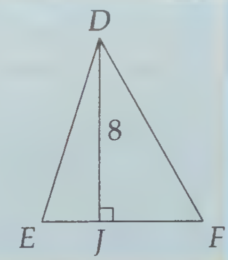
□

So that takes care of right triangles. But what if the triangle isn't a right triangle?

Problem 4.8:

(a) Find the area of $\triangle DEF$ given that J is on $\overline{EF}$, $DJ = 8$, $\overline{DJ} \perp \overline{EF}$, and $EF = 7$.

(b) Find a method to determine the area of any triangle.



Solution for Problem 4.8:

(a) We do know how to tackle right triangles, so we break $\triangle DEF$ into $\triangle DEJ$ and $\triangle DJF$:

$$[DEF] = [DEJ] + [DFJ] = \frac{(EJ)(DJ)}{2} + \frac{(JF)(DJ)}{2} = \frac{DJ}{2}(EJ + JF) = \frac{DJ}{2}(EF) = \left(\frac{8}{2}\right)(7) = 28.$$

(b) We call a perpendicular segment from a vertex of a triangle to the opposite side of the triangle an **altitude** of the triangle. Sometimes an altitude is also called a **height**. The terms 'altitude' and 'height' can also refer to the length of the altitude. The side to which the altitude is drawn is called the **base** that corresponds to the given altitude. For example, we say $\overline{DJ}$ is the altitude to base $\overline{EF}$ in $\triangle DEF$ above. (As with 'altitude', the term 'base' can also refer to the length of the base.) Point J gets a special name, too. The point at which an altitude meets the base to which it is drawn is sometimes called the **foot** of the altitude. Thus, J is the foot of the altitude from D to $\overline{EF}$.

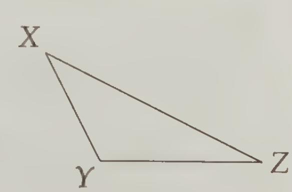
Our work so far strongly suggests that the area of a triangle is one-half the product of an altitude of a triangle and the length of the base to which the altitude is drawn. For example, in part (a), the area is one-half the product of EF and DJ, and $\overline{DJ}$ is the altitude from D to base $\overline{EF}$.

We have already shown that if our base is either leg of a right triangle, then the area is

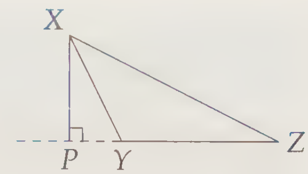
$$\frac{\text{base} \times \text{altitude}}{2}.$$

Our work on the first part of this problem can be used to show that this formula works whenever the two angles at the base are acute, as $\angle E$ and $\angle F$ are.

But what if one of the angles at the base is obtuse? In the diagram to the right, let $\overline{YZ}$ be the base. Here, we can't draw a line from X that is perpendicular to $\overline{YZ}$, so we extend side $\overline{YZ}$ and draw our altitude to the extension as shown on the next page. Now we need to check whether $[XYZ] = (XP)(YZ)/2$.



In our first part we added areas; here, we subtract:



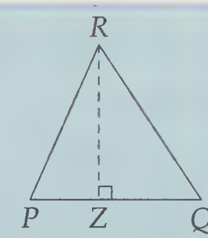
$$\begin{aligned}[XYZ] &= [XPZ] - [XPY] \\ &= \frac{(XP)(PZ)}{2} - \frac{(XP)(PY)}{2} \\ &= \frac{XP(PZ - PY)}{2} \\ &= \frac{(XP)(YZ)}{2}.\end{aligned}$$

We have to tweak our definition of 'altitude' a little, but our formula still works.

□

We can now summarize our findings about the area of a triangle.

Important: To find the area of a triangle, we select one side to be the base. The perpendicular segment from the vertex opposite the base to the base (extended if necessary) is the **altitude**. We then have:



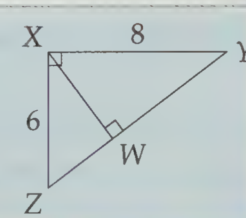
$$\text{Area} = \frac{\text{base} \times \text{altitude}}{2}.$$

For example, in the triangle shown, we have:

$$[PQR] = \frac{PQ \times RZ}{2}.$$

You'll get plenty of practice finding areas of triangles in the pages to come. But first, we see that unlike perimeter, which is rarely a useful problem solving tool, area can be used to solve problems that don't appear to have anything to do with area.

Problem 4.9: Given that $\angle ZXY = \angle XWY = 90^\circ$, $XY = 8$, $XZ = 6$, and $ZY = 10$, find XW.



Solution for Problem 4.9: This problem only has information about lengths and angles, and it doesn't even ask for area. However, all those perpendicular lines make us think about area. We can quickly see that

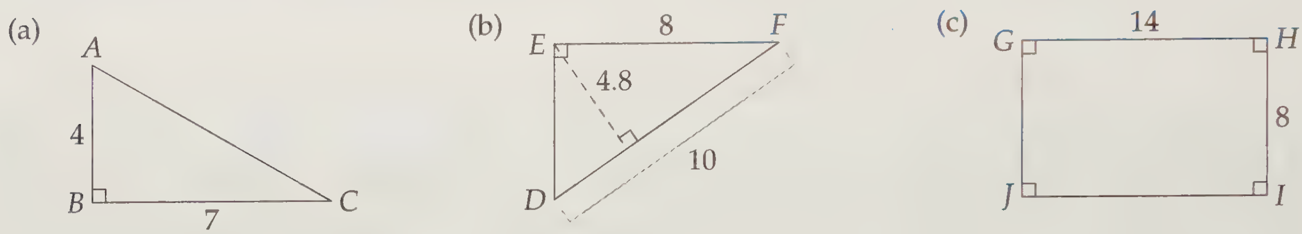
$$[XYZ] = \frac{(XY)(XZ)}{2} = 24.$$

$\overline{XW}$ is the altitude to side $\overline{YZ}$, so $[XYZ] = (XW)(YZ)/2$. Since $[XYZ] = 24$, we have the equation $24 = (XW)(10)/2 = 5XW$, so $XW = 24/5$. □

Sprinkled throughout this book will be many other uses of area as a problem solving tool.

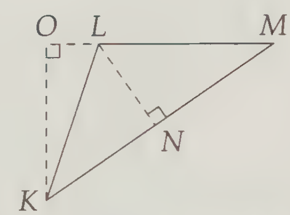
Exercises

4.2.1 Find the area of each figure shown below.



4.2.2 What is the area of square $ABCD$ if the perimeter of the square is 36?

4.2.3 In the diagram at right, $KO = 6$, $LM = 7$, and $KM = 12$.



(a) Find $[KLM]$.

(b) Find LN. **Hints:** 506

4.2.4 One gallon of paint covers 80 square feet of wall. How many gallons of paint do I have to buy if I am going to paint two rectangular walls that are each 24 feet long and 8 feet high?

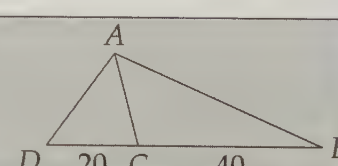
4.2.5 A square poster is replaced by a rectangular poster that is 2 inches wider and 2 inches shorter. What is the difference in the number of square inches between the area of the larger poster and the smaller poster? *(Source: MATHCOUNTS)* **Hints:** 102

4.2.6★ The perimeter of a square garden is 64 meters. The path surrounding the garden has uniform width and has an area of 228 square meters. How many meters of fencing are needed to surround the outer edge of the path? *(Source: MATHCOUNTS)* **Hints:** 494

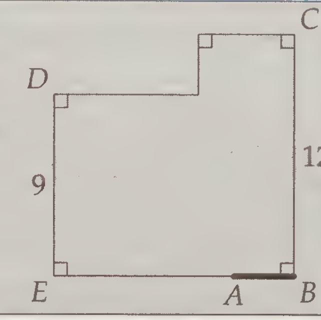
4.3 Same Base/Same Altitude

Problems

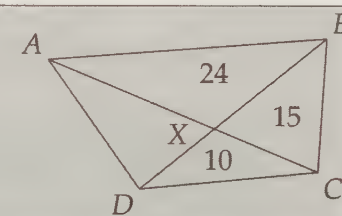
Problem 4.10: Given that $BC = 40$ and $CD = 20$ in the figure, what are $[ABC]/[ACD]$ and $[ABC]/[ABD]$?



Problem 4.11: Suba and Sam have been hired to paint a triangle with base $\overline{AB}$ on the weirdly shaped wall in the diagram. They are told to choose either C or D as the third vertex of the triangle. Suba thinks using D will make the triangle look cooler, but Sam thinks using C will result in a smaller triangle to paint. How can Suba convince Sam to choose D?

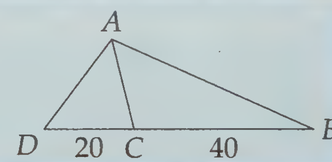


Problem 4.12: $\overline{AC}$ and $\overline{BD}$ meet at X as shown. Given $[ABX] = 24$, $[BCX] = 15$, and $[CDX] = 10$, find $[ADX]$.



In this section we examine a particularly useful triangle area concept.

Problem 4.10: Given that $BC = 40$ and $CD = 20$ in the figure, what are $[ABC]/[ACD]$ and $[ABC]/[ABD]$?



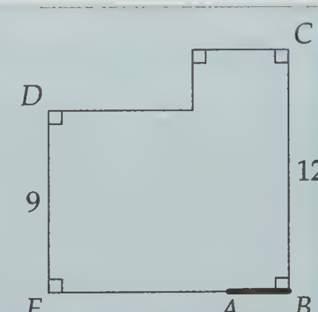
Solution for Problem 4.10: The altitude from A is the same for both $\triangle ACD$ and $\triangle ABC$. Let h be the length of this altitude. Hence, $[ABC] = (BC)(h)/2 = 20h$ and $[ACD] = (CD)(h)/2 = 10h$. Therefore, $[ABC]/[ACD] = 20h/10h = 2$. Notice that $[ABC]/[ACD] = (BC)/(CD)$.

Similarly, the altitude from A in each of $\triangle ABC$ and $\triangle ABD$ has length h. Therefore,

$$\frac{[ABC]}{[ABD]} = \frac{(BC)(h)/2}{(BD)(h)/2} = \frac{BC}{BD} = \frac{2}{3}.$$

A key step in our solution is noting that the three triangles have an altitude in common. Notice that in each case the ratio of the areas of the triangles equals the ratio of the lengths of the sides to which this common altitude is drawn. □

Problem 4.11: Suba and Sam have been hired to paint a triangle with base $\overline{AB}$ on the weirdly shaped wall in the diagram. They are told to choose either C or D as the third vertex of the triangle. Suba thinks using D will make the triangle look cooler, but Sam thinks using C will result in a smaller triangle to paint. How can Suba convince Sam to choose D?



Solution for Problem 4.11: To figure out which of $\triangle ABC$ or $\triangle ABD$ will use more paint, we must consider the areas of the two triangles. Since

$$[ABD] = \frac{(AB)(DE)}{2} \qquad \text{and} \qquad [ABC] = \frac{(AB)(BC)}{2},$$

we see that

$$\frac{[ABD]}{[ABC]} = \frac{(AB)(DE)/2}{(AB)(BC)/2} = \frac{DE}{BC} = \frac{3}{4}.$$

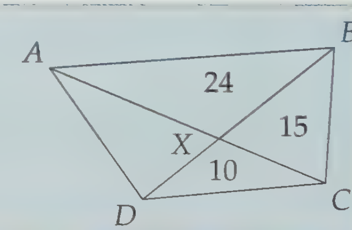
So, Suba should point out that $\triangle ABD$ will require less paint because its area is smaller than that of $\triangle ABC$.
□

The last two problems illustrate two useful problem solving principles:

Important:

1. If two triangles share an altitude, then the ratio of their areas is the ratio of the bases to which that altitude is drawn. This is particularly useful in problems in which two triangles have bases along the same line.
2. If two triangles share a base, then the ratio of their areas is the ratio of the altitudes to that base.

Problem 4.12: $\overline{AC}$ and $\overline{BD}$ meet at X as shown. Given $[ABX] = 24$, $[BCX] = 15$, and $[CDX] = 10$, find $[ADX]$.



Solution for Problem 4.12: Since $\triangle ABX$ and $\triangle CBX$ share an altitude from B, we have

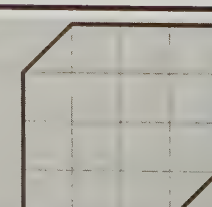
$$\frac{AX}{CX} = \frac{[ABX]}{[CBX]} = \frac{8}{5}.$$

Turning to triangles $\triangle ADX$ and $\triangle CDX$, we have

$$\frac{[ADX]}{[CDX]} = \frac{AX}{CX} = \frac{8}{5}.$$

Therefore, $[ADX] = (8/5)[CDX] = 16$. □

Extra! Can you cut the figure at right into two congruent pieces that can be arranged to form a rectangle?



Exercises

4.3.1 In the diagram at left below, $PT = 6$, $TR = 3$, and $QV = 4$.

(a) Find $[PQR]$, $[PTQ]$, and $[QTR]$.

(b) Find TR/PT and $[QTR]/[PTQ]$.

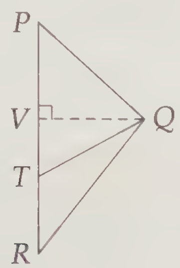


Figure 4.5: Diagram for Problem 4.3.1

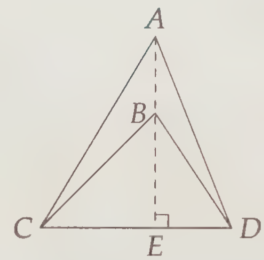


Figure 4.6: Diagram for Problem 4.3.2

4.3.2 In the diagram at right above, $AB = 2$ and $BE = 3$.

(a) Find $[BCD]/[ACD]$.

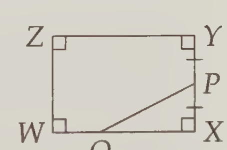
(b) Find $[ADBC]/[ADC]$. **Hints:** 255

(c) Find $[BCD]/[ACBD]$. **Hints:** 432

4.3.3 Euclid Apartment Building has a wall that is 30 feet long and 10 feet high. Jean is supposed to paint a triangle with one vertex at the top of the wall, and a base that runs 8 feet along the bottom of the wall. Jean wants to put the vertex at one corner and the base at the other side of the wall as shown on the left. George, the building owner, wants to save money on paint. He insists that the top vertex be right above the middle of the base, as shown on the right, so the triangle won't be so big. However, to show that he's a nice guy, he says that she can make the base 10 feet wide instead. Jean argues that George's design will use more paint than hers. Is she right?

4.3.4 By what factor is the area of triangle multiplied if the length of its base is doubled and the height is tripled? *(Source: MATHCOUNTS)*

4.3.5★ Find QX in the diagram at right given that $WX = 8$ and $[PQX] = [WXYZ]/6$.
Hints: 244, 390



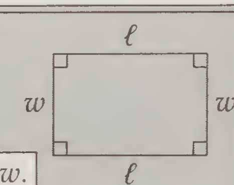
Extra! *For every complex problem there is an answer that is clear, simple, and wrong.*
–H. L. Mencken

4.4 Summary

Definition: The **perimeter** of a closed figure is how far you travel if you walk along its boundary all the way around it once.

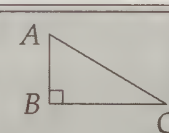
Definition: The **area** of a closed figure is the number of 1×1 squares (or pieces of squares) needed to exactly cover the figure. We sometimes use brackets to denote area, so that $[ABC]$ means the area of $\triangle ABC$.

Definition: A **rectangle** has four sides and four right angles, as shown. Furthermore, opposite sides of a rectangle equal each other in length.



Important: The area of a rectangle with consecutive sides of length ℓ and w is ℓw. Since a **square** is just a rectangle in which these lengths are the same, the area of a square equals the square of the length of one of its sides.

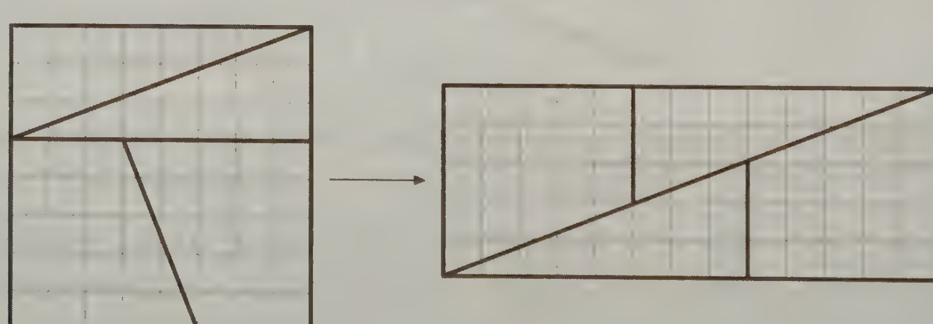
Definitions: A triangle that has a right angle as one of its angles is called a **right triangle**. The sides adjacent to the right angle of a triangle are called the **legs** of the triangle.



Important: The area of a right triangle is half the product of the legs of the triangle. For example, in the right triangle shown, we have

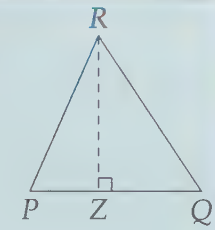
$$[ABC] = \frac{(AB)(BC)}{2}.$$

Extra! Tedrick, trying to trick me into believing that 8×8 is 65, showed me how he could cut an 8×8 square into 4 pieces that he could then reassemble into a 5×13 rectangle.



The 8×8 square has an area of 64, but the 5×13 rectangle has an area of 65. How is this possible?

Important: To find the area of a triangle, we select one side to be the base. The perpendicular segment from the vertex opposite the base to the base (extended if necessary) is the **altitude**. We then have:



$$\text{Area} = \frac{\text{base} \times \text{altitude}}{2}.$$

For example, in the triangle shown, we have:

$$[PQR] = \frac{PQ \times RZ}{2}.$$

Area can be a very powerful problem-solving tool. One particularly useful pair of principles is:

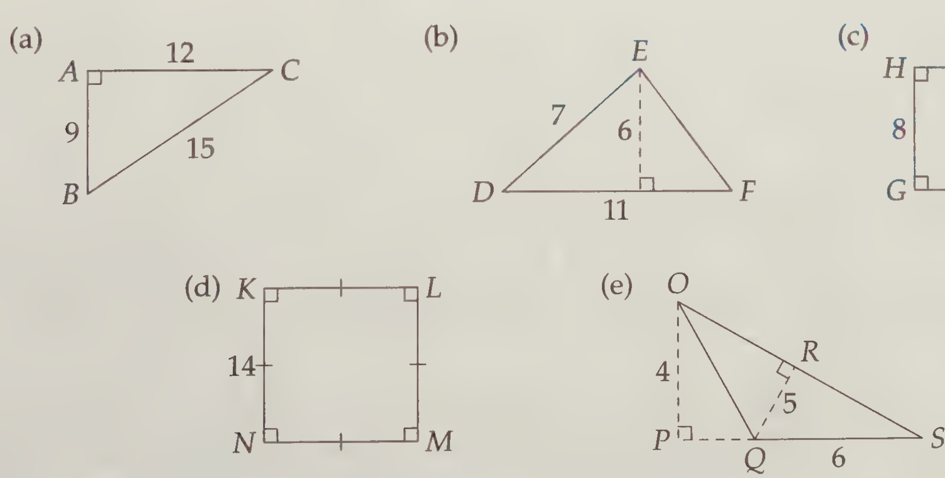
Important:

1. If two triangles share an altitude, the ratio of their areas is the ratio of the bases to which that altitude is drawn. This is particularly useful in problems in which two triangles have bases along the same line.

2. If two triangles share a base, then the ratio of their areas is the ratio of the altitudes to that base.

REVIEW PROBLEMS

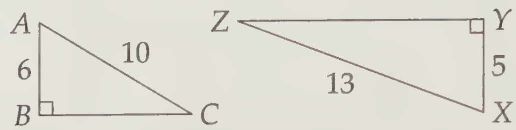
4.13 Find the area of each figure below.



4.14 A triangle has area 42 and one side of length 7. Find the length of the altitude to the side of length 7. Is it possible to find the perimeter of this triangle?

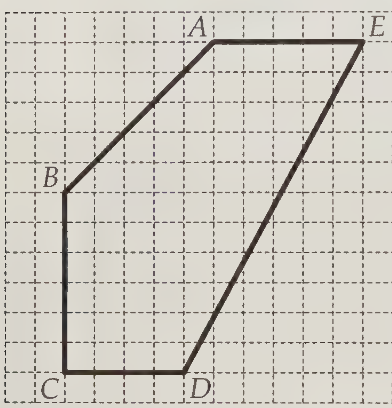
4.15 Find the perimeter of a square that has an area of 75.

4.16 Find the perimeters of $\triangle ABC$ and $\triangle XYZ$ shown below given that $[ABC] = 24$ and $[XYZ] = 30$.



4.17 The length of a given rectangle is 2 less than 4 times the width of the rectangle. Its perimeter is 51. What is its area?

4.18 I have a rectangular living room that is 24 feet long and 16 feet wide. I want to buy a carpet that costs \$2.50 per square foot, and I want to leave a 2 foot margin between the carpet and the wall all the way around the room. How much will the carpet cost?



4.19 What is the number of square units enclosed by the figure $ABCDE$?

4.20 A rectangle has a perimeter of 28 cm and an area of 48 cm^2. Find the dimensions of the rectangle.

4.21 If the height of a triangle is multiplied by 4, what must we do to the base of the triangle in order to leave the area unchanged?

4.22 The altitude from A to $\overline{BC}$ in $\triangle ABC$ has the same length as the altitude from B to $\overline{AC}$. Prove that $BC = AC$.

4.23 The area of rectangle $ABCD$ shown at left below is 36 and $DE = 2EC$.

(a) What is $[BCD]$?

(b) What is $[BED]$?

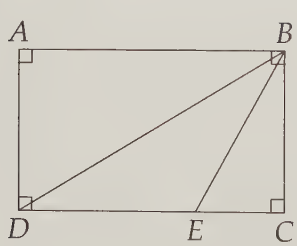


Figure 4.7: Diagram for Problem 4.23

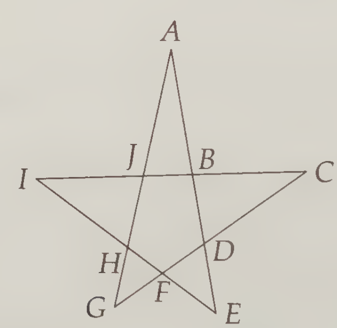


Figure 4.8: Diagram for Problem 4.24

4.24 Each of the outer triangles in the figure shown at right above has perimeter 23. The perimeter of $BDFHJ$ is 51. What is the perimeter of star $ABCDEFGHIJ$? **Hints:** 510

4.25 In the figure at left below, $AB = 12$ cm and $BC = AD = 8$ cm. $\overline{BC} \perp \overline{AB}$ and $\overline{DA} \perp \overline{AB}$. How many square centimeters are in the area of the shaded region? *(Source: MATHCOUNTS)*

Figure 4.9: Diagram for Problem 4.25

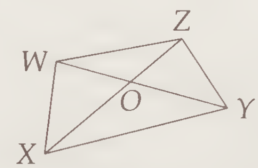
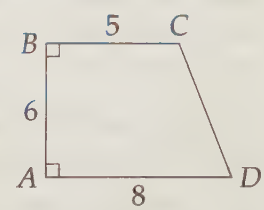


Figure 4.10: Diagram for Problem 4.26

4.26 In the figure at right above, $[WOZ] = 12$, $[ZOY] = 18$, and $[WXYZ] = 50$. Find $[WOX]$.

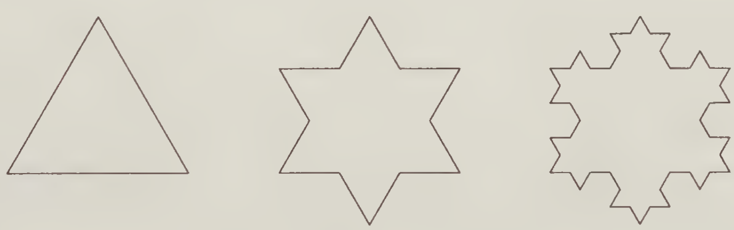
Challenge Problems

4.27 Find the area of $ABCD$ shown at right. **Hints:** 577



4.28 A gardener plans to build a fence to enclose a square garden plot. The perimeter of the plot is 96 feet, and he sets posts at the corners of the square. The posts along the sides are set 6 feet apart. How many posts will he use to fence the entire plot? *(Source: MATHCOUNTS)*

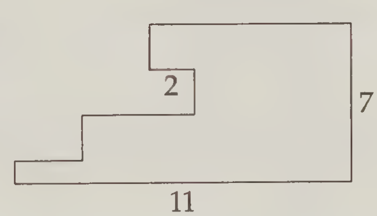
4.29 We begin with an equilateral triangle. We divide each side into three segments of equal length, and add an equilateral triangle to each side using the middle third as a base. We then repeat this, to get a third figure.



Given that the perimeter of the first figure is 12, what is the perimeter of the second figure? What is the perimeter of the third figure? **Hints:** 273, 141

4.30★ Suppose we continue the process described in Problem 4.29 forever. What is the perimeter of the resulting figure?

4.31 All sides of the building shown at right meet at right angles. If three of the sides measure 2 meters, 7 meters, and 11 meters as shown, then what is the perimeter of the building in meters? *(Source: Mandelbrot)*



4.32 In $\triangle ABC$ at left below, $CX = 2BX$ and $AY = 3BY$. Find $[BXY]$ given that $[ABC] = 144$. **Hints:** 285, 89

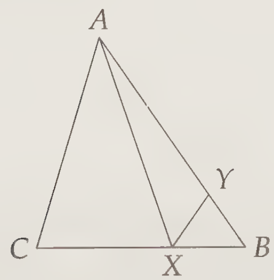


Figure 4.11: Diagram for Problem 4.32

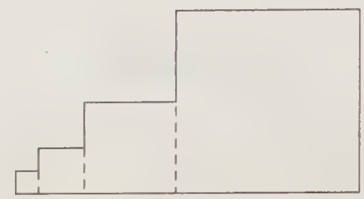


Figure 4.12: Diagram for Problem 4.33

4.33 The diagram at right above is formed by placing 4 squares together along a single line as shown. Each square has a side length that is 1/2 the side length of the next larger square. The outer perimeter of the figure is 115. What is the area of the whole figure? *(Source: ARML)* **Hints:** 329

4.34 Sue and Barry are trying to find the area of $\triangle ABC$. Sue mistakenly uses AB and the height from A (instead of the height from C), and Barry mistakenly uses BC and the height from C (instead of the height from A).

(a) Sue finds an area of 12 and Barry finds an area of 27. What is the area of $\triangle ABC$? **Hints:** 369, 38

(b)★ Suppose instead that Sue finds an area of 120 and Barry finds an area of 150. Now what is the area of $\triangle ABC$? **Hints:** 301

4.35 Show that $[WPX] = [WPZ]$ in the diagram at left below. **Hints:** 426, 108

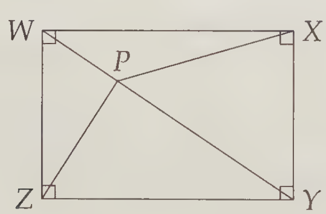


Figure 4.13: Diagram for Problem 4.35

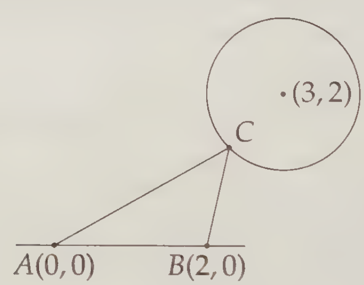


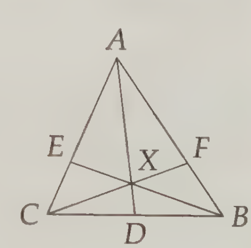
Figure 4.14: Diagram for Problem 4.36

4.36 In the diagram at right above, triangle ABC has vertex A at the origin, vertex B at the point $(2,0)$, and vertex C on the circle with center $(3,2)$ and radius 1. What is the maximum possible area for such a triangle? *(Source: Mandelbrot)* **Hints:** 75, 553

4.37★ $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ meet at X as shown at right. Prove that

$$\frac{[AXC]}{[BXC]} = \frac{AF}{FB}.$$

Hints: 231, 98



4.38★ In the diagram at left below, given $[PQRS] = 5[PQA]$ and $[PQRS] = 4[PBS]$, find $[ABP]/[PQRS]$. **Hints:** 583, 470, 361

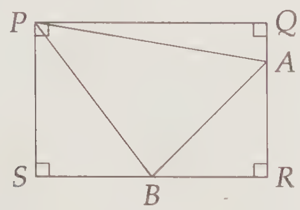


Figure 4.15: Diagram for Problem 4.38

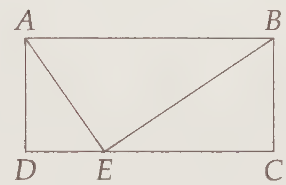
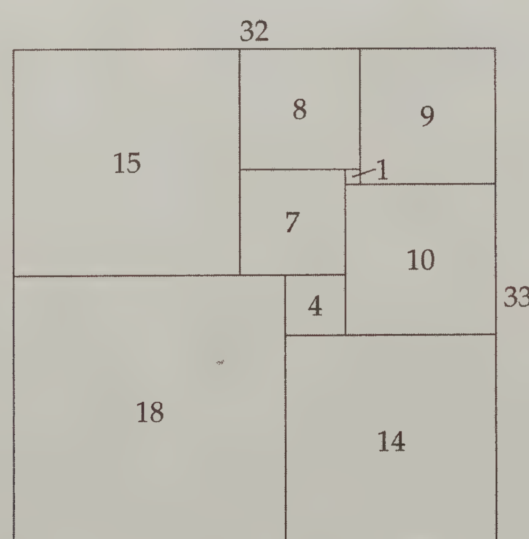


Figure 4.16: Diagram for Problem 4.39

4.39 In rectangle $ABCD$ at right above, point E is on side $\overline{CD}$ such that $[CBE] - [ADE] = [AEB] - [CBE]$. What is the ratio of the area of the largest region to the area of the smallest region? *(Source: ARML)* **Hints:** 324, 166

4.40★ For Joe's birthday, Will has bought a 7×7 inch square birthday cake with a flat top. It is a chocolate cake with banana frosting on the top and on the sides. It turns out that seven people will be present when the cake is cut, and each person will become quite envious if another person receives more cake or frosting. Find a way to divide the cake among seven people, so that each receives an equal amount of cake and frosting. **Hints:** 149, 67, 124

Extra! Perhaps surprisingly, it is possible to dissect some rectangles into squares of different sizes. The figure below shows a dissection of a 32×33 rectangle into squares, whose side lengths are indicated.

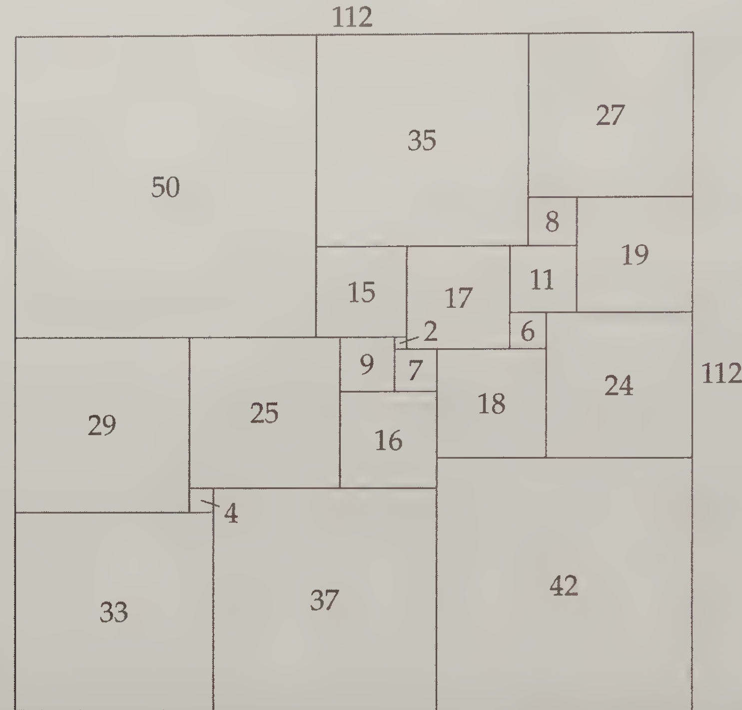


Is it possible to dissect a square into squares of different sizes?

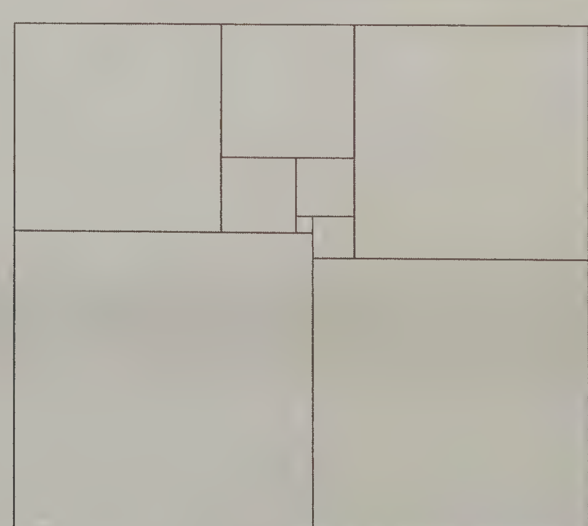
Continued on the next page. . .

Extra! *. . . continued from the previous page*

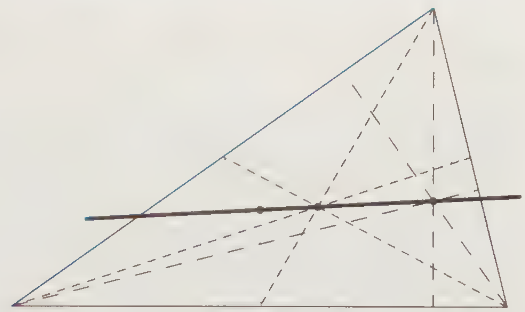
The answer is yes! The figure below shows such a dissection of a 112×112 square. (The numbers indicate side lengths of the squares.)



A problem from the 2000 American Invitational Mathematics Examination asks the following: "The diagram shows a rectangle that has been dissected into nine non-overlapping squares. Given that the width and the height of the rectangle are positive integers with greatest common divisor 1, find the perimeter of the rectangle."



Try to solve it!



The Euler Line

All cases are unique and very similar to others. – T. S. Eliot

CHAPTER 5 Similar Triangles

5.1 What is Similarity?

Problems

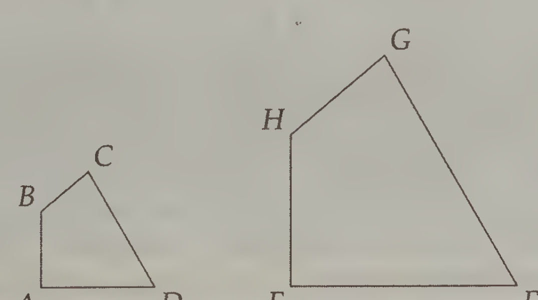
Problem 5.1:

(a) Use a ruler to approximate the following ratios in the figures below:

$$\frac{AB}{EH} \qquad \frac{BC}{HG} \qquad \frac{CD}{GF} \qquad \frac{DA}{FE} \qquad \frac{BD}{HF} \qquad \frac{AC}{EG}$$

(b) Measure angles $\angle A$ through $\angle H$.

(c) Do you find anything interesting in your answers to the first two parts?



We call two figures **similar** if one is simply a blown-up, and possibly rotated and/or flipped, version of the other. Our first problem gives us an example of similar figures.

Problem 5.1:

(a) Use a ruler to approximate the following ratios in Figure 5.1:

$$\frac{AB}{EH} \quad \frac{BC}{HG} \quad \frac{CD}{GF} \quad \frac{DA}{FE} \quad \frac{BD}{HF} \quad \frac{AC}{EG}$$

(b) Measure angles $\angle A$ through $\angle H$.

(c) Do you find anything interesting in your answers to the first two parts?

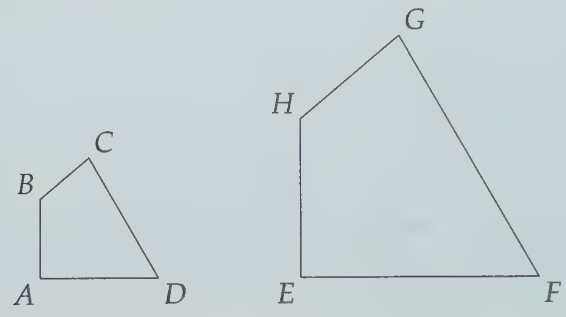


Figure 5.1: Two Similar Figures

Solution for Problem 5.1: Measuring each of the segments in the given ratios, we find that in each case, the ratio is 1/2. When we measure the angles, we find that the angles of $ABCD$ are equal to those in $EHGF$ (note the orders of the vertices!):

$$\begin{aligned} \angle A &= \angle E = 90^\circ \\ \angle B &= \angle H = 130^\circ \\ \angle C &= \angle G = 80^\circ \\ \angle D &= \angle F = 60^\circ \end{aligned}$$

□

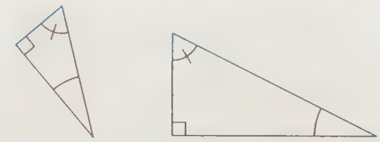
We write the similarity in Figure 5.1 as $ABCD \sim EHGF$ since $\angle A$ corresponds to $\angle E$, $\angle B$ corresponds to $\angle H$, etc. As with congruence, we have to be careful about the order of the vertices. For example, we would not write $ABCD \sim EFGH$ to describe Figure 5.1.

The ratio between corresponding lengths in similar figures is constant, and is equal to the ratio by which one figure is 'blown up' to get the other. In Figure 5.1, we have

$$\frac{AB}{EH} = \frac{BC}{HG} = \frac{CD}{GF} = \frac{DA}{FE}.$$

All corresponding lengths of $ABCD$ and $EHGF$ follow this ratio. For example, we could include BD/HF and AC/EG in that chain of equalities above.

As we saw in Problem 5.1, corresponding angles in similar figures are equal.



Similar figures do not need to have the same orientation. The diagram to the right shows two similar triangles with different orientations.

Speaking of triangles, we'll be spending the rest of this chapter discussing how to tell when two triangles are similar, and how to use similar triangles once we find them. Below are a couple Exercises that provide practice using triangle similarities to write equations involving side lengths.

Exercises

5.1.1 Given that $\triangle ABC \sim \triangle YXZ$, which of the statements below must be true?

(a) $AB/YX = AC/YZ$.

(b) $AB/BC = YX/XZ$.

(c) $AB/XZ = BC/YX$.

(d) $(AC)(YX) = (YZ)(BA)$.

(e) $BC/BA = XY/ZY$.

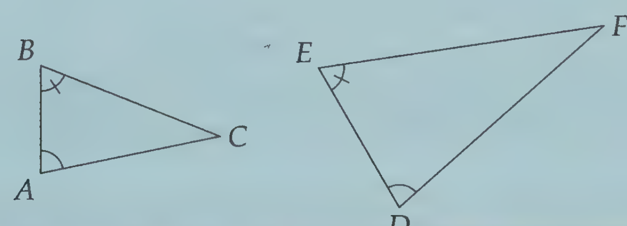
5.1.2 $\triangle ABC \sim \triangle ADB$, $AC = 4$, and $AD = 9$. What is AB? *(Source: MATHCOUNTS)* **Hints:** 113

5.2 AA Similarity

In our introduction, we stated that similar figures have all corresponding angles equal, and that corresponding sides are in a constant ratio. It sounds like a lot of work to prove all of that; however, just as for triangle congruence, we have some shortcuts to prove that triangles are similar. We'll start with the most commonly used method.

Important: **Angle-Angle Similarity (AA Similarity)** tells us that if two angles of one triangle equal two angles of another, then the triangles are similar. $\angle A = \angle D$ and $\angle B = \angle E$ together imply $\triangle ABC \sim \triangle DEF$, so

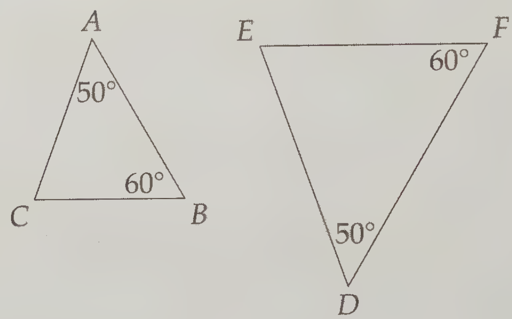
$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$



We'll explore why AA Similarity works in Section 5.5, but first we'll get some experience using it in some problems.

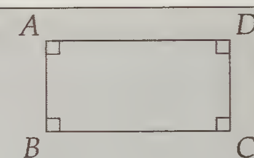
Problems

Problem 5.2: Below are two triangles that have the same measures for two angles.

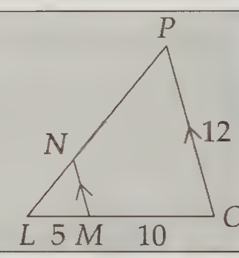


Find the third angle in each, and find the ratios AB/DF, AC/DE, BC/EF by measuring the sides with a ruler.

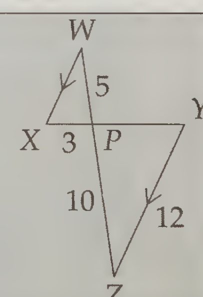
Problem 5.3: In this problem we try to extend AA Similarity to figures with more angles by considering figures with four angles. Can you create a figure $EFGH$ that has the same angles as $ABCD$ at right such that $EFGH$ and $ABCD$ are not similar? (In other words, can you create $EFGH$ so that the angles of $EFGH$ equal those of $ABCD$, but the ratio of corresponding sides between $EFGH$ and $ABCD$ is not the same for all corresponding pairs of sides?)



Problem 5.4: In the figure at right, $\overline{MN} \parallel \overline{OP}$, $OP = 12$, $MO = 10$, and $LM = 5$. Find MN.



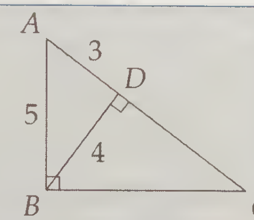
Problem 5.5: The lengths in the diagram are as marked, and $\overline{WX} \parallel \overline{YZ}$. Find PY and WX.



Extra! My dad was going to cut down a dead tree in our yard one day, but he was afraid it might hit some nearby power lines. He knew that if the tree were over 45 feet tall, the tree would hit the power lines. He stood 30 feet from the base of the tree and held a ruler 6 inches in front of his eye.

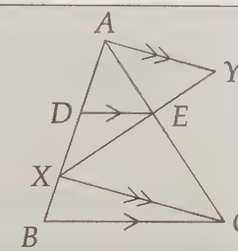
Continued on the next page. . .

Problem 5.6: Find BC and DC given $AD = 3$, $BD = 4$, and $AB = 5$.

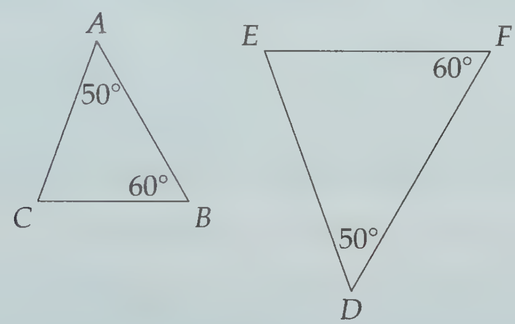


Problem 5.7: Given that $\overline{DE} \parallel \overline{BC}$ and $\overline{AY} \parallel \overline{XC}$, prove that

$$\frac{EY}{EX} = \frac{AD}{DB}.$$



Problem 5.2: Below are two triangles that have the same measures for two angles.



Find the third angle in each, and find the ratios AB/DF, AC/DE, BC/EF by measuring the sides with a ruler.

Solution for Problem 5.2: The last angle in each triangle is $180^\circ - 50^\circ - 60^\circ = 70^\circ$, so the angles of $\triangle ABC$ match those of $\triangle DFE$. In the same way, if we ever have two angles of one triangle equal to two angles of another, we know that the third angles in the two triangles are equal.

Measuring, we find that the ratios are each 1.5. It appears to be the case that if all the angles of two triangles are equal, then the two triangles are similar. □

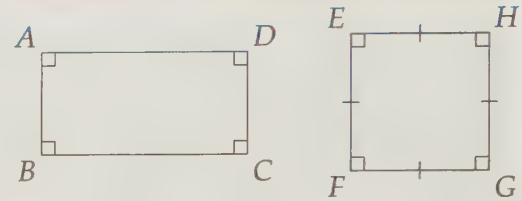
We might wonder if two figures with equal corresponding angles are always similar. So, we add an angle and see if it works for figures with four angles.

Extra! . . . *continued from the previous page*

He lined the bottom of the ruler up with the base of the tree, and saw that the top of the tree lined up with a point 8 inches high on the ruler. He then knew he could safely cut the tree down. How did he know?

Problem 5.3: Does your rule work for figures with more than 3 angles? Can you create a figure $EFGH$ that has the same angles as $ABCD$ at right such that $EFGH$ and $ABCD$ are not similar? (In other words, can you create $EFGH$ so that the angles of $EFGH$ equal those of $ABCD$, but the ratio of corresponding sides between $EFGH$ and $ABCD$ is not the same?)

Solution for Problem 5.3: We can quickly find such an $EFGH$. The diagram to the right shows a square $EFGH$ next to our initial rectangle. Clearly these figures have the same angles, but when we check the ratios, we find that

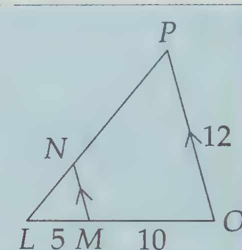


$$\frac{AB}{EF} < 1 < \frac{BC}{FG}.$$

$ABCD$ and $EFGH$ are not similar, so equal angles are not enough to prove similarity here. □

Let's return to triangles and tackle some problems using AA Similarity.

Problem 5.4: In the figure at right, $\overline{MN} \parallel \overline{OP}$, $OP = 12$, $MO = 10$, and $LM = 5$. Find MN.

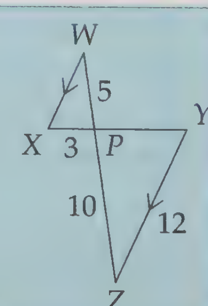


Solution for Problem 5.4: See if you can find the flaw in this solution:

Bogus Solution: Since $\overline{MN} \parallel \overline{OP}$, we have $\angle LMN = \angle LOP$ and $\angle LNM = \angle LPO$. Therefore, $\triangle LMN \sim \triangle LOP$, so $LM/MO = MN/OP$. Substituting our given side lengths gives $5/10 = MN/12$, so $MN = 6$.

Everything in this solution is correct except for $LM/MO = MN/OP$. $\overline{MO}$ is not a side of one of our similar triangles! The correct equation is $LM/LO = MN/OP$. Since $LO = LM + MO = 15$, we now have $5/15 = MN/12$, so $MN = 4$. □

Problem 5.5: The lengths in the diagram are as marked, and $\overline{WX} \parallel \overline{YZ}$. Find PY and WX.



Solution for Problem 5.5: Where does this solution go wrong:

Bogus Solution: Since $\overline{WX} \parallel \overline{ZY}$, we have $\angle W = \angle Z$ and $\angle X = \angle Y$. Therefore, $\triangle WPX \sim \triangle YPZ$, and we have

$$\frac{PX}{PZ} = \frac{WX}{YZ} = \frac{WP}{PY}$$

Substitution gives

$$\frac{3}{10} = \frac{WX}{12} = \frac{5}{PY}.$$

We can now easily find $YP = 50/3$ and $WX = 18/5$.

This solution doesn't get the vertex order in the similar triangles right, so it sets up the ratios wrong! $\overline{PX}$ and $\overline{PZ}$ are not corresponding sides. $\overline{PX}$ in $\triangle WPX$ corresponds to $\overline{PY}$ in $\triangle ZPY$ because $\angle W = \angle Z$.

Here's what the solution should look like. Pay close attention to the vertex order in the similarity relationship.

Since $\overline{WX} \parallel \overline{ZY}$, we have $\angle W = \angle Z$ and $\angle X = \angle Y$. Therefore, $\triangle WPX \sim \triangle ZPY$. Hence, we have

$$\frac{PX}{PY} = \frac{WX}{YZ} = \frac{WP}{PZ}.$$

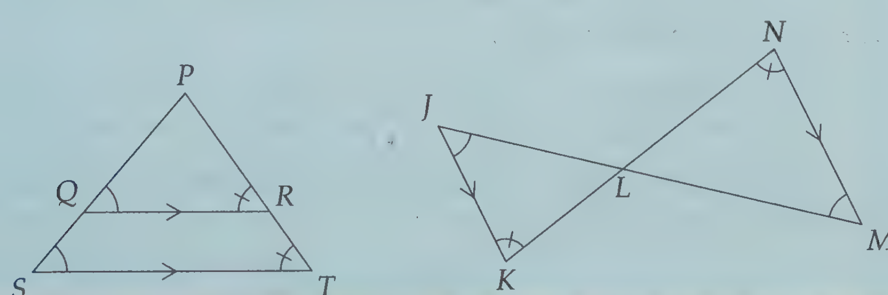
Substitution gives

$$\frac{3}{PY} = \frac{WX}{12} = \frac{5}{10}.$$

We can now easily find $PY = 6$ and $WX = 6$. □

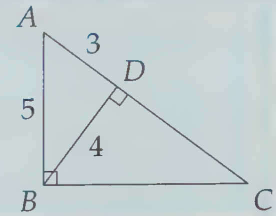
Perhaps you see a common thread in the last two problems. While you won't always find parallel lines in similar triangle problems, you'll almost always find similar triangles when you have parallel lines.

Important: Parallel lines mean equal angles. Equal angles mean similar triangles. The figures below show two very common set-ups in which parallel lines lead to similar triangles. Specifically, $\triangle PQR \sim \triangle PST$ and $\triangle JKL \sim \triangle MNL$.



WARNING!! Read the Bogus Solutions to Problems 5.4 and 5.5 again. These are very common errors; understand them so you can avoid them.

Problem 5.6: Find BC and DC given $AD = 3$, $BD = 4$, and $AB = 5$.



Solution for Problem 5.6: Since $\angle BAD = \angle CAB$ and $\angle BDA = \angle CBA$, we have $\triangle BAD \sim \triangle CAB$ by AA Similarity. Therefore, we have $BC/BD = AB/AD = 5/3$, so $BC = (5/3)(BD) = 20/3$.

We can use this same similarity to find AC, and then subtract AD to get CD. We could also note that $\angle BCD = \angle BCA$ and $\angle BDC = \angle CBA$, so $\triangle BCD \sim \triangle ACB$ by AA Similarity. Therefore, $CD/BD = BC/AB = (20/3)/5 = 4/3$, so $CD = (4/3)(BD) = 16/3$. □

Similar triangles – they're not just for parallel lines.

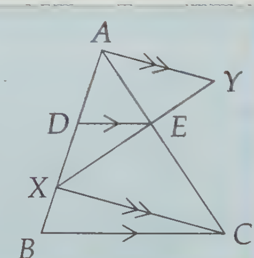
Important: Similar triangles frequently pop up in problems with right angles. The diagram in Problem 5.6 shows a common way this occurs. Make sure you see that

$$\triangle ABD \sim \triangle BCD \sim \triangle ACB.$$

As you'll see throughout the rest of the book, similar triangles occur in all sorts of problems, not just those with parallel lines and perpendicular lines. They're also an important step in many proofs.

Problem 5.7: Given that $\overline{DE} \parallel \overline{BC}$ and $\overline{AY} \parallel \overline{XC}$, prove that

$$\frac{EY}{EX} = \frac{AD}{DB}.$$



Solution for Problem 5.7: Parallel lines mean similar triangles. The ratios of side lengths in the problem also suggest we look for similar triangles.

Since $\overline{AY} \parallel \overline{XC}$, we have $\triangle AYE \sim \triangle CXE$. Now we look at what this means for our ratios. From $\triangle AYE \sim \triangle CXE$, we have $EY/EX = AE/EC$. All we have left is to show that $AE/EC = AD/DB$.

Since $\overline{DE} \parallel \overline{BC}$, we have $\triangle ADE \sim \triangle ABC$. Therefore, $AD/AB = AE/AC$, which is almost what we want! We break AB and AC into $AD + DB$ and $AE + EC$, hoping we can do a little algebra to finish:

$$\frac{AD}{AD + DB} = \frac{AE}{AE + EC}$$

If only we could get rid of the AD and AE in the denominators – then we would have $AD/DB = AE/EC$. Fortunately, we can do it. We can flip both fractions:

$$\frac{AD + DB}{AD} = \frac{AE + EC}{AE}.$$

Therefore, $\frac{AD}{AD} + \frac{DB}{AD} = \frac{AE}{AE} + \frac{EC}{AE}$, so $1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$, which gives us

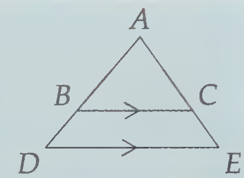
$$\frac{DB}{AD} = \frac{EC}{AE}.$$

Flipping these fractions back over gives us $AD/DB = AE/EC$. Therefore, we have $EY/EX = AE/EC = AD/DB$, as desired. □

Our solution to the previous problem reveals another handy relationship involving similar triangles:

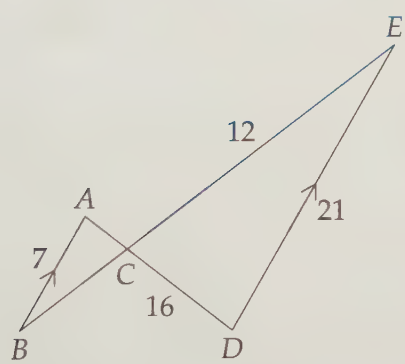
Important: If $\overline{BC} \parallel \overline{DE}$ and $\overleftrightarrow{BD}$ and $\overleftrightarrow{CE}$ meet at A as shown, then

$$\frac{AB}{BD} = \frac{AC}{CE}.$$

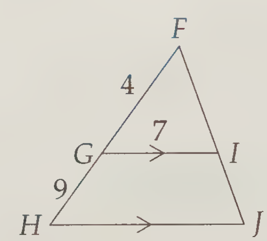


Exercises

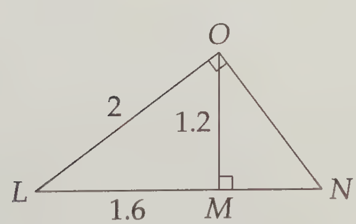
5.2.1



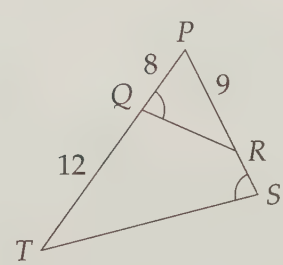
(a) Find AC and BC.



(b) Find HJ.



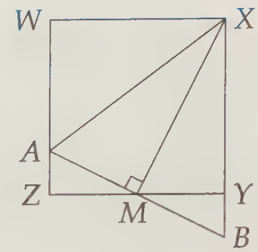
(c) Find ON and MN.



(d) Find RS.

5.2.2 If two isosceles triangles have vertex angles that have the same measure, are the two triangles similar? Why or why not?

5.2.3 In the diagram, $WXYZ$ is a square. M is the midpoint of $\overline{YZ}$, and $\overline{AB} \perp \overline{MX}$.



(a) Show that $\overline{WZ} \parallel \overline{XY}$. **Hints:** 182

(b) Prove that $AZ = YB$.

(c) Prove that $XB = XA$.

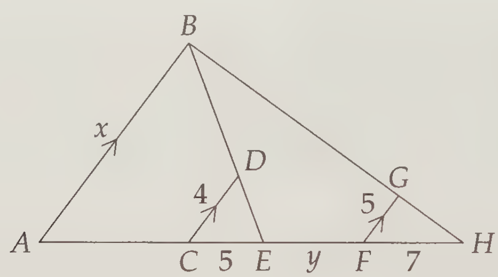
(d) Prove that $\triangle AZM \sim \triangle MYX$, and use this fact to prove $AZ = XY/4$.

5.2.4 In triangle ABC, $AB = AC$, $BC = 1$, and $\angle BAC = 36^\circ$. Let D be the point on side $\overline{AC}$ such that $\angle ABD = \angle CBD$.

(a) Prove that triangles ABC and BCD are similar.

(b)★ Find AB. **Hints:** 150

5.2.5★ Find x in terms of y given the diagram below. **Hints:** 258, 522



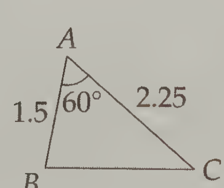
5.3 SAS Similarity

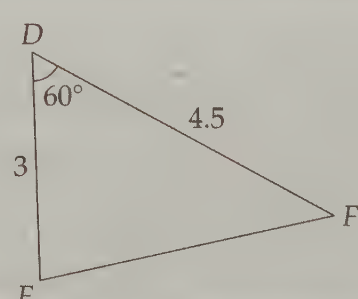
Problems

Problem 5.8:

(a) Measure $\overline{BC}$, $\overline{EF}$, and angles $\angle B$, $\angle C$, $\angle E$, and $\angle F$.

(b) Can you make a guess about how to use Side-Angle-Side for triangle similarity?

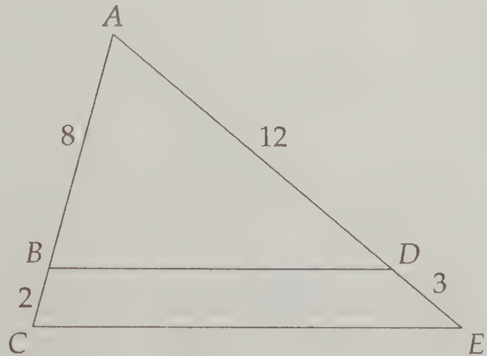


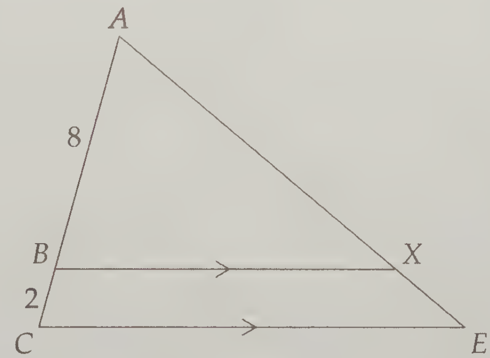


Extra! *Descartes commanded the future from his study more than Napoleon from the throne.*

–Oliver Wendell Holmes

Problem 5.9: In the figure below on the left, we have $\frac{AB}{AC} = \frac{AD}{AE} = \frac{4}{5}$, and clearly $\angle BAD = \angle CAE$. We wish to prove that $\triangle ABD \sim \triangle ACE$. (Note that we cannot assume that $\overline{BD} \parallel \overline{CE}$! We have to prove it.)





(a) Suppose we draw a line through B parallel to $\overline{CE}$ that hits $\overline{AE}$ at X as in the diagram on the right. What do we know about $\triangle ABX$ and $\triangle ACE$?

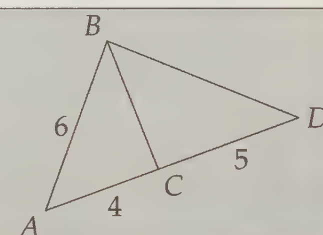
(b) Given that $AE = 15$ in both diagrams above, what is AX?

(c) What can we conclude about D and X?

(d) What can we conclude about $\triangle ABD$ and $\triangle ACE$?

(e) What similarity rule can we create from this investigation?

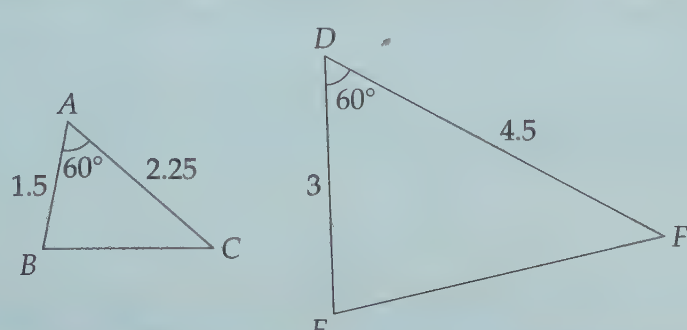
Problem 5.10: Given $AC = 4$, $CD = 5$, and $AB = 6$ as in the diagram, find BC if the perimeter of $\triangle BCD$ is 20. *(Source: Mandelbrot)*



Problem 5.8:

(a) Measure $\overline{BC}$, $\overline{EF}$, and angles $\angle B$, $\angle C$, $\angle E$, and $\angle F$.

(b) Can you make a guess about how to use Side-Angle-Side for triangle similarity?



Solution for Problem 5.8: We aren't surprised to find that BC appears to be half EF: BC is about 2 cm and EF is around 4 cm. We also aren't shocked to find that $\angle B$ appears to equal $\angle E$ and $\angle C$ appears to equal $\angle F$.

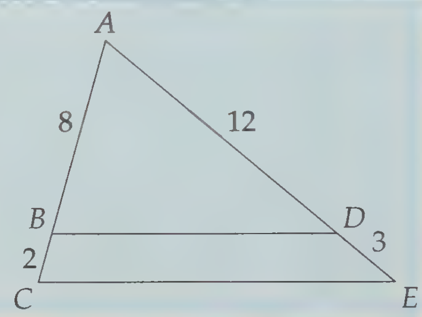
This example suggests that if two sides in one triangle are in the same ratio as two sides in another triangle (as $AB/AC = DE/DF$), and the angles between these sides are equal (as $\angle A = \angle D$), then the triangles are similar. □

No doubt, you know where this is headed. Time to develop a proof for our guess. As usual, we try to use what we already know, AA Similarity, to prove our guess for 'SAS Similarity'.

Problem 5.9: In the figure on the right, we have

$$\frac{AB}{AC} = \frac{AD}{AE} = \frac{4}{5},$$

and clearly $\angle BAD = \angle CAE$. Prove that $\triangle ABD \sim \triangle ACE$.



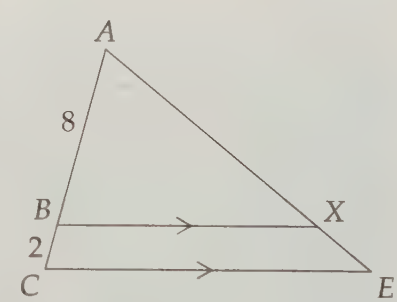
Solution for Problem 5.9: What did we do wrong here:

Bogus Solution: Since $\overline{BD} \parallel \overline{CE}$, we have $\angle ABD = \angle ACE$ and $\angle ADB = \angle AEC$, so $\triangle ABD \sim \triangle ACE$ by AA Similarity.

There's not a single false statement in that solution. However, the assertion that $\overline{BD} \parallel \overline{CE}$ needs to be proved, and our Bogus Solution merely states it without justification.

In the solution below, we take the clever tactic of considering the point X on $\overline{AE}$ such that $\overline{BX} \parallel \overline{CE}$. Then we prove that X is in fact D.

We'd like to prove that $\overline{BD} \parallel \overline{CE}$, but there's no obvious way to even start. We seem stuck, so we try to go a different direction. We create a point X on $\overline{AE}$ as shown at right, such that $\overline{BX} \parallel \overline{CE}$. Our goal now is to show that X must be D. Notice that we are not assuming that $\overline{BD} \parallel \overline{CE}$. We are taking some other point, X, such that $\overline{BX} \parallel \overline{CE}$, then trying to prove that X *must be* D.



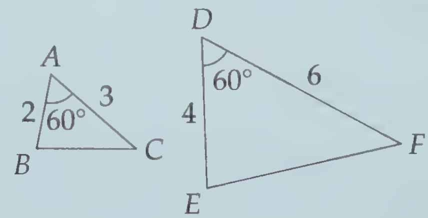
Since $\overline{BX} \parallel \overline{CE}$, we have $\angle ABX = \angle ACE$ and $\angle AXB = \angle AEC$, so $\triangle ABX \sim \triangle ACE$ by AA Similarity. Therefore,

$$\frac{AX}{AE} = \frac{AB}{AC} = \frac{4}{5},$$

so $AX = (4/5)(AE) = 12$. Hence, X is on $\overline{AE}$ 12 units from A. But that's where point D is! Therefore, D must be the same point as point X; i.e., D is the point on $\overline{AE}$ such that $\overline{BD} \parallel \overline{CE}$. Now that we've proved $\overline{BD} \parallel \overline{CE}$, we can conclude that $\triangle ABD \sim \triangle ACE$. □

We have established another way to prove two triangles are similar.

Important: **Side-Angle-Side Similarity (SAS Similarity)** tells us that if two sides in one triangle are in the same ratio as two sides in another triangle (as $AB/AC = DE/DF$ below), and the angles between these sides are equal (as $\angle A = \angle D$ below), then the triangles are similar.



Note that we can also write that ratio equality as the ratio of corresponding sides in the triangles: $AB/DE = AC/DF$.

You may be wondering how our solution to Problem 5.9 can be used to prove SAS Similarity in general, since Problem 5.9 only deals with the case of two triangles that share an angle, as $\triangle ABD$ and $\triangle ACE$ share $\angle A$. We can use this approach generally because if an angle in one triangle equals an angle in another, we can always slide (and/or flip) one triangle until it's on top of the other, as shown in Figure 5.2.

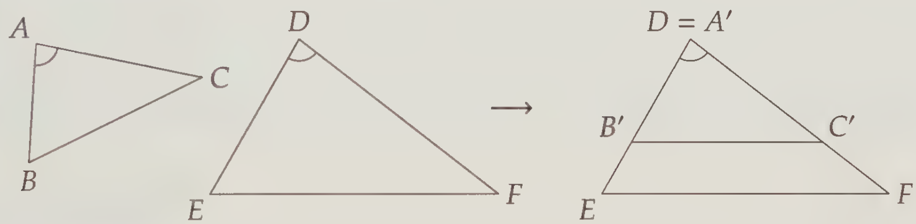
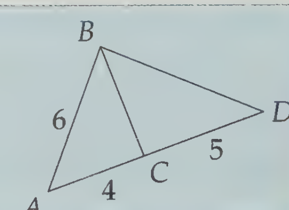


Figure 5.2: Sliding Triangles to Prove Similarity

SAS Similarity is most often used in diagrams like the one shown in Problem 5.9. However, it does come up in less obvious situations.

Problem 5.10: Given $AC = 4$, $CD = 5$, and $AB = 6$ as in the diagram, find BC if the perimeter of $\triangle BCD$ is 20. *(Source: Mandelbrot)*



Solution for Problem 5.10: Since

$$\frac{AC}{AB} = \frac{4}{6} = \frac{2}{3} \qquad \text{and} \qquad \frac{AB}{AD} = \frac{6}{9} = \frac{2}{3},$$

we have $\triangle ACB \sim \triangle ABD$ by SAS (since the angle between the sides in each ratio above is $\angle A$). Since the sides of $\triangle ABD$ are 3/2 the corresponding sides of $\triangle ACB$, we have $BD = 3BC/2$. Now we can use that perimeter information. Since $BC + CD + DB = 20$, we have

$$BC + 5 + \frac{3BC}{2} = 20.$$

Therefore, $BC = 6$. □

Exercises

5.3.1 Find DE in the figure at left below.

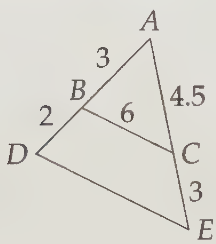


Figure 5.3: Diagram for Problem 5.3.1

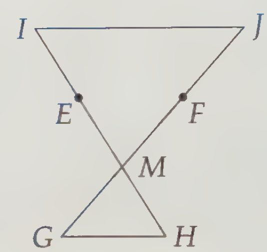


Figure 5.4: Diagram for Problem 5.3.2

5.3.2 In the figure at right above, M is the midpoint of $\overline{EH}$ and of $\overline{FG}$. E and F are midpoints of $\overline{IM}$ and $\overline{MJ}$, respectively. Prove that $\overline{IJ} \parallel \overline{GH}$.

5.3.3 Show that if $WZ^2 = (WX)(WY)$ in the diagram at left below, then $\angle WZX = \angle WYZ$. **Hints:** 147

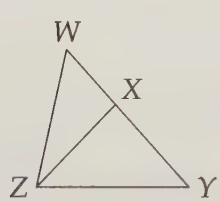


Figure 5.5: Diagram for Problem 5.3.3

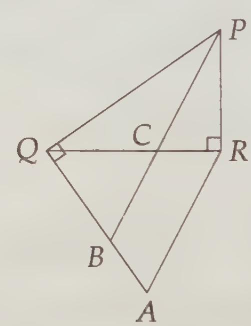


Figure 5.6: Diagram for Problem 5.3.4

5.3.4 In the diagram at right above, $\angle PRQ = \angle PQA = 90°$, $QR = QA$, and $\angle QPC = \angle RPC$.

(a) Prove $\angle QCB = \angle QBC$. **Hints:** 202

(b)★ Prove $\overline{RA} \parallel \overline{PB}$. **Hints:** 388

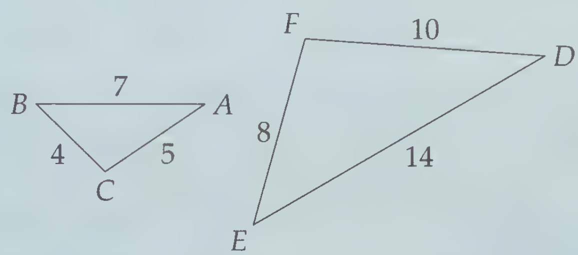
Extra! *I must study politics and war that my children may have liberty to study mathematics and philosophy. My children ought to study mathematics and philosophy, geography, natural history, naval architecture, navigation, commerce, and agriculture, in order to give their children a right to study painting, poetry, music, architecture, statuary, tapestry, and porcelain.*

–John Adams

5.4 SSS Similarity

We use SSS Similarity less often than AA and SAS.

Important: **Side-Side-Side Similarity (SSS Similarity)** tells us that if each side of one triangle is the same constant multiple of the corresponding side of another triangle, then the triangles are similar. (And therefore, their corresponding angles are equal.)



For example, in the diagram, we have

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF},$$

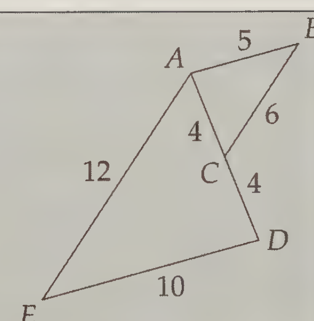
so

$$\triangle ABC \sim \triangle DEF.$$

Therefore, $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.

Problems

Problem 5.11: Given the side lengths shown in the diagram, prove that $\overline{AE} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DE}$.

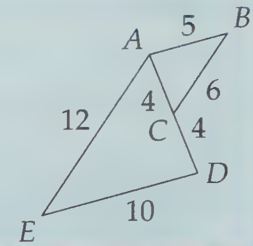


As we noted, few problems require SSS Similarity. We may, however, consider it in problems in which all we are given is lengths, but we have to prove something about angles.

Extra! *Mathematics is the art of giving the same name to different things.*

–Henri Poincaré

Problem 5.11: Given the side lengths shown in the diagram, prove that $\overline{AE} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DE}$.



Solution for Problem 5.11: We need to use angles to show the segments are parallel, but all we have are sides. We look for similarity, and see that

$$\frac{AB}{DE} = \frac{AC}{AD} = \frac{BC}{AE} = \frac{1}{2},$$

so $\triangle ABC \sim \triangle DEA$ by SSS Similarity. Therefore, $\angle BAC = \angle EDA$, so $\overline{AB} \parallel \overline{DE}$. Also, $\angle DAE = \angle ACB$, so $\overline{AE} \parallel \overline{BC}$. □

Exercises

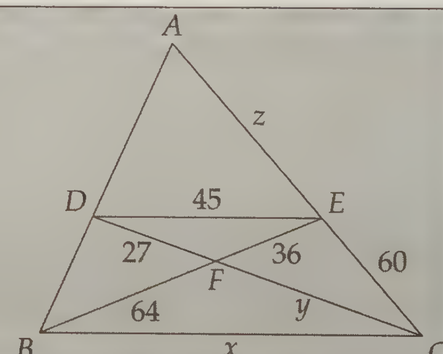
5.4.1 Two isosceles triangles have the same ratio of leg length to base length. Prove that the vertex angles of the two triangles are equal. **Hints:** 314

5.5 Using Similarity in Problems

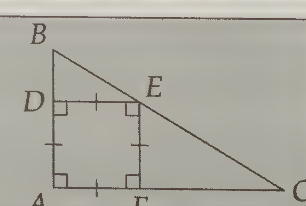
In this section we explore some challenging problems that are solved with similar triangles, and we discover why AA Similarity works.

Problems

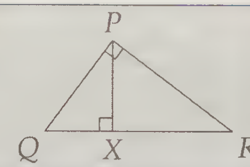
Problem 5.12: In the diagram, $\overline{DE} \parallel \overline{BC}$, and the segments have the lengths shown in the diagram. Find x, y, and z.



Problem 5.13: As shown in the diagram, $\angle A = 90°$ and $ADEF$ is a square. Given that $AB = 6$ and $AC = 10$, find AD.



Problem 5.14: In the diagram, $\overline{PX}$ is the altitude from right angle $\angle QPR$ of right triangle PQR as shown. Show that $PX^2 = (QX)(RX)$, $PR^2 = (RX)(RQ)$, and $PQ^2 = (QX)(QR)$.



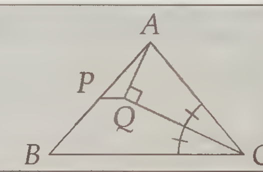
Problem 5.15: $\triangle ABC \sim \triangle XYZ$, $AB/XY = 4$, and $[ABC] = 64$. In this problem we will find $[XYZ]$.

(a) Let h_C be the altitude of $\triangle ABC$ to AB, and let h_Z be the altitude of $\triangle XYZ$ to XY. What is h_C/h_Z?

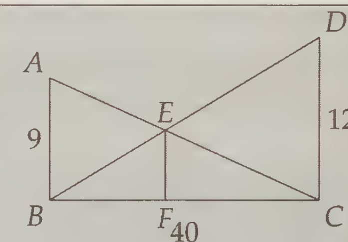
(b) Find $[XYZ]$.

Extra challenge: What general statement about the areas of similar triangles can you make?

Problem 5.16: In the diagram, $\angle ACQ = \angle QCB$, $\overline{AQ} \perp \overline{CQ}$, and P is the midpoint of $\overline{AB}$. Prove that $\overline{PQ} \parallel \overline{BC}$. **Hints:** 35, 417



Problem 5.17: Flagpole $\overline{CD}$ is 12 feet tall. Flagpole $\overline{AB}$ is 9 feet tall. Both flagpoles are perpendicular to the ground. A straight wire is attached from B to D, and another from A to C. The flagpoles are 40 feet apart, and the wires cross at E, which is directly above point F on the ground. We wish to find EF.



(a) Use similar triangles to find ratios of segments that equal EF/AB.

(b) Use similar triangles to find ratios of segments that equal EF/DC.

(c) Cleverly choose one ratio from each of the first two parts and add them to get an equation you can solve for EF.

Problem 5.18: In this problem we will explore why AA Similarity works. Do not use AA Similarity to solve the problem!

In the diagram below, we have two triangles ($\triangle ABE$ and $\triangle ACD$) with equal angles, and sides with lengths as marked. Our goal in this problem is to find BE and DE, and discover a process to prove that if the angles of one triangle equal those of another, then the corresponding sides of the two triangles are in constant proportion. We will make heavy use of the Same Base/Same Altitude principle we discovered in Section 4.3, so you might want to review that section if you get stuck.

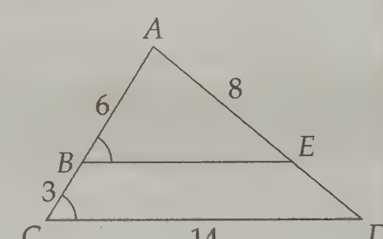
(a) What are $[ABE]/[ACE]$ and $[BEC]/[BED]$?

(b) Use the previous part to show that $[ACE] = [ABD]$.

(c) What is $[ABE]/[ABD]$?

(d) Use the previous part to find AD.

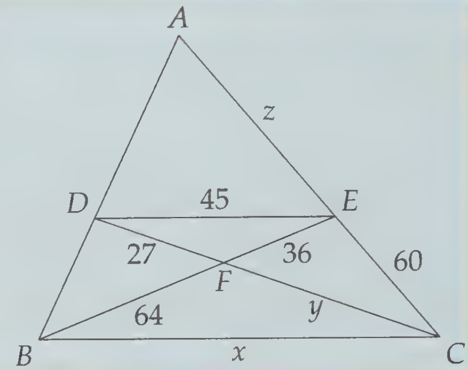
(e) What is BE?



(f) Can we use our work in this problem to prove that if two angles of one triangle equal those of another triangle, then the triangles are similar?

We start off with some warm-ups involving parallel and perpendicular lines.

Problem 5.12: In the diagram, $\overline{DE} \parallel \overline{BC}$, and the segments have the lengths shown in the diagram. Find x, y, and z.



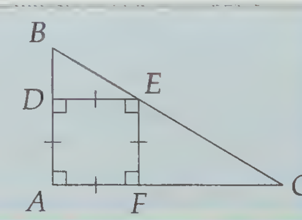
Solution for Problem 5.12: Since $\overline{ED} \parallel \overline{BC}$, we have $\triangle FBC \sim \triangle FED$ by AA Similarity. Therefore, we have

$$\frac{FC}{FD} = \frac{BC}{DE} = \frac{FB}{FE} = \frac{64}{36} = \frac{16}{9}.$$

Solving for x and y, we find $x = BC = (16/9)(DE) = 80$ and $y = FC = (16/9)(DF) = 48$.

Since $\triangle ADE \sim \triangle ABC$ by AA Similarity, we have $AE/AC = DE/BC = 45/80 = 9/16$. Since $AE = z$ and $AC = AE + EC = z + 60$, we have $z/(z+60) = 9/16$. Cross-multiplying gives $16z = 9z + 540$, so $z = 540/7$.
□

Problem 5.13: As shown in the diagram, $\angle A = 90^\circ$, and $ADEF$ is a square. Given that $AB = 6$ and $AC = 10$, find AD.



Solution for Problem 5.13: Since $\angle A = \angle EFC = 90^\circ$, we have $\overline{EF} \parallel \overline{AB}$; similarly, $\overline{DE} \parallel \overline{AC}$. Therefore, this problem has both right triangles and parallel lines. Our parallel lines quickly tell us that by AA, we have

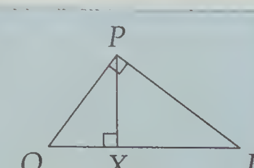
$$\triangle BDE \sim \triangle BAC \sim \triangle EFC.$$

If we let each side of $ADEF$ be x, we have $BD = 6 - x$ and $FC = 10 - x$. Our similar triangles can then be used to solve for x. From $\triangle BDE \sim \triangle EFC$, we have $BD/DE = EF/FC$. Substitution gives

$$\frac{6-x}{x} = \frac{x}{10-x}.$$

Cross-multiplying and solving the resulting equation for x gives $x = 15/4$. Therefore, $AD = x = 15/4$. □

Problem 5.14: In the diagram, $\overline{PX}$ is the altitude from right angle $\angle QPR$ of right triangle PQR as shown. Show that $PX^2 = (QX)(RX)$, $PR^2 = (RX)(RQ)$, and $PQ^2 = (QX)(QR)$.



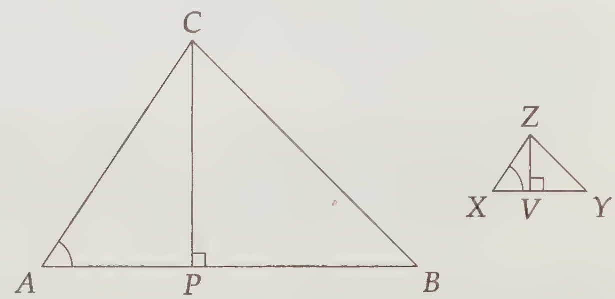
Solution for Problem 5.14: Right triangles mean similar triangles. $\angle PXR = \angle QPR$ and $\angle PRX = \angle PRQ$, so $\triangle PXR \sim \triangle QPR$. Therefore, we have $PR/RX = RQ/PR$, so $PR^2 = (RX)(RQ)$. Similarly, we can show $\triangle PQX \sim \triangle RQP$, so $PQ/QX = QR/PQ$, and we have $PQ^2 = (QX)(QR)$.

Combining the two triangle similarities (or by noting that $\angle XPQ = 90° - \angle XPR = \angle XRP$ and $\angle PXQ = \angle PXR$), we find $\triangle PXQ \sim \triangle RXP$. Therefore, $PX/QX = RX/PX$, so $PX^2 = (RX)(QX)$. □

The square root of the product of two numbers is called the **geometric mean** of the two numbers. The previous problem suggests where the name 'geometric mean' comes from. For example, what is the geometric mean of QX and RX?

Problem 5.15: Given that $\triangle ABC \sim \triangle XYZ$, $AB/XY = 4$, and $[ABC] = 64$, find $[XYZ]$.

Solution for Problem 5.15: Since $\triangle ABC \sim \triangle XYZ$ and $AB/XY = 4$, the ratio of corresponding lengths in the triangles is $4/1$. Therefore, the altitude of $\triangle ABC$ to $\overline{AB}$ is 4 times the corresponding altitude to $\overline{XY}$ in $\triangle XYZ$.



For a quick proof, consider the diagram to the right, in which we've drawn the aforementioned altitudes to $\overline{AB}$ and $\overline{XY}$. Since $\triangle ABC \sim \triangle XYZ$, we have $\angle A = \angle X$. Combining this angle equality with $\angle CPA = \angle ZVX$ gives $\triangle APC \sim \triangle XVZ$ by AA, so $CP/ZV = AC/XZ$. Since $\overline{AC}$ and $\overline{XZ}$ are corresponding sides of our original triangles, their ratio is $4/1$, so $CP/ZV = 4/1$.

Finally, we can find the ratio $[ABC]/[XYZ]$. Since both the base and the altitude of $\triangle ABC$ are 4 times the corresponding base and altitude of $\triangle XYZ$, we know that

$$[ABC]/[XYZ] = \frac{(AB)(CP)/2}{(XY)(ZV)/2} = \left(\frac{AB}{XY}\right)\left(\frac{CP}{ZV}\right) = \left(\frac{4}{1}\right)^2 = 16.$$

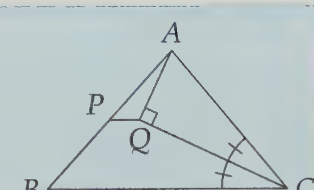
So, we have $[XYZ] = [ABC]/16 = 4$. □

The same procedure we used to solve this problem can be used to find an important relationship between the areas of two similar triangles.

Important: If two triangles are similar such that the sides of the larger triangle are k times the sides of the smaller, then the area of the larger triangle is k^2 times that of the smaller.

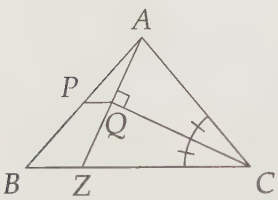
This relationship holds for any pair of similar figures, not just for triangles.

Problem 5.16: In the diagram, $\angle ACQ = \angle QCB$, $\overline{AQ} \perp \overline{CQ}$, and P is the midpoint of $\overline{AB}$. Prove that $\overline{PQ} \parallel \overline{BC}$.



Solution for Problem 5.16: If we could show that $\angle QPA = \angle B$, then we could use that to prove $\overline{PQ} \parallel \overline{BC}$.

Unfortunately, there are no obvious similar triangles or congruent triangles we can use to show that $\angle QPA = \angle B$.



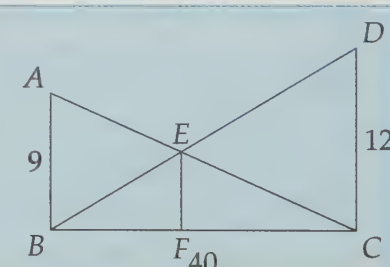
We extend segment $\overline{AQ}$ to point Z on $\overline{BC}$ because we'd like to create triangles that might be similar (namely, $\triangle APQ$ and $\triangle ABZ$). We'd also like to use the angle equalities at C, which we can now do by noting that $\angle AQC = \angle CQZ$, $CQ = CQ$, and $\angle ACQ = \angle QCZ$, so $\triangle CQZ \cong \triangle CQA$ by ASA. Therefore, we know that $AQ = QZ$, so $AQ = AZ/2$.

We might seem stuck here, but then we remember the last bit of information we haven't used. Since P is the midpoint of $\overline{AB}$, we have $AP = AB/2$, so $\triangle PAQ \sim \triangle BAZ$ by SAS Similarity. Thus, $\angle APQ = \angle B$, so $\overline{PQ} \parallel \overline{BC}$. □

Concept: When you're stuck on a problem, ask yourself, 'What piece of information have I not used?'

Concept: In many problems, there's more than meets the eye. Extending segments that seem to end abruptly (particularly in the middle of a triangle) sometimes gives useful information.

Problem 5.17: Flagpole $\overline{CD}$ is 12 feet tall. Flagpole $\overline{AB}$ is 9 feet tall. Both flagpoles are perpendicular to the ground. A straight wire is attached from B to D, and another from A to C. The flagpoles are 40 feet apart, and the wires cross at E, which is directly above point F on the ground. Find EF.



Solution for Problem 5.17: We start off by noticing that $\overline{AB} \parallel \overline{EF} \parallel \overline{CD}$ since all three are perpendicular to $\overline{BC}$. By now you know the drill: parallel lines mean similar triangles. We look first for similar triangles that include $\overline{EF}$, and we see $\triangle CEF \sim \triangle CAB$ and $\triangle EBF \sim \triangle DBC$. Therefore, we have

$$\frac{EF}{AB} = \frac{CF}{CB} = \frac{EC}{AC} \quad \text{and} \quad \frac{EF}{CD} = \frac{BF}{CB} = \frac{BE}{BD}.$$

We see CB in both groups, so we investigate the ratios involving CB more closely. We see that we have $CF + BF = CB$, so

$$\frac{EF}{AB} + \frac{EF}{CD} = \frac{CF}{CB} + \frac{BF}{CB} = \frac{CF + BF}{CB} = \frac{CB}{CB} = 1.$$

Now we can find EF:

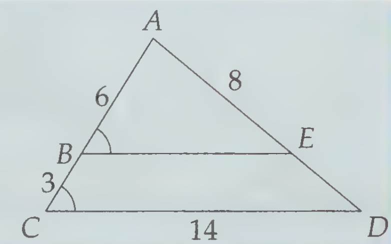
$$EF = \frac{1}{\frac{1}{AB} + \frac{1}{CD}} = \frac{1}{\frac{1}{9} + \frac{1}{12}} = \frac{36}{7}.$$

Notice that the length of $\overline{BC}$ is irrelevant! □

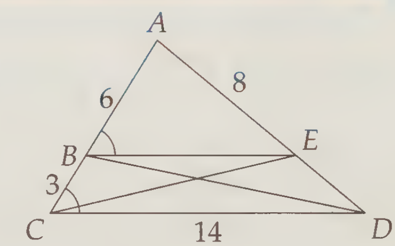
WARNING!! In the last solution we didn't spell out exactly why $\triangle EBF \sim \triangle DBC$, since we've gone through those steps several times already. When you are writing solutions for your class or for a contest, you should include the steps we left out here (cite which angles are equal and why, then invoke AA). Only start leaving out the simple steps if you are certain that it is O.K. to do so.

We finish this section by exploring why AA Similarity works.

Problem 5.18: In the diagram we have two triangles ($\triangle ABE$ and $\triangle ACD$) with equal angles, and sides with lengths as marked. Find BE and DE without using AA Similarity. Can you use your method to prove why AA Similarity works?



Solution for Problem 5.18: We would like to show that $\triangle ABE \sim \triangle ACD$, so we start thinking about side length ratios. The only ratio tool we have that doesn't depend on having similar triangles already is our Same Base/Same Altitude technique of Section 4.3, so we try that. We don't have any triangles to use our technique on, so we draw $\overline{BD}$ and $\overline{CE}$ as shown to the right.



$\triangle ABE$ and $\triangle AEC$ share an altitude from E, so

$$\frac{[ABE]}{[ACE]} = \frac{AB}{AC} = \frac{6}{9} = \frac{2}{3}. \tag{5.1}$$

Similarly, $\triangle ABE$ and $\triangle ABD$ share an altitude from B, so

$$\frac{[ABE]}{[ABD]} = \frac{AE}{AD} = \frac{8}{8+DE}. \tag{5.2}$$

We suspect that $AB/AC = AE/AD$ because we suspect $\triangle ABE \sim \triangle ACD$. From (5.1) and (5.2), we have

$$\frac{AE}{AD} = \frac{[ABE]}{[ABD]} \qquad \text{and} \qquad \frac{AB}{AC} = \frac{[ABE]}{[ACE]}.$$

Since the numerators in our area ratios are the same, we need only show that $[ABD] = [ACE]$. These two areas share $[ABE]$, so we need only show that $[BEC] = [BED]$.

Since $\angle ABE = \angle ACD$, we know $\overline{BE} \parallel \overline{CD}$. Therefore, the altitudes from C and D to $\overleftrightarrow{BE}$ must be the same. Hence, triangles BEC and BED have the same base ($\overline{BE}$) and the same length altitudes to that base, so $[BEC] = [BED]$.

Finally, we can find DE. We have:

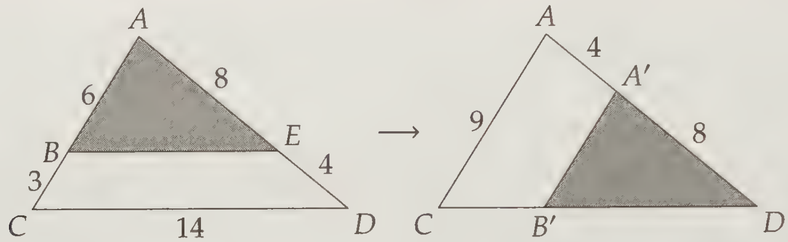
$$[AEC] = [AEB] + [BEC] = [AEB] + [BED] = [ABD],$$

so we can use our area ratios above. Since $[ABE]/[ACE] = [ABE]/[ABD]$, we have $AB/AC = AE/AD$, so

$$2/3 = 8/(8 + DE).$$

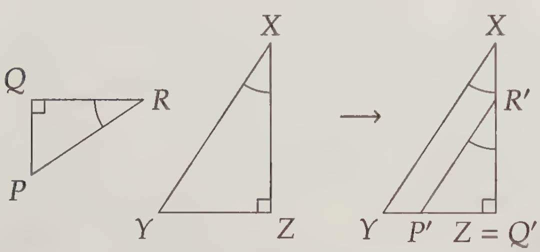
Solving this equation for DE, we find that $DE = 4$.

Now we very strongly suspect that $BE/CD = AE/AD$. To prove it, we use the same process we just followed.



We consider $\triangle A'B'D$ where A' is on $\overline{AD}$ and B' is on $\overline{CD}$ such that $A'D = AE$ and $BE = B'D$. Since $\overline{BE} \parallel \overline{CD}$, we have $\angle AEB = \angle ADC = \angle A'DB'$. Therefore, $\triangle A'B'D \cong \triangle ABE$ by SAS Congruence. (You can also think of $\triangle A'DB'$ as the result of sliding $\triangle ABE$ along $\overline{AD}$ until side $\overline{BE}$ is on $\overline{CD}$.) Since $A'D = AE$, we have $AA' = ED = 4$. We also have $\angle DA'B' = \angle EAB = \angle DAC$, so $\overline{AC} \parallel \overline{A'B'}$.

We can chase areas around as before to show that $B'D/CD = A'D/AD$, so $B'D = (2/3)(14) = 28/3$. Since $B'D = BE$, we have $BE = 28/3$. Note that because $B'D = BE$ and $A'D = AE$, we have shown that $BE/CD = AE/AD$, as suspected. □



Whenever we have a two triangles that have two angle measures in common, we can slide (and possibly flip) one triangle onto the other so that we get a diagram like that in Problem 5.18. For example, $\triangle PQR$ and $\triangle YZX$ in the diagram to the left have two angle measures in common (and consequently the third angles are equal, too). We can therefore move $\triangle PQR$ on top of $\triangle YZX$ such that two of the sides of the 'moved' triangle coincide with sides of $\triangle YZX$, as $\triangle P'Q'R'$ in the diagram shows.

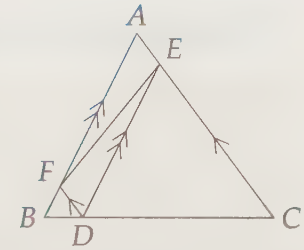
We can use the exact same approach as we used in Problem 5.18 to show that if two angles of one triangle equal the corresponding angles of the other, then each pair of corresponding lengths in the two triangles has the same ratio.

> **Concept:** Area can be a very useful problem solving tool even in problems that appear to have nothing to do with area.

Exercises

5.5.1 X and Y are on sides $\overline{PQ}$ and $\overline{PR}$, respectively, of $\triangle PQR$ such that $\overline{XY} \parallel \overline{QR}$. Given $XY = 5$, $QR = 15$, and $YR = 8$, find PY.

5.5.2 In the figure, the area of $\triangle EDC$ is 25 times the area of $\triangle BFD$.

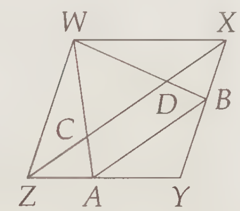


(a) Find CD/DB. **Hints:** 350

(b) Find $[EDC]/[ABC]$. **Hints:** 171

(c)★ Find $[AFE]/[ABC]$. **Hints:** 321

5.5.3 In the diagram, $\overline{WZ} \parallel \overline{XY}$ and $\overline{WX} \parallel \overline{ZY}$. $\overline{WA}$ and $\overline{WB}$ hit $\overline{XZ}$ at C and D, respectively, such that $ZC = XD$.



(a) Prove that $ZC/XC = AC/WC$.

(b) Prove that $XD/ZD = DB/WD$.

(c) Prove $\overline{CD} \parallel \overline{AB}$. **Hints:** 462

5.5.4 In the diagram at left below, $PQ = PR$, $\overline{ZX} \parallel \overline{QY}$, $\overline{QY} \perp \overline{PR}$, and $\overline{PQ}$ is extended to W such that $\overline{WZ} \perp \overline{PW}$.

(a) Show that $\triangle QWZ \sim \triangle RXZ$. **Hints:** 360

(b)★ Show that $YQ = ZX - ZW$. **Hints:** 172, 550

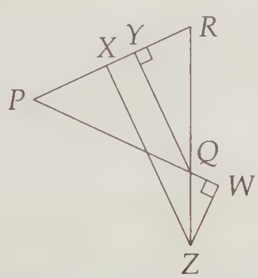


Figure 5.7: Diagram for Problem 5.5.4

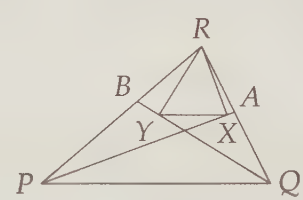


Figure 5.8: Diagram for Problem 5.5.5

5.5.5★ $\overline{PA}$ and $\overline{BQ}$ bisect angles $\angle RPQ$ and $\angle RQP$, respectively. Given that $\overline{RX} \perp \overline{PA}$ and $\overline{RY} \perp \overline{BQ}$, prove that $\overline{XY} \parallel \overline{PQ}$. **Hints:** 584, 152, 254

5.6 Construction: Angles and Parallels

In this section, we take a look at how to use our knowledge of similar triangles for geometric constructions. But first, since parallel lines and similar triangles are so closely related, we'll have to learn how to construct a parallel line.

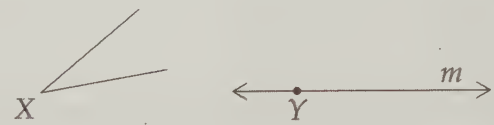
Don't forget the construction rules! Straightedge and compass only. No protractor. No measuring with your ruler.

Extra! *The shortest distance between two points is under construction.*

–Noelie Altito

Problems

Problem 5.19: Shown below are angle $\angle X$ and line m with point Y on it. Construct a line through Y that makes an angle with m that is equal to $\angle X$.



(a) Draw a circle with center X and another circle with the same radius and center Y.

(b) Use your circles from the first part to construct a point Z on the circle centered at Y such that $\overleftrightarrow{YZ}$ makes an angle with m equal to $\angle X$.

Problem 5.20: Given a line n and a point A not on n, construct a line through A that is parallel to n.

Problem 5.21: Draw a segment $\overline{AB}$. In this problem we learn how to divide $\overline{AB}$ into 3 equal pieces.

(a) Draw a line through A but not through B. Construct a segment on this line that has A as one endpoint. Call the other endpoint P.

(b) Construct a segment that has A as one endpoint, has P on the segment, and that is 3 times as long as $\overline{AP}$. Call the other endpoint of this segment R.

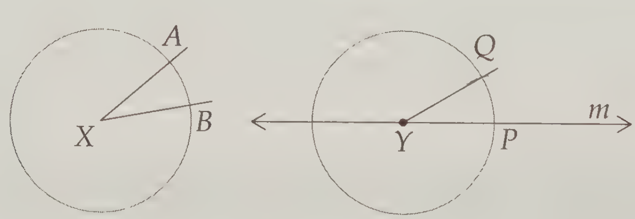
(c) Draw $\overline{RB}$. Construct point X on $\overline{AB}$ such that $AX/AB = 1/3$.

Equal angles are very important in our study of similar triangles and parallel lines, so we'll start our constructions by learning how to copy an angle.

Problem 5.19: Given $\angle X$ and line m with point Y on it, construct a line through Y that makes an angle with m that is equal to $\angle X$.

Solution for Problem 5.19: Since we measure angles by the portion of a circle that the angle cuts off, we start by making a circle with center X. Let the points where this circle hits the sides of $\angle X$ be A and B. We then make a circle with center Y *with the same radius*, since we want to cut off the same amount of this circle that $\angle X$ cuts off its circle. Let P be one of the points where this circle hits m.

We can't use a ruler to measure $\overline{AB}$ to tell how much of circle X that $\angle X$ cuts off, but we can use our compass! We open our compass to a width equal to AB, then draw an arc with center P and radius AB. Call the point where this arc meets circle Y point Q. Drawing $\overrightarrow{YQ}$ gives us angle $\angle QYP$ equal to $\angle X$.

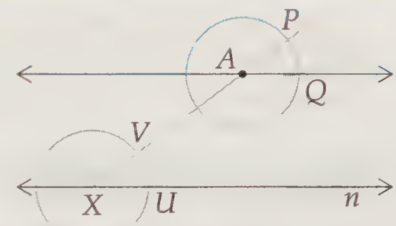


Are you convinced that this construction works? I'm not. We'll have to prove that $\angle Y = \angle X$ to see why the construction works. Since $\odot X$ and $\odot Y$ have the same radius, we have $XB = YP$ and $XA = YQ$. Since we opened the compass to width AB to make our arc centered at P, we have $AB = PQ$. Therefore, we have $\triangle BXA \cong \triangle PYQ$ by SSS, so we have $\angle X = \angle Y$! Now, I'm convinced. □

As we've seen, we can use equal angles to prove that two lines are parallel. Let's see if we can use equal angles to construct two parallel lines.

Problem 5.20: Given line n and point A not on n, construct a line through A that is parallel to n.

Solution for Problem 5.20: We know how to copy an angle, and equal angles give parallel lines, so we try copying an angle along line n to point A. But first, we'll need a line through A and an angle along n, so we draw a line through A that hits n at point X. We then copy angle X to point A just as we copied an angle in Problem 5.19. We draw circles with the same radii with centers A and X. We then set our compass to width UV and draw an arc centered at P with that radius. This arc hits $\odot A$ at point Q such that $\angle PAQ = \angle VXU$. Therefore, $\overleftrightarrow{AQ} \parallel n$. □



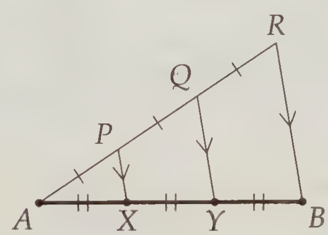
We haven't used parallel lines just for problems involving angles – we've also used them in problems involving similar triangles and ratios of lengths.

Problem 5.21: Given segment $\overline{AB}$, construct points X and Y on $\overline{AB}$ such that $AX = XY = YB = AB/3$.

Solution for Problem 5.21: We know how to divide a segment in half, but cutting it by a ratio like 1/3 calls for more advanced tools than simple midpoints. The best geometry tool we have for ratios is similar triangles, but it's not yet clear how we can use similar triangles.

Unsure how to deal with trisecting (dividing into three pieces) $\overline{AB}$, we try a simpler problem. Can we create a segment with A as an endpoint such that the segment is cut into three equal pieces? This isn't so hard – we start with a line through A, pick a different point P on the line, then copy $\overline{AP}$ twice along the line to get Q and R such that $AP = PQ = QR = AR/3$.

So, we have a trisected segment with A as an endpoint, but unfortunately, it isn't $\overline{AB}$ that we trisected. However, we do have that 1/3 ratio, so perhaps we can now use similar triangles. Similar triangles call for parallel lines. We draw $\overline{RB}$, then construct lines through P and Q that are parallel to $\overline{RB}$. These lines hit $\overline{AB}$ at X and Y.



We can use similar triangles to prove that X and Y are the points that divide $\overline{AB}$ into three equal pieces. Since $\overline{PX} \parallel \overline{QY} \parallel \overline{RB}$, we have $\triangle PAX \sim \triangle QAY \sim \triangle RAB$. Therefore $AX/AB = AP/AR = 1/3$ and $AY/AB = AQ/AR = 2/3$. Since $AX = AB/3$ and $AY = 2AB/3$, we have $AX = XY = YB = AB/3$, as desired. □

Notice that our first key step in finding this solution was thinking about an easier related problem.

Concept: When stuck on a problem, try solving an easier related problem. One way to do this with a construction problem is relaxing one of the constraints of the problem.

For example, in Problem 5.21, we relaxed the constraint that the trisected segment has both A and B as endpoints. Instead, we just created a trisected segment with A as an endpoint.

Although this is the end of the similar triangles chapter, this won't be the end of your study of similar triangles. You'll see them pop up in many more problems, and you'll find another whole chapter devoted to an application of similar triangles with the lofty name Power of a Point.

Exercises

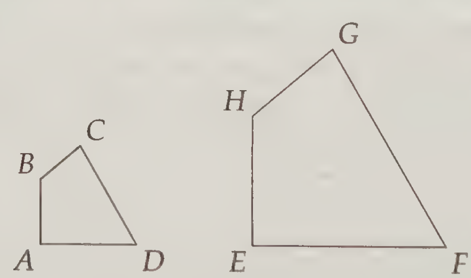
5.6.1 Given a segment of length 1, construct a segment with length 1/5. Construct a segment with length $2\frac{2}{3}$.

5.6.2 Given a triangle $\triangle ABC$, construct a triangle $\triangle XYZ$ that is similar to $\triangle ABC$, but has 9 times the area of $\triangle ABC$.

5.7 Summary

Definition: Two figures are **similar** if one is simply a blown-up, and possibly rotated and/or flipped, version of the other.

Important: Corresponding angles in similar figures are equal, and the ratio of the lengths of corresponding sides of similar triangles is always the same.



In similar quadrilaterals $ABCD$ and $EHGF$, we have $\angle A = \angle E$, $\angle B = \angle H$, $\angle C = \angle G$, and $\angle D = \angle F$. We also have

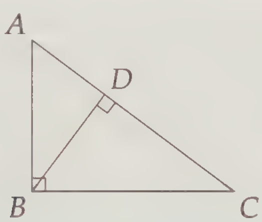
$$\frac{AB}{EH} = \frac{BC}{HG} = \frac{CD}{GF} = \frac{DA}{FE} = \frac{AC}{EG} = \frac{BD}{HF}.$$

We denote these figures as similar by writing $ABCD \sim EHGF$.

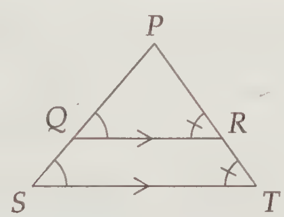
There are three main ways to show that two triangles are similar:

- **AA Similarity**. If two angles of one triangle equal two angles of another, then the triangles are similar. This is by far the most commonly used method to prove two triangles are similar. (Section 5.2)
- **SAS Similarity**. If two sides in one triangle are in the same ratio as two sides in another triangle, and the angles between the sides in each triangle equal each other, then the triangles are similar. (Section 5.3)
- **SSS Similarity**. If each side of one triangle is the same constant multiple of the corresponding side of another triangle, then the triangles are similar. (Section 5.4)

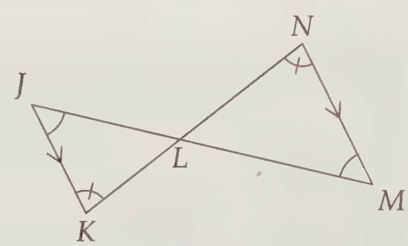
Parallel lines and perpendicular lines are clues to look for similar triangles. Three very common set-ups that contain similar triangles are shown below.



$\triangle ABC \sim \triangle BDC \sim \triangle ADB$



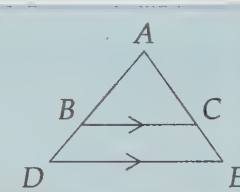
$\triangle PQR \sim \triangle PST$



$\triangle JLK \sim \triangle MLN$

Important: If $\overline{BC} \parallel \overline{DE}$ and $\overleftrightarrow{BD}$ and $\overleftrightarrow{CE}$ meet at A as shown, then

$$\frac{AB}{BD} = \frac{AC}{CE}.$$



Important: If two triangles are similar such that the sides of the larger triangle are k times the sides of the smaller, then the area of the larger triangle is k^2 times that of the smaller.

This relationship holds for any pair of similar figures, not just for triangles.

Problem Solving Strategies

Concepts:

- When you're stuck on a problem, ask yourself, 'What piece of information have I not used?'

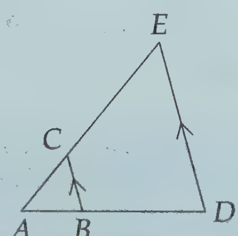
Continued on the next page. . .

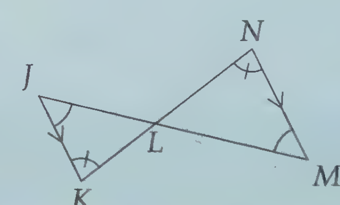
Concepts: *. . . continued from the previous page*

- In many problems, there's more than meets the eye. Extending segments that seem to end abruptly (particularly in the middle of a triangle) can often yield quick solutions.
- When stuck on a problem, try solving an easier related problem. For constructions, useful easier related problems often involve relaxing one of the constraints of the problem.
- Consider using similar triangles in problems involving ratios of segment lengths.

Things To Watch Out For!

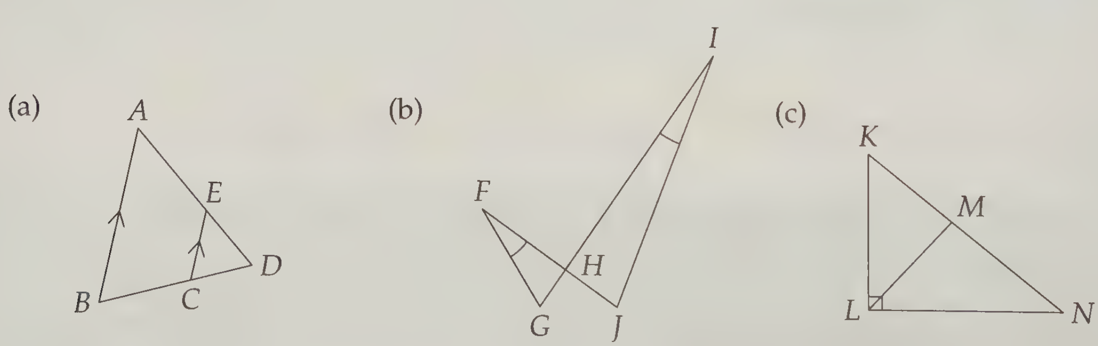
WARNING!! Below are shown two common situations that lead to mistakes. The diagram on the left may lead you to write '$\triangle ABC \sim \triangle ADE$, so $AB/BD = BC/DE$.' The one on the right might lead to '$\triangle JKL \sim \triangle NLM$, so $JL/NL = KL/ML$.' Both of these are **incorrect**! Make sure you see why!



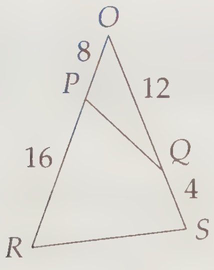


REVIEW PROBLEMS

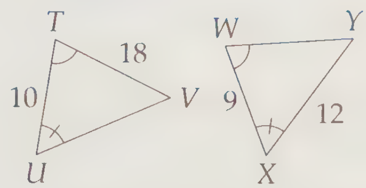
5.22 In each of the parts below, either identify all pairs of similar triangles or state that there are not any pairs of triangles that are necessarily similar. For each pair of similar triangles you find, state why the triangles are similar.



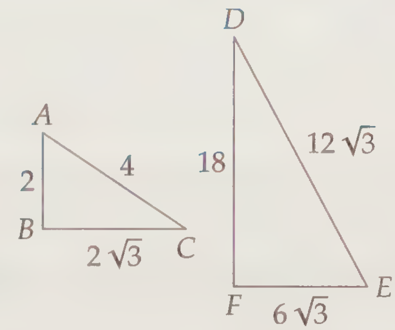
(d)



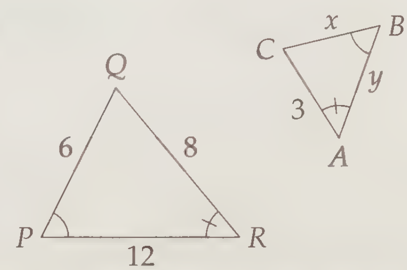
(e)



(f)



5.23 Find x and y in the diagram at right, given the angle equalities and side lengths shown in $\triangle PQR$ and $\triangle ABC$.



5.24 Points P and Q are on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{PQ} \parallel \overline{BC}$. Given $AB = 12$, $PB = 9$, and $AC = 18$, find QA.

5.25 The side lengths of a triangle are 4 centimeters, 6 centimeters, and 9 centimeters. One of the side lengths of a similar triangle is 36 centimeters. What is the maximum number of centimeters possible in the perimeter of the second triangle? *(Source: MATHCOUNTS)*

5.26 What's wrong with the diagram shown at left below?

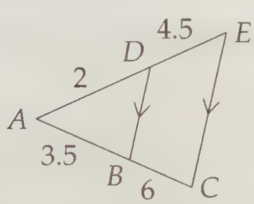


Figure 5.9: Diagram for Problem 5.26

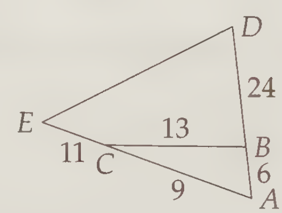


Figure 5.10: Diagram for Problem 5.27

5.27 Find DE in the diagram at right above.

5.28 Why is the diagram shown at left below impossible?

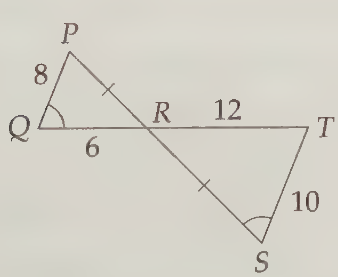


Figure 5.11: Diagram for Problem 5.28

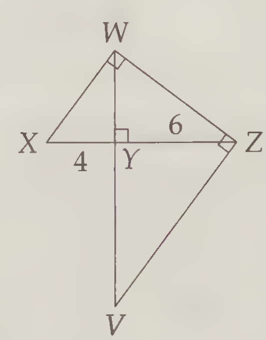
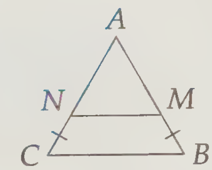


Figure 5.12: Diagram for Problem 5.29

5.29 In the diagram at right above, find WY and YV.

5.30 $\triangle ABC$ at right is equilateral. M is on $\overline{AB}$ and N on $\overline{AC}$ such that $BM = CN$.



(a) Prove that $AM = AN$.

(b) Prove that $\triangle AMN$ is equilateral.

5.31 Given $\triangle ABC \sim \triangle YZX$, $[ABC] = 40$, $[YZX] = 360$, $AB = 9$, and $BC = 12$, find the following:

(a) YZ.

(b) The length of the altitude to side $\overline{XZ}$ of triangle $\triangle YZX$.

5.32 Let $ABCD$ be a rectangle as shown at left below, with $AB = 25$ and $BC = 12$. Let E be a point on $\overline{AB}$, such that $AE < BE$ and triangles AED and BCE are similar. Find AE.

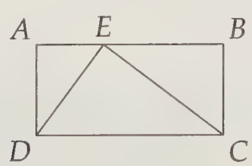


Figure 5.13: Diagram for Problem 5.32

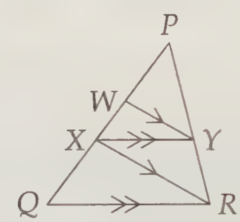


Figure 5.14: Diagram for Problem 5.33

5.33 In the diagram at right above, $PW = 6$ and $WX = 4$. Find QX.

5.34 *(Try this without looking back in the text first!)* In the diagram at left below, $\overline{AP} \parallel \overline{BQ} \parallel \overline{CR}$. Prove that

$$\frac{1}{CR} = \frac{1}{AP} + \frac{1}{BQ}.$$

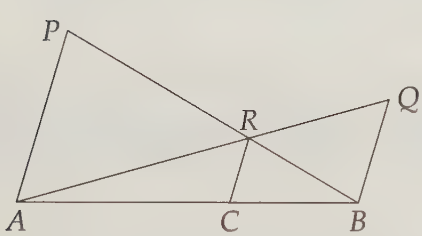


Figure 5.15: Diagram for Problem 5.34

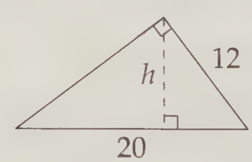


Figure 5.16: Diagram for Problem 5.35

5.35 Two of the sides in the right triangle at right above have length 12 cm and 20 cm, as shown. What is the number of centimeters in the length of the altitude h drawn to the side with length 20 cm? *(Source: MATHCOUNTS) (If you know the Pythagorean Theorem, try doing this problem without it!)*

5.36 Let ABC be a triangle, and let D and E be points on sides $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{DE} \parallel \overline{BC}$. Prove that

$$\frac{AD}{AE} = \frac{DB}{CE}.$$

(Try to do this one without looking back in the text for the proof!)

Challenge Problems

5.37 Let ABC be a triangle, and let D and E be points on $\overline{AB}$ and $\overline{AC}$, respectively, such that $AD/AE = BD/EC$. Prove that $\overline{DE} \parallel \overline{BC}$. *Make sure you see why this differs from the previous problem!* **Hints:** 363, 179

5.38 If the sum of one of the base angles and the vertex angle is the same for two different isosceles triangles, must the triangles be similar? **Hints:** 196

5.39 In the figure at left below, isosceles $\triangle ABC$ with base $\overline{AB}$ has altitude $CH = 24$ cm. $DE = GF$, $HF = 12$ cm, and $FB = 6$ cm. Find the area of $CDEFG$. *(Source: MATHCOUNTS)* **Hints:** 490

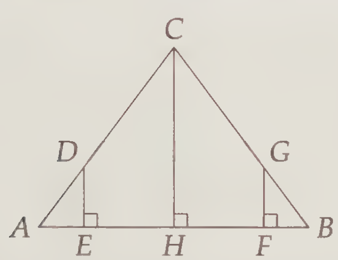


Figure 5.17: Diagram for Problem 5.39

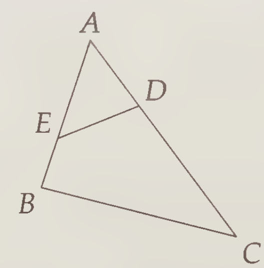


Figure 5.18: Diagram for Problem 5.40

5.40 In triangle ABC at right above, D and E are points on sides $\overline{AC}$ and $\overline{AB}$, respectively, such that $(AD)(AC) = (AE)(AB)$. Prove that $\angle CDE + \angle CBE = 180^\circ$ and $\angle ADB + \angle BEC = 180^\circ$. **Hints:** 269, 70

5.41 D, E, and F are on sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, of $\triangle ABC$ such that $\overline{DE} \parallel \overline{AB}$, $\overline{DF} \parallel \overline{AC}$, and $\overline{BC} \parallel \overline{EF}$. Prove that D, E, and F are the midpoints of the sides of $\triangle ABC$. **Hints:** 24

5.42 In the diagram at left below, $\overline{PS} \parallel \overline{QT}$ and $\overline{PQ} \parallel \overline{ST}$. Prove that $SU/SP = QP/QR$.

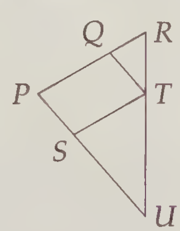


Figure 5.19: Diagram for Problem 5.42

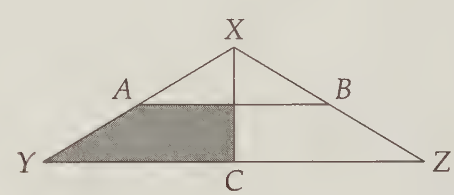


Figure 5.20: Diagram for Problem 5.43

5.43 The area of triangle XYZ at right above is 8 square inches. Points A and B are midpoints of congruent segments $\overline{XY}$ and $\overline{XZ}$. Altitude $\overline{XC}$ bisects $\overline{YZ}$. What is the area of the shaded region? *(Source: AMC 8)* **Hints:** 77, 162

5.44 The midpoints of the three sides of an equilateral triangle are connected to form a second triangle. A third triangle is formed by connecting the midpoints of the second triangle. This process is repeated until a tenth triangle is formed. What is the ratio of the perimeter of the tenth triangle to that perimeter of the third triangle? *(Source: MATHCOUNTS)* **Hints:** 72

5.45 In rectangle $ABCD$ at left below, $AB = 5$ and $BC = 3$. Points F and G are on $\overline{CD}$ so that $DF = 1$ and $GC = 2$. Lines $\overleftrightarrow{AF}$ and $\overleftrightarrow{BG}$ intersect at E. Find the area of $\triangle AEB$. *(Source: AMC 10)* **Hints:** 99

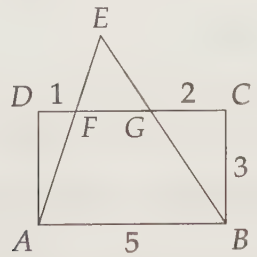


Figure 5.21: Diagram for Problem 5.45

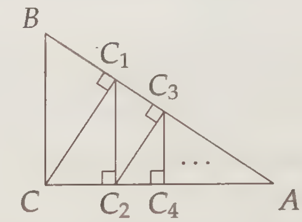
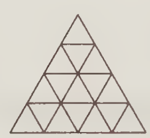


Figure 5.22: Diagram for Problem 5.46

5.46★ In triangle ABC at right above, $AC = 4$, $BC = 3$, $AB = 5$, and $\angle ACB = 90^\circ$. The infinite sequence of points $C_1, C_2, C_3, C_4, \ldots$ is generated as follows: C_1 is the foot of the altitude from C to side $\overline{AB}$, C_2 is the foot of the altitude from C_1 to side $\overline{AC}$, C_3 is the foot of the altitude from C_2 to side $\overline{AB}$, and so on. Calculate the sum $CC_1 + C_1C_2 + C_2C_3 + C_3C_4 + \cdots$. **Hints:** 138, 307

5.47★ In the figure shown, we have taken an equilateral triangle and divided each side into four segments of equal length. We have then connected these points to form smaller equilateral triangles.



Consider instead dividing each side into n segments of equal length, where n is some positive integer, then connecting these points as before to form smaller equilateral triangles. Use this dissection of the original equilateral triangle to prove that the sum of the first n positive odd integers is n^2. **Hints:** 367, 475

5.48★ Figure $ABCD$ at left below has sides $AB = 6$, $CD = 8$, $BC = DA = 2$, and $\overline{AB} \parallel \overline{CD}$. Segments are drawn from the midpoint P of $\overline{AB}$ to points Q and R on side $\overline{CD}$ so that $\overline{PQ}$ and $\overline{PR}$ are parallel to $\overline{AD}$ and $\overline{BC}$, as shown. Diagonal $\overline{DB}$ intersects $\overline{PQ}$ at X and $\overline{PR}$ at Y. Evaluate PX/YR. *(Source: Mandelbrot)* **Hints:** 58, 176

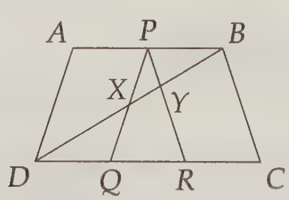


Figure 5.23: Diagram for Problem 5.48

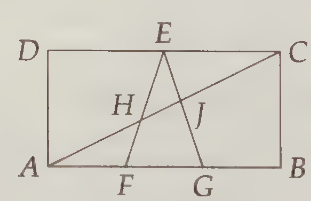


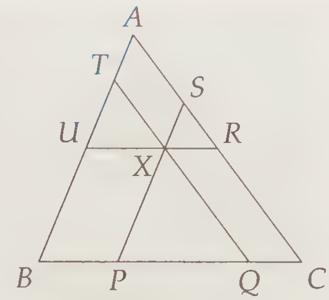
Figure 5.24: Diagram for Problem 5.49

5.49★ In rectangle $ABCD$ at right above, points F and G lie on $\overline{AB}$ so that $AF = FG = GB$ and E is the midpoint of $\overline{DC}$. Also, $\overline{AC}$ intersects $\overline{EF}$ at H and $\overline{EG}$ at J. The area of rectangle $ABCD$ is 70. Find the area of $\triangle AHF$. *(Source: AMC 12)* **Hints:** 257, 319, 393

5.50★ Given an acute triangle $\triangle ABC$, construct with straightedge and compass square $DEFG$ such that D and E are on $\overline{BC}$, G is on $\overline{AB}$, and F is on $\overline{AC}$. **Hints:** 201, 132

5.51★ Points P, Q, R, S, T, and U are on the sides of triangle ABC, as shown, such that line segments $\overline{UR}$, $\overline{QT}$, and $\overline{SP}$ all pass through point X, and are parallel to $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Prove that

$$\frac{PQ}{BC} + \frac{RS}{CA} + \frac{TU}{AB} = 1.$$



Hints: 353, 223, 261

Extra! A **polyomino**, a term coined by USC mathematics professor Solomon W. Golomb in 1953, is simply a piece consisting of a number of connected squares. Evidently, it is a generalization of a domino, a piece consisting of two connected squares.

The number of squares is indicated by the prefix, so a **monomino** consists of one square, a **domino** has two, a **triomino** has three, a **tetromino** has four, a **pentomino** has five, and so on. The five tetrominoes are shown below. (You might recognize them from the video game Tetris!)

The first obvious question to ask is, "How many polyominoes are there that contain a given number of squares?" The answer is partially answered by the following table.

Number of squares	Number of polyominoes
1	1
2	1
3	2
4	5
5	12
6	35
7	108
8	369
9	1285
10	4655

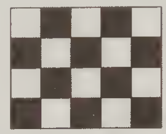
At the time of writing, the number of polyominoes is known up to 56 squares, but no general formula is known.

The next obvious question is to ask is, "What interesting shapes can these polyominoes make?" For example, can the five tetrominoes be used to cover a 4×5 chessboard? Try to use the tetrominoes to cover such a chessboard, or prove it is impossible to do so, before turning the page.

Continued on the next page. . .

Extra! *. . . continued from the previous page*

The answer turns out to be "no." But how can we prove this? Just saying that we have tried many different ways and given up is not a very satisfactory answer, and ultimately not rigorous. It turns out there is a simple proof, using a common problem solving technique. We color the 4×5 chessboard as shown.



Let's take another look at those tetrominoes again.

If we place the first piece on the rectangle, how many black and white squares will it cover? It's not hard to see that it will always cover two black squares and two white squares.

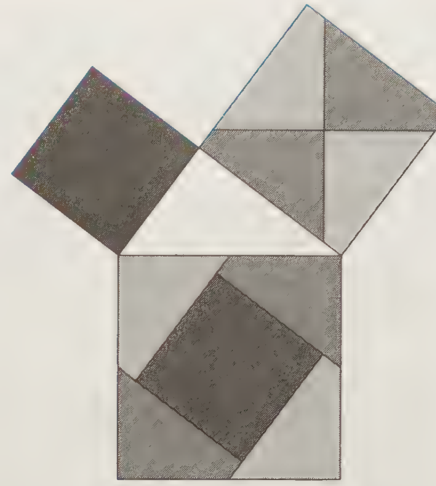
The same holds for the fourth and fifth pieces.

And there are two different combinations of colored squares the second piece can cover, but both still have two black squares and two white squares.

That means that these four pieces, together, will always cover eight black squares and eight white squares. But what about the third piece? It can be placed two different ways.

One way covers three white squares and one black square, and the other way covers one white square and three black squares. This means that the five pieces, together, will always cover eleven white squares and nine black squares, or nine white squares and eleven black squares, which means in particular they can never cover ten white squares and ten black squares, as shown. Thus, we have used **parity** (even-ness and odd-ness) and chessboard coloring to prove that a covering cannot exist.

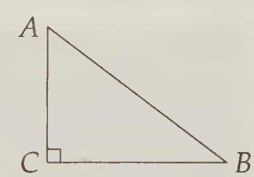
The twelve pentominoes, on the other hand, can fit snugly inside a 6×10 rectangle. See if you can figure out how. (Answer on page 285.)



The Pythagorean Theorem

Choose always the way that seems the best, however rough it may be; custom will soon render it easy and agreeable.
– Pythagoras

CHAPTER 6 Right Triangles



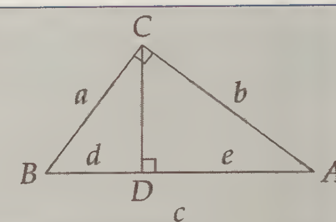
In this chapter we study right triangles. As a quick refresher, a right triangle is a triangle that has a right angle among its angles. The side opposite the right angle is called the **hypotenuse**, and the other two sides are called **legs**. In the figure to the right, $\overline{AB}$ is the hypotenuse, while $\overline{AC}$ and $\overline{BC}$ are both legs. As you'll see throughout the book, many problems are solved by building right triangles and using the principles you'll learn in this chapter, particularly one of the most famous math theorems of all: the Pythagorean Theorem.

Extra! It has long been a sport among lovers of mathematics to find new proofs of the Pythagorean Theorem. Most engaging are the proofs that can be expressed simply as a diagram, challenging the viewer to fill in the details and learn how the diagram provides a proof of the Pythagorean Theorem. Pure mathematicians are not the only people who get to join in this game; proofs have been attributed to President James A. Garfield and Leonardo da Vinci (among many others). We'll share several of these 'proofs without words' with you in this book. You can find a long list of Pythagorean Theorem proofs at Alexander Bogomolny's excellent Interactive Mathematics Miscellany website and in Roger Nelsen's two *Proofs Without Words* books.

6.1 Pythagorean Theorem

Problems

Problem 6.1: In this problem we will prove one of the most famous theorems of mathematics. The diagram shows right triangle $\triangle ABC$ with hypotenuse $AB = c$ and legs $AC = b$ and $BC = a$.

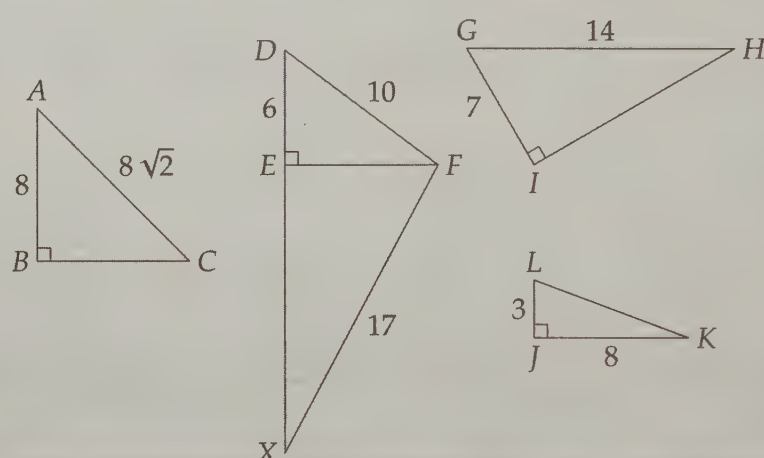


(a) Prove that $a^2 = cd$.

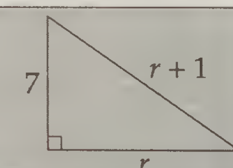
(b) Prove that $b^2 = ce$.

(c) Use the first two parts to show that $a^2 + b^2 = c^2$. This is the **Pythagorean Theorem**.

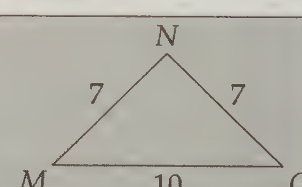
Problem 6.2: Find the missing sides in the triangles below.



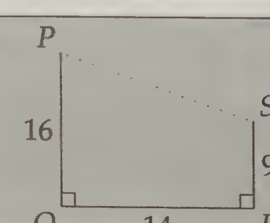
Problem 6.3: Find r.



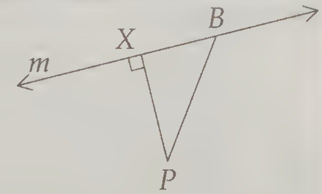
Problem 6.4: Isosceles triangle $\triangle MNO$ has $MN = NO = 7$ and $MO = 10$. Find $[MNO]$.



Problem 6.5: Two vertical poles are 14 feet apart. One is 16 feet tall, and the other is 9 feet tall. A rope extends straight from the top of one pole to the top of the other pole. How long is the rope?



Problem 6.6: Let P be a point and m be a line that doesn't pass through P. Let X be on m such that $\overline{PX} \perp m$. The distance from a point to a line is the length of the shortest segment from the point to the line. In this problem, we prove that the distance from P to m is PX.



(a) Let B be a point on m besides X. What kind of triangle is $\triangle PXB$?

(b) Use the Pythagorean Theorem to get an expression for PB^2.

(c) Why does the previous part show that PX is the shortest possible distance from P to line m?

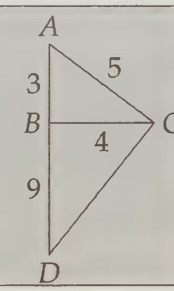
Problem 6.7: In this problem we investigate whether $\triangle ABC$ must be a right triangle given that $BC = a$, $AC = b$, $AB = c$, and $a^2 + b^2 = c^2$.

(a) Suppose we have a triangle $\triangle DEF$ such that $\angle F = 90^\circ$, $EF = a$, and $DF = b$. (These are the same lengths as BC and AC in $\triangle ABC$.) What is DE?

(b) What can we say about triangles $\triangle ABC$ and $\triangle DEF$?

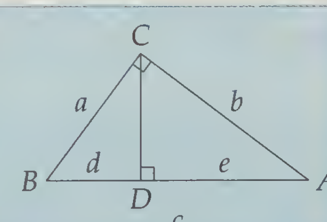
(c) Must $\triangle ABC$ be a right triangle?

Problem 6.8: The side lengths are as marked in the diagram. Find CD.



We start off with a proof of one of the most famous theorems in mathematics.

Problem 6.1: Right triangle $\triangle ABC$ has hypotenuse $AB = c$ and legs $AC = b$ and $BC = a$. Use similar triangles formed by altitude $\overline{CD}$ to prove that $a^2 + b^2 = c^2$.



Solution for Problem 6.1: As we showed in Problem 5.14, we can use $\triangle ADC \sim \triangle ACB$ to show that $b/e = c/b$, so $b^2 = ce$. Similarly, $\triangle BDC \sim \triangle BCA$ gives us $a/d = c/a$, so $a^2 = cd$. Adding these expressions for a^2 and b^2 gives us

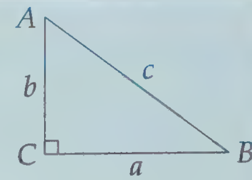
$$a^2 + b^2 = cd + ce = c(d + e) = c^2.$$

□

Extra! *Mighty is geometry; joined with art, resistless.*

–Euripides

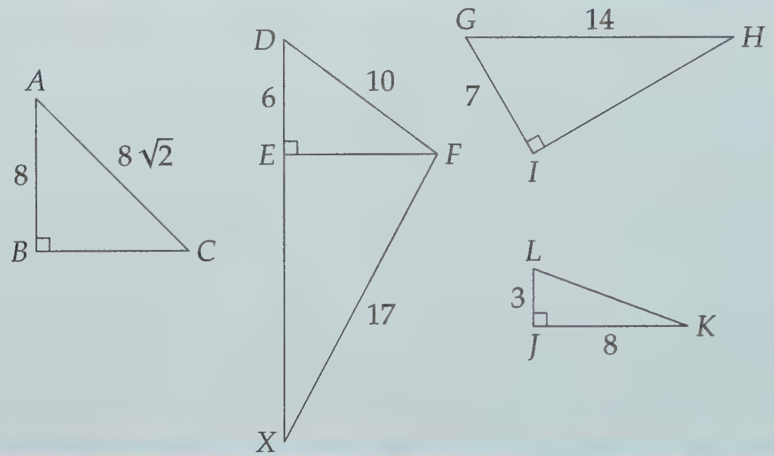
Important: In any right triangle, the sum of the squares of the legs equals the square of the hypotenuse. In the figure to the right, we have



$$a^2 + b^2 = c^2$$

This is the famous **Pythagorean Theorem**.

Problem 6.2: Find the missing sides in the triangles below.



Solution for Problem 6.2:

(a) $\triangle ABC$: $AB^2 + BC^2 = AC^2$, so $64 + BC^2 = 128$. Therefore, $BC^2 = 64$ and $BC = 8$.

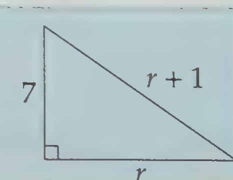
(b) $\triangle DEF$ & $\triangle FEX$: From $\triangle DEF$ we have $DE^2 + EF^2 = DF^2$, so $36 + EF^2 = 100$, and $EF = 8$. Now we can tackle EX with $\triangle FEX$: $EX^2 + FE^2 = FX^2$, so $EX^2 = FX^2 - EF^2 = 225$. So, $EX = 15$.

(c) $\triangle GHI$: $HI^2 + GI^2 = GH^2$, so $HI^2 = 196 - 49 = 147$. Therefore, $HI = \sqrt{147} = 7\sqrt{3}$.

(d) $\triangle JKL$: $KL^2 = LJ^2 + JK^2 = 9 + 64 = 73$. So, $KL = \sqrt{73}$.

□

Problem 6.3: Find r.



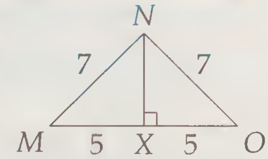
Solution for Problem 6.3: We have a right triangle, so we can apply the Pythagorean Theorem:

$$7^2 + r^2 = (r+1)^2.$$

Therefore, $49 + r^2 = r^2 + 2r + 1$, so $r = 24$. □

Sometimes it's not the sides of a right triangle we are missing, but the whole right triangle itself! In these cases, we have to build the right triangle ourselves, then apply the Pythagorean Theorem.

Problem 6.4: Isosceles triangle $\triangle MNO$ has $MN = NO = 7$ and $MO = 10$. Find $[MNO]$.



Solution for Problem 6.4: Finding an altitude will allow us to find the area. When we first studied isosceles triangles on page 67, we saw that we can cut an isosceles triangle in half with an altitude. We do so here in the figure to the right, in which altitude $\overline{NX}$ cuts $\overline{MO}$ in half so that $MX = XO = 5$. The Pythagorean Theorem applied to $\triangle MNX$ gives

$$MX^2 + NX^2 = MN^2,$$

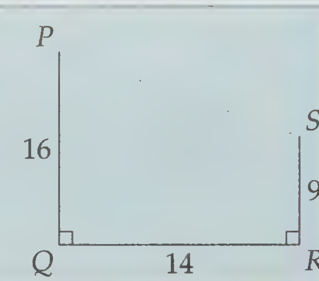
so $25 + NX^2 = 49$. Therefore, $NX = \sqrt{24} = 2\sqrt{6}$, and our area is

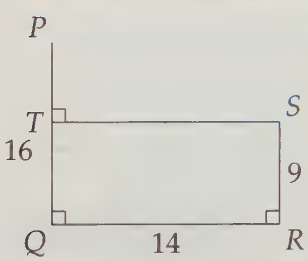
$$[MNO] = \frac{(NX)(MO)}{2} = 10\sqrt{6}.$$

□

In our next example, it's not as clear initially how to build a useful right triangle.

Problem 6.5: Two vertical poles are 14 feet apart. One is 16 feet tall, and the other is 9 feet tall. A rope extends straight from the top of one pole to the top of the other pole. How long is the rope?





Solution for Problem 6.5: We don't have a right triangle, but the right angles at Q and R have us thinking about building our own. We draw $\overline{ST}$ perpendicular to $\overline{PQ}$, and we have a right triangle with hypotenuse $\overline{PS}$! Since $\angle Q + \angle STQ = 180^\circ$, we have $\overline{QR} \parallel \overline{TS}$. Therefore, $\angle TSR + \angle R = 180^\circ$, so $\angle TSR = 90^\circ$, and $STQR$ is a rectangle. Since $STQR$ is a rectangle, we have $TS = QR = 14$ and $TQ = SR = 9$. Therefore, $PT = PQ - SR = 7$, and we can now use the Pythagorean Theorem to find PS:

$$PS^2 = PT^2 + TS^2 = 49 + 196 = 245.$$

Therefore, the length of our rope is $\sqrt{245} = 7\sqrt{5}$. □

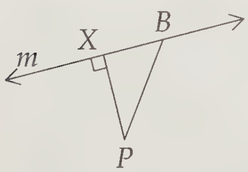
Concept: Building right triangles and applying the Pythagorean Theorem is one of the most common ways to find lengths. This is particularly true in problems that already involve right angles, such as Problem 6.5.

Problem 6.6: The distance from a point to a line is the length of the shortest segment from the point to the line. Prove that this shortest segment is perpendicular to the line.

Solution for Problem 6.6: This is a great example of proving the obvious. Sometimes the obvious is so obvious that people will propose solutions like this:

Bogus Solution: The shortest distance from P to the line m will be when we go straight from P to the line. This would make a right angle with m because we get a right angle when we go straight from the point to the line.

This 'proof' doesn't say anything at all. There aren't any false statements, but there aren't any useful statements either. It essentially argues 'the angle is a right angle because that's what you get when you make a right angle.' Here's why we get a right angle:



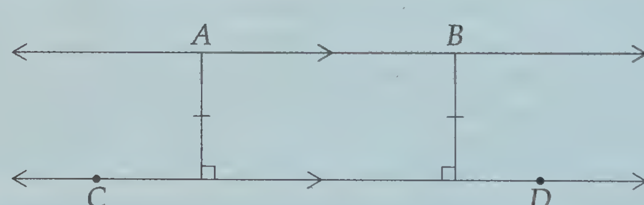
We start by drawing a diagram. Let P be our point and m be our line. Let X be on m such that $\overline{PX} \perp m$. We must show that $\overline{PX}$ is the shortest segment from P to m, so we must show that $\overline{PX}$ is shorter than any other segment from P to m. We let B be another point on m, and compare PX to PB.

Now we have a right triangle, $\triangle PXB$, so we can use the Pythagorean Theorem: $PB^2 = PX^2 + XB^2$. Since XB^2 is clearly greater than 0, we must have $PB^2 > PX^2$, so $PB > PX$. So, the perpendicular segment from P to m is shorter than any other segment from P to m. □

Important: The distance from a point to a line is the length of the segment from the point to the line that is perpendicular to the line.

You'll have a chance to use this definition of the distance from a point to a line to prove the following as an Exercise:

Important: If $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, then A and B are the same distance from $\overleftrightarrow{CD}$.



Throughout this section we have used the Pythagorean Theorem to find information about the sides of right triangles. Perhaps you're wondering if we can use the Pythagorean Theorem 'in reverse' to determine if a triangle is a right triangle.

Problem 6.7: Suppose we have $\triangle ABC$ such that $BC = a$, $AC = b$, $AB = c$, and $a^2 + b^2 = c^2$. Is $\triangle ABC$ necessarily a right triangle?

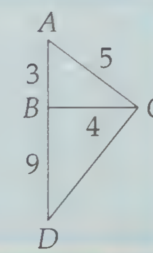
Solution for Problem 6.7: If $a^2 + b^2 = c^2$, we do know that there is some triangle with side lengths a, b, c that is a right triangle. Specifically, consider $\triangle DEF$ with $\angle F = 90^\circ$, $EF = a$, and $DF = b$. Then, the Pythagorean Theorem tells us that $DE^2 = EF^2 + DF^2 = a^2 + b^2$. Since we know that $a^2 + b^2 = c^2$, we have $DE = c$. Therefore, $\triangle DEF$ is a right triangle with sides of length a, b, and c. Since $\triangle ABC$ has the same side lengths as $\triangle DEF$, we have $\triangle ABC \cong \triangle DEF$ by SSS. Therefore, $\triangle ABC$ must also be a right triangle. □

Important: If the sides of a triangle satisfy the Pythagorean Theorem, then the triangle must be a right triangle. Any time you see triangle sides that satisfy the Pythagorean Theorem, you should mark the angle opposite the longest side as a right angle.

Sidenote: The ancient Egyptians used this method of determining a triangle is right in construction. They used loops of rope consisting of 12 rope segments of equal length. When pulled taut to form a triangle with sides equal to 3, 4, and 5 of these segments, they had a right angle opposite the 5-segment side! They could then use this to make sure angles on buildings or bricks, etc., were right angles.

Let's put this principle to work in a simple problem.

Problem 6.8: The side lengths are as marked in the diagram. Find CD.



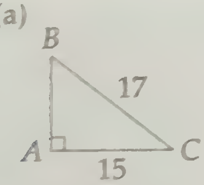
Solution for Problem 6.8: Since $AB^2 + BC^2 = AC^2$, we know that $\triangle ABC$ is a right triangle with right angle at B. Therefore, $\triangle CBD$ is a right triangle with hypotenuse $\overline{CD}$, so we can use the Pythagorean Theorem:

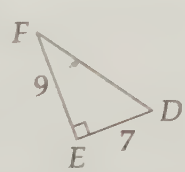
$$CD^2 = BC^2 + BD^2 = 16 + 81 = 97,$$

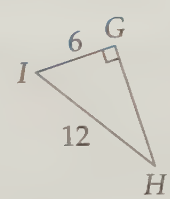
so $CD = \sqrt{97}$. □

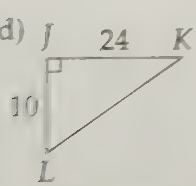
Exercises

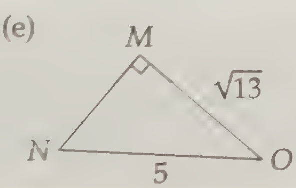
6.1.1 Find the missing side length in each of the diagrams below:

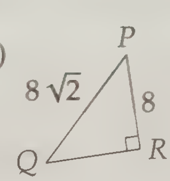
(a)


(b)


(c)


(d)


(e)


(f)


6.1.2 Prove that the two non-right angles of a right triangle are complementary.

6.1.3 One leg of a right triangle is 3 cm more than 3 times the other leg, and the hypotenuse is 1 cm longer than the longest leg. Find the area of the triangle. **Hints:** 276

6.1.4 Prove that in a right triangle, the hypotenuse is the longest side.

6.1.5 Which of the following groups of three numbers can be the side lengths of a right triangle?

(a) $6, 8, 10$.

(b) $4, 5, 6$.

(c) $9, 3\sqrt{3}, 6\sqrt{3}$.

(d) $5/8, 3/2, 13/8$.

(e) $2\sqrt{2}, 3\sqrt{2}, 5$.

(f) $1.2, 3.5, 3.7$.

6.1.6 Find the area of a triangle with sides of length $\sqrt{6}$, $\sqrt{7}$, and $\sqrt{13}$.

6.1.7 Find the area of an isosceles triangle with two legs of length 8 and base of length 6. **Hints:** 61

6.1.8 In this problem, we show that if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, then A and B are the same distance from $\overleftrightarrow{CD}$.

(a) Let X and Y be on $\overleftrightarrow{CD}$ such that $\overline{AX} \perp \overleftrightarrow{CD}$ and $\overline{BY} \perp \overleftrightarrow{CD}$. Show that $\angle XAB = 90^\circ$.

(b) Prove that $\triangle AXB \cong \triangle YBX$, and use this to prove that A and B are equidistant from $\overleftrightarrow{CD}$.

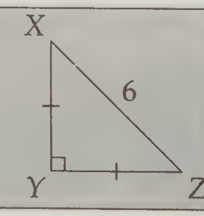
6.1.9 Initially, a 25-foot ladder rests against a vertical wall such that the top of the ladder is 24 feet from the ground. Then, Nathan moves the base of the ladder farther out from the wall so that the top of the ladder slides down until resting against the wall at a point 20 feet above the ground. Given that the wall is perpendicular to the ground, how far did Nathan move the base of the ladder?

6.1.10★ A triangle has sides measuring 13 cm, 13 cm, and 10 cm. A second triangle is drawn with sides measuring 13 cm, 13 cm and x cm, where x is a whole number other than 10. If the two triangles have equal areas, what is the value of x? *(Source: MATHCOUNTS)* **Hints:** 161, 326

6.2 Two Special Right Triangles

Problems

Problem 6.9: $\triangle XYZ$ is an isosceles right triangle as shown with $\angle Y = 90^\circ$. Given that $XZ = 6$, find $\angle X$ and XY.

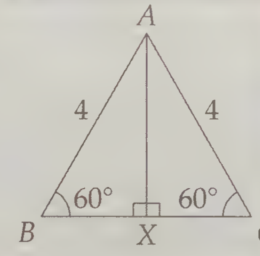


Problem 6.10:

(a) If a right triangle is isosceles, must it be the two legs that are the equal sides?

(b) Show that in any isosceles right triangle, the length of the hypotenuse is $\sqrt{2}$ times the length of a leg.

Problem 6.11: In this problem we will derive a formula for the area of an equilateral triangle. We'll start with $\triangle ABC$, an equilateral triangle with side length 4.



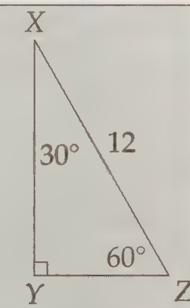
(a) Let $\overline{AX}$ be an altitude of $\triangle ABC$ as shown. Find BX.

(b) Find AX.

(c) Find the area of $\triangle ABC$.

(d) Let $\triangle PQR$ be an equilateral triangle with side length s. Find $[PQR]$ in terms of s.

Problem 6.12: Our goal in this problem is to find YZ and YX in the triangle shown.



(a) Suppose we extend $\overline{YZ}$ past Y to a point D such that $YD = YZ$. How are triangles $\triangle XYZ$ and $\triangle XYD$ related?

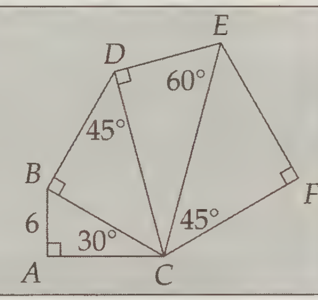
(b) What kind of triangle is $\triangle XZD$?

(c) Find YZ and YX.

(d) How are the sides of a triangle that have angle measures 30°, 60°, and 90° related?

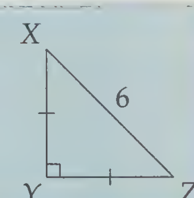
(e) Extra challenge: Suppose $\triangle PQR$ has sides 3, $3\sqrt{3}$, and 6. Can we deduce that the angles of $\triangle PQR$ are 30°, 60°, and 90°?

Problem 6.13: Find all the missing sides in the diagram.



We know a special side relationship when a triangle is isosceles, and another special relationship when a triangle is a right triangle. What if the triangle is both?

Problem 6.9: $\triangle XYZ$ is an isosceles right triangle with $\angle Y = 90°$. Given that $XZ = 6$, find $\angle X$ and XY.



Solution for Problem 6.9: Since the triangle is isosceles, the angles opposite the equal sides are equal. Therefore, $\angle X = \angle Z$. Since $\angle Y$ is a right angle, $\angle X$ and $\angle Z$ must add to 90°. So, $\angle X = \angle Z = 90°/2 = 45°$.

To find XY, we use the Pythagorean Theorem: $XY^2 + YZ^2 = XZ^2$. Since $YZ = XY$ and $XZ = 6$, we have $2(XY^2) = 36$, so $XY = \sqrt{18} = 3\sqrt{2}$. □

This problem suggests we might be able to find a general relationship among the sides in an isosceles right triangle.

Problem 6.10:

(a) If a right triangle is isosceles, must the two sides that are equal be the legs of the triangle?

(b) Show that in any isosceles right triangle, the length of the hypotenuse is $\sqrt{2}$ times the length of a leg.

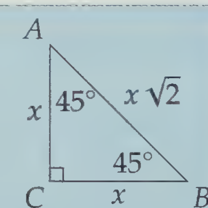
Solution for Problem 6.10:

(a) The Pythagorean Theorem quickly shows us that it must be the two legs that are equal in an isosceles right triangle. Suppose our triangle is $\triangle ABC$ with right angle at $\angle C$. The Pythagorean Theorem gives us $AC^2 + BC^2 = AB^2$. So, if AB equals AC or BC, then the other leg must be 0, which is impossible. Therefore, in any isosceles right triangle, it is the two legs that are equal.

(b) If legs AC and BC each equal x, then we can find hypotenuse AB with the Pythagorean Theorem: $AB^2 = x^2 + x^2 = 2x^2$. Therefore, $AB = x\sqrt{2}$, so the hypotenuse must be $\sqrt{2}$ times a leg.

□

Important: In any isosceles right triangle, the legs are equal and the hypotenuse is $\sqrt{2}$ times the leg. Because the angles of an isosceles right triangle are always 45°, 45°, and 90°, an isosceles right triangle is sometimes called a **45-45-90 triangle**. If we know the length of any side of such a triangle, we can quickly find the others using the side relationships shown in the figure at the right.



Sidenote: Due to the theorem that bears his name, **Pythagoras** may be an even more well-known name to the general public than the widely acknowledged 'father' of geometry, Euclid. Pythagoras lived during the 6th and 5th centuries BC (nearly 200 years before Euclid), and was the founder of a secret society known, unsurprisingly, as the Pythagoreans. It is unclear whether Pythagoras was the first to prove the Pythagorean Theorem, but he and his organization were definitely responsible for a number of other discoveries, including one that dramatically shook their understanding of the world.

The Pythagoreans had a deeply held belief that every phenomenon in nature could be described by whole numbers. However, their study of isosceles right triangles led them to the startling conclusion that the ratio of the length of the hypotenuse of an isosceles right triangle to the length of a leg of the triangle cannot be expressed as a ratio of two integers. As we have seen in this section, this ratio equals $\sqrt{2}$.

Continued on the next page. . .

Sidenote: *. . . continued from the previous page*

To show that $\sqrt{2}$ cannot be written as a ratio of two integers, we use **proof by contradiction**. This means we assume the opposite of what we want to prove, and show that this assumption leads to an impossible conclusion. Here's what the proof looks like:

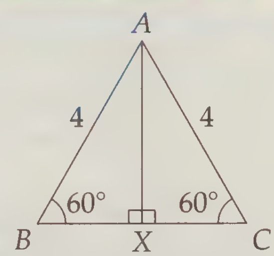
We assume that $\sqrt{2} = p/q$, where p and q are integers and p/q is in lowest terms. Squaring $\sqrt{2} = p/q$ gives $2 = p^2/q^2$, so we have $p^2 = 2q^2$. Therefore, p must be even, so we can write $p = 2m$ for some integer m. This gives $(2m)^2 = 2q^2$, so $2m^2 = q^2$. Therefore, q is even, too. But p and q can't both be even if p/q is in lowest terms!

Since our assumption that $\sqrt{2}$ can be written as a ratio of integers in lowest terms leads to the impossible conclusion that the ratio is also *not* in lowest terms, we have found our **contradiction** that forces us to conclude that our assumption is wrong. Therefore, $\sqrt{2}$ cannot be written as a ratio of integers.

We call a number that can be written as a ratio of integers a **rational number**. A number like $\sqrt{2}$ that cannot be so written is called an **irrational number**. Some sources claim that the Pythagorean responsible for proving the existence of irrational numbers, **Hippasus**, was rewarded for his insight by being murdered. Mathematical discoveries are considerably more well-rewarded now! (See page 216.)

Problem 6.11: Find the area of an equilateral triangle with side length 4, then find a formula for the area of an equilateral triangle with side length s.

Solution for Problem 6.11: To find the area, we will need to find the length of an altitude. Therefore, we draw altitude $\overline{AX}$ as shown. Since $\angle B = \angle C$, $\angle AXB = \angle AXC$, and $AB = AC$, we have $\triangle AXB \cong \triangle AXC$. Therefore, $BX = CX = BC/2 = 2$. We can now use the Pythagorean Theorem to find AX. Since $AX^2 + BX^2 = AB^2$, we have:



$$AX = \sqrt{AB^2 - BX^2} = \sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}.$$

The area of $\triangle ABC$ is

$$[ABC] = \frac{(BC)(AX)}{2} = \frac{(4)(2\sqrt{3})}{2} = 4\sqrt{3}.$$

We can use the exact same procedure to derive a formula for the area of an equilateral triangle with side length s. If we let the sides of $\triangle ABC$ each be s and draw altitude $\overline{AX}$ as before, we have $BX = CX = s/2$. The Pythagorean Theorem then gives

$$AX = \sqrt{AB^2 - BX^2} = \sqrt{s^2 - \frac{s^2}{4}} = \sqrt{\frac{3s^2}{4}} = \frac{s\sqrt{3}}{2}.$$

Now we can find our area:

$$[ABC] = \frac{(BC)(AX)}{2} = \frac{s(s\sqrt{3}/2)}{2} = \frac{s^2\sqrt{3}}{4}.$$

□

> **Important:** The area of an equilateral triangle with side length s is $s^2\sqrt{3}/4$. If you understand how we found this formula, you shouldn't need to memorize it!

Our derivation of the formula for the area of an equilateral triangle suggests a notable relationship among the sides of a triangle with angles 30°, 60°, and 90°.

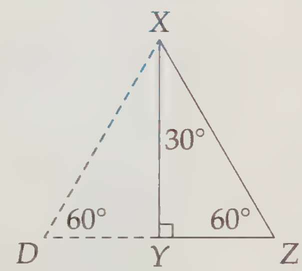
Problem 6.12:

(a) How are the sides of a triangle that have angle measures 30°, 60°, and 90° related?

(b) Suppose $\triangle PQR$ has sides 3, $3\sqrt{3}$, and 6. Can we deduce that the angles of $\triangle PQR$ are 30°, 60°, and 90°?

Solution for Problem 6.12:

(a) We found the area of an equilateral triangle by cutting the triangle into two right triangles that have acute angles of 30° and 60°. Here we go backwards – we build an equilateral triangle with two of these right triangles. In the diagram to the right, congruent right triangles $\triangle XYZ$ and $\triangle XYD$ together form equilateral triangle XDZ.

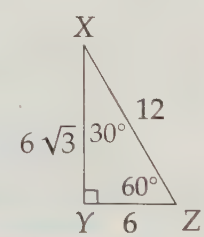


Suppose the length of $\overline{XZ}$ is s. Since $ZD = XZ = s$ and $DY = YZ$, we have $YZ = s/2$. As we saw in Problem 6.11, we can use the Pythagorean Theorem on $\triangle XYZ$ to find that $XY = s\sqrt{3}/2$. Therefore, the sides in any triangle with angles 30°, 60°, and 90° come in the ratio:

$$\text{Leg opposite } 30^\circ : \text{Leg opposite } 60^\circ : \text{Hypotenuse} = 1 : \sqrt{3} : 2.$$

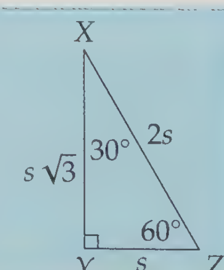
In other words, the hypotenuse is double the leg opposite the 30° angle, and the leg opposite the 60° angle is $\sqrt{3}$ times the other leg.

For example, if the hypotenuse is 12, then the leg opposite the 30° angle is $12/2 = 6$, and the leg opposite the 60° angle is $6\sqrt{3}$ as shown in the figure to the right.



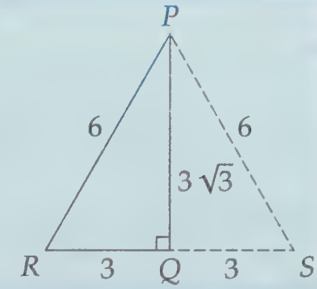
We have just discovered the relationship among the sides of another common type of right triangle.

> **Important:** In a right triangle with acute angles of 30° and 60°, the sides are in the ratio $1 : \sqrt{3} : 2$ as shown to the right. Such a triangle is often called a **30-60-90 triangle**.



(b) Having found this relationship among the sides given the angles, we wonder if all triangles that have sides in these ratios are 30-60-90 triangles. We'll consider one such triangle: $\triangle PQR$ with sides 3, $3\sqrt{3}$, and 6. What's wrong with this 'proof' that $\triangle PQR$ is a 30-60-90 triangle:

Bogus Solution: We extend $\overline{RQ}$ to S as shown such that $QS = QR = 3$. Since $QS = QR$ and $PQ = PQ$, we have $\triangle PQR \cong \triangle PQS$ by SAS Congruence. Therefore, $PS = PR = 6$. Clearly, $RS = 6$ also, so $\triangle PQS$ is equilateral. So, $\angle R = 60^\circ$ and $\angle RPQ = 90^\circ - \angle R = 30^\circ$, and $\triangle PQR$ is 30-60-90.



Unfortunately, we have a big problem here: 'Since $QS = QR$ and $PQ = PQ$, we have $\triangle PQR \cong \triangle PQS$ by SAS Congruence.' This statement assumes that $\angle PQR = \angle PQS$, which in turn assumes that $\angle PQR$ is a right angle. But we can't assume that, because that's what we're trying to prove!

There are a couple ways we can show that $\triangle PQR$ is a 30-60-90 triangle.

Solution 1: Patch the hole in the Bogus Solution. We show that $\triangle PQR$ is a right triangle by showing that its sides satisfy the Pythagorean Theorem:

$$3^2 + (3\sqrt{3})^2 = 9 + 27 = 36 = 6^2.$$

Therefore, $\triangle PQR$ is a right triangle with right angle opposite the side with length 6. From here, we can use the Bogus Solution to complete our proof that $\triangle PQR$ is a 30-60-90 triangle.

Solution 2: We can use triangle congruence another way to write an even simpler proof! We compare our $\triangle PQR$ to 30-60-90 triangle $\triangle ABC$, which has hypotenuse 6. Because $\triangle ABC$ is 30-60-90, we know that its legs have lengths 3 and $3\sqrt{3}$. Therefore, $\triangle ABC \cong \triangle PQR$ by SSS Congruence, so the angles of $\triangle PQR$ are the same as those of $\triangle ABC$. In other words, $\triangle PQR$ is a 30-60-90 triangle.

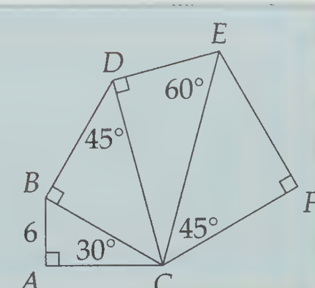
Important: If the side lengths of a triangle are in the ratio $1 : \sqrt{3} : 2$, then the triangle is a 30-60-90 triangle, with the right angle opposite the longest side and the 30° angle opposite the shortest side.

You've seen two proofs for one specific triangle above; you'll have a chance to write a proof for all such triangles in the Exercises. (You knew that was coming!)

□

Let's use our knowledge of 45-45-90 and 30-60-90 triangles to find some side lengths in a problem.

Problem 6.13: Find all the missing sides in the diagram.



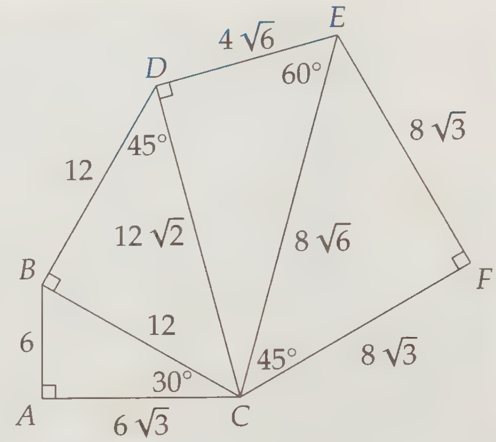
Solution for Problem 6.13: Because $\triangle ABC$ is a 30-60-90 triangle and $\overline{AB}$ is opposite the 30° angle, we have $AC = AB\sqrt{3} = 6\sqrt{3}$ and $BC = 2(AB) = 12$.

Because $\triangle BDC$ is a 45-45-90 triangle, we have $BD = BC = 12$ and $CD = BC\sqrt{2} = 12\sqrt{2}$.

Because $\triangle EDC$ is a 30-60-90 triangle and $\overline{CD}$ is the longer leg, we have $CD = DE\sqrt{3}$. Since $CD = 12\sqrt{2}$, we have

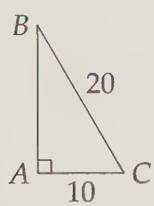
$$DE = \frac{CD}{\sqrt{3}} = \frac{12\sqrt{2}}{\sqrt{3}} = \frac{12\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{6}}{3} = 4\sqrt{6}.$$

We also have $EC = 2ED = 8\sqrt{6}$. Finally, $\triangle ECF$ is a 45-45-90 triangle with hypotenuse $EC = 8\sqrt{6}$. Therefore, the legs each have length $EC/\sqrt{2} = 8\sqrt{3}$. All our lengths are shown in the completed figure at right. □

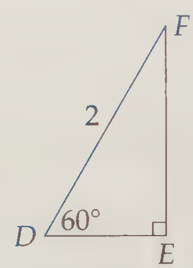


Exercises

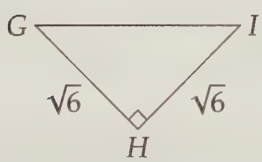
6.2.1



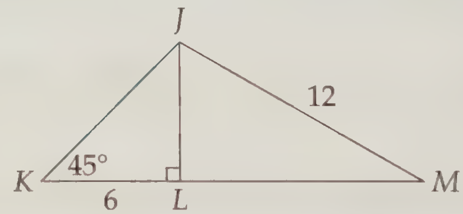
(a) Find AB, $\angle B$, and $\angle C$.



(b) Find DE, EF, and $\angle F$.



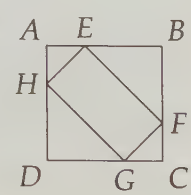
(c) Find GI, $\angle G$, and $\angle I$.



(d) Find JK, JL, LM, $\angle LJM$, and $\angle M$.

6.2.2 Prove that any triangle whose sides have lengths in the ratio $1 : \sqrt{3} : 2$ is a right triangle. **Hints:** 413

6.2.3 Let $ABCD$ be a square of side length 3, and let E, F, G, and H be points on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, such that $AE = CF = CG = AH = 1$.



(a) Prove that $EFGH$ is a rectangle by showing that all of its angles are right angles.

(b) Find the area of $EFGH$.

6.2.4 Find the area of $\triangle PQR$ given that $PQ = QR = 8$ and $\angle PQR = 120°$. **Hints:** 312

6.2.5★ In $\triangle ABC$, $AB = 20$, $\angle A = 30°$, and $\angle C = 45°$. Find BC. *(Source: MATHCOUNTS)* **Hints:** 469

6.3 Pythagorean Triples

A **Pythagorean triple** is a set of three integers that satisfy the Pythagorean Theorem. For example, $\{3, 4, 5\}$ is a Pythagorean triple because $3^2 + 4^2 = 5^2$. However, $\{1, 2, 3\}$ is not a Pythagorean triple since there's no way to add two of 1^2, 2^2, and 3^2 to get the third.

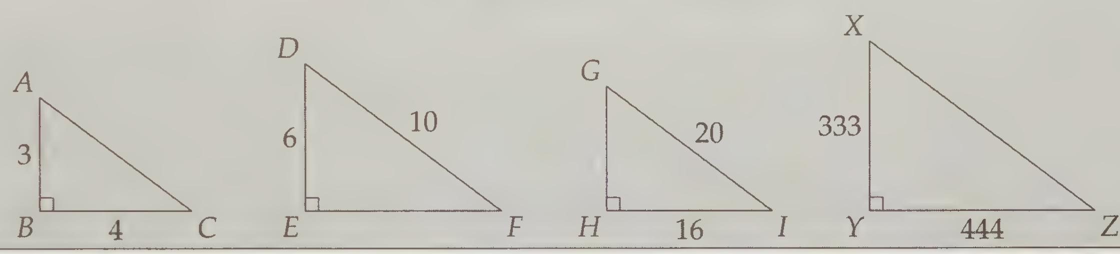
Before proceeding with the problems, play around a little and try to find some Pythagorean triples on your own. Can you find any patterns in your triples?

Problems

Problem 6.14:

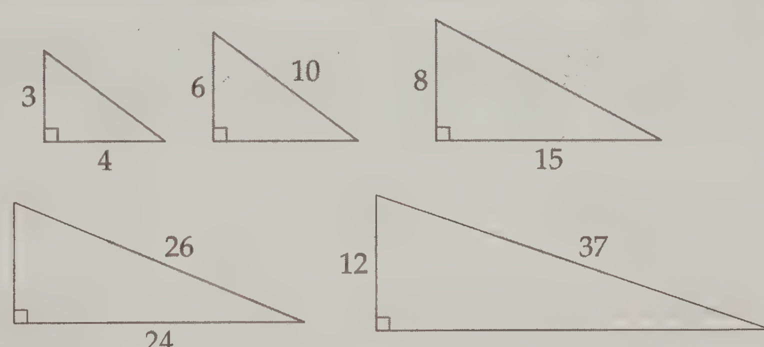
(a) Find the missing side length in each right triangle shown below.

(b) Given that $\{a, b, c\}$ is a Pythagorean triple, is $\{2a, 2b, 2c\}$ a Pythagorean triple? How about $\{na, nb, nc\}$, where n is a positive integer?



Problem 6.15:

(a) Find the last side in each right triangle below. Do you see a pattern?

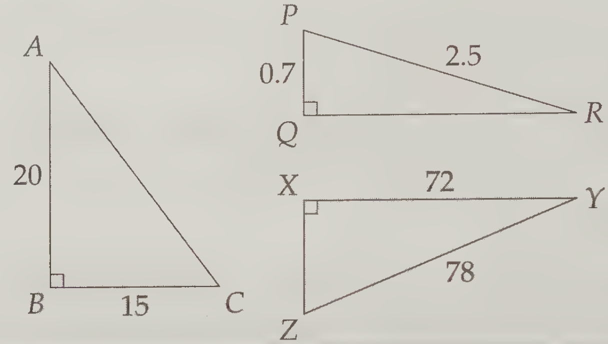


(b) Complete the table below.

n	$2n$	$n^2 - 1$	$n^2 + 1$
2	4	3	5
3			
4			
5			
6			

(c) Is a triangle with sides $2n$, $n^2 - 1$, and $n^2 + 1$, where $n > 1$, always a right triangle?

Problem 6.16: Find the missing side in each of the triangles below *without using the Pythagorean Theorem directly*. Just use your knowledge of Pythagorean triples!

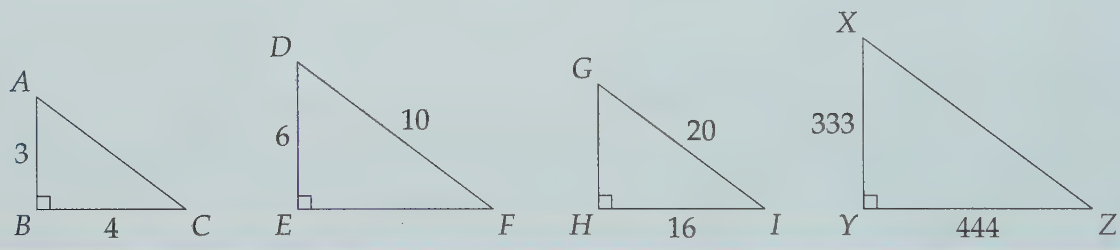


We begin our discussion of Pythagorean triples with the most common triple.

Problem 6.14:

(a) Find the missing side length in each right triangle shown below.

(b) Given that $\{a,b,c\}$ is a Pythagorean triple, is $\{2a,2b,2c\}$ also a Pythagorean triple? How about $\{na,nb,nc\}$, where n is a positive integer?



Solution for Problem 6.14:

(a) We can apply the Pythagorean Theorem to find

$$\begin{aligned} AC &= \sqrt{AB^2+BC^2} = \sqrt{9+16} = 5 \\ EF &= \sqrt{DF^2-DE^2} = \sqrt{64} = 8 \\ GH &= \sqrt{GI^2-HI^2} = \sqrt{144} = 12 \\ XZ &= \sqrt{XY^2+YZ^2} = \sqrt{\text{some big numbers}} = 555 \end{aligned}$$

OK, we got a little lazy on that last one. We noted in the first three that the sides are always in the ratio 3 : 4 : 5, so we took a stab at 555 for XZ. We're right, of course, but why?

There are a couple ways we can prove that we are right. First, we could note that $\triangle XYZ \sim \triangle ABC$ due to SAS Similarity. Second, we could use algebra, as we'll see in the second part of the problem.

Extra! *Mathematics consists of proving the most obvious thing in the least obvious way.*

–George Polya

(b) We are given that $\{a, b, c\}$ is a Pythagorean triple, so $a^2 + b^2 = c^2$. To test $\{2a, 2b, 2c\}$, we consider the sum of the squares of the first two terms:

$$\begin{aligned}(2a)^2 + (2b)^2 &= 4a^2 + 4b^2 \\ &= 4(a^2 + b^2) \\ &= 4c^2 \text{ (because } a^2 + b^2 = c^2) \\ &= (2c)^2.\end{aligned}$$

Therefore, $\{2a, 2b, 2c\}$ is a Pythagorean triple. There's nothing special about 2; we can run through this with any number:

$$(na)^2 + (nb)^2 = n^2a^2 + n^2b^2 = n^2(a^2 + b^2) = n^2c^2 = (nc)^2.$$

So, $\{na, nb, nc\}$ is also a Pythagorean triple. Notice also that n doesn't have to be an integer to produce side lengths of a right triangle! For example, since a triangle with sides $\{3, 4, 5\}$ is a right triangle, a triangle with sides $\{3(1/7), 4(1/7), 5(1/7)\} = \{3/7, 4/7, 5/7\}$ is a right triangle.

You will often see triangles with sides in this ratio referred to as 3-4-5 triangles. Triangles whose sides are in the ratios of other Pythagorean triples are sometimes similarly identified, such as 5-12-13 triangles or 7-24-25 triangles.

□

Important: If $\{a, b, c\}$ are the sides of a right triangle, then so are $\{na, nb, nc\}$ for any positive number n.

Therefore, we can generate whole groups of Pythagorean triples from a single triple:

$$\{3, 4, 5\} \to \{6, 8, 10\} \to \{9, 12, 15\} \to \cdots \to \{333, 444, 555\} \to \cdots$$

There's nothing special about integer multiples of our $\{3, 4, 5\}$ triple. We can generate all sorts of other sets of three numbers that are the sides of a right triangle. Here are a few:

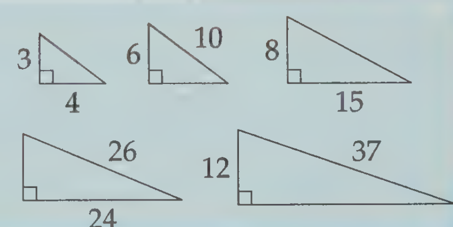
$$\{0.3, 0.4, 0.5\} \qquad \{3/11, 4/11, 5/11\} \qquad \{6\sqrt{5}, 8\sqrt{5}, 10\sqrt{5}\}$$

Recognizing common Pythagorean triples, or multiples of those triples, is a very useful problem solving tool. The following problem exhibits some common triples, as well as a neat pattern for generating more.

Problem 6.15:

(a) Find the last side in each right triangle.

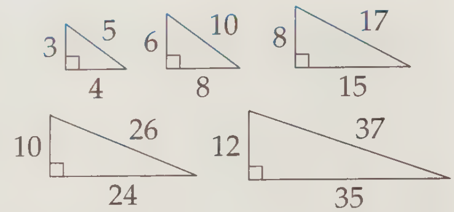
(b) Do you see a pattern? Can you prove the pattern always works?



Solution for Problem 6.15:

(a) We can use the Pythagorean Theorem to find the rest of the sides as shown below.

(b) If we take the smallest even leg, halve it, then square the result, we get the number that is one more than the other leg and one less than the hypotenuse. Putting this in math terms, we let our smallest even leg have length $2n$. Then the other two sides are $n^2 - 1$ and $n^2 + 1$. We can make a table that shows what happens for different values of the even leg.



	Smallest Even Leg	Other Leg	Hypotenuse
n	$2n$	$n^2 - 1$	$n^2 + 1$
2	4	3	5
3	6	8	10
4	8	15	17
5	10	24	26
6	12	35	37
7	14	48	50

All the triangles described in the table are indeed right triangles. We can show that any triangle with sides $2n$, $n^2 - 1$, and $n^2 + 1$, where $n > 1$, is a right triangle with some algebra:

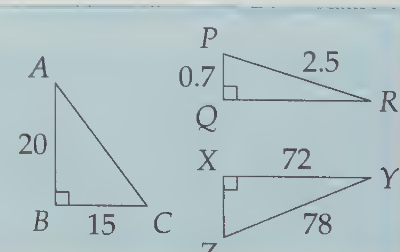
$$\begin{aligned}(2n)^2 + (n^2-1)^2 &= 4n^2 + n^4 - 2n^2 + 1\\ &= n^4 + 2n^2 + 1\\ &= (n^2+1)^2.\end{aligned}$$

□

Sidenote: Choose three positive integers k, m, and n, where $m > n$. Compute the three numbers $k(m^2 - n^2)$, $2kmn$, and $k(m^2 + n^2)$. Try it for several different values of k, m, and n. You should find that the three numbers you generate in each case form a Pythagorean triple. See if you can figure out why this works.

Even more surprising is the fact that every single Pythagorean triple fits this mold. In other words, for every single Pythagorean triple, can we always find some k, m, and n such that $\{2kmn, k(m^2 - n^2), k(m^2 + n^2)\}$ generates that triple.

Problem 6.16: Find the missing side in each of the triangles below *without using the Pythagorean Theorem directly.* Just use your knowledge of Pythagorean triples!



Solution for Problem 6.16: We note that the legs of $\triangle ABC$ are in the ratio $15/20 = 3/4$. Therefore, our legs are in the same ratio as the legs in a 3-4-5 triangle. We can then conclude that the hypotenuse is 5/3 of the smallest leg, or $(5/3)(15) = 25$.

We might also have found AC by noting that the legs of $\triangle ABC$ are $20 = 4(5)$ and $15 = 3(5)$, so this is a 3-4-5 triangle. Therefore, we know that $AC = 5(5) = 25$.

In $\triangle PQR$, the ratio of the given leg to the hypotenuse is $0.7/2.5 = 7/25$. This makes us think of the Pythagorean triple $\{7, 24, 25\}$. Since $0.7 = 7(0.1)$ and $2.5 = 25(0.1)$, we know the other leg is $24(0.1) = 2.4$.

In $\triangle XYZ$, the ratio of the given leg to the hypotenuse is $72/78 = 12/13$, which reminds us of the Pythagorean triple $\{5, 12, 13\}$. Since our leg is $72 = 12(6)$ and the hypotenuse is $78 = 13(6)$, the other leg must be $5(6) = 30$.

Notice our general tactic for finding the missing length in each triangle in Problem 6.16 – we take the ratio of the sides we know and then try to recognize a simple Pythagorean triple that has the numbers in our ratio. □

WARNING!! Make sure you see that it's important that the 2.5 length in $\triangle PQR$ of Problem 6.16 is the hypotenuse and not a leg. A triangle with legs 0.7 and 2.5 does not have third side 2.4. In fact, it doesn't match any simple Pythagorean triple. Therefore, you must be careful when using your knowledge of Pythagorean triples to make sure your given sides match corresponding sides (legs or hypotenuse) of the Pythagorean triple you would like to use.

Sidenote: Perhaps inspired by Pythagorean Triples, the great mathematician **Pierre de Fermat** explored the equation

$$x^n + y^n = z^n$$

for values of n besides $n = 2$ ($n = 2$ gives the Pythagorean Theorem, of course). That there are no integers x, y and z that satisfy this equation when n is an integer larger than 2 is known as **Fermat's Last Theorem**. Although it seems like a simple proposition, Fermat's Last Theorem evaded proof for hundreds of years until Andrew Wiles proved it in 1994. Alas, his proof is far too long to fit in the margins of this book.

Exercises

6.3.1 Which of the following are side lengths of a right triangle? (*Try using your knowledge of Pythagorean triples!*)

(a) 300, 400, 500.

(b) 36, 48, 60.

(c) $\sqrt{5}$, $\sqrt{12}$, $\sqrt{13}$.

(d) 20, 37.5, 42.5.

(e) 1.44, 1.96, 2.4.

(f) 15, 36, 39.

6.3.2

(a) Is there a right triangle such that all three side lengths are odd integers? **Hints:** 523

(b) Is there a right triangle such that two of the side lengths are even integers and the other is an odd integer? **Hints:** 143

(c) Is there a right triangle in which two side lengths are simple fractions (ratios of integers, such as 2/3 or 3/7), and the other side length is an integer?

(d) Is there a right triangle in which one of the side lengths is a simple fraction and the other two side lengths are integers?

6.3.3 Susie rides her bike north 3 miles, then east 8 miles, then south 13 miles, then east 16 miles. How far is she from where she started?

6.3.4 Use SSS Similarity to show that if $\{a, b, c\}$ is a Pythagorean triple, then so is $\{na, nb, nc\}$, where n is a positive integer.

6.3.5★ In this problem we find all Pythagorean triples in which 18 is one of the legs.

(a) Let b and c be the other two sides, where c is the hypotenuse. Show that $c^2 - b^2 = 18^2$.

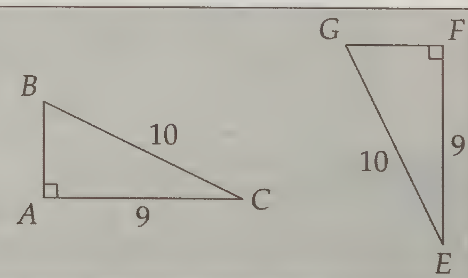
(b) Factor the right side of the equation from part (a) and use the factors of 18^2 to solve for all possible pairs of integers of integers (b, c) that satisfy the equation.

6.3.6★ Find the Pythagorean triple that has 97 as the length of the hypotenuse without using a computer or calculator. **Hints:** 101

6.4 Congruence and Similarity Revisited

Problems

Problem 6.17: Are the two triangles shown congruent? Why or why not?



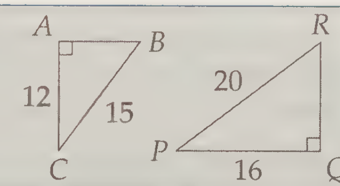
Problem 6.18: In this problem we will prove that if radius $\overline{OC}$ of $\odot O$ is perpendicular to chord $\overline{AB}$ of $\odot O$, then radius $\overline{OC}$ bisects chord $\overline{AB}$.

(a) Draw a diagram to use in your proof. Include both radius $\overline{OC}$ and chord $\overline{AB}$.

(b) Find a pair of congruent triangles that you can use to complete the proof.

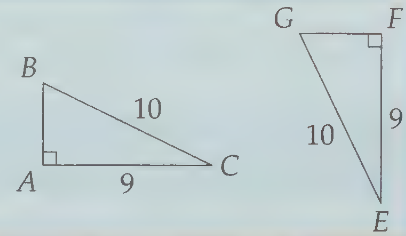
Problem 6.19: The center of a circle is 4 units away from a chord $\overline{PQ}$ of the circle. If $PQ = 12$, what is the radius of the circle?

Problem 6.20: Are the two triangles in the diagram similar? Why or why not?



In this section we discuss a special congruence theorem and a special similarity theorem that only work for right triangles.

Problem 6.17: Are the two triangles in the diagram congruent? Why or why not?



Solution for Problem 6.17: What's wrong here:

Bogus Solution: Two sides and an angle of one triangle equal two sides and an angle of the other, so the triangles are congruent.

The problem here is that SSA is not in general a valid congruence theorem, as we found in Section 3.5. However, if we note that the angles that are equal are right angles...

We can use the Pythagorean Theorem to find AB and FG. In both cases, the length is $\sqrt{100-81} = \sqrt{19}$. Therefore, by SSS (or SAS), we have $\triangle ABC \cong \triangle FGE$. □

In the same way, any time the hypotenuse and a leg of one right triangle equal those of another, we can use the Pythagorean Theorem to show that the third sides of each triangle are equal. In other words, SSA works for right triangles. Rather than give it an unwieldy name like 'SSA, but just for right triangles', we call it:

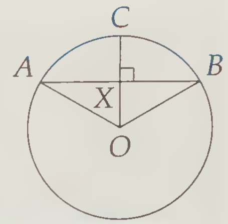
Important: **Hypotenuse-Leg Congruence** (**HL Congruence**) states that if the hypotenuse and a leg of one right triangle equal those of another, then the triangles are congruent.

As you might guess, there's an **Leg-Leg Congruence** (**LL Congruence**), but that's just SAS Congruence (make sure you see why), so there's really no need to give this a special name.

Problem 6.18: Prove that if radius $\overline{OC}$ of $\odot O$ is perpendicular to chord $\overline{AB}$ of $\odot O$, then radius $\overline{OC}$ bisects chord $\overline{AB}$.

Extra! *Common sense is the collection of prejudices acquired by age eighteen.*

–Albert Einstein



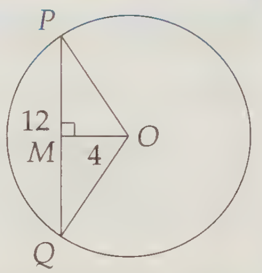
Solution for Problem 6.18: As usual, we start with a diagram, including both our radius and our chord. We add the radii to point A and B so we have right triangles to work with. Since $OA = OB$ and $OX = OX$, we have $\triangle OXA \cong \triangle OXB$ by HL Congruence. Therefore, $AX = BX$, so radius $\overline{OC}$ bisects chord $\overline{AB}$. □

Problem 6.18, combined with Problem 3.4 on page 54, gives us a very useful tool:

Important: If a radius of a circle bisects a chord of the circle, it is perpendicular to the chord. Conversely, if a radius of a circle is perpendicular to the chord, it bisects the chord.

We can use this tool on a variety of circle problems. For example:

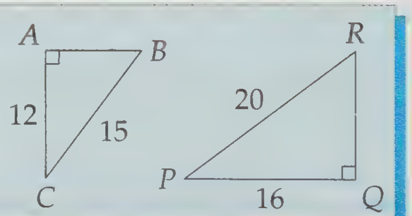
Problem 6.19: The center of a circle is 4 units away from a chord $\overline{PQ}$ of the circle. If $PQ = 12$, what is the radius of the circle?



Solution for Problem 6.19: We let point M be the point on $\overline{PQ}$ closest to the center of the circle, which we'll call O. Since M is the point on $\overline{PQ}$ closest to O, we have $\overline{OM} \perp \overline{PQ}$. Because $\overline{OM}$ is part of a radius that is perpendicular to chord $\overline{PQ}$, $\overline{OM}$ bisects chord $\overline{PQ}$. So, we have $PM = MQ = 6$. Finally, we apply the Pythagorean Theorem to $\triangle OPM$ to find $OP = \sqrt{PM^2 + OM^2} = \sqrt{36 + 16} = 2\sqrt{13}$. □

Unsurprisingly, the Pythagorean Theorem can be used to give us a Hypotenuse-Leg Similarity Theorem as well.

Problem 6.20: Are the two triangles in the diagram similar? Why or why not?



Solution for Problem 6.20: What's wrong with this:

Bogus Solution: Since $20/15 = 16/12$, we have two pairs of sides in a common ratio and $\angle A = \angle Q$, so by SAS Similarity, the triangles are similar.

SAS Similarity requires the angle to be *between* the two sides that have the constant ratios, so we can't use SAS Similarity like this.

We recognize these two triangles as 3-4-5 triangles (or we use the Pythagorean Theorem), and we find that $AB = 9$ and $QR = 12$. Therefore, we have

$$\frac{AB}{RQ} = \frac{AC}{PQ} = \frac{BC}{PR},$$

so $\triangle ABC \sim \triangle QRP$ by SSS Similarity. □

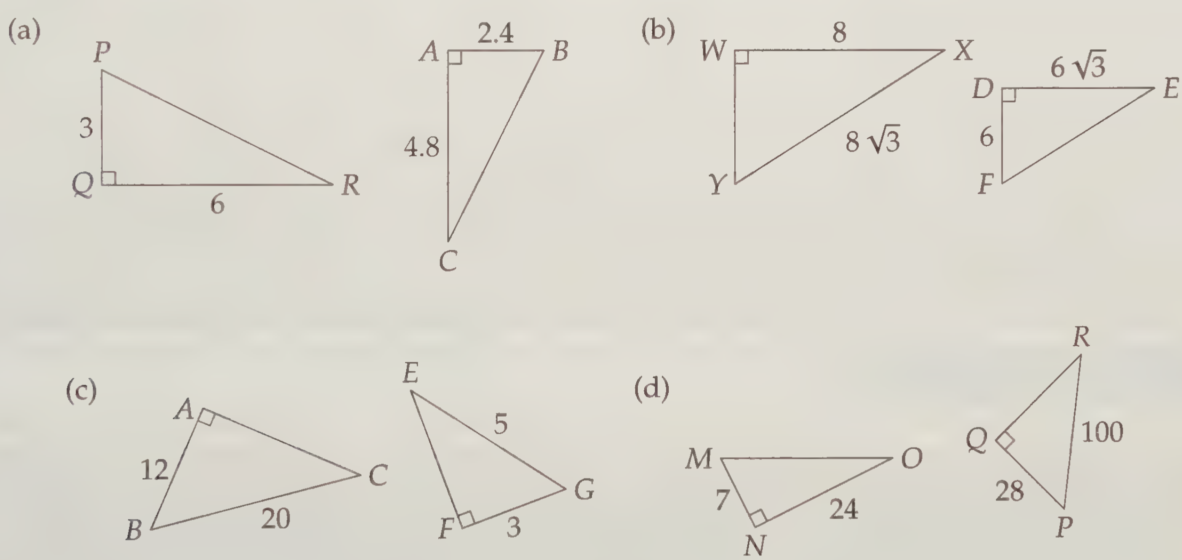
In much the same way, we can use the Pythagorean Theorem to show that if a hypotenuse and a leg of one right triangle are in the same ratio as the hypotenuse and leg of another right triangle, then the other legs are also in that ratio. So, the two triangles are similar. Therefore, we have a new similarity theorem just for right triangles.

> **Important:** **Hypotenuse-Leg Similarity** (**HL Similarity**) states that if the hypotenuse and a leg of one right triangle are in the same ratio as the hypotenuse and leg of another right triangle, then the triangles are similar.

As with congruence, there is a **Leg-Leg Similarity Theorem** that states that if the legs of one right triangle are in the same ratio as the legs of another right triangle, then the two triangles are similar. However, this is just the same thing as SAS Similarity. (Make sure you see why!)

Exercises

6.4.1 Which of the following pairs of triangles are similar? For the pairs that are similar, write both the similarity relationship and why the two triangles are similar.



6.4.2 Points A and B are on a circle with radius 9. Given that the center of the circle is 3 units from $\overline{AB}$, find AB.

6.4.3 In this problem we prove the HL Similarity Theorem we discovered in Problem 6.20. Let $\triangle ABC$ and $\triangle MNO$ be right triangles with right angles at $\angle A$ and $\angle M$. Furthermore, suppose $AB/BC = MN/NO$. We wish to prove that $\triangle ABC \sim \triangle MNO$.

(a) Use the Pythagorean Theorem to find expressions for AC^2 and MO^2.

(b) Use the given equation $AB/BC = MN/NO$ to find an expression for AB. Write an expression equal to AC^2/MO^2.

(c) Substitute your expression for AB into your expression for AC^2/MO^2. Prove that $AC^2/MO^2 = BC^2/NO^2$. **Hints:** 345

(d) Prove that $\triangle ABC \sim \triangle MNO$. **Hints:** 507

6.4.4 $\overline{AD}$ and $\overline{BC}$ are both perpendicular to $\overline{AB}$ in the diagram at left below, and $\overline{CD} \perp \overline{AC}$. If $AB = 4$ and $BC = 3$, find CD. *(Source: HMMT)*

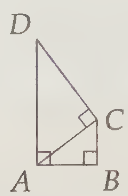


Figure 6.1: Diagram for Problem 6.4.4

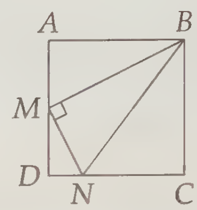


Figure 6.2: Diagram for Problem 6.4.5

6.4.5 $ABCD$ is a square in the diagram at right above. M is the midpoint of $\overline{AD}$ and $\overline{BM} \perp \overline{MN}$.

(a) Prove that $\triangle MDN \sim \triangle BAM$.

(b)★ Prove that $\angle ABM = \angle MBN$. **Hints:** 189

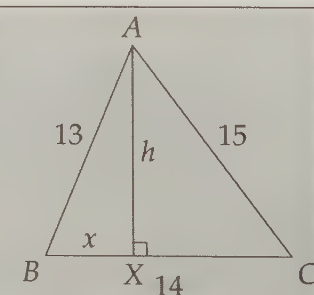
6.5★ Heron's Formula

As we saw when discussing SSS Congruence, all triangles with the same three side lengths are congruent. Therefore, they all have the same area. So, we wonder, 'Since the area is fixed given the three side lengths, is there a way to figure out what the area is?'

In this section we'll answer this question with the Pythagorean Theorem and a lot of algebra.

Problems

Problem 6.21: In this problem, we find the area of $\triangle ABC$, which has sides of length 13, 14, and 15. We start by drawing altitude $\overline{AX}$. Let it have length h and let $BX = x$.



(a) What is XC in terms of x?

(b) Apply the Pythagorean Theorem to $\triangle ABX$ to get an equation with h and x in it.

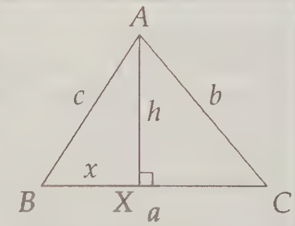
(c) Apply the Pythagorean Theorem to $\triangle ACX$ to get an equation with h and x in it.

(d) Subtract your equation in (c) from the one in (b) to find x. Use x to find h.

(e) Find the area of the triangle.

(f) How could you use your knowledge of Pythagorean triples to avoid all this algebra?

Problem 6.22: In this problem we will try to find a formula for the area of a triangle in terms of only its sides. As usual, we let $AB = c$, $BC = a$, and $AC = b$. We start again with altitude $AX = h$, and we let $BX = x$. Finally, we let $s = (a+b+c)/2$ to simplify our algebra in the problem. (Warning: There's a lot of algebra in this problem. You will use the difference of squares factorization many times: $y^2 - z^2 = (y-z)(y+z)$.)



(a) Apply the Pythagorean Theorem to $\triangle ABX$ to get an equation with a, b, c, h, and/or x in it.

(b) Apply the Pythagorean Theorem to $\triangle ACX$ to get an equation with a, b, c, h, and/or x in it.

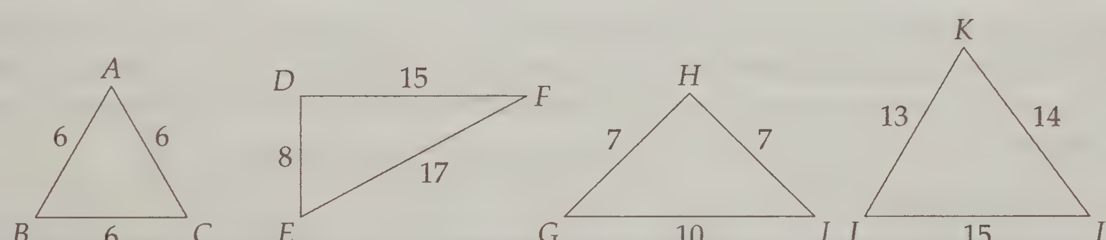
(c) Subtract your equation in (b) from the one in (a) to get an equation with just x, a, b, and c. Use this equation to show that $x = (a^2 + c^2 - b^2)/(2a)$.

(d) Substitute your expression from (c) into one of your earlier equations to show that

$$h = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}.$$

(e) Show that $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$.

Problem 6.23: Find the area of each triangle below using the formula you derived in the previous problem.



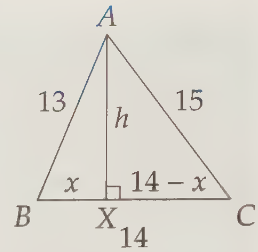
Before we tackle finding a formula for the area of a triangle in terms of its sides, we try finding the area of a specific triangle.

Concept: Often it's best to try a few examples before attempting to derive a general formula. The examples can be a good guide when you try to derive your formula.

Problem 6.21: Find the area of a triangle with sides of lengths 13, 14, and 15.

Extra! You've probably heard that music and math are related. You can thank Pythagoras for this discovery, too! Pythagoras is credited with discovering that pleasant sounds are produced by vibrating strings when the ratio of the lengths of the strings is a whole number. This discovery led to the development of the musical concept we know now as the **octave**.

Solution for Problem 6.21: We start by drawing altitude $AX = h$ of $\triangle ABC$ as shown. We let $BX = x$, so that $XC = 14 - x$. We have some right triangles, and lots of lengths, so we try the Pythagorean Theorem. Applying the Pythagorean Theorem to triangles $\triangle AXB$ and $\triangle AXC$ yields:



$$\begin{aligned} x^2 + h^2 &= 169 \\ (14-x)^2 + h^2 &= 225 \end{aligned}$$

Seeing h^2 on the left of both equations, we think to subtract the second equation from the first to eliminate h^2. We get

$$x^2 - (14-x)^2 = 169 - 225.$$

Therefore, $x^2 - 196 + 28x - x^2 = -56$, so $x = 5$. We can substitute $x = 5$ into either of our original equations to find $h = 12$. Therefore,

$$[ABC] = \frac{(BC)(AX)}{2} = \frac{(14)(12)}{2} = 84.$$

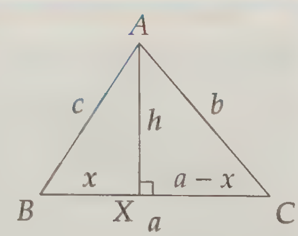
Seeing that x and h are integers, we wonder if we could have found them without the algebra. Triangles $\triangle ABX$ and $\triangle ACX$ share a leg and have 13 and 15 as their respective hypotenuses. This makes us think of the Pythagorean triples 5-12-13 and 9-12-15, which have the needed hypotenuses and a leg length in common. Fortunately, the two other legs, 5 and 9, add up to 14, the length of $\overline{BC}$. Therefore, when we glue a 5-12-13 triangle to a 9-12-15 along the leg with length 12, we get a triangle with sides 13, 14, 15. □

Unfortunately, we usually can't find Pythagorean triples that fit together nicely to make our triangle. However, we can follow our procedure from Problem 6.21 to find a formula for the area of a triangle in terms of its sides.

Put on your algebra hat...

Problem 6.22: In triangle ABC, let $AB = c$, $AC = b$, and $BC = a$ and let $s = (a+b+c)/2$. Find a formula for $[ABC]$ in terms of s, a, b, and c.

Solution for Problem 6.22: We set up the problem in the same way as the last problem, drawing altitude $\overline{AX}$ and letting $AX = h$ and $BX = x$, so $XC = a - x$.



We apply the Pythagorean Theorem to $\triangle ABX$ and $\triangle ACX$ to get:

$$\begin{aligned} x^2 + h^2 &= c^2 \\ (a-x)^2 + h^2 &= b^2 \end{aligned}$$

Again, we can subtract the second equation from the first to find x:

$$x^2 - (a-x)^2 = c^2 - b^2.$$

Therefore, $x^2 - a^2 + 2ax - x^2 = c^2 - b^2$, so $2ax = c^2 - b^2 + a^2$, and we have

$$x = \frac{a^2 + c^2 - b^2}{2a}.$$

That wasn't so bad, but finding h looks a little scary. We can make the algebra somewhat nicer by solving $x^2 + h^2 = c^2$ for h^2 and factoring using the difference of squares:

$$h^2 = c^2 - x^2 = (c - x)(c + x).$$

Substituting our expression for x into this equation gives

$$\begin{aligned} h^2 &= \left(c - \frac{a^2 + c^2 - b^2}{2a}\right)\left(c + \frac{a^2 + c^2 - b^2}{2a}\right) \\ &= \left(\frac{2ac - a^2 - c^2 + b^2}{2a}\right)\left(\frac{2ac + a^2 + c^2 - b^2}{2a}\right). \end{aligned}$$

Now we recognize $-(a - c)^2 = -a^2 + 2ac - c^2$ and $(a + c)^2 = a^2 + 2ac + c^2$ in the numerators:

$$\begin{aligned} h^2 &= \left(\frac{b^2 - (a - c)^2}{2a}\right)\left(\frac{(a + c)^2 - b^2}{2a}\right) \\ &= \left(\frac{[b - (a - c)][b + (a - c)]}{2a}\right)\left(\frac{(a + c - b)(a + b + c)}{2a}\right) \\ &= \frac{(b + c - a)(a + b - c)(a + c - b)(a + b + c)}{4a^2}. \end{aligned}$$

(Make sure you see how we used the factorization $y^2 - z^2 = (y - z)(y + z)$ to simplify the algebra.)

We can write h^2 even more briefly with the substitution $s = (a+b+c)/2$. We call this s the **semiperimeter**, i.e. one-half the perimeter. From this, we have $2s = a + b + c$, so $b + c - a = 2s - 2a$. Similarly, we can substitute for all the terms in the numerator of our h^2 expression:

$$h^2 = \frac{(2s - 2a)(2s - 2c)(2s - 2b)(2s)}{4a^2} = \frac{4s(s - a)(s - b)(s - c)}{a^2}.$$

Now that we have an expression for h, we can find the area:

$$[ABC] = \frac{(BC)(h)}{2} = \frac{a}{2}\sqrt{\frac{4s(s - a)(s - b)(s - c)}{a^2}} = \sqrt{s(s - a)(s - b)(s - c)}.$$

□

It's not always the nicest way to find the area of a triangle, but if we have the lengths of the three sides, we can find the area with:

Important: **Heron's Formula** tells us that if the sides of a triangle are a, b, and c, and we let $s = (a + b + c)/2$, then the area of the triangle is

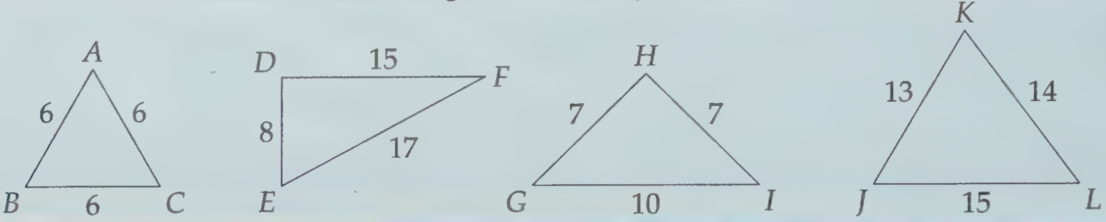
$$\sqrt{s(s - a)(s - b)(s - c)}.$$

Aside from algebra, the only tool we used in deriving Heron's formula was the Pythagorean Theorem.

Concept: When in doubt, build right triangles and use the Pythagorean Theorem if you're trying to find a length.

Here are a few triangles on which to try Heron's formula. You should compare your answers to the areas you have found with other methods.

Problem 6.23: Find the area of each triangle below using Heron's formula.



Solution for Problem 6.23: In each case, there are other ways to find the area of the triangle. See if you can find the area using these other ways and compare your result to Heron's formula.

$\triangle ABC$: $s = (6+6+6)/2 = 9$, so $[ABC] = \sqrt{9(9-6)(9-6)(9-6)} = 9\sqrt{3}$.

$\triangle DEF$: $s = (8+15+17)/2 = 20$, so $[DEF] = \sqrt{20(20-17)(20-15)(20-8)} = 60$.

$\triangle GHI$: $s = (7+7+10)/2 = 12$, so $[GHI] = \sqrt{12(12-10)(12-7)(12-7)} = 10\sqrt{6}$.

$\triangle JKL$: $s = (13+14+15)/2 = 21$, so $[JKL] = \sqrt{21(21-13)(21-14)(21-15)} = 84$. □

Lest you start to think Heron's is always the way to go, try finding the area of a triangle with sides $2\sqrt{3}$, 4, and $2\sqrt{7}$, or a triangle with sides $\sqrt{13}$, $\sqrt{13}$, and $\sqrt{6}$. Heron doesn't look so pretty in either case, but the answers can be found rather quickly with other methods. See if you can figure out how.

Exercises

6.5.1 Use Heron's Formula to find the areas of triangles with the following side lengths:

(a) 8, 8, 8.

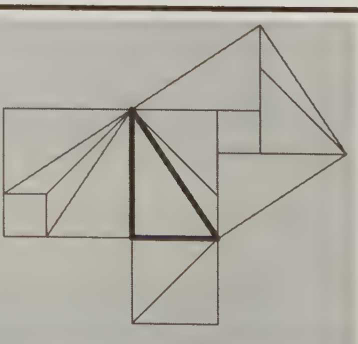
(b) 4, 5, 6.

(c) 12, 35, 37.

(d) $6\sqrt{2}, 7\sqrt{2}, 9\sqrt{2}$.

6.5.2 In $\triangle GHI$, $GH = 5$, $HI = 7$, $GI = 8$. Find the height of $\triangle GHI$ from G to $\overline{HI}$.

Extra! Chinese mathematician Liu Hui based a proof of the Pythagorean Theorem on the diagram at right in the third century A.D. (Of course, he probably didn't call the result the Pythagorean Theorem!) See if you can fill in the details!



6.6 Construction: Perpendicular Lines

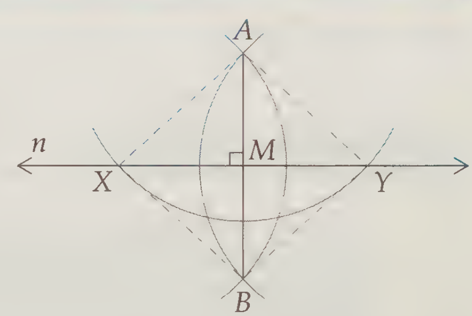
Problems

Problem 6.24: Construct a line through A that is perpendicular to line n. (Hint: See Problem 3.23.)

Problem 6.25: Suppose the shown segment has length 1. Construct a segment with length $\sqrt{10}$.

With all this talk about perpendicular lines and right triangles, we should wonder how to construct perpendicular lines with a ruler and compass.

Problem 6.24: Given line n and point A, construct a line through A that is perpendicular to n.



Solution for Problem 6.24: Back on page 73, we constructed a very specific perpendicular line given a segment. Maybe we can use what we learned there to make a line through A perpendicular to n. To do so, we'll have to find two points on n such that A is directly 'above' the midpoint of the segment connecting the two points. So, we draw a circle with center A that hits n in two points, X and Y. Therefore, $AX = AY$.

We can then mimic our construction on page 73 by drawing circles with centers X and Y and radius AY. Since $AY = AX$, these circles will meet at A. Let point B be the other point where these circles meet. Since $AX = AY$, $XB = YB$, and $AB = AB$, we have $\triangle AXB \cong \triangle AYB$. Therefore, $\angle MAX = \angle MAY$. Together with $AX = AY$ and $AM = AM$, this gives us $\triangle MAX \cong \triangle MAY$ by SAS Congruence. Finally, we have $\angle XMA = \angle YMA$, so these angles are each 90° because together they make up a straight angle.

Therefore, $\overleftrightarrow{AB}$ passes through A and is perpendicular to n. □

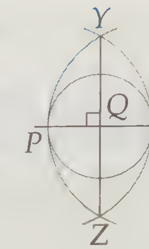
Now that we can create perpendiculars, we can use right triangles in our constructions. Let's give it a try.

Problem 6.25: Suppose the shown segment has length 1. Construct a segment with length $\sqrt{10}$.

Extra! *The mind is not a vessel to be filled, it is a fire to be kindled.*

–Plutarch

Solution for Problem 6.25: Seeing the square root, we think of the Pythagorean Theorem, since we've seen a lot of square root signs in our right triangle problems. Since $1^2+3^2 = 10$, we want to make a right triangle with legs of length 1 and 3. The hypotenuse of this triangle will have length $\sqrt{10}$. We start by extending $\overline{PQ}$, then constructing a line through Q perpendicular to $\overleftrightarrow{PQ}$ as described in Problem 6.24. Specifically, our steps for constructing the perpendicular line are:

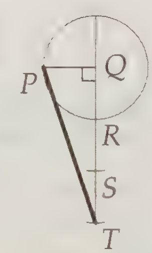


(a) Draw a circle with center Q and radius QP. Name the second point where this circle hits $\overleftrightarrow{PQ}$ point X.

(b) Draw circles with centers P and X, both with radius PX. Name the points where these circles meet Y and Z.

(c) Draw $\overleftrightarrow{YZ}$, which is the line through Q perpendicular to $\overline{PQ}$. (Make sure you see why this line must go through Q.)

Once we have this perpendicular line, we create a segment of length 3 along it by drawing a circle with center Q and radius QP to get point R on the perpendicular such that $QR = 1$. Then we can copy $\overline{QR}$ a couple times to get S and T such that $RS = ST = 1$, so $QT = 3$. Segment $\overline{PT}$ is then the hypotenuse of a right triangle with legs of lengths 1 and 3, so it has length $\sqrt{1^2+3^2} = \sqrt{10}$. □



Exercises

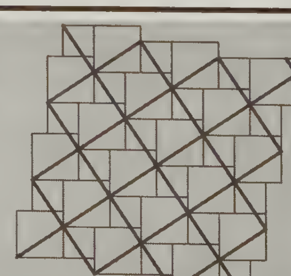
6.6.1 Construct a 45-45-90 triangle and a 30-60-90 triangle.

6.6.2 Given a segment of length 1, construct a segment of length $\sqrt{3}$.

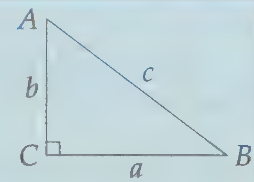
6.7 Summary

Definitions: A **right triangle** is a triangle that has a right angle among its angles. The side opposite the right angle is called the **hypotenuse** and the other two sides are the **legs**.

Extra! The dazzling tiling 'proof without words' of the Pythagorean Theorem shown at right comes from **Annairizi of Arabia** (circa 900 AD). See if you can figure out how it works! *Source: Proofs Without Words II by Roger Nelsen*



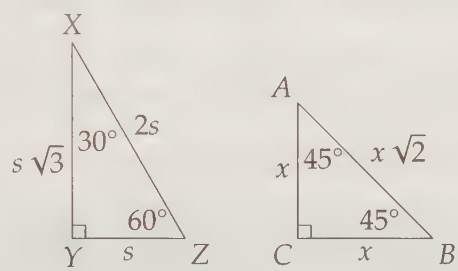
Important: In any right triangle, the sum of the squares of the legs equals the square of the hypotenuse. Or, in the figure to the right, we have



$$a^2 + b^2 = c^2.$$

This is the famous **Pythagorean Theorem**. The Pythagorean Theorem also runs in reverse: If the sides of a triangle satisfy the Pythagorean Theorem, then the triangle must be a right triangle. The longest side of such a triangle is the hypotenuse, and the angle opposite it is a right angle.

Two special right triangles, the 45-45-90 triangle and the 30-60-90 triangle, are shown below. The relationships among the sides of each triangle are as indicated.



Conversely, if you determine that the side lengths of a triangle are in the ratio $1 : 1 : \sqrt{2}$, you can deduce that the triangle is a 45-45-90 triangle. If the side lengths are in the ratio $1 : \sqrt{3} : 2$, then the triangle is a 30-60-90 triangle with the 30° angle opposite the shortest side.

Important: We used the Pythagorean Theorem to prove the following useful facts:

- The distance from a point to a line is the length of the segment from the point to the line that is perpendicular to the line.
- The area of an equilateral triangle with side length s is $s^2\sqrt{3}/4$. If you understand how we found this formula, you shouldn't need to memorize it!
- **Heron's Formula** tells us that if the sides of a triangle are a, b, and c, and we let $s = (a + b + c)/2$, then the area of the triangle is

$$\sqrt{s(s-a)(s-b)(s-c)}.$$

Definition: A **Pythagorean triple** is a set of three integers that satisfy the Pythagorean Theorem. Some common Pythagorean triples are:

$$\{3,4,5\} \qquad \{5,12,13\} \qquad \{7,24,25\} \qquad \{8,15,17\} \qquad \{9,40,41\}.$$

Important: If $\{a, b, c\}$ are the sides of a right triangle, then so are $\{na, nb, nc\}$ for any positive number n.

We also investigated two new congruence theorems and two new similarity theorems specifically for right triangles:

- **HL Congruence**. If the hypotenuse and a leg of one right triangle equal those of another, then the triangles are congruent.
- **LL Congruence**. If the legs of one right triangle equal those of another, then the triangles are congruent. (This is the same as SAS Congruence.)
- **HL Similarity**. If the hypotenuse and a leg of one right triangle are in the same ratio as the hypotenuse and leg of another right triangle, then the triangles are similar.
- **LL Similarity**. If the legs of one right triangle are in the same ratio as the legs of another right triangle, then the triangles are similar. (This is the same as SAS Similarity.)

We used HL Congruence to prove the following useful fact:

Important: If a radius of a circle bisects a chord of the circle, it is perpendicular to the chord. Conversely, if a radius of a circle is perpendicular to the chord, it bisects the chord.

Problem Solving Strategies

Concepts:

- Building right triangles and applying the Pythagorean Theorem is one of the most common ways to find lengths. This is particularly true in problems that already involve right angles.
- Often it's best to try a few examples before trying to derive a general formula – the examples can be a good guide when you try to derive your formula.

Things To Watch Out For!

WARNING!! Be careful when using your knowledge of Pythagorean triples to make sure your given sides match corresponding sides (legs or hypotenuse) of the Pythagorean triple you would like to use.

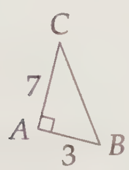
Extra! *In mathematics you don't understand things. You just get used to them.*

–John von Neumann

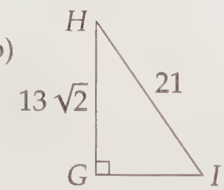
Review Problems

6.26 Find the missing side length in each of the triangles below:

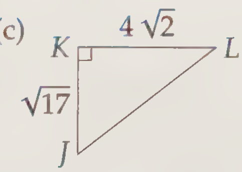
(a)



(b)

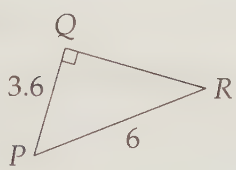


(c)

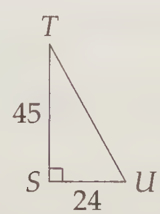


6.27 Find the missing side length in each of the triangles below by using your knowledge of Pythagorean triples:

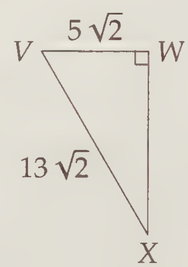
(a)



(b)

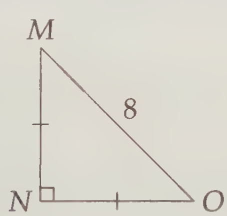


(c)

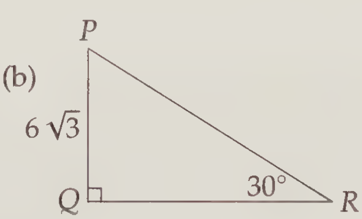


6.28 Find the missing side lengths in the triangles below:

(a)



(b)



6.29 For each of the following parts, state whether or not the three numbers listed could be the side lengths of a right triangle.

(a) 11, 16, 19.

(b) 1/5, 4/15, 1/3.

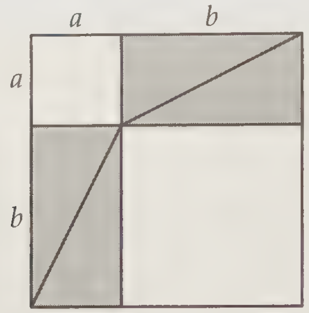
(c) $\sqrt{73}$, $2\sqrt{2}$, 9.

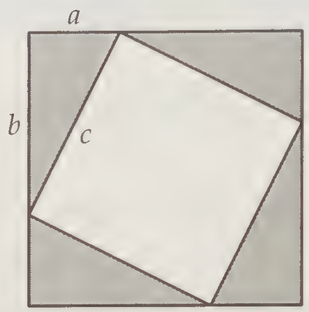
(d) $\sqrt{0.5}$, $\sqrt{1.2}$, $\sqrt{1.3}$.

(e) 0.77, 2.64, 2.75.

6.30 Given that $AB = 3$, $BC = 6$, and $\angle ABC = 60°$, why must $\triangle ABC$ be a right triangle? (*You cannot simply say 'It is a 30-60-90 triangle; you must prove that it is!*)

6.31 In this problem, we find yet another proof of the Pythagorean Theorem.



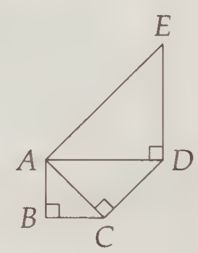


(a) Construct a square of side length $a + b$, and place four right triangles with legs of lengths a and b in it as shown in the left diagram. Let K be the area of each shaded triangle. Express the area of the whole square in terms of K and the areas of the two interior squares.

(b) Now place the four triangles as shown in the right diagram and let c be the length of the hypotenuse of each of the triangles. Express the area of the large square in terms of K and the area of the interior square.

(c) What do you conclude from parts (a) and (b)?

6.32 In the figure at right, $AB = BC = 1$. $\overline{AD} \parallel \overline{BC}$ and $\overline{CD} \parallel \overline{AE}$. Find AE. *(Source: MATHCOUNTS)*



6.33 Find the perimeter of an isosceles triangle with base length 10 and area 60.

6.34 Using any method you like, find the areas of the triangles with the side lengths below:

(a) 6, 8, 10.

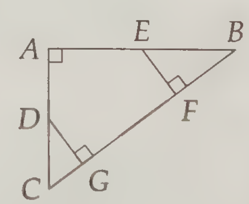
(b) 3, 4, 6.

(c) $\sqrt{6}$, $\sqrt{24}$, $\sqrt{30}$.

(d) 1, 24/7, 25/7.

(e) 5, 6, 7.

6.35 In the figure shown, E is the midpoint of $\overline{AB}$, D is the midpoint of $\overline{AC}$, $AB = 16$, and $AC = 12$. *(Source: MATHCOUNTS)*



(a) Show that $\triangle EFB \sim \triangle CAB$.

(b) Find DG and EF.

(c) What is the area of $AEFGD$?

6.36 Find the area of an equilateral triangle with a height of length 8.

6.37 In $\triangle MNO$, $MN = 13$, and $NO = 37$. P is on $\overline{MO}$ such that $\overline{NP} \perp \overline{MO}$ and $NP = 12$. Find the area of $\triangle MNO$. *(Source: MATHCOUNTS)*

6.38 Let ABC be a right triangle, with $\angle ACB = 90°$. Let D be the foot of the perpendicular from C to side $\overline{AB}$. Let $x = AD$ and $y = BD$. Prove that $CD = \sqrt{xy}$.

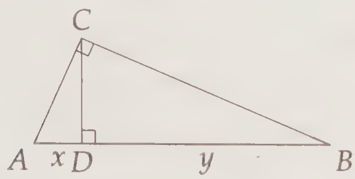


Figure 6.3: Diagram for Problem 6.38

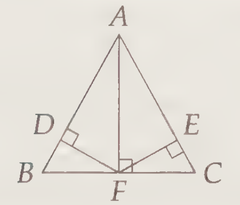
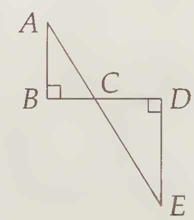


Figure 6.4: Diagram for Problem 6.39

6.39 In the diagram at right above, $AB = AC$, $\overline{AF} \perp \overline{BC}$, $\overline{FD} \perp \overline{AB}$, and $\overline{FE} \perp \overline{AC}$. Prove that $FD = FE$.

6.40 In the diagram at right, $BD = 6$, $AB = 3$, and $DE = 5$. What is the length of $\overline{AE}$? *(Source: MATHCOUNTS)*



6.41 The length of one leg of a right triangle is 22, and the other two sides also have integer lengths. Find the perimeter of the triangle.

6.42 Point A is on side $\overline{XZ}$ of $\triangle XYZ$ such that $XA = XY$. Given that $\angle YXZ = 90°$, $\angle YZX = 30°$, and $ZA = 6 - \sqrt{12}$, find the area of $\triangle XYZ$.

6.43 The sides of a triangle have lengths of 15, 20, and 25. Find the length of the shortest altitude of the triangle. *(Source: AMC 10)*

Challenge Problems

6.44 A right triangle has area 210 and hypotenuse 29. Find the perimeter of the triangle. **Hints:** 119, 392

6.45 What's wrong with the diagram at left below? **Hints:** 409

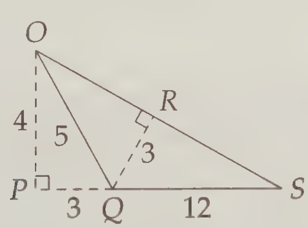


Figure 6.5: Diagram for Problem 6.45

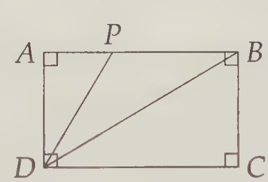
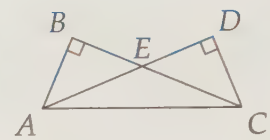


Figure 6.6: Diagram for Problem 6.46

6.46 In rectangle $ABCD$, $AD = 1$, P is on $\overline{AB}$, and $\overline{DB}$ and $\overline{DP}$ trisect $\angle ADC$. (An angle is trisected when it is divided in three equal angles.) What is the perimeter of $\triangle BDP$? *(Source: AMC 10)* **Hints:** 421

6.47 The hypotenuse of a right triangle has length 8. The triangle's area is also 8. Find the perimeter of the triangle. **Hints:** 16



6.48 In the diagram at right $\angle B = \angle D = 90^\circ$, $AB = DC = 24$, and $BC = AD = 32$. *(Source: ARML)*

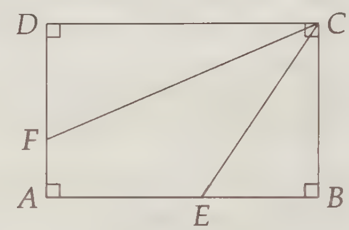
(a) Prove that $\triangle ABE \cong \triangle CDE$.

(b) Let M be the midpoint of $\overline{AC}$. Prove that $\overline{EM} \perp \overline{AC}$.

(c) Find the area of $\triangle AEC$. **Hints:** 296

6.49 Triangle ABC has $AB = 12$, $BC = 16$, and $AC = 20$. If D is on $\overline{AC}$ such that $AD = 12$, find the area of $\triangle ADB$. *(Source: ARML)* **Hints:** 116, 424

6.50 Riders on a Ferris wheel travel in a circle in a vertical plane. A particular wheel has radius 20 feet and revolves at the constant rate of one revolution per minute. How many seconds does it take a rider to travel from the bottom of the wheel to a point 10 vertical feet above the bottom? *(Source: AMC 10)* **Hints:** 465



6.51 Shown at right is rectangle $ABCD$. Angle C is trisected by $\overline{CF}$ and $\overline{CE}$, where E is on $\overline{AB}$, F is on $\overline{AD}$, $BE = 6$, and $AF = 2$. *(Source: AMC 12)*

(a) Find BC.

(b) Find DF.

(c) Find $[ABCD]$.

6.52 Initially, a fifty-foot ladder rests against a wall. As I start to climb it, the ladder slides down, finally stopping such that it touches the wall at a point 8 feet below where it originally touched the wall. During the slide, the base of the ladder slid 16 feet from its original position. How far is the top of the ladder from the ground after the slide, given that the wall is perpendicular to the ground? **Hints:** 137

6.53 Using only a pencil and paper, find the Pythagorean triple with 73 as the length of the hypotenuse. **Hints:** 40, 423

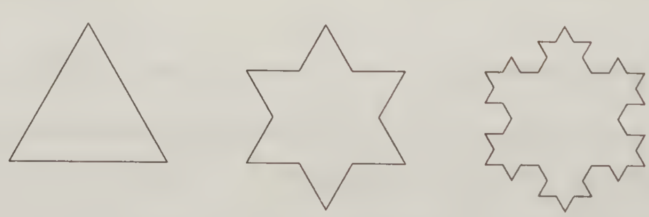
6.54★ If the lengths of the three sides of a right triangle are whole numbers, how many such distinct noncongruent right triangles exist having one leg with a length of 24 units?

6.55 In $\triangle RST$, $RS = 13$, $ST = 14$, and $RT = 15$.

(a) Find the length of the height from R to $\overline{ST}$.

(b)★ Let M be the midpoint of $\overline{ST}$. Find RM. **Hints:** 563

6.56★ We begin with an equilateral triangle with side length 1. We divide each side into three segments of equal length, and add an equilateral triangle to each side using the middle third as a base. We then repeat this, to get a third figure.



If we continue this process forever, what is the area of the resulting figure? **Hints:** 13, 398

6.57★ Let $\triangle XOY$ be a right-angled triangle with $\angle XOY = 90^\circ$. Let M and N be the midpoints of legs $\overline{OX}$ and $\overline{OY}$, respectively. Given that $XN = 19$ and $YM = 22$, find XY. *(Source: AMC 10)* **Hints:** 290, 505

6.58★ In triangle ABC, $AB = AC$, $BC = \sqrt{3} - 1$, and $\angle BAC = 30^\circ$. Find the length of AB. **Hints:** 384

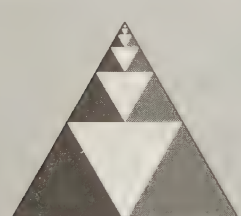
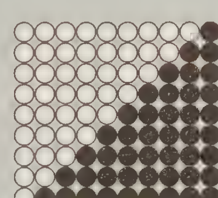
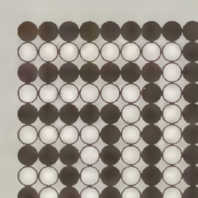
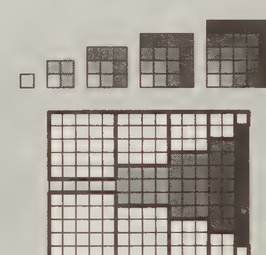
6.59★ In $\triangle ABC$, $AB = 6$ and $BC = 8$. Find AC in each of the following cases:

(a) $\angle B = 30^\circ$. **Hints:** 57

(b) $\angle B = 45^\circ$. **Hints:** 129

(c) $\angle B = 135^\circ$. **Hints:** 215

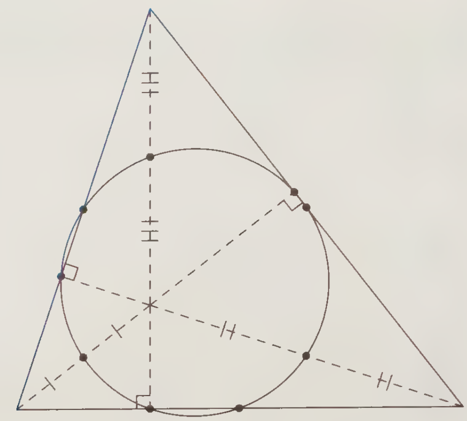
Extra! The great modern mathematician Paul Erdös was fond of describing particularly beautiful proofs as being from 'the book.' For sport, many aesthetically inclined math fans find beautiful 'proofs without words' by using diagrams designed to 'show' the truth of a mathematical statement rather than 'saying' it. In our various proofs without words of the Pythagorean Theorem, we've seen proofs without words applied to the most natural area of mathematics for them, geometry. However, the diagrams below show that the tools of geometry can be used to 'prove without words' statements from other fields of mathematics, as well. Perhaps you'll agree that these proofs are indeed 'from the book.'

Match the images above with the following mathematical statements:

- $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.
- $\dfrac{1}{4} + \left(\dfrac{1}{4}\right)^2 + \left(\dfrac{1}{4}\right)^3 + \left(\dfrac{1}{4}\right)^4 + \cdots = \dfrac{1}{3}$.
- $1 + 2 + 3 + 4 + \cdots + n = \dfrac{n(n+1)}{2}$.
- $1 + 3 + 5 + 7 + 9 + \cdots + (2n-1) = n^2$.

Make sure you see how each statement is illustrated by its matching image!
Source: Proofs Without Words by Roger Nelsen



The Nine-Point Circle

Why isn't there a special name for the tops of your feet? – Lily Tomlin

CHAPTER 7

Special Parts of a Triangle

Most of mastering geometry is mastering triangles. In this chapter we explore several special points, lines, and circles that can be found in any triangle.

In general, any line segment from a vertex of a triangle to a point on the opposite side is called a **cevian**. Before we dive into the special parts of a triangle, we'll investigate a couple of special lines.

7.1 Bisectors

The **perpendicular bisector** of a line segment is the line passing through the midpoint of the segment such that the line is perpendicular to the segment. In Figure 7.1, line k is the perpendicular bisector of $\overline{AB}$.

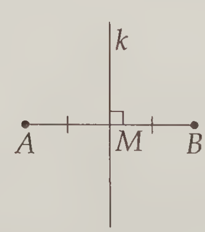


Figure 7.1: A Perpendicular Bisector

The **angle bisector** of an angle is the ray that divides the angle into two equal angles. In Figure 7.2, ray m is the angle bisector of $\angle AOB$.

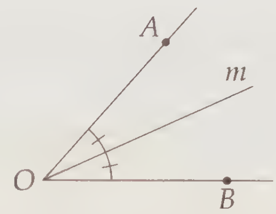
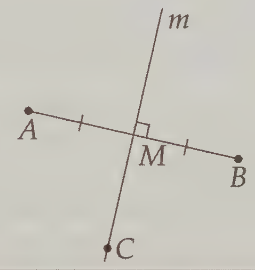


Figure 7.2: An Angle Bisector

Problems

Problem 7.1: Line m is the perpendicular bisector of $\overline{AB}$. Use congruent triangles to show that if C is on m, then $CA = CB$.



Problem 7.2: In this problem, we will show that if $AC = BC$, then C is on the perpendicular bisector of $\overline{AB}$.

(a) Let C be a point such that $AC = BC$ and let M be the midpoint of $\overline{AB}$. Draw a diagram.

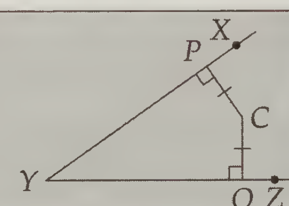
(b) Draw $\overline{CM}$, then use congruent triangles to prove that $\overline{CM} \perp \overline{AB}$. Why does this mean that C is on the perpendicular bisector of $\overline{AB}$?

Problem 7.3: In this problem we show that any point on the angle bisector of $\angle XYZ$ is equidistant from $\overrightarrow{YX}$ and $\overrightarrow{YZ}$.

(a) Let C be on the angle bisector of $\angle XYZ$. Draw $\angle XYZ$ and segments from C to the sides of $\angle XYZ$ that are perpendicular to $\overrightarrow{YX}$ and $\overrightarrow{YZ}$, respectively.

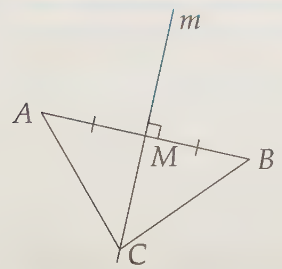
(b) Use congruent triangles to prove that the two segments from C to the sides of the angle are equal in length.

Problem 7.4: C is a point inside $\angle XYZ$ such that C is equidistant from $\overrightarrow{YZ}$ and $\overrightarrow{YX}$ as shown. Show that C must be on the angle bisector of $\angle XYZ$.



Before we start tackling problems involving special lines in triangles, we will use what we already know about congruent triangles to learn about these special lines. We'll start with perpendicular bisectors.

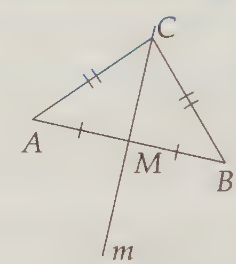
Problem 7.1: Line m is the perpendicular bisector of $\overline{AB}$. Show that if C is on m, then $CA = CB$.



Solution for Problem 7.1: Drawing $\overline{AC}$ and $\overline{CB}$, we see that $\triangle AMC \cong \triangle BMC$ by SAS Congruence since $AM = MB$, $\angle AMC = \angle BMC$, and $CM = CM$. Therefore, $AC = CB$. □

Having shown that every point on the perpendicular bisector of a segment is equidistant from from the endpoints of the segment, we should wonder if every point that is equidistant from these endpoints has to be on the perpendicular bisector of the segment connecting them.

Problem 7.2: Show that if $AC = BC$, then C is on the perpendicular bisector of $\overline{AB}$.



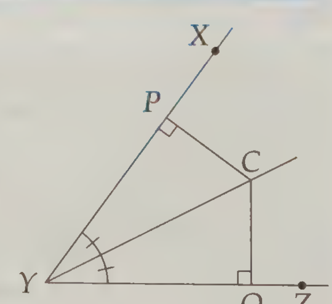
Solution for Problem 7.2: To show that C is on the perpendicular bisector of $\overline{AB}$, we connect C to the midpoint of $\overline{AB}$, which we'll call M, then show that $\overline{CM} \perp \overline{AB}$. Since M is the midpoint of $\overline{AB}$, we have $AM = MB$. We are given $AC = CB$, and obviously $CM = CM$, so we have $\triangle ACM \cong \triangle BCM$ by SSS Congruence. Therefore, $\angle AMC = \angle BMC$. Since these two angles must add to 180°, they must each equal 90°. So, $\overleftrightarrow{CM}$ is the perpendicular bisector of $\overline{AB}$. (This proof doesn't address the possibility that C is the midpoint of $\overline{AB}$. We take care of this by noting that the midpoint of $\overline{AB}$ is on the perpendicular bisector of $\overline{AB}$ by definition.) □

Putting these last two problems together tells us something very important about the perpendicular bisector of a segment.

Important: The perpendicular bisector of a segment is the straight line consisting of all points that are equidistant from the endpoints of the segment.

We can prove something very similar for angle bisectors.

Problem 7.3: Show that any point on the angle bisector of $\angle XYZ$ is equidistant from $\overrightarrow{YX}$ and $\overrightarrow{YZ}$.

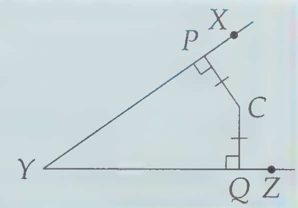


Solution for Problem 7.3: We start by drawing the lengths we need to show equal. These are the perpendicular segments $\overline{CP}$ and $\overline{CQ}$ from C to $\overrightarrow{YX}$ and $\overrightarrow{YZ}$ as shown. By AAS Congruence, we have $\triangle CPY \cong \triangle CQY$, so $CP = CQ$.

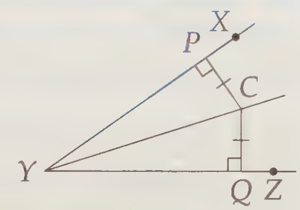
Therefore, any point on the angle bisector of an angle is equidistant from the sides of the angle. □

As with the perpendicular bisector, we can 'run this backwards,' deducing that any point that is equidistant from the sides of an angle must be on the angle bisector.

Problem 7.4: C is a point inside $\angle XYZ$ such that C is equidistant from $\overrightarrow{YZ}$ and $\overrightarrow{YX}$ as shown. Show that C must be on the angle bisector of $\angle XYZ$.



Solution for Problem 7.4: We have $\triangle CPY \cong \triangle CQY$ by HL Congruence because $CP = CQ$ and $CY = CY$. Therefore, $\angle CYP = \angle CYQ$, so $\overrightarrow{YC}$ bisects $\angle XYZ$. □



Putting the last two problems together tells us a little more about angle bisectors.

Important: The angle bisector of an angle consists of all points that are equidistant from the sides of the angle.

Exercises

7.1.1 Line m is the perpendicular bisector of both $\overline{AB}$ and $\overline{CD}$. Which of the following must be true and why (or why not)?

(a) $AB = CD$.

(b) $\overline{AB} \parallel \overline{CD}$.

(c)★ $AC = BD$. **Hints:** 460

7.1.2 Describe all the points that are equidistant from two intersecting lines.

7.1.3 Describe all the points that are equidistant from two parallel lines.

7.2 Perpendicular Bisectors of a Triangle

Problems

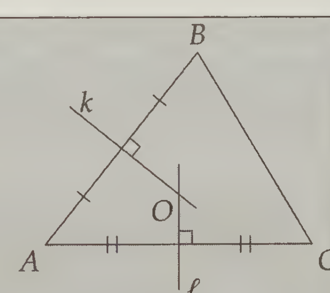
Problem 7.5: Let k and ℓ be perpendicular bisectors of the sides $\overline{AB}$ and $\overline{AC}$ of $\triangle ABC$ as shown.

(a) From what two points must every point on k be equidistant?

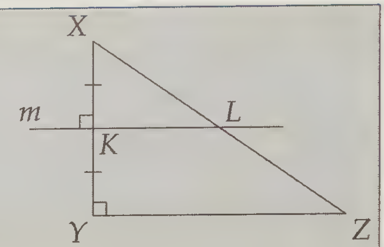
(b) From what two points must every point on ℓ be equidistant?

(c) Let O be the intersection of k and ℓ. Why must O be on the perpendicular bisector of $\overline{BC}$?

(d) If we draw a circle with center O and radius OA, will the circle go through B and C as well? Why or why not?



Problem 7.6: Line m is the perpendicular bisector of side $\overline{XY}$ of right triangle $\triangle XYZ$, and m meets $\overline{XZ}$ at L as shown.



(a) Show that $\triangle XKL \sim \triangle XYZ$.

(b) Use part (a) to show that L is the midpoint of $\overline{XZ}$.

(c) Where do the perpendicular bisectors of the sides of a right triangle meet?

(d) What is the radius of the circle that passes through X, Y, and Z?

Problem 7.7: Ariel and Zappa are playing 'NAME THAT CIRCLE.' In 'NAME THAT CIRCLE,' Ariel imagines a circle drawn on her sheet of paper. Zappa can then ask for points that are on the circle. Zappa wins when he can draw the whole circle.

(a) Can Zappa win the game if Ariel only gives him two points?

(b) Can Zappa win if Ariel gives him three points?

(c) Zappa asks for four points. Ariel isn't really paying attention and just picks out four points at random on the piece of paper and ends up with points shown in the diagram at right. Zappa tries and tries, but he can't NAME THAT CIRCLE. Finally, he insists that there is no such circle. Could he possibly be right or should he just try harder?

Ariel's points

Problem 7.8: In this problem we find the radius of a circle that passes through all three vertices of an equilateral triangle. Let $\triangle ABC$ be our triangle, with $AB = 6$. Let P be the midpoint of $\overline{BC}$ and Q the midpoint of $\overline{AC}$, and let O be the center of the circle through the vertices of $\triangle ABC$.

(a) Draw a diagram. Include the perpendicular bisectors of $\overline{AC}$ and $\overline{BC}$. Where must these perpendicular bisectors intersect?

(b) Prove that $\triangle OCP$ is a 30-60-90 right triangle.

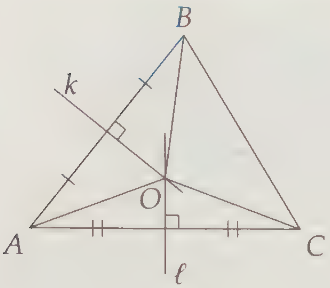
(c) Find CP, then find the radius of the circle.

Now that we've learned a little bit about perpendicular bisectors, we're ready to study the perpendicular bisectors of the sides of a triangle.

Problem 7.5: Show that the three perpendicular bisectors of the sides of a triangle are **concurrent**, meaning they meet at a single point. Show that there is a circle centered at this point that passes through all three vertices of the triangle.

Extra! *The mathematician's patterns, like those of the painter or the poet, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. There is no permanent place in the world for ugly mathematics.*

–G. H. Hardy



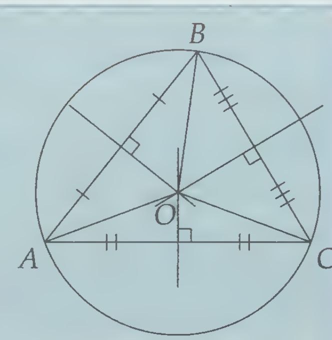
Solution for Problem 7.5: We start by thinking about just two of the perpendicular bisectors, k and ℓ, shown in the figure at left. Let k and ℓ meet at O. We wish to show that O is on the perpendicular bisector of $\overline{BC}$.

Since O is on the perpendicular bisector of $\overline{AB}$, it is equidistant from A and B, i.e. $OA = OB$. Similarly, since O is on the perpendicular bisector of $\overline{AC}$, we have $OA = OC$. So, $OA = OB = OC$. Since $OB = OC$, O must be on the perpendicular bisector of $\overline{BC}$ (see Problem 7.2), so the perpendicular bisectors of the sides of a triangle are concurrent.

Seeing $OA = OB = OC$, we note that if we draw a circle with center O and radius OA, the circle will pass through B and C as well. □

The point O and the circle described in the last problem are so useful in geometry problems that they have their own names.

Important: The perpendicular bisectors of the sides of a triangle are concurrent at a point called the **circumcenter**. The circle centered at the circumcenter that passes through the vertices of the original triangle is called the **circumcircle** of the triangle because it is **circumscribed** about the triangle (meaning it passes through all the vertices of the triangle).

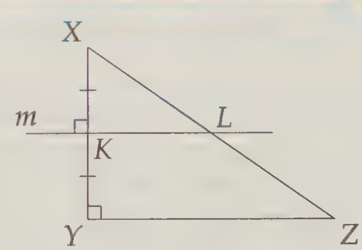


Finally, the radius of this circle is called the **circumradius**, the circumcenter is usually labeled with the letter O, and the circumradius is usually called R.

In right triangles, the circumcircle is particularly easy to describe.

Problem 7.6: Where is the circumcenter of a right triangle? What is the circumradius of a right triangle?

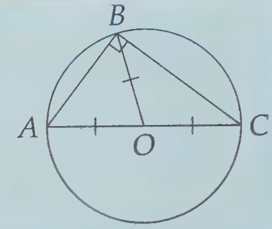
Solution for Problem 7.6: If we draw a few right triangles and their perpendicular bisectors, we will begin to suspect that the midpoint of the hypotenuse is the circumcenter.



We start our proof by drawing the perpendicular bisector of one of the legs, such as line m in the diagram. Let m meet $\overline{XZ}$ at L. Since m and $\overline{YZ}$ are both perpendicular to $\overline{XY}$, we have $m \parallel \overline{YZ}$. Therefore, $\triangle XKL \sim \triangle XYZ$, so $XL/XZ = XK/XY$. Since $XK/XY = 1/2$, we have $XL/XZ = 1/2$, so L is the midpoint of $\overline{XZ}$. Clearly, L is also on the perpendicular bisector of $\overline{XZ}$ (since L is the midpoint of $\overline{XZ}$), so L must be where the perpendicular bisectors of $\triangle XYZ$ intersect. Therefore, L is the circumcenter of $\triangle XYZ$.

The endpoints of the hypotenuse are vertices of the triangle, so by definition they are on the circumcircle. Since the midpoint of the hypotenuse of our right triangle is the circumcenter, the hypotenuse is a diameter of the circumcircle. Therefore, the circumradius is half the length of the hypotenuse of the right triangle. □

Important: The circumcenter of a right triangle is the midpoint of the hypotenuse, and the circumradius equals one-half the length of the hypotenuse.



Back in Problem 1.1, we saw that two points are enough to define a line. All this talk about circles going through the vertices of a triangle should make us wonder – how many points of a circle do we need to define a circle? Clearly, one won't do, since we can draw all sorts of circles through a given point.

Problem 7.7: Ariel and Zappa are playing 'NAME THAT CIRCLE.' In 'NAME THAT CIRCLE,' Ariel imagines a circle drawn on her sheet of paper. Zappa can then ask for points that are on the circle. Zappa wins when he can draw the whole circle.

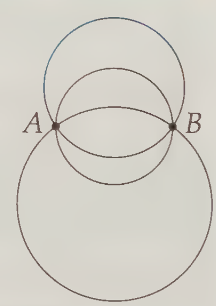
(a) Can Zappa win the game if Ariel only gives him two points?

(b) Can Zappa win if Ariel gives him three points?

(c) Zappa asks for four points. Ariel isn't really paying attention and just picks out four points at random on the piece of paper. Zappa tries and tries, but he can't NAME THAT CIRCLE. Finally, he insists that there is no such circle. Could he possibly be right or should he just try harder?

Solution for Problem 7.7:

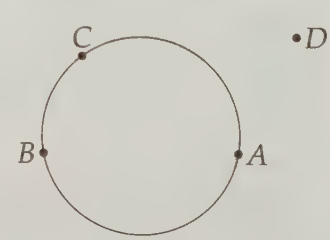
(a) Clearly two points isn't enough – we can make all sorts of different circles given two points. This is shown at right.



(b) Given three points, Zappa can NAME THAT CIRCLE! He can connect the points to make a triangle. Then, he can draw the perpendicular bisectors of two of the sides of the triangle. Where they meet is the circumcenter. He can then make the circumcircle. Notice that the three points must be **noncollinear**, meaning that it's impossible to draw a single line that goes contains all three points.

The center of any circle that passes through all three points must be on all three perpendicular bisectors of the sides of the triangle with these points as vertices. Therefore, the circumcenter is the only possible center of a circle through the three points. Therefore, the circumcircle is the *only* circle that passes through all three points.

(c) Let the points be A, B, C, and D. As described in the previous part, there is only one circle through A, B, and C. But this circle doesn't have to go through D! Therefore, if we pick out four points at random, there doesn't have to be a circle that passes through all four points. Zappa should give up once he sees that the circumcircle of $\triangle ABC$ doesn't pass through D. One such arrangement of points is shown at right.

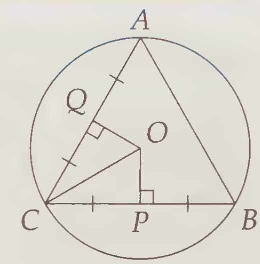


□

Important: Just as two points determine a line, we have now shown that three noncollinear points determine a circle. This means that given any three noncollinear points, there is exactly one circle that passes through all three.

Now let's try a problem involving the circumradius.

Problem 7.8: Find the circumradius of an equilateral triangle with side length 6.



Solution for Problem 7.8: Let $\triangle ABC$ be our equilateral triangle. We connect its circumcenter to a vertex and draw a couple perpendicular bisectors as shown. Since $CQ = CP$ and $OC = OC$, we know that $\triangle OCP \cong \triangle OCQ$ by HL Congruence. Therefore, $\angle OCQ = \angle OCP$, so $\overline{OC}$ must bisect $\angle ACB$. Hence, $\angle OCP = \angle ACB/2 = 30^\circ$ and $\triangle OCP$ is a 30-60-90 triangle. Since P is the midpoint of $\overline{BC}$, we have $CP = 3$, so

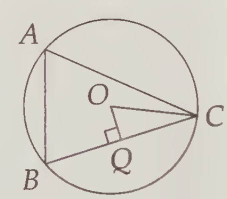
$$OC = CP\left(\frac{2}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} = 2\sqrt{3}.$$

□

Exercises

7.2.1 Where is the circumcenter of an acute triangle – in, on, or outside the triangle? How about an obtuse triangle? (You do not need to prove your answer – you'll learn tools later that will make these proofs easier.)

7.2.2 Find the radius of the shown circle with center O given that $BC = 8$ and $QO = 2$.



7.2.3 Find the circumradius of an equilateral triangle with side length 18.

7.2.4 What is the circumradius of a right triangle with legs of length 6 and 8?

7.3 Angle Bisectors of a Triangle

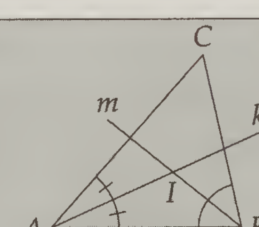
Problems

Problem 7.9: Let k be the angle bisector of $\angle CAB$ and m be the angle bisector of $\angle ABC$. Let k and m meet at I.

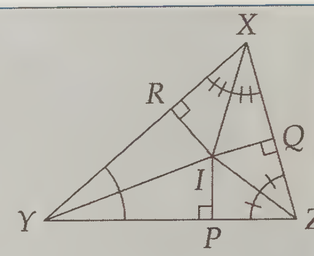
(a) Every point on k must be equidistant from what two lines?

(b) Every point on m must be equidistant from what two lines?

(c) Why must I be on the angle bisector of $\angle C$?



Problem 7.10: The angle bisectors of $\triangle XYZ$ meet at I, and perpendicular segments from I to the sides of the triangle are drawn as shown.

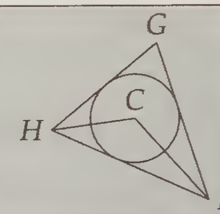


(a) Why is $IR = IQ = IP$?

(b) Suppose we draw a circle with center I and radius IP. Does the circle hit $\overline{YZ}$ at any point besides P?

(c) Show that the circle from (b) is tangent to sides $\overline{XZ}$ and $\overline{XY}$ at Q and R, respectively.

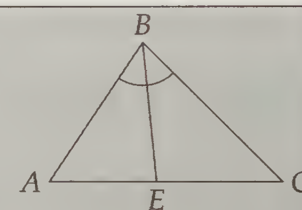
Problem 7.11: Segments $\overline{GH}$, $\overline{HI}$, and $\overline{IG}$ are tangent to the circle with center C as shown. Given that $\angle GHI = 70°$ and $\angle GIH = 50°$, find the following:



(a) $\angle CIH$.

(b) $\angle HCI$.

Problem 7.12: In the diagram, $\overline{BE}$ is the angle bisector of $\angle B$. In this problem, we will prove the Angle Bisector Theorem, which states that if $\overline{BE}$ is an angle bisector as shown, then $AB/AE = CB/CE$.

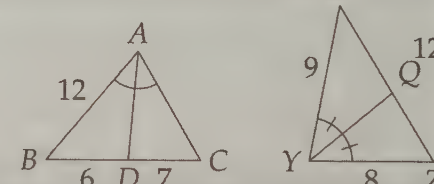


(a) Ratios make us think of similar triangles, which make us think of parallel lines. Draw a line through C parallel to $\overline{AB}$, then extend $\overline{BE}$ past E to hit this new line at X. What triangles are similar?

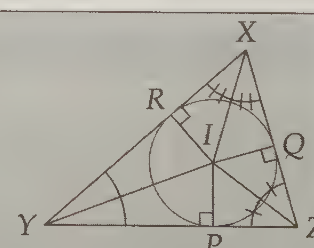
(b) What type of triangle is $\triangle BCX$?

(c) Use your similar triangles from (a) and your observation in (b) to prove that $AB/AE = CB/CE$.

Problem 7.13: Find AC and XQ in the triangles shown. ($XZ = 12$)



Problem 7.14: Back in Problem 7.10, we discovered a circle that is tangent to all three sides of $\triangle XYZ$. In this problem, we will discover a formula for the area of $\triangle XYZ$ in terms of the radius of this circle and the perimeter of the triangle.



(a) Break $\triangle XYZ$ into three triangles: $\triangle XYI$, $\triangle YZI$, and $\triangle ZXI$. Find the area of each triangle in terms of one of the sides of $\triangle XYZ$ and r, the radius of the circle in the diagram.

(b) Find the desired formula for $[XYZ]$ by adding $[XYI]$, $[YZI]$, and $[ZXI]$.

Problem 7.15: In this problem we find the radius of a circle that is tangent to all three sides of $\triangle ABC$ given that the sides of $\triangle ABC$ have lengths 7, 24, and 25.

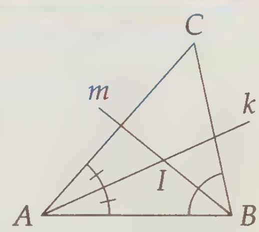
(a) What kind of triangle is $\triangle ABC$? What are the area and perimeter of $\triangle ABC$?

(b) What is the radius of the circle tangent to all three sides of $\triangle ABC$?

Having found that the perpendicular bisectors of the sides of a triangle are concurrent, we wonder if a similar proof can be used to show that the bisectors of the angles of a triangle are also concurrent.

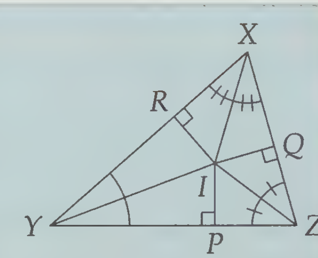
Problem 7.9: Prove that the angle bisectors of a triangle are concurrent.

Solution for Problem 7.9: We start much as we did with the perpendicular bisectors. We consider the intersection of two of our angle bisectors, point I in the diagram. Since I is on the bisector of $\angle CAB$, it is equidistant from $\overleftrightarrow{AC}$ and $\overleftrightarrow{AB}$. Since it is on the angle bisector of $\angle ABC$, it is equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$. Therefore, I is equidistant from all three sides of the triangle. Specifically, since I is equidistant from $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$, I is also on the angle bisector of $\angle ACB$, as we proved in Problem 7.4. □



We found a special circle centered at the intersection of the perpendicular bisectors. Let's see if the intersection of the angle bisectors is the center of a special circle, too!

Problem 7.10: The angle bisectors of $\triangle XYZ$ meet at I, and perpendicular segments from I to the sides of the triangle are drawn as shown.



(a) Why is $IR = IQ = IP$?

(b) Suppose we draw a circle with center I and radius IP. Does the circle hit $\overline{YZ}$ at any point besides P?

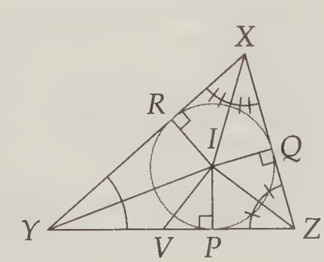
(c) Show that the circle from (b) is tangent to sides $\overline{XZ}$ and $\overline{XY}$ at Q and R, respectively.

Solution for Problem 7.10:

(a) In the previous problem, we found that I, the intersection of the angle bisectors of $\triangle XYZ$, is equidistant from the sides of $\triangle XYZ$. Since IR, IQ, and IP are the distances from I to the sides of $\triangle XYZ$, we have $IR = IQ = IP$.

(b) One way we can show that the circle with center I and radius IP doesn't hit $\overline{YZ}$ anywhere else is to show that all the other points on $\overline{YZ}$ are *outside* the circle.

As shown, point V is a point on $\overline{YZ}$ besides point P. From right triangle $\triangle IPV$, we have $IV^2 = IP^2 + VP^2$, so $IV > IP$. Our circle consists of all points exactly IP from I. Since V is farther than IP from I, it must be outside our circle. There's nothing special about V; all points on $\overline{YZ}$ besides P are outside our circle. Therefore, it is impossible for the circle with radius IP to hit $\overline{YZ}$ at a second point. So, the circle must be tangent to $\overline{YZ}$.

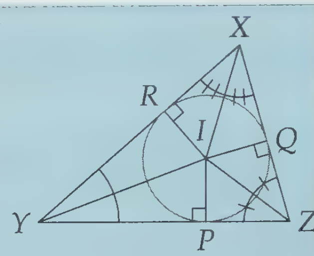


(c) Since $IP = IQ = IR$, the circle with center I and radius IP goes through Q and R, too. For the same reason as in the previous part, this circle is tangent to all three sides of $\triangle XYZ$.

□

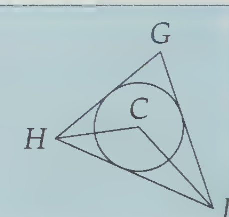
We have found the center of a special circle at the intersection of the angle bisectors of a triangle.

Important: The angle bisectors of a triangle are concurrent at a point called the **incenter**. This point is equidistant from the sides of the triangle. This common distance from the incenter to the sides of a triangle is called the **inradius**, because the circle with center I and this radius is tangent to all three sides of the triangle. This circle is unsurprisingly called the **incircle** because it is inscribed in the triangle (meaning it is tangent to all the sides of the triangle). The incenter is usually denoted I, and the inradius is usually written as r.



Each triangle has exactly one incircle. (You'll be asked to prove this in Problem 12.43.) Therefore, if a circle is tangent to all three sides of a triangle, its center is the intersection of the angle bisectors of the triangle.

Problem 7.11: Segments $\overline{GH}$, $\overline{HI}$, and $\overline{IG}$ are tangent to the circle with center C as shown. Given that $\angle GHI = 70^\circ$ and $\angle GIH = 50^\circ$, find $\angle HCI$.



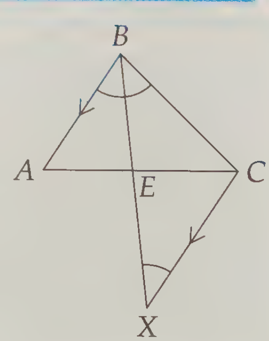
Solution for Problem 7.11: Since the circle is tangent to all three sides of $\triangle GHI$, it is the incircle of $\triangle GHI$. Therefore, C is the incenter of $\triangle GHI$, so $\overline{CI}$ and $\overline{CH}$ are bisectors of angles $\angle GIH$ and $\angle GHI$, respectively.

$\angle CHI = (\angle GHI)/2 = 35^\circ$ and $\angle CIH = (\angle GIH)/2 = 25^\circ$, so we have $\angle HCI = 180^\circ - 35^\circ - 25^\circ = 120^\circ$. □

The incenter and the incircle aren't the only useful aspects of angle bisectors.

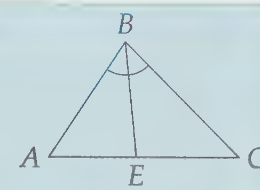
Problem 7.12: Given that E is on $\overline{AC}$ such that $\overline{BE}$ is the bisector of $\angle ABC$, prove that $AB/AE = CB/CE$.

Solution for Problem 7.12: Seeing ratios, we think of similar triangles. Similar triangles make us think of parallel lines. We draw a line through C parallel to $\overline{AB}$ and we extend $\overline{BE}$ past E to hit this new line at X. We pick this parallel line to draw because it lets us use the angles that are formed by the angle bisector. Specifically, we have $\angle BXC = \angle ABX$. $\angle ABX$ also equals $\angle XBC$, so $\triangle XCB$ is isosceles and $CB = CX$.



Our parallel line gives us similar triangles: $\triangle ABE \sim \triangle CXE$. This similarity gives us $AB/AE = CX/CE$. Since $CB = CX$, we have the desired $AB/AE = CB/CE$.
□

Important: The **Angle Bisector Theorem** states that if E is on $\overline{AC}$ such that $\overline{BE}$ is the angle bisector of $\angle B$ in triangle $\triangle ABC$, then

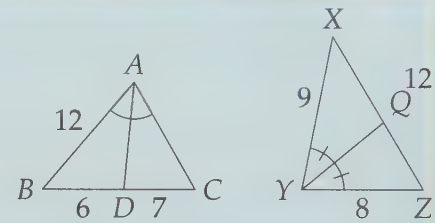


$$\frac{AB}{AE} = \frac{CB}{CE}.$$

We should think of this any time we have a problem involving lengths and angle bisectors.

For example, try using the Angle Bisector Theorem on this problem.

Problem 7.13: Find AC and XQ in the diagram. ($XZ = 12$)



Solution for Problem 7.13: From the Angle Bisector Theorem applied to $\triangle ABC$, we have $AB/BD = AC/CD$, so $12/6 = AC/7$ and $AC = 14$.

Finding XQ is a little more challenging. We let $XQ = x$, so $QZ = 12 - x$. The Angle Bisector Theorem tells us

$$\frac{XY}{XQ} = \frac{ZY}{ZQ},$$

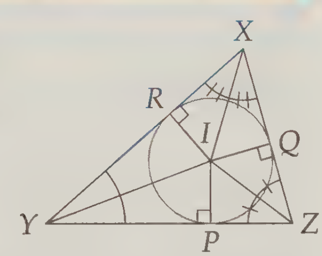
so we have

$$\frac{9}{x} = \frac{8}{12-x}.$$

Cross-multiplying gives $9(12 - x) = 8x$, so $x = 108/17$. Therefore, $XQ = x = 108/17$. Note that $XQ = (YX)(XZ)/(YX + YZ)$. Is this a coincidence? □

But wait, there's more. We can even relate an incircle to the area of its triangle!

Problem 7.14: Find the area of $\triangle XYZ$ in terms of its inradius and its side lengths.



Solution for Problem 7.14: Those radii of the incircle perpendicular to the sides make us think of altitudes. Indeed, if we break $\triangle XYZ$ into three triangles with I as a vertex, these radii are altitudes. Adding the areas of these three triangles gives us $[XYZ]$:

$$[XYZ] = [XYI] + [YZI] + [XZI] = \frac{(IR)(XY)}{2} + \frac{(IP)(YZ)}{2} + \frac{(IQ)(XZ)}{2}.$$

Since $IR = IP = IQ = r$, the inradius, and $XY + YZ + XZ = 2s$, where s is the semiperimeter of $\triangle XYZ$, we have:

$$[XYZ] = \frac{r(XY)}{2} + \frac{r(YZ)}{2} + \frac{r(XZ)}{2} = r\left(\frac{XY + YZ + XZ}{2}\right) = rs.$$

□

Important: The area of a triangle equals its inradius times its semiperimeter.

This $[ABC] = rs$ can be a useful tool in problems involving the area of a triangle or the inradius (or both).

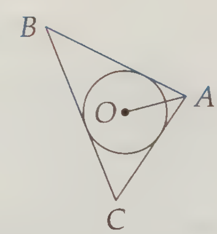
Problem 7.15: Find the radius of a circle that is tangent to all three sides of $\triangle ABC$ given that the sides of $\triangle ABC$ have lengths 7, 24, and 25.

Solution for Problem 7.15: Since $7^2 + 24^2 = 25^2$, the triangle is a right triangle with legs of length 7 and 24. (This problem is an example of why it's useful to recognize Pythagorean triples.) So, we can easily find $[ABC] = (7)(24)/2 = 84$. Since we can also find the semiperimeter, we can find the inradius. Because $s = (7 + 24 + 25)/2 = 28$, we have

$$r = \frac{[ABC]}{s} = \frac{84}{28} = 3.$$

□

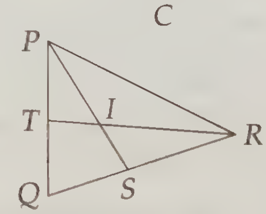
Exercises



7.3.1 $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ are tangent to $\odot O$. If $\angle A = 70°$, $\angle B = 72°$, and $\angle C = 38°$, what is the measure of $\angle OAB$? **Hints:** 228

7.3.2 Is the incenter of an obtuse triangle inside, outside, or on the triangle?

7.3.3 Can the incenter and the circumcenter of a triangle ever be the same point?



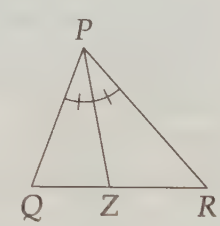
7.3.4 $\overline{PS}$ and $\overline{RT}$ are angle bisectors of $\triangle PQR$, and $\angle PIR = 130°$.

(a) Given $\angle RPQ = 30°$, find $\angle PRT$, $\angle PRQ$, and $\angle Q$.

(b) Given $\angle RPQ = 50°$, find $\angle PRT$, $\angle PRQ$, and $\angle Q$.

(c) Given $\angle RPQ = 80°$, find $\angle PRT$, $\angle PRQ$, and $\angle Q$.

(d) Do you notice anything unusual in your answers for $\angle Q$? Why does this unusual pattern occur?



7.3.5 Find RZ in the figure at right if $PR = 9$, $QZ = 4$, $PQ = 6$, and $\overline{PZ}$ bisects $\angle QPR$.

7.3.6 In $\triangle ABC$, $AB = 10$, $AC = 12$, and $BC = 8$. Point M is on side $\overline{BC}$ such that $\angle BAM = \angle CAM$. Find BM.

7.3.7 For each part below, find the length of the inradius of a triangle with the given numbers as side lengths.

(a) 3, 4, 5

(b) 6, 6, 6

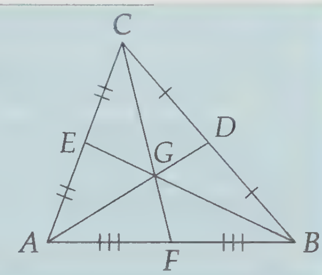
(c) 7, 7, 10

(d) 5, 6, 7

7.4 Medians

A **median** of a triangle is a segment from a vertex to the midpoint of the opposite side. In the figure below, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are all medians.

Important: The medians of a triangle are concurrent at a point called the **centroid** of the triangle. The centroid of the triangle is usually labeled G.



In each of the problems in this section, you can assume the medians of a triangle are concurrent. We'll prove that the medians of a triangle are concurrent in Chapter 17. You'll also have a chance to do it yourself in the Challenge Problems. (Hint: We suggest re-reading Section 4.3.)

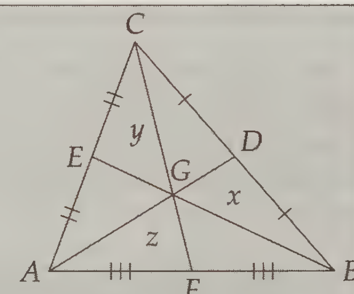
Problems

Problem 7.16: In this problem we will prove that the medians of a triangle divide the triangle into 6 triangles of equal area. We will make heavy use of the same base/same altitude principle we studied in Section 4.3.

(a) Let $[DGB] = x$, $[CGE] = y$, and $[AGF] = z$ as shown in the diagram. Prove that $[DGC] = x$, $[AGE] = y$, and $[BGF] = z$.

(b) Show that $[ABD] = [ACD]$ and use this to prove that $y = z$.

(c) Prove that the six triangles formed by drawing the medians of $\triangle ABC$ have equal area.



Problem 7.17: Use triangles $\triangle AGB$ and $\triangle DGB$ in the previous problem to prove that the centroid cuts each median in a $2:1$ ratio, specifically $AG/GD = 2$.

Problem 7.18: $\triangle PQR$ is an equilateral triangle with side length 12.

(a) Show that each angle bisector of an equilateral triangle is also a median.

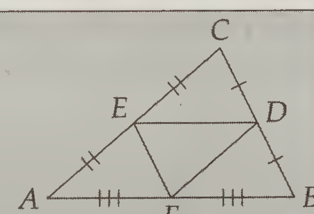
(b) Find the length of the inradius of $\triangle PQR$.

Problem 7.19: Points D, E, and F are the midpoints of the sides of $\triangle ABC$ as shown below.

(a) Show that $\triangle BDF \sim \triangle BCA$.

(b) Show that $\triangle DEF \sim \triangle ABC$.

(c) Show that $\overline{EF} \parallel \overline{CB}$.



(d) Show that each of the four small triangles in the diagram has area equal to $1/4$ the area of $\triangle ABC$.

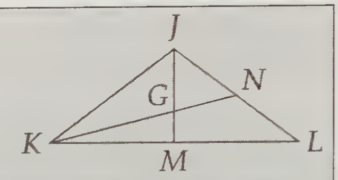
Problem 7.20: $\triangle ABC$ has sides of length $AB = 9$, $BC = 12$, and $CA = 15$. Find the length of the median from B to the midpoint of $\overline{AC}$. **Hints:** 229, 537

Problem 7.21: In $\triangle JKL$, $JK = JL = 10$ and $KL = 16$.

(a) Find the length of median $\overline{JM}$.

(b) Let medians $\overline{JM}$ and $\overline{KN}$ meet at G. Find KG.

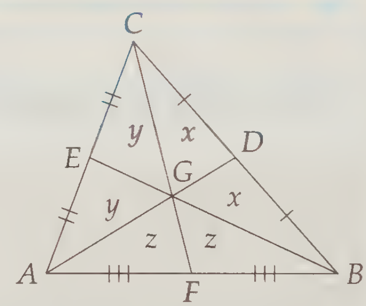
(c) Find the length of median $\overline{KN}$.



Medians create triangles with equal bases; therefore, they create triangles with equal areas. We can use these equal areas to learn even more about the medians of a triangle.

Problem 7.16: Prove that the medians of a triangle divide the triangle into six triangles with equal area.

Solution for Problem 7.16: Let our triangle be $\triangle ABC$ with midpoints D, E, and F, and with centroid G, as shown. We have $[BGD] = [CGD]$ because these triangles have the same altitude from G to equal bases $\overline{BD}$ and $\overline{CD}$. Similarly, $[AGE] = [CGE]$ and $[AFG] = [BFG]$. We assign variables to these areas as shown in the diagram at right.



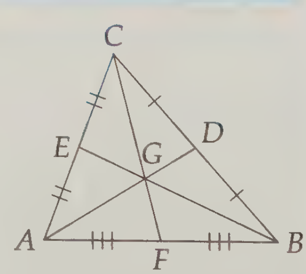
The little triangles aren't the only triangles that share altitudes. Since $\triangle ABD$ and $\triangle ACD$ have the same altitude from A to equal bases $\overline{BD}$ and $\overline{DC}$, we have $[ABD] = [ACD]$. Therefore, $x+2z = x+2y$, so $y = z$. Similarly, we can use $[AFC] = [BFC]$ to show that $x = y$, so we have $x = y = z$ and all six little triangles have the same area. □

Important: The medians of a triangle divide the triangle into six little triangles of equal area.

We can use this property of medians to discover an interesting property of the centroid.

Problem 7.17: Prove that the centroid of any triangle cuts each of the triangle's median in a 2 : 1 ratio, with the longer portion being the segment from the centroid to the vertex.

Solution for Problem 7.17: Our work with areas gives us the solution right away. Since $\triangle AGB$ consists of two of the little equal-area triangles and $\triangle GDB$ is only one of them, we have $[AGB] = 2[GDB]$. These two triangles share an altitude from B to bases $\overline{AG}$ and $\overline{GD}$, so

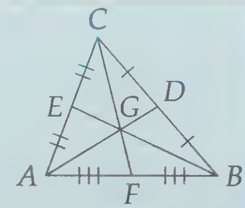


$$\frac{AG}{GD} = \frac{[AGB]}{[GDB]} = 2.$$

Clearly this works for any median. □

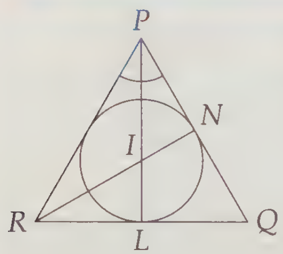
Important: The centroid of a triangle cuts its medians into a 2 : 1 ratio. For example, for the triangle shown, we have

$$\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = \frac{2}{1}.$$



This simple fact can be useful in a variety of problems. Here's an example.

Problem 7.18: $\triangle PQR$ is an equilateral triangle with side length 12. Find the length of the inradius of $\triangle PQR$.



Solution for Problem 7.18: Let $\overline{PL}$ and $\overline{RN}$ be angle bisectors of $\triangle PQR$. Since the angle bisectors of an equilateral triangle are also the medians of the triangle, L is the midpoint of $\overline{RQ}$ and the incenter, I, is also the centroid. So, I divides median $\overline{PL}$ such that $PI/IL = 2/1$. Therefore, $IL = PL/3$.

$\triangle PLR$ is a 30-60-90 triangle, so $RL = PR/2 = 6$ and $PL = RL\sqrt{3} = 6\sqrt{3}$. Finally, we have $IL = PL/3 = 2\sqrt{3}$. (Note: We could also have solved this problem by noting that $\triangle IRL$ is a 30-60-90 triangle.)□

Instead of connecting the midpoints to the opposite vertices, suppose we connect them to each other, as shown in Figure 7.3. $\triangle DEF$ is called the **medial triangle** of $\triangle ABC$.

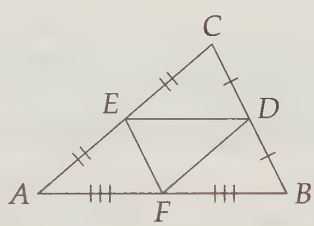


Figure 7.3: Medial Triangle

Problem 7.19: Prove that the four small triangles in Figure 7.3 are congruent, and that each is similar to $\triangle ABC$.

Solution for Problem 7.19: $\triangle FAE$ and $\triangle BAC$ have an angle in common, but we don't know anything about the other angles of the triangles. So, we look to SAS Similarity to prove similarity. Since $AF/AB = AE/AC = 1/2$, we have $\triangle FAE \sim \triangle BAC$ by SAS Similarity. Similarly, we can show $\triangle FBD \sim \triangle ABC$ and $\triangle CED \sim \triangle CAB$.

Since each of the small triangles above have lengths that are 1/2 the sides of $\triangle ABC$, we have

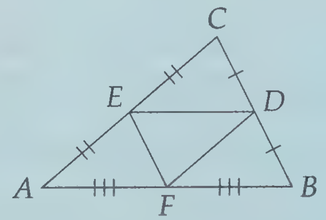
$$\frac{FE}{BC} = \frac{DE}{AB} = \frac{DF}{AC} = \frac{1}{2}.$$

Therefore, $\triangle DEF \sim \triangle ABC$ by SSS Similarity.

Since all four little triangles are similar to $\triangle ABC$ with the same side ratio, the four little triangles must be congruent. □

Notice that we can also use $\triangle AFE \sim \triangle ABC$ to show that $\angle AFE = \angle ABC$, from which we have $\overline{FE} \parallel \overline{BC}$. Similarly we can show that the other two sides of the medial triangle are parallel to their respective sides of the original triangle.

Important: Given that $\triangle DEF$ is the medial triangle of $\triangle ABC$ as shown below, we have:



I. $\triangle DEF \sim \triangle ABC$, $\triangle DEF \cong \triangle FBD \cong \triangle AFE \cong \triangle EDC$.

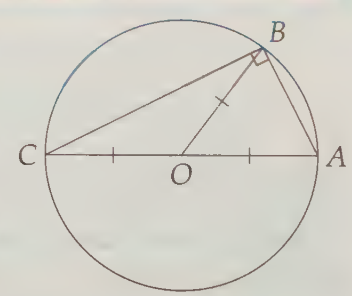
II. $\dfrac{EF}{BC} = \dfrac{DE}{AB} = \dfrac{DF}{AC} = \dfrac{1}{2}$.

III. $\overline{DF} \parallel \overline{AC}$, $\overline{EF} \parallel \overline{BC}$, and $\overline{DE} \parallel \overline{AB}$.

Together, facts II and III are sometimes called the **Midline Theorem**.

Let's try a couple problems using the properties of medians we have just found.

Problem 7.20: $\triangle ABC$ has sides of length $AB = 9$, $BC = 12$, and $CA = 15$. Find the length of the median from B to the midpoint of $\overline{AC}$.



Solution for Problem 7.20: First we notice that $\triangle ABC$ has side lengths in the ratio $3 : 4 : 5$, so it is a right triangle with hypotenuse $\overline{AC}$. Therefore, we are looking for the length of the median to the hypotenuse of a right triangle. Since the midpoint of the hypotenuse of a right triangle is also the circumcenter of the right triangle (see Problem 7.6), the median to the midpoint of the hypotenuse is a radius of the circumcircle as shown in the diagram. Therefore, the length of our median equals the circumradius, which is half the hypotenuse. So, our answer is $CA/2 = 15/2$. □

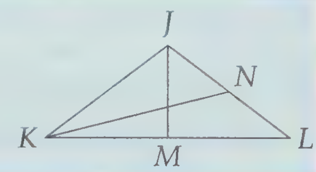
Concept: When given the lengths of the sides of a triangle in a problem, always take the time to check if it is a right triangle. Special properties of right triangles often simplify problems.

Important: The median to the hypotenuse of a right triangle is equal in length to half the hypotenuse.

Problem 7.21: In $\triangle JKL$, $JK = JL = 10$ and $KL = 16$.

(a) Find the length of median $\overline{JM}$.

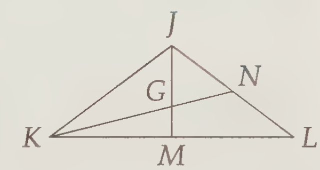
(b) Find the length of median $\overline{KN}$.



Solution for Problem 7.21:

(a) Since $JK = JL$, median $\overline{JM}$ is also an altitude (since $\triangle JKM \cong \triangle JLM$). Therefore, $\triangle JMK$ is a right triangle with hypotenuse $JK = 10$ and one leg of length $KM = 8$. So, $JM = 6$.

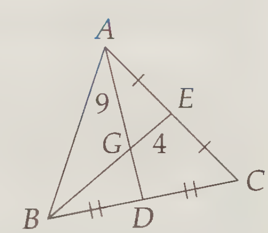
(b) Let $\overline{JM}$ and $\overline{KN}$ meet at G, which is the centroid of $\triangle JKL$. It's not obvious how to get KN, but we do have right triangle $\triangle KGM$, which we can use to find KG. Since G is the centroid and $\overline{JM}$ is a median, $GM = JM/3 = 2$. Therefore, $KG = \sqrt{GM^2 + KM^2} = \sqrt{4 + 64} = 2\sqrt{17}$. Since $\overline{KN}$ is a median and G the centroid, $KG/KN = 2/3$. Therefore, $KN = (3/2)KG = 3\sqrt{17}$.



□

Exercises

7.4.1 D and E are the midpoints of sides $\overline{BC}$ and $\overline{AC}$ of $\triangle ABC$ in the diagram at right. $\overline{AD}$ and $\overline{BE}$ meet at G, $AG = 9$, and $GE = 4$. Find GD and BG.



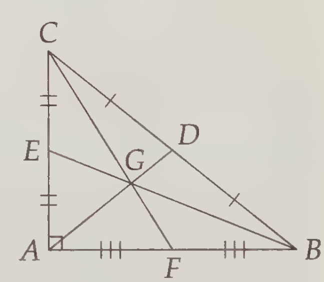
7.4.2 Right triangle $\triangle PQR$ has $PQ = 5$, $QR = 12$, and $RP = 13$. The midpoints of $\overline{QR}$, $\overline{PR}$, and $\overline{PQ}$ are A, B, and C, respectively.

(a) Draw a diagram. Mark the right angle.

(b) Find the length of median $\overline{QB}$.

(c) Find the lengths of medians $\overline{PA}$ and $\overline{RC}$.

7.4.3 The three medians of right triangle $\triangle ABC$ intersect at G as shown. Given that $AC = 8$ and $AB = 15$, find the following:



(a) $[ABC]$.

(b) $[AFC]$.

(c) $[ACD]$.

(d) $[AEG]$.

(e) $[EGDC]$.

(f) $[AFGE]$.

(g) $[DEF]$.

(h) $[BFD]$.

(i)★ CG. **Hints:** 403

7.4.4★ Medians $\overline{XA}$ and $\overline{YB}$ of $\triangle XYZ$ have the same length. Prove that $XZ = YZ$. **Hints:** 167, 375

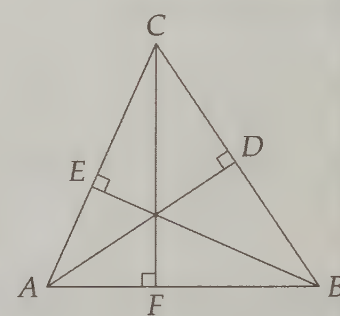
7.5 Altitudes

We've already seen altitudes in our investigation of triangle area. You can probably guess by now what we'll prove first about altitudes.

Problems

Problem 7.22: In this problem we will prove that the lines containing the altitudes of any triangle are concurrent. This is a pretty slick proof. Before trying it, you might want to review what we learned about the medial triangle on page 186.

(a) Our diagram shows altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ of $\triangle ABC$. Draw a line through A parallel to $\overline{BC}$, a line through B parallel to $\overline{AC}$, and a line through C parallel to $\overline{AB}$. The intersections of these three lines form another triangle. Call this triangle $\triangle JKL$, with A on $\overline{KL}$ and B on $\overline{JL}$.



(b) Show that $\triangle CAK \cong \triangle ACB$.

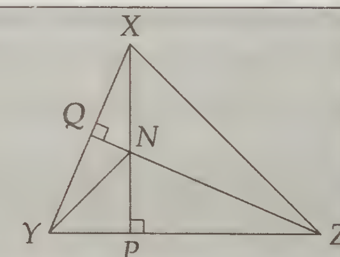
(c) Use part (b) and similar congruences to prove that A, B, and C are the midpoints of $\overline{KL}$, $\overline{JL}$, and $\overline{JK}$, respectively.

(d) How are lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ related to $\triangle JKL$?

(e) Prove that the altitudes of $\triangle ABC$ are concurrent.

Problem 7.23: Where do altitudes of a right triangle intersect – inside, outside, or on the triangle? How about an acute triangle? An obtuse triangle? (If necessary, extend the altitudes to the point where they all meet.)

Problem 7.24: Altitudes $\overline{QZ}$ and $\overline{XP}$ intersect at N as shown. Given that $\angle YXZ = 70°$ and $\angle XZY = 45°$, find each of the following:



(a) $\angle ZXP$.

(b) $\angle XZQ$.

(c) $\angle YXP$.

(d)★ $\angle NYZ$. **Hints:** 194

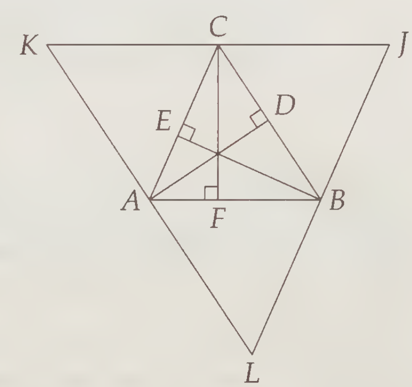
Problem 7.25: The altitudes of $\triangle ABC$ meet at point H. At what point do the altitudes of $\triangle ABH$ meet? How about the altitudes of $\triangle ACH$? $\triangle BCH$? (As usual, extend the altitudes if necessary.)

We start, as you probably guessed, by proving that the altitudes of a triangle are concurrent. This is a bit of a magical proof, and it uses what we have already learned about perpendicular bisectors and the medial triangle.

Problem 7.22: Prove that the lines containing the altitudes of a triangle are concurrent.

Solution for Problem 7.22: Our altitudes make us think of perpendicular lines. Concurrent perpendicular lines make us think of perpendicular bisectors. Unfortunately, there's no simple solution like we had with perpendicular bisectors. So, we put our wishful thinking hat on, and wonder if we could possibly show that the altitudes of our triangle are also the perpendicular bisectors of the sides of some other triangle. If so, we would know they are concurrent, since the perpendicular bisectors of our 'other triangle' must be concurrent. But what's our 'other triangle'?

We want our A, B, and C to be the midpoints of the sides of our other triangle (since then the altitudes of $\triangle ABC$ would be perpendicular bisectors of the sides of the other triangle). This means we want $\triangle ABC$ to be the medial triangle of some other triangle. Since we know that the sides of a triangle are parallel to the sides of its medial triangle, we know how to make a triangle starting with its medial triangle.



We draw lines $\overleftrightarrow{JK}$, $\overleftrightarrow{KL}$, and $\overleftrightarrow{JL}$ through the vertices of $\triangle ABC$ parallel to the sides of $\triangle ABC$ as shown. To prove that $\triangle ABC$ is indeed the medial triangle of $\triangle JKL$, we must show that the vertices of $\triangle ABC$ are the midpoints of the sides of $\triangle JKL$. Since $\overline{AB} \parallel \overline{JK}$, $\angle CAB = \angle ACK$. Similarly, $\angle CAK = \angle BCA$, so $\triangle CAK \cong \triangle ACB$. Likewise, $\triangle BLA \cong \triangle ACB$, so $\triangle BLA \cong \triangle CAK$. Therefore, $AL = AK$, so A is the midpoint of $\overline{LK}$.

In the same way, we find that B and C are the midpoints of $\overline{LJ}$ and $\overline{JK}$, respectively. Therefore, $\triangle ABC$ is the medial triangle of $\triangle JKL$. Since $\overline{KL} \parallel \overline{BC}$, altitude $\overline{AD}$ of $\triangle ABC$ is perpendicular to $\overline{KL}$. Since $\overleftrightarrow{AD}$ is perpendicular to $\overline{KL}$ and passes through its midpoint, it is the perpendicular bisector of $\overline{KL}$. Similarly, $\overleftrightarrow{CF}$ and $\overleftrightarrow{BE}$ are also perpendicular bisectors of the sides of $\triangle JKL$. Lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ are concurrent because they are the perpendicular bisectors of the sides of $\triangle JKL$. These lines also contain the altitudes of $\triangle ABC$, so we have proved that the lines containing the altitudes of $\triangle ABC$ are concurrent. □

Important: The altitudes of any triangle are concurrent at a point called the **orthocenter**. We usually denote the orthocenter with the letter H.

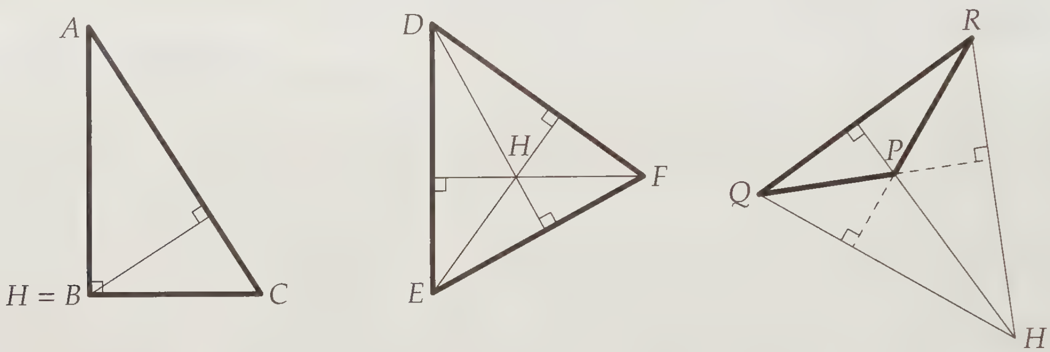
To get a feel for the orthocenter, we look for it in a few different types of triangles.

Problem 7.23: Where do altitudes of a right triangle intersect – inside, outside, or on the triangle? How about an acute triangle? An obtuse triangle?

Extra! Euclid's *Elements* used to be one of the cornerstones of a classical education, and was used by many self-educated people to hone their logic skills. Noted Abraham Lincoln biographer Carl Sandburg wrote that Lincoln 'bought the *Elements* of Euclid. . .[It] went into his carpetbag as he went out on the circuit. At night. . .he read Euclid by the light of a candle after others had dropped off to sleep.' (*Source: Journey Through Genius by William Dunham*)

You can follow in Lincoln's footsteps by reading Euclid's *Elements* online. A link to the online book is provided on our Links page described in the Website section on page viii.

Solution for Problem 7.23: We start with right triangle $\triangle ABC$ with right angle at B. Point B is the foot of the altitude from A to $\overline{BC}$ and the foot of the altitude from C to $\overline{AB}$; therefore, the orthocenter of a right triangle is the vertex of the right angle.



In acute $\triangle DEF$, we see that all the altitudes go inside the triangle, so the orthocenter is inside the triangle.

In obtuse $\triangle PQR$, the altitudes from the two acute angles are entirely outside the triangle (except where they pass through a vertex). Therefore, the orthocenter of an obtuse triangle is outside the triangle. □

Let's try using the orthocenter in a problem.

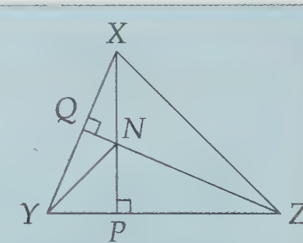
Problem 7.24: Altitudes $\overline{QZ}$ and $\overline{XP}$ of $\triangle XYZ$ intersect at N. Given that $\angle YXZ = 70^\circ$ and $\angle XZY = 45^\circ$, find the following:

(a) $\angle ZXP$.

(b) $\angle XZQ$.

(c) $\angle YXP$.

(d)★ $\angle NYZ$.



Solution for Problem 7.24:

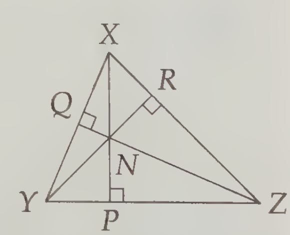
(a) From right triangle $\triangle XPZ$, we have $\angle ZXP = 90^\circ - \angle XZP = 45^\circ$.

(b) From right triangle $\triangle XZQ$, we have $\angle XZQ = 90^\circ - \angle ZXQ = 90^\circ - \angle ZXY = 20^\circ$.

(c) We could use either right triangle $\triangle XYP$, or we could use $\angle ZXP$ and $\angle YXZ$:

$$\angle YXP = \angle YXZ - \angle ZXP = 70^\circ - 45^\circ = 25^\circ.$$

(d) Since two altitudes meet at N, we know that N is the orthocenter. Therefore, if we continue $\overline{YN}$ to meet $\overline{XZ}$ at R, we know that $\overline{YR}$ is also an altitude. From right triangle $\triangle RYZ$, we find that $\angle NYZ = 90^\circ - \angle RZY = 45^\circ$.



□

Notice that recognizing the orthocenter was a key step in the solution to our last part. This is just one small example of how the awareness of the special points, lines, and circles of a triangle can help.

Concept: If we have the altitudes from two vertices of a triangle, then we know that the line through the intersection point of these two altitudes and the third vertex is also an altitude. Once you have two altitudes, you should almost always think about this third altitude.

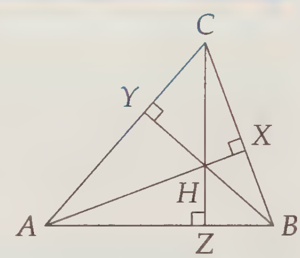
This, of course, also works with our other special lines – if you have two medians, you have the third; if you have two angle bisectors, you have the third, etc. With these other special lines you also get a little more, since we have special knowledge about the centroid (the 2 : 1 ratio), the incenter (center of the inscribed circle), and the circumcenter (center of the circumscribed circle).

Again, if you have two of any of these special lines, you should consider the third line and remember what you know about the intersection point.

Here's an interesting tidbit about altitudes and orthocenters:

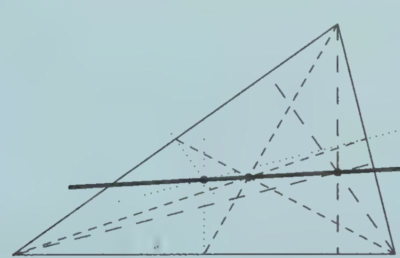
Problem 7.25: The altitudes of $\triangle ABC$ meet at point H. At what point do the altitudes of $\triangle ABH$ meet? How about $\triangle ACH$? $\triangle BCH$?

Solution for Problem 7.25: Our diagram shows that the altitudes of $\triangle ABH$ are $\overleftrightarrow{HZ}$ (perpendicular to side $\overline{AB}$), $\overleftrightarrow{AC}$ (perpendicular to $\overleftrightarrow{BH}$), and $\overleftrightarrow{BC}$ (perpendicular to $\overleftrightarrow{AH}$). These lines clearly all pass through C, so C is the orthocenter of $\triangle ABH$. Similarly, B is the orthocenter of $\triangle AHC$ and A is the orthocenter of $\triangle BHC$. □



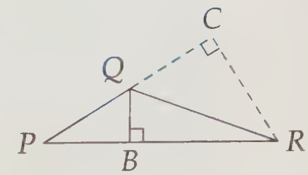
And here's an interesting fact about how the orthocenter, circumcenter, and centroid of a triangle are related:

Sidenote: In any triangle, the centroid, G, is on the line segment connecting the orthocenter, H, to the circumcenter, O, such that $2OG = GH$. This line through these three points is called the **Euler line**, after the great mathematician **Leonhard Euler**. One such line is shown below.



Euler was one of the most prolific mathematicians ever, producing notable work in nearly every field of mathematics, and creating new fields himself. The Euler line is just one of many mathematical results that have been named after Euler. His impact on mathematics was so great that much of our notation today was inspired by him, such as $f(x)$ for functions, i for $\sqrt{-1}$, e for the base of common logs, Σ for summation, and even π for, well, pi. He linked three of these with his famous statement $e^{\pi i} = -1$.

Exercises



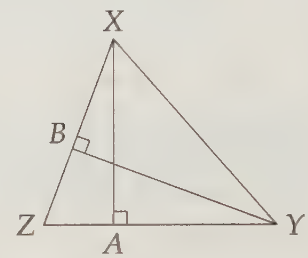
7.5.1 $\overline{RC}$ and $\overline{QB}$ are altitudes of $\triangle PQR$ as shown, $\angle QRP = 20°$, and $\angle QPR = 32°$. Find the following:

(a) $\angle PQB$.

(b) $\angle CRP$.

(c) $\angle CQR$.

(d) $\angle CQB$.



7.5.2 In the diagram, $YZ = 5$, $YB = 3$, and $XA = 4$.

(a) Find $[XYZ]$.

(b) Find BZ.

(c) Find XZ.

(d) Find XB.

(e) Find XY.

7.5.3 $\triangle ABC$ is isosceles with $AB = AC$. Must the altitudes from B and C have the same length? Why or why not?

7.5.4 Why must all the altitudes of an equilateral triangle have the same length?

7.5.5 Altitude $\overline{AD}$ of $\triangle ABC$ is also an angle bisector of $\triangle ABC$. Must D be the midpoint of $\overline{BC}$? Why or why not?

7.6★ Challenging Problems

In this section we tackle several challenging problems in which we use what we've learned in this chapter about the special points, lines, and circles of a triangle. You may wish to try your hand at the Review Problems at the end of this chapter before attempting the problems in this section.

Problems

Problem 7.26: In this problem, our goal is to find the area of $ABCD$, given that $AB = BC = 6$, $CD = 3\sqrt{2}$, $\angle ABC = 90°$, and that there is a circle passing through all four vertices of $ABCD$.

(a) Where is the center, O, of the circle that passes through all four vertices?

(b) What is the radius of this circle?

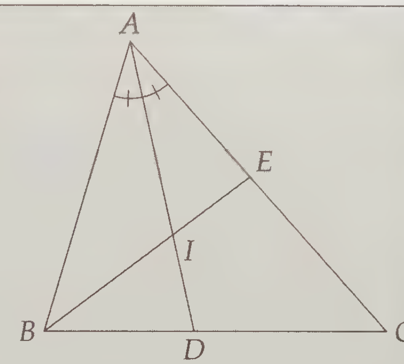
(c) What kind of triangle is $\triangle OCD$?

(d) What are the angles in $\triangle OAD$? What is $\angle CDA$?

(e) Find the area of $ABCD$.

Problem 7.27: Given $AB = BC = 10$ and $AC = 12$, find the circumradius and the inradius of $\triangle ABC$.
Hints: 39, 322, 549

Problem 7.28: In the diagram, $\overline{AD}$ bisects $\angle BAC$ and I is the incenter of $\triangle ABC$. Furthermore, $AB = 7$, $BC = 8$, and $AC = 11$. Find AI/ID.



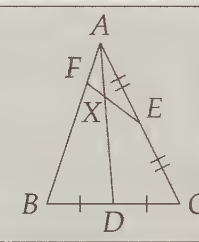
(a) Use the Angle Bisector Theorem to find another pair of segments that have a ratio equal to AI/ID.

(b) Use the Angle Bisector Theorem again to find another pair of lengths (besides AI and ID) that have a ratio equal to the ratio you found in part (a).

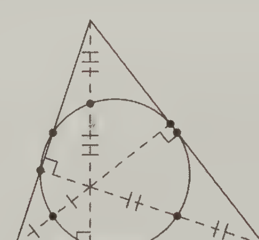
(c) Let $BD = x$. Set up and solve an equation for x using the ratios you found in the previous two parts. Then finish the problem by finding AI/ID.

Problem 7.29: Is it possible for there to be points E and F on side $\overline{BC}$ of $\triangle ABC$ such that $BE = EF = FC$ and $\angle BAE = \angle EAF = \angle FAC$? Why or why not? **Hints:** 377

Problem 7.30: Point E is the midpoint of $\overline{AC}$, and $\overline{AD}$ is a median of $\triangle ABC$. F is on $\overline{AB}$ such that $AF = AB/4$. $\overline{EF}$ and $\overline{AD}$ meet at X. Find AX/AD. **Hints:** 95



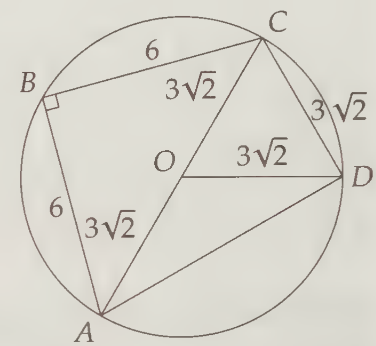
Extra! The incircle and the circumcircle are not the only notable circles that can be found in a triangle. For any triangle, a single circle passes through the midpoints of the sides, the feet of the altitudes, and the midpoints of the segments connecting the orthocenter to the vertices. This circle is cleverly called the **nine-point circle** of the triangle.



The nine-point circle is tangent to the incircle, has a radius equal to half the circumradius (you have the tools to prove this–see if you can!), and its center is the midpoint of the segment connecting the orthocenter and the circumcenter, which we discussed on page 191. We'll prove the existence of the nine-point circle, and some of its properties, during our deeper investigations of triangles in *Intermediate Geometry*.

Problem 7.26: Find the area of $ABCD$, given that $AB = BC = 6$, $CD = 3\sqrt{2}$, $\angle ABC = 90°$, and that there is a circle passing through all four vertices of $ABCD$.

Solution for Problem 7.26: We don't know much about figures with four sides at this point, but we now know a whole lot about triangles. We start with right triangle $\triangle ABC$. The area of this triangle is $(AB)(BC)/2 = 18$, so all we have to find to finish is $[ACD]$. Unfortunately, we don't have much information about $\triangle ACD$.



We do, however, know that the circumcenter of $\triangle ABC$ is the midpoint of $\overline{AC}$. Therefore the center of the circle, O, is the midpoint of $\overline{AC}$. Breaking the problem into triangles worked well once, so we try again by drawing $\overline{OD}$. Since $\triangle ABC$ is an isosceles right triangle, $AC = 6\sqrt{2}$. So, the radius of the circle is $3\sqrt{2}$. We label all the lengths we know, and we see that $\triangle OCD$ is an equilateral triangle.

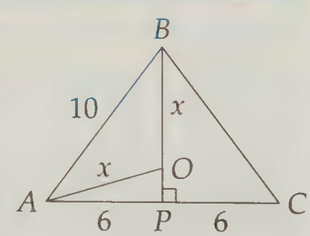
Since $\triangle OCD$ is equilateral, its angles are all $60°$. Furthermore, $\angle AOD = 180° - \angle COD = 120°$. Since $\triangle AOD$ is isosceles, $\angle OAD = \angle ODA = 30°$. Now, we see that $\triangle ACD$ is a 30-60-90 triangle, so $AD = CD\sqrt{3} = 3\sqrt{6}$. Finally, $[ACD] = (CD)(AD)/2 = 9\sqrt{3}$.

Finally, we have $[ABCD] = [ABC] + [ACD] = 18 + 9\sqrt{3}$. In Section 12.1, we will learn a much faster way to deduce that $\angle ADC$ is a right angle. □

In this solution, we seemed to stumble on a surprising equilateral triangle and a useful right angle. We were able to find them because we kept track of everything we learned in our diagram.

Concept: Label your diagram with all the lengths and angles you can find. This will help you find relationships in the diagram that might be hard to find otherwise.

Problem 7.27: Given $AB = BC = 10$ and $AC = 12$, find the circumradius and the inradius of $\triangle ABC$.



Solution for Problem 7.27: Since we're looking for the circumradius and the inradius, we're thinking about perpendicular bisectors and angle bisectors. Our triangle is isosceles, so we'll start with the special lines from the vertex angle ($\angle B$) since they're all the same line.

$\overline{BP}$ splits our triangle in two congruent halves, so $AP = 6$. Since $\triangle BPA$ is right, we have $BP = 8$ from the Pythagorean Theorem. To find the circumradius, we build a right triangle with the circumradius as a side by connecting the circumcenter, O, to A. Since $OB = x$ and $BP = 8$, we have $OP = 8 - x$ and we can apply the Pythagorean Theorem to $\triangle OAP$ to find $OA^2 = OP^2 + AP^2$. Substituting our expressions for the sides, we have

$$x^2 = (8 - x)^2 + 36.$$

A little algebra then yields $16x = 100$, so $x = 25/4$.

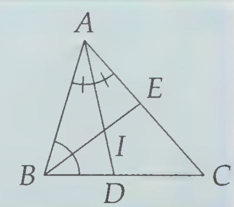
Since the area of a triangle is its semiperimeter times its inradius, we can quickly find the inradius of $\triangle ABC$:

$$r = \frac{[ABC]}{s} = \frac{(AC)(BP)/2}{(10+10+12)/2} = \frac{48}{16} = 3.$$

Challenge: See if you can find the inradius using a tactic like the one we used for the circumradius! □

Concept: Need to find a length in a problem? Try building right triangles.

Problem 7.28: In the diagram, $\overline{AD}$ bisects $\angle BAC$ and I is the incenter of $\triangle ABC$. Furthermore, $AB = 7$, $BC = 8$, and $AC = 11$. Find AI/ID.



Solution for Problem 7.28: We're looking for a ratio and we have an angle bisector, so we try using the Angle Bisector Theorem. Since I is the incenter of $\triangle ABC$, $\overline{BE}$ bisects $\angle ABC$. Therefore, $\overline{BI}$ is an angle bisector of $\triangle ABD$. From the Angle Bisector Theorem, we have

$$\frac{AI}{ID} = \frac{AB}{BD}.$$

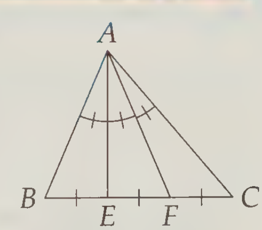
We know $AB = 7$, so if we can find BD, we're finished. The Angle Bisector Theorem also gives us $AB/BD = AC/CD$ when applied to angle bisector $\overline{AD}$ of $\triangle ABC$. Since $BD + DC = BC = 8$, we have:

$$\frac{7}{BD} = \frac{11}{8 - BD}.$$

Cross-multiplying gives $7(8 - BD) = 11BD$, from which we find $BD = 28/9$. Therefore, $AI/ID = AB/BD = 9/4$. □

Concept: Keep your eye on the ball! Often, working backwards from what you want will guide you to the solution.

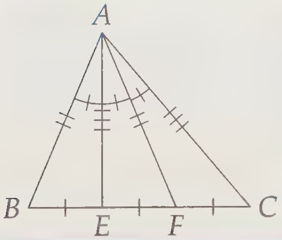
Problem 7.29: Is it possible for there to be points E and F on side $\overline{BC}$ of $\triangle ABC$ such that $BE = EF = FC$ and $\angle BAE = \angle EAF = \angle FAC$? Why or why not?



Solution for Problem 7.29: We have angle bisectors and lengths, so we try the Angle Bisector Theorem. $\overline{AE}$ and $\overline{AF}$ trisect $\angle BAC$, but we can also look at them as the angle bisectors of $\angle BAF$ and $\angle EAC$, respectively. The Angle Bisector Theorem applied to $\triangle BAF$ and $\triangle EAC$ gives us

$$\frac{AB}{AF} = \frac{BE}{EF} = 1 \qquad \text{and} \qquad \frac{AE}{AC} = \frac{EF}{FC} = 1.$$

Therefore, we have $AB = AF$ and $AE = EC$.

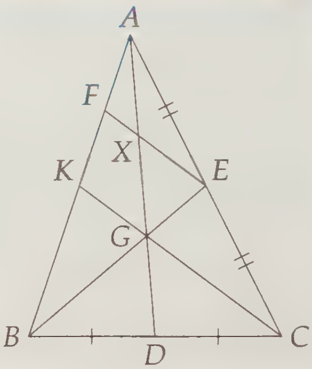


Marking these equalities in our diagram, congruent triangles jump out. Specifically, SSS tells us $\triangle ABE \cong \triangle AFE \cong \triangle AFC$. However, this means that $\angle AEB = \angle AEF = 90^\circ$ and $\angle AFE = \angle AFC = 90^\circ$. Therefore $\triangle AEF$ has two right angles, which is impossible.

So, it is not possible for there to be points E and F on side $\overline{BC}$ of $\triangle ABC$ such that $BE = EF = FC$ and $\angle BAE = \angle EAF = \angle FAC$. □

Problem 7.30: Point E is the midpoint of $\overline{AC}$, and $\overline{AD}$ is a median of $\triangle ABC$. F is on $\overline{AB}$ such that $AF = AB/4$. $\overline{EF}$ and $\overline{AD}$ meet at X. Find AX/AD.

Solution for Problem 7.30: We start with a diagram. We go ahead and draw in the other two medians, hoping we can use what we know about the centroid. Since K is the midpoint of $\overline{AB}$ and $AF = AB/4$, F is the midpoint of $\overline{AK}$. So, $\triangle AFE \sim \triangle AKC$ by SAS Similarity. Therefore, $\angle AFE = \angle AKC$, so $\overline{FE} \parallel \overline{KC}$. This gives us $\triangle AFX \sim \triangle AKG$, so



$$\frac{AX}{AG} = \frac{AF}{AK} = \frac{1}{2}.$$

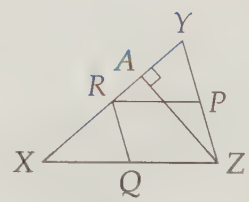
Since G is the centroid of $\triangle ABC$, we have $AG/AD = 2/3$, so

$$\frac{AX}{AD} = \left(\frac{AX}{AG}\right)\left(\frac{AG}{AD}\right) = \frac{1}{3}.$$

□

Exercises

7.6.1 P, Q, and R are the midpoints of sides $\overline{YZ}$, $\overline{XZ}$, $\overline{XY}$, respectively, and $\overline{ZA}$ is the altitude to side $\overline{XY}$.

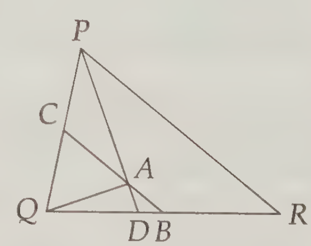


(a) Show that $AP = RQ$. **Hints:** 298

(b)★ Show that $\angle PRQ = \angle PAQ$. **Hints:** 47

7.6.2 In right triangle ABC, let F be the midpoint of hypotenuse $\overline{AB}$, and let D be the foot of the altitude from C to $\overline{AB}$. Let E be on $\overline{AB}$ such that $\overline{CE}$ is the angle bisector of $\angle ACB$. Prove that $\angle DCE = \angle ECF$. **Hints:** 136

7.6.3 In $\triangle PQR$, $PQ = 8$, $PR = 10$, and $QR = 9$. $\overline{PD}$ bisects $\angle QPR$, $\overline{QA} \perp \overline{PD}$, and $\overline{BC}$ passes through A such that $\overline{BC} \parallel \overline{PR}$.



(a) Prove that $\overline{AC}$ is a median of $\triangle PAQ$.

(b)★ Find DB. **Hints:** 532, 226, 125

Extra! *If at first the idea is not absurd, then there is no hope for it.*

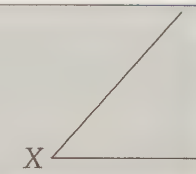
–Albert Einstein

7.7 Construction: Bisectors

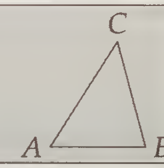
In this section we'll construct some of the special points, lines, and circles of a triangle.

Problems

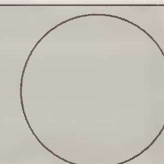
Problem 7.31: Given $\angle X$ shown, construct a ray from X that bisects $\angle X$.



Problem 7.32: Construct the incircle of $\triangle ABC$ shown.



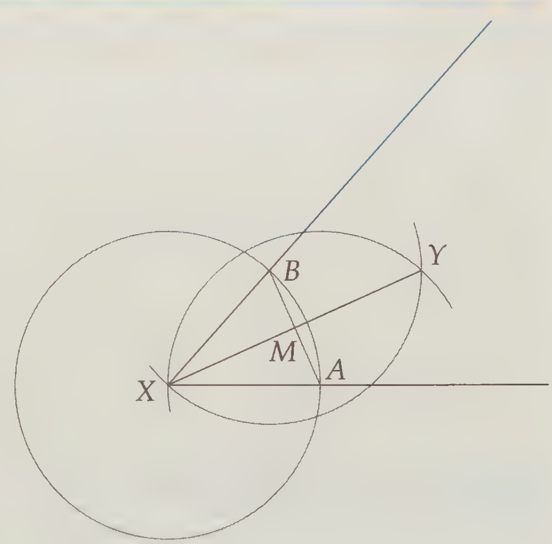
Problem 7.33: Yes, it's a circle. But you'll note that we haven't marked the center. Your job is to construct it.



We already know how to make perpendicular lines, midpoints, and perpendicular bisectors. But we haven't learned how to make angle bisectors, so we'll start there.

Problem 7.31: Given $\angle X$, construct a ray from X that bisects $\angle X$.

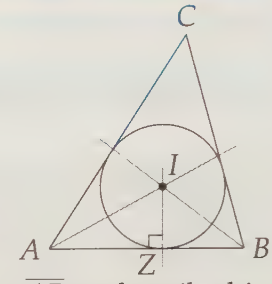
Solution for Problem 7.31: Since we measure angles as a portion of a circle, we start with a circle centered at X. We call the points where the circle meets the sides of the angle points A and B. Since $XA = XB$, $\triangle XBA$ is isosceles. This makes us happy because we know that the angle bisector of the vertex angle of an isosceles triangle is also a median, an altitude, and a perpendicular bisector.



Therefore, all we have to do is construct the perpendicular bisector of $\overline{AB}$ to have the angle bisector of $\angle X$. We can quickly construct this by drawing arcs centered at A and B with radius AX. These arcs meet at X and a point we'll call Y. $\overrightarrow{XY}$ bisects $\angle AXB$. We'll leave the proof that this construction works as an Exercise. □

Now that we can construct the special lines of a triangle, we can construct special circles. For example:

Problem 7.32: Given a triangle $\triangle ABC$, construct the incircle of the triangle.



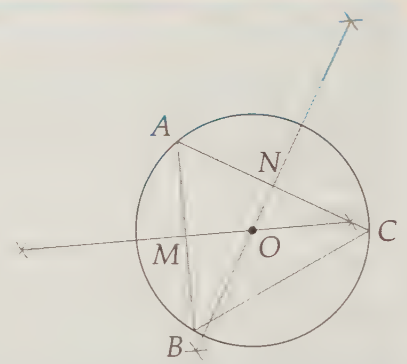
Solution for Problem 7.32: To construct a circle, we need both its center and its radius. We can find the center of the incircle by constructing the bisectors of two of the angles of the triangle. We can perform these constructions exactly as described in Problem 7.31. Where these rays meet is our center, point I.

Now that we have the center, we just need the radius. The radius of the incircle drawn to a point where the incircle touches a side of the triangle is perpendicular to that side. Therefore, we can find the radius by constructing this perpendicular segment from I. Specifically, we construct a line through I that is perpendicular to $\overline{AB}$ as described in Problem 6.24. Suppose this line meets $\overline{AB}$ at Z. IZ is the length of our radius.

We then construct our incircle by drawing a circle with center I and radius IZ. □

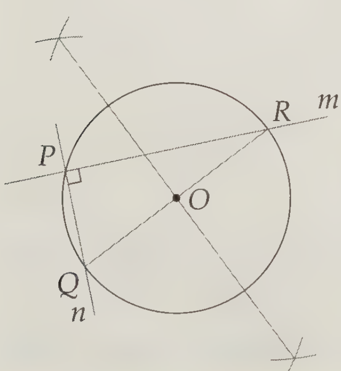
We'll finish this chapter with two clever solutions to a challenging construction problem.

Problem 7.33: Given a circle but not its center, construct the center of the circle.



Solution for Problem 7.33: Solution 1: We don't know a whole lot about constructing parts of a circle, but we know a ton about constructing parts of a triangle. Specifically, we know how to construct the circumcircle of a triangle – its center is just the intersection of the perpendicular bisectors of the sides.

Therefore, we just pick three points A, B, and C on the circumference of our circle. We then construct the perpendicular bisectors of two of the sides of $\triangle ABC$ as shown. The intersection of these perpendicular bisectors, point O, is the circumcenter of $\triangle ABC$. Since our starting circle is the circumcircle of $\triangle ABC$, the circumcenter of $\triangle ABC$ is the center of the starting circle.



Solution 2: Another useful circumcenter fact we know is that the circumcenter of a right triangle is the midpoint of its hypotenuse. Inspired by this observation, we create right $\triangle PQR$ by choosing point P on the circle, drawing line n through P to meet the circle again at Q, then constructing line m through P perpendicular to n. Where line m hits the circle again gives us point R.

Since $\triangle PQR$ is a right triangle, the midpoint of its hypotenuse is the center of its circumcircle. Therefore, we find the center of our original circle by constructing O, the midpoint of $\overline{RQ}$. □

Exercises

7.7.1 Given $\triangle ABC$, construct the circumcircle of the triangle.

7.7.2 Construct a 30° angle.

7.7.3 Construct a 45° angle.

7.7.4 Construct a 120° angle.

7.7.5 Given $\triangle ABC$, construct the orthocenter of the triangle.

7.7.6 Prove that the construction described in the solution to Problem 7.31 really does produce an angle bisector.

7.8 Summary

Definitions:

- The **perpendicular bisector** of a line segment is the line through the midpoint of the segment that is perpendicular to the segment.

  Important: The perpendicular bisector of a segment is a straight line consisting of all points that are equidistant from the endpoints of the segment.

- The **angle bisector** of an angle is the line that divides the angle into two equal angles.

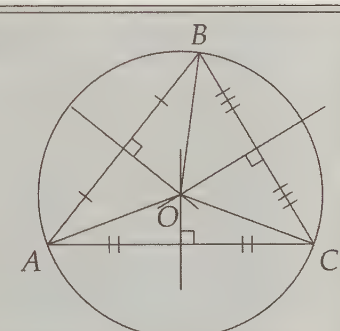
  Important: The angle bisector of an angle consists of all points that are equidistant from the sides of the angle.

Definition: A **cevian** is a line segment from a vertex of a triangle to a point on the line containing opposite side of the triangle.

We explored important properties of several sets of cevians:

Definitions: The perpendicular bisectors of the sides of a triangle are concurrent at a point called the **circumcenter**. The circle centered at the circumcenter that passes through the vertices of the original triangle is called the **circumcircle** of the triangle because it is **circumscribed** about the triangle (meaning it passes through all the vertices of the triangle).

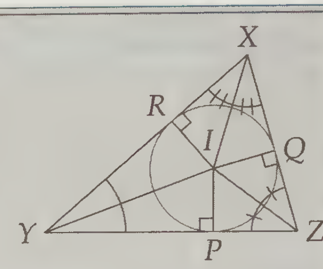
Finally, the radius of this circle is called the **circumradius**, the circumcenter is usually labeled with the letter O, and the circumradius is usually called R.



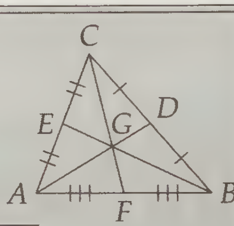
Extra! *There is geometry in the humming of the strings.*

–Pythagoras

Definitions: The angle bisectors of a triangle are concurrent at a point called the **incenter**. This point is equidistant from the sides of the triangle. This common distance from the incenter to the sides of a triangle is called the **inradius**, because the circle with center I and this radius is tangent to all three sides of the triangle. This circle is unsurprisingly called the **incircle** because it is inscribed in the triangle (meaning it is tangent to all the sides of the triangle). The incenter is usually denoted I, and the inradius is usually written as r.



Definitions: A **median** of a triangle connects a vertex of a triangle to the midpoint of the opposite side. The medians of a triangle are concurrent at a point called the **centroid** of the triangle. The centroid of the triangle is usually labeled G.

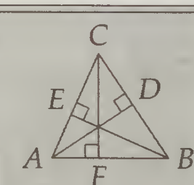


Important:

- The medians of a triangle divide the triangle into six little triangles with equal area.
- The centroid of a triangle cuts its medians into a 2 : 1 ratio. For example, for the triangle shown, we have

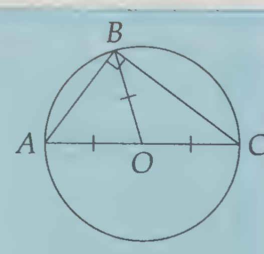
$$\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = \frac{2}{1}.$$

Definition: The altitudes of a triangle (extended if necessary) are concurrent at the **orthocenter** of the triangle, which is usually denoted H.



Important: The orthocenter of an acute triangle is inside the triangle. The orthocenter of a right triangle is the vertex of the right angle. The orthocenter of an obtuse triangle is outside the triangle.

Important: Right triangles offer a few important relationships.

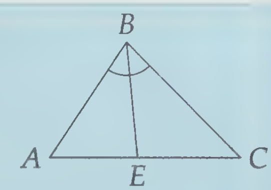


- The circumcenter of a right triangle is the midpoint of the hypotenuse, and the circumradius equals one-half the hypotenuse.
- The median to the hypotenuse of a right triangle is equal in length to half the hypotenuse.

Our exploration of these special points, lines, and circles uncovered a few more important facts:

Important: Given any three noncollinear points, there is exactly one circle that passes through all three.

Important: The **Angle Bisector Theorem** states that if E is on $\overline{AC}$ such that $\overline{BE}$ is the angle bisector of $\angle B$ in triangle $\triangle ABC$, then



$$\frac{AB}{AE} = \frac{CB}{CE}.$$

We should think of this any time we have a problem involving lengths and angle bisectors.

Important: The area of a triangle equals its inradius times half its perimeter.

Important: If we connect the midpoints of the sides of a triangle, we divide the triangle into four congruent triangles, each of which is similar to the original triangle. The central triangle is called the **medial triangle** of our original triangle.

Problem Solving Strategies

Concepts:

- If we have the altitudes from two vertices of a triangle, then we know that the line through the intersection point of these two altitudes and the third vertex is also an altitude. Once you have two altitudes, you should almost always think about this third altitude.

  This, of course, also works with our other special lines – if you have two medians, you have the third; if you have two angle bisectors, you have the third, etc. With these other special lines you also get a little more, since we have special knowledge about the centroid (the 2 : 1 ratio), the incenter (center of the inscribed circle), and the circumcenter (center of the circumscribed circle).

  Again, if you have two of any of these special lines, you should consider the third line and remember what you know about the intersection point.

Continued on the next page. . .

Concepts: *. . . continued from the previous page*

- When given the lengths of the sides of a triangle in a problem, always take the time to check if it is a right triangle. If it is, this fact may simplify the problem considerably.
- Label your diagram with all the lengths and angles you can find. This will help you find relationships in the diagram that might be hard to find otherwise.
- Need to find a length in a problem? Try building right triangles.
- Keep your eye on the ball! Often working backwards from what you want will guide you to the solution.

REVIEW PROBLEMS

7.34 In triangle $\triangle PQR$, $PQ = 12$, $\angle Q = 90^\circ$, and $QR = 16$. Find the area, the circumradius, and the inradius of $\triangle PQR$.

7.35 Points T and U are on $\odot K$ such that K is 2 units from $\overline{TU}$. If $TU = 14$, what is the radius of the circle?

7.36 $\triangle XYZ$ is an equilateral triangle with side length 12. M is the midpoint of side $\overline{YZ}$ and N is the midpoint of $\overline{XZ}$. $\overline{YN}$ and $\overline{XM}$ meet at E.

(a) What kind of triangle is $\triangle XYM$?

(b) Find XM.

(c) Find $[XYZ]$.

(d) Find XE/EM.

(e) Find the inradius of $\triangle XYZ$.

(f) Find the circumradius of $\triangle XYZ$.

7.37 Altitudes $\overline{AD}$ and $\overline{CF}$ of acute triangle $\triangle ABC$ meet at H. Prove that $\angle CHD = \angle ABC$.

7.38 If the altitudes of a triangle all have the same length, must the triangle be equilateral? Why or why not?

7.39 Point D is on side $\overline{YZ}$ of $\triangle XYZ$ such that $\angle YXD = \angle ZXD$. Given that $XZ = 8$, $YD = 3$, and $DZ = 4$, find XY.

7.40 Medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ meet at G. The area of $\triangle ABC$ is 48. Find the areas of $\triangle ADC$, $\triangle AGC$, $\triangle GFB$, $\triangle DEF$, and $\triangle AEF$.

7.41 $\triangle TUV$ is isosceles with $TU = UV = 30$ and $TV = 36$. $\overline{UY}$ bisects $\angle TUV$ and $\overline{TM}$ is a median of $\triangle TUV$. $\overline{UY}$ and $\overline{TM}$ meet at X.

(a) Show that Y is the midpoint of $\overline{TV}$.

(b) Show that $\overline{UY} \perp \overline{TV}$.

(c) Find TY and UY.

(d) Find XY and XU.

(e) Find XT and XM.

7.42 In Problem 7.8 we found the circumradius of an equilateral triangle with side length 6, while in Exercise 7.2.3, we found the circumradius of an equilateral triangle with side length 18. Given that the first circumradius is $2\sqrt{3}$, how could we have quickly found the second without going through the same laborious procedure we used to solve Problem 7.8?

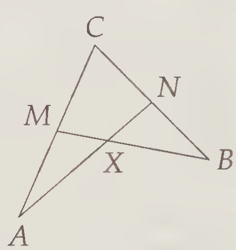
7.43 In triangle ABC, altitude $\overline{AD}$ intersects angle bisector $\overline{BE}$ at point X. If $\angle BAC = 117^\circ$ and $\angle ACB = 35^\circ$, then determine $\angle DXE$.

7.44 Point O is the circumcenter of $\triangle PQR$, $\angle QPR = 45^\circ$, and $\angle QPO = 23^\circ$.

(a) Find $\angle RPO$ and $\angle OQP$.

(b)★ Find $\angle OQR$.

7.45 M and N are the midpoints of $\overline{CA}$ and $\overline{CB}$, respectively, as shown at right. $\overline{BM}$ and $\overline{AN}$ meet at X. Given that $XM = 3.5$ and $XA = 7.2$, find XN and XB.



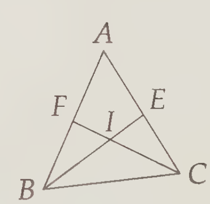
7.46 Medians $\overline{AX}$ and $\overline{BY}$ of triangle $\triangle ABC$ are perpendicular at point O. $AX = 12$ and $BC = 10$.

(a) Find AO and BY.

(b)★ Find the length of median $\overline{CZ}$.

7.47 Cevians $\overline{AD}$ and $\overline{BE}$ meet at point X inside $\triangle ABC$. $\overline{CX}$ extended meets $\overline{AB}$ at F. If $[AEX] = [CEX] = [CDX] = [BDX] = [BXF] = [AXF]$, must X be the centroid of $\triangle ABC$?

7.48 $\overline{BE}$ and $\overline{CF}$ are angle bisectors that meet at I as shown at right. $CE = 4$, $AE = 6$, and $AB = 8$.



(a) Prove that $\angle EIC = 90^\circ - \frac{\angle A}{2}$.

(b) Find BC.

(c) Find BF.

7.49

(a) Show that if the centroid and the orthocenter of a triangle are the same point, then the triangle is equilateral.

(b) Show that if the incenter and the centroid of a triangle are the same point, then the triangle is equilateral.

Challenge Problems

7.50 The angle bisector of $\angle G$ of $\triangle GHI$ passes through the circumcenter of $\triangle GHI$. Show that $GH = GI$. **Hints:** 454

7.51 Let H and O denote the orthocenter and circumcenter of acute triangle ABC, respectively. $\overleftrightarrow{AH}$ meets $\overleftrightarrow{BC}$ at P and $\overline{AO}$ meets $\overline{BC}$ at Q. Prove that $\angle BAP = \angle CAQ$. **Hints:** 499, 288

7.52 In $\triangle JKL$, we have $JK = JL = 25$ and $KL = 40$. Find the following:

(a) the area of $\triangle JKL$.

(b) the inradius of $\triangle JKL$.

(c) the circumradius of $\triangle JKL$. **Hints:** 305

7.53 Let I be the incenter of triangle ABC, and let $\overrightarrow{AI}$ meet $\overline{BC}$ at A'.

(a) Prove that $A'I/IA = BA'/AB$.

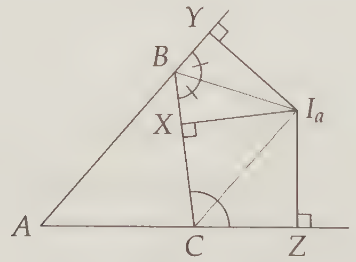
(b)★ Prove that $A'I/IA = BC/(AB+AC)$. **Hints:** 372

7.54 In triangle ABC, $\angle A = 100^\circ$, $\angle B = 50^\circ$, and $\angle C = 30^\circ$. H is on $\overline{BC}$ and M on $\overline{AC}$ such that $\overline{AH}$ is an altitude and $\overline{BM}$ is a median. Find $\angle MHC$. *(Source: AHSME)* **Hints:** 464

7.55 Is it possible for two angle bisectors in a triangle to be perpendicular? Why or why not? **Hints:** 509

7.56 In $\triangle ABC$, $\angle C$ is a right angle. Point M is the midpoint of $\overline{AB}$, point N is the midpoint of $\overline{AC}$, and point O is the midpoint of $\overline{AM}$. The perimeter of $\triangle ABC$ is 112 and $ON = 12.5$. What is the area of $MNCB$? *(Source: MATHCOUNTS)*

7.57 For a given triangle ABC, extend $\overline{AB}$ and $\overline{AC}$ as shown below. We will prove that there is a circle tangent to $\overline{BC}$ and the extensions of $\overline{AB}$ and $\overline{AC}$. We start by drawing the bisectors of two of the exterior angles of $\triangle ABC$ as shown with the grey lines below.



(a) Let P be a point on the external angle bisector of angle $\angle B$. Prove that P is equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$.

(b) Let Q be a point on the external angle bisector of angle $\angle C$. Prove that Q is equidistant from $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$.

(c) Let I_a be the intersection of the external angle bisectors of angles $\angle B$ and $\angle C$, and let X, Y, and Z be the feet of the altitudes from I_a to $\overline{BC}$, $\overline{AB}$, and $\overline{AC}$. Prove that $I_aX = I_aY = I_aZ$. What does this tell us about the circle with center I_a and radius I_aX? This circle is called an **excircle** of $\triangle ABC$; I_a is called an **excenter**, and r_a is an **exradius**. Every triangle has three excircles, one tangent to each side of the triangle.

(d) Prove that I_a lies on the internal angle bisector of angle $\angle A$.

(e)★ Let $r_a = I_aX$. Prove that $r_a = \dfrac{[ABC]}{s - BC}$, where s is the semiperimeter of $\triangle ABC$. **Hints:** 44, 474

7.58 Points M and N are the midpoints of sides $\overline{PA}$ and $\overline{PB}$ of $\triangle PAB$. As P moves along a line that is parallel to side $\overline{AB}$, which of the four quantities listed below change? *(Source: AMC 10)*

(a) the length of segment $\overline{MN}$.

(b) the perimeter of $\triangle PAB$.

(c) the area of $\triangle PAB$.

(d) the area of $ABNM$.

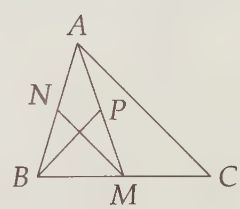
7.59 Let $\triangle XYZ$ have $\angle X = 60°$ and $\angle Y = 45°$. A circle with center P passes through A and B on side $\overline{XY}$, C and D on side $\overline{YZ}$, and E and F on side $\overline{ZX}$. Suppose $AB = CD = EF$.

(a) Prove that P is the incenter of $\triangle XYZ$.

(b) Find $\angle XPY$. *(Source: HMMT)*

7.60★ A ladder is initially resting vertically against a wall that is perpendicular to the ground. It begins to slip and fall to the ground, with the bottom of the ladder moving directly from the wall, and the top of the ladder always touching the wall. What path does the midpoint of the ladder trace? (Don't forget to prove that the midpoint of the ladder really does hit every point on the path!) **Hints:** 271, 347

7.61 Triangle $\triangle ABC$ has area 48. Let P be the midpoint of median $\overline{AM}$ and let N be the midpoint of side $\overline{AB}$. If G is the intersection of $\overline{MN}$ and $\overline{BP}$, find the area of $\triangle MGP$. *(Source: Mandelbrot)* **Hints:** 212, 407



7.62★ In this problem we will prove that the medians of a triangle are concurrent. Let $\triangle ABC$ be our triangle. Medians $\overline{AD}$ and $\overline{BE}$ meet at point G. $\overrightarrow{CG}$ meets $\overline{AB}$ at point F.

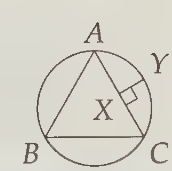
(a) Show that $[ACG] = [GCB] = [AGB]$. **Hints:** 557

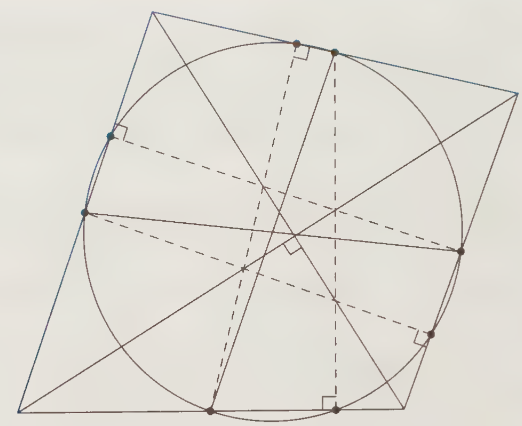
(b) Use your result from part (a) to show that $AF = FB$.

(c) Do the previous parts show that the medians of any triangle are concurrent?

7.63★ Point N is on hypotenuse $\overline{BC}$ of $\triangle ABC$ such that $\angle CAN = 45°$. Given $AC = 8$ and $AB = 6$, find AN. **Hints:** 448, 560, 585

7.64★ The circle at right has radius 1 and is circumscribed about equilateral triangle ABC. If X is the midpoint of $\overline{AC}$ and Y is on arc $\widehat{AC}$ such that $\angle YXA$ is right, then find length XY. *(Source: Mandelbrot)*





The Eight Point Circle Theorem

It is easier to square the circle than to get round a mathematician. – Augustus De Morgan

CHAPTER 8 Quadrilaterals

8.1 Quadrilateral Basics

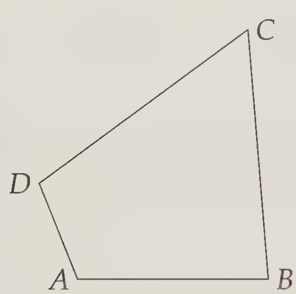


Figure 8.1: A Quadrilateral

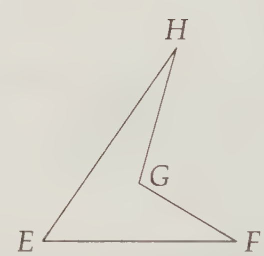


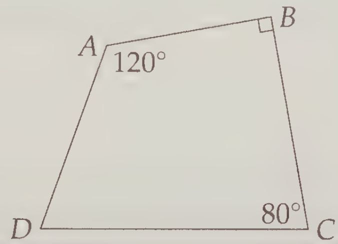
Figure 8.2: A Concave Quadrilateral

Figure 8.1 shows a **quadrilateral**, namely $ABCD$. A quadrilateral has four sides, four vertices, and four angles. All the quadrilaterals we will study in this text have all four vertices in the same plane. Nearly all quadrilaterals we deal with in this book are **convex**, like $ABCD$ above, meaning that all of the interior angles of the quadrilateral are less than $180°$. There are also **concave quadrilaterals**, like $EFGH$ in Figure 8.2.

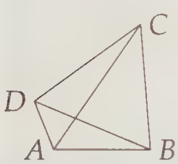
In this section we address basic definitions and facts that hold for all quadrilaterals. In later sections, we turn to specific types of quadrilaterals.

Problems

Problem 8.1: Use what you know about the sum of the angles in a triangle to find $\angle D$ in the figure below.



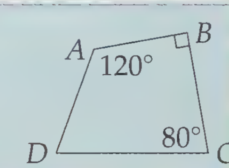
As you've probably already considered, we can connect two more pairs of vertices in a quadrilateral by connecting the opposite vertices.

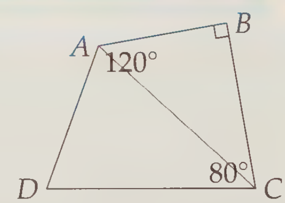


At left, we have added $\overline{AC}$ and $\overline{BD}$ to quadrilateral $ABCD$. We call these segments connecting the opposite vertices of a quadrilateral the **diagonals** of a quadrilateral.

Let's see how we can use a diagonal in a problem.

Problem 8.1: Find $\angle D$ in the figure.





Solution for Problem 8.1: We don't yet know anything about angles of a quadrilateral, but we know about triangles, so we add diagonal $\overline{AC}$, thus making two triangles, $\triangle ABC$ and $\triangle ADC$. The sum of the angles in each of these triangles is 180°, so the sum of the angles in the whole quadrilateral $ABCD$ is $2(180^\circ) = 360^\circ$. Now we can quickly find $\angle D$:

$$\angle D = 360^\circ - \angle DAB - \angle ABC - \angle BCD = 70^\circ.$$

□

Important: The interior angles of any quadrilateral add to 360°.

Concept: Triangles, triangles, triangles! As you'll see in the coming sections, nearly everything we can derive for specific types of quadrilaterals stems from triangles.

Exercises

8.1.1 In quadrilateral $ABCD$, $\angle A = 100^\circ$, $\angle B = 50^\circ$, and $\angle C$ is 30° more than $\angle D$. Find $\angle D$.

8.1.2 The angles of a quadrilateral are x, $3x - 10^\circ$, 27°, and $4x - 30^\circ$. Find the measure of the largest angle of the quadrilateral.

8.1.3

(a) If two of the angles of a quadrilateral are equal, must the other two be equal?

(b) Is it possible for three of the angles of a quadrilateral to equal each other, but the fourth angle be different?

(c) If all of the angles of a quadrilateral have the same measure, what is this measure?

8.1.4 How many different quadrilaterals can be formed by connecting the four points shown at right?

8.1.5 We showed earlier that the interior angles of a convex quadrilateral add to 360°. Show that the interior angles of a concave quadrilateral add to 360° as well. **Hints:** 391

8.2 Trapezoids

A **trapezoid** is a quadrilateral in which two sides are parallel. Figure 8.3 shows trapezoid $WXYZ$, in which $\overline{WX} \parallel \overline{ZY}$.

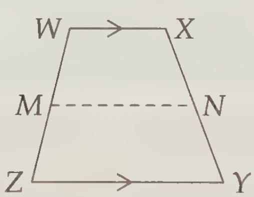


Figure 8.3: A Trapezoid

These parallel sides are often called the **bases** of the trapezoid, with the other sides called the **legs**. The segment that connects the midpoints of the legs is the **median** of the trapezoid; $\overline{MN}$ is the median of $WXYZ$ above.

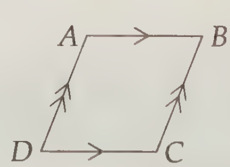


Figure 8.4: Is This a Trapezoid?

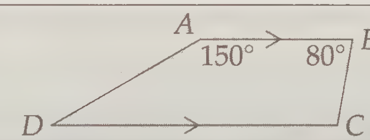
You might be wondering 'What if there are two pairs of opposite sides that are parallel like in Figure 8.4; is the quadrilateral still a trapezoid?' Good question! Unfortunately, there isn't a good answer. Some people define a trapezoid as having exactly one pair of opposite sides, so Figure 8.4

would not be a trapezoid. Other people define a trapezoid as having *at least* one pair of opposite sides; to these people, Figure 8.4 would be considered a trapezoid.

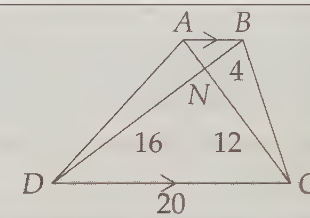
In this book, we will be careful to present proofs and definitions based on trapezoids as being valid for either of these definitions. In this section, we will focus on quadrilaterals with exactly one pair of parallel sides.

Problems

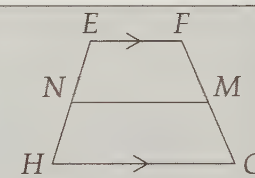
Problem 8.2: In trapezoid $ABCD$, we have $\overline{AB} \parallel \overline{CD}$, $\angle A = 150^\circ$, and $\angle B = 80^\circ$. Find $\angle C$ and $\angle D$.



Problem 8.3: In trapezoid $ABCD$, $\overline{AB} \parallel \overline{CD}$, diagonals $\overline{BD}$ and $\overline{AC}$ meet at N, and side lengths BN, CN, DN, and CD are as shown in the diagram. Find AN and AB.



Problem 8.4: In trapezoid $EFGH$, we have $\overline{EF} \parallel \overline{GH}$, $EF = 6$, $FG = 10$, $GH = 12$, and $EH = 8$. Our goal is to find the length of median $\overline{MN}$.



(a) Extend $\overline{EH}$ and $\overline{FG}$ so that the two meet at X.

(b) Use similar triangles to find XE and XF, then find XN and XM.

(c) Show that $\triangle XEF \sim \triangle XNM$. Is the median parallel to the bases of the trapezoid?

(d) Find MN.

Problem 8.5: Bases $\overline{PQ}$ and $\overline{SR}$ of trapezoid $PQRS$ have lengths 5 and 9, respectively, and are 8 units apart.

(a) Find $[PQS]$ and $[QRS]$.

(b) Find $[PQRS]$.

(c) Can you find a general formula for the area of a trapezoid given its base lengths and the distance between the bases?

Problem 8.6: In trapezoid $WXYZ$, we have $\overline{WX} \parallel \overline{YZ}$, $WX = 17$, $XY = 13$, $YZ = 21$, and $XZ = 20$. In this problem we seek the area of $WXYZ$.

(a) We build right triangles by drawing an altitude from X to $\overleftrightarrow{YZ}$. How do we know this altitude meets $\overleftrightarrow{YZ}$ between Y and Z?

(b) Let the length of the altitude we draw in part (a) be h, and let it split $\overline{YZ}$ into segments of lengths x and $21 - x$. Use either the Pythagorean Theorem and algebra or your knowledge of Pythagorean triples to find x and h.

(c) Find the area of $WXYZ$.

Problem 8.7: In trapezoid $ABCD$, we have $\overline{AD} \parallel \overline{BC}$, $\angle B = \angle C$, and $\angle B < 90^\circ$. Prove that $AB = CD$.
Hints: 131

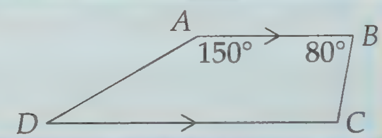
Problem 8.8: In trapezoid $GHIJ$, $\overline{GH} \parallel \overline{IJ}$, $GH = 8$, $HI = GJ = 6$, and $IJ = 10$. In this problem we will find HJ.

(a) Draw heights $\overline{HX}$ and $\overline{GY}$. Find JY, YX, and XI.

(b) Find HX, then find HJ.

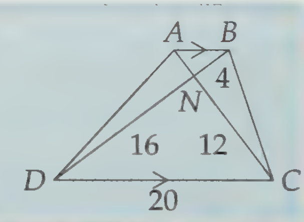
Many trapezoid problems are solved using the basic properties of parallel lines.

Problem 8.2: In trapezoid $ABCD$, we have $\overline{AB} \parallel \overline{CD}$, $\angle A = 150^\circ$, and $\angle B = 80^\circ$. Find $\angle C$ and $\angle D$.



Solution for Problem 8.2: Since $\overline{AB} \parallel \overline{CD}$, we have $\angle A + \angle D = 180^\circ$, so $\angle D = 30^\circ$. Similarly, $\angle C = 180^\circ - \angle B = 100^\circ$. □

Problem 8.3: In trapezoid $ABCD$, $\overline{AB} \parallel \overline{CD}$, diagonals $\overline{BD}$ and $\overline{AC}$ meet at N, and lengths BN, CN, DN, and CD are as shown in the diagram. Find AN and AB.



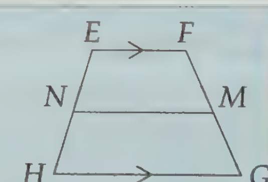
Solution for Problem 8.3: Parallel lines mean similar triangles: $\overline{AB} \parallel \overline{CD}$ gives $\triangle ABN \sim \triangle CDN$, so we have

$$\frac{AN}{CN} = \frac{BN}{DN} = \frac{AB}{CD}.$$

Since we have $BN/DN = 1/4$, we find $AN = CN/4 = 3$ and $AB = CD/4 = 5$. □

Now we turn our attention to the median and the area of a trapezoid. The median of a trapezoid connects the midpoints of the legs of the trapezoid.

Problem 8.4: In trapezoid $EFGH$, we have $\overline{EF} \parallel \overline{GH}$, $EF = 6$, $FG = 10$, $GH = 12$, and $EH = 8$. Find the length of median $\overline{MN}$.

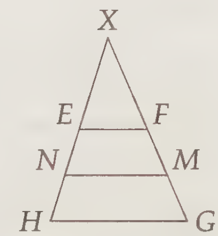


Solution for Problem 8.4: Make sure you understand why this solution is incomplete:

Bogus Solution: Since M is midway between the lines, and so is N, $\overline{MN}$ must be parallel to the bases and midway between the lengths of the bases. Therefore, MN is the average of the bases, so $MN = (EF + GH)/2 = 9$.

This solution has several problems. It consists of a series of statements without sufficient justification. For example, it doesn't explain what 'M is midway between the lines' means, nor why this leads to $\overline{MN}$ being parallel to the bases. We also have absolutely no justification for the claim that MN is the average of EF and GH.

Many students new to writing mathematics write solutions that look much like this Bogus Solution. Strive to explain all your steps in mathematical terms, and to justify your claims. Showing exactly why MN is the average of EF and GH is not a trivial task. $\overline{MN}$ looks parallel to the bases, but we'll have to go through a few steps to prove it. We want to use triangles since we know a lot about triangles, so we extend $\overline{EH}$ and $\overline{FG}$ to meet at X. With this inspiration as a starting point, here's what a complete solution looks like:



The extensions of the legs meet at X. Since $\triangle EXF \sim \triangle HXG$, we have

$$\frac{XE}{XH} = \frac{EF}{GH}.$$

So, substitution gives

$$\frac{XE}{XE+8} = \frac{1}{2}.$$

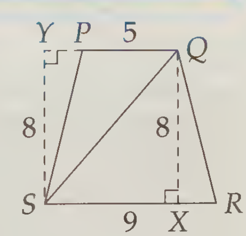
Solving this equation, we find $XE = 8$. Similarly, since $XF/XG = 1/2$, F is the midpoint of $\overline{XG}$, so $XF = FG = 10$. M is the midpoint of $\overline{FG}$, so $XM = XF + FG/2 = 15$. Similarly, we have $XN = 12$. Since $XN/XE = 12/8 = 3/2$ and $XM/XF = 15/10 = 3/2$, we have $\triangle XEF \sim \triangle XNM$ by SAS Similarity. Therefore, $NM/EF = XN/XE = 3/2$, so $NM = (3/2)(EF) = 9$. Notice also that $\angle XNM = \angle XEF$ from our similarity, so $\overline{NM} \parallel \overline{EF}$. □

Important: The median of a trapezoid is parallel to the bases of a trapezoid, and equal in length to the average of the lengths of the bases.

You'll have a chance at the end of this section to use the same process we used on Problem 8.4 to prove these assertions for all trapezoids.

We used triangles to learn about the median; now we use them to learn about the area of a trapezoid.

Problem 8.5: Bases $\overline{PQ}$ and $\overline{SR}$ have lengths 5 and 9, respectively, and are 8 units apart. What is the area of trapezoid $PQRS$?



Solution for Problem 8.5: We know how to find the area of triangles, so we divide the trapezoid into two triangles by drawing diagonal $\overline{QS}$. We then draw the altitudes $\overline{QX}$ and $\overline{SY}$ of the two triangles. Since the parallel lines are 8 units apart, $QX = SY = 8$. We can now find the area of the trapezoid by adding the areas of the triangles:

$$[PQRS] = [PQS] + [QRS] = \frac{(PQ)(SY)}{2} + \frac{(SR)(QX)}{2} = 8\left(\frac{PQ+SR}{2}\right) = 56.$$

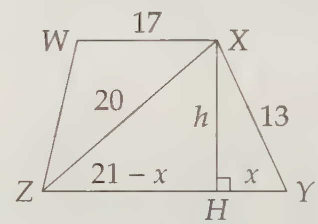
□

Since the median of a trapezoid is the average of the bases of the trapezoid, we can use our last problem to write a formula for the area of a trapezoid in terms of the height and median and a trapezoid.

Important: The area of a trapezoid equals the height of the trapezoid times the length of the median of the trapezoid.

Problem 8.6: In trapezoid $WXYZ$, we have $\overline{WX} \parallel \overline{YZ}$, $WX = 17$, $XY = 13$, $YZ = 21$, and $XZ = 20$. Find the area of $WXYZ$.

Solution for Problem 8.6: We need the height, so we try to add a height to the the diagram in a way that forms useful right triangles. The altitude from X to $\overleftrightarrow{YZ}$ is the most promising candidate, but we first have to figure out where the altitude from X will hit $\overleftrightarrow{YZ}$. It's not immediately clear if the foot of the altitude, which we'll call H, is on $\overline{YZ}$ or not. (It might be beyond Y on $\overrightarrow{ZY}$ for example.) Wherever H is on $\overleftrightarrow{YZ}$, $\triangle XZH$ is a right triangle. Since $\overline{XZ}$ is the hypotenuse and $\overline{ZH}$ a leg of this triangle, we have $HZ < XZ$. Since $XZ < YZ$, we know that $HZ < YZ$. Similarly, we have $HY < XY < YZ$. Since HZ and HY are both less than YZ, H must be on $\overline{YZ}$. Now we can confidently draw our diagram with H on $\overline{YZ}$.



We let $XH = h$ and $HY = x$, so that $ZH = 21 - x$ and we can now use the Pythagorean Theorem on $\triangle XHZ$ and $\triangle XHY$:

$$\begin{aligned} x^2 + h^2 &= 169 \\ (21-x)^2 + h^2 &= 400 \end{aligned}$$

Subtracting the first equation from the second allows us to eliminate h and find x. We get

$$[(21-x)^2 + h^2] - (x^2 + h^2) = 400 - 169.$$

Therefore, $x^2 - 42x + 441 + h^2 - x^2 - h^2 = 231$, so $-42x = -210$, which gives $x = 5$.

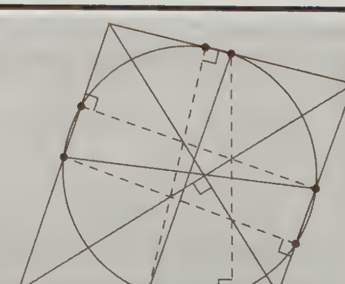
Substitution then gives us $h = 12$. Seeing such nice neat numbers, we realize that we could have guessed $h = 12$, since a leg of length 12 goes with hypotenuses $XZ = 20$ and $XY = 13$ nicely. (Notice that just because $h = 12$ fits these hypotenuses nicely doesn't mean that h must be 12. We still have to check that $ZY = 21$ if we guess $h = 12$.)

Now that we have $h = 12$, we can find $[WXYZ] = (h)(WX + YZ)/2 = (12)(38)/2 = 228$. □

Concept: Dropping well-chosen altitudes to build right triangles is a powerful problem-solving tool in trapezoid problems.

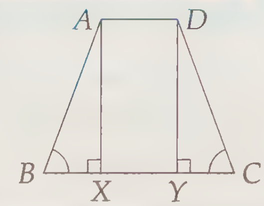
Extra! Consider a quadrilateral with perpendicular diagonals as shown at right. The **Eight-Point Circle Theorem** tells us that the midpoints of the sides and the feet of the perpendiculars from these midpoints to the opposite respective sides of the quadrilateral all lie on the same circle.

If you can't prove the Eight-Point Circle Theorem now, try coming back to it after you've tackled Problem 8.19. You might find the result of Problem 7.6 useful, as well.



We'll now take a look at a special class of trapezoids.

Problem 8.7: In trapezoid $ABCD$, we have $\overline{AD} \parallel \overline{BC}$, $\angle B = \angle C$, and $\angle B < 90^\circ$. Prove that $AB = CD$.



Solution for Problem 8.7: Once again, we introduce right triangles by drawing altitudes from A and D. Then $AX = DY$ because $\overline{AD} \parallel \overline{BC}$, so by AAS Congruence, we have $\triangle ABX \cong \triangle DCY$. Therefore, $AB = CD$.

Notice that we can easily deal with the case of $\angle B$ and $\angle C$ being obtuse by noting that if $\angle B = \angle C$, then $\angle A = 180^\circ - \angle B = 180^\circ - \angle C = \angle D$. If $\angle B$ and $\angle C$ are obtuse, then $\angle A$ and $\angle D$ are acute, and we can use essentially the same proof as above to prove that the legs of the trapezoid are equal in length. □

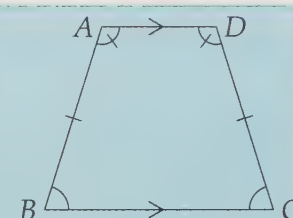
Two angles of a trapezoid that share one of the bases of the trapezoid as a side are together called **base angles** of the trapezoid. A trapezoid in which two base angles are equal is called an **isosceles trapezoid**. As an Exercise, you'll also prove that the diagonals of an isosceles trapezoid are congruent. Finally, in Section 8.7, you'll show that these relationships run 'backwards'; i.e., that if the legs are equal in length (and not parallel) or if the diagonals of a trapezoid are congruent, then both pairs of base angles are equal.

Important: In an isosceles trapezoid:

(a) The base angles come in two pairs of equal angles as shown at right.

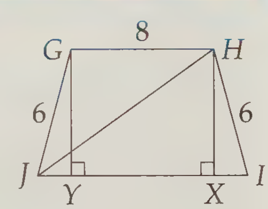
(b) The legs are equal.

(c) The diagonals are equal.



If any one of these items is true for a trapezoid with exactly one pair of parallel sides, then all the others must be true for that trapezoid.

Problem 8.8: Find HJ given that $GHIJ$ is an isosceles trapezoid with $GH = 8$, $HI = GJ = 6$, and $IJ = 10$.



Solution for Problem 8.8: We start by drawing altitudes $\overline{GY}$ and $\overline{HX}$, hoping to build useful right triangles. We note that since $GY = HX$ and $GJ = HI$, we have $\triangle GJY \cong \triangle HIX$ by HL Congruence. Therefore, $JY = XI$. Next, we see that $YX = GH$ because $\overline{GY} \parallel \overline{HX}$ and YX and GH are both the distance between $\overline{GY}$ and $\overline{HX}$. (Alternatively, we could note that $GHXY$ is a rectangle.) Therefore, $YX = 8$, so $JY = XI = (10-8)/2 = 1$.

Now we find HX from $\triangle HXI$: $HX = \sqrt{HI^2 - IX^2} = \sqrt{35}$. Finally, we use right triangle HXJ to get HJ: $HJ = \sqrt{HX^2 + XJ^2} = \sqrt{35+81} = \sqrt{116} = 2\sqrt{29}$. Building right triangles to find lengths strikes again. □

If you found this section a little tough, review it after reading the rest of this chapter. You'll find that the other quadrilateral types are much simpler to handle than trapezoids, primarily because our other types have restrictions on all the sides instead of just on one pair of opposite sides. Therefore, the other quadrilateral types don't offer as wide a variety of problems as trapezoids do.

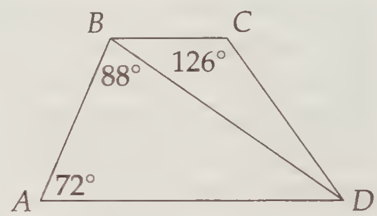
Exercises

8.2.1 In quadrilateral $PQRS$, $\overline{PQ} \parallel \overline{RS}$, $\overline{PS}$ is not parallel to $\overline{QR}$, $QR = PS$, and $\angle P = 83^\circ$. Find the rest of the angles of the trapezoid.

8.2.2 Find the area of a trapezoid with bases 44 and 24 and with height 18.

8.2.3 The area of trapezoid $ABCD$ is 96. One base is 6 units longer than the other, and the height of the trapezoid is 8. Find the length of the shorter base.

8.2.4 $ABCD$ shown at right is a trapezoid with $\overline{BC} \parallel \overline{AD}$, $\angle ABD = 88^\circ$, $\angle A = 72^\circ$, and $\angle C = 126^\circ$. Find $\angle ADC$ and $\angle CBD$.



8.2.5 Prove that the diagonals of an isosceles trapezoid are congruent.

8.2.6 In this problem we prove that if $ABCD$ is a trapezoid such that $\overline{AB} \parallel \overline{CD}$ and $AB < CD$, then the median of $ABCD$ is parallel to the bases of $ABCD$ and equals the average of the bases of the trapezoid. Use Problem 8.4 as your guide. (Note that this proof covers any trapezoid in which the legs are not parallel.)

(a) Let M and N be the midpoints of $\overline{AD}$ and $\overline{BC}$ respectively. Extend $\overline{AD}$ and $\overline{BC}$ to meet at X. Prove $\triangle XDC \sim \triangle XAB$.

(b) Prove that $AD/XA = BC/XB$, and use this to prove that $AM/XA = BN/XB$.

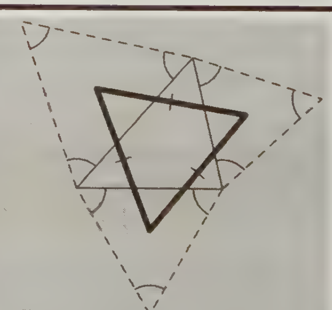
(c) Prove that $XM/XA = XN/XB$.

(d) Prove that $\overline{MN} \parallel \overline{AB}$.

(e)★ Prove that $MN = (AB + CD)/2$. **Hints:** 112, 153

8.2.7★ The bases of a trapezoid have lengths 50 and 75. Its diagonals have lengths 35 and 120. Find the area of the trapezoid. **Hints:** 339, 420

Extra! If we construct an equilateral triangle on the outside of each side of any triangle, the triangle formed by connecting the centroids of these equilateral triangles is called the **Outer Napoleon Triangle**. As the diagram strongly suggests, the outer Napoleon triangle is always equilateral. As you might guess, we can construct the equilateral triangles 'inward' and connect their centers to build an **Inner Napoleon Triangle**. Draw a triangle, then construct the Inner Napoleon Triangle. Does it look equilateral, too?



While the French emperor Napoleon Bonaparte is generally given credit for discovering these triangles, there's no solid proof that he actually did so.

8.3 Parallelograms

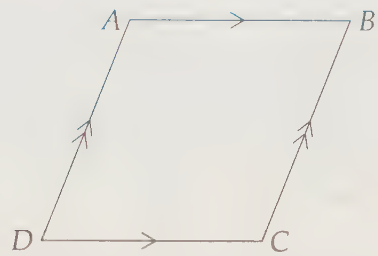
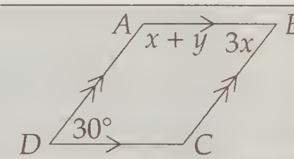


Figure 8.5: A Parallelogram

A quadrilateral is a **parallelogram** if both pairs of opposite sides are parallel. $ABCD$ in Figure 8.5 is a parallelogram because $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$.

Problems

Problem 8.9: Given that $\angle A = x + y$, $\angle B = 3x$, and $\angle D = 30°$ in parallelogram $ABCD$ as shown, find x, y, and $\angle C$.



Problem 8.10: In this problem we show that if $AB = CD$ and $AD = BC$, then $ABCD$ is a parallelogram.

(a) Draw a parallelogram $ABCD$ with diagonal $\overline{AC}$. Prove that $\triangle BCA \cong \triangle DAC$.

(b) Prove that $\overline{AB} \parallel \overline{CD}$.

(c) Prove that $ABCD$ is a parallelogram.

(d) If $ABCD$ is a parallelogram, must $AD = BC$ and $AB = CD$? Make sure you see how this is a different question from the one we have already tackled.

Problem 8.11:

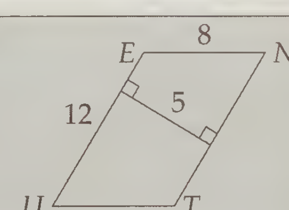
(a) Prove that if $EFGH$ is a parallelogram, then diagonals $\overline{EG}$ and $\overline{FH}$ bisect each other.

(b) Does this work in reverse? In other words, given that diagonals $\overline{WY}$ and $\overline{XZ}$ of quadrilateral $WXYZ$ bisect each other, is $WXYZ$ a parallelogram?

Problem 8.12: Sides $\overline{EU}$ and $\overline{NT}$ of parallelogram $ENTU$ are 5 units apart as shown. $EU = 12$ and $EN = 8$.

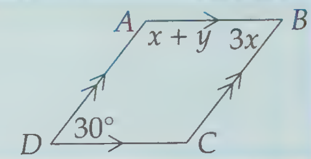
(a) Find the area of $ENTU$. (Hint: Remember the dissection we used to get the area of a trapezoid.)

(b) How far apart are sides $\overline{EN}$ and $\overline{TU}$?



As we'll see, parallelograms are a little easier to handle than quadrilaterals that have only one pair of parallel sides. Once again, our primary tools will be triangles and parallel lines.

Problem 8.9: Given that the measures of the angles of parallelogram $ABCD$ are as shown, find x, y, and $\angle C$.



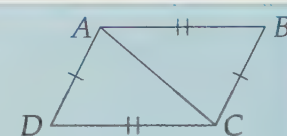
Solution for Problem 8.9: Since $\overline{AD} \parallel \overline{BC}$, we have $\angle D + \angle C = 180°$, so $\angle C = 150°$. Similarly, $\overline{AB} \parallel \overline{CD}$ means $\angle B + \angle C = 180°$, so $\angle B = 30°$. Therefore, $3x = 30°$, so $x = 10°$. Finally, $\angle A + \angle D = 180°$, so $\angle A = 150°$. Hence, $x + y = 150°$, so $y = 140°$. □

Notice that in our solution we found that $\angle B = \angle D$ and $\angle A = \angle C$. At the end of the section, you'll be asked to prove that the opposite angles of any parallelogram are equal.

Problem 8.10: Show that if $AB = CD$ and $AD = BC$, then $ABCD$ is a parallelogram.

Solution for Problem 8.10: Where did we go wrong here:

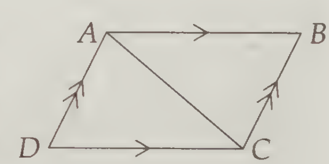
Bogus Solution: We cut $ABCD$ into triangles by drawing $\overline{AC}$. Since $\overline{AD} \parallel \overline{BC}$, we have $\angle DAC = \angle BCA$. Combining this with $AD = BC$ and $AC = AC$, we have $\triangle DAC \cong \triangle BCA$ by SAS Congruence. Therefore, $\angle BAC = \angle ACD$, so $\overline{AB} \parallel \overline{CD}$. Similarly, we can show that $\overline{AD} \parallel \overline{BC}$, so $ABCD$ is a parallelogram.



Our error is that we use $\overline{AD} \parallel \overline{BC}$ to prove that $\overline{AB} \parallel \overline{CD}$. However, we don't know that $\overline{AD} \parallel \overline{BC}$, so we can't use it yet. Fortunately, patching the solution is easy.

Still using our figure from the Bogus Solution above, we have $\triangle DAC \cong \triangle BCA$ by SSS Congruence. Therefore, $\angle BAC = \angle ACD$, so $\overline{AB} \parallel \overline{CD}$. Similarly, we have $\angle DAC = \angle ACB$, so $\overline{AD} \parallel \overline{BC}$ as well. Thus, $ABCD$ is a parallelogram.□

We can also quickly show that in any parallelogram, the opposite sides must be equal. We do so by again using diagonal $\overline{AC}$ of parallelogram $ABCD$. We have $\triangle ABC \cong \triangle CDA$ by ASA Congruence. These congruent triangles give us $AB = CD$ and $BC = AD$. Make sure you see why this addresses a different question from the one initially asked in this problem!



Extra! In the spirit of David Hilbert's famous 1900 lecture (see page 56), in 2000 the Clay Mathematics Institute offered $1,000,000 prizes for solutions to each of seven unsolved **Millennium Problems**. Russian mathematician **Grigori Perelman** put forth a solution to one of the Millennium problems, the **Poincaré Conjecture**, in 2003. His solution was so complex that it took roughly three years to validate! As of June, 2009, the Clay Mathematics Institute had not yet awarded the prize, since Perelman's proof has not been formally published in a peer-reviewed journal.

We've looked at the sides and the angles of a parallelogram; let's check out the diagonals.

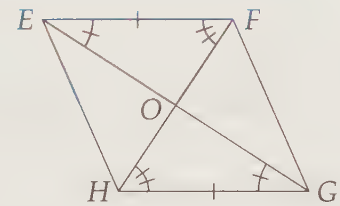
Problem 8.11:

(a) Prove that if $EFGH$ is a parallelogram, then diagonals $\overline{EG}$ and $\overline{FH}$ bisect each other.

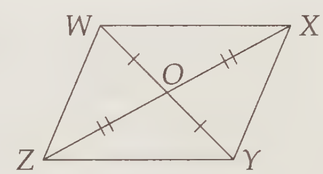
(b) Does this work in reverse? In other words, given that diagonals $\overline{WY}$ and $\overline{XZ}$ of quadrilateral bisect each other, is $WXYZ$ a parallelogram?

Solution for Problem 8.11:

(a) Since $EFGH$ is a parallelogram, we have $EF = GH$ and $\overline{EF} \parallel \overline{GH}$. Therefore, $\angle FEG = \angle EGH$ and $\angle EFH = \angle FHG$, so $\triangle EFO \cong \triangle GHO$ by ASA Congruence. Therefore, $EO = OG$ and $FO = OH$, so O is the midpoint of both diagonals of $EFGH$. Thus, the diagonals of a parallelogram bisect each other.



(b) Once again, we mark the information we're given – the fact that the diagonals bisect each other. We see that $\triangle WOZ \cong \triangle YOX$ by SAS Congruence, so $\angle ZWY = \angle WYX$, which means $\overline{WZ} \parallel \overline{XY}$. Similarly, $\triangle WOX \cong \triangle YOZ$, which we can use to show $\overline{WX} \parallel \overline{YZ}$. Hence, if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.



□

We can combine the information from the last three problems to see parallelograms the way an experienced geometer sees them.

Important: In parallelogram $ABCD$, the opposite sides are equal, the opposite angles are equal, and the diagonals bisect each other.

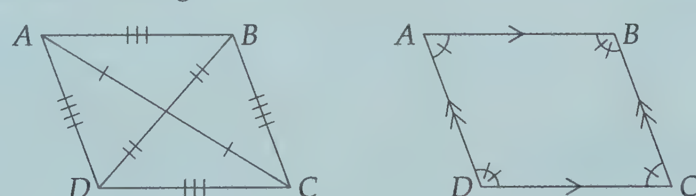


Figure 8.6: Side and Angle Equalities in Parallelograms

Conversely, $ABCD$ is a parallelogram if *any one* of the following are true:

(a) $AB = CD$ and $AD = BC$.

(b) $\angle A = \angle C$ and $\angle B = \angle D$.

(c) Diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other.

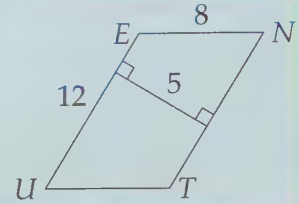
Proving one of these means the other two are true.

Finally, we tackle the area of a parallelogram. We could use the same tactic as we used with trapezoids, but there is an even easier approach.

Problem 8.12: Sides $\overline{EU}$ and $\overline{NT}$ of parallelogram $ENTU$ are 5 units apart as shown. $EU = 12$ and $EN = 8$.

(a) Find the area of parallelogram $ENTU$.

(b) How far apart are sides $\overline{EN}$ and $\overline{TU}$?



Solution for Problem 8.12:

(a) We can approach the area of a parallelogram the same way as we did with a trapezoid. If we draw diagonal $\overline{ET}$, we cut the parallelogram into two triangles, $\triangle EUT$ and $\triangle NET$, which have the same base ($EU = NT = 12$) and the same height, 5. Hence, we have

$$[ENTU] = [EUT] + [NET] = \frac{(EU)(5)}{2} + \frac{(NT)(5)}{2} = \frac{(EU)(5)}{2} + \frac{(EU)(5)}{2} = (EU)(5) = 60.$$

Notice that we can easily use this approach to show that the area of a parallelogram is its base times its height to that base.

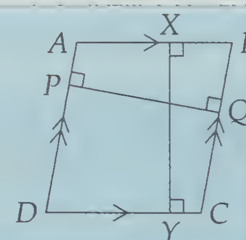
(b) We have the base, EN, and the area from the previous part. Since the area of $ENTU$ is 60, the height between $\overline{EN}$ and $\overline{TU}$ is

$$\frac{[ENTU]}{EN} = \frac{60}{8} = \frac{15}{2}.$$

□

Important: The area of a parallelogram is its base times its height.

$$[ABCD] = (AB)(XY) = (BC)(PQ).$$



Exercises

8.3.1 $TYUI$ is a parallelogram with $TY = 6$ and $YU = 8$.

(a) What is the perimeter of $TYUI$?

(b) Do we have enough information to find the area of $TYUI$?

8.3.2 $WXYZ$ is a parallelogram. Prove that $\angle W = \angle Y$.

8.3.3 In quadrilateral $WORK$, $WO = RK$ and $\overline{WO} \parallel \overline{RK}$. Prove that $WORK$ is a parallelogram.

8.3.4 Use a clever dissection of a parallelogram to turn it into a rectangle and prove that the area of the parallelogram is its base times its height. **Hints:** 394

8.3.5 Is it possible for a parallelogram $ENTU$ to have $EN = 4$, $EU = 12$, and for sides $\overline{EU}$ and $\overline{NT}$ to be 5 units apart? Why or why not? **Hints:** 528

8.3.6 In this Exercise, we give another way of deriving the length of the median and area of a trapezoid.

Let $ABCD$ be a trapezoid with bases $\overline{AB}$ and $\overline{CD}$.

(a) Extend $\overline{AB}$ past B to A' such that $BA' = CD$, and extend $\overline{DC}$ past C to D' such that $CD' = AB$. Prove that $AA'D'D$ is a parallelogram.

(b) Find the area of $[AA'D'D]$ in terms of AB, CD, and h, the distance between $\overline{AB}$ and $\overline{CD}$. Find the relationship between $[AA'D'D]$ and $[ABCD]$, and then find $[ABCD]$ itself. **Hints:** 341

(c) Let M, N, and M' be the midpoints of $\overline{AD}$, $\overline{BC}$, and $\overline{A'D'}$, respectively. Prove that $MM' = AB + CD$, and prove that $MN = MM'/2$. Conclude that $MN = (AB + CD)/2$.

8.3.7★ The diagonals of convex quadrilateral $ABCD$ meet at E. Prove that the centers of the circumcircles of $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$ are the vertices of a parallelogram. **Hints:** 80

8.4 Rhombi

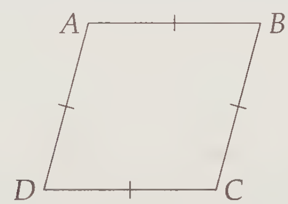


Figure 8.7: A Rhombus

A quadrilateral is a **rhombus** if all of its sides are equal.

Problems

Problem 8.13: Prove that every rhombus is a parallelogram.

Problem 8.14: $ABCD$ is a rhombus such that its diagonals $\overline{AC}$ and $\overline{BD}$ have lengths $AC = 30$ and $BD = 16$. In this problem, we will find the area of $ABCD$ and the side length of the rhombus.

(a) Let the diagonals intersect at X. Show that $\triangle AXB$, $\triangle CXB$, $\triangle CXD$, and $\triangle AXD$ are congruent.

(b) Prove that $\overline{AC} \perp \overline{BD}$.

(c) Find the area of $ABCD$.

(d) Find AB.

We can prove a whole lot about rhombi in one simple step.

Problem 8.13: Prove that every rhombus is a parallelogram.

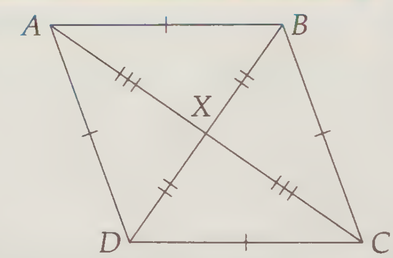
Solution for Problem 8.13: Let $ABCD$ be a rhombus. Since $AB = CD$ and $AD = BC$ (because all the sides of $ABCD$ are equal), we know that $ABCD$ is a parallelogram. □

Important: Every rhombus is a parallelogram. Therefore, all that is true about parallelograms is true about rhombi.

WARNING!! (Bet you saw this one coming.) Every parallelogram is *not* a rhombus. Therefore, if we prove a property of rhombi, this property is not necessarily true for all parallelograms – we'd have to prove the property for parallelograms separately.

To see some of the properties that distinguish a rhombus from a parallelogram that is not a rhombus, let's try a problem.

Problem 8.14: $ABCD$ is a rhombus such that its diagonals $\overline{AC}$ and $\overline{BD}$ have lengths $AC = 30$ and $BD = 16$. Find AB and $[ABCD]$.



Solution for Problem 8.14: We start with a diagram, in which we note that all the sides of $ABCD$ are equal, and that the diagonals bisect each other since rhombus $ABCD$ is also a parallelogram. All four little triangles in the diagram are congruent by SSS Congruence. Since all four angles at X must be equal and they add to 360°, they must be right angles. Hence, the diagonals of a rhombus not only bisect each other, but they are also perpendicular. Since each of these right triangles have legs of length 8 and 15, their hypotenuses (which are the sides of the rhombus) each have length 17.

All those congruent right triangles give us a quick approach to finding the area.

$$[ABCD] = 4[ABX] = 4\left(\frac{(AX)(BX)}{2}\right) = 2(AX)(BX) = 240.$$

Notice that since AX and BX are each half a diagonal of $ABCD$, we have

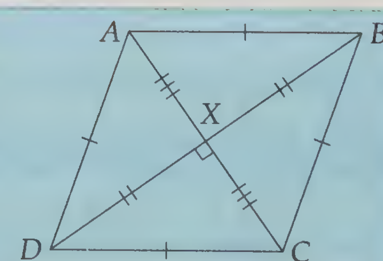
$$[ABCD] = 2(AX)(BX) = 2\left(\frac{AC}{2}\right)\left(\frac{BD}{2}\right) = \frac{(AC)(BD)}{2}.$$

□

Our solution gives us a few special rhombus facts to add to all the facts we know about rhombi because they are parallelograms.

Important: The diagonals of a rhombus are perpendicular. The area of a rhombus is half the product of its diagonals (of course, it is also base times height since a rhombus is a parallelogram).

$$[ABCD] = \frac{(AC)(BD)}{2}$$



Exercises

8.4.1 $PQRS$ is a rhombus with diagonals $PR = 6$ and $QS = 12$. Find the area and the perimeter of $PQRS$.

8.4.2 $WXYZ$ is a rhombus with $WX = 50$ and $WY = 96$.

(a) Find XZ.

(b) Find $[WXYZ]$.

(c) Find the distance between $\overline{WX}$ and $\overline{YZ}$. **Hints:** 586

8.4.3 Diagonals $\overline{AC}$ and $\overline{BD}$ of quadrilateral $ABCD$ are perpendicular. Prove that $[ABCD] = (AC)(BD)/2$. **Hints:** 438

8.4.4 $TUVW$ is a rhombus with $TU = 10$ and $\angle TUV = 60°$.

(a) Show that $\angle TUW = 30°$.

(b) Find $[TUVW]$.

8.5 Rectangles

We've looked at quadrilaterals in which all the sides are equal. Now we consider what happens if all the angles are equal. Such a quadrilateral is called a **rectangle**.

Problems

Problem 8.15: What are the measures of the angles in a rectangle?

Problem 8.16: Prove that all rectangles are also parallelograms.

Problem 8.17: Prove that the diagonals of a rectangle are congruent.

Problem 8.18: Find the length of a diagonal of a rectangle given that its perimeter is 44 and one side has length 10.

Problem 8.19: In this problem, we find two different proofs that the quadrilateral formed by connecting the midpoints of a rhombus is a rectangle.

(a) Start with a diagram. Draw rhombus $ABCD$ and let $EFGH$ be the quadrilateral formed by connecting the midpoints of its sides, with E on $\overline{AB}$, F on $\overline{BC}$, and so on.

(b) *Solution 1:* Show that $\angle AEH = \angle AHE$ and that $\triangle AEH \cong \triangle CGF$. Use this as a starting point to show that the angles of $EFGH$ are all equal.

(c) *Solution 2:* Show that $\overline{EF} \parallel \overline{AC}$. Show that the angles of $EFGH$ are right angles.

We know the angles of a rectangle are equal, but to what?

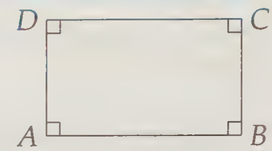
Problem 8.15: What are the measures of the angles in a rectangle?

Solution for Problem 8.15: The angles must add to 360° and they're all equal, so each of the four angles must be a right angle. □

As we did with rhombi, we can quickly get a wealth of information about rectangles.

Problem 8.16: Prove that all rectangles are also parallelograms.

Solution for Problem 8.16: Let $ABCD$ be our rectangle. Since $\angle A + \angle B = 180^\circ$, we know that $\overline{AD} \parallel \overline{BC}$. Similarly, $\angle B + \angle C = 180^\circ$, so $\overline{AB} \parallel \overline{CD}$. Therefore, $ABCD$ is a parallelogram. □

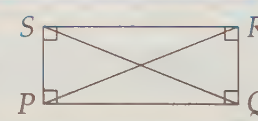


Among the many useful aspects of knowing that every rectangle is a parallelogram, the one that is probably most used is the fact that opposite sides are equal in length. When talking about rectangles, these two dimensions are often called the *length* and *width* of the rectangle. We have already seen that the area of a rectangle is the product of these two dimensions.

We know that the diagonals of a rhombus are perpendicular, so we wonder if there's a similar special property of the diagonals of a rectangle.

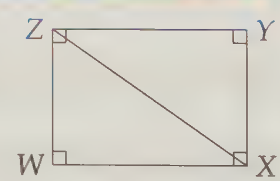
Problem 8.17: Prove that the diagonals of a rectangle are congruent.

Solution for Problem 8.17: As we have in many other problems, we turn to congruent triangles for a solution. Drawing our two diagonals, we have $\triangle PQR \cong \triangle QPS$ by SAS since $PS = QR$, $PQ = PQ$, and $\angle PQR = \angle QPS$. Therefore, $PR = QS$. □



Problem 8.18: Find the length of a diagonal of a rectangle given that its perimeter is 44 and one side has length 10.

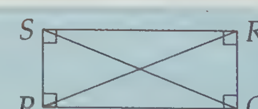
Solution for Problem 8.18: Let $WXYZ$ be our rectangle and let $XY = 10$. We know the diagonals have the same length, so we don't have to worry about which diagonal to find. Since $XY = 10$, we have $WZ = 10$ as well. Since the perimeter is 44, we have



$$WX + XY + YZ + ZW = 44.$$

Since $WX = YZ$ and $XY = WZ = 10$, we have $WX + 10 + WX + 10 = 44$, so $WX = 12$. Now we can use the Pythagorean Theorem to find $XZ = \sqrt{WX^2 + WZ^2} = \sqrt{100 + 144} = 2\sqrt{61}$ □

Important: The diagonals of a rectangle are equal to each other and equal to the square root of the sum of the squares of the length and the width of the rectangle.



$$QS = PR = \sqrt{QR^2 + SR^2}$$

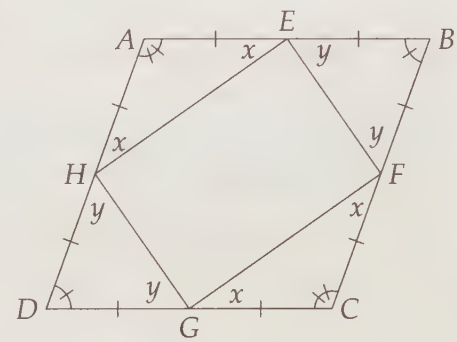
You shouldn't have to memorize how to find the diagonals of a rectangle: the right angles of a rectangle should make it clear to use the Pythagorean Theorem with the sides to get the diagonal.

Now we'll take a look at one way rectangles and rhombi are related, before moving on to a quadrilateral that is both a rhombus *and* a rectangle.

Problem 8.19: Prove that the quadrilateral formed by connecting the midpoints of the sides of a rhombus is a rectangle.

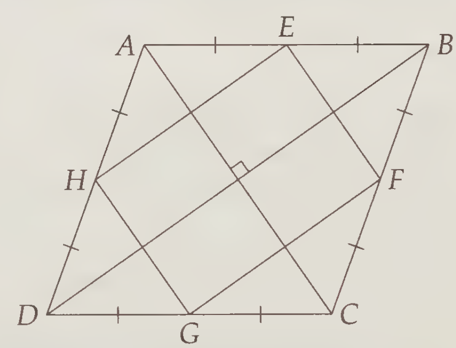
Solution for Problem 8.19: We draw rhombus $ABCD$ and let $EFGH$ be the quadrilateral formed by connecting the midpoints of the sides of $ABCD$, with E on $\overline{AB}$, F on $\overline{BC}$, and so on. We will take two different approaches to solving the problem.

(a) Since we want to show that $EFGH$ is a rectangle, we look for a way to find the measures of each angle of $EFGH$. Since its vertices are the midpoints of $ABCD$ and $ABCD$ is a rhombus, the vertices of $EFGH$ divide the sides of $ABCD$ into 8 equal segments as shown.



Since $ABCD$ is a parallelogram, $\angle A = \angle C$ and $\angle B = \angle D$. Therefore, $\triangle AEH \cong \triangle CGF$ and $\triangle BEF \cong \triangle DGH$. Furthermore, each of these small triangles is isosceles, so we can let $\angle AEH = x$ and $\angle DGH = y$ and identify equal angles as shown. Finally, we see that each angle of $EFGH$ has measure $180° - x - y$. Thus, the angles of $EFGH$ are equal and $EFGH$ must be a rectangle.

(b) For another approach, we might note that we want to prove that the angles of $EFGH$ are right angles. This gets us thinking about where we can find right angles in $ABCD$. Since $ABCD$ is a rhombus, its diagonals are perpendicular. To relate the sides of the rectangle to these diagonals, we find that $\triangle AEH \sim \triangle ABD$ by SAS Similarity since $AE/AB = AH/AD = 1/2$ and $\angle EAH = \angle BAD$. Therefore, $\angle AEH = \angle ABD$, so $\overline{EH} \parallel \overline{BD}$. Similarly, we can show that $\overline{FG} \parallel \overline{BD}$. Since $\overline{AC}$ is perpendicular to $\overline{BD}$, it must be perpendicular to $\overline{EH}$ and $\overline{FG}$ as well, because both $\overline{EH}$ and $\overline{FG}$ are parallel to $\overline{BD}$.



We can use exactly the same approach to show that $\overline{EF} \parallel \overline{AC} \parallel \overline{GH}$, from which we can conclude that $\overline{EF}$ and $\overline{GH}$ are perpendicular to $\overline{BD}$. Now we can tie all these parallel and perpendicular lines together. Since $\overline{EF} \perp \overline{BD}$ and $\overline{FG} \parallel \overline{BD}$, we have $\overline{EF} \perp \overline{FG}$. In exactly the same way, we can show that each of the other three angles of $EFGH$ is a right angle. Thus, $EFGH$ is a rectangle.

□

Exercises

8.5.1 $POST$ is a rectangle with $PO = 8$ and $OS = 12$.

(a) Find the perimeter of $POST$.

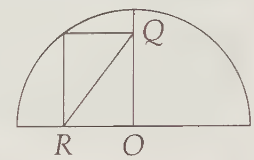
(b) Find PS.

(c) Find $[POST]$.

8.5.2 The length of a rectangle is one less than twice its width. If the perimeter of the rectangle is 36, what is the area of the rectangle?

8.5.3 Diagonals $\overline{WR}$ and $\overline{ET}$ of rectangle $WERT$ meet at Y. Given that $\angle WYE = x$, find $\angle ERY$ and $\angle YRT$ in terms of x.

8.5.4 A semicircle with center O has a radius of 9 cm. What is the number of centimeters in the length of $\overline{RQ}$, a diagonal of the rectangle shown? *(Source: MATHCOUNTS)*



8.5.5 I have a 36 inch by 24 inch rectangular painting. I would like to place a frame that is 2 inches wide around the painting. If the material for the frame costs \$1.50 per square inch, how much will the frame cost?

8.5.6 What kind of quadrilateral do we get when we connect the midpoints of the sides of a rectangle? (Prove your answer!)

8.5.7★ $EFGH$ is a rectangle with area 48. If $EGJI$ is a rectangle such that H is on $\overline{JI}$, what is the area of $EGJI$? **Hints:** 434

8.6 Squares

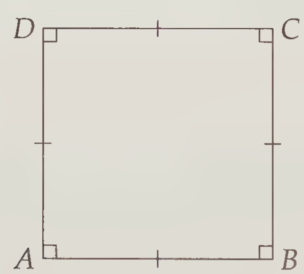


Figure 8.8: A Square

We've studied quadrilaterals with all their sides equal, and quadrilaterals with all their angles equal. What if a quadrilateral has both all sides equal and all angles equal? Such a quadrilateral is called a **square**, an example of which is shown in Figure 8.8.

As we'll see, squares are the easiest quadrilaterals to work with since everything that is true about rectangles, rhombi, and parallelograms is also true about squares.

Problems

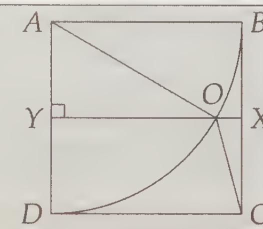
Problem 8.20: Find the perimeter and area of a square with side length 9.

Extra! *The composer opens the cage door for arithmetic, the draftsman gives geometry its freedom.*

–Arthur Cayley

Problem 8.21: Find the perimeter and area of a square that has a diagonal of length 8.

Problem 8.22: *ABCD* is a square. The circle with center *A* and radius *AB* meets the perpendicular bisector of $\overline{AD}$ in two points, of which *O* is the one inside the square. Find $\angle AOC$. **Hints:** 237



We've already studied all the tools we need for squares. How we use these tools in the following two problems should be unsurprising.

Problem 8.20: Find the perimeter and area of a square with side length 9.

Solution for Problem 8.20: Since all the sides of the square are equal, the perimeter of our square is $4 \times 9 = 36$. Since a square is a rectangle, its area is its length times its width. These dimensions are the same in a square, so the area is just the, um, 'square' of its side length, or $9^2 = 81$. (Now we see why multiplying a number by itself is called 'squaring' the number.) □

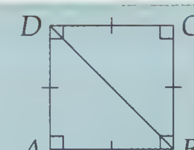
Problem 8.21: Find the perimeter and area of a square that has a diagonal of length 8.

Solution for Problem 8.21: A square is a rectangle, so its diagonals are congruent. It is also a rhombus, so its area is half the product of its diagonals, or $(8^2)/2 = 32$ in this case.

We can find a side of the square by noting that a diagonal of a square splits the square into two 45-45-90 triangles, so each side equals the diagonal divided by $\sqrt{2}$, or $8/\sqrt{2} = 4\sqrt{2}$. Alternatively, since the area is just the square of the side length, we can take the square root of the area we already found: $\sqrt{32} = 4\sqrt{2}$. Thus, the perimeter is $4(4\sqrt{2}) = 16\sqrt{2}$. □

These two examples give us the most useful length and area relationships for a square.

Important: Let s be the side length of a square, P be the perimeter, and A the area. We have



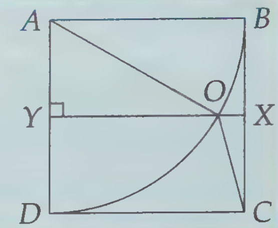
$$P = 4s \qquad\qquad A = s^2.$$

Drawing a diagonal creates two 45-45-90 triangles. Letting the length of a diagonal be d, we have

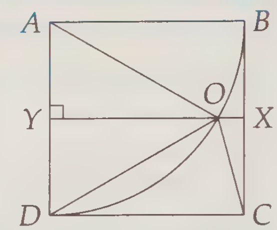
$$d = s\sqrt{2} \qquad\qquad P = d(2\sqrt{2}) \qquad\qquad A = \frac{d^2}{2}.$$

We often encounter squares in problems together with circles, since both involve segments with equal lengths (sides for the square, radii for the circle).

Problem 8.22: $ABCD$ is a square. The circle with center A and radius AB meets the perpendicular bisector of $\overline{AD}$ in two points, of which O is the one inside the square. Find $\angle AOC$.



Solution for Problem 8.22: There doesn't seem any easy way to get $\angle AOC$, so we try splitting it into pieces by drawing $\overline{OD}$. Since $\triangle AYO \cong \triangle DYO$ by SAS, we have $AO = OD$. Furthermore, $AD = AO$ because they are radii of the circle. Thus, $\triangle AOD$ is equilateral, which implies $\angle AOD = 60°$. Now we only have to find $\angle DOC$.



Since OD equals the side length of the square, we have $OD = DC$. Therefore, $\triangle DOC$ is isosceles, so $\angle DOC = \angle DCO$. Furthermore, we already know

$$\angle ODC = \angle ADC - \angle ADO = 90° - 60° = 30°.$$

From $\triangle DOC$ we have

$$\angle DOC + \angle DCO + \angle ODC = 180°,$$

so $2\angle DOC + 30° = 150°$. Therefore, $\angle DOC = 75°$. Finally, we have $\angle AOC = \angle AOD + \angle DOC = 135°$.

Note that you might also have found $\angle AOD = 60°$ by seeing that $AO = AD = 2AY$ and $\angle AYO = 90°$, so $\triangle AOY$ is a 30-60-90 triangle (and likewise for $\triangle DOY$.) □

Concept: Breaking a desired angle, length, or area into parts often makes it easier to find.

Exercises

8.6.1 The area of square $EFGH$ is 80. Find EF and EG.

8.6.2 M is the midpoint of $\overline{AB}$ on square $ABCD$. If $\overline{AC}$ and $\overline{BD}$ meet at O, and $MO = 4$, what is the area of square $ABCD$?

8.6.3 The diagonals of square $TYUI$ meet at M. Point K is on side $\overline{TY}$ such that $TK = TM$. Find $\angle MTK$ and $\angle TMK$.

8.6.4 Show that a rectangle with perpendicular diagonals must be a square.

8.6.5 $ABCD$ and $ACFG$ are squares. Find $[ACFG]/[ABCD]$. **Hints:** 236

8.6.6 Point E is inside square $ABCD$ such that $\triangle ABE$ is equilateral. Given that $AB = 4$, find the following:

(a) AE.

(b) $[ABCD]$.

(c) $[ABE]$.

(d) $\angle DAE$ and $\angle DEA$.

(e) The area inside $ABCD$ but outside $\triangle ABE$.

(f) CE.

8.7 If and Only If

In this section, we learn what 'if and only if' means by exploring what facts we need to classify a quadrilateral, and what facts we know given a quadrilateral's classification.

We already have a good idea of what 'if' means. For example, we can write 'An animal has four legs if it is a dog.' However, 'only if' is a little trickier – for example, the statement 'An animal has four legs only if it is a dog' is clearly false. Cats have four legs, too.

Many mathematical statements put 'if' and 'only if' together. Here's what it looks like in a non-mathematical context:

> A month has less than thirty days if and only if the month is a February.

This statement is equivalent to saying both of the following statements at the same time:

> Every month with less than thirty days is a February.
>
> Every February has less than thirty days.

In other words, to prove an 'if and only if' statement, we must prove two different things. In our example above, we would have to show that every month that has less than thirty days is a February, and we would have to show that every February has less than thirty days. Make sure you understand that these are two different statements to prove!

Important: Proving 'if and only if' statements requires proving two different statements.

Problems

Problem 8.23: Prove that *ABCD* is a parallelogram if and only if $\angle A = \angle C$ and $\angle B = \angle D$.

Problem 8.24:
(a) Prove that if *ABCD* is a square, then its diagonals bisect its angles.
(b) Is it true to say that *ABCD* is a square if and only if its diagonals bisect its angles?

Extra! *In the course of my law reading I constantly came upon the word demonstrate. I thought at first that I understood its meaning, but soon became satisfied that I did not... I consulted all the dictionaries and works of reference I could find... At last I said, 'Lincoln, you can never make a lawyer if you do not understand what demonstrate means,' and I left my situation in Springfield, went home to my father's house and stayed there till I could give any proposition in the six books of Euclid at sight. I then found out what demonstrate means and went back to my law studies.*

–Abraham Lincoln

Problem 8.25: Prove that the diagonals of a trapezoid are congruent if and only if the trapezoid is isosceles. **Hints:** 216

Problem 8.26: In this problem, we prove that the circumcenter of a triangle is the same point as the incenter if and only if the triangle is equilateral.

(a) First, we tackle the 'if' part: Show that in an equilateral triangle, the circumcenter and the incenter are the same point.

(b) In the next three parts, we tackle the 'only if' part of the problem by showing that if the incenter and the circumcenter are the same point, then the triangle is equilateral. Let X be this common point and $\triangle ABC$ be our triangle. What does X being the circumcenter tell us about X?

(c) What does X being the incenter tell us about X?

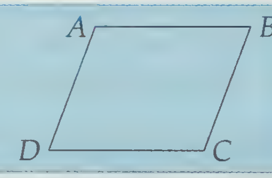
(d) Combine your answers from the previous two parts to find congruent triangles. Use these triangles to deduce that $\triangle ABC$ is equilateral.

Now we'll try our hand at a few proofs (or disproofs) of 'if and only if' statements.

Problem 8.23: Prove that $ABCD$ is a parallelogram if and only if $\angle A = \angle C$ and $\angle B = \angle D$.

Solution for Problem 8.23: What's missing from this solution:

Bogus Solution: Since $ABCD$ is a parallelogram, we have $\overline{AD} \parallel \overline{BC}$, so $\angle A + \angle B = 180^\circ$. Similarly, $\overline{AB} \parallel \overline{CD}$ means $\angle B + \angle C = 180^\circ$. Therefore, $\angle A = \angle C$. Similarly, we can show $\angle B = \angle D$.



This solution only does half the problem. 'If and only if' problems are two-part problems! We must prove two statements. First, we show:

$ABCD$ is a parallelogram if $\angle A = \angle C$ and $\angle B = \angle D$.

Since the angles of a quadrilateral add to 360°, we have $\angle A + \angle B + \angle C + \angle D = 360^\circ$. Since $\angle C = \angle A$ and $\angle D = \angle B$, we have $2(\angle A + \angle B) = 360^\circ$, so $\angle A + \angle B = 180^\circ$. Therefore, $\overline{AD} \parallel \overline{BC}$. Since $\angle A = \angle C$, we have $\angle B + \angle C = 180^\circ$ also. Therefore, $\overline{AB} \parallel \overline{CD}$ and $ABCD$ is a parallelogram.

Next, we must show:

$ABCD$ is a parallelogram only if $\angle A = \angle C$ and $\angle B = \angle D$.

In other words, we must show that every parallelogram $ABCD$ has $\angle A = \angle C$ and $\angle B = \angle D$. This is what we did in our Bogus Solution above, so we have already tackled this part.

Therefore, $ABCD$ is a parallelogram if and only if $\angle A = \angle C$ and $\angle B = \angle D$. □

Sometimes one part of 'if and only if' is clearly true, but the other half isn't so obvious. And sometimes the other half isn't even true!

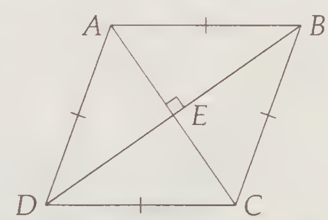
Problem 8.24:
(a) Prove that if $ABCD$ is a square, then its diagonals bisect its angles.
(b) Is it true to say that $ABCD$ is a square if and only if its diagonals bisect its angles?

Solution for Problem 8.24:

(a) We know that each diagonal of a square cuts the square into two isosceles right triangles. In other words, each diagonal cuts the right angles at the vertices into two 45° angles. So, if $ABCD$ is a square, its diagonals bisect its angles.

(b) We've already proved one direction of this statement. The other direction is

Prove that $ABCD$ is a square if its diagonals bisect its angles.



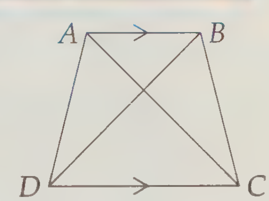
Unfortunately, try as we may, we can't prove this! For example, the diagonals of any rhombus bisect the angles of the rhombus. We can show this quickly by noting that $\triangle AED \cong \triangle CED$ in rhombus $ABCD$ at right. Hence, $\angle ADE = \angle CDE$ and diagonal $\overline{BD}$ bisects $\angle ADC$. Similarly, each of the four angles of rhombus $ABCD$ is bisected by a diagonal of $ABCD$. Therefore, if the diagonals of a quadrilateral bisect its angles, we *cannot* deduce that the quadrilateral is a square, since the quadrilateral may be a rhombus that is not a square.

□

Important: To prove a statement is true, you must have a proof that covers all possibilities. To disprove a statement, all you have to do is find one case in which the statement is false. It's much easier to break something than to build it!

Let's try a couple more challenging 'if and only if' statements.

Problem 8.25: Prove that the diagonals of a trapezoid are congruent if and only if the trapezoid is isosceles.



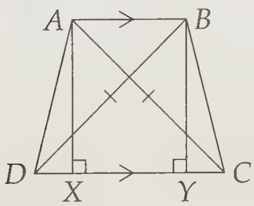
Solution for Problem 8.25: Our diagram shows trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$. We'll assume that $\angle C < 90^\circ$, because if $\angle C > 90^\circ$, then we note $\angle B < 90^\circ$ and we can then use essentially the same proof as for $\angle C < 90^\circ$. (What happens if $\angle C = 90^\circ$?) Since we have an if and only if statement, we must prove two items. First, we must show:

The diagonals of a trapezoid are congruent if the trapezoid is isosceles.

We must show that $AC = BD$ if $\angle ADC = \angle BCD$. Back in Problem 8.7, we showed that $\angle ADC = \angle BCD$ implies that $AD = BC$. Since we also have $\angle ADC = \angle BCD$ and $CD = CD$, we have $\triangle ADC \cong \triangle BCD$ by SAS. Therefore, $AC = BD$.

Next, we must show:

The diagonals of a trapezoid are congruent only if the trapezoid is isosceles.



In other words, we must show that if the diagonals are equal, the trapezoid is isosceles. So, we must start from $AC = BD$ and show that $\angle ADC = \angle BCD$. We would like to show that $\triangle ACD \cong \triangle BDC$, so we draw altitudes $\overline{AX}$ and $\overline{BY}$ as shown. Since $AC = BD$ and $AX = BY$ (because $\overline{AB}$ and $\overline{CD}$ are parallel), we have $\triangle AXC \cong \triangle BYD$ by HL Congruence. Hence, we have $\angle ACD = \angle BDC$. Since we also have $AC = BD$ and $CD = CD$, we have $\triangle ACD \cong \triangle BDC$ by SAS Congruence. Therefore, $\angle ADC = \angle BCD$.

Therefore, the diagonals of a trapezoid are congruent if and only if the trapezoid is isosceles. □

You might be wondering, 'How in the world would we think to draw the altitudes we drew in the second part?' We think to add these because we have to somehow use the fact that $\overline{AB} \parallel \overline{CD}$. We could also have used this fact by finding some equal angles (this path is a little longer), but we know that drawing altitudes often simplifies trapezoid problems.

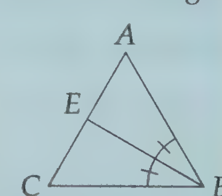
Concept: When stuck on a problem, think 'What information haven't I used yet?' Then, try to find some way to use that information.

Enough with quadrilaterals; let's go back to triangles.

Problem 8.26: Prove that the circumcenter of a triangle is the same point as the incenter if and only if the triangle is equilateral.

Solution for Problem 8.26: What's wrong with the following solution:

Bogus Solution: Let ABC be our triangle. We can easily take care of showing that the circumcenter and incenter are the same point in an equilateral triangle since the angle bisectors of an equilateral triangle are the same lines as the perpendicular bisectors of the sides of the triangle.



Going the other way is tougher. Let $\overline{BE}$ be the angle bisector of $\angle B$. Since $AB = BC$, $\angle ABE = \angle CBE$, and $BE = BE$, we have $\triangle EBC \cong \triangle EBA$. So, $AE = EC$ and $\angle AEB = \angle CEB = 90°$. Thus, angle bisector $\overline{BE}$ is part of the perpendicular bisector of $\overline{AC}$. Similarly, we can show that all the angle bisectors of $\triangle ABC$ are the same as the perpendicular bisectors, so the triangle must be equilateral.

This 'solution' has several missteps. First, we should be more explicit in the first part about showing why the angle bisectors and perpendicular bisectors are the same lines in an equilateral triangle. Second, and far more serious, is that our second part of the proof is logically flawed. We assume as a step in our 'proof' that $AB = BC$; however, we can't assume this because we don't know anything about the sides of $\triangle ABC$ in the second part yet – we are trying to prove that these are equal!

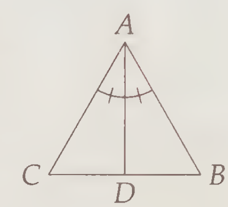
WARNING!! Be very careful that you do not assume what you are trying to prove as part of a proof!

Finally, our 'proof' ends by stating that since the angle bisectors and the perpendicular bisectors are the same, the triangle must be equilateral. This statement, while true, is not proved in our solution – it's what we are asked to prove!

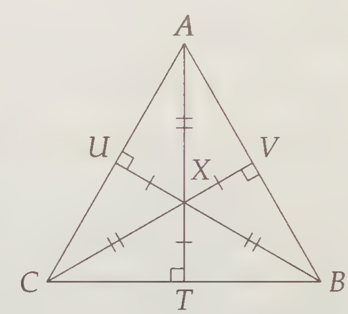
Let's try again:

Let $\triangle ABC$ be our triangle, O be the circumcenter, and I be the incenter.

First, we show that if a triangle is equilateral, then I and O are the same. We draw angle bisector $\overline{AD}$. Since $\triangle ABC$ is equilateral, we have $\triangle CAD \cong \triangle BAD$ by SAS. So, $CD = DB$ and $\angle CDA = \angle BDA = 90°$. Therefore, $\overleftrightarrow{AD}$ is the perpendicular bisector of $\overline{BC}$. Similarly, each angle bisector of $\triangle ABC$ is also a perpendicular bisector of one of the sides. Thus, the intersection of the angle bisectors, the incenter, is the same point as the intersection of the perpendicular bisectors, the circumcenter.



Next, we turn to the second part, proving that if the incenter and the circumcenter are the same point, then the triangle must be equilateral. We let this common point be X. Since X is the incenter, it is equidistant from all three sides. Since it is the circumcenter, it is equidistant from all three vertices. These two sets of length equalities are shown in the diagram. We quickly see that all six little right triangles are congruent by HL Congruence. Hence, we have $AV = VB = BT = TC = CU = UA$, so $AB = BC = CA$ and the triangle is equilateral. (Note: There are plenty of other ways to do this part; perhaps you can find another way!) □



Another phrase mathematicians will sometimes use to mean 'if and only if' is 'necessary and sufficient.' For example:

> It is necessary and sufficient for a quadrilateral's diagonals to bisect each other in order for the quadrilateral to be a parallelogram.

This is the same as saying:

> The diagonals of a quadrilateral bisect each other if and only if the quadrilateral is a parallelogram.

Unsurprisingly, we even have a symbol for 'if and only if', $\Leftrightarrow$:

> The diagonals of a quadrilateral bisect each other $\Leftrightarrow$ the quadrilateral is a parallelogram.

Usually this symbol is only used in brief statements. For example: $x^2 = 4 \Leftrightarrow x = \pm 2$. Finally, sometimes the phrase 'if and only if' is shortened to 'iff.'

Exercises

Prove or disprove each of the following statements.

8.7.1 $EFGH$ is a rhombus if and only if its diagonals are perpendicular.

8.7.2 $\triangle XYZ$ is equilateral if and only if $XY = YZ$ and $\angle X = 60^\circ$.

8.7.3 A quadrilateral is a parallelogram if and only if it is a rhombus.

8.7.4 A rhombus is a square if and only if it is also a rectangle.

8.7.5 $PQRS$ is a parallelogram if and only if $PQ = RS$ and $SP = QR$.

8.7.6 Trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$ is isosceles if and only if $\angle ABD = \angle BAC$. **Hints:** 466

8.7.7 $WXYZ$ is a rectangle if and only if $WX = YZ$, $\angle WXY = \angle XYZ$, and $\overline{WX} \parallel \overline{YZ}$.

8.7.8 Let M be the midpoint of $\overline{AB}$. $\triangle ABC$ is isosceles with $AC = BC$ if and only if $\overline{CM}$ bisects $\angle ACB$. **Hints:** 4

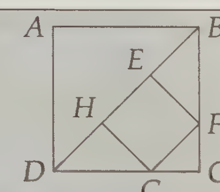
8.7.9 $ABCD$ is a square if and only if its diagonals are perpendicular and congruent.

8.8★ Quadrilateral Problems

In this section we will tackle a few more challenging problems using the quadrilateral principles we have learned.

Problems

Problem 8.27: Diagonal $\overline{BD}$ of square $ABCD$ is drawn. Square $EFGH$ is then inscribed in $\triangle BDC$ with two vertices on $\overline{BD}$ as shown. If $AB = 6$, find the area of $EFGH$.



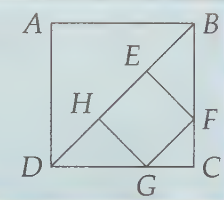
Problem 8.28: I have a rectangular painting that has a length 10 inches longer than its width. I frame it with a rectangular frame that is two inches wide all the way around. Given that the area of the frame is 152 square inches, what is the area of the painting?

Problem 8.29: Find the area of trapezoid $ABCD$ given that $\overline{AB} \parallel \overline{CD}$, $AB = 16$, $BC = 15$, $CD = 30$, and $AD = 13$.

Problem 8.30: Isosceles trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$ is inscribed in a circle with radius 10 such that the center of the circle is inside $ABCD$. Isosceles trapezoid $CDEF$, with $\overline{CD} \parallel \overline{EF}$, is inscribed in the same circle, but the center of the circle is not inside $CDEF$. Given that $AB = EF = 12$ and $CD = 16$, find the areas of $ABCD$ and $CDEF$.

Problem 8.31: Prove that the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides of the parallelogram.

Problem 8.27: Diagonal $\overline{BD}$ of square $ABCD$ is drawn. Square $EFGH$ is then inscribed in $\triangle BDC$ with two vertices on $\overline{BD}$ as shown. If $AB = 6$, find the area of $EFGH$.

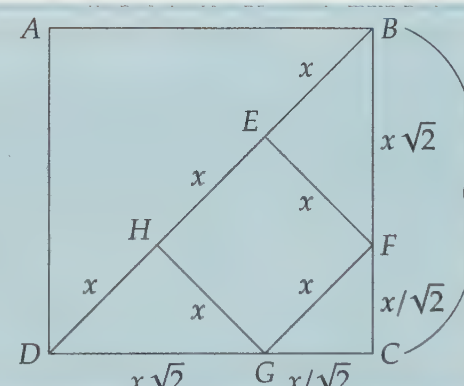


Solution for Problem 8.27: We need the side length of $EFGH$ to get its area, so we let the side length of $EFGH$ be x. Since $\triangle EBF$ is a 45-45-90 triangle, we have $BF = EF\sqrt{2} = x\sqrt{2}$. We also have $BE = EF = x$ (and from $\triangle DHG$, $DH = x$). Because $\overline{BD} \parallel \overline{FG}$, we have $\angle FGC = 45°$, so $\triangle FCG$ is also a 45-45-90 triangle. Therefore, $FC = FG/\sqrt{2} = x/\sqrt{2} = x\sqrt{2}/2$. We know $BC = 6$, so we have $BF + FC = BC = 6$. Substitution gives

$$x\sqrt{2} + \frac{x\sqrt{2}}{2} = 6.$$

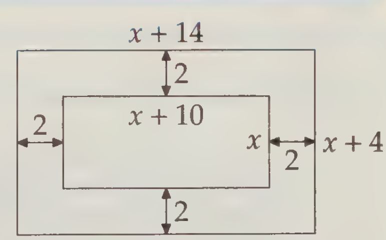
Therefore, $x\left(\frac{3\sqrt{2}}{2}\right) = 6$, so $x = 2\sqrt{2}$. So, the area of $EFGH$ is $EF^2 = x^2 = (2\sqrt{2})^2 = 8$. Alternatively, we could have noted that $BD = 3x = 6\sqrt{2}$, so $x = 2\sqrt{2}$. □

Concept: Don't just stare at a diagram. Set sides you seek equal to variables and find other lengths in terms of those variables. Label the diagram when you find these new lengths. Your final diagram for the last problem might look like the one at right.



Problem 8.28: I have a rectangular painting that has a length 10 inches longer than its width. I frame it with a rectangular frame that is two inches wide all the way around. Given that the area of the frame is 152 square inches, what is the area of the painting?

Solution for Problem 8.28: We start by making a sketch of our painting with its frame. We let the width of the painting without frame be x, so the length is $x + 10$. Since the frame is two inches thick, we find that the painting with frame is $x + 14$ inches long and $x + 4$ inches wide. We can then find the area of the frame in terms of x by subtracting the small rectangle (the picture) from the large one (picture plus frame). The area



of the picture plus frame is $(x+14)(x+4)$ and the area of the picture alone is $x(x+10)$, so we have:

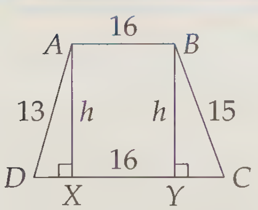
$$(x+14)(x+4) - x(x+10) = 152.$$

Expanding the left side gives $x^2 + 18x + 56 - x^2 - 10x = 152$, so $x = 12$.

Now we can find the area of the painting without the frame: $x(x+10) = 12(22) = 264$ square inches. □

Concept: A picture is worth a thousand words – making a sketch can greatly help you understand a geometric problem.

Problem 8.29: Find the area of trapezoid $ABCD$ given that $\overline{AB} \parallel \overline{CD}$, $AB = 16$, $BC = 15$, $CD = 30$, and $AD = 13$.



Solution for Problem 8.29: We have the bases, so we just need the height. We draw altitudes from A and B, thus forming a rectangle and two right triangles. (We like right triangles!) Since $XY = AB = 16$ and $CD = 30$, we know that $DX + CY = 14$. At this point, there are a couple approaches we could take to find the height.

Wishful thinking. We have the hypotenuses of $\triangle AXD$ and $\triangle BYC$, and we know they each have a leg of length h. The 15 and 13 make us think of the Pythagorean triples 5-12-13 and 9-12-15. These each have a leg of length 12, so perhaps our height is 12. This would make $DX = 5$ and and $CY = 9$, so $DX + CY = 14$, as desired. Our wishful thinking has been fruitful! The height must be 12, so we can now find the area.

Algebra. Perhaps we don't see the Pythagorean triples, or maybe they don't work out (the answers won't always be integers!); then, we have to do a little more work. We let $DX = x$, so $CY = 14 - x$. Now we can use the Pythagorean Theorem on $\triangle ADX$ and $\triangle BYC$ to build two equations in terms of h and x:

$$\begin{aligned} x^2 + h^2 &= 13^2 \\ (14-x)^2 + h^2 &= 15^2 \end{aligned}$$

We can subtract the first from the second to eliminate h^2:

$$(14-x)^2 - x^2 = 15^2 - 13^2.$$

Factoring both sides as a difference of squares gives

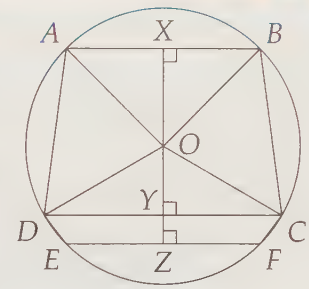
$$(14-x-x)(14-x+x) = (15-13)(15+13),$$

so $(14-2x)(14) = 56$, which gives $x = 5$. Now we can find $h = 12$ with either of our original equations.

Finally, we find our area: $h(AB+CD)/2 = 276$. □

Concept: Rectangles and right triangles are easier to work with than trapezoids. When stuck on a trapezoid problem, consider dropping altitudes.

Problem 8.30: Isosceles trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$ is inscribed in a circle with radius 10 such that the center of the circle is inside $ABCD$. Isosceles trapezoid $CDEF$, with $\overline{CD} \parallel \overline{EF}$, is inscribed in the same circle, but the center of the circle is not inside $CDEF$. Given that $AB = EF = 12$ and $CD = 16$, find the areas of $ABCD$ and $CDEF$.



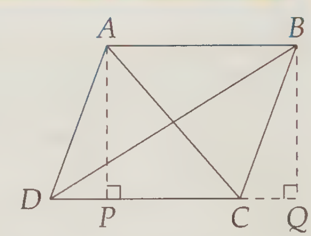
Solution for Problem 8.30: We'll tackle $ABCD$ first. In our diagram, we connect the center of the circle, O, to the vertices of the trapezoid, since we somehow want to use the fact that the trapezoid is inscribed in a circle. We have $OA = OB = OC = OD = 10$. We need a height, so we draw the height through O, since this will build right triangles in which we already know the hypotenuses. We let X and Y be the feet of the altitudes from O to $\overline{AB}$ and $\overline{CD}$, respectively.

Since $AO = BO$ and $XO = XO$, we have $\triangle AXO \cong \triangle BXO$ by HL Congruence. Therefore, X is the midpoint of $\overline{AB}$. Similarly, Y is the midpoint of $\overline{CD}$. Hence, $AX = 6$ and $DY = 8$. We can now use the Pythagorean Theorem on $\triangle XOA$ and $\triangle DOY$ to determine that $XO = 8$ and $YO = 6$. Thus, the height is $XO + YO = 14$, and our area is $(XY)(AB + CD)/2 = 196$.

Now we find the area of $CDEF$. We could break the problem into right triangles as before, or we could be a little more clever and apply what we learned about $\triangle AOB$ to $\triangle EOF$. Since $AO = EO$, $AB = EF$, and $BO = FO$, we have $\triangle AOB \cong \triangle EOF$. Therefore, the altitudes from O to $\overline{AB}$ and to $\overline{EF}$ are congruent. Letting Z be the foot of the altitude from O to $\overline{EF}$, we have $OZ = OX = 8$. Moreover, since $\overline{CD} \parallel \overline{EF}$ and $\overline{OZ} \perp \overline{EF}$, we have $\overline{OZ} \perp \overline{CD}$. Since Y is the foot of the altitude from O, $\overline{OZ}$ passes through Y. So, the height of $CDEF$ is $YZ = OZ - OY = 2$. Finally, the area of $CDEF$ is $(YZ)(CD + EF)/2 = 28$. □

We'll finish with a challenging proof.

Problem 8.31: Prove that the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides of the parallelogram.



Solution for Problem 8.31: Let $ABCD$ be our parallelogram. We wish to prove

$$AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2.$$

Since $AB = CD$ and $AD = BC$, we can simplify this to

$$AC^2 + BD^2 = 2(AD^2 + CD^2).$$

Seeing the sums of squares of sides, we reach for the Pythagorean Theorem. We have no right triangles, so we'll have to build them. We draw altitudes from A and B, thus building right triangles with our diagonals as hypotenuses. From right triangles $\triangle APC$ and $\triangle BQD$, we have

$$\begin{aligned} AC^2 &= PC^2 + AP^2 \\ BD^2 &= DQ^2 + BQ^2 \end{aligned}$$

We can add these to get an expression for the sum of the squares of the diagonals:

$$AC^2 + BD^2 = PC^2 + AP^2 + DQ^2 + BQ^2.$$

That looks like what we want, but we're not quite there. We have to get AD and CD in the equation somehow. We start by noting that $DQ = CD + CQ$ and $PC = CD - DP$. We also note that $AP = BQ$ and $AD = BC$, so $\triangle ADP \cong \triangle BCQ$ by HL Congruence. Thus, $DP = CQ$. Now our equation becomes:

$$\begin{aligned} AC^2 + BD^2 &= PC^2 + AP^2 + DQ^2 + BQ^2 \\ &= (CD - DP)^2 + AP^2 + (CD + CQ)^2 + BQ^2 \\ &= CD^2 - 2(CD)(DP) + DP^2 + AP^2 + CD^2 + 2(CD)(CQ) + CQ^2 + BQ^2 \\ &= CD^2 - 2(CD)(DP) + DP^2 + AP^2 + CD^2 + 2(CD)(DP) + DP^2 + AP^2 \\ &= 2(CD^2) + 2(DP^2 + AP^2). \end{aligned}$$

Since $DP^2 + AP^2 = AD^2$ from right triangle $\triangle DAP$, we have the desired

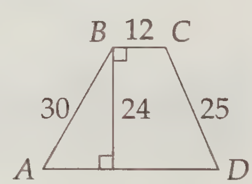
$$AC^2 + BD^2 = 2(CD^2 + AD^2).$$

As we noted earlier, $CD = AB$ and $AD = BC$, so we've shown that the sum of the squares of the sides of a parallelogram equals the sum of the squares of its diagonals. □

Exercises

8.8.1 In rectangle $ABCD$, H is the midpoint of $\overline{BC}$, E lies on $\overline{AD}$, and F lies on $\overline{AB}$. In rectangle $CEFG$, H lies on $\overline{FG}$ and $HG = 3$. Given $\angle DEC = 45°$, what is the positive difference between the areas of these two rectangles?

8.8.2 Find the area of trapezoid $ABCD$ shown at right.



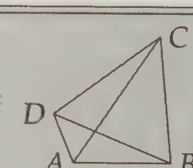
8.8.3 Quadrilateral $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{CD}$. We know $AB = 20$ and $CD = 12$. What is the ratio of the area of $\triangle ACB$ to the area of $ABCD$? *(Source: MATHCOUNTS)* **Hints:** 376

8.8.4 The diagonals of $EFGH$ are perpendicular. Prove that $EF^2 + GH^2 = FG^2 + EH^2$. (A quadrilateral with perpendicular diagonals is sometimes referred to as **orthodiagonal**.) **Hints:** 572

8.8.5★ In $\triangle ABC$, $AB = 6$, $BC = 7$, and $AC = 8$. Given that M is the midpoint of $\overline{AB}$, find CM. **Hints:** 240, 476, 117

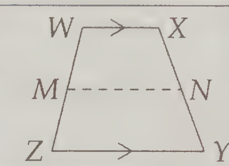
8.9 Summary

Definitions: A **quadrilateral**, such as $ABCD$ at the right, has four segments as sides, four vertices, and four angles. The segments connecting opposite vertices are called the **diagonals** of a quadrilateral.



Important: The interior angles of any quadrilateral add to 360°.

Definitions: A **trapezoid** is a quadrilateral with two parallel sides. The segment connecting the midpoints of the non-parallel sides is the **median** of the trapezoid, and the distance between the two parallel sides is the **height** of the trapezoid.



Important:

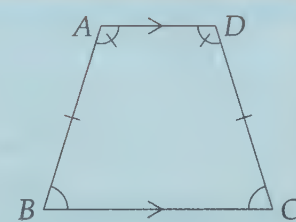
- The median of a trapezoid is parallel to the bases of a trapezoid, and equal in length to the average of the lengths of the bases.
- The area of a trapezoid equals the height of the trapezoid times the length of the median of the trapezoid.

Important: In an isosceles trapezoid:

(a) The base angles come in two pairs of equal angles as shown at right.

(b) The legs are equal.

(c) The diagonals are equal.

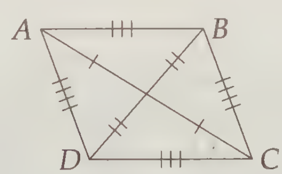


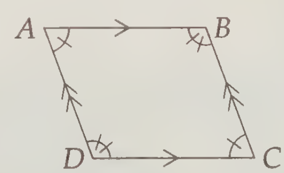
If any one of these items is true for a trapezoid with exactly one pair of parallel sides, then all the others must be true for that trapezoid.

Definitions: A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel.

Important:

- The area of a parallelogram is the product of a side length (the base) and the distance between that side and the opposite side of the parallelogram. This distance between opposite sides is called a **height** of the parallelogram.
- In parallelogram $ABCD$, the opposite sides are equal, the opposite angles are equal, and the diagonals bisect each other.





Conversely, $ABCD$ is a parallelogram if *any one* of the following are true:

(a) $AB = CD$ and $AD = BC$.

(b) $\angle A = \angle C$ and $\angle B = \angle D$.

(c) Diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other.

Therefore, proving one of these means the other two are true.

Definition: A quadrilateral is a **rhombus** if all of its sides are equal.

Important:

- Every rhombus is a parallelogram. Therefore, everything that is true about parallelograms is true about rhombi.
- The diagonals of a rhombus are perpendicular. The area of a rhombus is half the product of its diagonals (and also equals its base times its height).

Definition: A quadrilateral in which all angles are equal is a **rectangle**.

Important:

- All rectangles are parallelograms, so all that is true of parallelograms is true of rectangles.
- Let two consecutive sides of a rectangle have lengths ℓ and w. The area of the rectangle is ℓw, and the diagonals of the rectangle both have length $\sqrt{\ell^2 + w^2}$.

Definition: A quadrilateral in which all sides are equal and all angles are equal is a **square**.

Important:

- Each square is a parallelogram, a rectangle, and a rhombus so all that is true of a parallelogram, a rectangle, or a rhombus is true of a square.
- If the side length of a square is s and its diagonal is d, then $d = s\sqrt{2}$ and the area of the square is s^2, or $d^2/2$.

In proving various facts about quadrilaterals, we encountered the phrase 'if and only if.' Proving 'if and only if' statements requires proving two different statements.

Problem Solving Strategies

Concepts:

- Triangles, triangles, triangles. Although this chapter was about quadrilaterals, looking back you'll see that a great many of our solutions revolved around breaking the problems into triangles on which we can use all our triangle tools.
- Rectangles and right triangles are easier to work with than trapezoids. When stuck on a trapezoid problem, consider dropping altitudes to build rectangles and right triangles.

Continued on the next page. . .

Concepts: . . . *continued from the previous page*

- Breaking a desired angle, length, or area into parts often makes it easier to find.
- When stuck on a problem, think 'What information haven't I used yet?' Then, try to find some way to use that information.
- Don't just stare at a diagram. Set sides you seek equal to variables and find other lengths in terms of those variables. Label the diagram when you find these new lengths.
- A picture is worth a thousand words – making a sketch can greatly help you understand a visual problem.

Things To Watch Out For!

WARNING!!

- Although every rhombus is a parallelogram, every parallelogram is **not** a rhombus. Therefore, if we prove a property of rhombi, this property is not necessarily true for parallelograms. (The same is true of rectangles – rectangles are parallelograms, but not all parallelograms are rectangles, etc.)
- Be very careful that you do not assume what you are trying to prove as part of a proof!

REVIEW PROBLEMS

8.32 Fill in each of the boxes with the answer "Yes", "Sometimes", or "No".

Fact	Parallelogram	Rhombus	Rectangle	Square
Opposite sides have equal length				
All sides have equal length				
Opposite sides are parallel				
Opposite angles are equal				
All angles are equal				
Diagonals bisect each other				
Diagonals have equal length				
Area can be determined from sides				

8.33 In quadrilateral $ABCD$, $\angle A = \angle B = 128°$, and $\angle D$ is $10°$ less than 5 times $\angle C$. Find $\angle C$ and $\angle D$.

8.34 In isosceles trapezoid $ABCD$, $\angle A = x$ and $\angle B = 2x - 45°$. Find $\angle C$ and $\angle D$ in the following cases:

(a) $\overline{AB} \parallel \overline{CD}$.

(b) $\overline{AD} \parallel \overline{BC}$.

8.35 In which of the following quadrilaterals does a diagonal always divide the quadrilateral into two regions of equal area: Rhombus, Square, Rectangle, Trapezoid, Parallelogram?

8.36 If the numbers denoting the perimeter and area of a square are equal, what is the length of its diagonal?

8.37 If A, B, C, D are midpoints of the sides of square $EFGH$, what is the ratio of the area of triangle ABC to the area of square $EFGH$?

8.38 The bases of a trapezoid are $2x$ and $4x$. The height is $2x$. If the area is 48, what is x?

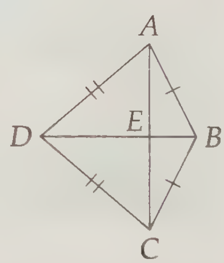
8.39 One of the angles of a rhombus is $120°$. If the shorter diagonal is 2, what is the area?

8.40 What is the number of centimeters in the length of a longer side of a rectangle that has a perimeter of 64 centimeters and an area of 192 square centimeters?

8.41 Quadrilateral $WXYZ$ is a parallelogram. Given that $WX = 2x - 3$, $XY = x + 7$, and $YZ = 3x - 8$, find the perimeter of $WXYZ$.

8.42 $WXYZ$ is a rhombus with $\angle X = 90°$. Prove that $WXYZ$ is a square.

8.43 One less well-known type of quadrilateral is the **kite**. The four sides of a kite can be split into two pairs of consecutive equal sides. For example, the figure at right shows kite $ABCD$ with $AB = BC$ and $CD = DA$. The diagonals of $ABCD$ meet at E as shown. Solve the following problems about kite $ABCD$.



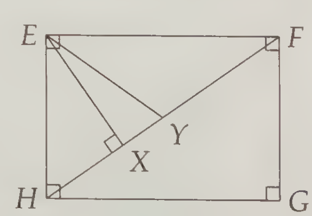
(a) Prove that $\angle ABD = \angle CBD$.

(b) Prove that $\overline{AC} \perp \overline{BD}$. **Hints:** 140, 249

8.44 Which of the following quadrilaterals is (are) a special case of a kite: parallelogram, trapezoid, rhombus, rectangle, square? (More than one answer may be correct.)

8.45 Each side of an equilateral triangle is 8 inches long. An altitude of this triangle is used as the side of a square. What is the number of square inches in the area of the square? *(Source: MATHCOUNTS)*

8.46 Shown at right is rectangle $EFGH$. Given that $\angle GHF = 31°$ and $EY = HY$, solve the following problems:



(a) Find $\angle XEH$.

(b) Find $\angle YEF$.

(c)★ Prove that $\overrightarrow{EY}$ passes through G. **Hints:** 366, 214

8.47 $EFGH$ is a rhombus. Given that $EF = EG = 6$, find the area of the rhombus. **Hints:** 404

8.48 Prove or disprove this statement: $ABCD$ is a square if and only if its sides are equal in length and $AC = BD$.

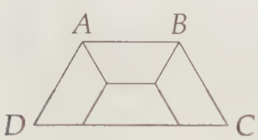
8.49 The angles of quadrilateral $WXYZ$ are such that $\angle W > \angle X > \angle Y > \angle Z$. The angles are in an arithmetic progression, meaning that $\angle W - \angle X = \angle X - \angle Y = \angle Y - \angle Z$.

(a) If $\angle W$ is four times the measure of $\angle Z$, what is $\angle W$?

(b) If $\angle W$ is two times $\angle Z$, what is $\angle W$?

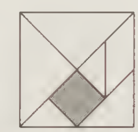
8.50 In quadrilateral $ABCD$, let E, F, G, and H be midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. Prove that $EFGH$ is a parallelogram.

8.51 A square and an equilateral triangle have the same perimeter. What is the ratio of the area of the square to the area of the equilateral triangle?



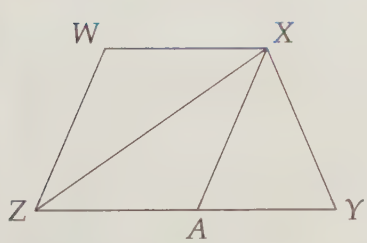
8.52 Trapezoid $ABCD$ is divided into four congruent trapezoids as shown. Given $AB = 4$ and $CD = 8$, find the sum of the lengths of all line segments in the figure. *(Source: ARML)*

8.53 What is the ratio of the area of the shaded square to the area of the large square in the figure at right? (The figure is drawn to scale.) *(Source: AMC 8)*



8.54 A street has parallel curbs 40 feet apart. A crosswalk bounded by two parallel stripes crosses the street at an angle. The length of the curb between the stripes is 15 feet and each stripe is 50 feet long. Find the distance between the stripes. *(Source: AMC 10)*

Challenge Problems



8.55 In the diagram at left, $AX = XY = 6$, $\angle AXY = 70^\circ$, $\angle W = 125^\circ$, $\angle XZA = 21^\circ$, and $\angle XZW = 34^\circ$.

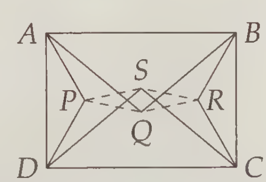
(a) Find $\angle ZXA$.

(b) Find WZ.

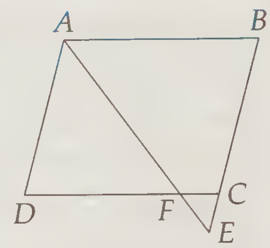
(c) Prove that $WY = XZ$. **Hints:** 328, 558

8.56 In rectangle $ABCD$, $AB = 16$ and $AD = 5$. F is on $\overline{AB}$ and G on $\overline{CD}$ such that $\overline{FG}$, $\overline{BD}$, and $\overline{AC}$ are concurrent at point O. Find $[FOB] + [GOC]$. *(Source: MATHCOUNTS)* **Hints:** 190

8.57 Each angle of rectangle $ABCD$ is trisected by a pair of segments. The angle trisectors meet at P, Q, R, and S, as shown in the diagram at right. Prove that $PQRS$ is a rhombus. **Hints:** 243



8.58 In rectangle $ABCD$, M is the midpoint of $\overline{BC}$. Points P and Q lie on $\overline{AB}$ and $\overline{DC}$, respectively, such that $PB = 4/3 \cdot BC$ and $\angle PMQ$ is a right angle. What is the ratio $PM : MQ$? *(Source: MATHCOUNTS)* **Hints:** 306



8.59 In the diagram, $ABCD$ is a parallelogram and E is on the extension of $\overline{BC}$ past C. $\overline{AE}$ and $\overline{CD}$ meet at F. Given $[ADF] = 64$ and $[CEF] = 4$, determine the following:

(a) CF/DF.

(b) $[BFC]$. **Hints:** 373

(c) $[ABCD]$. **Hints:** 213

(Source: ARML)

8.60 See Problem 8.43 for the definition of a kite.

(a) Find the area of a kite that has diagonals of length 10 and 12. **Hints:** 118, 299

(b)★ Find the area of kite $ABCD$ if $AC = 6\sqrt{2}$, $AB = 4\sqrt{3}$, and $BC = 2\sqrt{3}$. **Hints:** 193, 18, 527

8.61 Consider a square of side length 1. We draw four lines that each connect a midpoint of a side with a corner not on that side, such that each midpoint and each corner is touched by only one of these lines as shown at left below. Find the area of the shaded region. *(Source: HMMT)* **Hints:** 303

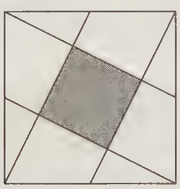


Figure 8.9: Diagram for Problem 8.61

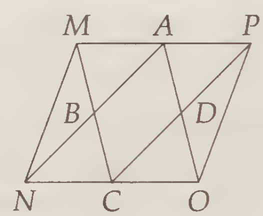
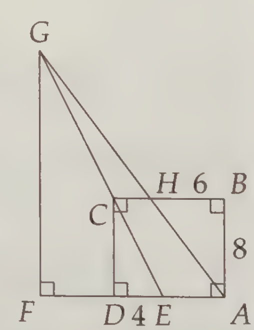


Figure 8.10: Diagram for Problem 8.62

8.62 Given $\overline{MP} \parallel \overline{NO}$, $\overline{PO} \parallel \overline{MN}$, $MA = AP$, and $NC = CO$ as in the diagram at right above, prove that $[ABCD] = [MNOP]/4$. **Hints:** 386

8.63 The diagonals of $WXYZ$ meet at A. Prove that the quadrilateral formed by connecting the incenters of $\triangle WXA$, $\triangle XYA$, $\triangle YZA$, and $\triangle ZWA$ has perpendicular diagonals. **Hints:** 368, 459

8.64 $EFGH$ is a rectangle with $EF = 12$ and area 192. $EGJI$ is a parallelogram such that H is on $\overline{JI}$. What are the possible values for the area of $EGJI$? **Hints:** 486



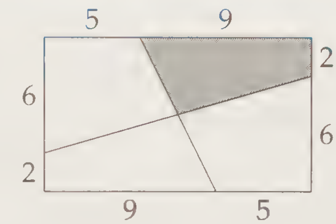
8.65 In rectangle $ABCD$, we have $AB = 8$, $BC = 9$, H is on $\overline{BC}$ with $BH = 6$, E is on $\overline{AD}$ with $DE = 4$, line $\overleftrightarrow{EC}$ intersects line $\overleftrightarrow{AH}$ at G, and F is on line $\overleftrightarrow{AD}$ with $\overline{GF} \perp \overline{AF}$. Find the length of $\overline{GF}$. *(Source: AMC 10)* **Hints:** 521, 379

8.66 In trapezoid $ABCD$, $\overline{AD} \parallel \overline{BC}$. $\angle A = \angle D = 45°$, while $\angle B = \angle C = 135°$. If $AB = 6$ and the area of $ABCD$ is 30, find BC. *(Source: HMMT)* **Hints:** 552

8.67 A circle is drawn through the vertices of square $ABCD$, and point X is on minor arc $\widehat{AB}$. Given that $[XAB] = 1$ and $[XCD] = 993$, find $[XAD] + [XBC]$. *(Source: Mandelbrot)* **Hints:** 412

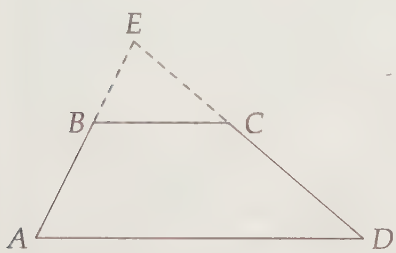
Extra! *A pupil from whom nothing is ever demanded that he cannot do, never does all he can.*

–John Stuart Mill



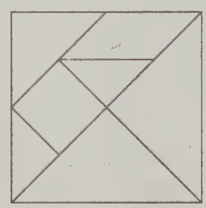
8.68★ The lengths indicated on the rectangle shown are in centimeters. What is the number of square centimeters in the area of the shaded region? *(Source: MATHCOUNTS)* **Hints:** 516

8.69★ P is inside rectangle $ABCD$. $PA = 2$, $PB = 3$, and $PC = 10$. Find PD. *(Source: HMMT)* **Hints:** 247, 381

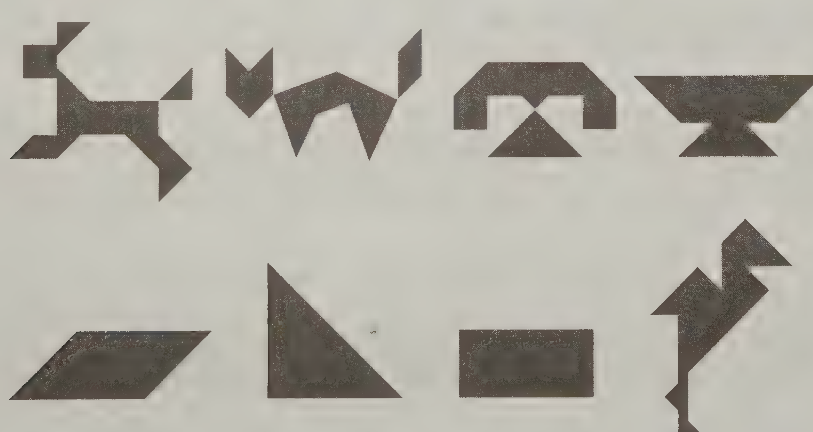


8.70★ In trapezoid $ABCD$ shown, $\overline{AD} \parallel \overline{BC}$, $AB = 6$, $BC = 7$, $CD = 8$, and $AD = 17$. The sides $\overline{AB}$ and $\overline{CD}$ are extended to meet at E. Prove that $\angle E = 90°$. *(Source: HMMT)* **Hints:** 511

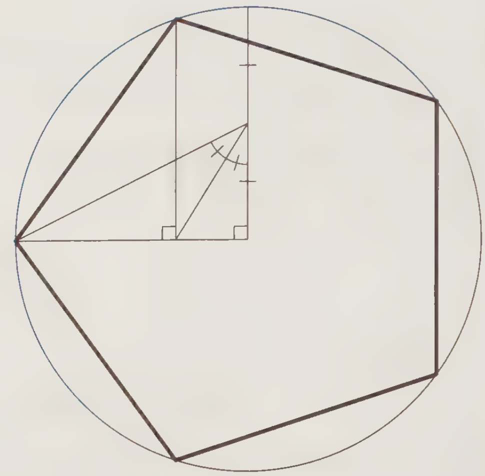
Extra! One of the oldest known puzzles is the Chinese *ch'i ch'iao t'u*, which means 'ingenious seven-piece plan.' In English, these are known as **tangrams**.



Tangrams involve rearranging the seven pieces that together make the square shown above into other given shapes. Some shapes that can be formed by these pieces are shown below. See if you can figure out how!



Perhaps you'll notice a couple things tangrams have in common with math problems. First, with experience, they get a lot easier. Second, often the simplest looking shapes end up being the hardest ones to solve! *Source: Martin Gardner's Mathematical Puzzles & Diversions*



Construction of a Regular Pentagon

Bees. . .by virtue of a certain geometrical forethought. . .know that the hexagon is greater than the square and the triangle, and will hold more honey for the same expenditure of material. – Pappus

CHAPTER 9 Polygons

9.1 Introduction to Polygons

We've tackled three sides, and we've handled four sides. Why stop there? Figure 9.1 shows several different **polygons**, which are closed planar figures with line segments as boundaries.

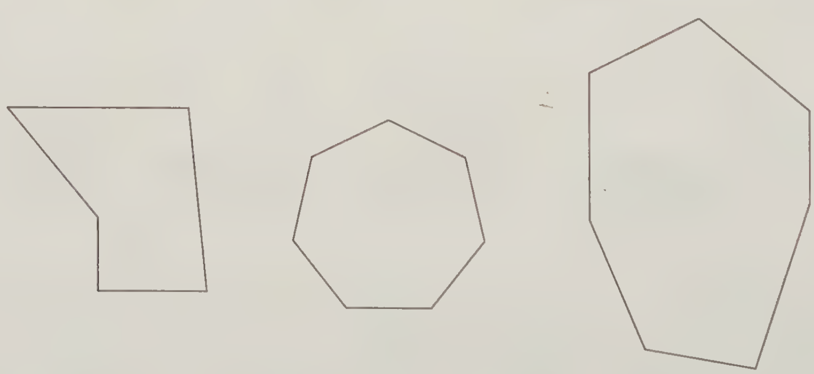


Figure 9.1: Some Polygons

All of the names of the parts of a general polygon are the same as in the polygons we've learned about so far (triangles and quadrilaterals). The segments that form the boundary are **sides**, which meet at **vertices** to form the **interior angles**. If we connect two vertices that are not adjacent on the polygon, we form a **diagonal**, such as diagonal $\overline{AC}$ in Figure 9.2.

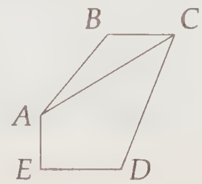


Figure 9.2: Polygon Diagonal

Polygons are classified by the number of sides they have. Here are some names given to different polygons.

Number of Sides	Polygon Name
3	triangle
4	quadrilateral
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon
10	decagon
12	dodecagon
n	n-gon

A polygon in which all the sides are equal and all the angles are equal is called a **regular polygon**. We've already seen two examples of regular polygons: equilateral triangles and squares. Figure 9.3 shows a few more regular polygons.

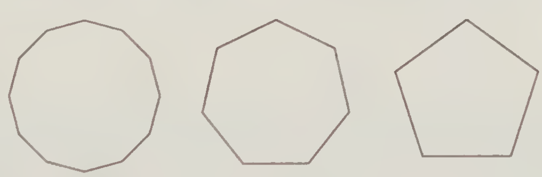


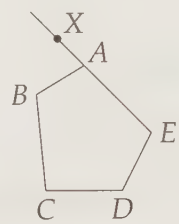
Figure 9.3: Some Regular Polygons

As with quadrilaterals, we'll be focusing on convex polygons in this chapter unless otherwise stated, which means that the interior angles of the polygons are all less than 180°.

Extra! *The most distinct and beautiful statement of any truth must at last take mathematical form.*
–Henry David Thoreau

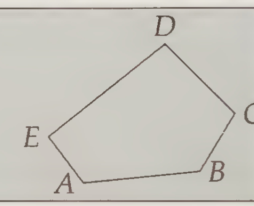
9.2 Angles in a Polygon

In this section, we explore not only the interior angles of a polygon, but also the **exterior angles**. Exterior angles are formed by extending the sides of the polygon. For example, $\angle BAX$ in the figure below is an exterior angle of $ABCDE$.



Problems

Problem 9.1: Find the sum of the measures of the interior angles of pentagon $ABCDE$ shown *without measuring the angles*. (Hint: See how we did Problem 8.1.)



Problem 9.2: In this problem we find a formula for the sum of the interior angles of a convex polygon with n sides. (By convex, we mean that all interior angles have measure less than 180°.)

(a) Draw all the diagonals from a single vertex of a polygon with n sides. In terms of n, how many triangles are formed?

(b) Use your dissection from part (a) to find a formula for the sum of the measures of the interior angles of a polygon.

Problem 9.3: The diagram shows the 10 exterior angles of a decagon. Find the sum of these angles. (Note: You cannot assume all the exterior angles have the same measure!)
Hints: 517

Problem 9.4: Find the number of sides in a regular polygon in which each interior angle measures 172°.

Problem 9.5: $ABCDEFGHIJKLMNO$ is a regular 15-gon.

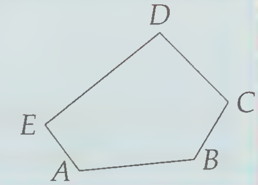
(a) Find $\angle ACB$.

(b) Find $\angle ACD$. **Hints:** 48

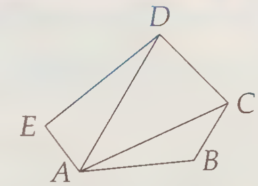
(c) Find $\angle ADE$. **Hints:** 250

We found the sum of the interior angles of a quadrilateral by using a diagonal to cut the quadrilateral into triangles. This strategy works for all polygons.

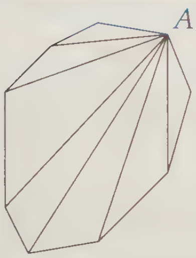
Problem 9.1: Find the sum of the measures of the interior angles of pentagon $ABCDE$ shown *without measuring the angles.*



Solution for Problem 9.1: We don't know anything about pentagons, but we know plenty about triangles. Therefore, we use diagonals to cut the pentagon into triangles. The sum of the angles in the pentagon is the same as the sum of the angles in the three triangles $\triangle ABC$, $\triangle ACD$, and $\triangle ADE$. Each triangle contributes $180°$ to the total, so the sum of the interior angles in a pentagon is $3(180°) = 540°$. □



Problem 9.2: Find a formula for the sum of the interior angles of a convex polygon with n sides.



Solution for Problem 9.2: As we did with the pentagon, we can break any polygon into triangles by drawing all the diagonals from one of the vertices. The sum of the angles in these triangles equals the sum of the angles in the polygon. Therefore, all we have to do is figure out how many triangles we form when we draw the diagonals. We can use our pentagon example for guidance. Let A be the common vertex of all our triangles. Each side of the polygon, except the two sides connected to A, is the side opposite A in exactly one of our triangles. So, there must be $n-2$ triangles formed. Since there are 180 degrees in each triangle the sum of the angles in an n-gon is $180(n-2)$ degrees. (It's a little more complicated to prove, but this result also holds for concave polygons.) □

Important: The sum of the interior angles in an n-sided polygon is $180(n-2)$ degrees. Therefore, the measure of each of the n interior angles in a regular n-gon is $180(n-2)/n$ degrees.

Now we turn our attention from the interior angles to the exterior angles.

Problem 9.3: The diagram shows the 10 exterior angles of a decagon. Find the sum of these angles. (Note: You cannot assume all the exterior angles have the same measure!)

Solution for Problem 9.3: Each of the exterior angles is supplementary to one of the interior angles, so we know that

$$\text{Sum of interior angles} + \text{Sum of exterior angles} = 10(180°) = 1800°,$$

since together, the interior and exterior angles make up 10 straight angles. Since the sum of the interior angles of a decagon is $(180°)(10-2) = 1440°$, we have

$$\text{Sum of exterior angles} = 1800° - 1440° = 360°.$$

□

In the Exercises at the end of the section, you'll be asked to work through this problem for a general n-gon. You'll thereby prove:

Important: The sum of the exterior angles in a convex polygon with n sides is $360°$. Therefore, the measure of each exterior angle in a regular n-gon is $360°/n$.

Let's try using our new knowledge about the angles of polygons on a couple problems.

Problem 9.4: Find the number of sides in a regular polygon in which each interior angle measures $172°$.

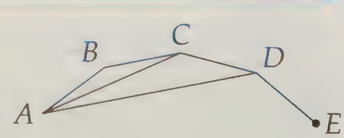
Solution for Problem 9.4: We let the polygon have n sides. There are two general approaches we can take to this problem:

Focus on the interior angles. Since the measure of one angle is $172°$, and the polygon is regular, the sum of all angles is $172n$ degrees. Since we know the sum is also given by $180(n-2)$ degrees, we can solve the equation $172n = 180(n-2)$ to find $8n = 360$, so $n = 45$.

Focus on the exterior angles. Since the measure of each interior angle is $172°$, each exterior angle is $180° - 172° = 8°$. The sum of the exterior angles is $360°$, so there must be $360/8 = 45$ of them. □

Concept: Sometimes thinking about the exterior angles of a polygon offers a simpler approach than thinking about the interior angles!

Problem 9.5: $ABCDEFGHIJKLMNO$ is a regular 15-gon. Find $\angle ACB$, $\angle ACD$, and $\angle ADE$.



Solution for Problem 9.5: We start with a diagram, then we find whatever angles we can determine. Since the polygon is a regular 15-gon, each exterior angle is $360/15 = 24$ degrees. Therefore, each interior angle is $180° - 24° = 156°$. Since $\triangle ABC$ is isosceles, $\angle BAC = \angle ACB$. The sum of the angles of $\triangle ABC$ gives us $\angle ABC + \angle BAC + \angle ACB = 180°$, so

$$156° + \angle ACB + \angle ACB = 180°.$$

Therefore, $2\angle ACB = 24°$, so $\angle ACB = 12°$. We also have

$$\angle ACD = \angle BCD - \angle ACB = 156° - 12° = 144°.$$

Turning to $\angle ADE$, we try the same strategy: find $\angle ADC$, then subtract it from $\angle CDE$. We note that $ABCD$ is an isosceles trapezoid in which $\overline{BC} \parallel \overline{AD}$ (make sure you see why), so $\angle ADC = 180° - \angle BCD = 24°$. So, $\angle ADE = 156° - 24° = 132°$. □

Exercises

9.2.1 Complete the table below for regular polygons.

Name	# of Sides	Sum of Interior ∠s	Int. ∠ Measure	Ext. ∠ Measure
pentagon	5	540°	108°	72°
hexagon	6			
heptagon	7			
octagon	8			
nonagon	9			
decagon	10			
dodecagon	12			
pentadecagon	15			
icosagon	20			
triacontagon	30			

9.2.2 Does the formula for the sum of the interior angles in an n-gon still work if the polygon is concave, as in the figure shown at right?

9.2.3 Each interior angle of a certain regular polygon has measure 160°. How many sides does the polygon have?

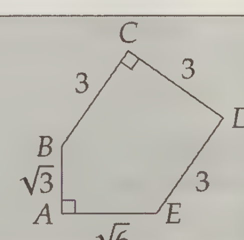
9.2.4 The measures of the angles of a pentagon are in the ratio of 3 : 3 : 3 : 4 : 5. What is the number of degrees in the measure of the largest angle? *(Source: MATHCOUNTS)* **Hints:** 587

9.2.5 The sum of the interior angles of a polygon is three times the sum of the exterior angles. How many sides does the polygon have?

9.3 Polygon Area

Problems

Problem 9.6: Find the area of pentagon $ABCDE$ given the side lengths and right angles shown. **Hints:** 538



Problem 9.7: In this problem we find the area of a regular hexagon with side length 8.

(a) Draw all three long diagonals of the hexagon; what kind of triangles do you form?

(b) Find the area of the hexagon by finding the areas of the triangles from (a).

Problem 9.8: In this problem we will find the area of regular octagon $ABCDEFGH$ with side length 4.

(a) Extend sides $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, and $\overline{GH}$ to meet at points W, X, Y, Z. What type of quadrilateral is $WXYZ$?

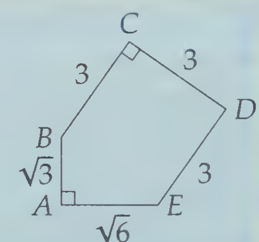
(b) Find the area of $WXYZ$.

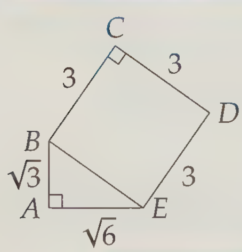
(c) Use $[WXYZ]$ to find the area of $ABCDEFGH$.

Problem 9.9: The center of a regular polygon is the point inside the regular polygon that is equidistant from all of the vertices of the polygon. Let the distance from the center of a regular n-gon to one of its sides be k and the length of each side of the n-gon be s. Break the polygon into triangles by connecting the center of the polygon to each vertex. Find a formula for the area of the n-gon in terms of k, n, and s.

We've seen all sorts of problems involving triangle and quadrilateral areas. You won't be at all surprised that we rely heavily on triangles and quadrilaterals when we find the areas of polygons with more sides.

Problem 9.6: Find the area of pentagon $ABCDE$ given the side lengths and right angles shown.



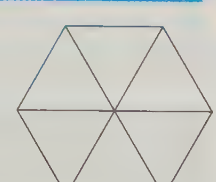


Solution for Problem 9.6: We start by drawing $\overline{BE}$ to separate right triangle $\triangle ABE$ from quadrilateral $BCDE$. We have $[ABE] = (\sqrt{3})(\sqrt{6})/2 = 3\sqrt{2}/2$, so all we need is $[BCDE]$. From the Pythagorean Theorem, we have $BE = \sqrt{AB^2 + AE^2} = 3$, so $BCDE$ is a rhombus. Since one of its angles is a right angle, it must also be a square. Therefore, $[BCDE] = 9$ and $[ABCDE] = [BCDE] + [ABE] = 9 + 3\sqrt{2}/2$. □

Concept: Break complicated areas into more manageable pieces.

Let's find the areas of a couple regular polygons.

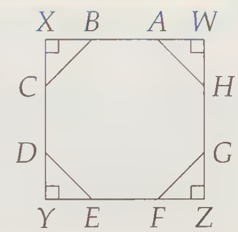
Problem 9.7: Find the area of a regular hexagon with side length 8.



Solution for Problem 9.7: We start by breaking the hexagon into pieces we can handle. Since the angles of a regular hexagon are each 120°, the long diagonals split each angle into two 60° angles. Therefore, each of the little six triangles formed by drawing all three long diagonals is equilateral. Thus, the area of the hexagon is 6 times the area of an equilateral triangle with side length 8. As we saw in Problem 6.11 on page 143, the area of such a triangle is $8^2\sqrt{3}/4 = 16\sqrt{3}$. So, the area of our regular hexagon is $6(16\sqrt{3}) = 96\sqrt{3}$. □

Concept: Many problems involving regular hexagons can be tackled by dissecting the hexagons into equilateral triangles.

Problem 9.8: Find the area of regular octagon $ABCDEFGH$ with side length 4.



Solution for Problem 9.8: We can find the area with a dissection (try drawing $\overline{AF}$, $\overline{BE}$, $\overline{CH}$, and $\overline{DG}$), but we can find a more elegant approach. We know each angle of a regular octagon is 135°. Since these angles are supplements of 45° angles, we start thinking about 45-45-90 triangles. This makes us think about squares, too. We extend the sides of the octagon, and we form square $WXYZ$. Our octagon is just $WXYZ$ with the four corners (the four 45-45-90 triangles) lopped off.

We can handle squares and 45-45-90 triangles. Since $AH = 4$, we have $WH = 4/\sqrt{2} = 2\sqrt{2}$. Similarly, each of the legs of the little right triangles has length $2\sqrt{2}$. Therefore, the area of each little triangle is $(2\sqrt{2})(2\sqrt{2})/2 = 4$. Since

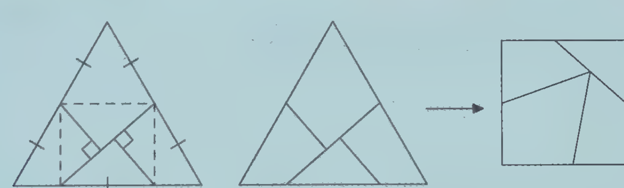
$$[WXYZ] = WZ^2 = (WH + HG + GZ)^2 = (4 + 4\sqrt{2})^2 = 48 + 32\sqrt{2},$$

the area of our octagon is

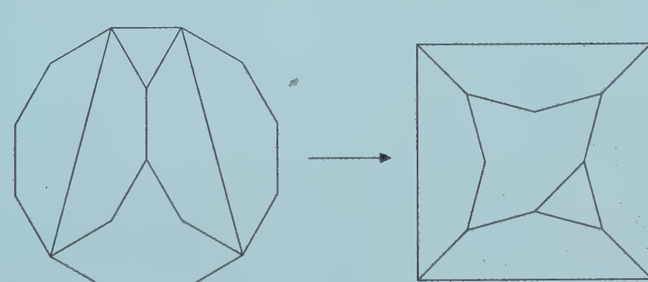
$$[ABCDEFGH] = [WXYZ] - 4[WAH] = 32 + 32\sqrt{2}.$$

□

Sidenote: The figure below shows how an equilateral triangle can be dissected into four pieces that can be reassembled to form a square.



Similarly, a regular dodecagon can be dissected into six pieces that can be reassembled to form a square.



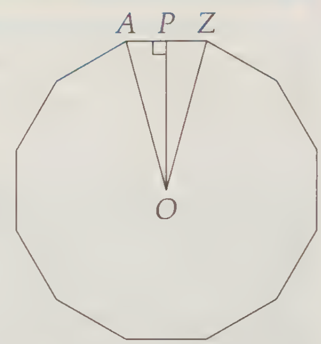
Can you find more regular polygons that can be dissected to form other regular polygons?

We now have another area-finding tool:

> **Concept:** Sometimes we can view our desired area as the 'leftover' portion from having simple pieces taken away from a simple starting figure.

Now we turn to the general problem of finding the areas of regular polygons.

> **Problem 9.9:** The center of a regular polygon is the point inside the regular polygon that is equidistant from all of the vertices of the polygon. Let the distance from the center of a regular n-gon to one of its sides be k and the length of each side of the n-gon be s. Find a formula for the area of the n-gon in terms of k, n, and s.



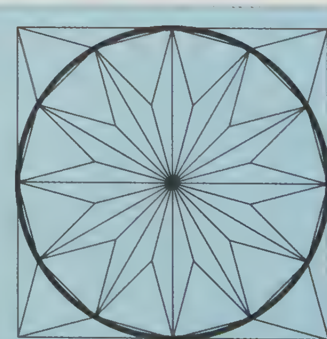
Solution for Problem 9.9: Let O be the center of the polygon. (You will be asked to prove that such a point exists as a Challenge Problem.) We can cut the regular polygon into n triangles by connecting the center to every vertex. One such triangle is $\triangle OAZ$, shown in the diagram. The altitude to side $\overline{AZ}$ of this triangle is the distance from O to a side of the polygon, given in the problem as k. Since AZ is also given as s, we have $[AZO] = (AZ)(OP)/2 = ks/2$. Finally, the polygon consists of n of these triangles, so the area of the whole polygon is $nks/2$. □

The distance from the center to the sides of a regular polygon is sometimes given the obscure name of the **apothem**. Noting that the perimeter of the polygon is ns, we can write our area formula as:

> **Important:** The area of a regular polygon is half its perimeter times the distance from the center of the polygon to a side.

As we have seen in earlier problems, this isn't always the fastest way to find the area. (Try it on the dodecagon above if you want to torment yourself.)

> **Sidenote:** One particularly fascinating dissection is that of a regular dodecagon shown at right. See if you can figure out how to use the dissection to prove that the area of the dodecagon is 3 times the square of the radius of the dodecagon's circumcircle, which is shown in bold in the diagram. This dissection of a regular dodecagon is used as the logo for the Art of Problem Solving Foundation, which supports problem solving activities for eager middle and high school students. You can find more information on the Foundation (and a graphical proof of the relationship described above) at www.artofproblemsolving.org.



Exercises

9.3.1 *ABCDEF* is a regular hexagon with side length 9.

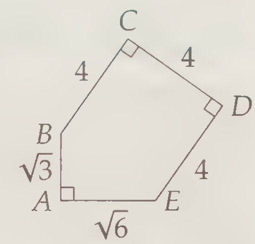
(a) Find AD.

(b) Find $[ABCDEF]$.

(c)★ Find AC.

9.3.2 Find a formula for the area of a regular hexagon with side length s.

9.3.3 What's wrong with pentagon *ABCDE* shown at right?



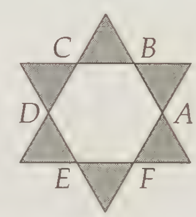
9.3.4 *ABCDEFGH* is a regular octagon with side length 8. Diagonals $\overline{AE}$ and $\overline{CG}$ meet at X. Point M is the midpoint of $\overline{AB}$.

(a) Find the area of *ABCDEFGH*.

(b) Find XM, the apothem of the octagon.

(c) Find XC. **Hints:** 22

9.3.5 Points A, B, C, D, E, and F are the vertices of a regular hexagon and also trisect the sides of the large equilateral triangles shown. Given that the area of *ABCDEF* is 24, what is the total area of the shaded regions? *(Source: MATHCOUNTS)*



9.3.6 We solved Problem 9.8 by extending the sides of a regular octagon to form a square. We didn't, however, prove that we form a square when we connect the points where these extensions meet. Fix this oversight by providing the proof. **Hints:** 564

9.3.7★ In this section we assumed that the long diagonals of a regular hexagon are concurrent. In this problem we fix this oversight by proving that these diagonals are concurrent.

(a) Let the hexagon be *ABCDEF* and let point O be the intersection of the bisectors of $\angle A$ and $\angle B$. Prove that $\triangle AOB$ is equilateral.

(b) Draw $\overline{OC}$. Prove that $\triangle BOC$ is equilateral.

(c) Prove that $\triangle COD$ is equilateral and that $\overrightarrow{AO}$ goes through D.

(d) Prove that the long diagonals of *ABCDEF* all meet at the same point.

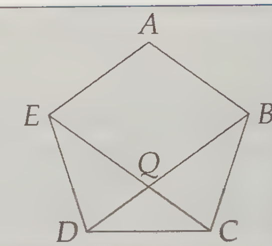
Sidenote: Back on page 251 we challenged you to find pairs of regular polygons such that the first can be dissected and rearranged to form the other. If you didn't find any, keep looking! Not only are there more than the pair we showed you, but in fact any two polygons with the same area can be dissected into the same set of pieces. Therefore, if we have two regular polygons with the same area, we can find some way to dissect one and rearrange the pieces to form the second. Of course, some of these dissections are very hard to find and may involve many pieces!

Many of these dissections are shown at Wolfram Research's MathWorld page; a link to these is provided on the links page described on page viii.

9.4 Polygon Problems

Problems

Problem 9.10: Diagonals $\overline{BD}$ and $\overline{CE}$ of regular pentagon $ABCDE$ meet at Q. Find $\angle BQE$.



Problem 9.11: In this problem we will find a formula for the number of diagonals in a polygon with n sides.

(a) How many diagonals does a square have?

(b) How many diagonals does a pentagon have?

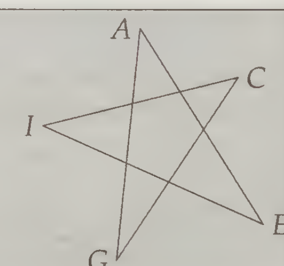
(c) How many diagonals does a hexagon have?

(d) Let A be a vertex of a polygon with n sides. How many diagonals of the polygon have A as a vertex?

(e) Use your answer to the previous part to find a formula for the number of diagonals in a polygon with n diagonals. Be sure to compare your result to your answers in the first 3 parts to check your formula.

Problem 9.12: $ABCDEFGH$ is a regular octagon. Find the ratio AE/BH. **Hints:** 422

Problem 9.13: In the star diagram, find the sum $\angle A + \angle C + \angle E + \angle G + \angle I$. (You cannot assume the pentagon in the middle of the star is regular!)



Problem 9.14: $ABCDEFGH$ is a regular octagon with side length 12. In this problem we will find the area of $ABDG$.

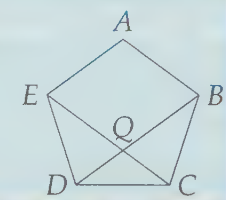
(a) What kind of quadrilateral is $ABDG$?

(b) Consider $FGDE$. Draw altitudes from F and E to $\overline{GD}$ to find GD. (And see if you can find GD in some other clever way!)

(c) Use your work from the last part as a guide to find the distance from B to $\overline{GD}$.

(d) Find $[ABDG]$.

Problem 9.10: Diagonals $\overline{BD}$ and $\overline{CE}$ of regular pentagon $ABCDE$ meet at Q. Find $\angle BQE$.



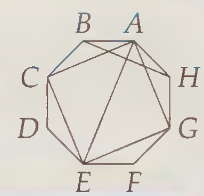
Solution for Problem 9.10: We start by finding the angles we can find. Each interior angle of a regular pentagon is 108°. Since $\triangle EDC$ is isosceles, $\angle DCE = \angle DEC = (180^\circ - 108^\circ)/2 = 36^\circ$. Similarly, we have $\angle CDB = 36^\circ$. Therefore, $\angle CQD = 180^\circ - \angle QDC - \angle QCD = 108^\circ$, so we have $\angle BQE = \angle CQD = 108^\circ$. □

Problem 9.11: Derive a formula for the number of diagonals in a polygon with n sides.

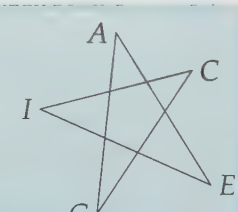
Solution for Problem 9.11: We consider a single vertex, A. A is connected to 2 other vertices by sides, leaving $n-3$ vertices to which A can be connected by diagonals. Since there are n vertices and each can be connected to $n-3$ others by diagonals, there appear to be $n(n-3)$ diagonals. However, this counts each diagonal twice, once for each endpoint. Therefore, we must divide by two to count each diagonal only once. This gives us $n(n-3)/2$ diagonals in an n-gon. □

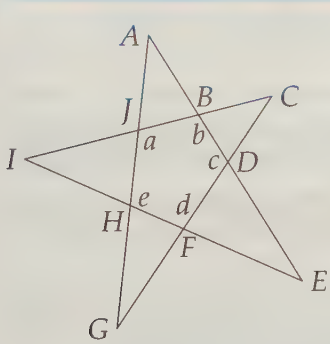
Problem 9.12: $ABCDEFGH$ is a regular octagon. Find the ratio AE/BH.

Solution for Problem 9.12: It's not immediately obvious how to compare BH to AE, but there are several other diagonals with the same length as $\overline{BH}$. Considering these, we see that four of them form square $AGEC$. Since $\overline{AE}$ is a diagonal of this square, we have $AE/BH = AE/AG = \sqrt{2}$. □



Problem 9.13: In the star diagram, find the sum $\angle A + \angle C + \angle E + \angle G + \angle I$. (You cannot assume the pentagon in the middle of the star is regular!)





Solution for Problem 9.13: All we have to work with is what we know about the sums of the angles in pentagons and in triangles. We label the angles in the pentagon as shown. We can find the non-star angles of the outer triangles in terms of these (for example, $\angle IJH = 180^\circ - a$). However, if we use a little ingenuity, we don't even need the exterior angles of $BDFHJ$. Consider the angles in each triangle like $\triangle AHE$. These must sum to 180°, so we have:

$$\begin{aligned}
\angle A + e + \angle E &= 180^\circ \\
\angle C + a + \angle G &= 180^\circ \\
\angle E + b + \angle I &= 180^\circ \\
\angle G + c + \angle A &= 180^\circ \\
\angle I + d + \angle C &= 180^\circ
\end{aligned}$$

We're looking for $\angle A + \angle C + \angle E + \angle G + \angle I$, and we know that $a + b + c + d + e = (180)(5-2) = 540°$, since these angles together make up the central pentagon. Therefore, our triangle equations beg to be added together, giving:

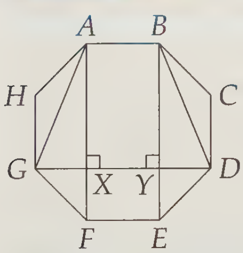
$$2(\angle A + \angle C + \angle E + \angle G + \angle I) + (a + b + c + d + e) = 5(180°).$$

Therefore, $2(\angle A + \angle C + \angle E + \angle G + \angle I) + 540° = 900°$, so

$$\angle A + \angle C + \angle E + \angle G + \angle I = 180°.$$

□

Problem 9.14: *ABCDEFGH* is a regular octagon with side length 12. Find the area of *ABDG*.



Solution for Problem 9.14: Our *ABDG* is a trapezoid (you'll prove this explicitly in the Exercises). We'd like to dissect the octagon into pieces we can handle, and we know altitudes of trapezoids can often be useful, so we draw diagonals $\overline{AF}$ and $\overline{BE}$. These are each perpendicular to $\overline{GD}$ and $\overline{FE}$ (you'll prove this fact in the Exercises, too).

Since $\overline{GD} \parallel \overline{FE}$, we have $\angle FGD = 180° - \angle GFE = 45°$, so $\triangle GXF$ is a 45-45-90 triangle. Therefore, $GX = GF/\sqrt{2} = 6\sqrt{2}$. Similarly, $YD = 6\sqrt{2}$. $XYEF$ is a rectangle, so $XY = FE = 12$. Therefore, base $\overline{GD}$ of our trapezoid has length $12 + 12\sqrt{2}$. Before we go through tricky computations to get AX, we can note that by symmetry we have $AF = GD$ (alternatively, we can use HL Congruence to show $\triangle AXG \cong \triangle DXF$), so $AX = AF - XF = DG - GX = 12 + 6\sqrt{2}$. Therefore, our area is:

$$\begin{aligned}[ABDG] = \frac{(AB + DG)(AX)}{2} &= \frac{(24 + 12\sqrt{2})(12 + 6\sqrt{2})}{2} \\ &= \frac{72(2 + \sqrt{2})(2 + \sqrt{2})}{2} = 36(6 + 4\sqrt{2}) = 216 + 144\sqrt{2}.\end{aligned}$$

□

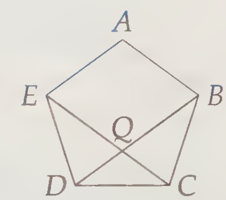
Concept: Most polygon problems are really quadrilateral or triangle problems. When stuck with a polygon, try dissecting it into smaller pieces.

In our last solution, we used **symmetry** to note that $AF = DG$. By symmetry, we essentially mean that the general properties of all the vertices are the same. For example, the distance from one vertex to a vertex three vertices away on the regular polygon (such as AF or DG) is the same no matter which initial vertex we are talking about. However, we couldn't say that '$AC = BE$ by symmetry' in the last problem, because AC is the distance between two vertices with one vertex between them, while BE is the distance between two vertices with two vertices (C and D) between them.

Concept: Symmetry can be a very useful tool in problems involving regular polygons!

In the Exercises that follow, you'll be asked to solve several problems that could be solved using symmetry. In addition to solutions involving symmetry, try to find explicit solutions using tools such as congruent and similar triangles.

Exercises



9.4.1 $ABCDE$ at right is a regular pentagon. Show that $AEQB$ is a rhombus.

9.4.2 $UVWXYZ$ is a regular hexagon with center O. $UOWABC$ is also a regular hexagon with $AB = 12$.

(a) Find $[UOWABC]$.

(b) Find $[UVWO]$.

9.4.3 All of the interior angles of octagon $EFGHIJKL$ are 135°, $EF = GH = IJ = KL = 2$ and $FG = HI = JK = LE = 6$.

(a) Find the perimeter of the octagon.

(b)★ Find the area of the octagon. **Hints:** 277

(c)★ Find EI. **Hints:** 406

9.4.4 Let $ABCDEF$ be a regular hexagon. Find the ratio of the area of $ABCE$ to the area of the hexagon. *(Source: ARML)* **Hints:** 74

9.4.5 $ABCDEFGH$ is a regular octagon. Prove that $ACEG$ is a square.

9.4.6 Explicitly prove that $ABDG$ of Problem 9.14 is a trapezoid.

9.4.7 Explicitly prove that $\angle FXG$ is a right angle in Problem 9.14.

9.4.8 The midpoints of the sides of a regular hexagon $ABCDEF$ are joined in order to form a smaller regular hexagon. What fraction of the area of $ABCDEF$ is enclosed by the smaller hexagon? *(Source: AMC 12)*

9.5 Construction: Regular Polygons

Problems

Problem 9.15: Construct a square.

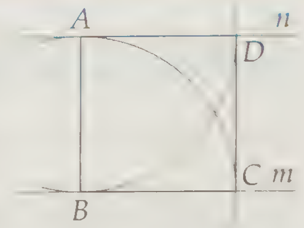
Problem 9.16: Construct a regular hexagon.

Problem 9.17: Construct a regular octagon.

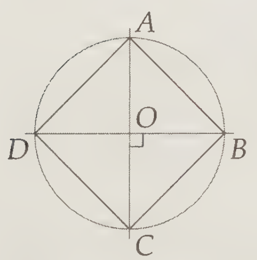
We already know how to construct a regular triangle, so we'll move on to a regular quadrilateral.

Problem 9.15: Construct a square.

Solution for Problem 9.15: Solution 1: A natural solution is to start with one side, $\overline{AB}$, then construct the others. The other two vertices must be on lines through A and B that are perpendicular to $\overline{AB}$, so we construct lines m and n perpendicular to $\overline{AB}$ as shown. We can then find C on m such that $AB = BC$ by constructing a circle with center B and radius AB. Where this circle meets m gives us C. Similarly, we construct a circle with center A and radius AB to find D on n such that $AD = AB$.



We'll leave the proof that this construction does indeed result in $ABCD$ being a square for an Exercise.



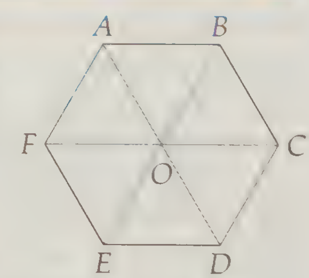
Solution 2: We can find an even slicker construction by thinking a little more about squares. For example, we know that the diagonals of a square are perpendicular and they meet at a point that is equidistant from all four vertices. In other words, the intersection of the diagonals of a square is the center of the circumcircle of the square.

We can use this observation to start with a circle to make our square. We draw a circle with center O. We then draw two perpendicular lines through O and label the points where these lines hit the circle A, B, C, and D. Since the four triangles that meet at O are congruent 45-45-90 triangles, we have $AB = BC = CD = DA$. Moreover, each angle of $ABCD$ equals $45° + 45° = 90°$. Therefore, $ABCD$ is a square. □

Let's try using this "circumcircle of a regular polygon" idea on another polygon.

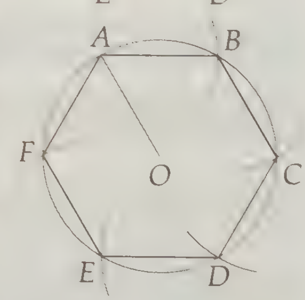
Problem 9.16: Construct a regular hexagon.

Solution for Problem 9.16: Solution 1: We already know how to make an equilateral triangle, and a regular hexagon can be built from six equilateral triangles. Therefore, we can start with equilateral $\triangle AOB$, then construct equilateral $\triangle OBC$ on $\overline{OB}$, then construct equilateral $\triangle OCD$ on $\overline{OC}$, and so on. After making 6 equilateral triangles, we'll have regular hexagon $ABCDEF$.



But that's an awful lot of work. There must be a faster way.

Solution 2: Since the circumcircle worked so well with the square, we'll try it with the hexagon. Since the long diagonal of a regular hexagon is equal both to twice the side length of the hexagon and to the diameter of the circumcircle of the hexagon, the radius of the circumcircle equals the length of a side of the regular hexagon. Therefore, if we draw a circle with radius $\overline{OA}$, the vertices of a regular hexagon inscribed in the circle will be at intervals of length OA around the circle.



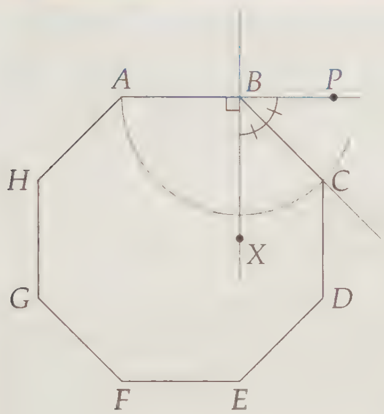
So, we draw an arc with center A and radius OA to find point B on the circle. Then we draw an arc with center B and radius OA to get C, and so on around the circle. The resulting $ABCDEF$ is a regular hexagon.

We can prove $ABCDEF$ is regular by noting that each of the triangles formed by connecting consecutive vertices to the center is equilateral. For example, $AO = BO = AB$ means $\triangle AOB$ is equilateral. Similarly, so is $\triangle BOC$, so $\angle ABC = 2(60^\circ) = 120^\circ$. In the same way, we see that each of the angles $ABCDEF$ is 120°, and each of the sides is equal (to the radius of the circle). Therefore, $ABCDEF$ is indeed regular. □

Concept: The circumcircle of a regular polygon can be extremely useful in working with the polygon.

We've already solved a few regular octagon problems, so you should be ready to construct one now.

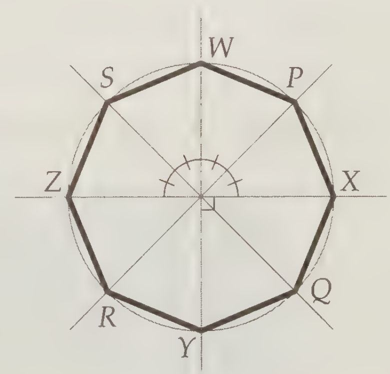
Problem 9.17: Construct a regular octagon.



Solution for Problem 9.17: Solution 1: Looking at two consecutive sides of a regular octagon, we see that if we can construct a 135° angle, we can construct a regular octagon. Since $135^\circ = 90^\circ + 45^\circ$ and we know how to make both a 90° and 45° angle, we can make a 135° angle.

Specifically, we start with $\overline{AB}$, which we continue to point P. We then draw a line through B perpendicular to $\overline{AB}$ and let point X be on this line inside the octagon as shown. We bisect $\angle XBP$ and let C be the intersection of this bisector and the circle centered at B with radius BA. Therefore, $AB = BC$ and $\angle ABC = \angle ABX + \angle XBC = 135^\circ$, so C is another vertex of our octagon. We can then copy $\angle ABC$ to make $\angle BCD$, and so on, completing the regular octagon.

But again, that's a lot of work!



Solution 2: Rather than look at the octagon from the point of view of the sides, let's look at it from the center's perspective. Again, we can use the circumcircle. We start by creating square $WXYZ$ by constructing perpendicular lines that meet at the center of the circle. Where these lines meet the circle give us the vertices of square $WXYZ$. The other four vertices of our regular octagon must also form a square, with its vertices at the midpoints of the arcs connecting the vertices of $WXYZ$. To construct these points, we just bisect the right angles formed by $\overline{WY}$ and $\overline{XZ}$. Where these bisectors meet the circle give us the remaining vertices of our regular octagon, $WPXQYRZS$. □

Exercises

9.5.1 Prove that our first solution to Problem 9.15 does produce a square.

9.5.2 Prove that our second solution to Problem 9.17 does produce a regular octagon.

9.5.3 Construct a regular dodecagon.

9.6 Summary

Definitions: A **polygon** is a closed planar figure with line segments as boundaries. As with triangles and quadrilaterals, the segments that form the boundaries are the **sides**, which meet at the **vertices** of the polygon. A **diagonal** is a segment that is not a side, but connects two vertices of a polygon. A **regular polygon** is a polygon in which all the angles have the same measure and all the sides have the same length.

Important:

- The area of a regular polygon is half its perimeter times the distance from the center of the polygon to a side.
- In a polygon with n sides, the measures of the interior angles have a sum of $(n-2)(180°)$ and the measures of the exterior angles have a sum of $360°$.

The names of common polygons and the interior and exterior angle measures for common regular polygons are below:

Name	# of Sides	Int. ∠ Measure	Ext. ∠ Measure
triangle	3	60°	120°
quadrilateral	4	90°	90°
pentagon	5	108°	72°
hexagon	6	120°	60°
octagon	8	135°	45°
nonagon	9	140°	40°
decagon	10	144°	36°
dodecagon	12	150°	30°

Problem Solving Strategies

Concepts:

- Sometimes thinking about the exterior angles of a polygon offers a simpler approach than thinking about the interior angles!
- Break complicated areas into pieces you can handle. Sometimes we can view a desired area as the 'leftover' portion from having simple pieces taken away from a simple starting figure.
- Many problems involving regular hexagons can be tackled by dissecting the hexagons into equilateral triangles.
- Symmetry can be a very useful tool in problems involving regular polygons.

Continued on the next page. . .

Concepts: . . . *continued from the previous page*

- Most polygon problems are essentially quadrilateral and triangle problems. When stuck with a polygon, try dissecting it into quadrilaterals and triangles.
- The circumcircle of a regular polygon can be extremely useful in working with the polygon.

REVIEW PROBLEMS

9.18 How many sides does a regular polygon with interior angles that measure 140° have?

9.19 What is the measure of each interior angle of a regular polygon that has 36 sides?

9.20 Find the area of a regular hexagon with side length 4.

9.21 How many sides does a regular polygon with exterior angles that each measure 6° have?

9.22 Find the area of a regular octagon with side length 6.

9.23 The measure of each exterior angle of a regular polygon is 1/8 the measure of an interior angle. How many sides does the polygon have?

9.24 *EFGHIJ* is a regular hexagon. Find EH/EG. **Hints:** 207

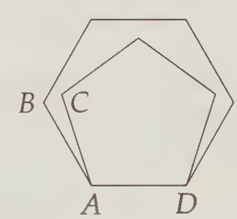
9.25 A square and a regular hexagon are drawn with the same side length. If the area of the square is $\sqrt{3}$, what is the area of the hexagon? *(Source: HMMT)*

9.26 In a given octagon, four angles have measure x degrees and two have measure $2x$ degrees. The other two angles are right angles. Is the octagon concave or convex?

9.27 A given polygon has 27 diagonals. How many sides does the polygon have?

9.28 *ABCDEFGH* is a regular octagon with $AB = 8$. Find AC, AD, and AE.

9.29 A regular pentagon and a regular hexagon share a common side $\overline{AD}$, as shown. What is the degree measure of $\angle BAC$? *(Source: MATHCOUNTS)*



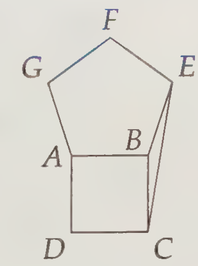
9.30 The number of diagonals in a regular polygon is equal to the number of sides. What is the number of degrees in the sum of all the interior angles of the polygon? *(Source: MATHCOUNTS)*

9.31 *ABCDEFGHIJ* is a regular decagon. Find appropriate congruent triangles (and prove they're congruent) to prove each of the following:

(a) $AC = AI$.

(b) $AD = AH$.

(c) $AE = CG$.



9.32 Let $ABCD$ be a square and let $ABEFG$ be a regular pentagon in the diagram at right. Find $\angle BCE$.

9.33 $ABCDEFGH$ is a regular octagon. Diagonals $\overline{AE}$ and $\overline{CG}$ meet at X. Point M is the midpoint of $\overline{AB}$.

(a) Find $\angle XAB$.

(b) Find $\angle MXA$.

(c) Find $\angle MXD$.

Challenge Problems

9.34 $A_1A_2A_3 \cdots A_n$ is a regular n-gon.

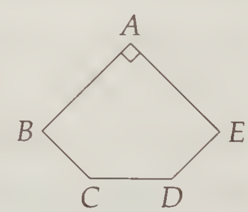
(a) Prove that $A_1A_2A_3A_4$ is a trapezoid.

(b) Prove that $\angle A_2A_1A_4 = 360^\circ/n$. *(Source: ARML)*

9.35 In our discussion of regular polygons, we assumed that every regular polygon has a point that is equidistant from all the vertices. We called this point the 'center' of the polygon.

(a) Prove that inside every regular polygon $\mathcal{P}$ there is a point O that is equidistant from all of the vertices of $\mathcal{P}$. **Hints:** 120

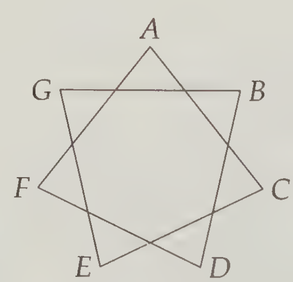
(b) Let O be the center of regular polygon $\mathcal{P}$, and A be one of the vertices of $\mathcal{P}$. Prove that $\overline{OA}$ bisects an interior angle of $\mathcal{P}$.



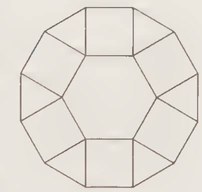
9.36 The pentagon $ABCDE$ at right has a right angle at A, $AB = AE$, and $ED = DC = CB = 1$. If $BE = 2$ and $\overline{BE} \parallel \overline{CD}$, what is the area of the pentagon? **Hints:** 191

9.37 The total number of degrees in the sum of the interior angles of two regular polygons is 1980. The sum of the number of diagonals in the two polygons is 34. What is the positive difference between the numbers of sides of the two polygons? *(Source: MATHCOUNTS)* **Hints:** 245

9.38 Find the sum of the angles at the points of the 7-pointed star $ABCDEFG$ shown. Do not assume that the heptagon in the center is regular. **Hints:** 408, 107



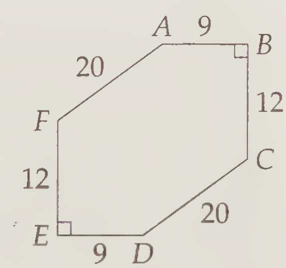
9.39 A regular hexagon and an equilateral triangle have the same perimeter. What is the ratio of the area of the hexagon to the area of the triangle?



9.40 In a patio, a pattern is determined by regular hexagonal tiles, square tiles and equilateral triangular tiles as shown in the diagram. If the area of each hexagonal tile is 96 in^2, what is the number of square inches in the area of each square tile? What is the area of the whole tiled region shown? *(Source: MATHCOUNTS)* **Hints:** 362

9.41 Given that $ABCDEF$ is a regular hexagon with side length 12, find the area of $\triangle ACE$.

9.42 The interior angles of a convex polygon form an arithmetic progression with a common difference of 4°. Determine the number of sides of the polygon if its largest interior angle is 172°. *(Source: USAMTS)* **Hints:** 123



9.43 Opposite sides of hexagon $ABCDEF$ shown are parallel. Given that $AD = 25$ and the side lengths of the hexagon are as shown, find $[ABCDEF]$. **Hints:** 15

9.44 Determine with proof the number of positive integers n such that a convex regular polygon with n sides has interior angles whose measures, in degrees, are integers. *(Source: USAMTS & AHSME)* **Hints:** 571

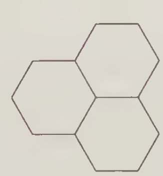
9.45

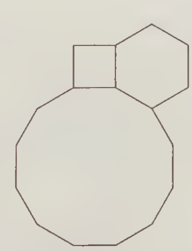
(a) What is the largest possible number of interior angles of a convex pentagon that can have measure 90°? **Hints:** 225

(b) What is the largest possible number of interior angles of a convex decagon that can have measure 90°?

(c) Find a formula in terms of n for the largest number of interior angles of a convex n-gon that can have measure 90°.

9.46★ It is possible to surround a point with three regular hexagons with side length 1 as shown at left below such that each hexagon shares a side with the other two hexagons. At right below, we see that it is also possible to surround a point with a regular hexagon, a square, and a regular dodecagon such that each has side length 1 and each of the polygons shares a side with the other two.

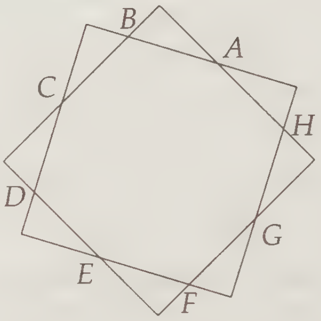




Find all other groups of three regular polygons with side length 1 that can surround a point such that each polygon shares a side with the other two. Prove that you have found all possible groups of three polygons. **Hints:** 68, 259

9.47★ Let ABC be a right triangle, with $\angle ACB = 90^\circ$. Let P_A, P_B, and P_C be regular pentagons, with side lengths BC, CA, and AB, respectively. Prove that $[P_A] + [P_B] = [P_C]$. **Hints:** 28, 414

9.48★ The two squares shown share the same center O and have sides of length 1. The length of $\overline{AB}$ is 43/99. Find the area of octagon $ABCDEFGH$ formed by connecting the 8 points where the two squares intersect as shown. *(Source: AIME)* **Hints:** 56, 442, 233

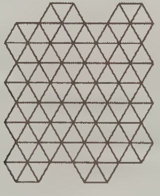


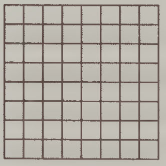
Extra!

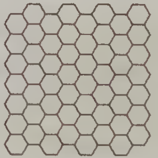
Have you ever seen a bathroom floor tiled with regular pentagons? How about with regular hexagons? Heptagons? Why do squares, triangles, and hexagons seem to show up everywhere, while the other regular polygons don't get much use?

Like lots of other things in geometry, it all boils down to angles. Now that you know how to find the angle in a regular polygon, you can answer this problem for yourself! Take a look at a tiling that does work, such as covering the plane with squares (like a piece of graph paper). At each vertex, exactly four squares meet. The angle of each square is 90°, so the four angles at each vertex add up to 360°.

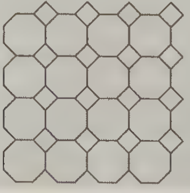
If you solved Problem 9.46, you should be able to quickly identify which regular polygons can be used to tile a plane all by themselves. We call such a regular **tiling** of the plane a **tessellation**. The three tessellations that use only a single regular polygon over and over are shown below. These are called **regular tessellations**.

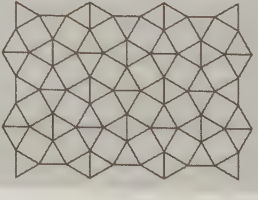




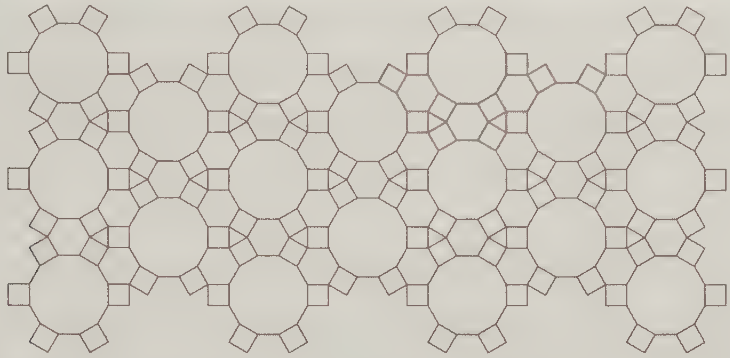


As we saw in Problem 9.46, we have many more possibilities if we allow ourselves to use more than one type of polygon. Shown below are two **semiregular tessellations**, in which the same set of regular polygons surround each vertex in the tessellation. There are eight such tessellations (not including the regular tessellations); see if you can find the other six.

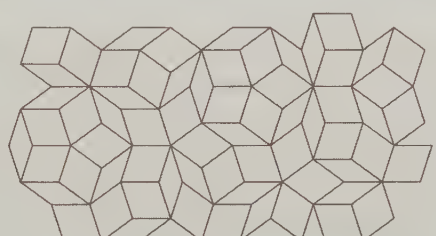




Extra! If we no longer restrict ourselves to having the same polygons around each point, we have **demiregular tessellations**, one of which is shown below.

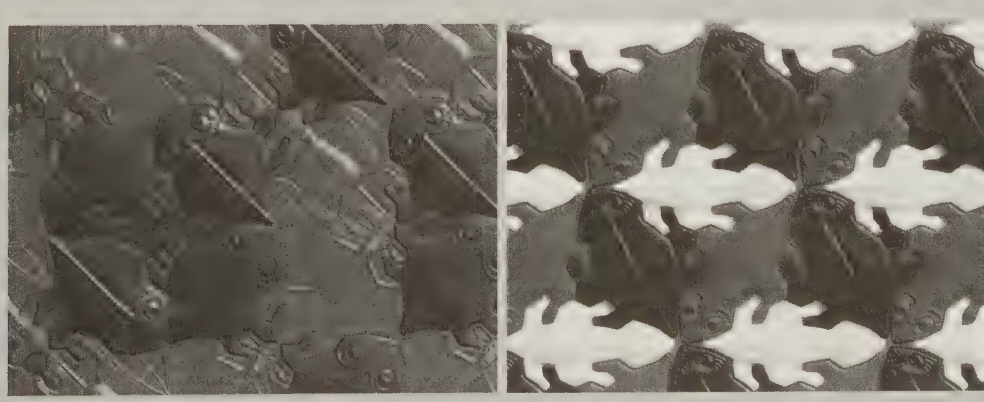


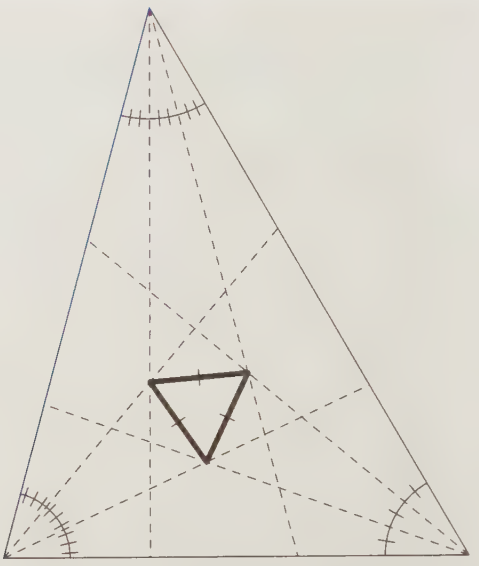
Of course, we need not only use regular polygons in building tilings. Moreover, tilings need not be periodic (meaning having the same pattern over and over). **Penrose tilings** consist of two quadrilaterals that are used as tiles to make fascinating non-periodic tilings. An example is shown below.



We don't have to restrict ourselves to simple geometric figures to tile a whole plane. The Dutch artist M. C. Escher produced thousands of tessellations using various animal shapes and other whimsical designs. Rather than being satisfied with just a tessellation, Escher often wove his tessellations into very engaging pieces of art. You'll find links to some of Escher's art on the links list described on page viii.

Tessellations have even made their way into puzzles and games, as the two pictures of the tessellation game *Bats and Lizards* below show. Perhaps you can see how these patterns were created from the regular tessellations on the previous page!





Morley's First Triangle

The worst form of inequality is to try to make unequal things equal. – Aristotle

CHAPTER 10 Geometric Inequalities

In this chapter we will explore three geometric inequalities involving triangles. First, we will examine how the order of the lengths of the sides is related to the order of the measures of the angles in a triangle. Then, we will learn how to tell if a triangle is acute or obtuse just by considering its side lengths. Finally, we will explore the most widely used geometric inequality of all – the Triangle Inequality. Like many of the most useful mathematical tools, the Triangle Inequality is so simple it's almost obvious, but it can be used to develop many non-obvious solutions to complex problems.

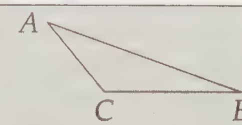
10.1 Sides and Angles of a Triangle

Problems

Problem 10.1: Draw a few triangles. Label each triangle $\triangle ABC$, and compare the order of the lengths AB, BC, and AC (from shortest to longest) to the order of the measures of the angles $\angle A$, $\angle B$, and $\angle C$. Make a guess about how the order of the lengths of the sides is related to the order of the measures of the angles in a triangle. Try to prove your guess always works before continuing this section! (Don't forget, you have to consider acute, right, and obtuse triangles.)

Problem 10.2: Prove that the hypotenuse of a right triangle is longer than each of the other two sides of the triangle.

Problem 10.3: In this problem, we will prove that in an obtuse triangle, the side opposite the obtuse angle is the longest side of the triangle. We start with $\triangle ABC$ with obtuse $\angle C$ as shown.



(a) We already have information about the longest side in a right triangle. Try building a right triangle with $\overline{AB}$ as a hypotenuse to show that $AB > BC$. Can you use a similar approach to show $AB > AC$?

(b) Try finding a second solution by drawing a right triangle with $\overline{BC}$ as a leg.

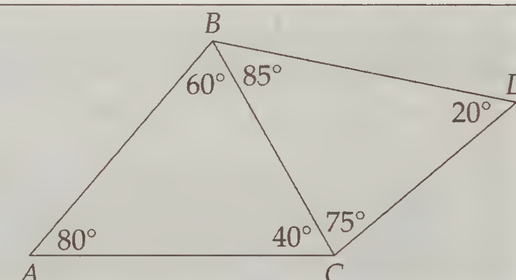
Problem 10.4: Let $\triangle ABC$ be acute with $\angle B > \angle C$. We will show that $AC > AB$.

(a) Draw altitude $\overline{AX}$ and find the point D on $\overleftrightarrow{BC}$ such that $AD = AC$ (C and D are different points). Show that $\angle DAX > \angle BAX$. Use this inequality to show that D is beyond B on $\overrightarrow{XB}$.

(b) Prove that $AC > AB$.

Problem 10.5: Let $\triangle ABC$ be acute. Show that if $AC > AB$, then $\angle B > \angle C$.

Problem 10.6: Given the angles shown in the diagram, order the lengths AB, BD, CD, BC, and AC from least to greatest. Note that the figure is not drawn to scale!



Problem 10.7: In $\triangle ABC$, the median $\overline{AM}$ is longer than $BC/2$. Prove that $\angle BAC$ is acute.

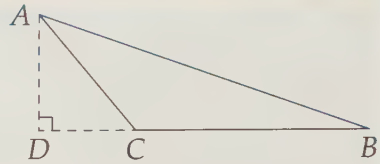
After looking at hundreds, if not thousands, of triangles while working to this point in the book, you probably think that the largest side of a triangle is opposite the triangle's largest angle, and the shortest side is opposite the smallest angle. Good instincts! We'll have to work through a few cases to prove it, though.

Problem 10.2: Prove that the hypotenuse of a right triangle is longer than each of the other two sides of the triangle.

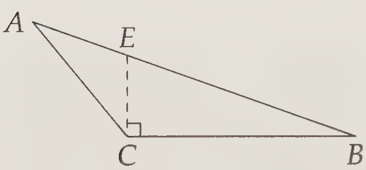
Solution for Problem 10.2: Let $\overline{AC}$ be the hypotenuse of $\triangle ABC$. The Pythagorean Theorem tells us $AC^2 = AB^2 + BC^2$. Since AB^2 and BC^2 are both positive, we must have $AC > AB$ and $AC > BC$. Hence, the hypotenuse of a right triangle is longer than each of the other two sides of the triangle. □

Taking care of the longest side in a right triangle was pretty easy. Now let's take a look at obtuse triangles.

Problem 10.3: Prove that in an obtuse triangle, the side opposite the obtuse angle is the longest side of the triangle.



Solution for Problem 10.3: Let $\angle C$ in $\triangle ABC$ be obtuse. We already know that the hypotenuse of a right triangle is the longest side of the triangle, so we try building a right triangle we can use. We want to show that $\overline{AB}$ is the longest side of our triangle, so we make it the hypotenuse of a right triangle by drawing altitude $\overline{AD}$ from A as shown. Since $\overline{AB}$ is the hypotenuse of $\triangle ABD$, we have $AB > BD$. Since $BD > BC$, we have $AB > BD > BC$. Similarly, we can draw an altitude from B to show that $AB > AC$.



We could also have built a right triangle with $\overline{BC}$ as a leg by drawing a line through C perpendicular to $\overline{BC}$ as shown. Since $\triangle ECB$ is a right triangle, we have $EB > BC$. Since $AB > EB$, we have $AB > EB > BC$. Similarly, we can show that $AB > AC$.

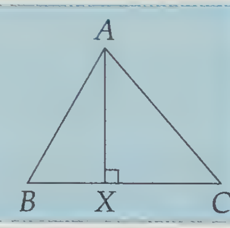
Therefore, in an obtuse triangle, the side opposite the obtuse angle is the longest side of the triangle. □

Now we're ready to deal with acute angles.

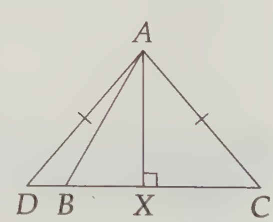
Problem 10.4: Let $\triangle ABC$ be acute with $\angle B > \angle C$. Show that $AC > AB$.

Solution for Problem 10.4: See if you can find what's missing in this 'proof':

Bogus Solution: We start by drawing altitude $\overline{AX}$. Since $\angle B > \angle C$, we have $\angle BAX < \angle CAX$. Therefore, $BX < CX$. Since $AB^2 = AX^2 + BX^2$ and $AC^2 = AX^2 + CX^2$, our $BX < CX$ tells us that $AB < AC$.



The problem here is that we haven't justified that $BX < CX$ claim. Why does $\angle BAX < \angle CAX$ mean that $BX < CX$ (and no, you can't just say 'It's obvious!')?



It's not clear how we can build a right triangle to directly compare AB and AC, but we can build an obtuse one. We draw altitude $\overline{AX}$ and locate point D on $\overleftrightarrow{BC}$ such that $AD = AC$. Since $AD = AC$, we have $\angle ADX = \angle C$. Therefore, $\angle D < \angle ABC$. Since $\angle D = 90^\circ - \angle DAX$ and $\angle ABC = 90^\circ - \angle BAX$, we have $90^\circ - \angle DAX < 90^\circ - \angle BAX$, so $\angle BAX < \angle DAX$. Therefore, D is beyond B on $\overrightarrow{CB}$ as shown at right.

Since $\angle ABC$ is acute, $\angle ABD$ is obtuse. This tells us $AD > AB$. Since $AD = AC$, we have the desired $AC > AB$. □

Problem 10.5: Let $\triangle ABC$ be acute. Show that if $AC > AB$, then $\angle B > \angle C$.

Solution for Problem 10.5: What's wrong with this solution:

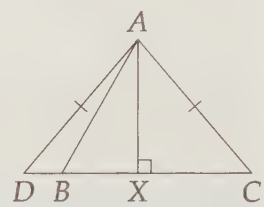
Bogus Solution: If $\angle B > \angle C$, then $AC > AB$. Since we know that $AC > AB$, we have $\angle B > \angle C$.

The problem with that 'proof' is that it is exactly the same as saying 'If I live in California, I live in America. Since I live in America, I live in California.' As we've noted before, we cannot assume that the converse of a true statement is true! We must prove a statement and its converse separately. Here, we will present two solutions.

Solution 1: Although our Bogus Solution was indeed quite bogus, we can still use what we already know about angles and sides. Specifically, consider three cases:

Case 1: $\angle B > \angle C$. As we have seen, if $\angle B > \angle C$, then $AC > AB$.
Case 2: $\angle B = \angle C$. If $\angle B = \angle C$, then $AC = AB$.
Case 3: $\angle B < \angle C$. As we have seen, if $\angle B < \angle C$, then $AC < AB$.

We are given that $AC > AB$. The only one of these cases that leads to $AC > AB$ is Case 1, so we can deduce that $\angle B > \angle C$. Make sure you see why this solution works, but the bogus one doesn't!



Solution 2: We can backtrack through our solution to Problem 10.4. We find point D on $\overrightarrow{CB}$ such that $AD = AC$. To show that D is beyond B on $\overrightarrow{CB}$, we first draw altitude $\overline{AX}$. Since we are given $AC > AB$ and we have both $AC^2 = AD^2 = AX^2 + XD^2$ and $AB^2 = AX^2 + XB^2$, we must have $XD > XB$.

Now we can use $\triangle ABD$ to show that $\angle C < \angle ABC$. From $\triangle ABD$, we have $\angle ADB + \angle ABD = 180^\circ - \angle DAB$. Therefore, $\angle ADB + \angle ABD < 180^\circ$. Since $\angle ADB = \angle C$ and $\angle ABD = 180^\circ - \angle ABC$, substitution into $\angle ADB + \angle ABD < 180^\circ$ gives us

$$\angle C + (180^\circ - \angle ABC) < 180^\circ,$$

so $\angle C < \angle ABC$ as desired. □

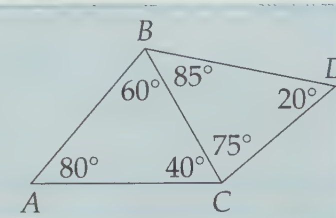
We can summarize all of our discoveries thus far in this section very simply:

Important: In any triangle, the longest side is opposite the largest angle and the shortest side is opposite the smallest angle. The middle side, of course, is therefore opposite the middle angle.

In other words, in $\triangle ABC$, $AB \geq AC \geq BC$ if and only if $\angle C \geq \angle B \geq \angle A$.

Let's try using these facts on a couple problems.

Problem 10.6: Given the angles as shown, order the lengths AB, BD, CD, BC, and AC from least to greatest. (Note: The diagram is not drawn to scale!)

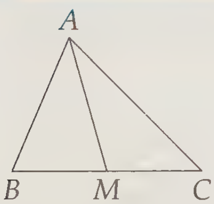


Solution for Problem 10.6: First we focus on $\triangle ABC$. Since $\angle A > \angle B > \angle C$, we have $BC > AC > AB$. Then, we turn to $\triangle BDC$. Since $\angle B > \angle C > \angle D$, we have $CD > BD > BC$. We got lucky! BC is the smallest in one inequality string and the largest in the other, so we can put the inequalities together:

$$CD > BD > BC > AC > AB.$$

□

Problem 10.7: In $\triangle ABC$ the median $\overline{AM}$ is longer than $BC/2$. Prove that $\angle BAC$ is acute.



Solution for Problem 10.7: We start with a diagram. We'd like to prove something about an angle, but all we are given is an inequality regarding lengths. So, we use the length inequality to get some angle inequalities to work with. Specifically, since $AM > BM$ in $\triangle ABM$ and $AM > MC$ in $\triangle ACM$ (because $AM > BC/2$ and $BM = MC = BC/2$), we have

$$\begin{aligned} \angle B &> \angle BAM \\ \angle C &> \angle CAM. \end{aligned}$$

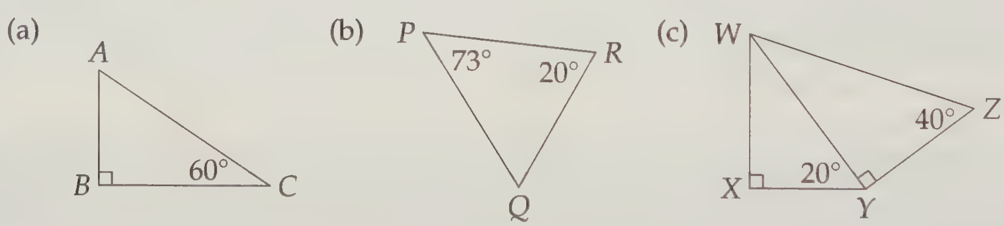
We want to prove something about $\angle BAC$, which equals $\angle BAM + \angle CAM$, so we add these two inequalities to give $\angle B + \angle C > \angle BAC$. Since $\angle B + \angle C + \angle BAC = 180°$, we can write $\angle B + \angle C > \angle BAC$ as

$$180° - \angle BAC > \angle BAC.$$

This gives us $180° > 2\angle BAC$, so $90° > \angle BAC$. Therefore, $\angle BAC$ is acute. □

Exercises

10.1.1 In each of the parts below, order all the segments in the diagram from longest to shortest. (The diagrams are not drawn to scale!)



10.1.2 In isosceles triangle $\triangle PQR$, we have $\angle P = 54°$ and $PQ = PR$. Which is longer, $\overline{PQ}$ or $\overline{QR}$?

10.1.3★ Point X is on $\overleftrightarrow{AB}$. Point C is given such that $\angle AXC = \angle ACB = 100°$. Show that X is on segment $\overline{AB}$, rather than beyond B on $\overrightarrow{AB}$ or beyond A on $\overrightarrow{BA}$. **Hints:** 579, 483

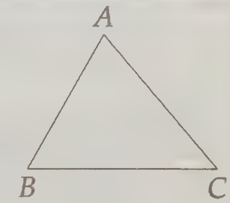
10.2 Pythagoras – Not Just For Right Triangles?

Back in Section 6.1, we learned that if $AC^2 + BC^2 = AB^2$, then $\angle ACB$ is a right angle. But what if $AC^2 + BC^2$ doesn't equal AB^2?

Problems

Problem 10.8: Draw lots of different triangles. Make a conjecture about what must be true if $AB^2+BC^2 > AC^2$, and what must be true if $AB^2 + BC^2 < AC^2$. Try to prove your conjectures before continuing!

Problem 10.9: Let $\angle C$ in $\triangle ABC$ be acute as shown. We will prove that



$$AC^2 + BC^2 > AB^2.$$

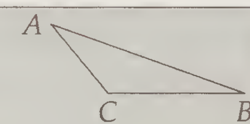
(a) Draw altitude $\overline{AX}$. Find an expression for AB^2.

(b) Show that $AC > AX$.

(c) Show that $AB^2 < AC^2 + BC^2$.

(d) Our diagram assumes that $\angle B$ is also acute. What if it isn't?

Problem 10.10: Let $\angle C$ in $\triangle ABC$ be obtuse as shown. We will prove that



$$AB^2 > AC^2 + BC^2.$$

(a) Draw altitude $\overline{AD}$ to $\overleftrightarrow{BC}$.

(b) Find an expression for AB.

(c) Find AD in terms of AC and CD, and find BD in terms of BC and CD.

(d) Use the previous two parts to show that $AB^2 > AC^2 + BC^2$.

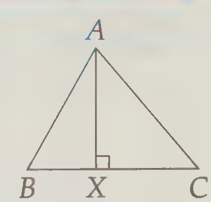
Problem 10.11: Prove that the converses of the facts we proved in Problems 10.9 and 10.10 are also true. In other words, prove that if $AB^2 > AC^2 + BC^2$, then $\angle C$ is obtuse, and prove that if $AB^2 < AC^2 + BC^2$, then $\angle C$ is acute.

Problem 10.12: In $\triangle XYZ$, we have $XY = 11$, $YZ = 14$. For how many integer values of XZ is $\triangle XYZ$ acute?

We know that if $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$, we have $AC^2+BC^2 = AB^2$. Intuitively, it seems that if $\triangle ABC$ is acute, then $AC^2+BC^2 > AB^2$, and if it is obtuse with $\angle C > 90°$, then $AC^2+BC^2 < AB^2$. But intuitively isn't good enough for us, is it?

Problem 10.9: Let $\angle C$ in $\triangle ABC$ be acute. Prove that $AC^2 + BC^2 > AB^2$.

Solution for Problem 10.9: Those sums of squares of sides send us hunting for right triangles. Drawing altitude $\overline{AX}$ gives us a couple. From right triangle $\triangle ABX$, we have $AB^2 = AX^2 + BX^2$. Clearly $BX < BC$, and right triangle $\triangle ACX$ gives us $AX < AC$. Therefore, we have

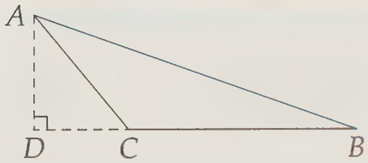


$$AB^2 = AX^2 + BX^2 < AC^2 + BC^2.$$

You might be wondering 'Where did we use the fact that $\angle C$ is acute?' Look closely – why must X be on $\overline{BC}$? If $\angle B$ and $\angle C$ are both acute, then altitude $\overline{AX}$ must meet $\overline{BC}$. Our proof, therefore, doesn't address the case where $\angle B$ is obtuse. However, in this case we have $AC > AB$, so obviously $AC^2 + BC^2 > AB^2$. □

Concept: One of the most useful approaches to solving new problems is to try to think of similar-looking problems you know how to handle. For example, our main step in the solution to Problem 10.9 was thinking to build right triangles once we saw the sum of squares of sides.

Problem 10.10: Let $\angle C$ in $\triangle ABC$ be obtuse. Prove that $AB^2 > AC^2 + BC^2$.



Solution for Problem 10.10: Once again, we start by drawing an altitude ($\overline{AD}$) to build right triangles. From right triangle $\triangle ABD$, we have $AB^2 = AD^2 + BD^2$. In order to get a BC term and an AC term in our equation, we note that $AD^2 = AC^2 - CD^2$ from $\triangle ADC$, and $BD = BC + CD$. Therefore, we have

$$\begin{aligned} AB^2 &= AD^2 + BD^2 \\ &= AC^2 - CD^2 + (BC + CD)^2 \\ &= AC^2 - CD^2 + BC^2 + 2(BC)(CD) + CD^2 \\ &= AC^2 + BC^2 + 2(BC)(CD) \end{aligned}$$

Since $2(BC)(CD)$ must be positive, we have $AB^2 > AC^2 + BC^2$. □

Make sure you see where in our solution we used the fact that $\angle ACB$ is obtuse.

Concept: If you don't use all the given information in a solution, proofread it closely to make sure you haven't made a mistake! Sometimes you won't need all the information you're given, but often if you haven't used it all, it's because you made a mistake somewhere.

Furthermore, when you're stuck on a problem, read it again and see if there is any information you haven't used yet!

In Problem 6.7 on page 138 we proved the converse of the Pythagorean Theorem; namely, that if the sides of a triangle satisfy $a^2 + b^2 = c^2$, then the triangle is right. Now we investigate if the converses of the inequalities we have just discovered are true.

Problem 10.11: Prove that the converses of the facts we proved in Problems 10.9 and 10.10 are also true. In other words, prove that if $AB^2 > AC^2 + BC^2$, then $\angle C$ is obtuse, and prove that if $AB^2 < AC^2 + BC^2$, then $\angle C$ is acute.

Solution for Problem 10.11: We can mirror our first solution to Problem 10.5 by considering three different cases:

Case 1: $\angle C$ is acute. If $\angle C$ is acute, then $AB^2 < AC^2 + BC^2$.
Case 2: $\angle C$ is right. If $\angle C$ is right, then $AB^2 = AC^2 + BC^2$.
Case 3: $\angle C$ is obtuse. If $\angle C$ is obtuse, then $AB^2 > AC^2 + BC^2$.

Each possible triangle $\triangle ABC$ must fall under exactly one of these cases. Only in the first case is $AB^2 < AC^2 + BC^2$, so if $AB^2 < AC^2 + BC^2$, we can conclude that $\angle C$ is acute. Similarly, only in the third case is $AB^2 > AC^2 + BC^2$, so we can conclude that $\angle C$ is obtuse whenever $AB^2 > AC^2 + BC^2$.

In the Exercises, you'll have a chance to provide a second 'geometric' argument, as we did in Solution 2 to Problem 10.5. □

We can now relate the squares of the sides of a triangle to the angles of the triangle:

Important: $\angle C$ of $\triangle ABC$ is acute if and only if $AB^2 < AC^2 + BC^2$.
$\angle C$ of $\triangle ABC$ is right if and only if $AB^2 = AC^2 + BC^2$.
$\angle C$ of $\triangle ABC$ is obtuse if and only if $AB^2 > AC^2 + BC^2$.

Let's try using some of this information on a problem.

Problem 10.12: In $\triangle XYZ$, we have $XY = 11$ and $YZ = 14$. For how many integer values of XZ is $\triangle XYZ$ acute?

Solution for Problem 10.12: In order for $\triangle XYZ$ to be acute, all three of its angles must be acute. Using the given side lengths and our inequalities above, we see that we must have

$$\begin{aligned} 121 + 196 &> XZ^2 \\ 196 + XZ^2 &> 121 \\ 121 + XZ^2 &> 196 \end{aligned}$$

Simplifying these three yields:

$$\begin{aligned} 317 &> XZ^2 \\ XZ^2 &> -75 \\ XZ^2 &> 75 \end{aligned}$$

The middle inequality is clearly always true. (We really didn't even have to include it – clearly 11 couldn't ever be the largest side!) Combining the other two and noting that we seek integer values of XZ, we have $9 \le XZ \le 17$ (since $8^2 < 75 < 9^2$ and $18^2 > 317 > 17^2$). So, there are 9 integer values of XZ such that $\triangle XYZ$ is acute. □

Exercises

10.2.1 Each of the following groups of three numbers are the lengths of the sides of a triangle. Identify each triangle as acute, right, or obtuse.

(a) $6, 8, 10$

(b) $6, 8, 11$

(c) $6, 8, 9$

(d) $\sqrt{13}, 3\sqrt{3}, \sqrt{15}$

(e) $2.1, 1.8, 3.1$

(f) $5/7, 9/14, 1$

10.2.2 In $\triangle ABC$, $AB = 17$ and $BC = 27$. For how many integer values of AC is $\triangle ABC$ acute?

10.2.3★ In this problem we will find a geometric proof that $\angle ACB$ is obtuse if $AB^2 > AC^2 + BC^2$.

(a) Let X be the point on $\overleftrightarrow{BC}$ such that $\overline{AX}$ is an altitude of $\triangle ABC$. Find an expression for AB^2 in terms of other segment lengths in the resulting diagram. (Note that at this point, we do not know if X is on $\overline{BC}$ or not. It might be on the extension of the segment instead.)

(b) Why must $\triangle ABC$ be obtuse if X is not on $\overline{BC}$, but is instead on $\overleftrightarrow{BC}$ such that C is between B and X?

(c) If X is on $\overline{BC}$, then how are BX, CX, and BC related?

(d) Solve your expression from part (c) for BX and substitute the result into your equation for part (a). Is it possible for AB^2 to be greater than $AC^2 + BC^2$? **Hints:** 488

(e) Complete the proof by concluding that X must be on $\overleftrightarrow{BC}$ such that C is between X and B, so $\angle ACB$ must be obtuse. **Hints:** 169

10.3 The Triangle Inequality

The Triangle Inequality answers the question 'When can three given segment lengths be the side lengths of a triangle?' The answer to this question is both more obvious and more powerful than the other inequalities we have explored in this chapter. We'll start with a proof of this simple inequality, then show why the inequality is so powerful by using it to solve a variety of problems.

Problems

Problem 10.13: In this problem we will prove the Triangle Inequality. Don't skip over this problem because it looks obvious! Try to find a mathematical proof for each part.

(a) Use the inequalities from the previous sections to prove that in any triangle, the sum of two sides is greater than the third side. This is the Triangle Inequality.

(b) The Triangle Inequality is often written as 'the sum of two sides is greater than or equal to the third side.' Why – where does the 'or equal to' come from?

(c) If a, b, and c are positive numbers such that $a + b > c$, $a + c > b$, and $b + c > a$, then show that there exists a triangle with side lengths a, b, and c.

Problem 10.14: In how many ways can we choose three different numbers from the set $\{1, 2, 3, 4, 5, 6\}$ such that the three could be the sides of a triangle? (Note: The order of the chosen numbers doesn't matter; we consider $\{3, 4, 5\}$ to be the same as $\{4, 3, 5\}$.)

Problem 10.15: Can the lengths of the altitudes of a triangle be in the ratio $2 : 5 : 6$? Why or why not?

Problem 10.16: Let $\overline{AM}$ be a median of $\triangle ABC$. Prove that $AM > (AB + AC - BC)/2$.

Problem 10.17: Circle O and circle P are tangent at point T such that neither circle passes through the interior of the other. In this problem we will prove that O, P, and T are collinear (i.e., that a line passes through all three).

(a) Prove that $OT + TP \geq OP$.

(b) Show that if $OT + TP > OP$, then $\odot O$ and $\odot P$ meet at a second point besides T.

(c) Show that O, P, and T must be collinear.

We'll start by 'proving the obvious.'

Problem 10.13:

(a) Prove that in any triangle, the sum of two sides is greater than the third side. This is the **Triangle Inequality**.

(b) The Triangle Inequality is often written as 'the sum of two sides is greater than or equal to the third side.' Why – where does the 'or equal to' come from?

(c) If a, b, and c are positive numbers such that $a + b > c$, $a + c > b$, and $b + c > a$, then show that there exists a triangle with side lengths a, b, and c.

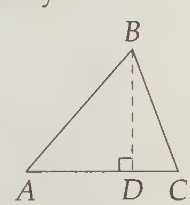
Solution for Problem 10.13:

(a) What's wrong with this solution:

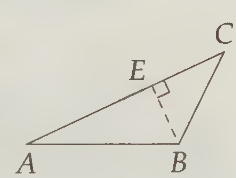
> **Bogus Solution:** Suppose we draw $\overline{AB}$ first. Clearly the farthest C could be from A is if we just draw $\overline{BC}$ in the same direction, so that $AB + BC = AC$. Any other way we draw $\overline{BC}$ will result in C being closer to A. Therefore, $AB + BC > AC$ if $\triangle ABC$ is a real triangle.

The second sentence is true, but we haven't proved it. Since we're dealing with an inequality, we reach for the inequalities we've already proved to tackle the Triangle Inequality.

We've had so much success building right triangles, we do so again here, drawing altitude $\overline{BD}$ as shown. We have two cases to consider:



Case 1: $\triangle ABC$ is acute. We draw altitude $\overline{BD}$, forming two right triangles. We thus have $AB > AD$ and $BC > CD$. Adding these gives the desired $AB + BC > AD + DC = AC$. We can do the same for each angle, showing that each side is less than the sum of the other two sides.



Case 2: $\triangle ABC$ is right or obtuse. If $\angle B \geq 90^\circ$ as shown, then clearly $AC > AB$ and $AC > BC$, so we definitely have $AB < AC + BC$ and $BC < AC + AB$. All we have left to prove is $AC < AB + BC$. Once again, right triangles come to the rescue. Drawing altitude $\overline{BE}$ gives us $AB > AE$ and $BC > EC$, so $AB + BC > AE + EC = AC$.

(b) We have seen above that if A, B, and C are not collinear, then there cannot be equality. However, when A, B, and C are collinear such that B is between A and C, we have $AB + BC = AC$. If A, B, and C are collinear, we call the resulting 'triangle' $\triangle ABC$ a **degenerate triangle**. (Typically the term 'triangle' only refers to **nondegenerate triangles**, i.e. those in which the vertices are not collinear.)

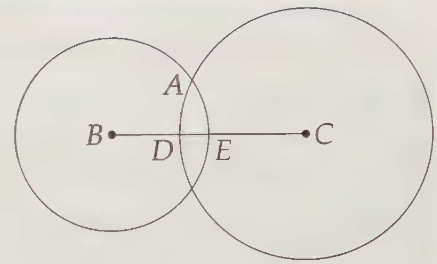
Note that when B is on $\overline{AC}$, we still have $AC + BC > AB$ and $AC + AB > BC$ in addition to $AB + BC = AC$, so the Triangle Inequality is still satisfied with a small modification:

Important: The Triangle Inequality states that for any three points, A, B, and C, we have

$$AB + BC \geq AC,$$

where equality holds if and only if B is on $\overline{AC}$. Therefore, for nondegenerate triangles, $AB + BC > AC$.

(c) Suppose $a \geq b \geq c$, and let our triangle be $\triangle ABC$ with $AB = c$, $AC = b$, and $BC = a$. We can start building our triangle by drawing $\overline{BC}$ with length a. We know that A must be on the circle with center B and radius $AB = c$, and on the circle with center C and radius $AC = b$. We draw both circles. Because $a \geq b \geq c$, we know that $BC \geq AC \geq AB$. Therefore, the radius of $\odot B$ is less than BC, so $\odot B$ intersects $\overline{BC}$. We call this point of intersection E. Similarly, $\odot C$ meets $\overline{BC}$ at point D as shown.



Since $b + c > a$, we know that $BE + DC = b + c > a = BC$. Since $BE + DC > BC$, we know that D cannot be on $\overline{EC}$. In other words, our two circles must meet! Where these circles meet gives us our final vertex of $\triangle ABC$.

□

Note that we started off our proof to the last part with 'Suppose $a \geq b \geq c$.' But what if that's not the case? Does our whole proof fall apart?

No, it doesn't. The whole proof still works; we just have to rearrange the letters a little. All the same logic still works.

Important: Mathematicians have a special way of saying 'All the different cases are essentially the same, so proving it for this one proves it for all of them.' To say this in our solution to the last part of the previous problem, a mathematician would have written, '**Without loss of generality**, let $a \geq b \geq c \ldots$.' Then, the mathematician doesn't need a whole separate proof for essentially equivalent cases such as $b \geq a \geq c$. Sometimes 'without loss of generality' is shortened to '**WLOG**'.

Now we'll use the Triangle Inequality to determine sets of specific side lengths that can be the sides of a triangle.

Problem 10.14: In how many ways can we choose three different numbers from the set $\{1, 2, 3, 4, 5, 6\}$ such that the three could be the sides of a nondegenerate triangle? (Note: The order of the chosen numbers doesn't matter; we consider $\{3, 4, 5\}$ to be the same as $\{4, 3, 5\}$.)

Solution for Problem 10.14: We first notice that if we have three numbers to consider as possible side lengths of a triangle, we only need to make sure that the sum of the smallest two is greater than the third. (Make sure you see why!) We could just start listing all the ones we see that work, but we should take an organized approach to make sure we don't miss any. We can do so by classifying sets of three numbers by the smallest number.

Case 1: Smallest side has length 1. No triangles can be made with three *different* lengths from our set if we include one of length 1.

Case 2: Smallest side has length 2. The other two sides must be 1 apart, giving the sets $\{2,3,4\}$, $\{2,4,5\}$, and $\{2,5,6\}$.

Case 3: Smallest side has length 3. There are only three possibilities and they all work: $\{3,4,5\}$, $\{3,4,6\}$, $\{3,5,6\}$.

Case 4: Smallest side has length 4. The only possibility is $\{4,5,6\}$, which works.

Adding them all up, we have 7 possibilities. □

We sometimes have to use some other tools in addition to the Triangle Inequality.

Problem 10.15: Can the lengths of the altitudes of a triangle be in the ratio $2:5:6$? Why or why not?

Solution for Problem 10.15: We don't know anything about how the lengths of the altitudes of a triangle are related to each other. We do, however, know a whole lot about how the lengths of the sides of a triangle are related to each other. Therefore, we turn the problem from one involving altitudes into one involving side lengths. We let the area be K, let the side lengths be a, b, c, and the lengths of the altitudes to these sides be h_a, h_b, h_c, respectively. Therefore, we have $K = ah_a/2 = bh_b/2 = ch_c/2$, so the sides of the triangle have lengths

$$\frac{2K}{h_a}, \frac{2K}{h_b}, \frac{2K}{h_c}.$$

If our heights are in the ratio $2:5:6$, then for some x, our heights are $2x$, $5x$, and $6x$. Then, our sides are

$$\frac{2K}{2x}, \frac{2K}{5x}, \frac{2K}{6x}.$$

However, the sum of the smallest two sides is then

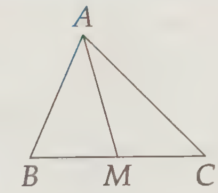
$$\frac{2K}{5x} + \frac{2K}{6x} = \frac{12K}{30x} + \frac{10K}{30x} = \frac{22K}{30x},$$

which is definitely less than the largest side, which is $2K/2x = K/x$. Therefore, the sides don't satisfy the Triangle Inequality, which means it is impossible to have a triangle with heights in the ratio $2:5:6$.□

Concept: When facing problems involving lengths of altitudes of a triangle, consider using area as a tool.

The Triangle Inequality is probably the most commonly used tool in geometric inequality proofs; here's an example of the Triangle Inequality in action.

Problem 10.16: Let $\overline{AM}$ be a median of $\triangle ABC$. Prove that $AM > (AB + AC - BC)/2$.



Solution for Problem 10.16: We can apply the Triangle Inequality to both $\triangle ABM$ and $\triangle ACM$, giving:

$$\begin{aligned} AM + BM &> AB \\ AM + CM &> AC. \end{aligned}$$

Noting that $BM + CM = BC$, we think to add these inequalities, which gives $2AM + BC > AB + AC$. Subtracting BC from both sides and dividing by 2 yields

$$AM > \frac{AB + AC - BC}{2}.$$

□

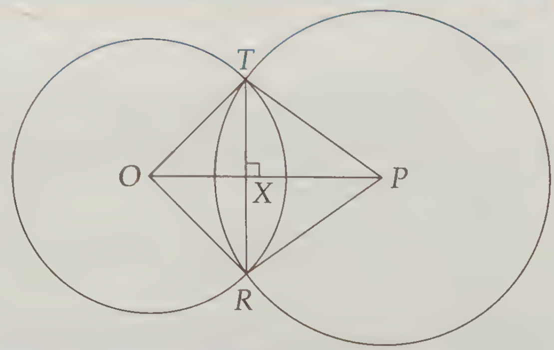
Concept: If you don't see the path to the solution immediately, don't just sit and stare at the problem! Make some observations. Write down statements you can prove that might be helpful. Perhaps you'll be able to combine these observations to complete your proof.

In Problem 10.16, even if we didn't see the solution immediately, the problem involves an inequality that has sums of lengths. This makes us think of using the Triangle Inequality to make observations. However, we don't simply write Triangle Inequality relationships blindly. We use the problem as a guide to make the observations. Specifically:

Concept: When trying to solve geometric inequalities, pay attention to which side of the inequality you want to prove each length is on. For example, we wouldn't want to start Problem 10.16 by observing $AM < AB + BM$, since this inequality has AM on the smaller side, but what we want to prove has AM on the larger side.

We end the chapter with an important application of the Triangle Inequality.

Problem 10.17: Circle O and circle P are tangent at point T such that neither circle passes through the interior of the other. Such circles are said to be **externally tangent**. Prove that O, P, and T are collinear.



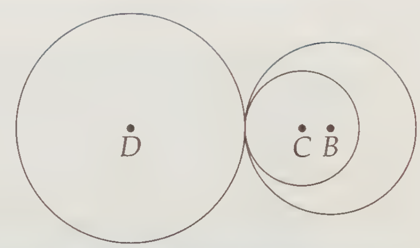
Solution for Problem 10.17: We don't know a whole lot about tangent lines at this point. However, in this section we have discovered a way to show that three points are on a line. Specifically, if $OT + TP = OP$, then T is on $\overline{OP}$, because otherwise the Triangle Inequality guarantees $OT + TP > OP$.

There doesn't seem to be any easy way to approach showing that $OT + TP = OP$. Instead, we try to show that it's impossible to have $OT + TP > OP$ and still have tangent circles. If we have $OT + TP > OP$, then T cannot be on $\overline{OP}$. Therefore, there is some point R such that $\overline{OP}$ is the perpendicular bisector of $\overline{TR}$. To find point R, we draw altitude $\overline{TX}$ from T to $\overline{OP}$, then we draw $\overline{RX}$ such that $RX = TX$ and $\overline{RX} \perp \overline{OP}$. Since $OX = OX$, $\angle TXO = \angle RXO$, and $RX = TX$, we have $\triangle TOX \cong \triangle ROX$. Therefore, $TO = RO$, so R is on $\odot O$. Similarly, we have $\triangle PTX \cong \triangle PRX$, so $PT = PR$ and R is on $\odot P$. Therefore, if $OT + TP > OP$, it is impossible for the circles to be tangent because there is a second point at which the circles meet.

However, our circles must be tangent, so we can't have $OT + TP > OP$. The Triangle Inequality tells us we can't have $OT + TP < OP$. Therefore, we must have $OT + TP = OP$, which means O, T, and P are collinear. □

Important: If $\odot O$ and $\odot P$ are tangent at point T, then O, P, and T are collinear. We'll often use this fact in problems with tangent circles by connecting the centers of the circles, knowing this line also goes through the point of tangency.

Notice that Problem 10.17 only tackles the case in which the circles are externally tangent. It's also true for circles that are **internally tangent**, i.e., when one circle is wholly inside the other except at the one point at which they are tangent. In the diagram at right, $\odot B$ and $\odot C$ are internally tangent, while $\odot D$ is externally tangent to both of the other circles.



Exercises

10.3.1 Which of the following sets of three numbers could be the side lengths of a triangle?

(a) 4, 5, 6

(b) 7, 20, 9

(c) 1/2, 1/6, 1/3

(d) 3.4, 11.3, 9.8

(e) $\sqrt{5}$, $\sqrt{14}$, $\sqrt{19}$

10.3.2 The lengths of two sides of a triangle are 7 cm and 3 cm. If the number of centimeters in the perimeter is a whole number, what is the number of centimeters in the positive difference between the greatest and least possible perimeters? *(Source: MATHCOUNTS)*

10.3.3 Prove that the sum of the diagonals of a quadrilateral is less than the quadrilateral's perimeter.

10.3.4★ $ABCD$ is a square and O is a point. Prove that the distance from O to A is not greater than the sum of the distances from O to the other three vertices, no matter which point we take to be point O. **Hints:** 97

10.3.5★ Prove that if a, b, and c are the sides of a triangle, then so are $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$. What about a^2, b^2, and c^2? **Hints:** 453, 36

Extra! *Calculating replaces thinking, while geometry stimulates it.*

–Jakob Steiner

10.4 Summary

Important: In any triangle, the longest side is opposite the largest angle and the shortest side is opposite the smallest angle. The middle side, of course, is therefore opposite the middle angle.

In other words, in $\triangle ABC$, $AB \geq AC \geq BC$ if and only if $\angle C \geq \angle B \geq \angle A$.

Important: $\angle C$ of $\triangle ABC$ is acute if and only if $AB^2 < AC^2 + BC^2$.
$\angle C$ of $\triangle ABC$ is right if and only if $AB^2 = AC^2 + BC^2$.
$\angle C$ of $\triangle ABC$ is obtuse if and only if $AB^2 > AC^2 + BC^2$.

Important: The Triangle Inequality states that for any three points, A, B, and C, we have

$$AB + BC \geq AC,$$

where equality holds if and only if B is on $\overline{AC}$. Therefore, for nondegenerate triangles (i.e., those in which the vertices are not collinear), $AB + BC > AC$.

Important: If $\odot O$ and $\odot P$ are tangent at point T, then O, P, and T are collinear. We'll often use this fact in problems with tangent circles by connecting the centers of the circles, knowing this line also goes through the point of tangency.

Problem Solving Strategies

Concepts:

- One of the most useful approaches to solving new problems is to try to think of similar-looking problems you know how to handle.
- If you don't use all the given information in a solution, proofread it closely to make sure you haven't made a mistake! Sometimes you won't need all the information you're given, but often if you haven't used it all, it's because you made a mistake somewhere.

  Furthermore, when you're stuck on a problem, read it again and see if there is any information you haven't used yet!
- When facing problems involving lengths of altitudes of a triangle, consider using area as a tool.

Continued on the next page. . .

Concepts: *... continued from the previous page*

- If you don't see the path to the solution immediately, don't just sit and stare at the problem! Make some observations – write down statements you can prove that might be helpful. Perhaps you'll be able to combine these observations to complete your proof.
- When trying to solve geometric inequalities, pay attention to which side of the inequality you want to prove each length is on. For example, we wouldn't want to start Problem 10.16 by observing $AM < AB + BM$, since this inequality has AM on the smaller side, but what we want to prove has AM on the larger side.

REVIEW PROBLEMS

10.18 For each of the groups of three numbers below, state whether the numbers could be the side lengths of a triangle or not. If they can be, identify whether or not the triangle is acute, obtuse, or right.

(a) $2, 3, 4$

(b) $2.1, 1.7, 3.9$

(c) $\sqrt{5}, 2, \sqrt{3}$

(d) $199, 401, 297$ (See if you can do it without squaring those numbers!)

(e) $1/2, 1/3, 1$

(f) $60, 24, 48$

10.19 Ari is solving a problem involving a right triangle with legs 119 and 120. He uses the Pythagorean Theorem and gets 261 as the hypotenuse. He immediately shakes his head and starts over. How did he know so quickly that he made a mistake?

10.20 In $\triangle ABC$, $AB = 5$ and $BC = 11$. For which integer values of AC is $\triangle ABC$ an obtuse triangle?

10.21 $A_1A_2A_3\cdots A_n$ is a regular polygon with $n > 3$. Prove that $A_1A_3 > A_1A_2$.

10.22 Prove that it is impossible for the length of a side of a triangle to be greater than half the triangle's perimeter.

10.23 The perimeter of an isosceles triangle is 38 centimeters and two sides of the triangle are whole numbers in the ratio 3 : 8. What is the number of centimeters in the length of the shortest side? *(Source: MATHCOUNTS)*

10.24 Find all positive integers x for which it is possible for $2x + 3$, $3x + 8$, and $6x + 7$ to be the side lengths of a nondegenerate triangle.

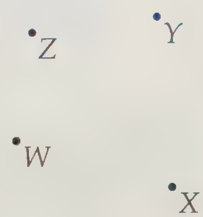
10.25 How many different obtuse triangles with integer side lengths and a perimeter of 20 can we draw such that no two of them are congruent?

10.26 The length of each leg of an isosceles triangle is $x+1$ and the length of the base is $3x-2$. Determine all possible values of x. (Your answer should be an inequality expressing the possible values of x.)

10.27 $\overline{YZ}$ is the base of isosceles triangle $\triangle XYZ$. Given that $YZ > XY$, show that $\angle X > 60°$. **Hints:** 551, 209

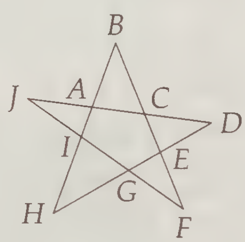
Challenge Problems

10.28 I have four stakes in my yard arranged like points W, X, Y, and Z shown. I wish to connect each stake to the same point in my yard with a string. What point should I choose to minimize the amount of string I must use? (Make sure to prove that your choice is the best possible!)

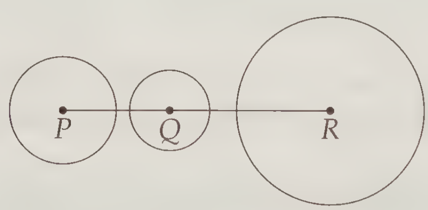


10.29 Orion chooses a positive number and Michelle chooses a positive number. Joshua chooses a positive number that is smaller than Orion's number. No matter what number Joshua chooses, we can always form a triangle with the three chosen numbers as side lengths. Show that Orion's and Michelle's numbers must be the same. **Hints:** 493

10.30 Show that it is impossible for any 5-pointed star like $ABCDEFGHIJ$ at right to have $AB > BC$, $CD > DE$, $EF > FG$, $GH > HI$, and $IJ > JA$ all be true.



10.31 As shown below, Q is on $\overline{PR}$ and circles $\odot P$, $\odot Q$, and $\odot R$ are drawn such that no two circles intersect and no one circle contains the other two. Suppose a fourth circle can be constructed that is externally tangent to all three of these circles.



(a) Use the Triangle Inequality to prove that the radius of our fourth circle is greater than the radius of $\odot Q$. **Hints:** 270

(b) Use the inequalities relating the order of sides in a triangle to the order of angles in a triangle to prove that the radius of our fourth circle is greater than the radius of $\odot Q$. **Hints:** 351

10.32 In convex quadrilateral $WXYZ$, we have $WX > WY$. Show that we must also have $XZ > YZ$. **Hints:** 447, 85

10.33 Let a, b, and c be three positive real numbers such that $a^2 + b^2 > c^2$, $a^2 + c^2 > b^2$ and $b^2 + c^2 > a^2$. Prove that a, b, and c can be the lengths of the sides of a triangle. **Hints:** 160

10.34 Prove that the sum of the lengths of the diagonals of a quadrilateral is greater than half the perimeter of the quadrilateral.

10.35★ In $\triangle XYZ$, we have $\angle X = 20^\circ$ and $XY = XZ$. Prove that $2YZ < XY < 3YZ$. **Hints:** 533, 84, 12, 395, 498

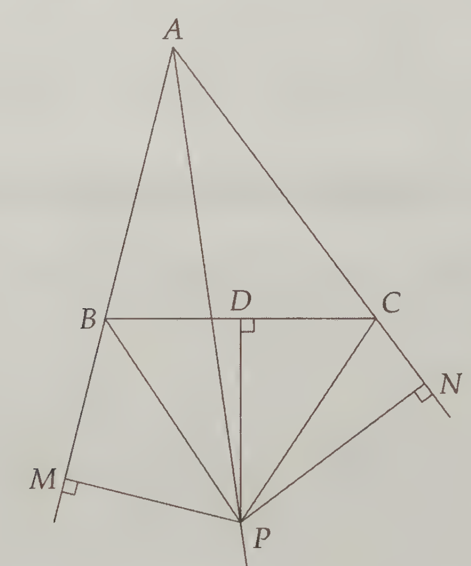
10.36★ In $\triangle ABC$ and $\triangle A'B'C'$, we have $AB = A'B'$, $AC = A'C'$, and $\angle BAC > \angle B'A'C'$. Prove that $BC > B'C'$. **Hints:** 266, 578

10.37★ Prove that the distance between any two points inside $\triangle ABC$ is not greater than half the perimeter of $\triangle ABC$. **Hints:** 430, 525, 437

Extra!

Proof That Every Triangle Is Isosceles?

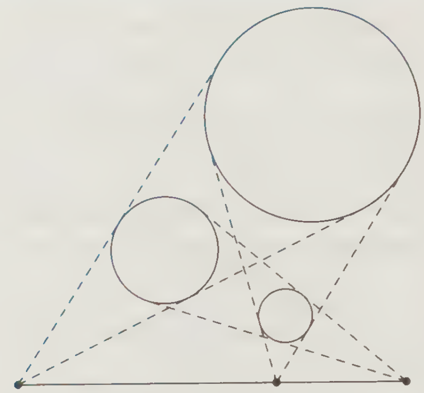
In triangle ABC below, let D be the midpoint of $\overline{BC}$. Let P be the intersection of the angle bisector of A and the perpendicular bisector of $\overline{BC}$. Let M be the foot of the perpendicular from P to $\overline{AB}$ and N be the foot of the perpendicular from P to $\overline{AC}$, as shown.



Since P lies on the angle bisector of A, $\angle PAM = \angle PAN$. Also, $\angle PMA = \angle PNA = 90^\circ$, so by AAS, triangles PAM and PAN are congruent. Therefore, $PM = PN$ and $AM = AN$.

P lies on the perpendicular bisector of $\overline{BC}$, so $PB = PC$. Also, $\angle PMB = \angle PNC = 90^\circ$, so triangles PMB and PNC are congruent by HL Congruence. Therefore, $MB = NC$. But $AM = AB + BM$, $AN = AC + CN$, and $AM = AN$, so $AB = AC$. Therefore, triangle ABC is isosceles.

It is obviously not true that every triangle is isosceles, so something is going wrong in the proof. But what?



Monge's Theorem

Probably no symbol in mathematics has evoked as much mystery, romanticism, misconception and human interest as the number pi. – William L. Schaaf

CHAPTER 11 Circles

11.1 Arc Measure, Arc Length, and Circumference

The portion of a circle connecting two points on the circle's circumference is called an **arc**. We find the measure of an arc of a circle by considering what fraction of the circle the arc is. For example, a whole circle is 360°, so an arc that is 1/6 of the circle has measure $(1/6)(360°) = 60°$.

However, not all 60° arcs are the same. Some are much longer than others, as shown in the figure to the right. Thus, we need more than just angle measure to classify arcs. We need a way to measure of the lengths of arcs.

We start with the length of an entire circle, which would be the perimeter of a circle. The perimeter of a circle is called the circle's **circumference**.

Before reading the rest of this chapter, put yourself in the sandals of the ancient Greeks and try a little experiment. Get a string, and find numerous circular objects. Measure the distance around each object by wrapping the string around it. Then measure the diameter of the object. Finally, for each object find the quotient

$$\frac{\text{Distance Around the Object}}{\text{Diameter of Object}}.$$

You should find that in each case the quotient is around 3.14. (If you get anything different, try measuring and dividing again!)

In every single circle, the ratio of circumference to diameter is the same. This ratio is called **pi**, and is given the symbol π. Its value is approximately 3.14. (If you ask a bunch of your math friends, you're likely to find someone who knows dozens more digits.)

Sidenote: For some reason, memorizing huge portions of pi is a bit of a sport among math-lovers. This should help you get started:

```
3.141592653589793238462643383279502884197169399375105820974944
5923078164062862089986280348253421170679821480865132823066470 9
```

(Please don't tell your parents you got this π-memorizing idea from us.)

Pi is an **irrational number**, which means that it cannot be expressed as a ratio of integers. Because pi is irrational, its decimal expansion does not terminate and does not become periodic. In other words, it does not get to a point where the same set of numbers is repeated over and over. So, there are no shortcuts to memorizing digits of pi!

Problems

Problem 11.1: The circumference of a circle is 32π. Find the circle's radius.

Problem 11.2: The length of a 72° arc of a circle is 15. What is the circumference of the circle?

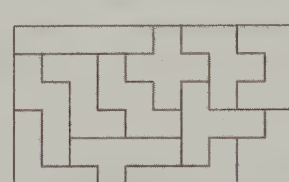
Problem 11.3: Chord $\overline{YZ}$ of a circle with center O has length 12. The circumference of the circle is 24π.

(a) What kind of triangle is $\triangle YOZ$?

(b) Find the radius of the circle.

(c) Find the length of $\widehat{YZ}$.

Extra! The twelve pentominoes (see page 132) fit inside a 6×10 rectangle as shown below.



They can also fit inside a 5×12, a 4×15, and even a 3×20 rectangle. Can you find how?

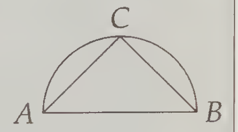
Problem 11.4: Chords $\overline{AB}$ and $\overline{CD}$ of $\odot O$ have the same length. We will prove that $\widehat{AB} = \widehat{CD}$.

(a) Connect A, B, C, and D to the center of the circle. Find some congruent triangles.

(b) Use your congruent triangles to prove that $\widehat{AB} = \widehat{CD}$.

Problem 11.5:

(a) Anna and Betty are 180 yards apart, at points A and B in the diagram, respectively. There is a semi-circular path from Anna to Betty. Chuck is at point C, the midpoint of this path. Anna must walk to Chuck and then to Betty. How much farther must Anna walk if she follows the path than if she 'cuts across,' walking straight to C then straight to B?



(b) What if the path is a 120° arc of a circle instead of a semi-circular arc? (Chuck is on the midpoint of this new arc, and we wish to see how much farther Anna must walk by staying on the path.)
Hints: 20, 569

Circumference is a pretty simple concept, once you've finished memorizing all the digits of pi.

Problem 11.1: The circumference of a circle is 32π. Find the circle's radius.

Solution for Problem 11.1: Let C be the circumference and d be the diameter. We know that $C/d = \pi$ and $C = 32\pi$. Therefore, $d = 32$. The radius is half the diameter, or $32/2 = 16$. □

Important: In a circle, let C be the circumference, d be the diameter, and r be the radius.

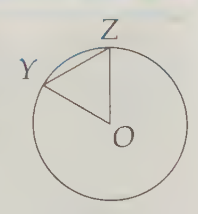
$$C = \pi d = 2\pi r.$$

Problem 11.2: The length of a 72° arc of a circle is 15. What is the circumference of the circle?

Solution for Problem 11.2: A 72° arc is $72°/360° = 1/5$ of an entire circle. Since this arc has length 15, the entire circumference of the circle has length $5(15) = 75$. □

Problem 11.3: Chord $\overline{YZ}$ of a circle with center O has length 12. The circumference of the circle is 24π. Find the length $\widehat{YZ}$.

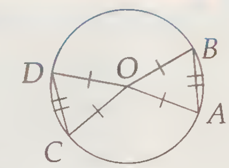
Solution for Problem 11.3: To find the length of the arc, we must find $\angle YOZ$. We don't have any information about angles, however, so we start by figuring out some lengths. Since the circumference is 24π, the diameter is 24 and the radius is 12. Hence, $OY = YZ = ZO$, so $\triangle YOZ$ is equilateral. Therefore, $\angle YOZ$ is 60°, and our arc is 1/6 of the circle. So, the length of $\widehat{YZ}$ is $(1/6)(24\pi) = 4\pi$. □



A **central angle** of a circle is an angle with the center of the circle as its vertex. Now that we know how to relate arcs to central angles, we can find a useful relationship between the arcs cut off by congruent chords of a circle.

Problem 11.4: Chords $\overline{AB}$ and $\overline{CD}$ of $\odot O$ have the same length. Prove that $\widehat{AB} = \widehat{CD}$.

Solution for Problem 11.4: We form two triangles by connecting A, B, C, and D to O. Since $AB = CD$, $AO = CO$, and $BO = DO$, we have $\triangle ABO \cong \triangle CDO$ by SSS Congruence. Therefore, $\angle AOB = \angle COD$, so arcs $\widehat{AB}$ and $\widehat{CD}$ are congruent. □

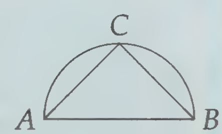


As you might suspect (and will prove as an Exercise), this relationship between congruent chords and congruent arcs works in reverse, too.

Important: If two chords of a circle are congruent, then the arcs they **subtend** (cut off) are congruent. Conversely, if two arcs of a circle are congruent, then the chords that subtend them are congruent.

Problem 11.5:

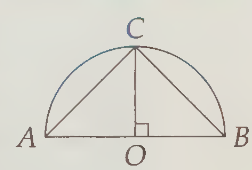
(a) Anna and Betty are 180 yards apart, at points A and B in the diagram, respectively. There is a semi-circular path from Anna to Betty. Chuck is at point C, the midpoint of this path. Anna must walk to Chuck and then to Betty. How much farther must Anna walk if she follows the path than if she 'cuts across,' walking straight to C then straight to B?



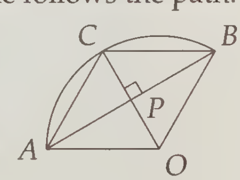
(b) What if the path is a 120° arc of a circle instead of a semi-circular arc? (Chuck is on the midpoint of this new arc, and we wish to see how much farther Anna must walk by staying on the path.)

Solution for Problem 11.5:

(a) We might immediately see that $\triangle ACB$ is a 45-45-90 triangle, but if we don't, we can start by connecting C to the center of the circle, O, as shown in the diagram. Since $\widehat{AC} = \widehat{CB} = 90°$, we have $\angle AOC = \angle COB = 90°$. $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ are all radii of the circle, so $\triangle AOC$ and $\triangle BOC$ are 45-45-90 triangles. Since $AB = 180$ yards, we have $AO = OB = OC = 180/2 = 90$ yards and $AC = BC = CO\sqrt{2} = 90\sqrt{2}$ yards. Since arc $\widehat{AB}$ is one-half a circle, its length is $(AB)\pi/2 = 90\pi$ yards. So, Anna must walk $90\pi - (AC + BC) = 90\pi - 180\sqrt{2}$ yards farther if she follows the path.



(b) We start by connecting A, B, and C to the center of the circle of which the path is a part, since we know more about radii than we do about arcs. Since $\widehat{AB}$ is 120°, and C is its midpoint, $\widehat{AC} = \widehat{BC} = 60°$. Therefore, $\angle AOC = \angle COB = 60°$. $OA = OC$ because they are radii of the same circle. Therefore, $\triangle AOC$ is isosceles with $\angle AOC = 60°$ and $AO = OC$, so $\triangle AOC$ is equilateral. Similarly, $\triangle BOC$ is also equilateral.



We need to find the radius of this circle to finish the problem. Let $\overline{AB}$ meet $\overline{CO}$ at P. Since $\triangle AOP \cong \triangle BOP$ by SAS, $\angle APO = \angle BPO$. These angles together make up a straight angle, so $\angle APO = \angle BPO = 90°$. So, $\triangle APO$ and $\triangle BPO$ are 30-60-90 triangles. Since $AP = AB/2 = 90$ yards, we have $AO = AP(2/\sqrt{3}) = 60\sqrt{3}$ yards. Since $AC = AO = BO = BC$, we have $AC+CB = AO+BO = 120\sqrt{3}$. The length of $\widehat{AB}$ is $(120°/360°)(2AO\pi) = 40\pi\sqrt{3}$. Therefore, by following the path, Anna walks $\widehat{AB} - (AC + BC) = 40\pi\sqrt{3} - 120\sqrt{3}$ farther. □

The key step in the second part of this problem is connecting the points on the path to the center of the circle. This is often an important step in challenging geometry problems.

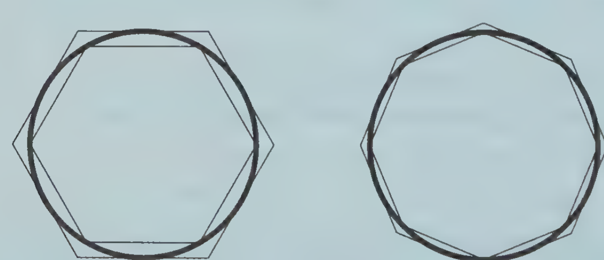
Concept: In complicated problems involving a circle, try connecting the center of a circle to important points on the circle.

Sidenote: We defined π as the ratio of the circumference of a circle to its diameter. But how do we know what π is numerically? In other words, how do we know that $\pi = 3.1415926535\ldots$, and so on?

Since π is one of the most important mathematical constants, much time and effort have been devoted to calculating π. At the present time, π is known to over one trillion decimal places. Why bother with so many digits? Certainly not for the sake of accuracy in everyday use – even 39 decimal places is sufficient to calculate the circumference of the known universe to within the radius of a hydrogen atom.

The answer, in part, is for the prestige of having set a new record number of places. The digits of π can also be used to check the speed and accuracy of computers. Finally, mathematicians are interested in seeing if they can find any patterns among these digits, such as whether the ten digits 0, 1, 2, 3, …, 9 appear equally often. However, despite all this effort, no patterns of note have been found.

So how do we calculate the value of π? One simple method is as follows: Suppose we take a circle of radius 1, and circumscribe it, and inscribe it, with a regular n-gon. The figure below shows the n-gons for $n = 6$ and $n = 8$.



The perimeter of the polygons approximate the circumference of the circle, which is 2π. The greater the number of sides in the polygon, the better the approximation. Let a_n denote the perimeter of the circumscribed n-gon (the outer polygon), and let b_n denote the perimeter of the inscribed n-gon (the inner polygon). Then it turns out that

$$a_{2n} = \frac{2a_nb_n}{a_n + b_n} \qquad \text{and} \qquad b_{2n} = \sqrt{a_{2n}b_n}.$$

Continued on the next page. . .

Sidenote: ... *continued from the previous page*

Thus, if we start with a_6 and b_6, we can calculate a_{12} and b_{12}, which can be used to calculate a_{24} and b_{24}, and so on. This was the observation of the ancient Greek mathematician **Archimedes**, after whom this algorithm is named. The following table shows some results of this algorithm. We begin with the values $a_6 = 4\sqrt{3}$ and $b_6 = 6$. (Make sure you see why $a_6 = 4\sqrt{3}$ and $b_6 = 6$; remember, the radius of the circle is 1.)

n	a_n	b_n	$a_n/2$	$b_n/2$
6	6.92820323	6.00000000	3.46410162	3.00000000
12	6.43078062	6.21165708	3.21539031	3.10582854
24	6.31931988	6.26525723	3.15965994	3.13262861
48	6.29217243	6.27870041	3.14608622	3.13935020
96	6.28542920	6.28206390	3.14271460	3.14103195
192	6.28374610	6.28290494	3.14187305	3.14145247
384	6.28332549	6.28311522	3.14166275	3.14155761
768	6.28322035	6.28316778	3.14161018	3.14158389

As we can see, the sequences $(a_n/2)$ and $(b_n/2)$ both converge to π. This is not the best algorithm for calculating the digits of π, but it illustrates how we can use a simple geometric idea to approximate π.

Exercises

11.1.1 Find the circumference of a circle with radius 4.

11.1.2 Find the radius of a circle with circumference 12π.

11.1.3 $\widehat{AC}$ of $\odot O$ has length 12π, and the circle has radius 18.

(a) Find OC.

(b) Find $\angle AOC$.

(c)★ Find AC. **Hints:** 471

11.1.4 Arcs $\widehat{WX}$ and $\widehat{YZ}$ of $\odot Q$ are congruent. Prove that $WX = YZ$.

Extra! Degrees are not the only way to measure angles! Just as there are different units for measuring length, like inch and centimeter, there are different units for angle measures. Besides the degree, the most commonly used unit for angle measure is the **radian**. Just as there are 360 degrees in a circle, there are 2π radians in a circle. Therefore, the measure in radians of a quarter-circle is $(2\pi)/4 = \pi/2$ radians, and of a semicircle is $(2\pi)/2 = \pi$ radians.

As you might guess, the usage of radians is inspired by circumference. You might also notice that every arc of a circle with radius 1 has its length in units equal to its measure in radians. However, radians are not very useful until our study of advanced trigonometry and calculus, so we'll stick with degrees for now.

11.2 Area

Problems

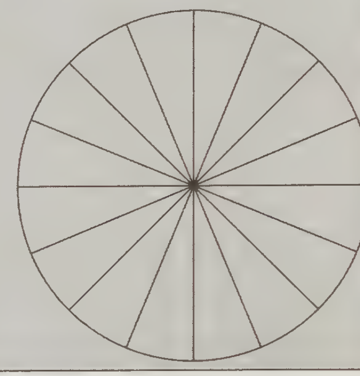
Problem 11.6: In this problem we will develop a formula for the area of a circle with radius r.

(a) The circle in the diagram has radius r and is divided into 16 equal pieces by drawing 8 diameters. These pieces are called **sectors**. Rearrange the sectors to look very similar to a quadrilateral whose area we can find.

(b) What is the length of the base of this 'quadrilateral' in terms of r?

(c) What is the height of this 'quadrilateral' in terms of r?

(d) What is the area of this 'quadrilateral' in terms of r?



Problem 11.7: Find the area of a circle with diameter 12.

Problem 11.8: $\triangle XYZ$ is equilateral with side length 10. A circle is constructed with center X and radius 10, thus passing through Y and Z. Find the area of sector YXZ of the circle. (In other words, find the area of the portion of the circle bounded by radii $\overline{XY}$ and $\overline{XZ}$ and by arc $\widehat{YZ}$.)

Problem 11.9: Farmer Tim has 50 feet of fence. He wants to enclose a semicircular area adjacent to his barn, thus using his barn as one side of the enclosure. What is the area of the space Farmer Tim can enclose?

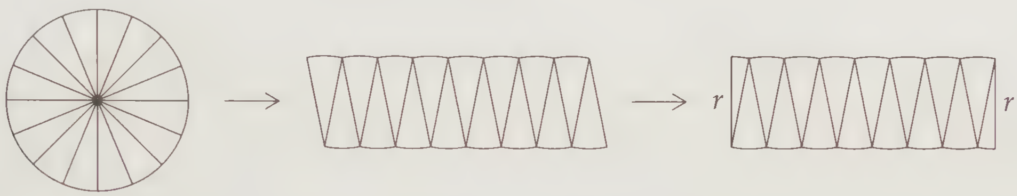
Having tackled the 'perimeter' of a circle, we now turn to its area. While proving the formula for the area of a circle would require far more advanced tools than we have now, we can develop an intuitive explanation for the formula.

Problem 11.6: Find a formula for the area of a circle by cutting up a circle and rearranging the pieces to look a lot like a rectangle.

Extra! GggXyZ the Martian has been told by the Martian King to make a rope to lasso the Earth. GggXyZ makes a rope that will fit snugly around the equator of the Earth. The Martian King decides he wants a little slack so he can put the rope around the Earth more easily. He wants the rope to fit around the equator such that there is one inch between the Earth and the rope all the way around the equator.

GggXyZ says this is no problem. She pulls out a short piece of rope and says she can just add this little piece to her rope to make the rope long enough. The Martian King laughs at her, saying she'll need miles of rope because the diameter of the Earth is almost 8,000 miles. Who is right? How much rope does she need?

Solution for Problem 11.6: We start by cutting the circle into equal slivers, which we call **sectors**, by drawing equally spaced radii as shown below. We then rearrange the sectors as shown in the second figure, thus forming a figure that resembles a parallelogram. Finally, we can take half of one of the end sectors and slide it to the other end as shown in the final diagram. Now we have a figure that strongly resembles a rectangle.



The 'width' of this 'rectangle' is clearly the radius of the circle, which we'll call r. The 'length' of the 'rectangle' is half the circumference of the circle, since the circumference of the circle is equally divided among the top and bottom of our 'rectangle.' Hence, the 'length' is $(2\pi r)/2 = \pi r$. Since the area of a rectangle is its length times its width, the area of our 'rectangle' is $(\pi r)(r) = \pi r^2$.

This is not a rigorous proof, but it should be clear that if we cut the circle into more and more sectors and do the described rearrangement, the 'rectangle' looks more and more like a real rectangle. □

We now know how to find the area of a circle.

> **Important:** The area of a circle with radius r is πr^2.

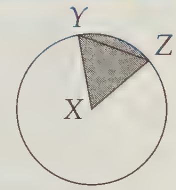
Let's use our newfound knowledge of how to find the area of a circle on a few problems.

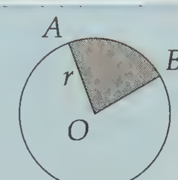
Problem 11.7: Find the area of a circle with diameter 12.

Solution for Problem 11.7: Since the diameter is 12, the radius is 6. Therefore, the area is $(6^2)\pi = 36\pi$. □

Problem 11.8: $\triangle XYZ$ is equilateral with side length 10. A circle is constructed with center X and radius 10, thus passing through Y and Z. Find the area of sector YXZ of the circle.

Solution for Problem 11.8: Our sector is shaded in the diagram. To find its area, we must determine what portion the sector is of the whole circle. Since $\triangle XYZ$ is equilateral, $\angle YXZ = 60°$. Since the whole circle is $360°$, this means that our sector is $60°/360° = 1/6$ of the entire circle. The whole circle has area $\pi r^2 = \pi(10^2) = 100\pi$, so the sector has area $(100\pi)(1/6) = 50\pi/3$. □



> **Important:** Following our solution to Problem 11.8, we can find the area of a sector of a circle of radius r given the central angle of the sector. In the figure to the right, $\angle AOB$ is $(\angle AOB)/360°$ of the whole circle. Therefore, we have:
>
> 
>
>
> $$\text{Area of sector } AOB = \left(\frac{\angle AOB}{360°}\right) r^2\pi.$$

You should not have to memorize this formula. Instead, understand that the area of a sector is simply related to the ratio of the sector's angle to the number of degrees in a whole circle.

Problem 11.9: Farmer Tim has 50 feet of fence. He wants to enclose a semicircular area adjacent to his barn, using his barn as one side of the enclosure. What is the area of the space Farmer Tim can enclose?

Solution for Problem 11.9: We need to find the radius of the semicircle in order to find its area. We start with a sketch of the problem. Our fence forms a semicircular arc, so the arc has length 50. Therefore, the semicircle is half of a circle that has circumference 100. The diameter of such a circle is the circumference divided by π, or $100/\pi$. So, the radius of our semicircle is $(100/\pi)/2 = (50/\pi)$.

The semicircle therefore has an area that is 1/2 the area of a circle with radius $(50/\pi)$, or $(1/2)(50/\pi)^2(\pi) = 1250/\pi$ square feet. □

Concept: Start geometric word problems with a sketch.

Exercises

11.2.1 Find the area of a circle with diameter 18.

11.2.2 Find the area of a circle that has circumference 12π.

11.2.3 Find the radius of a circle if its circumference is numerically equal to its area.

11.2.4 If a pizza that is 12 inches in diameter provides four full meals, how many meals are provided by a pizza that is 20 inches in diameter?

11.2.5 Points A and B are on the circumference of $\odot O$ such that $\angle AOB = 120^\circ$ and $OA = 12$.

(a) Find the area of $\odot O$.

(b) Find the area of sector AOB.

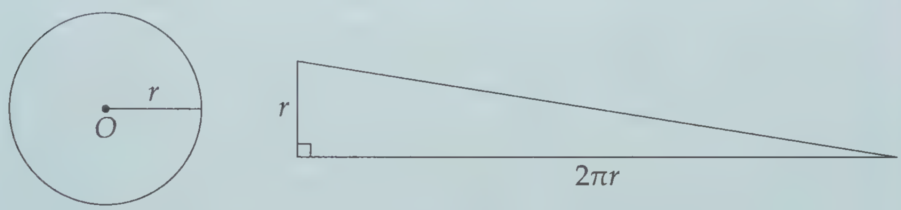
11.2.6 Sector XQZ of circle Q has area 30π. Given that the whole circle has area 100π, find $\angle XQZ$ and $\angle XZQ$. **Hints:** 342

11.2.7★ A man standing on a lawn is wearing a circular sombrero of radius 3 feet. Unfortunately, the hat blocks the sunlight so effectively that the grass directly under it dies instantly. If the man walks in a circle of radius 5 feet, what area of dead grass will result? *(Source: HMMT)* **Hints:** 81

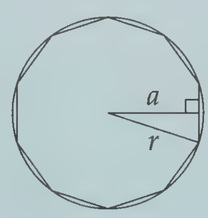
Extra! In 1897, the Indiana state legislature almost passed a bill that set the value of π to exactly 3.2. The House voted unanimously for it and it passed a first reading in the Senate. Fortunately, a math professor at Purdue University happened to be visiting the legislature at the same time and advised that the bill be postponed indefinitely, effectively killing it. If he hadn't, manholes in Indiana would look very strange.

Sidenote: The ancient Greeks didn't have calculus, so how did they find the formula for the area of a circle? While they didn't have calculus, they did have **Archimedes**, some of whose proofs came so close to calculus (which wouldn't be developed until nearly 2000 years after his death) that some suggest he might be called the father of calculus. Archimedes used two favorite tactics of the Greeks – contradiction (see page 41) and comparing a desired area to that of a figure whose area is easy to find.

Archimedes claimed that a circle has the same area as that of a right triangle with the circle's radius as one leg and its circumference as the other leg. He did so by proving that the area of the circle could be neither more nor less than that of the triangle. We'll show you his proof that the circle's area could not be greater than that of the triangle, and leave you to walk in his footsteps and supply the proof for the other half.



Let the radius of the circle be r, the area of the circle be A, and the area of the triangle be T. Then, we assume that $A > T$. As we saw in our discussion of calculating π, we can use regular polygons with more and more sides to approximate a circle. Specifically, at some point, the difference between the area of such a polygon, which we'll call P, and the area of the circle is less than $A - T$. Therefore, we have $A - P < A - T$, so $T < P$.



However, we showed on page 252 that the area of such a regular polygon is one-half the product of its perimeter, p, and its apothem, a. Since the perimeter of the polygon is smaller than the perimeter of the circle, we have $p < 2\pi r$. Moreover, the apothem of the polygon is clearly less than the radius of the circle. Therefore, we can use $p < 2\pi r$ and $a < r$ to write $P = ap/2 < (r)(2\pi r)/2 = T$. So, we have both $T < P$ and $P < T$. This is impossible, so we have reached a contradiction. We can thus conclude that our assumption $A < T$ was false.

See if you can complete the proof by showing that it is impossible to have $A > T$. As you might guess, you'll have to consider a polygon circumscribed about the circle!

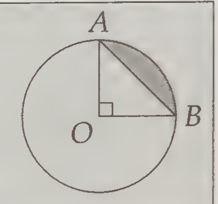
Source: Journey Through Genius by William Dunham

11.3 Funky Areas

We now know how to tackle the area of sectors and of triangles. We can put these two basic tools together to find the areas of all sorts of funky figures.

Problems

Problem 11.10: O is the center of the shown circle and $OA = 8$. The shaded region between chord $\overline{AB}$ and the circle is called a **circular segment**.

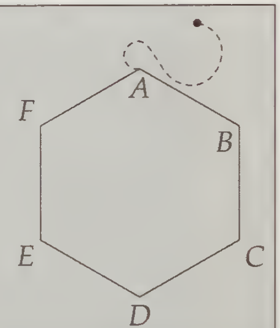


(a) Find the area of sector AOB.

(b) Find the area of $\triangle AOB$.

(c) Find the area of the shaded region.

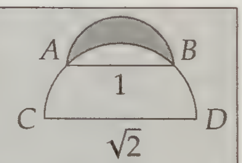
Problem 11.11: I have a barn that is a regular hexagon, as shown. Each side of the barn is 100 feet long. I tether my burro to point A with a 150 foot rope. In this problem we will find the area of the region in which my burro can graze.



(a) Imagine you are the burro. Sketch out the area in which you can graze. Pay close attention to what happens when you try to go past point B towards point C.

(b) Break your grazing region into sectors whose areas you can find, then find the area of the grazing region.

Problem 11.12: The shaded portion of the figure is called a **lune**. Given that $AB = 1$, $CD = \sqrt{2}$, and that $\overline{AB}$ and $\overline{CD}$ are diameters of the respective semicircles shown, we wish to find the area of the lune. *(Source: AMC 10)*

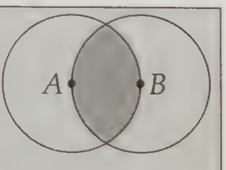


(a) Let O be the center of the larger semicircle. Draw $\overline{OA}$ and $\overline{OB}$. How long are these segments? What kind of triangle is $\triangle AOB$?

(b) Find the area of circular segment AB (i.e., the unshaded part of the small semicircle).

(c) Find the area of the lune.

Problem 11.13: Circle A passes through B and circle B passes through A. Given that $AB = 6$, we wish to find the area of the shaded region common to both circles.



(a) Let the two points where the circles intersect be X and Y. Connect A and B to each other and to X and Y.

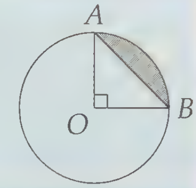
(b) What kind of triangles are $\triangle XAB$ and $\triangle YAB$?

(c) We know how to find the areas of circles, triangles, and circular segments. Find an expression for the shaded area in terms of pieces of the diagram whose areas you can find.

(d) Find the shaded area.

We know how to find the areas of circles, sectors of circles, triangles, and lots of other polygons. In this section we'll use that knowledge to find the area of even funkier shapes.

Problem 11.10: O is the center of the shown circle and $OA = 8$. The shaded region between chord $\overline{AB}$ and the circle is called a **circular segment**. Find the area of this circular segment.



Solution for Problem 11.10: We view the circular segment in terms of pieces whose areas we can find. The circular segment is what's left when we cut $\triangle AOB$ out of sector AOB. These areas are easy to find, so we have

$$\begin{aligned}\text{Area of circular segment } AB &= \text{Area of sector } AOB - \text{Area of } \triangle AOB\\ &= \left(\frac{90^\circ}{360^\circ}\right)\pi(8^2) - \frac{(AO)(BO)}{2}\\ &= 16\pi - 32.\end{aligned}$$

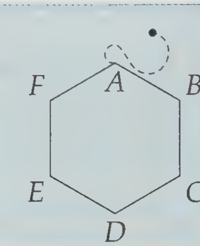
□

Concept: Nearly all funky area problems are solved by expressing the funky area as sums and/or differences of pieces whose areas we can easily find. The first step should be clearly expressing the funky area in terms of simple areas, as we did in Problem 11.10 when we wrote

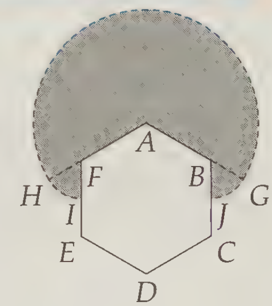
$$\text{Area of circular segment } AB = \text{Area of sector } AOB - \text{Area of } \triangle AOB.$$

Sometimes our region isn't already drawn for us.

Problem 11.11: I have a barn that is a regular hexagon, as shown. Each side of the barn is 100 feet long. I tether my burro to point A with a 150 foot rope. Find the area of the region in which my burro can graze.



Solution for Problem 11.11: First we have to figure out what the burro's region looks like. Say the burro is tethered at A and roams so far that the rope pulls taut. Suppose the burro keeps the rope taut and walks until part of the rope coincides with $\overline{AB}$. The burro then is at point G on $\overrightarrow{AB}$. Since $AB = 100$ feet and the rope is only 150 feet long, when the burro moves beyond G towards C, there's only 50 feet of rope for the burro to use. Hence, once the burro moves past G, the region he can reach is just sector JBG. We have the same situation on the other side, leading to sector HFI. Thus, our desired area is



$$\text{Area of major sector } HAG + \text{Area of sector } JBG + \text{Area of sector } HFI.$$

Since $\angle ABC = 120°$ (because $ABCDEF$ is a regular hexagon), we have $\angle JBG = 180° - 120° = 60°$. Therefore, sector JBG is $60°/360° = 1/6$ of a circle with radius 50 feet. Hence, its area is $(1/6)\pi(50^2) = 1250\pi/3$ square feet.

On the other end of the burro's range is sector HFI, which is exactly the same as sector JBG, so it contributes $1250\pi/3$ ft^2 to the burro's area. Finally we have the sector between major arc $\widehat{GH}$ and radii $\overline{AG}$ and $\overline{AH}$. Since $\angle FAB = 120°$, the sector is $(360° - 120°)/360° = 2/3$ of a circle. The radius of this sector is 150 feet, so its area is $(2/3)\pi(150^2) = 15000\pi$.

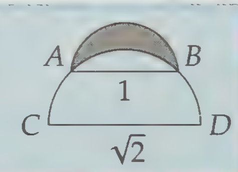
Combining these three parts gives us a grazing area of

$$15000\pi + 2\left(\frac{1250\pi}{3}\right) = \frac{47500\pi}{3} \text{ square feet.}$$

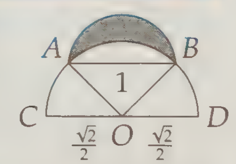
□

Once again, notice that one of our key steps is to express our desired region in terms of not-so-funky pieces we know how to tackle.

Problem 11.12: The shaded portion of the figure is called a **lune**. Given that $AB = 1$, $CD = \sqrt{2}$, and that $\overline{AB}$ and $\overline{CD}$ are diameters of the respective semicircles shown, find the area of the lune. *(Source: AMC 10)*



Solution for Problem 11.12: We start by noting that if we can find the area of circular segment AB, then we can find the area of the lune:



Area of lune = Area of small semicircle − Area of circular segment AB.

To get the area of the circular segment, we start by connecting its endpoints to the center, O, of its circle. Since

Area of circular segment AB = Area of sector AOB − Area of $\triangle AOB$,

we have

$$\begin{aligned}\text{Area of lune} &= \text{Area of small semicircle} - \text{Area of circular segment } AB\\ &= \text{Area of small semicircle} - (\text{Area of sector } AOB - \text{Area of } \triangle AOB)\\ &= \text{Area of small semicircle} - \text{Area of sector } AOB + \text{Area of } \triangle AOB.\end{aligned}$$

To find the area of sector AOB, we need $\angle AOB$. Since $CD = \sqrt{2}$, we have $OC = OA = OB = OD = \sqrt{2}/2$. Seeing those $\sqrt{2}$s makes us think of 45-45-90 triangles. We check and see that indeed $AO^2 + BO^2 = 1/2 + 1/2 = 1 = AB^2$, so $\angle AOB$ is a right angle since the sides of $\triangle AOB$ satisfy the Pythagorean Theorem.

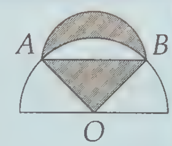
We can now find the areas of all of our pieces. The radius of the small semicircle is $1/2$, so its area is $\pi(1/2)^2/2 = \pi/8$. Sector AOB is $90°/360° = 1/4$ of a circle with radius $\sqrt{2}/2$, so its area is $(1/4)\pi(\sqrt{2}/2)^2 = \pi/8$. Finally, $[AOB] = (AO)(OB)/2 = 1/4$, so we have

$$\text{Area of lune} = \frac{\pi}{8} - \frac{\pi}{8} + \frac{1}{4} = \frac{1}{4}.$$

□

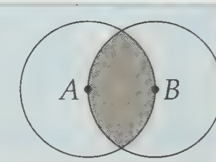
Again, look at our key step – expressing our funky area (the lune) in terms of not-so-funky areas (semicircle, sector, triangle).

Sidenote: You might have noticed that the area of the lune in Problem 11.12 equals the area of $\triangle AOB$. This isn't a coincidence! Moreover, the discovery that the area of the lune described in Problem 11.12 equals the area of $\triangle AOB$ led to one of the earliest notable proofs in math history.

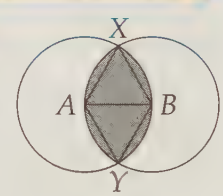


As we'll discuss on page 338, mathematicians tried in vain to square the circle (to 'square the circle' means construct a square with the same area as a given circle). However, the Greek mathematician Hippocrates succeeded in squaring the lune using the insight that the area of the lune of Problem 11.12 equals the area of the triangle with the lune's diameter and the center of the larger semicircle as vertices. See if you can walk in his footsteps by both re-creating the proof, and by performing the construction. *Source: Journey Through Genius by William Dunham*

Problem 11.13: Circle A passes through B and circle B passes through A. Given that $AB = 6$, find the area of the shaded region common to both circles.



Solution for Problem 11.13: We have a funky circular area, so we start by drawing some segments to cut the region into pieces we can handle. Some obvious candidates are $\overline{AB}$ connecting the centers, and the segments from the centers to the points where the circles meet. We thus form some triangles, sectors, and circular segments. We know how to handle these – we start by finding angles.



Since each of our segments is a radius of one (or both) of the circles, all of them are equal in length. Specifically, $AX = XB = AB = AY = YB = 6$. Therefore our two triangles are equilateral, so all of their angles are equal. We're now ready to express the shaded region in terms of pieces we can handle. We can do so in a few different ways. Here's one way:

$$\text{Shaded Area} = \text{Area of sector } XBY + \text{Area of sector } XAY - \text{Area of } XAYB.$$

We subtract $[XAYB]$ because when we add the sectors, we include the overlap, $[XAYB]$ twice. We could also have written

$$\text{Shaded Area} = [XAYB] + 4(\text{Area of circular segment } XB),$$

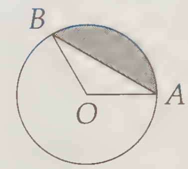
since all the circular segments are congruent. We'll take the first approach here since sectors are easier to deal with than circular segments. Since $\angle XBY = \angle XBA + \angle YBA = 120^\circ$, the area of sector XBY is $(120^\circ/360^\circ)\pi(6^2) = 12\pi$. Similarly, the area of sector XAY is 12π. Finally, $XAYB$ consists of two equilateral triangles with side length 6. Each has area $6^2\sqrt{3}/4 = 9\sqrt{3}$, so $[XAYB] = 2(9\sqrt{3}) = 18\sqrt{3}$. Finally, we have

$$\text{Shaded Area} = 24\pi - 18\sqrt{3}.$$

□

> **Concept:** In problems involving multiple circles, connecting the centers can be helpful. In problems involving intersecting circles, connecting the intersection points to the centers (and to each other) is often useful.

Exercises



11.3.1 Find the area of the shaded region given that O is the center of the circle, $\angle AOB = 120°$, and the radius of the circle is 6. **Hints:** 78

11.3.2 Each side of equilateral $\triangle XYZ$ has length 9. Find the area of the region inside the circumcircle of the triangle, but outside the triangle.

11.3.3 In the diagram at left below, $\triangle ABC$ is an equilateral triangle with side length 6. Arcs are drawn centered at the vertices connecting midpoints of consecutive sides, as shown. Find the area of the shaded region.

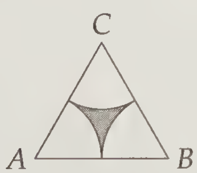


Figure 11.1: Diagram for Problem 11.3.3

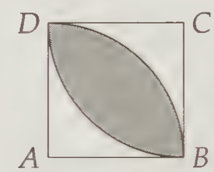


Figure 11.2: Diagram for Problem 11.3.4

11.3.4 In the diagram at right above, quarter-circles have been drawn centered at vertices A and C of square $ABCD$. Given that $AB = 6$, find the shaded area.

11.3.5★ In the diagram at left below, XOY is a quarter-circle. Semicircles are drawn with diameters $\overline{OX}$ and $\overline{OY}$ as shown. Find the area of the shaded region given that $XO = 4$. **Hints:** 405, 458

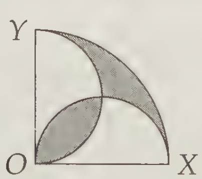


Figure 11.3: Diagram for Problem 11.3.5

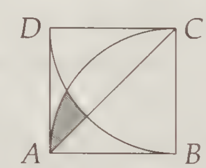


Figure 11.4: Diagram for Problem 11.3.6

11.3.6★ In the diagram at right above, $ABCD$ is a square of side length 4. Two quarter-circles and a diagonal are drawn as shown. Find the area of the shaded region. **Hints:** 121, 485, 450

Extra! 22/7 is a common approximation of π. 355/113 is an even better approximation. In 1914, the great Indian mathematician **Ramanujan** provided the uncanny approximation

$$\sqrt[4]{9^2 + \frac{19^2}{22}}.$$

How close is this to π? (And how in the world did he find this?)

11.4 Summary

Definitions:

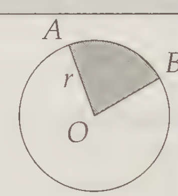
- The set of all points that are the same distance from a given point is a **circle**. The given point is the **center** of the circle, and the fixed distance is the **radius**.
- We often refer to a circle by its center using the symbol $\odot$: $\odot O$ refers to a circle with center O.
- A line that touches a circle in a single point is **tangent** to the circle, while a line that hits a circle in two points is a **secant**. A segment connecting two points on a circle is a **chord**, and a chord that passes through the center of its circle is a **diameter**.
- The portion of a circle that connects two points on the circle is an **arc**, which we denote with the endpoints of the arc: $\widehat{MN}$ is the shorter arc that connects M and N.
- The perimeter of a circle is called the circle's **circumference**.

Important: If a circle has diameter d and radius r, then:

- $d = 2r$.
- The circumference of the circle is πd, or $2\pi r$.
- The area of the circle is πr^2.

Definitions: A portion of a circle cut out by drawing two radii of the circle is called a **sector** of the circle. A portion of a circle between a chord and the arc of the circle connecting the endpoints of the chord is a **circular segment** of the circle.

Important: We can find the area of a sector of a circle of radius r given the central angle of the sector. In the figure to the right, since $\angle AOB$ is $(\angle AOB)/360°$ of the whole circle, we have:



$$\text{Area of sector } AOB = \left(\frac{\angle AOB}{360°}\right) r^2\pi.$$

We can find the area of a circular segment by first finding the area of the sector formed by drawing radii to the endpoints of the segment's chord, then subtracting the area of the triangle with these radii and the chord as sides.

Extra!

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \cdots$$

Problem Solving Strategies

Concepts:

- In complicated problems involving a circle, try connecting the center of the circle to important points on the circle.
- Start geometric word problems with a sketch.
- Nearly all funky area problems are solved by expressing the funky area as sums and/or differences of pieces whose areas we can easily find. The first step should be clearly expressing the funky area in terms of simple areas.
- In problems involving multiple circles, connecting the centers can be helpful. In problems involving intersecting circles, connecting the intersection points to the centers (and to each other) is often useful.

REVIEW PROBLEMS

11.14 Circle O has radius $3\sqrt{3}$. Points A and Z are on the circumference of the circle such that $\angle AOZ = 90^\circ$.

(a) Find the area of $\odot O$.

(b) Find the circumference of $\odot O$.

(c) Find the length of $\widehat{AZ}$.

(d) Find the area of sector AOZ.

(e) Find the area of circular segment AZ.

11.15 A giant earth-mover has rubber circular tires 11.5 feet in diameter. Given that there are 5280 feet in a mile, how many revolutions does each tire make during a 6-mile trip? (Answer to the nearest full revolution.) *(Source: MATHCOUNTS)*

11.16 Find the length of a 78° arc of a circle that has radius 14.

11.17 Find a formula that expresses the area, A, of a circle in terms of its circumference, C.

11.18 Radius $\overline{OA}$ of $\odot O$ is a diameter of $\odot B$. Radius $\overline{OB}$ of $\odot B$ is a diameter of $\odot C$.

(a) Find the ratio of the area of $\odot C$ to the area $\odot O$.

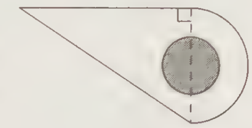
(b) Find CA/OA.

11.19 A 36° arc of $\odot C$ is 24π units long. Find the circumference and the area of the circle.

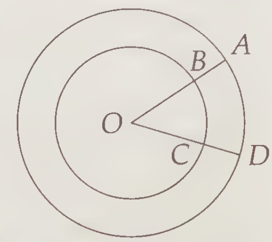
11.20 Regular hexagon $ABCDEF$ is inscribed in $\odot O$ with radius 6. What is the ratio of the circumference of the circle to the perimeter of the hexagon? **Hints:** 106

11.21 On each side of a right triangle a semicircle is constructed using that side as a diameter. How many square centimeters are in the area of the semicircle on the hypotenuse of a right triangle if the areas of the semicircles on the legs of the triangle are 36 and 64 square centimeters? *(Source: MATHCOUNTS)*

11.22 What is the number of square centimeters in the area that is not shaded in the diagram at right? The radius of the large semicircle is 1 centimeter, the radius of the small circle is 0.5 centimeters, and the length of the longer leg on the right triangle is 3 centimeters. *(Source: MATHCOUNTS)*



11.23 The larger circle at right has radius 1.5 times the smaller circle. Compute the ratio of the partial ring $ABCD$ to the area of sector BOC. *(Source: ARML)*

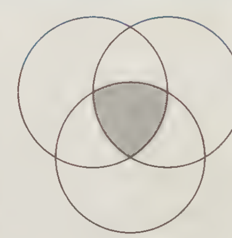


11.24 A rectangle that is 12 inches long and 5 inches wide is inscribed in a circle. What is the area of the region that is inside the circle but outside the rectangle? *(Source: MATHCOUNTS)*

11.25 Charlyn walks completely around the boundary of a square whose sides are exactly 5 km long. From any point on her path she can see exactly 1 km horizontally in all directions. What is the area of the region consisting of all points Charlyn can see during her walk? *(Source: AMC 10)*

Challenge Problems

11.26 Three circles of radius 12 lie in a plane such that each passes through the center of the other two. Find the area common to all three circles. **Hints:** 479



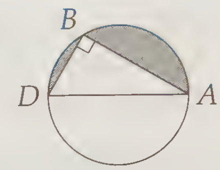
11.27 When my car has wheels with a diameter of 24 inches, its speedometer reports the correct speed of my car. I recently replaced my 24-inch wheels with 28-inch diameter wheels. I didn't change my speedometer, however. When the speedometer tells me the car is going 40 miles per hour (and I'm driving with my 28-inch wheels), how fast is my car really going? **Hints:** 515

11.28 Two congruent circular coins, $\mathcal{A}$ and $\mathcal{Z}$ are touching at point P. $\mathcal{A}$ is held stationary while $\mathcal{Z}$ is rolled around it one time in such a way that the two coins remain tangent at all times. How many times will $\mathcal{Z}$ revolve around its center? *(Source: MATHCOUNTS)* **Hints:** 568

11.29 The number of centimeters in the perimeter of a semicircle is numerically the same as the number of square centimeters in its area. What is the number of centimeters in the radius of the semicircle? *(Source: MATHCOUNTS)* **Hints:** 457

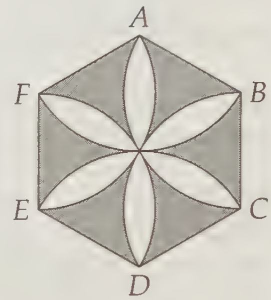
11.30 A cross-section of a river is a trapezoid with bases 10 ft and 16 ft and slanted sides of length 5 ft. At this section the water is flowing at π mph. A little ways downstream is a dam where the water flows through 4 identical circular holes at 16 mph. What is the radius of the holes? **Hints:** 383

Extra! March 14 (3/14) is known as Pi day.



11.31 Given that $\triangle ABD$ in the diagram at right is a right triangle with $BD = 8$ and $AB = 8\sqrt{3}$, find the total area of the shaded regions. **Hints:** 559

11.32 A circle of radius 3 passes through the center of a square with side length 2. Find the positive difference between the areas of the nonoverlapping portions of the figures. *(Source: HMMT)* **Hints:** 126



11.33 Six 120° arcs are drawn centered at the vertices of regular hexagon $ABCDEF$. Each arc connects two vertices of the hexagon as shown in the diagram at right. Given that $AF = 6$, find the total area of the shaded regions. **Hints:** 114

11.34 The number of inches in the perimeter of an equilateral triangle equals the number of square inches in the area of its circumscribed circle. What is the radius of the circle? *(Source: AMC 10)*

11.35 A circle has two parallel chords of length x that are x units apart. If the part of the circle included between the chords has area $2 + \pi$, find x. *(Source: HMMT)*

11.36★ Quarter circles are drawn centered at each vertex of square $ABCD$ as shown at left below. Given that $AB = 12$, find the area of the shaded region. **Hints:** 217, 272, 51

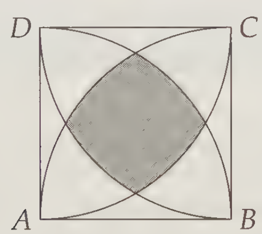


Figure 11.5: Diagram for Problem 11.36

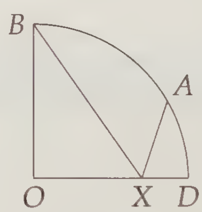


Figure 11.6: Diagram for Problem 11.37

11.37★ The figure at right above shows a quarter-circle of radius 1, with A on $\widehat{BD}$ such that $\angle AOD = 30^\circ$. What must the distance OX be such that the region bounded by $\overline{AX}$, $\overline{BX}$, and $\widehat{AB}$ occupies half the area of the quarter circle? **Hints:** 514, 174, 304

Extra! In the third century B.C., the Greek scholar **Eratosthenes** (the same man who invented the prime number sieve) was the director of the great library at Alexandria. At the time, other Greek scholars knew that the Earth was round, but had no idea how large it was. Eratosthenes learned of a deep well in Syene, a city south of Alexandria, whose bottom was sunlit at noon every year on the summer solstice. This meant that the sun was directly overhead. This gave Eratosthenes an ingenious idea: He could use this observation to measure the size of the Earth.

He assumed that since the sun was so far away, the rays coming from the sun were virtually parallel. Eratosthenes then measured a shadow (some say of a stick, some say of a tower) in Alexandria on the summer solstice, and found that the rays made an angle of approximately 7.2° with the object. Eratosthenes then required one more piece of data: the distance between Alexandria and Syene, which he estimated at 800 km.

Continued on the next page. . .

Extra! *. . . continued from the previous page*

Since there are 360° in a circle, Eratosthenes estimated the circumference of the Earth to be

$$800\text{ km} \times \frac{360°}{7.2°} = 40{,}000\text{ km}.$$

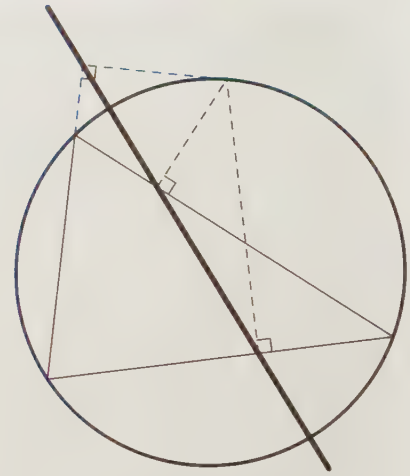
This is remarkably close to the actual figure, which is approximately 40,075.16 km.

Eratosthenes used an ancient measure of distance called a "stadium," whose actual length in today's units is uncertain. Also, Syene is 729 km, not 800 km, from Alexandria and slightly southwest instead of due south. Furthermore, Syene is not exactly on the Tropic of Cancer (where the sun is directly overhead on the summer solstice), but 55 km north. So a number of errors seem to have fortuitously cancelled out to give Eratosthenes such an accurate figure.

What is remarkable about this experiment is not the accuracy of Eratosthenes' estimate, but rather the conception of the experiment itself. As Dave Hanes, professor of astronomy at Queen's University, puts it on his website,

> The critical point is that Eratosthenes recognized the nature of the problem, found a method, and was able to derive an answer that was correct in spirit in the sense that **he correctly deduced that the Earth was an immense body that was very much larger in extent than the then-known lands of the Mediterranean basin, the home of Greek civilization at the time**. The **sense** of the discovery is the wonderful thing, not the mere accident that the numerical value was also correct.

Even if Eratosthenes had been off by a factor of two or more in his calculations, it would not have taken away from his brilliant insight into taking a simple observation of the sunlit well and deducing the circumference of the Earth, demonstrating the power of a simple idea.



The Simson Line

At a round table there is no dispute about place. – Italian Proverb

CHAPTER 12 Circles and Angles

In this chapter we will explore the many ways in which angles and arc measures are related.

12.1 Inscribed Angles

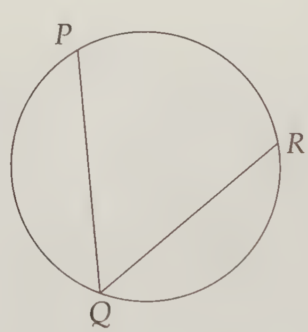
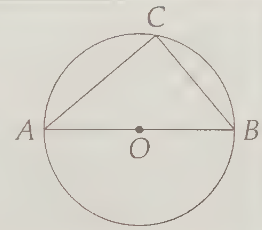


Figure 12.1: An Inscribed Angle

We say that $\angle PQR$ is **inscribed** in $\widehat{PR}$ in the diagram above because its vertex is on the circumference of the circle and its sides hit the circle at P and R. **Inscribed angles** are enormously useful tools in geometry; in this section we explore how inscribed angles are related to the arcs they cut off.

Problems

Problem 12.1: On page 175, we learned that the circumcenter of a right triangle is the midpoint of its hypotenuse. Thus, when we draw the circumcircle of a right triangle, the right angle is inscribed in a semicircle. In this problem, we investigate if it is true that any angle inscribed in a semicircle, such as $\angle ACB$ in the diagram at right, must be a right angle. Let O be the center of the circle.



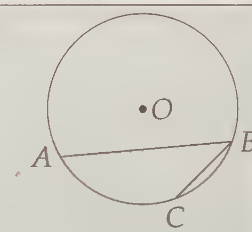
(a) Draw $\overline{OC}$. What do we know about $\triangle AOC$ and $\triangle BOC$?

(b) Let $\angle A = x$ and $\angle B = y$. What other angle measures x degrees? What other angle measures y degrees?

(c) Consider the sum of the angles in $\triangle ABC$ to show that $x + y = 90°$.

(d) Must $\angle ACB$ be a right angle?

Problem 12.2: In this problem we will find inscribed $\angle ABC$ shown at right, given that $\overset{\frown}{AC} = 80°$ and $\overset{\frown}{ACB} = 130°$.



(a) Draw $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$. Find $\angle AOC$ and $\angle COB$.

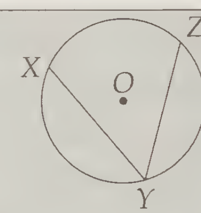
(b) Find $\angle OBC$ and $\angle OBA$.

(c) Use your answer to the previous part to find $\angle ABC$.

(d) Redo the problem with $\overset{\frown}{BC} = 64°$.

(e) Make a guess about how we can figure out $\angle B$ from the arcs without going through all the steps above. Can you prove your guess?

Problem 12.3: In Problem 12.1, we showed that if an angle is inscribed in a 180° arc (a semicircle), the measure of the angle is $180°/2 = 90°$. In this problem we will prove that when $\angle XYZ$ is inscribed in $\overset{\frown}{XZ}$ as shown in the diagram (with O, the center of the circle, inside $\angle XYZ$), then $\angle Y = \overset{\frown}{XZ}/2$.

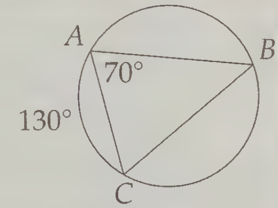


(a) Draw $\overline{OX}$, $\overline{OY}$, and $\overline{OZ}$. Let $\angle OXY = x$ and $\angle OZY = z$. Find $\angle XYZ$ in terms of x and z.

(b) Find $\angle XOY$, $\angle YOZ$, and $\angle XOZ$ in terms of x and z.

(c) Find $\overset{\frown}{XZ}$ in terms of x and z, then show that $\angle XYZ = \overset{\frown}{XZ}/2$.

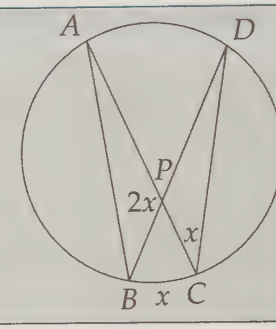
Problem 12.4: Given that $\triangle ABC$ is inscribed in the circle as shown, $\angle A = 70°$, and $\overset{\frown}{AC} = 130°$, find $\angle C$.



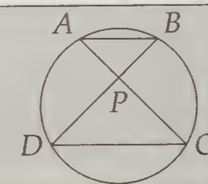
Extra! *The primary question is not what do we know, but how do we know it.*

–Aristotle

Problem 12.5: Find x given that $\angle APB = 2x$, $\angle ACD = x$, and $\widehat{BC} = x$.



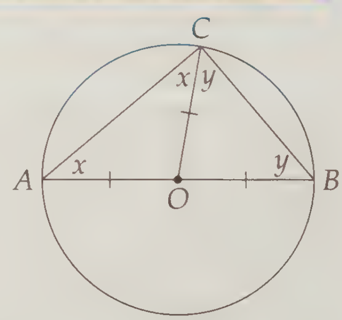
Problem 12.6: Chords $\overline{AC}$ and $\overline{BD}$ of a circle meet at P as shown. Given $AP = PB$, show that $\overline{AB} \parallel \overline{CD}$.



On page 175, we learned that when we draw the circumcircle of a right triangle, the right angle of the triangle cuts off a semicircle. We start this section by exploring whether the converse is true.

Problem 12.1: When an inscribed angle cuts off a 180° arc, we say that it is inscribed in a semicircle. Is such an angle necessarily a right angle?

Solution for Problem 12.1: Let $\angle ACB$ be inscribed in a semicircle. Since $\widehat{AB} = 180°$, $\overline{AB}$ is a diameter of the circle. Therefore, the midpoint of $\overline{AB}$, point O, is the center of the circle. Drawing $\overline{OC}$ forms two isosceles triangles since $AO = BO = CO$.



We suspect $\angle ACB$ is 90°, so we let $\angle ACO = x$ and $\angle BCO = y$ and try to show that $x + y = 90°$. Since $\triangle AOC$ and $\triangle BOC$ are isosceles, we have $\angle OAC = \angle OCA = x$ and $\angle OBC = \angle OCB = y$. From $\triangle ABC$, we have $\angle A + \angle ACB + \angle B = 180°$, so

$$x + (x + y) + y = 180°.$$

Therefore, $x + y = 90°$, which means $\angle ACB$ is a right angle. □

Sidenote: That any angle inscribed in a semicircle is a right angle is sometimes called **Thales Theorem**, after the mathematician **Thales of Miletus**, who is regarded by many as the father of the idea of 'proof'. He is credited by many with the first proofs that vertical angles are equal, that the sum of the angles in a triangle equals two right angles, and that the base angles of an isosceles triangle are equal.

Thales was an outstanding problem-solver. Legend has it that a farmer once complained to Thales, saying the farmer's donkey would lighten its load of salt on trips to the market by rolling in a river to dissolve the salt. Thales suggested the farmer load the donkey with sponges next time.
Source: Journey Through Genius by William Dunham

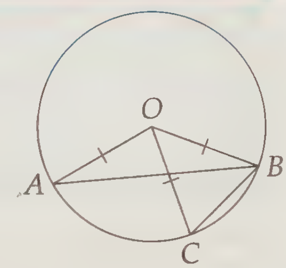
Important: Any angle inscribed in a semicircle is a right angle.

Having shown that any angle inscribed in a 180° arc has measure $180°/2 = 90°$, we should wonder if there's a general rule relating the measure of an inscribed angle to arcs in the circle.

Concept: When trying to prove something or find some general rule, investigating a few sample cases can be an excellent guide.

We'll try this strategy here.

Problem 12.2: Points A, B, and C are on $\odot O$ such that $\overset{\frown}{AC} = 80°$ and $\overset{\frown}{ACB} = 130°$. Find $\angle ABC$.



Solution for Problem 12.2: We know that the measure of an arc equals the angle formed by the radii that cut off the arc (we call such an angle a **central angle**). Therefore, we draw radii to A, B, and C, thus forming some isosceles triangles. Since $\angle BOC = \overset{\frown}{BC} = \overset{\frown}{AB} - \overset{\frown}{AC} = 50°$, we have

$$\angle OBC = \angle OCB = \frac{180° - 50°}{2} = 65°.$$

Similarly, $\angle AOB = \overset{\frown}{AB} = 130°$, so

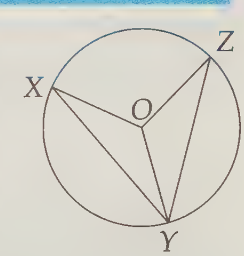
$$\angle OAB = \angle OBA = \frac{180° - 130°}{2} = 25°.$$

Therefore, $\angle ABC = \angle OBC - \angle OBA = 40°$.

We see that $\angle ABC = \overset{\frown}{AC}/2$ and we wonder if this is always the case. We can try changing $\overset{\frown}{BC}$ to see if that matters. If we let $\overset{\frown}{BC} = 64°$, we can go through the same series of calculations as above to find that, indeed, $\angle ABC$ is still 40°. □

Now that we have a specific case as a guideline, we'll try to prove that an inscribed angle is always half the arc it intercepts. Unfortunately, to completely prove this, we'll need a number of cases. We'll try one of these cases here, then leave the others for Exercises.

Problem 12.3: Prove that if $\angle XYZ$ is inscribed in a circle such that the center of the circle, O, is inside $\triangle XYZ$, then $\angle XYZ = \overset{\frown}{XZ}/2$.



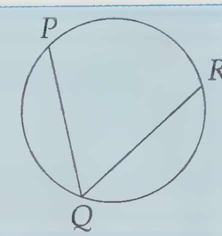
Solution for Problem 12.3: We use our specific case as a guide, and we connect the radii to X, Y, and Z to form isosceles triangles. We don't know any angles this time, so we'll have to use variables. Since we know we want to use our isosceles triangles, we let $\angle OYX = x$ and $\angle OYZ = z$, so that $\angle XYZ = x + z$. Isosceles triangles $\triangle XOY$ and $\triangle ZOY$ tell us that $\angle OXY = x$ and $\angle OZY = z$. Therefore, we have $\angle XOY = 180° - 2x$ from $\triangle OXY$, and $\angle ZOY = 180° - 2z$ from $\triangle OZY$. We can now find $\angle XOZ$ in terms of x and z:

$$\angle XOZ = 360° - \angle XOY - \angle YOZ = 2x + 2z = 2(x + z).$$

Therefore, $\overset{\frown}{XZ} = 2(x + z) = 2\angle XYZ$, so $\angle XYZ = \overset{\frown}{XZ}/2$. □

Important: The measure of an inscribed angle is one-half the measure of the arc it intercepts. For example,

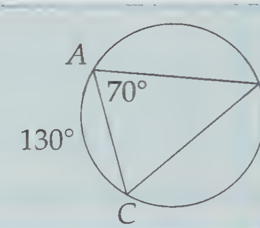
$$\angle PQR = \frac{\widehat{PR}}{2}.$$



Note that our proving that an angle inscribed in a semicircle is right is just a special case of this result. However, we draw special attention to the right angle case because right angles are so important in geometry.

Let's try using our newfound knowledge.

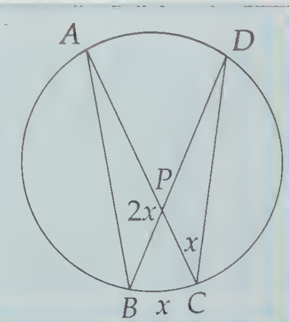
Problem 12.4: Given that $\triangle ABC$ is inscribed in the circle as shown, $\angle A = 70°$, and $\widehat{AC} = 130°$, find $\angle C$.



Solution for Problem 12.4: *Solution 1:* Since $\angle B$ is inscribed in $\widehat{AC}$, we have $\angle B = \widehat{AC}/2 = 65°$. Therefore, $\angle C = 180° - \angle A - \angle B = 45°$.

Solution 2: Since $\angle A$ is inscribed in $\widehat{BC}$, we have $\widehat{BC} = 2\angle A = 140°$. Therefore, we have $\widehat{AB} = 360° - \widehat{AC} - \widehat{BC} = 90°$. Since $\angle C$ is inscribed in $\widehat{AB}$, we have $\angle C = \widehat{AB}/2 = 45°$. □

Problem 12.5: Find x given that $\angle APB = 2x$, $\angle ACD = x$, and $\widehat{BC} = x$.



Solution for Problem 12.5: Since $\angle B$ and $\angle C$ are inscribed in the same arc, they must be equal (since each equals half the arc). Therefore, $\angle B = \angle C = x$. Since $\angle A$ is inscribed in $\widehat{BC}$, we have $\angle A = \widehat{BC}/2 = x/2$. Now we can use $\triangle APB$ to write an equation for x. Since

$$\angle A + \angle APB + \angle B = 180°,$$

we have

$$\frac{x}{2} + 2x + x = 180°.$$

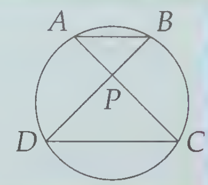
Solving this equation gives $x = 51\frac{3}{7}°$. □

We could have solved the last problem in many different ways, but our solution above illustrates a powerful principle that will be a crucial step in many problems when you move on to more advanced geometry.

Important: Any two angles that are inscribed in the same arc are equal.

We'll now use this important fact in a proof.

Problem 12.6: Chords $\overline{AC}$ and $\overline{BD}$ of a circle meet at P as shown. Given $AP = PB$, show that $\overline{AB} \parallel \overline{CD}$.



Solution for Problem 12.6: Since $AP = PB$, we have $\angle PBA = \angle PAB$. Since $\angle CAB$ and $\angle CDB$ are inscribed in the same arc, we have $\angle CDB = \angle CAB$. $\angle CAB$ is the same angle as $\angle PAB$, so we have $\angle CDB = \angle PAB = \angle PBA = \angle DBA$. Since $\angle CDB = \angle DBA$, we have $\overline{AB} \parallel \overline{CD}$. □

Exercises

In Exercises 12.1.5 and 12.1.6 below, do not assume that an inscribed angle equals half the arc it intercepts. You are asked to prove this fact for various cases in these two problems.

12.1.1 Points P, Q, and R are on $\odot O$ such that $\widehat{PQ} = 78^\circ$, $\widehat{QR} = 123^\circ$, and $\widehat{PQR} = 201^\circ$.

(a) Find $\angle QPR$.

(b) Find $\angle PQR$.

(c) Find $\angle PRQ$.

(d) Find $\angle POQ$.

(e) Find $\angle PQO$.

(f) Find $\angle POR$. (Be careful on this one!)

12.1.2 Points R and S are on $\odot E$ such that $\widehat{RS} = 50^\circ$. Point T is also on $\odot E$. Find all possible values of $\angle RTS$.

12.1.3 Points E, F and G are on $\odot O$ such that $\angle EFG = 48^\circ$ and $\angle GEF = 78^\circ$.

(a) Find $\angle EGF$.

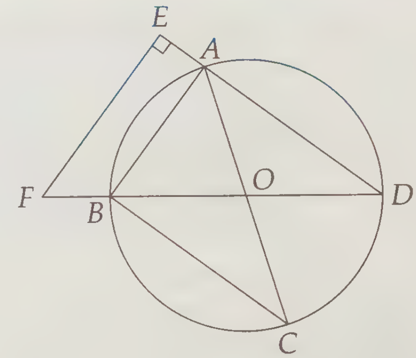
(b) Find $\widehat{EG}$.

(c) Find $\widehat{EFG}$.

Extra! *The advancement and perfection of mathematics are intimately connected with the prosperity of the state.*

–Napoleon Bonaparte

12.1.4 In the diagram, O is the center of the circle and $\angle ACB = 36°$.



(a) Find $\angle D$.

(b) Find $\widehat{AB}$.

(c) Find $\angle ABF$. (Do not assume $\overline{EF} \parallel \overline{AB}$ at this point.)

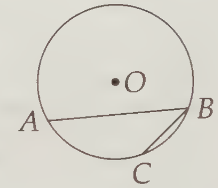
(d) Find $\angle EFB$. (Do not assume $\overline{EF} \parallel \overline{AB}$ at this point.)

(e) Find $\widehat{CD}$.

(f) Prove that $\overline{EF} \parallel \overline{AB}$.

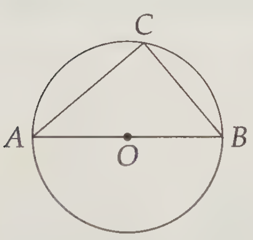
(g) Prove that $AD = BC$.

12.1.5 In Problem 12.2, we computed $\angle ABC$ in the diagram at right for specific measures of $\widehat{AC}$ and $\widehat{BC}$. We found that $\angle ABC = \widehat{AC}/2$.



(a) Prove that $\angle ABC = \widehat{AC}/2$ for all inscribed angles for which the center of the circle is outside the angle as shown in the figure at right. (Make sure you see how this problem is a little different from Problem 12.3.)

(b) Prove that $\angle ACB$ has measure equal to one-half major arc $\widehat{AB}$.



12.1.6 In Problem 12.1, we showed that an angle inscribed in a semicircle is a right angle. In this problem, we tackle the other angles that are formed when we connect the endpoints of a diameter of a circle to a point on the circle. $\overline{AB}$ is a diameter of $\odot O$ as shown. Prove that $\angle CAB = \widehat{BC}/2$. **Hints:** 83

12.1.7 Arcs $\widehat{WX}$ and $\widehat{YZ}$ of $\odot Q$ are congruent. Prove that either $\overline{WY} \parallel \overline{XZ}$ or $\overline{WZ} \parallel \overline{XY}$.

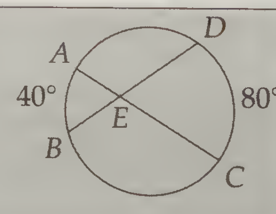
12.1.8 Back in Problem 9.5 of Chapter 9, we considered the regular 15-gon $ABCDEFGHILKLMNO$, and found $\angle ACD$ and $\angle ADE$. Find another solution with your new knowledge about angles and circles, by considering the circumcircle of the polygon to find $\angle ACD$ and $\angle ADE$.

12.2 Angles Inside and Outside Circles

A line that intersects a circle in two points is called a **secant**. In this chapter we'll explore relationships between the angles formed by intersecting secants and the arcs of circles these angles cut off.

Problems

Problem 12.7: Our goal in this problem is to find $\angle AEB$ given that $\widehat{AB} = 40°$ and $\widehat{DC} = 80°$.



(a) Draw $\overline{AD}$ to create inscribed angles. Find $\angle DAE$ and $\angle ADE$.

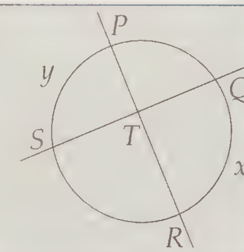
(b) Find $\angle AED$ and $\angle AEB$.

(c) How is $\angle AEB$ related to $\widehat{AB}$ and $\widehat{CD}$?

Problem 12.8: In this problem we will find a relationship between an angle inside a circle and the arcs the angle intercepts. In the diagram, let $\overset{\frown}{QR} = x$ and $\overset{\frown}{PS} = y$.

(a) Draw $\overline{PQ}$ and find $\angle PQT$ and $\angle QPT$ in terms of x and/or y.

(b) Find, with proof, $\angle QTR$ in terms of x and y.

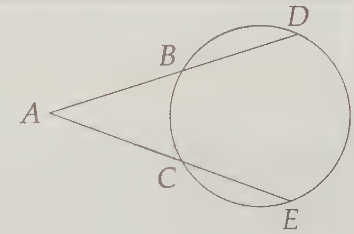


Problem 12.9: In this problem we will find $\angle DAE$ given $\overset{\frown}{DE} = 110°$ and $\overset{\frown}{BC} = 20°$.

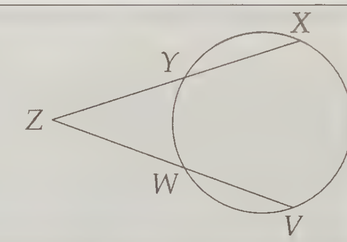
(a) Draw $\overline{BE}$ to create inscribed angles. Find $\angle BEC$ and $\angle DBE$.

(b) Find $\angle DAE$.

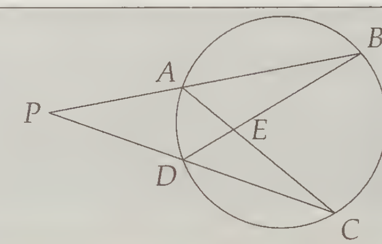
(c) Can you find a general relationship that must hold among $\angle A$, $\overset{\frown}{BC}$, and $\overset{\frown}{DE}$? (In other words, what if we replace 110° and 20° with x and y?)



Problem 12.10: Given that $\overset{\frown}{XV} = b$ and $\overset{\frown}{YW} = a$ (these are arc measures, not lengths), prove that $\angle Z = (b - a)/2$.



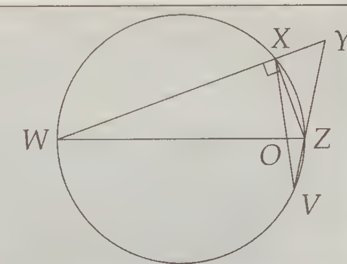
Problem 12.11: Given that $\overset{\frown}{AD} = 40°$, $\overset{\frown}{AB} = 120°$, and $\overset{\frown}{DC} = 100°$, find $\angle BPC$ and $\angle BEC$.



Problem 12.12: In the figure, $\angle Y = 58°$, $\angle W = 20°$, and $\overline{XZ} \perp \overline{WX}$.

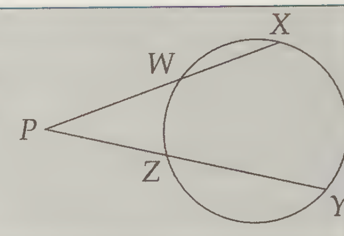
(a) Find $\angle WOV$.

(b) Find $\angle ZXV$.



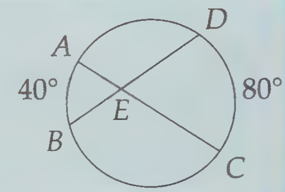
Extra! You'll sometimes see the letters 'Q.E.D.' at the end of a mathematical proof. These letters are an abbreviation for the Latin phrase *quod erat demonstratum*, which means 'which was to be proved.'

Problem 12.13: $\angle P$ meets the circle at four points as shown. The arcs $\overset{\frown}{WX}$, $\overset{\frown}{WZ}$, $\overset{\frown}{XY}$, and $\overset{\frown}{YZ}$ are in order from smallest to largest, and each of the last three is $42°$ larger than the arc listed before it. Find $\angle P$. **Hints:** 23 *(Source: Mandelbrot)*

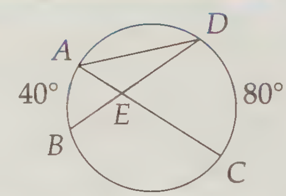


Having found a general formula for inscribed angles, we wonder how angles formed by secants are related to the arcs they intercept. We start, of course, with a specific example.

Problem 12.7: Find $\angle AEB$ given that $\overset{\frown}{AB} = 40°$ and $\overset{\frown}{DC} = 80°$.



Solution for Problem 12.7: We try to put what we just learned about inscribed angles to work by drawing $\overline{AD}$ to create some inscribed angles. We have $\angle DAC = \overset{\frown}{CD}/2 = 40°$ and $\angle ADB = \overset{\frown}{AB}/2 = 20°$. Therefore,



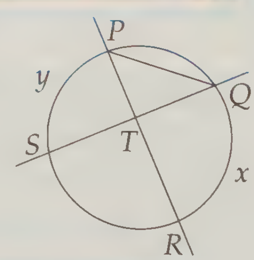
$$\angle AED = 180° - \angle EAD - \angle EDA = 120°,$$

so, $\angle AEB = 180° - \angle AED = 60°$. (Note that we could have seen that $\angle AEB = 20° + 40°$ since it is an exterior angle of $\triangle AED$.) □

With that basic case as a guide, we're ready to tackle finding a general formula.

Problem 12.8: Chords $\overline{QS}$ and $\overline{PR}$ intersect at T. Given that $\overset{\frown}{QR} = x$ and $\overset{\frown}{PS} = y$, find a formula for $\angle QTR$ in terms of x and y.

Solution for Problem 12.8: We form inscribed angles by adding $\overline{PQ}$ to our diagram. Since $\angle QPR$ is inscribed in $\overset{\frown}{QR}$, $\angle QPR = x/2$. Similarly, $\angle PQS = y/2$. Since $\angle QTR$ is an exterior angle of $\triangle PQT$, we have

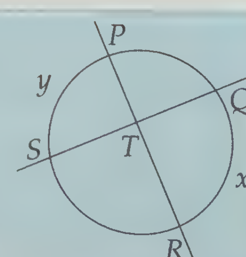


$$\angle QTR = \angle PQT + \angle QPT = \frac{x+y}{2}.$$

□

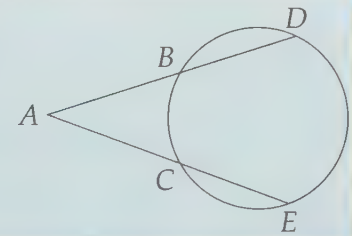
Important: The measure of the angle formed by two intersecting chords is the average of the measures of the arcs intersected by the chords. For example, in the diagram at right, we have

$$\angle PTS = \angle QTR = \frac{\overset{\frown}{PS} + \overset{\frown}{QR}}{2} = \frac{x+y}{2}.$$

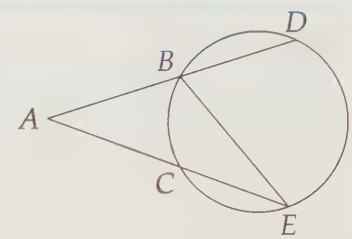


We've tackled angles on a circle and angles inside a circle. You know where we're headed now: outside the circle.

Problem 12.9: Find $\angle DAE$ given $\widehat{DE} = 110^\circ$ and $\widehat{BC} = 20^\circ$.



Solution for Problem 12.9: We succeeded with inscribed angles before, so we try using them again here by drawing $\overline{BE}$. We have $\angle BEC = 20^\circ/2 = 10^\circ$ and $\angle DBE = 110^\circ/2 = 55^\circ$. Thus, $\angle ABE = 180^\circ - \angle DBE = 125^\circ$, and we have



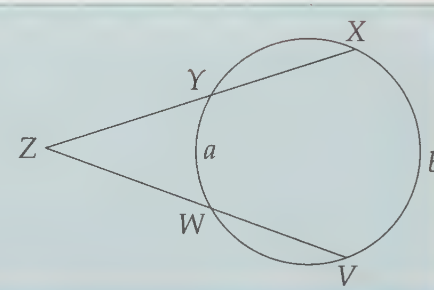
$$\angle DAE = 180^\circ - \angle ABE - \angle BEC = 180^\circ - 125^\circ - 10^\circ = 45^\circ.$$

Note that we could have seen that $\angle DBE$ is an exterior angle of $\triangle ABE$, so $\angle DBE = \angle BAE + \angle BEC$. As before, this gives us $\angle BAE = \angle DBE - \angle BEC = 45^\circ$. □

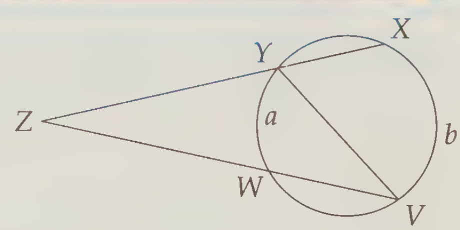
Now we have a clear path to prove the general formula.

Problem 12.10: Given that $\widehat{XV} = b$ and $\widehat{YW} = a$ (these are arc measures, not lengths) in the diagram, prove that

$$\angle Z = \frac{b-a}{2}.$$



Solution for Problem 12.10: We draw $\overline{YV}$ and have $\angle YVW = a/2$ and $\angle XYV = b/2$. Since $\angle XYV$ is an exterior angle of $\triangle VYZ$, we have $\angle XYV = \angle Z + \angle YVZ$. Therefore,

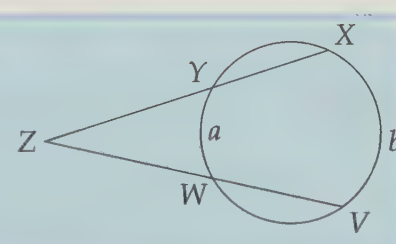


$$\angle Z = \angle XYV - \angle YVZ = \frac{b-a}{2}.$$

□

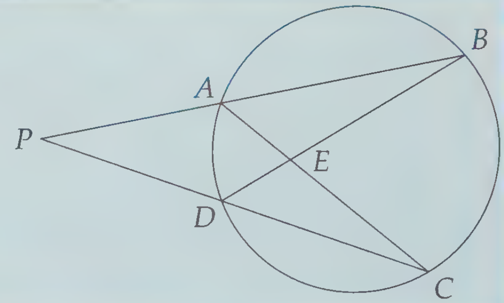
Important: Two secants that meet at a point outside a circle form an angle equal to half the difference of the arcs they intercept. For example, in the diagram we have

$$\angle Z = \frac{b-a}{2}.$$



Let's solve a few problems with our new relationships.

Problem 12.11: Given that $\widehat{AD} = 40°$, $\widehat{AB} = 120°$, and $\widehat{DC} = 100°$, find $\angle BPC$ and $\angle BEC$.



Solution for Problem 12.11: Since all the arcs together in a circle must sum to 360°, we have $\widehat{BC} = 360° - 120° - 40° - 100° = 100°$. Therefore, we have

$$\begin{aligned}\angle BEC &= \frac{\widehat{BC} + \widehat{AD}}{2} = 70° \\ \angle BPC &= \frac{\widehat{BC} - \widehat{AD}}{2} = 30°\end{aligned}$$

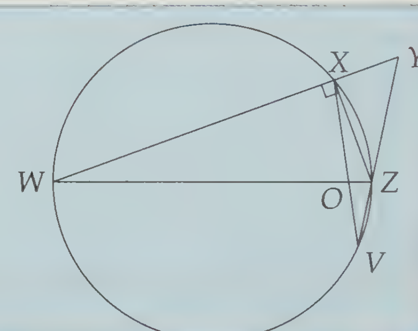
□

Not all problems can be solved in one or two steps. Here's an example in which we have to do a little more work.

Problem 12.12: In the figure, $\angle Y = 58°$, $\angle W = 20°$, and $\overline{XZ} \perp \overline{WX}$.

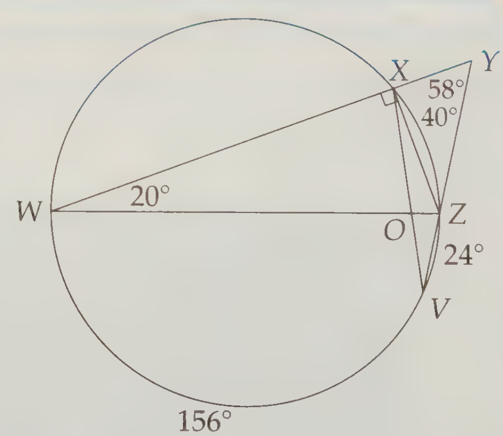
(a) Find $\angle WOV$.

(b) Find $\angle ZXV$.



Solution for Problem 12.12: We can't find the angles we want right away, but we have a circle with our angles, so we go hunting for measures of arcs. Since $\angle XWZ = 20°$, we know that $\widehat{XZ} = 40°$. Since $\angle Y = (\widehat{WV} - \widehat{XZ})/2$, we have

$$58° = \frac{\widehat{WV} - 40°}{2}.$$



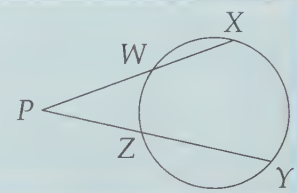
Solving for $\widehat{WV}$ gives $\widehat{WV} = 156°$. Now we can find $\angle WOV$:

$$\angle WOV = \frac{\widehat{WV} + \widehat{XZ}}{2} = \frac{156° + 40°}{2} = 98°.$$

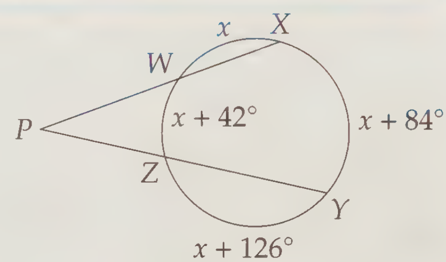
We can use $\widehat{VZ}$ to find $\angle ZXV$, so we focus on $\widehat{VZ}$. Since $\widehat{WVZ} = 2\angle ZXW = 180°$, we have $\widehat{VZ} = 180° - \widehat{WV} = 24°$. Therefore, $\angle ZXV = 12°$. □

Concept: Exploration can help unravel problems whose solutions require several steps. In geometry problems, effective exploration often requires drawing an accurate diagram and finding any angle measures or segment lengths you can. When you find any new information, you should label your diagram with your new information, as we did in the solution to Problem 12.12. This helps you see when you have the information you need to solve the problem.

Problem 12.13: $\angle P$ meets the circle at four points as shown. The arcs $\overset{\frown}{WX}$, $\overset{\frown}{WZ}$, $\overset{\frown}{XY}$, and $\overset{\frown}{YZ}$ are in order from smallest to largest, and each of the last three is 42° larger than the arc listed before it. Find $\angle P$. *(Source: Mandelbrot)*



Solution for Problem 12.13: We start by labeling our diagram with the given information. We let our smallest arc have measure x, and we are given that the other three arcs are $x + 42°$, $x + 84°$, and $x + 126°$. We could find x if we wanted to, but we don't have to, because



$$\angle P = \frac{\overset{\frown}{XY} - \overset{\frown}{WZ}}{2} = \frac{(x + 84°) - (x + 42°)}{2} = 21°.$$

□

Just because we use a variable in a problem doesn't mean we have to find its value. However, the variable did make finding the solution easier.

Concept: When you can't find any more lengths or angles in a problem but you still haven't solved it, try assigning a variable to one of the lengths or angles. Then, find other lengths or angles in terms of that variable. Finally, label your diagram with everything you find – this will make it easier to see when you have enough information to solve the problem.

Exercises

12.2.1 In the diagram at left below, $\overset{\frown}{AB} = 40°$, $\overset{\frown}{BC} = 103°$, and $\overset{\frown}{CD} = 83°$. Find $\angle AED$.

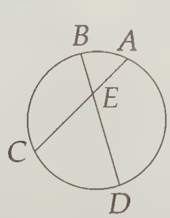


Figure 12.2: Diagram for Problem 12.2.1

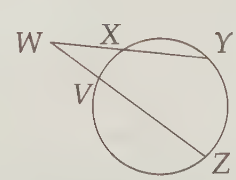


Figure 12.3: Diagram for Problem 12.2.2

12.2.2 Find $\overset{\frown}{YZ}$ and $\overset{\frown}{VZ}$ in the diagram at right above if $\overset{\frown}{XY} = 78°$, $\overset{\frown}{XV} = 32°$, and $\angle YWZ = 30.5°$.

12.2.3 Points P, Q, R, and S are in that order on $\odot O$. $\overline{PR}$ and $\overline{QS}$ meet at Z. Given that $\widehat{PQ} = 2\widehat{QR} = 3\widehat{RS} = 4\widehat{SP}$, find $\angle QZR$. **Hints:** 336

12.2.4 Given that $\widehat{XY} = 75°$, $\widehat{XYZ} = 117°$, and $\widehat{YZW} = 173°$ in the diagram at left below, find the following:

(a) $\angle YPZ$

(b) $\angle YOZ$

(c) $\angle XYW$

(d) $\angle OWZ$

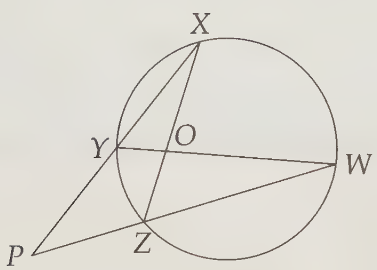


Figure 12.4: Diagram for Problem 12.2.4

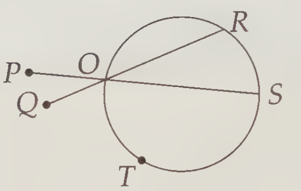


Figure 12.5: Diagram for Problem 12.2.5

12.2.5 Prove that $\angle POR = \dfrac{\widehat{OR} + \widehat{OTS}}{2}$ in the diagram at right above. **Hints:** 433

12.3 Tangents

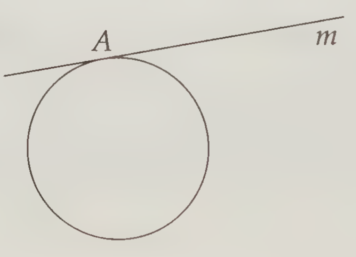
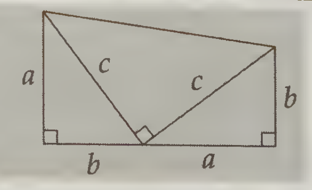


Figure 12.6: A Tangent Line

A line is **tangent** to a circle when it touches the circle in only one point. For example, line m is tangent to the circle at point A in the diagram above.

In this section we explore how a tangent line is related to the circle it touches.

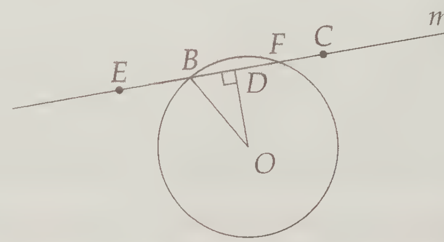
Extra! **President James A. Garfield** used the diagram at right to prove the Pythagorean Theorem. How did he do it? **Hints:** 5



Problems

Problem 12.14: Line m passes through point B on a circle with center O. In this problem, we will first show that if m is tangent to the circle, then $m \perp OB$. We do so by showing that if $\angle OBC$ is not $90°$, then m must hit the circle a second time. Then, we will show that if $m \perp \overline{OB}$, then m must be tangent to the circle.

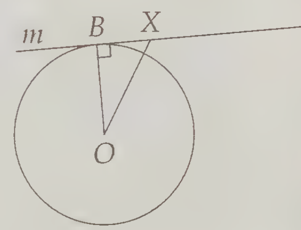
(a) Suppose $\angle OBC < 90°$ as shown at right. If so, there must be some other line through O besides $\overleftrightarrow{OB}$ that is perpendicular to m. Let this line hit m at point D. What kind of triangle is $\triangle OBD$?



(b) Show that $OD < OB$. Why does this mean that m must hit the circle a second time?

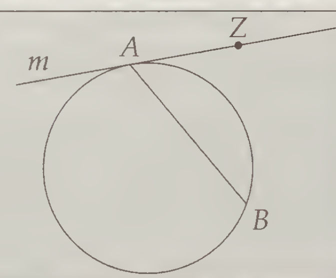
(c) Show that if $\angle OBC > 90°$, then m must hit the circle a second time. Conclude that if m is tangent to the circle at B, then $m \perp \overline{OB}$.

(d) Our first three parts show that a tangent line must be perpendicular to the radius drawn to the point of tangency. Now, we show that if a line m passes through a point B on a $\odot O$ such that $m \perp \overline{OB}$, then m must be tangent to the circle.



Suppose line m is meets $\odot O$ at B such that $m \perp \overline{OB}$. Let X be any point on m besides B. Consider $\triangle OBX$ and show that $OX > OB$. Does this prove that m can't hit the circle a second time if $m \perp \overline{OB}$?

Problem 12.15: Our goal in this problem is to find $\angle ZAB$ given that line m is tangent to the circle at A and $\overset{\frown}{AB} = 80°$.

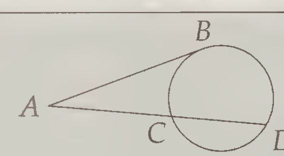


(a) Draw diameter $\overline{AC}$ and find $\overset{\frown}{BC}$.

(b) Find $\angle BAC$, $\angle ZAC$, and $\angle ZAB$.

(c) What relationship holds between $\angle ZAB$ and $\overset{\frown}{AB}$?

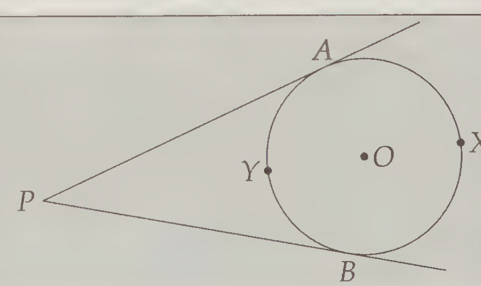
Problem 12.16: In this problem we will explore how an angle between a secant and a tangent is related to the arcs intercepted by the angle. In the diagram, let $\overset{\frown}{BD} = 110°$ and $\overset{\frown}{BC} = 40°$. $\overline{AB}$ is tangent to the circle at B.



(a) Draw $\overline{BD}$ and find $\angle BDC$ and $\angle DBA$.

(b) Find $\angle A$.

Problem 12.17: In this problem we investigate the relationship between the lengths of two tangent segments to a circle from the same point, as well as how to find the angle between them. $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are tangent to the circle, which has center O.



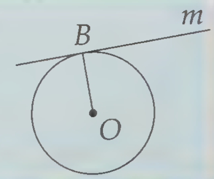
(a) Prove that $\triangle POB \cong \triangle POA$.

(b) Prove $\angle P = \dfrac{\overset{\frown}{AXB} - \overset{\frown}{AYB}}{2}$.

Problem 12.18: Point P is 25 centimeters from the center, O, of a circle with radius 10 centimeters. Point Q is on the circle such that $\overline{PQ}$ is tangent to the circle. Find PQ.

We start this section discovering a highly useful fact about tangents.

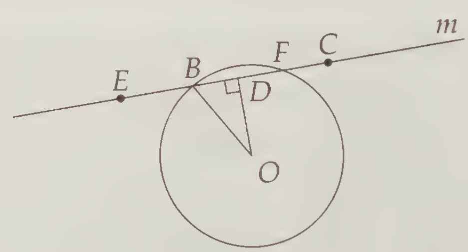
Problem 12.14: Given that line m is tangent to $\odot O$ at B, prove that radius $\overline{OB}$ is perpendicular to m. Conversely, prove that if a line m passes through point B on $\odot O$ such that $\overline{OB} \perp m$, then m is tangent to $\odot O$ at B.



Solution for Problem 12.14: This is one of those 'obvious' facts in geometry that isn't so easy to prove rigorously. Here's one 'obvious' solution that doesn't quite work:

Bogus Solution: Since the diagram is the same on both sides of $\overline{OB}$, the angles on both sides of B must be the same. These angles must also add to 180°, so they must be right angles.

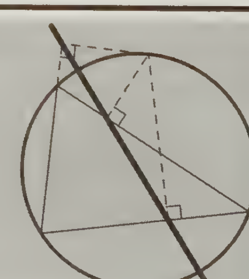
The problem with this solution is that the statement 'the diagram is the same on both sides of $\overline{OB}$' essentially assumes what we are trying to prove. Moreover, the solution is not clearly written – what do we mean by 'the angles on both sides of B'? We'll have to find another way.

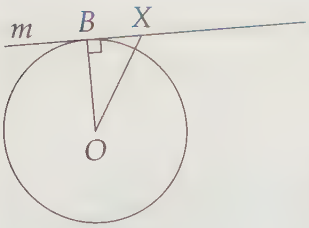


We let C be a point on m besides B. After trying and failing to directly show that $\angle OBC = 90°$, we decide to try proving it can't be anything else. First, we consider $\angle OBC < 90°$. Since $\angle OBC$ is not 90°, there must be some other point D on line m such that $\overline{OD} \perp m$. Hypotenuse $\overline{OB}$ of right triangle $\triangle OBD$ must be the largest side of $\triangle OBD$, so $OD < OB$. Since $\overline{OB}$ is a radius, this means that line m must go inside the circle if $\angle OBC < 90°$. Therefore, the line will hit the circle a second time. Specifically, consider the point F on m such that $BD = DF$ (i.e., D is the midpoint of $\overline{BF}$). By SAS Congruence we have $\triangle BDO \cong \triangle FDO$, so $OB = OF$. Therefore, point F is on $\odot O$, so if $\angle OBC < 90°$, then line m cannot be tangent to the circle since it will hit the circle twice.

Similarly, if $\angle OBC > 90°$, then $\angle OBE < 90°$, and we can use exactly the same logic to show that m cannot be a tangent line in this case either. Therefore, if m does not form a 90° angle with $\overline{OB}$, it cannot be a tangent line.

Extra! The feet of the perpendiculars from any point on the circumcircle of a triangle to the sides of the triangle are collinear, as shown at right. The line through these points is called the **Simson line**. See if you can prove that these three points are always collinear! **Hints:** 21



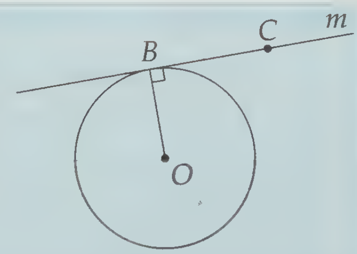


Unfortunately, we are not finished yet! We have shown that a line that makes any angle besides 90° can't be a tangent line, but we haven't shown that a line that makes a 90° angle is always a tangent. It might even be true that it's impossible for there to ever be a tangent line. Fortunately, it's not, and we'll prove it by considering line m at left, which hits the circle at B such that $m \perp \overline{OB}$. To show that m is a tangent line, we must show that it does not hit the circle again.

We consider some point on line m besides point B, and call this point X. Since $\overline{OX}$ is the hypotenuse of $\triangle OBX$, we must have $OX > OB$. Hence, point X must be outside the circle, since its distance from O is greater than the circle's radius. This is true for any point on line m besides point B, so all points on m besides B are outside the circle. Thus, line m is indeed tangent to the circle at point B. □

We therefore have a very non-obvious proof for our 'obvious' fact that:

Important: A line tangent to a circle is perpendicular to the radius drawn to the point of tangency. Conversely, a line drawn through a point on a circle that is perpendicular to the radius drawn to that point must be tangent to the circle. For example, for the diagram at right, we can write:

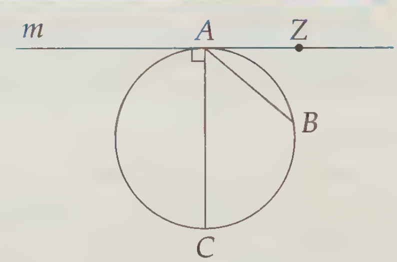


$\overleftrightarrow{BC}$ is perpendicular to radius $\overline{OB}$ if and only if $\overleftrightarrow{BC}$ is tangent to circle O.

You'll be seeing this 'radius perpendicular to a tangent at the point of tangency' again and again.

Problem 12.15: Line m is tangent to a circle at point A. Given point B on the circle such that $\widehat{AB} = 80^\circ$, find the acute angle formed by chord $\overline{AB}$ and line m.

Solution for Problem 12.15: We start by drawing diameter $\overline{AC}$, since we know this line is perpendicular to m at point A. Since $\widehat{AC}$ is a semicircle, $\widehat{BC} = 180^\circ - \widehat{AB} = 100^\circ$. Thus, $\angle BAC = \widehat{BC}/2 = 50^\circ$. Finally, $\angle ZAB = \angle ZAC - \angle BAC = 90^\circ - 50^\circ = 40^\circ$.

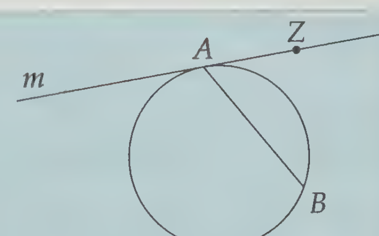


Note that $\angle ZAB = \widehat{AB}/2$. □

As an Exercise, you can follow these same steps to show:

Important: An angle formed by a tangent and a chord with the point of tangency as an endpoint equals one-half the arc intercepted by the angle. For example, in the figure at right, line m is tangent to the circle at A, so

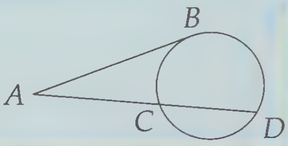
$$\angle ZAB = \frac{\widehat{AB}}{2}.$$



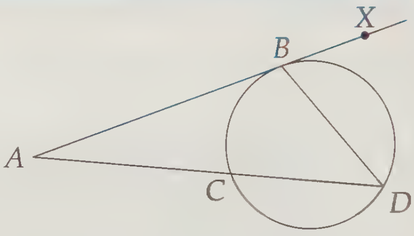
An angle between a tangent and a chord is also often referred to as an **inscribed angle**. Yes, this is the same name we give to an angle between two chords that share an endpoint; the measures of both types of angles equal half the measures of the arcs they cut off.

We now turn to angles formed by a tangent and a secant.

Problem 12.16: In the diagram, $\widehat{BD} = 110^\circ$ and $\widehat{BC} = 40^\circ$. $\overleftrightarrow{AB}$ is tangent to the circle at B. Find $\angle BAD$.



Solution for Problem 12.16: Since this looks so much like problems involving an angle between two secants (such as Problem 12.9), we try the same approach by drawing $\overline{BD}$. We have $\angle BDC = \widehat{BC}/2 = 20^\circ$ and $\angle XBD = \widehat{BD}/2 = 55^\circ$.

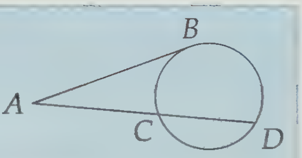


Finally, since $\angle XBD$ is an exterior angle of $\triangle ABD$, we have $\angle XBD = \angle BAD + \angle BDC$, so $\angle BAD = \angle XBD - \angle BDC = 55^\circ - 20^\circ = 35^\circ$.□

Unsurprisingly, the angle formed by a tangent and a secant has the same relationship to the arcs the angle intercepts as an angle formed by two secants does.

Important: The angle formed by a tangent and a secant is half the difference of the intercepted arcs. For example,

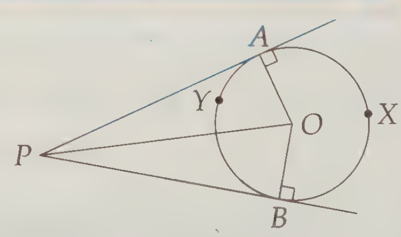
$$\angle BAD = \frac{\widehat{BD} - \widehat{BC}}{2}.$$



Next we consider two tangents from the same point to the same circle.

Problem 12.17: Segments $\overline{PA}$ and $\overline{PB}$ are tangent to the same circle at A and B, respectively. Prove that $PA = PB$, and that $\angle APB$ equals half the difference of major arc $\widehat{AB}$ and minor arc $\widehat{AB}$.

Solution for Problem 12.17: We start by drawing the radii to the points of tangency, thus forming right angles. We also draw $\overline{PO}$ to complete a pair of right triangles. Since $AO = OB$, we have $\triangle OAP \cong \triangle OBP$ by HL Congruence. Therefore, $PA = PB$.



We can find $\angle APB$ is a variety of ways. One is to use quadrilateral $PAOB$ to note that

$$\angle APB = 360^\circ - \angle PAO - \angle AOB - \angle OBP = 180^\circ - \angle AOB = 180^\circ - \widehat{AYB}.$$

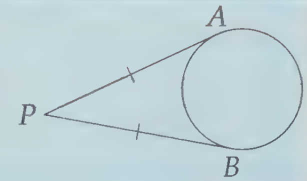
Since $\widehat{AXB} = 360^\circ - \widehat{AYB}$, we also have

$$\frac{\widehat{AXB} - \widehat{AYB}}{2} = \frac{360^\circ - 2\widehat{AYB}}{2} = 180^\circ - \widehat{AYB} = \angle APB.$$

□

The information about the angle between the tangents is unsurprising, as it's essentially the same as the angle between two secants. However, the information about the two tangents from the same point is new, and is one of the most useful tangent facts:

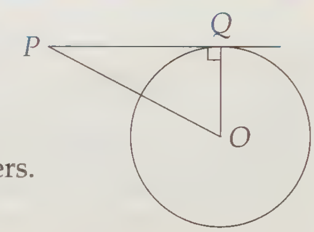
Important: As shown in the diagram at right, we can draw two tangent segments to a circle from a point outside the circle. These tangents are always equal to each other in length.



We'll finish with a very common problem involving tangents.

Problem 12.18: Point P is 25 centimeters from the center, O, of a circle with radius 10 centimeters. Point Q is on the circle such that $\overline{PQ}$ is tangent to the circle. Find PQ.

Solution for Problem 12.18: We start by drawing a radius to the point of tangency as shown. Since $\overline{PQ}$ is tangent to the circle, $\triangle POQ$ is a right triangle. Thus, $OP^2 = OQ^2 + PQ^2$, so



$$PQ = \sqrt{OP^2 - OQ^2} = \sqrt{25^2 - 10^2} = \sqrt{5^2(5^2 - 2^2)} = 5\sqrt{5^2 - 2^2} = 5\sqrt{21} \text{ centimeters.}$$

Notice how a little factoring makes our arithmetic easier. □

Exercises

12.3.1 Points U and I are on $\odot O$ such that major arc $\widehat{UI}$ is twice as long as minor arc $\widehat{UI}$. P is outside the circle such that $\overline{PU}$ and $\overline{PI}$ are tangent to $\odot O$.

(a) Find minor arc $\widehat{UI}$.

(b) Find $\angle UPI$.

(c) Find $\angle PIO$.

(d) Find $\angle IOU$.

(e) Find $\angle PIU$.

12.3.2 Point T is outside $\odot O$. X is on $\odot O$ such that $\overline{TX}$ is tangent to $\odot O$. The radius of $\odot O$ is 6 and $TX = 12$. Find TO.

12.3.3 Generalize Problem 12.15 by proving that the angle formed by a tangent and a chord that has the point of tangency as an endpoint is half the arc intercepted by the angle.

12.3.4 Y and Z are on $\odot G$ and X is outside $\odot G$ such that $\overline{XY}$ and $\overline{XZ}$ are tangent to $\odot G$. Given that $\angle YXZ = 51°$, find the measure of minor arc $\widehat{YZ}$.

12.3.5 Given a circle and a point outside the circle, it is intuitively clear that there are exactly two lines through the point that are tangent to the circle. However, intuitively isn't good enough – we're going to prove it. Let the point be P and the circle be $\odot O$.

(a) Consider the circle with diameter $\overline{OP}$. Call this circle C. Why must C hit $\odot O$ in at least two different points?

(b) Why is it impossible for C to hit $\odot O$ in *three* different points? **Hints:** 530

(c) Let the points where C hits $\odot O$ be A and B. Prove that $\angle PAO = \angle PBO = 90°$.

(d) Prove that $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$.

(e)★ Now for the tricky part – proving that these are the only two tangents. Suppose D is on $\odot O$ such that $\overline{PD}$ is tangent to $\odot O$. Why must D be on C? **Hints:** 478

(f) Why does the previous part tell us that $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are the only lines through P tangent to $\odot O$?

12.3.6★ $\odot O$ is tangent to all four sides of rhombus $ABCD$, $AC = 24$, and $AB = 15$.

(a) Prove that $\overline{AC}$ and $\overline{BD}$ meet at O (i.e., prove that the intersection of the diagonals of $ABCD$ is the center of the circle.) **Hints:** 439

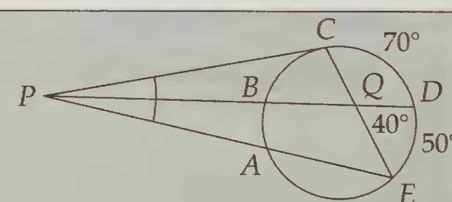
(b) What is the area of $\odot O$? **Hints:** 508

12.4 Problems

Problems

Problem 12.19: A quadrilateral is said to be a **cyclic quadrilateral** if a circle can be drawn that passes through all four of its vertices. Prove that if $ABCD$ is a cyclic quadrilateral, then $\angle A + \angle C = 180°$.

Problem 12.20: In the figure, $\overline{PC}$ is tangent to the circle and $\overline{PD}$ bisects $\angle CPE$. Furthermore, $\widehat{CD} = 70°$, $\widehat{DE} = 50°$, and $\angle DQE = 40°$. In this problem we determine the measure of the arc from A to E that does not include point C.



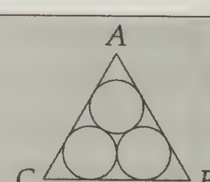
(a) Find $\widehat{BC}$.

(b) Find $\angle CPB$.

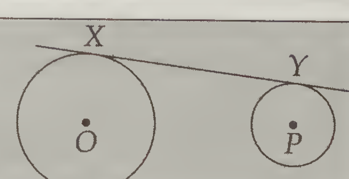
(c) Find $\widehat{AB}$.

(d) Finish the problem.

Problem 12.21: Three congruent circles with radius 1 are drawn inside equilateral $\triangle ABC$ such that each circle is tangent to the other two and to two sides of the triangle. Find the length of a side of $\triangle ABC$. **Hints:** 428

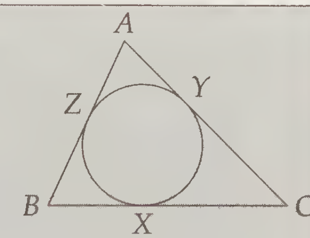


Problem 12.22: $\overleftrightarrow{XY}$ is tangent to both circle O and circle P. Given that $OP = 40$, and the radii of circles O and P are 31 and 7, respectively, find XY. **Hints:** 548, 267



Problem 12.23: Median $\overline{AM}$ of $\triangle ABC$ has length 8. Given that $BC = 16$ and $AB = 9$, find the area of $\triangle ABC$. **Hints:** 25, 187

Problem 12.24: The diagram shows the incircle of $\triangle ABC$. X, Y, and Z are the points of tangency where the incircle touches the triangle. In this problem we will find an expression for AZ in terms of the sides of the triangle.



(a) Find equal segments in the diagram and assign them variables.

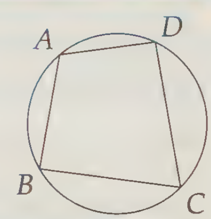
(b) Let $AB = c$, $AC = b$, and $BC = a$. Use your variables from the first part to write equations that include these lengths.

(c) Solve your resulting equations for AZ. **Hints:** 157

We'll now apply our knowledge of circles, angles, and tangents to more challenging problems and develop some useful geometric concepts.

Problem 12.19: A quadrilateral is said to be a **cyclic quadrilateral** if a circle can be drawn that passes through all four of its vertices. Prove that if $ABCD$ is a cyclic quadrilateral, then $\angle A + \angle C = 180°$.

Solution for Problem 12.19: Since $\angle A$ and $\angle C$ are inscribed angles, we have



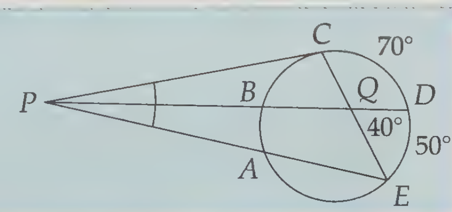
$$\angle A + \angle C = \frac{\widehat{BCD}}{2} + \frac{\widehat{BAD}}{2} = \frac{\widehat{BCD} + \widehat{BAD}}{2} = \frac{360°}{2} = 180°.$$

□

Important: A quadrilateral is a **cyclic quadrilateral** if a circle can be drawn that passes through all four of its vertices. Such a quadrilateral is said to be **inscribed** in the circle. The opposite angles of any cyclic quadrilateral sum to 180°.

You'll be seeing a lot more of cyclic quadrilaterals when you move into more advanced geometry.

Problem 12.20: In the figure, $\overline{PC}$ is tangent to the circle and $\overline{PD}$ bisects $\angle CPE$. If $\widehat{CD} = 70°$, $\widehat{DE} = 50°$, and $\angle DQE = 40°$, then determine the measure of the arc from A to E that does not include point C.



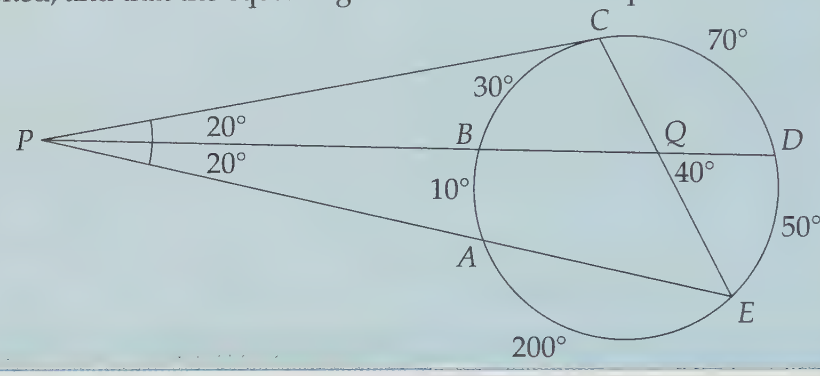
Solution for Problem 12.20: We can't directly find the desired arc, so we try finding whatever we can. First, we note that since $\angle DQE$ is the average of $\widehat{DE}$ and $\widehat{CB}$, we have $\widehat{CB} = 30°$. Now that we have $\widehat{CB}$, we can find $\angle CPD$:

$$\angle CPD = (\widehat{CD} - \widehat{CB})/2 = 20°.$$

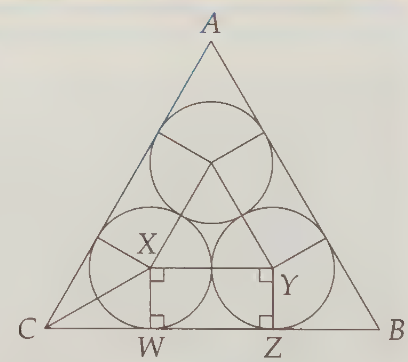
Since $\overline{DP}$ bisects $\angle CPE$, we know $\angle DPE = \angle CPD = 20°$. Because $\angle DPE = (\widehat{DE} - \widehat{AB})/2$, we find $\widehat{AB} = 10°$.

Now that we have all the other arcs of the circle, we can find our desired arc by subtracting from $360°$: $360° - 50° - 70° - 30° - 10° = 200°$. □

Concept: Label all the given information in the problem, and all the information you find as you find it. This will help you discover new facts about the problem. For example, your final diagram in the last problem might look like the figure below. Notice that our various arc measures and angles are marked, and that the equal angles at P are marked equal.



Problem 12.21: Three congruent circles with radius 1 are drawn inside equilateral $\triangle ABC$ such that each circle is tangent to the other two and to two sides of the triangle. Find the length of a side of $\triangle ABC$.

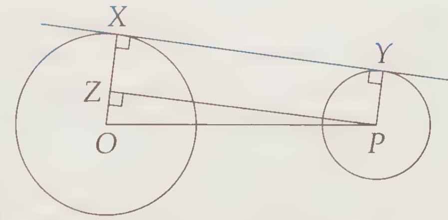


Solution for Problem 12.21: We need to create simple figures to work with, so we start by connecting the centers of our circles and drawing radii to tangent points. (Note that connecting the centers of the circles is the same as drawing radii to where the circles are tangent.) Since $WXYZ$ is a rectangle (because $XW = YZ$, $\overline{XW} \parallel \overline{YZ}$, and $\angle XWZ = 90°$), we have $WZ = XY = 2$. Hence, we need only find CW to finish, since BZ is the same as CW.

We draw $\overline{CX}$ to build a right triangle and note that this segment bisects $\angle ACB$ because circle X is tangent to both $\overline{AC}$ and $\overline{BC}$ (and hence its center is equidistant from them). Since $\angle XCW = (\angle ACB)/2 = 30°$, $\triangle CXW$ is a 30-60-90 triangle. Thus, $CW = XW\sqrt{3} = \sqrt{3}$. Finally, we have $BC = BZ + ZW + WC = 2 + 2\sqrt{3}$. □

Concept: When you have tangents in a problem, it's often very helpful to draw radii to points of tangency to build right triangles. When you have tangent circles, connect the centers. (In fact, if you have multiple circles in a problem, connecting the centers will sometimes help even when the circles aren't tangent.)

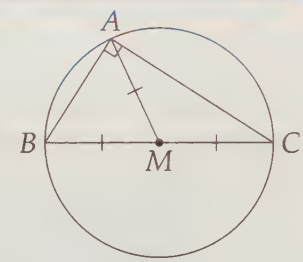
Problem 12.22: X is on circle O and Y on circle P such that $\overleftrightarrow{XY}$ is tangent to both circles. Given that $OP = 40$, and the radii of circles O and P are 31 and 7, respectively, find XY.



Solution for Problem 12.22: We start with the usual segments to draw: $\overline{XY}$, the radii to points of tangency, and the segment connecting the centers. We still don't have a right triangle to work with, but we do know that radii $\overline{OX}$ and $\overline{PY}$ are both perpendicular to tangent $\overline{XY}$ as shown. Since both $\overline{OX}$ and $\overline{PY}$ are perpendicular to the same line, they are parallel. We make a right triangle and a rectangle by drawing a line through P parallel to $\overline{XY}$. $ZX = PY = 7$, so $OZ = OX - ZX = 24$. Since $OP = 40$, we have $PZ = 32$ from right triangle $\triangle OZP$. Since $XYPZ$ is a rectangle, we have $XY = ZP = 32$, so the length of the common tangent is 32.

$\overline{XY}$ is called a common *external* tangent of the two circles. As an Exercise, you'll find the length of the common *internal* tangent, too. □

Problem 12.23: Median $\overline{AM}$ of $\triangle ABC$ has length 8. Given that $BC = 16$ and $AB = 9$, find the area of $\triangle ABC$.



Solution for Problem 12.23: When we draw the figure and label all our lengths, we see that $BM = AM = CM = 8$. Therefore, a circle centered at M with radius 8 goes through all three vertices of $\triangle ABC$. Since $\overline{BC}$ is a diameter of this circle, $\angle BAC$ is inscribed in a semicircle and therefore must be a right angle. So, $AC = \sqrt{BC^2 - AB^2} = 5\sqrt{7}$. $\triangle ABC$ is a right triangle, so its area is half the product of its legs:

$$[ABC] = \frac{(AB)(AC)}{2} = \frac{(9)(5\sqrt{7})}{2} = \frac{45\sqrt{7}}{2}.$$

□

Using this same reasoning, we can also prove this important fact:

Important: If the length of a median of a triangle is half the length of the side to which it is drawn, the triangle must be a right triangle. Moreover, the side to which this median is drawn is the hypotenuse of the right triangle.

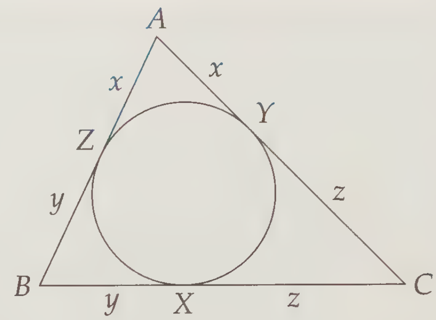
We can also look to this problem for some important problem solving techniques:

Concept: When stuck on a problem, always ask yourself 'Where have I seen something like this before?' In Problem 12.23, we have a median that is half the side to which it is drawn. This should make us think of right triangles, since the median to the hypotenuse of a right triangle is half the hypotenuse. Then we go looking for right triangles.

Concept: Always be on the lookout for right triangles.

Problem 12.24: Let Z be the point where the incircle of $\triangle ABC$ meets $\overline{AB}$. Find AZ in terms of the sides of $\triangle ABC$.

Solution for Problem 12.24: We start by labeling the equal tangents from the vertices as shown in the diagram. We want to relate x to the sides of the triangle, so we write the sides of the triangle in terms of x, y, and z. We let the sides of the triangle be $AB = c$, $AC = b$, and $BC = a$ and we have:



$$\begin{aligned} AB &= c = x+y \\ AC &= b = x+z \\ BC &= a = y+z \end{aligned}$$

We want x in terms of a, b, and c. Adding the three equations will give us $x+y+z$, which we can use with $y+z=a$ to find x:

$$a+b+c = 2(x+y+z) \qquad \text{so} \qquad x+y+z = \frac{a+b+c}{2}.$$

We can then subtract the equation $y+z=a$ from $x+y+z=(a+b+c)/2$ to find

$$x = \frac{a+b+c}{2} - a = s-a,$$

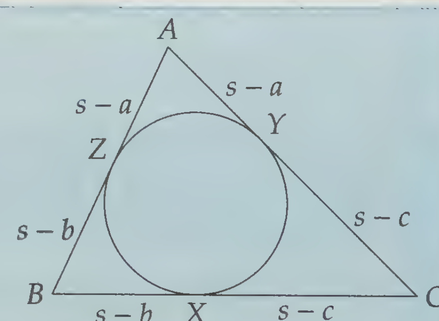
where s is the semiperimeter (half the perimeter) of the triangle. Similarly, $y = s-b$ and $z = s-c$. □

Concept: Symmetric systems of equations can often be easily solved by either multiplying all the equations or adding them.

Important: The lengths from the vertices of $\triangle ABC$ to the points of tangency of its incircle are given as follows:

$$\begin{aligned} AZ &= AY = s-a \\ BZ &= BX = s-b \\ CX &= CY = s-c \end{aligned}$$

where $AB = c$, $AC = b$, and $BC = a$, and the semiperimeter of $\triangle ABC$ is s.



Exercises

Problems 12.4.3, 12.4.4, 12.4.5, and 12.4.8 are very important relationships that you'll be seeing again in your study of more advanced geometry. Be sure to pay special attention to them.

12.4.1 Is every quadrilateral cyclic?

12.4.2 Prove the following about cyclic quadrilaterals:

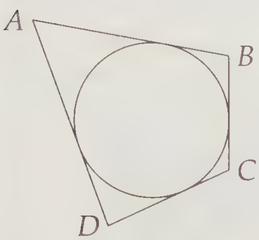
(a) A cyclic parallelogram must be a rectangle.

(b) A cyclic rhombus must be a square.

(c) A cyclic trapezoid must be isosceles.

12.4.3 Prove that if median $\overline{XM}$ of $\triangle XYZ$ has half the length of $\overline{YZ}$, then $\triangle XYZ$ is a right triangle with hypotenuse $\overline{YZ}$.

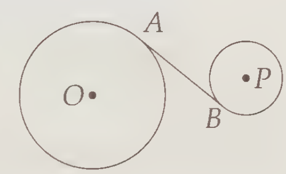
12.4.4 $ABCD$ is a cyclic quadrilateral. Prove that $\angle ACB = \angle ADB$.



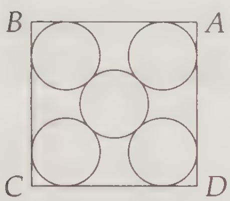
12.4.5 Quadrilateral $ABCD$ in the diagram at left is a **circumscribed quadrilateral**, meaning that it is circumscribed about a circle, so the circle is tangent to all four sides of $ABCD$. Show that $AB + CD = BC + AD$. **Hints:** 570

12.4.6 Does every quadrilateral have an inscribed circle (a circle tangent to all four sides), as $ABCD$ does in the previous problem?

12.4.7 In the figure at right, $\overline{AB}$ is tangent to both $\odot O$ and $\odot P$. The radius of $\odot O$ is 8, the radius of $\odot P$ is 4, and $OP = 36$. find AB. (A common tangent like $\overline{AB}$ is sometimes called the **common internal tangent** of two circles.) **Hints:** 591



12.4.8 Prove that the inradius of a right triangle with legs of length a and b and hypotenuse c is $(a + b - c)/2$.



12.4.9★ The five circles in the diagram are congruent and $ABCD$ is a square with side length 4. The four outer circles are each tangent to the middle circle and to the square on two sides as shown. Find the radius of each of the circles. **Hints:** 543, 416

12.5 Construction: Tangents

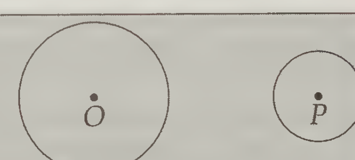
Now you'll use your newfound tangent knowledge to construct tangents to circles.

Problems

Problem 12.25: Given a circle with center O and point A on the circle, construct a line through A that is tangent to the circle.

Problem 12.26: Given a circle with center O and point P outside the circle, construct a line through P that is tangent to the circle.

Problem 12.27: Nonintersecting circles with centers O and P are shown. Construct a line that is tangent to *both* circles.



Extra! *Seek simplicity, and distrust it.*

–Alfred North Whitehead

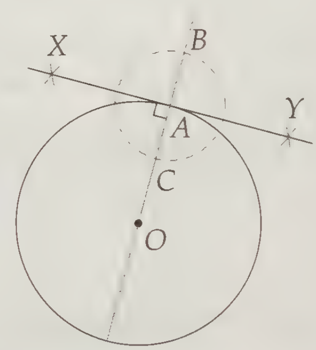
We'll start with constructing a tangent through a point on a circle.

Problem 12.25: Given a circle with center O and point A on the circle, construct a line through A that is tangent to the circle.

Solution for Problem 12.25: Make sure you see why this is not a correct solution:

> **Bogus Solution:** We place our straightedge on the paper so that it goes through the point A, then we slowly turn it, keeping it through A, until it only touches the circle in one point.

This Bogus Solution is *not* a Euclidean construction – it's just a sketch. We need to be able to prove our construction works. Here, the proof would essentially be 'Our line touches the circle once because we say so.' That's not good enough. We need to use geometric principles to prove our line is tangent, not just eyeball judgement.



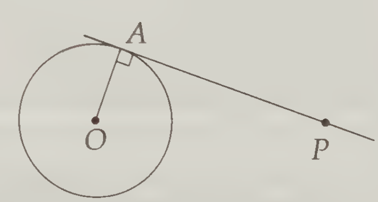
We know how to construct perpendiculars, and we know that a tangent is perpendicular to the radius drawn to the point of tangency. Putting these together, we have a pretty straightforward construction. We draw $\overleftrightarrow{OA}$, which includes the radius to point A, then construct the line line through A perpendicular to $\overline{OA}$. This line is our tangent line.

Our construction of this tangent line is indicated in the diagram at right. We draw a circle centered at A, which hits $\overleftrightarrow{OA}$ at B and C. We then draw two pairs of intersecting arcs with the same radius centered at B and C to find X and Y. $\overleftrightarrow{XY}$ is the perpendicular bisector of $\overline{BC}$, and is therefore our tangent line. □

So we can handle a tangent through a point on a circle; how about one through a point outside the circle?

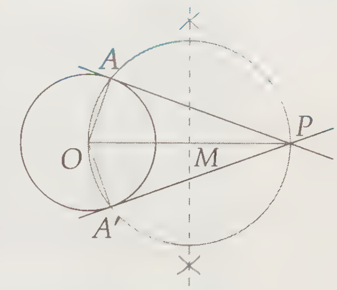
Problem 12.26: Given a circle with center O and point P outside the circle, construct a line through P that is tangent to the circle.

Solution for Problem 12.26: Unfortunately, we can't simply construct a radius, then draw a perpendicular line, because we don't know which radius to draw! So, feeling a little stuck, we look for a simpler problem to solve. But that doesn't get us anywhere, either – there's not any obvious simpler problem that will help us with this one. Therefore, we draw the completed diagram and look for clues how we might possibly construct it given only the circle and point P.



We go ahead and include the radius to the point of tangency because right angles are very useful, and we know how to construct perpendicular lines. If we can construct point A, where the tangent meets the circle, we'll be set. The only seemingly useful information we have is that $\angle OAP$ is a right angle, i.e., $\triangle OAP$ is a right triangle. So, we consider what we know about right triangles.

The circumcenter of any right triangle is the midpoint of its hypotenuse. Since A must be on the circumcircle of $\triangle OAP$, and $\triangle OAP$ must be a right triangle with hypotenuse $\overline{OP}$, we know that A is on the circle that has $\overline{OP}$ as a diameter. Therefore, to find A, we construct M, the midpoint of $\overline{OP}$. We then draw a circle with center M and radius MO. Where this circle hits $\odot O$ is the point A where our tangent touches the circle. $\overleftrightarrow{PA}$ is the tangent. (And the other point where the new circle hits $\odot O$ gives us the other tangent to $\odot O$ through P.) □

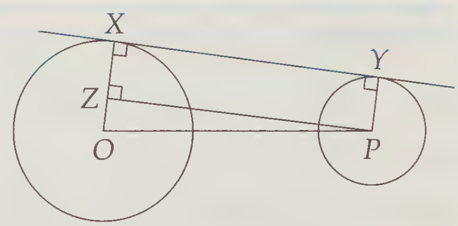


Concept: When tackling challenging construction problems, try drawing a completed diagram (without straightedge & compass construction). Then use observations about your completed diagram to figure out how to construct it with straightedge & compass alone.

Now that we've mastered tangent constructions, and have an intriguing new construction-finding strategy, let's try an even more challenging problem.

Problem 12.27: Given two nonintersecting circles with different radii, construct a line that is tangent to both.

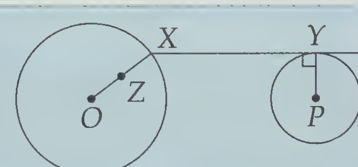
Solution for Problem 12.27: This construction looks pretty complicated, so we start from the completed diagram. We should probably add radii to the tangent points to give us perpendicular lines to work with. As we draw these, we realize, 'Hey, we've done a problem very much like this!' We think back to Problem 12.22, which we solved by creating the diagram shown at right.



Maybe this diagram will give us a clue. If we can find a way to construct any of X, Y, or even Z given only circles $\odot O$ and $\odot P$, we'll be able to construct a line tangent to both circles.

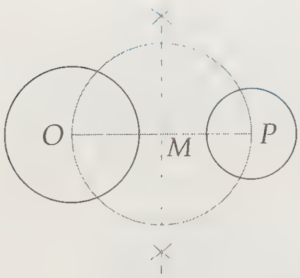
At this point, we might think:

Bogus Solution: Pick a point Z, draw $\overrightarrow{OZ}$ until it hits $\odot O$. That gives us point X. Then construct a line through X that is tangent to $\odot P$ using the construction from Problem 12.26.



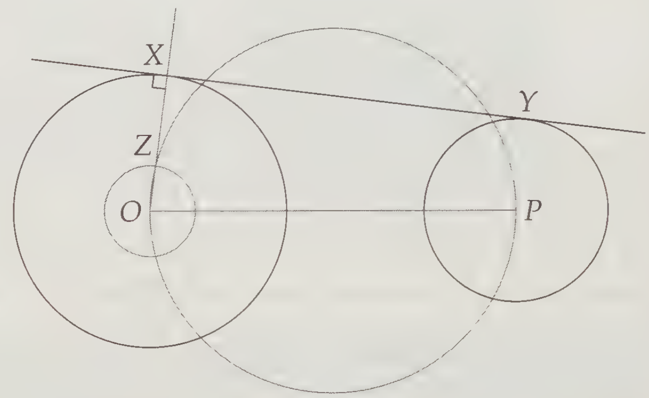
Unfortunately, this badly fails. If we just take any old Z, we might get a diagram like the one we want, but we probably won't. $\overleftrightarrow{XY}$ is indeed tangent to $\odot P$, since we constructed it to be tangent to $\odot P$. However, this line is not necessarily tangent to $\odot O$, as our diagram clearly points out.

So, we can't just pick any point to be Z (or X, or Y). And we can't just keep trying different possibilities for Z until we get one that works! We must find a specific one that we can prove works. Looking again at our 'working backwards' diagram we drew first, we see that Z is the vertex of the right angle of right triangle $\triangle OZP$. As we saw in Problem 12.26, this means that Z is on the circle that has $\overline{OP}$ as a diameter.



So, starting from just our circles and their centers, we at least have a first step: we construct a circle with $\overline{OP}$ as a diameter. Point Z must be on this circle. If we can find another circle or line with Z on it, we'll be finished. Thinking one more time about Problem 12.26 and our 'working backwards' diagram, we note that OZ is the difference between the radii of the two circles. Call this difference d. We can construct a segment with length d since we know the radii of the circles!

After constructing a segment with length d, we can draw a circle with center O and d as the radius. Since Z must be d away from O, it must be on this little circle. Therefore, the intersection of this circle and the circle with $\overline{OP}$ as diameter is Z! Extending $\overrightarrow{OZ}$ to hit $\odot O$ gives us X, and drawing the line perpendicular to $\overrightarrow{OZ}$ through X gives us the common tangent.



Of course, you'll get to prove that this final line is tangent to $\odot P$ as an Exercise. □

Exercises

12.5.1 Given two intersecting lines, construct a circle that is tangent to both of them.

12.5.2 Prove that our construction in Problem 12.27 does produce a line that is tangent to both circles.

12.5.3★ In Problem 12.27, we constructed the common external tangent of two circles. How can we construct a common internal tangent of two nonintersecting circles (i.e., a line tangent to both circles that intersects the line connecting the centers of the circles)? **Hints:** 29, 440, 378

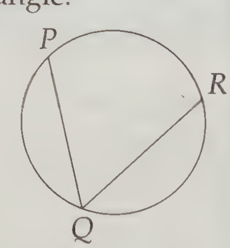
12.6 Summary

Definition: An angle formed by two chords of a circle is **inscribed** in the angle it cuts off.

Important:

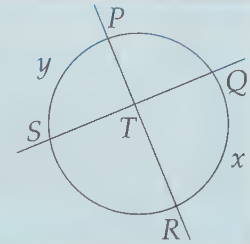
- Any angle inscribed in a semicircle is a right angle.
- An inscribed angle equals 1/2 the measure of the arc it intercepts. For example,

$$\angle PQR = \frac{\widehat{PR}}{2}.$$



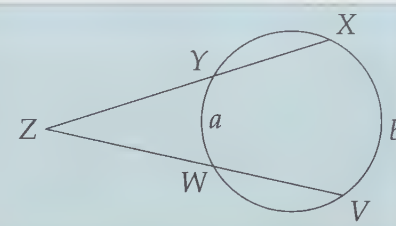
- Any two angles that are inscribed in the same arc are equal.

Important: The measure of the angle formed by two intersecting chords is the average of the measures of the arcs intersected by the chords. For example, in the diagram at right, we have



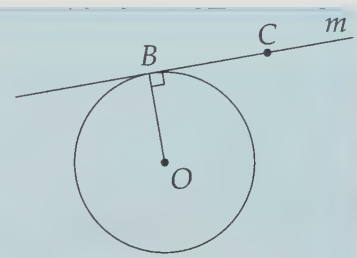
$$\angle PTS = \angle QTR = \frac{\widehat{PS} + \widehat{QR}}{2} = \frac{x+y}{2}.$$

Important: Two secants that meet at a point outside a circle form an angle equal to half the difference of the arcs they intercept. For example, in the diagram we have



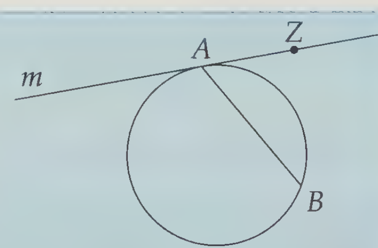
$$\angle Z = \frac{b-a}{2}.$$

Important: A line tangent to a circle is perpendicular to the radius drawn to the point of tangency. Conversely, a line drawn through a point on a circle that is perpendicular to the radius drawn to that point must be tangent to the circle. For example, for the diagram at right, we can write:



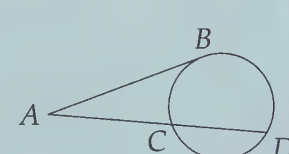
$\overleftrightarrow{BC}$ is perpendicular to radius $\overline{OB}$ if and only if $\overleftrightarrow{BC}$ is tangent to circle O.

Important: An angle formed by a tangent and a chord with the point of tangency as an endpoint equals one-half the arc intercepted by the angle. For example, in the figure at right, line m is tangent to the circle at A, so



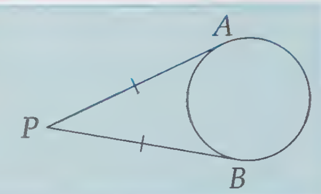
$$\angle ZAB = \frac{\widehat{AB}}{2}.$$

The angle formed by a tangent and a secant is half the difference of the intercepted arcs. For example,



$$\angle BAD = \frac{\widehat{BD} - \widehat{BC}}{2}.$$

Important: As shown in the diagram at right, we can draw two tangent segments to a circle from a point outside the circle. These tangents are always equal to each other in length.



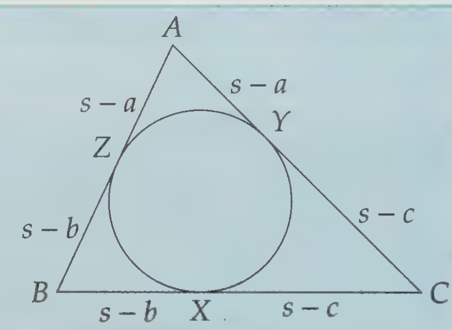
Important: A **cyclic quadrilateral** is a quadrilateral that can be inscribed in a circle. The opposite angles of any cyclic quadrilateral sum to 180°.

Important: If the length of a median of a triangle is half the length of the side to which it is drawn, the triangle must be a right triangle. Moreover, the side to which this median is drawn is the hypotenuse of the right triangle.

Important: The lengths from the vertices of $\triangle ABC$ to the points of tangency of its incircle are given as follows:

$$\begin{aligned} AZ &= AY = s-a \\ BZ &= BX = s-b \\ CX &= CY = s-c \end{aligned}$$

where $AB = c$, $AC = b$, and $BC = a$, and the semiperimeter of $\triangle ABC$ is s.



Important: The length of the inradius of a right triangle with hypotenuse of length c and legs of lengths a and b is

$$\frac{a+b-c}{2}.$$

Problem Solving Strategies

Concepts:

- When trying to prove something or find some general rule, investigating a few sample cases can be an excellent guide.
- When you can't find any more lengths or angles in a problem but you still haven't solved it, try assigning a variable to one of the lengths or angles. Then, find other lengths or angles in terms of that variable. Finally, label your diagram with everything you find – this will make it easier to see when you have enough information to solve the problem.

Continued on the next page. . .

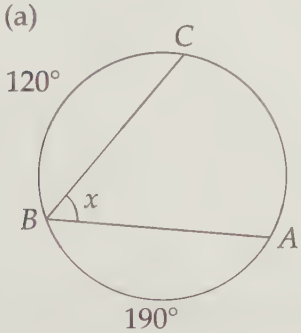
Concepts: *. . . continued from the previous page*

- When you have tangents in a problem, it's often very helpful to draw radii to points of tangency to build right triangles. When you have tangent circles, connect the centers. (In fact, if you have multiple circles in a problem, connecting the centers will sometimes help even when the circles aren't tangent.)
- When stuck on a problem, always ask yourself 'Where have I seen something like this before?'
- Always be on the lookout for right triangles.
- Symmetric systems of equations can often be easily solved by either multiplying all the equations or adding them.
- When tackling challenging construction problems, try drawing a completed diagram (without straightedge & compass construction), then using observations about your completed diagram to figure out how to construct it with straightedge & compass alone.

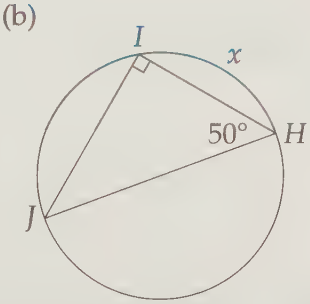
REVIEW PROBLEMS

12.28 Find x in each of the following

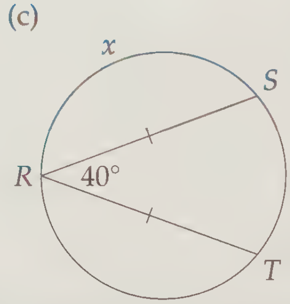
(a)



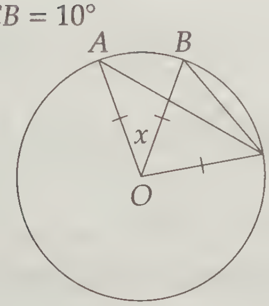
(b)



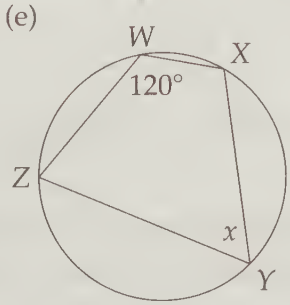
(c)



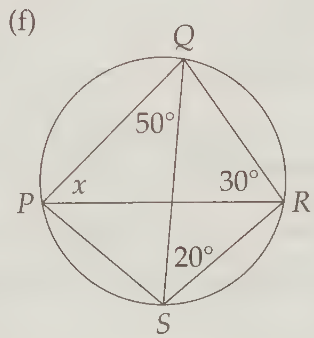
(d) $\angle ACB = 10°$



(e)

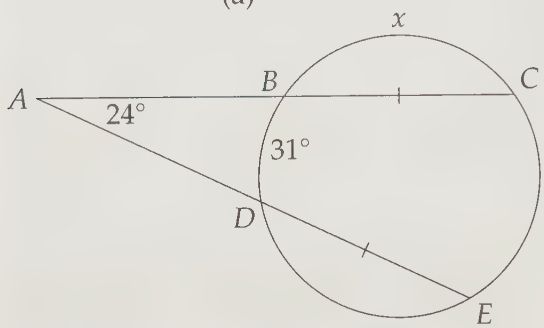


(f)

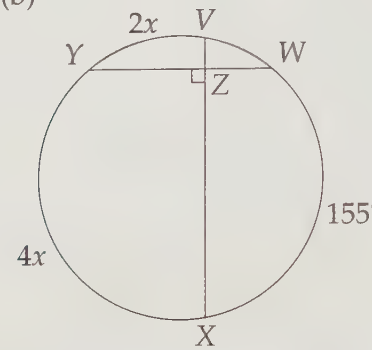


12.29 Find x in each of the following. (Assume that all lines that look like tangent lines are tangent lines.)

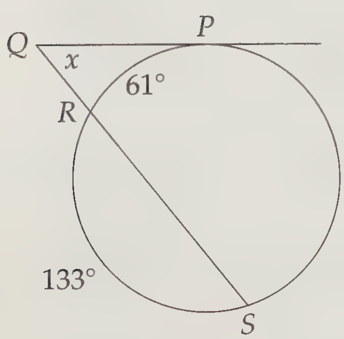
(a)



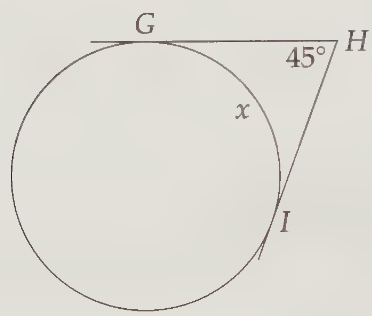
(b)



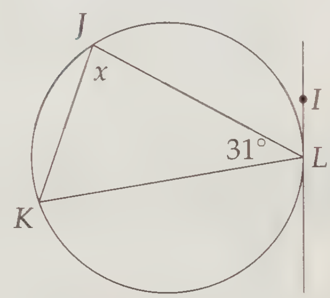
(c)



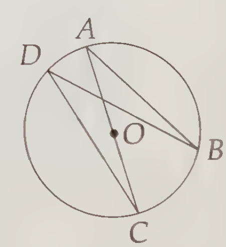
(d)



(e) $\angle JLI = 43°$



12.30 $\overline{AC}$ is a diameter of $\odot O$ in the diagram at left below. Given $\angle ACB = 58°$ and $BD = CD$, find the following:



(a) $\widehat{AB}$.

(b) $\angle BAC$.

(c) $\angle DBC$.

12.31 Points A, B, C, and D are on a circle in that order, and $\overline{AC}$ and $\overline{BD}$ meet at E. Given that $\widehat{AB} = 34°$, $\widehat{CD} = 102°$, $\widehat{DA} = 109°$, find $\angle BEC$.

12.32 Prove that if line m passes through a point B on circle O such that m is perpendicular to $\overline{OB}$, then m must be tangent to the circle. *(Try this one without looking back in the book for the proof!)* **Hints:** 302

12.33 Use your knowledge about inscribed angles to explain why the circumcenter of an obtuse triangle is outside the triangle.

12.34 Point T is on minor arc $\widehat{RS}$ of $\odot O$. Given that $\angle ROS = 53°$, find $\angle RTS$.

12.35 $\overline{KT}$ meets $\odot T$ at Y. Given that $YK = 6$ and $YT = 2$, find the length of the tangent segment from K to $\odot T$.

12.36 Point Y is on $\odot F$ and X outside $\odot F$ such that $\overline{XY}$ is tangent to F. $\overleftrightarrow{XF}$ intersects $\odot F$ at A and B such that $XA < XB$. Given that $\widehat{YB} = 134°$, find $\widehat{AY}$, $\angle YXF$, and $\angle XYA$.

12.37 $\overline{AT}$ and $\overline{BT}$ are tangent to $\odot O$ as shown at left below. Given $AT = 6$ and $\widehat{AB} = 120°$, find OT, $\angle ATB$, and $[OATB]$.

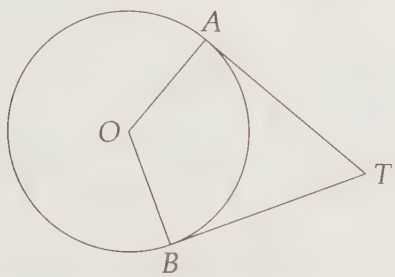


Figure 12.7: Diagram for Problem 12.37

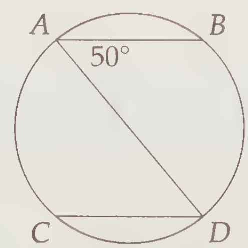


Figure 12.8: Diagram for Problem 12.38

12.38 In the circle at right above, $\overline{AB} \parallel \overline{CD}$, $\overline{AD}$ is a diameter of the circle, and $AD = 36$. What is the length of $\widehat{AB}$? *(Source: MATHCOUNTS)*

12.39 $ABCDEFGHIJKL$ is a regular dodecagon. Use the circumcircle of the dodecagon to find the following:

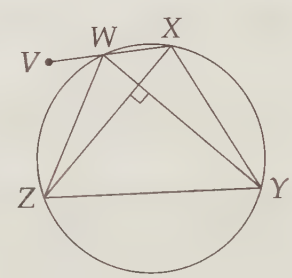
(a) $\angle ABC$.

(b) $\angle ACD$.

(c) $\angle ADJ$.

(d) the acute angle between $\overline{AF}$ and $\overline{BI}$.

12.40 Given $\angle VWZ = 81°$ and $\angle WZX = 38°$ in the diagram at right, find the following:



(a) $\angle VXZ$.

(b) $\angle WYZ$.

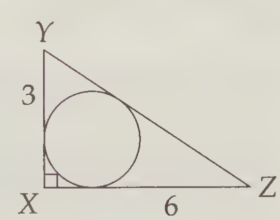
(c) $\angle XZY$.

(d) $\angle XYZ$.

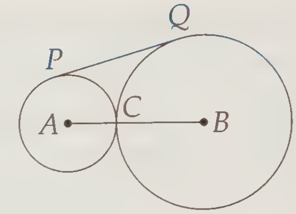
(e) $\angle WXY$.

12.41 $ABCD$ is a square with area 100. $\odot O$ is tangent to all four sides of the square. Diagonal $\overline{BD}$ meets the circle at X and Y, with X closer to D than to B. Find DX.

12.42 Right triangle $\triangle XYZ$ and its incircle are shown at right. As shown, Z is 6 units from the point where the incircle touches $\overline{XZ}$, and Y is 3 units from the point where the incircle touches $\overline{XY}$. Find YZ.



12.43 Prove that the incircle of a given triangle is unique. (In other words, prove that there is only one circle tangent to all three sides of a given triangle.)



12.44 Circles $\odot A$ and $\odot B$ are tangent at point C. P is on $\odot A$ and Q is on $\odot B$ such that $\overline{PQ}$ is tangent to both circles. Given $AC = 3$ and $BC = 8$, find PQ.

12.45 A circle is inscribed in a regular hexagon and another circle is circumscribed about the hexagon (i.e. it passes through all six vertices of the hexagon). Find the ratio of the area of the smaller circle to the area of the larger circle. **Hints:** 195

Challenge Problems

12.46 In the diagram at left below, lines ℓ and m are tangent to the circle. Prove that $\ell \parallel m$.

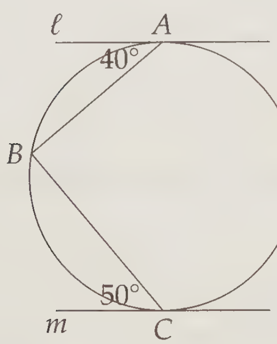


Figure 12.9: Diagram for Problem 12.46

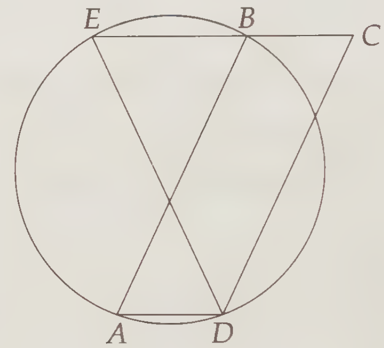


Figure 12.10: Diagram for Problem 12.47

12.47 $ABCD$ is a parallelogram. $\overline{BC}$ extended past B meets the circumcircle of $\triangle ABD$ at point E as shown at right above. Prove that $ED = CD$.

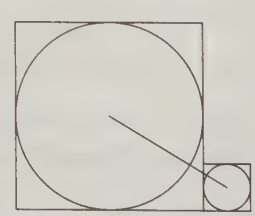
12.48 $\triangle ABC$ is a right triangle and the radius of its inscribed circle equals 5. If the perimeter of $\triangle ABC$ exceeds twice its hypotenuse by N, compute N. *(Source: ARML)* **Hints:** 449

12.49 Let A, B, C, and D be points on a circle ω, in that order. Let P and Q be points on ω such that $\overline{PA}$ bisects $\angle DAB$ and $\overline{QC}$ bisects $\angle BCD$. Prove that $\overline{PQ}$ is a diameter of ω.

12.50 Let the incircle of triangle ABC be tangent to sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ at points D, E, and F, respectively. Given that $\angle A = 32°$, find $\angle EDF$. **Hints:** 127

12.51 $\overline{AB}$ is a diameter of $\odot O$ and $\overline{AX}$ and $\overline{AY}$ are chords of the circle such that $AX = AY$. Prove that $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{XY}$. **Hints:** 210, 289

12.52 Find XZ in the right triangle described in Problem 12.42. **Hints:** 352



12.53 The areas of two adjacent squares are 256 square inches and 16 square inches, respectively, and their bases lie on the same line. What is the number of inches in the length of the segment that joins the centers of the two inscribed circles? *(Source: MATHCOUNTS)* **Hints:** 55, 456

12.54 Given a circle of radius 2, there are infinitely many line segments of length

2 that are tangent to the circle at their midpoints. Find the area of the region consisting of all such line segments. *(Source: AMC 12)* **Hints:** 49

12.55 Let C_1 and C_2 be two circles that intersect at two points, A and B. Let P be the point diametrically opposite A on C_1, and let Q be the point diametrically opposite A on C_2. Prove that P, B, and Q are collinear. **Hints:** 87

12.56 Chord $\overline{CE}$ is a segment of the perpendicular bisector of radius $\overline{OA}$ of $\odot O$. $\overline{AX}$ is a diameter of $\odot O$. Find $\angle OCX$. **Hints:** 264

12.57 Three of the sides of $ABCD$ are tangent to the circle shown at left below. Given that $AB = 6$, $BC = 7$, $CD = 8$, and $AD = 9$, find $CY + DX$. *(Source: Mandelbrot)* **Hints:** 173

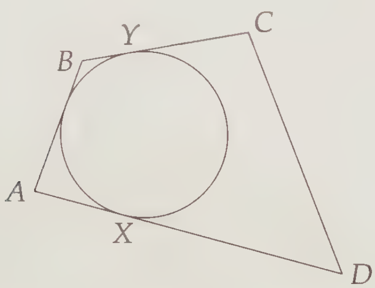


Figure 12.11: Diagram for Problem 12.57

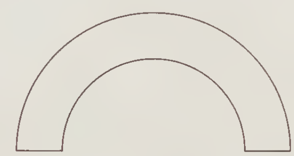


Figure 12.12: Diagram for Problem 12.58

12.58 A room is built in the shape of the region between two semicircles with the same center and parallel diameters such that the smaller semicircle is contained within the larger. The room is shown at right above. The farthest distance between two points with a clear line of sight is 12m. What is the area in m^2 of the room? *(Source: HMMT)* **Hints:** 316

12.59 The incircle of $\triangle ABC$ is tangent to $\overline{AB}$ at Z. Show that if the inradius of $\triangle ABC$ equals AZ, then the angle $\angle BAC$ is a right angle. **Hints:** 348

12.60 The sides of right triangle $\triangle ABC$ all have integer lengths. Prove that the inradius of $\triangle ABC$ also has integer length. **Hints:** 449

12.61 Let ω be the circumcircle of triangle ABC. Let A_1, B_1, and C_1 be the midpoints of arcs $\widehat{BC}$, $\widehat{CA}$, and $\widehat{AB}$, respectively, of circle ω. In this problem we prove that the incenter of triangle ABC is the orthocenter of triangle $A_1B_1C_1$.

(a) Let I be the incenter of $\triangle ABC$. Why must $\overrightarrow{AI}$ pass through A_1?

(b) Why must $\overline{AA_1}$ be perpendicular to $\overline{B_1C_1}$? **Hints:** 256

(c) Combine the previous parts to conclude that the incenter of triangle ABC is the orthocenter of triangle $A_1B_1C_1$.

12.62★ Prove that quadrilateral $ABCD$ is cyclic if $\angle A + \angle C = 180°$. (Note that this is the converse of what we discussed in Problem 12.19.) **Hints:** 331, 37

12.63★ We are given points A, B, C, and D in the plane such that $AD = 13$ while $AB = BC = AC = CD = 10$. Find $\angle ADB$. *(Source: Mandelbrot)* **Hints:** 82, 199

12.64★ Let $\overline{PQ}$ be a diameter of a circle and T be a point on the circle besides P and Q. The tangent to the circle through point Q intersects $\overleftrightarrow{PT}$ at R, and the tangent through T intersects $\overline{QR}$ at M. Prove that M is the midpoint of $\overline{QR}$. **Hints:** 222, 128

Extra! Most construction problems involve the two classical tools of geometry, the straightedge and compass. These tools can solve an array of construction problems, many of which are discussed in this book, but perhaps even more interesting are construction problems that are impossible to solve with these tools.

Three famous classical construction problems that the ancient Greeks couldn't solve are listed below.

(a) Squaring the Circle: Given a circle, construct a square of equal area.

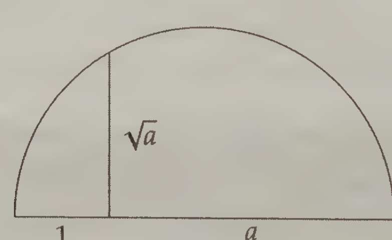
(b) Doubling the Cube: Given a cube, construct another cube with twice the volume of the first cube.

(According to Greek legend, the Athenians appealed to the Oracle for advice about a plague that was ravaging their city. The Oracle's answer was to have their cube-shaped altar to the god Apollo doubled in size. The Athenians then attempted to follow this advice by building a new altar that had twice the side length of the original altar, but this had the effect of increasing the volume of the altar by a factor of $2^3 = 8$, not 2, and as a result, the plague worsened.)

(c) Trisecting the Angle: Given an arbitrary angle, divide it into three equal angles.

We now know that the ancient Greeks couldn't solve these problems because all three of these constructions are impossible to perform with straightedge and compass.

Modern mathematicians proved these constructions are impossible by determining what lengths could be constructed if we are given segments of lengths a, b, and 1. With these segments as a starting point, the lengths $a+b$, $a-b$, ab, and a/b can be constructed. See if you can figure out how to construct ab and a/b. Even $\sqrt{a}$ can be constructed, as the diagram below suggests. (See if you can figure out how this construction works!)



This means that any of the following lengths can be constructed:

$$\sqrt{2},\quad \frac{1+\sqrt{5}}{2},\quad \sqrt{\frac{\sqrt{3}+8\sqrt{7}}{2}} - \sqrt{5} + \sqrt[4]{17-\sqrt{3}+\sqrt{2}}.$$

Continued on the next page. . .

Extra! *. . . continued from the previous page*

In other words, any length involving the integers and the arithmetic functions $+$, $-$, $\times$, $\div$, and $\sqrt{\ }$ can be constructed. Furthermore, these are the only lengths that can be so constructed. So how can this fact be used to solve the three problems?

The first problem, squaring the circle, is equivalent to constructing $\sqrt{\pi}$. However, it was shown by Lindemann in 1882 that π is transcendental, meaning that π is not the root of any polynomial with integer coefficients, which also implies the same for $\sqrt{\pi}$. Since every constructible number is the root of some polynomial with integer coefficients, we know that neither π nor $\sqrt{\pi}$ can be constructed.

The second problem, doubling the cube, is equivalent to constructing $\sqrt[3]{2}$. A branch of mathematics, known as Galois theory, tells us that $\sqrt[3]{2}$ cannot be expressed using arithmetic functions and square roots of integers. Hence, $\sqrt[3]{2}$ cannot be constructed.

For the third problem, trisecting the angle, certain angles can be trisected. For example, given a straight angle of 180°, a third of it would be 60°, which can be easily constructed. But can a 60° angle itself be trisected? In other words, can a 20° angle be constructed? It can be shown that if $x = \cos 20°$, then $8x^3 - 6x - 1 = 0$. Again, Galois theory tells us that such an x cannot be expressed using arithmetic functions and square roots. This counterexample of trisecting the 60° degree angle shows that trisection of angles cannot be done in general.

Despite the fact that these problems have been conclusively solved for years, many amateur mathematicians still attempt to find constructions for these problems and submit them to mathematical journals, hoping for fame. All these attempts are completely futile.

The same Galois theory that rules out these constructions also tells us which regular polygons can be constructed. The construction of an equilateral triangle and a regular hexagon are easy, but beyond that, which other regular n-gons are constructible?

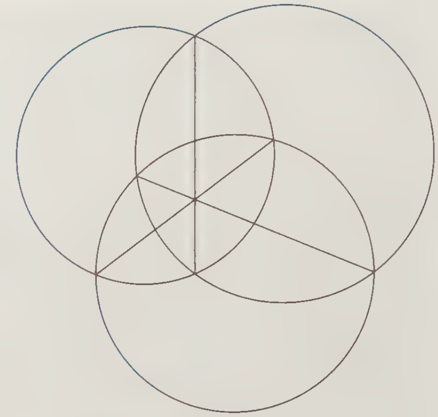
To answer this question, we introduce the **Fermat numbers**. The k^{th} Fermat number is $F_k = 2^{2^k} + 1$, so $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, and so on. A **Fermat prime** is a Fermat number that is also prime. At the time of writing, there are only five known Fermat primes, namely F_0, F_1, F_2, F_3, and F_4.

A regular n-gon is constructible if and only if n is a power of 2 or is of the form

$$n = 2^j F_{m_1} F_{m_2} \cdots F_{m_i},$$

where j is an arbitrary non-negative integer and $F_{m_1}, F_{m_2}, \ldots, F_{m_i}$ are *distinct* Fermat primes. So the regular polygon with $3 \cdot 17 = 51$ sides is constructible, but the regular polygons with 7 sides and 9 sides are not. Gauss first proved that the regular 17-gon is constructible, and was said to have been so proud of this fact that he requested that a regular 17-gon be etched into his tombstone.

It is surprising that either one of the straightedge or compass by itself can essentially do the same work as both of them put together. The **Mohr-Mascheroni Theorem** states that any construction that can be performed by straightedge and compass can be performed by compass alone. If the target of a construction is a line, the compass alone cannot produce the line; however, it can produce two points on the line, which defines the line correctly. Analogously, the **Poncelet-Steiner Theorem** states that any construction that can be performed by straightedge and compass can be performed by straightedge alone, as long as a single circle and its center are given.



Radical Axis Theorem

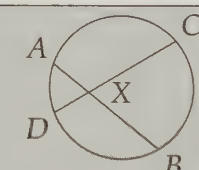
Power corrupts. Absolute power is kind of neat. – John Lehman, Secretary of the Navy, 1981-1987

CHAPTER 13 Power of a Point

In the previous chapter we explored the angles formed by lines that meet inside, outside, or on a circle. In this chapter, we explore the lengths of segments that meet inside or outside a circle. Together, the relationships we will prove and use in this chapter are given the lofty name 'Power of a Point.'

13.1 What is Power of a Point?

Problems



Problem 13.1: Chords $\overline{AB}$ and $\overline{CD}$ meet at point X as shown. In this problem we will prove that

$$(XA)(XB) = (XC)(XD).$$

(a) Draw $\overline{AD}$ and $\overline{BC}$. Why is $\angle XDA = \angle XBC$?

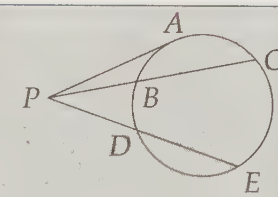
(b) Show that the two triangles in the diagram are similar. Use these similar triangles to prove that $XA/XC = XD/XB$.

(c) Complete the proof that $(XA)(XB) = (XC)(XD)$.

(d) Is every chord of the circle that passes through X split into two pieces that have a product equal to $(XA)(XB)$? (In other words, is there anything special about chord $\overline{CD}$ or will this proof work for every chord through X?)

Problem 13.2: Point P is outside a given circle and A is on the circle such that $\overline{PA}$ is tangent to the circle. C and E are on the circle such that secants $\overline{PC}$ and $\overline{PE}$ intersect the circle again at B and D, respectively. In this problem we prove that

$$PA^2 = (PB)(PC) = (PD)(PE).$$



(a) Draw $\overline{AB}$ and $\overline{AC}$. Why does $\angle ACP = \angle BAP$?

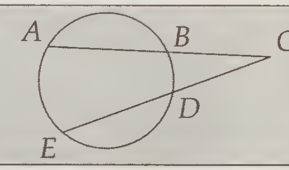
(b) Use the angle equality from the first part to identify two triangles that are similar.

(c) Use the triangle similarity to prove $PA^2 = (PB)(PC)$.

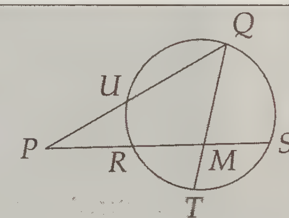
(d) How can we show that $(PD)(PE) = (PB)(PC)$?

Problem 13.3: Chords $\overline{TY}$ and $\overline{OP}$ meet at point K such that $TK = 2$, $KY = 16$, and $KP = 2(KO)$. Find OP.

Problem 13.4: In the diagram, $CB = 9$, $BA = 11$, and $CE = 18$. Find DE.



Problem 13.5: Points R and M trisect $\overline{PS}$, so $PR = RM = MS$. Point U is the midpoint of $\overline{PQ}$, $TM = 2$, and $MQ = 8$.



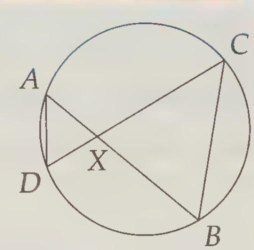
(a) Find RM and MS.

(b) Use the relationship you found in Problem 13.2 to find PU. (Don't forget that U is the midpoint of $\overline{PQ}$.)

We begin this section by proving the Power of a Point Theorem in two parts. First, we consider points inside a circle.

Problem 13.1: Given chords $\overline{AB}$ and $\overline{CD}$ that meet at point X, prove that

$$(XA)(XB) = (XC)(XD).$$



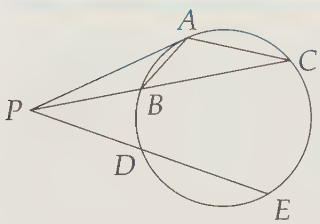
Solution for Problem 13.1: When we rearrange what we want to prove, we have $XA/XC = XD/XB$. The ratios suggest we look for similar triangles, so we draw $\overline{AD}$ and $\overline{BC}$ to form triangles. Since $\angle B$ and $\angle D$ are inscribed in the same arc, we have $\angle B = \angle D$. We also have $\angle AXD = \angle CXB$, so we have $\triangle AXD \sim \triangle CXB$ by AA Similarity. Therefore, we have $XA/XC = XD/XB$, so $(XA)(XB) = (XC)(XD)$.

Note that there's nothing special about chords $\overline{AB}$ and $\overline{CD}$. We can use the same exact approach as above to show that each of this circle's chords through X is divided into two pieces such that the product of the lengths of these pieces equals $(XA)(XB)$. □

Having finished with points inside the circle, we turn to points outside the circle.

Problem 13.2: Point P is outside a given circle and A is on the circle such that $\overline{PA}$ is tangent to the circle. C and E are on the circle such that secants $\overline{PC}$ and $\overline{PE}$ intersect the circle again at B and D, respectively. Prove that

$$PA^2 = (PB)(PC) = (PD)(PE).$$



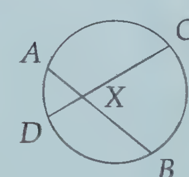
Solution for Problem 13.2: We start as we did in the previous problem, drawing $\overline{AB}$ and $\overline{AC}$ to make triangles. Since $\angle ACB = \widehat{AB}/2$ and $\angle PAB = \widehat{AB}/2$, we have $\triangle PAB \sim \triangle PCA$ by AA Similarity (since $\angle P$ is the same in both triangles as well). Therefore, $PA/PB = PC/PA$, so $PA^2 = (PB)(PC)$.

Just as there was nothing special about the chords we looked at in the previous problem, there's nothing special about secant $\overline{PC}$ here. By exactly the same reasoning, we have $(PD)(PE) = PA^2$, so $(PD)(PE) = (PB)(PC)$. Notice that once again there's nothing special about the tangent and the secants we have chosen except that they have point P in common. □

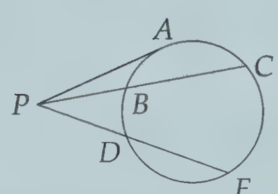
We can put the facts we've proved in the last two problems together in a single statement.

Important: Suppose a line through a point P intersects a circle in two points, U and V. The **Power of a Point Theorem** states that for all such lines, the product $(PU)(PV)$ is constant. We call this product the **power** of point P.

For example, in the figure at right, applying Power of a Point to X with respect to the circle shown gives



$$(XA)(XB) = (XC)(XD).$$



In the figure at left, the power of point P with respect to the circle gives us

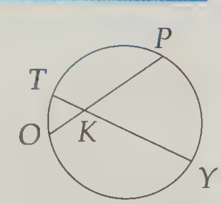
$$PA^2 = (PB)(PC) = (PD)(PE).$$

We think of Power of a Point whenever we have a problem involving intersecting segments and a circle.

Let's try using Power of a Point on a few problems.

Problem 13.3: Chords $\overline{TY}$ and $\overline{OP}$ meet at point K such that $TK = 2$, $KY = 16$, and $KP = 2(KO)$. Find OP.

Solution for Problem 13.3: A quick sketch suggests how to apply Power of a Point. From the power of point K, we have



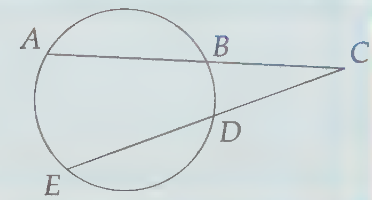
$$(KP)(KO) = (KT)(KY).$$

Substituting the given information in this equation yields

$$2(KO)(KO) = 2(16),$$

from which we find $KO = 4$. Therefore, $KP = 2KO = 8$, so $OP = KO + KP = 12$. □

Problem 13.4: In the diagram, $CB = 9$, $BA = 11$, and $CE = 18$. Find DE.

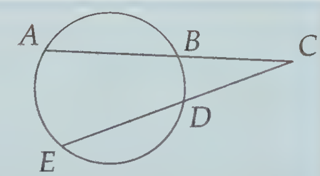


Solution for Problem 13.4: We have two intersecting secants, so we apply Power of a Point, which gives

$$(CD)(CE) = (CB)(CA).$$

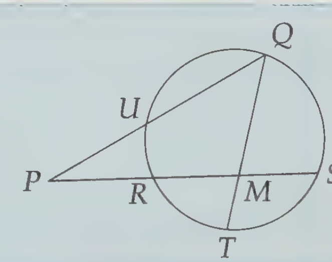
Therefore, $(CD)(18) = 9(20)$, so $CD = 10$ and $DE = CE - CD = 8$. □

WARNING!! A very common mistake in applying Power of a Point is to write $(BC)(BA) = (DC)(DE)$ when faced with the figure at right. This alleged equality is **not** what Power of a Point tells us! Note that whenever we write a Power of a Point relationship, the same point must appear in *all* of the segments in the equation, as point C does when we write the correct relationship for the secants in the figure above:



$$(CD)(CE) = (CB)(CA).$$

Problem 13.5: Points R and M trisect $\overline{PS}$, so $PR = RM = MS$. Point U is the midpoint of $\overline{PQ}$, $TM = 2$, and $MQ = 8$. Find PU.



Solution for Problem 13.5: Circles, chord lengths, and secant lengths. This is a job for Power of a Point. The power of point M gives us

$$(MR)(MS) = (MT)(MQ).$$

We know that $RM = MS$, so substitution gives $MR^2 = (2)(8)$, i.e., $MR = 4$. Therefore, $PR = MR = 4$ and $PS = 3(MR) = 12$. Since U is the midpoint of $\overline{PQ}$, we have $PQ = 2PU$. Now we can apply the power of point P to find:

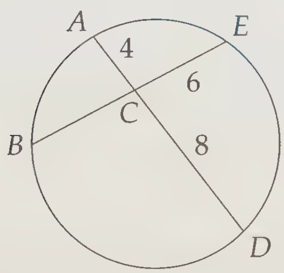
$$(PU)(PQ) = (PR)(PS).$$

Substitution gives $(PU)(2PU) = 4(12)$, so $PU = 2\sqrt{6}$. □

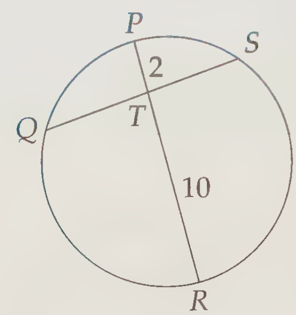
Extra! *Never say of a branch of mathematics, 'There's something I don't need to know.' It always comes back to haunt you.*
–Charles Rickart

Exercises

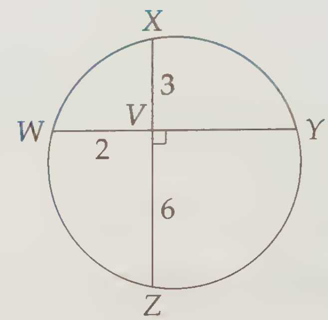
13.1.1



(a) Find BC.

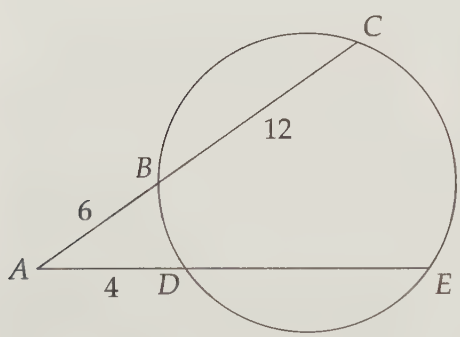


(b) $QS = 9$ and $TQ < TS$; find TQ.

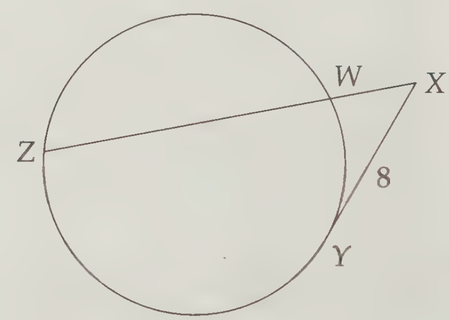


(c) Find ZY.

13.1.2



(a) Find DE.

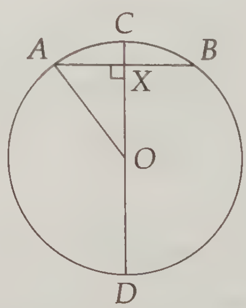


(b) Given $XZ = 12$, find WZ.

13.1.3 Chords $\overline{XY}$ and $\overline{AB}$ intersect at Q. Given $AQ = 4$, $BQ = 6$, and $XQ = 8$, find XY.

13.1.4 Chord $\overline{UV}$ bisects chord $\overline{ST}$ at point M. Given $ST = 12$ and $UV = 15$, find all possible values of UM.

13.1.5 In this problem we use Power of a Point to prove the Pythagorean Theorem.

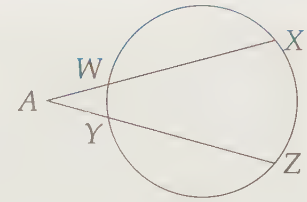


(a) Chord $\overline{AB}$ is perpendicular at point X to diameter $\overline{CD}$ of circle O. Let the radius of the circle be c, the length of $\overline{AB}$ be $2a$, and $OX = b$. Find CX and XD in terms of b and c.

(b) Use Power of a Point to show that $c^2 = a^2 + b^2$.

(c) How can you use this argument to prove the Pythagorean Theorem?
Hints: 3, 41

13.1.6★ In the figure at right, we have $WX = YZ$. Prove that $AW = AY$. **Hints:** 62

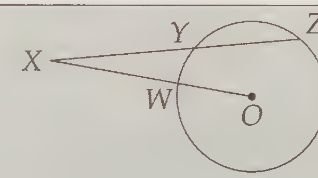


13.2 Power of a Point Problems

We now apply Power of a Point to some more challenging problems. As we'll see, Power of a Point can be particularly useful for proofs involving segment lengths when circles are part of the problem.

Problems

Problem 13.6: Given that $XY = 6$, $YZ = 5$, and that point X is 9 units from the center, O, of the circle, we wish to find the area of the circle.



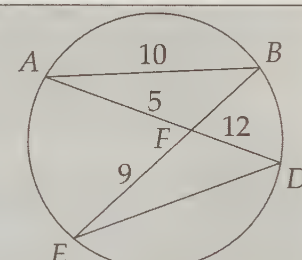
(a) Extend $\overline{XO}$ to meet the circle again at V.

(b) Let the radius of the circle be r; use the power of point X to find r^2, and hence, the area of the circle.

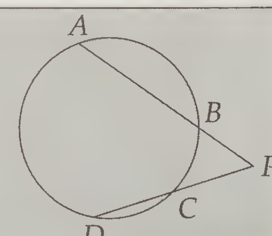
Problem 13.7:

(a) Find BF.

(b) Find ED.



Problem 13.8: In the diagram, we have $BP = 8$, $AB = 10$, $CD = 7$, and $\angle APC = 60^\circ$. *(Source: AHSME)*

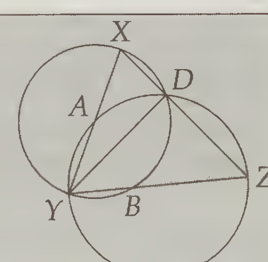


(a) Find CP.

(b) Show that $\triangle ACP$ is a 30-60-90 triangle. **Hints:** 317

(c) Find the area of the circle.

Problem 13.9: In this problem we wish to show that $XA = BZ$ in the diagram at right, given that $\overline{DY}$ bisects $\angle XYZ$.



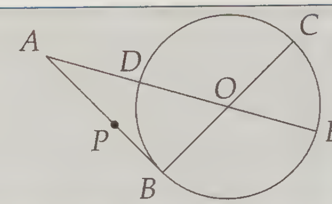
(a) Use Power of a Point to find expressions for XA and BZ in terms of other segments in the diagram.

(b) Use the Angle Bisector Theorem to prove that your expressions from the first part are equal.

Problem 13.10: Point O is the center of the circle, $\overline{AB} \perp \overline{BC}$, $AP = AD$, and $\overline{AB}$ has length twice the radius of the circle. Prove that

$$AP^2 = (PB)(AB).$$

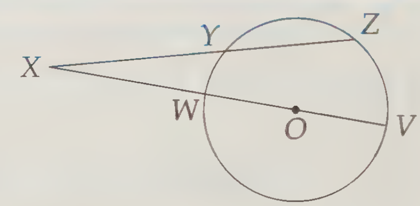
(*Source: AHSME*)



We start where we left off last section by finding lengths with Power of a Point in slightly more complicated problems.

Problem 13.6: Z is on $\odot O$ and secant $\overline{XZ}$ hits the circle at Y such that $XY = 6$ and $YZ = 5$. Given that X is 9 units from the center of this circle, find the area of the circle.

Solution for Problem 13.6: We start with a diagram. We continue segment $\overline{XO}$ to hit the circle again at V, so that we can use the power of point X: $(XW)(XV) = (XY)(XZ)$. If we let the radius of the circle be r, substitution gives $(XO-r)(XO+r) = 6(6+5)$. Since $XO = 9$, we have $(9-r)(9+r) = 66$, from which we find $r^2 = 15$. Therefore, the area of the circle is $\pi r^2 = 15\pi$. □

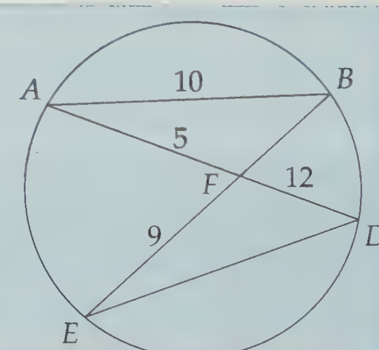


Concept: If a problem includes a segment that abruptly stops in the middle of a circle, consider continuing the segment until it hits the circle. Then see what Power of a Point gives you.

Problem 13.7:

(a) Find BF.

(b) Find ED.



Solution for Problem 13.7:

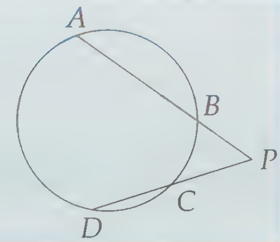
(a) We can find BF with Power of a Point: $(FE)(FB) = (FA)(FD)$, so $FB = (FA)(FD)/(FE) = 20/3$.

(b) While Power of a Point won't get us anywhere with ED, our proof of Power of a Point guides us to the answer. Specifically, since $\angle FDE = \angle FBA$ and $\angle AFB = \angle EFD$, we have $\triangle AFB \sim \triangle EFD$, so $ED/BA = EF/AF$. Hence, $ED = (EF)(BA)/AF = 18$.

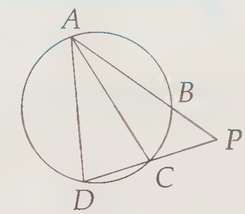
□

We used Power of a Point in combination with similar triangles in the previous problem. Now let's try using it with some of our other geometric tools.

Problem 13.8: In the diagram, we have $BP = 8$, $AB = 10$, $CD = 7$, and $\angle APC = 60^\circ$. Find the area of the circle. *(Source: AHSME)*



Solution for Problem 13.8: It's not immediately obvious how we will find the radius, so we start by finding what we can. The power of a point P gives us



$$(PC)(PD) = (PB)(PA),$$

so $(PC)(PC + 7) = 144$. Therefore, $PC^2 + 7PC - 144 = 0$, so $(PC + 16)(PC - 9) = 0$. PC must be positive, so $PC = 9$.

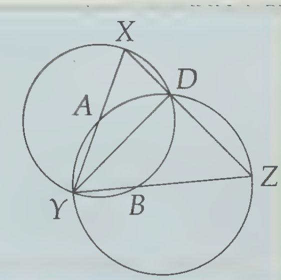
Seeing that $\angle APC = 60^\circ$ makes us wonder if there are any equilateral or 30-60-90 triangles lurking about. Since $CP = AP/2$ and the angle between these sides is 60°, the sides adjacent to the 60° angle in $\triangle ACP$ are in the same ratio as the sides adjacent to the 60° angle in a 30-60-90 triangle. Therefore, $\triangle APC$ is similar to a 30-60-90 triangle by SAS Similarity, so $\triangle APC$ must be a 30-60-90 triangle with right angle at $\angle ACP$!

Since $\angle ACD$ is right and inscribed in $\widehat{AD}$, we know $\widehat{AD}$ is a semicircle. Therefore, $\overline{AD}$ is a diameter of the circle. Since $AC = CP\sqrt{3} = 9\sqrt{3}$ from our 30-60-90 triangle, we have $AD = \sqrt{AC^2 + CD^2} = \sqrt{243 + 49} = 2\sqrt{73}$. Finally, the radius of the circle is $AD/2 = \sqrt{73}$, so the area is $(\sqrt{73})^2\pi = 73\pi$. □

Concept: If you see a 60° or 30° angle (or even a 120° angle) in a problem, you should be on the lookout for 30-60-90 triangles.

Now we will use Power of a Point in a couple proofs.

Problem 13.9: Given that $\overline{YD}$ bisects $\angle XYZ$ in the diagram at right, prove that $XA = ZB$.



Solution for Problem 13.9: This problem involves lengths, an angle bisector, chords, secants, and circles. Therefore, we should consider the Angle Bisector Theorem and Power of a Point. The Angle Bisector Theorem applied to angle bisector $\overline{YD}$ of $\triangle XYZ$ gives us

$$\frac{XD}{XY} = \frac{ZD}{ZY}.$$

However, we need something with XA and ZB. Therefore, we turn to the powers of points X and Z, which respectively give us

$$\begin{aligned}(XA)(XY) &= (XD)(XZ)\\ (ZD)(ZX) &= (ZB)(ZY).\end{aligned}$$

Seeing some common terms in these two, along with terms from our Angle Bisector Theorem equation, we know we're on the right track. We solve these two equations for XA and ZB, and we have

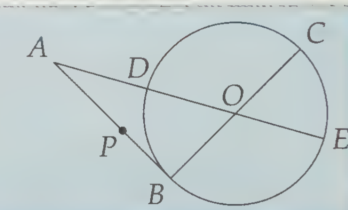
$$\begin{aligned}XA &= \frac{(XD)(XZ)}{XY}\\ ZB &= \frac{(ZD)(XZ)}{ZY}\end{aligned}$$

We need only show these two expressions are equal. We have XZ in common, and we already have $XD/XY = ZD/ZY$ from our Angle Bisector Theorem. We can put these together to show

$$XA = \frac{(XD)(XZ)}{XY} = XZ\left(\frac{XD}{XY}\right) = XZ\left(\frac{ZD}{ZY}\right) = \frac{(ZD)(XZ)}{ZY} = ZB.$$

□

Problem 13.10: Point O is the center of the circle, $\overline{AB} \perp \overline{BC}$, $AP = AD$, and $\overline{AB}$ has length twice the radius of the circle. Prove that $AP^2 = (PB)(AB)$. *(Source: AHSME)*



Solution for Problem 13.10: We have a lot of information here. What we want to prove looks a lot like Power of a Point when we have a tangent. $\overline{AB}$ sure looks tangent, but we have to prove it. Fortunately, since diameter $\overline{BC}$ is perpendicular to $\overline{AB}$ at point B on the circle, $\overline{AB}$ must be tangent to the circle. Therefore, $AB^2 = (AD)(AE)$.

This problem has a lot of different segments. We could get lost in a blizzard of AP's and AD's and DE's and so on. So, we have to stay organized. One way to do this is making a list of what we know and what we want to show. Letting the radius of the circle be r, such a table might look like this:

What We Know	What We Want
$AP = AD$ $AB^2 = (AD)(AE)$ $AB = 2r$	$AP^2 = (PB)(AB)$

This has the advantage of letting us work in two directions. We can work forwards from what we know to try to reach what we want, or we can work backwards from what we want. Let's try a little of both here. First, going forwards, we note that $BC = DE = 2r$ because they are diameters of the circle. Combining this with $AB = 2r$ gives $AB = BC = DE$.

Going backwards, we write our desired relationship in terms of fewer lengths. Specifically, since $PB = AB - AP$, we can write $AP^2 = (PB)(AB)$ in terms of just AB and AP. We can also note that $AB = 2r$, and write the equation we want in terms of r and AP. Now our table looks like:

What We Know	What We Want
$AP = AD$	$AP^2 = (PB)(AB)$
$AB^2 = (AD)(AE)$	$AP^2 = (AB - AP)(AB)$
$AB = 2r = DE = BC$	$AP^2 = (2r - AP)(2r)$

Going forwards with our new information, we don't see anything we can do with BC, but $\overline{DE}$ is part of $\overline{AE}$, so we have $AB^2 = (AD)(AE) = (AD)(AD + DE) = (AD)(AD + AB)$. All the stuff on our 'What We Want' side has AP's in it, so we use $AD = AP$ to write

$$AB^2 = (AD)(AD + AB) = (AP)(AP + AB) = AP^2 + (AP)(AB).$$

Now our table looks like this:

What We Know	What We Want
$AP = AD$	$AP^2 = (PB)(AB)$
$AB^2 = (AD)(AE)$	$AP^2 = (AB - AP)(AB)$
$AB = 2r = DE = BC$	$AP^2 = (2r - AP)(2r)$
$AB^2 = AP^2 + (AP)(AB)$	

Now we have an equation in our 'What We Know' that is *very* close to an equation in our 'What We Want.' A little rearranging turns one into the other. Subtracting $(AP)(AB)$ from both sides of

$$AB^2 = AP^2 + (AP)(AB)$$

gives $AB^2 - (AP)(AB) = AP^2$. Factoring then gives

$$(AB - AP)(AB) = AP^2,$$

which is the second equation in our 'What We Want' column.

Ah-ha! We have a path from something We Know to something We Want! However, we can't just say 'Done!' We have to write a nice solution. We do so by retracing our steps. We have to examine our work above to figure out three things:

(a) How we get from the given information to the equation $AB^2 = AP^2 + (AP)(AB)$.

(b) How we get from this equation to the one in our list of 'What We Want': $AP^2 = (AB - AP)(AB)$.

(c) How we get from $AP^2 = (AB - AP)(AB)$ to what we want to prove: $AP^2 = (PB)(AB)$.

After reviewing our work above, we write our solution:

Since diameter $\overline{BC}$ is perpendicular to $\overline{AB}$ at point B on the circle, $\overline{AB}$ must be tangent to the circle. Therefore, the power of point A gives us $AB^2 = (AD)(AE)$. We are given AB equals twice the radius of the circle of which $\overline{DE}$ is a diameter, so $AB = DE$. Since we are also given $AD = AP$, we have

$$AB^2 = (AD)(AE) = (AD)(AD + DE) = (AP)(AP + DE) = (AP)(AP + AB).$$

Expanding and rearranging $AB^2 = (AP)(AP + AB)$ gives $AB^2 - (AP)(AB) = AP^2$. Factoring yields $(AB - AP)(AB) = AP^2$. Since $AB - AP = PB$, we have the desired $(PB)(AB) = AP^2$.

Notice that we have some items on our 'What We Know' and 'What We Want' lists that we don't use. This will usually be the case for challenging geometry proofs. □

Important: When writing a solution, you should write a clean solution like what we have after 'After reviewing our work above, we write our solution.' All the work we did before that point was just our investigation to find the solution. This often involves working both forwards and backwards, as we saw. However, once we find our way, we write a nice clean solution *forwards*, starting from what we are given and ending at what we want to prove.

Concept: Organize your work on challenging geometry proofs. Keep track of what you know and what you can derive from what you know. Also keep track of what you need to prove, and work backwards from there to list more statements that, if proven, would mean you are finished. When you know how to get from a statement on your 'What We Know' list to a statement on your 'What We Want' list, then you can construct your proof.

Exercises

13.2.1 In the figure at left below, $\overline{AB}$ is tangent to $\odot O$. Given that $AB = 4$ and $BD = 12$, find the area of the circle.

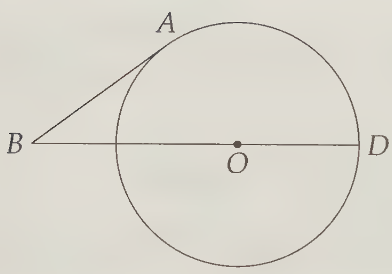


Figure 13.1: Diagram for Problem 13.2.1

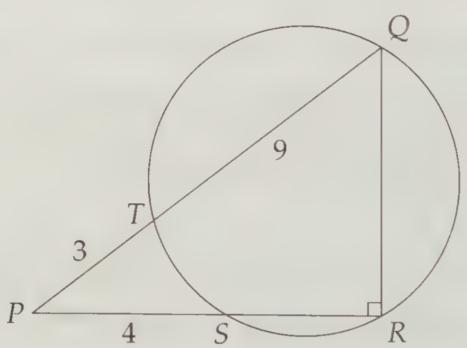


Figure 13.2: Diagram for Problem 13.2.2

13.2.2 Find the following in the figure at right above:

(a) $[PQR]$.

(b) ST. **Hints:** 69

(c) The area of the circle.

(d) $[QTR]$. **Hints:** 594

13.2.3 As shown in the diagram at left below, $\overline{XA}$ is tangent to the circle at A and a secant through X meets the circle at B and C, with $XB < XC$. Prove that $XB < XA < XC$. **Hints:** 91

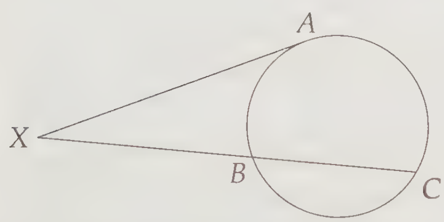


Figure 13.3: Diagram for Problem 13.2.3

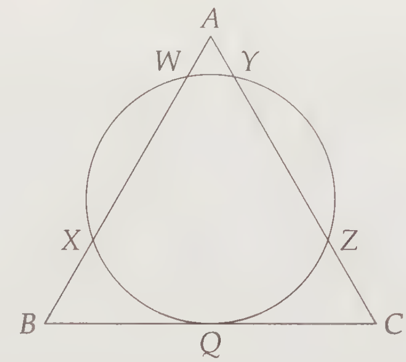


Figure 13.4: Diagram for Problem 13.2.4

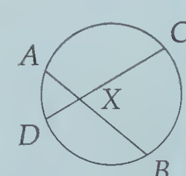
13.2.4 A circle is tangent to side $\overline{BC}$ of equilateral triangle $\triangle ABC$ at point Q as shown at right above. The circle intersects sides $\overline{AB}$ and $\overline{AC}$ in two points each, as shown. Given that $AW = AY$, prove that Q is the midpoint of $\overline{BC}$. **Hints:** 103

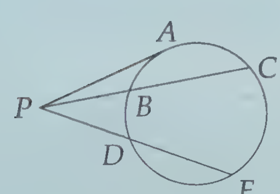
13.3 Summary

Important: Suppose a line through a point P intersects a circle in two points, U and V. The **Power of a Point Theorem** states that for all such lines, the product $(PU)(PV)$ is constant. We call this the **power** of point P.

For example, in the figure at right, applying Power of a Point to X with respect to the circle shown gives

$$(XA)(XB) = (XC)(XD).$$





In the figure at left, the power of point P with respect to the circle gives us

$$PA^2 = (PB)(PC) = (PD)(PE).$$

We think of these whenever we have a problem involving intersecting segments and a circle.

Important: While finding a solution often involves working both forwards and backwards, when we write our solution, we write a nice clean solution *forwards*, starting from what we are given and ending at what we want to prove.

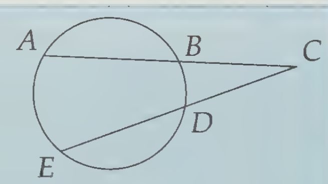
Problem Solving Strategies

Concepts:

- If a problem includes a segment that abruptly stops in the middle of a circle, consider continuing the segment until it hits the circle. Then see what Power of a Point gives you.
- If you see a 60° or 30° angle (or even a 120° angle) in a problem, you should be on the lookout for 30-60-90 triangles.
- Organize your work on challenging geometry proofs. Keep track of what you know and what you can derive from what you know. Also keep track of what you need to prove, and work backwards from there to list more statements that, if proven, would mean you are finished. When you know how to get from a statement on your 'What We Know' list to a statement on your 'What We Want' list, then you can construct your proof.

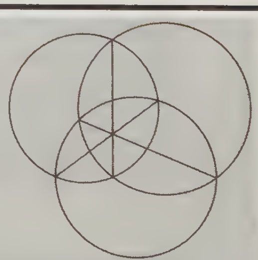
Things To Watch Out For!

WARNING!! A very common mistake in applying Power of a Point to a point outside a circle as in the figure at right is to write $(BC)(BA) = (DC)(DE)$. This alleged equality is **not** what Power of a Point tells us! Note that whenever we write a Power of a Point relationship, the same point must appear in *all* of the segments in the equation, as point C does when we write the correct relationship for the secants in the figure above:



$$(CD)(CE) = (CB)(CA).$$

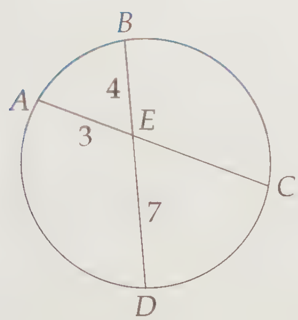
Extra! The set of all points that have the same power with respect to two given circles is a line called the **radical axis** of the two circles. One particularly important use of radical axes is the **Radical Axis Theorem**, which states that given three circles, the three radical axes of the three pairs of circles are concurrent. The diagram at right is an example of the Radical Axis Theorem for three intersecting circles. Do you see why the radical axis of two intersecting circles must include the chord connecting the points where the circles meet?



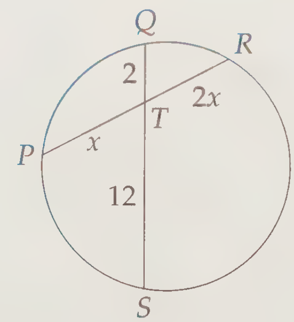
As an extra challenge, try proving first that the set of all points that have the same power with respect to two given circles is a line, then use this to prove the Radical Axis Theorem. (The first part is tougher than the second!)

Review Problems

13.11

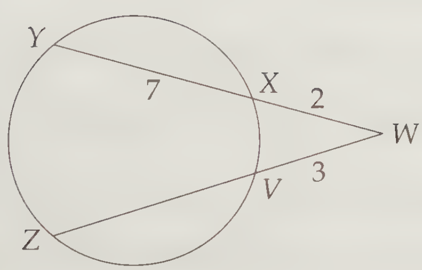


(a) Find EC.

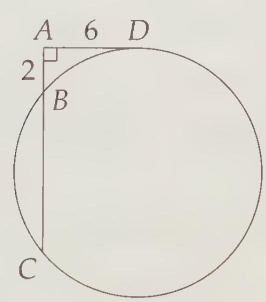


(b) Find PR.

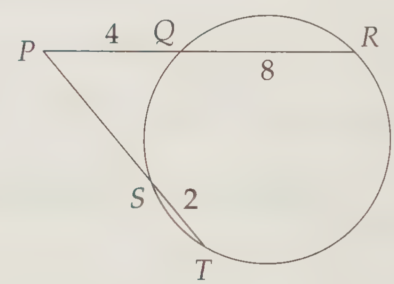
13.12



(a) Find VZ



(b) Find BC and CD.



(c) Find PS.

13.13 Chords $\overline{GH}$ and $\overline{IJ}$ of $\odot O$ are perpendicular at M. Given $GM = 2MH = 6IM = 12$, find GJ. **Hints:** 76

13.14 Lines m and n meet at P. $\odot O$ meets m at points A and B, and meets n at C and D, such that $PA = 3$, $AB = 9$, $PC = 4$, and $CD = 5$. Why must P be outside $\odot O$?

13.15 Earth is roughly 8000 miles in diameter.

(a) I'm riding in a hot air balloon 1 mile above the surface of Earth. Approximately how far away is the horizon? (In other words, how far away is the farthest point on the surface of Earth that I can see.)

(b) What if I'm in a plane 6 miles above the surface of the Earth?

(c) What if I'm in a spaceship 100 miles above the surface of Earth?

13.16 Given $ZY < ZW$ in the diagram at left below, show that $ZV < ZX$.

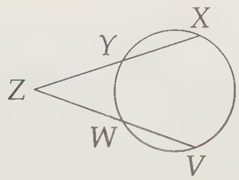


Figure 13.5: Diagram for Problem 13.16

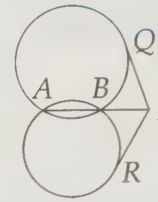
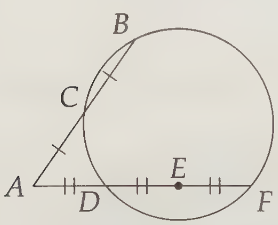


Figure 13.6: Diagram for Problem 13.17

13.17 The figure at right above shows two circles that intersect at A and B. P is on $\overleftrightarrow{AB}$, and $\overline{PQ}$ and $\overline{PR}$ are tangents as shown. Prove that $PQ = PR$.

13.18 Is it possible for chords $\overline{AB}$ and $\overline{CD}$ of a circle to intersect at X such that X is the midpoint of $\overline{AB}$, but is closer to C than to D?

13.19 Is it possible for two chords $\overline{AB}$ and $\overline{CD}$ of a circle to meet at a point X such that $AX = BX$, $CX = 2DX$, and $AB = CD$?

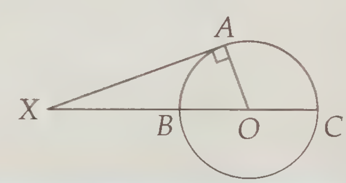


13.20 Find the ratio AC/AD in the figure at left.

13.21 Point A is on $\odot O$ such that $\overline{PA}$ is tangent to circle $\odot O$. Point B is on circle $\odot O$ such that $PB = PA$. Must $\overline{PB}$ be tangent to $\odot O$?

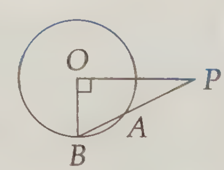
13.22 Let P be a point in the same plane as a circle centered at O with radius r. A secant through P intersects the circle at points A and B. Use $\overleftrightarrow{PO}$ to prove that if P lies outside the circle, then $PA \cdot PB = PO^2 - r^2$. What is the formula if P lies inside the circle?

13.23 Use the diagram in the figure at right to find a proof of the Pythagorean Theorem.



Challenge Problems

13.24 In circle O, $\overline{PO} \perp \overline{OB}$ and PO equals the length of the diameter of $\odot O$. Compute PA/AB. *(Source: ARML)* **Hints:** 294, 323



13.25 Chords $\overline{PQ}$ and $\overline{RS}$ of a circle meet at point X. Given that $PQ = RS$, show that $PX = RX$ or $QX = RX$ (or both). **Hints:** 364

13.26 Jake is working on a problem in which chords $\overline{AB}$ and $\overline{CD}$, when extended past B and D, respectively, meet at P. He is given PB, AB, and CD. He mistakenly uses Power of a Point improperly by using $(PB)(BA) = (PD)(DC)$. However, he still gets the right answer for PD. Prove that the question Jake was answering must have had $AB = CD$. **Hints:** 385

13.27 A and B are two points on a circle with center O, and C lies outside the circle, on ray $\overrightarrow{AB}$. Given that $AB = 24$, $BC = 28$, and $OA = 15$, find OC. **Hints:** 122

13.28 Circles C_1 and C_2 have the same center, O. The radius of C_1 is r_1 and the radius of C_2 is r_2, where $r_1 > r_2$.

(a) Prove that it is impossible for a point P to have the same power with respect to both circles if P is outside both circles. **Hints:** 104

(b) Prove that it is impossible for a point P to have the same power with respect to both circles if P is inside both circles.

(c) Show that it is possible for a point inside C_1 but outside C_2 to have the same power with respect to both circles, and find all such points that have equal power with respect to both circles.

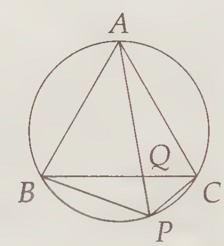
13.29 $\overline{WX}$ and $\overline{YZ}$ meet at P such that $(WP)(PX) = (YP)(PZ)$. Prove that the circumcircle of $\triangle XYZ$ goes through point W. **Hints:** 512

13.30 Two circles C_1 and C_2 intersect at two points, A and B. Let $\overline{PQ}$ be a chord of C_1 and $\overline{RS}$ a chord of C_2 such that they intersect on $\overline{AB}$. Prove that points P, Q, R, and S all lie on a circle. **Hints:** 540, 397

13.31 Given two segments with lengths a and b, construct a segment with length $\sqrt{ab}$. **Hints:** 415

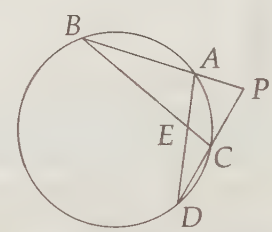
13.32 Equilateral triangle ABC is inscribed in a circle. Let P be a point on arc $\widehat{BC}$, and let $\overline{AP}$ intersect $\overline{BC}$ at Q. Prove that

$$\frac{1}{PQ} = \frac{1}{PB} + \frac{1}{PC}.$$



Hints: 382, 349

13.33 In the diagram at right, $AB = 8$, $AP = 2$, and $PC = 4$. *(Source: ARML)*



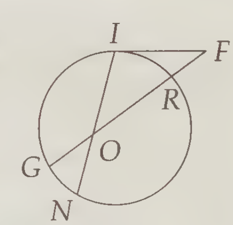
(a) Prove that $[ABE]/[CDE] = (AB/CD)^2$.

(b) Prove that $[PBC]/[PAD] = (PC/PA)^2$.

(c)★ Find $[PAEC]/[BAE]$. **Hints:** 309, 346

13.34★ An equilateral triangle is inscribed in a circle. Points D and E are midpoints of $\overline{AB}$ and $\overline{BC}$, respectively, and F is the point where $\overrightarrow{DE}$ meets the circle. Find DE/EF. *(Source: ARML)* **Hints:** 300, 252

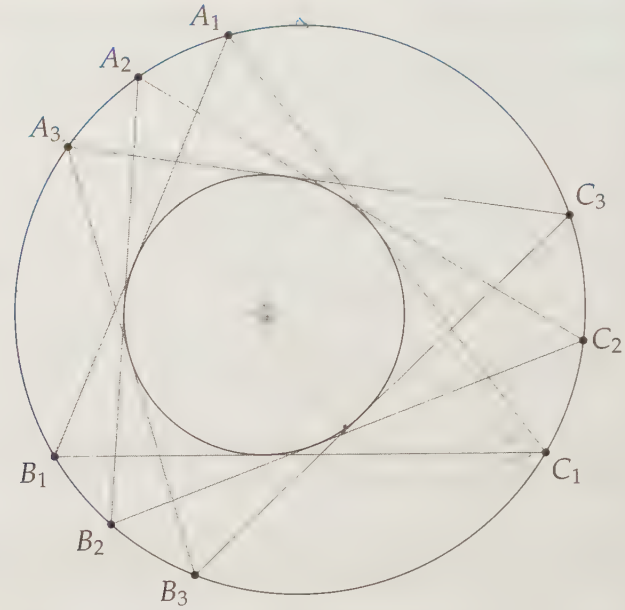
13.35★ Segment $\overline{IF}$ is tangent to the circle at point I as shown in the diagram at right. We are given that $IF = 21\sqrt{2}$, $IO = 20$, $ON = 12$, $RF = 18$, and $OR > GO$. Find OR. *(Source: Mandelbrot)* **Hints:** 188



13.36★ Chords $\overline{AB}$ and $\overline{CD}$ of $\odot O$ are perpendicular at point P. Given $CP = 2$, $AP = 3$, and $PD = 6$, find the following:

(a) OP. **Hints:** 31, 90

(b) The radius of $\odot O$. **Hints:** 387



Poncelet's Porism

Man's mind, stretched by a new idea, never goes back to its original dimensions. – Oliver Wendell Holmes

CHAPTER 14 Three-Dimensional Geometry

So far in this book we've confined ourselves to zero dimensions (points), one dimension (lines and line segments), or two dimensions (basically everything else in the book). However, we all know the world around us seems three-dimensional – that in addition to left and right, and forwards and backwards, there's up and down.

In this chapter we step off the page and into the 'real world' by considering three-dimensional objects. As we'll see, this isn't much of a step; much of our work in three-dimensional geometry will consist of reducing problems to two-dimensional situations we already know how to handle.

14.1 Planes

As mentioned in Section 1.2, a **plane** is a 'flat' two-dimensional surface that extends forever. For example, when your book is completely flat, this page is part of a plane. In this section, we'll explore planes, as well as how lines and planes can intersect in space.

Problems

Problem 14.1: We know that given any two points, we can draw exactly one line through the two points.

(a) Given two points, how many different planes pass through the two points?

(b) Given three points, is it always possible to find a plane that passes through all three points?

(c) Given any four points, is it always possible to find a plane that passes through all four points?

Problem 14.2:

(a) When two lines intersect, their intersection is a point. When two planes intersect, what kind of figure is their intersection?

(b) Is it possible for two planes to never intersect?

(c) Can there be a line and a plane that never intersect?

(d) Remember that two lines are parallel if they are *in the same plane* but do not intersect. Can two lines in space be such that they are not parallel but still do not intersect?

(e) Given any two intersecting lines, is there always a plane that contains both lines?

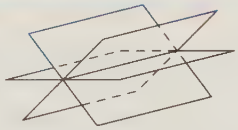
Problem 14.3:

(a) How can we tell if a line is perpendicular to a plane? Give an example in the 'real world' of a line and a plane that are perpendicular.

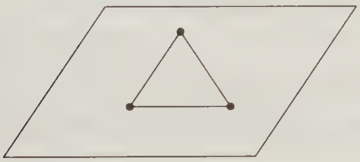
(b) How can we tell if two planes are perpendicular? Give an example in the 'real world' of two planes that are perpendicular.

We start with a fundamental question about planes.

Problem 14.1: How many points are needed to determine a unique plane?



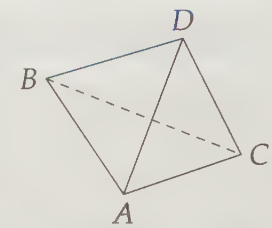
Solution for Problem 14.1: If we only have two points, we can find a line through them, but there are infinitely many planes that can pass through this line (and therefore pass through our initial two points). An example is shown in the figure at right, in which three planes are shown passing through the same line.



Given three or more collinear points (meaning they are all on the same line), we can still find infinitely many planes that pass through all three points, but if we have three points that are not collinear, then we have a triangle. The plane of this triangle is the unique plane that passes through the three points, as shown at left.

Sidenote: Why do most telescopes have three legs? Why do photographers use a tripod, which has three legs, to stabilize their cameras instead of something with four legs? The answer: three points determine a plane, so wherever you place a stand with three legs, it will be stable.

While any three points are **coplanar**, meaning there is a plane through all three of them, it is easy to find a set of four points that are **noncoplanar**. Three vertices of a triangle and a point not in the plane of the triangle give us an example in which no plane passes through four given points. Such a situation is depicted at right, where $\overline{BC}$ is dashed because it is in the 'back' of the figure. (We wouldn't be able to see $\overline{BC}$ if this figure were rendered in three dimensions.) □



Important: Three noncollinear points determine a plane.

Now we consider how lines and planes in space can be related.

Problem 14.2:

(a) When two lines intersect, their intersection is a point. When two planes intersect, what kind of figure is their intersection?

(b) Is it possible for two planes to never intersect?

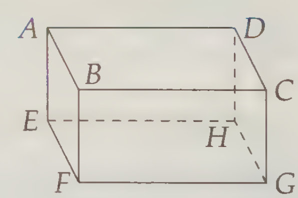
(c) Can there be a line and a plane that never intersect?

(d) Remember that two lines are parallel if they are *in the same plane* but do not intersect. Can two lines in space be such that they are not parallel but still do not intersect?

(e) Given any two intersecting lines, is there always a plane that contains both lines?

Solution for Problem 14.2:

(a) Consider a standard rectangular room as shown at right, where $EFGH$ is the 'floor' and $ABCD$ is the 'ceiling'. The walls are portions of planes. Look at where a wall meets the floor, and you'll see an example of two planes intersecting to form a line. Similarly, any two planes that intersect form a line.



(b) In a typical room, the ceiling is a constant distance from the floor. This means that if the ceiling and the floor extended forever in all directions, they would be planes that never intersect. Such planes are called **parallel**, just as lines in the same plane that never intersect are called parallel. For example, in our figure above, $ABCD$ and $EFGH$ are parallel planes.

(c) The intersection of a wall and the ceiling (line $\overleftrightarrow{AB}$, for example) will never meet the floor (plane $EFGH$). This is an example of a line that is parallel to a plane.

(d) Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CG}$ do not intersect and are not in the same plane. Since they are not coplanar, they are not parallel. But they still don't intersect! We call noncoplanar lines that do not intersect **skew lines**.

(e) Yes. Let X be the intersection point of the two lines, Y be a different point on one of the lines, and Z a different point on the other line. These three points determine a plane. Clearly, both $\overleftrightarrow{XY}$ and $\overleftrightarrow{XZ}$ are in the plane because $\overline{XY}$ and $\overline{XZ}$ are.

□

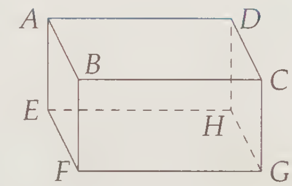
After dealing with parallels, you know what's next.

Problem 14.3:

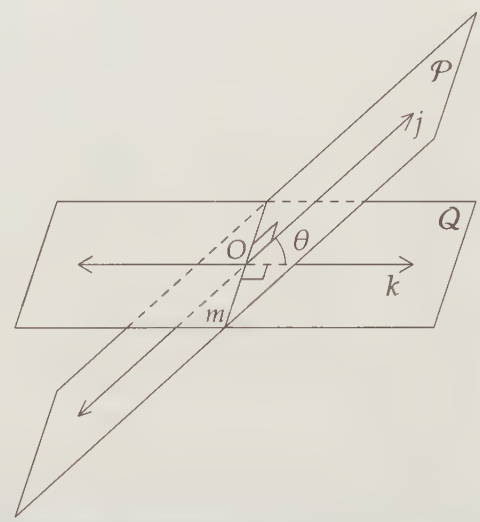
(a) How can we tell if a line is perpendicular to a plane? Give an example in the 'real world' of a line and a plane that are perpendicular.

(b) How can we tell if two planes are perpendicular? Give an example in the 'real world' of two planes that are perpendicular.

Solution for Problem 14.3: We'll use our trusty room for our 'real world' examples.



(a) Suppose line k intersects plane $\mathcal{P}$ at point X. If k is perpendicular to every line in $\mathcal{P}$ that passes through X, then we say that line k is perpendicular to the plane. The line of intersection between two adjacent walls in a typical room is perpendicular to the ceiling and the floor of the room. For example, $\overleftrightarrow{AE}$ is perpendicular to $ABCD$.



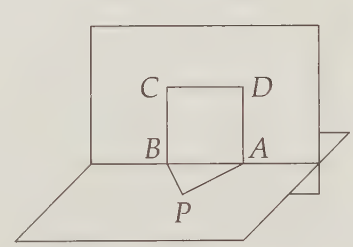
(b) Suppose planes $\mathcal{P}$ and Q intersect in line m, and let O be a point on line m. Consider lines j and k in $\mathcal{P}$ and Q, respectively, such that each of j and k is perpendicular to m at point O. The angle between j and k, which is marked with a θ in the diagram, is the angle between the planes. We call such an angle between planes a **dihedral angle**. If this angle is a right angle, the planes are perpendicular.

In a typical room, a wall is perpendicular to the ceiling and to the floor. For example, plane $ABCD$ is perpendicular to plane $BCGF$ in our 'room' above.

□

Exercises

14.1.1 Line m is perpendicular to plane $\mathcal{P}$ at point X. Line k is in plane $\mathcal{P}$ and passes through X. Are lines m and k necessarily perpendicular?



14.1.2 Triangle PAB and square $ABCD$ are in perpendicular planes at right. Given that $PA = 3$, $PB = 4$, and $AB = 5$, what is PD? *(Source: AMC 12)*

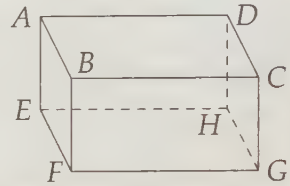
14.1.3 Is it true that the nearest point on line m in space to a point P in space is the foot of the perpendicular segment from P to m? Why or why not?

14.1.4 Is it true that the nearest point on plane $\mathcal{P}$ in space to a point X in space is the foot of the perpendicular segment from X to plane $\mathcal{P}$? Why or why not?

14.1.5 Two planes, $\mathcal{M}$ and $\mathcal{N}$ are each perpendicular to a third plane, $\mathcal{P}$. $\mathcal{M}$ and $\mathcal{N}$ intersect in line m. Is line m perpendicular to $\mathcal{P}$? (No proof necessary.)

14.2 Prisms

We move now from considering one- and two-dimensional figures in space to thinking about solid figures. Our 'room' was so useful in the previous section that we'll start with it here.



Mathematically speaking, $ABCDEFGH$ at the right is a **right rectangular prism**. Since all the boundaries of $ABCDEFGH$ are polygons, $ABCDEFGH$ can also be called a **polyhedron**. Each of these boundary polygons is a **face** of the polyhedron. The 'rectangular' part of 'right rectangular prism' refers to the fact that the top and bottom faces ($ABCD$ and $EFGH$) are rectangles. We typically call the 'top' and 'bottom' the **bases** of the prism. The 'right' part refers to the fact that all the edges connecting the two bases meet the bases at right angles.

Which leaves us with the 'prism' part. A **prism** is a three-dimensional solid figure with two congruent parallel faces, and with parallelograms as the other faces. The faces that are not bases are sometimes called the **sides** of the prism. In a **right prism** all of these sides are rectangles. As already noted, the congruent parallel faces are the bases of the prism. Hence, any pair of opposite sides of $ABCDEFGH$ above could be considered the bases of the prism.

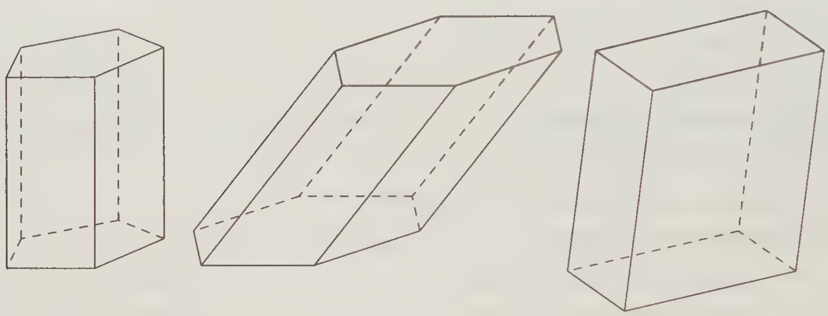


Figure 14.1: Some Prisms

Figure 14.1 shows some more prisms. The first is a right regular pentagonal prism because the base is a regular pentagon and the segments connecting the bases are perpendicular to the bases. The second is a hexagonal prism because the bases are hexagons (note that we don't call this one a 'right hexagonal prism'). The third has parallelograms as its bases, as well as all its sides. Such a prism is given the special name **parallelepiped**. As one last bit of vocabulary, if there's a 'regular' in the description of a prism, it means the base is a regular polygon.

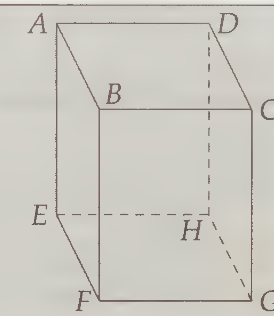
Make sure you understand what a prism is, but you don't have to worry about all those other names. One reason you don't have to worry too much is that they aren't used consistently. For example, the 'right' is often left out of the description, and you're meant to infer from the problem that the prism is indeed a right prism. Sometimes in this book we'll do this, such as by calling our original prism $ABCDEFGH$ a 'rectangular prism'.

Unfortunately, the vocabulary lesson isn't over yet. Just as we measure the region contained in a two-dimensional figure with area, we can measure the space inside a three-dimensional figure with **volume**. We do so in much the same way as we measure area. Instead of measuring how many 1×1 squares fit, we measure how many $1 \times 1 \times 1$ 'blocks' (or portions of blocks) fit inside the solid.

Finally, we can still use area to measure three-dimensional figures. The **total surface area** of a figure is the area of all of the surfaces that form the borders of the solid. The **lateral surface area** is the total area of all the faces that are not considered 'bases.' We often just ignore lateral surface area and say 'surface area,' which means the same as 'total surface area.'

Problems

Problem 14.4: *ABCDEFGH* shown is a rectangular prism with $AB = 3$, $BC = 4$, and $AE = 5$.



(a) Find BF and EH.

(b) Find the volume of *ABCDEFGH*.

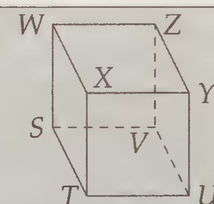
(c) Find the total surface area of *ABCDEFGH*.

(d) Find AC.

(e) What kind of triangle is $\triangle ACG$?

(f) Find AG.

Problem 14.5: A **cube** is a special rectangular prism in which all edges have the same length (i.e., its base is a square and its height has the same length as a side of the base).

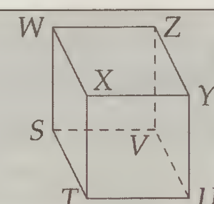


(a) Find the formula for the volume of a cube with side length s.

(b) Find the formula for the total surface area of a cube with side length s.

(c) Find a formula for the length of $\overline{SY}$ in the diagram given that the figure is a cube with side length s.

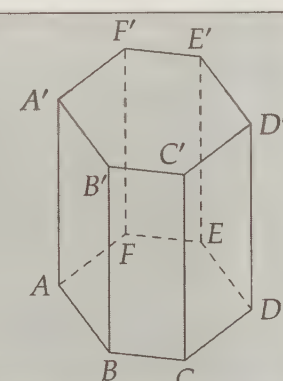
Problem 14.6: Once again, we consider cube *STUVWXYZ*.



(a) What geometric shape is *SXYV*?

(b) What is the area of *SXYV* if $SV = 4$?

Problem 14.7: The right regular hexagonal prism shown has sides of length 4 on base *ABCDEF* and a height of 8 units.



(a) Find the lateral surface area of the prism.

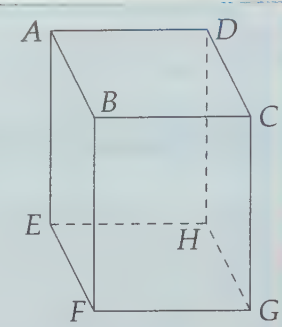
(b) Find the total surface area of the prism.

(c) Find the volume of the prism.

(d) Find AB', AC', and AD' by building the appropriate right triangles.

We start off by finding volumes, areas, and lengths of a (right) rectangular prism.

Problem 14.4: *ABCDEFGH* shown is a rectangular prism with $AB = 3$, $BC = 4$, and $AE = 5$.



(a) Find BF and EH.

(b) Find the volume of *ABCDEFGH*.

(c) Find the total surface area of *ABCDEFGH*.

(d) Find AC.

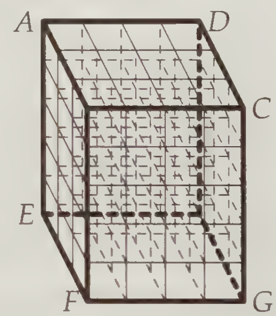
(e) What kind of triangle is $\triangle ACG$?

(f) Find AG.

Solution for Problem 14.4:

(a) All of the faces are rectangles. Since *AEFB* is a rectangle, $BF = AE = 5$. Similarly, rectangle *EFGH* tells us that $EH = FG$, and rectangle *BFGC* gives us $FG = BC = 4$. Therefore, $EH = 4$. We could also have seen that $EH = BC$ by noting that both represent the distance between parallel faces *AEFB* and *DHGC*.

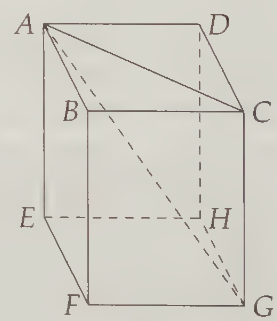
(b) We approach this much as we did finding the area of a rectangle with 1×1 squares on page 85. We can organize a layer of $(AB)(BC) = 12$ $1 \times 1 \times 1$ cubes on face *ABCD*. This plus four more such layers completely fills our prism, so the volume is $3 \times 4 \times 5 = 60$.



(c) To find the total surface area, we need to add the areas of all the faces. Each face is a rectangle, but rather than finding the areas of six different rectangles, we can note that the rectangles come in pairs of congruent rectangles. For example, $ABCD \cong EFGH$. Therefore, our total surface area is:

$$2([ABCD] + [BCGF] + [ABFE]) = 2[(3)(4) + (4)(5) + (3)(5)] = 94.$$

(d) $\overline{AC}$ is a diagonal of rectangle *ABCD* (or hypotenuse of triangle *ABC*), so it has length $\sqrt{3^2 + 4^2} = 5$. $\overline{AC}$ is sometimes called a **face diagonal** of the prism.



(e) Since $\overline{CG}$ is perpendicular to plane *ABCD*, it is perpendicular to $\overline{AC}$. Therefore, $\triangle ACG$ is a right triangle.

(f) From right $\triangle ACG$, we have

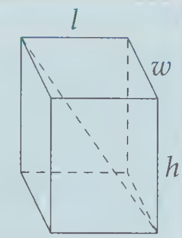
$$AG = \sqrt{GC^2 + AC^2} = \sqrt{25 + 25} = 5\sqrt{2}.$$

Note that $\sqrt{AB^2 + BC^2 + GC^2} = 5\sqrt{2} = AG$. Is this a coincidence?

□

Segment $\overline{AG}$ is called a **space diagonal** of the prism. The lengths AB, AD, and AE are sometimes referred to as the prism's **dimensions**. We might use these to describe the prism as a $3 \times 4 \times 5$ rectangular prism. We can follow the exact same logic we followed in Problem 14.4 to find formulas for the volume, total surface area, and the length of a space diagonal of a rectangular prism.

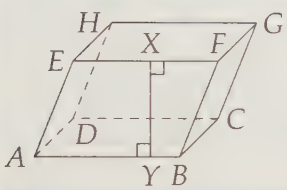
Important: The three dimensions of a rectangular prism are commonly called the length, l, the width, w, and the height, h. For such a prism, we have:



$$\begin{aligned}\text{Volume} &= lwh \\ \text{Surface area} &= 2(lw + wh + lh) \\ \text{Space Diagonal} &= \sqrt{l^2 + w^2 + h^2}\end{aligned}$$

Notice that our volume is simply the area of a base of the prism times the height of the prism. This is true for any prism.

Important: The volume of a prism equals the area of a base times the distance between the bases (i.e. the height).



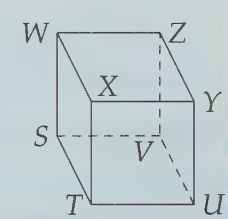
Notice that by 'distance between the bases', we do not necessarily mean the length of the edges connecting corresponding points on the bases. If the prism is not a right prism, then the height is the length of a segment from one base to the other that is perpendicular to both bases. For example, the height between bases $ABCD$ and $EFGH$ of the prism at left is XY, not FB.

Problem 14.5: A **cube** is a special rectangular prism that has all its dimensions the same (i.e., its base is a square and its height has the same length as a side of the base).

(a) Find the formula for the volume of a cube with side length s.

(b) Find the formula for the total surface area of a cube with side length s.

(c) Find a formula for the length of $\overline{SY}$ in the diagram given that the figure is a cube with side length s.

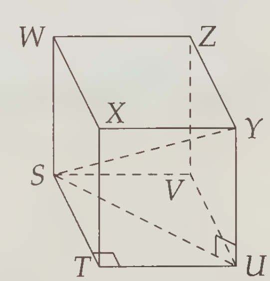


Solution for Problem 14.5:

(a) Since a cube is a rectangular prism, its volume is the product of all three dimensions. Since all three dimensions are the same, s, the volume is simply s^3.

(b) The faces of a cube are all squares. Therefore, each of the 6 faces has area s^2, so the total surface area is $6s^2$.

(c) Once again, we can simply use the formulas we already found for a rectangular prism. But to be a little more sporting, let's prove the formula directly for a cube. To find SY, we'd like to create a right triangle that has $\overline{SY}$ as a side, so we can use the Pythagorean Theorem. Hence, we draw $\overline{SY}$ and $\overline{SU}$, creating right triangle $\triangle SUY$. We have $UY = s$, so all we have to do is find $\overline{SU}$. $\overline{SU}$ is the hypotenuse of $\triangle STU$, so $SU = s\sqrt{2}$. Therefore,



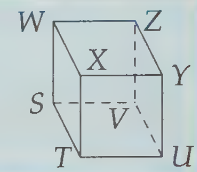
$$SY = \sqrt{SU^2 + UY^2} = \sqrt{2s^2 + s^2} = s\sqrt{3}.$$

□

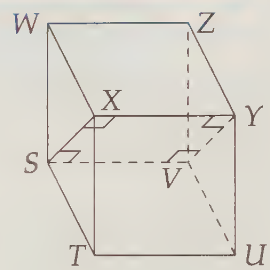
Important: A cube with side length s has:

$$\begin{aligned} \text{Volume} &= s^3 \\ \text{Surface area} &= 6s^2 \\ \text{Space Diagonal} &= s\sqrt{3} \end{aligned}$$

Problem 14.6: Once again, we consider cube $STUVWXYZ$. What is the area of $SXYV$ if $SV = 4$?



Solution for Problem 14.6: We start by drawing $SXYV$ so we can figure out what sort of shape it is. Since $\overline{SV}$ and $\overline{XY}$ are perpendicular to faces $STXW$ and $VUYZ$, they are both perpendicular to both $\overline{SX}$ and $\overline{YV}$. So, $SXYV$ is a rectangle. We already have one side of the rectangle, $SV = 4$. Therefore, we only need to find SX. Since $\overline{SX}$ is a diagonal of square $SWXT$, $SX = 4\sqrt{2}$, so $[SXYV] = (SV)(SX) = 16\sqrt{2}$. □



One way to view rectangle $SXYV$ of the previous problem is as the intersection of a plane and the cube, where the plane passes through vertices S, X, Y, and V of the cube. We call such an intersection of a plane and a solid a **cross-section** of the solid. Many three-dimensional geometry problems are solved by choosing the right cross-section of the problem to consider. Sometimes we even have to consider multiple cross-sections!

Lest you start to think that all prisms are rectangular ones, we'll try a problem involving a regular hexagonal prism.

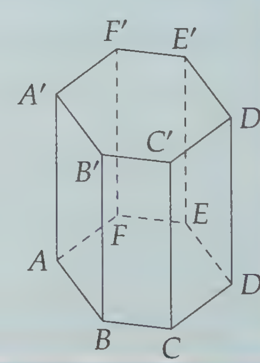
Problem 14.7: The right regular hexagonal prism shown has sides of length 4 on base $ABCDEF$ and a height of 8 units.

(a) Find the lateral surface area of the prism.

(b) Find the total surface area of the prism.

(c) Find the volume of the prism.

(d) Find AB', AC', and AD'.



Solution for Problem 14.7:

(a) Each lateral face of the prism is a 4×8 rectangle. There are 6 lateral faces, so the lateral surface area is $6(4 \times 8) = 192$.

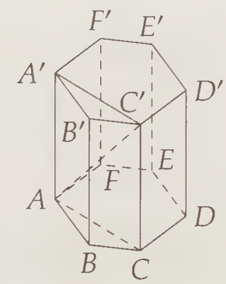
(b) To find the total surface area, we must add the area of the two bases to the lateral surface area. Each base is a regular hexagon. As we saw back in Problem 9.7, we can find the area of a regular hexagon of side length 4 by breaking it into six equilateral triangles with side length 4. Therefore, the area of each base is

$$6\left(\frac{4^2\sqrt{3}}{4}\right) = 24\sqrt{3}.$$

So, the total surface area of the prism is $192 + 2(24\sqrt{3}) = 192 + 48\sqrt{3}$.

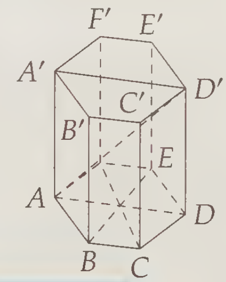
(c) We've already done the hard part! The volume is just the area of a base times the height, or $8(24\sqrt{3}) = 192\sqrt{3}$.

(d) Since $\overline{AB'}$ is a diagonal of rectangle $ABB'A'$, $AB' = \sqrt{4^2+8^2} = 4\sqrt{5}$.



The other two are a little trickier. For AC', we consider the cross-section $ACC'A'$ and notice that $\overline{AC'}$ is a diagonal of rectangle $ACC'A'$. We already know $CC' = 8$, so we only have to find AC. Since $\triangle ACD$ is a 30-60-90 triangle (make sure you see why), we have $AC = CD\sqrt{3} = 4\sqrt{3}$. Hence,

$$AC' = \sqrt{AC^2 + CC'^2} = \sqrt{48+64} = 4\sqrt{7}.$$



Similarly, $\overline{AD'}$ is a diagonal of rectangle $ADD'A'$. To find AD, we recall the dissection of a regular hexagon by its long diagonals into 6 equilateral triangles as shown, so $AD = 2(BC) = 8$. Therefore,

$$AD' = \sqrt{AD^2 + D'D^2} = \sqrt{64+64} = 8\sqrt{2}.$$

□

Concept: Building right triangles works in three dimensions every bit as well as it does in two dimensions.

Exercises

14.2.1 Describe as fully as possible a cross-section of a prism that is parallel to the bases of the prism.

14.2.2 A right rectangular prism has dimensions 2, 5, and $3\sqrt{2}$.

(a) Find the volume of the prism.

(b) Find the total surface area of the prism.

(c) Find the length of a space diagonal of the prism.

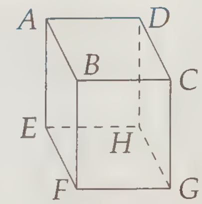
(d) Find the length of the longest face diagonals.

14.2.3 Find the volume of a rectangular prism that has a space diagonal of length 10 and two sides of length 3 and 8.

14.2.4 Find the volume of a cube given that its space diagonal has length 6.

14.2.5 How many different space diagonals does a cube have?

14.2.6 If all the dimensions of a right rectangular prism are doubled, what happens to the surface area? What happens to the volume?



14.2.7 Shown is right rectangular prism $ABCDEFGH$. Given that $AB = 4$, $BC = 3$, and $ABGH$ is a square, find the following:

(a) BD.

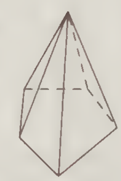
(b) AG.

(c) FD.

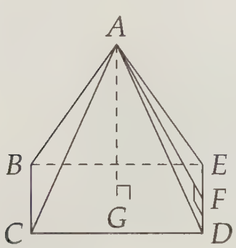
(d) the volume of $ABCDEFGH$.

14.2.8 Bridget places a box that is $4 \times 6 \times 8$ on the floor. She then places a $2 \times 3 \times 5$ box on top of the first box, forming a two-box tower. She will then paint all the surfaces she can paint without moving either of the boxes. She wants to paint as little as possible, so she places the boxes in a way that minimizes the amount she'll have to paint. What is the total area she has to paint? *(Source: ARML)*

14.3 Pyramids



If we connect all the vertices of a polygon to a point that is not in the same plane as the polygon, we form a **pyramid**. This point is called the **apex** of the pyramid and the polygon is the pyramid's **base**. As we can see at right, the non-base faces of a pyramid are all triangles. The lateral surface area of a pyramid is the sum of the areas of these triangles.



As with prisms, a pyramid is 'regular' if its base is a regular polygon. A pyramid is 'right' if the center of the base is the foot of the altitude from the apex to the base (i.e., the apex is directly 'over' the center of the base). Usually when we speak of a pyramid, we mean a right pyramid. The height of a pyramid is the distance from the apex to the base. For regular pyramids, we also define a **slant height**, which is the distance from the apex to a side of the base. For example, at left, AG is the height and AF is the slant height of the pyramid.

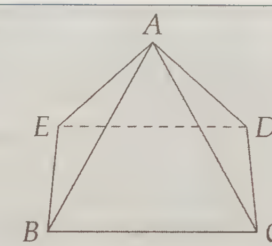
Important: The volume of a pyramid is one-third the product of the pyramid's height and the area of the pyramid's base.

Deriving this formula of a pyramid's volume is very challenging, and will be left for the Intermediate Geometry textbook. However, the formula is not hard to apply, so it is occasionally used in introductory three-dimensional geometry.

Extra! I have two fish tanks that are both rectangular prisms like the one shown to the right. I have a fish that is very picky, and will only be happy in a tank that has exactly 50 gallons of water. Unfortunately, my fish tanks are both 60-gallon tanks. Worse yet, I don't have anything I can use to measure the sides of the tank and mark off a 'fill line.' All I have is a hose to pour the water. How can I fill one of my tanks with exactly 50 gallons?

Problems

Problem 14.8: $ABCDE$ is a right square pyramid such that $BC = 6$ and $AC = 5$ as shown at right.



(a) Let O be the center of the square. Why is $\overline{AO}$ perpendicular to the base of the pyramid? Show that $AC = AD$.

(b) Find $[ACD]$ by first finding the length of an altitude from A to $\overline{CD}$.

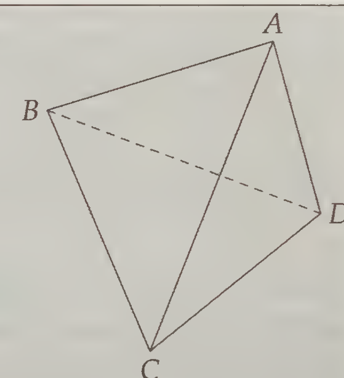
(c) Find the total surface area of $ABCDE$.

(d) Let M be the midpoint of $\overline{CD}$. What kind of triangle is $\triangle AOM$?

(e) Find AO, then find the volume of the pyramid.

Problem 14.9: Find a formula for the lateral surface area of a right pyramid whose base is a regular n-gon with side length s, and whose slant height is l.

Problem 14.10: A triangular pyramid is more commonly called a **tetrahedron**. A **regular tetrahedron** is a tetrahedron whose edges all have the same length. In the diagram, regular tetrahedron $ABCD$ has side length 6.



(a) Let G be the foot of the altitude from A to face $\triangle BCD$. What kind of triangles are $\triangle AGB$, $\triangle AGC$ and $\triangle AGD$?

(b) Use part (a) to show that G is the circumcenter of $\triangle BCD$. Why must it also be the centroid of $\triangle BCD$?

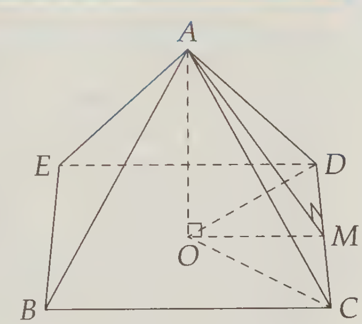
(c) Let M be the midpoint of $\overline{CD}$. Find BM and BG.

(d) Find AG.

(e) Find the volume of $ABCD$.

We'll start by investigating a square right pyramid.

Problem 14.8: $ABCDE$ is a square right pyramid such that $BC = 6$ and $AC = 5$. Find the total surface area and the volume of the pyramid.



Solution for Problem 14.8: We can find the area of the base quickly: $BC^2 = 36$. But to find the area of the triangular faces, we'll need an altitude of one of the faces. We start by drawing $\overline{AO}$, the altitude of the pyramid. Because the pyramid is right, O is the center of base $BCDE$. We suspect that each triangular face is isosceles because the pyramid is right. We can prove this by noting that since $\overline{AO}$ is perpendicular to the base, it is perpendicular to both $\overline{OC}$ and $\overline{OD}$. We have $OC = OD$ because O is the center of the square, so $\triangle AOC \cong \triangle AOD$ by SAS Congruence. This triangle congruence gives us $AC = AD$.

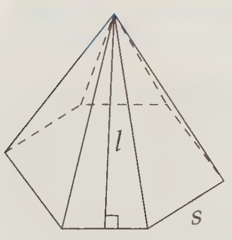
Therefore, $\triangle ACD$ is isosceles, so the median from A to $\overline{CD}$ is also an altitude. Since $AC = 5$ and $CM = CD/2 = 3$, we have $AM = 4$ from right triangle $\triangle AMC$. Therefore, $[ACD] = (AM)(CD)/2 = 12$. Our total surface area consists of four of these triangles plus the square, or $4(12) + 36 = 84$.

Turning to the volume, we need the height of the pyramid, AO. We can build a right triangle by connecting O to either M or to one of the base vertices. In the diagram, we have right triangle $\triangle AOM$. Since $OM = BC/2 = 3$ and $AM = 4$, we have $AO = \sqrt{AM^2 - OM^2} = \sqrt{7}$. Since the area of the base is 36, the volume of the pyramid is $(36)(\sqrt{7})/3 = 12\sqrt{7}$. □

Once again, we built right triangles to find the lengths we needed.

As we saw in that last problem, finding the lateral surface area of a regular right pyramid is easy once we know the slant height and the side length. We can even build a formula for it.

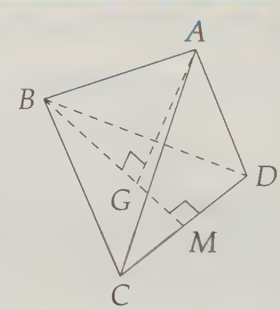
Problem 14.9: Find a formula for the lateral surface area of a right pyramid whose base is a regular n-gon with side length s, and whose slant height is l.



Solution for Problem 14.9: Each of the lateral faces of the pyramid is a triangle with height l and base s. Hence, each of these triangles has area $sl/2$. There is one triangle for each side of the base, so there are n triangles, which have a total area of $n(sl/2) = nsl/2$. Note that ns equals the perimeter of the base, so the lateral surface area of a regular pyramid is half the product of the slant height and the perimeter of the base. □

That formula is so much easier to derive than to memorize that we won't even put it in an 'Important' box.

Problem 14.10: A triangular pyramid is more commonly called a **tetrahedron**. A **regular tetrahedron** is a tetrahedron whose edges all have the same length. Find the volume of a regular tetrahedron that has sides of length 6.



Solution for Problem 14.10: To find the volume, we need an altitude, so we draw altitude $\overline{AG}$ from A to $\triangle BCD$. Since $\overline{AG}$ is perpendicular to plane BCD, triangles $\triangle AGB$, $\triangle AGC$, and $\triangle AGD$ are all right triangles. Because $AB = AC = AD$ and AG is obviously the same in all three triangles, we have $\triangle AGB \cong \triangle AGC \cong \triangle AGD$ by HL Congruence. Therefore, $BG = CG = DG$, which means that G is the circumcenter of $\triangle BCD$ because it is equidistant from the vertices of $\triangle BCD$. Since $\triangle BCD$ is an equilateral triangle, G is also the centroid of $\triangle BCD$.

We can build more right triangles by continuing $\overline{BG}$ to M. Since $\triangle BCD$ is equilateral, $\overline{BM}$ is a median and an altitude. Therefore, $DM = DC/2 = 3$ and $BM = 3\sqrt{3}$ (from 30-60-90 $\triangle BMD$). Since the centroid of a triangle divides its medians in a $2:1$ ratio, we have $BG = (2/3)BM = 2\sqrt{3}$. Finally, we can find AG from right triangle $\triangle AGB$:

$$AG = \sqrt{AB^2 - BG^2} = \sqrt{36 - 12} = 2\sqrt{6}.$$

Since the area of $\triangle BCD$ is $(DC)(BM)/2 = 9\sqrt{3}$, our volume is $([BCD])(AG)/3 = 18\sqrt{2}$.

Similarly, we can show that the volume of a regular tetrahedron with edge length s is $s^3\sqrt{2}/12$.□

Important: Problem 14.10 is a typical challenging three-dimensional geometry problem in that our general tactic is to reduce it to a series of two-dimensional problems. When you can work through this problem on your own, you're ready for some serious three-dimensional geometry.

Exercises

14.3.1 $ABCDE$ is a right square pyramid with base $ABCD$. Given that $AB = 4$ and $AE = 8$, find the following:

(a) The height of the pyramid.

(b) The slant height of the pyramid.

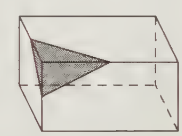
(c) The volume of the pyramid.

(d) The total surface area of the pyramid.

14.3.2 When we first created Problem 14.8, we let $BC = 8$ instead of $BC = 6$ in the problem statement. Why did we have to change it? **Hints:** 477

14.3.3 A regular pyramid with a square base has base edges of length 6 inches and height 4 inches. What is the ratio of the number of cubic inches in the volume of the pyramid to the number of square inches in its surface area? *(Source: MATHCOUNTS)*

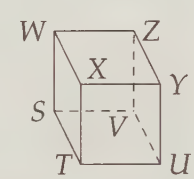
14.3.4 A $4'' \times 6'' \times 8''$ rectangular solid is cut by slicing through the midpoints of three adjacent sides as shown.



(a) Find the volume of the shaded piece that is cut off.

(b) Find the sum of the lengths of the edges of the shaded piece that is cut off. *(Source: MATHCOUNTS)*

14.3.5 $STUVWXYZ$ is a cube as shown with $ST = 4$.



(a) What are the volume and surface area of pyramid $STUW$?

(b) What are the volume and surface area of pyramid $STUX$? **Hints:** 524

(c) What are the volume and surface area of pyramid $STUZ$?

Extra! Back on page 253, we noted that any polygon can be dissected and rearranged to form any other polygon that has the same area. You might wonder if it is possible to dissect any polyhedron and rearrange the pieces to form any other polyhedron with the same volume. You wouldn't be the first to wonder this! In fact, this question was one of Hilbert's famous problems (see page 56). It was also the first to be solved, by **Max Wilhelm Dehn**.

Dehn used the powerful problem solving technique of **invariants** to solve the problem. He created a function of the edge lengths and the dihedral angles of a polyhedron and showed that this function must be equal for any two polyhedra that can be dissected into the same pieces. The function is different for a cube and a tetrahedron, so these two cannot be dissected into the same pieces.

14.4 Regular Polyhedra

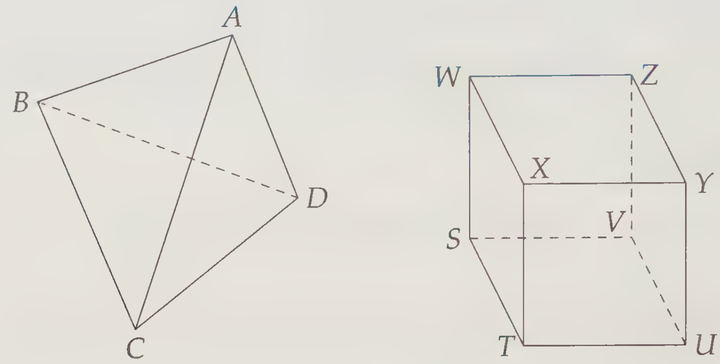


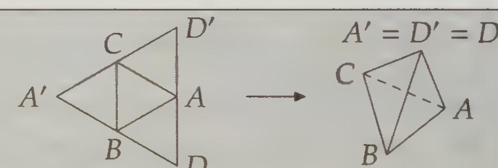
Figure 14.2: A Regular Tetrahedron and a Cube

We have already mentioned that a polyhedron is a solid figure that is bounded on all sides by polygons. A **regular polyhedron** is a convex polyhedron in which all the faces are congruent regular polygons, and there are the same number of edges at each vertex. We've already seen examples of two such regular polyhedra – a cube and a regular tetrahedron, examples of which are shown in Figure 14.2. The faces of the cube are congruent squares and those of the tetrahedron are congruent equilateral triangles.

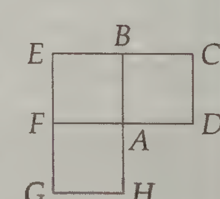
These aren't the only types of regular polyhedra, however. In this section we discover the other regular polyhedra, and we tackle a problem involving one of the types of regular polyhedra.

Problems

Problem 14.11: To make a regular tetrahedron, we might start with four equilateral triangles in a plane as shown at right. We then fold the triangles until A', D, and D' coincide as shown in the regular tetrahedron. As we'll see in this problem, thinking about 'folding up' regular polygons like this allows us to limit the possibilities of regular polyhedra that can be made.



(a) Consider three squares, $ABCD$, $ABEF$, and $AFGH$, arranged around a point as shown. If we 'fold' this arrangement so that D and H coincide, we'll be on our way to forming what type of regular polyhedron?



(b) Can we fit three regular pentagons around a point in a plane? Can we then fold the resulting figure to start to form a polyhedron?

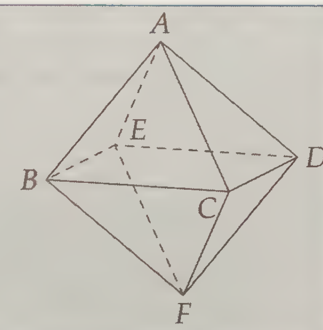
(c) Can we fit three regular hexagons around a point in a plane? Can we then fold the resulting figure to start to form a polyhedron?

(d) Can we fit four equilateral triangles around a point in a plane? Can we then fold the resulting figure to start to form a polyhedron? How about four squares?

(e) Can we fit five equilateral triangles around a point in a plane? Can we then fold the resulting figure to start to form a polyhedron? How about five squares?

(f) At most how many different types of regular polyhedra are there?

Problem 14.12: *ABCDEF* shown is a regular octahedron, which has eight congruent equilateral triangles as faces. We can also think of a regular octahedron as a pair of square pyramids glued together at their bases. In this problem, let $AB = 4$.



(a) Find BD.

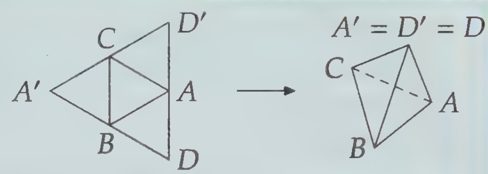
(b) Find AF.

(c) Find $[BCDE]$.

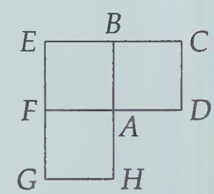
(d) Find the volume of $ABCDEF$.

There aren't many types of regular polyhedra. With a little physical intuition, we can discover which types of polyhedra might possibly exist.

Problem 14.11: To make a regular tetrahedron, we might start with four equilateral triangles in a plane as shown at right. We then fold the triangles until A, D, and D' coincide as shown in the regular tetrahedron. As we'll see in this problem, thinking about 'folding up' regular polygons like this allows us to limit the possibilities of regular polyhedra that can be made.



(a) Consider three squares, $ABCD$, $ABEF$, and $AFGH$, arranged around a point as shown. If we 'fold' this arrangement so that D and H coincide, we'll be on our way to forming what type of regular polyhedron?



(b) Can we fit three regular pentagons around a point in a plane? Can we then fold the resulting figure to start to form a polyhedron?

(c) Can we fit three regular hexagons around a point in a plane? Can we then fold the resulting figure to start to form a polyhedron?

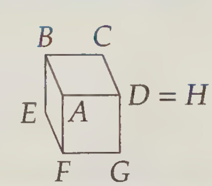
(d) Can we fit four equilateral triangles around a point in a plane? Can we then fold the resulting figure to start to form a polyhedron? How about four squares?

(e) Can we fit five equilateral triangles around a point in a plane? Can we then fold the resulting figure to start to form a polyhedron? How about five squares?

(f) At most how many different types of regular polyhedra are there?

Solution for Problem 14.11:

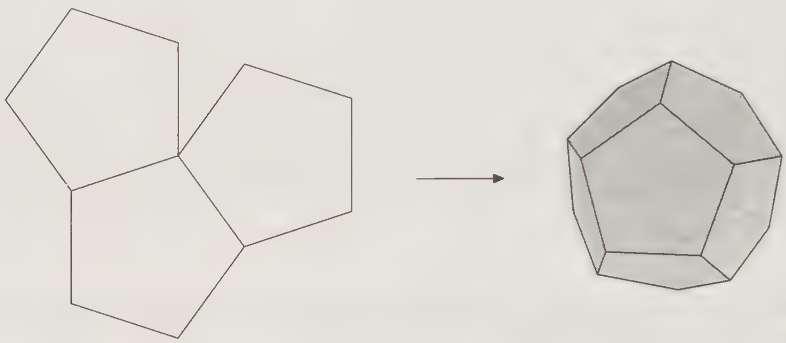
(a) Folding up our squares gives the figure shown, which is clearly well on its way to being a cube.



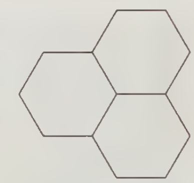
Extra! *We have to reinvent the wheel every once in a while, not because we need a lot of wheels, but because we need a lot of inventors.*

–Bruce Joyce

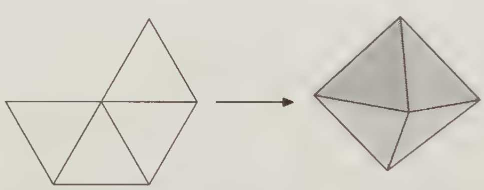
(b) We can fit three pentagons around a point with room to spare. Therefore, we can fold the pentagons to possibly form part of a regular polyhedron. Such a polyhedron does indeed exist, as shown below. This polyhedron is called a regular **dodecahedron**. It has 12 regular pentagons as its faces.



(c) We can fit three regular hexagons around a point, but since each angle of a regular hexagon is 120°, the three fit snugly. In other words, we can't bend the hexagons together as we've done earlier with the pentagons, squares and triangles. Therefore, there are no more regular polyhedra that have three of each polygon meeting at a point.

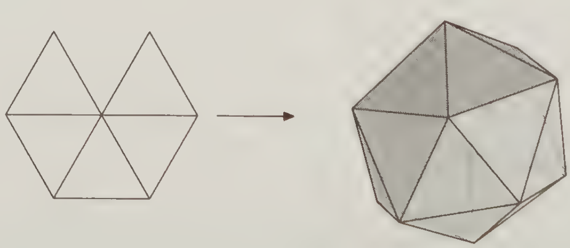


(d) Four equilateral triangles fit around a point with room to spare, so we can fold them together. The result is a square pyramid without its base. Gluing another square pyramid to this one gives us a polyhedron with eight congruent equilateral triangles as faces. This polyhedron, shown in the diagram below, is a regular **octahedron**.



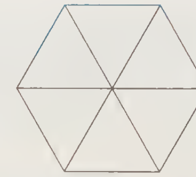
Four squares, of course, fit snugly around a point, since $4(90°) = 360°$. Hence, we can't fold the squares to make a polyhedron.

(e) As with three and four equilateral triangles, we can fit five triangles around a point, then fold them together. This gives us the start of a regular **icosahedron**, which has twenty congruent equilateral triangles as its faces.



Obviously, we can't fit five of any other regular polygon around a point.

(f) We can fit six equilateral triangles snugly around a point, so we can't 'fold' them into the start of a regular polyhedron. Obviously, any regular polyhedron must have at least three faces meet at each vertex, so we've exhausted all the possibilities for regular polyhedra.



□

Important: There are five regular polyhedra, which are described below.

Name	Face Type	# Faces	# Edges	# Vertices
Tetrahedron	Triangle	4	6	4
Cube	Square	6	12	8
Octahedron	Triangle	8	12	6
Dodecahedron	Pentagon	12	30	20
Icosahedron	Triangle	20	30	12

These five regular polyhedra are collectively named the **Platonic solids**, after the great Greek philosopher **Plato**.

Icosahedra and dodecahedra are pretty hard to work with, but since regular octahedra are just two square pyramids stuck together, they're much more manageable.

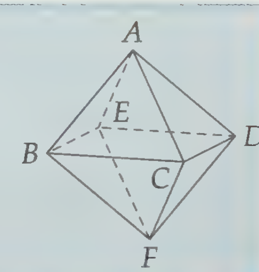
Problem 14.12: *ABCDEF* shown is a regular octahedron with $AB = 4$.

(a) Find BD.

(b) Find AF.

(c) Find $[BCDE]$.

(d) Find the volume of $ABCDEF$.



Solution for Problem 14.12:

(a) Since $BCDE$ is a square, $BD = BC\sqrt{2}$. All the sides of a regular polyhedron have the same length, so $BC = AB = 4$ and $BD = 4\sqrt{2}$.

(b) Instead of thinking of the octahedron as square pyramids $ABCDE$ and $FBCDE$ glued together, we can think of it as $BAEFC$ and $DAEFC$ glued together. Therefore, $ACFE$ is also a square, so $AF = BD = 4\sqrt{2}$. (We can also just note that since the polyhedron is regular, the distance between directly opposite vertices must be the same no matter which pair of opposite vertices we choose.)

(c) $BCDE$ is a square with side length 4, so its area is $4^2 = 16$.

(d) To find the volume of the regular octahedron, we think of it as two congruent square pyramids, $ABCDE$ and $FBCDE$. Since the altitudes from A and from F, respectively, of these two pyramids together make up $\overline{AF}$, the height of each pyramid is $AF/2 = (4\sqrt{2})/2 = 2\sqrt{2}$. The area of a base is $4^2 = 16$, so the volume of each pyramid is $(16)(2\sqrt{2})/3 = 32\sqrt{2}/3$. Therefore, our octahedron has volume $2(32\sqrt{2}/3) = 64\sqrt{2}/3$.

□

Exercises

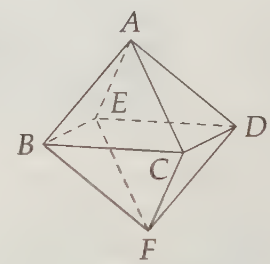
14.4.1 Compute the quantity

$$(\text{Number of vertices}) - (\text{Number of edges}) + (\text{Number of faces})$$

for each regular polyhedron (cube, tetrahedron, octahedron, dodecahedron, icosahedron). Notice anything interesting? Try it on other polyhedra, such as various types of prisms and pyramids and see if your interesting observation still holds.

14.4.2 The six vertices of a regular octahedron are snipped off, leaving a square face in place of each corner and a hexagonal face in place of each original face of the octahedron. How many vertices, faces, and edges will the new polyhedron have?

14.4.3★ $ABCDEF$ shown at right is a regular octahedron with side length 1. Let O be the center of face ABC, P be the center of face ACD, and Q be the center of face DEF.



(a) Find OP. **Hints:** 287

(b)★ Find OQ. **Hints:** 59

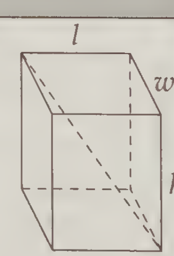
(c) Find $[OPQ]$. **Hints:** 580

14.5 Summary

Definitions: The **volume** of a three dimensional figure is a measure of the space inside the figure. The **total surface area** of a figure is the total area of all the surfaces that form a boundaries of the solid. The **lateral surface area** is the total area of all the surfaces that are not considered 'bases'.

Definitions: A **polyhedron** is a solid figure with polygons as its boundaries. A **prism** has two congruent parallel faces as **bases** and all remaining faces (called **sides**) are parallelograms. In a **right prism** all of these side faces are rectangles. The bases are used to describe the prism, as in 'right rectangular prism' (shown below) or 'hexagonal prism'.

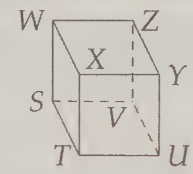
Important: The three dimensions of a right rectangular prism are commonly called the length, l, the width, w, and the height, h. For such a prism, we have:



$$\begin{aligned} \text{Volume} &= lwh \\ \text{Surface area} &= 2(lw + wh + lh) \\ \text{Space Diagonal} &= \sqrt{l^2 + w^2 + h^2} \end{aligned}$$

The volume of a prism equals the area of the base times the distance between the bases (i.e. the height).

Definition: A **cube** is a special right rectangular prism in which all the edge lengths are the same (i.e., its base is a square and its height has the same length as a side of the base).



Important: A cube with side length s has:

$$\begin{aligned} \text{Volume} &= s^3 \\ \text{Surface area} &= 6s^2 \\ \text{Space Diagonal} &= s\sqrt{3} \end{aligned}$$

Definitions: If we connect all the vertices of a polygon to a point that is not in the same plane as the polygon, we form a **pyramid**. This point is called the **apex** of the pyramid and the polygon is the pyramid's **base**. As we can see at right, the non-base faces of a pyramid are all triangles. The lateral surface area of a pyramid is the sum of the areas of these triangles. A **tetrahedron** is a pyramid with a triangular base.

The **height** of a pyramid is the distance from the apex to the base. If the base is a regular polygon, the pyramid is a **regular pyramid**. For regular pyramids, we also define a **slant height**, which is the distance from the apex to a side of the base.

Important:

- The volume of a pyramid is one-third the product of the pyramid's height and the area of the pyramid's base.
- The lateral surface area of a regular pyramid equals one-half the product of the slant height and the perimeter of the pyramid's base.

Definitions: A **regular polyhedron** is a polyhedron whose faces are all congruent regular polygons.

Important: There are five regular polyhedra, which are described below.

Name	Face Type	# Faces	# Edges	# Vertices
Tetrahedron	Triangle	4	6	4
Cube	Square	6	12	8
Octahedron	Triangle	8	12	6
Dodecahedron	Pentagon	12	30	20
Icosahedron	Triangle	20	30	12

Problem Solving Strategies

Concept:

- Building right triangles works in three dimensions every bit as well as it does in two dimensions.

REVIEW PROBLEMS

14.13 A diagonal of one of the faces of a given cube has length 4.

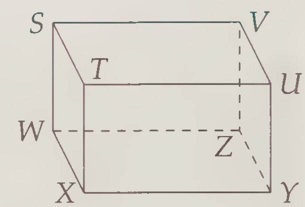
(a) Find the length of an edge of the cube.

(b) Find the length of a space diagonal of the cube.

(c) Find the total surface area of the cube.

(d) Find the volume of the cube.

14.14 Shown is right rectangular prism $ZYXWVUTS$. What type of quadrilateral is $ZYTS$ and why?



14.15 A right rectangular prism has space diagonal $3\sqrt{13}$ and two sides of length 3 and 7, respectively.

(a) Find the third dimension of the prism.

(b) Find the volume of the prism.

(c) Find the total surface area of the prism.

14.16 A space diagonal of cube $\mathcal{A}$ is an edge of cube $\mathcal{B}$.

(a) Find the ratio of the surface area of $\mathcal{A}$ to the surface area of $\mathcal{B}$.

(b) Find the ratio of the volume of $\mathcal{A}$ to the volume of $\mathcal{B}$.

14.17 $VWXYZ$ is a right square pyramid with square base $WXYZ$. Given $YZ = 10$ and $YV = 13\sqrt{2}$, find the following:

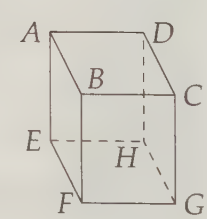
(a) the height of the pyramid.

(b) the slant height of the pyramid.

(c) the total surface area of the pyramid.

(d) the volume of the pyramid.

14.18 $M, N, O,$ and P are midpoints of edges $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, of right rectangular prism $ABCDEFGH$ shown. Q, R, S, and T are the midpoints of edges $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, and $\overline{HE}$, respectively. Given $AB = 8$, $BC = 10$, and $BF = 5$, find the following:



(a) MN.

(b) MO.

(c)★ the volume of $MNOPQRST$.

14.19 What figure is formed by connecting the centers of the faces of a:

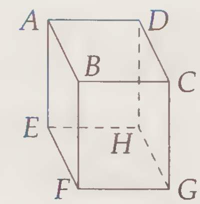
(a) cube?

(b) regular octahedron?

(c) regular tetrahedron?

(d) regular dodecahedron?

(e) regular icosahedron?

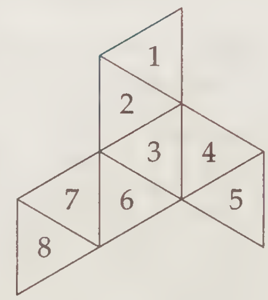


14.20 Shown is cube $ABCDEFGH$ with side length 8.

(a) Find $[ACF]$. **Hints:** 332

(b) Find $[ACG]$.

14.21 How many edges does a regular hexagonal prism have?



14.22 This shape at right is folded into a regular octahedron with the equilateral triangles shown as faces. What is the sum of the numbers on the faces sharing an edge with the face with a "1" on it? *(Source: MATHCOUNTS)*

14.23 $ABCDEFG$ is a right pyramid with regular hexagonal base $ABCDEF$ and apex G with $AB = 6$ and $AG = 6\sqrt{3}$.

(a) Find the volume of $ABCDEFG$.

(b) Find the total surface area of $ABCDEFG$.

14.24 Find the volume and the surface area of a regular tetrahedron with side length 9.

14.25 On the center of each face of a cube with side length 4 inches, we glue a cube of side length 1 inch (so that an entire face of each small cube is glued to the big cube). We then paint the entire resulting figure red. How many square inches of red paint will we use?

14.26 $WXYZ$ is a triangular pyramid with $XY = YZ = ZX = 9$ and $WX = WY = WZ = 18$. Let G be the centroid of $\triangle XYZ$ and M the midpoint of $\overline{XY}$.

(a) Prove that $\overline{WG}$ is perpendicular to plane XYZ. (If you're stuck, check out Problem 14.10.)

(b) Find WM.

(c) Find ZM.

(d) Find the volume of $WXYZ$.

14.27 In a cube with side length 6, what is the volume of the tetrahedron formed by any vertex and the three vertices connected to that vertex by edges of the cube? *(Source: HMMT)*

14.28 Each of the three lines k, m, and n is perpendicular to the other two lines. All three lines pass through point P. K is on line k, M is on line m, and N is on line n such that $KP = 7$, $KM = 9$, and $KN = 11$.

(a) Find the length of $\overline{MN}$.

(b) Find the volume of $KNMP$.

(c)★ Find the length of the segment from K to the midpoint of $\overline{MN}$. **Hints:** 554, 371

14.29 In tetrahedron $MNOP$, we have $MN = OP$, $MO = NP$, and $MP = NO$. Prove that $\angle MNO = \angle MPO$.

14.30 Find a formula for the volume of a regular octahedron with edge length s.

Challenge Problems

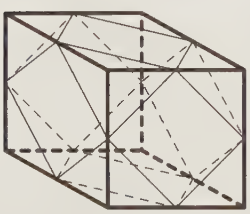
14.31 Base $EFGH$ of right prism $EFGHIJKL$ is a parallelogram with $EF = 8$, $FG = 6$, and $\angle EFG = 60^\circ$. Given that the height of the prism is 9, find the following:

(a) $[EFGH]$. **Hints:** 542

(b) the surface area of $EFGHIJKL$.

(c) the volume of $EFGHIJKL$.

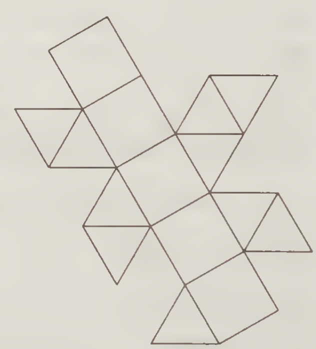
14.32 The altitude from vertex V of tetrahedron $VWXY$ meets face WXY at the circumcenter of $\triangle WXY$. Prove that $VW = VX = VY$. **Hints:** 425



14.33 Each side of the cube shown has length 10. What is the volume of the solid whose edges are formed by connecting all the midpoints of the edges of the cube as shown? **Hints:** 50

14.34 A soccer ball is constructed using 32 regular polygons with equal side lengths. Twelve of the polygons are pentagons, and the rest are hexagons. A seam is sewn wherever two edges meet. What is the number of seams in the soccer ball? *(Source: MATHCOUNTS)* **Hints:** 130

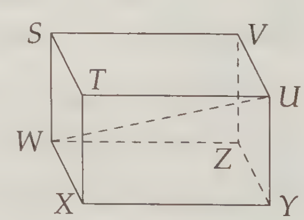
14.35 The figure at right with 5 square faces and 10 equilateral triangular faces is folded into a 15-faced polyhedron. How many edges does the polyhedron have? How many vertices does the polyhedron have? *(Source: MATHCOUNTS)*



14.36 An octahedron is formed by connecting the centers of the faces of a cube. What is the ratio of the volume of the cube to the volume of the octahedron? **Hints:** 164

14.37 An insect lives on the surface of a regular tetrahedron with edges of length 1. It wishes to travel on the surface of the tetrahedron from the midpoint of one edge to the midpoint of the opposite edge. What is the length of the shortest such trip? (Note: Two edges of a tetrahedron are opposite if they have no common endpoint.) *(Source: AMC 12)* **Hints:** 111

14.38 $STUVWXYZ$ is a right rectangular prism with $XY = 12$, $WX = 3$, and $TX = 4$.



(a) Find WU.

(b) Find the distance from Z to $\overline{WU}$. **Hints:** 64, 531

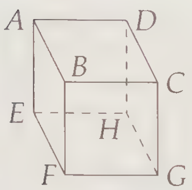
(c) Find the volume of $TWXY$.

(d)★ Find the volume of pyramid $VSUYW$. (V is the vertex.)

(e)★ Find the volume of $WXZU$. **Hints:** 155, 573

14.39 A cube of cheese $C = \{(x, y, z) | 0 \le x, y, z \le 1\}$ is cut along the planes $x = y$, $y = z$, and $z = x$. How many pieces are produced this way? *(Source: AHSME)* **Hints:** 181, 253

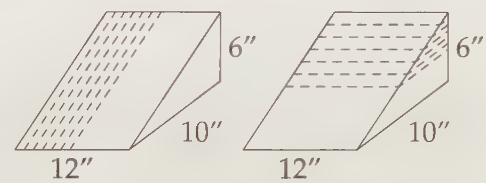
14.40 *EFGHIJKL* is a cube. How many different planes pass through at least three vertices of *EFGHIJKL*? **Hints:** 297



14.41 In cube *ABCDEFGH* shown, $AB = 12$. Find the distance from B to plane ACF. **Hints:** 93, 281

14.42 The sum of the lengths of the twelve edges of a rectangular box is 140, and the distance from one corner of the box to the farthest corner is 21. What is the total surface area of the box? *(Source: AMC 12)* **Hints:** 431

14.43 Congruent wedges A and B are sliced as shown. Each wedge has a pair of parallel triangular faces. Wedge A is cut vertically so that slices are made perpendicular to the base; wedge B is cut horizontally so that slices are made parallel to the base. Six slices 1/4 inch thick are cut from each wedge. What is the ratio of the remaining volume of wedge A to the remaining volume of wedge B? *(Source: MATHCOUNTS)* **Hints:** 45



Wedge *A* Wedge *B*

14.44 *ABCDEF* shown at left below is a regular octahedron with edge length 1. The midpoints of edges $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, and $\overline{AE}$ are M, N, O, and P, respectively. The midpoints of edges $\overline{BF}$, $\overline{CF}$, $\overline{DF}$, and $\overline{EF}$ are Q, R, S, and T, respectively. Find the volume of *MNOPQRST*. **Hints:** 96

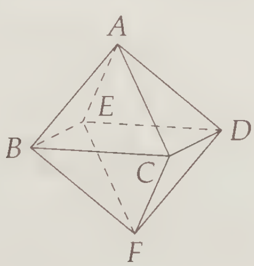


Figure 14.3: Diagram for Problem 14.44

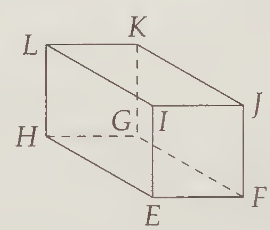


Figure 14.4: Diagram for Problem 14.45

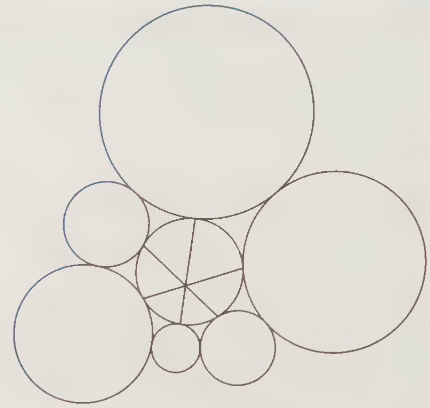
14.45★ Parallelogram *EFGH* is a base of right prism *EFGHIJKL* at right above. Given $EF = 8$, $FG = 6$, $EI = 9$, and $\angle EFG = 60°$, find FL and EK. **Hints:** 133

14.46★ A right square **frustum** is formed by cutting a right square pyramid with a plane parallel to its base. Suppose the original pyramid has base length 6 and height 9, and that the plane cutting the pyramid to form the frustum is 3 units from the base of the pyramid.

(a) Find the volume of the frustum. **Hints:** 185, 263

(b) Find the total surface area of the frustum.

14.47★ Let *ABCD* be a regular tetrahedron with side length 2. The plane parallel to edges $\overline{AB}$ and $\overline{CD}$ and lying halfway between them cuts *ABCD* into two pieces. Find the surface area of one of these pieces. *(Source: HMMT)* **Hints:** 211



The Seven Circles Theorem

Everything in nature adheres to the cone, the cylinder and the cube. – Paul Cezanne

CHAPTER 15 Curved Surfaces

15.1 Cylinders

Instead of using congruent polygons for bases as we do with a prism, suppose we use congruent circles. The resulting solid is called a **cylinder**, an example of which is shown below. The line connecting the centers of the bases is called the **axis** of the cylinder ($\overline{OO'}$ in the diagram) and the radius of a base is also considered the radius of the cylinder.

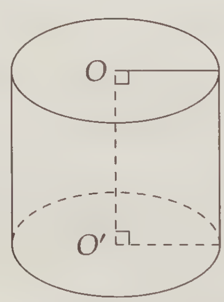


Figure 15.1: A Right Circular Cylinder

You won't be surprised to learn that if the axis is perpendicular to the bases, then the cylinder is a **right cylinder** as shown above. Furthermore, we can have cylinders in which the bases are any sort of wacky curves. Therefore, our simple cylinder above is more precisely called a **right circular cylinder**, where the 'circular' tells us that the bases are circles. When we speak of a cylinder, we nearly always mean a 'right circular cylinder,' instead of one of the wacky cylinders shown in Figure 15.2 below.

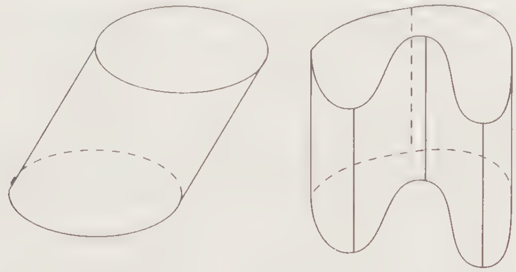


Figure 15.2: More Cylinders

Problems

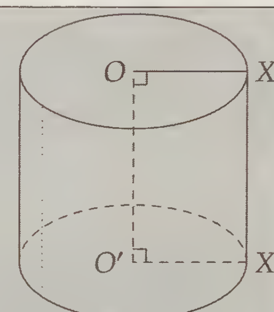
Problem 15.1:

(a) Suppose we take a cross-section of a cylinder that is perpendicular to the axis of the cylinder. What shape is this cross-section?

(b) What is the shape of a cross-section of a cylinder that contains the axis of the cylinder?

(c) What is the shape of a cross-section of a cylinder that is parallel to the axis of the cylinder?

Problem 15.2: The figure at right shows a right circular cylinder (a.k.a. a cylinder) with radius 3 and height 5.



(a) Find the volume of the cylinder.

(b) Suppose we cut the curved surface of the cylinder along the dotted line shown (assume the dotted line is perpendicular to the bases) and 'unroll' the surface into a plane. What type of figure results?

(c) What is the lateral surface area of the cylinder?

(d) What is the total surface area of the cylinder?

(e) Find formulas for the volume, lateral surface area, and total surface area of a cylinder with radius r and height h.

Problem 15.3: The radius of a cylinder is 2/3 its height. Find the total surface area of the cylinder if its volume is 96π.

We'll start our investigation of cylinders by investigating a few cross-sections of a cylinder.

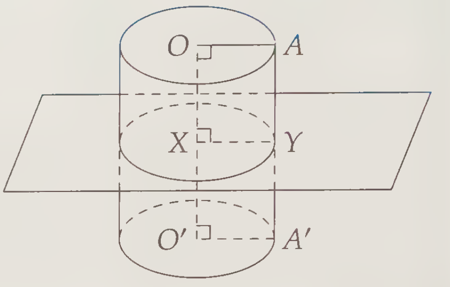
Problem 15.1:

(a) Suppose we take a cross-section of a cylinder that is perpendicular to the axis of the cylinder. What shape is this cross-section?

(b) What is the shape of a cross-section of a cylinder that contains the axis of the cylinder?

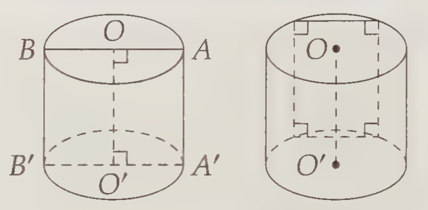
(c) What is the shape of a cross-section of a cylinder that is parallel to the axis of the cylinder?

Solution for Problem 15.1:



(a) Intuitively, it seems clear that the cross-section is a circle. To prove this, we consider A and A' on the bases such that $\overline{AA'}$ is parallel to the axis $\overline{OO'}$. We let X and Y be the points where $\overline{OO'}$ and $\overline{AA'}$, respectively, meet our cross-section plane. Since this plane is perpendicular to the axis, we have $\overline{XY} \perp \overline{OO'}$. So, $XYAO$ is a rectangle, and $XY = OA$. Hence, XY equals the radius of the cylinder.

But there's nothing special about the points A and A' we chose on the circumferences of the bases! In exactly the same way, we can show that any point on the intersection of our plane and the curved surface of the cylinder is exactly the cylinder's radius away from point X on the axis. Therefore, our cross-section is indeed a circle.



(b) Since $\overline{AA'}$ and $\overline{BB'}$ are both parallel to the axis of the cylinder, they are perpendicular to both bases. So, $AA'B'B$ is a rectangle.

(c) Since our cross-section is parallel to the axis of the cylinder, it must be perpendicular to both bases. Therefore, as in the previous part, our cross-section is a rectangle.

□

One result of part (a) above is that the set of all points that are a fixed distance from a given line in space is a cylindrical surface, as shown below. The surface is basically a cylinder without bases that continues forever along the line that is its axis.

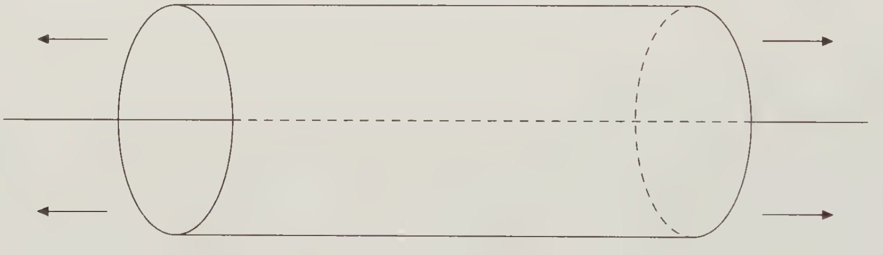
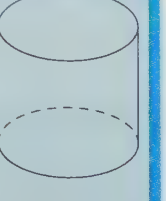


Figure 15.3: A Cylindrical Surface and Its Axis

Now that we have some sense for what a cylinder is like, let's investigate the area and volume of a cylinder.

Problem 15.2: The figure shows a right circular cylinder (a.k.a. a cylinder) with radius 3 and height 5.



(a) Find the volume of the cylinder.

(b) What is the lateral surface area of our cylinder?

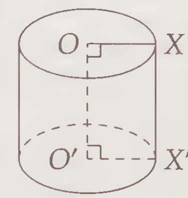
(c) What is the total surface area of our cylinder?

(d) Find formulas for the volume, lateral surface area, and total surface area of a cylinder with radius r and height h.

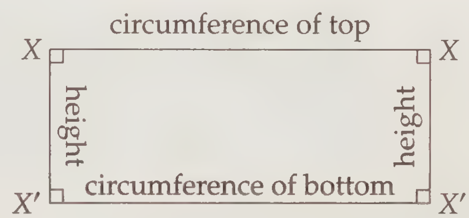
Solution for Problem 15.2:

(a) Our cylinder is just like a prism – its cross-sections have the same area from bottom to top. Therefore, its volume is just the area of a base times the height. The area of the base is $\pi(3^2) = 9\pi$, so the volume is $(9\pi)(5) = 45\pi$.

(b) We don't have any tools to deal with finding the area of a curved surface, so we'll have to do something clever. Suppose our cylindrical surface is made of paper, so we can cut along a segment like $\overline{XX'}$ that is parallel to both bases (and hence the same length as a height). Then we 'unroll' the surface. The top and the bottom are always 5 units apart (the height of the cylinder), so the unrolled surface is a rectangle.



The unrolled surface is shown at right. One dimension of this rectangle is the height of the cylinder; the other is the circumference of the cylinder. For our given cylinder, the height is 5 and the circumference is $2\pi r = 6\pi$. Therefore, our lateral surface area is $(5)(6\pi) = 30\pi$.



(c) For the total surface area, we add the bases to our lateral surface area. Each base has area 9π, so the total surface area is

$$30\pi + 2(9\pi) = 48\pi.$$

(d) We simply follow the steps we used to solve the preceding parts. The area of each base is πr^2, so the volume is $\pi r^2 h$. The curved surface unrolls to form a rectangle with dimensions h and $2\pi r$ (the circumference of each base), so it has area $2\pi rh$. Therefore, our lateral surface area is $2\pi rh$ and the total surface area is $2\pi rh + 2\pi r^2$.

□

Important: A cylinder with height h and radius r has:

$$\begin{aligned} \text{Volume} &= \pi r^2 h \\ \text{Lateral Surface Area} &= 2\pi rh \\ \text{Total Surface Area} &= 2\pi rh + 2\pi r^2 \end{aligned}$$

Don't memorize these formulas! If you take the time to understand them, they'll always be obvious to you.

Concept: Problems involving the curved surface of a cylinder can often be untangled by 'unrolling' the curved surface into a rectangle.

Let's try applying these formulas.

Problem 15.3: The radius of a cylinder is 2/3 its height. Find the total surface area of the cylinder if its volume is 96π.

Solution for Problem 15.3: We let the height be h, so the radius is $r = 2h/3$. Since the volume is 96π, we have $\pi r^2 h = 96\pi$. Substituting for r gives

$$\left(\frac{2h}{3}\right)^2 h\pi = 96\pi,$$

and solving this equation for h gives $h = 6$. Therefore, $r = 2h/3 = 4$. Now we can find our total surface area:

$$2\pi rh + 2\pi r^2 = 48\pi + 32\pi = 80\pi.$$

□

Concept: These basic area/volume word problems are often no different from other word problems. The key to solving them is to assign variables and use the area and volume information to set up equations with the given information.

Exercises

15.1.1 A cylinder has radius 8 and height 4.

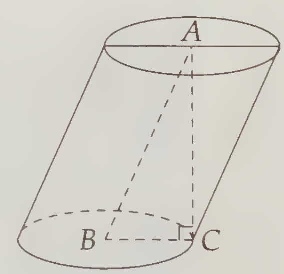
(a) Find the volume of the cylinder.

(b) Find the lateral surface area of the cylinder.

(c) Find the total surface area of the cylinder.

15.1.2 The diameter of a cylinder equals its height. The total surface area of the cylinder is 150π. What is the volume of the cylinder?

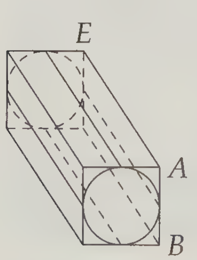
15.1.3 Shown at right is a circular cylinder that is *not* right. The bases have centers A and B as shown, and point C is on the circumference of a base. Given that $AB = 8$, $BC = 4$, and $\angle ACB = 90°$, find the following:



(a) The height of the cylinder. (Note that $\overline{AB}$ is not an altitude of the cylinder because it is not perpendicular to the bases! Remember, the height of a cylinder or prism is the distance between its bases.)

(b) The volume of the cylinder.

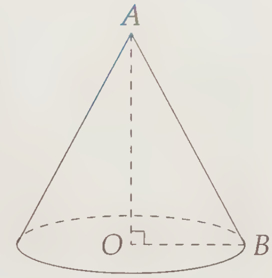
15.1.4 The lateral surface area of cylinder C equals the sum of the areas of its bases. What is the ratio of the radius of C to the height of C?



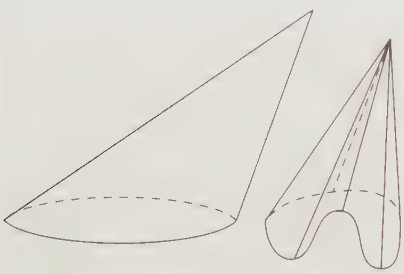
15.1.5 A cylinder is inscribed in a square prism. As shown at left, each of the bases of the cylinder is inscribed in a face of the prism, and the cylinder is tangent to the other four faces of the prism. Given that $AB = 4$ and $AE = 8$, find the volume of the cylinder.

15.1.6★ Does there exist a group of seven cylinders such that it is possible to arrange them so that each cylinder is tangent (i.e. touches at only one point) to the other six?

15.2 Cones



Just as turning the bases of a prism into circles makes a cylinder, we can turn the base of a pyramid into a circle to get a **cone**. The figure on the right shows a **right circular cone**. The point A at the tip of the cone is the **vertex** of the cone and the distance from the vertex to the base is the **height**. The line connecting the vertex to the center of the base is the **axis** of the cone. The radius of the base is considered the radius of the cone, and for right circular cones, the distance from the vertex to a point on the circumference of the base is the **slant height**.



As with cylinders, cones don't have to be 'right' or 'circular.' On the left are a couple of these less common cones. Despite the existence of these weird cones, when we simply say 'cone,' we almost always mean 'right circular cone.'

Having already learned that the volume of a pyramid equals one-third its height times the area of its base, you won't be surprised to learn that the same holds for a cone.

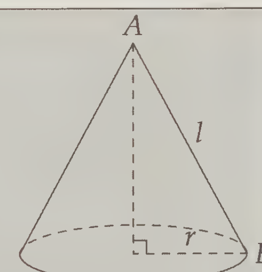
Important: The volume of a circular cone with height h and radius r is

$$\text{Volume} = \frac{1}{3}\pi r^2 h.$$

Problems

Problem 15.4:
(a) What is the shape of a cross-section of a cone that contains the axis of the cone?
(b) What is the shape of a cross-section of a cone that is perpendicular to the axis of the cone?

Problem 15.5: In this problem we will find a formula for the lateral surface area of a cone with base radius r and slant height l.



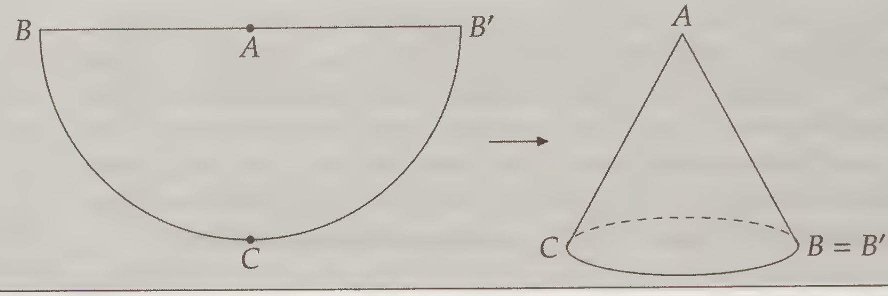
(a) Use the result of Problem 14.9 to take a guess at the formula.
(b) Let A be the vertex of the cone and B a point on the circumference of the cone. Suppose we cut the curved surface of the cone along $\overline{AB}$, then 'unroll' it into a flat figure. What type of figure do we thus form?
(c) How long is the curved portion of the figure you make in (b)? Use this information to find a formula for the lateral surface area of the cone.

Problem 15.6: Find the volume of a right circular cone that has radius 6 and slant height 8.

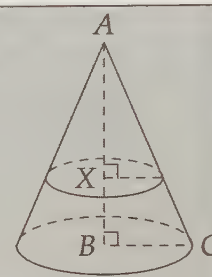
Extra! *The open secret of real success is to throw your whole personality at a problem.*

–George Polya

Problem 15.7: A semicircle is rolled up to form a cone as shown. If the radius of the semicircle is 8, what is the volume of the cone?



Problem 15.8: A cone with vertex A, height $AB = 9$, and radius $BC = 12$ is given. The cone is cut in two pieces by a plane perpendicular to $\overline{AB}$ at point X, where $AX = 6$.



(a) The top piece, of course, is a cone. What is the height of this cone? What is the volume of this cone?

(b) What is the volume of the other piece?

(c) What is the ratio of the volume of the little cone to the volume of the large cone? Is this surprising?

We start by getting a feel for cones by looking at some cross-sections.

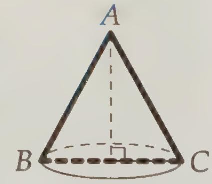
Problem 15.4:

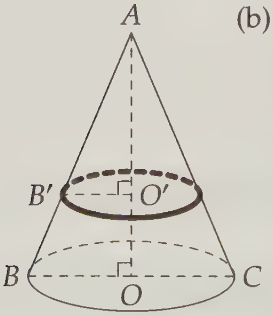
(a) What is the shape of a cross-section of a cone that contains the axis of the cone?

(b) What is the shape of a cross-section of a cone that is perpendicular to the axis of the cone?

Solution for Problem 15.4:

(a) A cross-section of a cone that contains the cone's axis consists of two segments of equal length connecting the vertex to two points on the circumference of the base of the cone, as well as the segment connecting these two points along the base of the cone. So, our cross-section is an isosceles triangle, such as $\triangle ABC$ in the diagram at right.





(b) Intuitively, it seems clear that a cross-section of a cone perpendicular to the axis of the cone is a circle. Suppose the plane of our cross-section meets the axis at O'. To prove our cross-section is a circle, we must show that every point where our plane hits the curved surface of the cone is equidistant from O'. Consider point B', the intersection of $\overline{AB}$ and our cross-section plane, as shown. Since $\overline{B'O'}$ and $\overline{BO}$ are each perpendicular to $\overline{AO}$, we have $\overline{B'O'} \parallel \overline{BO}$. Therefore, $\triangle AO'B' \sim \triangle AOB$.

Since $\triangle AO'B' \sim \triangle AOB$, we have $B'O'/BO = AO'/AO$. Therefore, we find $B'O' = (AO'/AO)(BO)$. Since BO is just the radius of the cone and AO

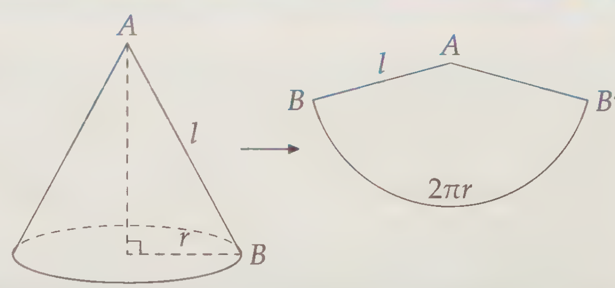
is the cone's height, we have $B'O' = (r/h)(AO')$. Similarly, we can show that all points of the cross-section are $(r/h)(AO')$ away from O'. Since AO' is fixed, all the points of our cross-section are the same distance from O'. Therefore, the cross-section must be a circle. (Make sure you see why every point on this circle must be in the cross-section.)

□

While the proof of the volume formula for cones requires some very advanced tools, we can find a formula for the area of the curved surface of a cone with techniques we already have.

Problem 15.5: Find a formula for the lateral surface area of a right circular cone with base radius r and slant height l.

Solution for Problem 15.5: Since cutting and unrolling the curved surface was so successful in finding the lateral surface area of a cylinder, we try it with a cone as well. We cut the curved surface of the cone along $\overline{AB}$, where A is the vertex and B is a point on the circumference of the base.



Since every point on the circumference of the cone's base is the same distance from the cone's vertex (the slant height), when we unroll the curved surface, these points will still all be the same distance from the vertex. Hence, our 'unrolled' surface is a sector of a circle as shown at right above. (B and B' coincide when the sector is rolled up to form a cone.)

The radius of this sector is AB, the slant height of the cone. To find the area of the sector, we must determine what portion of a whole circle the sector is. We know that the length of $\widehat{BB'}$ is just equal to the circumference of the cone's base. The base of the cone has radius r, so its circumference is $2\pi r$. Thus, the length of $\widehat{BB'}$ is $2\pi r$. Since a whole circle with radius $AB = l$ has circumference $2\pi l$, our sector is $(2\pi r)/(2\pi l) = r/l$ of a whole circle.

A full circle with radius l has area πl^2, so the area of a sector that is r/l of this circle is $(r/l)(\pi l^2) = \pi rl$.

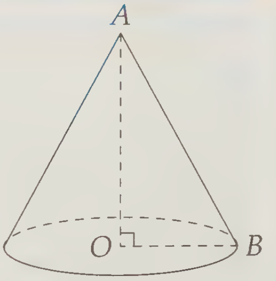
Recall from Problem 14.9 that we showed that the lateral surface area of a regular pyramid is half the product of the slant height and the perimeter of the base. The proof we used there wouldn't work for cones, since we don't have triangular faces as the sides of a cone. However, since cones are essentially just pyramids with circular bases, we expect the formula to work for cones, too. Trying it, we note that the perimeter of the base of a cone is $2\pi r$, so our formula gives us $(1/2)(2\pi r)(l) = \pi rl$ for the lateral surface area. Unsurprisingly, this matches the formula we already proved. □

Important: The lateral surface area of a right circular cone with radius r and slant height l is πrl.

Concept: As with cylinders, problems involving the curved surface of a cone can often be solved by 'unrolling' the curved surface. As we've seen, the result is a sector of a circle.

Problem 15.6: Find the volume of a right circular cone that has radius 6 and slant height 8.

Solution for Problem 15.6: We need the height in order to find the volume. As we did with a pyramid, we can build a right triangle to find the height. We draw $\overline{AO}$, the axis of the cone, and connect A and O to B, a point on the circumference of the base. Since $\overline{OB}$ is a radius and $\overline{AB}$ is a slant height, we have $AB = 8$ and $OB = 6$. Since the cone is 'right,' $\angle AOB$ is a right angle, so $AO = \sqrt{AB^2 - OB^2} = 2\sqrt{7}$. Therefore, we have:



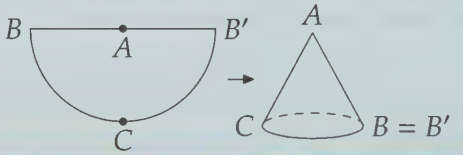
$$\text{Volume} = \frac{\pi r^2 h}{3} = \frac{\pi(6^2)(2\sqrt{7})}{3} = 24\pi\sqrt{7}.$$

□

As we've just seen, a radius, an altitude, and a slant height make a right triangle in a right circular cone. We could immortalize this in yet another formula, $l^2 = r^2 + h^2$, but there's no need to – this is just the Pythagorean Theorem!

Back in Problem 15.5, we unrolled the curved surface of a cone to get a sector. What happens if we go the other way?

Problem 15.7: A semicircle is rolled up to form a cone as shown. If the radius of the semicircle is 8, what is the volume of the cone?

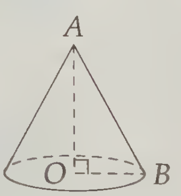


Solution for Problem 15.7: Here we need to find both the radius and the height of the cone. At first it seems that all we are given is the slant height; however, the information about the 'unrolled' curved surface can be used to find the radius in a couple different ways.

Method One: Roll it up! When we roll up the semicircle into a cone, the arc $\widehat{BCB'}$ becomes the circumference of the base of the cone. Since this arc of the semicircle is half the circumference of a circle with radius 8, it has length 8π. Therefore, the circumference of the base of the cone is 8π, so its radius must be 4.

Method Two: Use the lateral surface area! The semicircle has area $\pi(8^2)/2 = 32\pi$. When rolled up, this is the curved surface of the cone, which we know has area $\pi r l$, where r is the radius of the cone and l is its slant height. Since the radius of the semicircle is the slant height of the cone, we have $l = 8$. Hence, we can solve $\pi r l = 32\pi$ to find $r = 32/l = 4$.

Once we have the radius, we can find the height by using right triangle $\triangle AOB$ in the cone as before. Specifically, $h = \sqrt{8^2 - 4^2} = 4\sqrt{3}$. Thus, our volume is

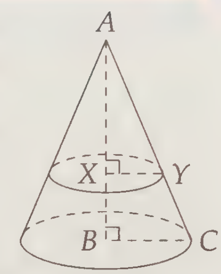


$$\frac{\pi r^2 h}{3} = \frac{64\pi\sqrt{3}}{3}.$$

□

Now that we have a good understanding of cone volumes and basic cross-sections of a cone, let's try putting them together.

Problem 15.8: A cone with vertex A, height $AB = 9$, and radius $BC = 12$ is given. The cone is cut in two pieces by a plane perpendicular to $\overline{AB}$ at point X, where $AX = 6$. Find the volume of the two smaller pieces thus formed.



Solution for Problem 15.8: We showed in Problem 15.4 that a cross-section of a cone perpendicular to its axis is a circle. So, one of our pieces is itself a cone. The other piece is called a **right circular frustum**. We don't have any tools to deal with a frustum, but we do know how to find the volume of a cone. The original cone has volume $\pi r^2 h/3 = \pi(12^2)(9)/3 = 432\pi$. We have the height of the smaller cone, $AX = 6$, so all we have to do is find the radius.

Since $\overline{XY}$ and $\overline{BC}$ are each perpendicular to $\overline{AB}$, we have $\angle AXY = \angle ABC$ and $\angle XAY = \angle BAC$, so $\triangle AXY \sim \triangle ABC$. (Notice that we are essentially considering the cross-section of the cone that contains $\triangle ABC$ here – three-dimensional problems are often just two-dimensional problems in disguise!) Therefore, $XY/AX = BC/AB$, so $XY = (BC/AB)(AX) = (12/9)(6) = 8$. Hence, our little cone has volume $\pi(XY^2)(AX)/3 = 128\pi$.

To get the volume of the other piece, we merely subtract the little cone from the big one, which yields $432\pi - 128\pi = 304\pi$.

Notice that

$$\frac{\text{Volume of small cone}}{\text{Volume of large cone}} = \frac{128\pi}{432\pi} = \frac{8}{27} = \left(\frac{2}{3}\right)^3 = \left(\frac{AX}{AB}\right)^3.$$

This shouldn't be a surprise, because our cones are similar figures. □

Important: Just as the ratio of the areas of similar two-dimensional figures is the square of the ratio of their corresponding sides, the ratio of the surface areas of similar three-dimensional figures is the square of the ratio of their corresponding side lengths. Moreover, the ratio of the volumes of similar three-dimensional figures is the cube of the ratio of their corresponding side lengths.

Exercises

15.2.1 A cone has height 5 and radius 2. Find the volume and total surface area of the cone.

15.2.2 A cone has a lateral surface area 54π and radius 6. Find the slant height, height, and volume of the cone.

15.2.3 A quarter-circle with radius 4 is rolled up to form a cone.

(a) Find the lateral surface area of the cone.

(b) Find the volume of the cone.

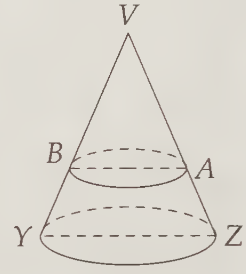
15.2.4 Cone $\mathcal{A}$ has twice the radius, but half the height, of cone $\mathcal{B}$. What is the ratio of the volume of $\mathcal{A}$ to the volume of $\mathcal{B}$?

15.2.5 Prove that the vertex of a right circular cone is equidistant from every point on the circumference of the cone's base. **Hints:** 310

15.2.6 Is it possible for the slant height of a cone to be smaller than the cone's radius?

15.2.7 How could we have used the fact that the two cones in Problem 15.8 are similar in order to find the volume of the small cone quickly?

15.2.8★ In this problem we will develop a formula for the volume of a right circular frustum, which was described in Problem 15.8. Call the bases $\mathcal{B}_1$ and $\mathcal{B}_2$ and let their radii be r_1 and r_2, respectively, with $r_1 < r_2$. Furthermore, let the distance between $\mathcal{B}_1$ and $\mathcal{B}_2$ be h. In Problem 15.8, we formed a frustum by cutting a cone with a plane parallel to the base. Let's try running this backwards – recreating a cone by extending the curved surface of the frustum up to a point. Call this point V, and let $\overline{AB}$ be a diameter of the small base and $\overline{YZ}$ be a diameter of a the large base as shown.



(a) What do we know about $\triangle VAB$ and $\triangle VZY$?

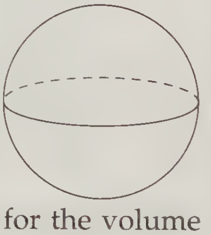
(b) Use part (a) to find the distance from V to $\mathcal{B}_1$ in terms of h, r_1, and r_2.

(c) What is the volume of the cone with vertex V and base $\mathcal{B}_1$ in terms of h, r_1, and r_2?

(d) What is the volume of the cone with vertex V and base $\mathcal{B}_2$ in terms of h, r_1, and r_2?

(e) What is the volume of the frustum in terms of h, r_1, and r_2?

15.3 Spheres



Just as a circle is the set of all points in a plane that are the same distance from a given point, a **sphere** is the set of all points in space that are equidistant from a given point. Most balls and globes are examples of spheres.

Fortunately, we can take a break from the vocabulary bombardment in this chapter. The only significant length we have in a sphere is the sphere's radius, which is just the distance from the center of the sphere to the surface of the sphere. The formulas for the volume and the surface area of a sphere are very challenging to derive (we won't derive them in this book; see page 407 for a hint as to how Archimedes did it). Archimedes considered his discovery of the formula for the volume of a sphere the greatest of all his accomplishments, and even requested a reference to it be inscribed on his tombstone.

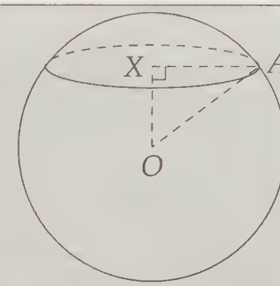
Important: A sphere of radius r has:

$$\begin{aligned} \text{Volume} &= \frac{4\pi r^3}{3} \\ \text{Surface Area} &= 4\pi r^2 \end{aligned}$$

Problems

Problem 15.9: The volume of a sphere is numerically equal to twice its surface area. What is the radius of the sphere?

Problem 15.10: In this problem, we show that if a plane passes through the interior of a sphere, then the intersection of the plane and the sphere is a circle. Let O be the center of the sphere, X be the foot of the perpendicular segment from O to a plane cutting the sphere, and A be a point on the intersection of the plane and the surface of the sphere (as shown).



(a) What kind of triangle is $\triangle OAX$?

(b) Find XA in terms of OA and OX.

(c) Why does your answer to part (b) tell you that the intersection of the plane and the sphere is a circle?

Problem 15.11: A plane cuts through a 26 cm diameter sphere, but the closest it gets to the center is 5 cm. What is the area of the intersection of the sphere and the plane?

Problem 15.12: A sphere is inscribed in a cube, meaning the sphere is tangent to all 6 faces of the cube. A second cube is then inscribed in the sphere, meaning that all 8 vertices of this second cube are on the surface of the sphere. Given that the radius of the sphere is 6, we will find the volume of each cube.

(a) Forget the small cube for a minute. Draw a cross-section of the large cube with the sphere inside that includes the center of the sphere and the four of the points where the sphere is tangent to the cube.

(b) Use your cross-section to find the length of an edge of the cube. Find the volume of the cube.

(c) Forget the large cube for a minute. Draw a cross-section of the sphere and small cube that includes four of the cube's vertices and the center of the sphere. What part of the small cube equals the diameter of the sphere?

(d) Use the radius of the sphere to find the length of an edge of the small cube, and then find the cube's volume.

We'll start our work with spheres by using the formulas for volume and surface area.

Problem 15.9: The volume of a sphere is numerically equal to twice its surface area. What is the radius of the sphere?

Solution for Problem 15.9: We let the radius of the sphere be r and convert the words in the problem to an equation we can solve: Volume = 2 × (Surface Area).

Extra! $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

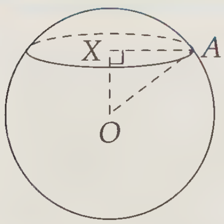
Therefore, we have

$$\frac{4\pi r^3}{3} = 2(4\pi r^2).$$

Solving this equation gives $r = 6$. □

We'll continue our exploration of spheres by considering a cross-section of a sphere.

Problem 15.10: Prove that if a plane passes through the interior of a sphere, then the intersection of the plane and the sphere is a circle.



Solution for Problem 15.10: Intuitively, the cross-section seems to be a circle. We'll prove it is a circle the same way we proved similar cross-sections are circles earlier. Let O be the center of our sphere, r be its radius, and $\mathcal{P}$ be our plane. Let X be the foot of the perpendicular segment from O to $\mathcal{P}$. X is the point we think is the center of our circle. Let A be a point where the surface of the sphere meets $\mathcal{P}$.

Since $\triangle OXA$ is a right triangle, we have $XA = \sqrt{OA^2 - OX^2}$. OA is just the radius of the sphere, so we have $XA = \sqrt{r^2 - OX^2}$. Similarly, every point where the surface of the sphere meets $\mathcal{P}$ is the same distance, $\sqrt{r^2 - OX^2}$, from X. Hence, the intersection of the plane and the sphere is a circle in plane $\mathcal{P}$ with center X and radius $\sqrt{r^2 - OX^2}$. Furthermore, every point in $\mathcal{P}$ that is on the circle with center X and radius XA is on the sphere. (Make sure you see why this is true, and why it is important!) □

We have now proved two very useful facts about cross-sections of a sphere.

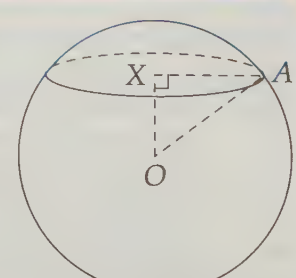
Important: Every cross-section of a sphere is a circle (or a point, when the cross-section plane is tangent to the sphere). The segment connecting the center of the sphere to the center of this circle is perpendicular to the plane of the cross-section.

A cross-section of a sphere that has the center of the sphere as its center is sometimes called a **great circle** of the sphere.

Let's try using our new cross-section knowledge on a problem.

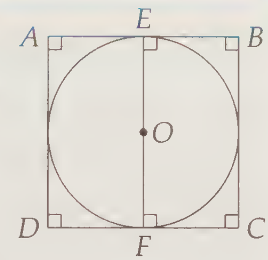
Problem 15.11: A plane cuts through a 26 cm diameter sphere, but the closest it gets to the center is 5 cm. What is the area of the intersection of the sphere and the plane?

Solution for Problem 15.11: In this problem we are simply considering a specific cross-section of a specific sphere. It's so close to the last problem that we can even use the same diagram. We are given that $AO = 26/2 = 13$ and $OX = 5$. From right triangle $\triangle AXO$ we find $AX = \sqrt{13^2 - 5^2} = 12$ (maybe you recognized the 5-12-13 Pythagorean triple). We know that this cross-section is a circle, so its area is $12^2\pi = 144\pi$ cm^2. □



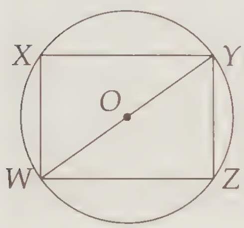
Now we'll use cross-sections to examine a problem that includes more than just spheres.

Problem 15.12: A sphere is inscribed in a cube, meaning the sphere is tangent to all 6 faces of the cube. A second cube is then inscribed in the sphere, meaning that all 8 vertices of this second cube are on the surface of the sphere. Given that the radius of the sphere is 6, find the volume of each cube.



Solution for Problem 15.12: To find the edge length of the larger cube, we consider a cross-section that includes the center of the sphere and four of the points of tangency where the sphere touches the cube at the centers of the faces of the cube. The cross-section of the sphere is a circle. Since the circle includes the center of the sphere, the radius of the circle equals the radius of the sphere. The cross-section of the cube is a square whose side lengths equal the edge lengths of the cube.

Drawing radii to the points of tangency at E and F, we see that $AEFD$ is a rectangle and $\overline{EF}$ is a diameter of the circle. Hence, $AD = EF = 2(OE) = 12$, and the volume of the large cube is $12^3 = 1728$. (Note that we assumed the 'obvious' fact that $\overline{EF}$ passes through O. For an extra challenge, prove it!)

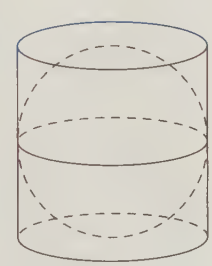


We have several ways to find the volume of the small cube. We'll start with a judiciously chosen cross-section. We want to include the center of the sphere since we know the radius of the sphere. Therefore, we include the center of the sphere and four vertices of the cube as shown at left. $\overline{XY}$ and $\overline{WZ}$ are diagonals of faces of the cube, while $\overline{WX}$ and $\overline{YZ}$ are edges of the cube. Therefore W and Y are opposite vertices of the cube, so that $\overline{WY}$ is a space diagonal of the cube. Since $\overline{WY}$ is also a diameter of the sphere, we know that $WY = 12$. Since $\overline{WY}$ is a space diagonal of the small cube, the side length of the small cube is $WY/\sqrt{3} = 4\sqrt{3}$. So, the volume of the small cube is $(4\sqrt{3})^3 = 192\sqrt{3}$. □

Concept: Cross-sections including the center of the sphere are often useful in problems involving a sphere. Typically, we want to include other important points in the cross-section as well, such as points where the sphere is tangent to other figures in the problem.

Exercises

15.3.1 The volume of a sphere is 36π. What is the surface area of the sphere?



15.3.2 A sphere is inscribed in a cylinder, meaning that it is tangent to both bases, and that one great circle of the sphere is along the curved surface of the cylinder.

(a) Find the ratio of the volume of the sphere to the volume of the cylinder. **Hints:** 354

(b) Find the ratio of the surface area of the sphere to the lateral surface area of the cylinder.

15.3.3 An ice cream cone has radius 1 inch and height 4 inches. What is the number of inches in the radius of a sphere of ice cream that has the same volume as the cone? *(Source: MATHCOUNTS)*

15.3.4 At the end of our solution to Problem 15.10, we noted that 'Every point in $\mathcal{P}$ that is on the circle with center X and radius XA is on the sphere.' Why is this true and why is our solution incomplete without this?

15.3.5 A ball was floating in a lake when the lake froze. The ball was removed (without breaking the ice), leaving a hole 24 cm across and 8 cm deep. Find the radius of the ball. *(Source: AHSME)*

15.3.6★ In this problem we explore facts about tangency and spheres.

(a) Sphere $\mathcal{S}$ is tangent to plane $\mathcal{P}$ at point T, and the center of sphere $\mathcal{S}$ is O. Let X be some point in plane $\mathcal{P}$ besides T. Consider the solution to Problem 12.14 on page 318, then prove that $\overline{OT} \perp \overline{TX}$.

(b) Spheres $\mathcal{S}_1$ and $\mathcal{S}_2$ are externally tangent at point T. The centers of the spheres are O_1 and O_2. Consider the solution to Problem 10.17 on page 278, then prove that $\overleftrightarrow{O_1O_2}$ passes through T.

(c) Suppose $\mathcal{S}_1$ is internally tangent to $\mathcal{S}_2$ at T, meaning $\mathcal{S}_1$ is inside $\mathcal{S}_2$ and both pass through T. Let the centers of $\mathcal{S}_1$ and $\mathcal{S}_2$ be O_1 and O_2, respectively. Must $\overleftrightarrow{O_1O_2}$ pass through T?

15.3.7★ A cube is inscribed in a sphere. Prove that each space diagonal is a diameter of the sphere. (Write a complete proof without invoking symmetry. Don't just say, 'It's obvious!' and move on.) **Hints:** 53, 100, 144

15.4 Problems

The following problems are challenging extensions of the material in this chapter and the previous chapter. Each problem illustrates useful problem solving techniques for three-dimensional geometry problems. Rather than write the terms 'two-dimensional' and 'three-dimensional' repeatedly, we'll frequently use '2-D' and '3-D' instead.

Problems

Problem 15.13: My company produces 6 inch tall cans of paint. The cans have a radius of 15 inches. (Yes, these are very oddly shaped cans!) We want to offer a super-size can of paint. Toby wants to leave the can just as wide as it is, but increase the height by x inches. Maryanna thinks the fat cans sell better, so she wants to make them even wider. She wants to increase the radius by x inches. After arguing for a while, they decide to use the design that produces the greatest increase in paint volume.

They reach for their scratch paper and figure out how much more paint is needed to fill their new can designs. They both use the same value of x, and they find that both designs require the *same* increase in paint volume! What value of x did they use?

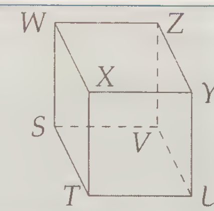
Problem 15.14: The areas of three of the faces of a rectangular prism are 24, 32, and 36. In this problem we will find the volume of the prism.

(a) Let the dimensions of the prism be x, y, and z. Use the information given to find three equations.

(b) In terms of x, y, and z, what is the volume of the prism?

(c) How can you combine your equations from the first part by adding, subtracting, and/or multiplying them to find the volume of the prism?

Problem 15.15: Cube $STUVWXYZ$ is shown. Find $\angle WYT$.



Problem 15.16: A ball with radius 6 inches is tangent to the wall and the floor in my house. Given that the wall is perpendicular to floor, what is the radius of the largest ball I can slide into the space between the ball and the corner where the wall meets the floor? **Hints:** 495, 175

Problem 15.17: Three congruent spheres are inside a cylinder such that each sphere is tangent to the other two spheres, to both bases of the cylinder, and to the curved surface of the cylinder. In this problem we will find the volume of the cylinder given that the radius of each sphere is 12.

(a) Find the height of the cylinder.

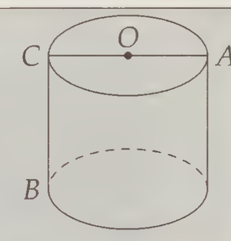
(b) Draw a cross-section of the figures in the problem that contains the centers of the spheres. What kind of triangle is formed by connecting the centers of the spheres?

(c) Let X be the point where your cross-section intersects the axis of the cylinder. How is X related to the triangle mentioned in part (b)?

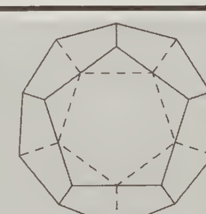
(d) Find the radius of the cylinder, then the volume of the cylinder.

Problem 15.18: Adam the Ant is at point A of cube $ABCDEFGH$ with edge length 1. He wishes to walk along the surface of the cube to point G, where G is the vertex of the cube such that $\overline{AG}$ is a space diagonal of the cube. What is the length of the shortest path Adam can take?

Problem 15.19: Adam the Ant is now on point A of the cylinder shown. He wishes to walk along the curved surface of the cylinder to point B, where $\overline{BC}$ is a height of the cylinder and C is on the top base such that $\overline{AC}$ is a diameter as shown. Given that the cylinder has height 4 and radius 3, what is the length of the shortest path Adam can take?



Extra! We saw on page 264 that squares, hexagons, and triangles are the only regular polygons that will tile the plane. Pentagons, with their quirky 108° angles, simply can't add up to 360°, no matter how many of them get together at a vertex. But, on a sphere, pentagons get their due! We can view the dodecahedron we discovered on page 372 (and shown at right) as a tiling of a sphere with regular pentagons.



Each of the other types of polyhedra can be considered a method of tiling a sphere with regular polygons. Notice that while there is only one way to tile a plane with equilateral triangles, there are three ways to tile a sphere with them!

We start off with a couple problems in which we apply algebra to geometry problems. Then we'll move on to challenging spacial geometry problems.

Problem 15.13: My company produces 6 inch tall cans of paint. The cans have a radius of 15 inches. (Yes, these are very oddly shaped cans!) We want to offer a super-size can of paint. Toby wants to leave the can just as wide as it is, but increase the height by x inches. Maryanna thinks the fat cans sell better, so she wants to make them even wider. She wants to increase the radius by x inches. After arguing for a while, they decide to go with the design that produces the greatest increase in paint volume.

They reach for their scratch paper and figure out how much more paint is needed to fill their new can designs. They both use the same value of x, and they find that both designs require the *same* increase in paint volume! What value of x did they use?

Solution for Problem 15.13: We are given

$$\text{Volume(Toby's design)} = \text{Volume(Maryanna's design)},$$

so we find each volume in terms of x:

$$\pi(15^2)(6+x) = \pi(15+x)^2(6).$$

After a few lines of algebra, we have $x(2x-15) = 0$. Since x must be nonzero (otherwise we wouldn't be making much of a change to the can design!), we have $x = 15/2$.

We can quickly check our work by considering the dimensions of the new cylinders. Toby's has radius 15 and height $6+15/2 = 27/2$, so it has volume $\pi(15^2)(27/2) = \pi(3^5 \cdot 5^2)/2$. Maryanna's has height 6 and radius $15+15/2 = 45/2$, so it has volume $\pi(45/2)^2(6) = \pi(3^5 \cdot 5^2)/2$. (Notice we don't have to bother multiplying all those numbers out. Looking at the prime factorizations allows us to quickly verify that the volumes are the same!)

So, they must have both been using a value of $15/2 = 7.5$ for x. □

Concept: For straightforward problems involving the volume and/or surface area of basic solids, usually we just turn the words of the problem into equations using volume and area formulas. Then we solve the equations, check our work, and make sure we answer the question that is asked.

Problem 15.14: The areas of three of the faces of a rectangular prism are 24, 32, and 36. Find the volume of the prism.

Solution for Problem 15.14: We start off by assigning variables and writing the information in the problem as equations. We let the dimensions of the prism be x, y, and z, so the given information about areas of faces becomes:

$$\begin{aligned} xy &= 24 \\ xz &= 32 \\ yz &= 36 \end{aligned}$$

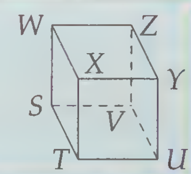
We seek the volume, which in terms of x, y, and z is just xyz. We notice from the nicely symmetric form of the left-hand sides of our equations (xy, yz, zx) that if we multiply all three, we'll get $x^2y^2z^2$. Specifically, we find that multiplying all three equations gives $x^2y^2z^2 = (24)(32)(36)$. Taking the square root of this yields $xyz = \sqrt{(24)(32)(36)} = 96\sqrt{3}$, which is the volume of the prism.

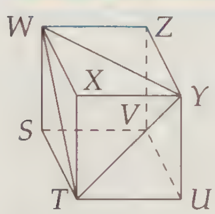
Notice that we didn't even find x, y and z! But now that we have xyz, we can find them quickly. You'll get a chance to do so as an Exercise. □

Concept: Sometimes turning a geometry problem into algebra results in a nice symmetric system of equations, as we saw in Problem 15.14. We often solve such symmetric systems by adding or multiplying all the given equations.

Now we turn to using our two-dimensional problem solving strategies on three-dimensional problems.

Problem 15.15: Cube $STUVWXYZ$ is shown. Find $\angle WYT$.





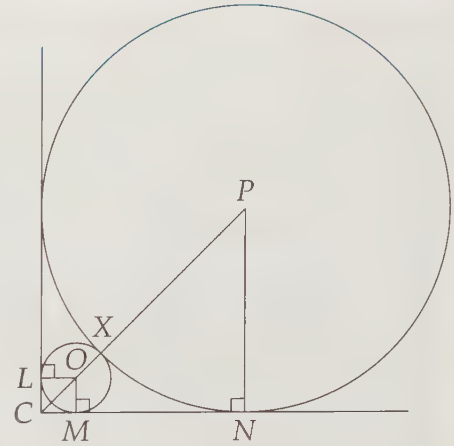
Solution for Problem 15.15: We don't really know how to handle angles in three dimensions, so we consider a two-dimensional figure that includes our angle. In other words, we consider the cross-section that contains $\angle WYT$. We form this cross-section by cutting through vertices W, Y, and T of the cube, as shown. Thus, our cross-section is a triangle. Moreover, we note that since each of the sides of $\triangle WYT$ is a diagonal of a face of the cube, the sides of $\triangle WYT$ are all equal. Hence, $\triangle WYT$ is equilateral, so each of its angles, including $\angle WYT$, is $60°$. □

Concept: Most 3-D problems are 2-D problems in disguise. Considering cross-sections that include important pieces of the problem often allows us to use our two-dimensional strategies to find the solution.

We'll try a couple more 3-D problems that can be solved with well-chosen cross-sections.

Problem 15.16: A ball with radius 6 inches is tangent to the wall and the floor in my house. Given that the wall is perpendicular to floor, what is the radius of the largest ball I can slide into the space between the ball and the corner where the wall meets the floor?

Extra! The mathematician Johannes Kepler wondered how densely spheres can fill space. The typical stacking you see for oranges at the grocery store fills just 74% of space. Is there a different arrangement that gets more oranges in the same space? It took nearly 400 years before mathematicians Thomas Hales and Samuel Ferguson were able to answer Kepler's question and prove in 1998 what grocers have known all along: there isn't a better way to pack oranges.



Solution for Problem 15.16: We have a 3-D problem involving spheres that are tangent to each other and to some planes, so we consider a cross-section including the centers of the spheres and points of tangency. Hopefully this allows us to build helpful right triangles. (Tangents mean right angles!)

O and P are the centers of our spheres, which are tangent at X. These spheres are tangent to the floor at M and N as shown. Sphere O is tangent to the wall at L. Since the spheres are tangent to the wall and the floor, O and P are equidistant from both. Therefore, both O and P are on the angle bisector of $\angle LCM$ as shown. Since $\angle LCM = 90°$, we have $\angle OCM = \angle PCN = 45°$. So, $\triangle OCM$ is an isosceles right triangle. Similarly, $\triangle PCN$ is an isosceles right triangle, so $PC = PN\sqrt{2} = 6\sqrt{2}$, and $XC = PC - PX = 6\sqrt{2} - 6$. If we let the radius of our small sphere be r, we now know that

$$OC = XC - r = 6\sqrt{2} - 6 - r.$$

However, from $\triangle OCM$, we also have

$$OC = OM\sqrt{2} = r\sqrt{2}.$$

Therefore, we have two expressions for OC, which we can set equal:

$$r\sqrt{2} = 6\sqrt{2} - 6 - r.$$

Therefore, $r + r\sqrt{2} = 6\sqrt{2} - 6$, so

$$r = \frac{6\sqrt{2} - 6}{1 + \sqrt{2}}.$$

We can rationalize the denominator by multiplying the top and bottom of the fraction by $1 - \sqrt{2}$:

$$r = \frac{6\sqrt{2} - 6}{1 + \sqrt{2}} \cdot \frac{1 - \sqrt{2}}{1 - \sqrt{2}} = \frac{-18 + 12\sqrt{2}}{-1} = 18 - 12\sqrt{2}.$$

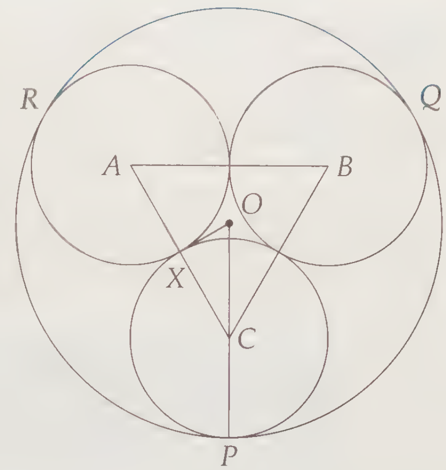
□

Concept: A great many geometry problems are solved by assigning a variable to a length, then finding two different expressions for some other length in terms of that variable. We can then set these expressions equal to solve for the variable.

Problem 15.17: Three congruent spheres are inside a cylinder such that each sphere is tangent to the other two spheres, to both bases of the cylinder, and to the curved surface of the cylinder. Find the volume of the cylinder given that the radius of each sphere is 12.

Solution for Problem 15.17: Since each sphere is tangent to the top and bottom base of the cylinder (and these bases are parallel), the height of the cylinder must equal the diameter of the sphere. Thus, the height of the cylinder is $2(12) = 24$.

To get the radius of the cylinder, we start with a well-chosen cross-section. We have spheres and tangency, so we take the cross-section that includes the centers of the three spheres, the points where the spheres touch each other, and the points where spheres touch the cylinders. Our cross-section is shown at right. $\triangle ABC$ connects the centers of our spheres, and O is where the cross-section meets the axis of the cylinder.



O is equidistant from P, Q and R because the cross-section of the cylinder is a circle with center O. Since $\odot A$, $\odot B$, and $\odot C$ are tangent to $\odot O$, rays $\overrightarrow{RA}$, $\overrightarrow{QB}$, and $\overrightarrow{PC}$ all pass through O. O is therefore equidistant from A, B, and C because $OR = OQ = OP$ and $AR = BQ = CP$. So, O is the circumcenter of $\triangle ABC$. Since $OP = OC + CP$, we only need to find OC to find the radius of the cylinder.

$\triangle ABC$ is equilateral because each side equals twice the radius of a sphere. Therefore, O is also the incenter and the centroid of $\triangle ABC$. Moreover, the points where the spheres' cross-sections touch each other are midpoints of the sides of $\triangle ABC$. Hence, $\triangle OXC$ is a 30-60-90 right triangle because $\overline{OC}$ bisects $\angle ACB$ and $\overline{OX} \perp \overline{AC}$ (make sure you see why). Therefore,

$$OC = 2OX = \frac{2XC}{\sqrt{3}} = \frac{24}{\sqrt{3}} = 8\sqrt{3}.$$

The radius of the cylinder is then $OC + CP = 8\sqrt{3} + 12$. Thus, our volume is:

$$\pi r^2 h = 24\pi(8\sqrt{3} + 12)^2 = 24\pi(4^2)(2\sqrt{3} + 3)^2 = (24 \cdot 16)\pi(21 + 12\sqrt{3}) = 8064\pi + 4608\pi\sqrt{3}.$$

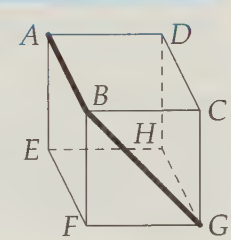
□

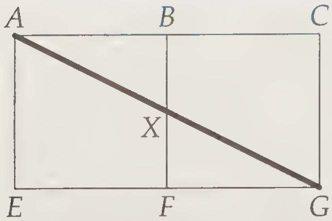
Concept: To solve 3-D problems involving a sphere, consider cross-sections including the center of the sphere. If you have multiple spheres in the problem, try finding a cross-section that includes multiple centers. If the sphere is tangent to anything, try including the point of tangency.

Sometimes the 2-D problem is more subtly hidden in our 3-D problem. In these cases we need more than a simple cross-section to coax it out.

Problem 15.18: Adam the Ant is at point A of cube $ABCDEFGH$ with edge length 1. He wishes to walk along the surface of the cube to point G, where G is the vertex of the cube such that $\overline{AG}$ is a space diagonal of the cube. What is the length of the shortest path Adam can take?

Solution for Problem 15.18: We might think that Adam should walk along edge $\overline{AB}$ and then along face diagonal $\overline{BG}$. However, if we put a big glob of jelly at G and put ants at A, the ants will quickly find a shorter path. They might go diagonally across face $ABFE$ to a point X on $\overline{BF}$, then on to point G. We suspect point X will be the midpoint of $\overline{BF}$, but how can we be sure? (And worse yet, what if $ABCDEFGH$ isn't a cube? You'll get a shot at dealing with that complication in the Exercises, of course!)

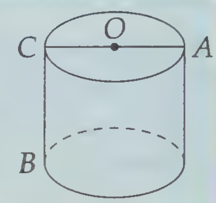




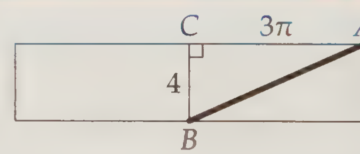
We make our 3-D problem into a 2-D problem by unfolding the cube and placing the two faces Adam crawls across in the same plane as shown. This doesn't change the distance Adam must crawl at all, but it does make our answer clear. Adam's shortest path is the straight segment from A to G. Since $AB = BC = CG = 1$, we have $AG = \sqrt{CG^2 + AC^2} = \sqrt{5}$. Therefore, the shortest path Adam can take has length $\sqrt{5}$. Notice that since $AB = FG$ and $\overline{AC} \parallel \overline{EG}$, we can use congruent triangles $\triangle ABX \cong \triangle GFX$ (by ASA) to show that $BX = FX$, i.e., that Adam's shortest path indeed goes through the midpoint of one of the edges of the cube. □

Concept: The 2-D problem lurking in a 3-D problem isn't always just a cross-section. Sometimes unfolding or unrolling a 3-D figure reveals a simpler 2-D problem.

Problem 15.19: Adam the Ant is now on point A of the cylinder shown. He wishes to walk along the curved surface of the cylinder to point B, where $\overline{BC}$ is a height of the cylinder and C is on the top base such that $\overline{AC}$ is a diameter as shown. Given that the cylinder has height 4 and radius 3, what is the length of the shortest path Adam can take?



Solution for Problem 15.19: When we found the lateral surface area of a cylinder, we discovered how to deal with problems involving the curved surface of a cylinder – we unroll the cylinder. We try that here and we get the rectangle shown at right.



BC still equals the height of the cylinder, so $BC = 4$. However, here $\overline{AC}$ is not a diameter of the original cylinder, but rather the distance from A to C along the circumference of the base. Therefore, AC is half the circumference of the base of our cylinder, or 3π. From right triangle $\triangle ABC$ we can now find $AB = \sqrt{AC^2 + BC^2} = \sqrt{16 + 9\pi^2}$. (Note that this is *not* the distance from A to B in space in our original cylinder. This is the length from A to B along the curved surface of the cylinder. Also, note that if Adam is willing to walk across a base of the cylinder, there is a shorter path he can take!) □

Exercises

15.4.1 The areas of three of the faces of a rectangular prism are 24, 32, and 36. Find the dimensions of the prism. **Hints:** 186

15.4.2 $ABCD$ is a regular tetrahedron. G is the centroid of $\triangle ABC$ and H is the centroid of $\triangle BCD$. Given $AB = 8$, find GH. **Hints:** 198

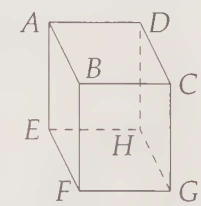
15.4.3 $WXYZ$ is a regular tetrahedron such that $WX = 4$. M is the midpoint of $\overline{WX}$ and N is the midpoint of $\overline{YZ}$.

(a) Prove that $\overline{MN} \perp \overline{WX}$.

(b)★ Find MN.

15.4.4 Three spherical balls are snugly in a row inside a cylindrical can such that the first is tangent to the lid of the can and the middle ball, the second is tangent to the other two balls, and the third is tangent to the bottom of the can and the middle ball. Each ball is fits snugly inside the can so that it is tangent to the curved surface of the can and cannot move side-to-side. Given that the radius of each ball is 2 inches, what is the volume of the can?

15.4.5 Annie the ant is on vertex A of right rectangular prism $ABCDEFGH$ shown. Given that $AB = 4$, $BC = 6$, and $AE = 8$, what is the shortest distance Annie can walk to reach vertex G?



15.4.6 Spheres $\mathcal{S}$ and $\mathcal{T}$ have radii 6 and 8, respectively, and centers O and P, respectively. $OP = 10$.

(a) Why must the spheres intersect?

(b) Let X be a point that is on both spheres. What is XO? What is XP?

(c) What kind of triangle is $\triangle XOP$?

(d) Let the altitude from X to $\overline{OP}$ meet $\overline{OP}$ at Y. What is XY?

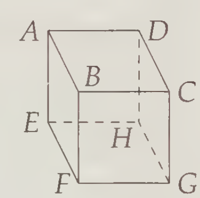
(e) What is OY?

(f) Suppose we pick another point Z that is on both spheres. Must $XY = ZY$?

(g) Use the previous part to show that the intersection of the two spheres is a circle.

(h) Find the area of this circle.

15.4.7 $ABCDEFGH$ is a cube as shown. M is the midpoint of $\overline{FG}$ and N is the midpoint of $\overline{GH}$. Given that $AB = 6$, find the following:



(a) MN.

(b) AM.

(c)★ $[AMN]$. **Hints:** 227

(d)★ the volume of $AEMN$. **Hints:** 265

15.4.8 The height of a cylindrical pole is 12 feet and its circumference is 2 feet. A rope is attached to a point on the circumference at the bottom of the pole. The rope is then wrapped tightly around the pole four times before it reaches a point on the top directly above the starting point at the bottom. What is the minimum number of feet in the length of the rope? *(Source: MATHCOUNTS)*

15.4.9★ Two regular square pyramids have all edges 12 cm in length. The pyramids have parallel bases and parallel edges, and each has a vertex at the center of the other pyramid's base. What is the total number of cubic centimeters in the volume of the solid of intersection of the two pyramids? *(Source: MATHCOUNTS)* **Hints:** 318

Extra! We've seen that regular hexagons can tile a plane, and regular pentagons can tile a sphere, giving us a dodecahedron. There's one well-known example of a tiling that uses both hexagons and pentagons – it's commonly known as a soccer ball. This fabulous structure has been around since before the invention of soccer, too, in the form of **buckminsterfullerene**, C_{60}, a recently discovered form of carbon. Drs. Richard Smalley and Robert Curl received the Nobel Prize in 1996 for that discovery.

15.5 Summary

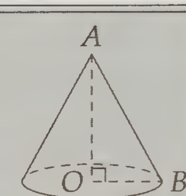
Definitions: If a prism has congruent curved figures as bases instead of polygons, we call it a **cylinder.** When we talk about a cylinder, we almost always mean a **right circular cylinder**, in which the bases are circles. The line connecting the centers of the bases is the **axis** of the cylinder.

Important: A cylinder with height h and radius r has:

$$\begin{aligned}\text{Volume} &= \pi r^2 h\\ \text{Lateral Surface Area} &= 2\pi rh\\ \text{Total Surface Area} &= 2\pi rh + 2\pi r^2\end{aligned}$$

Don't memorize these formulas! If you take the time to understand them, they'll always be obvious to you.

Definitions: A pyramid with a curved base is a **cone**. At the right is a **right circular cone**, which is usually what we mean when we write 'cone.' The point at the tip of the cone is the **vertex**. The distance from the vertex to the base is the **height** and the line connecting the vertex to the center of the base is the **axis** of the cone. In a right circular cone, the distance from the vertex to a point on the circumference of the base is the **slant height**.



Important: In a cone with radius r, slant height l, and height h, we have:

$$\begin{aligned}\text{Volume} &= \frac{1}{3}\pi r^2 h\\ \text{Lateral Surface Area} &= \pi rl\\ \text{Total Surface Area} &= \pi rl + \pi r^2\end{aligned}$$

Definition: The set of all points in space that are the same distance from a given point is a **sphere**.

Important: A sphere of radius r has:

$$\begin{aligned}\text{Volume} &= \frac{4\pi r^3}{3}\\ \text{Surface Area} &= 4\pi r^2\end{aligned}$$

Every cross-section of a sphere is a circle (or a point, when the plane is tangent to the sphere). The segment connecting the center of the sphere to the center of this circle is perpendicular to the plane of the cross-section.

Important: Just as the ratio of the areas of similar two-dimensional figures is the square of the ratio of their corresponding sides, the ratio of the surface areas of similar three-dimensional figures is the square of the ratio of their corresponding side lengths. Moreover, the ratio of the volumes of similar three-dimensional figures is the cube of the ratio of their corresponding side lengths.

Problem Solving Strategies

Concepts:

- Problems involving the curved surface of a cylinder can often be solved by 'unrolling' the curved surface into a rectangle.
- Similarly, problems involving the curved surface of a cone can often be solved by 'unrolling' the curved surface into a sector.
- Basic area/volume word problems are often no different from other word problems. The key to solving them is to assign variables and use the area and volume information to set up equations with the given information.
- To solve 3-D problems involving a sphere, consider cross-sections including the center of the sphere. If you have multiple spheres in the problem, try finding a cross-section that includes multiple centers. If the sphere is tangent to anything, try including the point of tangency.
- Most 3-D problems are 2-D problems in disguise. Usually considering cross-sections including important pieces of the problem allow us to use our two-dimensional strategies to find the solution.
- The most useful of these 2-D strategies is the Pythagorean Theorem. Building right triangles is just as powerful in three dimensions as in two.
- The 2-D problem lurking in a 3-D problem isn't always just a cross-section. Sometimes we have to manipulate our 3-D figures a little through unfolding or unrolling to discover our 2-D problem.
- A great many geometry problems are solved by assigning a variable to a length, then finding two different expressions for some other length in terms of that variable. We can then set these expressions equal to solve for the variable.

Extra!

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots$$

REVIEW PROBLEMS

15.20 The number of cubic inches in a sphere's volume equals the number of square inches in its surface area. What is the radius of the sphere?

15.21 A cone has slant height 7 and height 5.

(a) Find the volume of the cone.

(b) Find the total surface area of the cone.

15.22 A sphere has diameter 8. Find the surface area and the volume of the sphere.

15.23 A right circular cylinder has radius 3 and height 6.

(a) Find the volume of the cylinder.

(b) Find the lateral surface area of the cylinder.

(c) Find the total surface area of the cylinder.

15.24 A space diagonal of cube C is a diameter of sphere $\mathcal{S}$. The edges of the cube have length 8. Find the volume of the sphere.

15.25 The radius of cylinder $\mathcal{A}$ is three times the radius of cone $\mathcal{B}$, but the height of cylinder $\mathcal{A}$ is half the height of cone $\mathcal{B}$. Find the ratio of the volume of $\mathcal{A}$ to the volume of $\mathcal{B}$.

15.26 $ABCD$ is a square inscribed in one base of a cylinder, and $EFGH$ is a square inscribed in the other base of the cylinder. Given that $ABCDEFGH$ is a cube with side length 9, find the volume of the cylinder.

15.27 A 40° sector of a circle with radius 9 is rolled up to form a cone. Find the volume of the cone.

15.28 The intersection of a plane with sphere $\mathcal{G}$ is a circle with area 36π. Find the volume of the sphere if the center of this circle is 8 units from the center of $\mathcal{G}$.

15.29 Two spheres are inside a rectangular box such that each sphere is tangent to five faces of the box and to the other sphere. Each sphere has radius 4. Find the volume of the box.

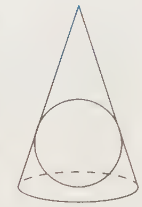
15.30 A cross-section of cylinder C is a square in a plane parallel to the axis of C. Given that the area of the square is 36 square units and that the plane of the cross-section is 4 units from the axis of the cylinder, find the volume of the cylinder.

15.31 A cone with radius 9 and height 12 is cut in two pieces by a plane parallel to the base of the cone such that the plane is 8 units from the base of the cone. Find the total surface area of each piece thus formed.

15.32 Sphere $\mathcal{S}$ is tangent to all 12 edges of a cube with edge length 8. Find the volume of the sphere.

Extra! *I don't like that sort of school ... where the bright childish imagination is utterly discouraged ... where I have never seen among the pupils, whether boys or girls, anything but little parrots and small calculating machines.*

–Charles Dickens



15.33 A sphere is inscribed in a cone as shown at right. The cone has radius 9 and height 12. Find the radius of the sphere.

15.34 In this problem we use volume to find the radius of a sphere inscribed in regular tetrahedron $ABCD$ with edge length 6. You may want to review Problem 7.14, since we use a similar tactic on this problem.

(a) Find the length of the altitude from D to face ABC of the tetrahedron.

(b) Find the volume of tetrahedron $ABCD$.

(c) Let the radius of the sphere inscribed in the tetrahedron be r and the center of this sphere be O. Why is O a distance of r away from each face of the tetrahedron?

(d) In terms of r, what is the volume of tetrahedron $OABC$? **Hints:** 402

(e) In terms of r, what are the volumes of tetrahedra $OABD$, $OACD$, and $OBCD$?

(f) Find r.

15.35 The height of right circular cone C equals the cone's radius in length. Given that A is the vertex of the cone and $\overline{XY}$ is a diameter of the cone's base, prove that $\angle XAY$ is a right angle.

Challenge Problems

15.36 Right circular cone C has vertex V, radius 6, and height 6. Sphere $\mathcal{S}$ has center V and is tangent to the base of cone C.

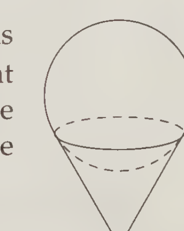
(a) What is the radius of the sphere?

(b) Let X be a point where $\mathcal{S}$ meets the curved surface of C. Find the distance from X to the axis of the cone. **Hints:** 11

(c) Show that the intersection of $\mathcal{S}$ and the curved surface of C is a circle.

(d) Find the area of the circle from the previous part.

15.37 A cylinder is inscribed in a right regular hexagonal prism such that each base of the cylinder is a circle that is tangent to all six sides of a base of the prism. Find the ratio of the volume of the cylinder to the volume of the prism.

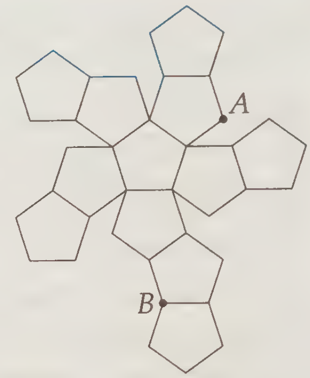


15.38 Dennis's scoop of ice cream has a radius of 2 cm. It rests in a cone that has a radius of $\sqrt{3}$ cm at the widest part, and the scoop is tangent to each line containing a slant height of the cone. He eats some of the ice cream and then finds that the remainder of the ice cream can be pushed down to fill the cone exactly. How many cubic centimeters of ice cream did he eat? *(Source: Mandelbrot)* **Hints:** 177

Extra! *Imagination is more important than knowledge.*

–Albert Einstein

15.39 An edgy spider walks along the edges from A to B of the dodecahedron formed by the folding up the figure shown at right. What is the number of edges in the shortest path that spider could take? *(Source: MATHCOUNTS)*



15.40 Two parallel planes intersect sphere $\mathcal{S}$, forming two congruent circles. Show that the center of the sphere is the midpoint of the segment connecting the centers of the two circles. **Hints:** 489, 556

15.41 Planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are perpendicular. Their intersection is line m. Plane $\mathcal{P}_1$ is 4 units from point O and plane $\mathcal{P}_2$ is 6 units from O. A sphere with radius 9 centered at O meets line m at points A and B. Find AB. **Hints:** 503, 567

15.42 X is on the circumference of one base of a cylinder and Y is on the other base such that $\overline{XY}$ is bisected by the axis of the cylinder. Given that the diameter of the cylinder is 8 and $XY = 12$, find the volume of the cylinder.

15.43 Arnav the ant is on the outside of a cylindrical glass, halfway up the glass. The glass is 8 inches tall and has a radius of 2 inches. Arnav wants to walk to a point on the inside of the glass that is diametrically opposite the point he's now at (i.e., halfway up the glass, on the inside surface of the glass, exactly opposite where he is now). What is the shortest distance he can walk? **Hints:** 218, 529

15.44 A spiral staircase with radius 3 feet turns 270° as it rises 10 feet. What is the number of feet in the length of the outer handrail? (The outer handrail is a curved rail along the entire staircase. It is everywhere 3 feet from the central axis of the staircase.) *(Source: MATHCOUNTS)* **Hints:** 66

15.45 A right circular cylinder with its diameter equal to its height is inscribed in a right circular cone. The cone has diameter 10 and altitude 12, and the axes of the cylinder and cone coincide. Find the radius of the cylinder. *(Source: AMC 10)* **Hints:** 429

15.46 $PQRS$ is a regular tetrahedron. The distance from the midpoint of $\overline{PQ}$ to the midpoint of $\overline{RS}$ is 6 units. Find the volume of $PQRS$. **Hints:** 565

15.47 The wire frame at left below consists of three mutually perpendicular segments $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$, each 3 cm in length. A quarter-circle of radius 3 is attached to each pair of segments as shown. The curved portions of the wire frame will fit snugly against a sphere of a certain size, so that the entire lengths of all three quarter circles make contact with the sphere. For what radius is this possible? *(Source: Mandelbrot)* **Hints:** 389, 452

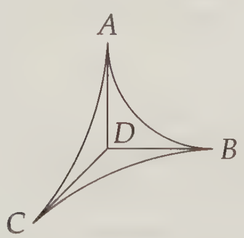


Figure 15.4: Diagram for Problem 15.47

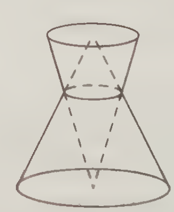


Figure 15.5: Diagram for Problem 15.48

15.48★ One cone of radius 4 and height 12 and another cone of radius 6 and height 12 intersect as shown at right above, so that the vertex of one coincides with the center of the base of the other. Find the volume of the intersection of the two cones. **Hints:** 139, 238

15.49★

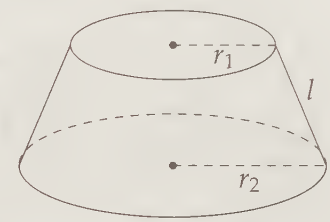
(a) In tetrahedron $ABCD$, let V be the volume of the tetrahedron and r the radius of the sphere that is tangent to all four faces of the tetrahedron. Let K_1, K_2, K_3, and K_4 denote the areas of faces BCD, ACD, ABD, and ABC, respectively. Prove that

$$r = \frac{3V}{K_1 + K_2 + K_3 + K_4}.$$

Hints: 184

(b) $WXYZ$ is a tetrahedron with $XY = YZ = ZX = 9$ and $WX = WY = WZ = 18$. Find the radius of the sphere that is inscribed in tetrahedron $WXYZ$. (You might remember working with this pyramid in Problem 14.26.)

15.50★ Find a formula for the total surface area of a right circular frustum with base radii r_1 and r_2 and slant height l. (The slant height of a frustum is the distance from a point on the circumference of one base to the nearest point on the circumference of the other base, as shown at right.) **Hints:** 219, 292

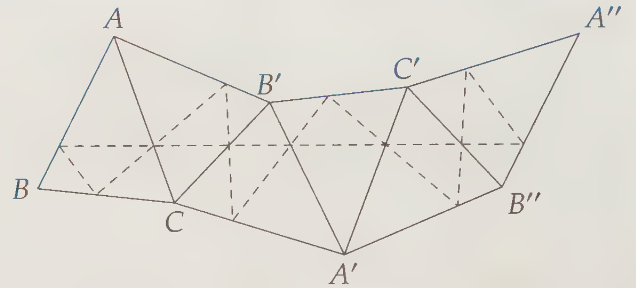


15.51★ A ball of radius 1 is in the corner of a room, tangent to two walls and the floor. A smaller ball is also in the corner, also tangent to both walls, the floor, and the larger ball. The walls are perpendicular to each other, and they are perpendicular to the floor. Find the radius of the smaller ball. **Hints:** 327

Extra! We've already had a glimpse of the brilliance of **Archimedes** n page 293 in his proof of the formula for the area of a circle. Although the mathematical and scientific accomplishments of Archimedes could fill several books, the feat he allegedly treasured most was his determination of the volume and surface area of a sphere. Legend has it that he even asked that his tombstone be inscribed with a sphere inscribed in a cylinder to commemorate his feat.

Archimedes used the tactics we saw in his area of a circle proof, although with considerably more complicated figures. He compared a sphere to figures he already knew how to handle: cones and cylinders. To get a feel for the volume formula, consider the figures below. Shown is a cone that shares a base with a cylinder such that both the cone and the cylinder have height equal to the radius of the base. Next to these is a **hemisphere**, or half a sphere, with the same radius as the base of the cylinder and cone.

Consider the cross-sections of each taken at the same height, h, from the base. Letting the radius of the hemisphere be r, find the areas of the cross-section of each figure in terms of r and h. Next, find the area inside the cross-section of the cylinder, but outside the cross-section of the cone. Notice anything interesting?



Fagnano's Problem

The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful. – Aristotle

CHAPTER 16 The More Things Change...

A **geometric transformation** is a rule we apply to a geometric figure that usually results in the figure being moved or changed in some way. In this chapter we will explore a few basic types of transformations by sliding, spinning, and flipping figures. As we'll see, these transformations change the locations of figures to which they are applied, as opposed to turning them into entirely different figures.

16.1 Translations

When we apply a **translation** to a figure, we simply slide it a specified distance in a given direction. For example A' and B' below are the result of translating A and B, respectively, in the direction and by the distance suggested by the arrows.

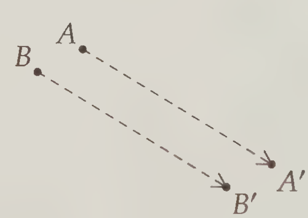


Figure 16.1: A Translation

We use some special terms when performing transformations. We say that A' is the **image** of A under the translation shown above. We can also say that the translation **maps** A to A'.

Of course, we can **translate** more than just points. We can translate any figure – a segment, a line, a triangle, etc. – and the result will be congruent to our original figure. This is one of those 'obvious' things that we won't worry about proving right now. We'll look at proving 'obvious' facts like these in the next book in this series, *Intermediate Geometry*.

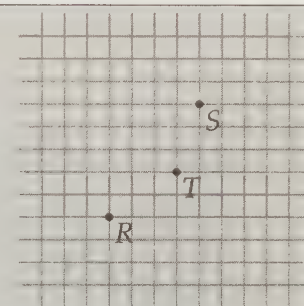
Problems

Problem 16.1: A **fixed point** of a transformation is a point that is its own image. The **identity** transformation is the transformation that maps every point to itself. In other words, the identity is the 'do nothing' transformation. Can a translation that is not the identity have a fixed point?

Problem 16.2:

(a) Locate point U on the grid shown such that $RSTU$ is a parallelogram.

(b) Is the U you found the only point that can be combined with R, S, and T to form the vertices of a parallelogram? If not, find the others.



Translations are the simplest transformations, so there aren't many interesting problems involving translations. We'll use these simple transformations to introduce a couple more terms.

Problem 16.1: A **fixed point** of a transformation is a point that is its own image. The **identity** transformation is the transformation that maps every point to itself. In other words, the identity is the 'do nothing' transformation. Can a translation that is not the identity have a fixed point?

Solution for Problem 16.1: In a translation, every point is moved the same distance in the same direction. If there is a fixed point, then that point is not moved at all. Since all points must move the same distance in the same direction, this means none of the points is moved by the translation. Hence, the only translation that has a fixed point is the identity. □

Concept: Understanding transformations is often more about learning what stays the same rather than what changes. In the three basic transformations we'll study in this chapter, everything will stay the same for each transformed figure except its location and orientation. That may sound like a lot to change, but it really isn't much. Line segments remain line segments, circles remain circles, points that are one unit apart remain one unit apart, and so on.

Here's one example of using translations to solve a problem:

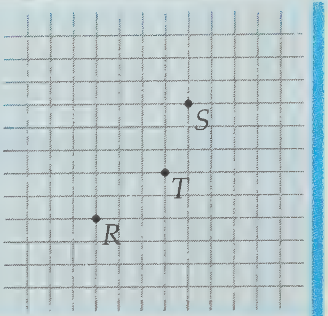
Extra! *"Obvious" is the most dangerous word in mathematics.*

–Eric Temple Bell

Problem 16.2:

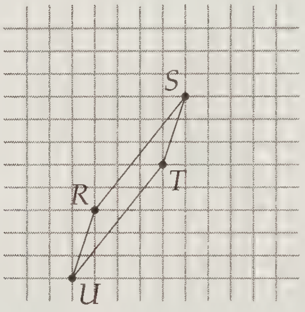
(a) Locate point U on the grid shown such that $RSTU$ is a parallelogram.

(b) Is the U you found the only point that can be combined with R, S, and T to form the vertices of a parallelogram? If not, find the others.



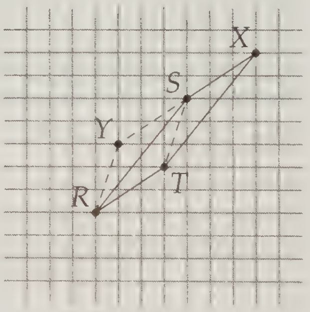
Solution for Problem 16.2:

(a) If $RSTU$ is a parallelogram, then $\overline{SR} \parallel \overline{TU}$ and $SR = TU$. In other words, the same translation that maps S to R must map T to U. Since we must go down 5 and left 4 to get from S to R, we must go down 5 and left 4 to get from T to U. Therefore, U is at the point shown in the diagram below.



(b) In the first part, we 'completed' the parallelogram that has $\angle RST$ as an angle. We could instead have chosen $\angle SRT$ or $\angle STR$ as an angle of the parallelogram. Let points X and Y, 'complete' parallelograms $SRTX$ and $STRY$, respectively.

To complete parallelogram $SRTX$, we note that we must go right 3 and up 2 to get from R to T. Therefore, we go right 3 and up 2 from S to get to X. Similarly, to complete $STRY$, we see that we go left 3 and down 2 to get from T to R, so we must go left 3 and down 2 from S to find Y. Our other two parallelograms are shown in the diagram.



□

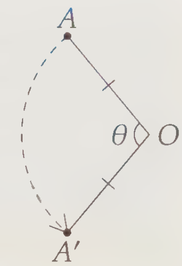
Exercises

16.1.1 $ABCDEF$ is a regular hexagon. Why must the translation that maps E to C also map F to B? **Hints:** 26, 32

Extra! *Somebody came up to me after a talk I had given, and said, "You make mathematics seem like fun." I was inspired to reply, "If it isn't fun, why do it?"*

–Ralph Boas

16.2 Rotations



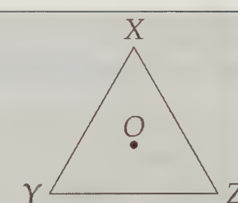
When we **rotate** a figure, we spin it by some angle about some point. In other words, the image of point A under a **rotation** of angle θ counterclockwise about point O is the point A' such that $OA = OA'$ and $\angle AOA' = \theta$, as shown at right. There are two points that satisfy this definition of A', which is why we must specify the direction of the rotation as clockwise or counterclockwise. Point O is the center of the rotation, and θ is the angle of rotation.

As with translations, the image of any figure upon rotation is congruent to the original figure. This is one of those 'obvious' facts that you might try proving on your own for specific shapes like line segments or circles.

Sidenote: The contributions of the ancient Greeks to mathematics have been immortalized in many ways. One of these is shown in our introduction of rotation. It's a common convention to use Greek letters as variables for angle measures. Above, we use the letter θ, or 'theta.' The letter ϕ is also often used for angles, and sometimes the Greek letters α, β, and γ are used to stand for the angles of a generic triangle $\triangle ABC$. Greek letters are also fun to write!

Problems

Problem 16.3: Let O be the center of equilateral $\triangle XYZ$.



(a) Through what angles between 0° and 360° can we rotate Y about Z such that the result is point X?

(b) Through what angles between 0° and 360° can we rotate X about O such that the result is point Y?

(c) Is there a rotation about Y that maps O to Z?

Problem 16.4: Through how many different positive angles θ less than 360° is it possible to rotate a regular dodecagon clockwise about its center such that its image coincides with the original dodecagon?

Problem 16.5: After I draw a figure, I rotate it 48° about some point P. If the image of this rotation coincides exactly with the original figure, is it true that the image of a rotation of the figure 72° about P must also coincide with the figure?

We'll explore rotations with a few basic problems. As you'll see, rotations are all about angles.

Extra! *The more things change, the more they are the same.*

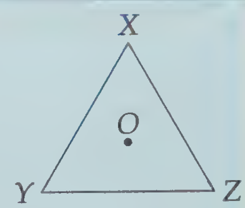
–Alphonse Karr

Problem 16.3: Let O be the center of equilateral $\triangle XYZ$.

(a) Through what angle(s) between $0°$ and $360°$ can we rotate Y about Z such that the result is point X?

(b) Through what angle(s) between $0°$ and $360°$ can we rotate X about O such that the result is point Y?

(c) Is there a rotation about Y that maps O to Z?



Solution for Problem 16.3:

(a) Since $YZ = XZ$ and $\angle YZX = 60°$, a $60°$ rotation clockwise about Z takes Y to X. Of course, a $300°$ rotation the other way will also do the job!

(b) Since O is the center of equilateral $\triangle XYZ$, $\overline{XO}$ and $\overline{YO}$ bisect $\angle ZXY$ and $\angle ZYX$, respectively. Therefore, $\angle YXO = \angle OYX = 30°$, so $\angle XOY = 180° - \angle YXO - \angle OYX = 120°$. Since $OX = OY$ and $\angle XOY = 120°$, if we rotate X $120°$ counterclockwise about O, it is mapped to point Y. Note that we can also rotate X $240°$ clockwise around O to get to Y.

(c) By the definition of rotation, every rotation about Y maps O to a point that is as far from Y as O is. Since $\overline{YO}$ and $\overline{YZ}$ are not equal in length (make sure you see why!), it is impossible for a rotation about Y to map O to Z.

□

Problem 16.4: Through how many different positive angles θ less than $360°$ is it possible to rotate a regular dodecagon clockwise about its center such that its image coincides with the original dodecagon?

Solution for Problem 16.4: Let $ABCDEFGHIJKL$ be our regular dodecagon and O its center. Each of the interior angles has measure $180° - 360°/12 = 150°$, and each segment from a vertex to the center bisects an interior angle, so $\angle OAB = \angle OBA = 75°$. Therefore, $\angle AOB = 30°$. So, if we rotate the dodecagon $30°$, every vertex will be 'shifted over' by one vertex, and the dodecagon will map to itself.

Because a rotation of $30°$ works, we know a rotation of $30° + 30° = 2(30°) = 60°$ will work too. A $60°$ rotation is just the result of two $30°$ rotations, and each of these maps the dodecagon to itself. We could also think of this as shifting each vertex over two vertices. Similarly, a rotation of each multiple of $30°$ will map the dodecagon to itself. This gives us 11 positive angle rotations less than $360°$ about the center that will take the dodecagon to itself.

It is impossible for two different clockwise rotations between $0°$ and $360°$ about O to map A to the same vertex, and we can't map A to itself with such a rotation. Therefore, we know that the 11 rotations we've found are the only clockwise rotations between $0°$ and $360°$ that map the dodecagon to itself. □

Problem 16.5: After I draw a figure, I rotate it $48°$ about some point P. If the image of this rotation coincides exactly with the original figure, is it true that the image of a rotation of the figure $72°$ about P must also coincide with the figure?

Solution for Problem 16.5: We know that rotating the figure 48° leaves the figure unchanged, so to think about what happens when we rotate by 72°, we start by finding what other rotations leave the figure unchanged. Spinning the figure by 48° twice will leave it unchanged, so a 96° rotation leaves the figure unchanged. Similarly, other angles of rotation about P that don't change the figure are $3(48^\circ) = 144^\circ, 4(48^\circ) = 192^\circ, 240^\circ, 288^\circ, 336^\circ, 384^\circ, 432^\circ, 480^\circ, \ldots$ Taking a look at those 'over 360°' rotations, we see that one of them is 432°. A 432° rotation is equivalent to a $432^\circ - 360^\circ = 72^\circ$ rotation. Therefore, a 72° rotation will also coincide with the original figure. □

Exercises

16.2.1 *ABCDEFGH* is a regular octagon, with the vertices labeled counterclockwise.

(a) A clockwise rotation with center B and angle θ maps A to C. Find θ such that $\theta < 360^\circ$.

(b) A clockwise rotation with center B and angle ϕ maps H to D. Find ϕ such that $\phi < 360^\circ$.

16.2.2 B' is the image of B rotated 90° about A. Given $AB = 5$, find BB'.

16.2.3 O is the intersection of the diagonals of quadrilateral $ABCD$. A 90° rotation about O maps $ABCD$ to itself, meaning each vertex of the image is also a vertex of the original quadrilateral. Must $ABCD$ be a square? **Hints:** 43

16.2.4★ Regular tetrahedron $ABCD$ is rotated 60° about its altitude from A to face BCD.

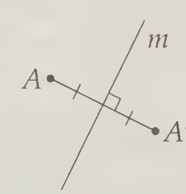
(a) What kind of shape is the intersection of the original tetrahedron and its image? **Hints:** 208

(b)★ Find the volume of intersection of the original tetrahedron and its image given that $AB = 6$. **Hints:** 561

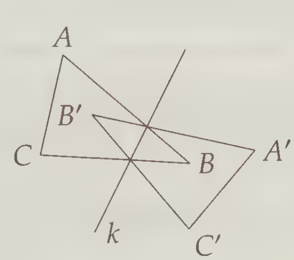
16.3 Reflections

We've tried sliding and we've tried spinning; now, we'll try flipping. If you've looked in a mirror, you have experienced **reflection**.

Mathematically speaking, the image when we **reflect** point A over line m is the point A' such that m is the perpendicular bisector of $\overline{AA'}$. In other words, if we folded our paper along line m, A and A' would coincide.



Of course, we can flip more than just points. In the diagram below, $\triangle A'B'C'$ is the image of $\triangle ABC$ upon reflection over line k.



If a figure maps to itself under a reflection over a certain line, that line is called a **line of symmetry** of the figure. For example, every diameter of a circle is part of a line of symmetry of a circle, and each diagonal of a square is part of a line of symmetry of the square.

Problems

Problem 16.6: What are the fixed points of a reflection over a given line m? In other words, what points are their own images when reflected over line m?

Problem 16.7: Given two different points, X and Y, describe all lines m such that the image of X upon reflection over m is point Y.

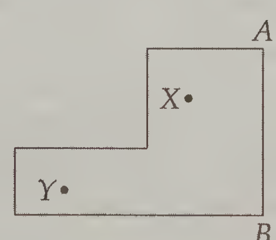
Problem 16.8: How many different lines of symmetry does a regular hexagon have?

Problem 16.9: I own a cabin in the woods that is 4 miles directly north of a river that runs east-west. I am currently out exploring the woods. I am 2 miles north of the river, and 3 miles west of my cabin. I want to walk to the river, then walk to the cabin. What is the length of the shortest possible route I can take?

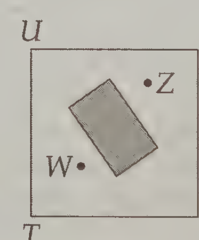
Problem 16.10: My miniature golf course has only two holes, which are shown below.

(a) On Hole 1, the ball starts at Y, and we want to hit it to X. Find the point on $\overline{AB}$ at which we should aim to hit the ball to X.

(b) On Hole 2, the ball starts at W, and we want to hit it to Z. There is a barrier between W and Z, as shown. Where should we aim on $\overline{UT}$ to hit the ball to Z?



Hole 1



Hole 2

One of the best ways to understand any geometric transformation is to analyze what stays the same when everything else is changing.

Problem 16.6: What are the fixed points of a reflection over a given line m?

Solution for Problem 16.6: Clearly every point on m maps to itself, since m is the 'fold.' To see that there are no other fixed points, we note that m divides the plane into two pieces. Any point on one side of m is mapped to a point on the other side. Since a point can't be on both sides of m, a point not on m can't be mapped to itself! □

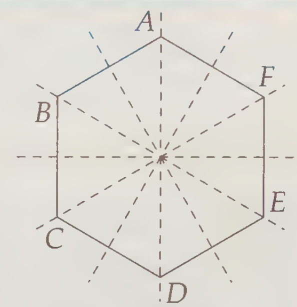
Suppose we are given a point and its image under a reflection. How do we find the line over which the point was reflected?

Problem 16.7: Given two different points, X and Y, describe all lines m such that the image of X upon reflection over m is point Y.

Solution for Problem 16.7: The image of a point A upon reflection over a line is the point A' such that the line is the perpendicular bisector of $\overline{AA'}$. Since the image of X in our problem is Y, the line over which we are reflecting must be the perpendicular bisector of $\overline{XY}$. □

Now that we've mastered reflecting points, let's reflect a more complicated figure.

Problem 16.8: How many different lines of symmetry does a regular hexagon have?

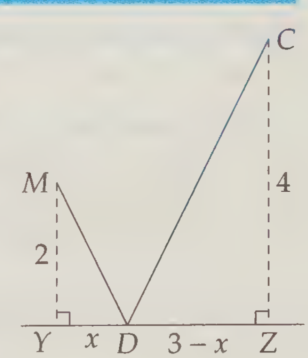


Solution for Problem 16.8: Intuitively, it seems like the only lines of symmetry are the six dashed lines shown that cut the regular hexagon in half, but how can we be sure these are the only ones?

To make sure there aren't any others, we note that if the reflection maps the hexagon to itself, then A must be mapped to some vertex of the hexagon. Furthermore, for any two different vertices, there is only one reflection that will map the vertices to each other – the reflection over the perpendicular bisector of the segment connecting the two vertices. Therefore, there's only one reflection that could possibly map A to B, and one that maps A to C, etc. Each of these reflections gives us a line of symmetry. This doesn't include lines of symmetry through A. $\overleftrightarrow{AD}$ is the only possible line of symmetry through A, since it's the only line through A that leaves two vertices on each side of the line. □

Now we'll use reflections to solve a couple of problems.

Problem 16.9: I own a cabin in the woods that is 4 miles directly north of a river that runs east-west. I am currently out exploring the woods. I am 2 miles north of the river, and 3 miles west of my cabin. I want to walk to the river, then walk to the cabin. What is the length of the shortest possible route I can take?

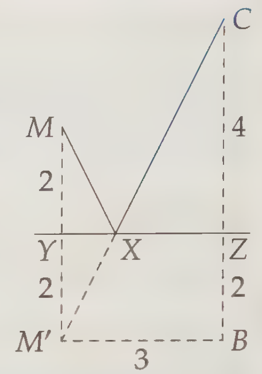


Solution for Problem 16.9: Point M is where we start, the cabin is at C, and $\overleftrightarrow{YZ}$ is the river. One approach we can take is to suppose we hit the river at point D, x miles east of where we start. We can then use the Pythagorean Theorem to make an expression in terms of x for how far we have to walk. Since $YD = x$, we have $DZ = 3 - x$, so from right triangles $\triangle MYD$ and $\triangle CZD$, we have

$$MD + CD = \sqrt{4 + x^2} + \sqrt{16 + (3 - x)^2}.$$

Figuring out the smallest possible value for this expression is going to be very hard. We need to find a more clever approach.

Since we're stuck, we try thinking of similar, simpler problems that we know how to solve. For example, if we didn't have to go to the river, the answer would be easy – we just go from M to C and we're done. Another simplified problem would be if we were on the other side of the river. Then we could still just go straight from our starting point to the cabin, and we'd still hit the river. Aha! We can cross the river.



Suppose we reflect ourselves over the river, to point M' as shown at right. Since $\overline{MX}$ is the mirror image of $\overline{M'X}$, we have $MX + XC = M'X + XC$. Therefore, finding the shortest total distance from M to the river to C is the same as finding the shortest total distance from M' to the river to C.

Any path from M' to C hits the river, so clearly our desired shortest distance is $M'C$. Since M is 2 miles north of the river, M' is 2 miles south of it. So, M' is $2 + 4 = 6$ miles south of the cabin. Since M' is also 3 miles west of the cabin, $M'C = \sqrt{36+9} = 3\sqrt{5}$.

Therefore, the shortest path I can take is $3\sqrt{5}$ miles long. □

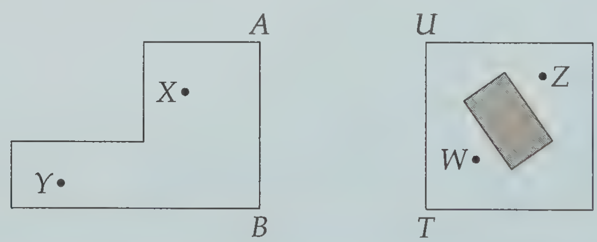
Concept: If you can't solve a problem, try thinking of a similar, simpler problem and solve that. Then try to use your solution to the simpler problem to solve the harder problem.

Next time you play miniature golf or shoot pool, you'll likely think of reflections. Here's why:

Problem 16.10: My miniature golf course has only two holes, as shown below.

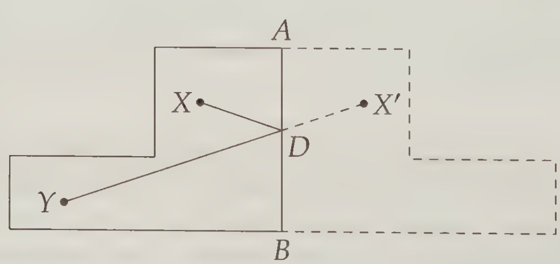
(a) On Hole 1, the ball starts at Y, and we want to hit it to X. Find the point on $\overline{AB}$ at which we should aim to hit the ball to X.

(b) On Hole 2, the ball starts at W, and we want to hit it to Z. There is a barrier between W and Z, as shown. Where should we aim on $\overline{TU}$ to hit the ball to Z?



Solution for Problem 16.10:

(a) For our first hole, we need to bounce the ball off $\overline{AB}$ so that it goes directly to X. To figure out where to aim the ball, we note that the path after the ball bounces is the reflection of what the path would be *if there were no wall* $\overline{AB}$ and the ball just kept going straight. Therefore, if we aim at the image of X upon reflection over $\overline{AB}$, the ball will bounce off $\overline{AB}$ and go straight to X.

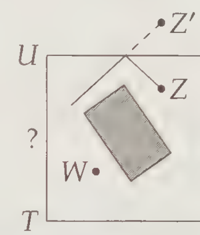


Hole 1

So, to find the point on $\overline{AB}$ to aim at, we reflect X over $\overline{AB}$ to get its image X'. The intersection of $\overline{YX'}$ and $\overline{AB}$ is the point we should target.

(b) On Hole 2, the barrier not only prevents us from hitting the ball straight from W to Z, but it also prevents us from just bouncing the ball off $\overline{TU}$ and having it go straight to Z. We'll have to go for a double-bank shot. Therefore, we need it to bounce off $\overline{TU}$, then hit the appropriate point on the top wall so that it continues to point Z.

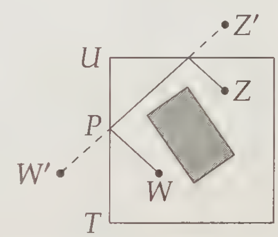
We still don't know where to aim along $\overline{TU}$, but we can figure out where we want the ball heading after it banks off of $\overline{TU}$. After coming off $\overline{TU}$, the ball should be heading towards the image of Z upon reflection over the top wall.



Hole 2

We now know where we want the ball heading once it comes off $\overline{TU}$, but we still don't know where to aim along $\overline{TU}$. Since we need the ball to go from W and bounce off $\overline{TU}$ so that it is heading straight towards Z', we think to try our reflection tactic on W as well.

Point W' is the image of W upon reflection over $\overline{TU}$. If we were at W' and there were no wall $\overline{TU}$, we would aim directly at Z'. The path $\overline{W'Z'}$ meets $\overline{TU}$ at P. Since $\overline{W'P}$ is the image of $\overline{WP}$, if we aim the ball at P, it will bounce off $\overline{TU}$, and head towards Z'. Since Z' is the image of Z upon reflection over the top wall, a ball heading from P towards Z' will bounce off the top wall and head straight to Z as desired.



Hole 2

Have fun beating your friends next time you play miniature golf!

□

Exercises

16.3.1 How many lines of symmetry does each of the following figures have?

(a) A rectangle that is not a square.

(b) A rhombus that is not a rectangle.

(c) A parallelogram that is not a rectangle or a rhombus.

(d) A regular pentagon.

(e) A regular dodecagon.

(f) A regular polygon with n sides.

16.3.2 Square $ABCD$ is reflected over $\overleftrightarrow{CD}$. A' and B' are the images of A and B, respectively. Given $CD = 4$, find $A'A$ and $A'B$.

16.3.3 Lines k and ℓ are not parallel, and line m is the image of line k upon reflection over ℓ. Show that ℓ bisects a pair of the angles formed by line m and k.

16.3.4 Given intersecting lines ℓ and m and point X not on either line, let Y be the image of X upon reflection over ℓ and let Z be the image of X upon reflection over m.

(a) Is it possible for the reflection of Y over m and the reflection of Z over ℓ to be the same point?

(b) Must the reflection of Y over m and the reflection of Z over ℓ be the same point?

16.3.5★ We can define reflections in space through a plane similar to the way we define reflections over a line. Specifically, given a point K and a plane $\mathcal{P}$, then the image of K upon reflection through $\mathcal{P}$ is the point K' such that $\mathcal{P}$ is perpendicular to $\overline{KK'}$ and bisects $\overline{KK'}$. A **plane of symmetry** of a figure is a plane such that the image upon reflecting the figure through the plane is the figure itself. For example, any plane through the center of a sphere is a plane of symmetry of the sphere.

(a) How many planes of symmetry does a right square pyramid have?

(b) How many planes of symmetry does a right rectangular prism have if the dimensions of the prism are all different?

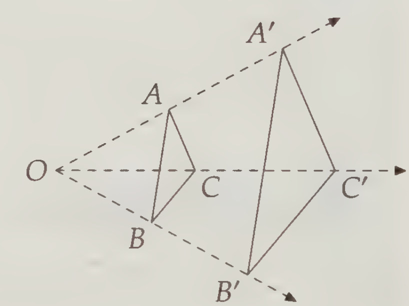
(c) How many planes of symmetry does a regular tetrahedron have? **Hints:** 154

(d) How many planes of symmetry does a cube have? **Hints:** 63

(e) Show that any plane of symmetry of a right circular cone must include the axis of the cone. **Hints:** 115

16.4 Dilation

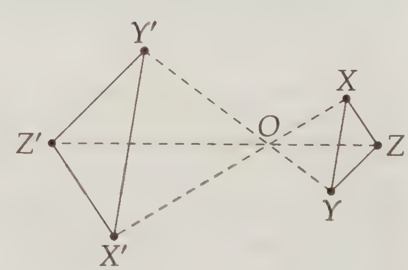
The last transformation we will explore is **dilation**. In plain English, a dilation results from stretching or shrinking a figure.



At right is an example of dilation. Point O is the **center of dilation**. The image of point A is the point A' on $\overrightarrow{OA}$ such that $OA'/OA = 2$. Similarly, the image of point B is the point B' on $\overrightarrow{OB}$ such that $OB'/OB = 2$, and the image of point C is the point C' on $\overrightarrow{OC}$ such that $OC'/OC = 2$. The image of the center of dilation is itself. We can think of the dilation as 'stretching' $\triangle ABC$ away from O.

In general, the image of a point P upon dilation with **scale factor** k and center O is the point P' on $\overrightarrow{OP}$ such that $OP' = k(OP)$ (when k is positive). In our example above, the scale factor is 2.

The scale factor need not be positive. At right we have an example in which the scale factor is -2. Here, the image of point X upon dilation about O is the point X' on the ray starting from O but *going in the opposite direction* from X such that $OX'/OX = 2$. In other words, point X' is the point such that $OX'/OX = 2$ and O is on $\overline{XX'}$.



Problems

Problem 16.11: Triangle $A'B'C'$ is the image of a dilation of triangle ABC about the centroid of triangle ABC. The dilation has a positive scale factor.

(a) Suppose $\triangle A'B'C'$ is entirely outside $\triangle ABC$. Is the scale factor greater than 1 or less than 1?

(b) Suppose $\triangle A'B'C'$ is entirely inside $\triangle ABC$. Is the scale factor greater than 1 or less than 1?

(c) What happens if the scale factor equals 1?

Problem 16.12: Square $A'B'C'D'$ is the image of a dilation of square $ABCD$ about A with scale factor 3. Suppose $AB = 2$.

(a) Find AA', AB', and BB'.

(b) Find BC' and CD'.

Problem 16.13: Let A' and B' be the images of points A and B, respectively, under a dilation with center O and positive scale factor k.

(a) Show that $\triangle AOB \sim \triangle A'O'B'$.

(b) Show that $\overline{AB} \parallel \overline{A'B'}$ and $A'B'/AB = k$.

Problem 16.11: Triangle $A'B'C'$ is the image of a dilation of $\triangle ABC$ about the centroid of triangle $\triangle ABC$.

(a) Suppose $\triangle A'B'C'$ is entirely outside $\triangle ABC$. Is the scale factor greater than 1 or less than 1?

(b) Suppose $\triangle A'B'C'$ is entirely inside $\triangle ABC$. Is the scale factor greater than 1 or less than 1?

(c) What happens if the scale factor equals 1?

Solution for Problem 16.11: Let G be the centroid of $\triangle ABC$. The centroid of a triangle is always inside the triangle, so G is inside $\triangle ABC$.

(a) Because A' is the image of A upon dilation about G, point A' is on $\overrightarrow{GA}$. If A' is outside the triangle on this ray, then it is beyond A on $\overrightarrow{GA}$. So, we must have $GA' > GA$, which means $GA'/GA > 1$. Therefore, the scale factor is greater than 1. An example of this case is shown at left below.

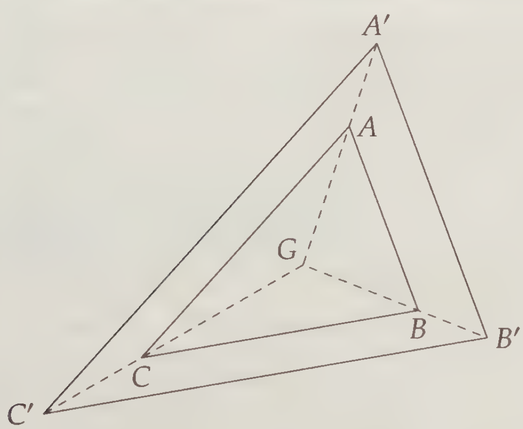


Figure 16.2: Scale Factor Greater Than 1

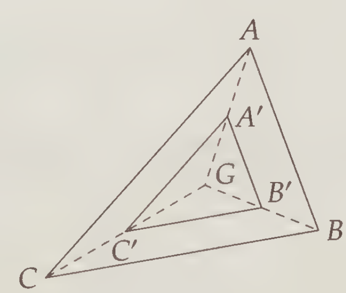


Figure 16.3: Scale Factor Less Than 1

(b) As before, we know that A' is on $\overrightarrow{GA}$. However, this time, A' is *inside* $\triangle ABC$, so we have $GA' < GA$. Therefore, we have $GA'/GA < 1$, so the scale factor is less than 1. An example of this case is shown at right above.

(c) If the scale factor equals 1, then the image of $\triangle ABC$ is itself.

□

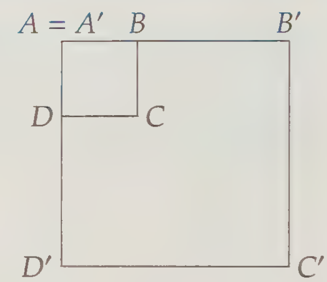
Important: If the scale factor of a dilation is greater than 1, then the dilation of a figure 'stretches' the figure. The resulting image is larger than the original figure. If the scale factor is positive but less than 1, then the dilation is a 'shrinking' of the original figure, which gives an image that is smaller than the original figure. A dilation of a figure with scale factor equal to 1 leaves the figure unchanged.

Problem 16.12: Square $A'B'C'D'$ is the image of a dilation of square $ABCD$ about A with scale factor 3. Suppose $AB = 2$.

(a) Find AA', AB', and BB'.

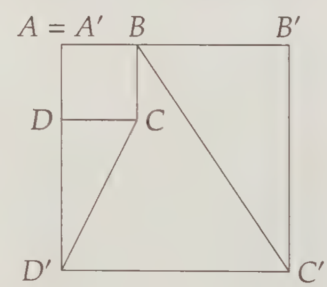
(b) Find BC' and CD'.

Solution for Problem 16.12:



(a) At right, we have the original square $ABCD$ and its image under the dilation with scale factor 3 and center A. The image of A upon a dilation about A is just itself. (The image of the center of a dilation is always the center itself.) Therefore, we have $AA' = 0$. (Yeah, that was a bit of a trick question!)

Because B' is the image of B upon a dilation about A with scale factor 3, we have $AB'/AB = 3$. In other words, B' is 3 times as far from A as B is. Since $AB = 2$, we have $AB' = 6$. Points A, B, and B' are collinear because B' is the image of B upon the dilation about A. So, we have $AB + BB' = AB'$. From this, we find that $BB' = AB' - AB = 4$.



(b) We seek two lengths and we have lots of right angles, so we look for right triangles on which we can use the Pythagorean Theorem. In the previous part, we found that $AB' = 6$, so square $AB'C'D'$ has side length 6. We also found that $BB' = 4$, so right triangle $BB'C'$ gives us

$$BC' = \sqrt{(BB')^2 + (CC')^2} = \sqrt{16+36} = 2\sqrt{13}.$$

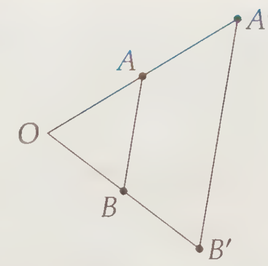
In the same way we found $BB' = 4$ in part (a), we can find that $DD' = 4$. Right triangle CDD' then gives us

$$CD' = \sqrt{CD^2 + D'D^2} = \sqrt{4+16} = 2\sqrt{5}.$$

□

This problem raises an interesting question about dilations. Points B' and C' are the images of B and C under the dilation, and we see that $\overline{B'C'} \parallel \overline{BC}$, and $B'C'/BC$ equals the scale factor. Note that we didn't prove these facts; we were told to assume that the image of $ABCD$ upon dilation is a square. Let's see if we can prove that these observations are not a coincidence.

Problem 16.13: Let A' and B' be the images of points A and B, respectively, under a dilation with center O and positive scale factor k. Show that $\overline{AB} \parallel \overline{A'B'}$ and $A'B'/AB = k$.



Solution for Problem 16.13: We want to prove something about a ratio, and we want to show that two lines are parallel. The dilation also gives us information about ratios. This looks like a job for similar triangles. Our diagram at right shows the set-up where $k > 1$. (The case $k < 1$ can be solved with the same steps we will use for $k > 1$.)

We're looking for similar triangles, and we need to find some way to use the fact that A' and B' are the images of A and B under the same dilation about O. The dilation information tells us that $\overrightarrow{OA}$ passes through A' and $\overrightarrow{OB}$ passes through B', as shown. But that's not enough! The dilation also gives us information about ratios. Specifically, it tells us that OA'/OA and OB'/OB both equal the scale factor, so

$$\frac{OA'}{OA} = \frac{OB'}{OB} = k.$$

Since we also have $\angle AOB = \angle A'OB'$, we have $\triangle AOB \sim \triangle A'OB'$ by SAS Similarity. From this triangle similarity, we have $A'B'/AB = OA'/OA = k$. We also have $\angle OAB = \angle OA'B'$, which tells us that $\overline{AB} \parallel \overline{A'B'}$.
□

Notice that we did not prove in Problem 16.13 that the image of $\overline{AB}$ is $\overline{A'B'}$. We only proved that $A'B'/AB = k$ and $\overline{AB} \parallel \overline{A'B'}$. You'll have a chance to prove that the image of a segment under dilation is a segment as a challenging Exercise. However, the parallel lines suggest something very useful about dilations.

Important: A figure and its image upon dilation are similar. The ratio of corresponding sides of the figure and its image equals the scale factor of the dilation.

You'll have a chance to prove this for triangles as an Exercise. This is just the beginning of interesting dilation properties. In Art of Problem Solving's *Intermediate Geometry*, we will build on these basics to find powerful and intriguing applications of dilation.

Exercises

16.4.1 In $\triangle XYZ$, we have $XY = 3$, $YZ = 4$, and $XZ = 5$. Suppose $\triangle X'Y'Z'$ is the image of $\triangle XYZ$ under a dilation with scale factor 4.

(a) What is $X'Y'$?

(b) What is the area of $\triangle X'Y'Z'$?

(c) Is it possible to determine XX' with the information given in the problem?

16.4.2 Point Q is the image of point P under a dilation with center O and scale factor 5. If $PQ = 20$, what is OP?

16.4.3 Show that if A', B', and C' are the images of A, B, and C, respectively, under a dilation with center O and scale factor k, then $\triangle ABC \sim \triangle A'B'C'$. (You can assume k is positive.) **Hints:** 6

16.4.4★ In this problem, we show that the image of a segment upon a dilation is itself a segment. Suppose that A' and B' are the images of A and B, respectively, upon dilation about O with scale factor k. (You can assume $k > 0$; the proof for $k < 0$ is very similar.)

(a) Let X be a point on $\overline{AB}$, and let X' be the image of X under the dilation. Show that $\angle OXA = \angle OX'A'$ and $\angle OXB = \angle OX'B'$.

(b) Use part (a) to show that $\angle A'X'B' = 180^\circ$. Why does this mean that the image of each point on $\overline{AB}$ is a point on $\overline{A'B'}$?

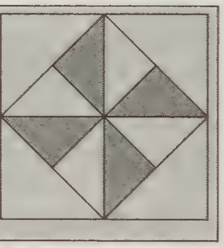
(c) Show that each point on $\overline{A'B'}$ is the image of some point on $\overline{AB}$ under the dilation. Why is this step necessary to complete the proof?

16.5 Changing the Question

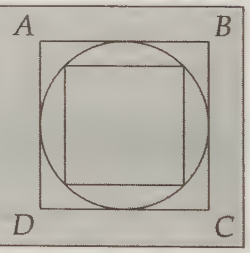
Sometimes problems can be quickly solved by rearranging the problem or by considering special cases of the problem. Typically this requires a little creativity and flexibility.

Problems

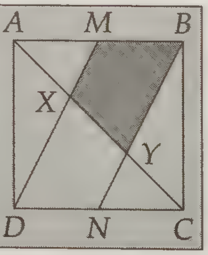
Problem 16.14: The side length of the largest square at right is 10 inches. The midpoints of the sides of the largest square are connected to form a smaller square. Opposite vertices and the midpoints of opposite sides of the smaller square are then connected as shown. What is the number of square inches in the area of the shaded region? *(Source: MATHCOUNTS)*



Problem 16.15: A circle is inscribed in a large square and circumscribed about a smaller square. The area of the larger square is 8 square meters. What is the number of square meters in the area of the smaller square? *(Source: MATHCOUNTS)*



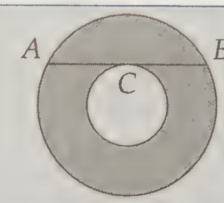
Problem 16.16: In the figure, $ABCD$ is a square with side length 1, and M and N are the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. Find the area of the shaded region. *(Source: Mandelbrot)*



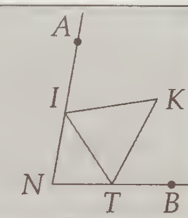
Extra! *Every new body of discovery is mathematical in form, because there is no other guidance we can have.*

–Charles Darwin

Problem 16.17: In the diagram we have two concentric circles, and chord $\overline{AB}$ of the large circle is tangent to the smaller circle. Given that $AB = 8$, find the area of the region between the two circles.

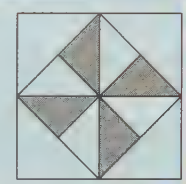


Problem 16.18: In the diagram, $\angle INT = 80^\circ$. The angle bisectors of two of the exterior angles of $\triangle TIN$ meet at point K, as shown. Compute $\angle IKT$.



Since rotations, reflections, and translations don't alter lengths or areas, we can often use them to simplify problems. Here are several examples.

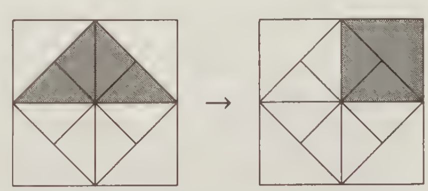
Problem 16.14: The side length of the largest square at right is 10 inches. The midpoints of the sides of the largest square are connected to form a smaller square. Opposite vertices and the midpoints of opposite sides of the smaller square are then connected as shown. What is the number of square inches in the area of the shaded region? *(Source: MATHCOUNTS)*



Solution for Problem 16.14: Solution 1: Each of the shaded regions is a 45-45-90 triangle, so we can find the area of each by finding one of the legs. One leg of each triangle is half a diagonal of one of the 5×5 quarters of the largest square. Therefore, each triangle has a leg of length $5\sqrt{2}/2$, so each triangle has an area of $(5\sqrt{2}/2)^2/2 = 25/4$. There are four such triangles, so the total shaded area is 25.

Solution 2: Instead of finding the leg lengths of each little triangle to find the area, we could have instead noted that each of these little triangles is 1/4 of one of the 5×5 squares. Hence, each has area 25/4, so the total shaded area is $4(25/4) = 25$.

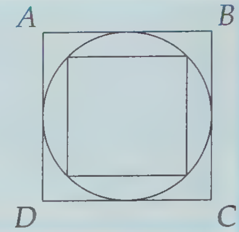
Solution 3: Since the area of a triangle doesn't change if we slide, spin, or flip it, we can manipulate the shaded pieces to make a region whose area we know how to find easily. We first slide the bottom triangles up to complete a large shaded isosceles right triangle. Then, we can slide half of this triangle to the right to complete one of the 5×5 squares as shown. Clearly, then, the area of the shaded region is $5^2 = 25$. □



Concept: In addition to our strategy of chopping areas into easy-to-handle pieces, we can also sometimes rearrange them to form an easy-to-handle whole.

Extra! **Fagnano's Problem** asks what is the triangle of smallest perimeter that can be formed by connecting a point on each side of an acute triangle. The answer and a dazzling solution is on page 425.

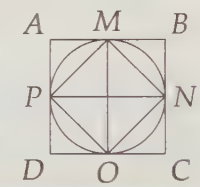
Problem 16.15: A circle is inscribed in a large square and circumscribed about a smaller square. The area of the larger square is 8 square meters. What is the number of square meters in the area of the smaller square?



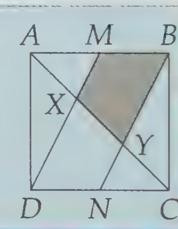
Solution for Problem 16.15: As with the last problem, we can work through several steps to the answer, or we can make a little transformation to simplify the problem.

Solution 1: Since our large square has area 8, each side has length $\sqrt{8} = 2\sqrt{2}$. Because the circle is tangent to all four sides of the large square, the diameter of the circle equals the side length of the large square. But the diagonal of the small square is also a diameter of the circle, so the diagonal of the small square has length $2\sqrt{2}$. Therefore, its sides are each $(2\sqrt{2})/\sqrt{2} = 2$ and its area is 4.

Solution 2: We can spin the small square without changing its area, so we spin it so that its vertices are the points where the circle is tangent to the large square. When we draw the diagonals of the small square $MNOP$, we divide the large square into four equal pieces. Half of each of these pieces is inside $MNOP$, so $[MNOP] = [ABCD]/2 = 4$. □



Problem 16.16: In the figure, $ABCD$ is a square with side length 1, and M and N are the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. Find the area of the shaded region. *(Source: Mandelbrot)*

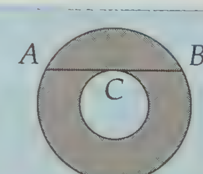


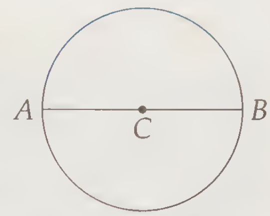
Solution for Problem 16.16: $MBYX$ is a trapezoid (why?), but it's not clear how we'll find the height or either of the bases. It's also not clear how we could chop $MBYX$ into easy-to-handle pieces. However, it looks like $MBYX$ is congruent to $NDXY$. We don't have any tools for proving quadrilaterals are congruent like we do for triangles. But if we can transform one quadrilateral into the other with flips, spins, or slides, then we'll know they are congruent.

We might think to reflect $\triangle ABC$ over $\overline{AC}$, but that won't map M to N. However, if we rotate $\triangle ABC$ 180° about the center of the square, we'll have the transformation we want. (You'll prove this as an Exercise.) This rotation maps $MBYX$ to $NDXY$, showing that these quadrilaterals are indeed congruent. Therefore, the area of $MBYX$ is half the area of $MBND$. $MBND$ is a parallelogram with base $ND = 1/2$ and height $BC = 1$, so its area is $1/2$. Therefore, the shaded region has area $(1/2)/2 = 1/4$. □

Transformations aren't the only ways we can manipulate diagrams to help us with problems.

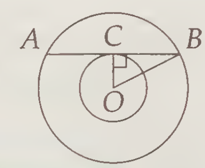
Problem 16.17: In the diagram we have two concentric circles, and chord $\overline{AB}$ of the large circle is tangent to the smaller circle. Given that $AB = 8$, find the area of the region between the two circles.





Solution for Problem 16.17: The problem doesn't tell us what the radius of either the larger circle or the smaller circle is. We only know the length of that one tangent chord. Therefore, the problem suggests that the shaded region between the circles has the same area *no matter what these two radii are*, as long as that tangent chord has length 8. We can get a quick answer for what that area is by picking a special case that's easy to work with. The simplest is to shrink the smaller circle down to a point. Then, our tangent chord becomes a diameter of the larger circle, and our shaded region is this whole circle. The area of this region, then, is just $(8/2)^2\pi = 16\pi$.

Notice that we have **not** proved that the shaded region is always 16π when our tangent chord is 8 units long. We've only found it for one case. However, if the area is always the same, we know it's 16π. To prove that it's always the same, we have to find some way to get information about the radii of the two circles. We have a tangent line, so naturally we build a right triangle by drawing radii as shown below.

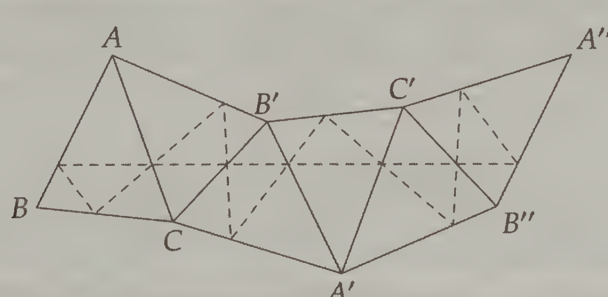


Since $\overline{OC} \perp \overline{AB}$, $\overline{OC}$ bisects chord $\overline{AB}$. Therefore, $BC = 4$. From right triangle $\triangle OBC$, we have $OB^2 - OC^2 = BC^2 = 16$. Since $\overline{OB}$ and $\overline{OC}$ are the radii of our circles, the area we seek is $(OB^2)\pi - (OC^2)\pi = (OB^2 - OC^2)\pi = 16\pi$. □

Concept: When working on a problem with a variable set-up, consider the extreme possibilities.

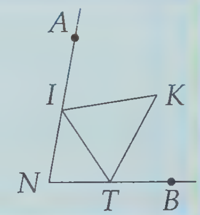
WARNING!! Before you start examining those extremes, make sure you really can vary the set-up. If you vary the set-up in a way that violates information given in the problem, then your conclusions don't necessarily apply to the original problem! For example, if we are asked to find the area of a triangle with sides of length 13, 14, and 15, we can't simply pretend two of the sides are perpendicular to get the answer!

Extra! In an acute triangle, the triangle formed by connecting the feet of the altitudes is the triangle with smallest perimeter that can be formed by connecting a point on each side of the triangle. This was originally proved by Fagnano using calculus, but mathematician HA Schwarz came up with the amazing proof without words offered below, using the property of reflection repeatedly. See if you can figure out how this proof works!



Let's try this technique on one more problem.

Problem 16.18: In the diagram, $\angle INT = 80°$. The angle bisectors of two of the exterior angles of $\triangle TIN$ meet at point K, as shown. Compute $\angle IKT$.



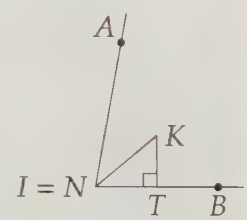
Solution for Problem 16.18: Since $\angle ITN$ can vary, we'll call it x. From $\triangle INT$, we find that $\angle TIN = 180° - 80° - x = 100° - x$. Therefore, $\angle AIT = 180° - \angle TIN = 80° + x$ and $\angle ITB = 180° - x$. Since $\overline{IK}$ and $\overline{TK}$ bisect these angles, we have $\angle KIT = 40° + x/2$ and $\angle KTI = 90° - x/2$. Finally, we can use $\triangle KIT$:

$$\angle K + \angle KIT + \angle KTI = 180°.$$

Substitution gives $\angle K + (40° + x/2) + (90° - x/2) = 180°$, from which we find $\angle K = 50°$.

Notice that in this problem, we can place points I and T anywhere on the sides of $\angle ANB$ without violating the problem. We only need $\angle INT = 80°$. Therefore, to quickly check the answer we found above, we can consider the extreme case of putting I at N.

When I is at N, $\angle AIT$ is $\angle N$ and $\overline{IK}$ is the bisector of $\angle N$. Therefore, $\angle KIT = 80°/2 = 40°$. Furthermore, $\angle ITB$ is a straight angle, so its bisector is perpendicular to $\overline{IT}$. Since $\triangle KIT$ is a right triangle, we then have $\angle IKT = 90° - 40° = 50°$. Notice that this is not a proof that $\angle IKT$ is always $50°$ – it's just a quick way to check the answer we found earlier. □

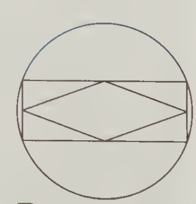


Concept: Considering extremes in problems with varying set-ups is a great way to check your answer. But remember this warning:

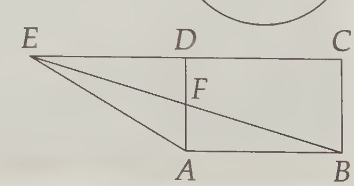
WARNING!! Considering extremes does not constitute a proof! You still have to show your answer holds in all cases.

Exercises

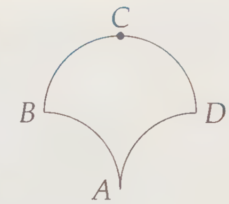
16.5.1 The rectangle at right is inscribed in the circle, and a rhombus is inscribed in the rectangle by connecting the midpoints of the sides of the rectangle as shown. If the radius of the circle is 4 ft, how many feet are in the perimeter of the rhombus? *(Source: MATHCOUNTS)*



16.5.2 In the diagram, D is the midpoint of $\overline{EC}$, and the area of $\triangle EDF$ is 4 square centimeters. What is the number of square centimeters in the area of rectangle $ABCD$? *(Source: MATHCOUNTS)*



16.5.3 Three circular arcs of radius 5 units bound the region shown. Arcs $\widehat{AB}$ and $\widehat{AD}$ are quarter-circles, and $\widehat{BCD}$ is a semicircle. What is the area, in square units, of the region?



16.5.4 In Problem 16.16, we asserted that $DNYX$ is the image of $BMXY$ upon rotation about the center of the square. Complete our solution by proving this fact. **Hints:** 419

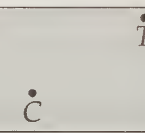
16.5.5★ In trapezoid $ABCD$, $\overline{AB}$ and $\overline{CD}$ are perpendicular to $\overline{AD}$, with $AB + CD = BC$ and $AD = 7$. What is $AB \cdot CD$? *(Source: AMC 10)* **Hints:** 497

16.6 Construction: Transformations

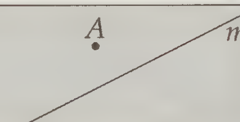
As a thorough test of your spinning and flipping abilities, you'll now have the chance to spin and flip some points given only a ruler and compass.

Problems

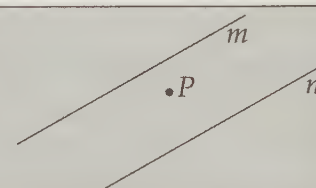
Problem 16.19: Given are points C and T. Construct point U such that U is the image of point T under a 30° rotation about C.



Problem 16.20: Given is point A and line m. Construct point B such that B is the image of point A upon reflection over m.



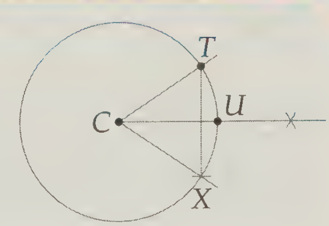
Problem 16.21: Given are parallel lines m and n and point P between the lines. Construct an equilateral triangle such that one vertex is P, one vertex is on m, and one vertex is on n.



Let's go for a little spin.

Problem 16.19: Given are points C and T. Construct point U such that U is the image of point T under a 30° rotation about C.

Solution for Problem 16.19: Since U is the image of T rotated 30° about C, we must have $CT = CU$ and $\angle TCU = 30°$. Since $CT = CU$, U must be just as far from C as T is. Therefore, U must be on the circle with center C and radius CT. So, we draw that circle. Since $\angle TCU = 30°$, U must be on a ray through C that forms a 30° angle with $\overline{CT}$. We construct this ray by first constructing an equilateral triangle, $\triangle CTX$, then bisecting $\angle C$ of this triangle.

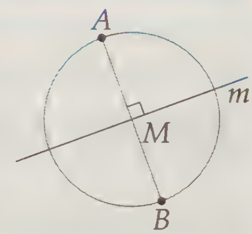


The intersection of this angle bisector and our circle is the desired point U such that $CT = CU$ and $\angle TCU = 30°$. Therefore, U is the image of T under a $30°$ rotation about C. □

Concept: In construction problems in which we must locate a single point, our general strategy is to find two figures, usually circles or lines, which must contain the point. The intersection of these two figures is the point we want.

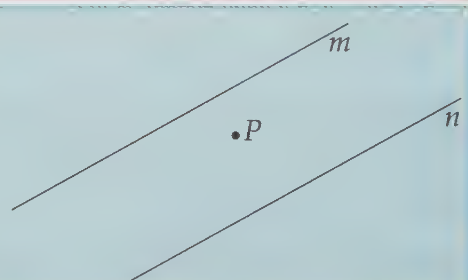
You'll flip over this one...

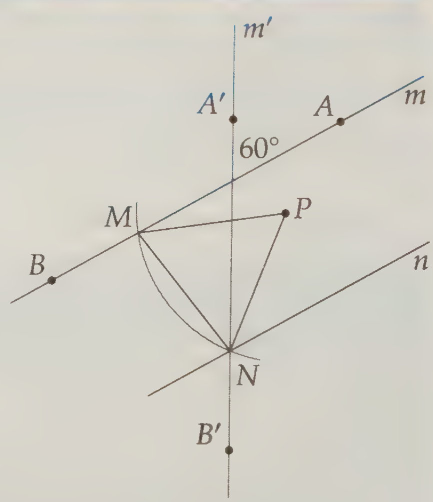
Problem 16.20: Given is point A and line m. Construct point B such that B is the image of point A upon reflection over m.



Solution for Problem 16.20: Since B is the image of point A upon reflection over m, we must find the point B such that m is the perpendicular bisector of $\overline{AB}$. Since $\overline{AB} \perp m$, B must be on the line through A perpendicular to m (see Problem 6.24 for this construction). Let this perpendicular line meet m at M. Since m bisects $\overline{AB}$, we must have $AM = BM$. Therefore, B is on the circle with center M and radius AM. Where this circle meets our earlier line through A is the desired point B such that m is the perpendicular bisector of $\overline{AB}$. □

Problem 16.21: Given are parallel lines m and n and point P between the lines. Construct an equilateral triangle such that one vertex is P, one vertex is on m, and one vertex is on n.





Solution for Problem 16.21: We seek point M on m and point N on n such that $\triangle PMN$ is equilateral. If we can find either M or N, we can easily construct our equilateral triangle, since once we have a side of the triangle, we can make the whole thing. Therefore, we'll focus on finding N.

We already have one figure that contains N – line n. Therefore, we only need to find one more line or circle that contains N. Point P and line n are not enough to find N; we need more. The only other piece of information we have is that M is on m, so we need to use m to find N.

Since $\triangle PNM$ must be equilateral, we know that a $60°$ rotation about P maps M to N. But we don't know where M is! However, we do know that M is on m, so the image of M upon rotating $60°$ about P must be on the image of m when m is rotated $60°$ about P. Since this image of point M is point N, all we have to do is rotate m $60°$ about point P to have another line that contains N!

To rotate m 60° about P, we simply pick two points on m and rotate them both. Connecting the images of these points upon rotation gives us m', the image of m upon 60° rotation about P. Since N is on both m' and n, where these lines meet gives us N! Now we can easily use $\overline{PN}$ to construct the equilateral triangle. □

One key step in finding our solution was thinking about what information we hadn't used yet, thereby focusing on using line m to find N. When we focused on how to use m to get N, we had to think about how to relate M to N given point P. This led us to a new problem solving tactic:

> **Concept:** When stuck on a problem involving equilateral triangles, think about 60° rotations!

Exercises

16.6.1 Given points A, A', and Q, construct point Q' such that Q' is the image of Q under the same translation that maps A to A'.

16.6.2 Given a point P and two intersecting lines, ℓ and m, construct the image of P upon reflection over ℓ. Let P' be the image of P upon reflection over ℓ. Construct the image of P' upon reflection over m. Try this for a variety of points (reflecting the point over ℓ, then reflecting its image over m). Notice anything interesting?

16.7 Summary

Definitions: In this chapter we explored three important geometric transformations. A figure's **image** under a transformation is the result of applying the transformation to that figure. We also say that a transformation **maps** a figure to its image. A **fixed point** of a transformation is a point that is its own image. Four important transformations are described below.

- When we slide a figure a given amount in a specified direction, we perform a **translation**.
- When we apply a **rotation** to a figure, we spin it by some angle about some point. The point is the **center** of rotation.
- The image of a figure upon **reflection** over a line is what results when the figure is flipped over the line. If a figure is its own image upon reflection over a line, the line is a **line of symmetry** of the figure.
- Loosely speaking, when we apply a **dilation** about a point O to a figure, we stretch the figure away from O or shrink it towards O.

Translations, rotations, and reflections preserve length, area, and angles, which means that the lengths, areas, and angles in a figure's image equal the corresponding quantities in the figure.

Problem Solving Strategies

Concepts:

- Understanding transformations is often more about learning what stays the same rather than what changes. Whenever you are learning about a new transformation, focus on what stays the same.
- If you can't solve a problem, try thinking of a similar, simpler problem and solve that. Then try to use your solution to the simpler problem to solve the harder problem.
- In addition to our strategy of chopping areas into easy-to-handle pieces, we can also sometimes rearrange them to form an easy-to-handle whole.
- When working on a problem with a variable set-up, consider the extreme possibilities. Considering extremes is also a great way to check your answer.
- In construction problems in which we must locate a single point, our general strategy is to find two figures, usually circles or lines, which must contain the point. The intersection of these two figures is the point we want.
- When stuck on a problem involving equilateral triangles, think about 60° rotations!

Things To Watch Out For!

WARNING!!

- Before you start examining extremes in a problem, make sure you really can vary the set-up. If you vary the set-up in a way that violates information given in the problem, then your conclusions don't necessarily apply to the original problem!
- Considering extremes does not constitute a proof! You still have to show your answer holds in all cases.

Extra! Suppose we have a circle $\odot O$ with radius r. The geometric transformation called **inversion** maps each point P other than O to the point P' on $\overrightarrow{OP}$ such that $OP \cdot OP' = r^2$. To make inversion work, we define the **point at infinity** to be the image of the center of inversion, O. Experiment with inversion yourself by finding out what happens to various points upon inversion. What happens to points inside the circle? Points outside the circle? Points on the circle? After playing with inversion for a while, turn to page 434 for more suggestions for exploring inversion.

Review Problems

16.22 $ABCDEFGH$ is a regular octagon. A translation maps G to D. To what point does this same translation map H?

16.23 $OPQRSTUVWXYZ$ is a regular dodecagon.

(a) A rotation of θ_1 degrees about U maps V to T. Given $\theta_1 < 180°$, find θ_1.

(b) A rotation of θ_2 degrees about U maps X to R. Given $\theta_2 < 180°$, find θ_2.

(c) A rotation of θ_3 degrees about U maps W to S. Given $\theta_3 < 180°$, find θ_3.

(d) A rotation of θ_4 degrees about U maps Y to Q. Given $\theta_4 < 180°$, find θ_4.

(e) A rotation of θ_5 degrees about U maps Z to P. Given $\theta_5 < 180°$, find θ_5.

16.24 All of the angles of octagon $GHIJKLMN$ are $135°$. $GH = IJ = KL = MN = 1$ and $HI = JK = LM = GN = 2$. How many lines of symmetry does $GHIJKLMN$ have?

16.25

(a) $\overline{GH}$ and $\overline{GI}$ are line segments such that $GH = GI$. Must there exist a reflection that maps $\overline{GH}$ to $\overline{GI}$?

(b) $\overline{XY}$ and $\overline{AB}$ are line segments such that $XY = AB$. Must there exist a reflection that maps $\overline{XY}$ to $\overline{AB}$?

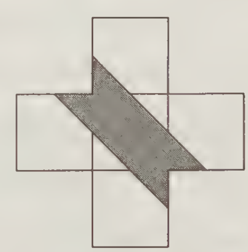
16.26 $ABCD$ is a square. M is the midpoint of $\overline{CD}$.

(a) Is there a rotation about M that maps A to B?

(b) Is there a rotation about M that maps A to C?

(c) Is there a reflection over some line that maps M to A?

16.27 In the figure at right, the shaded region is formed by drawing two parallel segments that connect the midpoints of congruent squares. Each square has side length 1 centimeter. What is the area of the shaded region? *(Source: MATHCOUNTS)*



16.28 The image upon reflection of regular hexagon $QWERTY$ over $\overleftrightarrow{WE}$ is regular hexagon $Q'W'E'R'T'Y'$.

(a) Why must W' and E' be W and E, respectively?

(b) Find $\angle QWQ'$.

(c)★ Find YT' if $QW = 6$. **Hints:** 435

16.29 Line ℓ is a line of symmetry of $\triangle ABC$.

(a) Must line ℓ pass through a vertex of $\triangle ABC$? Why or why not?

(b) Is it possible for $\triangle ABC$ to be scalene (i.e., all sides have different lengths)? Why or why not?

16.30

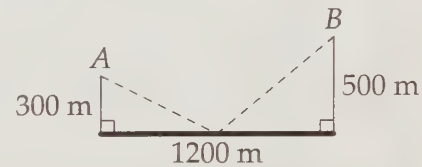
(a) Point R is the image of point S under a dilation with center T and scale factor 3. If $TR = 18$, what is TS?

(b) Point E is the image of point D under a dilation with center O and scale factor -3. If $ED = 9$, what is OD?

(c) Point X is the image of point Y under a dilation with center A and scale factor 0.25. If $XY = 20$, what is AY?

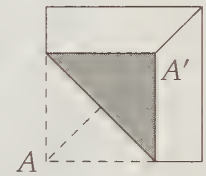
16.31 The image of circle $\odot O$ under a dilation about P with scale factor 4 is $\odot O'$. The area of $\odot O'$ is 48π. What is the area of $\odot O$?

16.32 The rules for a race require that all runners start at A, touch any part of the 1200-meter wall, and stop at B. What is the number of meters in the minimum distance a participant must run?



16.33 Lines ℓ and m are perpendicular at point X. Point A is not on either line. B is the image of A upon reflection over ℓ, and C is the image of B upon reflection over m. Show that $\angle ABC = 90^\circ$ and $XA = XB = XC$. Must $\overline{AC}$ pass through X?

16.34 A square sheet of paper has area 6 cm^2. The front is white and the back is shaded. When the sheet is folded so that point A rests on the diagonal as shown, the visible shaded area is equal to the visible white area. How many centimeters is A' from its original position, A? *(Source: MATHCOUNTS)* **Hints:** 335



16.35 A rotation about A maps X to Y. A different rotation about X maps A to Y. Show that $\triangle AXY$ is equilateral.

16.36 Point O is inside quadrilateral $ABCD$. A rotation of 180° about O maps the quadrilateral $ABCD$ to itself, meaning each vertex of the image is also a vertex of the original quadrilateral.

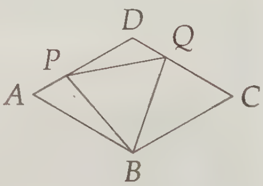
(a) Must $ABCD$ be a parallelogram?

(b) Must $ABCD$ be a rectangle?

(c) Must $ABCD$ be a square?

16.37 The image of point A upon a 90 degree clockwise rotation about point B is point C. The image of C upon a 90 degree rotation clockwise about point D is point A. Prove that $ABCD$ is a square.

16.38 Let $ABCD$ be a rhombus with $\angle BAD = 60^\circ$. Let P and Q be points on $\overline{AD}$ and $\overline{CD}$, respectively, such that $\angle PBQ = 60^\circ$. Find the other two angles of triangle PBQ.



16.39 $\triangle ABC$ has $AB = 3$, $BC = 4$, and $AC = 5$.

(a) A cone is formed by rotating the triangle around $\overline{AB}$. What is the volume of this cone?

(b) A cone is formed by rotating the triangle around $\overline{BC}$. What is the volume of this cone?

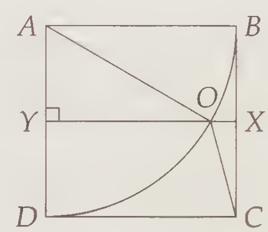
(c)★ A solid is formed by rotating the triangle around $\overline{AC}$. What is the volume of this solid?

16.40 Show that if X' and Y' are the images of X and Y under a dilation with center O and scale factor k, where k is *negative*, then we have $\overline{X'Y'} \parallel \overline{XY}$ and $X'Y'/XY = |k|$.

Challenge Problems

16.41 $\overline{CD}$ is the image of $\overline{AB}$ under some rotation. $\overline{AB}$ and $\overline{CD}$ have the same midpoint, M. Must M be the center of the aforementioned rotation that maps $\overline{AB}$ to $\overline{CD}$? **Hints:** 520

16.42 You might recognize the diagram at right from Problem 8.22 on page 226. It is indeed the same problem. $ABCD$ is a square. The circle with center A and radius AB intersects the perpendicular bisector of $\overline{AD}$ in two points, of which O is the one inside the square. Use the principles we have explored in this chapter to quickly find $\angle AOC$. **Hints:** 241, 180



16.43 Consider the octagon of Problem 16.24.

(a) Show that there is a point inside the octagon that is equidistant from all 8 vertices of the octagon. **Hints:** 204

(b) Show that a 90° rotation about the point found in part (a) maps the octagon to itself. **Hints:** 246

16.44 $\triangle ABC$ with vertices $A(2,4)$, $B(6,4)$ and $C(4,10)$ is graphed in a coordinate plane. What will the sum of the abscissas (x-coordinates) of the vertices be when $\triangle ABC$ is reflected over the line $x = 8$? *(Source: MATHCOUNTS)*

16.45 The image of the incircle of equilateral triangle $\triangle ABC$ upon a dilation about the center of the circle with scale factor k is the circumcircle of $\triangle ABC$. Find k.

16.46 Use a clever dissection of the section of regular octagon $ABCDEFGH$ that is traced in bold at left below to prove that $[ABCDEFGH] = (\sqrt{2})[ACEG]$. ($O$ is the center of the octagon.) **Hints:** 159, 410

Figure 16.4: Diagram for Problem 16.46

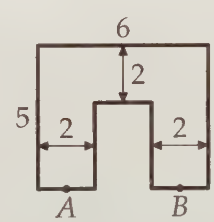


Figure 16.5: Diagram for Problem 16.47

16.47 A laser beam is fired from point A in the U-shaped room at right above. It reflects from each wall as from a perfect mirror. What is the minimum distance the beam travels before hitting a target at point B? (A and B are the midpoints of their respective sides.) *(Source: Mandelbrot)*

16.48 Suppose $\overline{AB} \parallel \overline{A'B'}$ and $A'B'/AB = k$, where $k \neq 1$. In this problem, we prove that there is a dilation that maps A to A' and B to B'.

(a) Why must $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ intersect?

(b) Let P be the intersection of $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$. Show that $\triangle PAB \sim \triangle PA'B'$.

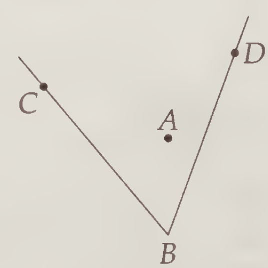
(c) Show that there is a dilation with center P that maps A to A' and B to B'.

16.49 Let $ABCD$ be a square, and let M and N be the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. Prove that $\overline{CM} \perp \overline{DN}$. **Hints:** 445

16.50 The center of the cue ball on my rectangular pool table is directly above point A on the table. I wish to bounce the cue ball off a rail such that after it bounces off the rail, the center of the ball will pass directly over point B on the table. The radius of the cue ball is 1 in. A is 6 inches from the rail, and B is 9 inches from the rail. A is 5 inches from B. How far from the nearest point on the rail to point B do I want the cue ball to hit the rail? **Hints:** 278, 234

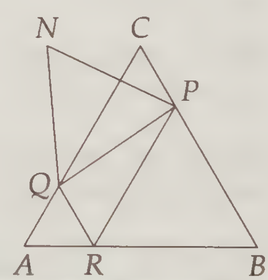
16.51★ What is the total length of the shortest path that goes from $(0,4)$ to a point on the x-axis, then to a point on the line $y = 6$, then to $(18,4)$? **Hints:** 262, 472

16.52★ I am standing at point A in the diagram at right. I wish to walk to a point on $\overrightarrow{BC}$, then to a point on $\overrightarrow{BD}$, then back to my starting point. Construct with straightedge and compass the shortest such path I can take. **Hints:** 338, 370



16.53★ Regular octahedron $ABCDEF$ has volume 360 cubic units. What is the volume of that portion of the octahedron that consists of all points that are closer to A than to any other vertex? **Hints:** 311

16.54★ In the diagram at right, $\triangle ABC$, $\triangle PBR$, $\triangle AQR$, and $\triangle PNQ$ are equilateral. Prove that $BQ = RN = AP$. *(Source: Mandelbrot)* **Hints:** 320

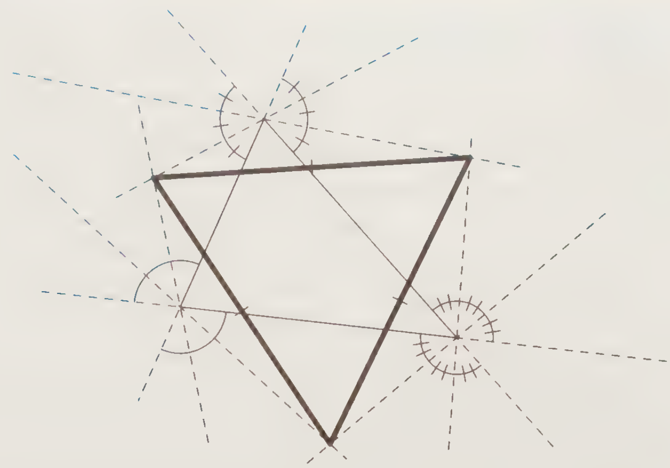


16.55★ Prove that the image of a circle under rotation about any point is also a circle. **Hints:** 590

Extra! Back on page 430, we introduced the geometric transformation called inversion. Here are some more questions to guide further exploration of inversion:

- What is the image of $\odot O$ upon inversion with respect to $\odot O$?
- What is the image of a line that passes through O? Of a line that does not pass through O?
- What is the image of a circle that passes through O? Of a circle that does not pass through O?

We'll explore these properties and more in later books in the Art of Problem Solving series.



Morley's Second Triangle

New facts often trigger new ideas. – Alex F. Osborn

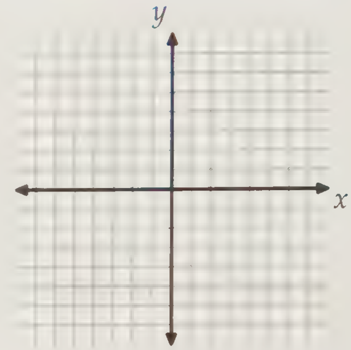
CHAPTER 17 Analytic Geometry

Much of the mathematics developed by the ancient Greeks was geometry. In fact, even much of the number theory and algebra that they developed was expressed in geometric terms. However, by the 16^{th} or 17^{th} century A.D., algebra was recognized as an extremely important field of mathematics in its own right, possibly equaling, or even exceeding, geometry.

In 1637, the great mathematician and philosopher René Descartes brought these two great fields of mathematics together when he described a general method to represent geometric figures with algebraic equations. This combination of algebra and geometry is often referred to as **analytic geometry**. In this chapter, we review how to describe lines and circles with equations, then we apply these methods to both algebra and geometry problems.

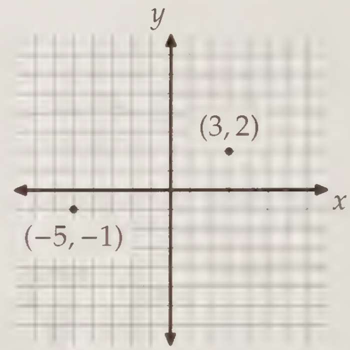
WARNING!! If you have not studied graphing lines and circles in the Cartesian plane, you may find the material in this chapter extremely difficult. If so, we recommend studying the fundamentals of graphing lines and circles in our *Introduction to Algebra* textbook.

Moreover, the algebraic manipulation required for parts of this chapter is considerably more complicated than in the rest of this text. Specifically, if you are not comfortable with completing the square, you may wish to study it in *Introduction to Algebra* (or another text) before returning to this chapter.



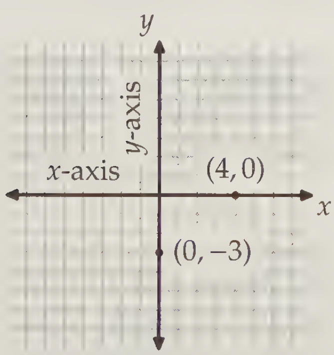
Descartes's great insight that unified geometry and algebra has become so important that we still use Descartes's name to describe the result. We use the **Cartesian plane** to describe geometric figures with algebraic equations.

At right is the Cartesian plane. The center of the plane, where the bold lines meet, is called the **origin**. On the Cartesian plane each point is represented by an **ordered pair** of numbers. These numbers denote the position of the point *relative to the origin*.



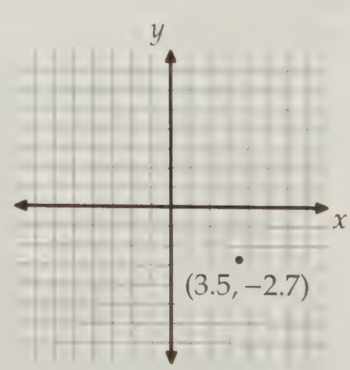
For example, we denote the point that is 3 steps to the right and 2 steps up from the origin with the ordered pair $(3, 2)$. The "ordered" part is very important! The first number always denotes how far the point is to the right (or left) of the origin, and the second number denotes how far the point is above (or below) the origin. We call the two numbers in an ordered pair the **coordinates** of the point. By convention, we call the horizontal (left-right) coordinate the **x-coordinate** and we call the vertical (up-down) coordinate the **y-coordinate**.

As you might have guessed, the x-coordinate of a point is negative when the point is to the left of the origin, and the y-coordinate is negative when the point is below the origin. For example, the point $(-5, -1)$ shown in the diagram is 5 steps to the left and one step below the origin. The point $(0, 0)$ is 0 steps to the right and 0 steps up from the origin. In other words, $(0, 0)$ is the origin itself.



When the point is neither above nor below the origin, its y-coordinate is 0. Such a point is directly to the left or right (or on) the origin. Therefore, the point must be on the bold horizontal line in the diagram at left. We call this line the **x-axis**. Similarly, the vertical line consisting of points directly above or below (or on) the origin is called the **y-axis**. We usually label the x-axis and y-axis with an x and a y, respectively, as shown in each of our diagrams.

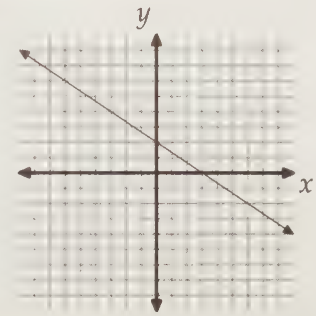
Finally, we aren't restricted to using integers for our coordinates. We can plot any point represented by an ordered pair of two real numbers on the Cartesian plane. For example, the diagram at right depicts the point $(3.5, -2.7)$, which is 3.5 steps to the right and 2.7 steps below the origin. Points that have integers for both coordinates are called **lattice points**.



17.1 Lines

In this section, we review how equations can be used to represent lines. We cover this material in much more detail in *Introduction to Algebra*. If you are unfamiliar with using equations to represent lines, you may wish to study the fundamentals covered in *Introduction to Algebra* before continuing.

The **graph** of an equation that has x and y as variables consists of all points (x, y) on the Cartesian plane that satisfy the equation. For example, the graph of the equation $2x + 3y = 6$ is shown at right. As we see, the graph is a line. Indeed, the graph of any equation of the form $Ax + By = C$, where A, B, and C are constants and A and B are not both 0, is a line. As a result, such an equation is often called a **linear equation**.



The points where a graph intersects the x-axis are called the ***x*-intercepts** of the graph, and the points where a graph intersects the y-axis are called the ***y*-intercepts** of the graph. For example, the only x-intercept of the graph of $2x + 3y = 6$ at right is $(3, 0)$ and the only y-intercept of the graph is $(0, 2)$.

One way we describe graphed lines is by indicating the 'direction' of the line, which we call the **slope** of the line. We define the slope, m, of the line that passes through the points (x_1, y_1) and (x_2, y_2) to be

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

A line with positive slope goes upward as it goes from left to right, and a line with negative slope goes downward as it goes from left to right.

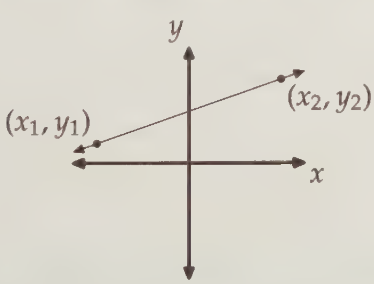


Figure 17.1: A Line With Positive Slope

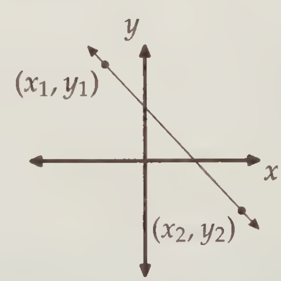
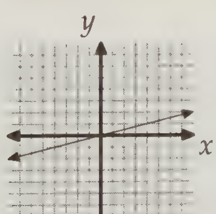
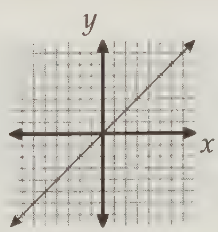


Figure 17.2: A Line With Negative Slope

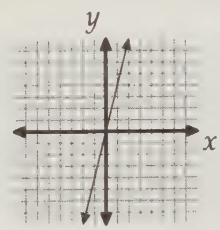
The slope of a horizontal line is equal to 0, and the slope of a vertical line is undefined. Moreover, the greater the magnitude of the slope of a line, the more 'steep' the line is.



Slope = 1/4



Slope = 1



Slope = 4

Problems

Problem 17.1: Find an equation whose graph is the line passing through the points $(-1,7)$ and $(-5,-3)$.

Problem 17.2: Find an equation whose graph is the line with slope 4 and y-intercept $(0,-3)$.

Problem 17.3:
(a) Find the length of the segment with endpoints $(-4,-3)$ and $(6,1)$.
(b) Find a formula for the distance between the points (x_1, y_1) and (x_2, y_2).

Problem 17.4: Let point A be (x_1, y_1) and point B be (x_2, y_2). Let M be the point with coordinates

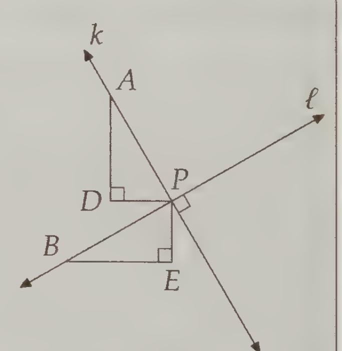
$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

In this problem we show that M is the midpoint of $\overline{AB}$.
(a) Show that $AM = MB$.
(b) Show that M is on $\overleftrightarrow{AB}$.

Problem 17.5: Explain why two different lines that have the same slope are parallel.

Problem 17.6: In this problem we show that if two lines are perpendicular, and neither line is vertical, then the product of their slopes is -1.

Shown at right are perpendicular lines k and ℓ. Let P be the point where these lines intersect. We construct two right triangles such that their hypotenuses are along lines k and ℓ, respectively, as shown. Moreover, each right triangle has a horizontal leg and a vertical leg.



(a) Show that $\triangle ADP \sim \triangle BEP$.
(b) Use the similar triangles from part (a) to show that

$$\frac{DA}{DP} \cdot \frac{EP}{EB} = 1.$$

Why does this tell us that the products of the slopes of k and ℓ is -1?

We start with a review of some different forms in which we can write equations whose graphs are lines.

Problem 17.1: Find an equation whose graph is the line passing through the points $(-1,7)$ and $(-5,-3)$.

Solution for Problem 17.1: The slope of the line through these two points is

$$\frac{7-(-3)}{-1-(-5)} = \frac{10}{4} = \frac{5}{2}.$$

Therefore, if (x, y) is any point on this line other than $(-1, 7)$, then the slope between (x, y) and $(-1, 7)$ must be $5/2$. So, we must have

$$\frac{y-7}{x-(-1)} = \frac{5}{2}.$$

Multiplying both sides of this equation by $x - (-1)$ gives us $y - 7 = \frac{5}{2}[x - (-1)]$. This is an example of the **point-slope form** of a linear equation.

Important: The graph of the equation

$$y - y_1 = m(x - x_1)$$

is a line through (x_1, y_1) with slope m. This is called a **point-slope form** of the equation.

While we often use point-slope form to find an equation of a line given the slope of the line and a point on the line, we often then convert the equation to **standard form**.

Important: The **standard form** of a linear equation is $Ax + By = C$, where, if possible, A, B, and C are integers, A is positive, and A, B, and C have no common factors besides 1.

We start converting $y - 7 = \frac{5}{2}[x - (-1)]$ to standard form by first multiplying both sides by 2 to get $2(y - 7) = 5(x + 1)$. Expanding both sides and rearranging gives $5x - 2y = -19$.

We can check our answer by making sure that the two points $(x, y) = (-1, 7)$ and $(x, y) = (-5, -3)$ satisfy our equation. Both do, so we know the equation $5x - 2y = -19$ represents the line through $(-1, 7)$ and $(-5, -3)$. □

One more form that is occasionally useful is **slope-intercept form**, which we'll investigate in the next problem.

Problem 17.2: Find an equation whose graph is the line with slope 4 and y-intercept $(0, -3)$.

Solution for Problem 17.2: Because the line passes through $(0, -3)$ and has slope 4, a point-slope form of the line is $y - (-3) = 4(x - 0)$. Isolating y on the left side then gives us $y = 4x - 3$. Notice that the coefficient of x equals the slope of our line and the constant term on the right equals the y-coordinate of the y-intercept. This is the slope-intercept form of the equation:

Important: The **slope-intercept form** of a linear equation is

$$y = mx + b,$$

where m is the slope of the line and b is the y-coordinate of the y-intercept.

As an Exercise, you'll prove that the graph of an equation of the form $y = mx + b$ has slope m. □

We'll now review some particularly useful relationships regarding segments and lines on the Cartesian plane.

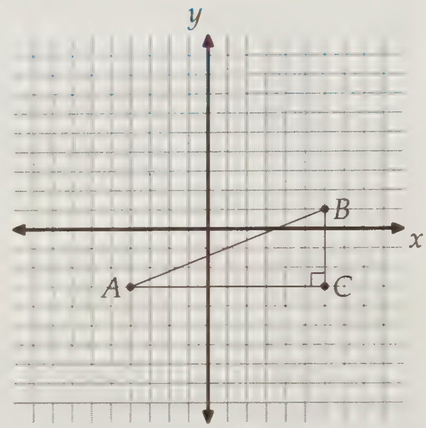
Problem 17.3:

(a) Find the length of the segment with endpoints $(-4,-3)$ and $(6,1)$.

(b) Find a formula for the distance between the points (x_1, y_1) and (x_2, y_2).

Solution for Problem 17.3:

(a) We start by labeling our points A and B such that A is $(-4,-3)$ and B is $(6,1)$.

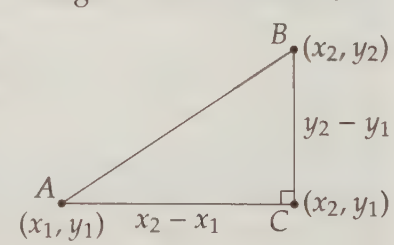


One of our favorite tactics for finding the length of a segment in a geometry problem is to build a right triangle with that segment as a side. Here, an easy way to build a right triangle with $\overline{AB}$ as a side is to extend segments horizontally from A and vertically from B to meet at point C, as shown. Since a horizontal line and a vertical line meet at a right angle, $\angle ACB$ is a right angle. So, if we can find AC and BC, we can use the Pythagorean Theorem to find AB.

The x-coordinate of C is the same as the x-coordinate of B and the y-coordinate of C is the same as the y-coordinate of A. So, point C is $(6,-3)$. Therefore, $AC = 6-(-4) = 10$ and $BC = 1-(-3) = 4$, and the Pythagorean Theorem gives us

$$AB = \sqrt{AC^2 + BC^2} = \sqrt{100+16} = \sqrt{116} = 2\sqrt{29}.$$

(b) Building a right triangle worked well when finding the distance between two specific points, so let's try using that tactic to find a formula for the distance between (x_1, y_1) and (x_2, y_2). Using part (a) as a guide, we let A be (x_1, y_1) and B be (x_2, y_2) and build a right triangle with $\overline{AB}$ as a side.



In the diagram at right, we assume that B is above and to the right of A. As before, we draw a vertical line down from B and a horizontal line to the right from A. We label the point where these lines meet C. Since C is directly below B, its x-coordinate is the same as B's, or x_2 (in other words, it is just as far horizontally from the y-axis as B is). Similarly, since C is directly to the right of A, its y-coordinate is the same as A's, y_1.

Now we're ready to use the Pythagorean Theorem. We can use our coordinates to see that C is $x_2 - x_1$ to the right of A and $y_2 - y_1$ below B. So, we can use the Pythagorean Theorem to find

$$AB^2 = AC^2 + BC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

We can take the square root of both sides of this equation to find a formula for the distance between the points (x_1, y_1) and (x_2, y_2):

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is often called the **distance formula**. Notice that if $x_1 = x_2$, then the distance between (x_1, y_1) and (x_2, y_2) is just the nonnegative difference between the y-coordinates. Let's see if that's what the distance formula gives us when $x_1 = x_2$:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_2 - x_2)^2 + (y_2 - y_1)^2} = \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|.$$

Indeed, the distance formula gives the expected result when $x_1 = x_2$. (We use absolute value signs when writing $\sqrt{(y_2 - y_1)^2} = |y_2 - y_1|$ because square roots, like distances, must be nonnegative.) Similarly, the distance formula tells us that the distance between (x_1, y_1) and (x_2, y_2) when $y_1 = y_2$ is the nonnegative difference between the x-coordinates.

□

Important: The distance in the plane between the points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is often referred to as the **distance formula**.

The distance formula is essentially the same thing as the Pythagorean Theorem. In fact, because you know the Pythagorean Theorem, you basically know the distance formula already, so you shouldn't have to memorize the distance formula.

Problem 17.4: Let point A be (x_1, y_1) and point B be (x_2, y_2). Let M be the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Show that M is the midpoint of $\overline{AB}$.

Solution for Problem 17.4: In order to show that M is the midpoint of $\overline{AB}$, we must show that $AM = MB$ and that M is on $\overleftrightarrow{AB}$. To show that $AM = MB$, we turn to the distance formula. We have

$$AM = \sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2} = \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2},$$

$$MB = \sqrt{\left(x_2 - \frac{x_1 + x_2}{2}\right)^2 + \left(y_2 - \frac{y_1 + y_2}{2}\right)^2} = \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}.$$

So, we have $AM = MB$.

To show that M is on $\overleftrightarrow{AB}$, we compare the slopes of $\overleftrightarrow{AM}$ and $\overleftrightarrow{MB}$. We have

$$\text{Slope of } \overleftrightarrow{AM} = \frac{\frac{y_1+y_2}{2} - y_1}{\frac{x_1+x_2}{2} - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

$$\text{Slope of } \overleftrightarrow{MB} = \frac{y_2 - \frac{y_1+y_2}{2}}{x_2 - \frac{x_1+x_2}{2}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

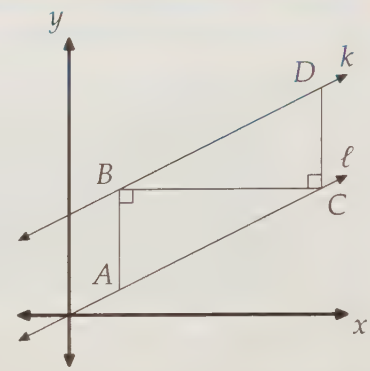
When $x_2 \neq x_1$, the slopes of $\overleftrightarrow{AM}$ and $\overleftrightarrow{MB}$ are the same, so A, M, and B are collinear. If $x_2 = x_1$, then A, M and B are on the same vertical line.

Since A, M, and B are collinear and $AM = MB$, we know that M is the midpoint of $\overline{AB}$. □

Important: The midpoint of the segment with endpoints (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Problem 17.5: Explain why two different lines with the same slope are parallel.



Solution for Problem 17.5: Slope is a measure of the direction of a line. Therefore, we expect two lines with the same slope to be parallel. We can prove this quickly with geometry. If our two lines both have slope 0, then they are both horizontal, and clearly are parallel. Otherwise, we can draw a horizontal line segment with one endpoint on each line, as shown in the diagram at right. We can build two right triangles by drawing two vertical line segments, as shown. If we show that $\angle ACB = \angle CBD$, then we can deduce that $k \parallel \ell$.

The only given information we have is that the lines have the same slope, so we need to find some way to express the slope of each line to allow us to use this information. Fortunately, our diagram gives us the answer. The slope of k is DC/BC (change in y-coordinate divided by change in x-coordinate), and the slope of line ℓ is AB/BC. Because the slopes of k and ℓ are equal, we have $DC/BC = AB/BC$, so $DC = AB$. We also have $\angle ABC = \angle BCD$, so we have $\triangle ABC \cong \triangle DCB$ by SAS Congruence. Therefore, we have $\angle ACB = \angle CBD$, so $k \parallel \ell$, as desired.

You'll have the chance to prove the converse, that two parallel lines have the same slope (if they are not vertical) as an Exercise. □

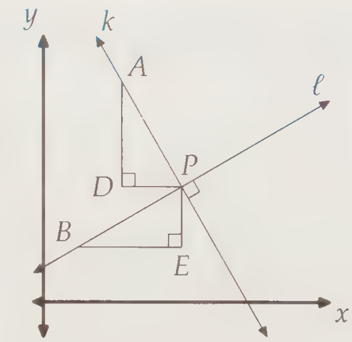
Important: If two lines have the same slope, then they are parallel. Conversely, if two non-vertical lines are parallel, then they have the same slope.

We've seen that the relationship between the slopes of two parallel lines is pretty straightforward. What about perpendicular lines?

Problem 17.6: Show that if two lines are perpendicular, and neither line is vertical, then the product of their slopes is -1.

Solution for Problem 17.6: Because the lines are perpendicular and neither is vertical, one line has positive slope and the other has negative slope. To see why, suppose the lines intersect at P, and that their slopes have the same sign. Then, the smaller angle formed by the lines is smaller than the right angle formed by the vertical line and horizontal line through P. So, if the slopes of the lines have the same sign, the lines cannot be perpendicular.

Let the line with negative slope be k and the line with positive slope be ℓ. Building right triangles worked so well in the last problem that we should try it again for this problem. We're also guided to build right triangles since these will give us a way to express the slopes of k and ℓ. We can use essentially this same proof for any configuration of k and ℓ in which the lines are perpendicular and neither line is vertical.



We make right triangles $\triangle ADP$ and $\triangle PBE$ such that each has a horizontal leg and a vertical leg. From $\triangle PBE$, we see that the slope of ℓ is EP/EB. From $\triangle ADP$, we see that the slope of k is $-DA/DP$. Make sure you see why we need the negative sign; the slope of k is clearly negative. Now that we have expressions for the slopes of k and ℓ, we can write an equation we wish to prove. We must show that

$$\left(\frac{EP}{EB}\right)\left(-\frac{DA}{DP}\right) = -1.$$

The ratios in this expression make us think of similar triangles. The right triangles are a pretty big clue, too. We already have $\angle ADP = \angle BEP$, so we only need to find one more pair of equal angles to prove the right triangles are similar. We probably need to use the fact that k and ℓ are perpendicular, so we focus on that. Since $\angle APB = \angle DPE = 90^\circ$, we have

$$\angle APD = 90^\circ - \angle DPB = \angle BPE.$$

Combining $\angle APD = \angle BPE$ with $\angle ADP = \angle BEP$ gives us $\triangle ADP \sim \triangle BEP$ by AA Similarity. This similarity gives us

$$\frac{DA}{DP} = \frac{EB}{EP}.$$

Multiplying both sides by -1 and by EP/EB gives us the desired $(EP/EB)(-DA/DP) = -1$. Therefore, the product of the slopes of k and ℓ is -1.

As an Exercise, you'll have the chance to prove the converse: if the product of the slopes of two lines is -1, then the two lines are perpendicular. □

> **Important:** If two non-vertical lines are perpendicular, then the product of their slopes is -1. Conversely, if the product of the slopes of two lines is -1, then the two lines are perpendicular.

Exercises

17.1.1 Find an equation, in standard form, whose graph is a line through $(-2,5)$ and $(-5,9)$.

17.1.2 Find the x-intercept and y-intercept of the line with slope 4 that passes through $(-7,2)$.

17.1.3 The distance between $(-4,3)$ and $(a,0)$ is 5. Find all possible values of a.

17.1.4 When finding the distance between two points using the distance formula, does it matter which point we call (x_1, y_1) and which we call (x_2, y_2)? Why or why not?

17.1.5 Point P is $(-5,2)$ and point Q is $(-8,8)$.

(a) Find PQ.

(b) Find the midpoint of $\overline{PQ}$.

(c) Find T such that Q is the midpoint of $\overline{PT}$.

17.1.6 Line k is perpendicular to the graph of $4x - 3y + 14 = 0$ and passes through $(-3, 2)$. Find an equation in standard form whose graph is line k.

17.1.7 Show that the graph of $y = mx + b$, where m and b are constants, has slope m.

17.1.8 Show that if the product of the slopes of lines k and ℓ is -1, then $k \perp \ell$.

17.1.9 Suppose A, B, and C are constants and $B \neq 0$. Show that the slope of the graph of the equation $Ax + By + C = 0$ is $-A/B$.

17.2 Circles

In this section, we use the distance formula to find equations whose graphs are circles.

Problems

Problem 17.7:

(a) Find an equation whose graph is a circle with center $(4, -5)$ and radius $3\sqrt{2}$.

(b) Explain why the graph of the equation

$$(x - h)^2 + (y - k)^2 = r^2,$$

where h, k, and r are constants with $r > 0$, is a circle with center (h, k) and radius r.

Problem 17.8: Find the area of the region that is enclosed by the graph of the equation

$$2x^2 - 8x + 2y^2 + 16y = 4.$$

Problem 17.7:

(a) Find an equation whose graph is a circle with center $(4, -5)$ and radius $3\sqrt{2}$.

(b) Explain why the graph of the equation

$$(x - h)^2 + (y - k)^2 = r^2,$$

where h, k, and r are constants with $r > 0$, is a circle with center (h, k) and radius r.

Solution for Problem 17.7:

(a) If a point (x, y) is on the circle, then it must be $3\sqrt{2}$ from the center of the circle, which is $(4, -5)$. So, the distance between (x, y) and $(4, -5)$ is $3\sqrt{2}$. The distance formula then gives us

$$\sqrt{(x - 4)^2 + (y + 5)^2} = 3\sqrt{2}.$$

Squaring both sides of this equation gives

$$(x - 4)^2 + (y + 5)^2 = 18.$$

We could multiply out both $(x-4)^2$ and $(y+5)^2$, but leaving the equation in this form makes it particularly easy to see that the graph of this equation is a circle with center $(4,-5)$ and radius $3\sqrt{2}$. In our next part, we'll see why.

(b) Taking the square root of both sides of $(x-h)^2+(y-k)^2=r^2$ gives us

$$\sqrt{(x-h)^2+(y-k)^2}=r.$$

Since h and k are constants and r is a positive constant, this equation tells us that the point (x,y) is a distance of r from the point (h,k). Therefore, every point (x,y) that satisfies this equation is on the circle with center (h,k) and radius r.

Conversely, if a point (x,y) is a distance of r from the point (h,k), then the distance formula tells us that

$$\sqrt{(x-h)^2+(y-k)^2}=r.$$

Squaring both sides of this equation gives us $(x-h)^2+(y-k)^2=r^2$, so we see that every point on the circle with center (h,k) and radius r satisfies the given equation. Therefore, the graph of the equation $(x-h)^2+(y-k)^2=r^2$ is a circle with center (h,k) and radius r.

□

Important: The **standard form** of an equation whose graph is a circle is

$$(x-h)^2+(y-k)^2=r^2,$$

where h, k, and r are constants with $r>0$. The center of the circle is (h,k) and the radius of the circle is r.

Sometimes we have to do a little work to put an equation in standard form.

Problem 17.8: Find the area of the region that is enclosed by the graph of the equation

$$2x^2-8x+2y^2+16y=4.$$

Solution for Problem 17.8: The $2x^2$ and $2y^2$ terms make us think of the standard form of a circle, because expanding the squares on the left side of

$$(x-h)^2+(y-k)^2=r^2$$

will give us an x^2 and y^2. So, we try to write the given equation in this form. Our first step is to divide both sides by 2, to make the coefficients of x^2 and y^2 both equal to 1. This gives us

$$x^2-4x+y^2+8y=2.$$

We then complete the square in both x and y by adding 4 and 16 to both sides:

$$x^2-4x+4+y^2+8y+16=2+4+16.$$

Since $x^2-4x+4=(x-2)^2$ and $y^2+8y+16=(y+4)^2$, we have

$$(x-2)^2+(y+4)^2=22.$$

This equation is in the desired standard form of a circle. Therefore, we know that the graph of this equation is a circle with center $(2, -4)$ and radius $\sqrt{22}$.

> **WARNING!!** Make sure you see why the center is not $(-2, 4)$ or $(2, 4)$. The standard form of a circle is
> $$(x-h)^2 + (y-k)^2 = r^2.$$
> Comparing this to
> $$(x-2)^2 + (y+4)^2 = 22$$
> gives us $h = 2$ and $k = -4$.

Because the graph is a circle with radius $\sqrt{22}$, its area is $(\sqrt{22})^2\pi = 22\pi$. □

Exercises

17.2.1 Find the center and the radius of the circle that is the graph of the equation $3x^2 - 12x + 3y^2 + 6y = 15$.

17.2.2 Find the standard form of the equation whose graph is a circle with center $(-2, 7)$ that passes through $(-5, 9)$.

17.2.3 What is the length of the longest chord of the graph of

$$x^2 + 2x + y^2 - 6y = 6$$

that passes through the point $(-2, 4.5)$?

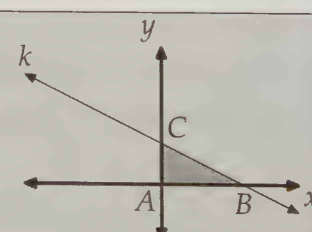
17.2.4★ Find the two points at which the graphs of $2x - y = 7$ and $x^2 - 10x + y^2 + 4y = -4$ intersect.

17.3 Basic Analytic Geometry Problems

In this section, we begin to explore the power of analytic geometry. Because analytic geometry gives us a tool to relate algebra and geometry, it gives us a way to apply geometric tools to algebraic problems, and to use algebraic tools to solve geometry problems.

Problems

Problem 17.9: The area of the shaded region between line k and the axes in the diagram at right is 36. If line k passes through $(12, 0)$, then what is the slope of k?



Problem 17.10: Let A be $(-5,6)$, B be $(-7,9)$, and C be $(1,10)$.

(a) Find the length of the median of $\triangle ABC$ from B to $\overline{AC}$.

(b) Find the slope of each side of $\triangle ABC$. Notice anything interesting?

(c) Find the area of $\triangle ABC$.

(d) Find the length of the altitude from A to $\overline{BC}$.

Problem 17.11: In this problem, we determine the number of points (a,b) such that $a^2+b^2=25$ and the area of the triangle with vertices $(-5,0)$, $(5,0)$, and (a,b) is 10. *(Source: MATHCOUNTS)*

(a) What does the fact that (a,b) satisfies $a^2+b^2=25$ tell us about where the point (a,b) is located on the coordinate plane?

(b) What does the fact that the area of the triangle with vertices $(-5,0)$, $(5,0)$ and (a,b) is 10 tell us about where the point (a,b) is located on the coordinate plane?

(c) How many points satisfy the conditions you found in parts (a) and (b)?

Problem 17.12: $ABCD$ is a rectangle with $AB=8$ and $AD=12$. Points X and Y are on $\overline{AB}$ and $\overline{CD}$, respectively, such that $AX=7/3$ and $DY=1$. $\overline{XY}$ meets diagonal $\overline{AC}$ at point P. In this problem, we find BP.

(a) Set the problem up on the Cartesian plane. Let A be the origin. What coordinates can we assign to each of B, C, and D?

(b) What are the coordinates of P?

(c) Find BP.

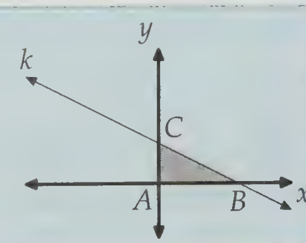
Problem 17.13: In $\triangle ABC$, we have $AB=BC=8$ and $\angle ABC=45^\circ$. In this problem, we find AC.

(a) Why should 45° make us think of the graph of $y=x$?

(b) What point on the line $y=x$ is 8 units from the origin?

(c) Let B be the origin. Find coordinates for A and C such that $\triangle ABC$ satisfies the problem. Find AC.

Problem 17.9: The area of the shaded region between line k and the axes in the diagram at right is 36. If line k passes through $(12,0)$, then what is the slope of k?



Solution for Problem 17.9: The shaded region is a right triangle. The vertex of the right angle of this triangle is the origin, $(0,0)$, which is labeled A in the diagram. Vertex B is $(12,0)$, so we know that leg $\overline{AB}$ has length 12. Because the area of the triangle is 36, we know that half the product of the lengths of the legs of the triangle is 36. Therefore, we have $(AB)(AC)=72$, so $AC=72/AB=6$. So, vertex C is $(0,6)$, and the slope of k is $(6-0)/(0-12) = -1/2$. □

Problem 17.10: Let A be $(-5,6)$, B be $(-7,9)$, and C be $(1,10)$.

(a) Find the length of the median of $\triangle ABC$ from B to $\overline{AC}$.

(b) Find the length of the altitude of $\triangle ABC$ from A to $\overline{BC}$.

Solution for Problem 17.10:

(a) The midpoint of $\overline{AC}$ is $\left(\frac{-5+1}{2}, \frac{6+10}{2}\right) = (-2,8)$. The distance from B to this point is

$$\sqrt{[-2-(-7)]^2+(8-9)^2} = \sqrt{25+1} = \sqrt{26}.$$

(b) *Solution 1: Find the coordinates of the foot of the altitude.* To find the length of the altitude from A to side $\overline{BC}$, we try to find the point P on $\overline{BC}$ such that $\overline{AP} \perp \overline{BC}$. First, we find the equation whose graph passes through B and C. The slope of this line is $(10-9)/[1-(-7)] = 1/8$. The line passes through $(1,10)$, so an equation of the line in point-slope form is $y - 10 = \frac{1}{8}(x-1)$. Rearranging this equation gives $x - 8y = -79$.

Let point P be (x,y). Since (x,y) is on $\overleftrightarrow{BC}$, it satisfies the equation $x-8y = -79$. Because $\overline{AP} \perp \overline{BC}$, the product of the slopes of these two segments must be -1. Since the slope of $\overline{BC}$ is $1/8$, we know the slope of $\overline{AP}$ is -8. So, $\overleftrightarrow{AP}$ passes through (x,y) and $(-5,6)$, and it has slope -8. This gives us

$$\frac{y-6}{x-(-5)} = -8.$$

Rearranging this equation gives $8x + y = -34$.

We now have the system of equations

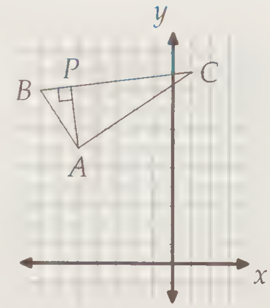
$$\begin{aligned} x - 8y &= -79, \\ 8x + y &= -34. \end{aligned}$$

Multiplying the second equation by 8 then adding the result to the first equation gives us the equation $65x = -351$, which yields $x = -27/5$. Substituting this into either equation above then gives us $y = 46/5$. Therefore, $P = (-27/5, 46/5)$, so we have

$$AP = \sqrt{\left[-\frac{27}{5}-(-5)\right]^2 + \left(\frac{46}{5}-6\right)^2} = \sqrt{\frac{4}{25}+\frac{256}{25}} = \sqrt{\frac{260}{25}} = \frac{2\sqrt{65}}{5}.$$

Solution 2: Use some geometric insight. The algebra in our first solution is a little messy. The system of equations we have to solve isn't very nice, and all the fractions give us plenty of opportunities to make mistakes. Once we see all that algebra coming our way, we might stop and think if there are any geometric insights we can use to simplify the problem.

Concept: When the algebra in an analytic geometry problem starts to get ugly, try using geometric facts to simplify the problem.



Here, we seek the length of the altitude from A to $\overline{BC}$. The altitude makes us think about area. We can easily find the length of $\overline{BC}$:

$$BC = \sqrt{(-7-1)^2 + (9-10)^2} = \sqrt{65}.$$

Therefore, if we can find $[ABC]$, we can easily find the length of $\overline{AP}$, since $(AP)(BC)/2 = [ABC]$.

Is $[ABC]$ easy to find? To answer this, we think about what kind of triangle ABC is. If it is isosceles or right, we can probably find the area easily. We can use slope to quickly check if it is right. We've already seen that the slope of $\overline{BC}$ is $1/8$. The slope of $\overline{AB}$ is $-3/2$ and the slope of $\overline{AC}$ is $2/3$. The product of the slopes of $\overline{AB}$ and $\overline{AC}$ is -1, so $\overline{AB} \perp \overline{AC}$.

Concept: When given the coordinates of the vertices of a triangle, check if the triangle is special in any way. Most notably, check if the triangle is a right triangle. If it is, this fact will probably simplify the problem, because we know so much about right triangles.

Now the area of $\triangle ABC$ is easy to find. Because $\angle BAC = 90^\circ$, we have $[ABC] = (AB)(AC)/2$. Using the distance formula, we find that $AB = \sqrt{13}$ and $AC = 2\sqrt{13}$, so

$$[ABC] = \frac{(AB)(AC)}{2} = \frac{(\sqrt{13})(2\sqrt{13})}{2} = 13.$$

Finally, we have

$$AP = \frac{2[ABC]}{BC} = \frac{26}{\sqrt{65}} = \frac{26}{\sqrt{65}} \cdot \frac{\sqrt{65}}{\sqrt{65}} = \frac{2\sqrt{65}}{5}.$$

A little geometry sometimes goes a long way in analytic geometry problems!

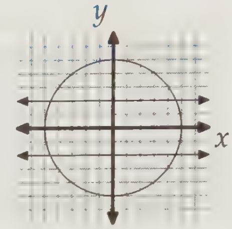
□

Problem 17.11: Find the number of points (a, b) such that $a^2 + b^2 = 25$ and the area of the triangle with vertices $(-5, 0)$, $(5, 0)$, and (a, b) is 10. *(Source: MATHCOUNTS)*

Solution for Problem 17.11: Let our triangle be PQR, with $P = (a, b)$, $Q = (-5, 0)$, and $R = (5, 0)$. Side $\overline{QR}$ is a horizontal segment with length 10. Because the area of $\triangle PQR$ is 10, we know that the length of the altitude from P to $\overline{QR}$ is 2. Since $\overline{QR}$ is along the x-axis, the fact that P is 2 units from $\overline{QR}$ means that P is either 2 units above or 2 units below the x-axis. Therefore, point P is on the line $y = -2$ or on the line $y = 2$. From here we present two solutions.

Solution 1: Algebra. Because P is on either $y = 2$ or $y = -2$, we must have either $b = 2$ or $b = -2$. When $b = 2$, the equation $a^2 + b^2 = 25$ gives us $a^2 = 25 - b^2 = 21$, so $a = \pm\sqrt{21}$. So, two possible points P are $(\sqrt{21}, 2)$ and $(-\sqrt{21}, 2)$. Similarly, letting $b = -2$ gives us two more points that satisfy the conditions in the problem, namely, $(\sqrt{21}, -2)$ and $(-\sqrt{21}, -2)$. Therefore, there are 4 different points that fit the description of the problem.

Solution 2: Geometry. Because (a, b) satisfies the equation $x^2 + y^2 = 25$, we know that (a, b) is on the circle centered at the origin with radius 5. We also know that (a, b) is either on the line $y = 2$ or the line $y = -2$. Each of these lines is 2 units from the center of the circle, so both lines go inside the circle. Therefore, both lines intersect the circle twice, as shown in the diagram at right. Each of the 4 points of intersection between a line and the circle gives us a point that satisfies the restrictions of the problem. □



In our second solution, we interpreted the equation $a^2 + b^2 = 25$ as telling us that the point (a, b) is on the circle with center $(0, 0)$ and radius 5. Any time we see an equation that is in the standard form of a circle, we might consider using analytic geometry. This gives us a new tool for tackling algebraic problems:

Concept: Interpreting an equation as a geometric figure on the Cartesian plane can often help us solve problems.

This street goes both ways! Sometimes it is useful to put a geometry problem on the Cartesian plane and use algebraic tools to solve the problem.

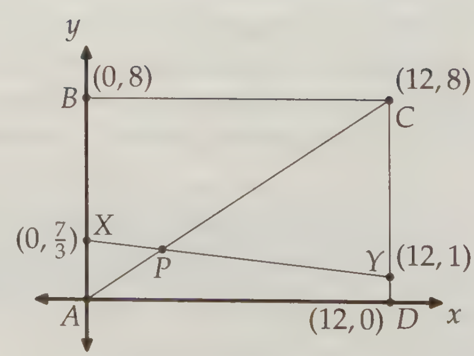
Problem 17.12: $ABCD$ is a rectangle with $AB = 8$ and $AD = 12$. Points X and Y are on $\overline{AB}$ and $\overline{CD}$, respectively, such that $AX = 7/3$ and $DY = 1$. $\overline{XY}$ meets diagonal $\overline{AC}$ at point P. Find BP.

Solution for Problem 17.12: We seek a length, so we might try building right triangles and using the Pythagorean Theorem. (Try doing so on your own.) Analytic geometry offers us another way to find lengths: if we can find the coordinates of two points, we can use the distance formula to find the distance between them. So, we try turning this problem into an analytic geometry problem. One reason we do so is that rectangles are particularly easy to describe with coordinates.

We start by letting A be the origin, and letting two sides of the rectangle be along the coordinate axes. This makes a lot of the coordinates in the problem equal to 0, and 0's are easy to deal with.

Concept: When setting a geometry problem up on the Cartesian plane, choose the origin and the axes in a convenient matter. Often this means letting the origin be the vertex of a right angle in the problem, so that the sides of the angle are along the axes.

In the diagram at right, we let $\overline{AB}$ be along the y-axis and $\overline{AD}$ be along the x-axis. Because $AB = 8$ and $AD = 12$, point B is $(0, 8)$ and point D is $(12, 0)$. Because X is on $\overline{AB}$ such that $AX = 7/3$, we have $X = (0, 7/3)$. Similarly, Y is 1 unit above D, so $Y = (12, 1)$.



Now, if we can find the coordinates of point P, we can use the distance formula to find BP. We know that P is on $\overline{AC}$. Because $\overleftrightarrow{AC}$ has slope $(8-0)/(12-0) = 2/3$ and it passes through the origin, the equation whose graph is $\overleftrightarrow{AC}$ is $y = 2x/3$. (This is why we like having so many 0's in our coordinates – the resulting equations for lines connecting points in the diagram are simpler.) Point P is also on $\overline{XY}$. The slope of $\overleftrightarrow{XY}$ is $(1-7/3)/(12-0) = -1/9$ and its y-intercept is $(0, 7/3)$, so the slope-intercept form of the equation of this line is

$$y = -\frac{1}{9}x + \frac{7}{3}.$$

Rearranging this equation gives $x + 9y = 21$. So, P is the intersection of the graphs of $y = 2x/3$ and $x+9y = 21$. Substituting $y = 2x/3$ into $x+9y = 21$ gives $x+6x = 21$, from which we find $x = 3$. Therefore, we have $y = 2x/3 = 2$, so P is $(3, 2)$. Finally, because B is $(0, 8)$, we have

$$BP = \sqrt{(3-0)^2 + (2-8)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}.$$

A little algebra sometimes goes a long way in geometry problems! □

WARNING!! Don't get carried away with using analytic geometry to solve geometry problems. While our solution to Problem 17.12 was pretty straightforward with analytic geometry, you'll find that most geometry problems are easier to solve with geometric methods than with analytic geometry.

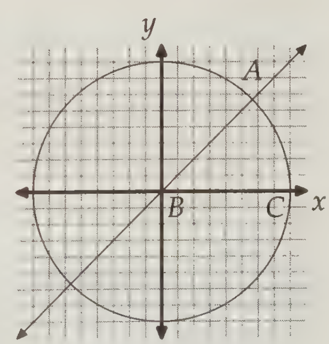
For example, suppose we wished to find the length of the median to the hypotenuse of a right triangle with sides of length 14, 48, and 50. It is much faster to use the simple geometric fact that the median to the hypotenuse of a right triangle is half the length of the hypotenuse than it is to assign coordinates to the vertices of the triangle, find the coordinates of the midpoint of the hypotenuse, then use the distance formula.

After finishing this chapter, try flipping back through this book and finding problems that you can solve with analytic geometry. You'll find some common trends among the ones you can solve, such as rectangles, midpoints, and right triangles. You'll also find some trends among problems that are hard to solve with analytic geometry, such as circles, angles that are not 45° or 90°, and complicated diagrams.

Our last sentence suggests that problems with 45° angles might be candidates for analytic geometry. Let's see why.

Problem 17.13: In $\triangle ABC$, we have $AB = BC = 8$ and $\angle ABC = 45°$. In this problem, we find AC.

Solution for Problem 17.13: We present both an analytic geometry solution and a 'pure geometry' solution.



Solution 1: Analytic geometry. The 45° angle might make us think about analytic geometry because the graph of the line $y = x$ makes a 45° angle with the coordinate axes. We also notice that $AB = BC = 8$ means that A and C are both on the circle with center B and radius 8. In other words, if we let B be the origin, then the coordinates of A and C are solutions to the equation

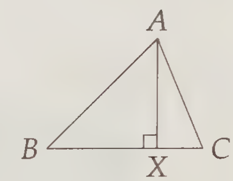
$$x^2 + y^2 = 64,$$

because the graph of this equation is the circle with center $(0, 0)$ and radius 8. We know that the graph of $y = x$ makes a 45° angle with the axes, so we let C be along the the x-axis, at $(8, 0)$, and A be on the graph of the line $y = x$.

We can find the coordinates of A either by drawing an altitude from A to the x-axis, or by using our equation for the circle. Because A is on both the circle and the graph of $y = x$, we can find the coordinates of A by substituting $y = x$ into our equation for the circle. This gives us $x^2 + x^2 = 64$, from which we find $x = 4\sqrt{2}$. (In our diagram, A is on the right side of the y-axis, so x must be positive.) Therefore, point A is $(4\sqrt{2}, 4\sqrt{2})$, and we can use the distance formula to find

$$AC = \sqrt{(4\sqrt{2}-8)^2 + (4\sqrt{2}-0)^2} = \sqrt{32 - 64\sqrt{2} + 64 + 32} = \sqrt{128 - 64\sqrt{2}} = 8\sqrt{2-\sqrt{2}}.$$

Solution 2: Geometry. Our analytic geometry solution might inspire a purely geometric solution. At the point where we were finding the coordinates for A, we suggested drawing an altitude from A to $\overline{BC}$. Suppose we do that right from the beginning. (We might also be inspired to do this by the 45° angle; drawing this altitude builds a 45-45-90 triangle).



From 45-45-90 triangle ABX, we have $AX = XB = AB/\sqrt{2} = 4\sqrt{2}$. Therefore, we have $CX = CB - BX = 8 - 4\sqrt{2}$. We can then apply the Pythagorean Theorem to $\triangle AXC$ to find $AC = 8\sqrt{2-\sqrt{2}}$, as before. □

Concept: Solving a problem in two ways is a good way to check your answer.

Our two solutions in the last problem also show that analytic geometry solutions and purely geometric solutions to the same problem are often closely related to each other.

Exercises

17.3.1 In Problem 17.10, we used slopes to determine that $\triangle ABC$ is a right triangle. How else could we have determined that $\triangle ABC$ is a right triangle?

17.3.2 Find the area of the triangle bounded by the graphs of the equations $y = x$, $y = -x$, and $y = 6$. *(Source: AMC 12)*

17.3.3 What is the shortest possible distance between a point on the graph of $x^2 + y^2 + 6x - 8y = 0$ and a point on the graph of $x^2 - 14x + y^2 + 10y + 65 = 0$?

17.3.4 In $\triangle ABC$, we have $AB = 8$, $BC = 12$, and $\angle ABC = 30°$. Find AC.

17.3.5★ Find the radius of each circle that passes through $(9, 2)$ and is tangent to both the x-axis and the y-axis. **Hints:** 7, 487

17.4 Proofs with Analytic Geometry

In this section, we learn how to prove geometric facts using analytic geometry.

Problems

Problem 17.14: In this problem, we prove that the midpoint of the hypotenuse of a right triangle is the circumcenter of the triangle.

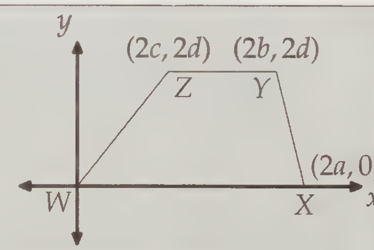
(a) Let the right triangle be $\triangle ABC$, with right angle at C. Why is it useful to choose C to be the origin of the Cartesian plane?

(b) Suppose C is the origin. Why should we choose the axes of the Cartesian plane such that the legs of $\triangle ABC$ are along the axes?

(c) Let A be $(a, 0)$ and B be $(0, b)$. What is the midpoint of the hypotenuse?

(d) Show that the midpoint of the hypotenuse is equidistant from the vertices of the triangle.

Problem 17.15: In this problem, we prove that the median of a trapezoid is parallel to the bases of the trapezoid, and that the length of the median is the average of the lengths of the bases. Let the trapezoid be $WXYZ$, where W is the point $(0,0)$, X is the point $(2a, 0)$, Y is the point $(2b, 2d)$, Z is the point $(2c, 2d)$, and we have $a > 0$, $b > c$, and $d > 0$.



(a) Why is $WXYZ$ a trapezoid? Which two sides are parallel?

(b) Find the coordinates of the midpoints of the legs of $WXYZ$. Why did we use $2a$, $2b$, $2c$, and $2d$ to describe the coordinates of the vertices of $WXYZ$, instead of using just a, b, c, and d?

(c) Show that the median of $WXYZ$ is parallel to the bases of $WXYZ$, and equal in length to the average of the lengths of the bases.

(d) Explain why our proof is valid by describing why for every trapezoid there is some choice of origin and axes such that the vertices of the trapezoid can be described with the coordinates $(0,0)$, $(2a,0)$, $(2b,2d)$, and $(2c,2d)$.

Problem 17.16: In this problem we prove that the medians of a triangle are concurrent. Let our triangle be ABC, where $A = (0,0)$, $B = (2a, 0)$, and $C = (2b, 2c)$.

(a) Find the coordinates of the midpoint of $\overline{AB}$. Find an equation whose graph is the line containing the median from C to $\overline{AB}$.

(b) Find an equation of the line whose graph is the line containing the median from B to $\overline{AC}$.

(c) Find the intersection of the lines you found in parts (a) and (b).

(d) Prove that the intersection point you found in part (c) is on the median from A to $\overline{BC}$.

Problem 17.17: Suppose line k divides rectangle $ABCD$ into two pieces of equal area. Prove that line k passes through the intersection of the diagonals of $ABCD$.

Problem 17.14: Use analytic geometry to prove that the midpoint of the hypotenuse of any right triangle is the circumcenter of the triangle.

Solution for Problem 17.14: Our first step is to set up the problem on the Cartesian plane. We must be very careful when we do so. See if you can figure out why the set-up in the Bogus Solution below is not sufficient:

Bogus Solution: We let our triangle be ABC, and we let $C = (0,0)$, $A = (4,0)$, and $B = (0,6)$. Legs $\overline{AC}$ and $\overline{BC}$ are along the axes, so they are perpendicular. Therefore, $\triangle ABC$ is a right triangle with hypotenuse $\overline{AB}$. The midpoint of $\overline{AB}$ is $(2,3)$. Let this midpoint be M. We have

$$\begin{aligned} AM &= \sqrt{(2-4)^2 + (3-0)^2} = \sqrt{13}, \\ BM &= \sqrt{(2-0)^2 + (3-6)^2} = \sqrt{13}, \\ CM &= \sqrt{(2-0)^2 + (3-0)^2} = \sqrt{13}. \end{aligned}$$

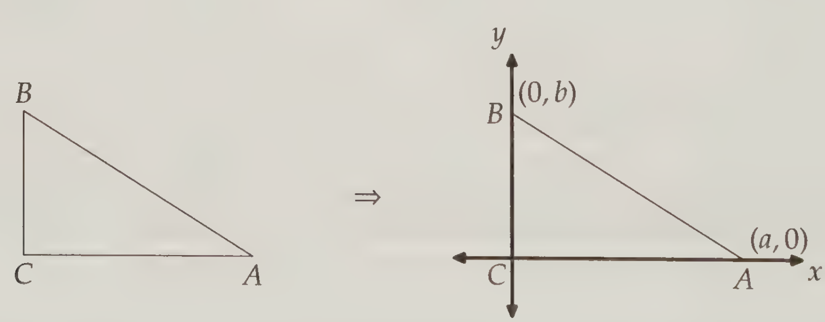
Since M is equidistant from the vertices of $\triangle ABC$, point M is the circumcenter of $\triangle ABC$.

This does indeed show why the midpoint of the hypotenuse of this particular right triangle is the circumcenter of this right triangle. However, what about other right triangles with different side lengths? Our 'proof' doesn't cover every possible right triangle.

WARNING!! When setting up a proof on the Cartesian plane, we must be very careful. Our proof must address all possible configurations of the problem, so our analytic geometry representation of the problem must cover all possible configurations.

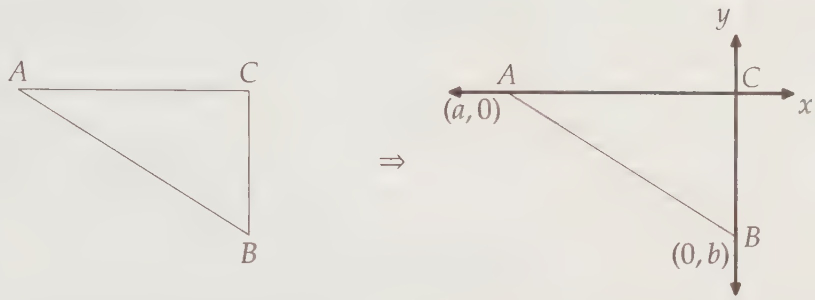
We might start by letting $A = (a,b)$, $B = (c,d)$, and $C = (e,f)$. But that's six variables! Hopefully we can find a simpler representation of $\triangle ABC$.

When setting up a geometry problem on the Cartesian plane, we can start with a diagram, then add the coordinate axes. We try to do so in a way that simplifies our problem. We can choose any point to be the origin, so we choose one of the vertices of the right triangle to be our origin.



We choose the vertex of the right angle, point C, to be the origin because this allows us to choose our axes so that the legs of the right triangle are along the axes. So, we let $A = (a,0)$ and $B = (0,b)$.

Notice that a and/or b could be positive or negative. For example, in the case below, both a and b are negative.



Since C is the vertex of the right angle, side $\overline{AB}$ is the hypotenuse. Let M be the midpoint of $\overline{AB}$, so that

$$M = \left(\frac{a}{2}, \frac{b}{2}\right).$$

To show that M is the circumcenter of $\triangle ABC$, we must show that M is equidistant from all three vertices. This is a job for the distance formula:

$$AM = \sqrt{\left(\frac{a}{2} - a\right)^2 + \left(\frac{b}{2} - 0\right)^2} = \sqrt{\frac{a^2 + b^2}{4}},$$
$$BM = \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(\frac{b}{2} - b\right)^2} = \sqrt{\frac{a^2 + b^2}{4}},$$
$$CM = \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(\frac{b}{2} - 0\right)^2} = \sqrt{\frac{a^2 + b^2}{4}}.$$

We have $AM = BM = CM$, so M is the circumcenter of $\triangle ABC$.

Notice that every step in our proof is valid even if a or b or both are negative. □

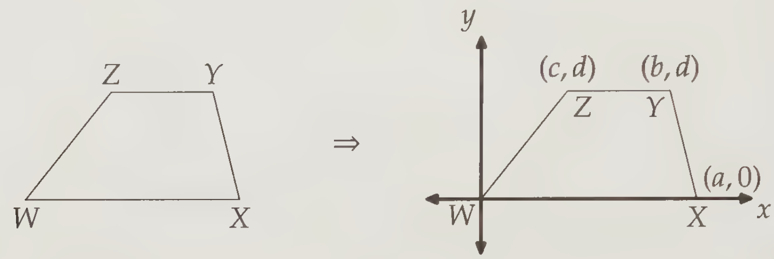
Important: When setting up a geometry problem on the Cartesian plane, choose your origin and your coordinate axes wisely. Typically, we do so in a way that makes the coordinates of important points in the problem as simple as possible.

Notice that our proof in Problem 17.14 only applies to right triangles, not to all triangles. The points we chose to represent the vertices, $(0,0)$, $(a,0)$, and $(0,b)$, are always the vertices of a right triangle (when a and b are nonzero). We cannot use these three points as our vertices to prove a fact about all triangles, since any triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$ must be a right triangle.

Problem 17.15: Use analytic geometry to prove that the median of a trapezoid is parallel to the bases of the trapezoid, and that the length of the median is the average of the lengths of the bases.

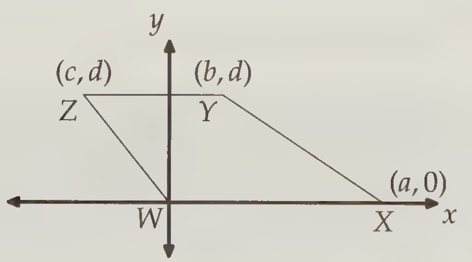
Solution for Problem 17.15: On page 211, we saw geometric proofs of these facts. Here, we try to find an analytic geometry proof. We might start by letting the vertices of our trapezoid be (a,b), (c,d), (e,f), and (g,h). Yikes. Eight variables! Maybe we can do better.

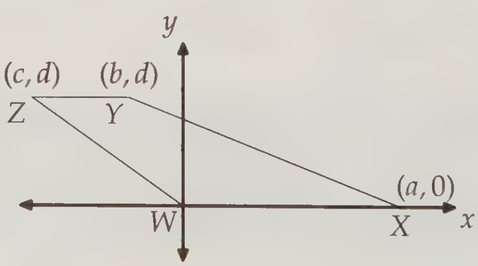
As noted in Problem 17.14, we can start with a diagram, then add the coordinate axes to set up a proof on the Cartesian plane. Here, we choose the x-axis to contain one of the bases such that the other base is above the x-axis. We then choose the leftmost of the two vertices on the x-axis to be the origin. So, one vertex is the origin and the other is $(a, 0)$ with $a > 0$. We'll call the origin W and the second vertex on the x-axis X, as shown below.



We placed the x-axis such that the endpoints of one base are on the x-axis because this forces the endpoints of the other base to have the same y-coordinate, since the bases are parallel. By choosing the 'lower' base to be x-axis, we force this common y-coordinate to be positive. Therefore, the other two vertices of the trapezoid can be represented by (b, d) and (c, d), where $d > 0$ and $b > c$. Finally, we let Y be (b, d) and Z be (c, d), so $\overline{XY}$ and $\overline{WZ}$ are the legs of the trapezoid. Now, we have represented our trapezoid with only 4 variables, instead of 8.

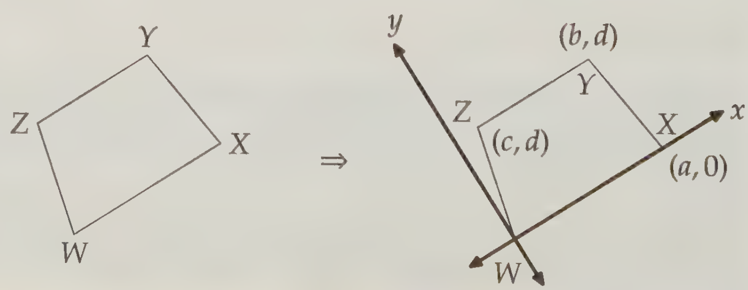
Notice that every trapezoid can be represented in this way. Because b and c can take on negative values, letting the vertices be $W = (0, 0)$, $X = (a, 0)$, $Y = (b, d)$, and $Z = (c, d)$ with $a > 0$, $d > 0$, and $b > c$ will also represent trapezoids in which Z and/or Y end up to the left of the y-axis. Two examples are shown below. In each case, we have W at the origin, X to the right of W on the x-axis, and $\overline{YX}$ and $\overline{ZW}$ as the legs (because $b > c$).



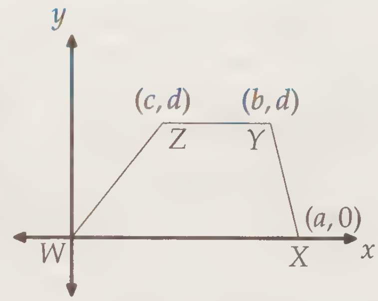


But what about trapezoids in which the bases are not horizontal?

There's no reason we have to make our x-axis and y-axis horizontal and vertical! They only have to be perpendicular. For example, consider the trapezoid with 'slanted' bases at right. We similarly make our axes 'slanted' so that the x-axis includes one base.



Now, we're ready for our proof. We'll use the diagram in which $\overline{YZ}$ is horizontal and both Y and Z are to the right of the y-axis, as shown at right. (However, our proof will be valid for all set-ups we have discussed.) The midpoint of leg $\overline{WZ}$ is $(\frac{c}{2}, \frac{d}{2})$ and the midpoint of leg $\overline{XY}$ is $(\frac{a+b}{2}, \frac{d}{2})$. The median connects these two points, so the median is on the line $y = \frac{d}{2}$. Therefore, the median is horizontal, so it is parallel to the bases. We can also use the coordinates of the midpoints of the legs to determine that the length of the median is



$$\frac{a+b}{2} - \frac{c}{2} = \frac{a+b-c}{2}.$$

The lengths of the bases are $WX = a$ and $YZ = b - c$, so the average of the lengths of the bases is also $(a + b - c)/2$. So, the length of the median equals the average of the lengths of the bases. Notice that every step of this proof is valid for all the other arrangements we showed.

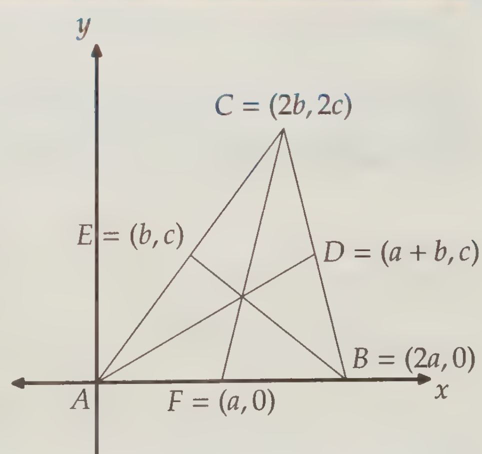
We also could have avoided the fractions in our solution by being a little clever about assigning variables to coordinates. Because we are going to have to work with midpoints to get information about the median, we know we'll have to divide expressions by 2. Therefore, we might make our coordinates $W = (0,0)$, $X = (2a,0)$, $Y = (2b,2d)$, and $Z = (2c,2d)$. Then, the endpoints of the median are $(a + b,d)$ and (c,d). No fractions! See if you can finish the problem from here. □

Concept: When we set up a geometry problem involving midpoints on the Cartesian plane, we often use $2a$, $2b$, $2c$, etc. for coordinates, rather than just a, b, c, etc. This helps us avoid expressions involving fractions when we find midpoints of segments in the problem.

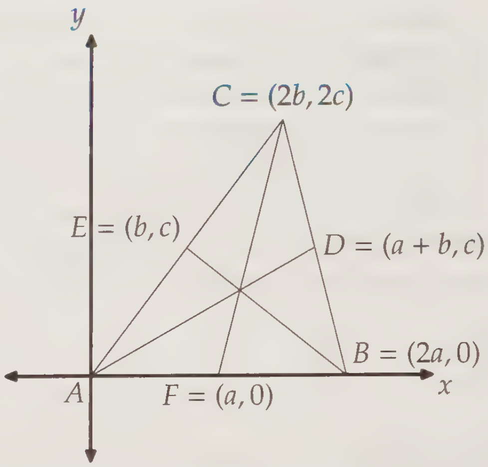
Speaking of midpoints, these are often a sign that an analytic geometry approach might work. This is because coordinates of midpoints are particularly easy to find.

Problem 17.16: Prove that the medians of any triangle are concurrent.

Solution for Problem 17.16: We must prove a statement about medians, so we're working with midpoints. This might make us consider analytic geometry. First, we have to set up the problem on the Cartesian plane. Let $\triangle ABC$ be our triangle, and let the medians be $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. We let A be the origin and choose our axes so that B is on the x-axis, as shown at right. Since we will have to find the coordinates of the midpoints of the sides of the triangle, we let A be $(0,0)$ and B be $(2a,0)$. We have no restrictions on C except that it cannot be on the x-axis, so we let C be $(2b,2c)$, where $c \neq 0$. Again, we use $2b$ and $2c$ because we will be finding the midpoints of the sides of the triangle. Specifically, the midpoints of the sides of $\triangle ABC$ are $D = (a + b,c)$, $E = (b,c)$, and $F = (a,0)$.



Now that we have the coordinates of two points on each median, we can find the linear equations whose graphs include the medians. We then must show that there is a point (x, y) that satisfies all three equations. First, we'll find the equation for $\overleftrightarrow{AD}$. The slope of $\overleftrightarrow{AD}$ is $c/(a+b)$ and $\overleftrightarrow{AD}$ passes through $(0, 0)$, so an equation whose graph is $\overleftrightarrow{AD}$ is



$$y - 0 = \frac{c}{a+b}(x - 0).$$

Notice that the right side of this equation is undefined if $a+b = 0$. Since the slope of $\overleftrightarrow{AD}$ is $c/(a+b)$, we only have $a + b = 0$ if $\overleftrightarrow{AD}$ is a vertical line. Specifically, if $a + b = 0$, then A and D are both on the y-axis.

WARNING!! When using analytic geometry for a proof, be extra careful not to overlook special cases.

To take care of this special case, we multiply both sides of the equation by $a + b$ to get

$$(a + b)y = c(x - 0).$$

If $\overleftrightarrow{AD}$ is vertical, then $a + b = 0$, and $\overleftrightarrow{AD}$ is the graph of $x = 0$. Our equation above becomes $x = 0$ when $a + b = 0$, so this equation describes $\overleftrightarrow{AD}$ even if $\overleftrightarrow{AD}$ is a vertical line.

Notice that we use $(0, 0)$ when writing a point-slope form of the equation of $\overleftrightarrow{AD}$ instead of using $(a + b, c)$. This makes our equation considerably simpler, and is one of the reasons we like to make as many of our coordinates 0 as possible.

Concept: Don't make analytic geometry proofs harder than they need to be!

Similarly, the slope of $\overleftrightarrow{BE}$ is $c/(b - 2a)$ and $\overleftrightarrow{BE}$ passes through $(2a, 0)$, so an equation whose graph is $\overleftrightarrow{BE}$ is

$$y - 0 = \frac{c}{b - 2a}(x - 2a).$$

Multiplying both sides by $b - 2a$ gives

$$(b - 2a)y = c(x - 2a).$$

If $b = 2a$, then $\overleftrightarrow{BE}$ is the graph of $x = 2a$, since $\overleftrightarrow{BE}$ must pass through $(2a, 0)$ and (b, c). If $b = 2a$, then $(b - 2a)y = c(x - 2a)$ becomes $0 = c(x - 2a)$, which simplifies to $x = 2a$. So, the equation $(b - 2a)y = c(x - 2a)$ describes $\overleftrightarrow{BE}$ even if $\overleftrightarrow{BE}$ is a vertical line.

And finally, the slope of $\overleftrightarrow{CF}$ is $2c/(2b - a)$ and $\overleftrightarrow{CF}$ passes through $(a, 0)$, so for this line we have

$$y - 0 = \frac{2c}{2b - a}(x - a).$$

Again, we multiply both sides by the denominator of the slope to get

$$(2b-a)y = 2c(x-a),$$

which describes $\overleftrightarrow{CF}$ even if $\overleftrightarrow{CF}$ is a vertical line.

We now have the three equations

$$\begin{array}{lrcl} \overleftrightarrow{AD}: & y(a+b) & = & xc, \\ \overleftrightarrow{BE}: & y(b-2a) & = & xc-2ac, \\ \overleftrightarrow{CF}: & y(2b-a) & = & 2xc-2ac. \end{array}$$

We can find the coordinates of the intersection of the graphs of the first two equations by substituting $xc = y(a+b)$ into $y(b-2a) = xc - 2ac$. This gives us

$$y(b-2a) = y(a+b) - 2ac.$$

Subtracting $y(a+b)$ from both sides gives us $y(b-2a) - y(a+b) = -2ac$. Simplifying the left side then gives $-3ay = -2ac$. We know that $a \neq 0$, so we can divide by $-3a$ to find $y = 2c/3$. Since we also have $xc = y(a+b)$, we find $x = y(a+b)/c = 2(a+b)/3$.

So, we know that $\overline{AD}$ and $\overline{BE}$ meet at $\left(\frac{2(a+b)}{3}, \frac{2c}{3}\right)$. All we have left is to see if this point is on $\overline{CF}$. We do so by substituting $x = 2(a+b)/3$ and $y = 2c/3$ into our equation for $\overleftrightarrow{CF}$ from above. Substituting these expressions into $y(2b-a) = 2xc - 2ac$ gives us a left side of

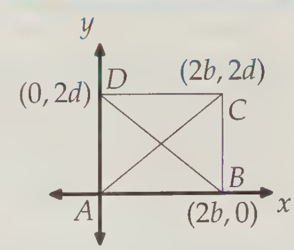
$$y(2b-a) = \frac{2c}{3}(2b-a) = \frac{4bc}{3} - \frac{2ac}{3},$$

and a right side of

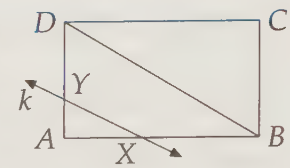
$$2xc - 2ac = 2\left[\frac{2(a+b)}{3}\right]c - 2ac = \frac{4ac}{3} + \frac{4bc}{3} - 2ac = \frac{4bc}{3} - \frac{2ac}{3}.$$

We see that the two sides are the same for the ordered pair $(x,y) = \left(\frac{2(a+b)}{3}, \frac{2c}{3}\right)$, so this point is on our third median, as well! Therefore, we can conclude that the medians of a triangle are concurrent. □

Problem 17.17: Suppose line k divides rectangle $ABCD$ into two pieces of equal area. Prove that line k passes through the intersection of the diagonals of $ABCD$.

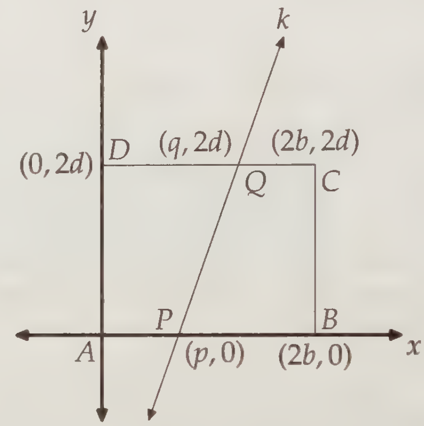


Solution for Problem 17.17: It's quite easy to describe a general rectangle with coordinates. We let A be the origin and choose the axes so that two sides are along the axes. Because $ABCD$ is a rectangle, it is also a parallelogram, so its diagonals bisect each other. This means we want to prove that k passes through the midpoint of $\overline{AC}$. Therefore, we let $B = (2b, 0)$ and $D = (0, 2d)$ (instead of using $B = (b, 0)$ and $D = (0, d)$). Point C is on the vertical line through B and on the horizontal line through D, so $C = (2b, 2d)$. This means the midpoint of $\overline{AC}$ is (b, d).



Having dealt with $ABCD$, it's time to take care of k. First, we have to think about ways in which k could intersect the rectangle. If k intersects two consecutive sides (excluding the vertices of the rectangle), then the area of the triangle it forms is smaller than the area of the triangle formed by drawing a diagonal. For example, in the diagram at right, we have $[AXY] < [ABD] = [ABCD]/2$. So, it is impossible for such a line to bisect the area of $ABCD$.

Following similar logic, if k passes through a vertex of $ABCD$ and bisects the area of $ABCD$, then line k must contain a diagonal of $ABCD$. So, line k clearly passes through the intersection point of the diagonals in this case.



The other possibility is that k intersects opposite sides of the rectangle (excluding the vertices). Shown at right is our set-up on the Cartesian plane. Line k intersects $\overline{AB}$ and $\overline{CD}$ at $P = (p, 0)$ and $Q = (q, 2d)$ as shown. If k bisects the area of $ABCD$, then it splits it into two trapezoids with equal area. The bases of trapezoid $APQD$ have lengths p and q, and $APQD$ has height $2d$, so we have

$$[APQD] = (2d)\left(\frac{p+q}{2}\right) = d(p+q).$$

The bases of $PBCQ$ have lengths $2b - p$ and $2b - q$, so we have

$$[PBCQ] = (2d)\left(\frac{2b-p+2b-q}{2}\right) = d(4b-p-q).$$

Setting these equal gives us $d(p+q) = d(4b-p-q)$. Dividing by d gives us $p+q = 4b-p-q$, so $2(p+q) = 4b$. Finally, we find that $p + q = 2b$. (We also could have used the fact the $[APQD] = \frac{1}{2}[ABCD]$ to show that $p + q = 2b$.)

But how does this help? We want to show that $\overleftrightarrow{PQ}$ passes through the midpoint of $\overline{AC}$. The midpoint of $\overline{AC}$ is (b, d). Now that we know that $p + q = 2b$, we see that the midpoint of $\overline{PQ}$ is

$$\left(\frac{p+q}{2}, \frac{0+2d}{2}\right) = (b, d).$$

Therefore, line k passes through the intersection of the diagonals of $ABCD$. □

Exercises

17.4.1 In part (a) of Problem 17.4.2, we will use analytic geometry to prove a fact about parallelograms. Which of the following groups of vertices can we use to represent the vertices of the parallelogram, to prove a fact for all parallelograms? If a group cannot be used, describe a parallelogram whose vertices cannot be represented by the given points. You can assume all variables represent nonzero numbers.

For example, if we wished to prove a result about all triangles, we cannot use $(0,0)$, $(a,0)$, and $(0,b)$ to represent the vertices of the triangle. This is because any triangle with these points as vertices is a right triangle. So, these points cannot represent the vertices of any triangle that is not a right triangle.

(a) $(0,0)$, $(a,0)$, (b,b), (b,c)

(b) $(a,0)$, $(0,b)$, $(-a,0)$, $(0,-b)$

(c) $(0,0)$, (b,b), $(b+a,b)$, $(a,0)$

(d) $(0,0)$, $(b-a,c)$, (b,c), $(a,0)$

17.4.2

(a) Use analytic geometry to prove that the diagonals of a parallelogram bisect each other.

(b)★ Use analytic geometry to prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

17.4.3 Use analytic geometry to show that if the diagonals of a quadrilateral are perpendicular and bisect each other, then the quadrilateral is a rhombus.

17.4.4★ Notice that in Problem 17.16, the sum of the equations we found for $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ is the equation for $\overleftrightarrow{CF}$. Why could we have used this observation to deduce that $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ are concurrent?

17.5 Distance Between a Point and a Line

The distance between a point and a line is the shortest distance from the point to a point on the line.

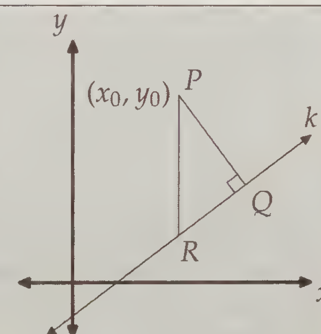
Problems

Problem 17.18: Find the distance between the point $(2,7)$ and the graph of the line $x - 3y = 1$.

Problem 17.19: In this problem, we find a formula for the distance between the point (x_0, y_0) and the line $Ax + By + C = 0$, where A, B, C, x_0, and y_0 are all constants. We'll find a formula assuming that $AB \neq 0$; you'll be asked to show that this formula also works when $AB = 0$ as an Exercise.

Let k be the line, P be the point (x_0, y_0), and Q be the foot of the altitude from P to k. Let R be the point on k that has x-coordinate x_0.

For the following parts, assume P is above line k on the Cartesian plane, and that k has a positive slope. (The proofs for all other possibilities are essentially the same.)



(a) Explain why $QR/PQ = -A/B$.

(b) Explain why $PR = (Ax_0 + By_0 + C)/B$. (Hint: Remember that R is on k, and has the same x-coordinate as P. By how much do we have to increase the y-coordinate of R to get the y-coordinate of P?)

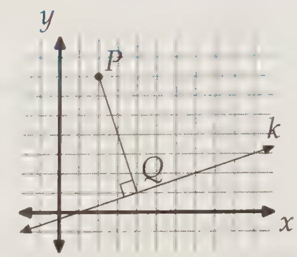
(c) Show that the distance from point P to line k is $\dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$.

Problem 17.20: Use your formula from Problem 17.19 to check your answer to Problem 17.18.

Problem 17.21: How many ordered pairs (x, y) satisfy both $14x - 48y + 49 = 0$ and $x^2 + y^2 = 1$?

We start by finding the distance between a specific point and a specific line.

Problem 17.18: Find the distance between the point $(2,7)$ and the graph of the line $x - 3y = 1$.



Solution for Problem 17.18: Let point P be $(2,7)$ and let line k be the graph of the line $x-3y=1$. Let Q be the point on k that is closest to P, so we have $\overleftrightarrow{PQ} \perp k$. Therefore, the product of the slopes of these lines is -1. Writing $x-3y=1$ in slope-intercept form gives $y = x/3 - 1/3$, so the slope of k is $1/3$. Therefore, the slope of $\overline{PQ}$ is -3. Since Q is on the line through $(2,7)$ with slope -3, it is on the graph of $y-7=-3(x-2)$. Rearranging this equation gives $3x+y=13$.

We now have two equations whose graphs pass through point Q, so we find the coordinates of Q by solving the system of equations

$$\begin{aligned} x-3y &= 1,\\ 3x+y &= 13. \end{aligned}$$

Solving this system of equations gives $(x,y)=(4,1)$, so point Q is $(4,1)$. As a quick check, we note that the slope of $\overline{PQ}$ is $(1-7)/(4-2)=-3$, so $\overline{PQ}$ is indeed perpendicular to k. (If this slope had not come out to -3, then we would have known we made a mistake.)

> **Concept:** Checking intermediate steps while working on problems will help you catch a lot of errors.

Now, we use the distance formula to find

$$PQ = \sqrt{(2-4)^2+(7-1)^2} = \sqrt{4+36} = 2\sqrt{10}.$$

□

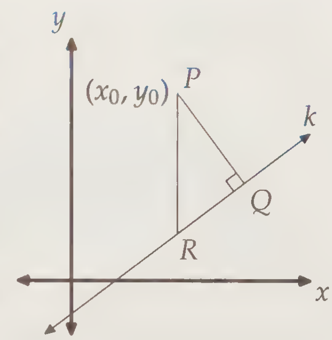
We could follow the procedure we used in Problem 17.18 to find the distance between any point and any line. However, the procedure requires a lot of steps, and often the numbers will get pretty ugly. Instead, let's see if we can find a formula to use to find the distance between a point (x_0, y_0) and the line that is the graph $Ax+By+C=0$.

> **Problem 17.19:** Find a formula for the distance between the point (x_0, y_0) and the line $Ax+By+C=0$, where A, B, C, x_0, and y_0 are all constants.

Solution for Problem 17.19: Let P be the point (x_0, y_0) and k be the graph of $Ax+By+C=0$. We could proceed as we did in our solution to Problem 17.18, by finding the slope of k, then finding the equation of the line through P that is perpendicular to k. Then, we find the intersection of k and this new line, and then...

That looks like a lot of work. Before diving into pages of algebra, let's see if we can use some geometric insights to simplify the problem.

> **Concept:** Algebra is not the only tool we have to solve problems about the Cartesian plane. Combining geometric insights with algebra can lead to very nice solutions.



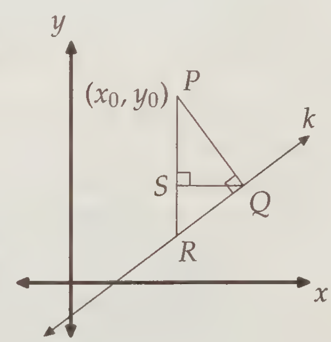
We start by drawing a diagram with P, k, and the perpendicular segment from P to k. Let point Q be the foot of the perpendicular from P to k. We seek a length, and we have a right angle, so we think about building right triangles. We build a right triangle by drawing a vertical segment from P to line k, meeting k at point R, as shown.

Since P and R are on the same vertical line, the x-coordinate of R is x_0. Since R is on k, its coordinates satisfy the equation $Ax + By + C = 0$. We can now use this equation to find the y-coordinate of R. Let y_R be the y-coordinate of R, so R is (x_0, y_R). Because R is on k, we must have

$$Ax_0 + By_R + C = 0.$$

Solving for y_R gives us

$$y_R = \frac{-Ax_0 - C}{B}.$$



It's still not clear how we can find the coordinates of Q. Right triangles make us think of similar triangles. Drawing the altitude $\overline{QS}$ to the hypotenuse of right triangle $\triangle PQR$ gives us plenty of similar triangles.

But what good are they? We haven't used any information about the equation of our line yet, so we focus on that. When we do so, we see that we can relate lengths of segments in our diagram to the slope of k. Specifically, we see that the slope of line k equals SR/QS. (In our diagram, we assume the slope of k is positive. Essentially the same approach will work if the slope of k is negative.) We can also write the slope of k in terms of A and B. Putting the equation $Ax + By + C = 0$ in slope-intercept form gives

$$y = -\frac{A}{B}x - \frac{C}{B},$$

so the slope of k is $-A/B$. Therefore, we have

$$\frac{SR}{QS} = -\frac{A}{B}.$$

Our similar triangles give us a way to relate $\triangle QSR$ to $\triangle PQR$. We have $\triangle QSR \sim \triangle PQR$, so

$$\frac{SR}{QS} = \frac{QR}{PQ}.$$

Combining this with $SR/QS = -A/B$ gives us $QR/PQ = -A/B$. Rearranging this gives $PQ = (-B/A)(QR)$.

Unfortunately, it's not so clear how to find QR. But we can find PR. Points P and R have the same x-coordinate, so PR equals the difference in the y-coordinates of P and R:

$$PR = y_0 - y_R = y_0 - \frac{-Ax_0 - C}{B} = \frac{Ax_0 + By_0 + C}{B}.$$

Now, we're close. We have PR, and we can relate PQ to QR. Finally, we use the Pythagorean Theorem to finish. We have $PR^2 = PQ^2 + QR^2$ and $QR = -(A/B)PQ$, so we have

$$\left(\frac{Ax_0 + By_0 + C}{B}\right)^2 = PQ^2 + \left(\frac{A^2}{B^2}\right) PQ^2.$$

Multiplying both sides by B^2 gives

$$(Ax_0 + By_0 + C)^2 = B^2 \cdot PQ^2 + A^2 \cdot PQ^2 = (A^2 + B^2)(PQ^2).$$

Dividing both sides by $A^2 + B^2$, then taking the square root of both sides, gives

$$PQ = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

We need the absolute value on the right side because length must be positive.

Our proof does not address the cases in which k is horizontal or vertical – you'll be asked to tackle these cases as an Exercise. Moreover, our proof assumes that the slope of k is positive, and that P is above the line. However, the proofs are essentially the same for other possible configurations in which k is neither horizontal nor vertical.

Also, our proof doesn't address the possibility that P is on k. We can quickly show that our formula works in this case. If $P = (x_0, y_0)$ is on the graph of $Ax + By + C = 0$, then we must have $Ax_0 + By_0 + C = 0$. When we substitute $Ax_0 + By_0 + C = 0$ in our formula, then our formula gives us a distance of 0, which is indeed the correct distance between point P and line k when P is on k. □

Important: The distance between the point (x_0, y_0) and the graph of the equation $Ax + By + C = 0$ is

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

Let's check our formula by using it to solve Problem 17.18.

Important: Whenever you derive a formula, you should test it on a specific case.

Problem 17.20: Use your formula from Problem 17.19 to check your answer to Problem 17.18 by using the formula to find the distance between $(2, 7)$ and the graph of $x - 3y = 1$.

Solution for Problem 17.20: See if you can figure out where we go wrong here:

Bogus Solution: The distance between the point $(2, 7)$ and the graph of the equation $x - 3y = 1$ is

$$\frac{|2 - 3 \cdot 7 + 1|}{\sqrt{1^2 + 3^2}} = \frac{18}{\sqrt{10}} = \frac{18}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{9\sqrt{10}}{5}.$$

Our answer in the Bogus Solution doesn't match our answer to Problem 17.18. Is our formula wrong? No! We applied it incorrectly.

WARNING!! In our formula for the distance between a point and a line, the linear equation is in the form $Ax + By + C = 0$, not $Ax + By = C$.

So, we write our linear equation as $x - 3y - 1 = 0$. The distance between $(2, 7)$ and the graph of this equation is

$$\frac{|2 - 3 \cdot 7 - 1|}{\sqrt{1^2 + 3^2}} = \frac{20}{\sqrt{10}} = \frac{20}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = 2\sqrt{10},$$

which matches our original answer. □

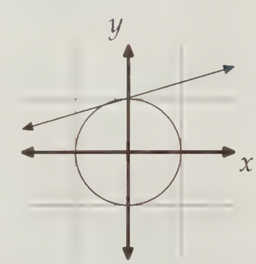
Our formula can also be used to solve less straightforward problems.

Problem 17.21: How many ordered pairs (x, y) satisfy both $14x - 48y + 49 = 0$ and $x^2 + y^2 = 1$?

Solution for Problem 17.21: We could solve the equation $14x - 48y + 49 = 0$ for x in terms of y and substitute the result into $x^2 + y^2 = 1$, but that will lead to some pretty ugly algebra. Before we do so, we think about analytic geometry, because the equation $x^2 + y^2 = 1$ is in the standard form of an equation whose graph is a circle. We like circles.

Concept: If an algebra problem has an equation whose graph is a circle, think about using analytic geometry.

The graph of the equation $x^2 + y^2 = 1$ is a circle with radius 1. The graph of the equation $14x - 48y + 49 = 0$ is a line, but does it intersect the circle? The graphs of both equations are shown at right. Unfortunately, we can't quite tell if the line hits the circle once or twice. And our graph might be off by just a little bit, so we can't even be sure if the line hits the circle at all!



Each point on the circle is 1 unit from the origin. So, if the line passes within 1 unit of the origin, it must intersect the circle twice, since it must go inside the circle. The distance between $(0, 0)$ and the graph of $14x - 48y + 49 = 0$ is

$$\frac{|14 \cdot 0 - 48 \cdot 0 + 49|}{\sqrt{14^2 + (-48)^2}} = \frac{49}{\sqrt{14^2 + (-48)^2}}.$$

Rather than multiplying out the squares in the denominator, we remember the Pythagorean triple $\{14, 48, 50\}$, so we know $14^2 + (-48)^2 = 50^2$. Therefore, the distance between the line and the origin is $49/50$. So, the line goes inside the circle, which means it intersects the circle twice. This tells us that there are two ordered pairs that satisfy both of the given equations. □

Exercises

17.5.1 Find the distance between $(3, 4)$ and the graph of the equation $4x - 3y + 7 = 0$.

17.5.2 Find the coordinates of the points on the graph of $y = 10$ that are a distance of 5 from the graph of $x + 3y = 9$.

17.5.3 Our proof of the formula for the distance between a point and a line does not address the cases in which the line is either horizontal or vertical. Complete the proof of the formula by showing that it holds if the line is horizontal or vertical.

17.5.4 Find the distance between the graphs of $3x = 4y + 8$ and $3x = 4y + 17$.

17.5.5 At how many points does the graph of $2x - 3y = 48$ intersect the graph of $(x-3)^2 + (y+2)^2 = 100$?

17.6 Advanced Analytic Geometry Problems

Problems

Problem 17.22: In this problem, we find the area of the region bounded by the lines $2x + 3y = 21$ and $5x + 2y = 25$ and the coordinate axes.

(a) Graph the two equations. Do we have a simple formula to find the area of the region?

(b) Dissect the region into pieces you know how to handle, and use these pieces to solve the problem.

Problem 17.23:

(a) What point is the image of rotating the point $(-1, 3)$ an angle of 180° about $(-6, -7)$?

(b) What point is the image of rotating the point $(4, 5)$ an angle of 90° clockwise about the point $(-4, 2)$?

(c) What point is the image of reflecting the point $(5, -3)$ over the line $y = 8$?

(d) What point is the image of reflecting the point $(3, -1)$ over the graph of the line $2x - y + 5 = 0$?

Problem 17.24: Triangles ABC and ADE have areas 2007 and 7002, respectively, with $B = (0, 0)$, $C = (223, 0)$, $D = (680, 380)$, and $E = (689, 389)$. In this problem, we find the sum of all possible x-coordinates of A. *(Source: AMC 12)*

(a) What does the information about the area of $\triangle ABC$ tell us about the possible locations of point A?

(b) What does the information about the area of $\triangle ADE$ tell us about the possible locations of point A?

(c) Combine your observations from the first two parts. How many possible points A are there?

(d) What figure is formed when you connect all the possible points A you found in part (c)?

(e) Use your observation in part (d) to find the sum of all possible x-coordinates of A, *without actually finding any of the points that A could be.*

Extra! *Well done is better than well said.*

–Benjamin Franklin

Problem 17.25: Medians $\overline{XA}$ and $\overline{YB}$ of $\triangle XYZ$ are perpendicular. In $\triangle XYZ$, we have both $\angle YXZ = 90^\circ$ and $XY = 8$. In this problem we find YZ.

(a) Set the problem up on the Cartesian plane. Choose your origin wisely, then find the coordinates of X, Y, Z, A, and B. If you choose your origin wisely, you should only need one variable.

(b) Use the information in the problem to find the value of the variable.

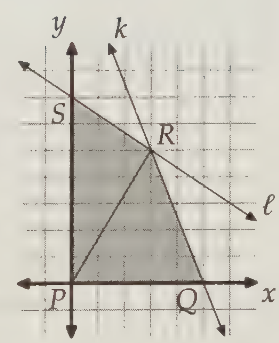
(c) Find YZ.

Problem 17.26: A point P lies in the same plane as a given square of side 1. Let the vertices of the square, taken counterclockwise, be A, B, C, and D. Also, let the distances from P to A, B, and C, respectively, be u, v, and w. What is the greatest distance that P can be from D if $u^2 + v^2 = w^2$? *(Source: AMC 12)*

Problem 17.22: Find the area of the region bounded by the lines $2x + 3y = 21$ and $5x + 2y = 25$ and the coordinate axes.

Solution for Problem 17.22: We start by graphing the lines and shading the bounded region as shown below.

Important: Just as with most geometry problems, our first step with many analytic geometry problems is drawing a diagram.



$PQRS$ is a quadrilateral, but it's not one of our special types of quadrilateral. So, it's not immediately obvious how to find the area of the shaded region. Therefore, we use a tactic from our study of funky areas: we divide the shaded region into pieces we know how to handle. We draw diagonal $\overline{PR}$, which cuts the shaded region into two triangles.

We could also have cut the region into two triangles by drawing $\overline{QS}$, but we can easily find the areas of $\triangle PQR$ and $\triangle PSR$. The graph of $5x + 2y = 25$ (line k in the diagram) intersects the x-axis at $(5, 0)$ and it intersects the graph of $2x+3y = 21$ (line ℓ in the diagram) at $(3, 5)$. So, $PQ = 5$ and the altitude from R to $\overline{PQ}$ is 5. This gives us $[PQR] = (5)(5)/2 = 25/2$.

Similarly, the graph of $2x + 3y = 21$ meets the y-axis at $(0, 7)$, so $PS = 7$. The altitude from R to $\overline{PS}$ has length 3, so $[PSR] = (3)(7)/2 = 21/2$. Finally, we have

$$[PQRS] = [PQR] + [PSR] = \frac{25}{2} + \frac{21}{2} = 23.$$

We also could have solved this problem by dissecting $PQRS$ into a trapezoid and a triangle. See if you can solve the problem this way, as well. □

Let's try some transformations on the Cartesian plane.

Problem 17.23:

(a) What point is the image of rotating the point $(-1,3)$ an angle of $180°$ about $(-6,-7)$?

(b) What point is the image of rotating the point $(4,5)$ an angle of $90°$ clockwise about the point $(-4,2)$?

(c) What point is the image of reflecting the point $(5,-3)$ over the line $y = 8$?

(d) What point is the image of reflecting the point $(3,-1)$ over the line $2x - y + 5 = 0$?

Solution for Problem 17.23:

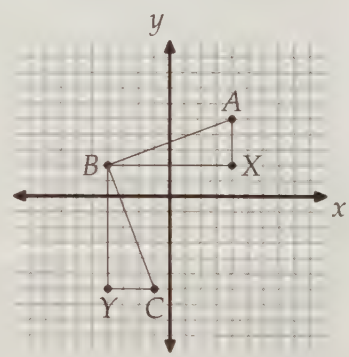
(a) Let X be $(-1,3)$, let Y be $(-6,-7)$, and let Z be the image of X upon a $180°$ rotation about Y. Because the angle of rotation is $180°$, we have $\angle XYZ = 180°$. So, point Y is on $\overline{XZ}$. Furthermore, because Z is the image of X upon rotation about Y, we have $XY = YZ$. Since Y is also on $\overline{XZ}$, we know that Y is the midpoint of $\overline{XZ}$. So, if Z is (a,b), we have

$$\left(\frac{a+(-1)}{2}, \frac{b+3}{2}\right) = (-6,-7).$$

From this, we find $a = -11$ and $b = -17$, so the image of X upon $180°$ rotation about Y is $(-11,-17)$.

We also could have solved this problem by noting that because Y is the midpoint of $\overline{XZ}$, we know that the translation that maps X to Y also maps Y to Z. To get from X to Y, we go left 5 units and down 10 units. So, to get from Y to Z, we also go left 5 units and down 10 units, to $(-6-5, -7-10) = (-11,-17)$.

(b) Let $(4,5)$ be point A and $(-4,2)$ be point B as shown in the diagram at right. If point C is the image of A upon a $90°$ clockwise rotation about B, then we must have $AB = CB$ and $\overline{AB} \perp \overline{CB}$. The slope of $\overline{AB}$ is $3/8$, so we know that the slope of $\overline{CB}$ is $-8/3$. This means that C is on the line through B with slope $-8/3$. But where on this line is it?



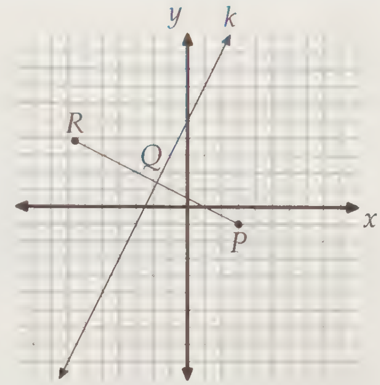
We can find point C by noting that to get from A to B, we go down 3 units and left 8 units. So, to get from point B to C, we go right 3 units and down 8 units, which means C is $(-4+3, 2-8) = (-1,-6)$. This is illustrated in the figure at right. We first build right triangle $\triangle ABX$ so that $\overline{AX}$ is vertical and $\overline{BX}$ is horizontal. When we rotate this triangle $90°$ clockwise, we get $\triangle BYC$, in which $BY = BX = 8$ and $CY = AX = 3$. Therefore, point C is 8 units below and 3 units to the right of B.

(c) The point $(5,-3)$ is 11 units below the horizontal line $y = 8$, so its image is 11 units directly above $y = 8$. Therefore, the image of $(5,-3)$ upon reflection over the graph of $y = 8$ is $(5, 8+11) = (5,19)$. We can quickly check this by noting that the graph of $y = 8$ is the perpendicular bisector of the segment with endpoints $(5,-3)$ and $(5,19)$. This tells us that $(5,19)$ is the image of $(5,-3)$ upon reflection over the graph of $y = 8$.

(d) Let line k be the graph of $2x - y + 5 = 0$. Let P be $(3,-1)$, and let R be its image upon reflection over k. Because R is the image of P upon reflection over k, $\overline{PR}$ and k are perpendicular. The slope of k is 2, so the slope of $\overline{PR}$ is $-1/2$. Therefore, R is on the line through P with slope $-1/2$. A point-slope form of this line's equation is

$$y - (-1) = -\frac{1}{2}(x-3).$$

But which point on the graph of this line is R? If we can find the point where $\overline{PR}$ meets k, we can use this point to find R, because this intersection point is the midpoint of $\overline{PR}$. Let the intersection point of $\overline{PR}$ and k be Q. We have the equations of both $\overleftrightarrow{PR}$ and k, so we can find Q by solving the system of equations



$$2x - y + 5 = 0,$$
$$y + 1 = -\frac{1}{2}(x - 3).$$

Adding these two equations gives $2x + 6 = -\frac{x}{2} + \frac{3}{2}$. Solving this equation gives $x = -9/5$. Substituting this into either of the above equations gives $y = 7/5$. So, point Q is $(-9/5, 7/5)$.

Now we're ready to find R. Let the coordinates of R be (x_R, y_R). Because Q is the midpoint of $\overline{PR}$, we must have

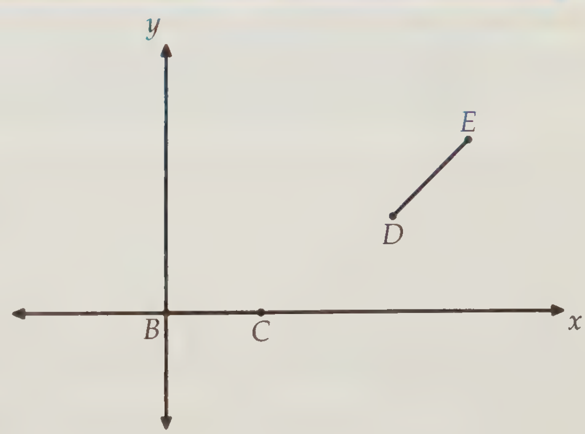
$$\left(\frac{3 + x_R}{2}, \frac{-1 + y_R}{2}\right) = \left(-\frac{9}{5}, \frac{7}{5}\right).$$

Solving $(3 + x_R)/2 = -9/5$ and $(-1 + y_R)/2 = 7/5$ gives us $(x_R, y_R) = (-33/5, 19/5)$. We could also have found the coordinates of R by noting that to get from P to Q, we go 24/5 units to the left and 12/5 units up. So, to get from Q to R, we go units 24/5 to the left and 12/5 units up from $(-9/5, 7/5)$ to $(-33/5, 19/5)$.

□

Problem 17.24: Triangles ABC and ADE have areas 2007 and 7002, respectively, with $B = (0, 0)$, $C = (223, 0)$, $D = (680, 380)$, and $E = (689, 389)$. Find the sum of all possible x-coordinates of A. *(Source: AMC 12)*

Solution for Problem 17.24: The huge numbers in the problem make it hard to draw our graph to scale, so we start with a rough sketch. We'll have to use the information about triangles ABC and ADE to determine the possible points A. We start with the fact that $[ABC] = 2007$. Base $\overline{BC}$ of this triangle has length 223, so if we let h_1 be the length of the altitude to side $\overline{BC}$ of $\triangle ABC$, we have



$$h_1 = \frac{2[ABC]}{BC} = \frac{2(2007)}{223} = 18.$$

This tells us that A is 18 units from $\overleftrightarrow{BC}$. Since $\overline{BC}$ is along the x-axis, we now know that A is on the graph of either $y = 18$ or $y = -18$.

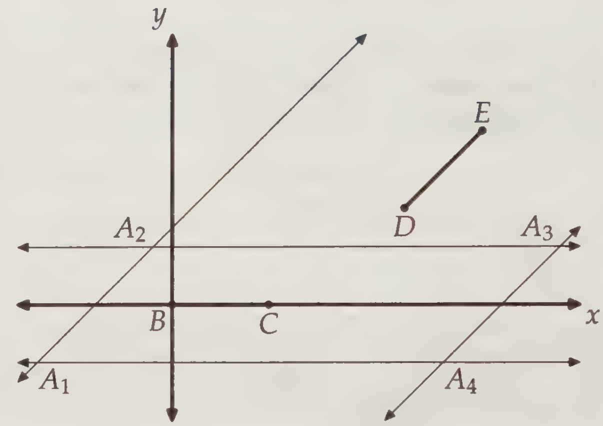
Triangle ABC gave us some information about A; let's take a look at $\triangle ADE$. The distance formula tells us that $DE = 9\sqrt{2}$, so if we let h_2 be the length of the altitude from A to $\overline{DE}$, we have

$$h_2 = \frac{2[ADE]}{DE} = \frac{2(7002)}{9\sqrt{2}} = \frac{2(778)}{\sqrt{2}} = 778\sqrt{2}.$$

So, point A is $778\sqrt{2}$ away from $\overleftrightarrow{DE}$. We could find the equations of the two lines that are $778\sqrt{2}$ away from $\overleftrightarrow{DE}$, but that looks like it might be pretty difficult. Instead, we go back to our diagram and see if we can find anything to simplify the problem.

Concept: One reason we draw diagrams for analytic geometry problems is that they might inspire geometric shortcuts.

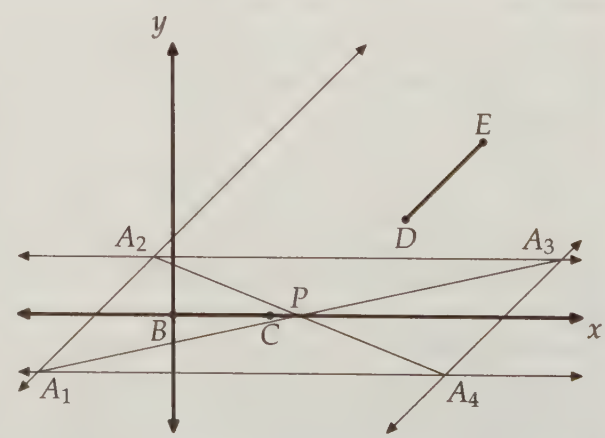
We've found that A must be on one of two lines parallel to $\overleftrightarrow{BC}$, and that these two lines are equidistant from $\overleftrightarrow{BC}$. Similarly, we found that A must be on one of two lines parallel to and equidistant from $\overleftrightarrow{DE}$. We add all four of these lines to our diagram. The four possible points A are where pairs of these four lines intersect. We label these A_1, A_2, A_3, and A_4, as shown. $A_1A_2A_3A_4$ is a parallelogram because the opposite sides of $A_1A_2A_3A_4$ are along parallel lines.



We could find the coordinates of the vertices of the parallelogram $A_1A_2A_3A_4$, but that looks like a lot of work. Instead, we think a little bit to see if we can avoid all that work. Because $A_1A_2A_3A_4$ is a parallelogram, we think about what's special about parallelograms, and how to use these special properties to find the sum of the x-coordinates of the vertices.

Specifically, we focus on parallelogram facts that might be used to discover something about this sum of x-coordinates. This gets us thinking about midpoints, since finding the midpoint of a segment requires adding coordinates. This, in turn, leads us to the fact that the diagonals of a parallelogram bisect each other, which means they have the same midpoint. In other words, if x_1, x_2, x_3, and x_4 are the x-coordinates of A_1, A_2, A_3, and A_4, respectively, then the x-coordinate of the intersection of diagonals $\overline{A_1A_3}$ and $\overline{A_2A_4}$ equals both $(x_1+x_3)/2$ and $(x_2+x_4)/2$. We want $x_1+x_2+x_3+x_4$, so we know we're close!

We let P be the intersection of diagonals $\overline{A_1A_3}$ and $\overline{A_2A_4}$, and let the coordinates of P be (x_P, y_P). We just saw above that $x_P = (x_1+x_3)/2 = (x_2+x_4)/2$. So, we have $x_1+x_3 = x_2+x_4 = 2x_P$, which means that our desired sum $x_1+x_2+x_3+x_4$ equals $4x_P$. All we have to do now is find P. We know P is the intersection of diagonals $\overline{A_1A_3}$ and $\overline{A_2A_4}$, but we would like to find a faster way to find P than finding all the vertices of $A_1A_2A_3A_4$. Our diagram gives us our slick approach. Because P is the midpoint of both diagonals, it is equidistant from opposite sides $\overline{A_1A_2}$ and $\overline{A_3A_4}$, and it is equidistant from opposite sides $\overline{A_2A_3}$ and $\overline{A_1A_4}$. Aha! $\overleftrightarrow{DE}$ is also equidistant from $\overline{A_1A_2}$ and $\overline{A_3A_4}$, and $\overleftrightarrow{BC}$ is equidistant from $\overline{A_2A_3}$ and $\overline{A_1A_4}$. So, P is the intersection point of $\overleftrightarrow{BC}$ and $\overleftrightarrow{DE}$. This point is easy to find!

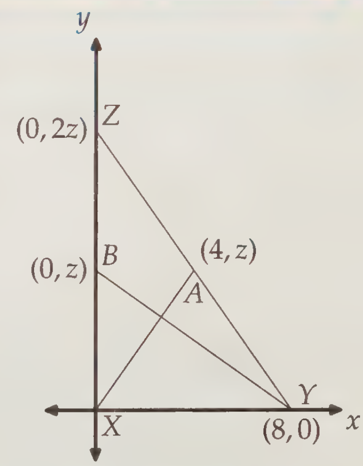


Because B and C are on the x-axis, $\overleftrightarrow{BC}$ is the x-axis. So, all points on $\overleftrightarrow{BC}$ satisfy $y = 0$. This means the y-coordinate of P is 0. The slope of $\overleftrightarrow{DE}$ is $(389 - 380)/(689 - 680) = 1$, and the line passes through $(680, 380)$, so $\overleftrightarrow{DE}$ is the graph of the equation $y - 380 = 1(x - 680)$. Rearranging this equation gives $x - y = 300$. Point P is on the graph of this line, and the y-coordinate of P is 0, so the x-coordinate of P is 300. Therefore, our desired sum of x-coordinates is $4(300) = 1200$. □

We've seen how geometric insights can help us with analytic geometry problems. Let's take a look at a couple examples of how we can use analytic geometry to tackle challenging geometric problems. Just as we saw when using analytic geometry for geometric proofs, one of the main keys in each problem is conveniently setting up the problem on the Cartesian plane.

Problem 17.25: Medians $\overline{XA}$ and $\overline{YB}$ of $\triangle XYZ$ are perpendicular. If $\angle YXZ = 90^\circ$ and $XY = 8$, find YZ.

Solution for Problem 17.25: We have two big clues to try analytic geometry: midpoints and perpendicular lines. We let X, the vertex of the right angle, be the origin. This allows us to place Y on the x-axis and Z on the y-axis. Because $XY = 8$, we can let Y be $(8, 0)$. Then, we can let Z be $(0, 2z)$ for some value of z. We use $2z$ instead of just z because we know we'll be working with midpoints in this problem. If we find z, we can use the distance formula to find YZ, since



$$YZ = \sqrt{(2z - 0)^2 + (0 - 8)^2} = \sqrt{4z^2 + 64} = 2\sqrt{z^2 + 16}.$$

We now can find the coordinates of A and B in terms of z. Because A is the midpoint of $\overline{YZ}$, its coordinates are $(4, z)$. Because B is the midpoint of $\overline{XZ}$, its coordinates are $(0, z)$. But now what?

Concept: When you're stuck on a problem, focus on information you haven't used yet.

We haven't used the fact that $\overline{XA} \perp \overline{YB}$. Because $\overline{XA} \perp \overline{YB}$, the product of the slopes of $\overleftrightarrow{XA}$ and $\overleftrightarrow{YB}$ is -1. The slope of $\overleftrightarrow{XA}$ is $z/4$ and the slope of $\overleftrightarrow{YB}$ is $-z/8$. Therefore, we have

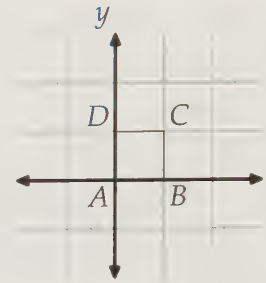
$$\left(\frac{z}{4}\right)\left(\frac{-z}{8}\right) = -1.$$

This gives us $-z^2/32 = -1$, so $z^2 = 32$. Now we can find YZ using our formula from above. We have

$$YZ = 2\sqrt{z^2 + 16} = 2\sqrt{32 + 16} = 2\sqrt{48} = 8\sqrt{3}.$$

□

Problem 17.26: A point P lies in the same plane as a given square of side 1. Let the vertices of the square, taken counterclockwise, be A, B, C, and D. Also, let the distances from P to A, B, and C, respectively, be u, v, and w. What is the greatest distance that P can be from D if $u^2 + v^2 = w^2$? *(Source: AMC 12)*



Solution for Problem 17.26: Squares are easy to represent on the Cartesian plane. We even know each side has length 1, so we can let the vertices of our square be $A = (0,0)$, $B = (1,0)$, $C = (1,1)$, and $D = (0,1)$.

Our given equation $u^2 + v^2 = w^2$ involves squares of distances. This is another clue that analytic geometry might be helpful: squaring the distances gets rid of the square root sign in the distance formula. We let the coordinates of P be (x, y), so we can use the distance formula to find:

$$\begin{aligned} u^2 &= PA^2 = (x-0)^2 + (y-0)^2, \\ v^2 &= PB^2 = (x-1)^2 + (y-0)^2, \\ w^2 &= PC^2 = (x-1)^2 + (y-1)^2. \end{aligned}$$

Substituting these into $u^2 + v^2 = w^2$ gives us

$$x^2 + y^2 + (x-1)^2 + y^2 = (x-1)^2 + (y-1)^2.$$

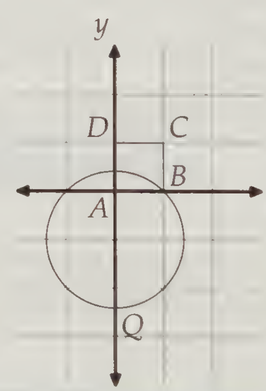
The $(x-1)^2$ terms cancel, and a little rearranging gives

$$x^2 + y^2 + 2y - 1 = 0.$$

The graph of this equation is a circle. To learn more about this circle, we complete the square in y on the left. This gives us

$$x^2 + (y+1)^2 = 2.$$

This means that (x, y) is on a circle with center $(0, -1)$ and radius $\sqrt{2}$. Moreover, we can reverse our steps above to see that any point on the circle satisfies the restrictions placed on point P in the problem. So, now our problem is to find the point on this circle that is farthest from point D.



We add this circle to our diagram at right. Because both the center of the circle and point D are on the y-axis, the point on the circle that is the farthest from D is on the opposite side of the circle from D. In other words, it is the 'bottommost' point of the circle, which we label as Q in the diagram. Because D is 2 units from the center of the circle, and the radius of the circle is $\sqrt{2}$, the distance between D and Q is $2 + \sqrt{2}$, which is therefore our desired greatest possible distance. □

Exercises

17.6.1 What is the image of $(5, 6)$ under each of the following transformations:

(a) A 180° rotation about the origin.

(b) The translation that maps $(2, -3)$ to $(0, 6)$.

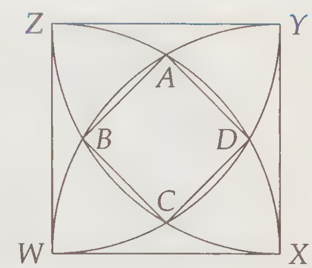
(c) Reflection over the graph of $x = -3$.

(d) A 90° counterclockwise rotation about $(-3, 2)$.

(e) Reflection over the graph of $y = x$.

(f) Reflection over the graph of $2x + 3y = -5$.

17.6.2 Two nonadjacent vertices of a rectangle are $(4, 3)$ and $(-4, -3)$, and the coordinates of the other two vertices are integers. How many different such rectangles are there? *(Source: AMC 12)*



17.6.3 $WXYZ$ is a square with $WX = 6$. Quarter circles centered at each vertex of $WXYZ$ are drawn inside the square as shown. The intersections of these quarter circles form square $ABCD$ shown.

(a) Find the area of $ABCD$ with analytic geometry.

(b) Find the area of $ABCD$ without analytic geometry.

17.6.4 Two parallel lines intersect the x-axis at points that are 5 apart. The same two lines intersect the y-axis at points that are 12 apart. How far apart are the two lines?

17.6.5 Let point C be $(3, -2)$. Points A and B are on the graph of $2x - y = -7$ such that $\triangle ABC$ is equilateral. Find AB.

17.6.6 Solve Problem 17.25 without using analytic geometry. **Hints:** 42

17.6.7★ Given that $x^2 + y^2 = 14x + 6y + 6$, what is the largest possible value that $3x + 4y$ can have? (*Source: AHSME*) **Hints:** 2

17.6.8★ Let $\overline{AC}$ and $\overline{BD}$ be two perpendicular chords of a circle with radius 8, and let the two chords intersect at P. Find all possible values of $PA^2 + PB^2 + PC^2 + PD^2$.

17.7 Summary

Important: The graph of the equation $y - y_1 = m(x - x_1)$ is a line through (x_1, y_1) with slope m. This is called a **point-slope form** of the equation.

The **standard form** of a linear equation is $Ax + By = C$, where, if possible, A, B, and C are integers, A is positive, and A, B, and C have no common factors besides 1.

The **slope-intercept form** of a linear equation is $y = mx + b$, where m is the slope of the line and b is the y-coordinate of the y-intercept.

Important: The distance in the plane between the points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is often referred to as the **distance formula**.

Important: The midpoint of the segment with endpoints (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Important: If two non-vertical lines have the same slope, then they are parallel. Conversely, if two non-vertical lines are parallel, then they have the same slope.

If two non-vertical lines are perpendicular, then the product of their slopes is -1. Conversely, if the product of the slopes of two lines is -1, then the two lines are perpendicular.

Important: The **standard form** of an equation whose graph is a circle is

$$(x-h)^2 + (y-k)^2 = r^2,$$

where h, k, and r are constants with $r > 0$. The center of the circle is (h, k) and the radius of the circle is r.

Problem Solving Strategies

Concepts:

- When the algebra in an analytic geometry problem starts to get ugly, try using geometric facts to simplify the problem.
- When given the coordinates of the vertices of a triangle, check if the triangle is special in any way. Most notably, check if the triangle is a right triangle. If it is, this fact will probably simplify the problem, because we know so much about right triangles.
- Interpreting an equation as a geometric figure on the Cartesian plane can often help us solve problems.
- When setting a geometry problem up on the Cartesian plane, choose the origin and the axes in a convenient matter. Often this means letting the origin be the vertex of a right angle in the problem, so that the sides of the angle are along the axes.
- Solving a problem in two ways is a good way to check your answer.
- Checking intermediate steps while working on problems will also help you catch a lot of errors.
- When we set up a geometry problem involving midpoints on the Cartesian plane, we often use $2a$, $2b$, $2c$, etc., for coordinates rather than just a, b, c, etc. This helps us avoid expressions involving fractions when we find midpoints of segments in the problem.

Continued on the next page. . .

Concepts: . . . *continued from the previous page*

- Algebra is not the only tool we have to solve problems about the Cartesian plane. Combining geometric insights with algebra can lead to very nice solutions.
- When stuck on a problem, focus on information that you haven't used yet.
- If an algebra problem has an equation whose graph is a circle, think about using analytic geometry.
- One reason we draw diagrams for analytic geometry problems is that they might inspire geometric shortcuts.

REVIEW PROBLEMS

17.27 Find an equation, in standard form, whose graph is the line through $(-5,2)$ that is perpendicular to the graph of $5x-2y+7=0$.

17.28 Find the area of the triangle with vertices $(3,4)$, $(8,4)$, and $(-6,-6)$.

17.29 Let A be $(-2,4)$, B be $(-7,1)$, and C be $(-1,-5)$.

(a) Find an equation whose graph is a line that contains the median from A to the midpoint of $\overline{BC}$.

(b) Find the length of the median from A to $\overline{BC}$.

(c) Find the length of the altitude from A to $\overline{BC}$.

(d) Find the area of $\triangle ABC$.

17.30 Find the constant a such that the graph of $(2a-3)x+(3a-1)y=3$ is parallel to the segment with endpoints $(2,-4)$ and $(-1,2)$.

17.31 Graph the equation $x^2-6x+y^2+4y=-3$. Find the area enclosed by your graph.

17.32 Let P be $(-2,5)$ and Q be $(8,-3)$. Find the standard form of the equation of the circle with $\overline{PQ}$ as a diameter.

17.33 Let point X be $(10,1)$, and let C be the graph of $(x-1)^2+(y+2)^2=10$.

(a) Let Y be the point on C that is closest to X. Find XY.

(b)★ Find the coordinates of Y.

17.34 Find all possible values of a such that the distance between the point $(a,3)$ and the graph of $5x-12y=8$ is 7.

17.35 Point A is $(-3,5)$. The midpoint of $\overline{AB}$ is $(4,2)$ and the midpoint of $\overline{AC}$ is $(3,1)$. What is BC?

17.36 Point O is the origin and point B is $(0, 13)$. Point A has positive coordinates such that $AO = 5$ and $AB = 12$. Find the coordinates of A.

17.37 What is the image of $(5, -2)$ when reflected over each of the following:

(a) The x-axis.

(b) The graph of $x = -3$.

(c)★ The graph of $2x - y = -1$.

17.38 Find the length of the segment that is a chord of both the graph of $x^2 + y^2 = 36$ and the graph of $x^2 + 2x + y^2 = 30$.

17.39 Use analytic geometry to show that if $AB = AC$ in $\triangle ABC$, then the altitude from A to $\overline{BC}$ is also the median from A to $\overline{BC}$.

17.40 Two adjacent vertices of a square are $(-4, 5)$ and $(-3, 7)$. What are all possible points besides these two that could be vertices of this square?

17.41 Show that the image of the point (a, b) upon reflection over the line $y = x$ is (b, a).

17.42 Point X is $(2, 3)$ and point Y is $(6, 3)$. Find all possible points P such that $\triangle XYP$ is equilateral.

17.43 Use analytic geometry to show that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of the lengths of the sides of the parallelogram.

17.44 Find the equation whose graph is the circle that passes through the points $(-1, 5)$, $(4, 4)$, and $(5, 9)$.

17.45 In triangle ABC, let D, E, F be the midpoints of $\overline{BC}$, $\overline{AC}$, $\overline{AB}$, respectively.

(a) Show that triangles ABC and DEF have the same centroid.

(b) Let G_1, G_2, G_3 be the centroids of triangles AFE, BDF, CED, respectively. Show that triangles ABC and $G_1G_2G_3$ have the same centroid.

17.46 Points X and Y are on $\overline{AB}$ such that $AX = XY = BY$. If A is $(0, 9)$ and B is $(4, 0)$, what is the slope of the line through X and the origin?

17.47 Point P is $(6, 2)$. Points Q and R are on the x-axis such that $\angle PQR = 30°$ and $\angle PRQ = 60°$. What is the area of $\triangle PQR$?

17.48 Point X is $(4, -3)$, point Y is $(-2, 5)$, and point Z is $(c, 3)$.

(a) For what values of c is $\triangle XYZ$ isosceles?

(b) For what values of c is $\triangle XYZ$ a right triangle?

Challenge Problems

17.49 Find an equation whose graph is a circle that passes through the origin and both the x-intercept and the y-intercept of the graph of $x - 3y = 18$.

17.50

(a) The graph of the equation $3x - y = 0$ is rotated 90° clockwise about the origin to produce line k. Find an equation whose graph is line k.

(b) The graph of the equation $3x - y = 6$ is rotated 90° clockwise about the origin to produce line k. Find an equation whose graph is line k.

17.51 The graphs of $x^2 + y^2 = 4 + 12x + 6y$ and $x^2 + y^2 = k + 4x + 12y$ intersect when k satisfies $a \le k \le b$, and for no other values of k. Find $b - a$. *(Source: AMC 12)*

17.52 How many points are equidistant from the x-axis, the y-axis, and the graph of the equation $x + y = 2$? *(Source: AMC 12)*

17.53 Point X is on diameter $\overline{PQ}$ of a circle. Prove that if $\overline{AB}$ is a chord of the circle parallel to $\overline{PQ}$, then $XA^2 + XB^2 = XP^2 + XQ^2$.

17.54 Let U be the point $(-4, 3)$ and V be the point $(2, -1)$.

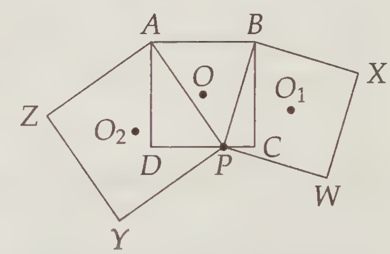
(a) Let A be $(-1, 3)$. Let A' be the image when A is rotated 90° clockwise about U. Let A'' be the image when A' is rotated 90° clockwise about V. Find A''.

(b) Let B be $(7, -2)$. Let B' be the image when B is rotated 90° clockwise about U. Let B'' be the image when B' is rotated 90° clockwise about V. Find B''.

(c) Find the midpoint of $\overline{AA''}$ and the midpoint of $\overline{BB''}$. Notice anything interesting?

(d)★ Let C be (a, b). Let C' be the image when C is rotated 90° clockwise about U. Let C'' be the image when C' is rotated 90° clockwise about V. Show that your observation in part (c) can be used to quickly find C'' for any a and b.

17.55 Let $ABCD$ be a square, and let P be a point on side $\overline{CD}$. Construct squares $BPWX$ and $APYZ$ externally on $\overline{BP}$ and $\overline{AP}$, respectively, as shown. Let O, O_1, and O_2 be the centers of squares $ABCD$, $BPWX$, and $APYZ$, respectively. Show that quadrilateral OO_1PO_2 is a parallelogram.

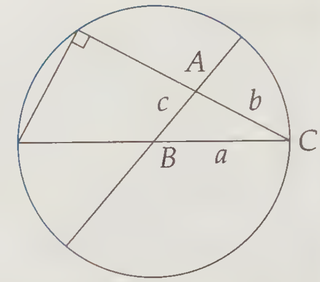


17.56★ Find the largest value of y/x for pairs of real numbers (x, y) that satisfy $(x - 3)^2 + (y - 3)^2 = 6$. *(Source: AHSME)*

17.57★ Let $\overline{A_1A_2}$ and $\overline{B_1B_2}$ be two perpendicular chords of a circle with radius r, each passing through a fixed point P. Show that $(A_1A_2)^2 + (B_1B_2)^2$ is independent of the position of the chords. (In other words, show that for all pairs of perpendicular chords of this circle that meet at P, the sum of the squares of the lengths of the chords is the same.)

Extra! *...fain would I turn back the clock and devote to French or some other language the hours I spent upon algebra, geometry, and trigonometry, of which not one principle remains with me. Stay! There is one theorem painfully drummed into my head which seems to have inhabited some corner of my brain since that early time: "The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides!" There it sticks, but what of it, ye gods, what of it?*

–Jessie B. Rittenhouse



Law of Cosines

New facts often trigger new ideas. – Alex F. Osborn

CHAPTER 18

Introduction to Trigonometry

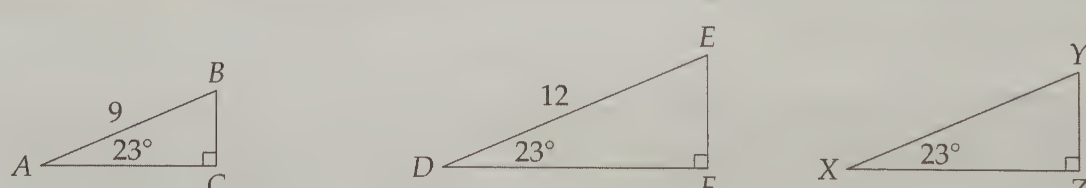
Trigonometry begins with the relationships among the angles and the sides of a triangle. We'll start by exploring these relationships for right triangles, and then we'll expand our study to all triangles.

The main purpose of this chapter is to give you a little taste of trigonometry, so that it won't be brand-new when you dive more deeply into the subject. We'll explore trigonometry in much more detail in later textbooks. Hopefully, at that time, this taste will make you a little more comfortable with trigonometry as you learn to apply it more broadly.

18.1 Trigonometry and Right Triangles

Problems

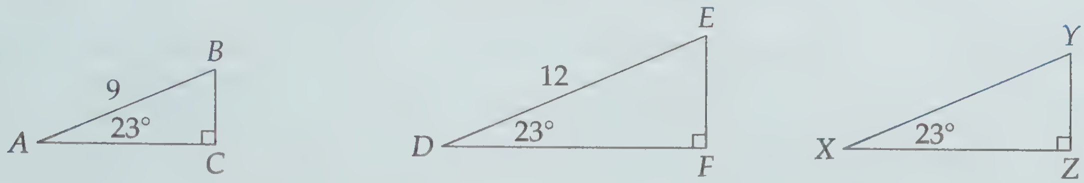
Problem 18.1: You may use a calculator on this problem.



(a) In the diagram above, $BC \approx 3.5$. What is EF to the nearest tenth?

(b) Use the given information about $\triangle ABC$ to find YZ/XY to the nearest 0.01.

Problem 18.1: You may use a calculator on this problem.



(a) In the diagram above, $BC \approx 3.5$. What is EF to the nearest tenth?

(b) Find YZ/XY to the nearest 0.01.

Solution for Problem 18.1:

(a) We have $\triangle ABC \sim \triangle DEF$ by AA Similarity. Therefore, we have $BC/AB = EF/DE$, so

$$EF = (DE)\left(\frac{BC}{AB}\right) = (12)\left(\frac{3.5}{9}\right) \approx 4.7.$$

(b) We have $\triangle ABC \sim \triangle XYZ$ by AA Similarity. Therefore, we have $YZ/XY = BC/AB \approx 0.39$. Notice that EF/DE is also approximately 0.39, which shouldn't be surprising because our triangle similarities mean that $YZ/XY = BC/AB = EF/DE$.

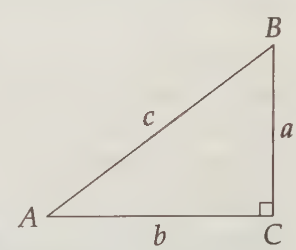
□

The ratio

$$\frac{\text{length of a leg opposite an acute angle of a right triangle}}{\text{length of the hypotenuse of the right triangle}}$$

is so common and important that we give it a name. This ratio is called the **sine** of the acute angle. We denote the sine of $\angle A$ as $\sin A$.

In $\triangle ABC$ at right, we have $\angle C = 90°$, $BC = a$, $CA = b$, and $AB = c$. So, our definition of sine gives us



$$\sin A = \frac{\text{length of leg opposite } A}{\text{length of hypotenuse}} = \frac{a}{c}.$$

In Problem 18.1, we used similar triangles to show that in all right triangles with an acute angle of 23°, the ratio of the length of the leg opposite the 23° angle to the length of the hypotenuse is the same. Specifically, we found that this ratio is approximately 0.39. So, we write $\sin 23° \approx 0.39$ to indicate that in any right triangle with an acute angle of 23°, the ratio

$$\frac{\text{length of the leg opposite the 23° angle}}{\text{length of the hypotenuse}}$$

is approximately 0.39.

We used similar triangles in Problem 18.1 to see that the ratio denoted by $\sin 23°$ is the same for an acute angle measuring 23° in any right triangle. Similarly, we can show that for any value of x between 0° and 90°, the ratio denoted by $\sin x$ is the same in any right triangle with an acute angle measuring x.

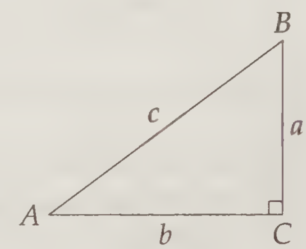
This is why we define sine for an angle measure, not for a specific triangle – in any right triangle with acute angle 23°, the side length ratio referred to by $\sin 23^\circ$ is the same.

As noted earlier, we denote the sine of $\angle A$ as simply $\sin A$, rather than writing $\sin \angle A$. Because sine is a function of an angle measure, we'll assume that the $\angle$ symbol is implied when writing $\sin A$. However, we typically include the angle symbol when identifying an angle with three letters, so we'll write $\sin \angle XYZ$ rather than $\sin XYZ$. These conventions are common but not universal. Some sources will write $\sin XYZ$ instead of $\sin \angle XYZ$, and some write $\sin \angle A$ instead of $\sin A$.

The ratio denoted by sine is not the only useful ratio in a right triangle. The **cosine** of an acute angle in a right triangle equals

$$\frac{\text{length of the leg adjacent to the angle}}{\text{length of the hypotenuse}}.$$

We denote the cosine of $\angle A$ as $\cos A$, so in our diagram at right, we have



$$\cos A = \frac{\text{length of the leg adjacent to the angle}}{\text{length of the hypotenuse}} = \frac{b}{c}.$$

Finally, the **tangent** of an acute angle in a right triangle is the ratio of the sine of the angle to the cosine of the angle. We denote the tangent of $\angle A$ as $\tan A$, so we have

$$\tan A = \frac{\sin A}{\cos A}.$$

Sine, cosine, and tangent are examples of **trigonometric functions**. We often refer to problems involving trigonometric functions as an area of math called **trigonometry**.

Just as we did with $\sin x$, we can show that the ratio denoted by $\cos x$ is the same in any right triangle with an acute angle with measure x, as is the ratio denoted by $\tan x$.

Therefore, the value $\sin 23^\circ$ is a number, just like $\sqrt{7}$ is a number. The same is true for any trigonometric function of an acute angle. For most angles, we need to resort to a calculator to quickly find the value of the sine, cosine, or tangent of the angle. But there are some angles for which we can use our understanding of special right triangles to find the sine, cosine, and tangent of the angle without a calculator.

Problems

Problem 18.2: Show that the tangent of an acute angle in a right triangle equals the ratio

$$\frac{\text{length of the leg opposite the angle}}{\text{length of the leg adjacent to the angle}}.$$

Problem 18.3: Suppose $\triangle ABC$ is a 45-45-90 right triangle with $\angle C = 90^\circ$. Use $\triangle ABC$ to find $\sin 45^\circ$, $\cos 45^\circ$, and $\tan 45^\circ$.

Problem 18.4:
(a) Find $\sin 30°$, $\cos 30°$, and $\tan 30°$.
(b) Find $\sin 60°$, $\cos 60°$, and $\tan 60°$.

Problem 18.5: In $\triangle PQR$, we have $\angle P = 90°$, $PQ = 3$, and $QR = 7$.
(a) Find $\sin Q$ and $\cos R$.
(b) Find $\tan Q$.
(c) Find $(\sin R)^2 + (\cos R)^2$. Note: We often write $(\sin R)^2$ as $\sin^2 R$, and we do likewise for other positive powers of trigonometric functions.

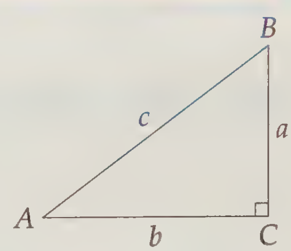
Problem 18.6: Suppose that $\angle A$ is an acute angle.
(a) Explain why $\sin A = \cos(90° - A)$.
(b) Explain why $\sin^2 A + \cos^2 A = 1$.

Problem 18.7: Let $\triangle ABC$ be a right triangle with $\angle C = 90°$.
(a) Show that $\sin A$ and $\cos A$ must be between 0 and 1.
(b) Show that $\tan A$ must be greater than 0, but that $\tan A$ has no upper bound. In other words, show that for any positive number x, there is an angle A such that $\tan A = x$.

Problem 18.2: Show that the tangent of an acute angle in a right triangle equals the ratio

$$\frac{\text{length of the leg opposite the angle}}{\text{length of the leg adjacent to the angle}}.$$

Solution for Problem 18.2: Let $\triangle ABC$ be right triangle with $\angle C = 90°$, $BC = a$, $AC = b$, and $AB = c$, as shown at right. Since tangent is the ratio of sine to cosine, we have



$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b} = \frac{\text{length of the leg opposite the angle}}{\text{length of the leg adjacent to the angle}}.$$

□

Putting the result of Problem 18.2 together with our definitions of sine and cosine, we have the following for right triangle $\triangle ABC$ above:

Important:

$$\sin A = \frac{\text{length of leg opposite } A}{\text{length of hypotenuse}} = \frac{a}{c},$$
$$\cos A = \frac{\text{length of leg adjacent to } A}{\text{length of hypotenuse}} = \frac{b}{c},$$
$$\tan A = \frac{\text{length of leg opposite } A}{\text{length of leg adjacent to } A} = \frac{a}{b}.$$

Since these are definitions, we have no choice but to simply memorize them. One common mnemonic for memorizing them is '**SOHCAHTOA**':

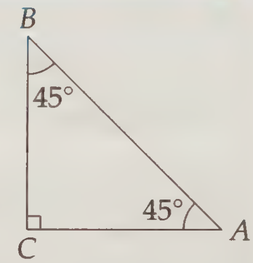
Sine = **O**pposite/**H**ypotenuse; **C**osine = **A**djacent/**H**ypotenuse; **T**angent = **O**pposite/**A**djacent.

We can't quite do SOHCAHTOA justice in print – it's a much more effective memorization tool when spoken, "So-cah-toe-ah."

We can find the values of trigonometric functions of some angles with our understanding of special right triangles.

Problem 18.3: Find $\sin 45^\circ$, $\cos 45^\circ$, and $\tan 45^\circ$.

Solution for Problem 18.3: At right is 45-45-90 triangle ABC with right angle at C. The ratio of a leg of a 45-45-90 triangle to its hypotenuse is $1/\sqrt{2}$, so we have



$$\sin 45^\circ = \sin A = \frac{BC}{AB} = \frac{1}{\sqrt{2}}.$$

We usually try to avoid writing fractions with a square root in the denominator. We can write $1/\sqrt{2}$ without a square root in the denominator by multiplying the numerator and denominator by $\sqrt{2}$:

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

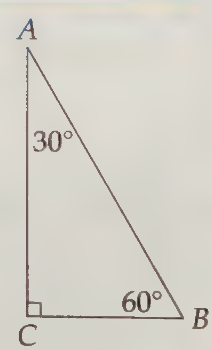
Similarly, we have $\cos 45^\circ = \cos A = \frac{AC}{AB} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Finally, since the legs of a 45-45-90 triangle have the same length, we have

$$\tan 45^\circ = \tan A = \frac{BC}{AC} = 1.$$

□

Problem 18.4: Find $\sin 30^\circ$, $\cos 30^\circ$, and $\tan 30^\circ$. Find $\sin 60^\circ$, $\cos 60^\circ$, and $\tan 60^\circ$.

Solution for Problem 18.4: We start with $\triangle ABC$ at right, in which $\angle A = 30^\circ$, $\angle C = 90^\circ$, and $\angle B = 60^\circ$. Because $\triangle ABC$ is a 30-60-90 right triangle, we have $BC/AB = 1/2$, so



$$\sin 30^\circ = \frac{BC}{AB} = \frac{1}{2} \quad \text{and} \quad \cos 60^\circ = \frac{BC}{AB} = \frac{1}{2}.$$

We also have $AC/AB = \sqrt{3}/2$, so

$$\sin 60^\circ = \frac{AC}{AB} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos 30^\circ = \frac{AC}{AB} = \frac{\sqrt{3}}{2}.$$

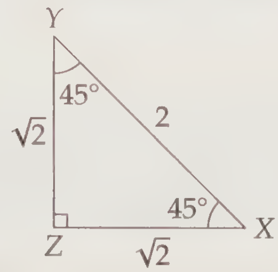
Since $AC/BC = \sqrt{3}/1$, we have

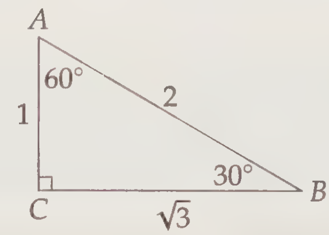
$$\tan 60^\circ = \frac{AC}{BC} = \sqrt{3} \quad \text{and} \quad \tan 30^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

□

The table below shows the values of sine, cosine, and tangent for the special angles we have explored in this section. The 45-45-90 triangle and the 30-60-90 triangle below exhibit the relationships among the sides of each triangle that we used to find these values.

Angle	sin	cos	tan
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$





You shouldn't have to memorize these! These are just another way of expressing the ratios of sides in 45-45-90 triangles and in 30-60-90 triangles.

If we have enough information about the sides of a not-so-special right triangle, we can evaluate the trigonometric functions of its angles, too.

Problem 18.5: In $\triangle PQR$, we have $\angle P = 90°$, $PQ = 3$, and $QR = 7$.

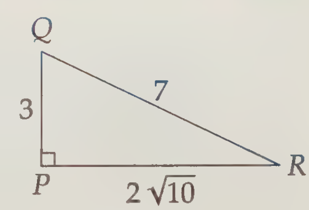
(a) Find $\sin Q$ and $\cos R$.

(b) Find $\tan Q$.

(c) Find $(\sin R)^2 + (\cos R)^2$.

Solution for Problem 18.5:

(a) We first use the Pythagorean Theorem to determine that $PR = 2\sqrt{10}$. Then, we draw our diagram with the sides labeled as at right. We have $\sin Q = PR/QR = 2\sqrt{10}/7$ and $\cos R = PR/QR = 2\sqrt{10}/7$.



(b) We have
$$\tan Q = \frac{PR}{PQ} = \frac{2\sqrt{10}}{3}.$$

(c) We first find $\sin R = PQ/QR = 3/7$. Then, we have
$$(\sin R)^2 + (\cos R)^2 = \left(\frac{3}{7}\right)^2 + \left(\frac{2\sqrt{10}}{7}\right)^2 = \frac{9}{49} + \frac{40}{49} = 1.$$

□

We typically write $(\sin R)^2$ and $(\cos R)^2$ as $\sin^2 R$ and $\cos^2 R$, respectively, and similarly for other positive integer powers of trigonometric functions. So, we'd write the result we found in part (c) as $\sin^2 R + \cos^2 R = 1$.

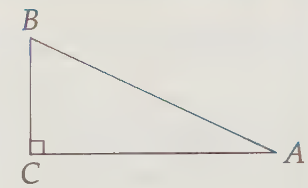
Parts (a) and (c) above suggest a couple interesting relationships between trigonometric functions. Let's investigate.

Problem 18.6: Suppose that $\angle A$ is an acute angle.

(a) Explain why $\sin A = \cos(90^\circ - A)$.

(b) Explain why $\sin^2 A + \cos^2 A = 1$.

Solution for Problem 18.6:



(a) At right is right triangle $\triangle ABC$ with acute angle $\angle A$. We have $\sin A = BC/AB$. We also have $\angle B = 90^\circ - \angle A$, so

$$\cos(90^\circ - A) = \cos B = \frac{BC}{AB}.$$

Therefore, we have $\sin A = \cos(90^\circ - A)$ for any acute angle $\angle A$.

(b) We have $\sin A = BC/AB$ and $\cos A = AC/AB$, so

$$\sin^2 A + \cos^2 A = \frac{BC^2}{AB^2} + \frac{AC^2}{AB^2} = \frac{BC^2 + AC^2}{AB^2}.$$

The Pythagorean Theorem gives us $BC^2 + AC^2 = AB^2$, so we have

$$\sin^2 A + \cos^2 A = \frac{BC^2 + AC^2}{AB^2} = \frac{AB^2}{AB^2} = 1.$$

□

A **trigonometric identity** is a statement involving trigonometric functions that is true for all values of the angles in the statement. In Problem 18.6, we proved two trigonometric identities for acute angles $\angle A$:

Important:

$$\sin^2 A + \cos^2 A = 1,$$
$$\sin A = \cos(90^\circ - A).$$

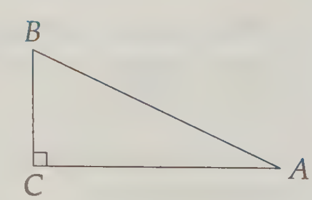
Once again, note that $\sin^2 A = (\sin A)^2$ (and likewise for all trigonometric functions raised to positive integer powers).

Problem 18.7: Let $\triangle ABC$ be a right triangle with $\angle C = 90^\circ$.

(a) Show that $\sin A$ and $\cos A$ must be between 0 and 1.

(b) Show that $\tan A$ must be greater than 0, but that $\tan A$ has no upper bound.

Solution for Problem 18.7:



(a) In $\triangle ABC$ at right, we have $\sin A = BC/AB$. Because $\overline{AB}$ is the hypotenuse of $\triangle ABC$, it is longer than both legs. Specifically, we have $AB > BC$, so $BC/AB < 1$. Furthermore, both lengths are positive, so $BC/AB > 0$. Combining these inequalities gives us $0 < \sin A < 1$.

Similarly, we have $\cos A = AC/AB$ and $AB > AC > 0$, so $0 < \cos A < 1$.

(b) Because $\tan A = BC/AC$ and BC and AC are positive, we have $\tan A > 0$. To see that $\tan A$ can take on any positive value, let $AC = 1$. Then, we have $\tan A = BC$. We have no restrictions on BC. For any positive number x, we can build a right triangle with legs $AC = 1$ and $BC = x$, so that $\tan A = x$. Therefore, $\tan A$ can take on any positive value, so there is no upper bound on the possible values of $\tan A$.

□

Important: If $\angle A$ is acute, then $0 < \sin A < 1$, $0 < \cos A < 1$, and $\tan A > 0$. There is no upper bound on the value of $\tan A$.

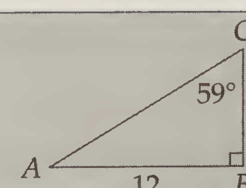
Note that while we proved that $\tan A$ can take on any positive value, we did not show that $\sin A$ and $\cos A$ can equal any positive number between 0 and 1. You'll have a chance to prove this in the Exercises.

We've seen that we can evaluate trigonometric functions for some special angles without a calculator. However, we use a calculator or a computer to find a value of the trigonometric functions of most angles. For the next set of problems, you are free to use your calculator.

WARNING!! Degrees are not the only units for angles. If you use your calculator to compute $\sin 30^\circ$ and you don't get 0.5 (or 1/2) as the result, then either you aren't using your calculator correctly, or your calculator is set to use units besides degrees for angles. If your calculator is set to use some other units of measure besides degrees, reset your calculator to degrees before continuing.

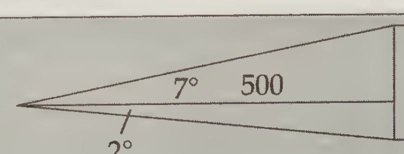
Problems

Problem 18.8: Find AC and BC in the diagram at right to the nearest tenth.

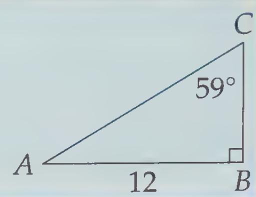


Problem 18.9: Right triangle ABC has $\angle ABC = 90^\circ$. Let the midpoint of $\overline{AB}$ be M. The perpendicular bisector of $\overline{AB}$ intersects $\overline{AC}$ at D, and $\sin \angle MDB = 1/5$. If $MD = 1$, then what is AC?

Problem 18.10: I am 500 feet from a building. When I look at the top of the building, I look upward at an angle of 7° from horizontal. When I look at the point where the building touches the ground, I look downward at an angle of 2° from horizontal. How tall is the building to the nearest foot?



Problem 18.8: Find AC and BC in the diagram at right to the nearest tenth.



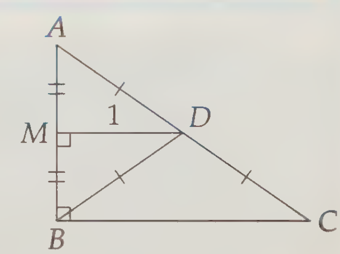
Solution for Problem 18.8: We have $\sin C = AB/AC$, so we have $AC = AB/(\sin C)$. Since $\sin C = \sin 59° \approx 0.857$, we have

$$AC = \frac{AB}{\sin C} \approx \frac{12}{0.857} \approx 14.0.$$

We can now use either the Pythagorean Theorem or trigonometry to find BC. Since we're in the trigonometry chapter, we'll use $\cos C$. Since $\cos C = BC/AC$, we have $BC = AC \cos C$. Since $C = 59°$, we have $\cos 59° \approx 0.515$, so $BC \approx (14.0)(.515) \approx 7.2$. □

Problem 18.9: Right triangle ABC has $\angle ABC = 90°$. Let the midpoint of $\overline{AB}$ be M. The perpendicular bisector of $\overline{AB}$ intersects $\overline{AC}$ at D, and $\sin \angle MDB = 1/5$. If $MD = 1$, then what is AC?

Solution for Problem 18.9: Oh no! There's no diagram! What will we do?



We start by drawing our own diagram. Don't let wordy, seemingly complicated geometry problems scare you. Take the time to draw a diagram for the problem, label it with information you know, then get to work. As you discover more about the diagram, add what you learn to the diagram.

Here, we start with the perpendicular bisector of $\overline{AB}$. The perpendicular bisectors of a triangle pass through the circumcenter of the triangle. The circumcenter of a right triangle is the midpoint of its hypotenuse. So, point D is the midpoint of $\overline{AC}$. Therefore, if we find AD, DC, or BD, we can easily find AC.

We focus on the information we haven't used yet: $MD = 1$ and $\sin \angle MDB = 1/5$. We'd like to find BD. From $\sin \angle MDB = 1/5$, we have

$$\frac{MB}{BD} = \frac{1}{5}.$$

Unfortunately, we don't know MB. However, we do know DM, and we know that

$$\frac{DM}{BD} = \cos \angle MDB.$$

So, if we can find $\cos \angle MDB$, we can find BD from this equation. We aren't given $\cos \angle MDB$, but we do know $\sin \angle MDB$, and we know that

$$\cos^2 \angle MDB + \sin^2 \angle MDB = 1.$$

Therefore, we have $\cos^2 \angle MDB = 1 - \sin^2 \angle MDB = 24/25$. We must have $\cos \angle MDB > 0$, so we have $\cos \angle MDB = 2\sqrt{6}/5$. From $DM/BD = \cos \angle MDB$, we have $BD/DM = 1/(\cos \angle MDB)$, so

$$BD = \frac{DM}{\cos \angle MDB} = \frac{5}{2\sqrt{6}} = \frac{5}{2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{5\sqrt{6}}{12}.$$

Finally, we have

$$AC = 2AD = 2BD = \frac{5\sqrt{6}}{6}.$$

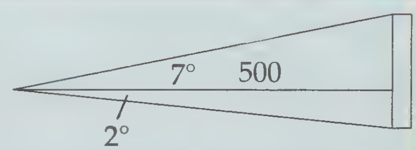
□

Our process of finding BD using $MD = 1$ and $\sin \angle MDB = 1/5$ in right triangle MDB is an example of the fact that:

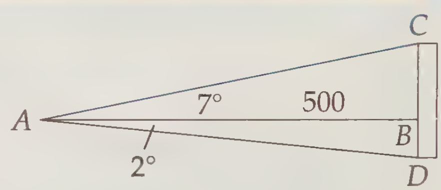
> **Important:** If we have the length of a side of a right triangle and the value of a trigonometric function of either of the acute angles of the triangle, then we can find all the side lengths of the triangle.

Trigonometric functions are used for surveying land and measuring large distances. Here's an example how.

Problem 18.10: I am 500 feet from a building. When I look at the top of the building, I look upward at an angle of 7° from horizontal. When I look at the point where the building touches the ground, I look downward at an angle of 2° from horizontal. How tall is the building to the nearest foot?



Solution for Problem 18.10: We label the diagram as shown at right. We seek the length of $\overline{CD}$. While we can't find CD directly, we can use trigonometry to find BC and BD. Specifically, we have $\tan \angle CAB = BC/BA$, so $BC = BA \tan 7^\circ \approx 61.4$. We also have $\tan \angle BAD = BD/BA$, so $BD = BA \tan 2^\circ \approx 17.5$. Therefore, to the nearest foot, the height of the building is $CD = CB + BD \approx 79$ feet. □



Sine, cosine, and tangent are not the only trigonometric functions. Three more trigonometric functions are **cosecant** (denoted csc), **secant** (denoted sec), and **cotangent** (denoted cot). These are defined as follows:

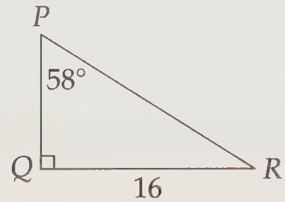
$$\begin{aligned}\csc x &= \frac{1}{\sin x}\\ \sec x &= \frac{1}{\cos x}\\ \cot x &= \frac{1}{\tan x}\end{aligned}$$

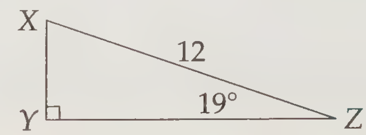
There's no need to memorize these right now – you'll have enough experience with them in later texts that you'll come to know them well.

Exercises

18.1.1 Let $\triangle ABC$ be a right triangle with $\angle ABC = 90^\circ$, $AB = 12$, and $AC = 16$. Find $\sin A$, $\cos A$, and $\tan A$.

18.1.2 Find the missing sides in the triangles below to the nearest 0.1.





18.1.3 Let x be a real number such that $0 < x < 1$. Prove that there exists an acute $\angle A$ such that $\sin A = x$, and that there exists an acute angle $\angle B$ such that $\cos B = x$. **Hints:** 1

18.1.4 Show that if θ is the measure of an acute angle, then $\cos\theta = \sin(90° - \theta)$.

18.1.5 In triangle ABC, we have $\angle B = 90°$ and $\sin A = 5/7$. Find $\tan C$.

18.1.6 Let $\overline{AB}$ be a diameter of $\odot O$, where $AB = 2$. Suppose $\overline{AX}$ is tangent to $\odot O$, and that $\overrightarrow{AY}$ bisects $\angle XAO$. The angle bisector of $\angle YAB$ intersects $\odot O$ at point Z (where A and Z are different points). Find AZ to the nearest hundredth.

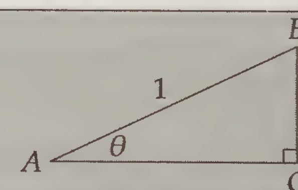
18.1.7★ Without using a calculator, evaluate $\sin 15°$. **Hints:** 513, 536

18.2 Not Just For Right Triangles

So far, we have only defined trigonometric functions for angles in right triangles. However, these functions are useful for far more than just right triangles. But to see why, we'll start with a special right triangle.

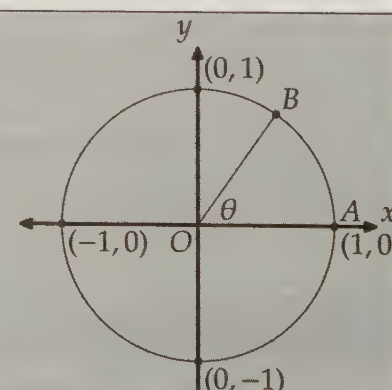
Problems

Problem 18.11: In $\triangle ABC$ at right, we have $\angle C = 90°$, $AB = 1$, and $\angle A = \theta$. Find expressions for AC and BC in terms of θ.



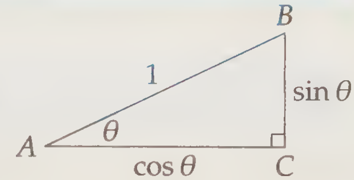
Problem 18.12: We commonly refer to the circle centered at the origin with radius 1 as the **unit circle**, which is shown at right.

Let O be the origin, A be the point $(1, 0)$ and B be a point on the unit circle such that both coordinates of B are positive and $\angle BOA = \theta$. Find the coordinates of B in terms of θ.



Problem 18.11: In $\triangle ABC$, let $\angle C = 90°$, $AB = 1$, and $\angle A = \theta$. Find expressions for AC and BC in terms of θ.

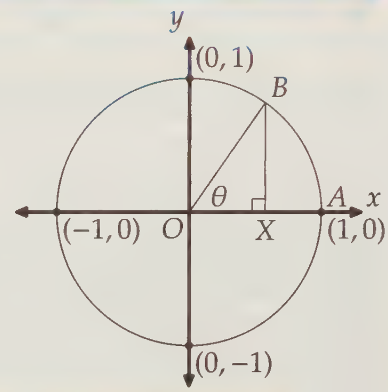
Solution for Problem 18.11: Because $AB = 1$, we have $\sin A = BC/AB = BC$. Therefore, $BC = \sin A = \sin\theta$. Similarly, we have $\cos A = AC/AB = AC$, so $AC = \cos A = \cos\theta$. These side lengths are shown in the diagram at right. □



We can extend our observation in Problem 18.11 to build a definition of sine and cosine for angles that are not acute. First, we'll take a slightly different look at depicting the cosine and sine of an acute angle.

Problem 18.12: We commonly refer to the circle centered at the origin with radius 1 as the **unit circle**. Let O be the origin, A be the point $(1, 0)$ and B be a point on the unit circle such that both coordinates of B are positive and $\angle BOA = \theta$. Find the coordinates of B in terms of θ.

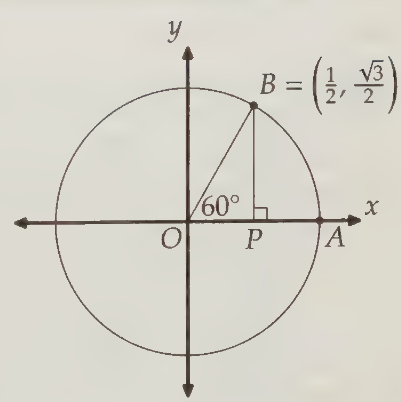
Solution for Problem 18.12: To find the coordinates of B in terms of θ, we draw altitude $\overline{BX}$ from B to the x-axis. Doing so builds a right triangle with an acute angle with measure θ. Furthermore, the hypotenuse, $\overline{OB}$, of this triangle is a radius of the unit circle, so $OB = 1$. Now, we have the same problem as in Problem 18.11, and we have $OX = \cos\theta$ and $XB = \sin\theta$. Therefore, the coordinates of B are $(\cos\theta, \sin\theta)$. □



Rather than using right triangles to define cosine and sine, we can extend our observation in Problem 18.12 to use the unit circle to define cosine and sine.

Important: Let point A be $(1, 0)$ and point B be on the unit circle such that B is θ degrees counterclockwise from A. Then, we define $\cos\theta$ and $\sin\theta$ to be the x-coordinate and y-coordinate, respectively, of point B.

For example, point B at right is on the unit circle, $60°$ counterclockwise from $(1, 0)$. Drawing altitude $\overline{BP}$ from B to the x-axis forms 30-60-90 triangle $\triangle BPO$, from which we have $BO = 1/2$ and $BP = \sqrt{3}/2$. Therefore, the coordinates of B are $(1/2, \sqrt{3}/2)$, so we have $\cos 60° = 1/2$ and $\sin 60° = \sqrt{3}/2$.



Let's see how this definition allows us to define sine and cosine of angles that are not acute.

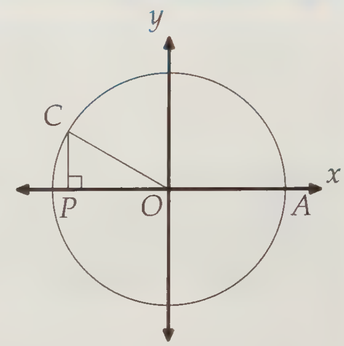
Problems

Problem 18.13: Let point A be $(1, 0)$ and let point C be on the unit circle such that $\overset{\frown}{AC} = 150°$ and such that C is $150°$ counterclockwise from A along the circle. Find the coordinates of C.

Problem 18.14: Let point A be $(1,0)$ and let point D be on the unit circle such that $\widehat{AD} = 225^\circ$ and such that D is 225° counterclockwise from A along the circle. Find the coordinates of D.

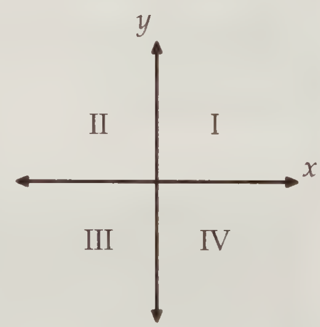
Problem 18.13: Find $\cos 150^\circ$ and $\sin 150^\circ$.

Solution for Problem 18.13: Following our definition of cosine and sine, we let A be $(1,0)$ and we let point C be the point on the unit circle that is 150° counterclockwise from A. The coordinates of C then are $(\cos 150^\circ, \sin 150^\circ)$. To find these coordinates, we first determine where point C is along the unit circle. The point $(0,1)$ is 90° counterclockwise from A and $(-1,0)$ is 180° counterclockwise from A, so C is on the minor arc from $(0,1)$ to $(-1,0)$. At right, we have our unit circle, the origin, O, and points A and C.



Once again, we build a right triangle by drawing the altitude from C to the x-axis. Because $\widehat{AC} = 150^\circ$, we have $\angle AOC = 150^\circ$. Therefore, we have $\angle COP = 180^\circ - 150^\circ = 30^\circ$. So, $\triangle COP$ is a 30-60-90 triangle. Since $OC = 1$ and $\angle COP = 30^\circ$, we have $OP = \sqrt{3}/2$ and $CP = 1/2$. Because C is to the left of the y-axis and above the x-axis, its x-coordinate is negative and its y-coordinate is positive. So, the coordinates of C are $(-\sqrt{3}/2, 1/2)$, which means $\cos 150^\circ = -\sqrt{3}/2$ and $\sin 150^\circ = 1/2$. □

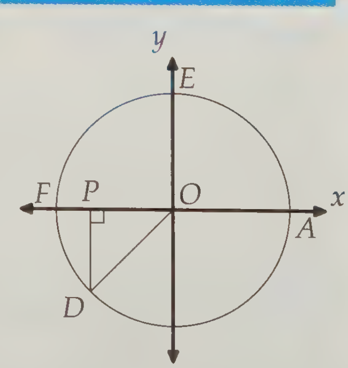
The coordinate axes divide the Cartesian plane into four **quadrants**. We usually label these quadrants with Roman numerals, as shown at right. We refer to them as the first, second, third, and fourth quadrant, respectively. So, point C in Problem 18.13 is in the second quadrant. All points in the second quadrant are to the left of the x-axis and above the y-axis, so all points in the second quadrant have a negative x-coordinate and a positive y-coordinate.



We sometimes refer to angle measures with quadrants, as well. When we say that a 150° angle is a 'second quadrant' angle, this means that the point that is 150° counterclockwise from $(1,0)$ on the unit circle is in the second quadrant. Similarly, because 60° is between 0° and 90°, a 60° angle is a 'first quadrant' angle.

Problem 18.14: Evaluate $\cos 225^\circ$ and $\sin 225^\circ$.

Solution for Problem 18.14: Again, we let A be $(1,0)$, and we let point D be the point on the unit circle that is 225° counterclockwise from A. Since $(-1,0)$ is 180° counterclockwise from A and $(0,-1)$ is 270° counterclockwise from A, we know that D is in quadrant III, as shown at right.



We draw altitude $\overline{DP}$ from D to the x-axis, forming right triangle DOP. Because $\widehat{AEF}$ is 180° and $\widehat{AED}$ is 225°, we have $\widehat{DF} = 45^\circ$, so $\triangle DOP$ is a 45-45-90 triangle. Since $OD = 1$, we have $OP = DP = \sqrt{2}/2$. Point D is $\sqrt{2}/2$ to the left of the y-axis and $\sqrt{2}/2$ below the x-axis, so its coordinates are $(-\sqrt{2}/2, -\sqrt{2}/2)$. Therefore, we have $\cos 225^\circ = \sin 225^\circ = -\sqrt{2}/2$. □

Now that we have a handle on sine and cosine, we can revisit tangent. Once we have defined sine and cosine in terms of the unit circle, we define the tangent of any angle θ for which $\cos\theta \neq 0$ as

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

If $\cos\theta = 0$, then $\tan\theta$ is undefined.

Problems

Problem 18.15:
(a) Use the unit circle to find $\sin 0^\circ$ and $\cos 0^\circ$, then evaluate $\tan 0^\circ$.
(b) Find $\sin 90^\circ$, $\cos 90^\circ$, and $\tan 90^\circ$.

Problem 18.16: Evaluate $\cos 300^\circ$, $\sin 300^\circ$, and $\tan 300^\circ$ without a calculator.

Problem 18.17: Explain why $\sin(180^\circ - \theta) = \sin\theta$ for any acute angle θ.

Problem 18.18: Show that for any triangle $\triangle ABC$, we have $[ABC] = \frac{1}{2}(AC)(BC)\sin C$.

Problem 18.15:
(a) Find $\sin 0^\circ$, $\cos 0^\circ$, then evaluate $\tan 0^\circ$.
(b) Find $\sin 90^\circ$, $\cos 90^\circ$, and $\tan 90^\circ$.

Solution for Problem 18.15:

(a) The point that is 0° counterclockwise from $(1,0)$ is simply $(1,0)$. Therefore, we have $\cos 0^\circ = 1$ and $\sin 0^\circ = 0$, so $\tan 0^\circ = (\sin 0^\circ)/(\cos 0^\circ) = 0$.

(b) The point that is 90° counterclockwise from $(1,0)$ is $(0,1)$. So, we have $\cos 90^\circ = 0$ and $\sin 90^\circ = 1$. Therefore, we have $\tan 90^\circ = (\sin 90^\circ)/(\cos 90^\circ) = 1/0 = \cdots$ Uh-oh! We can't divide by 0, so $\tan 90^\circ$ is undefined.

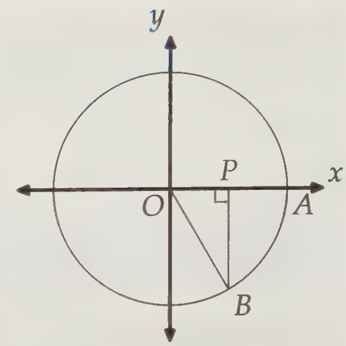
□

Sidenote: When we put the values of sine and cosine of 0° and 90° together with those of 30°, 45°, and 60°, a curious pattern emerges:

	0°	30°	45°	60°	90°
sin	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
cos	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$

Problem 18.16: Evaluate $\cos 300^\circ$, $\sin 300^\circ$, and $\tan 300^\circ$ without a calculator.

Solution for Problem 18.16: We let A be $(1,0)$, and we let point B be the point on the unit circle that is $300°$ counterclockwise from A. The point $(0,-1)$ is $270°$ counterclockwise from A, and going $360°$ counterclockwise from A takes us back to A, so we know that B is in quadrant IV, as shown at right. (We could also have determined that B is in quadrant IV by noting that $300°$ counterclockwise is the same as $60°$ clockwise.)



We draw the altitude from B to point P on the x-axis. Because B is $300°$ counterclockwise from A, we have $\widehat{AB} = 60°$, so $\angle BOA = 60$. From 30-60-90 triangle BPO, we have $BP = \sqrt{3}/2$ and $OP = 1/2$, so the coordinates of B are $(1/2, -\sqrt{3}/2)$. What's wrong with this conclusion:

> **Bogus Solution:** Because B is $(1/2, -\sqrt{3}/2)$, we have $\sin 300° = 1/2$ and $\cos 300° = -\sqrt{3}/2$.

We have sine and cosine backwards here! Cosine is the x-coordinate and sine is the y-coordinate.

> **WARNING!!** Be careful not to get sine and cosine backwards. Once you have found the appropriate point on the unit circle, an easy way to remember which coordinate is cosine and which is sine is to use alphabetical order: cosine is before sine and x is before y.

So, because B is $(1/2, -\sqrt{3}/2)$, we have $\cos 300° = 1/2$ and $\sin 300° = -\sqrt{3}/2$. Therefore, we have

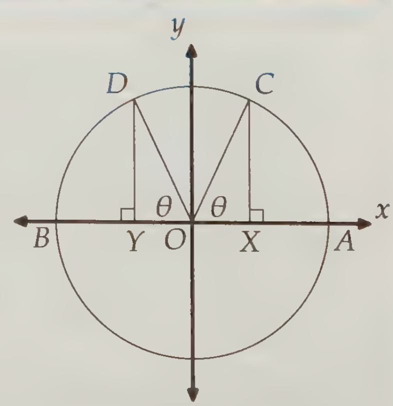
$$\tan 300° = \frac{\sin 300°}{\cos 300°} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}.$$

□

We can use the unit circle to 'see' trigonometric relationships.

Problem 18.17: Explain why $\sin(180° - \theta) = \sin\theta$ for any acute angle θ.

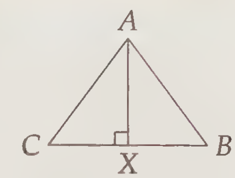
Solution for Problem 18.17: Let A be $(1,0)$ and B be $(-1,0)$. Let C be $(\cos\theta, \sin\theta)$, so that C is θ counterclockwise from A. Let D be the point $(\cos(180° - \theta), \sin(180° - \theta))$, so D is $180° - \theta$ counterclockwise from A. Because $(-1,0)$ is $180°$ counterclockwise from A and D is $180° - \theta$ counterclockwise from A, point D is θ *clockwise* from $(-1,0)$, as shown in the diagram. We draw altitudes $\overline{CX}$ and $\overline{DY}$ to the x-axis, since these lengths equal $\sin\theta$ and $\sin(180° - \theta)$, respectively. Because $OD = OC = 1$, we have $\triangle OYD \cong \triangle OXC$ by SA Congruence for right triangles. This gives us $CX = DY$, so $\sin\theta = \sin(180° - \theta)$. □



With a little bit of casework, we can extend this proof to show that $\sin(180° - \theta) = \sin\theta$ for all angles θ, not just acute angles. This relationship is not worth memorizing. After working with trigonometric functions and the unit circle more, this relationship, and others like it, will become automatic to you. We'll be exploring how in later books in the *Art of Problem Solving* series.

Problem 18.18: Show that for any triangle $\triangle ABC$, we have $[ABC] = \frac{1}{2}(AC)(BC)\sin C$.

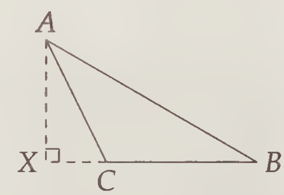
Solution for Problem 18.18: We consider separately the cases in which C is acute, right, or obtuse.



Case 1: $\angle C$ is acute. To find the area, we seek an expression for the altitude from A to $\overline{BC}$. So, we draw altitude $\overline{AX}$, forming right triangle $\triangle AXC$. From this triangle, we have $\sin C = AX/AC$, so $AX = AC\sin C$. Therefore, we have

$$[ABC] = \frac{1}{2}(AX)(BC) = \frac{1}{2}(AC)(BC)\sin C.$$

Case 2: $\angle C$ is right. If $\angle C$ is right, then we have both $\sin C = 1$ and $[ABC] = \frac{1}{2}(AC)(BC)$. So, we have $[ABC] = \frac{1}{2}(AC)(BC)\sin C$.



Case 3: $\angle C$ is obtuse. If $\angle C$ is obtuse, then altitude $\overline{AX}$ is outside $\triangle ABC$, as shown at right. We have $\sin\angle ACX = AX/AC$, so $AX = AC\sin\angle ACX$. Because $\angle ACX = 180^\circ - \angle ACB$, we have

$$[ABC] = \frac{1}{2}(AX)(BC) = \frac{1}{2}(AC)(BC)\sin\angle ACX = \frac{1}{2}(AC)(BC)\sin(180^\circ - \angle ACB).$$

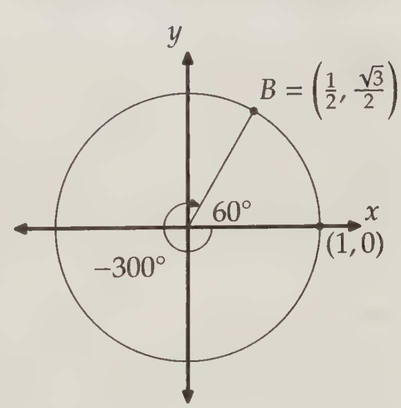
Because $\sin(180^\circ - \angle ACB) = \sin\angle ACB$ for any angle $\angle ACB$, we have $[ABC] = \frac{1}{2}(AC)(BC)\sin\angle ACB$, as desired.

We have covered all three possible cases for $\angle C$, so we have shown that

Important: Let $BC = a$ and $AC = b$ in $\triangle ABC$. Then, we have

$$[ABC] = \frac{1}{2}ab\sin C.$$

□



Using our unit circle, we can also evaluate trigonometric functions for angle measures greater than 360°, as well as for negative angle measures. For example, since going 360° counterclockwise from $(1,0)$ brings us back to $(1,0)$, the point that is $360^\circ + 60^\circ = 420^\circ$ counterclockwise from $(1,0)$ at right is point B shown. From the coordinates of B, we see that $\cos 420^\circ = 1/2$ and $\sin 420^\circ = \sqrt{3}/2$. To evaluate trigonometric functions of negative angles, we go clockwise from $(1,0)$ rather than counterclockwise. So, because B is 300° *clockwise* from $(1,0)$, we have $\cos(-300^\circ) = 1/2$ and $\sin(-300^\circ) = \sqrt{3}/2$.

Exercises

18.2.1 Evaluate each of the following:

(a) $\sin 120^\circ$ (c) $\tan 135^\circ$ (e) $\cos 210^\circ$

(b) $\sin 330^\circ$ (d) $\cos(-45^\circ)$ (f) $\tan 720^\circ$

18.2.2 Explain why $\sin^2\theta + \cos^2\theta = 1$ for any angle measure θ.

18.2.3 How many angle measures θ such that $0 \le \theta < 360^\circ$ satisfy the equation $\sin\theta = 0.31$?

18.2.4 Explain why $\cos(360^\circ + \theta) = \cos\theta$ for any angle θ.

18.2.5 Explain why $\cos(180^\circ - \theta) = -\cos\theta$ for any acute angle θ.

18.3 Law of Sines and Law of Cosines

SAS Congruence tells us that if two sides of one triangle and the angle between them are equal to the corresponding sides and angle of another triangle, then the two triangles are congruent. Therefore, the third sides of these two triangles have the same length. In other words, if we know two side lengths of a triangle and the measure of the angle between those sides, then there is only one possible length of the third side of the triangle. Hmmm... Can we find that length if we know the two side lengths and the angle measure?

Similarly, ASA and AAS Congruence tell us that if two angles and a side of a triangle equal the corresponding angles and side of another triangle, then the two triangles are congruent. In other words, if we know two angles and a side in one triangle, then there is only one possible value for each of the remaining two side lengths. Can we find these two side lengths if we know the angles of the triangle and the other side length?

In this section, we use trigonometry to answer both of these questions.

Note: you can use your calculator for the problems in this section.

Problems

Problem 18.19: In $\triangle ABC$, let $AC = 15$, $BC = 12$, and $\angle C = 34^\circ$. In this problem, we find AB to the nearest hundredth.

(a) Draw altitude $\overline{BX}$ from B to $\overline{AC}$. Find CX and BX to the nearest hundredth.

(b) Find XA to the nearest hundredth.

(c) Use parts (a) and (b) to find AB to the nearest hundredth.

Extra! *Mathematical knowledge adds vigor to the mind, frees it from prejudice, credulity, and superstition.*

–John Arbuthnot

Problem 18.20: Let $\triangle ABC$ be an acute triangle with $a = BC$, $b = AC$, and $c = AB$. In this problem, we prove that

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

(a) Draw altitude $\overline{BX}$ from B to $\overline{AC}$. Express BX and CX in terms of a, b, and/or trigonometric functions of $\angle C$.

(b) Express AX in terms of a, b, and/or trigonometric functions of $\angle C$.

(c) Use $\triangle ABX$ to show that $c^2 = a^2 + b^2 - 2ab\cos C$.

(d) What happens in the equation $c^2 = a^2 + b^2 - 2ab\cos C$ if $\angle C$ is a right angle?

Problem 18.21: In $\triangle PQR$, suppose $PR = 12$, $\angle QPR = 66^\circ$, and $\angle PRQ = 63^\circ$. In this problem, we find PQ and QR.

(a) Let T be the foot of the altitude from P to $\overline{QR}$. Find PT to the nearest hundredth.

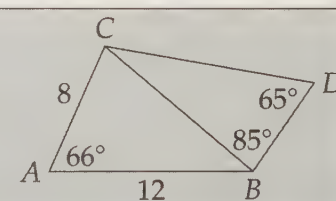
(b) Use PT to find PQ to the nearest hundredth.

(c) Find QR to the nearest hundredth.

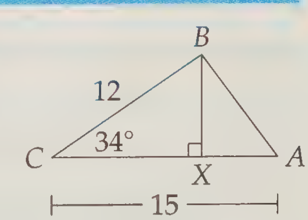
Problem 18.22: Suppose that $\triangle ABC$ is an acute triangle with $a = BC$, $b = AC$, and $c = AB$. Prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Problem 18.23: Find BC, CD, and BD to the nearest tenth in the diagram at right.



Problem 18.19: In $\triangle ABC$, let $AC = 15$, $BC = 12$, and $\angle C = 34^\circ$. Find AB to the nearest hundredth.



Solution for Problem 18.19: It's not immediately obvious how to find AB. We know how to use trigonometry to find side lengths in right triangles, so we start by drawing an altitude from B to $\overline{AC}$ as shown. This creates right triangle $\triangle BCX$ with the 34° angle as one of its acute angles. From right triangle $\triangle CBX$, we have

$$\frac{BX}{BC} = \sin C \approx 0.559,$$

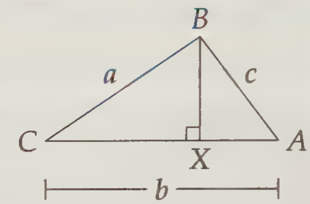
so $BX \approx 0.559(BC) \approx 6.71$. Similarly, we have $CX/BC = \cos C \approx 0.829$, so $CX \approx 9.95$. This doesn't tell us AB yet, but we now have the length of one leg of $\triangle BXA$. If we can find the other, we can use the Pythagorean Theorem to find AB. Fortunately, XA is easy to find: $XA = AC - CX \approx 5.05$. Now, we can use the Pythagorean Theorem to find

$$AB = \sqrt{BX^2 + XA^2} \approx 8.40.$$

□

In Problem 18.19, we were given two side lengths of a triangle, and the measure of the angle between these two sides. We then found the third side. There was nothing particularly special about the side lengths or the angle. We might be able to follow essentially the same process for any triangle. Let's give it a try.

Problem 18.20: Let $\triangle ABC$ be an acute triangle with $a = BC$, $b = AC$, and $c = AB$. Suppose we know a, b, and $\angle C$. Find a formula that we can use to find c.



Solution for Problem 18.20: We use Problem 18.19 as a guide. In fact, this problem is essentially the same as Problem 18.19, but with variables a, b, and $\angle C$ in place of the numbers that were given in that problem. We can use the same diagram, and we'll use the same steps. We draw altitude $\overline{BX}$ from B to $\overline{AC}$. Then, we have $\sin C = BX/BC$, so $BX = BC \sin C = a \sin C$. We also have $\cos C = CX/BC$, so $CX = BC \cos C = a \cos C$. Therefore, we have $AX = AC - CX = b - a\cos C$. Next we apply the Pythagorean Theorem to $\triangle ABX$ to find $AB^2 = BX^2 + AX^2$. Substituting our expressions for these three sides gives us

$$\begin{aligned} c^2 &= a^2 \sin^2 C + (b - a\cos C)^2 \\ &= a^2 \sin^2 C + b^2 - 2ab\cos C + a^2\cos^2 C \\ &= a^2(\sin^2 C + \cos^2 C) + b^2 - 2ab\cos C. \end{aligned}$$

Since $\sin^2 C + \cos^2 C = 1$, we have

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

□

We can test our new formula with the data from Problem 18.19.

Concept: Check formulas you prove by trying them on specific examples you have solved without the formula.

In Problem 18.19, we have $a = 12$, $b = 15$, and $\angle C = 34^\circ$, so we have

$$c^2 = a^2 + b^2 - 2ab\cos C \approx 70.546.$$

Taking the square root of both sides gives $c \approx 8.40$, which agrees with our answer from Problem 18.19.

With a little more casework (which you'll supply as an Exercise), we can show that this equation holds for any triangle ABC, not just for acute triangles.

Important: Let $a = BC$, $b = AC$, and $c = AB$ in $\triangle ABC$. The **Law of Cosines** states that

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

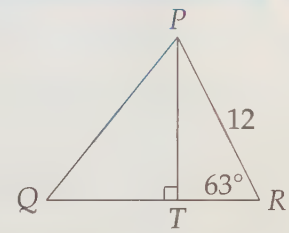
Notice that when $\angle C = 90^\circ$ in the Law of Cosines, we have $\cos C = 0$, so the law becomes $c^2 = a^2 + b^2$, which is just the Pythagorean Theorem.

Problem 18.19 is just a specific example of Problem 18.20, so we call Problem 18.20 a **generalization** of Problem 18.19.

Concept: Whenever we have to prove a general statement like the Law of Cosines, it's often useful to first consider a specific example of the general statement. If we can solve a specific example, we can sometimes use that solution as a guide to prove the generalization.

Now that you've learned the Law of Cosines, you're probably thinking there must be a Law of Sines, too. Of course, you've read the section title, so you know you're right. We'll discover the Law of Sines the same way we discovered the Law of Cosines – we'll start with a specific example.

Problem 18.21: Suppose $PR = 12$, $\angle QPR = 66^\circ$, and $\angle PRQ = 63^\circ$ in $\triangle PQR$. Find PQ and RQ to the nearest hundredth.



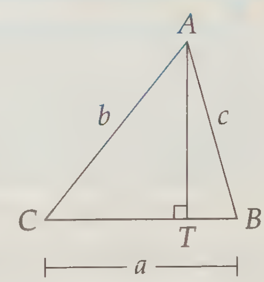
Solution for Problem 18.21: We start by building a right triangle, since we know how to use trigonometry to find lengths of sides in a right triangle. We draw altitude $\overline{PT}$ to side $\overline{QR}$. From right triangle $\triangle PTR$, we have $PT/PR = \sin R$, so $PT = PR\sin R \approx 10.69$. We also have $TR/PR = \cos R$, so $TR = PR\cos R \approx 5.45$.

Now, we can use right triangle $\triangle PQT$ to find lengths QT and PQ. First, we find that $\angle Q = 180^\circ - 66^\circ - 63^\circ = 51^\circ$. We have $PT/PQ = \sin Q$, so we have $PQ = PT/(\sin Q) \approx 13.76$. We also have $PT/QT = \tan Q$, so $QT = PT/(\tan Q) \approx 8.66$. Finally, we have $QR = QT + TR \approx 14.11$. □

Let's try using our approach in Problem 18.21 to produce another 'law.'

Problem 18.22: Suppose that $\triangle ABC$ is an acute triangle with $a = BC$, $b = AC$, and $c = AB$. Prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



Solution for Problem 18.22: We use Problem 18.21 as a guide. We draw altitude $\overline{AT}$ to $\overline{BC}$. From right triangle $\triangle ATC$, we have $\sin C = AT/AC$, so $AT = AC\sin C = b\sin C$. From right triangle $\triangle ABT$, we have $\sin B = AT/AB$, so $AT = AB\sin B = c\sin B$. These two expressions for AT must be equal, so we have $b\sin C = c\sin B$. Dividing this equation by $\sin B$ and by $\sin C$, we have

$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

We can follow essentially the same steps starting with the altitude from C to $\overline{AB}$ to show that $b/(\sin B) = a/(\sin A)$. So, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

□

Important: Let $a = BC$, $b = AC$, and $c = AB$ in $\triangle ABC$. The **Law of Sines** states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

As with the Law of Cosines, you'll tackle the cases in which $\triangle ABC$ is right or obtuse in the Exercises.

Sidenote: The Law of Sines gives us another look at why 'SSA Congruence' is not a valid theorem to prove triangle congruence. Suppose we have $\angle A = 30°$, $BC = 3$, and $AC = 4$ in $\triangle ABC$. The Law of Sines then gives us

$$\frac{BC}{\sin A} = \frac{AC}{\sin B},$$

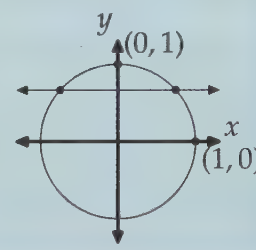
from which we find $\sin B = (AC/BC)\sin A = 2/3$.

Next, suppose we have $\angle D = 30°$, $EF = 3$, and $DF = 4$ in $\triangle DEF$. Again, the Law of Sines gives us

$$\frac{EF}{\sin D} = \frac{DF}{\sin E},$$

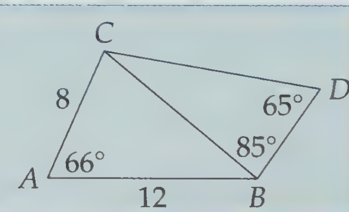
from which we find $\sin E = (DF/EF)\sin D = 2/3$. So, we have $\sin B = \sin E = 2/3$. Do we have $\angle B = \angle E$?

Not necessarily! There are two angles θ for which $0° < \theta < 180°$ and $\sin\theta = 2/3$. The two intersections of the graph of $y = 2/3$ with the unit circle correspond to these two angles; these are shown at right. So, we cannot deduce that $\angle B = \angle E$, and therefore we cannot conclude that $\triangle ABC \cong \triangle DEF$.



Let's give the Law of Cosines and the Law of Sines a try:

Problem 18.23: Find BC, CD, and BD to the nearest tenth in the diagram at right.



Solution for Problem 18.23: In $\triangle ABC$, we know two side lengths and the angle between these sides, so we can use the Law of Cosines to find BC. This gives us

$$BC^2 = 8^2 + 12^2 - 2(8)(12)\cos 66° \approx 129.9,$$

so $BC \approx 11.4$. Turning to $\triangle BCD$, we have $\angle BCD = 180° - 65° - 85° = 30°$. Now, we know one of the side lengths of $\triangle BCD$ and all of the angle measures of $\triangle BCD$, so we can use the Law of Sines to find the missing side lengths. Specifically, we have

$$\frac{BC}{\sin D} = \frac{BD}{\sin \angle BCD} = \frac{CD}{\sin \angle CBD},$$

which means we have

$$\frac{11.4}{\sin 65°} = \frac{BD}{\sin 30°} = \frac{CD}{\sin 85°}.$$

Solving for BD, we have

$$BD = \frac{11.4}{\sin 65^\circ} \cdot \sin 30^\circ \approx 6.3,$$

and solving for CD gives

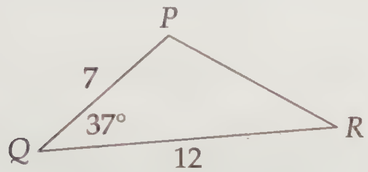
$$CD = \frac{11.4}{\sin 65^\circ} \cdot \sin 85^\circ \approx 12.5.$$

□

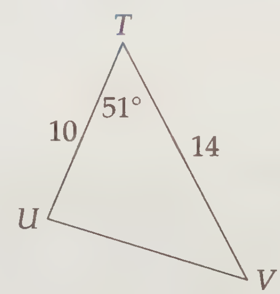
Exercises

18.3.1 Find the missing sides below to the nearest tenth.

(a)

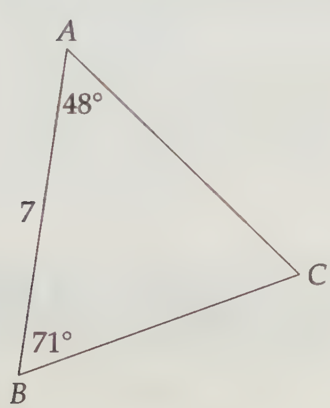


(b)

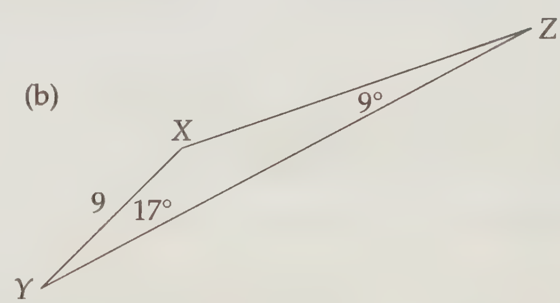


18.3.2 Find the missing sides below to the nearest tenth.

(a)



(b)



18.3.3 In this problem, we prove that the Law of Cosines works when the given angle is obtuse. Let $\triangle ABC$ be an obtuse triangle with $\angle C > 90^\circ$. Let $AB = c$, $CA = b$, and $BC = a$, as usual.

(a) Let D be the foot of the altitude from A to $\overleftrightarrow{BC}$. Find AD, BD, and CD in terms of a, b, c, and trigonometric functions of $\angle C$.

(b) Show that $c^2 = a^2 + b^2 - 2ab\cos C$.

18.3.4 Prove that the Law of Sines holds for right triangles and for obtuse triangles.

18.3.5 A surveyor is 3 kilometers from the base of a mountain. The mountain's face slopes up at an angle of 30° from horizontal. If the surveyor measures 10 degrees from horizontal to the top of the mountain, approximately how tall is the mountain to the nearest hundredth of a kilometer?

18.3.6 Let $\triangle ABC$ be a triangle with $BC = a$, $AC = b$, $AB = c$, and with circumradius R.

(a) Draw $\triangle ABC$ and its circumcircle. Let point D be on the circumcircle such that $\overline{AD}$ is a diameter. Find $\sin \angle ADB$ in terms of the side lengths of $\triangle ABD$.

(b) What angle of $\triangle ABC$ equals $\angle ADB$, and why?

(c) Show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

This is the **Extended Law of Sines**.

18.3.7★ Suppose $AC = 5$, $BC = 6$, and $AB = \sqrt{31}$. Find $\angle BCA$.

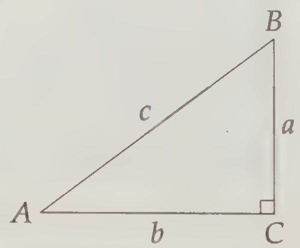
18.4 Summary

We write the cosine of an angle θ as $\cos\theta$ and the sine of θ as $\sin\theta$. We define sine and cosine using the **unit circle**, which is the circle on the Cartesian plane with radius 1 centered at the origin.

> **Important:** Let point A be $(1,0)$ and point B be on the unit circle such that B is θ degrees counterclockwise from A. Then, we define $\cos\theta$ and $\sin\theta$ to be the x-coordinate and y-coordinate, respectively, of point B.

We write the tangent of an angle θ as $\tan\theta$, and define it as $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. Sine, cosine, and tangent are examples of **trigonometric functions**.

For an acute angle $\angle A$ of a right triangle $\triangle ABC$, we have:



$$\begin{aligned}
\sin A &= \frac{\text{length of leg opposite } A}{\text{length of hypotenuse}} = \frac{a}{c},\\
\cos A &= \frac{\text{length of leg adjacent to } A}{\text{length of hypotenuse}} = \frac{b}{c},\\
\tan A &= \frac{\text{length of leg opposite } A}{\text{length of leg adjacent to } A} = \frac{a}{b}.
\end{aligned}$$

We proved several trigonometric identities, which are statements involving trigonometric functions of angles that are true for all angles. The most important is:

> **Important:**
> $$\sin^2 A + \cos^2 A = 1$$

While we proved several other identities, it is more important to understand how these identities are derived than to memorize all the identities.

We then used trigonometry to find three important triangle relationships:

Important: Let $BC = a$, $AC = b$, and $AB = c$ in $\triangle ABC$. Then, we have

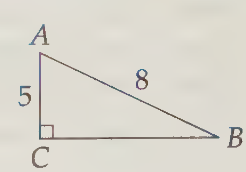
- $[ABC] = \frac{1}{2}ab\sin C$.
- $c^2 = a^2 + b^2 - 2ab\cos C$. This is called the **Law of Cosines**.
- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$. This is called the **Law of Sines**.

Problem Solving Strategies

Concepts:

- If we have the length of a side of a right triangle and the value of a trigonometric function of either of the acute angles of the triangle, then we can find all the side lengths of the triangle.
- Check formulas you prove by trying them on specific examples you have solved without the formula.
- Whenever we have to prove a general statement like the Law of Cosines, it's often useful to first consider a specific example of the general statement. If we can solve a specific example, we can sometimes use that solution as a guide to prove the generalization.

REVIEW PROBLEMS



18.24 In $\triangle ABC$ at right, find $\sin A$, $\cos A$, and $\tan A$.

18.25 In $\triangle PQR$, let $\angle Q = 90^\circ$, $\angle P = 71^\circ$, and $PR = 16$. Find PQ and QR to the nearest tenth.

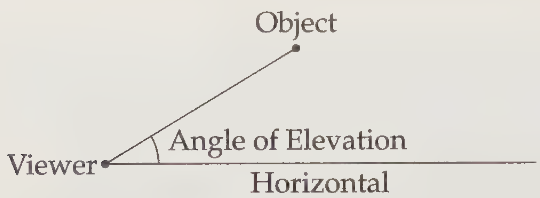
18.26 Show that if $\triangle XYZ$ is a right triangle with $\angle X = 90^\circ$, then $(\tan Y)(\tan Z) = 1$.

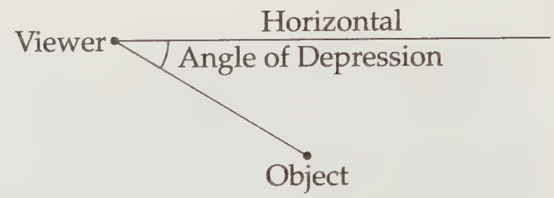
18.27 In $\triangle PQR$, we have $\angle Q = 90^\circ$ and $\sin P = 1/4$. Find $\sin R$.

18.28 Evaluate each of the following:

(a)	$\sin 135^\circ$	(c)	$\cos(-120^\circ)$	(e)	$\tan 300^\circ$
(b)	$\sin 630^\circ$	(d)	$\cos 315^\circ$	(f)	$\tan 150^\circ$

18.29 The **angle of elevation** is the angle above the horizontal at which a viewer must look to see an object that is higher than the viewer. Similarly, the **angle of depression** is the angle below the horizontal at which a viewer must look to see an object that is below the viewer.





Answer each of the following problems to the nearest foot.

(a) A surveyor measures the angle of elevation from her feet to the top of a building as 5°. The surveyor knows that the building is 500 feet tall. Assuming the ground is flat and level between the surveyor and the building, how far away is the surveyor from the building?

(b) I'm standing at the peak of a mountain that is 14,000 feet above sea level. The angle of depression from this peak to a nearby smaller peak is 4°. On my map, these two peaks are represented by points that are 1 inch apart. If each inch on my map represents 1.2 miles, and there are 5280 feet in a mile, then how many feet above sea level is the second peak?

(c)★ A bee is on a hill looking at a building. The building is 400 feet tall. The angle of elevation from the bee to the top of the building is 4° and the angle of depression from the bee to the bottom of the building is 2°. What is the shortest distance the bee will have to fly to reach the building?

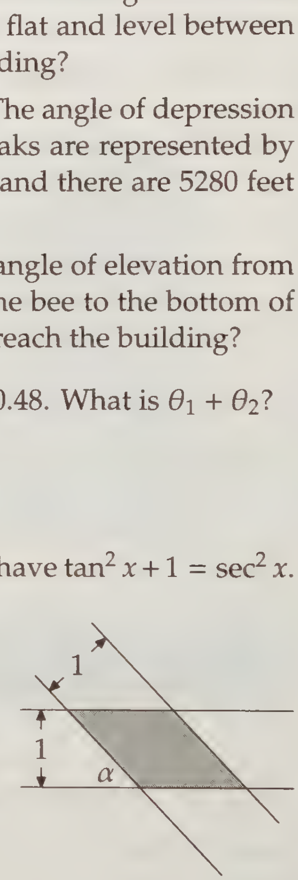
18.30 Let θ_1 and θ_2 be the two values of θ such that $0° < \theta < 180°$ and $\sin\theta = 0.48$. What is $\theta_1 + \theta_2$?

18.31 Suppose that $AB = 7$, $BC = 8$, and $\angle ABC = 45°$. Find $[ABC]$.

18.32 If $\tan\theta$ is negative, then which quadrant(s) could θ possibly be in?

18.33 Recall that $\sec x = \frac{1}{\cos x}$. Show that for any angle x for which $\cos x \neq 0$, we have $\tan^2 x + 1 = \sec^2 x$.

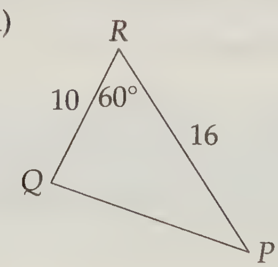
18.34 Two strips of width 1 overlap at an angle of α as shown. Show that the area of the overlap (shown shaded) is $1/(\sin\alpha)$.



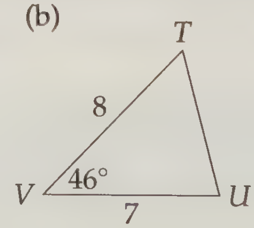
18.35 Point A is on the unit circle in the first quadrant such that A is θ degrees counterclockwise from $(1,0)$. Point B is on the x-axis such that $\overline{AB}$ is tangent to the unit circle. Show that $\tan\theta = AB$. (Do you now see why we use the name 'tangent' for this trigonometric function?)

18.36 Find the missing side lengths in the triangles below to the nearest tenth.

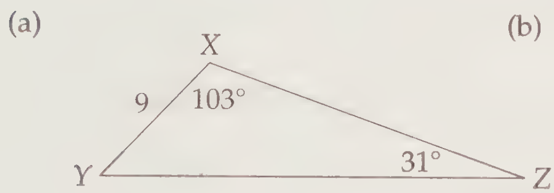
(a)

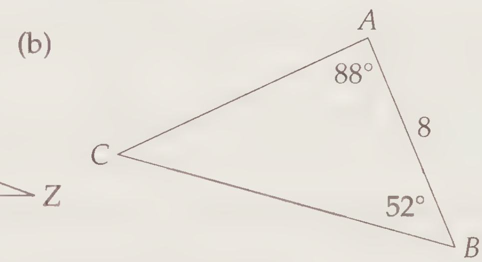


(b)



18.37 Find the missing side lengths in the triangles below to the nearest tenth.





18.38 If I walk north 3 miles, turn 32°, then walk another 2 miles, how far will I be from my initial starting point (to the nearest 0.1 mile)?

18.39 What's wrong with the following proof of the Pythagorean Theorem:

> In the text, we proved that if $a = BC$, $b = AC$, and $c = AB$, then $c^2 = a^2 + b^2 - 2ab\cos C$. If $\angle C = 90°$, then we have $\cos C = 0$. Therefore, if $\angle C = 90°$, we have $c^2 = a^2 + b^2$.

18.40 In $\triangle PQR$, we have $PQ = 3$, $QR = 5$, and $PR = 6$.

(a) Find $\cos P$.

(b)★ Find $\sin Q$.

Challenge Problems

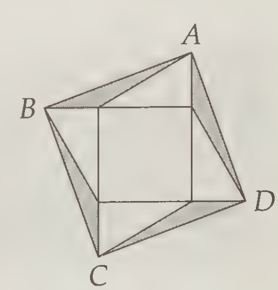
18.41 Show that if the sides of $\triangle ABC$ have lengths a, b, and c, and $\triangle ABC$ has circumradius R, then

$$[ABC] = \frac{abc}{4R}.$$

18.42 Two rays with common endpoint O form a 30° angle. Point A lies on one ray, point B on the other ray, and $AB = 1$. What is the maximum possible length of $\overline{OB}$? *(Source: AHSME)*

18.43 How many triangles have area 10 and vertices $(-5, 0)$, $(5, 0)$, and $(5\cos\theta, 5\sin\theta)$ for some angle θ? *(Source: AHSME)*

18.44 Four congruent 30-60-90 triangles are constructed on the sides of a square as shown at right. The hypotenuse of each of these triangles has length 2. The outer vertices of these triangles are connected as shown to form quadrilateral $ABCD$. What fraction of $ABCD$ is shaded?



18.45 A circle centered at O has radius 1 and contains the point A. Segment $\overline{AB}$ is tangent to the circle at A and $\angle AOB = \theta$. Suppose point C is on $\overline{OA}$ such that $\overline{BC}$ bisects $\angle ABO$. Show that $OC = 1/(1 + \sin\theta)$. *(Source: AMC 12)*

18.46 If $\sin x = 3\cos x$, then what is $(\sin x)(\cos x)$? *(Source: AHSME)*

18.47 Line ℓ intersects the x-axis at an angle of 50°. Line k makes an angle of 140° with the x-axis. The intersection of ℓ and k has a y-coordinate of 10. Find all possible values of the distance between the x-intercepts of ℓ and k.

18.48 Right triangle ABC has inradius 1 and $\sin A = 12/13$. Find the length of the hypotenuse of ABC. **Hints:** 593

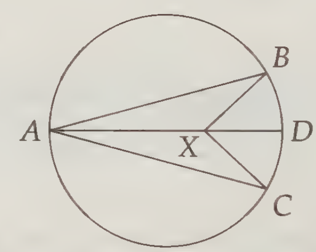
18.49 What's wrong with this proof of AA Similarity:

> Suppose we have $\triangle ABC$ and $\triangle DEF$ with $\angle A = \angle D$ and $\angle B = \angle E$. The Law of Sines gives us
>
> $$\frac{BC}{\sin A} = \frac{AC}{\sin B} \qquad \text{and} \qquad \frac{EF}{\sin D} = \frac{DF}{\sin E}.$$
>
> Therefore, we have $BC/AC = (\sin A)/(\sin B)$ and $EF/DF = (\sin D)/(\sin E)$. Since $\angle A = \angle D$ and $\angle B = \angle E$, we have $(\sin A)/(\sin B) = (\sin D)/(\sin E)$. So, we have
>
> $$\frac{BC}{AC} = \frac{\sin A}{\sin B} = \frac{\sin D}{\sin E} = \frac{EF}{DF}.$$
>
> Because $\angle A = \angle D$ and $\angle B = \angle E$, we have $\angle C = \angle F$. We can follow essentially the same steps as above to deduce $BC : AC : AB = EF : DF : DE$.

18.50★ Points A, B, C, and D are on a circle of diameter 1, and X is on diameter $\overline{AD}$. Suppose $BX = CX$ and $3\angle BAC = \angle BXC = 36°$. Show that

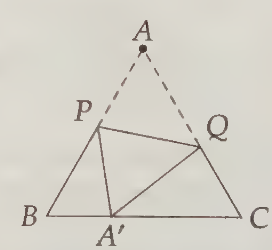
$$AX = (\cos 6°)(\sin 12°)/(\sin 18°).$$

(Source: AHSME) **Hints:** 589, 576

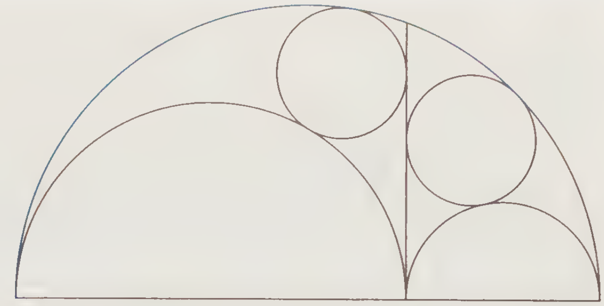


18.51★ An object moves 8 cm in a straight line from A to B, turns at an angle of α, where $0° < \alpha < 180°$, and then moves 5 cm in a straight line to C. What is the probability that $AC < 7$? **Hints:** 526

18.52★ Equilateral triangle ABC has been creased and folded so that vertex A now rests at A' on $\overline{BC}$ as shown. If $BA' = 1$ and $A'C = 2$, then what is the length of crease $\overline{PQ}$? *(Source: AMC 12)* **Hints:** 555, 544



Extra! *Since you are now studying geometry and trigonometry, I will give you a problem. A ship sails the ocean. It left Boston with a cargo of wool. It grosses 200 tons. It is bound for Le Havre. The mainmast is broken, the cabin boy is on deck, there are 12 passengers aboard, the wind is blowing East-North-East, the clock points to a quarter past three in the afternoon. It is the month of May. How old is the captain?*
– Gustave Flaubert



Archimedes' Circles

Let no one ignorant of geometry enter here. – Inscription above Plato's Academy

CHAPTER 19 Problem Solving Strategies in Geometry

In this chapter we review many of the most powerful geometry problem-solving strategies we have learned in the text by tackling some challenging problems with them. We also apply many of our geometric tools to proofs that require multiple insights.

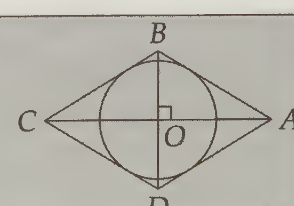
This chapter is meant to extend the lessons of this book, so many of the Exercises and Challenge Problems in this chapter are significantly more difficult than most of the problems elsewhere in this text. One excellent strategy for mastering the problems you cannot solve on your first try is to review the solutions, then try the problems again on your own a few days later.

19.1 The Extra Line

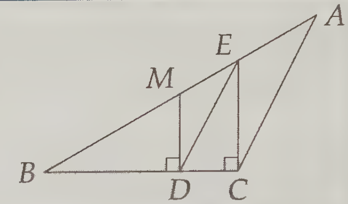
We cleverly added extra lines to diagrams to solve various problems throughout this book. In this section we reinforce the most common indications that extra lines might be helpful.

Problems

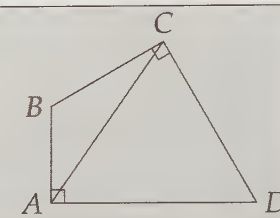
Problem 19.1: Find the radius of a circle that is inscribed in a rhombus that has diagonals of length 16 and 30.



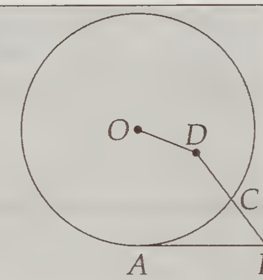
Problem 19.2: In obtuse triangle ABC shown, $AM = MB$, $\overline{MD} \perp \overline{BC}$, and $\overline{EC} \perp \overline{BC}$. If the area of $\triangle ABC$ is 24, what is the area of $\triangle BED$? *(Source: AMC 12)* **Hints:** 27, 541



Problem 19.3: In triangle ABC, $\angle ABC = 120°$, $AB = 3$ and $BC = 4$. If lines perpendicular to $\overline{AB}$ at A and to $\overline{BC}$ at C meet at D, then find CD. *(Source: AMC 12)*

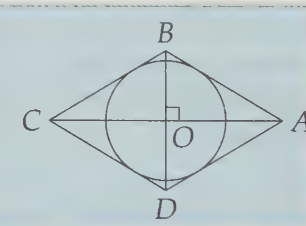


Problem 19.4: $\overline{AB}$ is tangent at A to the circle with center O, point D is interior to the circle, and $\overline{DB}$ intersects the circle at C. If $BC = CD = 3$, $OD = 2$, and $AB = 6$, then find the radius of the circle. *(Source: AMC 12)*

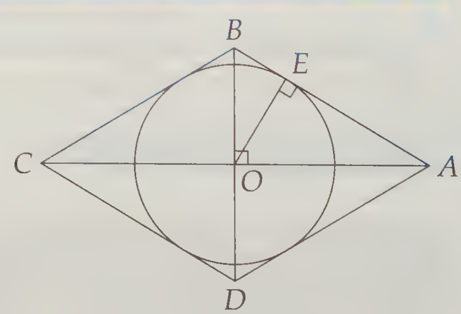


One very common type of 'extra line' we draw is a perpendicular from a point to a line. Usually, our goal in doing so is to build a useful right triangle. Here's a classic example.

Problem 19.1: Find the radius of a circle that is inscribed in a rhombus that has diagonals of length 16 and 30.



Solution for Problem 19.1: Circles + tangent lines = draw a radius to a point of tangency. We draw radius $\overline{OE}$ because $\overline{OE} \perp \overline{AB}$. Since the diagonals of a rhombus are perpendicular and bisect each other, $\triangle AOB$ is a right triangle with legs of length $OB = 8$ and $OA = 15$. Therefore, $AB = 17$. From here, we can solve the problem in several ways. Here are a couple:



Solution 1: Use similar triangles. Lots of right angles usually means there are similar triangles. Here, we have

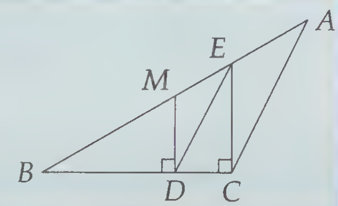
$$\triangle AOB \sim \triangle AEO \sim \triangle OEB.$$

Therefore, $OE/OA = OB/AB$, so $OE = (OA)(OB)/AB = 120/17$.

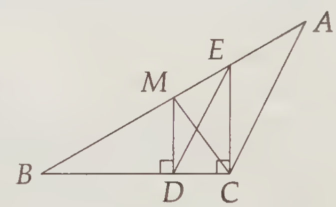
Solution 2: Use area. Since $[AOB] = (AO)(OB)/2 = (OE)(AB)/2$, we have $OE = (AO)(OB)/AB = 120/17$. (Yeah, I like the area approach better, too.) □

Concept: When in doubt, build right triangles. One very common way to do so is to draw radii to points of tangency. Nearly always try this when you have circles and tangents in a problem.

Problem 19.2: In obtuse triangle ABC shown, $AM = MB$, $\overline{MD} \perp \overline{BC}$, and $\overline{EC} \perp \overline{BC}$. If the area of $\triangle ABC$ is 24, what is the area of $\triangle BED$? *(Source: AMC 12)*



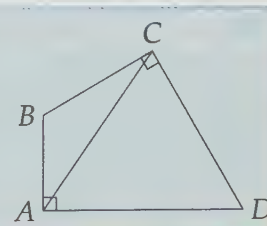
Solution for Problem 19.2: We have a problem about area, and we unfortunately can't find anything about $[BDE]$ easily. We do, however, know how to relate $[AMC]$ and $[BMC]$ to $[ABC]$. Therefore, we connect M to C. (We might also have thought to do this because $\overline{CM}$ is a median, and we know something about medians.)



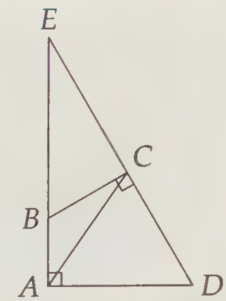
Since $\overline{CM}$ is a median of $\triangle ABC$, we have $[BMC] = [AMC] = 12$. Now we look for ways to relate either of these to what we want, $[BED]$. Triangles $\triangle BED$ and $\triangle BMC$ overlap. When we remove $\triangle BDM$ from both, we are left with two triangles that share side $\overline{DM}$: $\triangle EDM$ and $\triangle CDM$. Since $\overline{DM} \parallel \overline{EC}$, the altitudes to side $\overline{DM}$ of these triangles are the same. Therefore, $[EDM] = [CDM]$, which means $[BED] = [BMC] = 12$. □

Concept: Connecting points that are originally not connected in a diagram can be extremely useful! This doesn't mean you should connect everything in your diagram immediately, however. Look for segments to draw that will be helpful, particularly those connecting important points, or those that form segments, triangles, or angles you know something about immediately.

Problem 19.3: In triangle ABC, $\angle ABC = 120°$, $AB = 3$ and $BC = 4$. If lines perpendicular to $\overline{AB}$ at A and to $\overline{BC}$ at C meet at D, then find CD. *(Source: AMC 12)*



Solution for Problem 19.3: We start by noting that $\angle D = 360° - 120° - 90° - 90° = 60°$. Next, we might think to draw $\overline{BD}$, since that will give us a couple right triangles. However, we only have one side of those two right triangles, and we don't know anything about the acute angles of them. We'd like to make the 120° and 60° angles useful. This gets us thinking about 30-60-90 triangles.

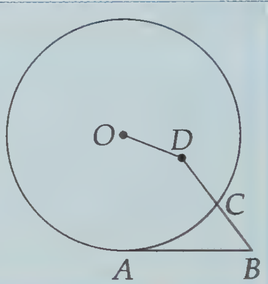


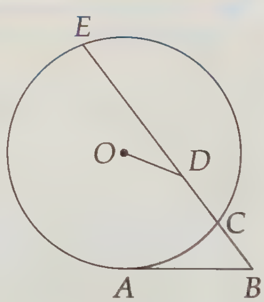
With a right angle at $\angle BAD$ and a 60° angle at $\angle D$, we see that extending $\overline{DC}$ past C and $\overline{AB}$ past B to meet at point E will give us a 30-60-90 triangle. Moreover, since $\angle CBE = 180° - 120° = 60°$ and $\angle BCE = 90°$, $\triangle CBE$ is also a 30-60-90 triangle. Now we can solve the problem.

From 30-60-90 $\triangle CBE$ we have $BE = 2BC = 8$ and $EC = BC\sqrt{3} = 4\sqrt{3}$. From $\triangle DAE$ we have $DE = AE(2/\sqrt{3}) = (AB + BE)(2/\sqrt{3}) = 22\sqrt{3}/3$. Therefore, $CD = ED - EC = 10\sqrt{3}/3$. □

Concept: 60°, 30°, and even 120° angles are often good clues to build 30-60-90 triangles by dropping altitudes or extending segments.

Problem 19.4: $\overline{AB}$ is tangent at A to the circle with center O, point D is interior to the circle, and $\overline{DB}$ intersects the circle at C. If $BC = CD = 3$, $OD = 2$, and $AB = 6$, then find the radius of the circle. *(Source: AMC 12)*





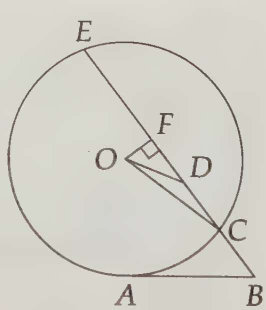
Solution for Problem 19.4: We have a problem with segment lengths and a circle, so we think of Power of a Point. We have a tangent, but only part of a secant. We continue $\overline{BD}$ past D until it hits the circle at E. We do this not only because it lets us use Power of a Point, but because $\overline{BD}$ seems to end rather abruptly in the middle of the circle. Segments that seem to end suddenly in a diagram (i.e. ones that, if continued, will hit an important circle or segment) are often candidates to be extended. The power of point B gives us

$$(BC)(BE) = BA^2,$$

and substitution gives

$$3(3 + CE) = 36.$$

Therefore, $CE = 9$. Here are a couple ways we can finish:



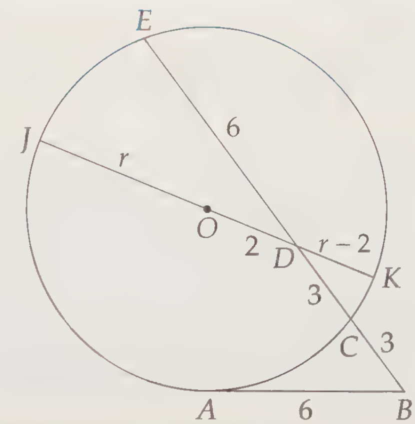
Solution 1: Build right triangles. We know the length of chord $\overline{CE}$ and we'd like to find the length of a radius. We draw radius $\overline{OC}$ and drop a perpendicular from O to $\overline{CE}$. This gives us a couple right triangles in $\triangle ODF$ and $\triangle OCF$. Since $\overline{OF}$ is part of a radius that is perpendicular to chord $\overline{CE}$, it bisects $\overline{CE}$. Therefore, $CF = 9/2$ and $DF = CF - CD = 3/2$. So,

$$OF = \sqrt{OD^2 - DF^2} = \frac{\sqrt{7}}{2}.$$

Finally, we have

$$OC = \sqrt{OF^2 + FC^2} = \sqrt{\frac{7}{4} + \frac{81}{4}} = \sqrt{22}.$$

Solution 2: Keep doing what worked. Extending a line that suddenly stopped worked once, so try it again! $\overline{OD}$ stops rather suddenly. So, we extend it in both directions to hit the circle at J and K, as shown. Now, we have two intersecting chords, so we can use Power of a Point! The power of point D gives



$$(JD)(DK) = (CD)(DE).$$

If we let the radius of the circle be r, we have

$$(r+2)(r-2) = (3)(9-3).$$

Solving for r, we find $r = \sqrt{22}$. □

Concept: Segments that stop suddenly inside figures (particularly triangles, quadrilaterals, or circles) are great candidates to be extended.

Concept: If you successfully use a certain tactic in a problem to get some information, but you still haven't solved the problem, try using that same tactic again in a different way. Maybe it still has more information to give!

Notice that our path to the solution is easy to see in the final diagram of the last solution. This is because the side lengths are labeled.

Concept: Label lengths in your diagram as you find them, even if you have to label them in terms of an important variable.

Exercises

19.1.1 A square is inscribed in a circle of radius 1 as shown at left below. Circles $\odot P$ and $\odot Q$ are the largest circles that can be inscribed in the indicated segments of the circle. The segment joining the centers of circles P and Q intersects the square at A and B. Find AB. *(Source: ARML)* **Hints:** 10

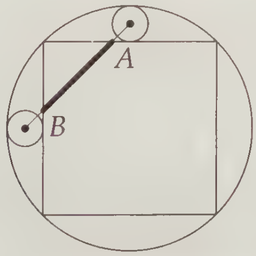


Figure 19.1: Diagram for Problem 19.1.1

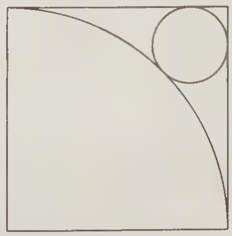


Figure 19.2: Diagram for Problem 19.1.2

19.1.2 In the diagram at right above, a quarter-circle centered at one vertex of the square connects two other vertices of the square. A small circle is tangent to the large circle and to two sides of the square as shown. Each side of the square is 4 units long. What is the radius of the small circle? **Hints:** 19

19.1.3 A round table is pushed into a corner as shown in the diagram. Point A is on the outer edge of the table and is 2 inches from one of the walls. Given that the radius of the table is 37 inches, how many inches is point A from the other wall? *(Source: MATHCOUNTS)* **Hints:** 30

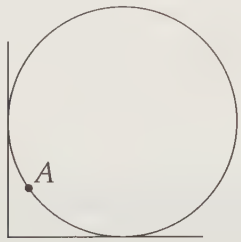


Figure 19.3: Diagram for Problem 19.1.3

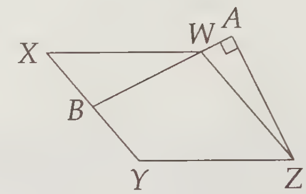


Figure 19.4: Diagram for Problem 19.1.4

19.1.4★ $WXYZ$ is a parallelogram, B is the midpoint of $\overline{XY}$, and A is on $\overleftrightarrow{WB}$ such that $\overline{ZA} \perp \overline{BA}$. Prove $AY = YZ$. **Hints:** 34

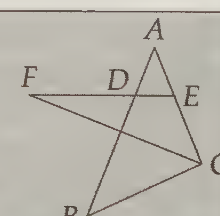
19.2 Assigning Variables

In many problems, finding the answer is not simply a matter of performing routine calculations. Often we need to discover relationships between angles or between lengths. Usually, these relationships will be equations of some sort. One very helpful way to keep track of this information is to assign variables to lengths and/or angles. We can then write equations that are easy to read, and we can use the variables to keep track of the information in our diagram.

As you'll see, we often use this strategy together with our 'extra line' tactic from the last section, particularly when we build right triangles to use the Pythagorean Theorem.

Problems

Problem 19.5: In the figure at right, $AD = AE$, and F is the intersection of $\overrightarrow{ED}$ with the bisector of $\angle C$. If $\angle B = 36°$, how many degrees are there in $\angle CFE$?

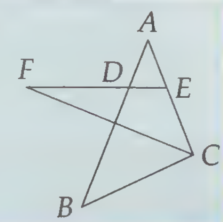


Problem 19.6: Points A, B, C, and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice the perimeter of $\triangle BEC$. Find AE. *(Source: AMC 10)*

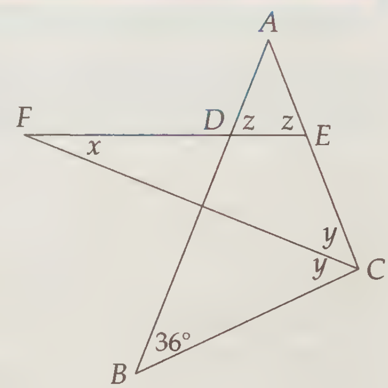
Problem 19.7: $WXYZ$ is a rectangular sheet of paper with $WX = 10$ and $XY = 12$. The paper is folded so that W coincides with the midpoint of $\overline{XY}$. What is the length of the fold?

Variables are great help in angle-chasing problems. As you might guess, the fact that the angles of a triangle add to 180° is our most commonly used equation-forming tool. Try assigning variables and chasing angles in this problem.

Problem 19.5: In the figure at right, $AD = AE$, and F is the intersection of $\overrightarrow{ED}$ with the bisector of $\angle C$. If $\angle B = 36^\circ$, how many degrees are there in $\angle CFE$?



Solution for Problem 19.5: We can't deduce the measures of any more angles directly from $\angle B = 36^\circ$, so we'll have to use relationships among the angles to learn more. We could start off by writing a bunch of equations like:



$$\begin{aligned} \angle B + \angle ACB + \angle A &= 180^\circ \\ \angle ADE &= \angle AED \\ \angle EDB + \angle DBC + \angle BCE + \angle DEC &= 360^\circ, \end{aligned}$$

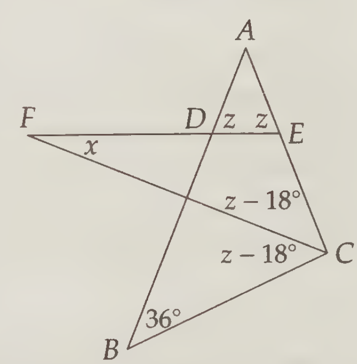
but there are so many different angles that it's hard to keep track of them all. Instead, we assign variables. We assign a variable to what we want, $\angle CFE = x$, and we also pick variables for angles that are easy to relate to other angles in the diagram. We are given that $\triangle ADE$ is isosceles, so we let $\angle ADE = \angle AED = z$. Similarly, our angle bisector gives us $\angle ACF = \angle FCB$, so we call these both y.

Now we look for other angles we can label in terms of x, y, or z. Triangle $\triangle ADE$ gives us $\angle A = 180^\circ - 2z$. Then, $\triangle ABC$ gives us $\angle A + \angle B + \angle ACB = 180^\circ$, so

$$180^\circ - 2z + 36^\circ + 2y = 180^\circ.$$

Therefore, $y = z - 18^\circ$.

Now we can reduce the number of variables in our diagram. If we don't see our path to solution now, we can redraw the diagram using only x and z. Since $\angle FEA$ is an exterior angle of $\triangle FEC$, we have



$$\angle CFE + \angle FCE = \angle DEA.$$

Therefore, $x + z - 18^\circ = z$, so $x = 18^\circ$.

We could also have written $\angle DEC = 180^\circ - z$ in our diagram, then used the sum of the angles of $\triangle FEC$ to solve the problem. □

Extra! *The essence of mathematics is in its freedom.*

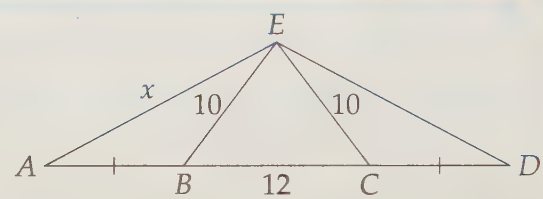
–Georg Cantor

Concept: Assign variables to important angles and use your mastery of basic geometry to express other angles in terms of those variables. Typically, you should assign variables to angles you seek and to angles you know a lot about first. Once you start chasing angles, reach for all the angle tools you know, most notably the facts you know about triangles, straight lines, parallel lines, and angles that intersect circles.

We can also chase lengths with equations. Our main tools here are the Pythagorean Theorem, similar triangles, and Power of a Point. As we've seen throughout this book, the Pythagorean Theorem is the most heavily used of these three for finding lengths.

Problem 19.6: Points A, B, C, and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice the perimeter of $\triangle BEC$. Find AE. *(Source: AMC 10)*

Solution for Problem 19.6: We start with a diagram. We draw everything we are given in the problem. Then we look for more lengths we can find in the diagram and we're almost immediately stuck. So, we start assigning variables. Since we want AE, we let $AE = x$ and look for other sides we can express in terms of x.



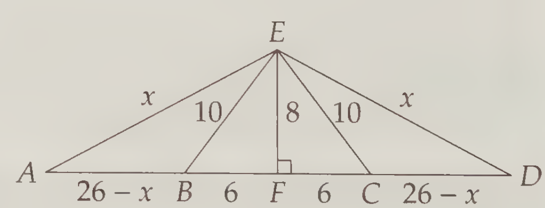
We suspect all these equal side lengths ($AB = CD$ and $BE = CE$) will mean $AE = DE$, so we try to prove it. Since $BE = CE$, $\angle EBC = \angle ECB$. Therefore, $\angle EBA = \angle ECD$, so $\triangle ABE \cong \triangle DCE$ by SAS Congruence. So, we have $DE = AE = x$, too.

This isn't enough, so we look for more lengths we can express in terms of x. AB and CD are all we have left. We look back at the problem for information we haven't used yet and see that bit about the perimeter of $\triangle AED$ being twice that of $\triangle BEC$. From this, we have

$$AE + ED + AB + BC + CD = 2(10 + 10 + 12).$$

Substitution gives $2x + 12 + 2AB = 64$, so $AB = 26 - x$.

Adding all this to our diagram, we're still stuck. The isosceles triangles are a clue to build right triangles and use the Pythagorean Theorem. We draw altitude $\overline{EF}$, thereby building right triangles $\triangle EFC$ and $\triangle EFA$. From $\triangle EFC$ we have $EF = 8$. Now we can build an equation for x by applying the Pythagorean Theorem to $\triangle EFA$. $EF^2 + AF^2 = AE^2$, so



$$64 + (32 - x)^2 = x^2.$$

Therefore, $64 = x^2 - (32 - x)^2$. We factor the right side of this equation as the difference of squares to find:

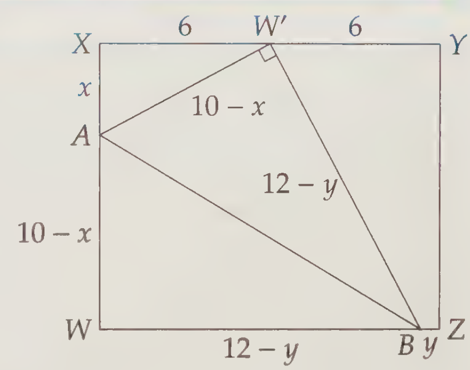
$$64 = [x - (32 - x)][x + (32 - x)] = (2x - 32)(32).$$

Solving for x gives $x = 17$.

Make sure you follow that nifty use of the difference of squares factorization. This manipulation can often save you a few extra steps of nasty algebra (and will reduce careless mistakes). □

Concept: If you need a length in a problem, try building right triangles. Some of the best tools for building right triangles are drawing altitudes of triangles and trapezoids, and drawing radii to points of tangency.

Problem 19.7: $WXYZ$ is a rectangular sheet of paper with $WX = 10$ and $XY = 12$. The paper is folded so that W coincides with the midpoint of $\overline{XY}$. What is the length of the fold?

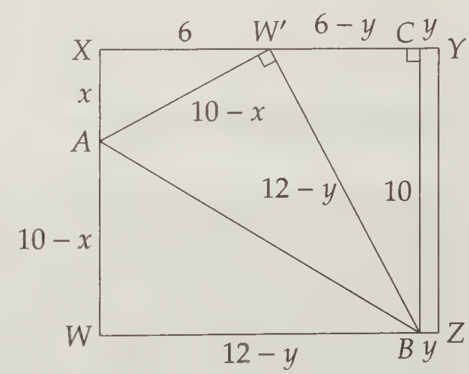


Solution for Problem 19.7: We start with a diagram. Instead of just drawing the folded paper, we also include the paper as it originally was, since we know so much about rectangles. Our diagram shows $WXYZ$, and the fold $\overline{AB}$ that leads to W coinciding with W', the midpoint of $\overline{XY}$. Since $\triangle W'AB$ is just the 'folded over' (i.e. reflected) version of $\triangle WAB$, we have $\triangle W'AB \cong \triangle WAB$. We only initially know $XW' = W'Y = 6$ and $XW = YZ = 10$. We'll have to assign some variables. We let $AX = x$ and $BZ = y$, so we then have $AW = 10 - x$ and $WB = 12 - y$.

We then use our congruent triangles to note that $AW' = AW = 10 - x$ and $W'B = WB = 12 - y$. Right triangle $\triangle XAW'$ gives us an equation for x:

$$x^2 + 6^2 = (10 - x)^2.$$

Solving this equation for x gives $x = 16/5$.



Unfortunately, we don't have a right triangle that quickly gives us y. So, we borrow a tactic from last section and build one. We draw $\overline{BC}$ such that $\overline{BC} \perp \overline{XY}$ as shown. Since $BCYZ$ is a rectangle, we have $CY = y$, $BC = 10$, and $CW' = 6 - y$. We could use the Pythagorean Theorem as before on $\triangle W'CB$. However, we can find the answer a little faster by noting that $\angle CW'B = 180^\circ - \angle AW'B - \angle AW'X = 90^\circ - \angle AW'X = \angle XAW'$. Therefore, right triangles $\triangle W'XA$ and $\triangle BCW'$ are similar. So, we have

$$\frac{AX}{W'C} = \frac{XW'}{BC}.$$

Substitution gives

$$\frac{16/5}{6 - y} = \frac{6}{10},$$

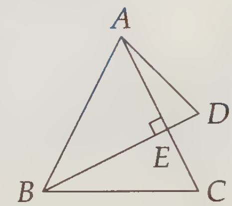
so $y = 2/3$.

Therefore, $WB = BW' = 12 - y = 34/3$. In right triangle $\triangle AWB$, we now have $BW = 34/3$ and $AW = 10 - x = 34/5$, so we can find AB:

$$AB = \sqrt{AW^2 + WB^2} = \sqrt{\left(\frac{34}{5}\right)^2 + \left(\frac{34}{3}\right)^2} = \sqrt{34^2\left(\frac{1}{5}\right)^2 + 34^2\left(\frac{1}{3}\right)^2} = 34\sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{34\sqrt{34}}{15}.$$

Notice once again that we use a little algebraic manipulation, in this case factoring out the 34, to simplify our work. □

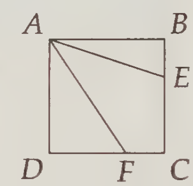
Exercises



19.2.1 Triangles ABC and ABD are isosceles with $AB = AC = BD$, and $\overline{BD}$ intersects $\overline{AC}$ at E. Given that $\overline{BD} \perp \overline{AC}$, find all possible values of $\angle C + \angle D$. *(Source: AMC 12)*

19.2.2 A point on a circle inscribed in a square is 1 and 2 units from the two closest sides of the square. Find the area of the square. *(Source: HMMT)*

19.2.3 $ABCD$ is a rectangular piece of paper with $AB = 8$ and $AD = 12$. We fold the paper so that B coincides with D. What is the length of the fold? **Hints:** 92



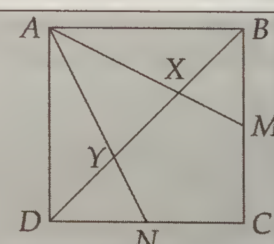
19.2.4★ Square $ABCD$ has side length 1. Point E is chosen on side $\overline{BC}$ so that $AE+EB = \frac{3}{2}$, and point F is chosen on side $\overline{CD}$ so that $\overline{AF}$ bisects $\angle DAE$. Find DF. *(Source: Mandelbrot)* **Hints:** 566, 588, 592

19.3 Proofs

Geometry proofs present a challenge by offering us so many different avenues of exploration. Our task is to narrow down the possible approaches to the ones that are most likely to bear fruit. Before trying the problems in this section, read through Problem 13.10 again, noting particularly how we keep track throughout the problem of What We Know and What We Want. Then, try applying this approach to these problems. We will also use some of our techniques from the first two sections of this chapter, especially adding extra lines.

Problems

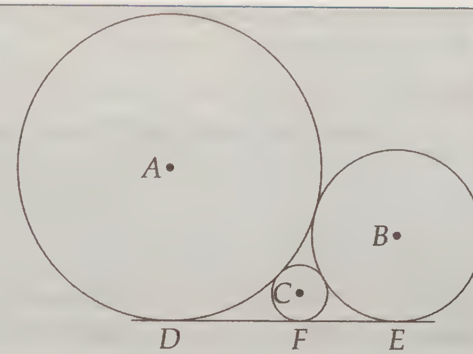
Problem 19.8: M and N are the midpoints of sides $\overline{BC}$ and $\overline{CD}$, respectively, of square $ABCD$. $\overline{AM}$ and $\overline{AN}$ meet $\overline{BD}$ at X and Y, as shown. Show that $BX = XY = YD$.



Extra! A 'proof without words' of the Pythagorean Theorem is shown at right. Alexander Bogomolny describes it as an 'unfolded variant' of a proof by abu' l'Hasan Thâbit ibn Qurra Marwân al'Harrani popularized by Monty Phister.

Problem 19.9: $\odot A$ and $\odot B$ are externally tangent. D is on $\odot A$ and E on $\odot B$ such that $\overleftrightarrow{DE}$ is a common external tangent of the two circles. $\odot C$ is tangent to $\odot A$, $\odot B$, and $\overline{DE}$ as shown. Given that a is the radius of $\odot A$, b is the radius of $\odot B$, and c is the radius of $\odot C$, prove that

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}.$$



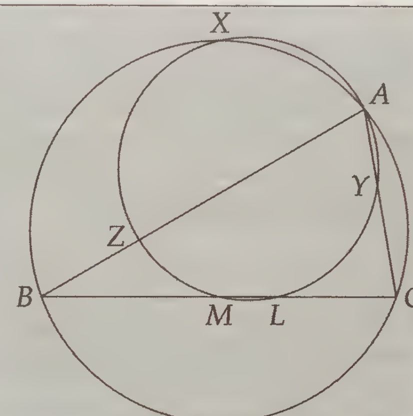
Problem 19.10: In $\triangle ABC$ let L be the foot of the angle bisector from A to $\overline{BC}$, and let M be the midpoint of $\overline{BC}$. We also draw the circumcircle of $\triangle ABC$. Finally, we construct the circle through points A, L, and M. This circle intersects the rest of the diagram in several remarkable ways, so we label points Z, Y, and X where the circle meets $\overline{AB}$, $\overline{AC}$ and the circumcircle of $\triangle ABC$, respectively.

(a) Prove that $BZ = CY$.

(b) Prove that $\triangle XBZ \cong \triangle XCY$.

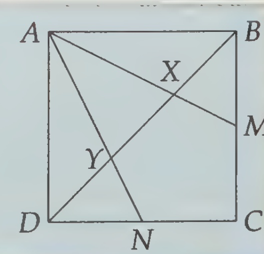
(c) Prove that $\overline{XL}$ is the diameter of the circumcircle of $\triangle ALM$.

(Source: Mandelbrot)



In some of the following problems, we will explore in search of the solution. After some of these problems, we will present brief solutions based on our discoveries. The purpose of the exploration is to give you some insight into how an experienced geometer thinks about problems. The purpose of the solutions following the exploration is to give you a feel for how to write up clear solutions once you find them.

Problem 19.8: M and N are the midpoints of sides $\overline{BC}$ and $\overline{CD}$, respectively, of square $ABCD$. $\overline{AM}$ and $\overline{AN}$ meet $\overline{BD}$ at X and Y, as shown. Show that $BX = XY = YD$.



Solution for Problem 19.8: We make our list of what we know and what we want. We want to prove a statement about lengths, so we start with information we know about lengths.

What We Know	What We Want
$AB = BC = CD = DA$	$BX = XY = XD$
$BM = MC = CN = DN$	

It sure seems reasonable that $BX = YD$, but we have to prove it. Our main tool for proving segments equal is congruent triangles, so we look for congruent triangles in our diagram. $\triangle ABX$ and $\triangle ADY$ look congruent, so we go after these two triangles.

Important: Draw precise diagrams. Triangle congruences, parallel lines, collinear points, and much, much more stand out in precise diagrams.

Since $ABCD$ is a square, we have $AD = AB$ and $\angle ADB = \angle ABD = 45°$. We just need another angle to prove congruence. The angles at X and Y don't look easy to work with, but angles $\angle XAB$ and $\angle DAY$ are also parts of right triangles $\triangle BAM$ and $\triangle DAN$. Since $AD = AB$ and $BM = DN$ (both are half the side length of the square), we have $\triangle BAM \cong \triangle DAN$ by LL Congruence (or by SAS Congruence). So, we have $\angle XAB = \angle MAB = \angle NAD = \angle YAD$, which means $\triangle ABX \cong \triangle ADY$ by ASA Congruence.

We have more information for our table now. Most importantly, *we've reduced our problem to just proving* $BX = XY$. We quickly brainstorm for a few other statements that are equivalent to $BX = XY$ and we see that $BY = 2BX$ or $BY = 2DY$ will also give us what we want. Also, if we show $BX = BD/3$ or $XY = BD/3$, we'll be finished.

What We Know	What We Want
$AB = BC = CD = DA$	$BX = XY$
$BM = MC = CN = DN$	$BY = 2BX$
$BX = XD$	$BY = 2DY$
$\triangle ADY \cong \triangle ABX$	$XY = BD/3$
	$BX = BD/3$

This gives us a lot to shoot for. We don't have any more congruent triangles that are interesting to investigate, but the ratios in our 'What We Want' list suggest looking for similar triangles. Parallel lines mean similar triangles. Specifically, we have $\triangle DYN \sim \triangle BYA$ because $\overline{DN} \parallel \overline{AB}$. Seeing $BY = 2DY$ in the 'What We Want' list, we focus on what $\triangle DYN \sim \triangle BYA$ tells us about $\overline{BY}$ and $\overline{DY}$:

$$\frac{DY}{BY} = \frac{DN}{AB} = \frac{CD/2}{AB} = \frac{1}{2}.$$

We've found something we want! Now we retrace our steps to write a nice solution:

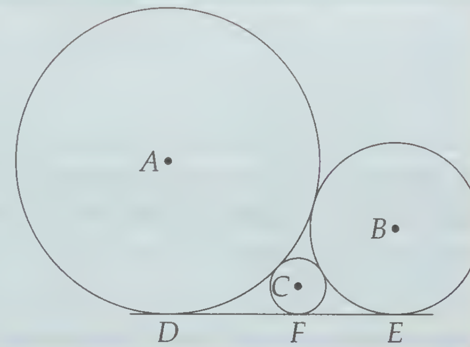
Since $AB = AD$, $\angle ABM = \angle ADN$ and $DN = BM$ (each is half the length of a side of the square), we have $\triangle ADN \cong \triangle ABM$ by SAS Congruence. Therefore, $\angle DAY = \angle DAN = \angle MAB = \angle XAB$. Together with $AB = AD$ and $\angle ABX = \angle YDA$ (since each is 45°), we have $\triangle ABX \cong \triangle ADY$ by ASA Congruence. Therefore, $BX = DY$.

Extra! Josey and Beth are standing 100 feet apart with a 100 foot rope connecting their ankles such that the rope is pulled taut 1 inch above the ground between them. Josey steps one foot towards Beth, so the rope is relaxed along the ground. Is it now possible to lift the center of the rope high enough that a person can walk under it without ducking or moving either of the girls? Is it possible for Josey to lift the rope above her head without her and Beth getting closer (and without her lifting her foot)?

Because $\overline{DN} \parallel \overline{AB}$, we have $\angle NDY = \angle YBA$ and $\angle YAB = \angle YND$, so $\triangle DYN \sim \triangle BYA$. Therefore, $DY/BY = DN/AB = (CD/2)/AB = 1/2$. Therefore, $DY = BY/2$, so $BX = BY/2$ and $XY = BY - BX = BY/2$ also. Thus, $BX = XY = YD$. (See if you can find other solutions!) □

Problem 19.9: $\odot A$ and $\odot B$ are externally tangent. D is on $\odot A$ and E on $\odot B$ such that $\overleftrightarrow{DE}$ is a common external tangent of the two circles. $\odot C$ is tangent to $\odot A$, $\odot B$, and $\overline{DE}$ as shown. Given that a is the radius of $\odot A$, b is the radius of $\odot B$, and c is the radius of $\odot C$, prove that

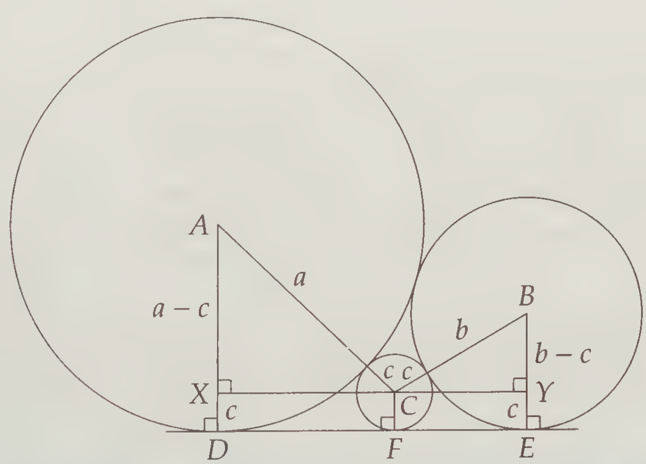
$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}.$$



Solution for Problem 19.9: There's not much working backwards we can do here, but we can at least get rid of the fractions by mutliplying our desired equation by $\sqrt{abc}$. This gives

$$\sqrt{ab} = \sqrt{bc} + \sqrt{ac}.$$

Going forwards, we can use all the information we know about tangent circles and lines. We connect centers and draw radii. The perpendicular lines thus formed and the square roots in our target equation both point in the same direction: the Pythagorean Theorem. In looking for a good right triangle to build, we remember Problem 12.22, in which we built a right triangle to find the length of a common tangent to two circles. Inspired by this, we draw a line through C perpendicular to $\overline{AD}$ and $\overline{BE}$. Rather than listing everything we know in a table, we add it to our diagram:



Concept: A large diagram in which we keep close track of what we discover is a *very* effective way to keep track of What We Know.

Now we have our right triangles. Since $XDFC$ and $CYEF$ are rectangles, we have $XD = CF = YE = c$. Therefore, $AX = a - c$ and $BY = b - c$. Since $AC = a + c$, we can apply the Pythagorean Theorem to $\triangle AXC$

to find

$$\begin{aligned} XC &= \sqrt{AC^2 - AX^2} \\ &= \sqrt{(a+c)^2 - (a-c)^2} \\ &= \sqrt{4ac} \\ &= 2\sqrt{ac}. \end{aligned}$$

Our result is part of the equation we want to prove! Since $XCFD$ is a rectangle, we have $DF = 2\sqrt{ac}$ as well. Therefore, the common external tangent of tangent circles with radii a and c has length $2\sqrt{ac}$. We can apply this result to our other two pairs of tangent circles to find $FE = 2\sqrt{bc}$ and $DE = 2\sqrt{ab}$. Since $DE = DF + FE$, we have

$$2\sqrt{ab} = 2\sqrt{ac} + 2\sqrt{bc}.$$

Dividing this equation by $2\sqrt{abc}$ gives the desired

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}.$$

Notice that we used both of our tactics from earlier in this chapter. We drew many extra lines to build right triangles and we labeled lengths we could find in terms of our variables. As we did so, we found lengths that are in our expression we sought. This brought us right to the solution. □

Concept: Always compare new problems to problems you have already solved. Problem 19.9 is very similar to Problem 12.22, in which we found the length of a common tangent. Thinking of this common external tangent problem gave us a quick path to the solution of our new problem.

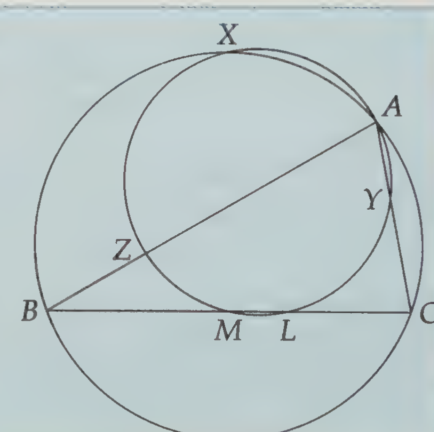
Problem 19.10: In $\triangle ABC$ let L be the foot of the angle bisector from A to $\overline{BC}$, and let M be the midpoint of $\overline{BC}$. We also draw the circumcircle of $\triangle ABC$. Finally, we construct the circle through points A, L, and M. This circle intersects the rest of the diagram in several remarkable ways, so we label points Z, Y, and X where the circle meets $\overline{AB}$, $\overline{AC}$ and the circumcircle of $\triangle ABC$, respectively.

(a) Prove that $BZ = CY$.

(b) Prove that $\triangle XBZ \cong \triangle XCY$.

(c) Prove that $\overline{XL}$ is the diameter of the circumcircle of $\triangle ALM$.

(Source: Mandelbrot)



Solution for Problem 19.10:

(a) We don't have an obvious pair of congruent triangles to go after to show that $BZ = CY$. (We suspect that we'll need $BZ = CY$ to tackle the triangles in the second part, so we don't go after those triangles immediately.) We pull out our other length tools that this diagram invites us to use. The circles suggest Power of a Point. Point B gives us $(BZ)(BA) = (BM)(BL)$ and C

gives $(CY)(CA) = (CL)(CM)$. These equations include our target lengths! The Angle Bisector Theorem gives us $AB/BL = AC/CL$, which has one of our target lengths, and the midpoint gives us $BM = CM$. We put all this information together:

What We Know	What We Want
$(BZ)(BA) = (BM)(BL)$	$BZ = CY$
$(CY)(CA) = (CL)(CM)$	
$AB/BL = AC/CL$	
$BM = CM$	

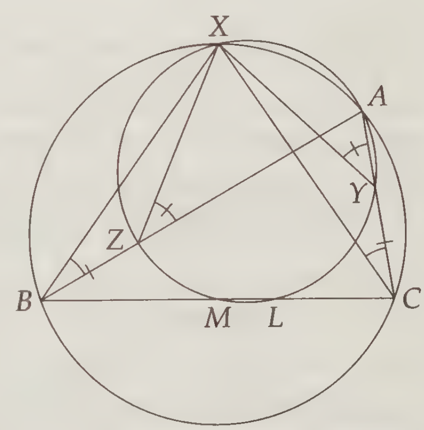
Solving our first two equations in the What We Know column for BZ and CY gives us $BZ = (BM)(BL)/(BA)$ and $CY = (CL)(CM)/(CA)$. We'd like the expressions on the right sides of these equations to be equal:

What We Know	What We Want
$(BZ)(BA) = (BM)(BL)$	$BZ = CY$
$(CY)(CA) = (CL)(CM)$	$(BM)(BL)/(BA) = (CL)(CM)/(CA)$
$AB/BL = AC/CL$	
$BM = CM$	

Since $CM = BM$, our second line of what we want reduces to $BL/BA = CL/CA$. This is just the reciprocal of the third line in What We Know! Now we have a path from What We Know to What We Want, so we can write our proof:

> From the power of point B, we have $(BZ)(BA) = (BM)(BL)$, or $BZ = (BM)(BL)/(BA)$. The Angle Bisector Theorem applied to bisector $\overline{AL}$ of $\triangle ABC$ gives $BL/BA = CL/CA$, so $BZ = (BM)(BL/BA) = (BM)(CL/CA)$. $BM = CM$ because M is the midpoint of $\overline{BC}$, so $BZ = (CM)(CL/CA)$. Finally, the power of point C gives us $(CY)(CA) = (CL)(CM)$, from which we have $CY = (CM)(CL/CA) = BZ$.

(b) The previous part gave us $BZ = CY$, so if we find two pairs of equal corresponding angles in $\triangle BZX$ and $\triangle CYX$, we will prove the triangles are congruent.

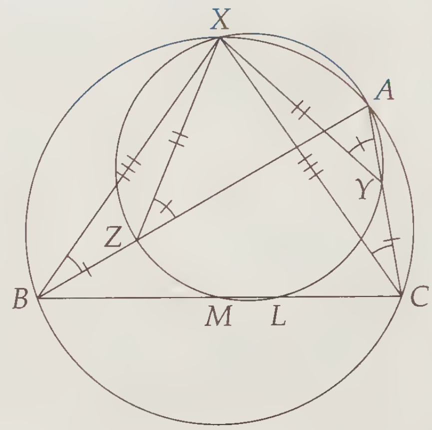


We have circles, so we look for angles that are inscribed in the same arc. Angles $\angle XBZ$ and $\angle XCY$ are the same as $\angle XBA$ and $\angle XCA$, respectively. These two angles are both inscribed in $\widehat{XA}$ of the larger circle, so we have $\angle XBZ = \angle XCY$. Unfortunately, we can't do the same for any other pair of angles of our triangles. We therefore start chasing angles, looking for a pair of equal angles that we might relate to the angles in our triangles.

We continue looking for equal inscribed angles, and find $\angle AZX = \angle AYX$ since both are inscribed in $\widehat{AX}$ of the smaller circle. We can quickly relate each to angles in our triangles:

$$\angle BZX = 180^\circ - \angle AZX = 180^\circ - \angle AYX = \angle XYC.$$

Therefore, by ASA Congruence, we have $\triangle XBZ \cong \triangle XCY$. Notice how marking equal angles and sides as we find them makes the congruent triangles stand out.



(c) We don't have a lot of tools for proving a segment is a diameter of a circle. One possible approach is to show that one of the angles inscribed in either $\widehat{XYL}$ or $\widehat{XZL}$ is a right angle, thus making the arc 180°. We consider the labeled points on the small circle, looking for a point to be the vertex of our sought-after right angle.

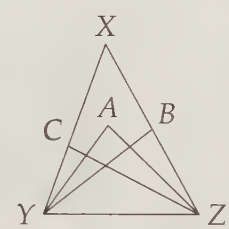
Marking the equal sides given by $\triangle XBZ \cong \triangle XCY$ helps guide us. Our congruent triangles tell us $BX = CX$, so $\triangle XBC$ is isosceles. Therefore, midpoint M of $\overline{BC}$ is also the foot of the altitude from X to $\overline{BC}$. Since $\angle XMC$ is a right angle, $\angle XML = \angle XMC$, and $\angle XML$ is inscribed in $\widehat{XYL}$, we have $\widehat{XYL} = 180°$. Therefore, $\overline{XL}$ is a diameter of the small circle.

As an Exercise, you'll find another approach to this part.

□

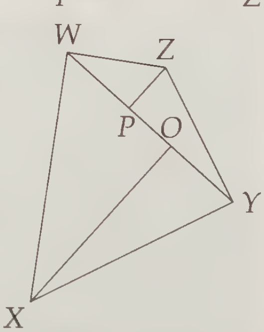
Exercises

19.3.1 In triangle ABC, let the bisector of angle BAC intersect $\overline{BC}$ at D and the circumcircle of triangle ABC at E. Prove that $AE \cdot DE = BE^2$. **Hints:** 260



19.3.2 In the diagram, $\overline{ZA}$ bisects $\angle BZC$, and $\overline{YA}$ bisects $\angle BYX$. Prove that $\angle YCZ + \angle YBZ = 2\angle YAZ$. **Hints:** 134

19.3.3 Prove $\angle LZX = \angle LYX$ in Problem 19.10 without using the fact that $\overline{XL}$ is a diameter of the small circle. Use this to find an alternate proof that $\overline{XL}$ is a diameter of the circumcircle of $\triangle ALM$.

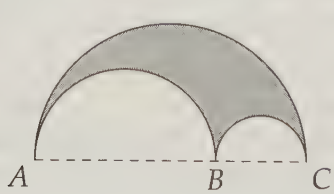


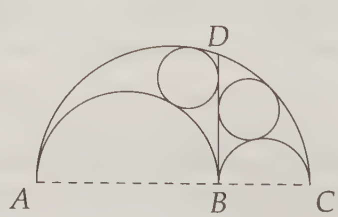
19.3.4 Quadrilateral $WXYZ$ has right angles at $\angle W$ and $\angle Y$ and an acute angle at $\angle X$. Altitudes are dropped from X and Z to diagonal $\overline{WY}$, meeting $\overline{WY}$ at O and P as shown. Prove that $WO = PY$.

19.3.5★ Let A, B, and C be points on a line, in that order. We draw semicircles on segments $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$. Then the figure they enclose is called an *arbelos*, which is shaded in the diagram below at left. The arbelos has many interesting properties, two of which we prove in this problem.

(a) Take point D on arc $\widehat{AC}$ such that $\overline{DB} \perp \overline{AC}$. Prove that the arbelos has the same area as the circle with diameter $\overline{BD}$. **Hints:** 340

(b)★ Line $\overline{BD}$ from the previous part divides the arbelos into two parts. Inscribe a circle in each part. Prove that the two circles have equal radius. **Hints:** 295, 330, 284, 221





19.4 Summary

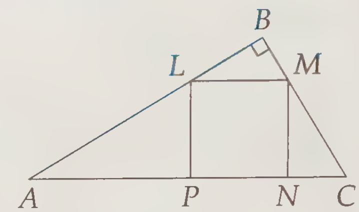
This entire chapter is about problem solving strategies. Here are several of the most useful approaches we used to solve problems throughout this book.

- When in doubt, build right triangles. One very common way to do so is to draw radii to points of tangency. Nearly always try this when you have circles and tangents in a problem. Altitudes of triangles and trapezoids also often produce useful right triangles.
- Connecting points that are originally not connected in a diagram can be extremely useful! This doesn't mean you should connect everything in your diagram immediately, however. Look for segments to draw that will be helpful, particularly those connecting important points, or those that form segments, triangles, or angles you know something about immediately.
- $60^\circ, 30^\circ$, and even 120° angles are good signs to try to build 30-60-90 triangles by dropping altitudes or extending segments.
- Consider extending segments that stop suddenly inside figures such as triangles, quadrilaterals, or circles.
- If you successfully use a certain tactic in a problem to get some information, but you still haven't solved the problem, try using that same tactic again in a different way. Maybe it still has more information to give!
- Label lengths as you find them, even if you have to label them in terms of an important variable.
- Assign variables to important angles and use your mastery of basic geometry to express other angles in terms of those variables. Typically, you should first assign variables to angles you seek and to angles you know a lot about.
- Once you start chasing angles, reach for all the angle tools you know, most notably the facts you know about triangles, straight lines, parallel lines, and angles that intersect circles.
- When trying to find a geometric proof, keep careful track of what you know and what you want.
- Draw large, precise diagrams – triangle congruences, parallel lines, collinear points, and much, much more stand out more clearly in large, precise diagrams than in small, scribbled ones.
- Compare new problems to problems you have already solved.

Challenge Problems

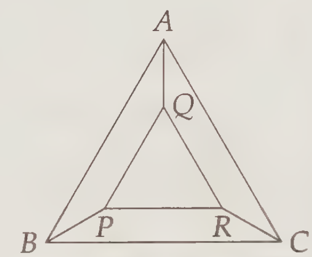
19.11 $\triangle XYZ$ is a right triangle with right angle $\angle X$. P is on $\overline{XZ}$. The triangle is folded over $\overline{YP}$ so that point X lands on side $\overline{YZ}$. Given that $XY = 6$ and $PY = PZ$, find, with proof, $\angle XZY$.

19.12 Square $LMNP$ is inscribed in right triangle ABC as shown. Given $PN = 6$, compute $(AP)(NC)$. *(Source: ARML)*



19.13 Four circles of radius 2 are arranged so that each is tangent to two others, and their centers are the vertices of a square of side length 4. A small circle is inside the square such that it is tangent to the four circles. What is the radius of the small circle?

19.14 $\triangle ABC$ and $\triangle PQR$ are equilateral in the diagram at right. $\overline{AQ}$, $\overline{RC}$, and $\overline{BP}$ are congruent and bisect their respective angles. The four interior regions $AQPB$, $AQRC$, $BPRC$, and QPR all have the same area. Given that $AB = 20$, what is PB? *(Source: MATHCOUNTS)*

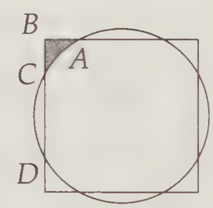


19.15 Point P is inside rectangle $ABCD$. Prove that $AP^2 + CP^2 = BP^2 + DP^2$. **Hints:** 344, 380

19.16 $WXYZ$ is a parallelogram. M is the midpoint of $\overline{WZ}$ and O is the midpoint of $\overline{YZ}$. $\overline{MO}$ and $\overline{XZ}$ meet at N. Find ZN/NX.

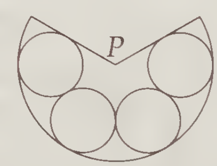
19.17 In $\triangle ABC$, $\angle BAC = 60°$, $\angle ACB = 45°$, and D is on $\overline{BC}$ such that $AD = 18$ and $\angle BAD = \angle CAD$. Find the area of $\triangle ABC$. **Hints:** 411, 443, 473

19.18 In the diagram at right, the circle and the square have the same center. If the area of shaded region ABC equals the area bounded by $\overline{CD}$ and minor arc $\widehat{CD}$, compute the ratio of the side of the square to the radius of the circle. *(Source: ARML)*



19.19 In quadrilateral $ABCD$, $\overline{AB} \parallel \overline{CD}$. $\overline{AC}$ and $\overline{BD}$ meet at E. Points M and N are the midpoints of $\overline{AE}$ and $\overline{DE}$, respectively. $\overline{BM}$ and $\overline{BE}$ trisect $\angle ABC$, and $\overline{CE}$ and $\overline{CN}$ trisect $\angle BCD$. Prove that $ABCD$ is a rectangle. **Hints:** 501, 539

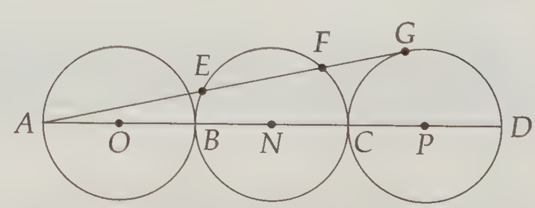
19.20 Four congruent circles are tangent to each other and tangent to the edges of a sector as shown. If the straight edges are joined to form a right circular cone with vertex at P, the radius of the base would be 2/3 the slant height of the cone. Compute the ratio of the radius of the sector to the radius of each circle. *(Source: ARML)* **Hints:** 562, 535



19.21 $ABCD$ is a square. Parallel lines m, n, and p pass through vertices A, B, and C, respectively. The distance between m and n is 7 units, and the distance between n and p is 9 units. Find the number of square units in the area of square $ABCD$. *(Source: MATHCOUNTS)* **Hints:** 502, 468

19.22 Chords $\overline{XY}$ and $\overline{TU}$ of $\odot O$ bisect each other. Furthermore, $XY = TU$. Prove that the two chords meet at the center of the circle.

19.23 In the figure, points B and C lie on line segment $\overline{AD}$, and $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are diameters of circles O, N, and P, respectively. Circles O, N, and P all have radius 15, and the line AG is tangent to circle P at G. If $\overline{AG}$ intersects circle N at points E and F, then find EF. *(Source: AHSME)* **Hints:** 502, 418



19.24 Find the volume of a sphere that is inscribed in a regular octahedron with side length 12. **Hints:** 337, 279, 242

19.25 In the diagram at left below, O is the center of the circle. Find the area of the shaded region given that $\angle AOC = 120°$ and $OM = MC = 6$.

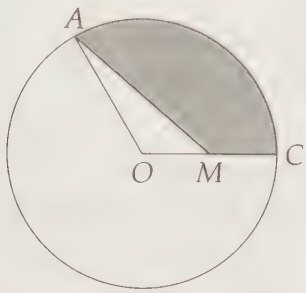


Figure 19.5: Diagram for Problem 19.25

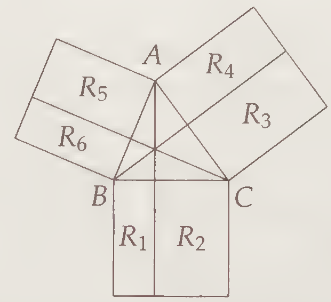


Figure 19.6: Diagram for Problem 19.26

19.26 Squares are erected externally on the sides of $\triangle ABC$ as shown at right above. The altitudes from vertices A, B, and C are drawn and extended to divide these squares into rectangles. Let R_i, for $1 \le i \le 6$, be the areas of the rectangles as shown. Prove that $R_2 = R_3$, $R_4 = R_5$, and $R_6 = R_1$. What does this result say if $\angle ACB = 90°$? **Hints:** 496, 446

19.27 Shown at left below is right rectangular prism $ABCDEFGH$. Prove that $\triangle ACF$ is acute. **Hints:** 396, 365

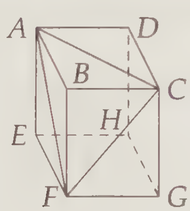


Figure 19.7: Diagram for Problem 19.27

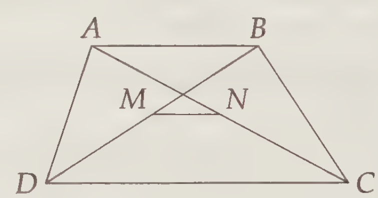


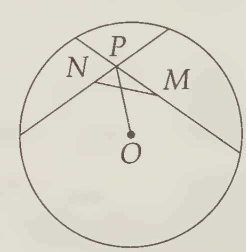
Figure 19.8: Diagram for Problem 19.28

19.28 M and N are the midpoints of diagonals $\overline{BD}$ and $\overline{AC}$, respectively, of trapezoid $ABCD$ at right above. $\overline{AB} \parallel \overline{CD}$, $AB = 8$, and $MN = 6$.

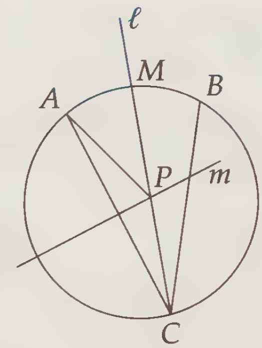
(a) Prove that $\overline{MN} \parallel \overline{CD}$. (Find a rigorous proof – this is one of those 'obvious' facts that requires a careful proof.) **Hints:** 333

(b) Find CD. **Hints:** 146, 492

19.29★ A circle in the plane has center O. Two chords, with midpoints M and N, intersect at P. Prove that $MN \le OP$. When does equality occur? **Hints:** 206, 239



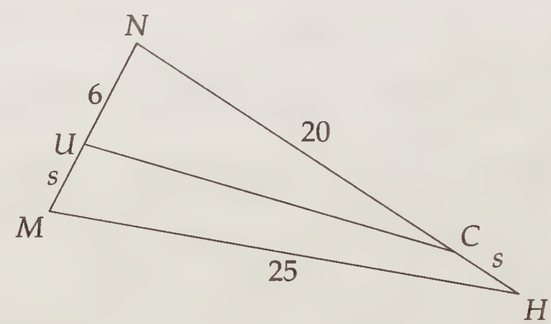
19.30★ Let R and S be points on the sides $\overline{BC}$ and $\overline{AC}$, respectively, of triangle ABC, and let P be the intersection of $\overline{AR}$ and $\overline{BS}$. Determine the area of triangle ABC if the areas of triangles APS, APB, and BPR are 5, 6, and 7, respectively. *(Source: USAMTS)* **Hints:** 286, 251



19.31 In the diagram to the right, points A and B are on $\odot O$. Point C is on major arc $\widehat{AB}$. Line ℓ is the angle bisector of $\angle ACB$ and line m is the perpendicular bisector of $\overline{AC}$. Lines ℓ and m meet at point P, point M is the midpoint of minor arc $\widehat{AB}$, and minor arc $\widehat{AB}$ has measure θ. Prove that $\angle APM = \theta/2$. *(Source: Mandelbrot)* **Hints:** 313

19.32 Sphere $\mathcal{Z}$ is tangent to all 6 edges of regular tetrahedron $ABCD$. Given that each edge of the tetrahedron has length 12, find the radius of sphere $\mathcal{Z}$. **Hints:** 205, 230, 268

19.33★ $\triangle ABC$ is an equilateral triangle with side length 6. Prove that for any point P we choose inside $\triangle ABC$, the sum of the distances from P to the sides of $\triangle ABC$ is the same. **Hints:** 110

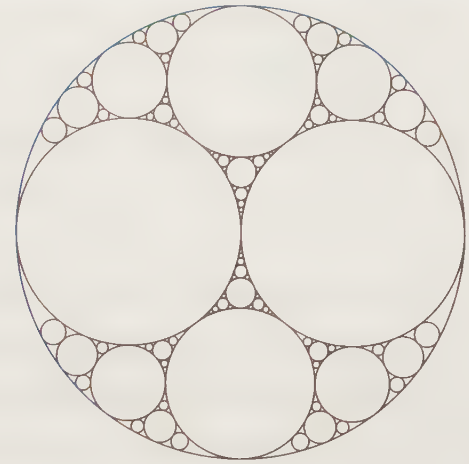


19.34★ U and C are points on the sides of $\triangle MNH$ such that $MU = s$, $UN = 6$, $NC = 20$, $CH = s$, $HM = 25$. If $\triangle UNC$ and $MUCH$ have equal areas, what is s? *(Source: HMMT)* **Hints:** 275

19.35★ Let C_1 and C_2 be two circles that are externally tangent, and let their centers be O_1 and O_2, respectively. Let P be a point on C_1 and let Q be a point on C_2 such that $\overline{PQ}$ is a common external tangent to circles C_1 and C_2, and let M be the midpoint of $\overline{PQ}$. Prove that $\angle O_1MO_2 = 90°$. **Hints:** 283, 308

19.36★ Four balls of radius 1 are all tangent to each other. What is the radius of the smallest sphere that encloses them? **Hints:** 334

19.37★ Diagonals $\overline{AC}$ and $\overline{BD}$ of regular heptagon $ABCDEFG$ meet at X. Prove that $AB + AX = AD$. **Hints:** 359



Hints to Selected Problems

1. Build a right triangle in which $\sin A = x$.
2. If a line is tangent to $\odot A$ at the point P, then the line is perpendicular to $\overline{AP}$.
3. Start with right triangle $\triangle TUV$ with right angle at U. Build a circle like the one we used in the previous parts.
4. To show that $AC = BC$ if $\overline{CM}$ bisects $\angle ACB$, try using the Angle Bisector Theorem.
5. Find the area of the whole figure two different ways.
6. What is $A'B'/AB$? What about $A'C'/AC$ and $B'C'/BC$?
7. The center of each circle must be on the graph of what equation?
8. Do you notice anything special about $\overline{AE}$ and $\overline{DF}$?
9. Consider the SSA examination we did in Problem 3.14.
10. Find the radii of the little circles. Draw segments from the center of the big circle to the points where the little circles are tangent to the big circle.
11. Let $\overrightarrow{VX}$ hit the base of the cone at Z. What kind of triangle is VXZ?
12. (For $3YZ > XY$.) Note that $\triangle XAB$ is equilateral. Apply the Triangle Inequality to $\triangle AYB$ and $\triangle YZB$.
13. How many little triangles do we add in the first step? In the second? In the third? In each step after the third? What is the ratio of the area of each triangle added to the area of each triangle added in the previous step?

14. Let the smallest exterior angle have measure x.
15. Why must $ACDF$ be a rectangle? (Don't forget to use $AD = 25$.)
16. Let the legs be a and b. And don't forget the expansion of $(a+b)^2$!
17. How did we find the sum of the exterior angles of a triangle? Can we try essentially the same tactic here?
18. Draw altitude $\overline{BX}$ of $\triangle ABC$. What kind of triangles are $\triangle ABX$ and $\triangle BXC$?
19. Draw the diagonal of the square that passes through the center of the little circle. Find the length of this diagonal in terms of the radius of the little circle.
20. Connect A, B, and C to the center of the circle of which the path is a part. Call this center O.
21. Let the triangle be $\triangle ABC$, and the point on the circumcircle be P. Let X and Y be the feet of the altitude from P to $\overline{AB}$ and $\overline{AC}$, respectively. Describe the circumcircles of $\triangle PXA$ and $\triangle PYA$. (Proving the existence of the Simson line is pretty tough! We'll explore it more in the next two volumes of this series.)
22. Build a right triangle with $\overline{XC}$ as one of the sides.
23. Let $\widehat{WZ} = x$. Label the other arcs in terms of x based on what we are given in the problem.
24. Prove that $\triangle BDF \cong \triangle EFD$. Can you show that any other triangles are congruent to $\triangle EFD$?
25. Notice that median $\overline{AM}$ is half as long as the side to which it is drawn.
26. If a translation maps E to C and F to B, then what must be true about $\overline{EC}$ and $\overline{FB}$?
27. Notice you have a midpoint of one side of $\triangle ABC$ – what do you know about midpoints and areas? What segment does this suggest drawing?
28. How are the sides of the pentagons related?
29. Try working backwards. Let our circles be $\odot O$ and $\odot P$ with Y on $\odot O$ and Z on $\odot P$ such that $\overline{YZ}$ is tangent to both circles. Furthermore, let r_O and r_P be the respective radii of the circles. Build right triangles.
30. Build a useful right triangle with the radius to A as hypotenuse.
31. Build more right angles.
32. Show that $EC = FB$ and $\overline{EC} \parallel \overline{FB}$.
33. Are our target angles corresponding angles of triangles we can prove are congruent?
34. Extend $\overline{AB}$ past B.
35. Extend $\overline{AQ}$ to point Z on $\overline{BC}$.
36. For the second part, notice that if we can find one set of a, b, and c for which it fails, then we are finished.

37. Suppose D is inside the circle. Extend $\overline{AD}$ and $\overline{CD}$ to meet the circle again at X and Y, respectively. In terms of arcs of the circumcircle, what is $\angle A + \angle C$? Can you prove this must be less than 180°? What if D is outside the circle?

38. What do we get when we multiply the two areas together? What lengths are included in this product?

39. What is $[ABC]$?

40. Start with the Pythagorean Theorem, $a^2 + b^2 = 73^2$. Solve for a^2. Can you factor the result? Use clever trial and error to make your resulting expression a perfect square.

41. Let T be the center of the circle. Extend $\overrightarrow{VU}$ to hit the circle again and extend $\overleftrightarrow{UT}$ to hit the circle twice. Use Power of a Point.

42. Let A be the midpoint of $\overline{YZ}$, B be the midpoint of $\overline{XZ}$, and G be the centroid. Let $AG = x$ and $BG = y$. Find other lengths in terms of x and/or y.

43. What kind of triangles must $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle DOA$ be?

44. How did we prove that the area of a triangle equals its inradius times its semiperimeter? Can we do something similar here?

45. Find the ratio of the volume of each remaining piece to the volume of its original wedge.

46. Let $\angle C = x$ and $\angle D = y$. Label as many angles as you can in terms of x and y until you can write an equation.

47. Prove $AQ = RP$. Can you find some congruent triangles now?

48. Can you find $\angle BCA$ by considering what you know about $\triangle ABC$?

49. Draw one of the segments. How far are the endpoints of the segment from the center of the circle?

50. Can you find the volume of each piece we cut off?

51. If you were given several pieces that were shaped like the region described in the first hint, and you were given one piece shaped like the shaded region, could you build the square?

52. Deal with the hour hand and the minute hand separately. Where does the minute hand point? The hour hand?

53. Must all the interior diagonals pass through the same point? How far is this point from each vertex of the cube?

54. Write some equations involving $\angle RQZ$. For example, it can be combined with $\angle ZQP$ to make $\angle RQP$.

55. Build some right triangles.

56. What do you know about the little triangles on the outside?

57. Build a 30-60-90 triangle by dropping a well-chosen altitude.

58. Find similar triangles to express PX and YR in terms of sides whose lengths we know.

59. The centers of all the faces of the octahedron together are the vertices of what kind of polyhedron?

60. Consider how we built a right triangle for Problem 19.1.3.

61. Draw the altitude to the side of length 6. Can you find the length of this altitude?

62. Start with the power of point A and note that $AX = AW + WX$ and $AZ = AY + YZ$. Let $WX = YZ$ as given. Expand, rearrange, and factor, factor, factor.

63. Why must each plane of symmetry intersect one pair of faces in lines that are lines of symmetry of those faces?

64. What cross-section should you consider?

65. Solve for a and c in terms of b and d, then use the Pythagorean Theorem.

66. Unroll the rail!

67. This problem is the same as asking: 'I have a 7 by 7 square whose sides are blue. How can I cut it into 7 pieces of equal area such that each piece has the same total length of blue segments from the original square in its perimeter?'

68. Let the three polygons have a, b, and c sides, respectively. If we add the measure of an interior angle of each, what should we get? What should we get if we add the exterior angles?

69. What is $\angle STQ$?

70. Find more similar triangles. Draw $\overline{DB}$ and $\overline{EC}$.

71. Find and mark equal angles. See any congruent triangles?

72. Prove that the perimeter of the second triangle is half the perimeter of the first triangle.

73. We did a very similar problem in the text. Go back and study it for guidance.

74. In terms of s, the side length of the hexagon, what are the areas of the regions inside the hexagon but outside $ABCE$?

75. No matter where C is, $\overline{AB}$ is always the same. Given that $AB = 2$, what must we determine to find the area of $\triangle ABC$?

76. What kind of triangle is $\triangle GJM$?

77. What is $[XYC]/[XYZ]$?

78. To find the area of $\triangle AOB$, draw altitude $\overline{OX}$ from O to $\overline{AB}$. What kind of triangle is $\triangle BOX$?

79. $\overline{OB}$ and $\overline{OA}$ are radii of the same circle.

80. Let P and Q be the circumcenters of triangles ABE and BCE. What is the relationship between $\overline{PQ}$ and $\overline{BD}$?

81. Draw a picture. To do so, figure out the distance from the center of the man's circle to the closest point of grass beneath the man's hat, then figure out the distance from this center to the farthest point of grass beneath the man's hat.

82. What does $BC = AC = DC$ tell us?

83. Look at how you solved Problem 12.1. Try drawing the same extra line here.

84. (For $3YZ > XY$.) Draw $\triangle XYA$ and $\triangle XZB$ outside $\triangle XYZ$ such that $\triangle XAY \cong \triangle XYZ \cong \triangle XZB$. What kind of triangle is $\triangle XAB$?

85. How are $\angle WYX$ and $\angle WXY$ related? How are $\angle WYX$ and $\angle ZYX$ related?

86. Is $\triangle ADF$ equilateral?

87. What is $\angle ABP$?

88. Do you see any triangles that look congruent?

89. What is $[BXY]/[BXA]$?

90. Connect O to the midpoints of $\overline{AB}$ and $\overline{CD}$.

91. Write the Power of a Point relationship in terms of ratios.

92. Be careful; this is a little different than the similar problem in the text. Specifically, the fold connects points on opposite sides of the rectangle!

93. Can you consider the desired length as the altitude of a tetrahedron?

94. Look at the diagrams for constructions in this section. See any 90° angles?

95. Length ratios and medians should make us think of using centroids. Draw median $\overline{CK}$.

96. Why is $\overline{MN} \parallel \overline{BC}$?

97. Consider triangles $\triangle ACO$ and $\triangle BDO$.

98. Write $[AXC]$ and $[BXC]$ in terms of the areas you used for your area ratios in the previous hint. Use some clever algebra and the ratio statements you came up with in the last hint to show that $[AXC]/[BXC] = AF/FB$.

99. Draw the altitude from E to $\overline{AB}$. See any similar triangles?

100. To show all the diagonals pass through the same point, consider the midpoint, O, of one of the interior diagonals. Show that this point is mid-way between each pair of opposite faces of the cube.

101. Start with the Pythagorean Theorem, $a^2 + b^2 = 97^2$. Solve for a^2. Can you factor the result? Can you find a value of b that makes both factors perfect squares?

102. Call the length of one side of the original poster x. What are the areas of the old poster and the new poster in terms of x?

103. Use Power of a Point to show that $WX = YZ$ and $BX = CZ$.

104. Consider Problem 13.22. Find the power of the point with respect to each circle. Can they possibly be the same?

105. When you draw BCD and its image $B'C'D'$, you should have 6 outer triangles and a hexagon in the middle. What do you know about those 6 outer triangles?

106. Problems involving regular hexagons can often be made easier by dissecting the hexagon into 6 equilateral triangles.

107. Let $\overline{AF}$ and $\overline{EG}$ meet at X. Look at the angles of $ACEX$.

108. What do we know about triangles $\triangle WXY$ and $\triangle YZW$ that might be useful?

109. Extend $\overline{AB}$ past B; call the point where this extension meets m point H.

110. Consider the areas of triangles PBC, PCA, and PAB.

111. Unfold the tetrahedron.

112. We want to show that $(AB + CD)/MN = 2$. Find ratios equal to AB/MN and CD/MN.

113. The order of the vertices in the statement $\triangle ABC \sim \triangle ADB$ is important! Use them to write an equation in terms of side lengths.

114. Connect the vertices of the hexagon to the center of the circle. What kind of triangles do you form?

115. Under reflection through a plane of symmetry, what is the image of the vertex of the cone? The base of the cone? The center of the base of the cone?

116. What kind of triangle is $\triangle ABC$?

117. Take another look at Problem 8.31.

118. What do you know about the diagonals of a kite?

119. Let our legs have lengths x and y. Write two equations for x and y.

120. Let the regular polygon be $A_1A_2A_3\cdots A_n$ and let O be the point where the angle bisectors of $\angle A_nA_1A_2$ and $\angle A_1A_2A_3$ meet. Can you prove $OA_1 = OA_2$? Can you use this to prove $OA_2 = OA_3$?

121. Let $\widehat{AC}$ and $\widehat{BD}$ meet at Y, and let $\widehat{BD}$ meet $\overline{AC}$ at X. Start with sector ABC and take out pieces you know how to handle.

122. Consider Problem 13.22.

123. What is the second angle? The third? The nth? What is the sum of these measures? What must this sum equal?

124. What is the 'blue' perimeter of each piece? Area of each piece?

125. $\overline{PD}$ is an angle bisector.

126. This problem is not nearly as hard as it looks. Don't try to find the nonoverlapping areas; try to find their difference.

127. In terms of arcs of the incircle, what is $\widehat{EDF}$? How about $\angle A$?

128. Show that MQ and MR both equal MT.

129. Build a 45-45-90 triangle by dropping a well-chosen altitude.

130. If we count all the sides of the pentagons, and all the sides of the hexagons, how many times do we count each seam?

131. Draw the altitudes from the endpoints of the shorter base.

132. Draw a line through B and the vertex of the square from the first hint that is not on a side of $\triangle ABC$. Let this line hit $\overline{AC}$ at F. Can you build the desired square with F as a vertex? (And more importantly, can you prove this is your desired square?)

133. Find EG and FH.

134. Find both $\angle YCZ + \angle YBZ$ and $\angle YAZ$ in terms of angles at Y and Z. (For example, consider $\triangle YCZ$.)

135. Let the smallest angle be x. What are the other two angles in terms of x?

136. Find $\angle ACD$ and $\angle BCF$ in terms of $\angle A$ and/or $\angle B$. (Don't forget what you know about the median to the hypotenuse of a right triangle!)

137. What Pythagorean triples have 50 as the hypotenuse? (We also strongly suggest trying to find an algebraic solution! Let x be the initial height of the top of the ladder and y the initial distance from the wall to the base.)

138. Forget about everything after C_1 and just find CC_1 and AC_1. Next, just look at right triangle CC_1A with altitude $\overline{C_1C_2}$ and just find C_1C_2. Then take another step and find C_2C_3.

139. Consider a cross-section that contains the common axis of the cones.

140. What kind of triangle is $\triangle ABC$?

141. Each segment in the first diagram is broken into how many segments in the second diagram? What is the ratio of the length of each segment in the first diagram to the length of each segment in the second?

142. Review the diagrams that showed us that SSA failed. See if you can play with the diagrams to find the cases where SSA succeeds. There's one tricky case that's not suggested in these diagrams. In what case could we determine another pair of corresponding angles are equal?

143. You have two cases to consider. Can the hypotenuse be odd if both legs are even? Can we have an even hypotenuse if one leg is even and the other is odd? (Use the Pythagorean Theorem!)

144. Let $ABCD$ and $EFGH$ be opposite faces of the cube, with $\overline{AE}$, $\overline{BF}$, $\overline{CG}$, and $\overline{DH}$ as edges of the cube. Let N be the midpoint of $\overline{EG}$. Show that $ON = AE/2$ and that $\overline{ON}$ is perpendicular to face $EFGH$.

145. Don't forget that $\overline{OA}$ and $\overline{CD}$ are parallel!

146. Let $\overline{AC}$ meet $\overline{BD}$ at E. Find AE/EN. Find EN/NC.

147. Can you find some similar triangles?

148. Don't forget there are three different angles that could be the vertex angle!

149. Reinterpret the problem in terms of area and perimeter.

150. Let $AB = x$. What is BD? What is AD? What is CD in terms of x?

151. Let the triangle be $\triangle ABC$, with $\angle A$ as the largest angle. Write an equation with the information given.

152. Extend $\overrightarrow{RY}$ and $\overrightarrow{RX}$ to meet $\overline{PQ}$ at C and D, respectively.

153. The parallel lines of the previous part give us similar triangles.

154. Is it possible for a plane of symmetry to pass through exactly 1 vertex of the tetrahedron? How about 3? 4? 0?

155. Is there another tetrahedron with the same volume?

156. Use your equations to write $\angle ZQR$ in terms of $\angle PQR - \angle PRQ$.

157. Try adding all your equations together.

158. Are $\overline{XN}$ and $\overline{YN}$ corresponding sides of congruent triangles? Which triangles? Why are they congruent?

159. Draw a square with opposite vertices B and O.

160. Note that $(a+b)^2 = a^2 + 2ab + b^2$.

161. What is the length of the altitude to the side of length 10 cm of the first triangle?

162. Let the point where $\overline{XC}$ and $\overline{AB}$ meet be M. What is $[XAM]/[XYC]$?

163. From the last hint, or from the diagram in Problem 3.14, you might have deduced that BC_1C_2 is isosceles. Show that $\angle BC_1A + \angle BC_2A = 180^\circ$, then consider the sum of the angles in your two possible triangles.

164. Let the cube have edge length s. What is the length of an edge of the octahedron?

165. Draw $\overline{BD}$. Do you have a pair of congruent triangles?

166. Use your equation to find DE/AB. Don't forget that $AB = CD = CE + DE$!

167. Are triangles XAY and YBX congruent?

168. Show that $\triangle AA'A'' \cong \triangle XX'X''$.

169. In the previous part, you should have shown that if X is on $\overline{BC}$, then $AB^2 \le AC^2 + BC^2$, which is the opposite of what we are given. Follow similar steps to show the same is true if X is beyond B on $\overrightarrow{CB}$. What does this tell us?

170. Find two pairs of similar triangles. You may need to use a little algebra!

171. There are lots of parallel lines. How are the two triangles in this part related?

172. Note that $RQ = RZ - QZ$, which looks a lot like what we want.

173. What is $AX + BY$?

174. The region described in the first hint is part of sector AOD.

175. Consider the cross-section that also includes the points where the spheres touch the wall and the floor.

176. Show that $DQ = AP = PB = CR$.

177. Let O be the center of the scoop, A be the vertex of the cone, and B be a point on the circumference of the cone. What kind of triangle is $\triangle OBA$? What is $\angle OAB$?

178. Can you prove that $VY = VZ$?

179. Show that $AD/AE = AB/AC$. Note that $AB = AD + BD$ and $AC = AE + EC$, and that you are given an equation you can use to get an expression for AD.

180. Consider the reflection of $\angle AOC$ over $\overline{XY}$.

181. Approach 1: Is there a single line through which all three cuts pass?

182. Look back at the ways to prove two lines are parallel. Then consider $\angle Z$ and $\angle XYZ$.

183. M is the midpoint of $\overline{BD}$ if $BM = MD$. Are these segments corresponding parts of congruent triangles?

184. In the Review Problems of this chapter, we found the radius of a sphere inscribed in a regular tetrahedron. Will the same approach work here?

185. Consider a cross-section that contains the apex of the original pyramid, the center of the base, and the midpoints of opposite sides of the base.

186. Let the dimensions of the prism be a, b, and c. Write equations for the given information, then find abc and use that to find each of the dimensions.

187. Consider a circle centered at M with radius 8.

188. Find FG first, then find RG.

189. Use the first part to find NM/BM. Compare this ratio to AM/AB. Do you have a pair of similar triangles?

190. How is $[FOB]$ related to $[DOG]$?

191. Draw $\overline{BE}$.

192. Let each of the marked angles have measure x. Find $\angle BAE$ in terms of x.

193. What do you know about the diagonals of a kite?

194. Continue $\overline{YN}$ to hit $\overline{XZ}$ at R. What do we know about $\overline{YR}$?

195. Draw a radius of the small circle to a point of tangency, and a radius of the large circle to a vertex of the hexagon. What kind of triangle can you thus form?

196. What do you know about the third angle of each triangle?

197. Do any triangles in the diagram look congruent? (You may have to consider points that aren't labeled!)

198. Show that $\triangle GEH \sim \triangle AED$, where E is the midpoint of $\overline{BC}$.

199. Draw a circle with center C and a judiciously chosen radius.

200. What are the exterior angles in terms of $\angle A$, $\angle B$, and $\angle C$?

201. Can you construct a square that has one side on $\overline{BC}$ and one vertex on $\overline{AB}$? Try using this square to find the square you want.

202. Chase both angles. Relate $\angle QCB$ and $\angle QBP$ to other angles in the diagram.

203. Draw a line segment and place the points on it. Label all the distances you can determine.

204. Let O be the intersection of the perpendicular bisectors of $\overline{HI}$ and $\overline{GH}$. Show that $\triangle OIJ \cong \triangle OGH$.

205. Let W, X, Y, and Z be the midpoints of $\overline{AB}$, $\overline{AC}$, $\overline{CD}$, and $\overline{BD}$, respectively. What kind of quadrilateral is $WXYZ$? What is WX?

206. What is $\angle OMP$?

207. What kind of triangle is $\triangle EGH$?

208. Focus on the base BCD. What happens when this is rotated 60° about its center?

209. What is $\angle X + \angle Y + \angle Z$? How are $\angle X$ and $\angle Y$ related?

210. Let $\overline{XY}$ meet $\overline{AB}$ at M. Show that $\triangle AMX \cong \triangle AMY$.

211. Each resulting piece has five faces. Carefully draw a diagram and use it to figure out what sort of shapes these faces are. (Don't forget what shape each face of the regular tetrahedron is!)

212. What is $[AMB]$?

213. What is $([ADF] + [BFC])/[ABCD]$?

214. Can you prove that Y is the midpoint of $\overline{FH}$?

215. Build a 45-45-90 triangle by dropping a well-chosen altitude. You may have to extend a side.

216. For the 'only if' part, draw altitudes from the short base to the long base and find congruent right triangles.

217. Let X be the leftmost point of the shaded region. Can you find the area of the region bound by $\widehat{AX}$, $\widehat{BX}$, and $\overline{AB}$?

218. Just unrolling the outside of the cylinder isn't enough; Arnav has to go inside the glass!

219. Look back at how we have done problems in this chapter with frustums.

220. Show that $\triangle XYL \cong \triangle XZL$. (Why must $\widehat{YL} = \widehat{ZL}$?)

221. Let O be the center of the largest semicircle. What are the following in terms of r, s, and t: the radius of the largest semicircle, OX, OB, and OY? What kind of triangle is $\triangle OXY$?

222. What is $\angle PTQ$?

223. Find $[PXQ]/[ABC]$ in terms of PQ and BC.

224. Did you see a similar problem in the text?

225. Let there be k right angles. Note that each of the other angles must be less than 180°. So, the sum of the angles must be less than what quantity?

226. Can you find QD?

227. Can you find the lengths of the sides of the triangle? What kind of triangle is it?

228. Since $\odot O$ is tangent to $\overline{AB}$ and $\overline{AC}$, what do we know about the segment connecting A to the center of the circle?

229. What kind of triangle is $\triangle ABC$?

230. The first hint was for the slick method. The rest of the hints are for a more mechanical route. How far is the center of the sphere from a face of the tetrahedron? (Note: This is the same as asking what the radius of the sphere inscribed in the tetrahedron is.)

231. What two area ratios do we know are equal to AF/FB?

232. Combine your expressions for $\angle ACE$ and $\angle BCD$ to find $\angle DCE$.

233. Still don't have it? Time for a different approach. Let O be the center of each square. Connect O to each vertex of $ABCDEFGH$. Can you find the areas of the triangles you form?

234. When the ball bounces off the rail, how far will the center be from the rail?

235. What kind of triangle is $\triangle AGB$? Can you find an expression for BG? How about CG and DG?

236. Draw the figure. Pay close attention to the vertices of the second square.

237. Draw $\overline{OD}$. What kind of triangle is $\triangle AOD$?

238. Your desired region is two cones with the same base. What segment in your cross-section corresponds to the radius of this common base of the two cones? What does your diagram for finding this radius have in common with Problem 5.17?

239. Consider the circle with diameter $\overline{OP}$. Through what other points in the diagram must this circle pass?

240. What general type of quadrilateral might be useful?

241. Is there an angle in the diagram equal to $\angle AOC$ that is easy to find?

242. Let ABC and ADE be faces of the octahedron with centers G and H, respectively. Let M and N be the midpoints of $\overline{BC}$ and $\overline{DE}$. Find the following: MN, AG/AM, GH.

243. Prove $\triangle SPD \cong \triangle QPA \cong \triangle SRC \cong \triangle QRB$.

244. What is $[WPX]/[WXYZ]$?

245. Let one polygon have a sides and the other have b sides. Write two equations using the given information.

246. Consider quadrilateral $OIHG$.

247. We have lots of right angles, and we're looking for an angle. Maybe the Pythagorean Theorem will help. Build right triangles.

248. Look back at Problem 2.14.

249. What type of cevian is $\overline{BE}$ in $\triangle ABC$?

250. What kind of quadrilateral is $ABCD$?

251. Let $[SPC] = x$ and $[CPR] = y$. What is $[SPC]/[BPC]$? Use this to build an equation.

252. $DF = DE + EF$.

253. Approach 2: How do our regions of cheese after the cuts correspond to orderings of x, y, and z?

254. Prove that $\triangle RYX \sim \triangle RCD$. (You may need to find some congruent triangles first!)

255. What do you get if you add this ratio to the ratio in the previous part?

256. Find the angle between $\overline{AA_1}$ and $\overline{B_1C_1}$ in terms of arcs of the circle. Don't forget that $\widehat{B_1A} = \widehat{B_1C}$, $\widehat{AC_1} = \widehat{BC_1}$, etc.

257. Find EH/EF using similar triangles.

258. Find two pairs of similar triangles. Write two equations involving x, y, and AC.

259. Don't just hunt blindly for solutions once you have an equation set up – some of the solutions are very surprising. Find an organized way. Here's a hint: If $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, and x, y, and z are positive integers, then at least one of these integers must be 3 or smaller (otherwise, the sum $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ will definitely be smaller than 1).

260. Find a pair of similar triangles such that the segments in $BE^2 = (AE)(DE)$ are among the sides of the triangles.

261. What is $[XPQ]/[XBC]$ in terms of PQ and BC?

262. Forget about A for a minute and pretend we know where B is. What is the shortest path that goes from B to a point on $y = 6$, then on to point D?

263. Similar triangles will help you find the side length of the second square base of the frustum.

264. Let $\overline{CE}$ meet $\overline{OA}$ at Y. What type of triangle is $\triangle OCY$?

265. Can you find the area of $\triangle MEN$?

266. Let $AB \leq AC$. (The proof for $AC < AB$ is essentially the same.) Arrange the triangles so that A and A' coincide and B and B' coincide. Does $\overline{AC'}$ intersect $\overline{BC}$? Why or why not?

267. Draw $\overline{OX}$ and $\overline{PY}$. Draw a line through P parallel to $\overline{XY}$. Consider the right triangle you thus form.

268. Consider the triangle formed by connecting the center of the sphere, the center of face ABC, and the midpoint of $\overline{AB}$. The hypotenuse of this triangle is the desired radius.

269. Find some similar triangles. You may want to rearrange the equation you're given.

270. Let $\odot O$ be the circle tangent to the four given circles. Apply the Triangle Inequality to $\triangle OPR$.

271. Let X be the point where the wall meets the floor and let M be the midpoint of the ladder. How is MX related to the length of the ladder?

272. The region described in the last hint is part of sector ABC.

273. Focus on a single side of the first triangle. What happens to it as we go from the first figure to the second figure?

274. Can you find three congruent triangles that have $\overline{DH}$, $\overline{HF}$, and $\overline{DF}$ as corresponding sides?

275. Draw $\overline{MC}$. What is $[MUC]/[NUC]$?

276. Let one of the legs have length x. Write an equation.

277. Extend the sides of the octagon; what type of quadrilateral is formed?

278. Make sure you track the path of the center of the ball, not the edge of the ball. You shouldn't be aiming at the reflection of B over the rail, because the ball will bounce before the center reaches the rail!

279. The points of tangency are the vertices of what figure?

280. Show that $\triangle ADE$, $\triangle EDF$, and $\triangle FDC$ are congruent.

281. What are two ways you can find the volume of tetrahedron $ABCF$?

282. For the hour hand, can you figure out how much of the distance from 11 to 12 the hour hand has covered by considering how much of the 11 o'clock hour has transpired?

283. Connect M to the midpoint of $\overline{O_1O_2}$.

284. You have to use the fact that the little circle is tangent to the largest semicircle somehow. What does this fact tell us? What segment should we draw to use this fact?

285. What is $[ABX]/[ABC]$?

286. Draw $\overline{CP}$.

287. Let M be the midpoint of $\overline{BC}$ and N be the midpoint of $\overline{CD}$. How is $\triangle AOP$ related to $\triangle AMN$?

288. Let $\angle ABO = x$, $\angle OBC = y$, and $\angle OAC = z$. Find other angles that equal these. Find $x + y + z$.

289. What arcs must you show are equal in order to show that $\angle XAM = \angle YAM$?

290. What kind of triangle is $\triangle MOY$? $\triangle NOX$?

291. Connect the endpoints of the chords to the center of the circle. What kind of triangles do you form? What is $ABCD$?

292. View the frustum as the result of chopping a small cone with radius r_1 off the top of a cone with radius r_2.

293. To show that $\overline{XL}$ is a diameter of the circle, note that $XZLY$ is a cyclic quadrilateral. What does this tell us about $\angle XZL + \angle XYL$?

294. Let the radius of the circle be r. Find PB in terms of r. How can you use Power of a Point?

295. Let X be the center of the small circle on the right, and let Y be the foot of the perpendicular from X to $\overline{AC}$. Let r be the radius of the little circle, and let s and t be the radii of the smaller semicircles. Find XY in terms of r, s, and t in two different ways.

296. What triangle is similar to $\triangle EMC$?

297. Break the problem into cases in terms of the types of cross-sections formed by the intersecting plane.

298. Prove that both AP and RQ equal $YZ/2$.

299. See Problem 8.4.3.

300. Prove that $DE/EF = DF/DB = DF/DE$.

301. If you answered the first part using trial and error, go back and read those hints!

302. Let point X be some point besides B on line m. Consider $\triangle OBX$ and show that $OX > OB$. Does this prove that m can't hit the circle a second time?

303. Lots of right angles and parallel lines mean lots of similar triangles. Use similar triangles to find as many lengths as you can.

304. To find the area of $\triangle AOX$, draw an altitude from A. What is $\angle AOX$?

305. Let O be the circumcenter of $\triangle JKL$. Why must point O be on the altitude from J to $\overline{KL}$? Let $OK = x$. Find some other lengths in terms of x.

306. You're looking for a ratio, so find some useful similar triangles.

307. How is CC_1 related to BC? How is C_1C_2 related to CC_1? And so on.

308. Let the midpoint of the last hint be N. What does MN equal?

309. $[PAEC] = [PAE] + [PEC]$.

310. Let V be the vertex of the cone, O be the center of the base, and X a point on the circumference. What kind of triangle is $\triangle VOX$? What is VX?

311. There's nothing special about A!

312. Draw an altitude from Q to $\overline{PR}$.

313. Show that $\angle PAC = \angle PCA = \theta/4$.

314. Are the triangles similar?

315. Find some congruent triangles with $\overline{YX}$ and $\overline{WZ}$ as corresponding sides.

316. Draw a segment representing one of the longest lines-of-sight. Build a right triangle by connecting the common center of the semicircles to the point of tangency of the segment to the smaller semicircle, and connecting the center to an endpoint of the segment.

317. Compare the sides of $\triangle ACP$ to those of a 30-60-90 triangle. What similarity theorem can you use?

318. Find the edge lengths of the area of intersection. What sort of shape is the intersection space?

319. What is $[AHF]/[AEF]$?

320. Consider a 60° rotation about R.

321. What's left over when you take $\triangle EDC$ and $\triangle FBD$ away from $\triangle ABC$? What portion of this is $\triangle AEF$?

322. The triangle is isosceles, so the perpendicular bisector of $\overline{AC}$ is also an altitude, median, and angle bisector. Can you find the length of the altitude from B to $\overline{AC}$?

323. Extend $\overrightarrow{PO}$ to hit the circle again at Y.

324. Write the given area equation in terms of sides (or parts of sides) of the rectangle.

325. Let $\overline{ZX}$ meet $\overline{AY}$ at B and $\overline{ZD}$ meet $\overline{AY}$ at C. Show that $\triangle ZBC \cong \triangle ZYC$.

326. Can you dissect the original triangle and rearrange it to make another triangle with two sides of length 13?

327. Look at how we did a very similar problem in the chapter, then try to find the distance from the corner of the room to the center of the sphere in two different ways.

328. What kind of quadrilateral is $WXYZ$?

329. Let the smallest square have side length x. Find all of the lengths in the diagram in terms of x.

330. Let P be the center of the semicircle with $\overline{BC}$ as diameter. What kind of triangle is $\triangle XYP$?

331. Start with the circumcircle of $\triangle ABC$.

332. What kind of triangle is $\triangle ACF$?

333. Show that both M and N are on the median of the trapezoid.

334. What sort of shape do we form if we connect the centers of the little spheres?

335. What other portions of the diagram have area equal to the shaded area?

336. Write each of the arcs in terms of $\widehat{PQ}$. Can you find the measures of the arcs now?

337. At what point on each face is the sphere tangent to the octahedron?

338. Pretend you already know what point on $\overrightarrow{BC}$ to go to. What point should you aim for to find the point on $\overrightarrow{BD}$ to go to? Once you know that, how can you figure out what point on $\overrightarrow{BC}$ to go to?

339. Parallel lines mean similar triangles. Let the diagonals meet at point E.

340. Let the radii of the small semicircles be s and t. Draw the full circle with diameter $\overline{AC}$. Continue $\overline{BD}$ to meet this circle again at E. What do we know about BE and BD? What tool considering lengths and circles can we use to find BD in terms of s and t?

341. Show that $ABCD$ and $A'BCD'$ are congruent trapezoids by showing that all the corresponding angles and sides of the two are the same.

342. What kind of triangle is $\triangle XQZ$?

343. Show that $PQ = PR$ and $PX = PY$. Use these to find another pair of equal lengths.

344. Squares of side lengths and a rectangle suggest building right triangles. Lots of them.

345. After substituting your expression for AB into your expression for AC^2/MO^2, you should be able to factor out BC^2 in the numerator. Try multiplying both the numerator and the denominator of the resulting expression by NO^2. You should then have an $NO^2 - MN^2$ term in both the numerator and denominator. Cancel them!

346. Triangles $\triangle AEB$ and $\triangle PEA$ share an altitude from E.

347. Let the ladder be $\overline{AB}$. What kind of triangle is $\triangle ABX$? How is $\overline{XM}$ related to it?

348. Let I be the incenter of $\triangle ABC$ and X be the point where the incircle is tangent to $\overline{AC}$. What are the lengths of the sides of $IXAZ$?

349. Use similar triangles to show that $CQ/PQ = AC/PB$.

350. Find similar triangles that have $\overline{CD}$ and $\overline{BD}$ as corresponding sides.

351. Let $\odot O$ be the circle that is tangent to our given three circles. Either $\angle OQR$ or $\angle OQP$ is at least 90°.

352. You should already have YZ. Let the distance from X to the point where the incircle touches $\overline{XZ}$ be x and don't forget that $\triangle XYZ$ is a right triangle.

353. Try using area.

354. Let the radius of the sphere be r. What is the radius of the cylinder? The height of the cylinder?

355. We know a lot about triangles. Consider building a triangle by extending one of the segments.

356. What is the angle between where the minute hand points and a line from the center of the clock to the '12' on the clock? How about the hour hand?

357. Let C_1 and C_2 be the two potential point Cs. What kind of triangle is BC_1C_2? What does this tell us about $\angle BC_1A$ and $\angle BC_2A$?

358. We know a whole lot about the angles of a triangle now. Build some triangles.

359. Rotate the heptagon so that the image of D is on $\overrightarrow{AC}$.

360. What angles equal $\angle ZQW$? What angles equal $\angle R$?

361. Find the ratios of $[PBS]$, $[PQA]$, and $[ARB]$ to $[PQRS]$. How is the sum of these ratios related to $[ABP]/[PQRS]$?

362. Let the side length of the hexagon be s. Find the ratio of the area of a square to the area of the hexagon. How about the area of one of the triangles? (Try doing these without actually finding s!)

363. What triangle similarity would allow you to deduce $\overline{DE} \parallel \overline{BC}$? What ratio of sides must you show is equal to AD/AE to deduce these triangles are similar?

364. After using Power of a Point, write everything in terms of PQ, PX, and RX. Rearrange and do some clever factoring.

365. Find the squares of the side lengths of $\triangle ACF$ in terms of edges of the prism.

366. Proving that $\overrightarrow{EY}$ will pass through G is the same as proving that $\overline{EG}$ passes through Y.

367. What is the ratio of the area of the large equilateral triangle to the area of one of the small equilateral triangles?

368. Must the diagonal connecting the incenter of $\triangle WXA$ to the incenter of $\triangle YZA$ pass through A?

369. Can we combine Sue's and Barry's answers in a way such that Sue's base is multiplied by Barry's altitude and vice versa?

370. Consider the reflection of A over $\overleftrightarrow{BD}$.

371. Connect P to the midpoint of $\overline{MN}$. How long is this new segment? (Remember, $\triangle MPN$ is a right triangle!)

372. Prove that $A'B/A'C = AB/AC$. What happens if you add 1 to both sides of that?

373. What is $[BFC]/[ADF]$?

374. At how many different points can two circles meet? How about two lines? How about a line and a circle?

375. Let the medians meet at G. What do we know about XG and YG?

376. What is $[ABC]/[ACD]$? What is $[ABC] + [ACD]$?

377. Angle bisectors and side lengths. Try the Angle Bisector Theorem.

378. Let X be the point where $\overline{OP}$ meets $\overline{YZ}$. In terms of our radii, what is OX/XP?

379. Parallel lines mean similar triangles!

380. Draw altitudes from P to the sides of $ABCD$.

381. Draw altitudes from P to each of the sides of the rectangle.

382. Find a couple pairs of similar triangles.

383. In any given hour, how is the total amount of water that flows through the trapezoid related to the combined total amount of water that flows through the holes?

384. We have a 30° angle. Build a useful 30-60-90 triangle by drawing the altitude from B to $\overline{AC}$.

385. Write the correct Power of a Point relationship. Combine it with Jake's accidentally correct equation.

386. What kind of quadrilateral is $ACNM$? (Don't just guess – you have to prove it!)

387. We know the power of point P.

388. What piece of information did you not use in the first part? Can you use this along with the result of the first part to find similar triangles?

389. Connect the center of the sphere to the centers of the circles of which the wires are arcs.

390. What is $[QPX]/[WPX]$?

391. Can we still cut the quadrilateral into two triangles?

392. Using your two equations, try to find two different approaches to get the solution. First, try guessing integer solutions (use the equation for xy first!) or use your knowledge of Pythagorean triples. Then, for an alternative solution, expand $(x + y)^2$ and use your equations.

393. What is $(AF)(AD)$?

394. Let the parallelogram be $ABCD$. Draw altitudes from A and B to $\overleftrightarrow{CD}$. See a rectangle?

395. (For $XY > 2YZ$.) Start a new diagram. We want $XY > 2YZ$, so if we pick a point Q on $\overline{XY}$ such that $YQ = YZ$, all we have left to show is that $XQ > YZ$.

396. What inequality can we use regarding the sides of a triangle in order to show that the triangle is acute?

397. What did you learn in the previous problem?

398. Show that in each step after the first step, the number of triangles added is 4 times the number of triangles added in the previous step. How much smaller is each triangle added in a given step than the triangles added in the previous step?

399. Mark equal angles and find lengths equal to DZ and lengths equal to CY.

400. Do we know the sum of the interior angles of the figure?

401. Find $\angle CAB$, $\angle ACD$, and $\angle BCD$ in terms of $\angle B$.

402. What is the length of the altitude from O to face ABC?

403. Find CF first.

404. What kind of triangle is $\triangle EFG$?

405. Let the semicircles have centers Q and R, and let S be the point besides O where they meet. Draw $\overline{QS}$ and $\overline{SR}$.

406. Can you build a useful right triangle with $\overline{EI}$ as hypotenuse?

407. What are segments $\overline{BP}$ and $\overline{MN}$ in $\triangle AMB$?

408. Look back at the tactic we took for a similar problem in the text.

409. Find OS a few different ways.

410. The remaining two pieces of the region enclosed by the bold line after you draw the square are triangles. Can you find any triangles congruent to these?

411. Find AC and the length of the altitude from B to $\overline{AC}$.

412. How is $[XAD] + [XBC]$ related to the area of $ABCD$? What about $[XCD] - [XAB]$?

413. Can you think of a right triangle that must be similar to a triangle that satisfies the conditions of the problem? (Make sure you include why the triangles are similar!)

414. The ratio of the sides of one pentagon to corresponding sides of another is always the same. Each of the angles of one pentagon equals the corresponding angles of the other. What does this suggest about the two pentagons?

415. Can you sketch a Power of a Point-like diagram in which we have segments of lengths a, b, and $\sqrt{ab}$? Now use your straightedge and compass to recreate this diagram.

416. Draw radii from the centers of the circles nearest A and C to the points where these circles are tangent to the square.

417. Can you find some congruent triangles?

418. Draw the perpendicular from N to $\overline{EF}$ and connect G to P.

419. What is the image of $\overline{BN}$ upon rotation about the center of the square?

420. Use similar triangles to find the distance from E to each vertex of the trapezoid.

421. What kind of triangles are $\triangle ABD$ and $\triangle BCD$? Mark all the angles you can find.

422. Are there other diagonals equal in length to $\overline{BH}$ that are easier to compare to AE?

423. Note that $(2n^2)(2m^2)$ is a perfect square.

424. What is $[ADB]/[ABC]$?

425. Let O be the circumcenter of $\triangle WXY$. What do we know about OW, OX, and OY?

426. These two triangles share a side. What do we have to show equal to conclude that their areas are equal?

427. Find all the angle measures you can. Label the diagram. Keep going until you get to x!

428. Connect the centers of the circles to points of tangency and to each other. Have you built figures you know how to handle?

429. Consider a cross-section that includes the axis of the cylinder and the cone. Note that the top base of the cylinder is parallel to the base of the cone.

430. Call our two points inside the triangle P and Q. If we draw $\overleftrightarrow{PQ}$, we will hit two sides of the triangle (possibly at a vertex). Let $\overleftrightarrow{PQ}$ hit sides $\overline{AB}$ and $\overline{AC}$ at points P' and Q', respectively. We want to show $PQ < (AB + BC + AC)/2$. Start by noticing that $PQ \leq P'Q'$.

431. Expand $(l + w + h)^2$.

432. Can you combine the ratios from the first two parts in some way to get the ratio in this part?

433. Use $\angle ROS$.

434. What is $[EHG]$?

435. What kind of triangle is $\triangle YTT'$?

436. Let $\angle BAC = x$. Find as many angles as you can in terms of x. Particularly, use the angles you find to get expressions for $\angle ABC$ and $\angle ACB$ in terms of x.

437. You can break quadrilateral $P'Q'CB$ into triangles. Can you then prove $P'Q'$ is less than the sum of the lengths of the other three sides?

438. Let E be the intersection of the diagonals. Can you find the areas of the four small triangles that each have a vertex at E?

439. Let X be the foot of the altitude from O to $\overline{AB}$ and Y be the foot of the altitude from O to $\overline{AD}$. Show that $\triangle AOX \cong \triangle AOY$. Why does this mean O is on the diagonal through A?

440. Create similar triangles that will allow you to construct a point X on $\overline{OP}$ such that $OX/XP = r_O/r_P$.

441. Can you write an equation involving $\angle PRQ$ and $\angle ZQR$? How about $\angle PZQ$ and $\angle ZQR$?

442. Let one side of a little triangle be x and the other y. Write some equations. And don't forget that $(x + y)^2 - (x^2 + y^2) = 2xy$.

443. What is $\angle B$? $\angle BDA$?

444. What do you know about the sum of the exterior angles of a triangle? If you don't remember, look back in the text, or try to figure out something about the sum on your own!

445. Is there a rotation that maps $\overline{CM}$ to $\overline{DN}$?

446. Let the foot of the altitude from A to $\overline{BC}$ be D and the foot of the altitude from B to $\overline{AC}$ be E. Find a triangle that is similar to $\triangle ADC$. What is R_2 in terms of segment lengths in your diagram?

447. Use your side length information to get information about angles.

448. What other segment lengths can you find? Draw some other significant segments that might be useful!

449. Look back for a useful Exercise in Section 12.4.

450. Here are some pieces to consider: sector ABC, sector YBC, segment BY of $\odot C$, sector XCB, and $\triangle ACB$. You should know how to handle these pieces. How can you assemble them to get the shaded area?

451. Let the image of A be A' and the image of B be B'. Let C be on $\overline{AB}$. Show that the image of C is on $\overline{A'B'}$. What type of quadrilateral is $AA'C'C$?

452. B is on the surface of the sphere.

453. What is $(\sqrt{a} + \sqrt{b})^2$?

454. Let O be the circumcenter of $\triangle GHI$. Can you prove that $\triangle GOH \cong \triangle GOI$?

455. Focus on $\overline{AP}$ and $\overline{RC}$. Are there any triangles that look congruent that have these as corresponding sides?

456. This problem has a lot in common with the problem in the text in which we found the length of a common external tangent. Here, we know what the length of the common tangent is and we want to find the distance between the centers. A similar tactic will probably work.

457. Don't forget that the diameter of the semicircle is part of the semicircle's perimeter!

458. Use $OQSR$ to find the area of the shaded region that has O on its boundary. How is the area of this shaded region related to the area of the other shaded region?

459. How is the diagonal in the first hint related to $\angle WAX$?

460. Let $AB \leq CD$. Draw altitudes from A and B to $\overline{CD}$. Use these altitudes, and the fact that $\overline{AB}$ and $\overline{CD}$ have the same perpendicular bisector, to find some congruent triangles. (Marking all the right angles in your diagram will help a lot!)

461. Find congruent triangles, then mark equal angles. Now find more congruent triangles!

462. What information given in the problem have you not used yet? Can you combine it with the first two parts to find more similar triangles?

463. Here's a starting point: Let $\triangle ABC$ and $\triangle DEF$ satisfy the AAS criteria, such that $\angle A = \angle D$, $\angle B = \angle E$, and $AC = DF$. Starting from here, prove these triangles are congruent using ASA Congruence.

464. M is the midpoint of hypotenuse $\overline{AC}$ of $\triangle AHC$.

465. Let O be the center of the wheel, A be the bottom, and B be the point on the wheel that is 10 feet above the ground. Draw the altitude from B to $\overline{OA}$ so you can use the information that B is 10 feet above the ground.

466. Let the diagonals meet at K. To show that $ABCD$ is isosceles if $\angle ABD = \angle BAC$, use $\triangle ABK$ and $\triangle CDK$ to show that $AC = BD$.

467. What is the length of $\overline{XZ}$? Now can you find congruent triangles?

468. Draw altitudes from A and C to n.

469. Draw an altitude from B to $\overline{AC}$.

470. Use the two area relationships we are given to find BS/RS and QA/QR. Can you use these to find $[ARB]/[PQRS]$?

471. Draw the altitude from O to $\overline{AC}$.

472. Reflect D over the line $y = 6$. This may not be the only reflection you need!

473. Draw altitudes from B and D to $\overline{AC}$.

474. Write the area of $\triangle ABC$ in terms of the areas of triangles I_aAC, I_aBC, and I_aAB

475. How many little triangles are in the top row of little triangles? How many in the next row down? The next row after that? And so on.

476. The diagonals of a parallelogram bisect each other!

477. Let O be the center of the base, as in our solution. If $BC = 8$, then what are the side lengths of $\triangle OCA$? What kind of triangle is $\triangle OCA$?

478. What is $\angle PDO$?

479. Connect the centers of the three circles. What kind of triangle do you form?

480. It's a word problem. Try assigning a variable to one of the angles.

481. Write all the angles you can find in terms of x and y. Can you use your resulting diagram to write some equations in terms of x and y?

482. Can you show that $AD = BC$? That $AD = BF$? That $AE = DC$?

483. To show that X cannot be beyond A on $\overrightarrow{BA}$, investigate what happens to the angles of $\triangle CXB$ if X is beyond A on $\overrightarrow{BA}$.

484. Argue by contradiction. Assume $XZ > XY$. What happens if $\angle Y > \angle Z$ or $\angle Y = \angle Z$?

485. If you're having trouble with the region bound by $\overline{AX}$, $\overline{AB}$, and $\widehat{BX}$, try finding $\angle ACB$.

486. What is the area of $\triangle EFG$?

487. Why must the center of each circle be (a, a) for some constant a? In terms of a, what is the radius of the circle?

488. Don't forget that $\triangle AXC$ is a right triangle.

489. Let O be the center of the sphere, and let C_1 and C_2 be the centers of the circles. First, why must $\overleftrightarrow{OC_1}$ be perpendicular to the planes of both circles? Does this show that $\overleftrightarrow{OC_1}$ goes through C_2?

490. Can you find $[ABC]$? Are there any other triangle areas you can find?

491. What is the sum of the measures of the exterior angles of a triangle?

492. N is the midpoint of $\overline{AC}$. Use this fact with your ratios from the first hint.

493. Suppose Orion and Michelle pick different numbers. Pretend you are Joshua – can you choose a number such that your three numbers together violate the Triangle Inequality?

494. What is the area of the garden? Of the garden and the path together?

495. Consider a cross-section that includes the centers of both spheres. What other useful points should you include in the cross-section?

496. Find some similar triangles.

497. What if $AB = CD$? (Make sure you go back and prove your answer works for all trapezoids that fit the problem! Try building a right triangle.)

498. (For $XY > 2YZ$.) Label all the angles you can find in your diagram.

499. Find both angles in terms of $\angle ABC$.

500. Draw $\overline{AC}$ and $\overline{AE}$.

501. Note that $\overline{CN}$ is a median and an angle bisector of $\triangle ECD$. What does that tell us about $\triangle ECD$?

502. Build right triangles.

503. Draw perpendiculars from O to $\mathcal{P}_1$ and to $\mathcal{P}_2$.

504. Look at the diagrams from constructions in this section. See any 30° angles?

505. Let $XO = x$ and $YO = y$. What is XY in terms of x and y?

506. $\overline{LN}$ is an altitude of $\triangle KLM$.

507. We used SSS Similarity in a similar problem in this chapter!

508. Find the radius by finding the area of $\triangle AOB$ in two ways.

509. Let $\triangle ABC$ be our triangle and I be its incenter such that $\overline{AI} \perp \overline{CI}$. Find $\angle BAC + \angle BCA$.

510. How is the perimeter we want to find related to the perimeters of all the triangles together?

511. Find EA and ED.

512. Let the point where $\overrightarrow{XP}$ meets the circumcircle of $\triangle XYZ$ be point Q. Where do $\overline{XQ}$ and $\overline{YZ}$ meet? What is QP in terms of PX, YP, and PZ?

513. Start with right triangle $\triangle ABC$ with $AB = 1$, $C = 90°$, and $\angle A = 15°$. What must we find in this diagram to evaluate $\sin 15°$?

514. Let $OX = x$. Find the area of the region bound by $\overline{AX}$, $\overline{DX}$, and $\widehat{AD}$ in terms of x.

515. When your wheels spin once, how far does your car go? How far does your speedometer think you go?

516. Cut the desired region into triangles.

517. What is the sum of the interior angles?

518. Can you prove $\angle VYZ = \angle VZY$?

519. Note that $\angle ABD = \angle BDE$. What does that tell you about $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$?

520. Show that the center of rotation is on the perpendicular bisector of $\overline{AC}$ and the perpendicular bisector of $\overline{BD}$.

521. Find CH and AE.

522. Solve one of your equations for AC in terms of x.

523. Use the Pythagorean Theorem, $a^2 + b^2 = c^2$. What happens if a and b are odd?

524. Describe in words each of the sides of $\triangle XUS$. What kind of triangle does that make $\triangle XUS$?

525. Use the Triangle Inequality a few times.

526. What is α when $AC = 7$?

527. Right triangles: Try the Pythagorean Theorem! Let $AX = x$ and $BX = y$. Write two equations and solve for x and y. Or, make a clever guess at x and y and see if your guess works.

528. $\overline{EU}$ and $\overline{NT}$ are 5 units apart. How does this restrict the length of $\overline{EN}$?

529. Unroll the outside and the inside. How should you place these unrolled surfaces next to each other so that Arnav's walking path can represent going over the edge of the glass, from outside to inside the glass?

530. Suppose the two circles meet in three points – what is the circumcircle of the triangle formed by connecting these three points?

531. What type of triangle is $\triangle WZU$?

532. What does the first part tell you about QC? About QB?

533. For proving $3YZ > XY$, note that $3(20°) = 60°$. Can you take advantage of this fact by constructing two more copies of $\triangle XYZ$, one on either side of the original triangle?

534. Is $AD = BF$?

535. Focus on one circle. What is the angle of the sector in which one circle is inscribed? Build a right triangle by drawing radii to points of tangency.

536. We don't know much about 15° angles or 75° angles. Can you break $\triangle ABC$ into an isosceles triangle and a right triangle that we know how to deal with?

537. Where is the circumcenter of a right triangle?

538. Draw $\overline{EB}$.

539. Prove that $\triangle BME \sim \triangle CNE$, and use this to show that $\angle ABC = \angle BCD$.

540. Find the power with respect to both circles of the point where the chords intersect.

541. Find $[BMC]$. How are $[BMC]$ and $[BED]$ related?

542. What kind of triangle do you form when you draw an altitude from G to $\overline{EF}$?

543. Draw diagonal $\overline{AC}$. Find AC in terms of the desired radius.

544. Notice that $\cos 60^\circ = 1/2$. This makes the Law of Cosines particularly easy to use on triangles with 60° angles.

545. Let A be a point on $\odot O$. Let A' and O' be the images of A and O, respectively. Show that $A'O' = AO$ (you can't assume this!). What type of quadrilateral is $A'AOO'$?

546. Draw a diagram. Mark equal lengths. See any isosceles triangles?

547. Make sure you investigate all possible diagrams!

548. Try to build a right triangle with $\overline{XY}$ as one side such that you know the lengths of the other two sides.

549. The circumcenter, O, is on the altitude from B to $\overline{AC}$. Let $OA = x$. What is OB? How far is O from the midpoint of $\overline{AC}$?

550. Solve for RZ and QZ in terms of RQ and the lengths in the expression we want to prove.

551. We are given an inequality in terms of sides, and want to prove an inequality in terms of angles. Use the side information to derive an inequality involving angles.

552. Draw altitudes from B and C to $\overline{AD}$.

553. How far can C get from $\overleftrightarrow{AB}$ and still be on the circle?

554. Build a right triangle with the desired segment as a hypotenuse.

555. Let $AP = x$ and $AQ = y$. Can you find all the rest of the side lengths in the diagram in terms of x and/or y?

556. Let X be on $\odot C_1$ and Y be on $\odot C_2$. Find some congruent triangles to prove $OC_1 = OC_2$.

557. Look at Problem 4.35.

558. What kind of trapezoid is $WXYZ$?

559. Where is the center of the circle?

560. We're looking for a length. Try building a useful right triangle.

561. The altitude from A of your intersection is the same as the altitude from A of tetrahedron $ABCD$ (look back to the 3-D geometry chapter if you don't remember how to find the length of this altitude).

562. First find the central angle of the sector.

563. Let H be the foot of the altitude from R to $\overline{ST}$. What kind of triangles are $\triangle RHS$ and $\triangle RHM$?

564. Look at the diagram in the solution to Problem 9.8. What kind of triangle is $\triangle BXC$?

565. Let M be the midpoint of $\overline{PQ}$ and N be the midpoint of $\overline{RS}$. Consider the cross-section containing $\triangle MRS$.

566. Find BE first by letting $BE = x$.

567. Draw perpendicular segments from the feet of the perpendicular segments you drew in the first hint to $\overleftrightarrow{AB}$. Why must these meet $\overleftrightarrow{AB}$ at the same point? Call this point Q. What is OQ?

568. Let the point on $\mathcal{Z}$ that is initially touching $\mathcal{A}$ be point P. Where on $\mathcal{Z}$ is P after $\mathcal{Z}$ has moved one-quarter of the way around $\mathcal{A}$?

569. What kind of triangle is $\triangle AOC$?

570. Label the points of tangency. Write each side of the equation you wish to prove in terms of segments with these points as endpoints.

571. If the interior angles have integer measure, so do the exterior angles.

572. Perpendicular lines mean right triangles, and what you want to prove has squares of lengths. What should you try?

573. Through what other point does plane WXZ pass? How does this help with finding the distance from Z to this plane?

574. Use $\triangle BEA$ to find $\angle BEA$. What does this mean $\angle DEB$ is?

575. Show that $\triangle BAF \cong \triangle EDF$.

576. What kind of triangle is $\triangle ABD$?

577. Cut the figure into two pieces you can find the area of: a right triangle and a rectangle.

578. Compare $\angle BC'C$ to $\angle BCC'$.

579. Show that X cannot be beyond B on $\overrightarrow{AB}$, and that X cannot be beyond A on $\overrightarrow{BA}$.

580. Why is $\triangle OPQ$ a right triangle?

581. Let the middle angle have measure x and the common difference between angles be y. Write an equation!

582. What is the measure of $\angle EBF$? Of $\angle EAD$?

583. $[PQA]$, $[PBS]$, $[ABP]$ are all part of the problem. What piece must we combine with these to make $[PQRS]$?

584. What strategy did we take for a similar looking problem in the text (look back in this section).

585. Draw the altitude from A to $\overline{BC}$.

586. Half the product of the diagonals is not the only way to find the area of a rhombus.

587. What is the sum of all the angles? What portion of this sum is the largest angle?

588. Let G be on $\overline{AE}$ such that $\overline{GF} \perp \overline{AE}$. Let $DF = y$.

589. Prove that $\overline{AD}$ bisects $\angle BXC$ and $\angle BAC$.

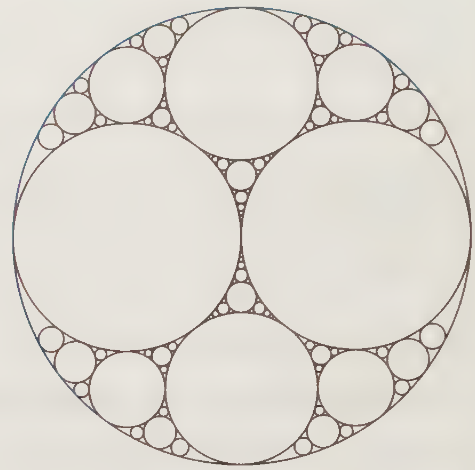
590. Let O be the center of the original circle and A be a point on the circle. Let the images of O and A under the rotation be O' and A', respectively. Find a pair of congruent triangles to show that $O'A' = OA$.

591. Build right triangles. Draw $\overline{OA}$, $\overline{OP}$ and $\overline{BP}$. See any similar right triangles?

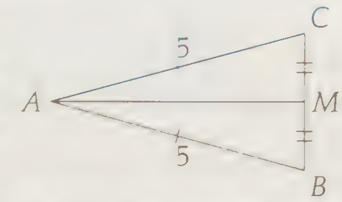
592. Find two expressions in terms of y for EF.

593. Let the hypotenuse of the triangle be $\overline{AC}$. Find BC and AB in terms of AC. To use the inradius information, draw the incircle and the inradius. Do you see any other segments equal in length to the inradius?

594. Does $\triangle QTR$ share an altitude or a base with another triangle whose area you know?



Index



Solution for Problem 3.15: We're a little thin on information. We have a side equality that seems important. But we only have one triangle, so there's no obvious way to use the side equality. We might look for congruent triangles, but we'll have to somehow make two triangles. Splitting $\angle ABC$ by connecting A to the midpoint, M, of $\overline{BC}$ gives us our congruent triangles.

Since $AB = AC$, $BM = CM$, and $AM = AM$, we have $\triangle AMB \cong \triangle AMC$ by SSS. This tells us that $\angle B = \angle C$, which is just enough to finish the problem. Since the angles of $\triangle ABC$ add up to 180°, we have

$$180^\circ = \angle A + \angle B + \angle C = 30^\circ + 2(\angle C).$$

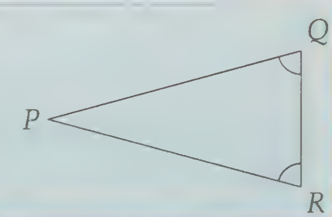
Solving for $\angle C$, we have $\angle C = (180^\circ - 30^\circ)/2 = 75^\circ$. □

A triangle in which two sides are equal is called an **isosceles triangle**. The equal sides are sometimes called the **legs** of the triangle, and the other side the **base**.

Our general approach in Problem 3.15 can be used to show that if two sides of a triangle are equal, then the angles opposite those sides are equal. These two equal angles are often called the **base angles** of the triangle, and the other angle the **vertex angle**. As we saw in the last problem, if $\triangle ABC$ is isosceles with $\angle B = \angle C$, we have $\angle B = \angle C = (180^\circ - \angle A)/2$.

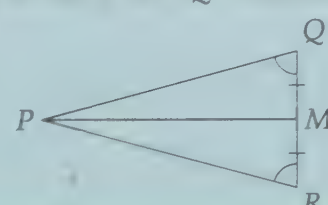
We might now wonder if this runs the other way: do equal angles imply equal sides?

Problem 3.16: Prove that if $\angle PQR = \angle PRQ$, then $PR = PQ$.



Solution for Problem 3.16: What's wrong with this solution?

Bogus Solution: We proceed as we did in Problem 3.15, by connecting P to M, the midpoint of $\overline{QR}$, as in Figure 3.6. Since $MR = MQ$, $\angle R = \angle Q$, and $PM = PM$, we have $\triangle PMR \cong \triangle PMQ$. Therefore, $PR = PQ$.



This failed solution uses 'SSA' Congruence, which we have shown in Section 3.5 doesn't work! We have to construct our triangles in a way that lets us use a valid congruence theorem.

Extra! *The essence of mathematics is not to make simple things complicated, but to make complicated things simple.*

–S. Gudder

the Art of Problem Solving

Introduction to Geometry

from the creators of **www.artofproblemsolving.com**

This book is probably very different from most of the math books that you have read before. We believe that the best way to learn mathematics is by solving problems. Lots and lots of problems. In fact, we believe that the best way to learn mathematics is to try to solve problems that you don't know how to do. When you discover something on your own, you'll understand it much better than if someone just tells it to you.

Richard Rusczyk is the founder of www.artofproblemsolving.com. He is co-author of *the Art of Problem Solving, Volumes 1 and 2*. He was a national MATHCOUNTS participant in 1985, a three-time participant in the Math Olympiad Summer Program, and a USA Math Olympiad winner in 1989.

ISBN 0977304531

90000>

9 780977 304530