the Art of Problem Solving

Prealgebra

Richard Rusczyk
David Patrick
Ravi Boppana

Acknowledgements

Contests

We would like to thank the following contests for allowing us to use a selection of their problems in this book:

- The **American Mathematics Competitions**, a series of contests for U.S. middle and high school students. The **AMC 8**, **AMC 10**, and **AMC 12** contests are multiple-choice tests that are taken by over 350,000 students every year. Top scorers on the AMC 10 and AMC 12 are invited to take the **American Invitational Mathematics Examination (AIME)**, which is a more difficult, short-answer contest. Approximately 10,000 students every year participate in the AIME. Then, based on the results of the AMC and AIME contests, about 500 students are invited to participate in the **USA Junior Mathematical Olympiad (USAJMO)** and **USA Mathematical Olympiad (USAMO)**, 2-day, 9-hour examinations in which each student must show all of his or her work. Results from the USA(J)MO are used to invite students to the Math Olympiad Summer Program, at which the U.S. team for the **International Mathematical Olympiad (IMO)** is trained. More information about the AMC contests can be found on the AMC website at http://www.maa.org/math-competitions.
- **MATHCOUNTS®**, the premier contest for U.S. middle school students. MATHCOUNTS is a national enrichment, coaching, and competition program that promotes middle school mathematics achievement through grassroots involvement in every U.S. state and territory, with over 160,000 students participating in 2013-14. President Barack Obama, and former Presidents Bush, Clinton, Bush and Reagan have all recognized MATHCOUNTS in White House ceremonies. The MATHCOUNTS program has also received two White House citations as an outstanding private sector initiative. More information is available at http://www.mathcounts.org.
- **Math Olympiads for Elementary and Middle Schools (MOEMS)**, an international program for students in grades 4-8. Created in 1977 by Dr. George Lenchner, MOEMS offers beginning problem solvers 5 contests each year. In 2013-14, nearly 150,000 students worldwide from 6,000 teams participated in MOEMS. More information is available at http://www.moems.org.

How We Wrote This Book

This book is a collaborative effort of the staff of the Art of Problem Solving. The three authors listed on the cover would like to particularly thank our colleagues listed below:

Jason Batterson, Jeremy Copeland, Larry Evans, and Shannon Rogers, for providing extensive proofreading, as well as valuable guidance on pedagogy and subject coverage;

Vanessa Rusczyk, for designing the cover and also contributing greatly to the interior design of the book;

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Josh Zucker, whose comments about how he learned mathematics inspired the questions-before-the-lessons approach of the text.

The print version of this book was written using the LaTeX document processing system, and the diagrams were prepared using Metapost and Asymptote. We thank the authors of the various LaTeX packages that we used while preparing this book, and also the brilliant authors of *The LaTeX Companion* for writing a reference book that is not only thorough but also very readable.

The source files for the print book were initially converted to this online book using a script written in the Python programming language. Palmer Mebane managed the conversion of this book from print to online, and Paul Salerno also wrote tools used in the conversion. Jason Batterson and the Jacob Tyler Creative Group designed the look and feel of this online book. James Fung, Shelley Garg, Kyle Guillet, Tasha Moyer, David Patrick, Shannon Rogers, Amy Szczepanski, Deven Ware, Phyllis Xu, and Laura Zehender all helped to review and edit the online book content.

Dedication

R.B.–I dedicate this book to my mom and dad (Krishna Rao and Jaganmohan Rao), my wife Ranu, and my daughter Meena. Thank you for always believing in me.

D.P.–Personal thanks to my most influential mathematical mentors:
my parents (when I was very young),
Gerry Rising (when I was moderately young),
and Michael Artin (when I was not-so-young).

R.R.–For my students, in the hopes that they learn from me as much as I learn from them.

Learn by Solving Problems

This book is probably very different from most of the math books that you have read before. We believe that the best way to learn mathematics is by solving problems. Lots and lots of problems. In fact, we believe that the best way to learn mathematics is to try to solve problems that you don't know how to do. When you discover something on your own, you'll understand it much better than if someone just tells it to you.

Most of the sections of this book begin with several problems. The solutions to these problems will be covered in the text, but try to solve the problems *before* reading the section. If you can't solve some of the problems, that's OK, because they will all be fully solved as you read the section. Even if you solve all of the problems, it's still important to read the section, both to make sure that your solution is correct, and also because you may find that the book's solution is simpler or easier to understand than your own.

If you find that the problems are too easy, this means that you should try harder problems. Nobody learns very much by solving problems that are too easy for them.

Navigating This Book

From any page in the book, you can click on the image of the book's cover in the top-left corner to view the table of contents. On large-screen devices, you can also click anywhere on the left-side navigation bar to jump to the corresponding section of the book. You can use the left and right arrows at the top of the page to move to the previous or next chapter or section of the book.

Most sections begin with all of the problems that appear in the section, as explained in the "Learn by Solving Problems" subsection above. Clicking on the *Jump to Solution* link will jump forward in the section to where that problem and its solution appear in the text.

Hovering over a paragraph will create a small icon in the left margin. Clicking on that icon will pop up a window with a permanent link to that paragraph. You can cut-and-paste this link into an email message or community post.

Interactive Features

There are several interactive features built into the book. All online books are linked to the AoPS community, so that students using the online book can discuss the book with other students. Click on the Community icon next to a section or problem to view all of the discussions about that section or problem. Click the New Topic icon to start a new discussion topic about the section or problem.

This book is also linked to *Alcumus*, Art of Problem Solving's innovative online learning system. Clicking on the Alcumus link at the top of a section takes you to Alcumus. This gives you the opportunity to work on additional practice problems that reinforce the material in your current section of the book. (Not all sections are linked to Alcumus, so the icon may not appear at the top of some sections.) To learn more about Alcumus, click here.

This book contains embedded video lessons. Just click on the "play" button on a video to watch the video. You can also view the entire Prealgebra video library.

Explanation of Icons

Throughout the book, you will see various shaded boxes and icons.

Concept:	This will be a general problem-solving technique or strategy. These are the "keys" to becoming a better problem solver!
Important:	This will be something important that you should learn. It might be a formula, a solution technique, or a caution.
WARNING!!	Beware if you see this box! This will point out a common mistake or pitfall.
Sidenote:	This box will contain material which, although interesting, is not part of the main material of the text. It's OK to skip over these boxes, but if you read them, you might learn something interesting!
Bogus Solution:	Just like the impossible cube shown to the left, there's something wrong with any "solution" that appears in this box.
Extra!	This is an "Extra!" and might be a quote, some biographical or historical background, or perhaps an interesting idea to think about.

Exercises, Review Problems, and Challenge Problems

Most sections end with several **Exercises**. These will test your understanding of the material that was covered in the section that you just finished. You should try to solve *all* of the exercises. Exercises marked with a ★ are more difficult.

Most chapters have a section containing **Review Problems**. These are problems which test your understanding of the material covered in the chapter. Your goal should be to solve most or all of the Review Problems for every chapter — if you're unable to do this, it means that you haven't yet mastered the material, and you should probably go back and read the chapter again.

All of the chapters end with a section containing **Challenge Problems**. These problems are generally more difficult than the other problems in the book, and will really test your mastery of the material. Some of them are very, very hard — the hardest ones are marked with a ★. Don't necessarily expect to be able to solve all of the Challenge Problems on your first try — these are difficult problems even for experienced problem solvers. If you are able to solve a large number of Challenge Problems, then congratulations, you are on your way to becoming an expert problem solver!

You can type your solution or notes for any Exercise, Review Problem, or Challenge Problem directly into the book. Your work will automatically be saved. You won't be able to view the solution to a problem until you type something in the solution box.

Many problems come with one or more **hints**. You can view any available hint by clicking on the *Hint* link after the problem statement. You can then hide the hint by again clicking on the *Hint* link. It is very important that you first try to solve the problem without peeking at the hints. Only after you've seriously thought about a problem and are stuck should you look at a hint. Also, for problems which have multiple hints, use the hints one at a time; don't go to the second hint until you've thought about the first one.

The **solutions** to all of the Exercises, Review Problems, and Challenge Problems are built into the book. Clicking the *Show Solution* button will display the solution, but you won't be able to view the solution until you've made an attempt to solve the problem and typed something into the solution box. Once you've viewed the solution, you can add notes to the solution in a separate box. You can also click the *Reset* button to clear your solution and notes and start fresh.

Here are some very important things to keep in mind about the solutions:

1. Make sure that you make a serious attempt at the problem before looking at the solution. You should think *hard* about a problem before deciding to give up and look at the solution. Remember, once you view a solution, you can't change what you typed for your solution.
2. After you solve a problem, it's usually a good idea to read the solution, even if you think you know how to solve the problem. Our solution might show you a quicker or more concise way to solve the problem, or it might have a completely different solution method that you might not have thought of.
3. If you have to look at the solution in order to solve a problem, make sure that you make a note of that problem. You can then come back to the problem in a week or two to make sure that you are able to solve it on your own without resorting to the solution.

Resources

Here are some other good resources for you to further pursue your study of mathematics:

- Art of Problem Solving has a complete library of books (both print and online) specifically designed for avid math students:
  - The *Introduction* series: *Introduction to Algebra*, *Introduction to Counting & Probability*, *Introduction to Number Theory*, and *Introduction to Geometry*, designed for students in grades 6-10.
  - The *Intermediate* series: *Intermediate Algebra*, *Intermediate Counting & Probability*, *Precalculus*, and *Calculus*, designed for students in grades 9-12.
  - The *Problem Solving* series: designed for students preparing for math competitions. In addition to our classics *the Art of Problem Solving, Volume 1: the Basics* (for students in grades 7-10 preparing for MATHCOUNTS and the AMC 8/10/12 contests) and *the Art of Problem Solving, Volume 2: and Beyond* (for students in grades 9-12 preparing for advanced contests such as the AIME), we also have *Competition Math for Middle School*.
  - *Beast Academy*: a full, rigorous, entertaining curriculum for aspiring math beasts in grades 2-5.
- The Art of Problem Solving website contains many other resources for students.
  - The AoPS Community has tens of thousands of members (if you are reading this online book, you're a member too!) and millions of posts on a variety of math, problem solving, and other fun topics.
  - Alcumus, our free adaptive online learning system containing over 13,000 practice problems.
  - For the Win!, our free interactive online game inspired by the MATHCOUNTS Countdown Round.
  - Our vast video library contains hundreds of videos featuring AoPS founder Richard Rusczyk.
  - Learn LaTeX, the mathematical typesetting system used by most professional mathematicians and scientists, from our widely-used LaTeX guide, and practice your LaTeX skills with the TeXeR.
  - AoPS Community members collaborate to build the AoPSWiki.
  - We have a collection of articles on a variety of problem-solving topics.
- You can hone your problem solving skills (and perhaps win prizes!) by participating in various math contests. Please see the **Acknowledgements** section of this book for more information.

A Note to Teachers and Parents

We believe that students learn best when they are challenged with hard problems that at first they may not know how to do. This is the motivating philosophy behind this book.

Rather than first introducing new material and then giving students exercises, we present problems at the start of each section that students should try to solve *before* the new material is presented. The goal is to get students to discover the new material on their own. Often, complicated problems are broken into smaller parts, so that students can discover new techniques one piece at a time. Then the new material is formally presented in the text, and full solutions to each problem are explained, along with problem-solving strategies.

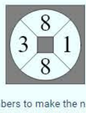
We hope that teachers will find that many students will discover most of the material in this book on their own by working through the problems. Other students may learn better from a more traditional approach of first seeing the new material, then working the problems. Teachers have the flexibility to use either approach when teaching from this book.

The book is linear in coverage. Generally, students and teachers should progress straight through the book in order, without skipping chapters. In general, chapters are not equal in length, so different chapters may take different amounts of classroom time.

After completing this book, students should be ready to continue with any book in the Art of Problem Solving's *Introduction* series of textbooks. The books in the *Introduction* series can be used in any order, although we generally recommend that *Introduction to Geometry* be used last.

Game: **The 24 Game**

See the four numbers on the card below?



Can you combine these four numbers to make the number 24? You may add, subtract, multiply, or divide. You may also use parentheses. You can't use any other symbols. You have to use each of the four numbers exactly once.

In the example above, one solution is $(3+1)\times 8-8$. Another solution is $(3-1)\times 8+8$. Yet another solution is $(8+1)\times 8\div 3$.

This game is called the **24th game**. For more information about it, visit `http://www.24game.com`.

For fun, this book has a few 24 cards at the top of each chapter. Give them a try! Some of them are hard, but each has at least one solution. If you get stuck, the button right below each card shows the solution.

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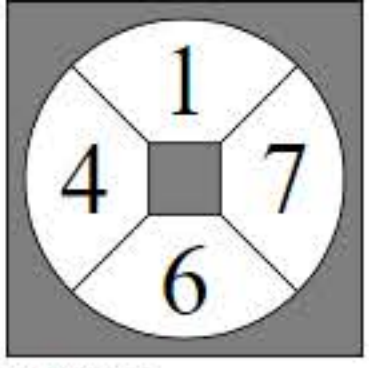
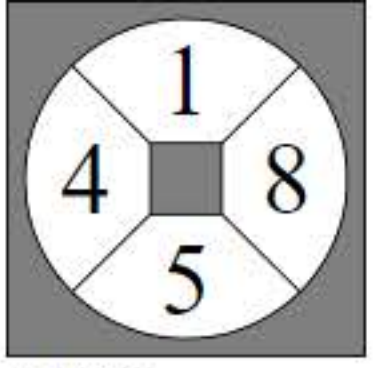
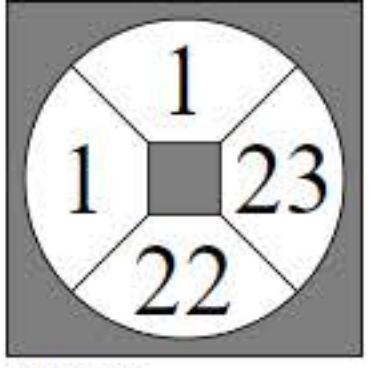


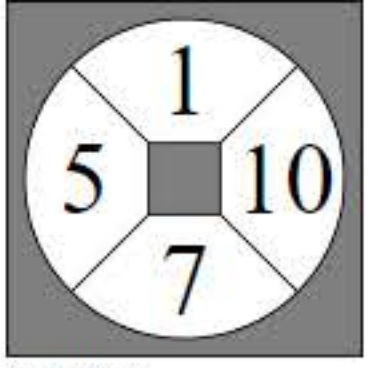
Solution:
$(7+1-4)\times 6$



Solution:
$(5+1)\times(8-4)$ or
$(5-1)\times 4+8$



Solution:
$(1+1)\times 23-22$



Solution:
$5\times 7-10-1$

Arithmetic is being able to count up to twenty without taking off your shoes. — Mickey Mouse

CHAPTER 1

Properties of Arithmetic

1.1 Why Start with Arithmetic?

You know how to add, subtract, multiply, and divide. In fact, you may already know how to solve many of the problems in this chapter. So why do we start this book with an entire chapter on arithmetic?

To answer this question, go back to the title of this book: ***Prealgebra***. What is prealgebra? Not everybody agrees on what "prealgebra" means, but we (the writers of this book) like to think of prealgebra as the bridge between *arithmetic* and *algebra*.

Arithmetic refers to the basics of adding, subtracting, multiplying, dividing, and (maybe) more exotic things like squares and square roots. You probably learned most of these basics already. The hardest thing that you usually do in arithmetic is "word problems" like "If Jenny has 5 apples and Timmy has 7 apples, then how many apples do they have together?" As you get older, the numbers get bigger, but the problems don't really get much harder. Arithmetic is great when trying to solve simple problems like counting apples. But when the problems get more complicated–like trying to compute a rocket's trajectory, or trying to analyze a financial market, or trying to count the number of ways a text message can be routed through a cellular phone network–we need a more advanced toolbox.

That toolbox is *algebra*. Algebra is the language of all advanced mathematics. Algebra gives us tools to take our concepts from arithmetic and make them *general*, meaning that we can use the concepts not just for arithmetic problems, but for other sorts of problems, too.

To take a simple example, you can use arithmetic to show that

$$2\times(3+5)=(2\times 3)+(2\times 5),$$

because the left side equals 2×8, which is 16, and the right side equals $6+10$, which is also 16. But algebra gives us the much more general tool that

$$a\times(b+c)=(a\times b)+(a\times c),$$

no matter what numbers a, b, and c are. And in higher mathematics, a, b, and c might not even be numbers as you recognize them now, but might be more complicated mathematical objects. (Even more generally, "$+$" and "$\times$" might not mean addition and multiplication as you think of them now, but instead might represent more complicated mathematical operations. But we're getting ahead of ourselves a little bit!)

So our initial goal, in Chapter 1, is to carefully lay down the rules of arithmetic, and to give you some ideas as to *why* these rules are true. Once you know the rules, you'll be ready to start thinking algebraically.

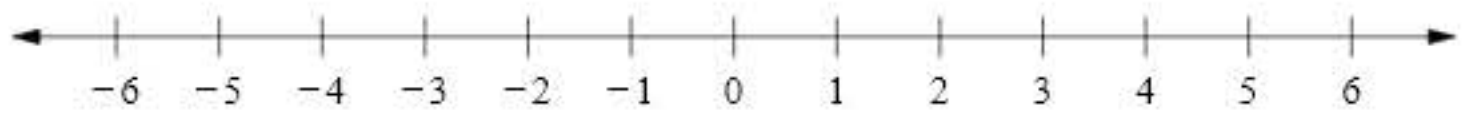
Also, by the time you start reading this book, you are mathematically mature enough to start thinking about not just *how* to perform various calculations, but *why* the techniques used in those calculations work. Understanding why mathematics works is the key to solving harder problems. If you only understand how techniques work but not why they work, you'll have a lot more difficulty modifying those techniques to solve more complicated problems. So, throughout this book, we will rarely just tell you how something works–we'll usually show you why it works.

By the end of this chapter, you should be able to explain *why* the following computations are true:

- $(-5)\times(-7)=35$ (and not -35)
- $(1990\times 1991)-(1989\times 1990)=3980$ (and be able to compute this in your head!)
- $8\div\frac{1}{7}=56$
- $(4\times 10\times 49)\div(2\times 5\times 7)=28$ (again, in your head!)

You'll know all these things not because you've blindly applied some calculation, or because you've memorized some formula–instead, you'll understand the mathematics behind all these expressions.

Unfortunately, different mathematicians and different textbooks may use slightly different words for the same concept, in the same way that what an American calls a "truck" is called a "lorry" by people in Great Britain. So, before we go any farther, we want to make sure that we all agree on some of the words that we're going to use.



The **number line** is shown above. It goes on forever in both directions. Every number that we will consider in this book is somewhere on the number line. The tick marks on the number line above indicate the **integers**. An integer is a number without a fractional part:

$$\ldots,-7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7,\ldots$$

(The symbol ... at either end of the above list means that the list goes on forever in that direction. The ... symbol is called an **ellipsis**.) However, as you know, there are many numbers on the number line other than integers. For instance, most fractions such as $\frac{1}{2}$ are not integers. (But some fractions are integers–we'll explore this further in Chapter 4.)

A number is called **positive** if it is to the right of 0 on the number line. In other words, a number is positive if it is greater than 0. A number is called **negative** if it is to the left of 0 on the number line. That is, a number is negative if it is less than 0. For example, 2 is positive, while -2 is negative. Note that 0 itself is neither positive nor negative, and that every number is either positive, negative, or 0.

A number is called **nonnegative** if it is not negative. In other words, a nonnegative number is positive or 0. Similarly, a number is called **nonpositive** if it is not positive. Finally, a number is called **nonzero** if it is not equal to 0. Note that 0 is nonnegative and nonpositive.

Sidenote: ♫

A lot of people use the term **whole number** to mean a nonnegative integer. In other words, a whole number is one of the numbers $0, 1, 2, \ldots$. These same people use the term **natural number** to mean a positive integer. In other words, a natural number is one of the numbers $1, 2, 3, \ldots$.

However–and this is the really irritating part–a lot of people use the term **natural number** to mean a nonnegative integer. In other words, a natural number is one of the numbers $0, 1, 2, \ldots$. These same people use the term **whole number** to mean a positive integer. In other words, a whole number is one of the numbers $1, 2, 3, \ldots$.

And some of both of these groups of people might also use the term **counting number** to mean either whole number or natural number.

These people have been arguing for centuries, and they will likely never agree. Since we don't like to argue, we will stick with the very clear terms **positive integers** for the numbers $1, 2, 3, \ldots$, and **nonnegative integers** for the numbers $0, 1, 2, \ldots$.

1.2 Addition

We'll start by exploring the simplest arithmetic operation **addition**. There's not a whole lot to explore, but as we'll see, a solid knowledge of the basic properties of addition makes complicated-looking calculations easy.

> **Important:** **How to use this book:** Most sections will begin with problems, like those shown below. You should first try to solve the problems. Then, continue reading the section, and compare your solutions to the solutions presented in the book.

Problems

Problem 1.1 Jump to Solution

Using the two pictures below, explain why $2 + 3 = 3 + 2$.

Problem 1.2 Jump to Solution

Using the two pictures below, explain why $(2 + 3) + 4 = 2 + (3 + 4)$

Reminder: The parentheses tell you what to compute first. For example, $(3 + 4) \times 5$ equals 7×5, whereas $3 + (4 \times 5)$ equals $3 + 20$.

Problem 1.3 Source: (b) MATHCOUNTS Jump to Solution

(a) Using the properties from Problems 1.1 and 1.2, explain why

$$472 + (219 + 28) = (472 + 28) + 219.$$

(b) Compute the sum $472 + (219 + 28)$.

Problem 1.4 Source: MATHCOUNTS Jump to Solution

Compute

$$(2 + 12 + 22 + 32) + (8 + 18 + 28 + 38).$$

Problem 1.5 Jump to Solution

Find the sum $1 + 2 + 3 + \cdots + 19 + 20$. Reminder: The ellipsis $\cdots$ means that we should include all the numbers in the pattern. So we are adding the positive integers from 1 to 20.

Problem 1.6 Jump to Solution

Using the picture below, explain why $2 + 0 = 2$.

Problem 1.1

Using the two pictures below, explain why $2 + 3 = 3 + 2$.

Solution for Problem 1.1: First let's look at the picture on the left. The first row has 2 squares; the second row has 3 squares. So in total, there are $2 + 3$ squares.

Now let's look at the picture on the right. The first row has 3 squares; the second row has 2 squares. In total, there are $3 + 2$ squares.

The picture on the right, however, is just an upside-down version of the picture on the left. Flipping a picture upside down doesn't change the number of squares. So we conclude that $2 + 3 = 3 + 2$. □

Whenever we add two numbers, the order of the numbers does not matter. For example, $5 + 17 = 17 + 5$ and $32 + 999 = 999 + 32$. There are an infinite number of such examples. Of course, we can't write down an infinite number of examples. Instead, we can write the equation

$$\text{first number} + \text{second number} = \text{second number} + \text{first number}.$$

This is a long equation to write. We can shorten it by letting a represent the first number and b represent the second number. Then our equation becomes

$$a + b = b + a.$$

Here a and b stand for any numbers (and possibly the same number). For instance, if we let $a = 2$ and $b = 3$, then we have our original example $2 + 3 = 3 + 2$. If we let $a = 100$ and $b = 200$, then we get $100 + 200 = 200 + 100$. Note that a has the same value throughout the equation; as does b. Both, however, may change from one problem to the next. Letters such as a and b that represent numbers are called **variables**.

The rule that $a + b = b + a$ for all numbers a and b is called the **commutative property** of addition.

> **Important:** **Addition is commutative:** Let a and b be numbers. Then
>
> $$a + b = b + a.$$

Commutative Property of Addition

In Problem 1.1, we explained why $a + b = b + a$ is true for one particular example, when $a = 2$ and $b = 3$. But one example doesn't prove that $a + b = b + a$ for all a and b. We don't have the tools in this book to explain why this must hold for any two numbers, but Problem 1.1 should give you good intuition for why it is true for positive integers.

Throughout the rest of this chapter, we will explore many more arithmetic rules. We will use examples and pictures to give intuition for why these rules work. These examples and pictures are not proofs, but they should give you a feel for why these rules must be true.

The commutative property is concerned with adding two numbers. What if we add three numbers?

Problem 1.2

Using the two pictures below, explain why $(2 + 3) + 4 = 2 + (3 + 4)$.

Reminder: The parentheses tell you what to compute first. For example, $(3 + 4) \times 5$ equals 7×5, whereas $3 + (4 \times 5)$ equals $3 + 20$.

Solution for Problem 1.2: For each picture, we will first count the number of light squares, then count the number of dark squares, and finally add the two counts.

Let's start with the picture on the left. It has $(2 + 3)$ light squares and 4 dark squares. So altogether it has $(2 + 3) + 4$ squares.

Next, let's look at the picture on the right. It has 2 light squares and $(3 + 4)$ dark squares. So altogether it has $2 + (3 + 4)$ squares.

What's the difference between the two pictures? The only difference is the color of the middle row. Changing the color doesn't change the number of squares. So we conclude that $(2 + 3) + 4 = 2 + (3 + 4)$. □

We get a similar equation for any three numbers: $(a + b) + c = a + (b + c)$. In other words, first adding a and b and then adding c is the same as adding a to $b + c$. This property is called the **associative property** of addition.

> **Important:** **Addition is associative:** Let a, b, and c be numbers. Then
>
> $$(a + b) + c = a + (b + c).$$

Associative Property of Addition

> **WARNING!!** Students sometimes mix up the names "commutative" and "associative." In the commutative property, the numbers are moved around ("commuted") on the two sides of the equation. In the associative property, the numbers stay in the same place, but are grouped ("associated") differently.

Together, the commutative and associative properties are sneakily powerful, as they let us add a list of numbers in any order. The next problem is an illustration of this **any-order principle**.

Problem 1.3 Source: (b) MATHCOUNTS

(a) Using the properties from Problems 1.1 and 1.2, explain why

$$472 + (219 + 28) = (472 + 28) + 219.$$

(b) Compute the sum $472 + (219 + 28)$.

Solution for Problem 1.3:

(a) Let's start with the left side of the equation:

$$472 + (219 + 28).$$

We will try to make it look like the right side. To do that, we need to switch the order of the 219 and the 28. We can do so by the commutative property. In other words, we replace $472 + (219 + 28)$ with the equal quantity

$$472 + (28 + 219).$$

This expression is close to what we want: $(472 + 28) + 219$ in fact, these two expressions are equal by the associative property.

Let's combine the pieces of our explanation into a nice chain of equations:

$$\begin{aligned} 472 &+ (219 + 28) \\ &= 472 + (28 + 219) && \text{commutative property} \\ &= (472 + 28) + 219. && \text{associative property} \end{aligned}$$

(b) By part (a), the quantities $472 + (219 + 28)$ and $(472 + 28) + 219$ are equal, so we can compute $(472 + 28) + 219$ instead. But this is easy to compute: the sum $472 + 28$ is 500, so we are left with $500 + 219$. The answer is 719.

□

The point of Problem 1.3 is that we can rearrange the numbers in our addition to make the addition easier to compute. It's easier to first compute $472 + 28$, and then compute $500 + 219$. From it would have been to start with $219 + 28$ and then add that sum to 472.

In a similar way, any addition problem can be rearranged without changing the sum. Usually we won't bother to write all the individual steps of the rearrangement, like we did in Problem 1.3. Instead, we'll use our knowledge of the commutative and associative properties to just go ahead and rearrange a sum in whatever way is best. Let's put this strategy to solve the next problem.

Problem 1.4 Source: MATHCOUNTS

Compute

$$(2 + 12 + 22 + 32) + (8 + 18 + 28 + 38).$$

Solution for Problem 1.4: We could start with 2, then add 12, then add 22, and so on, but that's too much work. Instead, let's try to rearrange the sum in a useful way. Let's pair up the numbers so that each pair has the same sum. Specifically, let's pair each number in the first group with a number in the second group:

$$(2 + 38) + (12 + 28) + (22 + 18) + (32 + 8).$$

The first pair of numbers adds up to 40, so does the second pair, the third pair, and the fourth pair. So our sum becomes

$$40 + 40 + 40 + 40.$$

The answer is 160. □

Let's use the any-order principle to compute a longer sum.

Problem 1.5

Find the sum $1 + 2 + 3 + \cdots + 19 + 20$. Reminder: The ellipsis $\cdots$ means that we should include all the numbers in the pattern. So we are adding the positive integers from 1 to 20.

Solution for Problem 1.5: We definitely don't want to add the 20 numbers one at a time. Instead, let's try again to rearrange the numbers into pairs, so that each pair has the same sum. We pair the smallest number with the largest, the second-smallest with the second-largest, and so on:

$$(1 + 20) + (2 + 19) + (3 + 18) + \cdots + (10 + 11).$$

We have grouped the 20 numbers into 10 pairs. Each pair adds up to 21. So our sum becomes

$$21 + 21 + 21 + 21 + 21 + 21 + 21 + 21 + 21 + 21.$$

Adding 10 copies of 21 is the same as multiplying 10 and 21. So the answer is 210. □

Sum the Numbers from 1 to 100

Finally, let's look at one more property of addition. What happens when we add zero to a number?

Problem 1.6

Using the picture below, explain why $2 + 0 = 2$.

Solution for Problem 1.6: On one hand, there are 2 squares. On the other hand, we can say there are 2 light squares and 0 dark squares. So we get the equation $2 + 0 = 2$. □

Adding zero to any number doesn't change the number.

> **Important:** **Adding zero:** Let a be a number. Then
>
> $$a + 0 = a.$$

Exercises

1.2.1:

Compute $99 + 99 + 99 + 101 + 101 + 101$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We can group each 99 with a 101:

$$\begin{aligned} &99 + 99 + 99 + 101 + 101 + 101 \\ &= (99 + 101) + (99 + 101) + (99 + 101) \\ &= 200 + 200 + 200 \\ &= \boxed{600}. \end{aligned}$$

1.2.2:

Compute

$$1999 + 2001 + 1999 + 2001 + 1999 + 2001 + 1999 + 2001.$$

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We can group the numbers into pairs without moving them:

$$\begin{aligned} &1999 + 2001 + 1999 + 2001 + 1999 + 2001 + 1999 + 2001 \\ &= (1999 + 2001) + (1999 + 2001) \\ &\quad + (1999 + 2001) + (1999 + 2001) \\ &= 4000 + 4000 + 4000 + 4000 = \boxed{16,000} \end{aligned}$$

1.2.3:

Compute

$$(3 + 13 + 23 + 33 + 43) + (7 + 17 + 27 + 37 + 47).$$

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We can pair each number in the first group with a number in the second group:

$$\begin{aligned} &(3 + 13 + 23 + 33 + 43) + (7 + 17 + 27 + 37 + 47) \\ &= (3 + 47) + (13 + 37) + (23 + 27) + (33 + 17) + (43 + 7) \\ &= 50 + 50 + 50 + 50 + 50 = \boxed{250}. \end{aligned}$$

1.2.4:

Compute

$$(1 + 2 + 3 + \cdots + 49 + 50) + (99 + 98 + 97 + \cdots + 51 + 50).$$

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Again let's match each number in the first group with a number in the second:

$$\begin{aligned} &(1 + 2 + 3 + \cdots + 49 + 50) + (99 + 98 + 97 + \cdots + 51 + 50) \\ &= (1 + 99) + (2 + 98) + (3 + 97) + \cdots + (49 + 51) + (50 + 50) \\ &= 100 + 100 + 100 + \cdots + 100 + 100 = 50 \cdot 100 = \boxed{5000}. \end{aligned}$$

50 numbers

1.3 Multiplying

Problems

Problem 1.1 ○ Hide Solution

Using the two pictures below, explain why $2 \times 3 = 3 \times 2$.

Problem 1.2 ○ Hide Solution

By counting the dots in the picture below in two different ways, explain why

$$(3 \times 5) + (3 \times 2) = 3 \times (5 + 2).$$

Problem 1.3 ○ Hide Solution

Using the picture below, explain why $2 \times 5 = 10$.

Problem 1.4 ○ Hide Solution

Using the picture below, explain why $2 \times 3 = 6$.

Problem 1.5 ○ Hide Solution

(a) Compute $(5 + 6) \times 7$.

(b) Compute $5 \times (6 \times 7)$.

Problem 1.6 ○ Hide Solution

What is the value of $(7 \times 11) \times 13 \times 19 \times 23$?

Problem 1.7 ○ Hide Solution

Consider the multiplication facts below:

$$5 \times 7 = 35$$
$$3 \times 7 = 21$$
$$2 \times 7 = 14$$
$$1 \times 7 = 7$$
$$0 \times 7 = ___$$

(a) Find a pattern in the multiplication answers.

(b) Assuming that the pattern continues, predict the answer to the last multiplication.

Problem 1.7 ○ 0/7

Using the two pictures below, explain why $2 \times 3 = 3 \times 2$.

Solution for Problem 1.7: The picture on the left has 3 rows. Each row has 2 square. So the total number of squares is 3×2, which is 6.

The picture on the right has 2 rows. Each row has 3 squares. So the total number of squares is 2×3 which is 6.

And the pictures of the square pieces are identical. So there are the same number of squares in both pictures. Thus, 2×3 and 3×2 are both equal to 6. We conclude that $2 \times 3 = 3 \times 2$.

Notice that the two pictures in this question reverse the numbers in multiplication just as we saw in addition. In other words, multiplication is commutative. In other words, it doesn't matter which order we multiply numbers, but we must know the product of all the numbers. Let us write down the product of two numbers as $a \times b$. Then, we can write the product of three numbers a, b, c as $a \times b \times c$. We call this **the commutative property of multiplication**.

Notice that $(a \times b) \times c$ means we first multiply a and b and then multiply the result by c. Alternatively, $a \times (b \times c)$ means another way to find the product. We know that $(a \times b) \times c$ and $a \times (b \times c)$ are equal. We call this **the associative property of multiplication**.

We will often go a step further, writing $a \cdot b$ and ab. This notation is never quite so easy to write, like each always leave out the dot. Though, for example, to express 7×5 with just numbers, we need the dot. Otherwise, we would write the number 75. In such cases, we will write either $7 \cdot 5$ or $5(7)$ to indicate the multiplication.

Let us write down the commutative property of multiplication:

Important:	**Multiplication is commutative** Let a and b be numbers. Then $$ab = ba.$$

Let us write down the associative property of multiplication:

Important:	**Multiplication is associative** Let a, b, and c be numbers. Then $$(ab)c = a(bc).$$

Together, the commutative and associative properties let us multiply numbers in any order just as we can add numbers in any order. Even when we have 3 numbers, the only order we pick matters. Let's see this in action.

Problem 1.8 ○ 0/7

By counting the dots in the picture below in two different ways, explain why

$$(3 \times 5) + (3 \times 2) = 3 \times (5 + 2).$$

Solution to Problem 1.8: On one hand, there are 3 rows. Each row has 5 dots. And there are 2 more dots. So the total number of dots is $(3 \times 5) + 2$.

On the other hand, there are 3 columns. Each column has 7 dots. So the total number of dots is $3 \times (5 + 2)$.

Both ways count the same number of dots. So we conclude that $(3 \times 5) + (3 \times 2) = 3 \times (5 + 2)$.

Problem 1.9 ○ 0/7

Using the picture below, explain why $7 \times (11 \times 13) = (7 \times 11) \times 13$.

Solution for Problem 1.9: On one hand, there are 7 rows. Each row has 11×13 dots. So the total number of dots is $7 \times (11 \times 13)$.

On the other hand, there are 11×13 columns. Each column has 7 dots. So the total number of dots is $(7 \times 11) \times 13$.

Both ways count the same number of dots. So we conclude that $7 \times (11 \times 13) = (7 \times 11) \times 13$.

Problem 1.10 ○ 0/7

Using the picture below, explain why $7 \times (11 \times 13) = (7 \times 13) \times 11$.

Solution for Problem 1.10: On one hand, there are 7 rows. Each row has 11×13 dots. So the total number of dots is $7 \times (11 \times 13)$.

On the other hand, there are 11 groups. Each group has 7×13 dots. So the total number of dots is $(7 \times 13) \times 11$.

Both ways count the same number of dots. So we conclude that $7 \times (11 \times 13) = (7 \times 13) \times 11$.

Problem 1.11 ○ 0/7

Consider the multiplication facts below:

$$1 \times 1 = 1$$
$$1 \times 2 = 2$$
$$1 \times 3 = 3$$
$$1 \times 4 = 4$$
$$1 \times 5 = 5$$
$$1 \times 6 = ___$$

(a) Find a pattern in the multiplication answers.

(b) Assuming that the pattern continues, predict the answer to the last multiplication.

Problem 1.12 ○ 0/7

Using the two pictures below, explain why $2 \times 3 = 3 \times 2$.

Solution for Problem 1.12: The picture on the left has 2 rows. Each row has 3 squares. So the total number of squares is 2×3, which is 6.

The picture on the right has 3 rows. Each row has 2 squares. So the total number of squares is 3×2 which is 6.

And the pictures of the square pieces are identical. So there are the same number of squares in both pictures. Thus, 2×3 and 3×2 are both equal to 6. We conclude that $2 \times 3 = 3 \times 2$.

Notice that the two pictures in this question reverse the numbers in multiplication just as we saw in addition. In other words, multiplication is commutative. In other words, it doesn't matter which order we multiply numbers, but we must know the product of all the numbers. Let us write down the product of two numbers as $a \times b$. Then, we can write the product of three numbers a, b, c as $a \times b \times c$. We call this **the commutative property of multiplication**.

Notice that $(a \times b) \times c$ means we first multiply a and b and then multiply the result by c. Alternatively, $a \times (b \times c)$ means another way to find the product. We know that $(a \times b) \times c$ and $a \times (b \times c)$ are equal. We call this **the associative property of multiplication**.

We will often go a step further, writing $a \cdot b$ and ab. This notation is never quite so easy to write, like each always leave out the dot. Though, for example, to express 7×5 with just numbers, we need the dot. Otherwise, we would write the number 75. In such cases, we will write either $7 \cdot 5$ or $5(7)$ to indicate the multiplication.

Let us write down the commutative property of multiplication:

Important:	**Multiplication is commutative** Let a and b be numbers. Then $$ab = ba.$$

Let us write down the associative property of multiplication:

Important:	**Multiplication is associative** Let a, b, and c be numbers. Then $$(ab)c = a(bc).$$

Together, the commutative and associative properties let us multiply numbers in any order just as we can add numbers in any order. Even when we have 3 numbers, the only order we pick matters. Let's see this in action.

Problem 1.13 ○ 0/7

By counting the dots in the picture below in two different ways, explain why

$$(3 \times 5) + (3 \times 2) = 3 \times (5 + 2).$$

Solution for Problem 1.13: On one hand, there are 3 rows. Each row has 5 dots. And there are 2 more dots. So the total number of dots is $(3 \times 5) + 2$.

On the other hand, there are 3 columns. Each column has 7 dots. So the total number of dots is $3 \times (5 + 2)$.

Both ways count the same number of dots. So we conclude that $(3 \times 5) + (3 \times 2) = 3 \times (5 + 2)$.

Problem 1.14 ○ 0/7

Compute $(7 \times 11) \times 13$.

Solution for Problem 1.14: Instead of separately computing 7×11 and 77×13, we can use the distributive property again:

$$(7 \times 11) \times 13 = (70 + 7) \times 13 = 910 + 91.$$

On the right side, the sum $910 + 91$ is easier to compute than $(7 \times 11) \times 13$. We conclude that $(7 \times 11) \times 13 = 910 + 91 = 1001$.

Instead of computing $(7 \times 11) \times 13$, we can use the distributive property again to make the computation simpler. Writing $11 = 10 + 1$ lets us compute:

$$(7 \times 11) \times 13 = (7 \times 10) + (7 \times 1) \times 13 = 910 + 13.$$

The advantage of using the distributive property is we can find the exact sum of 910 and 13 in your head. Or, we can use the property again to write:

$$910 + 13 = (900 + 10) + (10 + 3) = 900 + 20 + 3 = 923.$$

As we can see in Problem 1.13, we can use the distributive property "in either direction." Using the distributive property to rewrite $a(b + c)$ as $ab + ac$ is called **expanding**. Using the distributive property to rewrite $ab + ac$ as $a(b + c)$ is called **factoring**.

For example, in the first step of the solution to Problem 1.13, we factored $(3 \times 5) + (3 \times 2)$ to write it as the product $3(5 + 2)$. In the second step, we expanded $(7 \times 11) \times 13$ to write it as the product of sums $(70 + 7) \times 13$.

Concept:	**Factoring** means using the distributive property to rewrite something of the form $ab + ac$ as the product $a(b + c)$. For example, $$(3 \times 5) + (3 \times 2) = 3 \times (5 + 2).$$ In this example, the number 3 is called a **common factor** of both products on the left side of the equation. Factoring means finding a common factor of $ab + ac$. We can factor $ab + ac$ because both sums on the left side have a common factor of a.

So far, we have used the distributive property with some of the numbers. What about longer sums?

Problem 1.15 ○ 0/7

Compute $(5 \times 7) + (5 \times 11)$.

Distributive Property

Let's explore the distributive property further. What can this property do for us?

Problem 1.16 ○ 0/7

Compute $(7 \times 11) \times 13$.

Solution for Problem 1.16: Instead of separately computing 77 and 77×13, we can use the distributive property again:

$$(7 \times 11) \times 13 = (70 + 7) \times 13 = 910 + 91.$$

On the right side, the sum $910 + 91$ is easier to compute than $(7 \times 11) \times 13$. We conclude that $(7 \times 11) \times 13 = 910 + 91 = 1001$.

Instead of computing $(7 \times 11) \times 13$, we can use the distributive property again to make the computation simpler. Writing $11 = 10 + 1$ lets us compute:

$$(7 \times 11) \times 13 = (7 \times 10) + (7 \times 1) \times 13 = 910 + 13.$$

The advantage of using the distributive property is we can find the exact sum of 910 and 13 in your head. Or, we can use the property again to write:

$$910 + 13 = (900 + 10) + (10 + 3) = 900 + 20 + 3 = 923.$$

As we saw in Problem 1.15, we can use the distributive property "in either direction." Using the distributive property to rewrite $ab + ac$ as $a(b + c)$ is called **factoring**. Using the distributive property to rewrite $a(b + c)$ as $ab + ac$ is called **expanding**.

For example, in the first step of the solution to Problem 1.16, we factored $(7 \times 11) + (7 \times 13)$ to write it as the product $7(11 + 13)$. In the second step, we expanded $(7 \times 11) \times 13$ to write it as the product of sums $(70 + 7) \times 13$.

Concept:	**Factoring** means using the distributive property to rewrite something of the form $ab + ac$ as the product $a(b + c)$. For example, $$(7 \times 11) + (7 \times 13) = 7 \times (11 + 13).$$ In this example, the number 7 is called a **common factor** of both products on the left side of the equation. Factoring means finding a common factor of $ab + ac$. We can factor $ab + ac$ because both sums on the left side have a common factor of a.

Factoring and Expanding

What happens when we multiply a number by zero? You probably already know what happens, but let's try to see why it makes sense in the real world.

Problem 1.17 ○ 0/7

Consider the multiplication facts below:

$$5 \times 0 = 0$$
$$4 \times 0 = 0$$
$$3 \times 0 = 0$$
$$2 \times 0 = 0$$
$$1 \times 0 = 0$$
$$0 \times 0 = ___$$

(a) Find a pattern in the multiplication answers.

(b) Assuming that the pattern continues, predict the answer to the last multiplication.

Solution for Problem 1.17:

(a) The last number that we see on the right-hand side of an equation is 0 less than the number above it.

(b) The last number that we see on the right-hand side is 2 being down by another 2 brings us to 0. So we predict that $0 - 2 = 0$.

The idea that "multiplying by zero gives zero" is called the "zero property of multiplication" and it states that $0 \times a$ and $a \times 0$ equal 0. If you start with 7 dots and you take 7 of them and do this 0 times, you end up with 0 dots. So it makes sense that $7 \times 0 = 0$. Adding nothing gives us nothing, so again $7 + 0 = 7$. Multiplying by one gives the same number, so $7 \times 1 = 7$. It makes the rest of sense that $0 \times 0 = 0$.

The same idea shows that multiplying any number by 0 results in 0. It doesn't matter whether the starting number is small or large, a fraction or an integer, positive or negative, or even something crazy like multiplying a number by 0 raised to the π power. Multiplying by 0 "destroys" every number.

Important:	**Multiplying by zero** Let a be a number. Then $$0a = 0.$$

Exercises

1.3.1. Source: AMC8 ○ 0/7

What is the value of the product $(2^1 \cdot 3^1) \cdot (2^2 \cdot 3^2) \cdots (2^{10} \cdot 3^{10})$?

Hide Solution | Hide Hint | Reset

Your Submission: Enter your multiplication using an integer or fraction.

Solution: Enter your multiplication using an integer or fraction.

Solution: Also consider the numbers to group parts that are easy to multiply.

$$(2^1 \cdot 2^2 \cdot 2^3 \cdots 2^{10}) \cdot (3^1 \cdot 3^2 \cdots 3^{10}) = 2^{55} \cdot 3^{55} = \boxed{_____}$$

1.3.2. 0/7

Compute $1 \cdot 100 \cdot 2 \cdot 99 \cdot 3 \cdot 98 \cdots 50 \cdot 51$.

Hide Solution | Hide Hint | Reset

Your Submission: Enter your multiplication using an integer or fraction.

Solution: Hide Hint

Solution: We can pair the numbers without changing the product.

$$1 \cdot 100 \cdot 2 \cdot 99 \cdot 3 \cdot 98 \cdots 50 \cdot 51 = 100 \cdot 2 \cdot 99 \cdot 3 \cdot 98 \cdots 50 \cdot 51$$
$$= 100 \cdot 100 \cdot 100 \cdots 100$$
$$= \boxed{100^{50}}$$

1.3.3. 0/7

Compute $2 \cdot 3 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 3 \cdot 3$.

Hide Solution | Hide Hint | Reset

Your Submission: Enter your multiplication using an integer or fraction.

Solution: Hide Hint

Solution: We can pair the numbers without changing the product.

$$2 \cdot 3 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 6 \cdot 2 \cdot 6 \cdot 3 \cdot 6 \cdot 3 \cdot 3 \cdot 3$$
$$= 12 \cdot 18 \cdot 18 \cdot 27$$
$$= \boxed{104,976}$$

1.3.4. 0/7

Compute $2 \cdot 1000 \cdot 5$.

Hide Solution | Hide Hint | Reset

Your Submission: Enter your multiplication using an integer or fraction.

Solution: Hide Hint

Solution: $2 \cdot 1000 \cdot 5 = 2 \cdot 5 \cdot 1000 = \boxed{10,000}$

1.3.5. 0/7

Compute $6 \cdot 7 \cdot 11 \cdot 5 \cdot 3$.

Hide Solution | Hide Hint | Reset

Your Submission: Enter your multiplication using an integer or fraction.

Solution: Hide Hint

Solution: $6 \cdot 7 \cdot 11 \cdot 5 \cdot 3 = 5 \cdot 6 \cdot 3 \cdot 7 \cdot 11 = \boxed{6,930}$

1.3.6. 0/7

Using the distributive property, evaluate the following expressions.

(a) $52 \cdot 101 \cdot 101 \cdot 101$

Solution: We can factor out the 7 to get

$$52 \cdot 101 \cdot 101 \cdot 101 = 3,030 \cdot 101 \cdot 101 = 31 \cdot 100 \cdot \boxed{____}$$

(b) $52 \cdot 6 + 6 \cdot 19$

Solution: We use the commutative property to write $52 \cdot 6 + 6 \cdot 19$ and then factor out 6 to get

$$52 \cdot 6 + 6 \cdot 19 = 6 \cdot 52 + 6 \cdot 19 = 6(52 + 19) = 6 \cdot 71 = \boxed{426}$$

(c) $52 \cdot 101 + 101 \cdot 19$

Solution: Hide Hint

Solution: Hide Hint

Solution: Again, we factor.

$$52 \cdot 101 + 101 \cdot 19 = 52 \cdot 101 + 101 \cdot 19$$
$$= 101(52 + 19)$$
$$= 101 \cdot 71$$
$$= \boxed{7,171}$$

1.3.7. 0/7

Find values a, b, and c such that $a \cdot b = 2 \cdot c$ and $b \cdot c = 3 \cdot a$ and $c \cdot a = 5 \cdot b$. If other works, find an example to illustrate that a, b, and c are not all equal.

Hide Solution | Hide Hint | Reset

Your Submission: There are many choices of a, b, and c for which the two expressions are not equal. For instance, choose $a = 2$ and $b = 4$ and $c = 6$. Then the first expression is

$$a \cdot b = 2 \cdot 4 = 8 = 2 \cdot 6 = 2 \cdot c.$$

The second expression is

$$b \cdot c = 4 \cdot 6 = 24 = 3 \cdot 8 = 3 \cdot a.$$

The two numbers are indeed different.

1.3.8. 0/7

Compute

$$400 \times 200 \times 100 \times 400 \times 200 \times 100 \times 400 \times 200 \times 100.$$

Hide Solution | Hide Hint | Reset

Your Submission: Enter your multiplication using an integer or fraction.

Solution: Hide Hint

Solution: Multiplying by zero gives 0.

$$400 \times 200 \times 100 \times 400 \times 200 \times 100 \times 400 \times 200 \times 100 = \boxed{0}$$

1.3.9. Source: MATHCOUNTS ○ 0/7

What is the product of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ and 0?

Hide Solution | Hide Hint | Reset

Your Submission: Enter your multiplication using an integer or fraction.

Solution: Hide Hint

Solution: This is multiplication by zero.

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 0 = 0 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9$$
$$= \boxed{0}$$

1.3.10. 0/7

Compute $(2 \cdot 1,110 \cdot 37) \cdot 0 \cdot 411$.

Hide Solution | Hide Hint | Reset

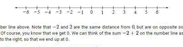
Your Submission: Enter your multiplication using an integer or fraction.

Solution: Hide Hint

Solution: Multiplying by zero gives 0, so

$$(2 \cdot 1,110 \cdot 37) \cdot 0 \cdot 411 = 0 \cdot 0 \cdot 411 = \boxed{0}$$

1.4 Negation



Take a look at the number line above. Note that -2 and 2 are the same distance from 0 but are on opposite sides of 0. What happens when we add -2 + 2? Of course, you know that we get 0. We can think of the sum -2 + 2 on the number line as starting at the position -2 and moving 2 units to the right, so that we end up at 0.

This idea of two numbers that sum to 0 is the key concept behind **negation**:

> **Definition:** The **negation** of a number x, written $-x$, is the number that we add to x to get 0. That is,
>
> $$x + (-x) = 0.$$
>
> The negation of x is also called the **opposite** of x or the **additive inverse** of x. We also sometimes pronounce $-x$ as "minus x" or "negative x."

For example, -1 is the negation of 1, since -1 + 1 = 0, and -2 is the negation of 2, since -2 + 2 = 0. The number line shows us that there is clearly a negation for any positive number. For example, -28 is the negation of 28, since -28 + 28 = 0.

You might ask: why do we bother with a new word "negation" when we already have negative numbers? The reason is that although so far we have mentioned the negations of positive numbers, we have not yet talked about the negations of zero and negative numbers. For example, the negation of 0 is -0, whatever that is. (We will find out soon enough.) The negation of -6 is $-(-6)$, whatever that is. (Again, we will soon find out.) Every number—positive, negative, or zero—has a negation.

Problems

Problem 1.14 Jump to Solution

Consider the negation facts below:

$$\begin{aligned}
&\text{The negation of } 4 \text{ is } -4.\\
&\text{The negation of } 3 \text{ is } -3.\\
&\text{The negation of } 2 \text{ is } -2.\\
&\text{The negation of } 1 \text{ is } -1.\\
&\text{The negation of } 0 \text{ is } ___.\\
&\text{The negation of } -1 \text{ is } ___.\\
&\text{The negation of } -2 \text{ is } ___.
\end{aligned}$$

(a) Consider where the numbers at the ends of the facts above are on the number line. After the first number, how is each number related to the number above it?

(b) Based on the pattern from part (a), what is the negation of 0?

(c) Based on the pattern from part (a), what is the negation of -1?

(d) Based on the pattern from part (a), what is the negation of -2?

Problem 1.17 Jump to Solution

Consider the multiplication facts below:

$$\begin{aligned}
&1 \cdot 3 = 12\\
&3 \cdot 3 = 9\\
&2 \cdot 3 = 6\\
&1 \cdot 3 = 3\\
&0 \cdot 3 = 0\\
&(-1) \cdot 3 = ___.
\end{aligned}$$

(a) Consider where the numbers at the ends of the facts above are on the number line. After the first number, how is each number related to the number above it?

(b) Based on the pattern from part (a), what is the negation of -1?

(c) Based on the pattern from part (a), what is the negation of -2?

Every number has a negation. What's the negation of zero? What's the negation of a negative number? Let's find out.

Problem 1.16

Consider the negation facts below:

$$\begin{aligned}
&\text{The negation of } 4 \text{ is } -4.\\
&\text{The negation of } 3 \text{ is } -3.\\
&\text{The negation of } 2 \text{ is } -2.\\
&\text{The negation of } 1 \text{ is } -1.\\
&\text{The negation of } 0 \text{ is } ___.\\
&\text{The negation of } -1 \text{ is } ___.\\
&\text{The negation of } -2 \text{ is } ___.
\end{aligned}$$

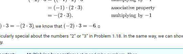
(a) Consider where the numbers at the ends of the facts above are on the number line. After the first number, how is each number related to the number above it?

(b) Based on the pattern from part (a), what is the negation of 0?

(c) Based on the pattern from part (a), what is the negation of -1?

(d) Based on the pattern from part (a), what is the negation of -2?

Solution for Problem 1.16:



(a) The numbers at the ends of the facts in the list are -4, -3, -2 and -1. After the first number in the list, each number is one unit to the right of the number before it.

(b) The pattern from part (a) suggests that the answer is 1 unit to the right of -1. That number is 0. So -0, the negation of 0, should be 0.

We can also see this from our definition of negation. The negation of 0 was defined as the number that we can add to 0 to get 0. That is, by our definition, the number -0 satisfies

$$-0 + 0 = 0.$$

But adding 0 to a number does nothing, so adding 0 to -0 gives -0:

$$-0 + 0 = -0.$$

Since -0 + 0 equals both -0 and 0, we know that -0 = 0.

(c) Starting from 0 and moving 1 unit to the right, we get to 1. So $-(-1)$ the negation of -1, is 1.

Again, we can use the definition of negation. We know that 1 and -1 sum to 0:

$$1 + (-1) = 0.$$

But we also know that $-(-1)$ is defined as the number whose sum with -1 is 0:

$$-(-1) + (-1) = 0.$$

Comparing the above two equations indicates that $-(-1) = 1$.

(d) Starting from 1 and moving 1 unit to the right, we get to 2. So $-(-2)$, the negation of -2, is 2. We also know that $2 + (-2) = 0$, and we must have $-(-2) + (-2) = 0$. So we conclude that $-(-2) = 2$.

Solution for Problem 1.17:

(a) The numbers at the ends of the facts in the list are 12, 9, 6, 3, and 0. On a number line, each number after the first is 3 units to the left of the number before it.

(b) The pattern from part (a) suggests that the answer is 3 units to the left of 0. That number is -3. So $(-1) \cdot 3 = -3$.

or

In words, multiplying a number by -1 appears to be the same as negating the number. This rule is handy. We can replace multiplication by -1 with negation if it helps us. We can also go the other way, replacing negation with multiplication—whichever way helps us solve a problem.

> **Important:** **Multiplying by** -1: Let x be a number. Then
>
> $$(-1)x = -x.$$

But just as before, a pattern is not a proof. So let's prove that $(-1)x$ is the negation of x. We can do this if we can show that

$$\begin{aligned}
&(-1)x + x = 0,
\end{aligned}$$

because if a number added to x gives 0, then our definition of negation says that the number is the negation of x. The clever idea here is to use that fact that $x = 1x$ (since multiplying by 1 doesn't change a number). Then, we have the following computation:

$$\begin{aligned}
&(-1)x + x\\
&= (-1)x + (1)x && \text{replacing } x \text{ by } (1)x\\
&= ((-1) + 1)x && \text{using the distributive property to factor } x\\
&= (0)x && \text{definition of the negation } -1\\
&= 0. && \text{multiplication by zero}
\end{aligned}$$

So indeed, $(-1)x$ is the negation of x, and therefore

$$(-1)x = -x.$$

We can use the fact that $-x = (-1)x$ to understand how to multiply by negative numbers.

Problems

Problem 1.18 Jump to Solution

Using the fact that $-x = (-1)x$, explain why

$$(-2) \cdot 3 = -(2 \cdot 3).$$

Problem 1.19 Jump to Solution

Using Problem 1.18, explain why

$$-2(-3) = 2 \cdot 3.$$

Problem 1.20 Jump to Solution

Using the fact that $-x = (-1)x$, explain why

$$-(4 + 5) = (-4) + (-5).$$

Problem 1.18

Using the fact that $-x = (-1)x$, explain why

$$(-2) \cdot 3 = -(2 \cdot 3).$$

Solution for Problem 1.18: We know that negation is the same as multiplying by -1, so let's use that fact twice:

$$\begin{aligned}
(-2) \cdot 3 &= ((-1) \cdot 2) \cdot 3 && \text{multiplying by } -1\\
&= (-1) \cdot (2 \cdot 3) && \text{associative property}\\
&= (-2 \cdot 3). && \text{multiplying by } -1
\end{aligned}$$

In particular, since $(-2) \cdot 3 = -(2 \cdot 3)$ we know that $(-2) \cdot 3 = -6$ $\checkmark$

There was nothing particularly special about the numbers "2" or "3" in Problem 1.18. In the same way, we can show that $(-x)y = -(xy)$ for any numbers x and y.

> **Important:** **Multiplying by negative:** Let x and y be numbers. Then
>
> $$\begin{aligned}(-x)y &= -(xy),\\ x(-y) &= -(xy).\end{aligned}$$

When x and y are positive, the equation $(-x)y = -(xy)$ says that a negative number times a positive number is negative, or "negative times positive is negative." By the commutative property, we can say the rule the other way around too: "positive times negative is negative."

How do we multiply a negative number by a negative number? Let's try an example.

Problem 1.19

Using Problem 1.18, explain why

$$-2(-3) = 2 \cdot 3.$$

Solution for Problem 1.19: We know how to multiply by a negative, so let's do that twice:

$$\begin{aligned}
-2(-3) &= -(2(-3)) && \text{multiplying by negative}\\
&= -(-(2 \cdot 3)) && \text{multiplying by negative}\\
&= 2 \cdot 3. && \text{negation of negation}
\end{aligned}$$

In particular, $-2(-3)$ is the positive number 6. $\checkmark$

In the same way, this works for any numbers x and y:

> **Important:** **Negation times negation:** Let x and y be numbers. Then
>
> $$(-x)(-y) = xy.$$

When x and y are positive, this equation says that a negative number times a negative number is positive. Students are often taught the chant "negative times negative is positive." Now you know why the chant is true!

How do we add negatives? Let's find out in the next problem.

Problem 1.20

Using the fact that $-x = (-1)x$, explain why

$$-(4 + 5) = (-4) + (-5).$$

Solution for Problem 1.20: Let's use the fact that negation is the same as multiplying by -1:

$$\begin{aligned}
-(4 + 5) &= -1(4 + 5) && \text{multiplying by } -1\\
&= (-1) \cdot 4 + (-1) \cdot 5 && \text{distributive property}\\
&= (-4) + (-5). && \text{multiplying by } -1
\end{aligned}$$

In particular, our work above tells us that $(-4) + (-5) = -9$. $\checkmark$

Similarly, the negation of $x + y$ equals $(-x) + (-y)$ in fancier words, negation distributes over addition.

> **Important:** **Negation of sum:** Let x and y be numbers. Then
>
> $$-(x + y) = (-x) + (-y).$$

Negation also distributes over longer sums. For example,

$$-(x + y + z) = (-x) + (-y) + (-z).$$

Exercises

1.4.1:

Compute $-631 + (114 + 631)$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: By rearranging,

$$\begin{aligned}
&-631 + (114 + 631) = (-631 + 631) + 114 = 0 + 114 = \boxed{114}.
\end{aligned}$$

1.4.2:

What is the sum of all of the negative integers that are greater than -5?

Preview: Solution

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Solution: The negative integers greater than -5 are -4, -3, -2, and -1. Their sum is

$$\begin{aligned}
&-4 + (-3) + (-2) + (-1) = -(4 + 3 + 2 + 1) = \boxed{-10}.
\end{aligned}$$

1.4.3:

What is the sum

$$\begin{aligned}
&-10 + (-9) + (-8) + \cdots + 9 + 10 + 11 + 12?
\end{aligned}$$

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Solution: Let's group numbers with their negatives:

$$\begin{aligned}
&-10 + (-9) + (-8) + \cdots + 9 + 10 + 11 + 12\\
&= (-10 + 10) + (-9 + 9) + \cdots + (-1 + 1) + 0 + 11 + 12\\
&= \underbrace{0 + 0 + \cdots + 0}_{10 \text{ zeros}} + 0 + 11 + 12\\
&= 0 + 0 + 11 + 12 = 11 + 12 = \boxed{23}.
\end{aligned}$$

1.4.4: Source: MATHCOUNTS

What is the value of $210 \cdot 5 + 105 \cdot (-9)$?

You may type any additional notes you have here.

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Reset

Your Submission: Solution

Solution:

$$\begin{aligned}
&210 \cdot 5 + 105 \cdot (-9) = 105 \cdot 2 \cdot 5 + 105 \cdot (-9)\\
&= 105 \cdot 10 + 105 \cdot (-9)\\
&= 105(10 + (-9))\\
&= 105 \cdot 1\\
&= \boxed{105}.
\end{aligned}$$

1.4.5: Source: MATHCOUNTS

What is

$$\begin{aligned}
&9342 + (-438)719 + (-9340) + (-436)(-719)?
\end{aligned}$$

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The two terms with 438 and 719 add up to zero:

$$\begin{aligned}
&9342 + (-438)719 + (-9340) + (-436)(-719)\\
&= 9342 + (-438 \cdot 719) + (-9340) + 438 \cdot 719\\
&= (9342 + (-9340)) + (-438 \cdot 719) + 438 \cdot 719)\\
&= (9342 - 9340) + 0 = 9342 - 9340 = \boxed{2}.
\end{aligned}$$

1.5 Subtraction

Also so we may be a subtraction such as $7 - 29$ of course, you know that $9 - 7 = 2$ but in order to generalize subtraction we want more complicated numbers such as negative or fractions, it will be useful to define subtraction as a combination of addition and negation. The idea is that instead of thinking about $7 - 29$ we can think about 7 plus the opposite of 29 and this combination is the closer to addition and negation.

Definition: Let x and y be any numbers. Then subtraction $x - y$ is pronounced "x minus y" and is defined as

$$x - y := x + (-y)$$

The subtraction $x - y$ is sometimes called the **difference** $x - y$.

In other words:

Subtracting y is better means adding its opposite.

For instance $3 - 5$ equals $3 + (-5)$ using our new definition. And if the subtraction is going to make any sense, we had better hope this equals -2. We can check this easily using the rules for addition. $3 + (-5)$ equals $-5 + 3$ which equals -2 as we hoped.

$$\begin{aligned} 3 - 5 &= 3 + (-5) && \text{definition of subtraction} \\ &= -5 + 3 && \text{we have } 3 \text{ and } -5 \\ &= 3 + (-5) && \text{commutative property} \\ &= -2 && \text{addition rule} \end{aligned}$$

Hence for new definition of subtraction and adding we can already know about subtracting positive integers! Specifically, using the definition, $5 - 2 = 3$ as we ought to since 5 and 2 3. 2 is one step to 3 and 5 is 2 in other words. $7 - 3$ is the number that fits the blank in the addition sentence $3 + __ = 7$.

$$__ + 3 = 7$$

So we can think of subtraction as the reverse of addition. This is the right way to think about subtraction.

Important: Let x, y and z be numbers. If $x + y = z$ then $z - x = y$ and $z - y = x$.

Our goal is to explain how subtraction works with zero and with negative numbers, and to explain some of the other properties of subtraction.

Important: Negative and subtraction look like symbols. But, as we saw, they are different operations. Negative takes one number and gives its opposite. Subtraction takes two numbers and gives their difference. Even though negative and subtraction use the same symbol (the minus sign), you should distinguish in your mind between negative and subtraction.

Problems

Problem 1.21 Jump to Solution

Use the definition of subtraction to answer the following questions:

(a) What is $5 - 3$?
(b) What is $27 - 29$?
(c) What is $-3 - 5$?

Problem 1.22 Jump to Solution

Use the definition of subtraction to explain the following equations:

(a) $10 - 2 = 10 + (-2) = 8$.
(b) $3 - 5 = 3 + (-5) = -2$.
(c) $10 - 10 = 10 + (-10) = 0$.

Problem 1.23 Jump to Solution

Find numbers x and y such that $x - y$ is not equal to $y - x$.

Problem 1.24 Jump to Solution

Find numbers x, y and z such that $(x - y) - z$ is not equal to $x - (y - z)$.

Problem 1.25 Jump to Solution

What is the value of $2\pi(2 - 2\pi)$ + $2\pi(2 - 2\pi)$?

Problem 1.26 Jump to Solution

What is the value of

$$2\pi + (2 - 4\pi) + 2\pi + (2 - 4\pi) = 0 \text{ ?}$$

Source: WOWC by Stanford

Problem 1.27 Jump to Solution

Source: WOWC by Stanford

(a) Explain why $(29 - 35) + 35 = 29 - 35 + 35$.
(b) Compute $29 - 35 + 35$.
(c) $29 - 35 + 35 = (29 - 35) + 35$.

Problem 1.28 Jump to Solution

Find the value for $(29 - 35) + 35$ in the problem WWC5(35).

Use the problem conditions to our choices about how subtract between with 0.

Problem 1.29 Jump to Solution

Use the definition of subtraction to answer the following questions:

(a) What is $5 - 0$?
(b) What is $-3 - 0$?
(c) What is $27 - 0$?

Problem 1.30 Jump to Solution

Use the definition of subtraction to explain the following equations:

(a) $10 - 0 = 10 + (-0) = 10$.
(b) $3 - 0 = 3 + (-0) = 3$.
(c) $10 - 0 = 10 + (-0) = 10$.

Find numbers x and y such that $x - y$ is not equal to $y - x$.

Find numbers x, y and z such that $(x - y) - z$ is not equal to $x - (y - z)$.

What is the value of $2\pi(2 - 2\pi)$ + $2\pi(2 - 2\pi)$?

What is the value of

$$2\pi + (2 - 4\pi) + 2\pi + (2 - 4\pi) = 0 \text{ ?}$$

Source: WOWC by Stanford

Explain why $(29 - 35) + 35 = 29 - 35 + 35$.

Compute $29 - 35 + 35$.

$29 - 35 + 35 = (29 - 35) + 35$.

Find the value for $(29 - 35) + 35$ in the problem WWC5(35).

Use the problem conditions to our choices about how subtract between with 0.

Problem 1.31 Jump to Solution

Use the definition of subtraction to explain the following equations:

(a) $10 - 10 = 10 + (-10) = 0$.
(b) $3 - 3 = 3 + (-3) = 0$.
(c) $10 - 10 = 10 + (-10) = 0$.

Find the value for $(29 - 35) + 35$ in the problem WWC5(35).

Use the problem conditions to our choices about how subtract between with 0.

Problem 1.32 Jump to Solution

Use the definition of subtraction to answer the following questions:

(a) What is $5 - (-3)$?
(b) What is $-3 - (-5)$?
(c) What is $27 - (-3)$?

Use the definition of subtraction to explain the following equations:

(a) $10 - (-2) = 10 + (-(-2)) = 12$.
(b) $3 - (-5) = 3 + (-(-5)) = 8$.
(c) $10 - (-3) = 10 + (-(-3)) = 13$.

Find numbers x and y such that $x - y$ is not equal to $y - x$.

Find numbers x, y and z such that $(x - y) - z$ is not equal to $x - (y - z)$.

What is the value of $2\pi(2 - 2\pi)$ + $2\pi(2 - 2\pi)$?

What is the value of

$$2\pi + (2 - 4\pi) + 2\pi + (2 - 4\pi) = 0 \text{ ?}$$

Source: WOWC by Stanford

Explain why $(29 - 35) + 35 = 29 - 35 + 35$.

Compute $29 - 35 + 35$.

$29 - 35 + 35 = (29 - 35) + 35$.

Find the value for $(29 - 35) + 35$ in the problem WWC5(35).

Use the problem conditions to our choices about how subtract between with 0.

Problem 1.33 Jump to Solution

Source: WOWC by Stanford

(a) Explain why $(29 - 35) + 35 = 29 - 35 + 35$.
(b) Compute $29 - 35 + 35$.
(c) $29 - 35 + 35 = (29 - 35) + 35$.

Find the value for $(29 - 35) + 35$ in the problem WWC5(35).

Use the problem conditions to our choices about how subtract between with 0.

Problem 1.34 Jump to Solution

Source: WOWC by Stanford

What is the value of $2\pi(2 - 2\pi)$ + $2\pi(2 - 2\pi)$?

What is the value of

$$2\pi + (2 - 4\pi) + 2\pi + (2 - 4\pi) = 0 \text{ ?}$$

Explain why $(29 - 35) + 35 = 29 - 35 + 35$.

Compute $29 - 35 + 35$.

$29 - 35 + 35 = (29 - 35) + 35$.

Find the value for $(29 - 35) + 35$ in the problem WWC5(35).

Use the problem conditions to our choices about how subtract between with 0.

Problem 1.35 Jump to Solution

Source: WOWC by Stanford

(a) Explain why $(29 - 35) + 35 = 29 - 35 + 35$.
(b) Compute $29 - 35 + 35$.
(c) $29 - 35 + 35 = (29 - 35) + 35$.

Find the value for $(29 - 35) + 35$ in the problem WWC5(35).

Use the problem conditions to our choices about how subtract between with 0.

Exercises

1.5.1. Jump to Solution

What is the sum of $(90.30 - 23.5) - (62.23 - 90)$?

Preview Solution

Hide Solution

Hint

You may type any additional notes you have here

Hide Solution

Hint

Your Submission: Solution

Solution: Subtracting a number from itself gives zero, so

$$(90.30 - 23.5) - (62.23 - 90) = (90 - 90) + (90.30 - 23.5) - (62.23 - 90)$$

1.5.2. Jump to Solution

Compute $(2000 + 1999) + (2000 - 1999)$.

Preview Solution

Hide Solution

Hint

You may type any additional notes you have here

Hide Solution

Hint

Your Submission: Solution

Solution: Rearranging the sum gives you,

$$(2000 + 1999) + (2000 - 1999) = (2000 + 2000) + (1999 - 1999)$$

1.5.3. Jump to Solution

Source: MATH1000FS

The city of Alexandria had a high temperature of 70° and a low temperature of 7° on the same day. So, how many degrees did the high temperature exceed the low temperature?

Illustration for Problem 1.

Having discussed the commutative and associative properties, we now turn to the distributive property. Remember that multiplication distributes over addition. Similar distributive property also holds for subtraction too! The next problem is a test case.

Problem 1.4 Jump to Solution

Source: MATH1000FS

(a) Explain why $(70 - 35) \times 4 = 70 \times 4 - 35 \times 4$.
(b) Compute $(70 - 35) \times 4$.

Find the value for $70 \times 4 - 35 \times 4$ in the problem WWC5(35).

Use the distributive property (one addition) to compute the value of $(70 - 35) \times 4$.

Use the distributive property (two additions) to compute the value of $(70 - 35) \times 4$.

(a) Explain why $(70 - 35) \times 4 = 70 \times 4 - 35 \times 4$.
(b) Compute $(70 - 35) \times 4$.

Find the value for $70 \times 4 - 35 \times 4$ in the problem WWC5(35).

Use the distributive property (one addition) to compute the value of $(70 - 35) \times 4$.

Use the distributive property (two additions) to compute the value of $(70 - 35) \times 4$.

Also Center Jurisdiction distributes over any combination of additions and subtractions. For example, we can expand $(70 - 7 - 4) \times 4$ into $(70 - 7 - 4) \times 4$.

The last problem have a surprising use of the distributive property.

Problem 1.5 Jump to Solution

Source: MATH1000FS

Use the fact that $999 = 1000 - 1$ to evaluate WWC5(1).

Answer the Problem 1.6 as proposed by the problem, we can write WWC5(1) as $1000 - 1$.

$$999 - 345 = (1000 - 1)(345)$$

Use the distributive property (one subtraction) to compute WWC5(1).

$$999 - 345 = 1000(345) - (1)(345),$$

Note now difference is just $(1)(345) = 345$, which is $999(345)$.

Illustration for Problem 1.5 Distributive Property.

Exercises

1.5.1. Jump to Solution

What is the sum of $(90.30 - 23.5) - (62.23 - 90)$?

Preview Solution

Hide Solution

Hint

You may type any additional notes you have here

Hide Solution

Hint

Your Submission: Solution

Solution: Subtracting a number from itself gives zero, so

$$(90.30 - 23.5) - (62.23 - 90) = (90 - 90) + (90.30 - 23.5) - (62.23 - 90)$$

1.5.2. Jump to Solution

Compute $(2000 + 1999) + (2000 - 1999)$.

Preview Solution

Hide Solution

Hint

You may type any additional notes you have here

Hide Solution

Hint

Your Submission: Solution

Solution: Rearranging the sum gives you,

$$(2000 + 1999) + (2000 - 1999) = (2000 + 2000) + (1999 - 1999)$$

1.5.3. Jump to Solution

Source: MATH1000FS

The city of Alexandria had a high temperature of 70° and a low temperature of 7° on the same day. So, how many degrees did the high temperature exceed the low temperature?

Illustration for Problem 1.

Having discussed the commutative and associative properties, we now turn to the distributive property. Remember that multiplication distributes over addition. Similar distributive property also holds for subtraction too! The last problem is a test case.

Problem 1.4 Jump to Solution

Source: MATH1000FS

(a) Explain why $(70 - 35) \times 4 = 70 \times 4 - 35 \times 4$.
(b) Compute $(70 - 35) \times 4$.

Find the value for $70 \times 4 - 35 \times 4$ in the problem WWC5(35).

Use the distributive property (one addition) to compute the value of $(70 - 35) \times 4$.

Use the distributive property (two additions) to compute the value of $(70 - 35) \times 4$.

(a) Explain why $(70 - 35) \times 4 = 70 \times 4 - 35 \times 4$.
(b) Compute $(70 - 35) \times 4$.

Find the value for $70 \times 4 - 35 \times 4$ in the problem WWC5(35).

Use the distributive property (one addition) to compute the value of $(70 - 35) \times 4$.

Use the distributive property (two additions) to compute the value of $(70 - 35) \times 4$.

1.5.4. Jump to Solution

Evaluate $5 - (-3) - (-5)$.

Preview Solution

Hide Solution

Hint

You may type any additional notes you have here

Hide Solution

Hint

Your Submission: Solution

Solution:

$$\begin{aligned} 5 - (-3) - (-5) &= 5 + (-(-3)) + (-(-5)) \\ &= 5 + (-(-3)) \\ &= 5 + 3 \\ &= 8 \\ &= -(-5) \\ &= 5 \end{aligned}$$

1.5.5. Jump to Solution

Find all positive numbers such that $y - x$ is equal to the value of $y - x$?

Preview Solution

Hide Solution

Hint

You may type any additional notes you have here

Hide Solution

Hint

Your Submission: Solution

Solution: $y - x = 2$ Hence $y = x + 2$ or $y = x - 2$.

1.5.6. Jump to Solution

Source: MATH1000FS

$$200 - 1 \quad 200 - 2 \quad 200 - 3 \quad 200 - 4 \quad 200 - 5 \quad 200 - 6 \quad 200 - 7 \quad 200 - 8 \quad 200 - 9 \quad 200 - 10$$

You may type any additional notes you have here

Hide Solution

Hint

Your Submission: Solution

Solution: We can't place the added parts first followed by the subtracted parts.

$$\begin{aligned} &(200 - 1) + 200 - 2 + 200 - 3 + 200 - 4 + 200 - 5 \\ &+ (200 + 200 + 200 + 200 + 200) - 6 - 7 - 8 - 9 - 10 \\ &= (10 - 2) + 2 + 3 + 4 + 5 + 10 + 10 + 10 + 10 + 10 \end{aligned}$$

1.5.7. Jump to Solution

Source: MATH1000FS

Compute

$$(1000 + 1000) + (1000 - 1000) + (1000 - 1000) + (1000 - 1000)$$

Preview Solution

Hide Solution

Hint

You may type any additional notes you have here

Hide Solution

Hint

Your Submission: Solution

Solution: We compute the number three times and we can't pair the numbers in the sum with number in the first part with a number in the second part.

$$\begin{aligned} &(1000 + 1000) + (1000 - 1000) + (1000 - 1000) + (1000 - 1000) \\ &= (1000 + 1000) + (1000 - 1000) + (1000 - 1000) + (1000 - 1000) \\ &= (1000 + 1000) + (1000 - 1000) + (1000 - 1000) + (1000 - 1000) \\ &= 2000 + 0 + 0 + 0 \\ &= 2000 \end{aligned}$$

1.5.10. Jump to Solution

Compute $999 - 345$ in your head.

Preview Solution

Hide Solution

Hint

You may type any additional notes you have here

Hide Solution

Hint

Your Submission: Solution

Solution: Let's rewrite this as $1000 - 1$.

$$\begin{aligned} 999 - 345 &= (1000 - 1) - 345 \\ &= 1000 - (1 + 345) \\ &= 1000 - 346 \\ &= 654 \end{aligned}$$

1.5.11. Jump to Solution

Source: MATH1000FS

The sum of the first 10,000 positive even numbers is two more than the sum of the first 10,000 positive odd numbers?

Preview Solution

Hide Solution

Hint

You may type any additional notes you have here

Hide Solution

Hint

Your Submission: Solution

Solution: The sum of the first 10,000 positive even numbers is

$$2 + 4 + 6 + \cdots + 19,998 + 20,000.$$

The sum of the first 10,000 positive odd numbers is

$$1 + 3 + 5 + \cdots + 19,997 + 19,999.$$

The first sum minus the second sum is

$$\begin{aligned} &(2 - 1) + (4 - 3) + (6 - 5) + \cdots + (19,998 - 19,997) \\ &+ 20,000 - 19,999 \\ &= 1 + 1 + 1 + \cdots + 1 \\ &= 10,000 \end{aligned}$$

1.6 Reciprocals

We learned back in Section 1.2 here that adding 0 to a number doesn't change the number:

$$0 + x = x$$

for any x. For this reason, we call 0 the **identity** for addition, in Section 1.4 we also learned that for any number x, there is a number called the negation of x (written $-x$) whose sum with x is this identity, 0. That is, $-x$ is the number that goes in the blank in the equation

$$___ + x = 0.$$

Does multiplication have an identity? It isn't 0, since multiplying any number by 0 gives 0, not the original number. Of course, 1 is the number we seek:

$$1 \cdot x = x$$

for any number x; so 1 is the identity for multiplication.

But the question we really want to ask is: given any number x, what number can we put in the blank to solve the equation

$$___ \cdot x = 17$$

The answer to this question is called the **reciprocal** of x.

> **Definition:** For any number x, the **reciprocal** of x, written as $\frac{1}{x}$, is the number such that
>
> $$\frac{1}{x} \cdot x = 1.$$
>
> This number is also called the **multiplicative inverse** of x, and we can say "$\frac{1}{x}$ over x."

Of course, you've seen this sort of number before: $\frac{1}{2}$ looks like a fraction. If $x = 2$, then $\frac{1}{2}$ is the fraction one-half; if $x = 3$, then $\frac{1}{3}$ is the fraction one-third, and so on. Later, in Chapter 4, we'll see that reciprocals of positive integers are exactly the same thing as the fractions that you already know. However, for now, we'll just treat a reciprocal as a new kind of number—a number that "magically" multiplies with another number to give a product of 1. In this section, we figure out some properties of reciprocals, and then in Chapter 4, we will use these properties to learn about fractions.

We're going to assume that every number, except for 0, has a reciprocal. However, the number 0 cannot have a reciprocal. We will see why in our first problem.

Problems

Problem 1.29 Jump to Solution

(a) What is the product of 0 and any number?

(b) Using part (a), explain why 0 doesn't have a reciprocal.

Problem 1.30 Jump to Solution

Explain why the reciprocal of 1 is 1.

Problem 1.31 Jump to Solution

(a) What is the product of $\frac{1}{2}$ and 2?

(b) Using part (a), explain why the reciprocal of $\frac{1}{2}$ is 2.

Problem 1.32 Jump to Solution

(a) What is the product of $5 \cdot 7$ and $\frac{1}{5} \cdot \frac{1}{7}$?

(b) Using part (a), explain why the reciprocal of $5 \cdot 7$ is $\frac{1}{5} \cdot \frac{1}{7}$.

Problem 1.33 Jump to Solution

(a) What is the product of -8 and $-\frac{1}{8}$?

(b) Using part (a), explain why the reciprocal of -8 is $-\frac{1}{8}$.

We'd like to assume that every number has a reciprocal. But we can't do this, because the number 0 is special:

Problem 1.29

(a) What is the product of 0 and any number?

(b) Using part (a), explain why 0 doesn't have a reciprocal.

Solution for Problem 1.29:

(a) We know that 0 times any number is 0. (If you've forgotten why this is true, go back and review Section 1.3 here.)

(b) If the number $\frac{1}{0}$ existed, then $\frac{1}{0} \cdot 0$ would have to equal 1. But recall part (a): the product of any number and 0 must be 0. So $\frac{1}{0}$ cannot exist! Therefore, 0 can't have a reciprocal.

□

Problem 1.29 tells us that we have to be careful with reciprocals. A number must be nonzero in order to have a reciprocal.

> **WARNING!!** The reciprocal of 0 is undefined.

Every nonzero number does in fact have a reciprocal. Let's now look at some properties of reciprocals. The "nicest" number for multiplication is 1, so it makes sense to first look at $\frac{1}{1}$.

Problem 1.30

Explain why the reciprocal of 1 is 1.

Solution for Problem 1.30: By definition, the reciprocal of 1 is the number that goes in the blank to solve the equation

$$___ \cdot 1 = 1.$$

But we know that $1 \cdot 1 = 1$. So the reciprocal of 1 is 1. □

Problem 1.30 confirms the "obvious" fact that $\frac{1}{1} = 1$. We also saw a general strategy for dealing with reciprocals:

> **Concept:** **Reciprocal strategy:** To show that two numbers are reciprocals of each other, multiply them and check that their product is 1.

Remember that the negation of the negation of a number is the original number. Is there a similar result for reciprocals?

Problem 1.31

(a) What is the product of $\frac{1}{2}$ and 2?

(b) Using part (a), explain why the reciprocal of $\frac{1}{2}$ is 2.

Solution for Problem 1.31:

(a) By the definition of reciprocal, the product of $\frac{1}{2}$ and 2 is 1.

(b) From part (a), we see that the number that fills in the blank to solve

$$___ \cdot \frac{1}{2} = 1$$

is 2. But that number, by definition, is the reciprocal of $\frac{1}{2}$. So the reciprocal of $\frac{1}{2}$ is 2.

□

We just showed that the reciprocal of the reciprocal of 2 is 2. In the same way, for every nonzero number x, we can show that the reciprocal of $\frac{1}{x}$ is x: we can write that property as

$$\frac{1}{\left(\frac{1}{x}\right)} = x.$$

> **Important:** **Reciprocal of reciprocal:** Let x be a nonzero number. Then $\frac{1}{x}$ is nonzero and its reciprocal is x.

How do we multiply reciprocals? The next problem shows how.

Problem 1.32

(a) What is the product of $5 \cdot 7$ and $\frac{1}{5} \cdot \frac{1}{7}$?

(b) Using part (a), explain why the reciprocal of $5 \cdot 7$ is $\frac{1}{5} \cdot \frac{1}{7}$.

Solution for Problem 1.32:

(a) Let's multiply:

$$\begin{aligned} &(5 \cdot 7)\left(\frac{1}{5} \cdot \frac{1}{7}\right) && \text{commutative and associative properties} \\ &= \left(\frac{1}{5} \cdot 5\right)\left(\frac{1}{7} \cdot 7\right) && \text{definition of reciprocal (twice)} \\ &= 1 \cdot 1 && \text{multiplying by 1} \\ &= 1. \end{aligned}$$

So the product is 1.

(b) From part (a), we see that $5 \cdot 7$ and $\frac{1}{5} \cdot \frac{1}{7}$ are reciprocals of each other because their product is 1. So the reciprocal of $5 \cdot 7$ is $\frac{1}{5} \cdot \frac{1}{7}$. As an equation, we have

$$\frac{1}{5} \cdot \frac{1}{7} = \frac{1}{5 \cdot 7}.$$

□

There was nothing special about the numbers 5 and 7 in Problem 1.32—the same result holds for any two nonzero numbers. We conclude that the reciprocal of a product is the product of reciprocals.

> **Important:** **Reciprocal of product:** Let x and y be nonzero numbers. Then xy is nonzero and its reciprocal is $\frac{1}{x} \cdot \frac{1}{y}$. That is,
>
> $$\frac{1}{x} \cdot \frac{1}{y} = \frac{1}{xy}.$$

A similar result holds for longer products. For example,

$$\frac{1}{3} \cdot \frac{1}{5} \cdot \frac{1}{7} = \frac{1}{3 \cdot 5 \cdot 7} = \frac{1}{105}.$$

How do we find the reciprocal of a negative number? The next problem shows the way.

Problem 1.33

(a) What is the product of -8 and $-\frac{1}{8}$?

(b) Using part (a), explain why the reciprocal of -8 is $-\frac{1}{8}$.

Solution for Problem 1.33:

(a) Let's multiply the two numbers:

$$\begin{aligned} \left(\frac{-1}{8}\right) \cdot (-8) &= \frac{1}{8} \cdot 8 && \text{negation times negation} \\ &= 1. && \text{reciprocal property} \end{aligned}$$

So the product is 1.

(b) From part (a), -8 and $-\frac{1}{8}$ are reciprocals of each other. So the reciprocal of -8 is $-\frac{1}{8}$, as an equation,

$$\frac{1}{-8} = -\frac{1}{8}.$$

□

In a similar way, we can take the reciprocal of any negative. The reciprocal of a negation is the negation of the reciprocal.

> **Important:** **Reciprocal of negation:** Let x be a nonzero number. Then
>
> $$\frac{1}{-x} = -\frac{1}{x}.$$

What's a Reciprocal?

Exercises

1.6.1:

What is the reciprocal of -2?

Preview: Solution

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Your Submission: Solution

Solution: By the "reciprocal of negation" property, and because the reciprocal of 1 is 1, we have

$$\frac{1}{-1} = -\frac{1}{1} = \boxed{-1}.$$

1.6.2:

What number is *not* the reciprocal of any number?

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Solution: The only number that is not a reciprocal is $\boxed{0}$. Zero times any number is 0, not 1, so zero can't be the reciprocal of a number.

In contrast, any number besides zero is a reciprocal. Namely, if x is a nonzero number, then x is the reciprocal of $\frac{1}{x}$.

1.6.3:

What is the product of any nonzero number and twice its reciprocal?

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Solution: Let x be the nonzero number. Twice its reciprocal is $2 \cdot \frac{1}{x}$. So the product of x and twice its reciprocal is

$$x\left(2 \cdot \frac{1}{x}\right) = 2\left(\frac{1}{x} \cdot x\right) = 2 \cdot 1 = \boxed{2}.$$

1.6.4:

Multiply the negation of a positive number by the reciprocal of that same positive number. What is the product?

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Solution: Let x be the positive number. When we multiply the reciprocal of x by the negation of x, we get

$$\frac{1}{x}(-x) = -\left(\frac{1}{x} \cdot x\right) = \boxed{-1}.$$

1.6.5: Source: AMC 8

Compute $(2 \cdot 3 \cdot 4)\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$.

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Your Submission: Solution

Solution: Let's use the distributive law:

$$\begin{aligned} &(2 \cdot 3 \cdot 4)\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \\ &= (2 \cdot 3 \cdot 4)\frac{1}{2} + (2 \cdot 3 \cdot 4)\frac{1}{3} + (2 \cdot 3 \cdot 4)\frac{1}{4} \\ &= \left(\frac{1}{2} \cdot 2\right)(3 \cdot 4) + \left(\frac{1}{3} \cdot 3\right)(2 \cdot 4) + \left(\frac{1}{4} \cdot 4\right)(2 \cdot 3) \\ &= 1(3 \cdot 4) + 1(2 \cdot 4) + 1(2 \cdot 3) \\ &= 3 \cdot 4 + 2 \cdot 4 + 2 \cdot 3 \\ &= 12 + 8 + 6 \\ &= \boxed{26}. \end{aligned}$$

1.7 Division

Sometimes people find it hard to do division. Here is a useful way to think about it: $\frac{a}{b} = c$ means that $b \times c = a$. If we want to know what c is, we ask "What times b equals a?" Sometimes we can do division in our head. Sometimes we need to do it on paper. Sometimes we need a calculator or computer.

Definition: Let a and b be numbers such that b is not zero. Then the **quotient** $a \div b$ (pronounced "a divided by b") is defined by:

$$a \div b = c \iff b \times c = a$$

In other words, dividing by a number means multiplying by its reciprocal. For instance, $10 \div 2$ equals $10 \cdot \frac{1}{2}$. But for the definition to be correct, we need to be careful about what we mean by "reciprocal" when we are dealing with numbers other than 1 and 0.

IMPORTANT: Division by zero is undefined. For instance, $1 \div 0$ is undefined because there is no number c such that $0 \times c = 1$. It is not enough that the reciprocal of 0 is undefined.

SUPPORT: You can't divide by zero. Division by zero is undefined. Before dividing by a number, be sure that the number is nonzero.

Problems

Problem 1.7.1 Using the definition

Using our definition of division and the fact that $\frac{1}{2} \cdot 6 = 3$, explain why $\frac{1}{2} \div 3 = \frac{1}{6}$.

Problem 1.7.2 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Problem 1.7.3 Using the definition

(a) Find positive numbers a and b such that $a \div b$ is not equal to $b \div a$.

(b) Find positive numbers a, b, and c such that $a \div (b \div c)$ is not equal to $(a \div b) \div c$.

Problem 1.7.4 Using the definition

Source: AMC8AMC 8 Problem 16

What is

$$\frac{20 \cdot 20 \cdot 20 \cdot 20 \cdot 20}{20 \cdot 20 \cdot 20 \cdot 20 \cdot 20}?$$

Problem 1.7.5 Using the definition

Compute $(1.96) \div (2.00)$. (R1)

Problem 1.7.6 Using the definition

(a) Using the definition of division, explain why

$$(12) \div (4) \neq (3) \div (4) \div (3).$$

(b) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

Problem 1.7.7 Using the definition

Use the exercise on division of numbers by dividing that our formal definition of subtraction gives us the expected result $9 - 2 = 7$.

Important: Let a, b, and c be nonzero numbers. Prove that $a \div (b \div c) = (a \div b) \div c$.

Problem 1.7.8 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.8

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.9 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.9

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.10 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.11 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.11

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.12 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.13 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.13

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.14 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.15 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.15

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.16 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.17 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.17

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.18 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.19 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.19

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.20 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.21 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.21

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.22 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.23 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.23

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.24 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.25 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.25

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.26 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.27 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.27

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.28 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.29 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.29

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.30 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.31 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.31

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.32 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.33 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.33

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.34 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Problem 1.7.35 Using the definition

Let a be any nonzero number. Use the definition of division to answer the following questions.

(a) What is $0 \div a$?

(b) What is $a \div a$?

(c) What is $a \div 1$?

(d) What is $a \div 0$?

Solution for Problem 1.7.35

(a) Remember that division is multiplication by the reciprocal.

$$\begin{aligned} 0 \div a &= 0 \cdot \frac{1}{a} \\ &= 0. \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} a \div a &= a \cdot \frac{1}{a} \\ &= 1. \end{aligned}$$

(c) Again, in a similar way,

$$\begin{aligned} a \div 1 &= a \cdot \frac{1}{1} \\ &= a. \end{aligned}$$

(d) Now, once, you have

$$\begin{aligned} 1 \div a &= 1 \cdot \frac{1}{a} \\ &= \frac{1}{a}. \end{aligned}$$

So, the result of dividing a nonzero number a into 1 is the reciprocal of a.

Important: Let a be any number.

Division by zero: $a \div 0$ is not defined if a is nonzero. Part (d) is

Self-division: $0 \div 0$ is indeterminate. Part (b) is

Dividing by 1: $a \div 1 = a$. Part (c) is

Dividing by 0: $0 \div a$ is nonzero. Part (a) is

Feel free to use these in your homework and in class.

Problem 1.7.36 Using the definition

(a) $\div(1 \div (2) \div (3)) \neq \div(1) \div (2) \div (3)$

(b) $\div(1 \div (2) \div (3)) \neq (1) \div (2) \div (3)$

Exercises

1.7.1

Simplify

$$(1 \div 2) \div (3 \div 4) = ((1 \div 2) \div 3) \div (4 \div 3)$$

Your Step-by-step additional notes go here here

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Solution: Dividing a nonzero number by itself gives 1.

$$\begin{aligned} (1 \div 2) \div (3 \div 4) &= ((1 \div 2) \div 3) \div (4 \div 3) \\ &= (1 \div 2) \div (3 \div 4) \\ &= ((1 \div 2) \div 3) \div (4 \div 3) \end{aligned}$$

1.7.2

In the form $\frac{(a)(b)}{(c)}$, (R1), what is the result?

$$\frac{(20)(-20)(-20)(-20)(-20)}{(20)(20)(20)(20)(20)}$$

Your Step-by-step additional notes go here here

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Solution: You can cancel a factor of 100 from both numbers.

$$\frac{(20)(-20)(-20)(-20)(-20)}{(20)(20)(20)(20)(20)} = \boxed{-1}$$

1.7.3

What is $\div(\div(100) \div (50))$?

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Solution: You can subtract a factor of 100 from both numbers.

$$\div(\div(100) \div (50)) = \boxed{-100 \div (-50)}$$

1.7.4

Simplify

$$(20)(20)(20)(20) \div (20)(20)(20)(20)$$

Your Step-by-step additional notes go here here

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Solution: The first addition is 10 times the second number, so the quotient is

$$\begin{aligned} &(20)(20)(20)(20) \div (20)(20)(20)(20) \\ &= (20)(20)(20)(20) \div (20)(20)(20)(20) \\ &= \boxed{1} \end{aligned}$$

1.7.5

What is $\div(\div(\div(1)))$?

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Solution: The exponent of $\div(\div(\div(1)))$ is $\boxed{-1}$.

1.7.6 Source: AMC8AMC 8 Problem 16

What number is 10 times the quotient when 70 is divided by $\frac{7}{5}$?

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Solution: The exponent of $\div(\div(\div(1)))$ is $\boxed{50}$.

1.7.7 Source: AMC8AMC 8 Problem 16

What is the value of $\frac{\frac{1}{2}}{\frac{3}{4}}$?

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Solution: Division is rather cumbersome to do, especially in a hard copy format. We can compute the divisions first with ease.

$$\begin{aligned} &\frac{\frac{1}{2}}{\frac{3}{4}} \\ &= \frac{\frac{1}{2}}{\frac{3}{4}} \\ &= \frac{\frac{1}{2}}{\frac{3}{4}} \\ &= \boxed{\frac{2}{3}} \end{aligned}$$

1.7.8

Simplify

$$(37 \cdot 37) \div (37 \cdot 37) \div (37 \cdot 37) \div (37 \cdot 37)$$

Your Step-by-step additional notes go here here

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Solution: We can cancel the common factor of 37, 36 times.

$$\begin{aligned} &(37 \cdot 37) \div (37 \cdot 37) \div (37 \cdot 37) \div (37 \cdot 37) \\ &= (37 \cdot 37) \div (37 \cdot 37) \div (37 \cdot 37) \div (37 \cdot 37) \\ &= 37 \div 37 \\ &= \boxed{1} \end{aligned}$$

1.7.9

Compute $(20 \cdot 20) \div (5 \cdot 5) \div (2 \cdot 2)$.

Your Step-by-step additional notes go here here

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Solution: We can cancel the common factor of 5, 20 times.

$$\begin{aligned} &(20 \cdot 20) \div (5 \cdot 5) \div (2 \cdot 2) \\ &= (20 \cdot 20) \div (5 \cdot 5) \div (2 \cdot 2) \\ &= (20 \div 5) \cdot (20 \div 5) \div (2 \cdot 2) \\ &= (20 \div 5) \cdot (20 \div 5) \div (2 \cdot 2) \\ &= (20 \div 5) \cdot (20 \div 5) \div (2 \cdot 2) \\ &= \boxed{100} \end{aligned}$$

1.7.10

Compute $(10, 20, 30) \div (5, 10, 15)$.

Your Step-by-step additional notes go here here

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Solution: $(10, 20, 30) \div (5, 10, 15)$

$$\begin{aligned} &= (10 \div 5, 20 \div 10, 30 \div 15) \\ &= (2, 2, 2) \\ &= \boxed{(2, 2, 2)} \end{aligned}$$

1.7.11

Compute

$$\begin{aligned} &(10 \div 20) \div (40 \div 50) \\ &= (10 \div 20) \div (40 \div 50) \\ &= (10 \div 20) \div (40 \div 50) \end{aligned}$$

Your Step-by-step additional notes go here here

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Solution: We can cancel the common factor of 10, 20 times.

$$\begin{aligned} &(10 \div 20) \div (40 \div 50) \\ &= (10 \div 20) \div (40 \div 50) \\ &= (10 \div 20) \div (40 \div 50) \\ &= (10 \div 20) \div (40 \div 50) \\ &= \boxed{1} \end{aligned}$$

1.8 Summary

In this chapter, we explored the basic arithmetic operations of addition, negation, subtraction, multiplication, reciprocation, and division. Our goal was to derive as many properties about these operations as possible.

Important:

Order of operations: Perform the operations in an expression in the following order.

1. Evaluate expressions inside parentheses first.
2. Compute powers. (We cover powers in Chapter 2.)
3. Multiply and divide from left to right.
4. Add and subtract from left to right.

We described that the following properties of arithmetic hold for all numbers:

Important:

Let a, b, and c be numbers.

Addition is commutative: $a + b = b + a$.
Addition is associative: $(a + b) + c = a + (b + c)$.
Adding zero: $a + 0 = a$.
Multiplication is commutative: $ab = ba$.
Multiplication is associative: $(ab)c = a(bc)$.
Multiplying by 1: $1a = a$.
Multiplication distributes over addition: $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.
Negation property: $-a + a = 0$.
Reciprocal property: If a is nonzero, then $\frac{1}{a} \cdot a = 1$.

These nine rules are the only assumptions that we need! Starting from these few properties, we can figure out a surprising number of other useful facts.

We defined subtraction and division in terms of addition and multiplication.

Definitions:

Let a and b be numbers.

Subtraction: $a - b = a + (-b)$.

Division: If b is nonzero, then $a \div b = a \cdot \frac{1}{b}$.

From these few basic properties and definitions, we proved dozens of other important properties of arithmetic. You shouldn't have to memorize these properties. Instead, you should be comfortable with *why* each of the following rules of arithmetic are true. If you aren't, go back and review the appropriate section of the chapter.

Important:

Let a, b, and c be numbers.

Negation of negation: $-(-a) = a$.
Negation of sum: $-(a + b) = (-a) + (-b)$.
Multiplying by zero: $0a = 0$.
Multiplying by -1: $(-1)a = -a$.
Multiplying by negation: $(-a)b = -(ab)$ and $a(-b) = -(ab)$.
Negation times negation: $(-a)(-b) = ab$.
Subtracting from zero: $0 - a = -a$.
Self subtraction: $a - a = 0$.
Subtracting zero: $a - 0 = a$.
Subtraction of negation: $a - (-b) = a + b$.
Subtraction from negation: $-a - b = -(a + b)$.
Negation of subtraction: $-(a - b) = b - a$.
Multiplication distributes over subtraction: $a(b - c) = ab - ac$ and $(b - c)a = ba - ca$.
Reciprocal of reciprocal: If a is nonzero, then the reciprocal of $\frac{1}{a}$ is a.
Reciprocal of negation: If a is nonzero, then $\frac{1}{-a} = -\frac{1}{a}$.
Reciprocal of product: If a and b are nonzero, then $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$.
Dividing into zero: If a is nonzero, then $0 \div a = 0$.
Self division: If a is nonzero, then $a \div a = 1$.
Dividing by 1: $a \div 1 = a$.
Dividing into 1: If x is nonzero, then $1 \div x = \frac{1}{x}$.
Dividing by reciprocal: If b is nonzero, then $a \div \frac{1}{b} = ab$.
Dividing into negation: If b is nonzero, then $(-a) \div b = -(a \div b)$.
Dividing by negation: If b is nonzero, then $a \div (-b) = -(a \div b)$.
Negation divided by negation: If b is nonzero, then $(-a) \div (-b) = a \div b$.
Cancel common factor: If a and c are nonzero, then $(ab) \div (ac) = b \div c$.
Division by a number distributes over addition: If c is nonzero, then $(a + b) \div c = a \div c + b \div c$.
Division by a number distributes over subtraction: If c is nonzero, then $(a - b) \div c = a \div c - b \div c$.

WARNING!!

Keep these arithmetic warnings in mind:

- Subtraction is neither commutative nor associative. That is, $a - b$ and $b - a$ are **not** necessarily equal, and neither are $a - (b - c)$ and $(a - b) - c$.
- You can't divide by zero. Division by zero is undefined. Before dividing by a number, be sure that the number is nonzero.

WARNING!!

Watch out for these, too:

- Division is neither commutative nor associative. That is, $a \div b$ and $b \div a$ are **not** necessarily equal, and neither are $a \div (b \div c)$ and $(a \div b) \div c$.
- We can use the distributive property to divide a sum by a number, but we can't use the distributive property to divide a number by a sum. In other words, $a \div (b + c)$ is **not** necessarily equal to $a \div b + a \div c$. Similarly, $a \div (b - c)$ is **not** the same as $a \div b - a \div c$.

Challenge Problems

1.62:

What is the sum of the first sixty-one positive integers?

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Solution: We know that we can sum an even number of consecutive integers by grouping the integers into pairs. So, we'll first add the first 60 positive integers, and then add 61 to the resulting sum. Grouping the first 60 integers into 30 pairs gives

$$\begin{aligned} 1+2+\cdots+59+60 &= (1+60)+(2+59)+\cdots+(30+31) \\ &= 61+61+\cdots+61 \quad \text{(30 clauses)} \\ &= 30\cdot 61 = 1830. \end{aligned}$$

So, the sum of the first 61 positive integers is $1830+61=\boxed{1891}$.

1.64:

What is the value of the sum $5+10+15+\cdots+95+100$?

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Your Submission: Solution

Solution: Let's group the 20 numbers into 10 pairs:

$$\begin{aligned} &5+10+15+\cdots+95+100 \\ &=(5+100)+(10+95)+(15+90)+\cdots+(50+55) \\ &=105+105+105+\cdots+105 \quad \text{(10 clauses)} \\ &=10\cdot 105=\boxed{1050}. \end{aligned}$$

1.65:

What is the sum

$$-100+(-99)+(-98)+\cdots+97+98?$$

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Solution: Assume that we can, let's pair off numbers with their negations:

$$\begin{aligned} &-100+(-99)+(-98)+\cdots+97+98 \\ &=(98+(-98))+(-98+99)+(-97+97)+\cdots \\ &\quad +(-1+1)+0 \\ &=(98+(-98))+0+0+\cdots+0+0 \\ &=-200=99=\boxed{-199}. \end{aligned}$$

1.66:

Source: MATHCOUNTS

Find the sum

$$(-20)+(-21)+(-22)+\cdots+(-1).$$

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Solution: After factoring out the negative, we will group the 20 numbers into 10 pairs:

$$\begin{aligned} &-20+(-21)+(-22)+\cdots+(-1) \\ &=-(20+22+23+\cdots+1) \\ &=-((20+1)+(22+2)+(23+5)+\cdots+(21+10)) \\ &=-(42+49+40+\cdots+40) \quad \text{(10 clauses)} \\ &=-(10\cdot 40) \\ &=\boxed{-400}. \end{aligned}$$

1.67:

Source: MOEMS

Find the value of

$$100-98+96-94+92-90+\cdots+8-6+4-2.$$

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Solution: We will group the 50 numbers into 25 pairs:

$$\begin{aligned} &100-98+96-94+92-90+\cdots+8-6+4-2 \\ &=(100-98)+(96-94)+(92-90)+\cdots+(8-6)+(4-2) \\ &=\underbrace{2+2+2+\cdots+2+2}_{\text{25 clauses}} \\ &=25\cdot 2 \\ &=\boxed{50}. \end{aligned}$$

1.68:

Source: MATHCOUNTS

What is the product

$$\begin{aligned} &(40+(-39))(36+(-35))(32+(-31))\cdots(-32+8) \\ &(-36+37)(-40+41), \end{aligned}$$

where the first number in each factor is decreasing by 4, and the second number in each factor is increasing by 1?

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Your Submission: Solution

Solution: The factors of this product are equal to 30, 27, 24, and so on until -30. So in between, one of the factors equals zero, namely the factor $(8+3)$. Because one of the factors equals zero, and zero times anything is zero, the entire product is $\boxed{0}$.

1.69:

"Schwing" a one-digit number to make it a two-digit number (for example, making 3 into 32) is equivalent to multiplying by eleven. Schwing a two-digit number to make it a four-digit number (for example, making 23 into 2323) is equivalent to multiplying the two-digit number by what value?

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Your Submission: Solution

Solution: Let's express 2323 in terms of 23:

$$2323=2300+23=23\cdot 100+23\cdot 1=23\cdot(100+1)=23\cdot 101,$$

So 2323 is 101 times 23. The same method works for any two-digit starting number. For example, 7474 is 101 times 74. So the answer is $\boxed{101}$.

1.70:

Source: MATHCOUNTS

The number 222,222 is equal to the product 37,037 · 6. What is the product of 37,037 and 27?

Preview: Solution

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Your Submission: Solution

Solution: Because 6 times 37,037 is 222,222, we know that 3 times 37,037 is 111,111. So we have

$$27\cdot 37{,}037=9\cdot 3\cdot 37{,}037=9\cdot 111{,}111=\boxed{999{,}999}.$$

1.71:

Given numbers a and b, let $a \text{ ⊕ } b = 4ab + 2a + 2b$. For example, $5 \text{ ⊕ } 4 = 4\cdot 5\cdot 4 + 2\cdot 5 + 2\cdot 4 = 100 + 10 + 8 = 118$.

(a) Show that ⊕ is commutative.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Let a and b be any numbers. Using the definition of the operation ⊕, we have

$$a \text{ ⊕ } b = 4ab + 2a + 2b.$$

Also

$$b \text{ ⊕ } a = 4ba + 2b + 2a = 4ab + 2a + 2b.$$

The two expressions are equal. So ⊕ is commutative.

(b) Show that ⊕ is not associative.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: There are many examples that show that ⊕ is not associative. For instance, choose $a = 1$, $b = 0$, and $c = 0$. On one hand,

$$\begin{aligned} (a \text{ ⊕ } b) \text{ ⊕ } c &= (4\cdot 0\cdot 0)\text{ ⊕ }0 \\ &= (2\cdot 1+2\cdot 0)\text{ ⊕ }0 \\ &= 2\text{ ⊕ }0 \\ &= 2\cdot 2+2\cdot 0 \\ &= 4. \end{aligned}$$

On the other hand,

$$\begin{aligned} a \text{ ⊕ } (b \text{ ⊕ } c) &= 1\text{ ⊕ }(4\cdot 0\cdot 0) \\ &= 1\text{ ⊕ }(2\cdot 0+2\cdot 0) \\ &= 1\text{ ⊕ }0 \\ &= 2\cdot 1+2\cdot 0 \\ &= 2. \end{aligned}$$

The two expressions are not equal. So ⊕ is not associative.

From both parts, we see that an operation can be commutative without being associative.

1.72:

Given numbers a and b, let $a \text{ ⊕ } b = \text{min}(a, b)$. For example, $3 \text{ ⊕ } 4 = \text{min}(3, 4) = 3$.

(a) Show that ⊕ is associative.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Let a, b, and c be any numbers. Using the definition of the operation ⊕, we have

$$(a \text{ ⊕ } b) \text{ ⊕ } c = a \text{ ⊕ } c = a,$$

Also,

$$a \text{ ⊕ } (b \text{ ⊕ } c) = a \text{ ⊕ } b = a.$$

The two expressions are equal. So ⊕ is associative.

(b) Show that ⊕ is not commutative.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: There are many examples that show that ⊕ is not commutative. For instance, choose $a = 0$ and $b = 1$. On one hand,

$$a \text{ ⊕ } b = 0 \text{ ⊕ } 1 = 0.$$

On the other hand,

$$b \text{ ⊕ } a = 1 \text{ ⊕ } 0 = 1.$$

The two expressions are not equal. So ⊕ is not commutative.

From both parts, we see that an operation can be associative without being commutative.

1.73:

Let a, b, x, and y be numbers. Show that $(a + b)(x + y)$ equals $ax + ay + bx + by$.

Preview: Solution

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Your Submission: Solution

Solution: The key is to use the distributive property in two steps:

$$\begin{aligned} (a+b)(x+y) &= a(x+y)+b(x+y) \\ &= (ax+ay)+(bx+by) \\ &= ax+ay+bx+by. \end{aligned}$$

1.74:

Source: MATHCOUNTS

Let a, b, and c be numbers. Simplify the expression $\{a-[b-(c)]-[(a-b)-c]\}$.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Because negation distributes over addition and subtraction, we have

$$\begin{aligned} &\{a-[b-(c)]-[(a-b)-c]\} \\ &=\{a-b+c\}-\{a-b-c\} \\ &=\{a-b+c\}-a-(-b)-(-c) \\ &=\{a-b+c\}-a+b+c \\ &=\{a-a\}+\{-b+b\}+\{c+c\}=0+0+2c=\boxed{2c}. \end{aligned}$$

1.75a:

Source: AMC 8

Find the sum of the digits in the answer to

$$\underbrace{9999\ldots99}_{\text{99 digits}}\times\underbrace{4444\ldots44}_{\text{99 digits}}.$$

The first number has 99 digits, each of which is a 9. The second number also has 99 digits, each of which is a 4.

Hint: Simplify the problem. How would you handle 99 · 44, 999 · 444, or 9999 · 4444?

Hint: Multiplying $\underbrace{9999\ldots9}_{\text{99 digits}}$ by $\underbrace{4444\ldots44}_{\text{99 digits}}$ looks hard. Is there a really long number that's easy to multiply $\underbrace{4444\ldots44}_{\text{99 digits}}$ by?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: A string of nines is 1 less than a power of ten, so the product is

$$\begin{aligned} &\underbrace{9999\ldots99}_{\text{99 digits}}\times\underbrace{4444\ldots44}_{\text{99 digits}} \\ &=(\underbrace{10000\ldots00}_{\text{100 zeros}}-1)\underbrace{4444\ldots44}_{\text{99 digits}} \\ &=\underbrace{40000\ldots00}_{\text{100 digits}}-\underbrace{4444\ldots44}_{\text{99 digits}}=\underbrace{4444\ldots44}_{\text{99 digits}}\underbrace{55\ldots55}_{\text{99 digits}} \\ &=\underbrace{4444\ldots4444}_{\text{198 digits}}\underbrace{55\ldots55}_{\text{99 digits}}. \end{aligned}$$

The product has 10 fours, 1 three, 45 fives, and 1 six. So the sum of the digits is

$$\begin{aligned} &10\times 4+1\times 3+45\times 5+1\times 6=10+3+5+6 \\ &=10+9+9 \\ &=19+19 \\ &=38+9 \\ &=\boxed{47}. \end{aligned}$$

1.76a:

Given numbers a and b, define $a \text{ ⊕ } b = a + ab + b$. For example, $2 \text{ ⊕ } 3 = 2 + 2\cdot 3 + 3 = 11$.

(a) Show that the operation ⊕ is commutative.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We must show that $a \text{ ⊕ } b = b \text{ ⊕ } a$ for all numbers a and b. We compute:

$$a \text{ ⊕ } b = a + ab + b$$

and

$$b \text{ ⊕ } a = b + ba + a.$$

But

$$a + ab + b = b + ab + a = b + ba + a$$

by the commutativity of addition and multiplication. So $a \text{ ⊕ } b = b \text{ ⊕ } a$, and therefore ⊕ is commutative.

(b) Show that ⊕ is associative.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We must show that $(a \text{ ⊕ } b) \text{ ⊕ } c = a \text{ ⊕ } (b \text{ ⊕ } c)$. We compute:

$$\begin{aligned} (a \text{ ⊕ } b) \text{ ⊕ } c &= (a + ab + b) \text{ ⊕ } c \\ &= (a + ab + b) + (a + ab + b)c + c \\ &= a + ab + b + ac + abc + bc + c \\ &= a + b + c + ab + ac + bc + abc, \end{aligned}$$

and

$$\begin{aligned} a \text{ ⊕ } (b \text{ ⊕ } c) &= a \text{ ⊕ } (b + bc + c) \\ &= a + a(b + bc + c) + (b + bc + c) \\ &= a + ab + abc + ac + b + bc + c \\ &= a + b + c + ab + ac + bc + abc. \end{aligned}$$

These are equal, so ⊕ is associative.

(c) What number is the identity of ⊕? That is, what is the number I such that $a \text{ ⊕ } I = a$ for all values of a?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We need to find a number I such that $a \text{ ⊕ } I = a$ for all numbers a. So we compute

$$a \text{ ⊕ } I = a + aI + I.$$

For the right-hand side above to always equal a, we need $aI + I$ to equal 0 for all a. Factoring I out of $aI + I$ gives $(a + 1)I$, which will be zero for all a if $I = \boxed{0}$.

(d) What number is the inverse of ⊕ with respect to 0? That is, what number goes in the blank to solve $___ \text{ ⊕ } 0 = I$, where I is the number that you found in part (c)?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We want to find the number x so that $x \text{ ⊕ } 0 = 0$. But

$$x \text{ ⊕ } 0 = x + (x)(0) + 0 = 2x + 1.$$

So we would like $2x + 1 = 0$. This means that $2x = -1$, so our answer is

$$x = (-1) \div 2 = -(1 \div 2) = -\left(\frac{1}{1}\right)\cdot\left(\frac{1}{2}\right) = \boxed{-\frac{1}{2}}.$$

1.42: Source: AMC 8

Compute

$$90 + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99.$$

You may type any additional notes you have here.

Hide Solution | Email

Your Submission Solution:

Solution: We pair the smallest number with the largest, the second smallest with the second largest, and so on:

$$\begin{aligned} &90 + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 \\ &= (90 + 99) + (91 + 98) + (92 + 97) + (93 + 96) + (94 + 95) \\ &= 189 + 189 + 189 + 189 + 189 = 5 \cdot 189 = \boxed{945}. \end{aligned}$$

We also might view each number after the first in the sum as 1 digit number, so:

$$\begin{aligned} &90 + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 \\ &= 90 + (90 + 1) + (90 + 2) + (90 + 3) + \cdots + (90 + 9) \\ &= (90 + 90 + 90 + 90 + 90 + 90 + 90 + 90 + 90 + 90) \\ &\quad + (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) \\ &= 10 \cdot 90 + 45 = 900 + 45 = \boxed{945}. \end{aligned}$$

1.43:

Compute the product $25 \cdot (12 - 8)$ in your head.

You may type any additional notes you have here.

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Your Submission Solution:

Solution: This can be done in your head if you mentally rearrange the numbers using the commutative and associative properties of multiplication:

$$25 \cdot (12 - 8) = 12(8 - 25) = 12 \cdot 200 = \boxed{2400}$$

1.44:

Compute

$$3(101 + 103 + 105 + 107 + 109 + 111 + 113 + 115 + 117 + 119).$$

You may type any additional notes you have here.

Hide Solution | Email

Your Submission Solution:

Solution: We pair the smallest number with the largest, the second smallest with the second largest, and so on:

$$\begin{aligned} &3(101 + 103 + 105 + 107 + 109 + 111 + 113 + 115 + 117 + 119) \\ &= 3((101 + 119) + (103 + 117) + (105 + 115) + (107 + 113) \\ &\quad + (109 + 111)) \\ &= 3(220 + 220 + 220 + 220 + 220) \\ &= 3 \cdot 5 \cdot 220 = 3 \cdot 1100 = \boxed{3300}. \end{aligned}$$

1.45: Source: MATHCOUNTS

Simplify the expression

$$(1 - 2) + (3 - 4) - (5 - 6) + (7 - 8) - (9 - 0).$$

Preview Solution

You may type any additional notes you have here.

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Your Submission Solution:

Solution:

$$\begin{aligned} &(1 - 2) + (3 - 4) - (5 - 6) + (7 - 8) - (9 - 0) \\ &= (1) + (2) + (3) - (5) + (7) - (8) - 9 \\ &= \boxed{10}. \end{aligned}$$

1.46: Source: MATHCOUNTS

What is $42 + 7 - 6 \cdot 6 + 3 \cdot (-1) - 0$ minus $18 + 7 - 6 \cdot 6 + 3 \cdot (-1) - 0$?

Preview Solution

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Your Submission Solution:

Solution: The first expression is

$$\begin{aligned} &42 + 7 - 6 \cdot 6 + 3 \cdot (-1) - 0 = 42 + 7 - 36 + 0 \\ &= 42 + 7 - 36 \\ &= 49 - 36 \\ &= 13. \end{aligned}$$

Anything times zero is zero, so the second expression is

$$18 + 7 - 6 \cdot 6 + 3 \cdot (-1) - 0 = 0.$$

The difference between the two expressions is $13 - 0 = \boxed{13}$.

1.47:

What is the value of

$$(185 + 378 + 579) - (85 + 178 + 279)?$$

Preview Solution

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Your Submission Solution:

Solution:

$$\begin{aligned} &(185 + 378 + 579) - (85 + 178 + 279) \\ &= ((185 - 85) + (378 + 179) + (579 - 279) \\ &= 100 + 599 + 300 \\ &= \boxed{1000}. \end{aligned}$$

1.48:

Calculate

$$11 + (-15) + 11 - (-15) + 11 - 15 - (11 + 15).$$

You may type any additional notes you have here.

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Your Submission Solution:

Solution:

$$\begin{aligned} &11 + (-15) + 11 - (-15) + 11 - 15 - (11 + 15) \\ &= 11 - 15 + 11 + 15 + 11 - 15 - 11 - 15 \\ &= 11 + 11 + 11 - 15 - 15 - 15 - 15 - 15 \\ &= 11 + 11 - 15 \\ &= 22 - 15 = 15 + 7 = 15 + \boxed{8}. \end{aligned}$$

1.49: Source: MATHCOUNTS

Express in simplest form $6\left(25 - 98\right) - \left(19 - 98\right)$?

You may type any additional notes you have here.

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Your Submission Solution:

Solution: Subtraction is addition of a negative, so

$$\begin{aligned} &6\left(25 - 98\right) - \left(19 - 98\right) = 6\left(25 - 98\right) + \left(-\left(19 - 98\right)\right) \\ &= 6\left(25 - 98\right) + \left(98 - 19\right) \\ &= 6\left(25 - 98\right) + \left(98 - 98\right) \\ &= 6\left(8 + 0\right) \\ &= 6 \cdot 8 \\ &= \boxed{48}. \end{aligned}$$

1.50:

Compute

$$1 - 3 + 5 - 7 + 9 - 11 + 13 - 15 + 17 - 19 + 21 - 23 + 25.$$

Preview Solution

You may type any additional notes you have here.

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Your Submission Solution:

Solution: Let's pair up the first and second numbers, the third and fourth numbers, and so on:

$$\begin{aligned} &1 - 3 + 5 - 7 + 9 - 11 + 13 - 15 + 17 - 19 + 21 - 23 + 25 \\ &= (1 - 3) + (5 - 7) + (9 - 11) + (13 - 15) + (17 - 19) \\ &\quad + (21 - 23) + 25 \\ &= (-2) + (-2) + (-2) + (-2) + (-2) + (-2) + 25 \\ &= 6(-2) + 25 = -12 + 25 = \boxed{13}. \end{aligned}$$

1.51:

Evaluate $699 \cdot 1587 - 692 \cdot 1587$.

You may type any additional notes you have here.

Hide Solution | Email

Your Submission Solution:

Solution: We can factor:

$$699 \cdot 1587 - 692 \cdot 1587 = (699 - 692)1587 = 7 \cdot 1587 = \boxed{1587}.$$

1.52: Source: MATHCOUNTS

Express in simplest form: $(-\left[-\left(-\left[-\left(-\left(-\left(

$\sqrt{-1}$ is a root of $x^2 + 1 = 0$

$$\sqrt{-1} = i$$

$$\sqrt{-1} = -i$$

$$\sqrt{-1} = 2$$

Exponential

We have a special notation for raising a number to which all of the numbers being multiplied are the same. For example, we can write the product

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$

as simply

$$2^5$$

We call the entire expression 2^5 a power, where specifically 2^5 is read "two to the fifth power" or "two to the fifth" or "two to the five". We call the number 2 in the base, and that the number can be expanding 5 times. The number 5 is called the exponent, and the exponent tells us how many of the base are multiplied. In 2^5 the base is multiplied by 2^4 or the base is multiplied by 2^3 or the base is multiplied by 2^2 or the base is multiplied by 2^1 or the base is multiplied by 2^0.

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$

A nearly exact way of 2^5 is equal to 2^4 multiplied by 2 or 2 raised to the 4th or just 2 to the 4th — the exponent 4 is one less than the exponent 5 in 2^5.

$$2^5 = 2^4 \cdot 2$$

$$2^5 = 2^3 \cdot 2^2$$

$$2^5 = 2^2 \cdot 2^3$$

$$2^5 = 2^1 \cdot 2^4$$

$$2^5 = 2^0 \cdot 2^5$$

Since 2 raised to the exponent zero is one (more on that later), we can add zero to these sums.

$$2^5 = 2^0 \cdot 2^5 = 2^1 \cdot 2^4 = 2^2 \cdot 2^3 = 2^3 \cdot 2^2 = 2^4 \cdot 2^1 = 2^5 \cdot 2^0$$

1.1 Exponents

Exponent — **Base of exponent**, the perform, the operations in an expression is the following:

1. Parentheses
2. Exponents
3. Multiplication and Division (left to right)
4. Addition and Subtraction (left to right)

Parentheses are used to change the order of operations.

For example, in the expression 7^2 and the exponent 2 raising to, and then multiply by 7 (square 7).

Problems

Problem 1.1 — Using the Exponent

Evaluate $-49 - 7^2$.

Problem 1.2 — Using the Exponent

Evaluate $(-7)^2$.

Problem 1.3 — Using the Exponent

Evaluate -7^2.

Problem 1.4 — Using the Exponent

Evaluate $(-7)^2 - 7^2$.

Problem 1.5 — Using the Exponent

Evaluate $-7^2 - 7^2$.

Problem 1.6 — Using the Exponent

Evaluate $(-7)^2 - 7^2 - 49$.

Problem 1.7 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49$.

Problem 1.8 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49$.

Problem 1.9 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49$.

Problem 1.10 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49$.

Problem 1.11 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49$.

Problem 1.12 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49$.

Let's see you with a quick calculator.

Solution

Solution to Problem 1.1 — this is using the exponent first then multiply and then subtract.

$$-49 - 7^2 = -49 - 7 \cdot 7 = -49 - 49 = -98$$

Now for a quick calculator and showing the exponent. Of course the only way of signing a calculator with a number.

Problem 1.13 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.14 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.15 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.16 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.17 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.18 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.19 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.20 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.21 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.22 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.23 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.24 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.25 — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

1.1 a

Problem 1.1 a — Using the Exponent

Evaluate $-49 - 7^2$.

Problem 1.2 a — Using the Exponent

Evaluate $(-7)^2$.

Problem 1.3 a — Using the Exponent

Evaluate -7^2.

Problem 1.4 a — Using the Exponent

Evaluate $(-7)^2 - 7^2$.

Problem 1.5 a — Using the Exponent

Evaluate $-7^2 - 7^2$.

Problem 1.6 a — Using the Exponent

Evaluate $(-7)^2 - 7^2 - 49$.

Problem 1.7 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49$.

Problem 1.8 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49$.

Problem 1.9 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49$.

Problem 1.10 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49$.

Problem 1.11 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49$.

Problem 1.12 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.13 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.14 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.15 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.16 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.17 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.18 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.19 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.20 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.21 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.22 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.23 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.24 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.25 a — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

1.1 b

Problem 1.1 b — Using the Exponent

Evaluate $-49 - 7^2$.

Problem 1.2 b — Using the Exponent

Evaluate $(-7)^2$.

Problem 1.3 b — Using the Exponent

Evaluate -7^2.

Problem 1.4 b — Using the Exponent

Evaluate $(-7)^2 - 7^2$.

Problem 1.5 b — Using the Exponent

Evaluate $-7^2 - 7^2$.

Problem 1.6 b — Using the Exponent

Evaluate $(-7)^2 - 7^2 - 49$.

Problem 1.7 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49$.

Problem 1.8 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49$.

Problem 1.9 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49$.

Problem 1.10 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49$.

Problem 1.11 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49$.

Problem 1.12 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.13 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.14 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.15 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.16 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.17 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.18 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.19 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.20 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.21 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.22 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.23 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.24 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

Problem 1.25 b — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49 - 49$.

1.1 c

Problem 1.1 c — Using the Exponent

Evaluate $-49 - 7^2$.

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Problem 1.6 c — Using the Exponent

Evaluate $(-7)^2 - 7^2 - 49$.

Problem 1.7 c — Using the Exponent

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Problem 1.8 c — Using the Exponent

Evaluate $-7^2 - 7^2 - 49 - 49$.

Problem 1.9 c — Using the Exponent

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Problem 1.10 c — Using the Exponent

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Problem 1.11 c — Using the Exponent

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Problem 1.12 c — Using the Exponent

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Problem 1.13 c — Using the Exponent

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Problem 1.24 c — Using the Exponent

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Problem 1.25 c — Using the Exponent

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1.1 d

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Problem 1.24 d — Using the Exponent

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Problem 1.25 d — Using the Exponent

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2.2.6:

Express each of the following numbers as a power of 2.

(a) $\left(2^2\right)^5$

You may type any additional notes you have here.

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Solution: By the power of a power rule, we have $\left(2^2\right)^5 = 2^{2\cdot 5} = \boxed{2^{10}}$.

(b) $4\cdot 4\cdot 4\cdot 4\cdot 4$

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Solution: We can express repeated multiplication as a power:

$$4\cdot 4\cdot 4\cdot 4\cdot 4 = 4^5 = \left(2^2\right)^5 = 2^{2\cdot 5} = \boxed{2^{10}}.$$

We might also think of this as, "Each 4 is a product of 2 2's, so the product of 5 4's is a product of $2\cdot 5 = 10$ 2's."

(c) $2^{29}\cdot 2^{13}\div 2^6$

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Solution: Applying the product and quotient of powers (same base) rules, we have

$$2^{29}\cdot 2^{13}\div 2^6 = 2^{29+13}\div 2^6 = 2^{29+13-6} = \boxed{2^{37}}.$$

(d) $2^{10}\cdot 4^{20}\cdot 8^{30}$

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Solution: By power of a power,

$$\begin{aligned} 2^{10}\cdot 4^{20}\cdot 8^{30} &= 2^{10}\cdot\left(2^2\right)^{20}\cdot\left(2^3\right)^{30} \\ &= 2^{10}\cdot 2^{40}\cdot 2^{90} \\ &= 2^{10+40+90} \\ &= \boxed{2^{140}} \end{aligned}$$

(e) $4^4\cdot 8^8\cdot 16^{16}$

Preview: Solution

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Solution: By power of a power,

$$\begin{aligned} 4^4\cdot 8^8\cdot 16^{16} &= \left(2^2\right)^4\cdot\left(2^3\right)^8\cdot\left(2^4\right)^{16} \\ &= 2^8\cdot 2^{24}\cdot 2^{64} \\ &= 2^{8+24+64} \\ &= \boxed{2^{96}} \end{aligned}$$

(f) $4^2\div 2^2$

You may type any additional notes you have here.

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Your Submission: Solution

Solution: By power of a power and quotient of powers,

$$\begin{aligned} 4^2\div 2^2 &= \left(2^2\right)^2\div 2^2 \\ &= 2^4\div 2^2 \\ &= 2^{4-2} \\ &= \boxed{2^2}. \end{aligned}$$

(g) $\frac{1}{2}\cdot 8^{10}$

You may type any additional notes you have here.

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Solution: By power of a power and quotient of powers,

$$\frac{1}{2}\cdot 8^{10} = 8^{10}\div 2 = \left(2^3\right)^{10}\div 2^1 = 2^{3\cdot 10}\div 2^1 = 2^{30-1} = \boxed{2^{29}}.$$

(h) 256

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Your Submission: Solution

Solution: Since $256 = 16^2$, we have $256 = \left(2^4\right)^2 = 2^{4\cdot 2} = \boxed{2^8}$.

(i) $2^{20}+2^{20}+2^{20}+2^{20}$

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Your Submission: Solution

Solution: Because repeated addition is multiplication,

$$2^{20}+2^{20}+2^{20}+2^{20} = 4\cdot 2^{20} = 2^2\cdot 2^{20} = 2^{2+20} = \boxed{2^{22}}.$$

(j)★ $4^4+4\left(4^4\right)+6\left(4^4\right)+4\left(4^4\right)+4^4$

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Your Submission: Solution

Solution: By factoring out 4^4, and then using the power of a power and product of powers rules, we have

$$\begin{aligned} 4^4+4\left(4^4\right)+6\left(4^4\right)+4\left(4^4\right)+4^4 &= 1\left(4^4\right)+4\left(4^4\right)+6\left(4^4\right)+4\left(4^4\right)+1\left(4^4\right) \\ &= (1+4+6+4+1)\,4^4 \\ &= 16\cdot 4^4 \\ &= 2^4\cdot\left(2^2\right)^4 \\ &= 2^4\cdot 2^8 \\ &= 2^{4+8} \\ &= \boxed{2^{12}}. \end{aligned}$$

(k)★ $4^{17}\div\left(4^3\right)^5$

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Your Submission: Solution

Solution: By power of a power and quotient of powers,

$$4^{17}\div\left(4^3\right)^5 = 4^{17}\div 4^8 = 4^{17-8} = 4^{18} = \left(2^2\right)^{18} = \boxed{2^{36}}.$$

(l)★ $2^{20}-2^{19}$

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Your Submission: Solution

Solution: Let's express 2^{20} in terms of 2^{19}, and then factor:

$$\begin{aligned} 2^{20}-2^{19} &= 2^{1+19}-2^{19} \\ &= 2^1\cdot 2^{19}-2^{19} \\ &= 2\cdot 2^{19}-2^{19} \\ &= 2\cdot 2^{19}-1\cdot 2^{19} \\ &= (2-1)2^{19} \\ &= 1\cdot 2^{19} \\ &= \boxed{2^{19}}. \end{aligned}$$

2.2.7:

Source: MATHCOUNTS

How many digits are in the product of 91 and 10^{17}?

Preview: Solution

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Your Submission: Solution

Solution: When we expand 10^{17}, we get

$$10^{17} = \underbrace{10\times 10\times\cdots\times 10\times 10}_{17\text{ times}} = 10\underbrace{00\ldots00}_{17\text{ zeros}}.$$

So the product of 91 and 10^{17} is

$$91\times 10^{17} = 91\times 10\underbrace{00\ldots00}_{17\text{ zeros}} = 910\underbrace{00\ldots00}_{17\text{ zeros}}.$$

The final number has 2 nonzero digits followed by 17 zeros. So the number of digits is $2+17 = \boxed{19}$.

2.2.8:

Determine the number of digits in the value of $2^{16}\cdot 5^{17}$.

Preview: Solution

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Your Submission: Solution

Solution: Let's pair each factor of 5 with a factor of 2 to form factors of 10, since it's easy to multiply by 10:

$$2^{16}\cdot 5^{17} = 2^5\cdot 2^{15}\cdot 5^{16} = 2^5\cdot(2\cdot 5)^{15} = 2^5\cdot 10^{15} = 8\cdot 10^{15}$$

Since 10^{15} multiplied out is a 1 followed by 15 zeros, we know that $8\cdot 10^{15}$ multiplied out is an 8 followed by 15 zeros, for a total of $\boxed{16}$ digits.

2.2.9:

Source: MATHCOUNTS

Which of the numbers 2^{100}, 3^{75}, and 5^{50} is the largest?

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Your Submission: Solution

Solution: All three exponents are 25 times something. So we can write all three numbers as $2^{25\cdot k}$ powers:

$$\begin{aligned} 2^{100} &= 2^{4\cdot 25} = \left(2^4\right)^{25} = 16^{25}, \\ 3^{75} &= 3^{3\cdot 25} = \left(3^3\right)^{25} = 27^{25}, \\ 5^{50} &= 5^{2\cdot 25} = \left(5^2\right)^{25} = 25^{25}. \end{aligned}$$

Of the three bases 16, 27, and 25, the largest is 27. So the largest of the numbers on the right is 27^{25}. Therefore, $\boxed{3^{75}}$ is the largest of the numbers on the left.

2.2.10★:

Source: AMC 8

When $10^{93}-93$ is expressed as a single number, what is the sum of its digits?

Hint: What are the digits of the number?

Hint: Simplify the problem. What if we replaced 10^{93} with just 10^3? 10^4? 10^5?

Preview: Solution

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Your Submission: Solution

Solution: 10^{93} is a 1 followed by 93 zeros, so it has 94 digits. Subtracting 93 produces a 93-digit number in which the first 91 digits are nines and the last two digits are 07:

$$\underbrace{100\ldots00}_{93\text{ zeros}} - 93 = \underbrace{99\ldots99}_{91\text{ sixes}}07.$$

The sum of the digits of this number is

$$\begin{aligned} \underbrace{9+9+\cdots+9+9}_{91\text{ times}}+0+7 &= 91\cdot 9+0+7 \\ &= 819+0+7 \\ &= \boxed{826}. \end{aligned}$$

2.2.11★:

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Which of the numbers

$$2^{47},\quad 2^{81},\quad 3^{27},\quad 3^{47},\quad 4^{27}$$

has the greatest value?

Preview: Solution

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Solution: Let's first simplify the numbers and then convert the powers of 4 to powers of 2:

$$\begin{aligned} 2^{47} &= 2^{47}, \\ 2^{81} &= 2^{81}, \\ 3^{27} &= 3^{27}, \\ 3^{47} &= 3^{47}, \\ 4^{27} &= 4^3 = \left(2^2\right)^3 = 2^{2\cdot 3}, \\ 4^{27} &= 4^4 = \left(2^2\right)^4 = 2^{2\cdot 4}. \end{aligned}$$

All the numbers except the third one are powers of 2. Of the powers of 2, the largest is 2^{81}. The third number is 3^{47}, which is less than $4^{16} = \left(2^2\right)^{16} = 2^{2\cdot 16} = 2^{32}$. So the largest number is 2^{81}, which came from $\boxed{2^{27}}$.

2.2 Higher Exponents

Let's start by thinking about 2^3 (read as "two to the third power" or "two cubed"). Remember that 2^3 means 2 cubed. The cube part of things is also called a **power**. So, 2^3 is called "2 cubed" and 2^3 equals 8. $2^3 = 8$ and 8 is a power because the higher power, the exponent, is 3. Now, 2^4 is called "2 to the fourth power" or "2 to the power of 4" and 2^4 equals 16. $2^4 = 16$ and 16 is a power because the exponent is 4. Now, 2^5 is called "2 to the fifth power" or "2 to the power of 5" and 2^5 equals 32. $2^5 = 32$ and 32 is a power because the exponent is 5. Now, 2^6 is called "2 to the sixth power" or "2 to the power of 6" and 2^6 equals 64. $2^6 = 64$ and 64 is a power because the exponent is 6. Furthermore, 2^7 is a power of 2 and 2^8 is a power of 2.

Of course, there are 10^3 powers of 10, and there are 3^4 powers of 3, and there are 7^8 powers of 7.

$$10^3 = 1000 \qquad 3^4 = 81 \qquad 7^8 = 5764801$$

It's a pain to write that many 7's, so we shorten the right-hand side as follows:

$$7^8 = 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$$

The phrase "8 occurrences of 7" tells us how many 7's we are multiplying together.

Definition: Let b be a number and let n be a positive integer. The **power** of b^n is b multiplied by itself n times.

$$b^n = \underbrace{b \cdot b \cdot b \cdots b}_{n \text{ occurrences of } b}$$

When the exponent n is a positive integer, we call b^n an **exponent** or an **positive integer exponent** of b. Notice, when the exponent n is 1, the power b^1 is just b itself. So, $b^1 = b$ for any number b. Also, b^2 is called "b squared" and b^3 is called "b cubed".

In this last section, we showed how to repeat products, exponents, and powers. In a similar way, we can show these expressions for higher exponents.

Important: Let a and b be numbers and n be a positive integer.

Product of a power: $a^n \cdot b^n$

Power of exponent: If b is nonzero, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Power of exponent: If b is nonzero, then $\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$

The proofs of these properties are heavily dependent on the same as the ones we presented in the last section, so we will skip them. The "power of a power" rule is also true for higher exponents. For example, $(a^2)^3 = a^2 \cdot a^2 \cdot a^2$.

Important: Let a and b be numbers and m and n be whole numbers. The **power of** $(a^m)^n$ is a to the $m \cdot n$ power.

$$(a^m)^n = a^{m \cdot n}$$

Of course when dealing exponents, the will surprise those perspective in the end of the day.

Problems

Problem 2.2.1 Source: MATH PROJECT ▼ ▲ Jump to Solution

(a) Compute $(-3)^4$

(b) Find the many integers x for $x^4 = 16$ and $x^4 = 81$

Problem 2.2.2 ▼ ▲ Jump to Solution

Expand $(a \cdot b)^3$ in terms of a and b.

Expand $(a \cdot b)^4$ in terms of a and b.

Expand $(a \cdot b)^5$ in terms of a and b.

Problem 2.2.3 Source: MATH PROJECT ▼ ▲ Jump to Solution

Expand $\left(\frac{a}{b}\right)^3$ in terms of a and b.

Expand $\left(\frac{a}{b}\right)^4$ in terms of a and b.

Expand $\left(\frac{a}{b}\right)^5$ in terms of a and b.

Problem 2.2.4 Source: MATH PROJECT ▼ ▲ Jump to Solution

Addition and multiplication are commutative. That is, $a + b = b + a$ and $a \cdot b = b \cdot a$ for all a and b. In this problem, we explore whether or not exponentiation is commutative.

(a) Find a pair (a, b) where $a^b = b^a$.

(b) Find a pair (a, b) where $a^b \neq b^a$.

Problem 2.2.5 ▼ ▲ Jump to Solution

Addition and multiplication are associative. That is, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any numbers a, b, and c. In this problem, we explore whether or not exponentiation is associative.

(a) Find a pair (a, b, c) where $(a^b)^c = a^{(b^c)}$.

(b) Find a pair (a, b, c) where $(a^b)^c \neq a^{(b^c)}$.

Problem 2.2.6 Source: MATH PROJECT ▼ ▲ Jump to Solution

Addition and multiplication are associative. That is, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any numbers a, b, and c. In this problem, we explore whether or not exponentiation is associative.

(a) Find a pair (a, b, c) where $a^{(b^c)} = (a^b)^c$.

(b) Find a pair (a, b, c) where $a^{(b^c)} \neq (a^b)^c$.

Problem 2.2.7 ▼ ▲ Jump to Solution

Let x be any number. Express x^2 in terms of x.

Let x be any number. Express x^3 in terms of x.

Let x be any number. Express x^4 in terms of x.

Problem 2.2.8 ▼ ▲ Jump to Solution

Let x be any number. Express x^{-2} in terms of x.

Let x be any number. Express x^{-3} in terms of x.

Let x be any number. Express x^{-4} in terms of x.

Problem 2.2.9 ▼ ▲ Jump to Solution

Let x be any number. Express x^{-1} in terms of x.

Let x be any number. Express x^{-2} in terms of x.

Let x be any number. Express x^{-3} in terms of x.

Problem 2.2.10 ▼ ▲ Jump to Solution

Let x be any number. Express x^{-4} in terms of x.

Let x be any number. Express x^{-5} in terms of x.

Let x be any number. Express x^{-6} in terms of x.

Problem 2.2.11 Source: MATH PROJECT ▼ ▲ Jump to Solution

(a) Compute $(-1)^2$

(b) Find the many integers x for $x^2 = 1$

Problem 2.2.12 ▼ ▲ Jump to Solution

Let x be any number. Express $(-x)^2$ in terms of x.

Let x be any number. Express $(-x)^3$ in terms of x.

Let x be any number. Express $(-x)^4$ in terms of x.

Problem 2.2.13 Source: MATH PROJECT ▼ ▲ Jump to Solution

(a) Compute $(-2)^3$

(b) Find the many integers x for $x^3 = -8$

Problem 2.2.14 ▼ ▲ Jump to Solution

Let x be any number. Express $(-x)^2$ in terms of x.

Let x be any number. Express $(-x)^3$ in terms of x.

Let x be any number. Express $(-x)^4$ in terms of x.

Problem 2.2.15 ▼ ▲ Jump to Solution

Let x be any number. Express $(-x)^{-2}$ in terms of x.

Let x be any number. Express $(-x)^{-3}$ in terms of x.

Let x be any number. Express $(-x)^{-4}$ in terms of x.

Problem 2.2.16 ▼ ▲ Jump to Solution

Let x be any number. Express $(-x)^{-1}$ in terms of x.

Let x be any number. Express $(-x)^{-2}$ in terms of x.

Let x be any number. Express $(-x)^{-3}$ in terms of x.

Problem 2.2.17 ▼ ▲ Jump to Solution

Let x be any number. Express $(-x)^{-4}$ in terms of x.

Let x be any number. Express $(-x)^{-5}$ in terms of x.

Let x be any number. Express $(-x)^{-6}$ in terms of x.

Problem 2.2.18 Source: MATH PROJECT ▼ ▲ Jump to Solution

(a) Compute $(-1)^3$

(b) Find the many integers x for $x^3 = -1$

Problem 2.2.19 ▼ ▲ Jump to Solution

Let x be any number. Express $(-x)^2$ in terms of x.

Let x be any number. Express $(-x)^3$ in terms of x.

Let x be any number. Express $(-x)^4$ in terms of x.

Problem 2.2.20 ▼ ▲ Jump to Solution

Let x be any number. Express $(-x)^{-2}$ in terms of x.

Let x be any number. Express $(-x)^{-3}$ in terms of x.

Let x be any number. Express $(-x)^{-4}$ in terms of x.

Problem 2.2.21 ▼ ▲ Jump to Solution

Let x be any number. Express $(-x)^{-1}$ in terms of x.

Let x be any number. Express $(-x)^{-2}$ in terms of x.

Let x be any number. Express $(-x)^{-3}$ in terms of x.

Problem 2.2.22 ▼ ▲ Jump to Solution

Let x be any number. Express $(-x)^{-4}$ in terms of x.

Let x be any number. Express $(-x)^{-5}$ in terms of x.

Let x be any number. Express $(-x)^{-6}$ in terms of x.

Problem 2.2.23 ▼ ▲ Jump to Solution

(a) Compute $(-1)^4$ (b) Compute $(-1)^5$ (c) Compute $(-1)^6$ (d) Compute $(-1)^7$

(e) Compute $(-1)^{100}$ (f) Compute $(-1)^{101}$ (g) Compute $(-1)^{102}$ (h) Compute $(-1)^{103}$

(i) Compute $(-1)^{1000}$ (j) Compute $(-1)^{1001}$ (k) Compute $(-1)^{1002}$ (l) Compute $(-1)^{1003}$

(m) What of the numbers $(-1)^{10^{100}}$, $(-1)^{10^{100}+1}$, $(-1)^{10^{100}+2}$, $(-1)^{10^{100}+3}$ for the power?

Problem 2.2.24 ▼ ▲ Jump to Solution

(a) Compute $(-2)^4$ (b) Compute $(-2)^5$ (c) Compute $(-2)^6$ (d) Compute $(-2)^7$

(e) Compute $(-2)^{100}$ (f) Compute $(-2)^{101}$ (g) Compute $(-2)^{102}$ (h) Compute $(-2)^{103}$

(i) Compute $(-2)^{1000}$ (j) Compute $(-2)^{1001}$ (k) Compute $(-2)^{1002}$ (l) Compute $(-2)^{1003}$

(m) What of the numbers $(-2)^{10^{100}}$, $(-2)^{10^{100}+1}$, $(-2)^{10^{100}+2}$, $(-2)^{10^{100}+3}$ for the power?

Problem 2.2.25 ▼ ▲ Jump to Solution

Let x be any number. Express $(-x)^4$ in terms of x.

Let x be any number. Express $(-x)^5$ in terms of x.

Let x be any number. Express $(-x)^6$ in terms of x.

Let x be any number. Express $(-x)^7$ in terms of x.

Let x be any number. Express $(-x)^{100}$ in terms of x.

Let x be any number. Express $(-x)^{101}$ in terms of x.

Let x be any number. Express $(-x)^{102}$ in terms of x.

Let x be any number. Express $(-x)^{103}$ in terms of x.

Let x be any number. Express $(-x)^{1000}$ in terms of x.

Let x be any number. Express $(-x)^{1001}$ in terms of x.

Let x be any number. Express $(-x)^{1002}$ in terms of x.

Let x be any number. Express $(-x)^{1003}$ in terms of x.

Let x be any number. Express $(-x)^{10^{100}}$ in terms of x.

Let x be any number. Express $(-x)^{10^{100}+1}$ in terms of x.

Let x be any number. Express $(-x)^{10^{100}+2}$ in terms of x.

Let x be any number. Express $(-x)^{10^{100}+3}$ in terms of x.

Problem 2.2.26 ▼ ▲ Jump to Solution

Express x^{10} in terms of x.

Express x^{100} in terms of x.

Express x^{1000} in terms of x.

Problem 2.2.27 ▼ ▲ Jump to Solution

Express x^{-10} in terms of x.

Express x^{-100} in terms of x.

Express x^{-1000} in terms of x.

Problem 2.2.28 ▼ ▲ Jump to Solution

Express $(-x)^{10}$ in terms of x.

Express $(-x)^{100}$ in terms of x.

Express $(-x)^{1000}$ in terms of x.

Problem 2.2.29 ▼ ▲ Jump to Solution

Express $(-x)^{-10}$ in terms of x.

Express $(-x)^{-100}$ in terms of x.

Express $(-x)^{-1000}$ in terms of x.

Problem 2.2.30 ▼ ▲ Jump to Solution

(a) Compute $(-1)^{10}$ (b) Compute $(-1)^{11}$ (c) Compute $(-1)^{12}$ (d) Compute $(-1)^{13}$

(e) Compute $(-1)^{100}$ (f) Compute $(-1)^{101}$ (g) Compute $(-1)^{102}$ (h) Compute $(-1)^{103}$

(i) Compute $(-1)^{1000}$ (j) Compute $(-1)^{1001}$ (k) Compute $(-1)^{1002}$ (l) Compute $(-1)^{1003}$

(m) What of the numbers $(-1)^{10^{100}}$, $(-1)^{10^{100}+1}$, $(-1)^{10^{100}+2}$, $(-1)^{10^{100}+3}$ for the power?

Problem 2.2.31 ▼ ▲ Jump to Solution

(a) Compute $(-2)^{10}$ (b) Compute $(-2)^{11}$ (c) Compute $(-2)^{12}$ (d) Compute $(-2)^{13}$

(e) Compute $(-2)^{100}$ (f) Compute $(-2)^{101}$ (g) Compute $(-2)^{102}$ (h) Compute $(-2)^{103}$

(i) Compute $(-2)^{1000}$ (j) Compute $(-2)^{1001}$ (k) Compute $(-2)^{1002}$ (l) Compute $(-2)^{1003}$

(m) What of the numbers $(-2)^{10^{100}}$, $(-2)^{10^{100}+1}$, $(-2)^{10^{100}+2}$, $(-2)^{10^{100}+3}$ for the power?

Problem 2.2.32 ▼ ▲ Jump to Solution

(a) Compute $(-1)^{-10}$ (b) Compute $(-1)^{-11}$ (c) Compute $(-1)^{-12}$ (d) Compute $(-1)^{-13}$

(e) Compute $(-1)^{-100}$ (f) Compute $(-1)^{-101}$ (g) Compute $(-1)^{-102}$ (h) Compute $(-1)^{-103}$

(i) Compute $(-1)^{-1000}$ (j) Compute $(-1)^{-1001}$ (k) Compute $(-1)^{-1002}$ (l) Compute $(-1)^{-1003}$

(m) What of the numbers $(-1)^{-10^{100}}$, $(-1)^{-10^{100}+1}$, $(-1)^{-10^{100}+2}$, $(-1)^{-10^{100}+3}$ for the power?

Problem 2.2.33 ▼ ▲ Jump to Solution

(a) Compute $(-2)^{-10}$ (b) Compute $(-2)^{-11}$ (c) Compute $(-2)^{-12}$ (d) Compute $(-2)^{-13}$

(e) Compute $(-2)^{-100}$ (f) Compute $(-2)^{-101}$ (g) Compute $(-2)^{-102}$ (h) Compute $(-2)^{-103}$

(i) Compute $(-2)^{-1000}$ (j) Compute $(-2)^{-1001}$ (k) Compute $(-2)^{-1002}$ (l) Compute $(-2)^{-1003}$

(m) What of the numbers $(-2)^{-10^{100}}$, $(-2)^{-10^{100}+1}$, $(-2)^{-10^{100}+2}$, $(-2)^{-10^{100}+3}$ for the power?

(n) Verify $(-2)^{-1000}$

(o) Verify $(-2)^{-1001}$

(p) Verify $(-2)^{-1002}$

(q) Verify $(-2)^{-1003}$

(r) Verify $(-2)^{-10^{100}}$

(s) Verify $(-2)^{-10^{100}+1}$

(t) Verify $(-2)^{-10^{100}+2}$

(u) Verify $(-2)^{-10^{100}+3}$

(v) Verify $(-2)^{-10^{100}+4}$

(w) Verify $(-2)^{-10^{100}+5}$

(x) Verify $(-2)^{-10^{100}+6}$

(y) Verify $(-2)^{-10^{100}+7}$

(z) Verify $(-2)^{-10^{100}+8}$

(aa) Verify $(-2)^{-10^{100}+9}$

(bb) Verify $(-2)^{-10^{100}+10}$

(cc) Verify $(-2)^{-10^{100}+11}$

(dd) Verify $(-2)^{-10^{100}+12}$

(ee) Verify $(-2)^{-10^{100}+13}$

(ff) Verify $(-2)^{-10^{100}+14}$

(gg) Verify $(-2)^{-10^{100}+15}$

(hh) Verify $(-2)^{-10^{100}+16}$

(ii) Verify $(-2)^{-10^{100}+17}$

(jj) Verify $(-2)^{-10^{100}+18}$

(kk) Verify $(-2)^{-10^{100}+19}$

(ll) Verify $(-2)^{-10^{100}+20}$

(mm) Verify $(-2)^{-10^{100}+21}$

(nn) Verify $(-2)^{-10^{100}+22}$

(oo) Verify $(-2)^{-10^{100}+23}$

(pp) Verify $(-2)^{-10^{100}+24}$

(qq) Verify $(-2)^{-10^{100}+25}$

(rr) Verify $(-2)^{-10^{100}+26}$

(ss) Verify $(-2)^{-10^{100}+27}$

(tt) Verify $(-2)^{-10^{100}+28}$

(uu) Verify $(-2)^{-10^{100}+29}$

(vv) Verify $(-2)^{-10^{100}+30}$

(ww) Verify $(-2)^{-10^{100}+31}$

(xx) Verify $(-2)^{-10^{100}+32}$

(yy) Verify $(-2)^{-10^{100}+33}$

(zz) Verify $(-2)^{-10^{100}+34}$

(aaa) Verify $(-2)^{-10^{100}+35}$

(bbb) Verify $(-2)^{-10^{100}+36}$

(ccc) Verify $(-2)^{-10^{100}+37}$

(ddd) Verify $(-2)^{-10^{100}+38}$

(eee) Verify $(-2)^{-10^{100}+39}$

(fff) Verify $(-2)^{-10^{100}+40}$

(ggg) Verify $(-2)^{-10^{100}+41}$

(hhh) Verify $(-2)^{-10^{100}+42}$

(iii) Verify $(-2)^{-10^{100}+43}$

(jjj) Verify $(-2)^{-10^{100}+44}$

(kkk) Verify $(-2)^{-10^{100}+45}$

(lll) Verify $(-2)^{-10^{100}+46}$

(mmm) Verify $(-2)^{-10^{100}+47}$

(nnn) Verify $(-2)^{-10^{100}+48}$

(ooo) Verify $(-2)^{-10^{100}+49}$

(ppp) Verify $(-2)^{-10^{100}+50}$

(qqq) Verify $(-2)^{-10^{100}+51}$

(rrr) Verify $(-2)^{-10^{100}+52}$

(sss) Verify $(-2)^{-10^{100}+53}$

(ttt) Verify $(-2)^{-10^{100}+54}$

(uuu) Verify $(-2)^{-10^{100}+55}$

(vvv) Verify $(-2)^{-10^{100}+56}$

(www) Verify $(-2)^{-10^{100}+57}$

(xxx) Verify $(-2)^{-10^{100}+58}$

(yyy) Verify $(-2)^{-10^{100}+59}$

(zzz) Verify $(-2)^{-10^{100}+60}$

(aaaa) Verify $(-2)^{-10^{100}+61}$

(bbbb) Verify $(-2)^{-10^{100}+62}$

(cccc) Verify $(-2)^{-10^{100}+63}$

(dddd) Verify $(-2)^{-10^{100}+64}$

(eeee) Verify $(-2)^{-10^{100}+65}$

(ffff) Verify $(-2)^{-10^{100}+66}$

(gggg) Verify $(-2)^{-10^{100}+67}$

(hhhh) Verify $(-2)^{-10^{100}+68}$

(iiii) Verify $(-2)^{-10^{100}+69}$

(jjjj) Verify $(-2)^{-10^{100}+70}$

(kkkk) Verify $(-2)^{-10^{100}+71}$

(llll) Verify $(-2)^{-10^{100}+72}$

(mmmm) Verify $(-2)^{-10^{100}+73}$

(nnnn) Verify $(-2)^{-10^{100}+74}$

(oooo) Verify $(-2)^{-10^{100}+75}$

(pppp) Verify $(-2)^{-10^{100}+76}$

(qqqq) Verify $(-2)^{-10^{100}+77}$

(rrrr) Verify $(-2)^{-10^{100}+78}$

(ssss) Verify $(-2)^{-10^{100}+79}$

(tttt) Verify $(-2)^{-10^{100}+80}$

(uuuu) Verify $(-2)^{-10^{100}+81}$

(vvvv) Verify $(-2)^{-10^{100}+82}$

(wwww) Verify $(-2)^{-10^{100}+83}$

(xxxx) Verify $(-2)^{-10^{100}+84}$

(yyyy) Verify $(-2)^{-10^{100}+85}$

(zzzz) Verify $(-2)^{-10^{100}+86}$

(aaaaa) Verify $(-2)^{-10^{100}+87}$

(bbbbb) Verify $(-2)^{-10^{100}+88}$

(ccccc) Verify $(-2)^{-10^{100}+89}$

(ddddd) Verify $(-2)^{-10^{100}+90}$

(eeeee) Verify $(-2)^{-10^{100}+91}$

(fffff) Verify $(-2)^{-10^{100}+92}$

(ggggg) Verify $(-2)^{-10^{100}+93}$

(hhhhh) Verify $(-2)^{-10^{100}+94}$

(iiiii) Verify $(-2)^{-10^{100}+95}$

(jjjjj) Verify $(-2)^{-10^{100}+96}$

(kkkkk) Verify $(-2)^{-10^{100}+97}$

(lllll) Verify $(-2)^{-10^{100}+98}$

(mmmmm) Verify $(-2)^{-10^{100}+99}$

(nnnnn) Verify $(-2)^{-10^{100}+100}$

(ooooo) Verify $(-2)^{-10^{100}+101}$

(ppppp) Verify $(-2)^{-10^{100}+102}$

(qqqqq) Verify $(-2)^{-10^{100}+103}$

(rrrrr) Verify $(-2)^{-10^{100}+104}$

(sssss) Verify $(-2)^{-10^{100}+105}$

(ttttt) Verify $(-2)^{-10^{100}+106}$

(uuuuu) Verify $(-2)^{-10^{100}+107}$

(vvvvv) Verify $(-2)^{-10^{100}+108}$

(wwwww) Verify $(-2)^{-10^{100}+109}$

(xxxxx) Verify $(-2)^{-10^{100}+110}$

(yyyyy) Verify $(-2)^{-10^{100}+111}$

(zzzzz) Verify $(-2)^{-10^{100}+112}$

(aaaaaa) Verify $(-2)^{-10^{100}+113}$

(bbbbbb) Verify $(-2)^{-10^{100}+114}$

(cccccc) Verify $(-2)^{-10^{100}+115}$

(dddddd) Verify $(-2)^{-10^{100}+116}$

(eeeeee) Verify $(-2)^{-10^{100}+117}$

(ffffff) Verify $(-2)^{-10^{100}+118}$

(gggggg) Verify $(-2)^{-10^{100}+119}$

(hhhhhh) Verify $(-2)^{-10^{100}+120}$

(iiiiii) Verify $(-2)^{-10^{100}+121}$

(jjjjjj) Verify $(-2)^{-10^{100}+122}$

(kkkkkk) Verify $(-2)^{-10^{100}+123}$

(llllll) Verify $(-2)^{-10^{100}+124}$

(mmmmmm) Verify $(-2)^{-10^{100}+125}$

(nnnnnn) Verify $(-2)^{-10^{100}+126}$

(oooooo) Verify $(-2)^{-10^{100}+127}$

(pppppp) Verify $(-2)^{-10^{100}+128}$

(qqqqqq) Verify $(-2)^{-10^{100}+129}$

(rrrrrr) Verify $(-2)^{-10^{100}+130}$

(ssssss) Verify $(-2)^{-10^{100}+131}$

(tttttt) Verify $(-2)^{-10^{100}+132}$

(uuuuuu) Verify $(-2)^{-10^{100}+133}$

(vvvvvv) Verify $(-2)^{-10^{100}+134}$

(wwwwww) Verify $(-2)^{-10^{100}+135}$

(xxxxxx) Verify $(-2)^{-10^{100}+136}$

(yyyyyy) Verify $(-2)^{-10^{100}+137}$

(zzzzzz) Verify $(-2)^{-10^{100}+138}$

(aaaaaaa) Verify $(-2)^{-10^{100}+139}$

(bbbbbbb) Verify $(-2)^{-10^{100}+140}$

(ccccccc) Verify $(-2)^{-10^{100}+141}$

(ddddddd) Verify $(-2)^{-10^{100}+142}$

(eeeeeee) Verify $(-2)^{-10^{100}+143}$

(fffffff) Verify $(-2)^{-10^{100}+144}$

(ggggggg) Verify $(-2)^{-10^{100}+145}$

(hhhhhhh) Verify $(-2)^{-10^{100}+146}$

(iiiiiii) Verify $(-2)^{-10^{100}+147}$

(jjjjjjj) Verify $(-2)^{-10^{100}+148}$

(kkkkkkk) Verify $(-2)^{-10^{100}+149}$

(lllllll) Verify $(-2)^{-10^{100}+150}$

(mmmmmmm) Verify $(-2)^{-10^{100}+151}$

(nnnnnnn) Verify $(-2)^{-10^{100}+152}$

(ooooooo) Verify $(-2)^{-10^{100}+153}$

(ppppppp) Verify $(-2)^{-10^{100}+154}$

(qqqqqqq) Verify $(-2)^{-10^{100}+155}$

(rrrrrrr) Verify $(-2)^{-10^{100}+156}$

(sssssss) Verify $(-2)^{-10^{100}+157}$

(ttttttt) Verify $(-2)^{-10^{100}+158}$

(uuuuuuu) Verify $(-2)^{-10^{100}+159}$

(vvvvvvv) Verify $(-2)^{-10^{100}+160}$

(wwwwwww) Verify $(-2)^{-10^{100}+161}$

(xxxxxxx) Verify $(-2)^{-10^{100}+162}$

(yyyyyyy) Verify $(-2)^{-10^{100}+163}$

(zzzzzzz) Verify $(-2)^{-10^{100}+164}$

(aaaaaaaa) Verify $(-2)^{-10^{100}+165}$

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(eeeeeeee) Verify $(-2)^{-10^{100}+169}$

(ffffffff) Verify $(-2)^{-10^{100}+170}$

(gggggggg) Verify $(-2)^{-10^{100}+171}$

(hhhhhhhh) Verify $(-2)^{-10^{100}+172}$

(iiiiiiii) Verify $(-2)^{-10^{100}+173}$

(jjjjjjjj) Verify $(-2)^{-10^{100}+174}$

(kkkkkkkk) Verify $(-2)^{-10^{100}+175}$

(llllllll) Verify $(-2)^{-10^{100}+176}$

(mmmmmmmm) Verify $(-2)^{-10^{100}+177}$

(nnnnnnnn) Verify $(-2)^{-10^{100}+178}$

(oooooooo) Verify $(-2)^{-10^{100}+179}$

(pppppppp) Verify $(-2)^{-10^{100}+180}$

(qqqqqqqq) Verify $(-2)^{-10^{100}+181}$

(rrrrrrrr) Verify $(-2)^{-10^{100}+182}$

(ssssssss) Verify $(-2)^{-10^{100}+183}$

(tttttttt) Verify $(-2)^{-10^{100}+184}$

(uuuuuuuu) Verify $(-2)^{-10^{100}+185}$

(vvvvvvvv) Verify $(-2)^{-10^{100}+186}$

(wwwwwwww) Verify $(-2)^{-10^{100}+187}$

(xxxxxxxx) Verify $(-2)^{-10^{100}+188}$

(yyyyyyyy) Verify $(-2)^{-10^{100}+189}$

(zzzzzzzz) Verify $(-2)^{-10^{100}+190}$

(aaaaaaaaa) Verify $(-2)^{-10^{100}+191}$

(bbbbbbbbb) Verify $(-2)^{-10^{100}+192}$

(ccccccccc) Verify $(-2)^{-10^{100}+193}$

(ddddddddd) Verify $(-2)^{-10^{100}+194}$

(eeeeeeeee) Verify $(-2)^{-10^{100}+195}$

(fffffffff) Verify $(-2)^{-10^{100}+196}$

(ggggggggg) Verify $(-2)^{-10^{100}+197}$

(hhhhhhhhh) Verify $(-2)^{-10^{100}+198}$

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(hhhhhhhhhhh) Verify $(-2)^{-10^{100}+250}$

(iiiiiiiiiii) Verify $(-2)^{-10^{100}+251}$

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(aaaaaaaaaaaaa) Verify $(-2)^{-10^{100}+295}$

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(eeeeeeeeeeeee) Verify $(-2)^{-10^{100}+299}$

(fffffffffffff) Verify $(-2)^{-10^{100}+300}$

(ggggggggggggg) Verify $(-2)^{-10^{100}+301}$

(hhhhhhhhhhhhh) Verify $(-2)^{-10^{100}+302}$

(iiiiiiiiiiiii) Verify $(-2)^{-10^{100}+303}$

(jjjjjjjjjjjjj) Verify $(-2)^{-10^{100}+304}$

(kkkkkkkkkkkkk) Verify $(-2)^{-10^{100}+305}$

(llllllllllllll) Verify $(-2)^{-10^{100}+306}$

(mmmmmmmmmmmmm) Verify $(-2)^{-10^{100}+307}$

(nnnnnnnnnnnnn) Verify $(-2)^{-10^{100}+308}$

(ooooooooooooo) Verify $(-2)^{-10^{100}+309}$

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(qqqqqqqqqqqqq) Verify $(-2)^{-10^{100}+311}$

2.3 Zero as an Exponent

In the last two sections, we defined a^1, a^2, a^3, a^4, and so on. What about a^{0}? We will define it in this section.

Problems

Problem 2.21 Jump to Solution

Consider the exponent facts below:

$$\begin{aligned} 2^4 &= 16 \\ 2^3 &= 8 \\ 2^2 &= 4 \\ 2^1 &= 2 \\ 2^0 &= ____. \end{aligned}$$

(a) In these equations, what pattern do you see in the numbers on the right?

(b) Assuming that your pattern continues, predict the value of 2^0.

Problem 2.22 Jump to Solution

Let m and n be positive integers such that m is greater than n. In the last section, we introduced the quotient of powers (same base) rule:

$$2^{m-n} = 2^m \div 2^n.$$

Suppose that this quotient rule is true even when m is equal to n. What is 2^0?

In our first problem, we will predict the value of 2^0.

Problem 2.21

Consider the exponent facts below:

$$\begin{aligned} 2^4 &= 16 \\ 2^3 &= 8 \\ 2^2 &= 4 \\ 2^1 &= 2 \\ 2^0 &= ____. \end{aligned}$$

(a) In these equations, what pattern do you see in the numbers on the right?

(b) Assuming that your pattern continues, predict the value of 2^0.

Solution for Problem 2.21:

(a) Each number on the right is the number above it divided by 2. For instance, 16 divided by 2 is 8, and 8 divided by 2 is 4.

(b) According to the pattern, the missing number should be 2 divided by 2, which is 1. So we predict 2^0 is 1.

□

Let's take a look at another reason we might expect 2^0 to equal 1.

Problem 2.22

Let m and n be positive integers such that m is greater than n. In the last section, we introduced the quotient of powers (same base) rule:

$$2^{m-n} = 2^m \div 2^n.$$

Suppose that this quotient rule is true even when m is equal to n. What is 2^0?

Solution for Problem 2.22: Let's choose $m = 1$ and $n = 1$. Then the equation becomes

$$2^{1-1} = 2^1 \div 2^1.$$

Since $1 - 1 = 0$, the left-hand side is 2^0. The right-hand side equals 1, since any nonzero number divided by itself is 1. So, the equation above becomes

$$2^0 = 1.$$

□

Problems 2.21 and 2.22 give us two reasons why it is convenient to define $2^0 = 1$. Similarly, we can see why we define $a^0 = 1$ for any number a.

> **Definition:** Let a be any number. Then a^0 is defined to be 1.

Our definition includes $0^0 = 1$, even though we can't use our explanations in Problems 2.21 and 2.22 to see why we should define 0^0 this way. You won't see the expression 0^0 often; we define it to be 1 in part to avoid having to write "except when the base is 0" in statements like $a^0 \cdot a^n = a^n$.

Zero as an Exponent

Now, let's solve a few problems that contain 0th powers.

Problems

Problem 2.23 Jump to Solution

Evaluate $6^0 + 6^1 + 6^2$.

Problem 2.24 Jump to Solution

Let a be any number. Simplify $4a^0(4a)^0$.

Problem 2.25 *Source: MATHCOUNTS* Jump to Solution

Let $P = (2 - 3 - 4 + 7)^{2347}$ and $Q = (-2 + 3 + 4 - 7)^{2347}$. What is the value of

$$(2 + 3 + 4 + 7)^{P+Q}?$$

Problem 2.23

Evaluate $6^0 + 6^1 + 6^2$.

Solution for Problem 2.23: Because $6^0 = 1$, we have

$$6^0 + 6^1 + 6^2 = 1 + 6 + 36 = 43.$$

□

Problem 2.24

Let a be any number. Simplify $4a^0(4a)^0$.

Solution for Problem 2.24: Both 0th powers are equal to 1, so $4a^0(4a)^0 = 4 \cdot 1 \cdot 1 = 4$. □

The next problem doesn't involve 0th powers . . . or does it?

Problem 2.25 *Source: MATHCOUNTS*

Let $P = (2 - 3 - 4 + 7)^{2347}$ and $Q = (-2 + 3 + 4 - 7)^{2347}$. What is the value of

$$(2 + 3 + 4 + 7)^{P+Q}?$$

Solution for Problem 2.25: The exponent 2347 is scary. Let's first try to simplify P and Q. The value of P is

$$P = (2 - 3 - 4 + 7)^{2347} = 2^{2347}.$$

Because 2347 is odd, the value of Q is

$$Q = (-2 + 3 + 4 - 7)^{2347} = (-2)^{2347} = -2^{2347}.$$

Aha! The values of P and Q are negations of each other. In other words, $P + Q$ is zero. So the expression we want is

$$(2 + 3 + 4 + 7)^{P+Q} = (2 + 3 + 4 + 7)^0 = 1.$$

□

Exercises

2.3.1:

Evaluate the following expressions.

(a) $56 \div 4 \div 3 \cdot 2^0$

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: We evaluate the exponent first:

$$56 \div 4 \div 3 \cdot 2^0 = 56 \div 4 \div 3 \cdot 1 = 14 \div 3 \cdot 1 = 14 \div 3 = \boxed{17}.$$

(b) $7^4(8 - 2^3) + 11^{6(8-32)}$

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: We evaluate inside the parentheses first:

$$\begin{aligned} 7^4(8 - 2^3) + 11^{6(8-32)} &= 7^4(8 - 8) + 11^{6(8-32)} \\ &= 7^4 \cdot 0 + 11^{6(8-32)}. \end{aligned}$$

We don't have to compute 7^4; the product $7^4 \cdot 0$ is 0 no matter what 7^4 is. So, we have

$$\begin{aligned} 7^4 \cdot 0 + 11^{6(8-32)} &= 0 + 11^{6(8-32)} \\ &= 11^{6(8-32)} \\ &= 11^{52-32} \\ &= 11^0 \\ &= \boxed{1}. \end{aligned}$$

(c) $7^0 + 3^2 \cdot 4 - 2(14 - 8 \div 2)$

Preview: Solution

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: We evaluate inside the parentheses first, and then the exponents:

$$\begin{aligned} 7^0 + 3^2 \cdot 4 - 2(14 - 8 \div 2) &= 7^0 + 3^2 \cdot 4 - 2(14 - 4) \\ &= 7^0 + 3^2 \cdot 4 - 2 \cdot 10 \\ &= 1 + 9 \cdot 4 - 2 \cdot 10 \\ &= 1 + 36 - 20 \\ &= 37 - 20 \\ &= \boxed{17}. \end{aligned}$$

2.3.2: Source: MATHCOUNTS

Consecutive powers of 3 are added to form this sequence: $3^0, 3^0 + 3^1, 3^0 + 3^1 + 3^2$, and so on. What is the value of the fourth term of this sequence?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: The fourth term is $3^0 + 3^1 + 3^2 + 3^3$; its value is

$$\begin{aligned} 3^0 + 3^1 + 3^2 + 3^3 &= 1 + 3 + 9 + 27 \\ &= \boxed{40}. \end{aligned}$$

2.3.3:

When $x = 2$ and $y = -2$, what is the value of $x^{x+y} + y^{x-y}$?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: Let's replace each x with 2 and each y with -2:

$$\begin{aligned} x^{x+y} + y^{x-y} &= 2^{2+(-2)} + (-2)^{2-(-2)} \\ &= 2^0 + (-2)^4 \\ &= 1 + 16 \\ &= \boxed{17}. \end{aligned}$$

2.3.4:

Let n be a number. Evaluate $3n^0 \cdot (7n)^0$.

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: Anything raised to the 0th power is 1, so $3n^0 \cdot (7n)^0 = 3 \cdot 1 \cdot 1 = \boxed{3}$.

2.3.5: Source: MATHCOUNTS

Let x be a number. Simplify $6^0x^2 + 6x^2$. Express your answer as a number times a power of x.

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: By replacing 6^0 with 1 and then factoring out x^2, we get

$$6^0x^2 + 6x^2 = 1x^2 + 6x^2 = (1 + 6)x^2 = \boxed{7x^2}.$$

2.4 Negative Exponents

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Problems

Problem 1 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

(1) - Is these equations, what pattern do you see in the numbers on the right?

(2) - Assuming the pattern continues, what is the value of 2^{-3} and 2^{-4}?

(3) - Write the pattern for 2^{-n} in terms of 2^n and $2^{-(n-1)}$.

Problem 2 (1 pts) [show ex 2.4 Negative Exponents]

(1) - Write the pattern for 2^{-n} in terms of 2^n and $2^{-(n-1)}$.

(2) - Assuming the pattern continues, what is the value of 2^{-3} and 2^{-4}?

(3) - Is these equations, what pattern do you see in the numbers on the right?

$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

$2^{-4} = \frac{1}{2^4} = \frac{1}{16}$

Problem 3 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^{-4}}$
- $-x^4$
- x^4
- $\frac{1}{x^4}$

Problem 4 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 5 (1 pts) [show ex 2.4 Negative Exponents]

Simplify x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 6 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 7 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 8 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 9 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 10 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 11 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 12 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 13 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 14 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 15 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 16 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 17 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 18 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 19 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 20 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 21 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 22 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 23 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 24 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 25 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 26 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 27 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 28 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 29 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 30 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 31 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 32 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 33 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 34 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 35 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 36 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 37 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 38 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 39 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 40 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 41 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 42 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 43 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 44 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 45 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 46 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 47 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 48 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 49 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 50 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 51 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 52 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 53 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 54 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 55 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 56 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 57 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 58 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 59 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 60 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 61 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 62 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 63 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 64 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 65 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 66 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 67 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 68 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 69 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 70 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 71 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 72 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 73 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 74 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 75 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 76 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 77 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 78 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 79 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 80 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 81 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 82 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 83 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 84 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 85 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 86 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 87 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 88 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 89 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 90 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 91 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
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Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 93 (1 pts) [show ex 2.4 Negative Exponents]

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- $\frac{1}{x^4}$
- x^4
- $-x^4$
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Problem 94 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 95 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 96 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
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Answer x^{-4} in terms of positive exponents only.

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Problem 99 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 100 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 101 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 102 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 103 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 104 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 105 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 106 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 107 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
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Problem 108 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 109 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
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Problem 110 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 111 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 112 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 113 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 114 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 115 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
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Problem 116 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 117 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
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Problem 118 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 119 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
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Problem 120 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

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Problem 121 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 122 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 123 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 124 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^{-n} = \frac{1}{x^n}$. Take only positive terms in the answer.

Answer x^{-4} in terms of positive exponents only.

Answer x^{-4} in terms of positive exponents only.

Problem 125 (1 pts) [show ex 2.4 Negative Exponents]

Simplify the expression x^{-4} below.

- $\frac{1}{x^4}$
- x^4
- $-x^4$
- $\frac{1}{x^{-4}}$

Problem 126 (1 pts) [show ex 2.4 Negative Exponents]

Assume that $x \neq 0$. x^{-n} is the inverse of x^n and negative exponents work like $x^

2.5 Summary

In this chapter, we introduced the concept of exponents.

Definition: Let a be any number. Let n be an integer. If n is positive, then

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ copies of } a}.$$

For instance, $a^4 = a \cdot a \cdot a \cdot a$. If n is zero, then $a^n = 1$. If a is nonzero and n is positive, then

$$a^{-n} = \frac{1}{a^n}.$$

The expression a^n is called a **power**, with **base** a and **exponent** n.

Important: Let a and b be numbers. Let m and n be integers.

Product of powers (same base): $a^m a^n = a^{m+n}$.
Product of powers (same exponent): $a^n b^n = (ab)^n$.
Quotient of powers (same base): $a^m \div a^n = a^{m-n}$.
Quotient of powers (same exponent): $a^n \div b^n = (a \div b)^n$.
Power of power: $(a^m)^n = a^{mn}$.
Power of reciprocal: $\left(\frac{1}{a}\right)^n = \frac{1}{a^n}$.
Negation in exponent: $a^{-n} = \frac{1}{a^n}$.
Power of negation: If n is even, then $(-a)^n = a^n$. If n is odd, then $(-a)^n = -a^n$.

Before using each property, make sure that the powers in it are defined. For example, in the product rule $a^m a^n = a^{m+n}$, we can't choose $a = 0$ and $m = -1$, because 0^{-1} is undefined.

Challenge Problems

2.57

What is the sum of two consecutive positive integers differ by 47. What is the smallest of the two integers?

Hint: Write one as x and then express the second in terms of x and the difference.

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2.58. Source: MATHCOUNTS ★ ★

Let x be the multiplicative inverse Sphenogram (1987-1989) from Part Zero (or the first different positive integer $x \in \mathbb{Z}$ and $2|x$ and $3|x$ and $5|x$ and $7|x$ and $11|x$ and $13|x$ and $17|x$ and $19|x$ and $23|x$ and $29|x$ and $31|x$ and $37|x$ and $41|x$ and $43|x$ and $47|x$ and $x \equiv 1 \pmod{2}$ and $x \equiv 1 \pmod{3}$ and $x \equiv 1 \pmod{5}$ and $x \equiv 1 \pmod{7}$ and $x \equiv 1 \pmod{11}$ and $x \equiv 1 \pmod{13}$ and $x \equiv 1 \pmod{17}$ and $x \equiv 1 \pmod{19}$ and $x \equiv 1 \pmod{23}$ and $x \equiv 1 \pmod{29}$ and $x \equiv 1 \pmod{31}$ and $x \equiv 1 \pmod{37}$ and $x \equiv 1 \pmod{41}$ and $x \equiv 1 \pmod{43}$ and $x \equiv 1 \pmod{47}$). Find the smallest positive integer x for which $x \equiv 1 \pmod{2}$ and $x \equiv 1 \pmod{3}$ and $x \equiv 1 \pmod{5}$ and $x \equiv 1 \pmod{7}$ and $x \equiv 1 \pmod{11}$ and $x \equiv 1 \pmod{13}$ and $x \equiv 1 \pmod{17}$ and $x \equiv 1 \pmod{19}$ and $x \equiv 1 \pmod{23}$ and $x \equiv 1 \pmod{29}$ and $x \equiv 1 \pmod{31}$ and $x \equiv 1 \pmod{37}$ and $x \equiv 1 \pmod{41}$ and $x \equiv 1 \pmod{43}$ and $x \equiv 1 \pmod{47}$.

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Solution: The first five positive rates are 1, 6, 31, 44, 100, 214, 340, 512, 735, 1000, 1301, and 1728. We can square 1728 and get 2985984 and 2985984 is the smallest square that is divisible by 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47. Consider the sum of positive perfect cubes up to 1728: $1^3 + 2^3 + 3^3 + \cdots + 1728^3$ and $6 \mid 1^3$, $6 \mid 2^3$, and $6 \mid 3^3$, so $6 \mid$ sum of cubes. Therefore, then sum $\boxed{\text{YES}}$.

2.59.

Express $x^2 - 2^2$ as a power of x.

Hint: Let's find expressions for the number as a power of 2, and then convert 2 to a power of x.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let's find expressions for the number as a power of 2, and then convert 2 to a power of x.

$$\begin{aligned} 2^1 &= 2^1 \\ 2^2 &= 2^2 \\ 2^3 &= 2^3 \\ 2^4&= 2^4 \\ &\vdots \\ 2^{10} &= 2^{10} \\ &\vdots \\ 2^{20} &= 2^{20} \\ &\vdots \\ 2^{30} &= 2^{30} \\ &\vdots \\ 2^{40} &= 2^{40} \\ &\vdots \\ 2^{50} &= 2^{50} \\ &\vdots \\ 2^{60} &= 2^{60} \\ &\vdots \\ 2^{70} &= 2^{70} \\ &\vdots \\ 2^{80} &= 2^{80} \\ &\vdots \\ 2^{90} &= 2^{90} \\ &\vdots \\ 2^{100} &= 2^{100} \end{aligned}$$

2.60.

What five-digit positive integer with an 8 in the ten-thousands place is the cube of an integer?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We are looking for a cube between 80,000 and 90,000. The cube of 40 is 64,000 and the cube of 50 is 125,000. In the middle, the cube of 45 is 91,125. A little lower, the cube of 44 is $\boxed{\text{YES}}$.

2.61.

Express $\frac{10^{10}}{2^{10}}$ as a power of $\frac{5}{2}$.

Hint: Can you express 10 as a power of 5?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Because $\frac{10}{2} = 5$, we have

$$\frac{10^{10}}{2^{10}} = \left(\frac{10}{2}\right)^{10} = 5^{10}.$$

Because $5^{10} = \boxed{5^{10}}$, we have

$$5^{10} = 5^{10} = \left(5^2\right)^5 = 5^5,$$

$$5^{10} = 5^{10} = \left(5^5\right)^2 = 5^2,$$

$$5^{10} = 5^{10} = \left(5^{10}\right)^1 = \boxed{5^{10}}.$$

2.62. Source: MATHCOUNTS ★ ★

Express $x^2 + x^4 + x^6 + x^8 + \cdots + 100x^{100}$ as a power of x.

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: By contending the product of squares to the square of a product, and then expressing each number as a power of x, we get

$$\begin{aligned} &x^2 + x^4 + x^6 + x^8 + \cdots + 100x^{100} \\ &= x^2 + x^4 + x^6 + x^8 + \cdots + 10x^{50} \\ &= (1 + 1) + (1 + 1) + \cdots + 10(x^{50}) \\ &= x^2 \cdot (1 + 1) + x^4 \cdot (1 + 1) + \cdots + x^{50} \\ &= x^{50} \cdot (1 + 1) + \cdots + x^{50} \\ &= (x^{50})^2 \\ &= (x^2)^{25} \\ &= \boxed{x^{50}}. \end{aligned}$$

2.63. Source: MATHCOUNTS ★ ★

What the expression $50^{10} \cdot 5^{10}$ be multiplied out how many digits does the number have?

Hint: Zeros are fun! Let's see how many consecutive zeros there make you think about multiplying by

and now can you get 10's in the product?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We can use the power of 5 to a power of 2, and then combine the 2's and 5's to form a power of 10.

$$\begin{aligned} 50^{10} \cdot 5^{10} &= (2 \cdot 5^2)^{10} \cdot 5^{10} \\ &= 2^{10} \cdot 5^{20} \cdot 5^{10} \\ &= 2^{10} \cdot 5^{30} \\ &= 2^{10} \cdot (5 \cdot 5)^{15} \\ &= 2^{10} \cdot (5 \cdot 5)^{15} \\ &= 2^{10} \cdot 10^{15} \\ &= 2^{10} \cdot 10^{15} \\ &= \underbrace{2^{10}}_{\text{4 digits}} \cdot \underbrace{10^{15}}_{\text{16 digits}} \end{aligned}$$

So the number of digits is $5 + 16 = \boxed{21}$.

2.64.

For what value of x does $100{,}000^2 \cdot 100{,}000^2 = 10^x$?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The left side of the equation simplifies as follows:

$$\begin{aligned} 100{,}000^2 \cdot 100{,}000^2 &= (100{,}000 \cdot 100{,}000)^2 \\ &= (10^5 \cdot 10^5)^2 \\ &= (10 \cdot 10)^{10} \\ &= (10 \cdot 10)^{10} \\ &= (10^2 \cdot 10^2 \cdot 10^2 \\ &= (10^2 \cdot 10^2 \cdot 10^2 \\ &= 10^{10}. \end{aligned}$$

So the equation is $\boxed{x = 10}$.

2.65.

Find the number n such that $n \cdot 3^2 = 3^6$.

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We can expand the right side of the equation as follows:

$$\begin{aligned} 3^6 &= (3 \cdot 3^2 = 3^2 \cdot 3^2, \\ &= (3 \cdot 3^2)^2 \\ &= 3^2 \cdot 3^2 \cdot 3^2, \\ &= 3^2 \cdot (3 \cdot 3)^2 \\ &= 3^2 \cdot 3^2 \cdot 3^2 \\ &= 3^2 \cdot 3^2 \cdot 3^2 \\ &= 3^2 \cdot 3^2 \cdot 3

Review Problems

2.31 Hide Solution

Evaluate the following expressions:

(a) $-4(2^3 - 5)^2 - (-4-6)$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We evaluate inside the parenthesis first

$$-4(2^3-5)^2-(-4-6) = -4\cdot 3^2 - (-10) = -4\cdot 9 + 10 = -36+10$$
$$= -26$$

(b) $(5-3)^2 \div (5-3)^3$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We evaluate inside parenthesis, then compute the powers, then multiply, then subtract

$$5-3^2 - (5-3)^2 \div 5-3^3 = 5-3^2 \div 2^2 - 3^3 = 5 - 9 \div 4 - 27 = -22 - 9 \div 4 = -\boxed{\frac{97}{4}}$$

(c) $5 \div 5 + 5^2 \div 5^3$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We compute $2.$ 5^2 inside the parenthesis first.

$$5 \div 5 + 5^2 \div 5^3 = 5 \div 5 + 5^2 \div 5^3 = 5 \div 5 + 5^2 \div 5^3 = 1 + 5^2 \div 5^3$$

Then, we compute

$$5^2 \div 5^3 = 5^2 \div 5^3 = 5^2 \div 5^3 = 5^{-1} = \frac{1}{5}$$

so

$$5 \div 5 + 5^2 \div 5^3 = 5 \div 5 + 5^2 \div 5^3 = 1 + 5^2 \div 5^3 = 1 + \frac{1}{5} = \boxed{\frac{6}{5}}$$

2.32 Hide Solution

Which was from P years?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Since the number $N^P = 100$, the second number is N^{P-100}. Since the first number exceeds the second number by $N^P - 100 = \boxed{N^P - 100}$

2.33 Hide Solution | Source: MATHCOUNTS

What is the value of the sum

$$\sqrt{1000+1} + \sqrt{1000-1} + \sqrt{1000-2}$$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Because $\sqrt{1000-1} = \sqrt{1000+1}$ we have $\sqrt{1000-1} = 31$. Because 1000 is odd, we have $\sqrt{1000-2} = 31$. Since any power of 31 is odd, then $\sqrt{1000+1}$ and $\sqrt{1000-1}$ is odd. So we find that

$$\sqrt{1000+1} + \sqrt{1000-1} + \sqrt{1000-2} = 31 + 31 + 31 = 3\cdot 31 = \boxed{93}$$

2.40 Hide Solution | Source: MATHCOUNTS

Find a positive integer n $n^2 = 2401$. For n

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Inspecting n^2 and n we find the square of 49 is 2401. Since 49^2 is less than and 50^2, is too high, the square root is $\sqrt{2401} = \boxed{49}$

2.41 Hide Solution

What are the 10th twenty (the years 1951 through 2000) over a perfect square?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Inspecting n^2 and n we find the square of 44 is 1936. Since 50^2 is less than and 51^2, is too high, the square root is $\sqrt{2500} = \boxed{50}$

2.42 Hide Solution

What is the value of $x^2 - 2x$ when $x = 3$?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Substituting 3 for x we get

$$x^2 - 2x = 3^2 - 2\cdot 3 = 9 - 2\cdot 3 = 9 - 6 = \boxed{3}$$

2.43 Hide Solution

What is the value of $3x^2 - 2x^3$ when $x = 2$?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Inspecting x and x we get

$$-2x^3 - 3x^2 = -2\cdot 2^3 - 3\cdot 2^2$$
$$= -2\cdot 8 - 3\cdot 4$$
$$= -16 - 12$$
$$= \boxed{-28}$$

2.44 Hide Solution

Find the sum of the positive perfect squares less than 10,000

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: $\boxed{10,000}$ is exactly 100^2. So the positive perfect squares less than $10,000$ are $1^2, 2^2, 3^2, \ldots, 99^2$. The number of such squares is $\boxed{99}$

2.45 Hide Solution | Source: MATHCOUNTS

Let n be a square and $n^2 = 625$. What is the value of N?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: The value of N is 100, which is far too low, and the value of N is 2000, which is too high. In the middle, the value of N is 1275. A little higher, the value of N is 1410. So $N = \boxed{1275}$

2.47 Hide Solution | Source: MATHCOUNTS

Mentally calculate the sum of the first three terms of the sequence whose nth term is $n^2 - n$. Explain how you mentally do the "calculating" that is displays a number that is greater than 20^2?

Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: The sides of 1 is and the sum of 1+1=2. The sides of 2 is between 20^2 (which is 125,000,000) and 30^2 (which is 135,000,000). So $20^2 + 30^2 = 1,000,000 + 2,000,000 = 3,000,000$. So $20^2 + 30^2 = 3,000,000$.

Next, we consider the cube of 20^2. So we ask by the cube of 20^2 greater than 20^2, we note that 20^2 is greater than 100^2. The cube of 100^2 is $1,000,000,000,000$. So 20^2 is greater than 100^2. So the cube of 20^2 is greater than 20^2. Hence the cube of 20^2 is greater than 20^2. Hence the cube of 20^2 is greater than 20^2.

2.48 Hide Solution

Two integers m & n are relatively prime if the only common factor they share is 1. Find the smallest integer n that is relatively "coprime" to the integer m m n m n n $n

$\frac{3}{3}\frac{3}{3}\frac{3}{3}\frac{3}{3}$

Solution: $3 \times 3 \times 3 - 3$

$\frac{3}{3}\frac{3}{3}\frac{3}{3}\frac{3}{3}$

Solution: $(3 \times 5 - 7) \times 3$

$\frac{3}{3}\frac{4}{4}\frac{8}{8}$

Solution: $(7 - 3) \times 4 + 8$

$\frac{3}{3}\frac{5}{5}\frac{13}{13}$

Solution: $(5 \times 13 + 7) \div 3$

I was interested on the Israeli radio for five minutes and I said that more than 2000 years ago, Euclid proved that there are infinitely many primes. Immediately the host interrupted me and asked, "Are there still infinitely many primes?"

— Noga Alon

CHAPTER 3 Number Theory

Number theory is the study of integers.

3.1 Multiples

We know that 12 equals 3 times 4. In other words, 12 equals some integer times 4. For that reason, we say that 12 is a **multiple** of 4.

> **Definition:** Let a and b be numbers. We say that a is a **multiple** of b if a equals b times some integer. In other words, a is a multiple of b if there is an integer n such that $a = bn$.

For instance, 7 is not a multiple of 4, because we cannot write 7 as the product of 4 and an integer. Note that -12 is a multiple of 4, because -12 equals -3 times 4. Similarly, 0 is a multiple of 4, because 0 equals 0 times 4.

In this chapter, we'll talk about division differently than we did in Chapter 1. We'll use the "quotient and remainder" concept of division, which you probably used when you first learned about division. As an example, when we divide 13 by 4, the quotient is 3 and the remainder is 1.

Using this view of division, we can say that an integer a is a multiple of an integer $b \neq 0$ if we divide a by b has remainder 0. So, for example, 12 is a multiple of 4 because 12 divided by 4 has remainder 0, while 13 is not a multiple of 4 because 13 divided by 4 has remainder 1.

Problems

Problem 3.1 ⇧ Jump to Solution

(a) Both 147 and 357 are multiples of 7. Is $147 + 357$ a multiple of 7?

(b) Suppose a is a multiple of 7. Must $7a$ be a multiple of 7?

(c) Suppose r and s are multiples of 7. Must $r + s$ be a multiple of 7?

(d) Suppose a is a multiple of 7. Is it possible for $k \cdot a$ to be a multiple of 7?

Problem 3.2 *Source: MODMO* ⇧ Jump to Solution

What number between 100 and 200 is both a perfect square and a multiple of 7?

Problem 3.3 *Source: MATHCOUNTS* ⇧ Jump to Solution

What is the greatest three-digit number that is a multiple of 13?

Problem 3.4 ⇧ Jump to Solution

(a) How many integers between 2 and 1001 are multiples of 5?

(b) How many integers between 150 and 300 are multiples of 9?

Problem 3.5 ⇧ Jump to Solution

(a) Must every multiple of 15 also be a multiple of 3?

(b) Must every multiple of 3 also be a multiple of 15?

Problem 3.1

(a) Both 147 and 357 are multiples of 7. Is $147 + 357$ a multiple of 7?

(b) Suppose a is a multiple of 7. Must $k \cdot a$ be a multiple of 7?

(c) Suppose r and s are multiples of 7. Must $r + s$ be a multiple of 7?

(d) Suppose a is a multiple of 7. Is it possible for $k \cdot a$ to be a multiple of 7?

Solution for Problem 3.1:

(a) We have $147 = 7 \cdot 21$ and $357 = 7 \cdot 51$. We can add these two using the distributive property:

$$147 + 357 = 7 \cdot 21 + 7 \cdot 51 = 7 \cdot (21 + 51) = 7 \cdot 72,$$

So, $147 + 357$ can be written as 7 times an integer, which means that $147 + 357$ is a multiple of 7.

(b) Because k is a multiple of 7, there is an integer a for which $k = 7a$. So, adding 7 to k gives $k + 7 = 7a + 7$. We can then factor out 7 to write

$$k + 7 = 7a + 7 = 7 \cdot a + 7 \cdot 1 = 7(a + 1).$$

Since a is an integer, so is $a + 1$. Therefore, we can write $k + 7$ as 7 times some integer, which means that $k + 7$ is indeed a multiple of 7. Moreover, our work above shows that adding 7 to a multiple of 7 produces the next multiple of 7.

(c) We can use our solution to part (b) of as a guide. Because r is a multiple of 7, there is an integer m such that $r = 7m$. Similarly, because s is a multiple of 7, there is an integer n such that $s = 7n$. We then have $r + s = 7m + 7n$, and factoring out 7 gives

$$r + s = 7m + 7n = 7(m + n).$$

Because m and n are integers, $m + n$ is an integer. So the equation $r + s = 7(m + n)$ says that $r + s$ is 7 times some integer. In other words, $r + s$ is a multiple of 7.

(d) Part (b) tells us that the sum of any two multiples of 7 is a multiple of 7. So, if k is a multiple of 7, then $k + 21$ and $k + 28$ are both multiples of 7. Moreover, because $k + 21$ and $k + 28$ are 7 apart, they are consecutive multiples of 7. So, there are no multiples of 7 between $k + 21$ and $k + 28$. Specifically, there are no multiples of 7 between k and $k + 7$.

We might also have thought about division to see why $k + 28$ is not a multiple of 7. Since $k + 21$ is a multiple of 7 and $k + 28$ is 2 greater than $k + 21$, we know that $k + 28$ divided by 7 has a remainder of 2. So, $k + 28$ is not a multiple of 7.

□

Part (b) of Problem 3.1 tells us that if we start at any multiple of 7 and count by 7s, we'll generate multiples of 7 from our starting point onward. Similarly, we can also generate multiples of 7 by starting from some multiple of 7 and subtracting 7 over and over. For example, the following are all multiples of 7:

$$700, 693, 686, 679, 672, 665, 658, 651, \ldots.$$

> **Important:** If we start from any multiple of n and count either upward or downward by n's, all the numbers we hit will be multiples of n.

Part (c) of Problem 3.1 is an example of a neat property of multiples. If we add two multiples of a number, we get another multiple of the same number. Similarly, the difference between any two multiples of a number is another multiple of that same number.

> **Important:** If a and b are multiples of c, then both $a + b$ and $a - b$ are multiples of c.

Finally, part (d) of Problem 3.1 is an example of why we know that we don't skip any multiples of 7 when we count by 7s starting from a multiple of 7. We can go through basically the same steps to see that if we add a multiple of 7 and a number that is not a multiple of 7, then the resulting sum is not a multiple of 7.

> **Important:** If we start with a multiple of n and add (or subtract) a number that is not a multiple of n, then the resulting sum (or difference) is not a multiple of n.

"Multiplication Facts"

Problem 3.2 *Source: MODMO*

What number between 100 and 200 is both a perfect square and a multiple of 7?

> **WARNING!** In this book, when we say "between 100 and 200," we don't include 100 and 200.

Solution for Problem 3.2: The perfect squares between 100 and 200 are 121, 144, 169, and 196. When we divide each of these squares by 7, we find that only 196 is a multiple of 7. To be specific, 196 is 28 times 7. So the answer is 196.

Here's another way to see that 196 is a multiple of 7. Note that 196 is 14 times 14, and 14 is a multiple of 7. Specifically, we have $14 = 2 \cdot 7$, so

$$196 = 14 \cdot 14 = 14 \cdot 2 \cdot 7 = 28 \cdot 7.$$

Therefore, 196 is a multiple of 7. □

Problem 3.3 *Source: MATHCOUNTS*

What is the greatest three-digit number that is a multiple of 13?

Solution for Problem 3.3: The three-digit numbers are the integers from 100 to 999. We are looking for the greatest such integer that is a multiple of 13. So let's divide 999 by 13. We get a quotient of 76 and a remainder of 11. In other words, 999 is 11 more than a multiple of 13. So the multiple of 13 we want is $999 - 11$, which is 988.

We also can solve this problem using our insights about multiples from Problem 3.1. We find a multiple of 13 near 1000, and then add or subtract multiples of 13 to find the largest three-digit multiple of 13. For example, we know that 1000 and 100 are multiples of 10, so $1000 - 100 = 1170$ is a multiple of 13, as is $1170 - 100 = 1080$. Therefore,

$$1040 = 2(13) + 1040 = 2(13) + 1040 - 39 = 1001$$

is a multiple of 13, which means that the largest multiple of 13 less than 1000 is $1001 - 13 = 988$. □

Let's try some counting problems involving multiples.

Problem 3.4 *Source: MATHCOUNTS*

(a) How many integers between 2 and 1001 are multiples of 5?

(b) How many integers between 150 and 300 are multiples of 9?

Solution for Problem 3.4:

(a) The multiples of 5 between 2 and 1004 are

$$5, 10, 15, \ldots, 995, 1000.$$

How many numbers is that? To make the counting easier, let's express each number as an integer times 5:

$$1 \cdot 5, 2 \cdot 5, 3 \cdot 5, \ldots, 199 \cdot 5, 200 \cdot 5.$$

In other words, we are multiplying 5 by 1, 2, 3, up through 200. So the number of such multiples is 200.

(b) Let's simplify the problem a bit by temporarily ignoring the 150, which makes this problem look like part (a).

> **Concept:** One approach a difficult problem is to relate the problem to an easier problem you already know how to do.

The positive multiples of 9 less than 300 are

$$9, 18, \ldots, 297.$$

Let's express these numbers as an integer times 9:

$$1 \cdot 9, 2 \cdot 9, \ldots, 33 \cdot 9.$$

So there are 33 such multiples.

We need to remove the numbers less than 150. The positive multiples of 9 less than 150 are

$$9, 18, \ldots, 144.$$

We can express each as an integer times 9:

$$1 \cdot 9, 2 \cdot 9, \ldots, 16 \cdot 9.$$

So there are 16 such multiples.

If we remove these 16 multiples of 9 less than 150 from the 33 multiples of 9 less than 300, we are left with $33 - 16 = 17$ multiples between 150 and 300.

□

Problem 3.5

(a) Must every multiple of 15 also be a multiple of 3?

(b) Must every multiple of 3 also be a multiple of 15?

Solution for Problem 3.5:

(a) Suppose that m is a multiple of 15, so we have $m = 15n$ for some integer n. Since $15 = 3 \cdot 5$, we can write $m = 3 \cdot 5n$, which means m is 3 times the integer $5n$. Therefore, m is a multiple of 3, as well.

(b) Not every multiple of 3 is a multiple of 15. For example, 3, 6, 9, 12, and 18 are all multiples of 3, but none of these numbers is a multiple of 15.

□

Part (a) of Problem 3.5 tells us that because 15 is a multiple of 3, every multiple of 15 is also a multiple of 3. This is an example of another neat property of multiples:

> **Important:** If a is a multiple of b, then every multiple of a is also a multiple of b.

Exercises

3.1.1:

What are the 39 smallest positive multiples of 6?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We can list positive multiples of 6 by starting from 6 and counting by 6's:

$$\boxed{6, 12, 18, 24, 30, 36, 42, 48, 54, 60}$$

3.1.2: *Source: AMC 8*

There are many positive two-digit multiples of 7, but only two of these multiples have a digit sum of 10. (The **digit sum** of an integer is the sum of its digits.) What are these two multiples of 7?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We find the positive multiples of 7 less than 100 by starting from 7 and counting by 7's. We compute the sum of the digits of each, and find that the two with digit sum 10 are $\boxed{28 \text{ and } 91}$.

3.1.3:

What is the sum of all positive integers less than 100 that are multiples of 13?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The desired multiples of 13 are 13, 26, 39, 52, 65, 78, and 91. These have sum

$$13 \cdot 1 + 13 \cdot 2 + 13 \cdot 3 + 13 \cdot 4 + 13 \cdot 5 + 13 \cdot 6 + 13 \cdot 7$$

$$= 13(1 + 2 + 3 + 4 + 5 + 6 + 7) = 13(28) = \boxed{364}.$$

3.1.4:

What is the least positive number that can be added to 173 so that the result is a multiple of 20?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The sum of the desired number and 173 must be the smallest multiple of 20 that is greater than 173, which is 180. So, the desired number is $180 - 173 = \boxed{7}$.

3.1.5:

What is the smallest positive four-digit multiple of 11?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since 1000 divided by 17 leaves remainder 14, we know that 1000 is 14 more than a multiple of 17. Therefore, $1000 - 14 = 986$ is a multiple of 17, and $986 \div 17 = \boxed{1003}$ is the smallest four-digit multiple of 17. (We also might have noticed that since 1000 divided by 17 leaves a remainder of 14, we know that 1000 is 3 less than a multiple of 17.)

3.1.6:

What is the greatest four-digit multiple of 18?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The greatest four-digit number is 9999. Dividing 9999 by 18 leaves a remainder of 9, so $9999 - 9 = \boxed{9990}$ is the largest four-digit multiple of 18.

Here's a quick way to see that 9999 divided by 18 has remainder 9. The number 9999 is clearly a multiple of 9, since $9999 = 9 \cdot 1111$. But 9999 is not a multiple of 2. The multiples of 9 alternate between even numbers that are multiples of 18 (like 0, 18, 36, 54) and odd numbers that are 9 more than multiples of 18 (like 9, 27, 45). So, 9999 must be 9 more than a multiple of 18.

3.1.7: *Source: MATHCOUNTS*

What is the greatest three-digit multiple of 33 that can be written using three different digits?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: First, we find the greatest multiple of 33. Since 990 divided by 33 has remainder 0, we know that 990 is the largest three-digit multiple of 33. But the digits of 990 are not all different. So, we consider the next smaller multiple of 33, which is $990 - 33 = 957$. The digits of 957 are all different, so $\boxed{957}$ is the greatest three-digit multiple of 33 that can be written with three different digits.

3.1.8: *Source: MATHCOUNTS*

How many integers between 17 and 2678 are multiples of 11?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The smallest multiple of 11 between 17 and 2678 is $2 \cdot 11 = 22$. But what is the largest? Dividing 2678 by 11 gives quotient 243 and remainder 5, so the largest multiple of 11 between 17 and 2678 is $243 \cdot 11$. Therefore, the integers we must count are $2 \cdot 11, 3 \cdot 11, 4 \cdot 11, \ldots, 243 \cdot 11$. This list has all of the first 243 positive multiples of 11 except 11 itself, so there are $\boxed{242}$ integers between 17 and 2678 that are multiples of 11.

3.2 Divisibility Tests

Another way to say that 52 is a multiple of 4 is to say that 52 is **divisible by** 4.

Definition: Let a be a number, and let b be a non-zero number. We say that a is **divisible** by b if a is a multiple of b.

So, for example, 12 is divisible by 3 because 12 is 4 equals the integer 3. Similarly, 0 is divisible by every nonzero integer, since 0 divided by any nonzero integer is the integer 0.

The phrase "divisible" is a little bit more flexible than the phrase "multiple" in that it allows us to say that a and b are both b, then "a is divisible by b" and "a is a multiple of b" say the same thing. If $a = b = 0$ then a is still a multiple of b, but it is not divisible by b since we can't divide by 0.

In this section, we explore some shortcuts for determining whether or not one number is divisible by another. You've probably familiar with several of these shortcuts already, but if you are like most people, you don't also understand why these shortcuts work.

Problem

Problem 3.2.1 ▷ Jump to Solution

(a) Note that

$$12222 - 12000 = 1200 + 22.$$

How can we use this fact to tell that 12222 is divisible by 2?

(b) How can we quickly tell that 301827 is not divisible by 2?

Problem 3.2.2 ▷ Jump to Solution

How can we tell at a glance whether or not a number is a multiple of 10?

Problem 3.2.3 ▷ Jump to Solution

(a) Is every multiple of 10 also a multiple of 5?

(b) How can we tell at a glance whether or not a number is a multiple of 5?

(c) How can we tell at a glance whether or not a number is a multiple of 2?

Problem 3.2.4 ▷ Jump to Solution

(a) Which of the following numbers are divisible by 3?

$$12, \quad 312, \quad 512, \quad 2512, \quad 42427$$

(b) Is every multiple of 100 also a multiple of 4?

(c) Use your answer to part (b) to explain why 5427422100 is divisible by 4.

(d) Use your answer to part (b) to explain why 4242512 is not divisible by 4.

Problem 3.2.5 ▷ Jump to Solution

In the problem, we discover a method for determining whether or not a number is divisible by 3. Notice that

$$765 = 7 \cdot 100 + 6 \cdot 10 + 5$$
$$= 7 \cdot (99 + 1) + 6 \cdot (9 + 1) + 5$$
$$= 7 \cdot 99 + 7 + 6 \cdot 9 + 6 + 5$$
$$= 7 \cdot 99 + 6 \cdot 9 + (7 + 6 + 5)$$

(a) As shown above, 765 equals $7 \cdot 99 + 6 \cdot 9 + (7 + 6 + 5)$ Notice that $7 + 6 + 5$ is divisible by 3. Explain why this tells us that 765 is divisible by 3.

(b) Notice that $9 = 3 \cdot 3 + 0$. Use this fact to show that $9(3k+1)$ is also a multiple of 3.

(c) Determine whether or not 1561 is divisible by 3 without dividing 1561 by 3.

Problem 3.2.11 ▷ Jump to Solution

In the previous problem, we noted that

$$765 = 7 \cdot 99 + 6 \cdot 9 + (7 + 6 + 5).$$

(a) Are both 99 and 9 divisible by 3?

(b) Is 765 divisible by 3?

(c) Is 765 divisible by 9?

(d) Explain why 1456 is not divisible by 9 without dividing 1456 by 9.

Problem 3.2.12 ▷ Jump to Solution

(a) Is the prime digit in the four-digit number 5453, 5 4 5 3 divisible by 3 and by 9, then what are all the possible values of $?$

Problem 3.2.13 ▷ Jump to Solution

(a) Note that $12222 + 10000 = 22222$. How can we use this fact to tell that 12222 is divisible by 11?

(b) How can we quickly tell that 301827 is not divisible by 11?

Hint: Consider the number 1001.

(c) Since 13 is a multiple of 3, we know that 13000, 13001 and 13 are all multiples of 3. From our work in Section 3.1, we know that the sum of multiples is a multiple of 3. So, we know that $13000 + 13001 + 13 = 26014$ is a multiple of 3. Explain why 26014 is not a multiple of 11.

(d) Looking back at the previous part, we know that $26000 = 26000 + 3001 \cdot 0$ is a multiple of 3 because 26 is a multiple of 3. Since 26 is not 11 more than a multiple of 11, we know that dividing 26000 by 11 will give a remainder of 1. Therefore, 26 is not divisible by 11.

Our work in Problem 3.2.6 gives us two strategies for testing whether a number is divisible by some other number b.

- Write out a sum of numbers that are divisible by b. We did this when we wrote 12222 as $12000 + 1200 + 22$.
- Compare a nonzero number that we are testing for divisibility by b to the nearest multiple of b which we can notice that 301827 is 1 greater than 301826.

For the rest of this section, we will use these two tactics to develop strategies for testing for divisibility by specific numbers.

Problem 3.2.14 ▷ Jump to Solution

How can we tell at a glance whether or not a number is a multiple of 10?

Solution for Problem 3.2.14: We compute the product of 10 and an integer by placing a 0 at the end of the integer. For example, $10 \cdot 3 = 30$, $10 \cdot 4 = 40$, $10 \cdot 5 = 50$, $10 \cdot 6 = 60$, $10 \cdot 7 = 70$, $10 \cdot 8 = 80$, $10 \cdot 9 = 90$. Every multiple of 10 ends in a 0 and, conversely, every 0 ends in a 0 is a multiple of 10, since it equals 10 times the integer that remains when its only digit is removed $90 \div 10 = 9$.

Problem 3.2.15 ▷ Jump to Solution

(a) Is every multiple of 10 also a multiple of 5?

(b) How can we tell at a glance whether or not a number is a multiple of 5?

(c) How can we tell at a glance whether or not a number is a multiple of 2?

Solution for Problem 3.2.

(a) Because 10 is a multiple of 5, every multiple of 10 is also a multiple of 5.

(b) The multiples of 10 are the integers that end in 0. So, we can tell if a number is a multiple of 5 by seeing if its last digit is 0 and 5. But there are other multiples of 5, such as 25, 35, 45, 55, 65, 75, 85, 95, 105, 115, 125, 135, 145, 155, 165, 175, 185, 195, 205, 215, 225, 235, 245, 255, 265, 275, 285, 295, 305, 315, 325, 335, 345, 355, 365, 375, 385, 395, 405, 415, 425, 435, 445, 455, 465, 475, 485, 495, 505, 515, 525, 535, 545, 555, 565, 575, 585, 595, 605, 615, 625, 635, 645, 655, 665, 675, 685, 695, 705, 715, 725, 735, 745, 755, 765, 775, 785, 795, 805, 815, 825, 835, 845, 855, 865, 875, 885, 895, 905, 915, 925, 935, 945, 955, 965, 975, 985, 995.

(c) Every multiple of 10 is a multiple of 2, so 100 is a multiple of 2. Since 5 is obviously a multiple of 2, the sum $100 + 5$ is the sum of two multiples of 2. The sum of two multiples of 2 must be a multiple of 2. So 105 is a multiple of 2.

We can do the same with any integer. That is, we can write any integer in the form $10 \cdot a$ plus a multiple of 10 and the units digit of the original integer. The multiple of 10 is a multiple of 2, so the same is a multiple of 2 whenever the units digit is also a multiple of 2. The units digit that is a multiple of 2 are 0, 2, 4, 6, 8. So, we can say that the number that is 0 or 2 or 4 or 6 or 8 is a multiple of 2.

Any number that does not have 0 or 2 or 4 or 6 or 8 as its units digit is not divisible by 2. Such a number cannot be a multiple of 2. For example, 41 is not divisible by 2 because $41 = 40 + 1$ and 40 is a multiple of 2.

(d) Our key observation in the previous part is that 10 is a multiple of 5. But 10 is also a multiple of 2. So, we can go through the same logic as in the previous part to see that every multiple of 10 is also a multiple of 5. For example, we can write 100 as $10 \cdot 10$. Since 10 is a multiple of 10, we know 10 is a multiple of 2 and 5 and so both multiples of 2, we know that $100 + 5 = 105$ is a multiple of 2.

Any integer that has an even units digit $(0, 2, 4, 6, 8)$ is a multiple of 2. Any integer that has an odd units digit $(1, 3, 5, 7, 9)$ is not a multiple of 2.

Problem 3.2.

(a) Which of the following numbers are divisible by 4

$$12, \quad 312, \quad 512, \quad 2512, \quad 42427$$

(b) Is every multiple of 100 also a multiple of 4?

(c) Use your answer to part (b) to explain why 5427422100 is divisible by 4.

(d) Use your answer to part (b) to explain why 4242512 is not divisible by 4.

Problem 3.2.19 ▷ Jump to Solution

In this problem, we discover a method for determining whether or not a number is divisible by 9. Notice that

$$765 = 7 \cdot 100 + 6 \cdot 10 + 5$$
$$= 7 \cdot (99 + 1) + 6 \cdot (9 + 1) + 5$$
$$= 7 \cdot 99 + 7 + 6 \cdot 9 + 6 + 5$$
$$= 7 \cdot 99 + 6 \cdot 9 + (7 + 6 + 5)$$

(a) As shown above, 765 equals $7 \cdot 99 + 6 \cdot 9 + (7 + 6 + 5)$ Notice that $7 + 6 + 5$ is divisible by 9. Explain why this tells us that 765 is divisible by 9.

(b) Notice that $9 = 9 \cdot 1 + 0$. Use this fact to show that $9(3k+1)$ is also a multiple of 9.

(c) Determine whether or not 1561 is divisible by 9 without dividing 1561 by 9.

Solution for Problem 3.2.19

(a) Since 9 is a multiple of 9, so is $7 \cdot 99$. Therefore, the sum $7 \cdot 99 + 6 \cdot 9 + (7 + 6 + 5)$ is the sum of three multiples of 9, namely, $7 \cdot 99 \cdot 9$, $6 \cdot 9$ and $7 + 6 + 5$. This means that the sum itself, which equals 765, is a multiple of 9.

(b) Let

$$765 = 7 \cdot 99 + 6 \cdot 9 + (7 + 6 + 5)$$

we have

$$9(3k+1) = 0 \cdot 1000 + 0 \cdot 100 + 0 \cdot 10 + 9$$
$$= 0 \cdot (999 + 1) + 0 \cdot (99 + 1) + 0 \cdot (9 + 1) + 9$$
$$= 0 \cdot 999 + 0 + 0 \cdot 99 + 0 + 0 \cdot 9 + 0 + 9$$
$$= 0 \cdot 999 + 0 \cdot 99 + 0 \cdot 9 + (0 + 0 + 9)$$

We have $9 = 9 \cdot 1 + 0 = 9$, which is a multiple of 9. Each of $0 \cdot 999$, $0 \cdot 99$, and $0 \cdot 9$ also a multiple of 9. So, we have written $9(3k+1)$ as the sum of multiples of 9. The sum of multiples is a multiple of 9.

Notice that we say they are to write $9(3k+1)$ as the sum of several multiples of 9 plus the sum of the digits of $9(3k+1)$.

$$9(3k+1) = 999 \cdot 0 \cdot 99 \cdot 0 \cdot 9 \cdot 0 + 0 + 9$$

Similarly, we can write any positive integer as the sum of multiples of 9 plus the sum of the digits of the integer. The sum of the digits then tells us whether or not the original integer is a multiple of 9.

Important: If the sum of the digits of an integer is a multiple of 9, then the integer is divisible by 9. Otherwise, the integer is not divisible by 9.

(c) The sum of the digits of 1561 is $5 \cdot 1 + 9 \cdot 1 + 6 \cdot 1 = 21$. This sum is divisible by 9, so 1561 is divisible by 9.

Problem 3.2.11 ▷ Jump to Solution

In the previous problem, we noted that

$$765 = 7 \cdot 99 + 6 \cdot 9 + (7 + 6 + 5).$$

(a) Are both 99 and 9 divisible by 3?

(b) Is 765 divisible by 3?

(c) Is 765 divisible by 9?

(d) Explain why 1456 is not divisible by 9 without dividing 1456 by 9.

Solution for Problem 3.2.11:

(a) Yes because $99 = 3 \cdot 33$ and $9 = 3 \cdot 3$, so both 99 and 9 are divisible by 3.

(b) In Problem 3.1.10, we noted $765 = 255 \cdot 3$.

$$765 = 7 \cdot 99 + 6 \cdot 9 + (7 + 6 + 5)$$

$$\begin{aligned}
&\overset{67242}{\cdot} \\
&= 6 \cdot (1000 + 7 \cdot 1000 + 3 \cdot 100 + 4 \cdot 10 + 2 \\
&= 6 \cdot (999 + 1 + 7 \cdot (999 + 1) + 3 \cdot (99 + 1) + 4 \cdot (9 + 1) + 2 \\
&= 6 \cdot 999 + 6 + 7 \cdot 999 + 7 + 3 \cdot 99 + 3 + 4 \cdot 9 + 4 + 2 \\
&= 6 \cdot 999 + 7 \cdot 999 + 3 \cdot 99 + 4 \cdot 9 + (6 + 7 + 3 + 4 + 2)
\end{aligned}$$

any number consisting only of 9s is a multiple of 3, and so therefore a multiple of 3. So, each product in the sum $6 \cdot 999 \cdot 7 \cdot 999 \cdot 3 \cdot 99 \cdot 4 \cdot 9$ is a multiple of 3. The sum is a multiple of 3. This means that this sum is a multiple of 3. Since $6 + 7 + 3 + 4 + 2 = 22$ is also a multiple of 3, we know that 765 is the sum of multiples of 3. Therefore, 765 is a multiple of 3.

(c) We have the sum of the digits of an integer is a multiple of 9 then the integer is divisible by 9.

(d) When we follow the same process as in part (c), we get

$$6148 = 6 \cdot 1000 + 1 \cdot 100 + 4 \cdot 10 + 8$$
$$= 6 \cdot (999 + 1) + 1 \cdot (99 + 1) + 4 \cdot (9 + 1) + 8$$
$$= 6 \cdot 999 + 6 + 1 \cdot 99 + 1 + 4 \cdot 9 + 4 + 8$$

Since $6 \cdot 999 \cdot 1 \cdot 99$ and $4 \cdot 9$ are multiples of 3, we know $6 \cdot 999 + 1 \cdot 99 + 4 \cdot 9$ is a multiple of 3. So, we can that it is $6 + 1 + 4 + 8 = 19$ more than a multiple of 3. The only two numbers in the form which the last two digits form a multiple of 3 are 12 and 15. Because 19 is not a multiple of 3, we know that 6148 is not a multiple of 3. Therefore, 6148 is not a multiple of 9.

Our work in Problem 3.1 suggests the following rule for divisibility by 3, which looks a lot like our rule for divisibility by 9.

Important: If the sum of the digits of an integer is a multiple of 3, then the integer is divisible by 3. Otherwise, the integer is not divisible by 3.

Problem 3.2.12 ▷ Jump to Solution

(a) Is the prime digit in the four-digit number 5453, 5 4 5 3 divisible by 3 and by 9, then what are all the possible values of $?$

Solution for Problem 3.2.12: Since the number is divisible by 3, the number formed by its two last digits, 53, must be divisible by 3. The only two-digit multiples of 3 with 3 as the units digit are 33 and 63. The only two-digit numbers whose sum is 9 are 63 and 27.

The four-digit number 5453 must be divisible by 3. Its check for divisibility by 3 is the sum of the digits of 5453 and 12. Adding $5 + 4 + 5 + 3 + 12$ gives 29. The only multiples of 3 that are 29 or more are 30 and 33. So we have $5 + 4 + 5 + 3 + 29$ which is not divisible by 3. Therefore the only possible value of $?$ is 3.

Problem 3.2.13 ▷ Jump to Solution

(a) Is the two digit in the five-digit number 21575, 2 1 5 7 5 divisible by 3, then what is the possible values of $?$

Solution for Problem 3.2.13: Since the number is divisible by 3, the sum of its digits is divisible by 3. The sum of the digits of the number is $2 + 1 + 5 + 7 + 5 = 20$. The only two-digit numbers whose sum is 20 are 11 and 29. The only two-digit multiples of 3 are 12 and 21. The number formed by the two-digits of 21575 is 15. Since 15 is not divisible by 3, we know that 21575 is not divisible by 3.

Here is a summary of all the divisibility tests we learned in this section.

Number	Divisible under which rule divisible by the number
2	Units digit of 0 or 2, 4, 6 or 8
3	Sum of the digits is a multiple of 3
4	Number formed by last two digits of 4 is a multiple of 4
5	Units digit of 0 or 5
9	Sum of the digits of 9 is a multiple of 9
10	Units digit of 0

Divisibility Rules for 2, 5, 9 and 10

Divisibility Rules for 3 and 4

Exercises

3.2.1:

Which of the following numbers are divisible by 3?

$$48,624, \quad 580,335, \quad 60,201,060 \quad 9,071,1187$$

You may type any additional notes you have here.

Hide Solution

Save

Solution Numbers that end in 0 or 5 are divisible by 5, all other numbers are not divisible by 5. So, $580,335$ and $60,201,060$ are the only numbers in the list that are divisible by 5.

3.2.2:

Which of the following numbers are divisible by 4?

$$48,624, \quad 580,335, \quad 60,201,060 \quad 9,071,1187$$

You may type any additional notes you have here.

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Your Solution: Solution

Solution: If the last two digits of a number form a number that is a multiple of 4, then the original number is a multiple of 4. Otherwise, the original number is not a multiple of 4. The only two numbers in the list for which the last two digits form a multiple of 4 are $48,624$ and $60,201,060$.

3.2.3:

Which of the following numbers are divisible by 9?

$$48,624, \quad 580,335, \quad 60,201,060 \quad 9,071,1187$$

You may type any additional notes you have here.

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Your Solution: Solution

Solution: To test if a number is divisible by 9, we turn the digits of the number into the first

$$48,624 = 4 + 8 + 6 + 2 + 4 = 24$$
$$580,335 = 5 + 8 + 0 + 3 + 3 + 5 = 24,$$
$$60,201,060 = 6 + 0 + 2 + 0 + 1 + 0 + 6 + 0 = 15,$$
$$9,071,1187 = 9 + 0 + 7 + 1 + 1 + 8 + 7 = 33$$

If the sum is a multiple of 9, then the original number is divisible by 9. Otherwise, the original number is not divisible by 9. So, the only two numbers that are divisible by 9 are $580,335$ and $9,071,1187$.

3.2.4:

Which of the following numbers are not divisible by 7:

$$7,000,014, \quad 14,455, \quad 7,777,726,$$
$$12,720,891, \quad 99,767$$

You may type any additional notes you have here.

Hide Solution

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Your Solution: Solution

Solution: Four of the five numbers can be written as the sum of numbers that are obviously multiples of 7.

$$7,000,014 = 7,000,000 + 14,$$
$$14,455 = 14,000 + 455,$$
$$7,777,726 = 7,000,000 + 777,726,$$
$$12,720 = 10,500 + 720 + 63.$$

Any sum of multiples of 7 must also be a multiple of 7, so these four numbers are multiples of 7. The remaining number in the list is $12,720,891$.

$$12,720,891 = 12,700,000 + 700,000 + 21,000 + 191.$$

The numbers on the right side are clearly divisible by 7 so $12,720,891$ is a multiple of 7. Since $12,720,891$ is less than $12,721,000$, we know that $12,720,891$ is not divisible by 7.

3.2.5:

How many numbers from 1 through 100 have a 2 in the units place (ones place) and are divisible by 4?

You may type any additional notes you have here.

Hide Solution

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Your Solution: Solution

Solution: Four of the five numbers can be written as the sum of numbers that are obviously multiples of 7.

$$\begin{aligned}
&\{10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, \\
&42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, \\
&70, 72, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100\}
\end{aligned}$$

3.2.6: Review: $\boxed{\text{Review: 3.2.6}}$

2001^{2001} and 2006 are three-digit numbers such that $2001^{2001} - 2006 = 200$ 2 0 2 0 is divisible by 9, then what number does 200 represent?

Hide Solution

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Your Solution: Solution

Solution: Since 2006 is divisible by 9, the sum of its digits must be divisible by 9. The sum of the digits of 2006 is $2 + 0 + 0 + 6 = 8$. The only digit for 2001^{2001} is 2. So, it is easy to see that $2001^{2001} - 2006$ is not divisible by 9. Since $2001^{2001} - 2006 = 200$ 2 0 2 0, we know that 2 is not divisible by 9.

3.2.7:

What is the largest digit D for which the number $211,4D7$ is divisible by 7?

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Your Solution: Solution

Solution: In order for $211,4D7$ to be divisible by 3, the sum of its digits must be divisible by 3. The sum is $2 + 1 + 1 + 4 + D + 7 = 15 + D$. The only digit for D that will make the sum of $15 + D$ divisible by 3 are 0, 3 and 6. The largest of these is $\boxed{6}$.

3.2.8a:

Consider the rules we found for test for divisibility by 2 and by 4. Can you find a similar rule for divisibility by 8?

You may type any additional notes you have here.

Hide Solution

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Your Solution: Solution

Solution: We found that a number is divisible by 2 $\Leftrightarrow$ 2^1 if its final 1 digit is divisible by 2. We found that a number is divisible by $4 = 2^2$ if its final 2 digits form a number that is divisible by 4. We might guess that a number is divisible by $8 = 2^3$ if its final 3 digits form a number that is divisible by 8.

Consider the number $1,000,000$. Its last digit is 0 and its last two digits form a number that is divisible by 4. But $1,000,000$ is not divisible by 8. Since $1,000,000$ is not divisible by 8, we know that $1,000,000$ is not divisible by 2^3. The key step in our proof for the divisibility by 4 was the fact that 100 is divisible by 2. The key step in our proof for the divisibility by 8 is the fact that $1000 = 10^3$ is divisible by 8. So, we expect that a key step in proving the rule for divisibility by 8 is the fact that $1000 = 10^3$ is divisible by 8.

$1,000,000$ is equal to $1000 \cdot 1,000$ so $1,000,000$ is divisible by 8. Therefore, any multiple of 1000 is a multiple of 8. Any number can be written as the sum of a multiple of 1000 and a number between 0 and 999. For example, we can write

$$\begin{aligned}
&41,673 = 41,000 + 673, \\
&8,012 = 8,000 + 012, \\
&1,562,951 = 1,562,000 + 951.
\end{aligned}$$

The multiple of 1000 is such a sum is a multiple of 8. So if the number is such a sum is also a multiple of 8, then the original number is a multiple of 8. Otherwise, the original number is not a multiple of 8, so it is not divisible by 8.

So, the final 3 digits of a number form a multiple of 8, then the original number is a multiple of 8. Otherwise, the original number is not a multiple of 8.

3.3 Prime Numbers

Every positive integer is divisible by 1 and by itself. Some numbers are not divisible by any other positive integers. For example, 5 is divisible by 1 and 5, but not by any other positive integer. Mathematicians refer to such numbers as **prime numbers**.

Definition: A **prime number** is a positive integer that is divisible by exactly two positive integers: 1 and the number itself. A **composite number** is a positive integer that is divisible by more than two positive integers. This means that a number is composite if it is divisible by some positive integer besides 1 and the number itself.

For example, 2 is divisible by 1 and by 2, but it is not divisible by any other positive number. Therefore, 2 is prime. Meanwhile, 12 is divisible by 3, so 12 is composite. The number 1 is the only positive integer that is neither prime nor composite. Each integer greater than 1 is either prime or composite.

Mathematicians use the word "prime" as a noun, as in, "2 is a prime."

Problems

Problem 3.14 Jump to Solution

List every prime that is less than 20.

Problem 3.15 Jump to Solution

Meena is trying to determine whether or not 113 is prime. She's doing so by checking 113 for divisibility by each positive integer starting with 2; she sees that 113 is not divisible by 2, so she moves on to 3. She then checks if 113 is divisible by 3. Since $1 + 1 + 3 = 5$ is not divisible by 3, she knows that 113 is not divisible by 3.

(a) Meena says, "Since 113 is not divisible by 2, I know that 113 isn't divisible by 4." Is she right? Why or why not?

(b) How does Meena know at a glance that 113 is not divisible by 5?

(c) She then says, "Since 113 is not divisible by 2, I know that it isn't divisible by 6." Is she right? Why or why not?

(d) Meena divides 113 by 7 and finds a remainder of 1, so she knows that 113 is not divisible by 7. She then says, "I don't have to test 8, 9, or 10. I know that 113 isn't divisible by any of those." How does she know without testing?

(e) Meena then says, "Since $11^2 \approx 121$, I don't have to test any numbers higher than 10. I now know that 113 is prime." Why doesn't Meena have to test any more numbers?

Problem 3.16 Jump to Solution

Describe each of the following numbers as prime or composite:

(a) 61

(b) 91

(c) 143

(d) 157

Problem 3.17 Jump to Solution

Find the largest two-digit composite number in which both digits are prime.

Problem 3.18 Jump to Solution

Find all pairs of primes whose sum is 61.

Problem 3.19 Jump to Solution

Start with the grid of numbers on the right, and perform the following process:

(a) Circle the smallest number that is not already circled or crossed out.

(b) Cross out all the multiples of the number you circled in Step 1, except for the circled number itself.

(c) If every number is either circled or crossed out, then stop. Otherwise, go back to Step 1.

When you finish, do you notice anything interesting about the circled numbers?

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Problem 3.14

List every prime that is less than 20.

Solution for Problem 3.14: In the introduction, we noted that 1 is not prime and 2 is prime.

The number 3 is divisible by 1 and by 3, but not by 2. A positive number cannot be divisible by a number greater than itself, so we don't have to check if 3 is divisible by 4 or by any other number greater than 3. Therefore, we know that 3 is prime.

Since 4 is divisible by 2, we know that 4 is composite. Similarly, 6, 8, 10, 12, 14, 16, and 18 are all divisible by 2 and therefore are composite. Of course, you probably recognize these as even numbers. We know that 5 is prime, because it is divisible by 1 and itself, but it is not divisible by 2 or by 3 and 5 is prime. Every even number is divisible by 2, the only even prime is 2. So, we don't have to check any more even numbers.

Continuing in this manner, we find that the only primes less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. The only other numbers, which are not prime, are 1 that is not prime by definition, all the other even numbers are divisible by 2, and both 9 and 15 are multiples of 3.

Problem 3.15

Meena is trying to determine whether or not 113 is prime. She's doing so by checking 113 for divisibility by each positive integer starting with 2; she sees that 113 is not divisible by 2, so she moves on to 3. She then checks if 113 is divisible by 3. Since $1 + 1 + 3 = 5$ is not divisible by 3, she knows that 113 is not divisible by 3.

(a) Meena says, "Since 113 is not divisible by 2, I know that 113 isn't divisible by 4." Is she right? Why or why not?

(b) How does Meena know at a glance that 113 is not divisible by 5?

(c) She then says, "Since 113 is not divisible by 2, I know that it isn't divisible by 6." Is she right? Why or why not?

(d) Meena divides 113 by 7 and finds a remainder of 1, so she knows that 113 is not divisible by 7. She then says, "I don't have to test 8, 9, or 10. I know that 113 isn't divisible by any of those." How does she know without testing?

(e) Meena then says, "Since $11^2 \approx 121$, I don't have to test any numbers higher than 10. I now know that 113 is prime." Why doesn't Meena have to test any more numbers?

Solution for Problem 3.15:

(a) Yes, she is right. Every multiple of 4 is also a multiple of 2, since 4 is a multiple of 2. But Meena already knows that 113 is not a multiple of 2, so she knows that 113 cannot possibly be a multiple of 4.

(b) She had digit 113 is not 0 or 5, so she knows that 113 is not divisible by 5.

(c) Yes, she is right. As in part (a), every multiple of 6 is also a multiple of 2, since 6 is a multiple of 2. Meena knows that 113 is not a multiple of 2, so she knows that 113 cannot be a multiple of 6.

(d) She doesn't have to test 8 or 10 because these are both multiples of 2. Since 113 is not a multiple of 2, Meena knows that 113 is not divisible by any even number.

Every multiple of 9 is also a multiple of 3. Since Meena knows that 113 is not a multiple of 3, she knows that 113 cannot be a multiple of 9, either. Therefore, Meena doesn't have to test 9, 9, or 10.

(e) Meena knows that 113 is not a multiple of any integer from 2 through 10. So, if 113 is the product of two integers, and neither integer is 1, then both of the integers must be greater than 11.

The smallest possible product of two integers greater than 10 is $11 \cdot 11 = 121$, which is larger than 113. Any other product of two integers greater than 10 will be even larger. So, 113 cannot be written as the product of two integers that are greater than 11. This means that 113 is divisible only by 1 and itself, so 113 is prime.

Our work in Problem 3.15 revealed two very important points about testing whether or not a number is prime.

Important: If a number is not divisible by a prime, then it is not divisible by any multiple of that prime. So, in testing whether or not a number is prime, we only need to test if that number is divisible by prime numbers.

Important: To check if a number is prime, we only need to test if it is divisible by primes whose squares are less than or equal to the number.

For example, to test whether or not 59 is prime, we only need to check if it is divisible by 2, 3, 5, or 7. Since $11^2 \approx 121$ is greater than 59, we don't need to test any other primes.

How to Tell if a Number is Prime

Problem 3.16

Describe each of the following numbers as prime or composite:

(a) 61

(b) 91

(c) 143

(d) 157

Solution for Problem 3.16:

(a) Since $11^2 \approx 121$ is greater than 61, we only need to test for divisibility by 2, 3, 5, and 7. Since 61 is not divisible by any of these primes, we know that 61 is prime.

(b) $91 = 7 \cdot 13$, so 91 is composite.

(c) $143 = 11 \cdot 13$, so 143 is composite.

(d) Since $13^2 = 169$ is greater than 157, we only need to test for divisibility by 2, 3, 5, 7, and 11. Since 157 is not divisible by any of these primes, we know that 157 is prime.

Problem 3.17

Find the largest two-digit composite number in which both digits are prime.

Solution for Problem 3.17: The one-digit primes are 2, 3, 5, and 7. The largest two-digit number we can form with these digits is 77, which is indeed composite because $77 = 7 \cdot 11$.

Problem 3.18

Find all pairs of primes whose sum is 61.

Problem 3.19

Start with the grid of numbers on the right, and perform the following process:

(a) Circle the smallest number that is not already circled or crossed out.

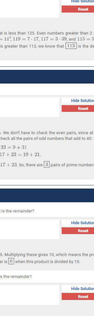
(b) Cross out all the multiples of the number you circled in Step 1, except for the circled number itself.

(c) If every number is either circled or crossed out, then stop. Otherwise, go back to Step 1.

When you finish, do you notice anything interesting about the circled numbers?

Solution for Problem 3.19: The smallest number in the grid is 2, so we start by circling 2 in Step 1. Step 2 then tells us to cross out all of the multiples of 2 except for 2, so we cross out all the even numbers except 2. Moving on to Step 1, we still see that plenty of numbers that are not circled or crossed out, so we go back to Step 1.

The smallest number that was not circled or crossed out already is 3, so we circle 3. We then cross out all the other multiples of 3. As we go, we notice that 6 and 12 are already crossed out, so we skip over them. Once we finish, we see that we also multiples of 2. After crossing out all the multiples of 3 besides 3, we still have many numbers that are not circled or crossed out. Back to Step 1 we go.

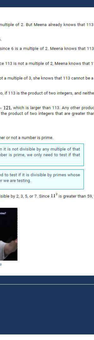


The smallest number that was not circled or crossed out already is 5, so we circle 5. We then cross out all the other multiples of 5. As we go, we notice that 10, 15, and 20 are already crossed out. Once we finish, we see that we also multiples of 2, 3, and 5. After crossing out all the multiples of 5 besides 5, we still have many numbers that are not circled or crossed out. Back to Step 1 we go.

Now the smallest number that was not circled or crossed out already is 7, so we circle 7. We then cross out all of the other multiples of 7. There are still plenty of numbers to go. Back to Step 1.

The smallest number that was not circled or crossed out is now 11, so we circle that and then cross out the other multiples of 11. Still, a bunch of numbers to go.

The next number left circle is 11. But there are no other multiples of 11 in the grid that are not circled or crossed out. This is because we have circled or crossed out all the multiples of 2, 3, 5, and 7 already. That means we have crossed out $2 \cdot 11$, $3 \cdot 11$, $5 \cdot 11$, and $7 \cdot 11$. We circled 11 and crossed out $11 \cdot 11$. So, we have circled or crossed out multiples of 2, and we crossed out $7 \cdot 11$ when we crossed out multiples of 7.



We circle 11 and still have plenty of numbers left. But for each number we circle, there are no more uncrossed-out multiples of the circled number left in the grid. So, as we repeat the three steps over and over, we circle all the remaining numbers in the grid. In the end, we have the grid at the right.

The circled numbers are the primes less than 100! Why?

Each time we cross out numbers in Step 2, the numbers we cross out are multiples of the number we just circled in Step 1. So, all of the crossed out numbers are composite. We'll never cross out a prime number, because a prime number isn't a multiple of any of the numbers in the grid besides itself. Therefore, eventually each prime number will become the smallest available number in the grid in Step 1. It will then be circled and all its multiples will be crossed out. Since we cross out all the multiples of each prime, we know that each composite number will get crossed out in the step after we circle the smallest prime that evenly divides the composite number. So, we must have all primes in the grid circled and all composite numbers crossed out when we finish.

This process is called the **Sieve of Eratosthenes** after the ancient Greek mathematician who developed it. There's nothing special about 100 in our example above of the Sieve of Eratosthenes. We could have started with a much larger grid to find many more primes.

Sidenote: The Sieve of Eratosthenes is a very efficient way to generate long lists of prime numbers. For example, if you wanted to write a computer program to list of all the primes less than 1,000,000, the Sieve of Eratosthenes would be a great way to do it. Eratosthenes's other claim to fame involves checking each number one at time with the process we used in Problems 3-15.

Exercises

3.3.1.

What is the sum of all the prime numbers between 60 and 90?

You may type any additional notes you have here.

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Solution: We can read the primes between 60 and 90 off of the Sieve of Eratosthenes in the text. Or, we can test each odd number between 60 and 90 to see if it's prime. (The even numbers are obviously not primes.) 61 and 67 are divisible by 1, and 65 is divisible by 5. Both 63 and 69 are prime, and their sum is $\boxed{132}$.

3.3.2. Source: MATHCOUNTS

Find every number between 70 and 80 that is not prime and is not a multiple of 2, 3, or 5.

Preview: Solution

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Solution: All of the even numbers of 2, and 70 is a multiple of 5. That leaves 71, 73, 77, and 79. We find that $77 = 7 \cdot 11$ is not prime and is not a multiple of 2, 3, or 5. Meanwhile, 71, 73, and 79 are all prime. Therefore, $\boxed{77}$ is the only number between 70 and 80 that is not prime and is not a multiple of 2, 3, or 5.

3.3.3.

What is the largest prime number p such that it strikes p% less than 1000?

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Solution: 1000 divided by 8 is 125. So, we seek the largest prime number that is less than 125. Even numbers greater than 2 aren't prime, so we only have to check odd numbers. We have $123 = 3 \cdot 41$, $121 = 11^2$, $119 = 7 \cdot 17$, $117 = 3 \cdot 39$, and $115 = 5 \cdot 23$. So these are all composite. Since 113 isn't divisible by 2, 3, 5, or 7, and 17^2 is greater than 113, we know that $\boxed{113}$ is the desired prime.

3.3.4.

How many pairs of primes are there such that sum of the pair is 99?

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Solution: If two integers add to 99, then either both are odd or both are even. We don't have to check the even pairs, since at least one of two even numbers that add to 99 is a composite multiple of 2. So, we check all the pairs of odd numbers that add to 99:

$$99 = 1 + 98 = 3 + 97 = 5 + 95 = 7 + 93 = \ldots = 97 + 2$$

$$= 11 + 88 = 13 + 86 = 15 + 84 = 17 + 82 = \ldots = 49 + 50$$

The only sums that consist of two prime numbers are $3 + 97$, $11 + 88$ and $17 + 82$. So, there are $\boxed{3}$ pairs of prime numbers that add to 40.

3.3.5.

(a) The product of all prime numbers between 1 and 80 is divisible by 10. What is the remainder?

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Solution: The product of all prime numbers between 1 and 80 includes 2 and 5. Multiplying these gives 10, which means the product of all the primes between 1 and 80 is a multiple of 10. Therefore, the remainder is $\boxed{0}$ when this product is divided by 10.

(b) ★ The product of all prime numbers between 1 and 80 is divisible by 4. What is the remainder?

You may type any additional notes you have here.

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Solution: The product of all prime numbers between 1 and 80 includes exactly one 2, and has no other even numbers. So, we know that the product is even, but is not a multiple of 4 (which is 2). When an odd number is divided by 4, the remainder is 1, or 3, and when an even number is divided by 4, the remainder is 0 or 2. The remainder is only 0 if the number is a multiple of 4. We know that the product in the problem is even but not a multiple of 4, so the remainder is $\boxed{2}$ when the product is divided by 4.

3.3.6.

How many groups of three prime numbers add to 20? Note: (2, 3, 5) is considered the same group of three primes as (3, 2, 5).

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Solution: The sum of three integers is even only if the three integers are even or if two of the integers are odd and the other even. The only even prime is 2, so the only integer that is the sum of three primes is $2 + 2 + 2 = 6$. So, we must have 2 and two odd primes that add to 20. The pairs of odd primes that add to 20 are

$$20 = 1 + 19 = 3 + 17 = 5 + 15 = 7 + 13 = 9 + 11.$$

Two of these sums consist of two prime integers ($3 + 17$ and $7 + 13$), so there are $\boxed{2}$ groups of three primes whose sum is 22.

3.3.7★: Source: MOEMS

Suppose P and Q both represent whole numbers, and

$$5P + 7Q = 105.$$

Find the value of the prime P.

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Solution: The sum of two odd numbers is even, so $5P$ and $7Q$ cannot both be odd. Therefore, either $5P$ or $7Q$ is even, which means P or Q is 2.

If P is 2, then the equation becomes $10 + 7Q = 105$. This means that $7Q$ must be 95. But 95 is not a multiple of 7, so there is no integer Q for which $7Q$ is 95.

If Q is 2, then the equation becomes $5P + 14 = 105$. This means that $5P$ is 91, so P must be $91 \div 5 = \boxed{19}$.

Extra: Finding large prime numbers is a thrill for some mathematicians and computer scientists. As of this writing, the largest known prime is

$$2^{74,207,281} - 1,$$

which has 22,338,618 digits! This prime was discovered in 2016 by the **Great Internet Mersenne Prime Search (GIMPS)**, which is a collaborative internet-based search for large primes. Visit http://www.mersenne.org/ to join the search!

3.4 Prime Factorization

Every number can be written as a product of integers. So, if a number is composite, then we can factor it by writing it as the product of two smaller integers. For example, we can factor 12 as $12 = 2 \cdot 6$. If 6 is not a composite, we also can factor it, writing it as the product of primes:

$$12 = 2 \cdot 6 = 2 \cdot 2 \cdot 3$$

Since 2 and 3 are primes, we cannot write either of them as the product of smaller positive integers. We have expressed 12 as a product of primes, so we say that $2 \cdot 2 \cdot 3$ is the prime factorization of 12. The prime factorization of a number is unique, so we can write $12 = 2 \cdot 2 \cdot 3$ and not worry about other factorizations.

We usually write prime factorizations using powers. For example, the prime factorization of 12 is $2^2 \cdot 3^1$. Notice that we write the primes in increasing order. In the chapter, we will often write prime factorizations in this way, without explanation.

It's easy to keep track of your work when finding the prime factorization of a small number like 12. When finding the prime factorization of a large number, it helps to keep a table of your work. For example:

We'll make a table to find the prime factorization of 12. We start by writing 12. We factor 12 as $2 \cdot 6$. Since 6 is not prime, we write this in the factor tree by splitting 12 into 2 and 6. We also produce an annotated tree in the tree, so we can keep track of which numbers are prime and which numbers are composite. We continue factoring 6 into $2 \cdot 3$. Since 2 and 3 are prime, we circle them.

We can now read the prime factorization of 12 from the tree. Since 2 and 3 are prime, we can write $12 = 2 \cdot 6$ and $6 = 2 \cdot 3$ to get $12 = 2 \cdot 2 \cdot 3$. Writing this in exponent form, we get $12 = 2^2 \cdot 3^1$. This is the prime factorization of 12 from our factor tree.

Problems

Problem 3.20 Going to textbook

Find the prime factorization of each of the following numbers:

(a) 14 (b) 15

(c) 16 (d) 18

Problem 3.21 Going to textbook

Find the prime factorization of each of the following perfect squares:

(a) 36 (b) 49

(c) 64 (d) 81

Problem 3.22 Going to textbook

How can we use the prime factorization of a number to tell whether or not the number is a perfect square?

Problem 3.23 Going to textbook

What prime integer square equals 961? 1681

Problem 3.24 Going to textbook

Find multiples two positive integers and gets a product of 1600. If neither of these integers ends in 0, then what is the sum of these integers?

Problem 3.25 Going to textbook

(a) Let N be the product of the 10 smallest prime numbers. Explain why the prime factorization of $N + 1$ must include a prime that isn't among the 10 smallest prime numbers.

(b) Let N be the product of the 100 smallest prime numbers. Explain why the prime factorization of $N + 1$ must include a prime that isn't among the 100 smallest prime numbers.

(c) Explain why there is not a largest prime number. That is, explain why there are infinitely many primes.

Problem 3.26 Going to textbook

Find the prime factorization of each of the following numbers:

(a) 100 (b) 1000

(c) 10000 (d) 100000

Solution for Problem 3.26:

(a) We have $100 = 10 \cdot 10$ and $10 = 2 \cdot 5$, which gives us the factor tree above on the left. We write this tree as:

$$10 = 2 \cdot 5$$

so the prime factorization of 100 is:

$$100 = 2^2 \cdot 5^2$$

(b) Of course, this isn't the only factor tree we could have made for 1000. We could have started with $100 = 10 \cdot 10$ and produced the factor tree above at left. This factor tree also produces the prime factorization:

$$1000 = 2^3 \cdot 5^3$$

We also could have used our result from part (a). There, we found that $100 = 2^2 \cdot 5^2$. Therefore:

$$1000 = 10 \cdot 100 = 10 \cdot 2^2 \cdot 5^2 = 2^3 \cdot 5^3$$

(c) We can easily divide 100 by 2, so we start with $100 = 2 \cdot 50$. We then have $50 = 2 \cdot 25$ and $25 = 5 \cdot 5$. We write this as the factor tree on the right. Writing this in exponent form, we get:

$$10000 = 2^4 \cdot 5^4$$

(d) We think there to start off dividing by 2. We then have 50 as one of the digits of 100. So, we have $100 = 2 \cdot 50$. We then have $50 = 2 \cdot 25$. We have $25 = 5 \cdot 5$. We have $5 = 5 \cdot 1$. We left-align the factor tree, so we can see that we have $100 = 2 \cdot 50$. We then have $50 = 2 \cdot 25$. We left-align the factor tree, so we can see that we have $25 = 5 \cdot 5$. We then have $5 = 5 \cdot 1$. We left-align the factor tree, so we can see that we have $5 = 5 \cdot 1$. We then have $1 = 1 \cdot 1$. We left-align the factor tree, so we can see that we

3.3 Least Common Multiple

For a positive integer n, a positive multiple is a multiple of n that is n or larger. We call the first two multiples a **common multiple** of n and m if they are positive multiples of both n and m. We call the smallest such multiple the **least common multiple** of n and m. Similarly, d is a common multiple of n and m and the least common multiple of n and m is d if:

- d is a common multiple of n and m, and
- d is the smallest positive integer, that is, $d \geq 1$.

We denote the least common multiple of n and m by $\mathrm{lcm}(n, m)$.

- What is the least common multiple of n and m? Give us the common multiples you found and (or) related to the least common multiple?

Discuss with your neighbor

Find the prime factorizations of the two smallest positive common multiples of 12 and 18. Compare those prime factorizations to the prime factorizations of 12 and 18.

Discuss with your neighbor

How can we use the prime factorizations of 12 and 18 to find $\mathrm{lcm}(12, 18)$?

Discuss with your neighbor

Consider each of the following:

- $\mathrm{lcm}(20, 100)$
- $\mathrm{lcm}(20, 90)$
- $\mathrm{lcm}(2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3)$
- $\mathrm{lcm}(n, n)$

Discuss with your neighbor

Consider each of the following:

- $\mathrm{lcm}(20, 100)$
- $\mathrm{lcm}(20, 90)$
- $\mathrm{lcm}(2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3)$
- $\mathrm{lcm}(n, n)$

Discuss with your neighbor

Find the prime factorizations of the two smallest positive common multiples of 20. Compare those prime factorizations to the prime factorizations of 20.

Discussion Questions 3.3.1 Source: MATH2000 F. Jones

Consider each of the following:

- $\mathrm{lcm}(20, 100)$
- $\mathrm{lcm}(20, 90)$
- $\mathrm{lcm}(2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3)$
- $\mathrm{lcm}(n, n)$

Discussion Questions 3.3.1 Source: MATH2000 F. Jones

Find the prime factorizations of the two smallest positive common multiples of 20. Compare those prime factorizations to the prime factorizations of 20.

Discussion Questions 3.3.1 Source: MATH2000 F. Jones

Find the prime factorizations of the two smallest positive common multiples of 12. Compare those prime factorizations to the prime factorizations of 12.

Discussion Questions 3.3.1 Source: MATH2000 F. Jones

Find the prime factorizations of the two smallest positive common multiples of 18. Compare those prime factorizations to the prime factorizations of 18.

Discussion Questions 3.3.1 Source: MATH2000 F. Jones

Consider each of the following:

- $\mathrm{lcm}(20, 100)$
- $\mathrm{lcm}(20, 90)$
- $\mathrm{lcm}(2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3)$
- $\mathrm{lcm}(n, n)$

Discussion Questions 3.3.1 Source: MATH2000 F. Jones

Find the prime factorizations of the two smallest positive common multiples of 12. Compare those prime factorizations to the prime factorizations of 12.

Discussion Questions 3.3.1 Source: MATH2000 F. Jones

Find the prime factorizations of the two smallest positive common multiples of 18. Compare those prime factorizations to the prime factorizations of 18.

Discussion Questions 3.3.1 Source: MATH2000 F. Jones

Find the prime factorizations of the two smallest positive common multiples of 20. Compare those prime factorizations to the prime factorizations of 20.

Discussion Questions 3.3.1 Source: MATH2000 F. Jones

Find the prime factorizations of the two smallest positive common multiples of 12. Compare those prime factorizations to the prime factorizations of 12.

Discussion Questions 3.3.1 Source: MATH2000 F. Jones

Find the prime factorizations of the two smallest positive common multiples of 18. Compare those prime factorizations to the prime factorizations of 18.

Discussion Questions 3.3.1 Source: MATH2000 F. Jones

Consider each of the following:

- $\mathrm{lcm}(20, 100)$
- $\mathrm{lcm}(20, 90)$
- $\mathrm{lcm}(2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3)$
- $\mathrm{lcm}(n, n)$

3.6 Divisors

In Section 3.1, we learned that 12 is a multiple of 4, because 12 equals 4 times an integer. We can also express this fact by saying, "4 is a **divisor** of 12."

> **Definition:** Let a be a nonzero integer and b be an integer. We say that a is a **divisor** of b if b is a multiple of a. In other words, b is a times some integer.

We can also say that 4 is a **factor** of 12. The nouns "divisor" and "factor" mean the same thing. For example, the positive factors of 6 are 1, 2, 3, and 6. The negative factors of 6 are $-1, -2, -3$, and -6.

One way we sometimes think of divisors is as "a is a divisor of b if dividing b by a leaves no remainder." So, because 60 divided by 6 has no remainder, 6 is a divisor of 60. Since 72 divided by 5 has remainder 2, we know that 5 is not a divisor of 72.

> **Important:** If a and b are nonzero integers, then all of the following statements mean the same thing:
>
> - a is a divisor of b.
> - a is a factor of b.
> - b is a multiple of a.
> - b is divisible by a.

Problems

Problem 3.34 (Jump to Solution)

Find all of the positive divisors of 84.

Problem 3.35 (Jump to Solution)

(a) Count the positive divisors of each integer from 8 to 18. Which of these integers have an odd number of positive divisors?

(b) There is one integer between 20 and 30 that has an odd number of positive divisors. Which one?

(c) Which positive integers have an odd number of positive divisors?

Problem 3.36 (Jump to Solution)

Note that 20 is a divisor of 60.

(a) Must every divisor of 20 also be a divisor of 60?

(b) Must every divisor of 60 also be a divisor of 20?

Problem 3.37 (Jump to Solution)

(a) Suppose 3 is a divisor of k; is 3 a divisor of $k+3$?

(b) Suppose 3 is a divisor both of b and of c. Must 3 be a divisor of $b+c$?

(c) Suppose 3 is a divisor of $b+c$. Must 3 be a divisor both of b and of c?

Problem 3.34

Find all of the positive divisors of 84.

Solution for Problem 3.34: We start off testing each positive integer, starting with 1.

is 1 a divisor of 84? Yes, because $84 = 1 \cdot 84$. This equation shows that both 1 and 84 are divisors of 84.

Is 2 a divisor of 84? Yes, because $84 = 2 \cdot 42$. Both 2 and 42 are divisors of 84.

In the same way, we can find other pairs that multiply to 84:

$$3 \cdot 28 = 84, \qquad 4 \cdot 21 = 84, \qquad 6 \cdot 14 = 84, \qquad 7 \cdot 12 = 84.$$

None of 5, 8, or 9 is a divisor of 84. Rather than continuing to test the numbers from 10 up to 84 to see if they are divisors of 84, we notice that $10 \cdot 10 > 100$ greater than 84. So, it's impossible to find two numbers greater than 9 whose product is 84. We've already tested all the numbers from 1 to 9, so we have found all the pairs of integers whose product is 84.

In summary, 84 has 12 positive divisors: 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, and 84. □

As we saw in Problem 3.34, when hunting for pairs of positive integers that multiply to n, we only have to test numbers up to the first integer whose square is greater than n.

Problem 3.35

(a) Count the positive divisors of each integer from 8 to 18. Which of these integers have an odd number of positive divisors?

(b) There is one integer between 20 and 30 that has an odd number of positive divisors. Which one?

(c) Which positive integers have an odd number of positive divisors?

Solution for Problem 3.35:

(a) Below is a table of the positive divisors of each integer from 8 to 18, as well as a count of the positive divisors of each integer:

Number	Divisors	Count	Number	Divisors	Count
8	1,2,4,8	4	14	1,2,7,14	4
9	1,3,9	3	15	1,3,5,15	4
10	1,2,5,10	4	16	1,2,4,8,16	5
11	1,11	2	17	1,17	2
12	1,2,3,4,6,12	6	18	1,2,3,6,9,18	6
13	1,13	2			

The only integers from 8 to 18 that have an odd number of positive divisors are 9 and 16:

(b) The perfect squares 9 and 16 are the only numbers in part (a) that have an odd number of positive divisors. So, we guess that the perfect square 25 is the number between 20 and 30 that has an odd number of positive divisors. Checking, we find that the only positive divisors of 25 are 1, 5, and 25. Indeed, 25 is the integer between 20 and 30 that has an odd number of positive divisors.

(c) From the previous two parts, we suspect that perfect squares have an odd number of positive divisors, and the rest of the positive integers have an even number of positive divisors.

The positive divisors of an integer come in pairs of numbers whose product is that integer. If an integer is not a perfect square, the two numbers in a such a pair cannot ever be the same. When we count the positive divisors of a non-square integer, they come in pairs, so the count is even:

$$12 = 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4.$$

If an integer is a perfect square, then one pair has two numbers that are the same. All the rest of the positive divisors of the integer come in pairs of different numbers:

$$36 = 1 \cdot 36 = 2 \cdot 18 = 3 \cdot 12 = 4 \cdot 9 = 6 \cdot 6.$$

So, when we count the positive divisors of a perfect square, we count by twos for the pairs with different numbers, and then count only one divisor for the pair that has two numbers that are the same. This final divisor makes the count of positive divisors odd.

□

Problem 3.36

Note that 20 is a divisor of 60.

(a) Must every divisor of 20 also be a divisor of 60?

(b) Must every divisor of 60 also be a divisor of 20?

Solution for Problem 3.36:

(a) Yes. If an integer d is a divisor of 20, then 20 is a multiple of d. This means that every multiple of 20 is also a multiple of d. Specifically, 60 is a multiple of 20, so 60 is a multiple of d. Therefore, d is a divisor of 60.

(b) No. For example, 30 is a divisor of 60, but 30 is not a divisor of 20.

□

> **Important:** If a is a divisor of b, then every divisor of a is also a divisor of b.

Problem 3.37

(a) Suppose 3 is a divisor of k; is 3 a divisor of $k+3$?

(b) Suppose 3 is a divisor both of b and of c. Must 3 be a divisor of $b+c$?

(c) Suppose 3 is a divisor of $b+c$. Must 3 be a divisor both of b and of c?

Solution for Problem 3.37:

(a) 3 is a divisor of k, so k is a multiple of 3. Adding 3 to a multiple of 3 gives another multiple of 3, so $k+3$ is a multiple of 3. Therefore, 3 is a divisor of $k+3$.

(b) Since 3 is a divisor of both b and c, we know that both b and c are multiples of 3. The sum of two multiples of 3 must also be a multiple of 3. (See Problem 3.1 if you don't remember why.) Therefore, $b+c$ is a multiple of 3, which means that 3 is a divisor of $b+c$.

(c) No. Suppose $b = 2$ and $c = 1$. Then, 3 is a divisor of $b+c$, but 3 is not a divisor of b or c.

□

The principles we explored in Problem 3.37 are the same as the rules we learned about multiples in Problem 3.1. Here are the rules written in terms of "divisors" instead of "multiples":

> **Important:** Let a, b, and c be integers. If c is a divisor of a and of b, then c is a divisor of $a+b$ and of $a-b$.

A Challenging Divisor Problem

Exercises

3.6.1:

What is the product of all positive factors of 6?

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Solution: We have $6 = 1 \cdot 6 = 2 \cdot 3$, so the product of the positive factors is $1 \cdot 6 \cdot 2 \cdot 3 = 6 \cdot 6 = 6^2 = \boxed{36}$.

3.6.2:

What is the sum of the positive divisors of 18?

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Solution: We have $18 = 1 \cdot 18 = 2 \cdot 9 = 3 \cdot 6$, so the sum of the positive divisors of 18 is $1 + 2 + 3 + 6 + 9 + 18 = \boxed{39}$.

3.6.3:

How many positive factors does 32 have?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We have $32 = 1 \cdot 32 = 2 \cdot 16 = 4 \cdot 8$, so 32 has $\boxed{6}$ divisors.

3.6.4:

For how many integers n is $28 \div n$ an integer?

You may type any additional notes you have here.

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Solution: The values of n such that $28 \div n$ is an integer are the divisors of 28. However, we must be careful; the problem didn't specify that n must be positive. So, we must count both the positive and the negative divisors. The negative divisors are simply the negations of the positive divisors, so we'll count the positive divisors and double the total. We have

$$28 = 1 \cdot 28 = 2 \cdot 14 = 4 \cdot 7,$$

so 28 has 6 positive divisors. This gives us $2 \cdot 6 = \boxed{12}$ values of n for which $28 \div n$ is an integer.

3.6.5: Source: MATHCOUNTS

The product of two positive integers is 2005. If neither integer is 1, what is the sum of the two integers?

You may type any additional notes you have here.

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Solution: We have $2005 = 5 \cdot 401$. Since 401 is not divisible by any of the primes less than 20, and 23^2 is greater than 401, we know that 401 is prime. So, the only possibility for the desired sum is $5 + 401 = \boxed{406}$.

3.6.6: Source: MOEMS

The product of the three-digit number ABC and the single-digit number D is 1673. If A, B, C, and D represent different digits, then what three-digit number does ABC represent?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: D is a one-digit divisor of 1673. We know that D isn't 1 because D times a three-digit number equals 1673. So, we seek a one-digit divisor of 1673 besides 1. Since 1673 is odd, we know that D can't be even. We can also use divisibility tests to quickly eliminate 3, 5, and 9 as candidates for D. That leaves 7. We have $1673 \div 7 = 239$, so the desired ABC is $\boxed{239}$.

3.6.7: Source: AMC 8

A lucky year is one in which at least one date, when written in the form month/day/year, has the following property: the product of the month times the day equals the last two digits of the year. For example, 1956 is a lucky year because it has the date 7/8/56 and $7 \cdot 8 = 56$. Which of the following is NOT a lucky year: 1990, 1991, 1992, 1993, or 1994?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Let A be the units digit in the year $199A$. In order for $199A$ to be lucky, the two-digit number $9A$ must be the product of two integers in which one integer is from 1 to 12 (the month) and the other integer is no greater than the number of days in the corresponding month.

We have $90 = 3 \cdot 30$, so 1990 is lucky. (3/30/90)

We have $91 = 7 \cdot 13$, so 1991 is lucky. (7/13/91)

We have $92 = 4 \cdot 23$, so 1992 is lucky. (4/23/92)

We have $93 = 3 \cdot 31$, so 1993 is lucky. (3/31/93)

That leaves $\boxed{1994}$ as the year that is not lucky. To see that 1994 is not lucky, note that the only pairs of positive divisors of 94 that multiply to 94 are $94 = 1 \cdot 94 = 2 \cdot 47$. Neither of these pairs leads to a valid date.

3.6.8:

What is the greatest positive integer less than 100 that has an odd number of positive divisors?

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Perfect squares have an odd number of positive divisors, and integers that are not perfect squares have an even number of positive divisors. So, we seek the largest square that is less than 100, which is $\boxed{81}$.

3.6.9: Source: MATHCOUNTS

What is the greatest integer less than 10000 that is a factor of $11000 + 1100 + 11$?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The sum is 12,111. The largest divisor of 12,111 is 12,111 itself, but that's larger than 10,000. The largest divisor of 12,111 and the smallest positive divisor have product 12,111. This gives us the pair of divisors 12,111 and 1. Similarly, the product of the next largest divisor and the next smallest positive divisor is 12,111. So, we can find the next largest divisor by finding the next smallest divisor.

Since 12,111 is odd, 2 is not a divisor. But the sum of the digits of 12,111 is a multiple of 3, so we know that 3 is a divisor of 12,111. We have $12{,}111 = 3 \cdot 4{,}037$, so the greatest factor that is less than 10,000 is $\boxed{4{,}037}$.

3.7 Greatest Common Divisor

Let a and b be two integers. Then the number d is called the **common divisor** of a and b. Common divisors are also referred to as **common factors**.

Let a and b be two integers. Then the number d is called the **greatest common divisor** of a and b, denoting as the common divisor of a and b that is the largest. Common divisors are also referred to as **common factors**. The greatest common divisor of a and b is denoted as $\gcd(a, b)$.

The greatest positive integer that is a divisor of both a and b is called the **greatest common divisor** of a and b, denoting as the common divisor of a and b that is the largest. Common divisors are also referred to as **common factors**. The greatest common divisor of a and b is denoted as $\gcd(a, b)$.

Problems

Problem 3.1 ▶ Jump to Solution

(a) Find the positive divisors of 30.

(b) Find the positive divisors of 45.

(c) Find the positive common divisors of 30 and 45.

(d) What is the greatest common divisor of 30 and 45?

Problem 3.2 ▶ Jump to Solution

Find the greatest common divisor of each of the positive divisors of 24. Compare these prime factorizations to the prime factorization of 24. Do you see a connection?

Problem 3.3 ▶ Jump to Solution

Find the factorization of 2004 in $\mathbb{Z}^+$ (there are three numbers and three prime factorizations). In each part, use the prime factorization of the number to determine the greatest common divisor.

(a) $\gcd(2004, 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$

(b) $\gcd(2004, 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$

(c) $\gcd(2004, 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$

Problem 3.4 ▶ Jump to Solution

(a) $\gcd(2^3 \cdot 5^2, 2^2 \cdot 5^1 \cdot 7^1)$

(b) $\gcd(2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1, 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1)$

(c) $\gcd(2^1 \cdot 3^1 \cdot 5^1, 2^1 \cdot 3^1 \cdot 5^1)$

Problem 3.5 ▶ Jump to Solution

(a) $\gcd(2^3 \cdot 5^2 \cdot 7^1, 2^2 \cdot 5^1 \cdot 7^1 \cdot 11^1)$

(b) $\gcd(2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1, 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$

(c) $\gcd(2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1, 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$

Problem 3.6 ▶ Jump to Solution

How can we use the prime factorization of $\gcd(a, b)$?

Problem 3.7 ▶ Jump to Solution

Compute each of the following:

(a) $\gcd(27, 36)$

(b) $\gcd(100, 55)$

(c) $\gcd(2^3 \cdot 3^1 \cdot 5^1, 2^2 \cdot 3^2 \cdot 7^1)$

(d) $\gcd(70, 154, 196)$

Problem 3.8 ▶ Jump to Solution

(a) $\gcd(2^3, 3^2, 5^1, 7^1)$ $\gcd(2^2, 3^1, 5^1, 7^1)$

(b) $\gcd(2^1, 3^1, 5^1, 7^1)$ $\gcd(2^1, 3^1, 5^1, 7^1)$

(c) $\gcd(70, 154, 196)$

Problem 3.9 ▶ Jump to Solution

(a) $\gcd(2^3, 3^2, 5^1, 7^1)$ $\gcd(2^2, 3^1, 5^1, 7^1)$

(b) $\gcd(2^1, 3^1, 5^1, 7^1)$ $\gcd(2^1, 3^1, 5^1, 7^1)$

(c) $\gcd(70, 154, 196)$

Problem 3.10 ▶ Jump to Solution

(a) $\gcd(2^3, 3^2, 5^1, 7^1)$ $\gcd(2^2, 3^1, 5^1, 7^1)$

(b) $\gcd(2^1, 3^1, 5^1, 7^1)$ $\gcd(2^1, 3^1, 5^1, 7^1)$

(c) $\gcd(70, 154, 196)$

Problem 3.11 ▶ Jump to Solution

Let a and b be integers. Then $\gcd(a, b) = \gcd(|a|, |b|)$.

Problem 3.12 ▶ Jump to Solution

Let a and b be positive integers such that $\gcd(a, b) = 1$. Explain why $\gcd(a, b) = 1$.

Problem 3.13 ▶ Jump to Solution

(a) Show a number that is a multiple of a and b but is multiple of $\gcd(a, b)$.

(b) Show a number that is a multiple of a and b but is multiple of $\gcd(a, b)$.

(c) Find the $\gcd(2, 3)$ and that $\gcd(2, 3)$ is a multiple of 2.

Problem 3.14 ▶ Jump to Solution

(a) Let the positive divisors of 18.

(b) Let the positive divisors of 24.

(c) Let the positive divisors of 36.

(d) What is the greatest common divisor of 18 and 24?

Problem 3.15 ▶ Jump to Solution

Find the prime factorization of 2004.

(a) Find $\gcd(2004, 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$.

(b) Find $\gcd(2004, 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$.

(c) Find $\gcd(2004, 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$.

Problem 3.16 ▶ Jump to Solution

(a) $\gcd(2^3, 5^2)$, $\gcd(2^2, 5^1, 7^1)$

(b) $\gcd(2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1, 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1)$

(c) $\gcd(2^1 \cdot 3^1 \cdot 5^1, 2^1 \cdot 3^1 \cdot 5^1)$

Problem 3.17 ▶ Jump to Solution

(a) $\gcd(2^3, 5^2, 7^1)$, $\gcd(2^2, 5^1, 7^1, 11^1)$

(b) $\gcd(2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1, 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$

(c) $\gcd(2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1, 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$

Problem 3.18 ▶ Jump to Solution

(a) $\gcd(2^3, 3^2, 5^1, 7^1)$, $\gcd(2^2, 3^1, 5^1, 7^1)$

(b) $\gcd(2^1, 3^1, 5^1, 7^1)$, $\gcd(2^1, 3^1, 5^1, 7^1)$

(c) $\gcd(70, 154, 196)$

Problem 3.19 ▶ Jump to Solution

Let a and b be integers such that $\gcd(a, b) = 1$. Explain why $\gcd(a, b) = 1$.

Problem 3.20 ▶ Jump to Solution

(a) Show a number that is a multiple of a and b but is multiple of $\gcd(a, b)$.

(b) Show a number that is a multiple of a and b but is multiple of $\gcd(a, b)$.

(c) Find the $\gcd(2, 3)$ and that $\gcd(2, 3)$ is a multiple of 2.

Problem 3.21 ▶ Jump to Solution

(a) Let the positive divisors of 18.

(b) Let the positive divisors of 24.

(c) Let the positive divisors of 36.

(d) What is the greatest common divisor of 18 and 24?

Problem 3.22 ▶ Jump to Solution

Find the prime factorization of each of the positive divisors of 24. Compare these prime factorizations to the prime factorization of 24. Do you see a connection?

Problem 3.23 ▶ Jump to Solution

Find the factorization of 2004 in $\mathbb{Z}^+$ (there are three numbers and three prime factorizations). In each part, use the prime factorization of the number to determine the greatest common divisor.

(a) $\gcd(2004, 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$

(b) $\gcd(2004, 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$

(c) $\gcd(2004, 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$

Problem 3.24 ▶ Jump to Solution

(a) $\gcd(2^3, 5^2)$, $\gcd(2^2, 5^1, 7^1)$

(b) $\gcd(2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1, 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1)$

(c) $\gcd(2^1 \cdot 3^1 \cdot 5^1, 2^1 \cdot 3^1 \cdot 5^1)$

Problem 3.25 ▶ Jump to Solution

(a) $\gcd(2^3, 5^2, 7^1)$, $\gcd(2^2, 5^1, 7^1, 11^1)$

(b) $\gcd(2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1, 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$

(c) $\gcd(2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1, 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1)$

Problem 3.26 ▶ Jump to Solution

(a) $\gcd(2^3, 3^2, 5^1, 7^1)$, $\gcd(2^2, 3^1, 5^1, 7^1)$

(b) $\gcd(2^1, 3^1, 5^1, 7^1)$, $\gcd(2^1, 3^1, 5^1, 7^1)$

(c) $\gcd(70, 154, 196)$

Problem 3.27 ▶ Jump to Solution

Let a and b be integers such that $\gcd(a, b) = 1$. Explain why $\gcd(a, b) = 1$.

Problem 3.28 ▶ Jump to Solution

(a) Show a number that is a multiple of a and b but is multiple of $\gcd(a, b)$.

(b) Show a number that is a multiple of a and b but is multiple of $\gcd(a, b)$.

(c) Find the $\gcd(2, 3)$ and that $\gcd(2, 3)$ is a multiple of 2.

Exercises

5.1.1.

Find the following greatest common divisors:

(a) $\gcd(30, 45)$

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(b) $\gcd(100, 55)$

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(c) $\gcd(2^3 \cdot 3^2 \cdot 5^1, 2^2 \cdot 3^2 \cdot 7^1)$

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5.1.2.

Find the $\gcd(70, 154, 196)$.

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5.1.3.

Let a and b be positive integers such that $\gcd(a, b) = 1$. Explain why $\gcd(a, b) = 1$.

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Let a and b be integers. Then $\gcd(a, b) = \gcd(|a|, |b|)$.

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3.8 Summary

Definition: A **prime number** is a positive integer that is divisible by exactly 2 positive integers: 1 and the number itself. A **composite number** is a positive integer that is divisible by some positive integer besides 1 and the number itself.

Important: **Fundamental Theorem of Arithmetic**. Every integer greater than 1 can be written as the product of one or more primes in exactly one way (disregarding the order of the primes in the product).

Definition: Let a and b be integers. We say that a is a **multiple** of b if a equals b times some integer. In other words, a is a multiple of b if there is an integer n such that $a = bn$. If a is a multiple of b and b is nonzero, then we say that b is a **divisor**, or **factor**, of a, and that a is **divisible** by b.

Important: Let a, b, and c be numbers.

- If a and b are multiples of c, then both $a + b$ and $a - b$ are multiples of c.
- If a is a multiple of b, then every multiple of a is also a multiple of b.

We developed several useful divisibility tests:

Number	Condition under which n is divisible by the number
2	Units digit of n is 0, 2, 4, 6, or 8
3	Sum of the digits of n is a multiple of 3
4	Number formed by last two digits of n is a multiple of 4
5	Units digit of n is 0 or 5
6	Divisible by 2 and by 3
9	Sum of the digits of n is a multiple of 9
10	Units digit of n is 0

If a number is a multiple of both a and b, then we say that the number is a **common multiple** of a and b. The smallest positive integer that is a multiple of both a and b is called the **least common multiple** of a and b. We refer to the least common multiple of a and b as $\operatorname{lcm}[a, b]$. Every multiple of $\operatorname{lcm}[a, b]$ is a common multiple of a and b, and every common multiple of a and b is a multiple of $\operatorname{lcm}[a, b]$.

Important: To find the prime factorization of the least common multiple of a group of numbers, we first find the prime factorization of each of the numbers. The prime factorization of the least common multiple is the product of the highest power of each prime factor that appears in the prime factorizations of the numbers.

If a number is a divisor of both a and b, then we say that the number is a **common divisor** of a and b. The greatest positive integer that is a divisor of both a and b is called the **greatest common divisor** of a and b. We refer to the greatest common divisor of a and b as $\gcd(a, b)$. Every divisor of $\gcd(a, b)$ is a common divisor of a and b, and every common divisor of a and b is a divisor of $\gcd(a, b)$. We say that two integers are **relatively prime** if their greatest common divisor is 1.

Important: We can find the greatest common divisor of a group of numbers with the following process:

Step 1: Find the prime factorization of each number.

Step 2: Identify the primes that appear in *all* of the prime factorizations. If no primes appear in all of the prime factorizations, then the greatest common divisor of the numbers is 1.

Step 3: Among the prime factorizations, find the smallest power of each prime from Step 2.

Step 4: Multiply the powers of primes found in Step 3.

Important: For any positive integers a, b, and n, we have

$$\operatorname{lcm}[na, nb] = n\operatorname{lcm}[a, b],$$
$$\gcd(na, nb) = n\gcd(a, b).$$

3.72: Source: MATHCOUNTS ⚙ ⏱ ⏱

Find the single-digit number $\overline{5CB4A}$ is divisible by 4, the remainder is 4. List both possible values of the digit C.

Hint: What's the nearest multiple of 4?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Since $\overline{5CB4A}$ is 4 more than a multiple of 4, we know that $\overline{5CB4A} - 4 = \overline{5CB4A} - 4 = \overline{5CB0A}$ is a multiple of 4. Since $\overline{5CB0A} = 5000 + 100C + 100B + 0 = 5000 + 100C + 100B$ is a multiple of 4, the number $\overline{5CB0A}$ is a multiple of 4 only when the hundred number $C + B$ is a multiple of 4. The only two-digit multiples of 4 with 5 as the ten digit are 50 and 54, so the possible values of $C + B$ are $\boxed{50}$ and $\boxed{54}$. Since that is not generally true that a number is divisible by 4 if the number formed by the last two digits is. This fact only applies to numbers of the form $\overline{5CB0A}$ which is divisible by 4, in general a number is divisible by 4 if the number formed by its last three digits is divisible by 4.

3.73: Source: MATHCOUNTS ⚙ ⏱ ⏱

The greatest integer's less a greatest common divisor of 2 and a least common multiple of $\overline{CB}$, what are the possible values of the larger of the two integers?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: If two positive integers are relatively prime, then the product of the integers equals their least common multiple. So, the product of the two numbers is $\overline{CB}$. Since $\overline{CB}$ is the product of primes 2 · 19, there are only two pairs of positive integers whose product is $\overline{CB}$. These are numbers 1 and $\overline{CB}$ and 19 and 2. So the two possible values of the larger number are $\boxed{\overline{CB}}$ and $\boxed{19}$.

3.74: Source: MATHCOUNTS ⚙ ⏱ ⏱

Find the largest multiple of 13 that can be written using each digit 1, 3, 5, ... 9 exactly once?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The multiples of 13 are the numbers that are multiples of both 3 and 4. A number is a multiple of 3 if the sum of its digits is a multiple of 3. Since

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

is a multiple of 3, any number written using each digit 1, 3, 5, ... 9 exactly once is a multiple of 3. So, we only have to construct the largest such number that is divisible by 4. This means we must make the final two digits of the number form a multiple of 4. We want the largest such number, so we want the first digit to be as large as possible. The largest digit is 9, so we want the second digit to be as large as possible. We want the second multiple of 4 we can form. Digits must be this to be further to the left of the number, so $\overline{98}$ is not a multiple of 4. $\overline{97}$ is not a multiple of 4 so that is the two digits, since 100 is not a multiple of 4. And we can put 8 and 1 in the final two digits. $\overline{981}$ is $\boxed{\text{Yes}}$.

3.75: Source: MATHCOUNTS ⚙ ⏱ ⏱

Find the single number $\overline{AGCD}$ is divisible by 36. What is the sum of A and D?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We have $36 = 4 \cdot 9$, and 4 and 9 are relatively prime, so we can test for divisibility by 36 through testing for divisibility by 4 and by 9. We know that $\overline{AGCD}$ is a multiple of both 4 and 9. In order for $\overline{AGCD}$ to be a multiple of 4, the two-digit number $\overline{CD}$ must be a multiple of 4. The only possible values of the digit D are 0, 4 and 8. In order for $\overline{AGCD}$ to be a multiple of 9 and 4, the sum $A + G + C + D$ must be a multiple of 9. The only possible values of the digits A and G are 1, 2, 3, 4, 5, 6, 7, 8, 9. So, we have $A + G + C + D = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$. The only two-digit multiples of 9 in the form $\overline{CD}$ is 18, which means $D = 8$.

Now, we find the sum A of the value $\overline{AGCD8}$ is a multiple of 9. The sum of the digits of $\overline{AGCD8}$ is $A + G + C + D + 8 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 8 = 62$. The only digit A for which this is true is $A = 6$, which gives us $A + D = 6 + 8 = 14$. Alternatively, we know that $A + G + C + D = 45$, so $A + D = 45 - (G + C) = 45 - (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 14$. Since $\overline{CD} = 18$, the only three-digit multiple of 4 in the form $\overline{7CD}$ is 736, which means $D = 6$.

Now, we find the sum A of the value $\overline{AGCD6}$ is a multiple of 9. The sum of the digits of $\overline{AGCD6}$ is $A + G + C + D + 6 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 6 = 62$. The only digit A for which this is true is $A = 6$, which gives us $A + D = 6 + 6 = 12$. So, the sum of the digits A and D is $\boxed{12}$.

3.76: Source: MATHCOUNTS ⚙ ⏱ ⏱

Find the sum $A+D+E$ is a six-digit integer in which A and D are digits, and the number is divisible by $\overline{72}$. Find the value of A and the value of D.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We have $\overline{72} = 8 \cdot 9$, and 8 and 9 are relatively prime, so we can test for divisibility by $\overline{72}$ through testing for divisibility by 8 and by 9. We know that $\overline{A+D+E}$ is a multiple of both 8 and 9. In order for $\overline{A+D+E}$ to be a multiple of 8, the three-digit number $\overline{E0A}$ must be a multiple of 8. The only possible values of the digit E are 1,6 and 9. In order for $\overline{A+D+E}$ to be a multiple of 9, the sum $A + D + E$ must be a multiple of 9. The only possible values of $A + D$ are 1,8 and 9. So, our number is $\overline{A+D+E}$.

Now, we find the value A of the value $\overline{A+D+E}$ is a multiple of 9. The sum of the digits of $\overline{A+D+E}$ is $A + D + E = 1 + 6 + 9 = 16$. The only digit A for which this is true is $A = 7$, so the number is $\overline{7+D+E}$. Since $A + D = 1$ or $A + D = 8$, we have $D = 1 - A = 1 - 7 = -6$ or $D = 8 - A = 8 - 7 = 1$. So, the sum of the digits A and D is $\boxed{1}$.

3.77: Source: MATHCOUNTS ⚙ ⏱ ⏱

Find the largest factor of 2187 that is not divisible by 3.

Hint: How must the power of 3 in the factorization of 2187 be related to the prime factorization of 2187?

Hint: If a number is not a multiple of 3, then what must be true about its prime factorization?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We think about the prime factorization of 2187 and its divisors, since we can use prime factorization to tell if one number is a divisor of another. The prime factorization of 2187 is

$$3^7$$

So, the only primes that can appear in the prime factorization of a divisor of 2187 is 3, so $3, 9, 27, 81, 243, 729$ and 2187 are divisors of 2187. The largest divisor of 2187 is 2187 the largest prime power of 3 is 3^7, and the largest prime power of 3^7 is 3^7 itself.

We want the divisor to be as large as possible without being divisible by 3, including 3^7 and 3^6 in the prime factorization of the divisor doesn't affect whether or not the divisor is divisible by 3, so we include both. Since the divisor cannot be a multiple of 3, we cannot the divisor can't be a multiple of 3 and 3. This means the divisor can't have both 3 and 3 in the prime factorization. So, when we include the largest power of 3, we can't include the largest power of 3. So, we include the larger of 3^6 and 3^5. Since $3^6 > 3^5$, we include 3^6.

Therefore, the largest divisor that is not a multiple of 3 is $3^6 \cdot 3^7 = \boxed{1458}$.

3.78: Source: MATHCOUNTS ⚙ ⏱ ⏱

Find the sum of the digits of three different prime numbers. Is it possible for the sum of the digits of each product to be 10? Why or why not?

Hint: What do you know about a number whose digits sum to 10?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: $\boxed{\text{No}}$. A number's digits sum to 10, then the number must be a multiple of 9, which is 3. A product of different primes can only have factor of 3, so each product can't be a multiple of 9. Therefore, the sum of the digits of each product can't be 10.

3.79: Source: MATHCOUNTS ⚙ ⏱ ⏱

The least common multiple of 12, 15, and x is 420. What is the least possible value of the positive integer x?

Hint: How do you find the least common multiple of a group of numbers? What must be true about x in order for $\text{lcm}[12, 15, 20, x] = \text{lcm}[12, 15, 20]$ to be 420?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We can find the least common multiple of a group of numbers by finding the prime factorizations of those numbers, and taking the product of the prime factorizations of the first numbers, and of the remaining prime factorizations. So, we start with the prime factorizations of the first three numbers, and of the remaining prime factorizations.

$$12 = 2^2 \cdot 3^1,$$
$$15 = 3^1 \cdot 5^1,$$
$$20 = 2^2 \cdot 5^1,$$
$$420 = 2^2 \cdot 3^1 \cdot 5^1,$$

We know that

$$\text{lcm}[12, 15, 20] = 420 = 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1,$$

In forming the least common multiple, we can get the 2^2 and the 5^1 from the prime factorization of 12, and the 3^1 from the prime factorization of 15. So, we need the 7^1 and x from the prime factorization of 420. So, x is 20. It must come from 4. Therefore, it must a positive multiple of 7, which means the smallest possible value of x is $\boxed{7}$.

3.80: Source: MATHCOUNTS ⚙ ⏱ ⏱

For each of the following pairs of numbers, find the product of the numbers, and then find the product of the greatest common divisor and least common multiple of the numbers.

(a) 18, 24

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Your Submission: Solution

Solution: We have $18 = 2^1 \cdot 3^2$ and $24 = 2^3 \cdot 3^1$ so $\gcd(18, 24) = 2^1 \cdot 3^1 = 6$ and $\text{lcm}[18, 24] = 2^3 \cdot 3^2 = 72$. Therefore, we have

$$18 \cdot 24 = (2^1 \cdot 3^2) \cdot (2^3 \cdot 3^1)$$
$$= (2^1 \cdot 3^1) \cdot (2^3 \cdot 3^2)$$
$$= 2^4 \cdot 3^3,$$
$$\gcd(18, 24) \cdot \text{lcm}[18, 24] = (2^1 \cdot 3^1) \cdot (2^3 \cdot 3^2)$$
$$= (2^1 \cdot 3^1) \cdot (2^3 \cdot 3^2)$$
$$= 2^4 \cdot 3^3,$$

Both products equal $2^4 \cdot 3^3 = 432$.

(b) 35, 42

Preview Solution

Your Submission: Solution

Solution: We have $35 = 5^1 \cdot 7^1$ and $42 = 2^1 \cdot 3^1 \cdot 7^1$ so $\gcd(35, 42) = 7^1 = 7$ and

$$\text{lcm}[35, 42] = 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 210.$$

Therefore, we have

$$35 \cdot 42 = (5^1 \cdot 7^1) \cdot (2^1 \cdot 3^1 \cdot 7^1)$$
$$= 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^2,$$
$$\gcd(35, 42) \cdot \text{lcm}[35, 42] = (5^1 \cdot 7^1) \cdot (2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1)$$
$$= 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^2.$$

Both products equal $2^1 \cdot 3^1 \cdot 5^1 \cdot 7^2 = 1470$.

(c) 84, 84

Hide Solution | Reset

Your Submission: Solution

Solution: We have $84 = 2^2 \cdot 3^1 \cdot 7^1$ and $84 = 2^2 \cdot 3^1 \cdot 7^1$ so $\gcd(84, 84) = 2^2 \cdot 3^1 \cdot 7^1 = 84$ and

$$\text{lcm}[84, 84] = 2^2 \cdot 3^1 \cdot 7^1 = 84.$$

Therefore, we have

$$84 \cdot 84 = (2^2 \cdot 3^1 \cdot 7^1) \cdot (2^2 \cdot 3^1 \cdot 7^1)$$
$$= 2^4 \cdot 3^2 \cdot 7^2,$$
$$\gcd(84, 84) \cdot \text{lcm}[84, 84] = (2^2 \cdot 3^1 \cdot 7^1) \cdot (2^2 \cdot 3^1 \cdot 7^1)$$
$$= 2^4 \cdot 3^2 \cdot 7^2.$$

Both products equal $2^4 \cdot 3^2 \cdot 7^2 = 1764$.

(d) 84, 84

Hide Solution | Reset

Your Submission: Solution

Solution: We have $84 = 2^2 \cdot 3^1 \cdot 7^1$ and $84 = 2^2 \cdot 3^1 \cdot 7^1$ so $\gcd(84, 84) = 2^2 \cdot 3^1 \cdot 7^1 = 84$ and

$$\text{lcm}[84, 84] = 2^2 \cdot 3^1 \cdot 7^1 = 84.$$

Therefore, we have

$$84 \cdot 84 = (2^2 \cdot 3^1 \cdot 7^1) \cdot (2^2 \cdot 3^1 \cdot 7^1)$$
$$= 2^4 \cdot 3^2 \cdot 7^2,$$
$$\gcd(84, 84) \cdot \text{lcm}[84, 84] = (2^2 \cdot 3^1 \cdot 7^1) \cdot (2^2 \cdot 3^1 \cdot 7^1)$$
$$= 2^4 \cdot 3^2 \cdot 7^2.$$

Both products equal $2^4 \cdot 3^2 \cdot 7^2 = 1764$.

(e) Do you see an interesting pattern in your answers? Will that pattern hold for any two positive integers you start with? If so, why?

Preview Solution

Your Submission: Solution

Solution: We have $\text{lcm}[12, 15] = 2^2 \cdot 3^1 \cdot 5^1$ and $\text{lcm}[12, 20] = 2^2 \cdot 3^1 \cdot 5^1$ so $\text{lcm}[\text{lcm}[12, 15], \text{lcm}[12, 20]] = 2^2 \cdot 3^1 \cdot 5^1 = 60$.

Therefore, we have

$$\text{lcm}[\text{lcm}[12, 15], \text{lcm}[12, 20]] = 2^2 \cdot 3^1 \cdot 5^1 = 60$$

Hide Solution | Reset

Your Submission: Solution

Solution: In each part, we started with two positive integers, a and b, and found that $ab = \gcd(a, b) \cdot \text{lcm}(a, b)$. We saw that these products were the same by seeing that the two prime factorizations of a and b and $\gcd(a, b) \cdot \text{lcm}(a, b)$ were the same. To see why this must always happen for any a and b, let's think about a single prime factor. There are three possibilities for each prime factor. 1. The prime factor appears in the prime factorization of a and b. 2. The prime factor is in the prime factorization of a but it does not show up in the prime factorization of b. For $\text{lcm}(a, b)$, the $\text{lcm}(a, b)$ will appear in either of a or b. $\gcd(a, b) \cdot \text{lcm}(a, b)$

3. The prime appears in the prime factorization of b and not in the prime factorization of a and $\text{lcm}(a, b)$ the prime is p and the power of p appears in the prime factorization of a but not b. The p^1 is in the prime factorization of a and $\text{lcm}(a, b)$ and $\gcd(a, b)$ times $\text{lcm}(a, b)$ times p^1 is the prime factorization of a and b. So, the prime factorization of a and b is the product of the prime factorization of $\gcd(a, b)$, $\text{lcm}(a, b)$ and $\gcd(a, b) \cdot \text{lcm}(a, b)$. Hence $\gcd(a, b)$ must be a multiple of a. Therefore, p^1 appears in the prime factorizations of both a and $\gcd(a, b)$. $\gcd(a, b)$

Consider the prime factorization of a. Suppose p^1 is the prime factorization of a and p^1 is the prime factorization of b. Then, the prime factorization of a is the product of the prime factorization of p^1 and p^1. Where we can use a prime factorization, we multiply the exponents in the prime factorizations by 1. For example, the value of $3^1 \cdot 3^1$ is

$$3^{1+1} = 3^2 = (3^1 \cdot 3^1)$$
$$= (3^1 \cdot 3^1) \cdot 3^1 \cdot 3^1$$
$$= 3^{1+1} \cdot 3^1 \cdot 3^1$$
$$= 3^1 \cdot 3^1 \cdot 3^1$$
$$= 3^1 \cdot 3^1 \cdot 3^1.$$

We can put each of the 3's into a factor of 3 to make a factor of 9.

$$3^1 \cdot 3^1 \cdot 3^1 = (3^1 \cdot 3^1) \cdot 3^1$$
$$= 3^1 \cdot 3^1 \cdot (3^1)$$
$$= 3^1 \cdot 3^1 \cdot (3 \cdot 3)^1$$
$$= 3^1 \cdot 3^1 \cdot 9^1$$

Hide Solution | Reset

Your Submission: Solution

Solution: We multiply a number by 10 by adding a zero to the end of the number, which multiplies by two 10's, so 10^2 by adding five zeros to the end of the number. So, to figure out how many zeroes there are at the end $80^{10} \cdot 75^{12}$, we write the number as the product of powers of primes and find the number of zeroes.

We get factors of 10 when we combine factors of 2 and factors of 5. So, we'll start by writing the prime factorization of $80^{10} \cdot 75^{12}$ to see how many factors of each we have. Since $80^{10} = 16 \cdot 5^{10}$, $2 \cdot 5 = 10$ and $75 = 3 \cdot 25 = 3 \cdot 5^2$ we have

$$80^{10} \cdot 75^{12} = (16 \cdot 5^{10}) \cdot (3 \cdot 5^2)^{12}$$
$$= (2^4 \cdot 5^{10}) \cdot 3^{12} \cdot (5^2)^{12}$$
$$= 2^{4+0} \cdot 5^{10} \cdot 3^{12} \cdot 5^{2 \cdot 12}$$
$$= 2^4 \cdot 5^{10} \cdot 3^{12} \cdot 5^{24}$$
$$= 2^4 \cdot 3^{12} \cdot 5^{34}.$$

We can pair each of the 2's 10 with a factor of 5 to make a factor of 10.

$$2^4 \cdot 3^{12} \cdot 5^{34} = (2^4 \cdot 5^{10}) \cdot 3^{12} \cdot 5^{24}$$
$$= (2^4 \cdot 5^{10}) \cdot 3^{12} \cdot 5^{10} \cdot 5^{14}$$
$$= (2^4 \cdot 5^{10}) \cdot 3^{12} \cdot 10^{10} \cdot 5^{14}$$
$$= 10^{10} \cdot 2^4 \cdot 3^{12} \cdot 5^{14}.$$

The number has 10 zeroes, so $\boxed{\text{Yes}}$.

3.81: Source: MATHCOUNTS ⚙ ⏱ ⏱

How many terminal zeroes does the integer equal to $80^{10} \cdot 75^{12}$ have?

Hint: Multiplying an integer by 10 means you add a zero to the end of the integer. How do you get factors of 10?

Hint: How might the prime factorization of $80^{10} \cdot 75^{12}$ look?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We multiply a number by 10 by adding a zero to the end of the number, which multiplies by two 10's, so 10^2 by adding five zeros to the end of the number. So, to figure out how many zeroes there are at the end $80^{10} \cdot 75^{12}$, we write the number as the product of powers of primes and find the number of zeroes.

We get factors of 10 when we combine factors of 2 and factors of 5. So, we'll start by writing the prime factorization of $80^{10} \cdot 75^{12}$ to see how many factors of each we have. Since $80^{10} = 16 \cdot 5^{10}$, $2 \cdot 5 = 10$ and $75 = 3 \cdot 25 = 3 \cdot 5^2$ we have

$$80^{10} \cdot 75^{12} = (16 \cdot 5^{10}) \cdot (3 \cdot 5^2)^{12}$$
$$= (2^4 \cdot 5^{10}) \cdot 3^{12} \cdot (5^2)^{12}$$
$$= 2^{4+0} \cdot 5^{10} \cdot 3^{12} \cdot 5^{2 \cdot 12}$$
$$= 2^4 \cdot 5^{10} \cdot 3^{12} \cdot 5^{24}$$
$$= 2^4 \cdot 3^{12} \cdot 5^{34}.$$

We can pair each of the 2's 10 with a factor of 5 to make a factor of 10.

$$2^4 \cdot 3^{12} \cdot 5^{34} = (2^4 \cdot 5^{10}) \cdot 3^{12} \cdot 5^{24}$$
$$= (2^4 \cdot 5^{10}) \cdot 3^{12} \cdot 5^{10} \cdot 5^{14}$$
$$= (2^4 \cdot 5^{10}) \cdot 3^{12} \cdot 10^{10} \cdot 5^{14}$$
$$= 10^{10} \cdot 2^4 \cdot 3^{12} \cdot 5^{14}.$$

The number has 10 zeroes, so $\boxed{\text{Yes}}$.

3.82: Source: MATHCOUNTS ⚙ ⏱ ⏱

The number 56 is both a perfect square, since $7^2 = 56$ and $2^3 \cdot 7 = 56$. What is the next larger number that is both a perfect square and a perfect cube?

Hint: All of the exponents in the prime factorization of a perfect square must be even. What similar fact is true about the exponents in the prime factorization of a perfect cube?

Hide Solution | Reset

Your Submission: Solution

Solution: The exponents in the prime factorization of a perfect square must be even. Let's see if there is similar fact for the exponents in the prime factorization of a perfect cube. We know that the exponents in the prime factorizations of the first two cubes to see if we find anything interesting.

$$1^3, \quad 2^3, \quad 3^3 = (3^1)^3 = 3^{1 \cdot 3}, \quad 4^3, \quad 5^3 = (5^1)^3 = 5^{1 \cdot 3},$$

It looks like all of exponents in the prime factorization of a perfect cube must be multiples of 3. The examples of 3^3 and 5^3 above show why this occurs. To get the prime factorization of 3^3, we multiply 3^1 by itself three times. So, we get 3^3. When we cube such a prime factorization, we multiply the exponents in the prime factorizations by 3. For example, the cube of 3^1 is

$$(3^1)^3 = 3^{1 \cdot 3} = 3^3.$$

Therefore, each of the exponents in the prime factorization of a cube must be a multiple of 3.

Since all the exponents in the prime factorization of a square must be a multiple of 2 and each of the exponents in the prime factorization of a cube must be a multiple of 3. Therefore, if a number is a square and a cube, then each of the exponents in its prime factorization must be a multiple of 2 and 3. So, the next number after 56 that is both a perfect square and a perfect cube must be a sixth power. Because $5 \cdot 6 = 30$, the next number after $\boxed{56}$ is $\boxed{\text{Yes}}$ $7^6 = (7^2)^3 = 7^3 \cdot 7^3$ and $7^6 = (7^3)^2 = 7^3 \cdot 7^3$ so 7396 is a perfect square and a cube.

3.83: Source: MATHCOUNTS ⚙ ⏱ ⏱

Suppose that n and m are positive integers with n greater than 5, and with $\text{lcm}(n, 5) = 480$ and $\gcd(n, 5) = 10$ Find the largest prime factor of n that is not a prime factor of m.

Hint: How do you usually find the least common multiple and the greatest common divisor of two numbers? Can you use that process here to find the prime factors of $\text{lcm}(n, 5)$ and $\gcd(n, 5)$, if it is later about n and m?

Hint: What do the prime factorizations of $\text{lcm}(n, 5)$ and $\gcd(n, 5)$ tell you about the prime factorizations of n and m?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We know that we can find the prime factorizations of $\text{lcm}(n, 5)$ and $\gcd(n, 5)$, if we know the prime factorizations of n and 5. So, we find the prime factorizations of the given values of $\text{lcm}(n, 5)$ and $\gcd(n, 5)$, hoping these will help us learn about the prime factorizations of n and 5 in turn.

$$\text{lcm}(n, 5) = 480 = 2^5 \cdot 3^1 \cdot 5^1, \quad \gcd(n, 5) = 10 = 2^1 \cdot 5^1.$$

Since 2^5 is the prime factorization of $\text{lcm}(n, 5)$ and $\gcd(n, 5)$ is 2^1, we know that 2^4 both the largest and the smallest power of 2 that appears in the prime factorizations of n and 5. Therefore, 2^4 appears in the prime factorizations of both n and 5. Similarly, 5^1 is the prime factorizations of $\text{lcm}(n, 5)$ and $\gcd(n, 5)$, so 5^1 appears in the prime factorizations of n and 5.

Since 3^1 is a factor of $\text{lcm}(n, 5)$ but not $\gcd(n, 5)$, we know that 3^1 is in the prime factorization of one or of n and 5, but not the other. So, 3^1 is in the prime factorization of n but not in the prime factorization of m.

The prime factorizations of n and 5 both include $3^1 \cdot 5^1$. The prime factorizations of exactly one of n and 5 contains 2^1. Also, the prime factorizations of exactly one of n and 5 includes 7^1 (possibly the same one that includes 2^1). Neither has any other prime factor besides these four, since no other primes appears in the prime factorizations of $\text{lcm}(n, 5)$. Since n is greater than 5, it must be n that has the prime factorization. Therefore, $\boxed{7}$ is the largest prime that is a factor of n but not of m.

3.84: Source: MATHCOUNTS ⚙ ⏱ ⏱

What is the smallest positive multiple of 49 that has only 0's and 1's as digits?

Hint: We don't know a divisibility rule for 49, but we do know divisibility rules for the factors of 49.

Preview Solution

Your Submission: Solution

Solution: We have $\boxed{10}$ $\boxed{\text{No}}$ $\boxed{\text{No}}$ $\boxed{\text{No}

Review Problems

3.47.

Find the prime factorization of each of the following numbers:

(a) 392

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: See here

$$392 = 2 \cdot 196 = 2 \cdot 2 \cdot 98 = 2 \cdot 2 \cdot 2 \cdot 49 = 2^3 \cdot 7^2$$

We also might have noticed that

$$392 = 196 \cdot 2 = 7 \cdot 56 = 14 \cdot 28 = 28 \cdot 14 = 7 \cdot 2^3 \cdot 7 = 2^3 \cdot 7^2.$$

(b) 5655

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: See here

$$5655 = 5 \cdot 1131 = 5 \cdot 3 \cdot 377 = 5 \cdot 3 \cdot 13 \cdot 29 = 3 \cdot 5 \cdot 13 \cdot 29$$

(c) 30100

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: See here

$$30100 = 301 \cdot 100$$
$$= 29 \cdot 10 \cdot 10$$
$$= 2 \cdot 5 \cdot 5 \cdot 29 \cdot 2^2$$
$$= 2^3 \cdot 5^2 \cdot 29$$

3.48.

Find the exponent over 2 $\mathbb{N}(2^n)$ in $\mathbb{N}(100!)$.

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Since 99 is a multiple of 3, so is $\mathbb{N}(100!)$. Therefore, the remainder is $\boxed{0}$ when we divide $\mathbb{N}(100!)$ by 3.

3.49.

What is the sum of the two smallest multiples of 6 that are greater than 100?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Since 102 divided by 6 leaves remainder 3, we know that 102 is a multiple of 6. So, the next two multiples of 6 are $102 + 6 = 108$ and $108 + 6 = 114$. Thus our answer is $108 + 114 = \boxed{222}$.

3.50.

What is the largest multiple of 7 that is less than 2000?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: A number is divisible by 7 if the rest of its digits is divisible by 7. Otherwise, the number is not divisible by 7. Assuming the digit of thousands of the numbers is the digit 1, we have:

$$\begin{aligned} 0123 &= 1 \cdot 7 + 4 \\ 1123 &= 1 \cdot 7 + 5 \\ 2123 &= 2 \cdot 7 + 1 \\ 3123 &= 3 \cdot 7 + 2 \\ 4123 &= 4 \cdot 7 + 3 \\ 5123 &= 5 \cdot 7 + 4 \\ 6123 &= 6 \cdot 7 + 5 \\ 7123 &= 7 \cdot 7 + 6 \\ 8123 &= 8 \cdot 7 + 7 \\ 9123 &= 9 \cdot 7 + 6 \end{aligned}$$

The only number whose digits sum to a multiple of 7 is 9123 (9+1+2+3). So the only number that is divisible by 7 is $\boxed{9123}$ (duh).

3.51.

Which of the following numbers is divisible by 7

45,624, 346,556, 90,226,992 9,015,0417

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: A number is divisible by 7 if the rest of its digits is divisible by 7. Otherwise, the number is not divisible by 7. Assuming the digit of thousands of the numbers is the digit 1, we have:

$$\begin{aligned} 45{,}624 &= 4 \cdot 7 + 1 \\ 346{,}556 &= 49{,}556 \cdot 7 + 3 \\ 90{,}226{,}992 &= 9 \cdot 7 + 2 \cdot 6 + 2 \cdot 5 + 2 \cdot 4 + 2 \cdot 3 + 1 \\ 9{,}015{,}0417 &= 9 \cdot 7 + 1 \cdot 6 + 5 \cdot 5 + 1 \cdot 3 \end{aligned}$$

The only number whose digits sum to a multiple of 7 is 45,624 (24). So the only number that is divisible by 7 is $\boxed{\text{45,624}}$.

3.52.

Which one of the following four-digit numbers is not divisible by 4: 2044, 2064, 2084, 2104, or 2124?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: A number is divisible by 4 if its final two digits form a multiple of 4. Therefore, 2044, 2064, 2084, and 2124 are multiples of 4. However, 2104 is not divisible by 4 because the digits of 2 form the original number is not a multiple of 4. Since 2104 is not a multiple of 4, we have that $\boxed{2104}$ is not a multiple of 4.

3.53.

Any base10 that is written in divisible by 3 if its units digit is 0 or 3. What is a similar rule that we can use to find if a number is divisible by 9?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We can get a similar rule by looking at the first several positive multiples of 9:

$$9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, 117, 126, 135, 144, 153, 162, 171, 180.$$

It looks like every multiple of 10 ends in a digit that is one more than a multiple of 9. Thus, we can say a number is the sum of a multiple of 9 and its units digit. Now, we can find the sum of the digits of original number. For example:

$$\begin{aligned} 4{,}761{,}522&= 4{,}761{,}518 + 4 \\ &= 4{,}761{,}522 = 4{,}761{,}522 \end{aligned}$$

Since any multiple of 10 is a multiple of 9, we know that 4,761,518 is a multiple of 9 and the new digit sum is also a multiple of 9. The only numbers we can form with two digits that are multiples of 9 are 90, 18, and 72. So, if a number ends in 90, 18, 72, the digit sum is divisible by 9. Therefore, we have shown that 4,761,522 is divisible by 9.

3.54.

Answer the question for the following:

(a) $\log_2(16, 8)$

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $2^4 = \boxed{16}^1$ and $2^3 = \boxed{8}^1$. Thus

$$\log_2(16, 8) = 2^4 \cdot 2^3 = \boxed{128}$$

We also might have noticed that

$$\log_2(16, 8) = \log_2(2 \cdot 2, 2, 2) = 3 \log_2(2, 2) = 3 \cdot 10 = 130.$$

(b) $\log_3(36, 72)$

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have

$$36 = 10 \text{ in } 4 \cdot 3 = 2^2 \cdot 3^2 \text{ so } \boxed{2^2}^1 \cdot \boxed{3^2}^1$$

and $72 = 9 \cdot 8 = 3^2 \cdot 2^3$ so $\boxed{3^2}^1 \cdot \boxed{2^3}^1$.

We also might have noticed that

$$\log_3(36, 72) = \log_3(4 \cdot 3, 9 \cdot 8) = \log_3(4, 3) + \log_3(9, 8) = 2 \cdot 1 + 2 \cdot 1 = 4.$$

(c) $\log_5(15, 35)$

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $15 = \boxed{3^1} \cdot \boxed{5^1}$ and $35 = \boxed{5^1} \cdot 7^1$ so

$$\log_5(15, 35) = 3^1 \cdot 5^1 \cdot 7^1 = \boxed{105}$$

3.55.

Compute each of the following:

(a) $\log_2(16, 32)$

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $2^4 = \boxed{16}^1$ and $2^5 = \boxed{32}^1$ so

$$\log_2(16, 32) = 2^4 \cdot 2^5 = \boxed{128}$$

We also might have noticed that

$$\log_2(16, 32) = \log_2(2 \cdot 2, 2, 2) = 3 \log_2(2, 2) = 3 \cdot 10 = 130.$$

(b) $\log_3(36, 72)$

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $36 = 10$ in $4 \cdot 3 = 2^2 \cdot 3^2$ so $\boxed{2^2}^1 \cdot \boxed{3^2}^1$ and

$$72 = 9 \cdot 8 = 3^2 \cdot 2^3 \text{ so } \boxed{3^2}^1 \cdot \boxed{2^3}^1.$$

We also might have noticed that

$$\log_3(36, 72) = 2^1 \cdot 3^1 \cdot 2^1 = (2 \cdot 3) \cdot 2 = \boxed{12}.$$

(c) $\log_5(15, 35)$

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $15 = \boxed{3^1} \cdot \boxed{5^1}$ and $35 = 5 \cdot 7 = 5^1 \cdot 7^1$ so

$$\log_5(15, 35) = 3^1 \cdot 5^1 \cdot 7^1 = (3 \cdot 5) \cdot 7 = \boxed{105}$$

3.56.

Compute each of the following:

(a) $\gcd(24, 36)$

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $24 = 8 \cdot 3 = 2^3 \cdot 3^1$ and $36 = 9 \cdot 4 = 3^2 \cdot 2^2$ so $\boxed{2^2 \cdot 3^1}$.

We also might have noticed that

$$\gcd(24, 36) = \gcd(2, 3, 2) = (\gcd(2, 3) = 2) \cdot 2 = \boxed{12}.$$

(b) $\gcd(144, 360)$

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $144 = 12^2 = (2^2 \cdot 3)^2 = \boxed{2^4 \cdot 3^2}$ and

$$360 = 10 \cdot 3 = 2^3 \cdot 5^1 = (2^2 \cdot 3 = 3^1 \cdot 5^1.$$

so $\gcd(144, 360) = \boxed{2^2 \cdot 3^1}$.

(c) $\gcd(1234, 4567)$

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Since $1234 = 2 \cdot 7 \cdot 17 \cdot 37$, $4567 = 7 \cdot 17 \cdot 37$ and

$$\gcd(1234, 4567) = 7^1 \cdot 17^1 \cdot 37^1 = \boxed{4{,}811}$$

3.57.

If the $\gcd(p, q) = 7$ and $p \cdot q = 490$, find the $\gcd$ and the product of the $\gcd(p, q)$

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We know that if we know the numbers that are multiples of both 2 and 3. The number 70 which is $\gcd$ is a multiple of 2 in order for a number to be a multiple of 2 and 3 the sum of the digits must be a multiple of 3. Thus, we know that the digits of 70 $\boxed{7 + 0 = 7}$ $\boxed{7 + 2 = 9}$ $\boxed{7 + 3 = 10}$ the digits of the which $\boxed{7}$ is a multiple of 3 so, 7 is $\gcd$ so the desired $\gcd$ product is $\boxed{490}$ $\boxed{7}$.

3.58.

Which of the following numbers is not divisible by 6

$$\boxed{6{,}024}, \quad \boxed{459{,}048}, \quad \boxed{9{,}000{,}006}, \quad \boxed{720{,}002}, \quad \boxed{64{,}507}$$

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: The sum of multiples of 9 must also be a multiple of 9. We can write four of the numbers as sums of multiples of 9:

$$\begin{aligned} 6{,}024 &= 6{,}000 + 24, \\ 459{,}048 &= 450{,}000 + 9{,}000 + 48, \\ 9{,}000{,}006 &= 9{,}000{,}000 + 6, \\ 720{,}002 &= 720{,}000 + 2. \end{aligned}$$

For the remaining number, we have

$$\begin{aligned} 9{,}000{,}000 &= 9{,}000{,}000 + 540{,}000 = 9{,}000 + 54, \\ 540{,}000 &= 540{,}000 + 9{,}000{,}000 = 9{,}000 + 9{,}000{,}000 \end{aligned}$$

The first three numbers in the right-hand sides of the last two sums, so it is true that a multiple of 9. Therefore, we see that

$$9{,}000{,}000 = 9{,}000{,}000 + 540{,}000 = 9{,}000 + 54$$

So a multiple of 9. Since $\boxed{64{,}507}$ is a power that is a multiple of 9 cannot be a multiple of 9. Therefore, the only number in the list that is not a multiple of 9 is $\boxed{64{,}507}$.

3.59. Source: AMC 8 $\quad$ ⚑

Which digit (ones place) digit of the product of all odd positive integers between 1 and 10?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Only one group of six consecutive integers between 1 and 10 is a multiple of 2 and a multiple of 5 and a factor of 3 appears in the product, which means that the product is a multiple of $2 \cdot 5 = 10$. This means the units digit of the product is $\boxed{0}$.

3.60.

What is the smallest prime factor of $7^4 - 7^2$?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: $7^4 - 7^2$ and 7^2 are odd, so their sum is even. Therefore, $7^4 - 7^2$ is divisible by 2, since it is the smallest prime number, $\boxed{2}$ is the smallest prime factor of $7^4 - 7^2$.

3.61.

What is the sum of its prime factor of positive multiples of 12?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Since it is a multiple of 12, and 12 is a multiple of and of 12's divisors, we know that it must be a multiple of each of 12's divisors. Thus, 12's divisors 12 are 1, 2, 3, 4, and 6. So the prime factors number sum of the divisors 12 are 1, 2, 3, 2, 2, and 3. So the prime factors number sum is $\boxed{13}$.

3.62.

What is the greatest prime factor of 12,012?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Let n be the largest prime factor, and let k be the factor of 12,012 such that

$$nk = 12{,}012.$$

The prime factorization of n and k is multiplied to give us the prime factorization of 12,012. We can also think of this as splitting the prime factorization of 12,012 into two parts, one for n and one for k.

We make n as large as possible by putting all the odd primes into its prime factorization. But k is must be odd, so we must include all of the 2's in the prime factorization of 12,012. Thus, we can write the prime factorization of 12,012 as

$$12{,}012 = 2^2 \cdot 6012 = 2^2 \cdot 3006 = 2^3 \cdot 1503.$$

All of the odd primes in the prime factorization of 12,012 are included in the prime factorization of 1503, so the desired largest odd factor is $\boxed{1503}$.

3.63. Source: AMC 8 $\quad$ ⚑

What is the sum of the two digits of the product of all two consecutive positive integers?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Only one group of six consecutive integers between 1 and 10 is a multiple of 2 and a multiple of 5 and a factor of 3 appears in the product, which means that the product is a multiple of $2 \cdot 5 = 10$. This means the units digit of the product is $\boxed{0}$.

3.64.

What is the smallest prime factor of $7^4 - 7^2$?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: $7^4 - 7^2$ and 7^2 are odd, so their sum is even. Therefore, $7^4 - 7^2$ is divisible by 2, since it is the smallest prime number, $\boxed{2}$ is the smallest prime factor of $7^4 - 7^2$.

3.65.

What is the sum of its prime factor of positive multiples of 12?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Since it is a multiple of 12, and 12 is a multiple of and of 12's divisors, we know that it must be a multiple of each of 12's divisors. Thus, 12's divisors 12 are 1, 2, 3, 4, and 6. So the prime factors number sum of the divisors 12 are 1, 2, 3, 2, 2, and 3. So the prime factors number sum is $\boxed{13}$.

3.66.

What is the greatest prime factor of 12,012?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Let n be the largest prime factor, and let k be the factor of 12,012 such that

$$nk = 12{,}012.$$

The prime factorization of n and k is multiplied to give us the prime factorization of 12,012. We can also think of this as splitting the prime factorization of 12,012 into two parts, one for n and one for k.

We make n as large as possible by putting all the odd primes into its prime factorization. But k is must be odd, so we must include all of the 2's in the prime factorization of 12,012. Thus, we can write the prime factorization of 12,012 as

$$12{,}012 = 2^2 \cdot 6012 = 2^2 \cdot 3006 = 2^3 \cdot 1503.$$

All of the odd primes in the prime factorization of 12,012 are included in the prime factorization of 1503, so the desired largest odd factor is $\boxed{1503}$.

3.67.

A six-digit number (base10) of the form $\overline{abcdef}$ is divisible by 7 if $a + 3b + c + 2d + e + 3f$ is divisible by 7.

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Find each of the following numbers is not divisible by 4: 2044, 2064, 2084, 2104, or 2124?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: A number is divisible by 4 if its final two digits form a multiple of 4. Therefore, 2044, 2064, 2084, and 2124 are multiples of 4. However, 2104 is not divisible by 4 because the digits of 2 form the original number is not a multiple of 4. Since 2104 is not a multiple of 4, we have that $\boxed{2104}$ is not a multiple of 4.

3.68.

Any base10 that is written in divisible by 3 if its units digit is 0 or 3. What is a similar rule that we can use to find if a number is divisible by 9?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: We can get a similar rule by looking at the first several positive multiples of 9:

$$9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, 117, 126, 135, 144, 153, 162, 171, 180.$$

It looks like every multiple of 10 ends in a digit that is one more than a multiple of 9. Thus, we can say a number is the sum of a multiple of 9 and its units digit. Now, we can find the sum of the digits of original number. For example:

$$\begin{aligned} 4{,}761{,}522 &= 4{,}761{,}518 + 4 \\ &= 4{,}761{,}522 = 4{,}761{,}522 \end{aligned}$$

Since any multiple of 10 is a multiple of 9, we know that 4,761,518 is a multiple of 9 and the new digit sum is also a multiple of 9. The only numbers we can form with two digits that are multiples of 9 are 90, 18, and 72. So, if a number ends in 90, 18, 72, the digit sum is divisible by 9. Therefore, we have shown that 4,761,522 is divisible by 9.

3.69. Source: MATHCOUNTS $\quad$ ⚑

Answer Section

Your Answer:

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Find each of the following numbers is not divisible by 4: 2044, 2064, 2084, 2104, or 2124?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: A number is divisible by 4 if its final two digits form a multiple of 4. Therefore, 2044, 2064, 2084, and 2124 are multiples of 4. However, 2104 is not divisible by 4 because the digits of 2 form the original number is not a multiple of 4. Since 2104 is not a multiple of 4, we have that $\boxed{2104}$ is not a multiple of 4.

3.70. Source: MATHCOUNTS $\quad$ ⚑

Any base10 that is written in divisible by 3 if its units digit is 0 or 3. What is a similar rule that we can use to find if a number is divisible by 9?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We can get a similar rule by looking at the first several positive multiples of 9:

$$9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, 117, 126, 135, 144, 153, 162, 171, 180.$$

It looks like every multiple of 10 ends in a digit that is one more than a multiple of 9. Thus, we can say a number is the sum of a multiple of 9 and its units digit. Now, we can find the sum of the digits of original number. For example:

$$\begin{aligned} 4{,}761{,}522 &= 4{,}761{,}518 + 4 \\ &= 4{,}761{,}522 = 4{,}761{,}522 \end{aligned}$$

Since any multiple of 10 is a multiple of 9, we know that 4,761,518 is a multiple of 9 and the new digit sum is also a multiple of 9. The only numbers we can form with two digits that are multiples of 9 are 90, 18, and 72. So, if a number ends in 90, 18, 72, the digit sum is divisible by 9. Therefore, we have shown that 4,761,522 is divisible by 9.

3.71.

Let a and b be positive integers such that a is a divisor of b.

(a) What is the greatest common divisor of a and b?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The greatest divisor a is itself, so it is also a divisor of b, we have $\gcd(a, b) = \boxed{a}$.

(b) What is the least common multiple of a and b?

Answer Section

Your Answer:

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The positive multiple of b is b. Since a is a divisor of b, we know that b also a multiple of a. Therefore, we have $\operatorname{lcm}(a, b) = \boxed{b}$.

Section **4** **Fractions**

4.1 What is a Fraction?

In Section 1.1, we discussed division. For example, $12 \div 3 = 4$ and $18 \div 6 = 3$. But what is $2 \div 3$? In a number box, though not an integer, the other side the number equal to $2 \div 3$ is $\boxed{\frac{2}{3}}$.

Definition: If a is a number and b is a non-zero number, then the **fraction** $\frac{a}{b}$ equals $a \div b$.

Suppose we have two three pans: a top (pan), a bottom (b), and a line in the middle called a **fraction line**. We call the top the **numerator**, and we call the bottom the **denominator**. When typing, we often write Fractions in horizontal form, such as $\frac{a}{b}$, that a slash separates the numerator and denominator.

Fraction is a number, so we can put the entire fraction on the number line. This is way of the fraction equals an integer. For example, drop $\frac{6}{3} = 2 \div 3 = 6$ the fraction $\frac{6}{3}$ is at the number line. Similarly, we have $\frac{4}{2} = 2 \div 3 = 4$ and $\frac{2}{1} = 2 \div 3 = 2$.

Where $\frac{2}{3}$ is 1 times 1 between 0 and 1, or pass that $\frac{2}{3}$ between $\boxed{\text{pan}}$. The Fraction $\frac{2}{3}$ number 1 divided by 3, its is, to be $\frac{2}{3}$ on the number line, we divide the number line between 0 and 1 into 3 equal pieces. The fraction $\frac{2}{3}$ is the point at the right end of the first piece, is the point at the right end of the second piece, and $\frac{3}{3}$ is the point at the right end of the third piece.

Of course, you've already seen fraction notation in this book, when we talked about reciprocals. When a is nonzero, the reciprocal $\frac{1}{a}$ is simply $\frac{1}{a} = a$.

Because every fraction is a division, we can use the properties of division we learned in Chapter 1 to work with fractions. Here are some properties we've seen in this section, written with fraction notation:

Important:

- **Dividing into zero:** If a is not zero, then $\frac{0}{a} = 0$.
- **Self division:** If a is not zero, then $\frac{a}{a} = 1$.
- **Dividing by 1:** $\frac{a}{1} = a$.
- **Dividing into a negative:** If b is not zero, then $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.
- **Dividing by a negative:** If a is not zero, then $\frac{-a}{-b} = \frac{a}{b}$.
- **Negative divided by negative:** If b is not zero, then $\frac{-a}{-b} = \frac{a}{b}$.

Jessica Ruscello

above, we repeatedly want that denominators of fractions cannot be 0. This is because $\frac{a}{0}$ or $\frac{0}{0}$ and division by 0 is not defined.

Problems

Problem 4.1 ▶ Jump to Solution

Simplify each of the following fractions.

(a) $\frac{6}{2}$
(b) $\frac{8}{4}$
(c) $\frac{10}{5}$
(d) $\frac{12}{3}$
(e) $\frac{12}{4}$
(f) $\frac{12}{6}$
(g) $\frac{12}{12}$
(h) $\frac{0}{12}$
(i) $\frac{12}{1}$
(j) $\frac{12 \times 12}{12}$

Solution for Problem 4.1

Problem 4.2 ▶ Jump to Solution

For what values of a between 1 and 10 is $\frac{a}{2}$ an integer?

Solution for Problem 4.2

Problem 4.3 ▶ Jump to Solution

Suppose a is between two consecutive integers on the number line below. b divided into five equal pieces. For each relevant b, find a number that corresponds to that point on the number line.

Solution for Problem 4.3

Problem 4.4 ▶ Jump to Solution

(a) Use the number line to explain why $\frac{1}{2} = \frac{2}{4}$.
(b) Without what two consecutive integers is $\frac{7}{4}$?

Solution for Problem 4.4

Problem 4.5 ▶ Jump to Solution

The points A, B, C and D on the number line below correspond to the numbers $\frac{7}{3}$, $\frac{10}{3}$, $\frac{13}{3}$ and $\frac{16}{3}$ in some order. Match each point to the correct number.

Solution for Problem 4.5

Problem 4.6 ▶ Jump to Solution

Use the number line to explain why $\frac{1}{3} = \frac{2}{6}$.

Solution for Problem 4.6

Problem 4.7 ▶ Jump to Solution

Find two fractions that are equal to $\frac{1}{2}$.

Solution for Problem 4.7

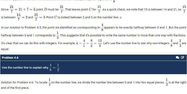
Problem 4.8 ▶ 🔖

For what values of a between 1 and 10 is $\frac{a}{3}$ an integer?

Solution for Problem 4.8 Since $\frac{a}{3} = a \div 3$, the number a must be a multiple of 3 in order for $\frac{a}{3}$ to be an integer. The only multiples of 3 between 1 and 10 are 3, 6, 9. So $\frac{a}{3}$ is an integer when $a = 3$, $a = 6$, or $a = 9$.

Problem 4.9 ▶ 🔖

Suppose a is between two consecutive integers on the number line below. b divided into five equal pieces. For each relevant b, find a number that corresponds to that point on the number line.



Solution for Problem 4.9 Because the dots divide the number line between a and $a + 1$ into 5 equal pieces, the right end of the first of these pieces is $\frac{1}{5}$ the right end of the second pieces is $\frac{2}{5}$ the third pieces is $\frac{3}{5}$ and so on.

Doing is the one direction from a to the right, we have to correspondingly negative another, by starting from a and going leftward, the left end of the first pieces is $-\frac{1}{5}$. The left end of the second pieces is $-\frac{2}{5}$ the third pieces is $-\frac{3}{5}$ and so on. Continuing in this way, we can label the number line as shown below.

We find that, $a + \frac{3}{5} = a + \frac{6}{10}$ and $a - \frac{4}{5} = a - \frac{8}{10}$.

Problem 4.10 ▶ 🔖

(a) Use the number line to explain why $\frac{1}{2} = \frac{2}{4}$.
(b) Without what two consecutive integers is $\frac{9}{4}$?

Solution for Problem 4.10

(a) We locate the number $\frac{1}{2}$ on the number line by dividing the segment between 0 and 1 into 2 equal pieces. $\frac{1}{2}$ is the point at the right end of the first of these pieces.

Since $\frac{1}{2}$ is the right end of the benefit of these pieces, we see that $\frac{1}{2}$ between $\boxed{\text{pan}}$ is 1 between 0 and 1.

Similarly, if the numerator and denominator of a fraction are positive, we can quickly compare the fractions 1.

Important:

Suppose a and b are positive.

- a less than 1 if a is less than b.
- equals 1 if $a = b$.
- a greater than 1 if a is greater than b.

(b) We can write a fraction $\frac{a}{b}$ as an integer if a is a multiple of b. So, to determine which consecutive integers $\frac{9}{4}$ is between, we must find the consecutive multiples of 4 that sit between. Since 8 is between 4 and 12, we know that $\frac{8}{4}$ is between $\frac{4}{4}$ and $\frac{12}{4}$.

Therefore, $\frac{9}{4}$ is between 2 and 3.

Problem 4.11 ▶ 🔖

The points A, B, C, and D on the number line below correspond to the numbers $\frac{2}{3}$, $\frac{5}{3}$, $\frac{8}{3}$ and $\frac{11}{3}$ in some order. Match each point to the correct number.

Solution for Problem 4.11 Point A is the only labeled point to the left of 0, so it must correspond to the only negative number in the list, which is $-\frac{1}{3}$.

Point B is between 0 and 1. Of the three positive numbers in our list, only $\frac{2}{3}$ has a numerator that is less than its denominator. Therefore $\frac{2}{3}$ is the only positive number in the list that is less than 1. So, point B corresponds to $\frac{2}{3}$.

Since $\frac{5}{3} = 1 + \frac{2}{3}$ and $\frac{8}{3} = 2 + \frac{2}{3}$, that leaves point C for $\frac{5}{3}$. As a quick check, we note that $\frac{5}{3}$ is between 1 in and 2, so $\frac{5}{3} = 1 + \frac{2}{3} = 2 + \frac{2}{3}$ is between 1 and 2 and is the number in.

In our solution to Problem 4.3, the point we identified as $\frac{7}{3}$ is apparently the only number in the list that is not a multiple of 3. Since $\frac{7}{3}$ is not a multiple of 3, it is reasonable to guess that it is possible to write the same number in more than one way with fractions. We show that we can do this with integers. For example, $\frac{7}{3} = \frac{14}{6}$ and we can write the fraction $\frac{14}{6}$ in two ways. We can write $\frac{14}{6}$ as $\frac{7}{3}$ or $\frac{14}{6}$ itself.

Problem 4.12 ▶ 🔖

Use the number line to explain why $\frac{1}{3} = \frac{2}{6}$.

Solution for Problem 4.12 To locate $\frac{1}{3}$ on the number line, we divide the number line between 0 and 1 into two equal pieces. $\frac{1}{3}$ is the right end of the first piece.

So, the right end of the first of these eight equal pieces between 0 and 1 is the same as the right end of the first of the two equal pieces we used to locate $\frac{1}{3}$. Therefore, $\frac{1}{3}$ and $\frac{2}{6}$ are the same point on the number line, and $\frac{1}{3} = \frac{2}{6}$.

Problem 4.13 ▶ 🔖

Find two fractions that are equal to $\frac{1}{2}$.

Solution for Problem 4.13 We can look each of these three equal pieces between 0 and 1 into two equal pieces, so that we have four equal pieces between 0 and 1. The point corresponding to $\frac{1}{2}$ at the right end of the 2nd of these 4 pieces, so we see that $\frac{1}{2} = \frac{2}{4}$.

Similarly, we could look each of our original three equal pieces into three equal pieces, so that we have 9 equal pieces between 0 and 1. The point corresponding to $\frac{1}{2}$ at the right end of the 3rd of these 9 pieces, so we see that $\frac{1}{2} = \frac{3}{9}$.

Therefore, both $\frac{2}{4}$ and $\frac{3}{9}$ are two fractions that equal $\frac{1}{2}$. For example, $\frac{1}{2} = \frac{100}{200}$ and $\frac{1}{2} = \frac{1000}{2000}$ are other ways to write $\frac{1}{2}$ as a fraction.

Exercises

4.1.1. ▶ 🔖

Find the value of each of the following:

(a) $\frac{-6}{3}$
Hide Solution
Reset
You may type any additional notes you have here.
Preview: Solution
Your Submission: Solution: Applying our rule for dividing into negative, we have $\frac{-6}{3} = -6 \div (3) = \boxed{-2}$.

(b) $\frac{-10}{2}$
Hide Solution
Reset
You may type any additional notes you have here.
Preview: Solution
Your Submission: Solution: $\frac{-10}{2} = -10 \div 2 = \boxed{-5}$

(c) $\frac{-12}{-3}$
Hide Solution
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You may type any additional notes you have here.
Preview: Solution
Your Submission: Solution: Any integer number divided by itself equals 1, so $\frac{-12}{-3} = \boxed{4}$.

(d) $\frac{0}{-5}$
Hide Solution
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You may type any additional notes you have here.
Preview: Solution
Your Submission: Solution: Any integer number divided by 1 equals a, so $\frac{0}{-5} = \boxed{0}$

4.1.2. ▶ 🔖

Compute $\frac{10 \times 5}{10 \times 5}$.

Hide Solution
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You may type any additional notes you have here.
Preview: Solution
Your Submission: Solution: $\frac{10 \times 5}{10 \times 5} = 10 \div 5 = \boxed{2}$

4.1.3. Source: MATHCOUNTS ▶ 🔖

Compute $\frac{1 \times 2 \times 4 \times 5 \times 7 \times 11}{2 \times 4 \times 5 \times 7 \times 11}$.

Hide Solution
Reset
You may type any additional notes you have here.
Preview: Solution
Your Submission: Solution: $\frac{1 \times 2 \times 4 \times 5 \times 7 \times 11}{2 \times 4 \times 5 \times 7 \times 11} = 10 \div 5 = \boxed{2}$

4.1.4. ▶ 🔖

Compute $\frac{0}{-3}$ is equal to $\boxed{}$.

Hide Solution
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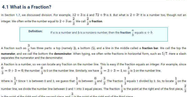
You may type any additional notes you have here.
Preview: Solution
Your Submission: Solution: We first find the multiples of 7 that sit in between. We have $\frac{21}{7} = 3$. $\frac{0}{-3} = 0$ and $\frac{21}{7} = 3$. So $\frac{0}{-3}$ is between $\frac{21}{7}$ in 3 and $\frac{21}{7}$ in 3. We show it on the number line and notice that $\frac{0}{-3}$ is closer to $\boxed{\frac{21}{7}}$ than to $\boxed{\frac{21}{7}}$.

Another way to think about this problem is to consider the location of $\frac{0}{-3}$ on the number line. Suppose we break the number line between each pair of consecutive integers into 7 equal pieces. The fraction $\frac{0}{-3}$ which equals 0, is at the right end of the 21st piece, to the right of 0. The fraction $\frac{21}{7}$ is at the right end of the 21st piece, so it is closer to $\boxed{\frac{21}{7}}$ than to any other integer.



4.1.5. ▶ 🔖

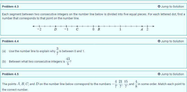
$\frac{0}{-3} = (-3)\text{ary} = -3$ Define value of $\frac{0}{-3}$.

Hide Solution
Reset
You may type any additional notes you have here.
Preview: Solution
Your Submission: Solution: Step 1: $\frac{0}{-3} = -3$ Define value of $\frac{0}{-3}$.

Step 2: $\frac{0}{-3} = \frac{0}{-3} \times \frac{-1}{-1} = \frac{0 \times (-1)}{(-3) \times (-1)} = \frac{0}{3} = \boxed{0}$

4.1.6. ▶ 🔖

The points A, B, C, and D on the number line below correspond to the numbers $\frac{2}{3}$, $\frac{5}{3}$, $\frac{8}{3}$ and $\frac{11}{3}$ in some order. Match each point to the correct number.



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You may type any additional notes you have here.
Preview: Solution
Your Submission: Solution: Point A is the only point to the left of 0, so this must be the only number that corresponds to negative fractions. Since $-\frac{1}{3}$ is a point B corresponds to $-\frac{1}{3}$. Therefore, C corresponds to the other negative fraction, $\frac{-2}{3}$. Notice that point C is between 0 and -1, but closer to the point -1.

Turning to our positive fractions, because $\frac{2}{3}$ is greater than 1, we know that $\frac{2}{3}$ is greater than $\frac{5}{3}$. We could also have seen this by dividing the number line between each pair of consecutive integers into six equal pieces. $\frac{2}{3}$ is at the right end of the $\frac{2}{3}$rd piece, to the right of 1, while $\frac{5}{3}$ is at the right end of the 5th piece, hence, $\frac{2}{3}$ corresponds to $\boxed{\frac{2}{3}}$.

$\frac{5}{3}$ is at the right end of the 5th piece, so it corresponds to $\boxed{\frac{5}{3}}$.

$\frac{8}{3}$ is at the right end of the 8th piece, hence, $\frac{8}{3}$ corresponds to $\boxed{\frac{8}{3}}$.

$\frac{11}{3}$ is at the right end of the 11th piece, therefore, $\frac{11}{3}$ corresponds to $\boxed{\frac{11}{3}}$.

4.1.7. ▶ 🔖

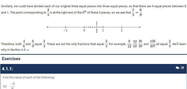
Find three fractions that are equal to $\frac{1}{2}$.

Hide Solution
Reset
You may type any additional notes you have here.
Preview: Solution
Your Submission: Solution: We locate $\frac{1}{2}$ on the number line by dividing the number line between 0 and 1 into 2 equal pieces. $\frac{1}{2}$ is at the right end of the 1st piece.

If we divide each of these 2 equal pieces into 3 equal pieces, then $\frac{1}{2}$ is at the right end of the 3rd of 6 equal pieces, so $\frac{1}{2} = \frac{3}{6}$.

If we divide each of our original 2 equal pieces into 4 equal pieces, then $\frac{1}{2}$ is at the right end of the 2nd of 4 equal pieces, so $\frac{1}{2} = \frac{2}{4}$.

If we divide each of our original 2 equal pieces into 5 equal pieces, then $\frac{1}{2}$ is at the right end of the 5th of 10 equal pieces, so $\frac{1}{2} = \frac{5}{10}$.



So, three fractions that equal $\frac{1}{2}$ are $\boxed{\frac{3}{6}}$ and $\boxed{\frac{2}{4}}$. These are not the only possible answers. There are infinitely many other possible answers.

4.1.8. ▶ 🔖

For how many positive integer values of a is $\frac{a}{3}$ an integer?

Hide Solution
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You may type any additional notes you have here.
Preview: Solution
Your Submission: Solution: The expression $\frac{a}{3}$ is a positive integer if and only if $a \div 3$. Division ends into 0. Therefore, $a \div 3$ must be a positive divisor of 3. The positive divisors of 3 are 1, 3. Hence, a is 1, 3, and 9. Therefore, the problem asks for the number of positive values of a for which a is 1, 3, and 9. The number of positive values of a for which this satisfy the problem.

4.2 Multiplying Fractions

We often use the word 'of' to mean multiplication. For example, suppose there are 12 apples in each apple basket in the grocery store; much how many total apples there are in 2 of these baskets, we compute $2 \cdot 12 = 24$. We often avoid using 'of' with integers, saying the basket rather than '2 of the baskets.' But with fractions, we frequently use 'of' rather $\frac{1}{2}$ of a basket' instead of $\frac{1}{2}$ basket.

How many apples are in $\frac{1}{2}$ of a basket? There are 12 apples in each basket! Since '$\frac{1}{2}$ of' means multiply, there are $\frac{1}{2} \cdot 12$ apples. To compute, $\frac{1}{2} \cdot 12$ we need to multiply fractions.

Here are a couple of properties we learned in Chapter 1 that will be helpful in learning how to multiply fractions.

Recall: **Definition of division:** If $\frac{a}{b}$ is a real number, then $\frac{a}{b} = a \cdot \frac{1}{b}$.

Reciprocal of product: If a and b are real numbers, then $\frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}$.

Problems

Problem 4.2

Jump to Solution

Simplify $\frac{1}{2} \cdot \frac{1}{3}$.

Problem 4.3

Jump to Solution

In this problem, we compute $\frac{1}{5} \cdot \frac{1}{7}$.

(a) Use our definition of division to write $\frac{1}{5} \cdot \frac{1}{7}$ as the product of an integer and the reciprocal of an integer.

(b) Use the appropriate properties of multiplication and your result from part (a) to compute $\frac{1}{5} \cdot \frac{1}{7}$. By "compute," we mean find a number that equals $\frac{1}{5} \cdot \frac{1}{7}$.

Problem 4.4

Jump to Solution

(a) Explain why $\frac{\text{RANDOM}}{\text{RANDOM}} \cdot \frac{1}{\text{RANDOM}}$.

(b) Compute $\frac{\text{RANDOM}}{\text{RANDOM}} \cdot \frac{1}{\text{RANDOM}}$.

Problem 4.5

Jump to Solution

Simplify $\frac{1}{2} \cdot \frac{1}{3}$.

Problem 4.6

Jump to Solution

In this problem, we compute $\frac{1}{5} \cdot \frac{1}{7}$.

(a) Use the definition of division to write $\frac{1}{5} \cdot \frac{1}{7}$ as the product of an integer and the reciprocal of an integer.

(b) Use the appropriate properties of multiplication and your result from part (a) to compute $\frac{1}{5} \cdot \frac{1}{7}$.

Problem 4.7

Jump to Solution

Compute $\frac{\text{RAND}}{\text{RAND}} \cdot \frac{\text{RAND}}{\text{RAND}}$.

(a) What is $\frac{1}{2} \cdot \frac{1}{3}$?

(b) What is $\frac{1}{5} \cdot \frac{1}{7}$?

Wait, what's with 1st pancake, 2nd pancake, 3rd pancake, 4th pancake, 5th pancake, 6th pancake, 7th pancake, 8th pancake, 9th pancake, 10th pancake, 11th pancake, 12th pancake, 13th pancake, 14th pancake, 15th pancake, 16th pancake, 17th pancake, 18th pancake, 19th pancake, 20th pancake, 21st pancake, 22nd pancake, 23rd pancake, 24th pancake, 25th pancake, 26th pancake, 27th pancake, 28th pancake, 29th pancake, 30th pancake, 31st pancake, 32nd pancake, 33rd pancake, 34th pancake, 35th pancake, 36th pancake, 37th pancake, 38th pancake, 39th pancake, 40th pancake, 41st pancake, 42nd pancake, 43rd pancake, 44th pancake, 45th pancake, 46th pancake, 47th pancake, 48th pancake, 49th pancake, 50th pancake, 51st pancake, 52nd pancake, 53rd pancake, 54th pancake, 55th pancake, 56th pancake, 57th pancake, 58th pancake, 59th pancake, 60th pancake, 61st pancake, 62nd pancake, 63rd pancake, 64th pancake, 65th pancake, 66th pancake, 67th pancake, 68th pancake, 69th pancake, 70th pancake, 71st 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4.3 Dividing by a Fraction

Recall that we define division in terms of multiplication:

$$a \div b = a \cdot \frac{1}{b}$$

In this section, we combine this definition with the rule for multiplying fractions to learn how to divide by fractions.

Problems

Problem 4.16 Jump to Solution

What is $\frac{1}{2} \div \frac{3}{5}$?

Problem 4.17 Jump to Solution

(a) What is the reciprocal of $\frac{3}{5}$?

(b) If a and b are nonzero, then what is the reciprocal of $\frac{a}{b}$?

Problem 4.18 Jump to Solution

What is $3 \div \frac{5}{7}$?

Problem 4.19 Jump to Solution

What is $\frac{3}{5} \div \frac{5}{7}$?

Problem 4.20 Jump to Solution

What is $\frac{12/5}{7/3}$?

Problem 4.21 Jump to Solution

(c) $\frac{1}{3}$ of what number?

Problem 4.22 Jump to Solution

Each panel of fencing material is $\frac{3}{2}$ feet long. How many panels do I need to build a 60-foot fence?

Problem 4.16 ☆ ✎

What is $\frac{1}{2} \div \frac{3}{5}$?

Solution for Problem 4.16: Since $a \div b = a \cdot \frac{1}{b}$, we have

$$\frac{1}{2} \div \frac{3}{5} = \frac{1}{2} \cdot \frac{5}{3} = \frac{1 \cdot 5}{2 \cdot 3} = \frac{5}{6}.$$

Hint

What about dividing by a fraction? Our definition of division tells us that to divide by a fraction, we multiply by the reciprocal of the fraction. So, we'll have to figure out how to find the reciprocal of a fraction.

Problem 4.17 ☆ ✎

(a) What is the reciprocal of $\frac{3}{5}$?

(b) If a and b are nonzero, then what is the reciprocal of $\frac{a}{b}$?

Solution for Problem 4.17:

(a) We seek a number such that the product of $\frac{3}{5}$ and the number is 1. Suppose that this number is a fraction, $\frac{m}{n}$. Applying the rule for multiplying fractions, we have

$$\frac{3}{5} \cdot \frac{m}{n} = \frac{3m}{5n}.$$

The fraction $\frac{3m}{5n}$ equals 1 if the numerator and denominator are equal. One easy way to accomplish this is by letting $m = 5$ and $n = 3$:

$$\frac{3m}{5n} = \frac{3 \cdot 5}{5 \cdot 3} = \frac{3 \cdot 5}{3 \cdot 5} = 1.$$

So, $\frac{5}{3}$ is the reciprocal of $\frac{3}{5}$.

$$\frac{3}{5} \cdot \frac{5}{3} = \frac{3 \cdot 5}{5 \cdot 3} = \frac{3 \cdot 5}{3 \cdot 5} = 1.$$

We also could have found the answer by applying the definition of division together with the rule for the product of reciprocals:

$$\frac{3}{5} = \frac{3}{5} \div \frac{1}{1} = \frac{3}{5} \cdot \frac{1}{1}.$$

Since the reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$, we have

$$\frac{1}{1} \cdot \frac{5}{3} = \frac{5}{3}.$$

(b) In part (a), we found that the reciprocal of $\frac{3}{5}$ is formed by swapping the numerator and denominator to get $\frac{5}{3}$. Maybe we can find the reciprocal of any fraction the same way! Let's check if this formula works:

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{ab}{ab} = 1.$$

Since $\frac{b}{a} \cdot \frac{a}{b} = 1$, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

As in part (a), we could have applied the definition of division together with the rule for the product of reciprocals:

$$\frac{a}{b} = \frac{a}{b} \div \frac{1}{1} = \frac{a}{b} \cdot \frac{1}{1}.$$

Since $\frac{1}{1} \cdot \frac{b}{a} = \frac{b}{a}$, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

Important: If a and b are nonzero, then the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. In other words,

$$\frac{1}{\frac{a}{b}} = \frac{b}{a}.$$

Problem 4.18 ☆ ✎

What is $3 \div \frac{5}{7}$?

Solution for Problem 4.18: The definition of division tells us that $3 \div \frac{5}{7}$ equals 3 times the reciprocal of $\frac{5}{7}$. The reciprocal of $\frac{5}{7}$ is $\frac{7}{5}$, so we have

$$3 \div \frac{5}{7} = 3 \cdot \frac{7}{5} = \frac{3 \cdot 7}{5} = \frac{21}{5}.$$

Hint

Usually, we have not the intermediate step $\frac{3}{1}$ above and just write $3 \div \frac{5}{7} = \frac{3}{1} \cdot \frac{7}{5}$.

Problem 4.19 ☆ ✎

What is $\frac{3}{5} \div \frac{5}{7}$?

Solution for Problem 4.19: Since the reciprocal of $\frac{5}{7}$ is $\frac{7}{5}$, we have

$$\frac{3}{5} \div \frac{5}{7} = \frac{3}{5} \cdot \frac{7}{5} = \frac{3 \cdot 7}{5 \cdot 5} = \frac{21}{25}.$$

Hint

Similarly, we can have write a rule for dividing by a fraction:

Important: If b, c, and d are nonzero, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Fraction Division

Problem 4.20 ☆ ✎

What is $\frac{12/5}{7/3}$?

Solution for Problem 4.20: Here we have a fraction in which the numerator and the denominator are both fractions, all this means is that we are dividing the fraction in the numerator, $\frac{12}{5}$, by the fraction in the denominator, $\frac{7}{3}$. In order to divide by $\frac{7}{3}$, we need to find the reciprocal of a negative fraction. Fortunately, in section 1.4 here we learned the rule for the reciprocal of a negative:

$$\frac{1}{-\frac{a}{b}} = -\frac{b}{a}.$$

So, we have $\frac{1}{-\frac{7}{3}} = -\frac{3}{7}$. Now, we can perform our division:

$$\frac{12/5}{7/3} = \frac{12}{5} \div \left(-\frac{7}{3}\right) = \frac{12}{5} \cdot \left(-\frac{3}{7}\right) = -\frac{12 \cdot 3}{5 \cdot 7} = -\frac{36}{35}.$$

Since 14 is a multiple of 2 and 5 is a multiple of 3, we can simplify the final computation:

$$-\frac{12 \cdot 3}{5 \cdot 7} = -\frac{12 \cdot 3}{5 \cdot 3 \cdot 7} = -\frac{12}{5} \cdot \frac{3}{3 \cdot 7} = -\frac{12}{5} \cdot 1 \cdot \frac{3}{7} = -\frac{12}{5} \cdot 1 \cdot 3 = -\frac{12}{5} \cdot 3 = -\frac{36}{35}.$$

Hint

Problem 4.21 ☆ ✎

(c) $\frac{1}{3}$ of what number?

Solution for Problem 4.21: Since "of" means "multiply," the question asks, "(c) equals $\frac{1}{3}$ times what number?" This means we seek the number that fits the box in the equation

$$\frac{1}{3} \cdot \boxed{} = 32.$$

If we still don't know what to do to fill in the box here, we can use a very powerful problem-solving strategy:

Concept: If you don't know how to solve a problem at first, try solving a simpler version of the problem.

It's probably the fraction that makes this problem difficult, so we think about how to solve a similar problem without a fraction. Imagine instead that the problem were to find the number that fills the box in the equation

$$3 \cdot \boxed{} = 6.$$

This is more familiar. We learned back in section 1.7 that we can think of division as the reverse of multiplication, and the number that fills this box is $\frac{6}{3} = 2$.

Let's return to the equation with the fraction:

$$\frac{1}{3} \cdot \boxed{} = 32.$$

Now, we could have used division to find the number that fills the box:

$$\frac{1}{3} \cdot \boxed{} = 32 \quad \text{so} \quad \boxed{} = \frac{32}{\frac{1}{3}} = \frac{32}{1} \div \frac{1}{3} = 32 \cdot 3 = 96.$$

Hint

We also could have used some number sense to solve this problem. Since $\frac{1}{3}$ of a number is 32, we know that $\frac{1}{3}$ of that same number is half of 32, or $\frac{32}{2} = 16$. Since $\frac{1}{3}$ times 16 equals 32, the number is 3 times 16, or 48. Notice that in this solution, we divided 32 by 2 and then multiplied the result by 3. Compare these two steps to the steps we used to compute $\frac{32}{\frac{1}{3}}$ above—they're the same steps!

Problem 4.22 ☆ ✎

Each panel of fencing material is $\frac{3}{2}$ feet long. How many panels do I need to build a 60-foot fence?

Solution for Problem 4.22: Once again, we can start by simplifying the problem to get a better hands on it. Suppose the problem were instead:

Each panel of fencing material is 2 feet long. How many panels do I need to build a 60-foot fence?

Now it's more clear that we have to divide. Each panel is 2 feet, so I need $60 \div 2 = 30$ panels to have all feet of fence.

Returning to our original problem, we see that if each panel $\frac{3}{2}$ feet, then the number of panels needed for a 60-foot fence is

$$60 \div \frac{3}{2} = \frac{60}{1} \div \frac{3}{2} = \frac{60 \cdot 2}{1 \cdot 3} = \frac{60 \cdot 2}{3 \cdot 1} = 20 \cdot 2 = 3 \cdot 20 = 60.$$

As a check, we confirm that 3 panels do indeed provide $\frac{3}{2} \cdot \frac{20}{1} = \frac{3 \cdot 20}{2 \cdot 1} = \frac{180}{2} = 90$ feet of fence. ✓

Exercises

4.3.1: ☆ ✎

Evaluate each of the following:

(a) $\frac{1}{2} \div \frac{3}{7}$.

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Solution: $\frac{1}{2} \div \frac{3}{7} = \frac{1}{2} \cdot \frac{7}{3} = \boxed{\frac{7}{6}}$

(b) $7 \div \frac{4}{5}$.

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Solution: $7 \div \frac{4}{5} = 7 \cdot \frac{5}{4} = \frac{7 \cdot 5}{4} = \boxed{\frac{35}{4}}$

(c) $\frac{14/3}{5/2}$.

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Solution: The quotient of two negative numbers is positive, so we have

$$\left(-\frac{14}{3}\right) \div \left(-\frac{5}{2}\right) = \frac{14}{3} \div \frac{5}{2} = \frac{14}{3} \cdot \frac{2}{5} = \frac{14 \cdot 2}{3 \cdot 5} = \boxed{\frac{28}{15}}.$$

(d) $\left(-\frac{4}{7}\right) \div \left(-\frac{17}{7}\right)$.

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Solution: The quotient of two negative numbers is positive, so we have

$$\left(-\frac{4}{7}\right) \div \left(-\frac{17}{7}\right) = \frac{4}{7} \div \frac{17}{7} = \frac{4}{7} \cdot \frac{7}{17} = \frac{4 \cdot 7}{7 \cdot 17} = \boxed{\frac{4}{17}}.$$

4.3.2: ☆ ✎

What is the quotient when $\frac{3}{5}$ is divided by $\frac{7}{2}$?

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Solution: When $\frac{3}{5}$ is divided by $\frac{7}{2}$ the result is $\frac{3}{5} \div \frac{7}{2} = \frac{3}{5} \cdot \frac{2}{7} = \boxed{\frac{6}{35}}$.

4.3.3: ☆ ✎

If $x = \frac{7}{2}$ what is the value of $\frac{36}{x}$?

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Solution: $\frac{36}{x} = 36 \div x = 36 \div \frac{7}{2} = 36 \cdot \frac{2}{7} = \frac{36 \cdot 2}{7} = \boxed{\frac{72}{7}}$.

4.3.4: ☆ ✎

(a) $40 \div \frac{5}{8}$ of what number?

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Solution: We must determine what number is in the blank in $\frac{5}{8} \cdot \underline{\quad\quad}$, so we divide $\frac{5}{8} \div \frac{5}{8} = \frac{5}{8} \cdot \frac{8}{5} = 1$ so $\boxed{64}$

(b) $\frac{3}{4} \div \frac{9}{10}$ of what number?

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Solution: We must determine what number is in the blank in $\frac{9}{10} \cdot \underline{\quad\quad}$, so we divide $\frac{3}{4} \div \frac{9}{10} = \frac{3}{4} \cdot \frac{10}{9} = \boxed{\frac{5}{6}}$

4.3.5: ☆ ✎

(a) Dividing $\frac{6}{7}$ by what number gives $\frac{2}{7}$?

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Solution: To get to thanks to these questions, we first think about the problem without fractions. Suppose we are asked, "Dividing 6 by what number gives 2?" Then, the answer is 3, which is $6 \div 2$. Similarly, the answer to "Dividing 40 by what number is 8?" is 5, which is $40 \div 8$. So, it seems like the answer to the question,

"Dividing a by what number equals b?"

is $a \div b$. So, we expect that $x \div \frac{2}{7}$ is $\frac{6}{7}$. To see that this is correct, we write $x \div \frac{2}{7}$ as $\frac{x}{1} \div \frac{2}{7}$ and we have

$$x \div \left(\frac{x}{1} \div \frac{2}{7}\right) = x \cdot \frac{7}{2} = \frac{x \cdot 7}{2} = \frac{7x}{2}.$$

So, indeed, $x \div 2$ is the number that we must divide x by to get 2.

Using what we just learned, we must divide $\frac{6}{7} \div \frac{2}{7} = \frac{6 \cdot 7}{7 \cdot 2} = \frac{6 \cdot 7}{2 \cdot 7} = \boxed{3}$. Checking our answer, we find that $\frac{6}{7} \div 3 = \frac{6}{7} \div \frac{3}{1} = \frac{6}{7} \cdot \frac{1}{3} = \frac{2}{7}$, as expected!

(b) Dividing $\frac{3}{4}$ by what number gives $\frac{5}{6}$?

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Your Submission: Solution

Solution: Looking at the solution in part (a) and again using what we just learned, we must divide $\frac{3}{4} \div \frac{5}{6} = \frac{3}{4} \cdot \frac{6}{5} = \frac{3 \cdot 6}{4 \cdot 5} = \frac{3 \cdot 6}{2 \cdot 2 \cdot 5} = \frac{3 \cdot 6}{2 \cdot 2 \cdot 5} = \frac{3 \cdot 3}{2 \cdot 5} = \boxed{\frac{9}{10}}$.

Checking our answer, we find $\frac{9}{10} \div \frac{5}{6} = \frac{9}{10} \cdot \frac{6}{5} = \frac{9 \cdot 6}{10 \cdot 5} = \frac{9 \cdot 6}{2 \cdot 5 \cdot 5} = \frac{3 \cdot 3}{5 \cdot 5} = \frac{9}{25}$, as required.

(c) Dividing $\frac{3}{4}$ by what number gives $\frac{9}{10}$?

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Your Submission: Solution

Solution: We must divide $\frac{3}{4} \div \frac{9}{10} = \frac{3}{4} \cdot \frac{10}{9} = \frac{3 \cdot 10}{4 \cdot 9} = \frac{3 \cdot 10}{2 \cdot 2 \cdot 9} = \boxed{\frac{5}{6}}$.

4.3.6: ☆ ✎

Dividing $\frac{1}{2}$ into what number gives a quotient of 20?

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Your Submission: Solution

Solution: Once again, we get a handle on the problem by trying a problem without fractions. Suppose the problem were:

"Dividing 2 into what number gives a quotient of 20?"

Now, it's more clear what the answer is: we divide 2 into 40 to get a quotient of 20. Notice that $40 \div \frac{1}{2} = 20$. So, it looks like the answer to the question, "Dividing a into what number gives a quotient of b?" is simply $a \cdot b$. We can quickly check this by simplifying $(a \cdot b) \div a$:

$$(a \cdot b) \div a = (a \cdot b) \cdot \frac{1}{a} = \frac{a \cdot b}{a} = b.$$

This makes sense: if we divide the product of a and b by a, we get b. That's how division works, it undoes multiplication!

Returning to the original problem, to get a quotient of 20, we divide $\frac{1}{2} \div 20 = \frac{1}{2} \cdot \frac{1}{20} = \frac{1}{2 \cdot 20} = \boxed{\frac{1}{40}}$. Checking our answer, we have $\frac{1}{2} \div \left(\frac{1}{40}\right) = \frac{1}{2} \cdot \frac{40}{1} = 1 \cdot 20$ as required.

4.3.7: Source: AMC 8 ☆ ✎

Multiplying a number by $\frac{3}{7}$ and then dividing the result by $\frac{3}{7}$ has the same effect as multiplying the original number by what number?

Preview: Solution

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Your Submission: Solution

Solution: Suppose our original number is x. Multiplying x by $\frac{3}{7}$ gives $\left(x \cdot \frac{3}{7}\right)$. Dividing this number by $\frac{3}{7}$ gives

$$\left(x \cdot \frac{3}{7}\right) \div \frac{3}{7} = \left(x \cdot \frac{3}{7}\right) \cdot \frac{7}{3} = x \cdot \left(\frac{3}{7} \cdot \frac{7}{3}\right) = x \cdot \left(\frac{7}{7} \cdot \frac{3}{3}\right) = x \cdot 1 = x.$$

So, the result is the same as multiplying the original number by $\boxed{1}$.

4.3.8: ☆ ✎

I have a scoop that holds $\frac{3}{4}$ cup of flour. If my recipe calls for $5\frac{1}{2}$ cups of flour, how many scoops do I need?

Preview: Solution

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Your Submission: Solution

Solution: If we multiply the number of scoops by the size of the scoop, we get the total amount of flour.

[Teacher of scoops] · [Size of scoops] = Total amount of flour.

So, to find the number of scoops, we divide the total amount of flour by the size of each scoop. Therefore, I need $5\frac{1}{2} \div \frac{3}{4} = \frac{11}{2} \div \frac{3}{4} = \frac{11}{2} \cdot \frac{4}{3} = \boxed{\frac{22}{3}}$ scoops.

4.4 Raising Fractions to Powers

Problems

Problem 4.23 — Jump to Solution

(a) Compute $\left(\frac{2}{5}\right)^4$.

(b) Let b be nonzero and let m be a positive integer. Explain why $\left(\frac{b}{5}\right)^m = \frac{b^m}{5^m}$.

Problem 4.24 — Jump to Solution

(a) Evaluate $\left(\frac{3}{5}\right)^{-2}$.

(b) Evaluate $\left(\frac{b}{5}\right)^{-n}$.

Problem 4.25 — Jump to Solution

Suppose a and b are nonzero, and n is a positive integer. Explain why

$$\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}.$$

Problem 4.26 — Jump to Solution

Compute $\frac{(25/36)^3(25/21)^2}{(21/36)^{-2}}$.

Problem 4.23

(a) Compute $\left(\frac{2}{5}\right)^4$.

(b) Let b be nonzero and let m be a positive integer. Explain why $\left(\frac{b}{5}\right)^m = \frac{b^m}{5^m}$.

Solution for Problem 4.23:

(a) We apply the rule for fraction multiplication:

$$\left(\frac{2}{5}\right)^4 = \frac{2}{5}\cdot\frac{2}{5}\cdot\frac{2}{5}\cdot\frac{2}{5} = \frac{2\cdot2\cdot2\cdot2}{5\cdot5\cdot5\cdot5} = \frac{2^4}{5^4} = \frac{16}{625}.$$

(b) If m is a positive integer, then $\left(\frac{b}{5}\right)^m$ is a product of m copies of $\frac{b}{5}$. Applying the rule for multiplying fractions tells us that such a product equals $\frac{b^m}{5^m}$:

$$\left(\frac{b}{5}\right)^m = \underbrace{\frac{b}{5}\times\frac{b}{5}\times\cdots\times\frac{b}{5}}_{m \text{ copies}} = \frac{\overbrace{b\times b\times\cdots\times b}^{m \text{ copies}}}{\underbrace{5\times5\times\cdots\times5}_{m \text{ copies}}} = \frac{b^m}{5^m}.$$

□

> **Important:** If m is a positive integer and b is not 0, then
>
> $$\left(\frac{b}{5}\right)^m = \frac{b^m}{5^m}.$$

This isn't something new. It's just the quotient of powers rule (i.e. $\frac{b^m}{5^m} = m^b \div 5^m$) written using fractions.

Problem 4.24

(a) Evaluate $\left(\frac{3}{5}\right)^{-2}$.

(b) Evaluate $\left(\frac{b}{5}\right)^{-n}$.

Solution for Problem 4.24:

(a) Back in Section 3.4, we defined a^{-n} for any nonzero a to be the reciprocal of a. Since the reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$, we have

$$\left(\frac{3}{5}\right)^{-2} = \frac{5}{3}.$$

Similarly, we have:

> **Important:** If a and b are nonzero, then
>
> $$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}.$$

This is just another way to write our earlier rule for finding the reciprocal of a fraction.

(b) In Section 3.4, we also defined a^{-n} for any nonzero a and any positive integer n as follows:

$$a^{-n} = \frac{1}{a^n}.$$

So, $\left(\frac{b}{5}\right)^{-n}$ equals the reciprocal of $\left(\frac{b}{5}\right)^n$. Applying the rule we discovered in this section here for raising a fraction to a positive integer power gives

$$\left(\frac{b}{5}\right)^{-n} = \frac{1}{(b/5)^n} = \frac{1}{b^n/5^n} = \frac{5^n}{b^n} = \frac{27}{64}.$$

Often, a more helpful way to think about negative powers of fractions is to apply the exponent rule

$$a^{-n} = (a^{-1})^n,$$

which we learned in Section 2.4 here. Applying this rule to $\left(\frac{b}{5}\right)^{-n}$, together with the rule for finding the reciprocal of a fraction, we have

$$\left(\frac{b}{5}\right)^{-n} = \left(\left(\frac{b}{5}\right)^{-1}\right)^n = \left(\frac{5}{b}\right)^n = \frac{5^n}{b^n} = \frac{27}{64}.$$

□

Our second solution to part (b) is the way we usually compute negative powers of fractions.

Problem 4.25

Suppose a and b are nonzero, and n is a positive integer. Explain why

$$\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}.$$

Solution for Problem 4.25: We have

$$\begin{aligned}\left(\frac{a}{b}\right)^{-n} &= \left(\left(\frac{a}{b}\right)^{-1}\right)^n && \text{power of a power}\\ &= \left(\frac{b}{a}\right)^n && \text{reciprocal of a fraction}\\ &= \frac{b^n}{a^n}, && \text{power of a fraction}\end{aligned}$$

□

> **Important:** If a and b are nonzero and n is a positive integer, then
>
> $$\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}.$$

Rather than memorizing this as a separate formula, we usually think of this with the intermediate step

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}.$$

As an Exercise, you'll use $\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$ to explain why we can apply the same rule to negative powers of fractions that we use for positive powers of fractions:

> **Important:** If a and b are nonzero and n is any integer, then
>
> $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Problem 4.26

Compute $\frac{(25/36)^3(25/21)^2}{(21/36)^{-2}}$.

Solution for Problem 4.26: We could compute this the long way, multiplying out each of the powers of fractions. But instead, we notice that $\frac{21}{36}$ appears in both the numerator and the denominator, so we can use an exponent law to simplify:

$$\begin{aligned}\frac{(25/36)^3(25/21)^2}{(21/36)^{-2}} &= \frac{(25/36)^3}{(21/36)^{-2}}\cdot(25/21)^2\\ &= (25/36)^3\cdot(21/36)^2\cdot(25/21)^2\\ &= (25/36)^3\cdot(25/21)^2.\end{aligned}$$

Here are two solutions from this point:

Solution 1: Use the rule for raising fractions to powers. We have

$$\left(\frac{25}{36}\right)^3\cdot\left(\frac{25}{21}\right)^2 = \frac{25^3}{36^3}\cdot\frac{25^2}{21^2} = \frac{25^5\cdot25^2}{36^3\cdot21^2}.$$

The numerator and denominator of this final fraction are equal, so the fraction equals 1.

Solution 2: Notice that the powers are the same. Since both fractions in $(25/36)^3\cdot(25/21)^2$ are raised to the same power, we apply an exponent law:

$$\left(\frac{25}{36}\right)^3\cdot\left(\frac{25}{21}\right)^2 = \left(\frac{25}{36}\cdot\frac{25}{21}\right)^3.$$

Since $\frac{21}{36}$ is the reciprocal of $\frac{36}{21}$, the product $\frac{25}{36}\cdot\frac{25}{21}$ equals 1. This means our expression equals 1^3, which is 1. □

Exercises

4.4.1:

Compute each of the following:

(a) $\left(\frac{2}{3}\right)^3$.

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Your Submission: Solution

Solution: $\left(\frac{2}{3}\right) = \frac{2^3}{3^3} = \boxed{\frac{8}{27}}$

(b) $\left(-\frac{3}{7}\right)^0$.

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Your Submission: Solution

Solution: Any number raised to the power 0 equals 1, so $\left(-\frac{3}{7}\right)^0 = \boxed{1}$

(c) $\left(\frac{5}{4}\right)^6$.

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Your Submission: Solution

Solution:

$$\left(\frac{5}{4}\right)^6 = \left(\left(\frac{5}{4}\right)^{-1}\right)^6 = \left(\frac{4}{5}\right)^6 = \frac{4^6}{5^6} = \boxed{\frac{4}{5}}$$

(d) $\left(-\frac{3}{2}\right)^5$.

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Your Submission: Solution

Solution:

$$\left(-\frac{3}{2}\right)^5 = \frac{(-3)^5}{2^5} = \frac{-243}{32} = \boxed{-\frac{243}{32}}$$

(e) $\left(\frac{1}{1.25}\right)^3$.

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Your Submission: Solution

Solution: Since $\left(\frac{1}{1}\right) = \frac{1^3}{1^3} = \frac{1^3}{125} = \frac{1}{125}$, we have $\frac{1}{1/125} = \frac{125}{1} = \boxed{125}$

(f) $\frac{(3/8)^2}{(5/3)^4}$.

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Your Submission: Solution

Solution:

$$\frac{(3/8)^2}{(5/3)^4} = \frac{3^2/8^2}{5^4/3^4} = \frac{9/64}{625/81} = \frac{9}{64}\cdot\frac{81}{625} = \frac{9}{1}\cdot\frac{81}{625} = \boxed{\frac{9}{625}}$$

4.4.2:

(a) $\frac{3}{2}$ raised to what power equals $\frac{27}{64}$?

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Your Submission: Solution

Solution: We might recognize 27 as 3^3 and 64 as 4^3, but if we don't, we can simply compute a few powers of $\frac{3}{2}$ and hope we get lucky. We have $\left(\frac{3}{2}\right)^1 = \frac{3^1}{2^1} = \frac{3}{2}$ and $\left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4}$, so $\left(\frac{3}{2}\right)^3 = \frac{27}{8}$. So, we raise $\frac{3}{2}$ to the power $\boxed{3}$ to get $\frac{27}{64}$.

(b) $\frac{3}{2}$ raised to what power equals $\frac{16}{9}$?

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Your Submission: Solution

Solution: We see that 9 is 3^2 and 16 is 4^2, so $\frac{16}{9} = \frac{4^2}{3^2} = \left(\frac{4}{3}\right)^2$. But we need $\frac{16}{9}$ as a power of $\frac{3}{2}$ not as a power of $\frac{4}{3}$. Since $\frac{4}{3}$ is the reciprocal of $\frac{3}{4}$, we can use negative exponents:

$$\left(\frac{4}{3}\right)^2 = \left(\left(\frac{3}{4}\right)^{-1}\right)^2 = \left(\frac{3}{4}\right)^{-2}.$$

Therefore, we raise $\frac{3}{2}$ to the power $\boxed{-2}$ to get $\frac{16}{9}$.

4.4.3:

Compute $\frac{(2/1641)^3}{(3/1641)^2}$.

Preview: Solution

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Your Submission: Solution

Solution 1: Raise both fractions to the 4th power. It's a little scary to raise 1641 to the 4th power, but maybe we'll get lucky and not have to. We have

$$\frac{(2/1641)^3}{(3/1641)^2} = \frac{2^3/1641^3}{3^2/1641^2} = \frac{2^3}{3^2}\cdot\frac{1641^2}{1641^3} = \frac{2^3}{3^2}\cdot\frac{1641^2}{1641^3}.$$

Phew! Since the numerator and denominator $\frac{1641^2}{1641^3}$ are the same, we have $\frac{1641^2}{1641^3} = 1$. So, we have $\frac{2^3}{3^2}\cdot\frac{1641^2}{1641^3} = \frac{2^3}{3^2}\cdot\frac{1}{1} = \boxed{\frac{8}{9}}$.

Solution 2: Use the fact that $\frac{2}{3} = \left(\frac{3}{2}\right)^{-1}$. The best here is that a and b themselves are fractions! We have

$$\frac{(2/1641)^3}{(3/1641)^2} = \left(\frac{3/1641}{2/1641}\right)^{-1} = \left(\frac{3}{2}\right)^{-1} = \boxed{\frac{2}{3}}.$$

Conveniently, once again the 1641's cancel out and we have $\left(\frac{3}{2}\right)^{-1} = \frac{2^1}{3^1} = \boxed{\frac{2}{3}}$.

4.4.4:

Compute $\frac{(5/8)^4(5/3)^7}{(5/3)^5}$.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: All three fractions are the same, so we can apply the exponent laws $a^b\cdot a^c = a^{b+c}$ and $a^b/a^c = a^{b-c}$:

$$\frac{(5/8)^4(5/3)^7}{(5/3)^5} = \frac{(5/8)^{4+7}}{(5/3)^5} = \frac{(5/8)^{11}}{(5/3)^5} = (5/8)^{11-5} = (5/8)^6 = \boxed{\left(\frac{5}{8}\right)^6}$$

4.4.5:

Compute $\left(\frac{2}{3}\right)^4\left(\frac{3}{4}\right)^3\left(\frac{4}{3}\right)^2$.

Preview: Solution

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Your Submission: Solution

Solution: One of the fractions is not the same as the other two, so it looks like we can't use the same strategy as the previous problem. However, $\frac{3}{4}$ is the reciprocal of $\frac{4}{3}$, so we can use negative exponents! Since $\frac{3}{4} = \left(\frac{4}{3}\right)^{-1}$, we have $\left(\frac{3}{4}\right)^3 = \left(\left(\frac{4}{3}\right)^{-1}\right)^3 = \left(\frac{4}{3}\right)^{-3}$. Then, we have

$$\begin{aligned}\left(\frac{2}{3}\right)^4\left(\frac{3}{4}\right)^3\left(\frac{4}{3}\right)^2 &= \left(\frac{2}{3}\right)^4\left(\frac{4}{3}\right)^{-3}\left(\frac{4}{3}\right)^2\\ &= \left(\frac{2}{3}\right)^4\left(\frac{4}{3}\right)^{-1}\\ &= \left(\frac{2}{3}\right)^4\left(\frac{3}{4}\right)\\ &= \boxed{\frac{3}{8}}\end{aligned}$$

□

4.4.6★:

Suppose a and b are nonzero, and n is a positive integer. In the text, we showed that $\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$. Explain why

$$\left(\frac{a}{b}\right)^n = \frac{a^{-n}}{b^{-n}}$$

by showing that $\frac{a^{-n}}{b^{-n}}$ also equals $\frac{b^n}{a^n}$.

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Your Submission: Solution

Solution: We have

$$\begin{aligned}\frac{a^{-n}}{b^{-n}} &= a^{-n}\div b^{-n} && \text{definition of fraction}\\ &= \frac{1}{a^n}\div\frac{1}{b^n} && \text{negative in exponent}\\ &= \frac{1}{a^n}\cdot\frac{b^n}{1} && \text{division by a fraction}\\ &= \frac{b^n}{a^n}. && \text{product of fractions}\end{aligned}$$

Since $\left(\frac{a}{b}\right)^{-n}$ and $\frac{a^{-n}}{b^{-n}}$ both equal $\frac{b^n}{a^n}$, we must have $\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}}$.

4.5 Simplest Form of a Fraction

A fraction is **simplest form** if its numerator and denominator have no positive common factors other besides 1. For example, the numerator and the denominator of $\frac{5}{8}$ have 2 as a common divisor, so $\frac{5}{8}$ is not in simplest form. The only positive common divisor of the numerator and denominator of $\frac{3}{7}$ is 1, so $\frac{3}{7}$ is in simplest form. We saw back in Problem 0.8 that the fractions $\frac{3}{7}$ and $\frac{5}{8}$ are equal. So, we can write $\frac{5}{8}$ in simplest form as $\frac{3}{7}$.

Problems

Problem 4.27 ↻ 7

Write $\frac{12}{17}$ in simplest form.

$\frac{12}{17}$ is simplest form.

Problem 4.28 ↻ 7

One of the properties of division that we learned in Section 1.7 was the "cancel common factor" property: if b and c are not zero, then

$$(ma) \div (mb) = a \div b$$

Use fraction multiplication to explain why the cancel common factor property works.

Problem 4.29 ↻ 7

Write $\frac{105}{210}$ in simplest form.

$\frac{105}{210}$ is simplest form.

Problem 4.30 ↻ 7

Compute each of the following in simplest form:

(a) $\frac{6}{9} \cdot \frac{12}{7}$

(b) $\left(\frac{3}{5}\right) \div \left(\frac{20}{9}\right)$

(c) $\left(\frac{2}{3}\right) \div \left(\frac{10}{9}\right)$

(d) $\frac{56}{15} \div \frac{14}{5}$

(e) $\frac{56}{15} \div \frac{14}{15}$

Problem 4.31 ↻ 7

What fraction of 96 is 64? Answer as a fraction in simplest form.

Problem 4.32 ↻ 7

(a) Simplify $\frac{2^3 \cdot 3^2}{2^2 \cdot 3^3}$.

(b) Simplify $\frac{2^3 \cdot 3^2}{2^2 \cdot 3^3 \cdot 5}$.

(c) Suppose x is nonzero. Simplify $\frac{2x^3}{3x^2}$.

Problem 4.33 ↻ 7

If x and d are not zero, then simplify $\frac{dx^2d^3}{d^2x}$.

Problem 4.27 ↻ 7

Write $\frac{12}{17}$ in simplest form.

Solution for Problem 4.27: Since 12 and 20 have 4 as a common divisor, we have

$$\frac{12}{20} = \frac{3 \cdot 4}{5 \cdot 4} = \frac{3}{5}.$$

2 and 5 have only 1 as a positive common divisor, so $\frac{3}{5}$ is simplest form. ✓

Fraction Simplification

Problem 4.28 ↻ 7

One of the properties of division that we learned in Section 1.7 was the "cancel common factor" property: if b and c are not zero, then

$$(ma) \div (mb) = a \div b$$

Use fraction multiplication to explain why the cancel common factor property works.

Solution for Problem 4.28: We start by writing $(ma) \div (bc)$ in the fraction $\frac{ma}{bc}$. Even our discussions of multiplying fractions, we know that $\frac{ma}{bc} = \frac{m}{b} \cdot \frac{a}{c}$. Since $\frac{m}{m} = 1$ we have $\frac{m}{b} = \frac{m}{b} \cdot \frac{m}{m} = \frac{m^2}{bm}$ and we have $(ma) \div (bc) = a \div b$. ✓

When the numerator and the denominator of a fraction have a positive common factor besides 1, we can use the fact that $\frac{mx}{mb} = \frac{x}{b}$ to simplify the fraction.

Problem 4.29 ↻ 7

Write $\frac{105}{210}$ in simplest form.

Solution for Problem 4.29: It is a common divisor of 210 and 140, and we have $\frac{105}{140} = \frac{105 \cdot 2}{140 \cdot 2} = \frac{21}{28}$. $\frac{21}{28}$ is not in simplest form because 49 and 108 are both divisible by 9:

$$\frac{21}{28} = \frac{3 \cdot 9}{4 \cdot 9} = \frac{3}{4}.$$

The only positive common divisor of 3 and 12 is 1, so $\frac{3}{4}$ is simplest form. ✓

We also could have used the prime factorizations of 105 and 140 to help us find common divisors. Since $105 = 3^1 \cdot 5^1$ and $140 = 2^2 \cdot 7^1$, it is clear that 210 and 140 have two factors of 2 and one factor of 5 in common, so

$$\frac{105}{210} = \frac{3^1 \cdot 5^1}{2^1 \cdot 2^1 \cdot 2^1 \cdot 5^1} = \frac{3^1 \cdot 5^1}{2^1 \cdot 2^1 \cdot 2^1 \cdot 5^1} = \frac{3}{4}.$$

✓

Concept: We can use the prime factorizations of the numerator and denominator of a fraction to simplify the fraction.

Fraction Calculations

Problem 4.30 ↻ 7

Compute each of the following in simplest form:

(a) $\frac{6}{9} \cdot \frac{12}{7}$

(b) $\left(\frac{3}{5}\right) \div \left(\frac{20}{9}\right)$

(c) $\left(\frac{2}{3}\right) \div \left(\frac{10}{9}\right)$

(d) $\frac{56}{15} \div \frac{14}{5}$

(e) $\frac{56}{15} \div \frac{14}{15}$

Solution for Problem 4.30:

(a) We also could have simplified $\frac{6}{9}$ and $\frac{12}{7}$ before multiplying. Since $\frac{6}{9} = \frac{2 \cdot 3}{3 \cdot 3} = \frac{2}{3}$ and $\frac{12}{7} = \frac{2 \cdot 6}{1 \cdot 7}$ we have

$$\frac{6}{9} \cdot \frac{12}{7} = \frac{2}{3} \cdot \frac{2 \cdot 6}{1 \cdot 7}$$

(b) The product of two negative numbers is positive, we have

$$\left(-\frac{3}{5}\right) \div \left(-\frac{20}{9}\right) = \frac{3}{5} \div \frac{20}{9}.$$

We still have to compute $\frac{3}{5} \div \frac{20}{9}$. We could multiply our $3 \div 20$ and $5 \div 9$, and we can then turn for common factors. However, we can save a lot of work by simplifying the two fractions before multiplying. We have $\frac{3}{5} = \frac{3 \cdot 3}{5 \cdot 3} = \frac{9}{15}$ and $\frac{20}{9} = \frac{20 \cdot 5}{9 \cdot 5} = \frac{100}{45}$.

We say that we **cancelled** a common divisor d in the final step above. We sometimes express this cancellation with a slash through the numbers being cancelled:

$$\frac{3}{5} \div \frac{20}{9} = \frac{\cancel{3}}{\cancel{5}} \div \frac{\cancel{20}}{\cancel{9}}$$

(c) The product of a positive number and a negative number is negative, we have

$$\left(\frac{2}{3}\right) \div \left(-\frac{10}{9}\right) = -\left(\frac{2}{3} \div \frac{10}{9}\right).$$

We still must compute $\frac{2}{3} \div \frac{10}{9}$. We can't simplify either fraction. However, the numerator of each fraction has divisors in common with the denominator of the other fraction. We can perform a clever manipulation to allow us to take advantage of these common factors:

$$-\left(\frac{2}{3} \div \frac{10}{9}\right) = -\left(\frac{2 \cdot 15}{3 \cdot 15} \div \frac{10 \cdot 3}{9 \cdot 3}\right) = -\left(\frac{2 \cdot 15}{15 \cdot 3} \div \frac{10 \cdot 3}{9 \cdot 3}\right)$$

Since $\frac{10}{9} = \frac{10 \cdot 3}{9 \cdot 3} = \frac{2 \cdot 5 \cdot 3}{3 \cdot 3 \cdot 3}$ we have

$$-\left(\frac{2 \cdot 15}{15 \cdot 3} \div \frac{10 \cdot 3}{9 \cdot 3}\right) = -\left(\frac{2 \cdot 15}{15 \cdot 3} \div \frac{2 \cdot 5 \cdot 3}{3 \cdot 3 \cdot 3}\right) = -\left(\frac{2}{3} \div \frac{10}{9}\right) = -\frac{2}{3} \div \frac{10}{9}$$

(d) First, we write the division as a multiplication: $\frac{56}{15} \div \frac{14}{5} = \frac{56}{15} \cdot \frac{5}{14}$. Now, we notice that the denominator of $\frac{56}{15}$ and the numerator of $\frac{5}{14}$ have 5 as a common divisor. Rearranging the product will allow us to cancel this common divisor:

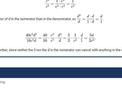
$$\frac{56}{15} \cdot \frac{5}{14} = \frac{56}{15} \cdot \frac{5}{14} = \frac{56}{3 \cdot 5} \cdot \frac{5}{14} = \frac{56}{3} \cdot \frac{5}{14}$$

We have $\frac{56}{3} \cdot \frac{5}{14} = \frac{8 \cdot 7}{3} \cdot \frac{5}{2 \cdot 7} = \frac{8}{3} \cdot \frac{5}{2} = \frac{8}{3} \cdot \frac{5}{2} = \frac{4}{3} \cdot \frac{5}{1} = \frac{20}{3}$.

As you get more comfortable with fraction multiplication, you need have to rearrange products in the numerator and denominator in order to cancel common factors. For example, in the product $\frac{56}{15} \cdot \frac{5}{14}$ above, you might cancel out the common factor of 7 in 56 and 14, and cancel out the common factor of 5 in 15 and 5:

$$\frac{56}{15} \cdot \frac{5}{14} = \frac{8 \cdot \cancel{7}}{3 \cdot \cancel{5}} \cdot \frac{\cancel{5}}{2 \cdot \cancel{7}}$$

Concept: When finding the product of fractions in simplest form, we usually cancel common divisors as much as possible before computing our product. Moreover, we can cancel a common divisor from the numerator of one fraction and the denominator of another.



Multiplying Fractions the Easy Way

Problem 4.31 ↻ 7

What fraction of 96 is 64? Answer as a fraction in simplest form.

Solution for Problem 4.31: Because "of" means "multiply," we seek the fraction that goes in the blank in the equation

$$96 \cdot ___ = 64.$$

We know how to handle equations like this. Thinking of division as the reverse of multiplication, the number that goes in the blank is $64 \div 96$. Expressing this as a fraction, we have

$$64 \div 96 = \frac{64}{96} = \frac{2 \cdot 32}{3 \cdot 32}.$$

To check our answer, we compute that $\frac{2}{3}$ of 96 is

$$\frac{2}{3} \cdot 96 = \frac{2 \cdot 96}{3} = \frac{2 \cdot 32 \cdot 3}{3} = 64.$$

✓

Problem 4.32 ↻ 7

(a) Simplify $\frac{2^3 \cdot 3^2}{2^2 \cdot 3^3}$.

(b) Simplify $\frac{2^3 \cdot 3^2}{2^2 \cdot 3^3 \cdot 5}$.

(c) Suppose x is nonzero. Simplify $\frac{2x^3}{3x^2}$.

Solution for Problem 4.32:

(a) Since 3^2 is a common divisor of the numerator and denominator, we have

$$\frac{2^3 \cdot 3^2}{2^2 \cdot 3^3} = \frac{2^3 \cdot 3^2}{2^2 \cdot 3^2 \cdot 3} = \frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 3} = \frac{2 \cdot 2}{3}.$$

(b) Since 2^2 is a common divisor of the numerator and denominator, we have

$$\frac{2^3 \cdot 3^2}{2^2 \cdot 3^3 \cdot 5} = \frac{2^3 \cdot 3^2}{2^2 \cdot 3^2 \cdot 3 \cdot 5} = \frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 3 \cdot 5} = \frac{2}{3 \cdot 5}.$$

Of course, if x were even, we could simplify $\frac{2x^3}{3x^2}$ further. But we won't even a value of x, so we can't cancel it with anything number. We can only cancel a value of x. So, we cannot simplify any further.

Problem 4.33 ↻ 7

If x and d are not zero, then write $\frac{dx^2d^3}{d^2x}$ in simplest form.

Solution for Problem 4.33: Writing $\frac{dx^2d^3}{d^2x}$ as $\frac{d^1 \cdot x^2 \cdot d^3}{d^2 \cdot x^1}$ allows us to simplify piece by piece.

We have $\frac{d^1}{d^2} = \frac{1}{d^1}$ and $\frac{x^2}{x^1} = x^1$.

To simplify $\frac{d^3}{d^2}$, we note that there are two more factors of d in the denominator than in the numerator, so

$$\frac{d^3}{d^2} = \frac{d \cdot d \cdot d}{d \cdot d} = \frac{d}{1} = d.$$

$\frac{d^3}{d^2}$ there is one more factor of d in the numerator than in the denominator, so $\frac{d^3}{d^2} = \frac{d}{1} = d$.

Combining these, we have

$$\frac{dx^2d^3}{d^2x} = \frac{d^1 \cdot x^2 \cdot d^3}{d^2 \cdot x^1} = \frac{1}{d^1} \cdot x^1 \cdot d^1 = \frac{x \cdot d}{d}.$$

We can't simplify $\frac{x \cdot d}{d}$ any further, since neither the x nor the d in the numerator can cancel with anything in the denominator. ✓

Exercises

4.5.1. ↻ 7

Simplify each of the following:

(a) $\frac{27}{27}$

Hide Solution

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Your Submission: Solution

Solution: $\frac{27}{27} = \frac{3 \cdot 9}{3 \cdot 9} = \frac{1}{1}$

(b) $\frac{250}{300}$

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Good

You may type any additional notes you have here.

Your Submission: Solution

Solution: We simplify this fraction in steps:

$$\frac{250}{300} = \frac{5 \cdot 50}{6 \cdot 50} = \frac{5 \cdot 50}{6 \cdot 50} = \frac{5 \cdot 25 \cdot 2}{6 \cdot 50} = \frac{5 \cdot 25 \cdot 2}{6 \cdot 25 \cdot 2} = \frac{5 \cdot 25}{6 \cdot 25} = \frac{5 \cdot 5 \cdot 5}{6 \cdot 5 \cdot 5} = \frac{5}{6}$$

(c) $\frac{1600}{1200}$

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Your Submission: Solution

Solution:

$$\frac{1600}{1200} = \frac{16 \cdot 100}{12 \cdot 100} = \frac{16}{12} = \frac{4 \cdot 4}{3 \cdot 4} = \frac{4}{3}$$

(d) $\frac{1200}{1600}$

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Your Submission: Solution

Solution:

$$\frac{1200}{1600} = \frac{12 \cdot 100}{16 \cdot 100} = \frac{12}{16} = \frac{3 \cdot 4}{4 \cdot 4} = \frac{3}{4}$$

Using our divisibility rule for 3, we see that the sum of the digits of 270 is 9, so is the sum of the digits of 252. So, 270 and 252 are both divisible by 9, and we have

$$\frac{252}{270} = \frac{9 \cdot 28}{9 \cdot 30} = \frac{9}{9} \cdot \frac{28}{30} = \frac{28}{30}.$$

4.5.2. ↻ 7

Compute each of the following. Express your answer in simplest form.

(a) $\frac{25}{36} \cdot \frac{28}{35}$

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Your Submission: Solution

Solution: We simplify before multiplying. We have $\frac{25}{36} \cdot \frac{28}{35} = \frac{5 \cdot 5}{4 \cdot 9} \cdot \frac{4 \cdot 7}{5 \cdot 7} = \frac{5}{9} \cdot \frac{7}{7} = \frac{5}{9}$.

$$\frac{25}{36} \cdot \frac{28}{35} = \frac{5 \cdot 5}{4 \cdot 9} \cdot \frac{4 \cdot 7}{5 \cdot 7} = \frac{5}{9} \cdot \frac{7}{7} = \frac{5}{9}$$

(b) $\frac{35}{36} \div \frac{42}{54}$

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Your Submission: Solution

Solution: We have $\frac{35}{36} \div \frac{42}{54} = \frac{35}{36} \cdot \frac{54}{42}$.

$$\frac{35}{36} \cdot \frac{54}{42} = \frac{5 \cdot 7}{4 \cdot 9} \cdot \frac{9 \cdot 6}{6 \cdot 7} = \frac{5}{4} \cdot \frac{7}{7} \cdot \frac{9}{9} \cdot \frac{6}{6} = \frac{5}{4} \cdot 1 \cdot 1 \cdot 1 = \frac{5}{4}$$

(c) $\left(-\frac{5}{6}\right) \div \frac{35}{42}$

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You may type any additional notes you have here.

Your Submission: Solution

Solution: The product of a negative number and a positive number is negative, so we have

$$\left(-\frac{5}{6}\right) \div \frac{35}{42} = -\left(\frac{5}{6} \div \frac{35}{42}\right).$$

We can't simplify either of these fractions, but each numerator has divisors besides 1 in common with the other fraction's denominator. So, we write

$$-\left(\frac{5}{6} \div \frac{35}{42}\right) = -\left(\frac{5 \cdot 7}{6 \cdot 7} \div \frac{35 \cdot 6}{42 \cdot 6}\right) = -\left(\frac{5 \cdot 7}{7 \cdot 6} \div \frac{35 \cdot 6}{42 \cdot 6}\right)$$

We have $\frac{35}{42} = \frac{5 \cdot 7}{6 \cdot 7} = \frac{5 \cdot 7}{2 \cdot 3 \cdot 7} = \frac{5 \cdot 7}{2 \cdot 3 \cdot 7} = \frac{5}{2 \cdot 3}$. So, we have

$$-\left(\frac{5 \cdot 7}{7 \cdot 6} \div \frac{35 \cdot 6}{42 \cdot 6}\right) = -\left(\frac{5 \cdot 7}{7 \cdot 6} \div \frac{5 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 7 \cdot 6}\right) = -\left(\frac{5}{6} \div \frac{5}{2 \cdot 3}\right) = -\frac{5}{6} \div \frac{5}{2 \cdot 3}$$

(d) $\frac{400}{1600} \div \frac{1600}{400}$

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You may type any additional notes you have here.

Your Submission: Solution

Solution: We simplify this fraction in steps:

$$\frac{400}{1600} \div \frac{1600}{400} = \frac{400}{1600} \cdot \frac{400}{1600} = \frac{400}{400} \cdot \frac{400}{1600} = 1 \cdot \frac{400}{1600} = \frac{400}{1600} = \frac{4 \cdot 100}{16 \cdot 100} = \frac{4}{16} = \frac{1 \cdot 4}{4 \cdot 4} = \frac{1}{4}$$

4.5.3. ↻ 7

Write in simplest form.

(a) $\frac{6x^3y}{12xy^2}$

Hide Solution

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You may type any additional notes you have here.

Your Submission: Solution

Solution:

$$\frac{6x^3y}{12xy^2} = \frac{6 \cdot x^3 \cdot y^1}{12 \cdot x^1 \cdot y^2} = \frac{1}{2} \cdot x^{3-1} \cdot y^{1-2}$$

(b) $\frac{8xy^2z^3}{12xz^2y^3}$

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Your Submission: Solution

Solution:

$$\frac{8xy^2z^3}{12xz^2y^3} = \frac{2 \cdot 4 \cdot x^1 \cdot y^2 \cdot z^3}{3 \cdot 4 \cdot x^1 \cdot z^2 \cdot y^3} = \frac{2}{3} \cdot \frac{4}{4} \cdot \frac{x^1}{x^1} \cdot \frac{y^2}{y^3} \cdot \frac{z^3}{z^2}$$

4.5.4. ↻ 7

Express the product $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{7}{8} \cdot \frac{8}{9} \cdot \frac{9}{10}$ in simplest form.

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Your Submission: Solution

Solution: The sequence 4 through 7 appears as factors in both the numerator and denominator, so they cancel out conveniently:

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{7}{8} \cdot \frac{8}{9} \cdot \frac{9}{10}$$

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{7}{8} \cdot \frac{8}{9} \cdot \frac{9}{10}$$

Cancelling the common factor leaves $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{8} \cdot \frac{1}{9} \cdot \frac{1}{10}$.

4.5.5. ↻ 7

Evaluate $\frac{x+y}{x^2+y^2}$ where $x = \frac{1}{2}$ and $y = \frac{2012}{2013}$.

Hint: It's easier to substitute into simple expressions than to substitute into complicated expressions.

Preview Solution

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Your Submission: Solution

Solution: We simplify the fraction we wish to evaluate before we substitute:

$$\frac{x+y}{x^2+y^2} = \frac{x+y}{(x+y)^2 - 2xy} = \frac{x+y}{(x+y)(x+y) - 2xy} = \frac{x+y}{(x+y)(x+y) - 2xy} = \frac{1}{(x+y) - \frac{2xy}{x+y}}$$

When $x = \frac{1}{2}$ and $y = \frac{2012}{2013}$, we have $\frac{1}{(x+y) - \frac{2xy}{x+y}} = \frac{1}{\frac{1}{2} + \frac{2012}{2013} - \frac{2 \cdot \frac{1}{2} \cdot \frac{2012}{2013}}{\frac{1}{2} + \frac{2012}{2013}}}$

4.6 Comparing Fractions

In this section, we reverse our fraction simplification strategies from the previous section to compare values of fractions.

Problems

Problem 4.34 Jump to Solution

Which is greater, $\frac{3}{5}$ or $\frac{4}{7}$?

Problem 4.35 Jump to Solution

In this problem, we determine which is greater, $\frac{3}{5}$ or $\frac{7}{10}$.

(a) What fraction with denominator 10 is equal to $\frac{3}{5}$?

(b) Which is greater, $\frac{3}{5}$ or $\frac{7}{10}$?

Problem 4.36 Jump to Solution

(a) Which is greater, $\frac{5}{11}$ or $\frac{8}{11}$?

(b) Which is greater, $\frac{5}{6}$ or $\frac{7}{9}$?

Problem 4.37 Jump to Solution

Which is greater, $\frac{2}{5}$ or $\frac{3}{7}$?

Problem 4.38 Jump to Solution

Which is greater, $\frac{-17}{14}$ or $\frac{-41}{35}$?

Problem 4.39 Jump to Solution

Which is greater, $\frac{541}{539}$ or $\frac{399}{401}$?

Problem 4.34

Which is greater, $\frac{4}{7}$ or $\frac{3}{5}$?

Solution for Problem 4.34: We can see that $\frac{4}{7}$ is greater than $\frac{3}{5}$ by considering the number line. To locate both fractions on the number line, we divide the number line between 0 and 1 into 5 equal pieces. Since $\frac{3}{5}$ is at the right end of the 3rd piece and $\frac{4}{7}$ is at the right end of the 4th piece, we conclude that $\frac{4}{7}$ is to the right of $\frac{3}{5}$.

Therefore, $\frac{4}{7}$ is greater than $\frac{3}{5}$.

In general, if two fractions have the same positive denominator, then the fraction with the greater numerator is the greater of the two fractions.

Problem 4.35

Which is greater, $\frac{3}{5}$ or $\frac{7}{10}$?

Solution for Problem 4.35: We know how to compare two fractions that have the same denominator, so it would be convenient if we could express the two fractions with the same denominator. To do so, we use our fraction simplification strategy from Section 4.3 in reverse. Multiplying $\frac{3}{5}$ by 1 doesn't change the value of the fraction, so we multiply $\frac{3}{5}$ by $\frac{2}{2}$ which equals 1, to express $\frac{3}{5}$ with 10 as the denominator:

$$\frac{3}{5} = \frac{3}{5} \cdot \frac{2}{2} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10}.$$

Since 7 is greater than 6, we know that $\frac{7}{10}$ is greater than $\frac{6}{10}$, so $\frac{7}{10}$ is greater than $\frac{3}{5}$.

We often leave out the "multiplying by 1" step when writing a fraction with a new denominator. Instead, we go straight to multiplying the numerator and denominator by the same factor:

$$\frac{3}{5} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10}.$$

When we write multiple fractions with the same denominator, we say that we write the fractions with a **common denominator**.

Problem 4.36

(a) Which is greater, $\frac{5}{11}$ or $\frac{8}{11}$?

(b) Which is greater, $\frac{5}{6}$ or $\frac{7}{9}$?

Solution for Problem 4.36:

(a) We know how to compare fractions that have the same denominator. Unfortunately, neither of our denominators in this part is a multiple of the other. So, we change both denominators to make them the same. When writing two fractions with a common denominator, we can always use the product of the fractions' denominators as the common denominator. Writing each fraction with $7 \cdot 11$ as the denominator gives

$$\frac{5}{7} = \frac{5 \cdot 11}{7 \cdot 11} = \frac{55}{77}, \qquad \frac{8}{11} = \frac{8 \cdot 7}{11 \cdot 7} = \frac{56}{77}.$$

Since $\frac{56}{77}$ is greater than $\frac{55}{77}$, we know that $\frac{8}{11}$ is greater than $\frac{5}{7}$.

(b) Once again, we can use the product of the denominators as our common denominator. Writing each fraction with $6 \cdot 9$ as the denominator gives

$$\frac{5}{6} = \frac{5 \cdot 9}{6 \cdot 9} = \frac{45}{54}, \qquad \frac{7}{9} = \frac{7 \cdot 6}{9 \cdot 6} = \frac{42}{54}.$$

Since $\frac{45}{54}$ is greater than $\frac{42}{54}$, we know that $\frac{5}{6}$ is greater than $\frac{7}{9}$.

Thinking back to Section 3.5, we realize that we could have used a smaller common denominator. The common denominator must be a multiple of both 6 and 9. The least common multiple of 6 and 9 is 18, so we can write both $\frac{5}{6}$ and $\frac{7}{9}$ with a denominator of 18:

$$\frac{5}{6} = \frac{5 \cdot 3}{6 \cdot 3} = \frac{15}{18}, \qquad \frac{7}{9} = \frac{7 \cdot 2}{9 \cdot 2} = \frac{14}{18}.$$

Once again, we see that $\frac{5}{6}$ is greater than $\frac{7}{9}$.

+

The **least common denominator** of two or more fractions is the least common multiple of their denominators. In part (a) of Problem 4.36, the least common denominator is 77, while in part (b), the least common denominator is 18.

Problem 4.37

Which is greater, $\frac{2}{5}$ or $\frac{3}{7}$?

Solution for Problem 4.37: The least common denominator of $\frac{2}{5}$ and $\frac{3}{7}$ is 35. Writing both fractions with this denominator gives

$$\frac{2}{7} = \frac{2 \cdot 5}{7 \cdot 5} = \frac{10}{35}, \qquad \frac{3}{5} = \frac{3 \cdot 7}{5 \cdot 7} = \frac{14}{35},$$

so we see that $\frac{3}{5}$ is greater than $\frac{2}{7}$.

We also could have considered the number line. To locate $\frac{2}{7}$ on the number line, we split the number line between 0 and 1 into 7 equal pieces, and $\frac{2}{7}$ is at the right end of the 2nd piece. Similarly, to locate $\frac{3}{5}$ on the number line, we split the number line between 0 and 1 into 5 equal pieces, and $\frac{3}{5}$ is at the right end of the 3rd piece. We make larger pieces when we make 5 pieces than we do when we make 7 pieces, so $\frac{3}{5}$ is farther from 0 than $\frac{2}{7}$ is from 0. Therefore, $\frac{3}{5}$ is greater than $\frac{2}{7}$.

In general, if two positive fractions have the same numerator, then the fraction with the smaller denominator is the greater fraction.

Comparing Fractions Part 1

Problem 4.38

Which is greater, $\frac{-17}{14}$ or $\frac{-41}{35}$?

Solution for Problem 4.38: Before we even start thinking about common denominators, we note that the numbers in this problem are negative. So, we first have to think about how comparing negative numbers is different from comparing positive numbers. When comparing two positive numbers, the number that is farther from 0 is the greater number. When comparing two negative numbers, this is reversed! The number closest to 0 is the greater number. For example, -2 is greater than -10.

Now, we're ready to compare the fractions by finding a common denominator. The least common multiple of 14 and 35 is 70, so we write each fraction with 70 as the denominator:

$$\frac{-17}{14} = \frac{-17 \cdot 5}{14 \cdot 5} = \frac{-85}{70}, \qquad \frac{-41}{35} = \frac{-41 \cdot 2}{35 \cdot 2} = \frac{-82}{70}.$$

Since $-\frac{82}{70}$ is closer to 0, it is the greater of these two negative numbers. Therefore, $\frac{-41}{35}$ is greater than $-\frac{17}{14}$.

Problem 4.39

Which is greater, $\frac{541}{539}$ or $\frac{399}{401}$?

Solution for Problem 4.39: Writing these fractions with a common denominator looks like a pain. Fortunately, we don't have to! Since the numerator of $\frac{541}{539}$ is greater than its denominator, the fraction $\frac{541}{539}$ is greater than 1. Meanwhile, the numerator of $\frac{399}{401}$ is less than its denominator, so $\frac{399}{401}$ is less than 1. Putting these together, $\frac{541}{539}$ must be greater than $\frac{399}{401}$.

Comparing Fractions Part 2

Exercises

4.6.1:

In each of the following pairs of numbers, which number is smaller?

(a) $\frac{3}{4}, \frac{7}{5}$

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Solution: Writing the two fractions with a common denominator gives us $\frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{15}{20}$ and $\frac{7}{5} = \frac{7 \cdot 4}{5 \cdot 4} = \frac{14}{20}$. Since 14 is smaller than 15, we know that $\frac{14}{20}$ is smaller than $\frac{15}{20}$. This means $\boxed{\frac{7}{5}}$ is the smaller fraction.

(b) $\frac{2}{3}, \frac{3}{4}$

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Solution: Every negative number is less than 0, and 0 is less than every positive number. So, every negative number is less than every positive number, which means that $\boxed{-\frac{2}{3}}$ is the smaller number.

(c) $-\frac{2}{3}, -\frac{3}{5}$

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Solution: Since the numbers are negative, the number that is farther from 0 on the number line is the lesser number. To locate $-\frac{2}{3}$ on the number line, we go two steps of length $\frac{1}{3}$ to the left of 0. To get to $-\frac{3}{5}$, we must take one more step leftward of length $\frac{1}{5}$. So, $-\frac{2}{3}$ is to the left of $-\frac{3}{5}$ on the number line, which means that $\boxed{-\frac{3}{5}}$ is the smaller number.

4.6.2:

For each of the following lists of numbers, arrange the numbers in decreasing order (from largest to smallest).

(a) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{12}$

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Solution: We write all three fractions with a common denominator. The least common multiple of 2, 4, and 12 is 12. We have $\frac{1}{2} = \frac{1 \cdot 6}{2 \cdot 6} = \frac{6}{12}$, $\frac{3}{4} = \frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12}$, $\frac{5}{6} = \frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12}$. So the numbers from largest to smallest are $\boxed{\frac{5}{6}, \frac{3}{4}, \frac{1}{2}, \frac{7}{12}}$.

(b) $\frac{3}{2}, \frac{5}{3}, \frac{9}{5}, \frac{8}{3}$

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Your Submission: Solution

Solution: The least common denominator of these three fractions is 24. We have $\frac{3}{4} = \frac{6}{8}$, $\frac{18}{24} = \frac{3}{8}$, $\frac{16}{24} = \frac{2}{3}$, $\frac{5}{3} = \frac{15}{24}$, so the numbers from largest to smallest are $\boxed{\frac{3}{2}, \frac{5}{3}, \frac{9}{5}}$.

(c) $-\frac{5}{4}, -\frac{7}{2}, -\frac{13}{3}$

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Since $\frac{3}{2}$ is the only positive number in the list, it is the largest number. To compare $-\frac{5}{4}$ and $-\frac{13}{3}$, we can think about what pair of consecutive integers each is between. Since 5 is between 4 and 8, we know that $-\frac{5}{4}$ is between $\frac{4}{4} = -1$ and $\frac{8}{4} = -2$. So, $-\frac{5}{4}$ is between -2 and -1, which means $-\frac{5}{4}$ is larger than -3. Similarly, since 13 is between 12 and 15, we know that $\frac{13}{3}$ is between $\frac{12}{3} = 4$ and $\frac{15}{3} = 5$. This means that $-\frac{13}{3}$ is between -5 and -4, which means $-\frac{13}{3}$ is less than -3. Therefore, from largest to smallest, the numbers are $\boxed{-\frac{5}{4}, -3, -\frac{13}{3}}$.

4.6.3: Source: AMC 8

Which one of these numbers is less than its reciprocal?

$$-2, \quad -1, \quad 0, \quad 1, \quad 2$$

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The reciprocal of -2 is $-\frac{1}{2}$, which equals $-\frac{1}{2}$. Since -2 is less than -1 and $-\frac{1}{2}$ is greater than -1 (because $-\frac{1}{2}$ is between -1 and 0), we know that $\boxed{-2}$ is less than $-\frac{1}{2}$.

The reciprocal of -1 is $\frac{1}{-1} = -1$, which is not greater than -1. The number 0 does not have a reciprocal. The reciprocal of 1 is 1, which is not greater than 1. The reciprocal of 2 is $\frac{1}{2}$, which is less than 2. So, none of the other numbers in the list is less than its reciprocal.

4.6.4:

Which number is greater, $\frac{3}{2011}$ or $\frac{3}{2012}$?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Writing the two fractions with a common denominator gives $\frac{3}{2011} = \frac{3 \cdot 2012}{2011 \cdot 2012}$ and $\frac{3}{2012} = \frac{3 \cdot 2011}{2011 \cdot 2012}$. Since $3 \cdot 2012$ is greater than $3 \cdot 2011$, we see that $\boxed{\frac{3}{2011}}$ is greater than $\frac{3}{2012}$.

We might also have thought about the number line. To locate $\frac{3}{2011}$ on the number line, we split the number line between 0 and 1 into 2011 equal pieces, and $\frac{3}{2011}$ is at the right end of the 3rd piece. Similarly, to locate $\frac{3}{2012}$, we split the number line between 0 and 1 into 2012 equal pieces, and $\frac{3}{2012}$ is at the right end of the 3rd piece. We make smaller pieces when we make 2012 pieces than we do when we make 2011 pieces. Therefore, $\frac{3}{2012}$ is not as far from 0 as $\frac{3}{2011}$, which means $\boxed{\frac{3}{2011}}$ is greater than $\frac{3}{2012}$.

4.6.5:

Which number is greater, $\frac{19}{30}$ or $\frac{23}{35}$?

Hide Solution | Reset

Your Submission: Solution

Solution: Since the least common multiple of 30 and 35 is 210, the least common denominator of $\frac{19}{30}$ and $\frac{23}{35}$ is 210. We have $\frac{19}{30} = \frac{19 \cdot 7}{30 \cdot 7} = \frac{133}{210}$ and $\frac{23}{35} = \frac{23 \cdot 6}{35 \cdot 6} = \frac{132}{210}$. So $\boxed{\frac{19}{30}}$ is the greater fraction.

4.6.6:

Which number is greater, $\frac{506}{101}$ or $\frac{509}{102}$?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We see that the numerators of the fractions are close to 5 times the denominators. So, we compare each fraction to 5. Since $\frac{505}{101} = 5$, we see that $\frac{506}{101}$ is greater than 5. Since $\frac{510}{102} = 5$, we see that $\frac{509}{102}$ is less than 5. Combining these, we know that $\boxed{\frac{506}{101}}$ is greater than $\frac{509}{102}$.

4.7 Adding and Subtracting Fractions

Adding and subtracting fractions is easy if the denominator is the same in all cases. Not surprisingly, it is also easy if one denominator is a multiple of the other. For example, 1/2 + 1/8 = 5/8 since 2 × 4 = 8. In this case, the 2 in the denominator is a factor of 8. In other words, we can rewrite the first fraction as 4/8 and then add, as in the case above.

Writing these relationships in terms of factors, we get the following:

$$\frac{1}{2} + \frac{1}{8} = \frac{1 \times 4}{2 \times 4} + \frac{1}{8} = \frac{4}{8} + \frac{1}{8} = \frac{5}{8}$$

Multiplying the numerator and denominator by 4 to get an equivalent fraction

Adding the numerators

Reducing to a common denominator (we used only one, 8, in this case, there could be more)

Half could be a crowding

WARNING! The multiplier fractions are multiplied the numerators and multiplied the denominators. We do not multiply the numerator by the denominator or the denominator by the numerator!

Put in language you can use to describe what we do when we add fractions:

$$\frac{\text{Numerator}}{\text{Denominator}} + \frac{\text{Numerator}}{\text{Denominator}} = \frac{\text{Numerator} \times \text{Multiplier}}{\text{Denominator} \times \text{Multiplier}} + \frac{\text{Numerator}}{\text{Denominator}} = \frac{\text{Numerator}}{\text{Denominator}} + \frac{\text{Numerator}}{\text{Denominator}} = \frac{\text{Numerator}}{\text{Denominator}}$$

If we add the numerators, we get the common denominator. Now we can add the fractions.

We did not need to do anything to the denominator this time. We could, if we had to, but we don't have to. We can leave it as it is and we get the right answer.

Put in language you can use to describe what we do when we add fractions:

$$\frac{\text{Numerator}}{\text{Denominator}} + \frac{\text{Numerator}}{\text{Denominator}} = \frac{\text{Numerator}}{\text{Denominator}} + \frac{\text{Numerator}}{\text{Denominator}} = \frac{\text{Numerator}}{\text{Denominator}}$$

Problems

Problem 4.7.1 Solve by Solution

Compute and reduce the following in simplest form:

$$\frac{1}{3} + \frac{1}{6} =$$

Problem 4.7.2 Solve by Solution

Compute and reduce the following in simplest form:

$$\frac{1}{4} + \frac{1}{8} =$$

Problem 4.7.3 Solve by Solution

Compute and reduce the following in simplest form:

$$\frac{1}{5} + \frac{1}{10} =$$

Problem 4.7.4 Solve by Solution

Compute and reduce the following in simplest form:

$$\frac{1}{6} + \frac{1}{12} =$$

Problem 4.7.5 Solve by Solution

Compute and reduce the following in simplest form:

$$\frac{1}{7} + \frac{1}{14} =$$

Problems 4.7.1 and 4.7.2 suggest an easy trick. When we're multiplying, the new fraction is the product of the old fractions. When we're adding, the new fraction is the sum of the old fractions. Problems 4.7.3 and 4.7.4 suggest that this trick doesn't work in general.

Problem 4.7.6 Solve by Solution

Which problems, if any, do not work because of a reduction, and why? Is the trick true, in a contradiction. When these two problems, the two 5/10 fractions of the two problems, the two 5/10 fractions of the two problems, the two 5/10 fractions of the

4.6 Mixed Numbers

A proper fraction, such as $\frac{3}{4}$, is a fraction in which the numerator is less than the denominator. The improper fraction, written as $\frac{5}{4}$, or $\frac{7}{3}$, or $\frac{11}{2}$, is a fraction in which the numerator is greater than or equal to the denominator. The improper fraction $\frac{5}{4}$ can be written as $1\frac{1}{4}$, which is read as "one and one-fourth." The improper fraction $\frac{7}{3}$ can be written as $2\frac{1}{3}$, which is read as "two and one-third." The improper fraction $\frac{11}{2}$ can be written as $5\frac{1}{2}$, which is read as "five and one-half." The number $1\frac{1}{4}$ is called a **mixed number**, and it consists of a whole number and a proper fraction. The whole number is the **integer** part, and the proper fraction is the **fractional** part. In the mixed number $2\frac{1}{3}$, the integer part is 2, and the fractional part is $\frac{1}{3}$. The fractional part of a mixed number is always a proper fraction.

Do the mixed numbers "three and a quarter" and "three and a dozen" have the same meaning?

Correct	No, "three and a dozen" is a mixed number.

Which number is written in the form of mixed numbers, not usually second or a mixed number rather than a fraction.

Problem 4.6.1 Answer by Solution

Convert $\frac{5}{2}$ to a mixed number.

Correct	$2\frac{1}{2}$

Problem 4.6.10 Answer by Solution

1st	Correct
2nd	Incorrect: $\frac{5}{2}$ is a mixed number.

Problem 4.6.40 Answer by Solution

Explain why and the following are a mixed number or not the negative of a mixed number.

(a) $\frac{1}{2}$

1st	Correct
2nd	Correct

(b) $\frac{3}{4}$

1st	Correct
2nd	Incorrect: $\frac{3}{4}$ is a mixed number.

(c) $\frac{5}{2}$

1st	Correct
2nd	Incorrect: $\frac{5}{2}$ is a mixed number.

(d) $\frac{7}{3}$

1st	Correct
2nd	Incorrect: $\frac{7}{3}$ is a mixed number.

(e) $\frac{11}{2}$

1st	Correct
2nd	Incorrect: $\frac{11}{2}$ is a mixed number.

Problem 4.6.41 Answer by Solution

Explain why and the following are a mixed number or not the negative of a mixed number.

(a) $1\frac{1}{2}$

1st	Correct
2nd	Correct

(b) $1\frac{3}{4}$

1st	Correct
2nd	Correct

(c) $2\frac{5}{2}$

1st	Correct
2nd	Incorrect: $2\frac{5}{2}$ is a mixed number.

(d) $2\frac{7}{3}$

1st	Correct
2nd	Incorrect: $2\frac{7}{3}$ is a mixed number.

(e) $5\frac{11}{2}$

1st	Correct
2nd	Incorrect: $5\frac{11}{2}$ is a mixed number.

Problem 4.6.42 Answer by Solution

Because what two consecutive integers $\frac{19}{2}$ lies?

Correct	$9 < \frac{19}{2} < 10$

Problem 4.6.43 Answer by Solution

Because what two consecutive integers $\frac{19}{3}$ lies?

Correct	$6 < \frac{19}{3} < 7$

Problem 4.6.44 Answer by Solution

Because what two consecutive integers $\frac{19}{4}$ lies?

Correct	$4 < \frac{19}{4} < 5$

Problem 4.6.45 Answer by Solution

Because what two consecutive integers $\frac{19}{5}$ lies?

Correct	$3 < \frac{19}{5} < 4$

Problem 4.6.46 Answer by Solution

Because what two consecutive integers $\frac{19}{6}$ lies?

Correct	$3 < \frac{19}{6} < 4$

Problem 4.6.47 Answer by Solution

Because what two consecutive integers $\frac{19}{7}$ lies?

Correct	$2 < \frac{19}{7} < 3$

Problem 4.6.48 Answer by Solution

Because what two consecutive integers $\frac{19}{8}$ lies?

Correct	$2 < \frac{19}{8} < 3$

Problem 4.6.49 Answer by Solution

Because what two consecutive integers $\frac{19}{9}$ lies?

Correct	$2 < \frac{19}{9} < 3$

Problem 4.6.50 Answer by Solution

Because what two consecutive integers $\frac{19}{10}$ lies?

Correct	$1 < \frac{19}{10} < 2$

Problem 4.6.51 Answer by Solution

Because what two consecutive integers $\frac{19}{11}$ lies?

Correct	$1 < \frac{19}{11} < 2$

Problem 4.6.52 Answer by Solution

Because what two consecutive integers $\frac{19}{12}$ lies?

Correct	$1 < \frac{19}{12} < 2$

Problem 4.6.53 Answer by Solution

Because what two consecutive integers $\frac{19}{13}$ lies?

Correct	$1 < \frac{19}{13} < 2$

Problem 4.6.54 Answer by Solution

Because what two consecutive integers $\frac{19}{14}$ lies?

Correct	$1 < \frac{19}{14} < 2$

Problem 4.6.55 Answer by Solution

Because what two consecutive integers $\frac{19}{15}$ lies?

Correct	$1 < \frac{19}{15} < 2$

Problem 4.6.56 Answer by Solution

Because what two consecutive integers $\frac{19}{16}$ lies?

Correct	$1 < \frac{19}{16} < 2$

Problem 4.6.57 Answer by Solution

Because what two consecutive integers $\frac{19}{17}$ lies?

Correct	$1 < \frac{19}{17} < 2$

Problem 4.6.58 Answer by Solution

Because what two consecutive integers $\frac{19}{18}$ lies?

Correct	$1 < \frac{19}{18} < 2$

Problem 4.6.59 Answer by Solution

Because what two consecutive integers $\frac{19}{19}$ lies?

Correct	$1 < \frac{19}{19} < 2$

Problem 4.6.60 Answer by Solution

Because what two consecutive integers $\frac{19}{20}$ lies?

Correct	$0 < \frac{19}{20} < 1$

Problem 4.6.61 Answer by Solution

Because what two consecutive integers $\frac{19}{21}$ lies?

Correct	$0 < \frac{19}{21} < 1$

Problem 4.6.62 Answer by Solution

Because what two consecutive integers $\frac{19}{22}$ lies?

Correct	$0 < \frac{19}{22} < 1$

Problem 4.6.63 Answer by Solution

Because what two consecutive integers $\frac{19}{23}$ lies?

Correct	$0 < \frac{19}{23} < 1$

Problem 4.6.64 Answer by Solution

Because what two consecutive integers $\frac{19}{24}$ lies?

Correct	$0 < \frac{19}{24} < 1$

Problem 4.6.65 Answer by Solution

Because what two consecutive integers $\frac{19}{25}$ lies?

Correct	$0 < \frac{19}{25} < 1$

Problem 4.6.66 Answer by Solution

Because what two consecutive integers $\frac{19}{26}$ lies?

Correct	$0 < \frac{19}{26} < 1$

Problem 4.6.67 Answer by Solution

Because what two consecutive integers $\frac{19}{27}$ lies?

Correct	$0 < \frac{19}{27} < 1$

Problem 4.6.68 Answer by Solution

Because what two consecutive integers $\frac{19}{28}$ lies?

Correct	$0 < \frac{19}{28} < 1$

Problem 4.6.69 Answer by Solution

Because what two consecutive integers $\frac{19}{29}$ lies?

Correct	$0 < \frac{19}{29} < 1$

Problem 4.6.70 Answer by Solution

Because what two consecutive integers $\frac{19}{30}$ lies?

Correct	$0 < \frac{19}{30} < 1$

Problem 4.6.71 Answer by Solution

Because what two consecutive integers $\frac{19}{31}$ lies?

Correct	$0 < \frac{19}{31} < 1$

Problem 4.6.72 Answer by Solution

Because what two consecutive integers $\frac{19}{32}$ lies?

Correct	$0 < \frac{19}{32} < 1$

Problem 4.6.73 Answer by Solution

Because what two consecutive integers $\frac{19}{33}$ lies?

Correct	$0 < \frac{19}{33} < 1$

Problem 4.6.74 Answer by Solution

Because what two consecutive integers $\frac{19}{34}$ lies?

Correct	$0 < \frac{19}{34} < 1$

Problem 4.6.75 Answer by Solution

Because what two consecutive integers $\frac{19}{35}$ lies?

Correct	$0 < \frac{19}{35} < 1$

Problem 4.6.76 Answer by Solution

Because what two consecutive integers $\frac{19}{36}$ lies?

Correct	$0 < \frac{19}{36} < 1$

Problem 4.6.77 Answer by Solution

Because what two consecutive integers $\frac{19}{37}$ lies?

Correct	$0 < \frac{19}{37} < 1$

Problem 4.6.78 Answer by Solution

Because what two consecutive integers $\frac{19}{38}$ lies?

Correct	$0 < \frac{19}{38} < 1$

Problem 4.6.79 Answer by Solution

Because what two consecutive integers $\frac{19}{39}$ lies?

Correct	$0 < \frac{19}{39} < 1$

Problem 4.6.80 Answer by Solution

Because what two consecutive integers $\frac{19}{40}$ lies?

Correct	$0 < \frac{19}{40} < 1$

Problem 4.6.81 Answer by Solution

Because what two consecutive integers $\frac{19}{41}$ lies?

Correct	$0 < \frac{19}{41} < 1$

Problem 4.6.82 Answer by Solution

Because what two consecutive integers $\frac{19}{42}$ lies?

Correct	$0 < \frac{19}{42} < 1$

Problem 4.6.83 Answer by Solution

Because what two consecutive integers $\frac{19}{43}$ lies?

Correct	$0 < \frac{19}{43} < 1$

Problem 4.6.84 Answer by Solution

Because what two consecutive integers $\frac{19}{44}$ lies?

Correct	$0 < \frac{19}{44} < 1$

Problem 4.6.85 Answer by Solution

Because what two consecutive integers $\frac{19}{45}$ lies?

Correct	$0 < \frac{19}{45} < 1$

Problem 4.6.86 Answer by Solution

Because what two consecutive integers $\frac{19}{46}$ lies?

Correct	$0 < \frac{19}{46} < 1$

Problem 4.6.87 Answer by Solution

Because what two consecutive integers $\frac{19}{47}$ lies?

Correct	$0 < \frac{19}{47} < 1$

Problem 4.6.88 Answer by Solution

Because what two consecutive integers $\frac{19}{48}$ lies?

Correct	$0 < \frac{19}{48} < 1$

Problem 4.6.89 Answer by Solution

Because what two consecutive integers $\frac{19}{49}$ lies?

Correct	$0 < \frac{19}{49} < 1$

Problem 4.6.90 Answer by Solution

Because what two consecutive integers $\frac{19}{50}$ lies?

Correct	$0 < \frac{19}{50} < 1$

Problem 4.6.91 Answer by Solution

Because what two consecutive integers $\frac{19}{51}$ lies?

Correct	$0 < \frac{19}{51} < 1$

Problem 4.6.92 Answer by Solution

Because what two consecutive integers $\frac{19}{52}$ lies?

Correct	$0 < \frac{19}{52} < 1$

Problem 4.6.93 Answer by Solution

Because what two consecutive integers $\frac{19}{53}$ lies?

Correct	$0 < \frac{19}{53} < 1$

Problem 4.6.94 Answer by Solution

Because what two consecutive integers $\frac{19}{54}$ lies?

Correct	$0 < \frac{19}{54} < 1$

Problem 4.6.95 Answer by Solution

Because what two consecutive integers $\frac{19}{55}$ lies?

Correct	$0 < \frac{19}{55} < 1$

Problem 4.6.96 Answer by Solution

Because what two consecutive integers $\frac{19}{56}$ lies?

Correct	$0 < \frac{19}{56} < 1$

Problem 4.6.97 Answer by Solution

Because what two consecutive integers $\frac{19}{57}$ lies?

Correct	$0 < \frac{19}{57} < 1$

Problem 4.6.98 Answer by Solution

Because what two consecutive integers $\frac{19}{58}$ lies?

Correct	$0 < \frac{19}{58} < 1$

Problem 4.6.99 Answer by Solution

Because what two consecutive integers $\frac{19}{59}$ lies?

Correct	$0 < \frac{19}{59} < 1$

Problem 4.6.100 Answer by Solution

Because what two consecutive integers $\frac{19}{60}$ lies?

Correct	$0 < \frac{19}{60} < 1$

Problem 4.6.101 Answer by Solution

Because what two consecutive integers $\frac{19}{61}$ lies?

Correct	$0 < \frac{19}{61} < 1$

Problem 4.6.102 Answer by Solution

Because what two consecutive integers $\frac{19}{62}$ lies?

Correct	$0 < \frac{19}{62} < 1$

Problem 4.6.103 Answer by Solution

Because what two consecutive integers $\frac{19}{63}$ lies?

Correct	$0 < \frac{19}{63} < 1$

Problem 4.6.104 Answer by Solution

Because what two consecutive integers $\frac{19}{64}$ lies?

Correct	$0 < \frac{19}{64} < 1$

Problem 4.6.105 Answer by Solution

Because what two consecutive integers $\frac{19}{65}$ lies?

Correct	$0 < \frac{19}{65} < 1$

Problem 4.6.106 Answer by Solution

Because what two consecutive integers $\frac{19}{66}$ lies?

Correct	$0 < \frac{19}{66} < 1$

Problem 4.6.107 Answer by Solution

Because what two consecutive integers $\frac{19}{67}$ lies?

Correct	$0 < \frac{19}{67} < 1$

Problem 4.6.108 Answer by Solution

Because what two consecutive integers $\frac{19}{68}$ lies?

Correct	$0 < \frac{19}{68} < 1$

Problem 4.6.109 Answer by Solution

Because what two consecutive integers $\frac{19}{69}$ lies?

Correct	$0 < \frac{19}{69} < 1$

Problem 4.6.110 Answer by Solution

Because what two consecutive integers $\frac{19}{70}$ lies?

Correct	$0 < \frac{19}{70} < 1$

Problem 4.6.111 Answer by Solution

Because what two consecutive integers $\frac{19}{71}$ lies?

Correct	$0 < \frac{19}{71} < 1$

Problem 4.6.112 Answer by Solution

Because what two consecutive integers $\frac{19}{72}$ lies?

Correct	$0 < \frac{19}{72} < 1$

Problem 4.6.113 Answer by Solution

Because what two consecutive integers $\frac{19}{73}$ lies?

Correct	$0 < \frac{19}{73} < 1$

Problem 4.6.114 Answer by Solution

Because what two consecutive integers $\frac{19}{74}$ lies?

Correct	$0 < \frac{19}{74} < 1$

Problem 4.6.115 Answer by Solution

Because what two consecutive integers $\frac{19}{75}$ lies?

Correct	$0 < \frac{19}{75} < 1$

Problem 4.6.116 Answer by Solution

Because what two consecutive integers $\frac{19}{76}$ lies?

Correct	$0 < \frac{19}{76} < 1$

Problem 4.6.117 Answer by Solution

Because what two consecutive integers $\frac{19}{77}$ lies?

Correct	$0 < \frac{19}{77} < 1$

Problem 4.6.118 Answer by Solution

Because what two consecutive integers $\frac{19}{78}$ lies?

Correct	$0 < \frac{19}{78} < 1$

Problem 4.6.119 Answer by Solution

Because what two consecutive integers $\frac{19}{79}$ lies?

Correct	$0 < \frac{19}{79} < 1$

Problem 4.6.120 Answer by Solution

Because what two consecutive integers $\frac{19}{80}$ lies?

Correct	$0 < \frac{19}{80} < 1$

Problem 4.6.121 Answer by Solution

Because what two consecutive integers $\frac{19}{81}$ lies?

Correct	$0 < \frac{19}{81} < 1$

Problem 4.6.122 Answer by Solution

Because what two consecutive integers $\frac{19}{82}$ lies?

Correct	$0 < \frac{19}{82} < 1$

Problem 4.6.123 Answer by Solution

Because what two consecutive integers $\frac{19}{83}$ lies?

Correct	$0 < \frac{19}{83} < 1$

Problem 4.6.124 Answer by Solution

Because what two consecutive integers $\frac{19}{84}$ lies?

Correct	$0 < \frac{19}{84} < 1$

Problem 4.6.125 Answer by Solution

Because what two consecutive integers $\frac{19}{85}$ lies?

Correct	$0 < \frac{19}{85} < 1$

Problem 4.6.126 Answer by Solution

Because what two consecutive integers $\frac{19}{86}$ lies?

Correct	$0 < \frac{19}{86} < 1$

Problem 4.6.127 Answer by Solution

Because what two consecutive integers $\frac{19}{87}$ lies?

Correct	$0 < \frac{19}{87} < 1$

Problem 4.6.128 Answer by Solution

Because what two consecutive integers $\frac{19}{88}$ lies?

Correct	$0 < \frac{19}{88} < 1$

Problem 4.6.129 Answer by Solution

Because what two consecutive integers $\frac{19}{89}$ lies?

Correct	$0 < \frac{19}{89} < 1$

Problem 4.6.130 Answer by Solution

Because what two consecutive integers $\frac{19}{90}$ lies?

Correct	$0 < \frac{19}{90} < 1$

Problem 4.6.131 Answer by Solution

Because what two consecutive integers $\frac{19}{91}$ lies?

Correct	$0 < \frac{19}{91} < 1$

Problem 4.6.132 Answer by Solution

Because what two consecutive integers $\frac{19}{92}$ lies?

Correct	$0 < \frac{19}{92} < 1$

Problem 4.6.133 Answer by Solution

Because what two consecutive integers $\frac{19}{93}$ lies?

Correct	$0 < \frac{19}{93} < 1$

Problem 4.6.134 Answer by Solution

Because what two consecutive integers $\frac{19}{94}$ lies?

Correct	$0 < \frac{19}{94} < 1$

Problem 4.6.135 Answer by Solution

Because what two consecutive integers $\frac{19}{95}$ lies?

Correct	$0 < \frac{19}{95} < 1$

Problem 4.6.136 Answer by Solution

Because what two consecutive integers $\frac{19}{96}$ lies?

Correct	$0 < \frac{19}{96} < 1$

Problem 4.6.137 Answer by Solution

Because what two consecutive integers $\frac{19}{97}$ lies?

Correct	$0 < \frac{19}{97} < 1$

Problem 4.6.138 Answer by Solution

Because what two consecutive integers $\frac{19}{98}$ lies?

Correct	$0 < \frac{19}{98} < 1$

Problem 4.6.139 Answer by Solution

Because what two consecutive integers $\frac{19}{99}$ lies?

Correct	$0 < \frac{19}{99} < 1$

Problem 4.6.140 Answer by Solution

Because what two consecutive integers $\frac{19}{100}$ lies?

Correct	$0 < \frac{19}{100} < 1$

Exercises

4.6.1 Find the sum of the following improper fractions. Write your sum as a mixed number.

(a) $\frac{3}{2} + \frac{1}{2}$

Hint: Fractions

Your Answer: $\frac{4}{2}$

Your Interpretation: Solution

Your Explanation: Solution

Solution:

$$\frac{3}{2} + \frac{1}{2} = \frac{3+1}{2} = \frac{4}{2}$$

(b) $\frac{5}{3} + \frac{2}{3}$

Hint: Fractions

Your Answer: $\frac{7}{3}$

Your Interpretation: Solution

Your Explanation: Solution

Solution:

$$\frac{5}{3} + \frac{2}{3} = \frac{5+2}{3} = \frac{7}{3}$$

Sums, such as the fourth. The number $\frac{7}{3}$ is a proper fraction, but it is a mixed number, that is $2\frac{1}{3}$.

(c) $\frac{7}{4} + \frac{3}{4}$

Hint: Fractions

Your Answer: $\frac{10}{4}$

Your Interpretation: Solution

Your Explanation: Solution

Solution:

$$\frac{7}{4} + \frac{3}{4} = \frac{7+3}{4} = \frac{10}{4}$$

The sum is an improper fraction $\frac{10}{4}$ representing the number five. Separate we break the number into between 2 and 3 in $\frac{10}{4}$ $\frac{10}{4} = 2\frac{2}{4}$ $\frac{10}{4} = 2\frac{2}{4}$ $\frac{10}{4} = 2\frac{2

4.9 Summary

Definition: If a is a number and b is a nonzero number, then the **fraction** $\frac{a}{b}$ equals $a \div b$. In the fraction $\frac{a}{b}$, the **numerator** is a and the **denominator** is b.

A fraction is in **simplest form** if its numerator and denominator have no positive common divisor besides 1.

We perform multiplication, division, and exponentiation with fractions according to the following rules:

- **Multiplication of fractions**. If b and d are nonzero, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.
- **Reciprocation**. If a and b are nonzero, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, so $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$.
- **Division by a fraction**. If b, c, and d are nonzero, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.
- **Exponentiation**. If a and b are nonzero, and n is an integer, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.
- **Negation in exponent**. If a and b are nonzero, and n is an integer, then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.

We perform fraction addition and subtraction by writing the fractions with a **common denominator**, and then applying the distributive property. That is, if c is nonzero, we have

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \qquad \text{and} \qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

We sometimes write the sum of a positive integer and a fraction between 0 and 1 as a **mixed number**, which consists of the integer immediately followed by the fraction. For example, $3 + \frac{1}{3}$ can be written as $3\frac{1}{3}$.

Challenge Problems

4.60:

Make sure of the following problems without writing anything.

(a) Which is greater, $\frac{23}{24}$ or $\frac{25}{26}$?

Hint: Are there any simple fractions nearby?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Edit

Your Submission Solution:

Since $\frac{23}{24}$ is $\frac{1}{24}$ away from $\frac{24}{24} = 1$ and $\frac{25}{26}$ is $\frac{1}{26}$ away from $\frac{26}{26} = 1$, $\frac{23}{24}$ is greater than $\frac{25}{26}$. Similarly, $\frac{23}{24} = \frac{1}{1 + \frac{1}{23}}$ and $\frac{25}{26} = \frac{1}{1 + \frac{1}{25}}$ and $\frac{23}{24}$ is greater than $\frac{25}{26}$. Therefore, to compare $\frac{23}{24}$ and $\frac{25}{26}$ we only have to compare $\frac{1}{23}$ and $\frac{1}{25}$.

(b) Which is greater, $\frac{97}{98}$ or $\frac{98}{97}$?

Preview Solution

You may type any additional notes you have here.

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Your Submission Solution:

Since $\frac{97}{98} = \frac{1}{1 + \frac{1}{97}}$ and $\frac{98}{97} = \frac{1}{1 - \frac{1}{98}}$. Just as we saw that $\frac{1}{97}$ is greater than $\frac{1}{98}$ in the previous part, we can see that $\frac{1}{97}$ is greater than $\frac{1}{98}$. Therefore, we subtract more from 1 to get $\frac{97}{98}$ than we do to get $\frac{98}{97}$. This means that $\frac{97}{98}$ is greater than $\frac{98}{97}$.

4.61:

Evaluate in simplest form: $\frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2}$.

You may type any additional notes you have here.

Hide Solution | Edit

Your Submission Solution:

Solution: We could just multiply everything out, but that looks scary. Instead, we notice that $25 = 5^2$ and $36 = 6^2$, so we can use some exponent laws:

$$\frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2}$$

Now, we notice that we have $\frac{1}{2}$ in the numerator and denominator, so we can apply some more exponent laws:

$$\frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2}.$$

Now we see that we can cancel:

$$\left(\frac{1}{2}\right)^2 \left(\frac{1}{3}\right)^2 = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2} = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2}.$$

Now we see that we can cancel:

$$\left(\frac{1}{2}\right)^2 \left(\frac{1}{3}\right)^2 = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^2}

Review Problems

4.2.1.

Calculate $\det(A^{-1})$ for the following 4 singular forms.

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Choices: Solution

Hide solution

Show all

Text Submission: Solution

Solution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Choices: the singular matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Choices: the permutation $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: We can go to Gauss-Jordan this matrix, which will be the inverse of the original matrix Minus.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Choices: the singular matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4.2.2.

Calculate $\det(A^{-1})$ for the following 4 invertible matrices.

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: We can go to Gauss-Jordan this matrix, which will be the inverse of the original matrix Minus.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4.2.3.

Calculate $\det(A^{-1})$ for the following 4 invertible matrices.

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: We can go to Gauss-Jordan this matrix, which will be the inverse of the original matrix Minus.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4.2.4.

Calculate $\det(A^{-1})$ for the following 4 invertible matrices.

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: We can go to Gauss-Jordan this matrix, which will be the inverse of the original matrix Minus.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4.2.5.

Find the inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ where $a = -a$.

Hide solution

Show all

Text Submission: Solution

Solution: The determinant of each Pascal's Cofactor is the Pascal's Cofactor, the. Pascal's Cofactor equals

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

4.2.6.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.7. Points: MATH1553: 7

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: The determinant of each Pascal's Cofactor is the Pascal's Cofactor, the. Pascal's Cofactor equals

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

4.2.8.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.9. Points: MATH1553: 7

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: The determinant of each Pascal's Cofactor is the Pascal's Cofactor, the. Pascal's Cofactor equals

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

4.2.10.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.11.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.12.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.13.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.14.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.15.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.16.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.17.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.18.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.19. Points: MATH1553: 7

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

4.2.20.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.21. Points: MATH1553: 7

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Hide solution

Show all

Text Submission: Solution

Solution: Choices: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

4.2.22.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.23.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.24.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.25.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.26.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.27.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.28.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.29.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.30.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.31.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.32.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.33.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.34.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.35.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.36.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.37.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.38.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.39.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.40.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.41.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.42.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.43.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.44.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.45.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.46.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.47.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.48.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.49.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.50.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.51.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.52.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.53.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.54.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.55.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.56.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.57.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.58.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.59.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.60.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.61.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.62.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.63.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.64.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.65.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.66.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.67.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.68.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

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Text Submission: Solution

Choices: Solution

4.2.69.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.70.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.71.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

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Text Submission: Solution

Solution: Choices: Solution

4.2.72.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.73.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.74.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.75.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.76.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.77.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.78.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.79.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.80.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.81.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.82.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.83.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.84.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.85.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.86.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

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Text Submission: Solution

Choices: Solution

4.2.87.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

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Text Submission: Solution

Solution: Choices: Solution

4.2.88.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

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Text Submission: Solution

Choices: Solution

4.2.89.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

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Text Submission: Solution

Solution: Choices: Solution

4.2.90.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

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Text Submission: Solution

Choices: Solution

4.2.91.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.92.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

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Text Submission: Solution

Choices: Solution

4.2.93.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.94.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Choices: Solution

4.2.95.

Find the cofactor C_{12} of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Hide solution

Show all

Text Submission: Solution

Solution: Choices: Solution

4.2.96.

Find the cofactor C_{12} of $\begin{bmatrix}

Exercises 5

5.1 Equations

When we combine numbers or variables using mathematical operations, we form a collection of **expressions**. For example, the following expressions are equivalent:

$$2 + 2 \qquad 2 \times 2 \qquad 2^2 \qquad 4$$

A **value** is a collection of numbers and variables joined by some operation. We say that the number 4 is a value for the **coefficients** of the polynomial $x^2 - 4x + 4$ because it is the value we get when we substitute 2 for x. We say that the expression x^2 and $2x$ are the **coefficients** of x^2 and $2x$ in the polynomial $x^2 + 2

5.2 Solving Linear Equations I

An **equation** states that two quantities are equal. The most basic type of equation comes from arithmetic. For example,

$$2 + 6 = 8 + 0.$$

You've already seen many examples of this sort of equation.

So far in this book, nearly every equation with variables has been used to say that two expressions are equivalent, such as

$$a + b = b + a.$$

In this section, we introduce equations with a variable such that the equation is true for only some values of the variable. Unfortunately, we still use the same "=" to mean that the two expressions are equivalent and to write equations that are only true for some values of a variable.

For example, the equation $x + 3 = 9$ does not tell us that $x = 3$ or $x = 6$ for all values of x. If $x = 3$, then $x + 3 = 6$, not 9, so the equation $x + 3 = 9$ is not true when $x = 3$. However, if $x = 6$, then $x + 3 = 9$, so the equation $x + 3 = 9$ is true when $x = 6$. The **solutions** to an equation are the values of the variable that make the equation true. So $x = 6$ is a solution to $x + 3 = 9$.

$x + 3 = 9$ is an equation, and we can find all solutions to this equation by asking questions about it. The two most important tactics we use to solve equations are:

1. We can replace any expression with an equivalent expression. For example, in the equation

$$5x - 6x + 3 = 14,$$

we can simplify the left-hand side to $x + 3$, so the equation becomes

$$x + 3 = 14.$$

2. We can perform the same mathematical operation to both sides of the equation. For example, starting with the equation $x + 3 = 14$, we can subtract 3 from both sides of the equation to get

$$x + 3 - 3 = 14 - 3.$$

Simplifying both sides of the equation then gives $x = 11$, and we have found the solution to the equation. Looking back to the original equation, $5x - 6x + 3 = 14$, we see that when we have $x = 11$ we get $5 \cdot 11 - 6 \cdot 11 + 3 = 5 \cdot 11 - 3 = 52$, which is indeed true equation.

Important: If you add, subtract, multiply or divide the expression on one side of the equation by something, then you have to do the same to the expression on the other side of the equation.

We often solve equations with one variable by performing operations on both sides of the equation and simplifying expressions until the variable is alone on one side of the equation. When we do this, we say that we **isolate** the variable.

In this section, we focus on solving **linear equations**. An equation is a linear equation if every term in the equation is a constant term or is a constant times the first power of the variable. So,

$$x + 4 = 5 \qquad 3 - 6x = 7 \qquad \text{and} \qquad 2y + 7 = 3 - 2y$$

are linear equations. The equations

$$x^2 = 36 \qquad \text{and} \qquad \frac{3}{y - 1} = 19$$

are not linear equations.

Problems

Problem 5.0 Jump to Solution

Consider the equation $x - 12 = 289$. We will solve this equation in several different ways.

(a) Use your understanding of fractions to find a value of x that makes the equation true.

(b) Use the number line to find a value of x that makes the equation true.

(c) What number can be added to both sides of the equation to give an equation in which x is alone on the left side?

(d) Use part (c) to solve the equation.

Problem 5.5 Jump to Solution

Solve the following equations:

(a) $x - 4 = \frac{13}{3}$

(b) $4 - \frac{5}{2} = 3x + 3 - x + \frac{1}{2}$

Problem 5.10 Jump to Solution

Consider the equation $31x = 713$.

(a) By what number can we divide both sides of the equation to give an equation in which x is alone on the left side?

(b) Solve the equation.

Problem 5.11 Jump to Solution

Solve the following equations:

(a) $5x = -15$

(b) $24 = -75y$

(c) $\frac{3}{7} = \frac{9}{14}$

(d) $\frac{2x}{3} = \frac{8}{9} - \frac{15}{6}$

Problem 5.9

Solve the equation $x - 12 = 289$.

Solution for Problem 5.0 We present three different solutions.

Arithmetic. The equation means that 12 less than x equals 289. Since 289 is 12 less than x, we know that x must be 12 more than 289. Therefore, x equals $289 + 12$, which is 301.

Number Line. If we draw a number line for the equation $x - 12 = 289$ with x so that 289 is 12 steps to the left of x. This means that x is 12 steps to the right of 289, so x is $289 + 12 = 301$.

Algebra. To solve the equation, we manipulate it until it reads $x =$ (a base number). Therefore we must get x alone on one side of the equation. To do so, we eliminate the -12 on the left side by adding 12 to both sides of the equation:

$$\begin{aligned} x - 12 &= 289 \\ + 12 &= +12 \\ x &= 301 \end{aligned}$$

We have therefore isolated x on the left side of the equation. We can now see that the solution to the equation $x - 12 = 289$ is $x = 301$.

Whichever method we use to solve the equation, we can check our answer by substituting our solution, $x = 301$, back in to the original equation, $x - 12 = 289$, to get $301 - 12 = 289$. This equation is true, so our solution works. x.

Perhaps you noticed that each of our three solution approaches comes down to the same two step, adding 12 to 289 to get our answer. The first uses words, the second uses pictures, the third uses algebra. While logic and pictures are sometimes helpful in solving equations, algebraic manipulations are by far the most generally useful tools to solve equations. Try using algebra to solve the following equations.

Problem 5.3

Solve the following equations:

(a) $x - \frac{5}{2} = \frac{9}{4}$

(b) $4 - \frac{5}{2} = 3x + 3 - x + \frac{1}{2}$

Solution for Problem 5.9

(a) We isolate x by adding $\frac{5}{2}$ to both sides:

$$\begin{aligned} x - \tfrac{5}{2} &= \tfrac{9}{4} \\ + \tfrac{5}{2} &= + \tfrac{5}{2} \\ x &= \tfrac{19}{4} \end{aligned}$$

We finish by adding the mixed numbers on the right side.

We have therefore isolated x on the left side of the equation. We can now see that the solution to the equation $x - 12 = 289$ is $x = 301$.

Whichever method we use to solve the equation, we can check our answer by substituting our solution, $x = 301$, back in to the original equation, $x - 12 = 289$, to get $301 - 12 = 289$. This equation is true, so our solution works. x.

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Solution for Problem 5.9

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Solve the following equations:

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(b) $4 - \frac{5}{2} = 3x + 3 - x + \frac{1}{2}$

Solution for Problem 5.9

(a) We isolate x by adding $\frac{5}{2}$ to both sides:

$$\begin{aligned} x - \tfrac{5}{2} &= \tfrac{9}{4} \\ + \tfrac{5}{2} &= + \tfrac{5}{2} \\ x &= \tfrac{19}{4} \end{aligned}$$

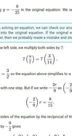
We finish by adding the mixed numbers on the right side.

We have therefore isolated x on the left side of the equation. We can now see that the solution to the equation $x - 12 = 289$ is $x = 301$.

Whichever method we use to solve the equation, we can check our answer by substituting our solution, $x = 301$, back in to the original equation, $x - 12 = 289$, to get $301 - 12 = 289$. This equation is true, so our solution works. x.

Perhaps you noticed that each of our three solution approaches comes down to the same two step, adding 12 to 289 to get our answer. The first uses words, the second uses pictures, the third uses algebra. While logic and pictures are sometimes helpful in solving equations, algebraic manipulations are by far the most generally useful tools to solve equations. Try using algebra to solve the following equations.

Concept Isolate; isolate; isolate. The key to solving most equations is to get the variable alone on one side of the equation.



Solving Linear Equations Part 1

Addition and subtraction are not the only tools we can use to solve linear equations.

Problem 5.10

Solve the equation $31x = 713$.

Solution for Problem 5.10 We divide both sides of the equation by 31. This leaves x alone on the left:

$$\frac{31x}{31} = \frac{713}{31}.$$

Since $31x/31 = x$ and $713/31 = 23$, we have $x = 23$.

In this solution we used division to change the coefficient of x from 31 to 1. We could also have viewed this as multiplying both sides of the equation by the reciprocal of the coefficient of $31x$ to give $\frac{1}{31} \cdot 31x = \frac{1}{31} \cdot 713$. The $\frac{1}{31}$ and 31 cancel on the left, and we have $x = \frac{713}{31} = 23$.

Problem 5.11

Solve the following equations:

(a) $5x = -15$

(b) $24 = -75y$

(c) $\frac{3}{7} = \frac{9}{14}$

(d) $\frac{2x}{3} = \frac{8}{9} - \frac{15}{6}$

Solution for Problem 5.11

(a) We isolate $5x$ dividing both sides of the equation by 5:

$$\frac{5x}{5} = \frac{-15}{5}.$$

Since $\frac{5x}{5}$ simplifies to $1x$, we have $x = \frac{-15}{5}$, as our solution.

(b) We divide both sides by -75:

$$\frac{24}{-75} = \frac{-75y}{-75}.$$

$\frac{-75y}{-75} = y$. We usually write the variable first, so we can write this equation as

$$y = \frac{24}{-75}.$$

We finish by simplifying the right-hand side:

$$y = \frac{24}{-75} = \frac{24}{-25} = -\frac{8}{25}.$$

Therefore, the solution is $y = -\frac{8}{25}$.

We can check our answer by substituting $y = -\frac{8}{25}$ in the original equation. We see that $-75 \cdot \left(-\frac{8}{25}\right)$ does equal 24, so our answer is correct.

Important: When solving an equation, we can check our answer by substituting our answer back into the original equation. If the original equation is not satisfied by our answer, then we probably made a mistake and should solve the equation again.

(c) To get rid of the 7 in the denominator on the left side, we multiply both sides by 7:

$$7 \cdot \left(\frac{3}{7}\right) = 7 \cdot \left(\frac{9}{14}\right).$$

We have $7 \cdot \left(\frac{3}{7}\right) = \frac{7}{1} \cdot \frac{3}{7} = \frac{21}{7} = 3$. So the equation above simplifies to $3 = \frac{9}{2}$.

(d) At first, it might look like we can't isolate x with one step. But if we write $-\frac{15}{6}$ as $\left(-\frac{5}{2}\right) \cdot 3$, we have

$$\left(-\frac{5}{2}\right) \cdot x = \frac{8}{9}.$$

Now, we can isolate x by multiplying both sides of the equation by the reciprocal of the coefficient of x. The reciprocal of $-\frac{5}{2}$ is $-\frac{2}{5}$ and multiplying both sides of the equation by $-\frac{2}{5}$ gives

$$\left(-\frac{2}{5}\right) \cdot \left(-\frac{5}{2}\right) \cdot x = \left(-\frac{2}{5}\right) \cdot \frac{8}{9}.$$

The product of a number and its reciprocal is 1, so the left side simplifies to x, as planned. We therefore have

$$x = \left(-\frac{2}{5}\right) \cdot \frac{8}{9} = \frac{8}{-5} \cdot \frac{2}{9} = \frac{-16}{-45}.$$

Checking our work, we find that when $x = \frac{-16}{-45}$, we have

$$\frac{2x}{3} = \frac{2 \cdot \frac{-16}{-45}}{3} = \frac{-32/5}{3} = \left(\frac{-32}{5}\right) \cdot \frac{1}{3} = \frac{-32}{15}, \quad \frac{8}{9} - \frac{15}{6} = \frac{8}{9} - \frac{5}{2} = \frac{-16}{18} - \frac{45}{18} = \frac{-32}{15}.$$

So, the equation is indeed satisfied when $x = \frac{-16}{-45}$.

Solving Linear Equations Part 2

Exercises

5.2.1:

Solve each of the following equations:

(a) $x + 235 = 137$

You may type any additional notes you have here.

Hide Solution / Reset

Your Submission: Solution

Solution: Subtracting 235 from both sides gives $x = 137 - 235 = \boxed{-98}$.

(b) $x + \frac{5}{7} = \frac{-2}{3}$

You may type any additional notes you have here.

Hide Solution / Reset

Your Submission: Solution

Solution: Subtracting $\frac{5}{7}$ from both sides gives $x = -\frac{2}{3} - \frac{5}{7} = \boxed{\frac{-31}{21}}$.

(c) $-\frac{3}{2}(x - 15) = x$

Preview: Solution

You may type any additional notes you have here.

Hide Solution / Reset

Your Submission: Solution

Solution: Adding 14 to both sides gives $11 - \frac{3}{2}(x - 15) = x$, so $x = \boxed{\frac{75}{5}}$.

(d) $-3y + 2\frac{1}{3} = 3y + \frac{1}{2}$

You may type any additional notes you have here.

Hide Solution / Reset

Your Submission: Solution

Solution: Simplifying the left side gives $y + \frac{7}{3} = \frac{1}{2}$. Subtracting $\frac{7}{3}$ from both sides gives

$$y = \frac{1}{2} - \frac{7}{3} = \frac{3}{6} - \frac{14}{6} = \frac{-11}{6}.$$

5.2.2:

Solve each of the following equations:

(a) $-2y = 510$

You may type any additional notes you have here.

Hide Solution / Reset

Your Submission: Solution

Solution: Dividing both sides by -2 gives $y = 510/(-7) = \boxed{-85}$.

(b) $16x = 5\frac{1}{2}$

Preview: Solution

You may type any additional notes you have here.

Hide Solution / Reset

Your Submission: Solution

Solution: We find write the right side as a fraction instead of a mixed number:

$$16x = \frac{5}{2}.$$

Multiplying both sides by $\frac{1}{16}$ (which is the same as dividing both sides by 16) gives

$$x = \frac{1}{16} \cdot \frac{5}{2} = \boxed{\frac{5}{32}}.$$

(c) $\frac{x}{3} = \frac{9}{10}$

You may type any additional notes you have here.

Hide Solution / Reset

Your Submission: Solution

Solution: Multiplying both sides by 3 cancels the 3 in the denominator on the left and gives $x = \frac{9}{10} = \boxed{\frac{9}{10}}$.

(d) $\frac{3y}{2} = \frac{7}{15}$

You may type any additional notes you have here.

Hide Solution / Reset

Your Submission: Solution

Solution: We can write the left side as $\frac{3}{2} \cdot y$. We can eliminate the coefficient of y in the equation by multiplying both sides by the reciprocal of the coefficient:

$$\left(\frac{2}{3}\right) \cdot \left(\frac{3}{2}\right) \cdot y = \left(\frac{2}{3}\right) \cdot \left(\frac{7}{15}\right).$$

The product of the fractions on the left is 1, and we have $y = \left(\frac{2}{3}\right) \cdot \left(\frac{7}{15}\right) = \boxed{\frac{14}{45}}$.

5.2.3:

Solve the equation $\frac{3}{2} \cdot y = 18\frac{2}{3}$.

You may type any additional notes you have here.

Hide Solution / Reset

Your Submission: Solution

Solution: Subtracting $\frac{3}{2}$ from both sides gives $-y = 18\frac{2}{3} - 1\frac{1}{2}$. Since $-y = 1\frac{1}{2}$, we have $y = \boxed{-1\frac{1}{2}}$.

5.2.4:

Solve the equation $\frac{x - 2}{3} = 2$.

You may type any additional notes you have here.

Hide Solution / Reset

Your Submission: Solution

Solution: We don't have a single step that will get us x. However, since $x - 2$ divided by 3 equals 2, and 14 is the number we divide by 7 to get 2, we know that $x - 2$ equals 14. Therefore, $x = \boxed{16}$.

We also could have substituted $x = 2\times 3 + 2$ in 2 right away. Since $x = 2$ is a solution to $\frac{x-2}{3} = 2$, we have $\frac{2-2}{3} = 2$. Multiplying both sides by x gives $2 = 2x$, and dividing by 2 gives $x = \boxed{1}$.

5.2.5:

Solve the equation $3(x - 7) = 21$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution / Reset

Your Submission: Solution

Solution: As in the last problem, we can't isolate x in one step. However, since 3 times $x - 7$ equals 24, and 8 is the number we multiply by 3 to get 24, we know that $x - 7$ equals 8. Therefore, $x = \boxed{15}$.

5.2.6:

Find the value of x such that $x - 2$ is a solution to the equation $\frac{x}{3} = 5$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution / Reset

Your Submission: Solution

Solution: Just as we can multiply both sides of the equation $\frac{x}{3} = 5$ by 3 to get $x = 3 \cdot 5$, we can multiply both sides of $\frac{x-2}{3} = 5$ by 3 to get $x - 2 = 3 \cdot 5$. When $x = 2$ this equation becomes $2 = 3 \cdot 5$, and dividing both sides by 3 gives $x = \boxed{1}$.

5.3 Solving Linear Equations II

Problems

Problem 5.3.1 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = 5$

(b) $\frac{x}{-2} = 7$

(c) $\frac{x}{4} = 0$

(d) $\frac{x}{-3} = 0$

Problem 5.3.2 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2}{5}$

(b) $\frac{x}{-2} = \frac{7}{3}$

(c) $\frac{x}{4} = \frac{3}{2}$

(d) $\frac{x}{-3} = \frac{1}{2}$

Problem 5.3.3 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2}$

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3}$

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2}$

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2}$

Problem 5.3.4 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + 1$

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + 2$

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + 3$

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + 4$

Problem 5.3.5 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2}$

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3}$

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2}$

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2}$

Problem 5.3.6 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - x$

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3} - x$

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - x$

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - x$

Problem 5.3.7 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2}$

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3}$

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2}$

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2}$

Problem 5.3.8 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 1$

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + 2$

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 3$

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 4$

Problem 5.3.9 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2}$

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + \frac{1}{3}$

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2}$

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2}$

Problem 5.3.10 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - x$

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + \frac{1}{3} - x$

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - x$

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - x$

Problem 5.3.11 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2}$

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3}$

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2}$

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2}$

Problem 5.3.12 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 1$

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + 2$

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 3$

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 4$

Problem 5.3.13 ▶ Jump to Solution

Find all values of x that satisfy $5x + 3 = 2x + x + 3 = 2x + 3$.

Problem 5.3.14 ▶ Jump to Solution

Find all values of x that satisfy $5x + 3 = 2x + x + 3 = 2x + 3$.

Problem 5.3.15 ▶ Jump to Solution

Find all values of x that satisfy $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 1$. Whose the same solution for x?

In the last section, we used addition and subtraction to solve some equations, and used multiplication and division to solve others. To solve the more complicated equations, however, we'll have to use a combination of all four.

Problem 5.3.16 ▶ Jump to Solution

Solve the equation $7 - 3 = 10$.

Solution: Subtracting 3 from both sides gives $7 = 13$. Dividing both sides by 7 gives $x = \frac{13}{7}$.

Concept

When solving an equation that isn't exactly the an equation you have solved before, it is helpful to think of how you should move from left side.

See if you can apply this strategy to the following problem.

Problem 5.3.17 ▶ Jump to Solution

Solve the equation $7 + 3 = 10 + 5 - x$.

Solution: We want to get the variable x by itself on the left side of the equation. By grouping the terms, the left-hand side of the original equation becomes

$$7 + 3 = 10 + (5 - x) \Rightarrow 10 + (5 - x)$$

The right-hand side of the original equation becomes

$$10 + 5 - x = (10 + 5) - x = 15 - x$$

Combining these results simplifies the original equation to

$$10 + 5 - x = 15 - x$$

We haven't solved any equations in which the variable appears on both sides. We know how to handle an equation if the variable only appears on one side. So, we add x to both sides to eliminate the variable on the right-hand side. This gives us

$$10 + 5 - x + x = 15 - x + x$$

Now we have an equation we know how to solve! We add 5 to both sides to get $15 = 20$. We then divide by 15 to find $x = \frac{20}{15} = \frac{4}{3}$.

Problem 5.3.18 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 1 - x$

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + 2 - x$

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 3 - x$

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 4 - x$

Solution: For Problem 5.3.18

(a) Adding $\frac{x}{2}$ to both sides leaves the variable term on the left side, putting all the constant terms on the right:

$$\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 1 - x$$

Simplifying the right-hand side gives

$$\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + 1 + 1 + 1 - \frac{x}{2} - \frac{x}{2} - x$$

so we now have

$$\frac{x}{3} = \frac{2x}{5}$$

Multiplying both sides by $\frac{5}{2}$ gives the same as dividing both sides by $\frac{2}{5}$ gives

$$x = \frac{2}{5} \cdot 2x = 4x$$

Finally, we get all the terms with x on one side of the equation and the constant terms on the other side. Subtracting $4x$ from both sides gives $-3x = 0$. Dividing by -3 finishes the job and gives $x = 0$. Finally, dividing both sides by $\frac{1}{2}$ gives $x = 0$.

(b) Adding $\frac{x}{3}$ to both sides leaves the variable term on the left side, putting all the constant terms on the right:

$$\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + 2 - x$$

Simplifying the right-hand side gives

$$\frac{x}{-2} = \frac{7}{3} + \frac{1}{3} + \frac{1}{3} + 2 - \frac{x}{3} - \frac{x}{3} - x$$

so we now have

$$\frac{x}{-2} = \frac{13}{3} - \frac{2x}{3}$$

Multiplying both sides by -2 gives $x = -\frac{2}{3} \cdot \frac{13}{3} + \frac{2x}{3} \cdot 2 = -\frac{26}{9} + \frac{4x}{3}$. Finally, we get all the terms with x on one side of the equation and the constant terms on the other side. Subtracting $\frac{4x}{3}$ from both sides gives $-\frac{26}{9} = -\frac{26}{9}$. Dividing by $-\frac{26}{9}$ finishes the job and gives $x = \frac{26}{9}$.

(c) Adding $\frac{x}{2}$ to both sides leaves the variable term on the left side, putting all the constant terms on the right:

$$\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 3 - x$$

Simplifying the right-hand side gives

$$\frac{x}{4} = \frac{3}{2} + \frac{1}{2} + \frac{1}{2} + 3 - \frac{x}{2} - \frac{x}{2} - x$$

so we now have

$$\frac{x}{4} = 5 - 2x$$

Multiplying both sides by $\frac{4}{x}$ gives $\frac{4}{x} \cdot \frac{x}{4} = \frac{4}{x} \cdot 5 - \frac{4}{x} \cdot 2x = \frac{20}{x} - 8$. Finally, adding 8 to both sides gives $\frac{4}{x} = \frac{20}{x} - 8 + 8 = \frac{20}{x}$.

(d) Adding $\frac{x}{2}$ to both sides leaves the variable term on the left side, putting all the constant terms on the right:

$$\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 4 - x$$

Simplifying the right-hand side gives

$$\frac{x}{-3} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 4 - \frac{x}{2} - \frac{x}{2} - x$$

so we now have

$$\frac{x}{-3} = 5 - 2x$$

Multiplying both sides by $\frac{-3}{x}$ gives $\frac{-3}{x} \cdot \frac{x}{-3} = \frac{-3}{x} \cdot 5 - \frac{-3}{x} \cdot 2x = \frac{-15}{x} + 6$. Finally, adding $\frac{15}{x}$ to both sides gives $\frac{-3}{x} = \frac{-15}{x} + 6 + \frac{15}{x} = \frac{-15}{x}$.

Concept

If you find the dividing one fractions, you can eliminate fractions from a linear equation by multiplying both sides of the equation by a constant that cancels the denominators of the fractions.

Let's practice this strategy.

Problem 5.3.19 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 1 - x$

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + 2 - x$

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 3 - x$

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 4 - x$

Solution: For Problem 5.3.19

(a) Adding $\frac{x}{2}$ to both sides leaves the variable term on the left side, putting all the constant terms on the right:

$$\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 1 - x$$

Simplifying the right-hand side gives

$$\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + 1 + 1 + 1 - \frac{x}{2} - \frac{x}{2} - x$$

so we now have

$$\frac{x}{3} = \frac{2x}{5}$$

Multiplying both sides by $\frac{5}{2}$ gives the same as dividing both sides by $\frac{2}{5}$ gives

$$x = \frac{2}{5} \cdot 2x = 4x$$

Finally, we get all the terms with x on one side of the equation and the constant terms on the other side. Subtracting $4x$ from both sides gives $-3x = 0$. Dividing by -3 finishes the job and gives $x = 0$. Finally, dividing both sides by $\frac{1}{2}$ gives $x = 0$.

(b) Adding $\frac{x}{3}$ to both sides leaves the variable term on the left side, putting all the constant terms on the right:

$$\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + 2 - x$$

Simplifying the right-hand side gives

$$\frac{x}{-2} = \frac{7}{3} + \frac{1}{3} + \frac{1}{3} + 2 - \frac{x}{3} - \frac{x}{3} - x$$

so we now have

$$\frac{x}{-2} = \frac{13}{3} - \frac{2x}{3}$$

Multiplying both sides by -2 gives $x = -\frac{2}{3} \cdot \frac{13}{3} + \frac{2x}{3} \cdot 2 = -\frac{26}{9} + \frac{4x}{3}$. Finally, we get all the terms with x on one side of the equation and the constant terms on the other side. Subtracting $\frac{4x}{3}$ from both sides gives $-\frac{26}{9} = -\frac{26}{9}$. Dividing by $-\frac{26}{9}$ finishes the job and gives $x = \frac{26}{9}$.

(c) Adding $\frac{x}{2}$ to both sides leaves the variable term on the left side, putting all the constant terms on the right:

$$\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 3 - x$$

Simplifying the right-hand side gives

$$\frac{x}{4} = \frac{3}{2} + \frac{1}{2} + \frac{1}{2} + 3 - \frac{x}{2} - \frac{x}{2} - x$$

so we now have

$$\frac{x}{4} = 5 - 2x$$

Multiplying both sides by $\frac{4}{x}$ gives $\frac{4}{x} \cdot \frac{x}{4} = \frac{4}{x} \cdot 5 - \frac{4}{x} \cdot 2x = \frac{20}{x} - 8$. Finally, adding 8 to both sides gives $\frac{4}{x} = \frac{20}{x} - 8 + 8 = \frac{20}{x}$.

(d) Adding $\frac{x}{2}$ to both sides leaves the variable term on the left side, putting all the constant terms on the right:

$$\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 4 - x$$

Simplifying the right-hand side gives

$$\frac{x}{-3} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 4 - \frac{x}{2} - \frac{x}{2} - x$$

so we now have

$$\frac{x}{-3} = 5 - 2x$$

Multiplying both sides by $\frac{-3}{x}$ gives $\frac{-3}{x} \cdot \frac{x}{-3} = \frac{-3}{x} \cdot 5 - \frac{-3}{x} \cdot 2x = \frac{-15}{x} + 6$. Finally, adding $\frac{15}{x}$ to both sides gives $\frac{-3}{x} = \frac{-15}{x} + 6 + \frac{15}{x} = \frac{-15}{x}$.

Exercises

5.3.1. ▶ Jump to Solution

Solve the following equations:

(a) $3x = 5 + 10$

Show Solution Solution

Solution: Subtracting 5 from both sides gives $3x = 5$. Dividing both sides by 3 gives $x = \frac{5}{3}$.

(b) $\frac{x}{-2} = \frac{1}{2} - 5x$

Show Solution Solution

Solution: Adding $\frac{x}{2}$ to both sides gives

$$-5x = \frac{1}{2} + \frac{x}{2}$$

Now we multiply x by $-\frac{2}{5}$. Dividing both sides by $-\frac{2}{5}$ gives $x = \frac{1}{5}$.

(c) $-7 + 3 = 4x$

Show Solution Solution

Solution: Subtracting 3 from both sides gives $-7 = 4x$. Dividing both sides by 4 gives $x = \frac{-7}{4}$.

5.3.2. ▶ Jump to Solution

Solve the following equations:

(a) $3x + 9 = 5x + 1$

Show Solution Solution

Solution: Subtracting $5x$ from both sides gives $-2x + 9 = 1$. Subtracting 9 from both sides gives $-2x = -8$. Dividing -2 from both sides gives $x = 4$.

(b) $\frac{x}{-2} + 3 = x + 5x + 12$

Show Solution Solution

Solution: Subtracting $5x$ from both sides gives $\frac{x}{-2} + 3 = 6x + 12$. Subtracting 3 from both sides gives $\frac{x}{-2} = 6x + 9$. Multiplying -2 from both sides gives $-2 \cdot \frac{x}{-2} = -2 \cdot 6x + 9$. Subtracting $12x$ from both sides gives $x = -12x + 9$. Dividing -12 from both sides gives $x = \frac{-9}{12} = \frac{-3}{4}$.

5.3.3. ▶ Jump to Solution

If $8x - 2 = 6$, find what is the value of $5x + 10$?

Show Solution Solution

Solution: Adding 2 to both sides gives $8x = 10$. Dividing both sides by 8 gives $x = \frac{5}{4}$. Finally, we find that $5x + 10$ is equal to $\frac{5}{4} \cdot 5 + 10 = \frac{25}{4} + 10 = \frac{65}{4}$.

5.3.4. ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + 1$

Show Solution Solution

Solution: Subtracting $\frac{2x}{5}$ from both sides gives

$$\frac{x}{3} - \frac{2x}{5} = -\frac{x}{2} + 1$$

Take away $\frac{x}{3}$ by multiplying both sides by $\frac{-3}{x}$ gives

$$\frac{-3}{x} \cdot \left(\frac{x}{3} - \frac{2x}{5}\right) = \frac{-3}{x} \cdot \left(-\frac{x}{2} + 1\right)$$

Show Solution

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + 2$

Show Solution Solution

Solution: We get rid of the fractions by multiplying both sides by 6. This gives $6 \cdot \left(\frac{x}{-2}\right) = 6 \cdot \left(\frac{7}{3} - \frac{x}{3} + 2\right)$. Simplifying the left-hand side gives $-3x$. Simplifying the right-hand side gives $14 - 2x + 12$. We now have $-3x = 26 - 2x$. Subtracting $2x$ from both sides gives $-x = 26$. Dividing both sides gives $x = -26$.

Preview: Solution

Show Solution Solution

Solution: Multiplying both sides by a number that gets rid of all of the fractions. The common denominator of all of the fractions is 6. So we multiply both sides by 6. This gives

$$6 \cdot \left(\frac{x}{-2}\right) = 6 \cdot \left(\frac{7}{3} - \frac{x}{3} + 2\right)$$

Multiplying the right-hand side of the original equation by 6 gives

$$6 \cdot \left(\frac{7}{3} - \frac{x}{3} + 2\right) = \frac{6 \cdot 7}{3} - \frac{6 \cdot x}{3} + 6 \cdot 2$$

$$= \frac{42}{3} - \frac{6x}{3} + 12$$

$$= 14 - 2x + 12$$

$$= 26 - 2x$$

Therefore, multiplying both sides of the original equation by 6 gives $-3x = 26 - 2x$. Adding $3x$ to both sides gives $-x = 26$. Finally, dividing by -1 gives $x = \frac{26}{-1} = -26$.

5.3.5. ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 1$

Show Solution Solution

Solution: Adding $\frac{x}{2}$ to both sides gives $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 1 + \frac{x}{2}$. Simplifying the left-hand side gives $\frac{x}{3}$. Simplifying the right-hand side gives $\frac{2x}{5} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 1 + \frac{x}{2} = \frac{2x}{5} - \frac{x}{2} + 1 + 1 + 1 - \frac{x}{2} - \frac{x}{2} + \frac{x}{2}$. We now have $\frac{x}{3} = \frac{2x}{5} - \frac{x}{2} + 3$. Subtracting $\frac{2x}{5}$ from both sides gives $\frac{x}{3} - \frac{2x}{5} = -\frac{x}{2} + 3$. Multiplying both sides gives $\frac{-3}{x} \cdot \left(\frac{x}{3} - \frac{2x}{5}\right) = \frac{-3}{x} \cdot \left(-\frac{x}{2} + 3\right)$.

Show Solution

(b) $\frac{x}{-2} = \frac{7}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + 2$

Show Solution Solution

Solution: Multiplying both sides by 6 gives $6 \cdot \left(\frac{x}{-2}\right) = 6 \cdot \left(\frac{7}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + \frac{1}{3} - \frac{x}{3} + 2\right)$. Simplifying the left-hand side gives $-3x$. Simplifying the right-hand side gives $\frac{6 \cdot 7}{3} - \frac{6 \cdot x}{3} + \frac{6 \cdot 1}{3} - \frac{6 \cdot x}{3} + \frac{6 \cdot 1}{3} - \frac{6 \cdot x}{3} + 6 \cdot 2 = 14 - 2x + 2 - 2x + 2 - 2x + 12$. Simplifying both sides gives $-3x = 30 - 6x$. Subtracting $6x$ from both sides gives $-3x + 6x = 30$. Subtracting 30 from both sides gives $-3x + 6x - 30 = 0$. Dividing -3 from both sides gives $x = \frac{-30}{-3} = 10$.

Show Solution

(c) $\frac{x}{4} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 3$

Show Solution Solution

Solution: Subtracting $\frac{x}{2}$ from both sides gives $\frac{x}{4} - \frac{x}{2} = \frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 3 - \frac{x}{2}$. Simplifying the left-hand side gives $-\frac{x}{4}$. Simplifying the right-hand side gives $\frac{3}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 3 - \frac{x}{2} = \frac{3}{2} + \frac{1}{2} + \frac{1}{2} + 3 - \frac{x}{2} - \frac{x}{2} - \frac{x}{2}$. We now have $-\frac{x}{4} = 5 - \frac{3x}{2}$. Multiplying both sides gives $\frac{4}{x} \cdot \left(-\frac{x}{4}\right) = \frac{4}{x} \cdot \left(5 - \frac{3x}{2}\right)$.

Show Solution

(d) $\frac{x}{-3} = \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 4$

Show Solution Solution

Solution: Multiplying both sides by 6 gives $6 \cdot \left(\frac{x}{-3}\right) = 6 \cdot \left(\frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + \frac{1}{2} - \frac{x}{2} + 4\right)$. Simplifying the left-hand side gives $-2x$. Simplifying the right-hand side gives $\frac{6 \cdot 1}{2} - \frac{6 \cdot x}{2} + \frac{6 \cdot 1}{2} - \frac{6 \cdot x}{2} + \frac{6 \cdot 1}{2} - \frac{6 \cdot x}{2} + 6 \cdot 4 = 3 - 3x + 3 - 3x + 3 - 3x + 24$. Simplifying both sides gives $-2x = 33 - 9x$. Adding $9x$ to both sides gives $-2x + 9x = 33$. Subtracting 33 from both sides gives $-2x + 9x - 33 = 0$. Dividing -7 from both sides gives $x = \frac{-33}{-7} = \frac{33}{7}$.

Show Solution

5.4 Word Problems

Most word problems can be solved using the following general method:

1. Read the problem carefully. Well, I don't actually know what it means.

2. Convert the words to math.
3. Solve the math.
4. Convert your answer back to words.
5. Check your work (or at least make sure it makes sense in the context of the question that was asked).

Problems

Problem 5.18 Jump to Solution

Seven more than twice what number equals thirty-five?

Problem 5.19 Jump to Solution

Six plus half of a number equals five plus one-third of the same number. What is the number?

Problem 5.20 Source: MATHOUNDTV Jump to Solution

When you add 13 to a number and then divide the sum by 13, you get the same result as when you subtract 13 from the number and then divide the difference by 13. What is the number?

Problem 5.21 Jump to Solution

My sister and I are buying a television for our room. Because I am older, I will pay $45 more than my sister. If the television costs $335, then how much does my sister have to pay?

Problem 5.22 Jump to Solution

I bought a new comic book at the Comic Book Shoppe and paid entirely using quarters. If I had instead paid using only dimes, I would have needed 9 more coins. How much did the comic book cost?

Problem 5.23 Jump to Solution

A garage has 17 cars and motorcycles. Altogether, there are 56 wheels. How many of each type of vehicle are there?

Problem 5.24 Jump to Solution

Three years ago, I was two-thirds as old as I will be eight years from now. How old am I now?

Problem 5.25 Jump to Solution

In softball, a fielder is worth 2 points and a glove is worth 3 points. Kumquare and the Wazzits recently played for the Intergalactic Starball Championship. During the game, Kumquare scored eight more fielders than the Wazzits, but scored five fewer gloves than the Wazzits. Together the two teams scored 93 points total. What was the final score?

Problem 5.18 ⏱ ✓

Seven more than twice what number equals thirty-five?

Solution for Problem 5.18 The first step in turning many word problems into math is assigning a variable to an unknown quantity. Here, we let x be the unknown number. Now, we can rewrite the problem as:

Seven more than twice a number equals thirty-five.

We can write "Seven more than twice x" as $7 + 2x$, so we can rewrite the problem as:

$$7 + 2x \text{ equals thirty-five.}$$

Now, it's clear how to write this as an equation:

$$7 + 2x = 35.$$

Subtracting 7 from both sides gives $2x = 28$, and dividing both sides by 2 gives $x = 14$.

Checking our answer, we see that seven more than twice 14 does indeed equal 35. Therefore, the desired number is 14. ◻

Problem 5.19 ⏱ ✓

Six plus half of a number equals four plus one-third of the same number. What is the number?

Solution for Problem 5.19 We again start by assigning a variable, x, to the unknown number. This makes our problem:

Six plus half of a equals four plus one-third of a.

Converting this sentence into an equation gives

$$6 + \frac{1}{2}x = 4 + \frac{1}{3}x.$$

Subtracting $\frac{1}{3}x$ from both sides gives

$$6 + \frac{1}{6}x = 4.$$

so $6 + \frac{1}{6}x = 4$. Subtracting 6 from both sides gives $\frac{1}{6}x = -2$. Multiplying both sides by 6 gives $x = -12$

We finish by checking our answer: Six plus half of -12 equals four plus one-third of -12 also equals 6. So, the number is -12. ◻

Problem 5.20 Source: MATHOUNDTV ⏱ ✓

When you add 13 to a number and then divide the sum by 13, you get the same result as when you subtract 13 from the number and then divide the difference by 13. What is the number?

Solution for Problem 5.20 Let's mix it up a little bit. We'll use n for the number this time. We'll also go straight from the words in the problem to the equation:

Concept: When you write an equation to represent a problem, take a moment to check that your equation does correctly represent the problem before solving the equation.

Multiplying both sides by 12 and by 13 to cancel the denominators gives

$$13(12 + n) = 12(n - 13).$$

Expanding both sides gives

$$156 + 12n = 12n - 156.$$

Subtracting $12n$ from both sides gives $156 = -156$ adding 156 to both sides gives $n = 312$.

Checking our answer, we see that adding 12 to 312 and dividing the sum by 13 gives 25. Subtracting 13 from 312 and then dividing the difference by 12 also gives 25. So, the desired number is indeed 312. ◻

Word Problems Part 1

Of course, you won't often be confronted with problems written in terms of "a number" you must find. Instead, you'll usually be seeking a more meaningful unknown quantity.

Problem 5.21 ⏱ ✓

My sister and I are buying a television for our room. Because I am older, I will pay $45 more than my sister. If the television costs $335, then how much does my sister have to pay?

Solution for Problem 5.21 We don't know how much I pay or how much my sister pays. To which of these quantities should we assign a variable?

Concept: When assigning a variable in a problem with multiple unknown quantities, we usually assign the variable to the unknown quantity we care most about.

We wish to know how much my sister pays, so let s be the number of dollars she pays. We then express how much I pay in terms of my sister's variable:

Concept: We can often express multiple unknown quantities in terms of the same variable.

Since I pay $45 more than my sister does, I must pay $s + 45$ dollars. Together, we spend $335, so we must have

$$(s + 45) + s = 335.$$

Simplifying the left side gives $2s + 45 = 299$. Subtracting 45 from both sides gives $2s = 254$. Dividing both sides by 2 gives $s = 127$. Therefore, my sister pays $127.

Checking our answer, I must pay $127 + $45 = $172. Combining this with the $127 my sister pays gives $172 + $127 = $299, as expected. ◻

Word Problems Part 2

Sometimes it isn't immediately obvious what quantity the variable should represent or a problem.

Problem 5.22 ⏱ ✓

I bought a new comic book at the Comic Book Shoppe and paid entirely using quarters. If I had instead paid using only dimes, I would have needed 9 more coins. How much did the comic book cost?

Solution for Problem 5.22 At first, it looks like we should assign a variable to the cost of the comic book. But it's not immediately clear how we'd relate that to the information about the number of quarters and dimes.

Concept: Sometimes you might not find a way to use your first choice for assigning a variable to make an equation. If assigning a variable to the quantity you seek doesn't seem to work, try assigning a variable to a quantity you have information about. This is especially true when you can relate this quantity to what you seek.

We know something about the number of quarters I paid. Also, if we find the number of quarters I paid, then we can figure out how much the comic book costs. So, let q be the number of quarters I paid. In order to pay for the comic, I would have needed $q + 9$ dimes. Both q quarters and $q + 9$ dimes must equal the price of the comic book. Since q quarters is $25q$ cents, and $q + 9$ dimes is $10(q + 9)$ cents, we must have

$$25q = 10(q + 9).$$

Expanding the right side gives $25q = 10q + 90$. Subtracting $10q$ from both sides gives $15q = 90$, and dividing by 15 gives $q = 6$. Therefore, I paid 6 quarters for the comic book, which means the comic book cost $1.50. To check our answer, we note that $6 + 9 = 15 dimes is also $1.50. ◻

Word Problems Part 3

Problem 5.23 ⏱ ✓

A garage has 17 cars and motorcycles. Altogether, there are 56 wheels. How many of each type of vehicle are there?

Solution for Problem 5.23 Let c be the number of cars. Since there are 17 cars and motorcycles total, there are $17 - c$ motorcycles. Since each car has 4 wheels and each motorcycle has 2 wheels, the total number of wheels is $4c + 2(17 - c)$. Therefore we must have

$$4c + 2(17 - c) = 56.$$

Expanding the product on the left side gives $4c + 34 - 2c = 56$, and simplifying gives $2c + 34 = 56$. Subtracting 34 from both sides gives $2c = 22$ so $c = 11$. This means that there are 11 cars and $17 - 11 = 6$ motorcycles. Checking, we find that 11 cars and 6 motorcycles together have $11 \cdot 4 + 6 \cdot 2 = 56$ wheels. ◻

We also might have solved this problem with a little clever insight. If all 17 vehicles were motorcycles, then there are $17 \cdot 2 = 34$ wheels total. That's $56 - 34 = 22$ wheels too fast. Each time we replace a motorcycle with a car, the number of wheels increases by 2. So, if we start with 17 motorcycles, which together have 22 wheels too few, then we need to replace $22/2 = 11$ motorcycles with 11 cars in order to have 56 wheels total. ◻

Problem 5.24 ⏱ ✓

Three years ago, I was two-thirds as old as I will be eight years from now. How old am I now?

Solution for Problem 5.24 Let my age now be n. Three years ago, my age was $n - 3$, and eight years from now, my age will be $n + 8$. What's wrong with this next step:

Begin Solution: Converting the words in the problem to an equation gives

$$\frac{2}{3}(n - 3) = n + 8.$$

We set the equation up incorrectly. The problem tells us that

My age three years ago — two-thirds my age eight years from now.

Since my age three years ago is $n - 3$, and my age eight years from now is $n + 8$, we have the equation

$$n - 3 = \frac{2}{3}(n + 8).$$

Multiplying both sides by 3 gives $3(n - 3) = 2(n + 8)$. Expanding both sides gives $3n - 9 = 2n + 16$. Subtracting $2n$ from both sides gives $n - 9 = 16$, and adding 9 to both sides gives $n = 25$. Therefore, I'm 25 years old now.

Checking, we see that three years ago I was 22, and eight years from now I'll be 33. Since $\frac{2}{3}(33) = 22$, our answer is correct. ◻

In our Bogus Solution to Problem 5.24, we started with the equation $\frac{2}{3}(n - 3) = n + 8$. Suppose we hadn't realized that we wrote the wrong equation, and proceeded to solve the equation. Multiplying both sides by 3 to get rid of the fraction gives

$$3 \cdot \frac{2}{3}(n - 3) = 3(n + 8),$$

so $2(n - 3) = 3n + 24$. Expanding both sides gives $2n - 6 = 3n + 24$. Subtracting $2n$ from both sides gives $-6 = n + 24$, and subtracting 24 from both sides gives $-30 = n$. Clearly this is ridiculous. My age can't be negative! This is a strong clue that we made an error somewhere and need to check our work.

WARNING! Always take a moment to consider whether or not your final answer makes sense.

Problem 5.25 ⏱ ✓

In softball, a fielder is worth 2 points and a glove is worth 3 points. Kumquare and the Wazzits recently played for the Intergalactic Starball Championship. During the game, Kumquare scored eight more fielders than the Wazzits, but scored five fewer gloves than the Wazzits. Together the two teams scored 93 points total. What was the final score?

Solution for Problem 5.25 Once again, it isn't obvious what quantity a variable should represent. We do know that the two teams together scored 93 points. So, if we let f represent the number of fielders, then the Wazzits' score was $93 - f$. But how will we build an equation?

What else do we know about the quantities of fielders and gloves? Kumquare scored eight more fielders, which is $f + 8$. Since the Wazzits, the Bogus Kumquare scored five fewer gloves, which is $5 - 5 = 20$ gloves, then the Wazzits. So, Kumquare scored 5 fewer gloves than the Wazzits, which means that the Wazzits scored 5 more gloves than Kumquare. We now have two expressions for the same quantity, the Wazzits' score, so we can write an equation setting them equal:

$$4 + 9 = 93 - 8.$$

Concept: If you find two different expressions that represent the same quantity, then you can set an equation.

Adding 8 and subtracting 9 from both sides of $4 + 9 = 93 - 8$ gives $28 = 85$. Dividing by 2 gives $f = 42$, which means Kumquare scored 42 points and the Wazzits scored $42 + 9 = 51$ points. So, the final score was the Wazzits 51 points and Kumquare 42 points. ◻

A key step in our solution to Problem 5.25 was assigning a variable to Kumquare's score even though it wasn't immediately clear how doing so would lead to an equation.

Concept: Do something! Don't wait until you know how to build an equation to assign variables and start thinking algebraically. You may not get to the solution immediately, but you'll almost certainly do better by trying something than by not trying anything.

Exercises

5.4.1: ⏱ ✓

Battle Ninka of a number, then double the number, and then multiplies the result by 3. If her final number is 65 more than her original number, then what was her original number?

You may type any additional notes you have here. Hide Solution Reset

Your Solution

Solution: Let x be Kefka's number. Double her number is $2x$, and triple this is $6x$. Since this is 65 greater than her original number, we have $6x = x + 65$. Subtracting x from both sides gives $5x = 65$, and dividing by 5 gives $x = \boxed{13}$.

5.4.2: Source: MOMS ⏱ ✓

If I add 5 to $\frac{1}{3}$ of a number, the result is $\frac{2}{3}$ of the number. What is the number?

Preview Solution

You may type any additional notes you have here. Hide Solution Reset

Your Solution

Solution: Let n be the number. Adding 5 to $\frac{1}{3}$ of the number gives $5 + \frac{1}{3}n$, so we must have $5 + \frac{1}{3}n = \frac{2}{3}n$. Subtracting $\frac{1}{3}n$ from both sides gives $5 = \frac{1}{3}n$, so $\left(\frac{3}{1}\right) \cdot 5 = \frac{1}{3}n \cdot \left(\frac{3}{1}\right)$. Multiplying both sides of $5 = \frac{1}{3}n$ by 3 gives $n = \boxed{15}$.

5.4.3: ⏱ ✓

One of my dogs is 25 pounds heavier than the other and the two together weigh 137 pounds. How much does the heavier dog weigh?

You may type any additional notes you have here. Hide Solution Reset

Your Solution

Solution: Let x be the weight of the heavier dog in pounds, so the lighter dog weighs $x - 25$ pounds. Since the two dogs together weigh 137 pounds, we have $x + (x - 25) = 137$. Adding 25 to both sides gives $2x = 162$. Dividing both sides by 2 gives $x = 81$, so the heavier dog weighs $\boxed{81 \text{ pounds}}$.

5.4.4: Source: MATHOUNDTV ⏱ ✓

What integer is triplet when ten added to three-fourths of it?

You may type any additional notes you have here. Hide Solution Reset

Your Solution

Solution: Let n be the integer. Triple the integer is $3n$. The sum of 9 and three-fourths of the integer is $9 + \frac{3}{4}n$, so we have $3n = 9 + \frac{3}{4}n$. Subtracting $\frac{3}{4}n$ from both sides gives $3n - \frac{3}{4}n = 9$, so $\frac{9}{4}n = 9$. Multiplying both sides by $\frac{4}{9}$ gives $n = \boxed{4}$.

5.4.5: Source: MOMS ⏱ ✓

The sum of the ages of three children is 22. The age of the oldest is twice the age of the youngest. The two older children differ by three years. What is the age of the youngest child?

You may type any additional notes you have here. Hide Solution Reset

Your Solution

Solution: Let the age of the youngest child be y. The oldest is twice as old as the youngest, so the oldest is $2y$ years old. The two older children differ by three years, so the middle child is $2y - 3$ years old. The sum of their ages is 22, so $y + 2y + (2y - 3) = 22$. Simplifying the left side gives $5y - 3 = 22$, so $5y = 25$, which means $y = 7$. Therefore, the youngest child is $\boxed{7 \text{ years old}}$.

5.4.6: ⏱ ✓

If the sum of six consecutive even integers is 282, then what is the largest of the integers?

You may type any additional notes you have here. Hide Solution Reset

Your Solution

Solution: The sum of the integers from -6 to 5 is the sum of the six integers in $0, 2, 4, 6, 8, 10$ (i.e. $0 + 2 + 4 + 6 + 8 + 10 = 30$). Simplifying this expression gives $6n = 282$. We are given that the numbers add to 282, so $6n = 282 = 282$. Subtracting 30 from both sides gives $6n = 252$. Dividing by 6 gives $n = 42$. Therefore, the largest of the six integers is $n + 10 = \boxed{52}$.

5.4.7: ⏱ ✓

Bobby's Bike Shack orders tires each week for its two-wheel bikes and three-wheel bikes. They order tires for all 47 of their bikes this week. If they ordered 101 tires, how many bikes do the Bike Shack have?

You may type any additional notes you have here. Hide Solution Reset

Your Solution

Solution: Let t be the number of two-wheel bikes they have. Since they have 47 bikes total, they must have $47 - t$ three-wheel bikes. The two-wheel bikes need 2t tires total and the three-wheel bikes need $3(47 - t)$ tires total, so we must have $2t + 3(47 - t) = 101$. Expanding the product on the left gives $2t + 141 = 101$. Subtracting 141 from both sides gives $t = -40$. Simplifying the left side gives $-t = 40$. Dividing by -1 gives $t = 40$.

We also could have noted that if each of the 47 bikes has 2 wheels, then there are 94 wheels total. That's $101 - 94 = 15$ wheels too few. Each time we replace a two-wheel bike with a three-wheel bike, we gain one wheel. So, if we start with 47 two-wheel bikes, which have too few wheels, and then replace 15 of them with three-wheel bikes, we'll have $94 + 15 = 109$ wheels total and $47 - 15 = \boxed{32}$ two-wheel bikes.

5.4.8: ⏱ ✓

In Problem 5.22, we solved the following problem:

I bought a new comic book at the Comic Book Shoppe and paid entirely using quarters. If I had instead paid using only dimes, I would have needed 9 more coins. How much did the comic book cost?

We first considered assigning a variable to the cost of the comic book, but instead found a solution by assigning a variable to the number of quarters I said to be the comic book. However, it's possible to find the solution by assigning a variable to the cost of the comic book. How?

Write d to be the cost of the comic book in cents, then what other quantities in the problem can you express in terms of d?

You may type any additional notes you have here. Hide Solution Reset

Your Solution

Solution: If we let c be the number of cents in the cost of the comic book, then $\frac{c}{25}$ is the number of quarters I use to buy the comic book and $\frac{c}{10}$ is the number of dimes I would have used to buy the comic book. Since I would need 9 more dimes than quarters, we have

$$\frac{c}{10} = \frac{c}{25} + 9.$$

Multiplying both sides by 50 gives

$$50\left(\frac{c}{10}\right) = 50\left(\frac{c}{25}\right) + 50 \cdot \frac{9}{1}.$$

so $5c = 450 + 5c$. Solving this equation gives $c = 150$, which means the comic book cost $\boxed{\$1.50}$.

5.4.9: ⏱ ✓

The Phillies won 3 of their first 21 games. How many games in a row after these 21 games do the Phillies have to win in order to have won exactly $\frac{2}{3}$ of the games they have played?

You may type any additional notes you have here. Hide Solution Reset

Your Solution

Solution: Let g be the number of games in a row the Phillies win. After these games, they have won $3 + g$ games total out of their $21 + g$ games. Since they must win $\frac{2}{3}$ of their games, we must have $\frac{3+g}{21+g} = \frac{2}{3}$. Multiplying both sides by 3 gives $3(3 + g) = 2(21 + g)$. Expanding both sides gives $9 + 3g = 42 + 2g$. Subtracting $3g$ from both sides gives $15 = 2g + 42$, so $2g = 45$, which gives $g = \boxed{22}$.

5.5 Inequalities

Inequality is a statement that one quantity is greater than or less than another quantity.

The symbol "greater than" is "$>$". For example, $5 > 3$ means "5 is greater than 3".

The symbol "less than" is "$<$". For example, $3 < 5$ means "3 is less than 5".

Both of the inequalities above are **strict inequalities**, since one side must be larger than the other. We can also use **nonstrict inequalities**, which include equality, and use the symbols "greater or equal" ($\geq$) and "less or equal" ($\leq$).

Unless $x = 0$, the inequality $x \neq 0$ means x is not equal to 0.

The symbol "not equal" ($\neq$) is used when we don't care about the relative order of the quantities, but just care that they are not equal.

The inequality $x \neq 0$ means x is not equal to 0. Similarly, the $\neq$ symbol means "less than or equal to".

Just as with equations, we can add and subtract from both sides of an inequality.

$$x + 3 > 5$$

$$x + 3 - 3 > 5 - 3$$

$$x > 2$$

This tells us that x is greater than 2. For example, x could be $x = 3$ or $x = 4$. We can graph the values of x that satisfy the inequality on the number line, as shown below:

We draw an open circle at 2 on the number line to indicate that $x = 2$ is not a valid solution to the inequality. $x > 2$ is not a valid solution because x must be greater than 2, not just equal to 2. We draw the line to the right of 2 to show that all numbers greater than 2 are valid solutions to the inequality. We can also use the open circle on the number line to show the solution of the strict inequality $x < 2$. We draw the line to the left of 2 to show that all numbers smaller than 2 are valid solutions to the inequality.

Unlike with equations, we can multiply both sides by the same positive x and inequality stays the same.

$$x \cdot 2 > 3 \cdot 2$$

$$x \cdot 2 > 6$$

$$x > 3$$

Unlike with equations, we can multiply both sides by the same negative x and inequality flips.

$$x \cdot (-2) < 3 \cdot (-2)$$

$$x \cdot (-2) < -6$$

$$x > -3$$

Problems

Problem 5.29 ★ Lesson 5 Problem 5

(a) Write an inequality that describes all real numbers x that are less than -3.

(b) Write an inequality that describes all real numbers x that are greater than -3.

(c) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.30 ★ Lesson 5 Problem 6

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) Write an inequality that describes all real numbers x that are at most -3.

(c) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.31 ★ Lesson 5 Problem 7

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.32 ★ Lesson 5 Problem 8

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.33 ★ Lesson 5 Problem 9

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.34 ★ Lesson 5 Problem 10

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.35 ★ Lesson 5 Problem 11

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.36 ★ Lesson 5 Problem 12

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.37 ★ Lesson 5 Problem 13

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.38 ★ Lesson 5 Problem 14

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.39 ★ Lesson 5 Problem 15

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.40 ★ Lesson 5 Problem 16

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.41 ★ Lesson 5 Problem 17

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.42 ★ Lesson 5 Problem 18

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.43 ★ Lesson 5 Problem 19

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.44 ★ Lesson 5 Problem 20

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.45 ★ Lesson 5 Problem 21

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.46 ★ Lesson 5 Problem 22

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.47 ★ Lesson 5 Problem 23

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.48 ★ Lesson 5 Problem 24

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.49 ★ Lesson 5 Problem 25

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.50 ★ Lesson 5 Problem 26

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.51 ★ Lesson 5 Problem 27

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.52 ★ Lesson 5 Problem 28

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.53 ★ Lesson 5 Problem 29

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.54 ★ Lesson 5 Problem 30

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.55 ★ Lesson 5 Problem 31

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.56 ★ Lesson 5 Problem 32

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.57 ★ Lesson 5 Problem 33

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.58 ★ Lesson 5 Problem 34

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.59 ★ Lesson 5 Problem 35

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.60 ★ Lesson 5 Problem 36

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.61 ★ Lesson 5 Problem 37

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.62 ★ Lesson 5 Problem 38

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.63 ★ Lesson 5 Problem 39

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.64 ★ Lesson 5 Problem 40

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.65 ★ Lesson 5 Problem 41

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.66 ★ Lesson 5 Problem 42

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.67 ★ Lesson 5 Problem 43

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.68 ★ Lesson 5 Problem 44

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.69 ★ Lesson 5 Problem 45

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.70 ★ Lesson 5 Problem 46

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.71 ★ Lesson 5 Problem 47

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.72 ★ Lesson 5 Problem 48

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.73 ★ Lesson 5 Problem 49

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.74 ★ Lesson 5 Problem 50

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.75 ★ Lesson 5 Problem 51

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.76 ★ Lesson 5 Problem 52

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.77 ★ Lesson 5 Problem 53

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.78 ★ Lesson 5 Problem 54

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.79 ★ Lesson 5 Problem 55

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.80 ★ Lesson 5 Problem 56

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.81 ★ Lesson 5 Problem 57

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.82 ★ Lesson 5 Problem 58

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.83 ★ Lesson 5 Problem 59

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.84 ★ Lesson 5 Problem 60

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.85 ★ Lesson 5 Problem 61

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.86 ★ Lesson 5 Problem 62

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.87 ★ Lesson 5 Problem 63

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.88 ★ Lesson 5 Problem 64

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.89 ★ Lesson 5 Problem 65

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.90 ★ Lesson 5 Problem 66

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.91 ★ Lesson 5 Problem 67

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.92 ★ Lesson 5 Problem 68

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.93 ★ Lesson 5 Problem 69

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.94 ★ Lesson 5 Problem 70

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.95 ★ Lesson 5 Problem 71

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.96 ★ Lesson 5 Problem 72

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.97 ★ Lesson 5 Problem 73

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.98 ★ Lesson 5 Problem 74

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.99 ★ Lesson 5 Problem 75

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.100 ★ Lesson 5 Problem 76

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.101 ★ Lesson 5 Problem 77

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.102 ★ Lesson 5 Problem 78

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.103 ★ Lesson 5 Problem 79

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.104 ★ Lesson 5 Problem 80

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.105 ★ Lesson 5 Problem 81

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.106 ★ Lesson 5 Problem 82

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.107 ★ Lesson 5 Problem 83

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.108 ★ Lesson 5 Problem 84

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.109 ★ Lesson 5 Problem 85

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.110 ★ Lesson 5 Problem 86

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.111 ★ Lesson 5 Problem 87

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.112 ★ Lesson 5 Problem 88

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.113 ★ Lesson 5 Problem 89

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.114 ★ Lesson 5 Problem 90

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.115 ★ Lesson 5 Problem 91

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.116 ★ Lesson 5 Problem 92

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.117 ★ Lesson 5 Problem 93

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.118 ★ Lesson 5 Problem 94

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.119 ★ Lesson 5 Problem 95

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.120 ★ Lesson 5 Problem 96

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.121 ★ Lesson 5 Problem 97

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.122 ★ Lesson 5 Problem 98

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.123 ★ Lesson 5 Problem 99

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.124 ★ Lesson 5 Problem 100

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.125 ★ Lesson 5 Problem 101

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.126 ★ Lesson 5 Problem 102

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.127 ★ Lesson 5 Problem 103

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.128 ★ Lesson 5 Problem 104

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.129 ★ Lesson 5 Problem 105

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.130 ★ Lesson 5 Problem 106

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.131 ★ Lesson 5 Problem 107

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.132 ★ Lesson 5 Problem 108

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.133 ★ Lesson 5 Problem 109

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.134 ★ Lesson 5 Problem 110

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.135 ★ Lesson 5 Problem 111

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.136 ★ Lesson 5 Problem 112

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.137 ★ Lesson 5 Problem 113

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.138 ★ Lesson 5 Problem 114

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.139 ★ Lesson 5 Problem 115

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.140 ★ Lesson 5 Problem 116

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.141 ★ Lesson 5 Problem 117

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.142 ★ Lesson 5 Problem 118

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.143 ★ Lesson 5 Problem 119

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at least -3.

Problem 5.144 ★ Lesson 5 Problem 120

(a) Write an inequality that describes all real numbers x that are at most -3.

(b) What are the values for $\frac{x}{2}$?

(c) What are the values for $\frac{x}{-2}$?

(d) Write an inequality that describes all real numbers x that are at most -3.

Problem 5.145 ★ Lesson 5 Problem 121

(a) Write an inequality that describes all real numbers x that are at least -3.

(b) What are

5.6 Summary

In this chapter, we learned how to solve **one-variable linear equations**. By "solving" an equation, we mean finding all values of the variable for which the equation is true. The "one-variable" in "one-variable linear equation" means that only one variable appears in the equation, though it may appear multiple times. The "linear" means that every term in the equation is a constant term or is a constant times the first power of the variable.

We say that we **solve** an equation when we find all values of the variable that make the equation true. The two most important tactics we use to solve equations are:

1. *We can replace any expression with an equivalent expression*. For example, in the equation

$$5x - 4x + 3 = 14,$$

we can simplify the left-hand side to $x + 3$, so the equation becomes

$$x + 3 = 14.$$

2. *We can perform the same mathematical operation to both sides of the equation*. For example, starting with the equation $x + 3 = 14$, we can subtract 3 from both sides of the equation to get

$$x + 3 - 3 = 14 - 3.$$

Simplifying both sides of the equation then gives $x = 11$, and we have found the solution to the equation. Looking back to the original equation, $5x - 4x + 3 = 14$, we see that when we have $x = 11$, we get $5 \cdot 11 - 4 \cdot 11 + 3 = 14$, which is indeed a true equation.

Important:	When solving an equation, we can check our answer by substituting it back into the original equation. If the original equation is not satisfied by our answer, then we made a mistake.

Important:	If a linear equation can be manipulated into an equation that is never true (such as $-1 = -5$), then there are no solutions to the equation. Similarly, if a linear equation can be manipulated into an equation that is always true (such as $4r + 5 = 4r + 5$), then all possible values of the variable are solutions to the original equation.

We can often solve word problems by turning them into linear equations.

Important:	The key to solving word problems is converting the words into the language of mathematics. To do so, assign a variable to be a quantity you seek. Then, try to build an equation to solve for that variable.

WARNING!!	When solving a word problem, define your variable clearly and use it exactly as you've defined it.

The statement $x > y$ means that x is greater than y. Similarly, $x < y$ means that x is less than y. Both $x > y$ and $x < y$ are **inequalities**. More specifically, they are strict inequalities, because in both cases we cannot have $x = y$.

We can also write nonstrict inequalities, such as $x \geq y$, which means that x is greater than or equal to y. Similarly, $x \leq y$ means that x is less than or equal to y.

Important:	Here are several useful rules regarding ways in which we can manipulate inequalities: ▪ If $a > b$ and $b > c$, then $a > c$. ▪ If $x > y$, then $x + c > y + c$ for any number c. If we also have $a > b$, then $x + a > y + b$. ▪ If $x > y$ and $a > 0$, then $xa > ya$.

Important:	If we multiply or divide an inequality by a negative number, we must reverse the direction of the inequality. For example, if $x > y$ and $a < 0$, then $xa < ya$.

Similar rules hold for nonstrict inequalities. For example, if $a \geq b$ and $b \geq c$, then $a \geq c$.

WARNING!!	Inequality rules that work when all the variables are positive don't always work when some of the variables are negative! Be careful when dealing with negative numbers (or expressions that can be negative) in inequalities.

Challenge Problems

5.56:

Source: MATHCOUNTS

Solve for x: $\frac{x}{x-1} = \frac{11}{10}$.

Hint: Consider a simpler problem. How would you handle the problem if you replaced the mixed numbers with integers?

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Solution Find, we write the mixed numbers as fractions, which gives

$$\frac{x}{x-1} = \frac{11}{10}.$$

Therefore, we have $\frac{25}{10} = \frac{3}{2} = \frac{x}{x-1} = \frac{11}{10}$. Multiplying both sides by 2x gives $\frac{5}{2}x = \frac{11}{5}x$ or $\frac{5}{2}x = \frac{11}{5}x$. Subtracting $\frac{11}{5}x$ from $\frac{5}{2}x$ gives $\frac{3}{10}x = 0$, and dividing by 3 gives $x = \boxed{0}$

5.57:

What values of x satisfy the inequality $\frac{1}{x} \geq 0$?

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Solution: We want to be tempted to simply multiply both sides by x and write $1 \geq 0x$, so $\frac{1}{x} \geq 0$. However, we have to be careful. First, x cannot be 0, since we cannot divide by 0. Second, if x is negative, then we have to reverse the inequality. Multiplying by x yields $1 \leq 0x$. Dividing by 0 gives $\frac{1}{0} \geq x$, but we must remember that x has to be positive. So, the full solution is $0 < x \leq \frac{1}{0}$.

[all positive numbers less than or equal to $\frac{1}{0}$]

5.58:

Kayla adds the same number to both the numerator and denominator of the fraction $\frac{1}{10}$. Her resulting fraction equals $\frac{1}{3}$. What number did she add to both the numerator and denominator of her original fraction?

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Solution Let x be Kayla's number, so we must have $\frac{1+x}{10+x} = \frac{1}{3}$. Multiplying both sides by 3 and by $10 + x$ gives

$$3 \cdot (10 + x) = \frac{1+x}{10+x} \cdot 3 \cdot \frac{3}{3} \cdot (10 + x).$$

The right-hand side is simply $2(10 + x)$ while the left simplifies to

$$3 \cdot (10 + x) = \frac{1+x}{10+x} \cdot 3 \cdot \frac{(10+x)(1+x)}{10+x}$$
$$= \frac{10+x}{10+x} \cdot (1 + x)$$
$$= 3(1 + x).$$

So, our equation is now $3(1 + x) = 2(10 + x)$. Expanding both sides gives $3 + 3x = 20 + 2x$. Subtracting 2x and 3 from both sides gives $x = \boxed{17}$

5.59:

Douglas writes down his favorite number, which is a two-digit positive integer. He then turns the number into a three-digit number by writing a 1 at the end of the number. This new number is 189 more than Douglas's favorite number. What is Douglas's favorite number?

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Solution Let x be Douglas's favorite number. Next, we have to figure out how to represent the number Douglas makes when he writes a 7 at the end of his favorite number. We know how to represent a new number that results when we write a 6 at the end of a number. That's just 10 times the original number. So, if Douglas put a 6 at the end of his favorite number, he would make the number $10x$. The number he makes when he puts a 7 at the end of his favorite number is 7 greater than the number he would make by putting a 6 at the end of his favorite number. So, the number he makes when he puts a 7 at the end of his favorite number is $10x + 7$. The number is 189 more than Douglas's favorite number, so we must have $10x + 7 = x + 189$. Subtracting x and 7 from both sides gives $9x = 378$, and dividing by 9 gives $x = \boxed{42}$

5.60:

Source: MATHCOUNTS

Solve for y: $\frac{y}{36} = \frac{m+8}{84} + \frac{y-m}{120}$.

Hint: One step at a time. What variable can you solve for immediately?

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Solution We start with the first two fractions. $\frac{y}{36}$. Multiplying both sides by 72 gives $y = \frac{m}{2} + 72 = 60$. Substituting this value in for y gives

$$\frac{5}{6} = \frac{m+8}{84} + \frac{m+60}{120} = \frac{y-m}{120}.$$

So, now we have $\frac{5}{6} = \frac{m+8}{84}$. Multiplying both sides by 84 gives $m + 8 = \frac{5}{6} \cdot 84 = 70$, so $m = 19$. Substituting this value in for m gives

$$\frac{5}{6} = \frac{60}{120} = \frac{19+60}{120} = \frac{y-19}{120}.$$

So, now we have $\frac{5}{6} = \frac{y-19}{120}$. Multiplying both sides by 120 gives $y - 19 = \frac{5}{6} \cdot 120 = 100$, so $y = \boxed{119}$

5.61:

Suppose that $\jmath$ is nonzero.

(a) Express $\frac{1}{\jmath} + \frac{\jmath}{\jmath^2}$ as a single fraction.

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Solution: The denominators of the fractions are the same, so we can apply the distributive property:

$$\frac{1}{\jmath} + \frac{\jmath}{\jmath^2} = \frac{1+\jmath}{\jmath^2} = \boxed{\frac{1+\jmath}{\jmath^2}}$$

(b) Express $\frac{5}{\jmath^2} + \frac{3\jmath}{\jmath^2}$ as a single fraction.

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Solution: Again, we can apply the distributive property. The second denominator is twice the first, so we multiply the numerator and denominator of the second fraction by 2:

$$\frac{5}{\jmath^2} + \frac{3\jmath}{2\jmath^2} = \frac{5}{\jmath^2} + \frac{3 \cdot 2}{2\jmath^2} = \frac{5}{\jmath^2} + \frac{6}{2\jmath^2} = \boxed{\frac{11}{2\jmath^2}}$$

(c) Express $\frac{7}{2\jmath^2} - \frac{3}{2\jmath}$ as a single fraction.

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Solution: The least common multiple of 1k and 1k is 6k, and we have

$$\frac{7}{2\jmath^2} - \frac{3}{2\jmath} = \frac{7}{2\jmath^2} - \frac{3}{2\jmath} \cdot \frac{3\jmath}{3\jmath}$$
$$= \frac{7}{2\jmath^2} - \frac{9\jmath}{6\jmath^2}$$
$$= \frac{7 - 9\jmath}{2\jmath^2}$$

5.62:

Solve the equation $\frac{1}{x-1} + \frac{1}{x} = \frac{2}{x-1}$.

Hint: How have we dealt with fractions in equalities in the past?

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Solution: We notice two of the fractions have the same denominator, so we will be easy to add or subtract. Subtracting $\frac{1}{x-1}$ from both sides gives

$$\frac{1}{x} = \frac{2}{x-1} - \frac{1}{x-1} = \frac{2-1}{x-1} = \frac{1}{x-1}.$$

Therefore, we have $\frac{1}{x} = \frac{1}{x-1}$. Multiplying both sides by 2 gives $5 = \frac{1}{x-1}$ and multiplying both sides of this equation by $x - 1$ gives $5(x - 1) = 6$. Expanding the left side gives $5x - 5 = 6$, so $5x = 11$ and $x = \boxed{\frac{11}{5}}$

5.63:

Find all values of t for which $(12 - t)^2 = (2 + 2t)^2$.

Hint: If the squares of two numbers are equal, then what do we know about the two numbers? (Remember, the numbers don't have to be positive!)

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Solution: If the squares of two numbers are equal, then either the two numbers are equal or they are opposites. So, we must have either $12 - t = 2 + 2t$ or $12 - t = -(2 + 2t)$. Solving the first equation gives $t = 5$. Turning to the second equation, we get $12 - t = -(2 + 2t)$ so $12 - t = -2 - 2t$. Solving this equation gives $t = -10$. So, the two solutions to the equation is $\boxed{5 \text{ and } -10}$

5.64:

For what integers a and b do we have $a = 28 + 2b$ and $a + b = 28$?

Hint: When solving word problems, one of our most useful strategies is finding two expressions for the same quantity, and then setting them equal!

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Solution: Since a must equal both $28 + 2b$ and $b - 28$, we must have $28 + 2b = b - 28$. Subtracting b and 28 from both sides gives $b = -56$. Substituting this into $a + b = 28$, we have $a = -77$. So, the integers are $\boxed{a = -77 \text{ and } b = -56}$

5.65:

For what number x does the equation $7x + 3c = 9x + 15 - 2y$ holds for any y?

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Solution: Simplifying the right side gives $7x + 3c = 9x + 15 - 2y$. Subtracting 7x from both sides gives $3c = 15$. In order for the original equation to have infinitely many solutions for y, the equation $3c = 15$ must always be true. Therefore, we must have $c = \boxed{5}$

5.66:

Paula can't quite read the boxed in her math class. She writes down the equation she reads on the board as $2x - 7 = 23$. She correctly defines her variable she wants but is surprised to hear the teacher say the answer is 3 less than the answer Paula found. When Paula asks the teacher to check her work, the teacher says that Paula copied the coefficient of x incorrectly (but copied everything else correctly). What was the coefficient of x have been?

Hint: Approach the problem in pieces. What can you figure out right away from the given information?

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Solution: Find, we solve the equation Paula wrote down. Adding 7 to both sides of $2x - 7 = 23$ gives $2x = 30$, so $x = 15$. Therefore, the correct answer must have been $15 - 3 = 10$. Let a be the number that was supposed to have been the coefficient of x. So Paula should have written $ax - 7 = 23$. Since the answer to the problem was supposed to be $x = 10$, we must have $a(10) - 7 = 23$. Therefore, we must have $10a = 30$, so $a = \boxed{3}$

5.67a:

Graph on a number line the values of x that satisfy $7 < x \geq 3 > 10 - x$.

Hint: The inequality chain $3 > 5 > x$ means that $x > 3$ and $5 > x$ both hold.

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Solution: In order have $7 < x \geq 3 > 10 - x$, we must have both $7 < x \geq 3$ or $3 \geq x \geq 3 > 10 - x$. So, we tackle these two inequalities separately and then combine the results. First, we solve $7 < x \geq 3$. Subtracting x and 3 from both sides gives $4 < 0 \geq x - 3$. Adding 3 to all sides, $7 < 3 \geq x$. Finally, we have to solve $3 \geq x \geq 3 > 10 - x$. We have $3 \geq x \geq 3$. We then combine the results of these two inequalities. We must have $3 \leq x$ and $x > 3$ so x can be any number that is greater than 3 and less than or equal to 4. We can write this as $3 < x \leq 4$, and graph the solutions as shown below:

5.68a:

Graph on the number line all values of x that satisfy $\frac{2x+2}{x+1} \geq 0$.

Hint: If you are given two numbers, how can you tell if their quotient will be positive?

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Solution: If $\frac{2x+2}{x+1} \geq 0$, then the fraction must either be positive or equal 0. If the fraction equals 0, then we must have $2x + 2 = 0$ so $x = -1$. If the fraction is positive, then the numerator and denominator have the same sign (positive or negative). We investigate these cases separately.

Case 1: Both are positive. If $2x + 2 > 0$, then $2x > -2$ so $x > -1$ if $x + 1 > 0$, then $x > -1$. So, in order for both $2x + 2$ and $x + 1$ to be positive, we must have both $x > -1$ and $x > -1$. Therefore, all values of x greater than -1 make both expressions positive.

Case 2: Both are negative. If $2x + 2 < 0$, then $2x < -2$ so $x < -1$ if $x + 1 < 0$, then $x < -1$. So, in order for both $2x + 2$ and $x + 1$ to be negative, we must have both $x < -1$ and $x < -1$. Therefore, all values of x less than -1 make both expressions negative.

So, the inequality $\frac{2x+2}{x+1} \geq 0$ is satisfied if $x < -1$ if $x > -1$, or if $x = -1$. We can combine the first two as $x \geq -1$ and graph the solutions on the number line as shown below:

5.69a:

Source: MATHCOUNTS

A number x is more than its reciprocal. What is x^2?

Hint: Simplify the problem. First x^2

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Solution: Since the number x is less its reciprocal, we must have $x = \frac{1}{x}$. Multiplying both sides by x gives $x \cdot x = 2 \cdot \frac{1}{x} \cdot x$, so $x^2 = 2$. Thus, no integer whose square is 2, but we can still answer the question! We want the value of x^2, and $x^2 = x^2 + x^2 \frac{1}{x^2}$. Since $x^2 = 2$, we have $x^2 + (x^2)^{-1} = 2^2 + \frac{1}{2^2} = \boxed{\frac{9}{2}}$

5.70:

(a) Find the value of $\frac{1}{1} + \frac{1}{2} + \frac{1}{3}$.

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Solution: Subtracting from both sides gives $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{1}{6}$ so $n = \boxed{6}$

(b) Find the value of $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$.

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Solution: Subtracting from both sides gives $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{1}{n}$ so $n = \boxed{12}$

(c) Find the value of $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$.

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Solution: Subtracting from both sides $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{1}{n}$ so $n = \boxed{20}$

(d) Find the value of $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$.

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Solution: Subtracting $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{1}{n}$ from both sides gives

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{1}{n}$$

so $n = \boxed{\frac{1000}{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}} = \frac{1000}{1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{5} + \frac{1}{6}} = \frac{1000}{2 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6}} = \frac{1000}{2 + \frac{10}{30} + \frac{6}{30} + \frac{5}{30}} = \frac{1000}{2 + \frac{21}{30}} = \frac{1000}{\frac{81}{30}} = \frac{30000}{81}}$

(e) Evaluate the sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$$

Hint: Compare the first 3 terms of the sum in this part to your answers in the first 3 parts.

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Solution: We see a pattern in our first four parts. Specifically, it appears that $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{1}{n}$ has $n = m(m + 1)$. For example, in the first part, the solution $\frac{1}{1} + \frac{1}{2} = \frac{1}{n}$ is $n = 2(2 + 1)$, and in the second part, the solution $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{1}{n}$ is $n = 3(3 + 1)$ Let's see why this is true. Subtracting $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{1}{n}$ from both sides of $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{1}{m}$ gives

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{1}{m}$$
$$\frac{1}{m} = \frac{1}{5}$$
$$\frac{1}{m} = \frac{1}{m+1} + \frac{1}{m}$$
$$\frac{1}{m+1} = \frac{1}{m}$$
$$m + 1 = m$$

Since $\frac{1}{m+1} = \frac{1}{m(m+1)}$, we have $n = m(m + 1)$.

Therefore, we can write any fraction of the form $\frac{1}{m(m+1)}$. When we do this with the fractions in our sum, we have

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \frac{1}{6 \cdot 7} + \frac{1}{7 \cdot 8} + \frac{1}{8 \cdot 9} + \frac{1}{9 \cdot 10} + \frac{1}{10 \cdot 11} + \frac{1}{11 \cdot 12} + \frac{1}{12 \cdot 13} + \frac{1}{13 \cdot 14} + \frac{1}{14 \cdot 15} + \frac{1}{15 \cdot 16} + \frac{1}{16 \cdot 17} + \frac{1}{17 \cdot 18} + \frac{1}{18 \cdot 19} + \frac{1}{19 \cdot 20}$$

When we add all the left sides of these equalities, we get our desired sum. This must equal the sum of all the right sides. In the sum of all the right sides, every fraction except $\frac{1}{1 \cdot 2}$ and $\frac{1}{19 \cdot 20}$ is added once and subtracted once, so they all cancel. We are left with $\frac{1}{1 \cdot 2} - \frac{1}{19 \cdot 20} = \boxed{\frac{95}{19}}$

We call the sum in this part a **telescoping** sum, since we can write the terms in the sum such that part of each term cancels with part of another term.

Series Problems

1.1

Simplify each of the following expressions.

(a) $\sum_{k=1}^{n} \frac{1}{k(k+1)}$ (b) $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)}$

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Show Fact Sheet

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k}$$

$$= \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \sum_{k=1}^{n} \frac{1}{2} \left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

$$= \frac{1}{2} \left(\sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=1}^{n} \frac{1}{2k+1}\right)$$

$$= \frac{1}{2} \left(\sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=2}^{n+1} \frac{1}{2k-1}\right)$$

$$= \frac{1}{2} \left(\frac{1}{1} - \frac{1}{2n+1}\right) = \frac{n}{2n+1}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.2

Evaluate the following sums.

(a) $\sum_{k=1}^{n} (2k-1) = n^2$ (b) $\sum_{k=1}^{n} (2k-1)^2 = \frac{n(4n^2-1)}{3}$

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Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} (2k-1)$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} (2k-1) = \frac{n(2n-1)}{2} + n = n^2$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} (2k-1)^2$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} (2k-1)^2 = \frac{n(2n-1)(2n+1)}{3} + n^2 = \frac{n(4n^2-1)}{3}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.3

Evaluate the following sums.

(a) $\sum_{k=1}^{n} (-1)^k k = \frac{(-1)^n(2n+1)-1}{4}$ (b) $\sum_{k=1}^{n} (-1)^k k^2 = \frac{(-1)^n n(2n+1)+1}{2}$

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Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} (-1)^k k$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} (-1)^k k = \frac{(-1)^n(2n+1)-1}{4}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} (-1)^k k^2$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} (-1)^k k^2 = \frac{(-1)^n n(2n+1)+1}{2}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.4

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$ (b) $\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$

Hide solution

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Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{2^k}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{k}{2^k}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.5

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$ (b) $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+3)} = \frac{n}{3(2n+3)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{k(k+2)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+3)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+3)} = \frac{n}{3(2n+3)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.6

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$ (b) $\sum_{k=1}^{n} \frac{k^2}{(k+1)!} = 2 - \frac{n+2}{(n+1)!}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{k}{(k+1)!}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{k^2}{(k+1)!}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{k^2}{(k+1)!} = 2 - \frac{n+2}{(n+1)!}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.7

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$ (b) $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+3)} = \frac{n}{3(2n+3)}$

Hide solution

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+3)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+3)} = \frac{n}{3(2n+3)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.8

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \frac{n}{2(n+2)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} = \frac{n}{3(n+3)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \frac{n}{2(n+2)}$$

Proof Sketch

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} = \frac{n}{3(n+3)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.9

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+3)} = \frac{1}{2} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+4)} = \frac{1}{4} - \frac{1}{2(n+3)} - \frac{1}{2(n+4)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+3)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+3)} = \frac{1}{2} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+4)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+4)} = \frac{1}{4} - \frac{1}{2(n+3)} - \frac{1}{2(n+4)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.10

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)} = \frac{n}{6(n+3)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)} = \frac{n}{12(n+4)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)} = \frac{n}{6(n+3)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)} = \frac{n}{12(n+4)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.11

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)} = \frac{1}{2} - \frac{1}{(n+2)} + \frac{1}{2(n+3)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)} = \frac{1}{6} - \frac{1}{(n+3)} + \frac{1}{2(n+4)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)} = \frac{1}{2} - \frac{1}{(n+2)} + \frac{1}{2(n+3)}$$

Proof Sketch

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)} = \frac{1}{6} - \frac{1}{(n+3)} + \frac{1}{2(n+4)}$$

Proof Sketch

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1.12

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)} = \frac{n}{6(n+4)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)} = \frac{n}{24(n+5)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)} = \frac{n}{6(n+4)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)} = \frac{n}{24(n+5)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.13

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)} = \frac{1}{3} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{3(n+4)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)} = \frac{1}{12} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{12(n+5)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)} = \frac{1}{3} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{3(n+4)}$$

Proof Sketch

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)} = \frac{1}{12} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{12(n+5)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.14

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)} = \frac{n}{24(n+5)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)} = \frac{n}{120(n+6)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)} = \frac{n}{24(n+5)}$$

Proof Sketch

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)} = \frac{n}{120(n+6)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.15

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)} = \frac{1}{4} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{(n+4)} + \frac{1}{4(n+5)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)} = \frac{1}{20} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{(n+5)} + \frac{1}{20(n+6)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)} = \frac{1}{4} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{(n+4)} + \frac{1}{4(n+5)}$$

Proof Sketch

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)} = \frac{1}{20} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{(n+5)} + \frac{1}{20(n+6)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.16

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)} = \frac{n}{120(n+6)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)} = \frac{n}{720(n+7)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)} = \frac{n}{120(n+6)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)} = \frac{n}{720(n+7)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.17

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)} = \frac{1}{5} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{(n+4)} + \frac{1}{(n+5)} - \frac{1}{5(n+6)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)} = \frac{1}{30} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{(n+5)} + \frac{1}{(n+6)} - \frac{1}{30(n+7)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)} = \frac{1}{5} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{(n+4)} + \frac{1}{(n+5)} - \frac{1}{5(n+6)}$$

Proof Sketch

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)} = \frac{1}{30} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{(n+5)} + \frac{1}{(n+6)} - \frac{1}{30(n+7)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.18

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)} = \frac{n}{720(n+7)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)} = \frac{n}{5040(n+8)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)} = \frac{n}{720(n+7)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)} = \frac{n}{5040(n+8)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.19

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)} = \frac{1}{6} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{(n+4)} + \frac{1}{(n+5)} - \frac{1}{(n+6)} + \frac{1}{6(n+7)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)} = \frac{1}{42} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{(n+5)} + \frac{1}{(n+6)} - \frac{1}{(n+7)} + \frac{1}{42(n+8)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)} = \frac{1}{6} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{(n+4)} + \frac{1}{(n+5)} - \frac{1}{(n+6)} + \frac{1}{6(n+7)}$$

Proof Sketch

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)} = \frac{1}{42} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{(n+5)} + \frac{1}{(n+6)} - \frac{1}{(n+7)} + \frac{1}{42(n+8)}$$

Proof Sketch

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1.20

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)} = \frac{n}{5040(n+8)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)} = \frac{n}{40320(n+9)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)} = \frac{n}{5040(n+8)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)} = \frac{n}{40320(n+9)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.21

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)} = \frac{1}{7} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{(n+4)} + \frac{1}{(n+5)} - \frac{1}{(n+6)} + \frac{1}{(n+7)} - \frac{1}{7(n+8)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)} = \frac{1}{56} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{(n+5)} + \frac{1}{(n+6)} - \frac{1}{(n+7)} + \frac{1}{(n+8)} - \frac{1}{56(n+9)}$

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)} = \frac{1}{7} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{(n+4)} + \frac{1}{(n+5)} - \frac{1}{(n+6)} + \frac{1}{(n+7)} - \frac{1}{7(n+8)}$$

Proof Sketch

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Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)} = \frac{1}{56} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{(n+5)} + \frac{1}{(n+6)} - \frac{1}{(n+7)} + \frac{1}{(n+8)} - \frac{1}{56(n+9)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.22

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)} = \frac{n}{40320(n+10)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)} = \frac{n}{362880(n+11)}$

Hide solution

Show Fact Sheet

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)} = \frac{n}{40320(n+10)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)} = \frac{n}{362880(n+11)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.23

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)} = \frac{1}{8} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{(n+4)} + \frac{1}{(n+5)} - \frac{1}{(n+6)} + \frac{1}{(n+7)} - \frac{1}{(n+8)} + \frac{1}{8(n+9)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)(k+12)} = \frac{1}{72} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{(n+5)} + \frac{1}{(n+6)} - \frac{1}{(n+7)} + \frac{1}{(n+8)} - \frac{1}{(n+9)} + \frac{1}{72(n+10)}$

Hide solution

Show Fact Sheet

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)} = \frac{1}{8} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{(n+4)} + \frac{1}{(n+5)} - \frac{1}{(n+6)} + \frac{1}{(n+7)} - \frac{1}{(n+8)} + \frac{1}{8(n+9)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)(k+12)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)(k+12)} = \frac{1}{72} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{(n+5)} + \frac{1}{(n+6)} - \frac{1}{(n+7)} + \frac{1}{(n+8)} - \frac{1}{(n+9)} + \frac{1}{72(n+10)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.24

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)(k+12)} = \frac{n}{362880(n+12)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)(k+12)(k+13)} = \frac{n}{3628800(n+13)}$

Hide solution

Show Fact Sheet

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)(k+12)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)(k+12)} = \frac{n}{362880(n+12)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)(k+12)(k+13)}$ with Maxima and verifying, one can see that, and

$$\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)(k+12)(k+13)} = \frac{n}{3628800(n+13)}$$

Proof Sketch

Show step-by-step derivation in Maxima (see Help Menu)

1.25

Evaluate the following sums.

(a) $\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)(k+12)(k+13)} = \frac{1}{9} - \frac{1}{(n+2)} + \frac{1}{(n+3)} - \frac{1}{(n+4)} + \frac{1}{(n+5)} - \frac{1}{(n+6)} + \frac{1}{(n+7)} - \frac{1}{(n+8)} + \frac{1}{(n+9)} - \frac{1}{9(n+10)}$ (b) $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)(k+8)(k+9)(k+10)(k+11)(k+12)(k+13)(k+14)} = \frac{1}{90} - \frac{1}{(n+3)} + \frac{1}{(n+4)} - \frac{1}{(n+5)} + \frac{1}{(n+6)} - \frac{1}{(n+7)} + \frac{1}{(n+8)} - \frac{1}{(n+9)} + \frac{1}{(n+10)} - \frac{1}{90(n+11)}$

Hide solution

Show Fact Sheet

Show step-by-step derivation in Maxima (see Help Menu)

Fact Submission Submitted

Statement Evaluating $\sum_{k=1}^{

Exercise 6 is accompanied by an interactive module. Complete Parts 1-4 in the module, which will prompt you to return here.

6.1 Arithmetic with Decimals

Consider the following decimal numbers. Some of the numbers have an exact binary representation in base 2. For numbers which do have an exact binary representation, write the binary representation in the form $1.b_1b_2b_3 \times 2^{b_4b_5b_6}$ where $b_1, \ldots, b_6$ are binary digits.

$$0.5,\ 0.75,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.5,\ 2.0,\ 2.5,\ 3.0,\ 4.0,\ 5.0,\ 6.0,\ 7.0,\ 8.0,\ 9.0,\ 10.0,\ 11.0,\ 12.0,\ 13.0,\ 14.0,\ 15.0,\ 16.0,\ 17.0,\ 18.0,\ 19.0,\ 20.0$$

Enter the binary representation of the decimal numbers in the order they appear above, separated by commas. For example, if the decimal numbers were 0.5, 0.75, 0.1, 0.2, 0.3, 0.4, 0.6, 0.8, 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0, 11.0, 12.0, 13.0, 14.0, 15.0, 16.0, 17.0, 18.0, 19.0, 20.0 and the binary representation of 0.5 was 1.000×2^{000}, the binary representation of 0.75 was 1.100×2^{000}, and the binary representation of 0.1 was $1.100110011001100110011 \times 2^{000}$, you would enter:

1.000 × 2^000, 1.100 × 2^000, 1.100110011001100110011 × 2^000, ...

Enter the binary representation of the decimal numbers in the order they appear above, separated by commas.

1.000 × 2^000, 1.100 × 2^000, 1.100110011001100110011 × 2^000, 1.100110011001100110011 × 2^001, 1.100110011001100110011 × 2^010, 1.100110011001100110011 × 2^011, 1.100110011001100110011 × 2^100, 1.100110011001100110011 × 2^101, 1.100110011001100110011 × 2^110, 1.100110011001100110011 × 2^111, 1.100110011001100110011 × 2^1000, 1.100110011001100110011 × 2^1001, 1.100110011001100110011 × 2^1010, 1.100110011001100110011 × 2^1011, 1.100110011001100110011 × 2^1100, 1.100110011001100110011 × 2^1101, 1.100110011001100110011 × 2^1110, 1.100110011001100110011 × 2^1111, 1.100110011001100110011 × 2^10000, 1.100110011001100110011 × 2^10001

Consider the following decimal numbers. Some of the numbers have an exact representation in base 10. For numbers which do have an exact representation, write the representation in the form $1.d_1d_2d_3 \times 10^{d_4d_5d_6}$ where $d_1, \ldots, d_6$ are decimal digits.

$$0.5,\ 0.75,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.5,\ 2.0,\ 2.5,\ 3.0,\ 4.0,\ 5.0,\ 6.0,\ 7.0,\ 8.0,\ 9.0,\ 10.0,\ 11.0,\ 12.0,\ 13.0,\ 14.0,\ 15.0,\ 16.0,\ 17.0,\ 18.0,\ 19.0,\ 20.0$$

Enter the decimal representation of the decimal numbers in the order they appear above, separated by commas. For example, if the decimal numbers were 0.5, 0.75, 0.1, 0.2, 0.3, 0.4, 0.6, 0.8, 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0, 11.0, 12.0, 13.0, 14.0, 15.0, 16.0, 17.0, 18.0, 19.0, 20.0 and the decimal representation of 0.5 was 1.000×10^{000}, the decimal representation of 0.75 was 1.000×10^{000}, and the decimal representation of 0.1 was 1.000×10^{000}, you would enter:

1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, ...

Enter the decimal representation of the decimal numbers in the order they appear above, separated by commas.

1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000

Problems

Problem 1 — Value 5 Points

Consider the following equations:

$$x = 0.1 \times 10^{-1}$$
$$y = 0.1 \times 10^{-2}$$
$$z = 0.1 \times 10^{-3}$$

Which of the following equations is correct?

- $x + y = z$
- $x - y = z$
- $x \times y = z$
- $x \div y = z$

Problem 2 — Value 5 Points

Consider the following equations:

$$x = 0.1 \times 10^{-1}$$
$$y = 0.1 \times 10^{-2}$$
$$z = 0.1 \times 10^{-3}$$

Which of the following equations is correct?

- $x + y = z$
- $x - y = z$
- $x \times y = z$
- $x \div y = z$

Problem 3 — Value 5 Points

Consider the following equations:

$$x = 0.1 \times 10^{-1}$$
$$y = 0.1 \times 10^{-2}$$
$$z = 0.1 \times 10^{-3}$$

Which of the following equations is correct?

- $x + y = z$
- $x - y = z$
- $x \times y = z$
- $x \div y = z$

Problem 4 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 5 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 6 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 7 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 8 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 9 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 10 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 11 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 12 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 13 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 14 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 15 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 16 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 17 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 18 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 19 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 20 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

6.2 Fractions

Consider the following fractions:

$$\frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \frac{1}{9},\ \frac{1}{10},\ \frac{1}{11},\ \frac{1}{12},\ \frac{1}{13},\ \frac{1}{14},\ \frac{1}{15},\ \frac{1}{16},\ \frac{1}{17},\ \frac{1}{18},\ \frac{1}{19},\ \frac{1}{20}$$

Enter the binary representation of the fractions in the order they appear above, separated by commas. For example, if the fractions were $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{19}, \frac{1}{20}$ and the binary representation of $\frac{1}{2}$ was 1.000×2^{000}, the binary representation of $\frac{1}{3}$ was 1.100×2^{000}, and the binary representation of $\frac{1}{4}$ was $1.100110011001100110011 \times 2^{000}$, you would enter:

1.000 × 2^000, 1.100 × 2^000, 1.100110011001100110011 × 2^000, ...

Enter the binary representation of the fractions in the order they appear above, separated by commas.

1.000 × 2^000, 1.100 × 2^000, 1.100110011001100110011 × 2^000, 1.100110011001100110011 × 2^001, 1.100110011001100110011 × 2^010, 1.100110011001100110011 × 2^011, 1.100110011001100110011 × 2^100, 1.100110011001100110011 × 2^101, 1.100110011001100110011 × 2^110, 1.100110011001100110011 × 2^111, 1.100110011001100110011 × 2^1000, 1.100110011001100110011 × 2^1001, 1.100110011001100110011 × 2^1010, 1.100110011001100110011 × 2^1011, 1.100110011001100110011 × 2^1100, 1.100110011001100110011 × 2^1101, 1.100110011001100110011 × 2^1110, 1.100110011001100110011 × 2^1111, 1.100110011001100110011 × 2^10000, 1.100110011001100110011 × 2^10001

Consider the following fractions:

$$\frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \frac{1}{9},\ \frac{1}{10},\ \frac{1}{11},\ \frac{1}{12},\ \frac{1}{13},\ \frac{1}{14},\ \frac{1}{15},\ \frac{1}{16},\ \frac{1}{17},\ \frac{1}{18},\ \frac{1}{19},\ \frac{1}{20}$$

Enter the decimal representation of the fractions in the order they appear above, separated by commas. For example, if the fractions were $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{19}, \frac{1}{20}$ and the decimal representation of $\frac{1}{2}$ was 1.000×10^{000}, the decimal representation of $\frac{1}{3}$ was 1.000×10^{000}, and the decimal representation of $\frac{1}{4}$ was 1.000×10^{000}, you would enter:

1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, ...

Enter the decimal representation of the fractions in the order they appear above, separated by commas.

1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000, 1.000 × 10^000

Problems

Problem 1 — Value 5 Points

Consider the following equations:

$$x = 0.1 \times 10^{-1}$$
$$y = 0.1 \times 10^{-2}$$
$$z = 0.1 \times 10^{-3}$$

Which of the following equations is correct?

- $x + y = z$
- $x - y = z$
- $x \times y = z$
- $x \div y = z$

Problem 2 — Value 5 Points

Consider the following equations:

$$x = 0.1 \times 10^{-1}$$
$$y = 0.1 \times 10^{-2}$$
$$z = 0.1 \times 10^{-3}$$

Which of the following equations is correct?

- $x + y = z$
- $x - y = z$
- $x \times y = z$
- $x \div y = z$

Problem 3 — Value 5 Points

Consider the following equations:

$$x = 0.1 \times 10^{-1}$$
$$y = 0.1 \times 10^{-2}$$
$$z = 0.1 \times 10^{-3}$$

Which of the following equations is correct?

- $x + y = z$
- $x - y = z$
- $x \times y = z$
- $x \div y = z$

Problem 4 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 5 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 6 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 7 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 8 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 9 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 10 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 11 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 12 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 13 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 14 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 15 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 16 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 17 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 18 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 19 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

Problem 20 — Value 5 Points

Consider the following numbers:

$$0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 2.0$$

Which of the following numbers, if rounded to nearest, would round to the decimal number 0.5?

6.2 Rounding

When we **round** an integer to a particular power of 10, we mean that we approximate our number by the closest number that's a multiple of our chosen power of 10. For example, the number 16,392 is nearer to 16,000 than to 17,000, so 16,392 rounds to the nearest thousand as 16,000. Similarly, 16,392 rounds to the nearest hundred as 16,400 because 16,392 is closer to 16,400 than to 16,300.

We always round to the nearest whole multiple of the power of 10 that we're rounding to. If a number is exactly halfway between two whole multiples of the power of 10 that we're rounding to, then we always pick the larger one. For example, the number 2,500 gets rounded to the nearest thousand as 3,000, even though 2,500 is equally close to 2,000 and 3,000.

We can also round decimals. The simplest sort of rounding for a decimal is to round it to the nearest integer. Remember that 1 is a power of 10, since $1 = 10^0$, so rounding to an integer is essentially "rounding to a multiple of 1." For example, 2.18 rounds to the nearest integer as 2 and 5.891 rounds to the nearest integer as 6.

We can also round a decimal to the nearest tenth, nearest hundredth, or to any decimal place that we choose. For instance, 3.419 rounds to the nearest tenth to 3.4 or to the nearest hundredth as 3.42. We also sometimes say that 3.419 rounds to 3.4 "to one decimal place" or rounds to 3.42 "to two decimal places." Again, if a number is exactly halfway between the two nearest multiples of the power of 10 that we're rounding to, we always pick the larger one. For example, 11.35 rounded to the nearest tenth is 11.4, even though 11.35 is exactly halfway between 11.3 and 11.4.

We often round numbers in our everyday life because it's usually easier for us to think about "round" numbers rather than exact numbers. For example, the high temperature today might be 78.39 degrees, but the weather forecast will probably just say "high of 78" and your mom might tell you that it's "about 80 degrees." Another common use of rounding is with large numbers that are hard to measure exactly. For example, according to the 2010 census, the population of Los Angeles is 3,792,621 people, but it is silly to think that this number can be exactly measured. It is more meaningful, and easier to process, if we round and say that the population of Los Angeles is about 3.8 million people.

We also use rounding to simplify or check calculations. For instance, if you wanted to multiply 2,049 and 6,892, we could instead multiply the numbers rounded to the nearest thousand, 2,000 and 7,000. This multiplication is just

$$\begin{aligned} 2{,}000 \cdot 7{,}000&= (2 \cdot 1{,}000) \cdot (7 \cdot 1{,}000) \\ &= (2 \cdot 7) \cdot (1{,}000 \cdot 1{,}000) \\ &= 14 \cdot 1{,}000{,}000 \\ &= 14{,}000{,}000. \end{aligned}$$

So we expect that $2{,}049 \cdot 6{,}892$ should be close to 14,000,000. (In fact it is 14,121,708.)

Problems

Problem 6.6 Jump to Solution

(a) Round 697 to the nearest hundred.

(b) Round $-2{,}712$ to the nearest thousand.

(c) Round 1.651 to the nearest tenth.

(d) Round 0.00282 to the nearest thousandth.

(e) Round 0.03 to the nearest tenth.

(f) Round 0.1972 to the nearest hundredth.

(g) Round -2.35 to the nearest tenth.

(h) Round 1.995 to the nearest hundredth.

Problem 6.7 Jump to Solution

(a) Round 4.73, 4.739, and 4.7395 each to the nearest tenth.

(b) When rounding a positive number to the nearest tenth, which digit determines whether the number you round to is larger or smaller than the original number?

Problem 6.8 Jump to Solution

Suppose you are told that the number x rounds to 2.7 when rounded to the nearest tenth. What can you conclude about x?

Problem 6.9 Jump to Solution

Suppose your friend tells you that 5,192 · 7,832 equals 51,663,744. Is this reasonable? Why or why not?

Problem 6.6

(a) Round 697 to the nearest hundred.

(b) Round $-2{,}712$ to the nearest thousand.

(c) Round 1.651 to the nearest tenth.

(d) Round 0.00282 to the nearest thousandth.

(e) Round 0.03 to the nearest tenth.

(f) Round 0.1972 to the nearest hundredth.

(g) Round -2.35 to the nearest tenth.

(h) Round 1.995 to the nearest hundredth.

Solution for Problem 6.6:

(a) 697 is closer to 700 than to 600, so 697 rounded to the nearest hundred is 700.

(b) $-2{,}712$ is closer to $-3{,}000$ than to $-2{,}000$, so $-2{,}712$ rounded to the nearest thousand is $-3{,}000$. Note also that 2,712 rounded to the nearest thousand is 3,000. Is it true that $-x$ always rounds in the same way as x? (By the time we finish all the parts of this problem, we'll know the answer.)

(c) 1.651 is slightly closer to 1.7 than to 1.6. We can tell because 1.65 is exactly halfway between them, and 1.651 is a little bit larger than 1.65. So 1.651 rounded to the nearest tenth is 1.7.

(d) 0.00282 is closer to 0.003 than to 0.002, so 0.00282 rounded to the nearest thousandth is 0.003.

(e) 0.03 is closer to 0 than to 0.1, so 0.03 rounded to the nearest tenth is 0. Many people would write the answer as 0.0 to emphasize that it is rounded to the tenths decimal place, but of course 0 and 0.0 are the same number.

(f) 0.1972 is closer to 0.20 than to 0.19, so 0.1972 rounded to the nearest hundredth is 0.2. Again, many would write this as 0.20 to emphasize that we are worried about the hundredths digit.

(g) -2.35 is exactly halfway between -2.3 and -2.4, so by rule, it rounds to the greater number, or -2.3. Note that 2.35, by the same rule, would round to 2.4, so here is an example of a number x for which x and $-x$ round differently.

(h) 1.995 is exactly halfway between 1.99 and 2, so by rule, it rounds to the larger quantity, which is 2. Often we would write this as 2.00 (that is, write the "unnecessary" 0's in the tenths and hundredths digits) to emphasize the fact that we've rounded to the nearest hundredth.

□

Problem 6.7

(a) Round 4.73, 4.739, and 4.7395 each to the nearest tenth.

(b) When rounding a positive number to the nearest tenth, which digit determines whether the number you round to is larger or smaller than the original number?

Solution for Problem 6.7:

(a) Each of these numbers is between 4.7 and 4.8, and is closer to 4.7 than to 4.8 because each is less than 4.75. Therefore, each rounded to the nearest tenth is 4.7.

(b) Part (a) suggests that only the hundredths digit matters when rounding to the nearest tenth, because even adding a digit of 9 to go from 4.73 to 4.739 doesn't change the fact that we round down. Indeed, any number of the form $4.7D$, where D is one of the digits 0, 1, 2, 3, or 4, will satisfy

$$4.7 \le 4.7D < 4.75.$$

(Here we write $4.7D$ not to mean 4.7 multiplied by D, but instead to mean the number with 4 as the units digit, 7 as the tenths digit, and D as the hundredths digit.) Moreover, adding additional digits after the D will still keep the number below 4.75. So any number of the form $4.7D\ldots$, where D is one of the digits 0, 1, 2, 3, or 4, is rounded down to the nearest tenth to 4.7. (Actually, there is one technical exception to this rule, which we will discuss in Section 6.4.)

On the other hand, if D is one of the digits 5, 6, 7, 8, or 9, then the number $4.7D$ satisfies

$$4.75 \le 4.7D < 4.8,$$

and adding additional digits after the D will still keep the number between 4.75 and 4.8. So $4.7D\ldots$, where D is one of the digits 5, 6, 7, 8, or 9, is rounded up to the nearest tenth to 4.8. This also suggests why we round 4.75 up to 4.8, even though it is exactly halfway between 4.7 and 4.8: it makes "look at the hundredths digit" a consistent rule for rounding positive numbers to the nearest tenth.

□

Problem 6.8

Suppose you are told that the number x rounds to 2.7 when rounded to the nearest tenth. What can you conclude about x?

Solution for Problem 6.8: We know that x rounds to 2.7 if x is closer to 2.7 than to any other multiple of 0.1. This can only happen if x is between 2.6 and 2.7, or if x is between 2.7 and 2.8.

If x is between 2.6 and 2.7, then it rounds up to 2.7 only if it is halfway between (that is, if $x = 2.65$) or greater. So $2.65 \le x < 2.7$.

If x is between 2.7 and 2.8, then it rounds down to 2.7 only if it is less than halfway between. So $2.7 < x < 2.75$.

And, of course, we could have $x = 2.7$ to begin with. Putting all these cases together, we conclude that $2.65 \le x < 2.75$. □

One use of rounding is as a quick check of the validity of a solution to a lengthy computation, as the following example:

Problem 6.9

Suppose your friend tells you that 5,192 · 7,832 equals 51,663,744. Is this reasonable? Why or why not?

Solution for Problem 6.9: The following argument is not quite good enough:

> **Bogus Solution:** 5,192 rounded to the nearest thousand is 5,000, and 7,832 rounded to the nearest thousand is 8,000. Since 1 thousand times 1 thousand is 1 million, the product of the two given numbers rounds, to the nearest million, as
>
> $$\begin{aligned} 5{,}000 \cdot 8{,}000 &= (5 \cdot 10^3) \cdot (8 \cdot 10^3) \\ &= (5 \cdot 8) \cdot 10^6 \\ &= 40 \cdot 10^6 \\ &= 40{,}000{,}000. \end{aligned}$$
>
> But my friend's answer rounded to the nearest million is 52,000,000, so it cannot be correct.

This is the right idea, but not quite precise enough. It is not necessarily true that the product xy rounds in the same way that the product of x rounded and y rounded do. For example, 6 and 7 each round to the nearest ten as 10, but their product $6 \cdot 7 = 42$ does not round to the nearest hundred as $10 \cdot 10 = 100$, since 42 rounded to the nearest hundred is 0.

But we can still use thoughts to make an estimate of the product. We note that

$$\begin{aligned} 5{,}000 &< 5{,}192 &< 6{,}000, \\ 7{,}000 &< 7{,}832 &< 8{,}000, \end{aligned}$$

So the product of 5,192 and 7,832 must be between the product of the lower estimates and the product of the higher estimates. That is,

$$(5{,}000)(7{,}000) < (5{,}192)(7{,}832) < (6{,}000)(8{,}000),$$

which means that the product is between 35,000,000 and 48,000,000. But the answer given is larger than 48 million, so it cannot be correct. □

Decimals and Estimation

Exercises

6.2.1:

Round 28.2508 to

(a) the nearest ten.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 28.2508 is between 20 and 30, and is closer to 30 than to 20, so 28.2508 rounded to the nearest ten is 30.

(b) the nearest tenth.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 28.2508 is between 28.2 and 28.3, and is closer to 28.3 than to 28.2, so 28.2508 rounded to the nearest tenth is 28.3.

(c) the nearest hundredth.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 28.2508 is between 28.25 and 28.26, and is closer to 28.25 than to 28.26, so 28.2508 rounded to the nearest hundredth is 28.25.

6.2.2:

Round -0.155 to

(a) the nearest integer.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: -0.155 is between -1 and 0, and is closer to 0 than to -1, so -0.155 rounded to the nearest integer is 0.

(b) the nearest tenth.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: -0.155 is between -0.1 and -0.2. It is 0.055 from -0.1 and 0.045 from -0.2, so -0.155 is closer to -0.2 than to -0.1. Therefore, -0.155 rounded to the nearest tenth is -0.2.

(c) the nearest hundredth.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: -0.155 is exactly halfway between -0.16 and -0.15, so by rule, it rounds to the larger quantity, which is -0.15.

6.2.3:

Round 7.6397 to the nearest thousandth.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 7.6397 is between 7.639 and 7.640, and is closer to 7.640 than to 7.639, so 7.6397 rounded to the nearest thousandth is 7.640 which is the same as 7.64.

6.2.4:

Find a number x such that x rounded to the nearest tenth is 1.8, x rounded to the nearest hundredth is 1.82, and x rounded to the nearest thousandth is 1.819.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Because x rounded to the nearest tenth is 1.8, we have $1.75 \le x < 1.85$. Because x rounded to the nearest hundredth is 1.82, we have $1.815 \le x < 1.825$. Because x rounded to the nearest thousandth is 1.819 we have $1.8185 \le x < 1.8195$.

Both bounds of $1.8185 \le x < 1.8195$ are inside the bounds of $1.75 \le x < 1.85$ and $1.815 \le x < 1.825$. So, any number x that satisfies $1.8185 \le x < 1.8195$ will automatically satisfy the other two. For example, $\boxed{1.819}$ satisfies the problem. Any other number x that satisfies $1.8185 \le x < 1.8195$ is also a valid answer.

6.2.5: Source: AMC 8

The fraction $\frac{401}{205}$ is closest to which of the following numbers:

$$.2,\ .2,\ .20,\ .200,\ \text{or } .2000\,?$$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We multiply the numerator and the denominator by 10^5, so that we aren't dividing by a decimal:

$$\frac{401}{0.205} = \frac{401 \cdot 10^5}{0.205 \cdot 10^5} = \frac{401 \cdot 10^5}{205} = \frac{401}{205} \cdot 10^5.$$

Since 401 is very close to $2 \cdot 205 = 410$, we know that $\frac{401}{205}$ is nearly 2, which means that among the choices given, $\frac{401}{205}$ is closest to $\boxed{2000}$.

To be sure that $\frac{401}{205} \cdot 10^5$ is closer to 2000 than to 200, we note that $\frac{401}{205} = 1\frac{196}{205}$ is greater than 1.5, so $\frac{401}{205} \cdot 10^5$ is greater than $1.5 \cdot 10^5 = 1500$. This means that $\frac{401}{205} \cdot 10^5$ is between 1500 and 2000, which means it is closer to 2000 than to 200.

6.3 Decimals and Fractions

We have two ways to express a number that is not an integer: as a fraction or as a decimal. Since we may want to use the same number in more than one format, we need to be able to convert numbers back and forth between fractions and decimals.

Problems

Problem 6.10 Jump to Solution

Our goal is to write 0.2 as a fraction.

(a) Write 0.2 in terms of a power of 10.

(b) Write the power of 10 from part (a) as a fraction.

(c) Write 0.2 as a fraction.

Problem 6.11 Jump to Solution

Write the following decimals as fractions:

(a) 0.4

(b) 0.1

(c) 0.04

(d) 0.125

(e) -1.75

(f) 2.5625

Problem 6.12 Jump to Solution

Write the following fractions as decimals:

(a) $\frac{1}{2}$

(b) $\frac{1}{5}$

(c) $\frac{1}{25}$

(d) $\frac{11}{20}$

(e) $\frac{13}{50}$

Problem 6.13 Jump to Solution

Write 13/400 as a fraction in simplest form.

Problem 6.14 Jump to Solution

Find the reciprocal of 2.5 (express your answer as a decimal).

Problem 6.15 Rate it

Write the following decimals as fractions:

(a) 0.6

(b) 0.1

(c) 0.04

(d) -1.75

(e) 2.5625

Solution for Problem 6.11:

(a) We have

$$0.4 = 0 \cdot 10^{-1} + 4 \cdot \frac{1}{10^1} = \frac{4}{10} = \frac{2}{5}.$$

This matter, perfect sense: 0.4 is read aloud as "tenth tenths." The digit 0 in the tenths place of 0.4, and the equivalent fraction is $\frac{4}{10}$.

(b) We have

$$0.1 = 5 \cdot 10^{-1} = \frac{5}{10} = \frac{1}{2}.$$

We could also notice that 0.1 is exactly halfway between 0 and 1, so $0.1 = \frac{1}{2}$.

(c) We have $0.04 = 4 \cdot 10^{-2} = \frac{4}{100} = \frac{1}{25}$.

(d) We can expand this decimal and convert each part of the expansion to a fraction, then add:

$$\begin{aligned} 0.125 &= 1 \cdot 10^{-1} + 2 \cdot 10^{-2} + 5 \cdot 10^{-3} = \frac{1}{10} + \frac{2}{100} + \frac{5}{1000} \\ &= \frac{100}{1000} + \frac{20}{1000} + \frac{5}{1000} \\ &= \frac{125}{1000} \\ &= \frac{125}{1000} \\ &= \frac{1}{8}. \end{aligned}$$

However, it is easier to perform this computation if we write the decimal as an integer times a power of 10, as $0.125 = 125 \cdot 0.001 = 125 \cdot 10^{-3}$. Then, we immediately have

$$0.125 = 125 \cdot 10^{-3} = \frac{125}{1000} = \frac{125}{1000} = \frac{1}{8}.$$

(e) We'll use the technique from part (d):

$$-1.75 = -17.5 \cdot 0.1 = -\frac{175}{10} = -\frac{35}{2}.$$

(f) We have $2.5625 = \frac{5625}{10000}$. Since $\frac{5625}{10000} = \frac{5 \cdot 1125}{2 \cdot 1125} = \frac{5}{2}$, we have

$$2.5625 = \frac{5}{2} = \frac{5}{2} = \frac{41}{16}.$$

Converting Decimals to Fractions

We also need to be able to convert fractions to decimals. To do this, we try to write our fraction with a denominator that is a power of 10.

Problem 6.16 Rate it

Write the following fractions as decimals:

(a) $\frac{1}{2}$

(b) $\frac{1}{5}$

(c) $\frac{1}{25}$

(d) $\frac{29}{20}$

(e) $\frac{13}{50}$

(f) $\frac{41}{16}$

Solution for Problem 6.12

(a) We can do this directly by converting the denominator to a power of 10:

$$\frac{1}{2} = \frac{1}{10^1} = \frac{5}{10} = 0.5.$$

(b) This is also pretty easy to write with the denominator as a power of 10:

$$\frac{1}{5} = \frac{2}{10} = 0.4.$$

(c) The denominator is already a power of 10:

$$\frac{11}{20} = \frac{55}{100} = 0.55.$$

(d) This one is a little trickier. To get the denominator to be a power of 10, we want to find the smallest power of 10 that is a multiple of 5. Since $5 = 5^1$ and $10^2 = 5^2 \cdot 2^2$, we can multiply the numerator and denominator of the fraction by $5^1 = 125$ to get

$$\frac{7}{25} = \frac{7 \cdot 125}{25 \cdot 125} = \frac{875}{3125} = 875 \cdot 0.001 = 0.875.$$

(e) Multiplying the numerator and denominator by 5 will do the trick:

$$\frac{13}{50} = \frac{13 \cdot 5}{50 \cdot 5} = \frac{65}{250} = 65 \cdot 0.01 = -0.55.$$

(f) Since $20 = 2^2$, we can multiply by $5^2 = 25$ to get 20 in the denominator:

$$\frac{29}{20} = \frac{29 \cdot 25}{20 \cdot 25} = \frac{725}{500} = 725 \cdot 0.001 = 0.580725.$$

Converting Decimals to Fractions

Some fractions, such as $\frac{1}{3}$, cannot be written as an equivalent fraction with an integer numerator and a denominator that's a power of 10. We will see how to convert these fractions to decimals in Section 6.4.

Problem 6.17 Rate it

Write 13/500 as a fraction in simplest form.

Solution for Problem 6.13: Since moving the decimal point 4 places to the right gives us the integer 13500, we know that $13.500 = 13500 \cdot 10^{-4}$. Thus, dividing by 10^4 will give us the decimal:

$$\frac{13500}{10000} = \frac{135 \cdot 100}{100 \cdot 100} = \frac{135}{100}.$$

Clearly 135/100 is not a multiple of 5, but we can try to cancel all four of the powers of 2 in the denominator by seeing if 16 is a factor of 13500. Indeed, we have $13500 = 16 \cdot 1125$, so the fraction simplifies as

$$\frac{13500}{10000} = \frac{16 \cdot 1125}{16 \cdot 625} = \frac{1125}{625} = \frac{225 \cdot 5}{125 \cdot 5} = \frac{225}{125} = 1.8.$$

Problem 6.18 Rate it

Find the reciprocal of 2.5 (express your answer as a decimal).

Solution for Problem 6.14: We usually seem to find reciprocals of fractions than of decimals, so let's first convert 2.5 to a fraction:

$$2.5 = \frac{25}{10} = \frac{5}{2}.$$

Its reciprocal is then $\frac{2}{5}$, and as a decimal this is $\frac{4}{10} = 0.4$. Like a check, note that $(2.5)(0.4) = 1$. i.e.

Exercises

6.3.1. Rate it

Express each of the following as a decimal:

(a) $\frac{1}{2}$

Your Submission: Solution

Solution: Multiplying 1/2 by 4 produces a power of 10, and we have $\frac{1}{2} = \frac{2 \cdot 4}{2 \cdot 4} = \frac{8}{10} = \boxed{0.5}$.

(b) $\frac{1}{5}$

Your Submission: Solution

Solution: Since $5^2 = 2^2$, we see that multiplying 1/5 by 2 produces a power of 10:

$$\frac{1}{5} = \frac{1 \cdot 2^2}{5 \cdot 2^2} = \frac{1 \cdot 4}{5 \cdot 4} = \frac{4}{10} = \frac{0.4}{1} = \boxed{0.4}.$$

(c) $-\frac{11}{4}$

Your Submission: Solution

Solution: Multiplying 4 by 25 gives a power of 10, and we have

$$-\frac{11}{4} = -\frac{11 \cdot 25}{4 \cdot 25} = -\frac{275}{100} = \boxed{-2.75}.$$

(d) $\frac{41}{1000}$

Your Submission: Solution

Solution: The denominator is already a power of 10: $\frac{41}{1000} = 41 \cdot 0.001 = \boxed{0.041}$.

(e) $\frac{17}{2^2 5^3}$

Your Submission: Solution

Solution: Multiplying 4 by 25 gives a power of 10, and we have

$$\frac{17}{2^2 5^3} = \frac{17 \cdot 25}{2^2 \cdot 5^3 \cdot 25} = \frac{425}{10^3} = \boxed{0.425}.$$

We might also have noticed that $\frac{17}{2^2} = \frac{17}{4} = 4.25$, and

$$\frac{17}{2^2 5^3} = \frac{17}{5^3} = (4.25) \cdot \frac{1}{25} = \boxed{0.425}.$$

(f) $\frac{3}{2048}$

Your Submission: Solution

Solution: The denominator is already a power of 10: $\frac{3}{2048} = 3 \cdot 0.0005 = \boxed{0.0015}$.

(g) $\frac{17}{2^2 5^3}$

Your Submission: Solution

Solution: Multiplying 4 by 25 gives a power of 10, so multiplying 17/4 by 25 gives a power of 10:

$$\frac{17}{2^2 5^3} = \frac{17 \cdot 25}{2^2 \cdot 5^3 \cdot 25} = \frac{425}{10^3} = \frac{425}{1000}.$$

We might also have noticed that $\frac{17}{2^2} = \frac{17}{4} = 4.25$, and

$$\frac{17}{2^2 5^3} = \frac{17}{5^3} = (4.25) \cdot \frac{1}{25} = \boxed{0.425}.$$

(h) $\frac{1}{2048}$

Your Submission: Solution

Solution: The denominator is already a power of 10. Dividing by 2048 is the same as moving the decimal point 4 places to the left. We are careful to include zeros in the appropriate places after the decimal point: $\frac{1}{2048} = \boxed{0.0005}$.

6.3.2. Source: AMC 8, '99 Rate it

Express each of the following decimals as a fraction in simplest form:

(a) -0.2

Your Submission: Solution

Solution: $-0.2 = \boxed{-\frac{1}{5}}$.

(b) 0.0130

Your Submission: Solution

Solution: There are four places past the decimal point, so the denominator of our fraction is 10^4:

$$0.0130 = \frac{130}{10^4} = \frac{13}{1000} = \boxed{\frac{65}{500}}.$$

(c) 0.875

Your Submission: Solution

Solution:

$$0.875 = \frac{875}{1000} = \frac{5 \cdot 175}{5 \cdot 200} = \frac{175}{200} = \boxed{\frac{7}{8}}.$$

(d) 5.11

Your Submission: Solution

Solution:

$$5.11 = \frac{511}{100} = 5 + \frac{11}{100} = 5 + \frac{100 \cdot 11}{100 \cdot 100} = \boxed{\frac{511}{100}}.$$

(e) 0.002

Your Submission: Solution

Solution: There are three places past the decimal point, so the denominator of our fraction is 10^3:

$$0.002 = \frac{2}{10^3} = \frac{2}{1000} = \boxed{\frac{1}{500}}.$$

(f) 2.5

Your Submission: Solution

Solution:

$$2.5 = \frac{5}{2} = \frac{5 \cdot 5}{2 \cdot 5} = \frac{25}{10} = \frac{5}{2}.$$

We also could have measured $2.5 = 10 \cdot \frac{1}{4} = \boxed{\frac{5}{2}}$.

6.3.4. Source: AMC 8, '99 Rate it

Express the product $0.1 \cdot 20 \cdot 0.2 \cdot 1.00$ as a fraction.

Your Submission: Solution

Solution: We write each decimal as a fraction, and we have

$$\begin{aligned} 0.1 \cdot 20 \cdot 0.2 \cdot 1.00&= \frac{1}{10} \cdot 20 \cdot \frac{2}{10} \cdot \frac{100}{100} \\ &= 2 \cdot \frac{2}{10} \cdot \frac{100}{100} \\ &= 2 \cdot \frac{2}{10} \cdot 1 \\ &= \frac{4}{10} \\ &= \boxed{\frac{2}{5}}. \end{aligned}$$

6.3.5. Source: MATHCOUNTS Rate it

Express the reciprocal of 2.2 as a fraction.

Your Submission: Solution

Solution: We have $2.2 = \frac{22}{10} = \frac{11}{5}$. So the reciprocal of 2.2 is the reciprocal of $\frac{11}{5}$, which is $\boxed{\frac{5}{11}}$.

6.3.6. Source: MATHCOUNTS Rate it

How many eighths are in 7.375?

Your Submission: Solution

Solution: We have $7.375 = \frac{59}{8}$. Is $\frac{59}{8}$ an integer? No. There are 8 · 80 eighths in 5. Since $\frac{59}{8} > 5$, there are 5 eighths in $\frac{59}{8}$. So, there are $\boxed{59 \text{ eighths in } 7.375}$.

6.3.7. Source: MATHCOUNTS Rate it

A wall is propped from a height of 2 meters. On its first bounce it rises to a height of 2 meters. It keeps falling and bouncing to a $\frac{3}{4}$ of the height it reached in the previous bounce. On which bounce will it first not rise to a height of 0.5 meters?

Your Submission: Solution

Solution: We list the heights (in meters) that the ball reaches on each bounce for the first several bounces by repeatedly multiplying by $\frac{3}{4}$:

$$2, \frac{3}{2}, \frac{9}{8}, \frac{27}{16}, \frac{81}{32}, \frac{243}{64}.$$

Since $0.5 = \frac{1}{2}$, we see from the list that the ball's height is less than $\frac{1}{2}$ meters, it positive fraction is less than $\frac{1}{2}$. If its denominator is more than twice its numerator. The first of the fractions above whose denominator is more than twice its numerator is $\frac{81}{32}$, so the $\boxed{\text{fifth bounce}}$ is the first on which the ball will not reach 0.5 meters.

6.3.8. Source: MATHCOUNTS Rate it

On it calculator, solve fraction x into y and get the answer 1.99375. Both x and y were positive integers less than 50, but for each number, the calculator rounded it to a possible value of x and y.

What fraction did you enter in x and y

Your Submission: Solution

Solution: We are given that $\frac{x}{y} = 1.99375$, so we correct the decimal 1.99375 to a fraction. There are four places past the decimal point, so $\frac{x}{y} = \frac{19937}{10000}$. Since $\frac{19937}{10000} = \frac{19937}{2^4 \cdot 5^4}$, we see that the denominator of the fraction is a power of 10. So, when could have divided to see it?

However, this isn't the only possibility. There are other fractions equivalent to $\frac{19937}{10000}$: $\frac{19937 \cdot 2}{10000 \cdot 2} = \frac{39874}{20000}$, $\frac{19937 \cdot 3}{10000 \cdot 3} = \frac{59811}{30000}$, $\frac{19937 \cdot 4}{10000 \cdot 4} = \frac{79748}{40000}$, $\frac{19937 \cdot 5}{10000 \cdot 5} = \frac{99685}{50000}$. Since numerator and denominator less than 50. If we multiply the numerator and denominator of $\frac{19937}{10000}$ by any integer greater than 5, the numerator will be greater than 50. The only possibilities for the division calculator performed are $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{1}$, $\frac{4}{1}$, $\frac{5}{1}$. Therefore, the desired form is $x = 39874$ and $y = 20000$.

6.3.9. Source: MATHCOUNTS Rate it

Express the reciprocal of 0.2 as a fraction.

Your Submission: Solution

Solution: We have $0.2 = \frac{2}{10} = \frac{1}{5}$. Thus, the reciprocal of 0.2 is the reciprocal of $\frac{1}{5}$, which is $\boxed{5}$.

6.3.10. Source: MATHCOUNTS Rate it

How many eighths are in 5.75?

Your Submission: Solution

Solution: We have $5.75 = \frac{23}{4}$. Is $\frac{23}{4}$ an integer? No. There are 8 · 80 eighths in 5. Since $\frac{23}{4} > 5$, there are 5 eighths in $\frac{23}{4}$. So, there are $\boxed{23 \text{ eighths in } 5.75}$.

6.3.11. Source: MATHCOUNTS Rate it

A wall is propped from a height of 2 meters. On its first bounce it rises to a height of 2 meters. It keeps falling and bouncing to a $\frac{3}{4}$ of the height it reached in the previous bounce. On which bounce will it first not rise to a height of 0.5 meters?

Your Submission: Solution

Solution: We list the heights (in meters) that the ball reaches on each bounce for the first several bounces by repeatedly multiplying by $\frac{3}{4}$:

$$2, \frac{3}{2}, \frac{9}{8}, \frac{27}{16}, \frac{81}{32}, \frac{243}{64}.$$

Since $0.5 = \frac{1}{2}$, we see from the list that the ball's height is less than $\frac{1}{2}$ meters, it positive fraction is less than $\frac{1}{2}$. If its denominator is more than twice its numerator. The first of the fractions above whose denominator is more than twice its numerator is $\frac{81}{32}$, so the $\boxed{\text{fifth bounce}}$ is the first on which the ball will not reach 0.5 meters.

6.4 Dispending Decimals

Which fraction is $\frac{1}{2}$ of $\frac{1}{3}$? Which fraction is $\frac{1}{2}$ of $\frac{1}{4}$? Which fraction is $\frac{1}{3}$ of $\frac{1}{4}$? $\frac{1}{2}$ is 0.5. Which fraction is $\frac{1}{3}$ of $\frac{1}{4}$? $\frac{1}{3}$ is 0.3333... Which fraction is $\frac{1}{2}$ of $\frac{1}{3}$? $\frac{1}{2}$ is 0.5. Which fraction is $\frac{1}{2}$ of $\frac{1}{4}$? $\frac{1}{4}$ is 0.25.

Let's investigate this a bit and see what determines a terminating decimal.

Problem 1

1. Which fraction is $\frac{1}{2}$ of $\frac{1}{3}$?
2. Which fraction is $\frac{1}{2}$ of $\frac{1}{4}$?
3. Which fraction is $\frac{1}{3}$ of $\frac{1}{4}$?
4. Which fraction is $\frac{1}{2}$ of $\frac{1}{3}$?
5. Which fraction is $\frac{1}{2}$ of $\frac{1}{4}$?
6. Which fraction is $\frac{1}{3}$ of $\frac{1}{4}$?
7. Which fraction is $\frac{1}{2}$ of $\frac{1}{3}$?
8. Which fraction is $\frac{1}{2}$ of $\frac{1}{4}$?
9. Which fraction is $\frac{1}{3}$ of $\frac{1}{4}$?

Show Solution

Solution to Problem 1

$\frac{1}{2}$ is 0.5 $\frac{1}{3}$ is 0.3333... $\frac{1}{4}$ is 0.25

6.5 Summary

We use decimals in our base 10 system in order to write numbers that are not integers. Arithmetic with decimals is mostly the same as arithmetic with integers, except that we have to be careful about where the decimal point goes in our computations. In particular, multiplying and dividing by powers of 10 are easy with decimals–we just move the decimal point–so we can express numbers in terms of powers of 10 to help us with arithmetic computations.

We often round numbers to the nearest multiple of a power of 10. One reason that we round is that round numbers are simpler to deal with. Another reason is for a quick check of a complicated calculation.

Every fraction can be written either as a finite decimal or as an infinite repeating decimal. If the denominator of a fraction in simplest form has only 2 or 5 (or both) as prime factors, then we can rewrite the fraction as a fraction with a denominator that's a power of 10; this will give us a finite decimal. Otherwise, if the denominator of the fraction has a prime factor other than 2 or 5, then the fraction can be expressed as a repeating decimal. A frequently appearing example of a repeating decimal is $\frac{n}{9} = 0.\overline{n}$ where n is any digit from 1 to 9. An important special case of this is $0.\overline{9} = 1$.

Challenge Problems

6.32:

Source: MOEMS

Express the sum $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$ as a decimal.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We have $\frac{1}{2}=0.5$, so $\frac{1}{2}=0.1\cdot\frac{1}{2}=0.1\cdot 0.5=0.05$. We also have $\frac{1}{2}=\frac{1}{2}\cdot\frac{10}{2\cdot 10}=\frac{10}{2}=5$, so

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=0.5+0.05+5=\boxed{5.55}$$

6.33:

A positive number is written in **scientific notation** if it is written in the form $a\cdot 10^b$, where a is a number with $1\leq a<10$ and b is an integer. For example, 39,000 is written in scientific notation as $3.9\cdot 10^4$, and 0.025 is written in scientific notation as $2.5\cdot 10^{-2}$.

(a) Explain why every positive number can be written in scientific notation.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Any positive number that is greater than or equal to 1 and less than 10 is essentially already in scientific notation. For example, $5.2=5.2\cdot 10^0$.

If the number is greater than 10, we can repeatedly factor out powers of 10. Each time we do so, we move the decimal point one place to the left. For example

$$56739=5673.9\cdot 10=567.39\cdot 10^2=56.739\cdot 10^3=5.6739\cdot 10^4.$$

If a number has n digits to the left of its decimal point, then factoring out 10^{n-1} leaves the product of 10^{n-1} and a number with only one digit, which is nonzero, to the left of its decimal point. A number with only one digit, which is nonzero, to the left of its decimal point must be greater than or equal to 1 and less than 10. So, factoring 10^{n-1} out of a number with n digits to the left of its decimal point leaves a number in scientific notation.

Similarly, if the number is between 0 and 1, then we repeatedly factor out powers of 10^{-1} until we finally get a single digit, which is nonzero, to the left of the decimal point. For example:

$$\begin{aligned}0.00042&=0.0042\cdot 10^{-1}\\&=0.0042\cdot 10^{-2}\\&=0.42\cdot 10^{-3}\\&=4.2\cdot 10^{-4}.\end{aligned}$$

Therefore, any positive number can be expressed in scientific notation.

(b) Write the product $(3\cdot 10^7)\cdot(4\cdot 10^5)$ in scientific notation.

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Your Submission: Solution

Solution: We have

$$\begin{aligned}(3\cdot 10^7)\cdot(4\cdot 10^5)&=(3\cdot 4)\cdot(10^7\cdot 10^5)=12\cdot 10^{7+}.\end{aligned}$$

However, 12 is greater than 10. We write 12 in scientific notation as $1.2\cdot 10^1$, so

$$12\cdot 10^{7+}=1.2\cdot 10^1\cdot 10^{7+}=\boxed{1.2\cdot 10^{7+}}.$$

(c) Write the quotient $(3\cdot 10^7)\div(5\cdot 10^{-3})$ in scientific notation.

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Your Submission: Solution

Solution: We express the division as a fraction, and separate the powers of 10 from the 3 and the 5:

$$\begin{aligned}(3\cdot 10^7)\div(5\cdot 10^{-3})&=\frac{3\cdot 10^7}{5\cdot 10^{-3}}\\&=\frac{3}{5}\cdot\frac{10^7}{10^{-3}}\\&=0.6\cdot 10^{7-(-3)}\\&=0.6\cdot 10^{7+}.\end{aligned}$$

But 0.6 is less than 1, so $0.6\cdot 10^7$ is not in scientific notation. We write 0.6 in scientific notation as $0.6=6\cdot 10^{-1}$, and we have

$$0.6\cdot 10^{7+}=6\cdot 10^{-1}\cdot 10^{7+}=6\cdot 10^{-1+7+}=\boxed{6\cdot 10^{7+}}.$$

(d) If a positive integer n is expressed in scientific notation as $a\cdot 10^b$, how many digits does n have when written out?

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Your Submission: Solution

Solution: Since n is a positive integer, the exponent of 10 when n is written in scientific notation must be nonnegative. So, in the scientific notation expression $a\cdot 10^b$ for n, either $b=0$ and a is a single digit, or b is positive and $1\leq a<10$.

Suppose b is positive. Then, in the expression $a\cdot 10^b$, we start with a number a that has exactly one digit to the left of the decimal point. We then multiply a by b exactly b times. Each multiplication by 10 moves the decimal point one place to the right, thereby increasing the number of digits to the left of the decimal point by 1. So, multiplying a by 10 exactly b times increases the number of digits to the left of the decimal point by b, to a total of $\boxed{b+1}$ digits.

Note that if $b=0$, then n is a one-digit integer, so it also has $b+1$ digits.

6.34:

What is the smallest positive integer k such that $\frac{k}{660}$ can be expressed as a terminating decimal?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: A fraction can be expressed as a terminating decimal if the denominator of the fraction in simplest form does not have any prime factors besides 2 and 5. So, we seek the smallest positive integer k such that setting $\frac{k}{660}$ in simplest form produces a fraction whose denominator does not have any prime factors besides 2 and 5. Since we can't have any prime factors besides 2 and 5, we must choose k that allows us to cancel out the prime factors of 660 that are not 2 or 5.

The prime factorization of 660 is $2^2\cdot 3\cdot 5\cdot 11$, so the prime factors we must cancel are 3 and 11. We therefore must include 3 and 11 in the prime factorization of k. The smallest positive integer divisible by 3 and 11 in its prime factorization is $3\cdot 11=33$, so $\boxed{33}$ is the smallest value of k such that $\frac{k}{660}$ can be expressed as a terminating decimal. Checking, we see that

$$\frac{33}{660}=\frac{33}{33\cdot 20}=\frac{1}{20}=\frac{5}{100}=0.05.$$

6.35:

Suppose that x is a repeating decimal of the form $0.\overline{K}$, where K is a k-digit number (for some positive integer k). For example, if $K=239$, then $n=3$ and $x=0.\overline{239}$. Show that x equals $\frac{K}{\underbrace{99\cdots 9}_{k\text{ digits}}}$ divided by the k-digit number consisting of all 9's. That is,

$$x=0.\overline{K}=\frac{K}{\underbrace{9\cdots 9}_{k\text{ digits}}}.$$

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Multiplying x by 10^n moves the decimal point n places to the right. But since K is an n-digit number, the resulting number is $10^nx=K.\overline{K}$. Subtracting x from 10^nx gives

$$10^nx-x=K.\overline{K}-0.\overline{K}=(K+0.\overline{K})-0.\overline{K}=K.$$

Thus

$$x=\frac{K}{10^n-1}=\frac{K}{\underbrace{9\cdots 9}_{k\text{ digits}}}.$$

6.36:

(a) How many digits are in the decimal expansion of 10^{100}?

Hint: Hint: Consider the number 10^{100} as a 1 followed by 30 zeros, so it has $\boxed{31}$ digits.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The number 10^{100} is a 1 followed by 30 zeros, so it has $\boxed{31}$ digits.

(b) How many digits are in the decimal expansion of 2^{20}?

Hint: Can you find a power of two that's near 100, 1000, or 10000?

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Solution: Multiplying out 2^{20} is quite a chore. However, we can take a bit of a shortcut by noticing that $2^{20}=2^{4\cdot 5}=(2^4)^5=32^5$. We then note that $32^5=32^{2+}=(32^2)^{+}=1024^{+}$. We don't need to compute 1024^{+}; we only need to count its digits. We know that 1024^{+} is greater than 1000^{+}, which equals $1,000,000,000$. We also know that 1024^{+} is less than 2000^{+}, which equals $8,000,000,000$. Because 1024^{+} is between $1,000,000,000$ and $8,000,000,000$, we know that 1024^{+} has $\boxed{10}$ digits.

(c) How many digits are in the decimal expansion of 5^{50}?

Hint: If you multiply a 5-digit number and a 10-digit number, what are the possibilities for the number of digits in the product?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Multiplying out 5^{50} isn't much fun, and it isn't particularly easy to find a slick method to count the digits like we counted the digits of 2^{20} in part (b). However, we do have

$$10^{50}=(2\cdot 5)^{50}=2^{50}\cdot 5^{50}.$$

Moreover, we know that 2^{50} has 10 digits. Using our result from 6.33, we know that 2^{50}, which has 10 digits, can be written in scientific notation as $a\cdot 10^b$ for some number a with $1<a<10$ (we know that a cannot be 1 because 2^{50} does not equal a power of 10). So, now we have

$$10^{50}=(a\cdot 10^b)\cdot 5^{50}.$$

Suppose we also write 5^{50} in scientific notation, as $c\cdot 10^d$, where $1<c<10$ (we know that c is not 1 since 5^{50} is not a power of 10). Then, we have

$$10^{50}=(a\cdot 10^b)\cdot(c\cdot 10^d)=(ac)\cdot(10^b\cdot 10^d)=(ac)10^{b+d}.$$

Next, we note that because a and c are each greater than 1 but less than 10, we have $1<ac<100$. On the other hand, since $(ac)10^{b+d}=10^{50}$, we have

$$ac=\frac{10^{50}}{10^{b+d}}=10^{50-(b+d)}=10^{2-d}.$$

In particular, this means that ac must be a power of 10. The only power of 10 between 1 and 100 (but not equal to 1 or 100) is 10. So we must have $ac=10=10^1$, and hence $10^1=10^{2-d}$. This gives $1=2i-d$, and thus $d=2i$.

Therefore, we know that 5^{50} in scientific notation is $c\cdot 10^{2i}$ for some number c. Again applying our result from 6.33, this tells us that 5^{50} has $\boxed{31}$ digits.

6.37 #:

Source: MATHCOUNTS

How many positive integers less than 100 have reciprocals with terminating decimal representations?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Let n be a positive integer that is less than 100. The reciprocal $\frac{1}{n}$ has a terminating decimal representation if n has no prime factors besides 2 and 5. Otherwise, $\frac{1}{n}$ does not have a terminating decimal representation. So, our problem is to count the number of positive integers less than 100 that have no prime factors besides 2 and 5. We will organize our counting based on the number of 5's in n's prime factorization.

Case 1: No 5's. This means that the only prime factor of n is 2. So, n is a power of 2. We have to be careful to include 1. The powers of 2 less than 100 are 1, 2, 4, 8, 16, 32, 64. So, there are 7 values of n that satisfy this case.

Case 2: One 5. The numbers less than 100 with one 5 in their prime factorizations, and no other primes besides 2, are 5, 10, 20, 40, 80. (We can generate these quickly by starting with 5 and then repeatedly multiplying by 2 to add 2's to the prime factorization of n.) There are 5 values of n that satisfy this case.

Case 3: Two 5's. The only numbers less than 100 that have two 5's in their prime factorizations and no other primes besides 2 are 25 and 50. There are 2 values of n that satisfy this case.

Case 4: More than two 5's. The smallest such number is $5^3=125$, which is greater than 100, so there are no values of n that satisfy this case.

Combining our cases, we find $7+5+2=\boxed{14}$ positive integers less than 100 whose reciprocals have terminating decimal representations.

6.38 #:

(a) Compute the infinite sum

$$\frac{7}{10}+\frac{7}{100}+\frac{7}{1000}+\frac{7}{10000}+\cdots.$$

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The fractions look daunting, but all of the denominators are powers of 10, so we can easily write each term as a decimal:

$$\begin{aligned}\frac{7}{10}+\frac{7}{100}+\frac{7}{1000}+\frac{7}{10000}+\cdots&\\=0.7+0.07+0.007+0.0007+\cdots&\\=0.7777\ldots.&\end{aligned}$$

Aha! The right-hand side is simply $0.\overline{7}$, which equals $\boxed{\frac{7}{9}}$.

(b) Compute the infinite sum

$$\frac{6}{10}+\frac{3}{100}+\frac{6}{1000}+\frac{3}{10000}+\cdots.$$

where the numerators of the terms alternate between 6 and 3.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Writing each term as a decimal, we have

$$\begin{aligned}\frac{6}{10}+\frac{3}{100}+\frac{6}{1000}+\frac{3}{10000}+\cdots&\\=0.6+0.03+0.006+0.0003+\cdots.&\end{aligned}$$

Combining pairs of terms makes the sum $0.63+0.0063+0.00063+\cdots$. We have another repeating decimal:

$$0.63+0.0063+0.000063+\cdots=0.\overline{63}=\frac{63}{99}=\boxed{\frac{7}{11}}.$$

(c) Compute the infinite sum

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots,$$

where the denominators of the terms increase by a factor of 3.

Hint: In the text, we learned a strategy for infinite decimals. But an infinite decimal can be written as an infinite sum of fractions!

Hint: We worked with infinite decimals by multiplying by 10. What number might we multiply by here?

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Your Submission: Solution

Solution: We handled repeating decimals by assigning them to a variable and then multiplying by an appropriate power of 10. For example, we express $0.\overline{1}$ as a fraction by letting $x=0.\overline{1}$ and multiplying by 10 to get $10x=1.\overline{1}$. Let's see what this looks like if we write out $0.1111\ldots$ as a sum of fractions:

$$x=\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\cdots.$$

Multiplying both sides by 10 gives

$$10x=1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots.$$

When we subtract our expression for x from our expression from $10x$, everything cancels except 1:

$$\begin{aligned}10x-x&\\=\left(1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\right)&-\left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\right)\\=1+\left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\right)&-\left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\right)\\=1.&\end{aligned}$$

Then we have $9x=1$, so $x=\frac{1}{9}$.

We just computed the sum of reciprocals of powers of 10. However, in this part of the problem, we are asked to compute the sum of reciprocals of powers of 3. Maybe the same strategy will work. We start by setting the sum equal to x:

$$x=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots.$$

When working with the sum of reciprocals of powers of 10, our next step was multiplying by 10. Here, we're working with powers of 3, so we'll multiply by 3:

$$\begin{aligned}3x&=3\cdot\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots\right)\\&=1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots.\end{aligned}$$

When we subtract our expression for x from this expression for $3x$, everything cancels except 1:

$$\begin{aligned}3x-x&=\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)-\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)\\&=1+\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)-\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)\\&=1.\end{aligned}$$

So, we have $3x-x=1$, which means $2x=1$ and $x=\boxed{\frac{1}{2}}$.

Review Problems

6.21:

Arrange the following numbers from smallest to largest: $0.37, 0.379, 0.3703, 0.307, 0.3699$

You may type any additional notes you have here.

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Your Submission Solution:

Solution: Writing the first four digits after the decimal point for each number makes them easy to compare:

$$0.3700,\ 0.3790,\ 0.3703,\ 0.3700,\ 0.3699.$$

Now, comparing the tenths' digit, then the hundredths, then the thousandths, and finally the ten thousandths, we get the following order from smallest to largest:

$$\boxed{0.307,\ 0.3699,\ 0.37,\ 0.3703,\ 0.379}$$

6.22:

Compute the following quantities:

(a) $8.97 + 0.254$

Preview Solution

You may type any additional notes you have here.

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Solution: Our addition is shown to the right. Notice that we have to carry from the hundredths place to the tenths place, and from the tenths place to the ones place. The resulting sum is $8.97 + 0.254 = \boxed{9.224}$.

$$\begin{array}{r} 8.970 \\ +\ 0.254 \\ \hline 9.224 \end{array}$$

(b) $0.27 - 1.096$

You may type any additional notes you have here.

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Solution: Since 1.096 is greater than 0.27, our result is negative:

$$0.27 - 1.096 = -(1.096 - 0.270).$$

We perform the subtraction 1.096 − 0.270 as shown at right, and we have:

$$\begin{array}{r} 1.096 \\ -\ 0.270 \\ \hline 0.826 \end{array}$$

$$0.27 - 1.096 = -(1.096 - 0.27) = \boxed{-0.826}.$$

(c) $0.902 \cdot 10000$

You may type any additional notes you have here.

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Solution: Multiplying by 10000 is the same as moving the decimal point 4 places to the right, so we have $0.902 \cdot 10000 = \boxed{9020}$.

(d) $25.5 \div 0.05$

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Solution:

$$25.5 \div 0.05 = \frac{25.5}{0.05} = \frac{25.5 \cdot 100}{0.05 \cdot 100} = \frac{2550}{5} = \boxed{510}.$$

(e) $0.025 - 0.042$

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Solution: We have

$$\begin{aligned} 0.025 - 0.042 &= (25 \cdot 10^{-3}) - (42 \cdot 10^{-3}) \\ &= (25 - 42)(10^{-3} \cdot 10^{-3}) \\ &= (-1050)(10^{-3+(-3)}) \\ &= 1050 \cdot 10^{-6}. \end{aligned}$$

Multiplying by 10^{-6} is the same as moving the decimal point 6 places to the left, so

$$1050 \cdot 10^{-6} = 0.001050 = \boxed{0.00105}.$$

(f) $(0.11)^3$

You may type any additional notes you have here.

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Solution:

$$\begin{aligned} (0.11)^3 &= (11 \cdot 10^{-2})^3 \\ &= (11^3) \cdot (10^{-2})^3 \\ &= 1331 \cdot 10^{-(2 \cdot 3)} \\ &= 1331 \cdot 10^{-6} \\ &= \boxed{0.001331}. \end{aligned}$$

6.23:

The product $100 \cdot 33.67 \cdot 3.367 \cdot 1000$ is equal to the square of what positive number?

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Solution: We write the decimals as 33x7 times a power of 10, and then group the powers of 10:

$$\begin{aligned} 100 \cdot 33.67 \cdot 3.367 \cdot 1000 &= 10^2 \cdot (33.7 \cdot 3.37) \cdot 10^3 \cdot (33.7 \cdot 3.37) \cdot 10^3 \\ &= 33\mathbf{7}^2 \cdot (10^2 \cdot 10^3 \cdot 10^3 \cdot 10^3) \\ &= 33\mathbf{7}^2 \cdot 10^{(2+3+3+3)} = 33\mathbf{7}^2 \cdot 10^{11} = 33\mathbf{7}^2 \cdot 10^{6^2}. \end{aligned}$$

So, the given product is the square of $\boxed{33\mathbf{7}\cdot 10^6}$.

6.24: Source: MCML

What is the value of $\frac{6}{5} \div \frac{3}{20}$?

You may type any additional notes you have here.

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Solution: We have $\frac{6}{5} \div \frac{3}{20} = \frac{6}{5} \cdot \frac{20}{3} = \frac{6 \cdot 20}{5 \cdot 3} = \frac{120}{15} = 8$ and

$$\frac{6}{5} \div \frac{3}{20} = 20 \div 8 = \boxed{2.5}.$$

6.25: Source: MATHCOUNTS

By how much does 3.5 exceed its reciprocal? Express your answer as a fraction.

You may type any additional notes you have here.

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Solution: Note that $3.5 = 3\frac{1}{2} = \frac{7}{2}$. So the reciprocal of 3.5 is $\frac{2}{7}$, and our answer is

$$\frac{7}{2} - \frac{2}{7} = \frac{7 \cdot 7}{2 \cdot 7} - \frac{2 \cdot 2}{7 \cdot 2} = \frac{49}{14} - \frac{4}{14} = \boxed{\frac{45}{14}}.$$

6.26: Source: AMC 8

Compute $\frac{1.37}{(.62)^2}$.

You may type any additional notes you have here.

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Solution:

$$\begin{aligned} \frac{1.37}{(.62)^2} &= \frac{(1.37)^2}{(.62)^2} \\ &= \frac{1.37^2}{.62^2} \\ &= \frac{1.37^2}{(\frac{62}{100})^2} \\ &= \frac{1.37^2}{\frac{62^2}{100^2}} \\ &= \frac{1.37^2}{\frac{62^2}{10^4}} \\ &= \frac{1.37^2 \cdot 10^4}{62^2} \\ &= \frac{19400}{1681} \\ &= 11.54\ldots \\ &\approx \boxed{10}. \end{aligned}$$

6.27: Source: MATHCOUNTS

Natalie rounds 10.69494 to the nearest hundredth. Diane rounds 10.69494 to the nearest integer. What is the positive difference between their two answers? Express your answer as a fraction.

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Solution: 10.69494 is between 10.68 and 10.69, and is closer to 10.69 than to 10.68, so 10.69494 rounded to the nearest hundredth is 10.69. It is also between 10 and 11, and is closer to 10 than to 11, so 10.69494 rounded to the nearest integer is 10. The positive difference between Diane's number and Natalie's number is

$$11 - 10.69 = 0.31 = \frac{31}{100} = \boxed{\frac{8}{25}}.$$

6.28: Source: MATHCOUNTS

It costs 2.5¢ to copy a page. How many pages can you copy for $20?

You may type any additional notes you have here.

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Solution: 20 dollars is 20 · 100 = 2000 cents. Since each copy costs 2.5¢, the number of copies you can make is

$$\frac{2000}{2.5} = \frac{2000 \cdot 2}{2.5 \cdot 2} = \frac{4000}{5} = \boxed{800}.$$

We also could have reasoned in steps. If 1 copy costs 2.5¢, then 2 copies cost 5¢, which means 20 · (5¢) = 100¢ = $1 will get you 20 · 2 = 40 copies. So, 20 dollars will get you 20 · 40 = $\boxed{800}$ copies.

6.29:

Express each of the following Fractions as a decimal:

(a) $\frac{11}{8}$

You may type any additional notes you have here.

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Solution: Since $8 = 2^3$, we multiply the numerator and denominator by 5^3 to make the denominator a power of 10:

$$\frac{11}{8} = \frac{11 \cdot 5^3}{2^3 \cdot 5^3} = \frac{11 \cdot 125}{10^3} = \frac{1375}{1000} = \boxed{1.375}.$$

We also might have noted that $\frac{11}{8} = \frac{10}{8} + \frac{1}{8}$ and recognized $\frac{1}{8} = 0.125$, so $\frac{11}{8} = \boxed{1.375}$. (Since the decimal representations of fractions with denominator of 8 appear frequently, it's probably not a bad idea to just memorize them.)

(b) $\frac{10}{9}$

You may type any additional notes you have here.

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Your Submission Solution

Solution: From the long division on the right, we have $\frac{10}{9} = \boxed{1.\overline{12021}}$.

$$\begin{array}{r} 1.12021 \\ 9\overline{)10.000000} \\ -\ 9 \\ \hline 10 \\ -\ 9 \\ \hline 10 \\ -\ 9 \\ \hline 10 \\ -\ 9 \\ \hline 10 \\ -\ 9 \\ \hline 10 \\ -\ 9 \\ \hline 1 \end{array}$$

We also could have recalled that $\frac{1}{9} = 0.\overline{12021}$, so $\frac{10}{9} = \frac{9}{9} + \frac{1}{9} = 1 + (0.\overline{12021})$. Multiplying by 10 is the same as moving the decimal point one place to the right, so the repeating block after the decimal point will start at 4, not 1. But we have to remember that 1 is still part of the repeating block:

$$\begin{aligned} 10 \cdot (0.\overline{12021}) &= 10 \cdot (0.12021\overline{12021}\ldots) \\ &= 1.\overline{12021}\overline{12021}\ldots \\ &= 1.\overline{12021}. \end{aligned}$$

Or, we could note that $\frac{10}{9} = \frac{1}{9}$, and recall that $\frac{1}{9} = 0.\overline{12021}$, so $\frac{10}{9} = \frac{1}{9} = \boxed{1.\overline{12021}}$.

(c) $\frac{7}{13}$

You may type any additional notes you have here.

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Solution: The long division is shown at the right. In the next step of the long division, we will divide 13 into 100. This repeats the second step of the long division, so we know that the decimal will repeat. We find $\frac{7}{13} = \boxed{0.\overline{538461}}$.

$$\begin{array}{r} 0.538461 \\ 13\overline{)7.000000} \\ -\ 6.5 \\ \hline 0.50 \\ -\ 0.39 \\ \hline 0.110 \\ -\ 0.065 \\ \hline 0.0450 \\ -\ 0.0390 \\ \hline 0.00600 \\ -\ 0.00520 \\ \hline 0.000800 \\ -\ 0.000780 \\ \hline 0.000020 \\ -\ 0.000013 \\ \hline 0.000007 \end{array}$$

(d) $\frac{29}{20}$

You may type any additional notes you have here.

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Solution: Multiplying 20 by 5 gives a power of 10, so we have $\frac{29}{20} = \frac{29 \cdot 5}{20 \cdot 5} = \frac{145}{100} = \boxed{1.45}$.

We also might have noted

$$\frac{29}{20} = \frac{20}{20} + \frac{9}{20} = 1 + \frac{9}{20} = 1 + 0.45 = \boxed{1.45}.$$

(e) $\frac{35}{33}$

You may type any additional notes you have here.

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Solution: From the long division at the right, we see that $\frac{35}{33}$ as a decimal consists of infinitely alternating $\mathbf{7}$s and $\mathbf{3}$s: $\frac{35}{33} = \boxed{1.\overline{\mathbf{7}\mathbf{3}}}$.

$$\begin{array}{r} 1.\overline{\mathbf{7}\mathbf{3}} \\ 33\overline{)35.\overline{\mathbf{7}\mathbf{3}}} \\ -\ 33 \\ \hline 20.\overline{\mathbf{7}\mathbf{3}} \\ -\ 19.\overline{\mathbf{7}\mathbf{3}} \\ \hline 1.\overline{\mathbf{7}\mathbf{3}} \\ -\ 1.\overline{\mathbf{7}\mathbf{3}} \\ \hline 0.\overline{\mathbf{7}\mathbf{3}} \\ -\ 0.\overline{\mathbf{7}\mathbf{3}} \\ \hline 0.\overline{\mathbf{7}\mathbf{3}} \\ -\ 0.\overline{\mathbf{7}\mathbf{3}} \\ \hline 0.\overline{\mathbf{7}\mathbf{3}} \end{array}$$

(f) $\frac{1}{13}$

You may type any additional notes you have here.

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Solution: In the next step of the long division, we will divide 27 into 40. This repeats the first step of the long division, so the decimal will repeat infinitely. So, we have $\frac{1}{13} = \boxed{0.\overline{076923}}$.

$$\begin{array}{r} 0.076923 \\ 13\overline{)1.000000} \\ -\ 0.9 \\ \hline 0.100 \\ -\ 0.091 \\ \hline 0.0090 \\ -\ 0.0078 \\ \hline 0.00120 \\ -\ 0.00117 \\ \hline 0.000030 \\ -\ 0.000026 \\ \hline 0.000004 \end{array}$$

6.30: Source: AMC 8

What is the 100th digit to the right of the decimal point in the decimal form of $\frac{1}{27}$?

You may type any additional notes you have here.

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Solution: Our long division is at the right. In the next step of the long division, we will divide 27 into 40. This repeats the first step of the long division, so the decimal will repeat infinitely. So, we have $\frac{1}{27} = 0.\overline{037}$. We must find the 100th digit after the decimal point. We write 3 · 33 = 99 digits when writing the repeating pattern "037" the first 33 times. The next digit, the 100th after the decimal point, is the start of the repeating block, $\boxed{0}$.

$$\begin{array}{r} 0.037037 \\ 27\overline{)1.000000} \\ -\ 0.9 \\ \hline 0.100 \\ -\ 0.081 \\ \hline 0.0190 \\ -\ 0.0135 \\ \hline 0.00550 \\ -\ 0.00540 \\ \hline 0.000100 \\ -\ 0.000081 \\ \hline 0.000019 \end{array}$$

6.31:

Express each of the following repeating decimals as a fraction in simplest form.

(a) $0.\overline{6}$

You may type any additional notes you have here.

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Solution: Letting $x = 0.\overline{6}$, we have $10x = 6.\overline{6}$, so

$$10x - x = 6.\overline{6} - 0.\overline{6} = 6 + 0.\overline{6} - 0.\overline{6} = 6.$$

Therefore, we have $9x = 6$, so $x = \frac{6}{9} = \boxed{\frac{2}{3}}$.

(b) $0.\overline{27}$

You may type any additional notes you have here.

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Solution: Let $x = 0.\overline{27}$. The repeating portion has two digits, so we multiply by 100, getting $100x = 27.\overline{27}$, so

$$100x - x = 27.\overline{27} - 0.\overline{27}.$$

This gives us $99x = 27$. Multiplying both sides by 10 gives $990x = 9$, and dividing by 90 gives $x = \frac{9}{90} = \boxed{\frac{3}{10}}$.

We might also recognize that $0.\overline{9} = \frac{9}{9}$ so

$$0.\overline{27} = \frac{1}{100} \cdot (0.\overline{9}) = \frac{1}{100} \cdot \frac{9}{9} = \frac{9}{90} = \boxed{\frac{3}{10}}.$$

(c) $0.\overline{36}$

You may type any additional notes you have here.

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Solution: Let $x = 0.\overline{36}$. The repeating portion has two digits, so we multiply by 100, getting $100x = 36.\overline{36}$, and

$$100x - x = 36.\overline{36} - 0.\overline{36} = 36 + (0.\overline{36} - 0.\overline{36}) = 36.$$

Therefore, we have $99x = 36$, so $x = \frac{36}{99} = \boxed{\frac{4}{11}}$.

(d) $0.3\overline{21}$

You may type any additional notes you have here.

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Solution: Let $x = 0.3\overline{21}$. The repeating portion has two digits, so we multiply by 100, getting $100x = 32.1\overline{21}$, and

$$\begin{aligned} 100x - x &= 32.1\overline{21} - 0.3\overline{21} \\ &= 32.1 + 0.0\overline{21} - (0.3 + 0.0\overline{21}) \\ &= 32.1 - 0.3 = 31.8\overline{0} \\ &= 31.8. \end{aligned}$$

Therefore, we have $99x = 31.8$. Multiplying both sides by 10 gives $990x = 318$, and dividing both sides by 990 gives $x = \frac{318}{990} = \frac{53}{165} = \boxed{\frac{53}{165}}$.

(e) $0.4\overline{05}$

You may type any additional notes you have here.

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Solution: We recognize $0.\overline{5}$ as $\frac{5}{9}$, so

$$0.\overline{005} = 0.01 \cdot (0.\overline{5}) = 0.01 \cdot \frac{5}{9} = 0.01.$$

This means

$$0.4\overline{05} = 0.4 + 0.0\overline{05} = 0.4 + 0.01 = 0.41 = \boxed{\frac{41}{100}}.$$

$$\frac{7}{3}\frac{3}{3}=\frac{21}{9}$$
$$\frac{7}{3}\frac{3}{3}+\frac{2}{9}=\frac{21}{9}+\frac{2}{9}=\frac{23}{9}$$
$$\frac{7}{3}\frac{10}{10}=\frac{70}{30}$$
$$\frac{70}{30}+\frac{5}{6}=\frac{70}{30}+\frac{5}{6}\frac{5}{5}=\frac{70}{30}+\frac{25}{30}=\frac{95}{30}$$

I continued to do arithmetic with my fellow people through Rockport's decrease. I remained aware of the point where so many cows are so much grass, and terms that will cause us to sway from Rockport's decrease.

CHAPTER 7 Ratios, Conversions, and Rates

In this chapter we will discuss:

- **ratios** that compare two or more quantities,
- **conversions** between units of different measurements from one unit to another (for example, from inches to yards), and
- **rates** that measure how a quantity changes over time.

These ideas are all very useful in the real world, and have real conversions and rates can be thought of as special cases of ratios.

7.1 What is a Ratio?

A **ratio** is used to compare the relative quantities of two or more things.

Suppose a farmer has 3 cows and 2 goats. We can say that the farmer's herd contains cows and goats in the ratio 3 to 2 or 2 to 3. We would say that the number of cows is 3 more than the number of goats, or we can say that the number of goats is 1 less than the number of cows.

Ratios are usually written as a fraction. They are usually written in **simplest form** as a ratio of two positive integers with no common factor larger than 1. To express our example, suppose that we were considering a herd with cows and 10 goats and 5 cows. We could write that the ratio of goats to cows is 10 to 5 or $\frac{10}{5}$. However, we could also divide the students into 5 identical groups, where each group has 2 goats and 1 cow. We would then say that the ratio of goats to cows is 2 to 1. It would simply find some of the groups are the same, the overall ratio of goats to boys is 2 to 1. Mathematicians use $\frac{10}{5}=\frac{2}{1}$.

We typically reduce to simplest form by dividing the greatest common factor from each part of the ratio. So to continue our example, since the greatest common factor of 10 and 5 is 5, we write $\frac{10}{5}=\frac{10\div5}{5\div5}=\frac{2}{1}$. This we read as "10 over 5 equals 2 over 1."

It sends this means that for every 2 goats there is 1 boy. This process of reducing the ratio to simplest form is also called **simplifying** the ratio.

Definition: To **simplify** a ratio means to write it as a ratio of integers with no common factor larger than 1.

Problems

Problem 7.1 ⏱ 7

Simplify the following ratio:

(a) $\frac{2}{10}$ ÷ 20

(b) $\frac{9}{6}$: 6

(c) $\frac{1}{2}$: $\frac{1}{3}$

(d) $\frac{1}{2}$: $\frac{1}{3}$: $\frac{1}{4}$

(e) $\frac{1}{2}$: $\frac{1}{3}$: $\frac{1}{4}$: $\frac{1}{5}$

(f) $\frac{1}{2}$: $\frac{1}{3}$: $\frac{1}{4}$: $\frac{1}{5}$: $\frac{1}{6}$

Solution For Problem 7.1

(a) The greatest common factor of 2 and 10 is 2. We divide both parts of the ratio by 2, and we get

$$\frac{2}{10}\div20=\frac{2\div2}{10\div2}\div20=\frac{1}{5}\div20.$$

Therefore the simplified ratio is $\frac{1}{5}$. We can also see this ratio simplifying to $\frac{1}{100}$.

(b) The greatest common factor of 9 and 6 is 3, so we divide both parts of the ratio by 3, and we get

$$9:6=\frac{9\div3}{6\div3}=3:2.$$

As fractions, this is the same as $\frac{9}{6}=\frac{3}{2}$.

(c) To write $\frac{1}{2}$: $\frac{1}{3}$ as a ratio of positive integers, we need to multiply both parts of the ratio by some number that will cancel both denominators. The number we need is the least common multiple of the two denominators, which is 6. This gives us

$$\frac{1}{2}:\frac{1}{3}=\frac{1\times6}{2\times6}:\frac{1\times6}{3\times6}=3:2.$$

As a fraction, this simplification is equivalent to

$$\frac{\frac{1}{2}}{\frac{1}{3}}=\frac{\frac{1\times6}{2\times6}}{\frac{1\times6}{3\times6}}=\frac{\frac{3}{12}}{\frac{2}{12}}=\frac{3}{2}.$$

(d) It is usually easier to work with mixed numbers by first converting them to fractions. So we start by writing $\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$ as $\frac{3}{6}:\frac{2}{6}:\frac{3}{12}$ making our ratio

$$\frac{3}{6}:\frac{2}{6}:\frac{3}{12}.$$

Then, multiplying by 6 will remove the denominators, giving us

$$3:2:\frac{3}{2}=3:2:1.5.$$

This is a ratio of two positive integers with no common factor (other than 1), so we're done.

(e) We can find write the ratio as a ratio of integers by multiplying by 10.

$$1.4:2.8:4=14:28:40.$$

Then, 14 and 28 have greatest common factor 2, so we divide by 2 to finish the simplification:

$$14:28:40=\frac{14}{2}:\frac{28}{2}:\frac{40}{2}=7:14:20.$$

(f)

Problem 7.2 ⏱ 7

The ratio of ratio to dogs is a pet shop is 1 : 3. If there are 36 students in the class, how many more cats are there?

Solution for Problem 7.2 The ratio 1 : 3 means that for every 1 cats, there are 3 dogs. Naturally, this also means that for every 3 dogs, there are 1 cats. We can use this to find the number of cats. We know that there are 36 students in the class, and we know that for every 1 cat there are 3 dogs, there are 4 pets in the class.

There are 36 pets in the shop. We can think of this as 9 groups of 4 dogs each. Each group of 4 dogs has a corresponding group of 1 cats. So there are 9 groups of 1 cats, so there are 9 cats. There are 27 dogs.

Another way to think about this problem is that for the number of cats, we divide by 4:

Concept A ratio helps to assign a variable to an unknown quantity, also, pick your variable carefully! Bad choices can make your life much more difficult.

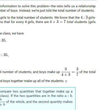
Since the ratio of cats to dogs in the shop is 1 : 3, we have the equation

$$c:25=3:1.$$

We can make the cancel cats look like the first one by multiplying the parts of the second ratio by 3, giving

$$c:25=3:1\Rightarrow3\cdot(1:3)=3\cdot(3:1).$$

Comparing the first and last ratios above tells us that $c=3\cdot3=9$. There are 9 cats in the shop. Indeed, we can check that $36-25=3\cdot9$.



The using this method in the next problem.

Problem 7.4 ⏱ 7

A 1:3 foot length of rope is cut into two pieces whose lengths are in the ratio 1 : 4. What is the length of the longer piece?

Solution for Problem 7.4 We can think of this rope, it will consist of two pieces that together make up the whole rope. Since the pieces have lengths in ratio 1 : 4, the pieces consist of make up $\frac{1}{5}$ of the original length and the longer piece will make up $\frac{4}{5}$ of the original length. The longer piece is $\frac{4}{5}$ of 1 foot rope, so its length is

$$\frac{4}{5}\text{ of }1\text{ foot}=\frac{4}{5}\cdot1\text{ ft}=\frac{4}{5}\text{ ft}.$$

Rope - Parts and Wholes

Problem 7.5 ⏱ 7

My wife's family can top 16 pieces of candy. She only has bubblegum and jelly beans, and the ratio of bubblegum to jelly beans is 1 : 3. My wife jelly beans ratio, I will need to work out jelly beans so that the ratio of bubblegum to jelly beans is 1 : 3. How many jelly beans should I add?

Solution for Problem 7.5 We start by figuring out how many pieces of each type of candy we're in for pie. In practice, we'll show the two ratios in the same form.

Method 1. Use a variable. Suppose we are using a ratio to compare two quantities that together make up a group (such as girls and boys in a class). If the ratio gives us fractions, we can then find exact quantity multiply up $\frac{1}{2}$ of the whole, and the second quantity multiply up $\frac{1}{3}$ of the whole. The second quantity multiply up $\frac{1}{4}$ of the whole.

The using this method in the next problem.

Problem 7.6 ⏱ 7

A 1:3 foot length of rope is cut into two pieces whose lengths are in the ratio 1 : 4. What is the length of the longer piece?

Solution for Problem 7.6 We start by figuring out how many pieces of each type of candy we're in for pie. In practice, we'll show the two ratios in the same form.

Method 1. Use a variable. Suppose we are using a ratio to compare two quantities that together make up a group (such as girls and boys in a class). If the ratio gives us fractions, we can then find exact quantity multiply up $\frac{1}{2}$ of the whole, and the second quantity multiply up $\frac{1}{3}$ of the whole. The second quantity multiply up $\frac{1}{4}$ of the whole.

Exercises

7.1.1. ⏱ 7

Simplify the following ratio:

(a) 20 : 15

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Hint

Detail

Your Free Hint

Your Free Solution

Your Free Solution

Solution: The greatest common factor of 20 and 15 is 5. Dividing both this factor, we have $20:15=\frac{20\div5}{15\div5}=4:3$.

(b) $6^2:6^4$

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Your Free Solution

Solution: Since $6^2=(2\cdot3)^2=2^2\cdot3^2$ and $6^4=(2\cdot3)^4=2^4\cdot3^4$, the greatest common factor of 6^2 and 6^4 is $2^2\cdot3^2$. Dividing by this common factor, we have

$$6^2:6^4=\frac{2^2\cdot3^2}{2^2\cdot3^2}:\frac{2^4\cdot3^4}{2^2\cdot3^2}=1:2^2\cdot3^2=1:36.$$

(c) 150 : 500

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Solution: Dividing both 150 and 500 by 150 gives $150:500=\frac{150}{150}:\frac{500}{150}=1:\frac{10}{3}$.

(d) $\frac{1}{2}:\frac{1}{3}$

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Solution: Multiplying both fractions by 12 turns each into an integer:

$$\frac{1}{2}:\frac{1}{3}=\frac{1\cdot12}{2\cdot12}:\frac{1\cdot12}{3\cdot12}=\frac{12}{24}:\frac{12}{36}=12:24.$$

(e) 150 : 500

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Solution: First, we write both mixed numbers as fractions, which gives us $\frac{1}{2}:\frac{1}{3}$. Then, multiplying both by 6 turns both into integers:

$$\frac{1}{2}:\frac{1}{3}=\frac{1\cdot6}{2\cdot6}:\frac{1\cdot6}{3\cdot6}=\frac{6}{12}:\frac{6}{18}=6:18.$$

(f) 172 : 0

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Solution: Dividing both parts of the ratio by 172 gives a simplified ratio $172:0$.

7.1.2. ⏱ 7

There are 5 boys in a class of 9 students. What is the ratio of girls to boys?

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Your Free Solution

Solution: There are $25-9=25$ girls, so the ratio of girls to boys is $25:25$. We can divide by 5 to simplify the ratio as $25:25=\frac{25}{5}:\frac{25}{5}=5:5$.

7.1.3. ⏱ 7 **Source: MATHCOUNTS** ⊕ Jump to Solution

Jane J. Jetsons 2 resolutions for every 3 resolutions (year 2 makes 2 resolutions 3 resolutions) 4 minutes, then how many resolutions does year 2 make?

Solution For Problem 7.3. What are the ratio of girls to boys?

Free View Solution

Hide Solution

Hint

Detail

Your Free Hint

Your Free Solution

Your Free Solution

Solution: The ratio of the number of resolutions 2 makes to the number of resolutions 3 makes is $2:3$. We multiply both parts of the ratio by 10 because 2 makes 10 resolutions.

$$2:3=(2\cdot10):(3\cdot10)=20:30,$$

So, 20 makes $\frac{20}{30}$ resolutions when 3 makes 30 resolutions.

We also could have seen the $2:3$ ratio to note that 20 makes as many resolutions as 3, when 4 makes 10 resolutions, 20 makes $\frac{20}{30}=\frac{2}{3}$ resolutions.

7.1.4. ⏱ 7 **Source: MATHCOUNTS** ⊕ Jump to Solution

The ratio of girls to boys participating in rehearsal rollerball at Jedland Middle School is 7:6. There are 65 girls in the program. What is the total number of students?

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Solution: Since the ratio of girls to boys is 7:6, the ratio of girls to boys is $7:6$. Since there are 65 girls, the total number of students is $65+\frac{6}{7}\cdot65=\frac{13}{7}\cdot65=\frac{845}{7}=121$ students.

7.1.5. ⏱ 7

Two numbers are in the ratio 3:4. Their sum is 84. What is the greater of the two numbers?

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Solution: Since the ratio of the smaller number to the larger number is $3:4$, the numbers are $3n$ and $4n$ for some value of n. Therefore, we must have $3n+4n=84$, so $7n=84$, so $n=12$. Dividing by 12 gives $n=4$, so the larger number is $4\cdot4=16$.

7.1.6. ⏱ 7

An 8-inch-long submarine sandwich is cut into two pieces whose lengths are in the ratio of 1 to 3. How long is the shorter piece?

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Solution: Since the ratio of the longer piece to the shorter piece is $1:3$, the shorter piece is $\frac{3}{1+3}=\frac{3}{4}$ of the whole sandwich. Since the sandwich is 8 inches long, the length of the shorter piece is $\frac{3}{4}\cdot8$ inches $=\frac{3}{4}\cdot\frac{8}{1}$ inches $=\frac{24}{4}$ inches $=6$ inches.

7.1.7. ⏱ 7 **Source: MATHCOUNTS** ⊕ Jump to Solution

A father left 200 acres of land to be divided among his sons of and that is the ratio 1 : 3, respectively. How many acres should it contain?

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Your Free Solution

Solution: At recreation $\frac{1}{4}$ is the best, so he gets $\frac{1}{4}\cdot200=\frac{200}{4}=50$ acres.

7.1.8. ⏱ 7

Two positive numbers are in the ratio of 1 : 9. Their difference is 16. What is the sum of the two numbers?

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Your Free Hint

Your Free Solution

Your Free Solution

Solution: Since the numbers are in the ratio $1:9$, the numbers are n and $9n$ for some value of n. Therefore, we have $9n-n=16$ so $8n=16$. Dividing by 8 gives $n=2$, so the sum of the numbers is $n+9n=10n=10\cdot2=20$.

7.1.9. ⏱ 7 **Source: MATHCOUNTS** ⊕ Jump to Solution

The ratio of teachers to students in a particular school is 1 to 11. The ratio of female students to the total number of students is 4 to 5. If there are 240 female students, then how many teachers are there?

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Your Free Solution

Solution: Since the ratio of teachers to students is $1:11$, the ratio of female students to total number of students is $4:5$. If the total number of students is n the number of female students. Therefore, the total number of students is $\frac{240}{4}\cdot5=300$. Noticing that $240=60\cdot4$ allows us to compute this as quickly.

$$\frac{240}{4}\cdot5=\frac{240}{4}\cdot\frac{5}{1}=\frac{240\cdot5}{4\cdot1}=\frac{240}{4}\cdot5=60\cdot5=300.$$

Next, we count the teachers. Since the ratio of teachers to students is $1:11$, the number of teachers is $\frac{1}{11}$ the number of students. There are 300 students, so there are $\frac{1}{11}\cdot300=\frac{300}{11}$ teachers.

7.1.10. ⏱ 7 **Source: MATHCOUNTS** ⊕ Jump to Solution

The ratio of boxes to wins for Kylo's team is 3 to 2. If we have played the same number of games, but had one more win as many of games, then what could be the number of games, have it would have won?

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Your Free Solution

Solution: The ratio of boxes to wins for Kylo's team is $3:2$, which means that the team won 2 out of every 5 games. If it had won one more game, but played the same number of games, have it would have won 3 out of every 5 games. This means 1 out of every 5 games is the x-factor, so the ratio of boxes to wins is $\frac{5}{1}:\frac{5}{1}$.

7.1.11a. ⏱ 7

The ratio of girlness to Brian is a pi to 2. 1 and there are a total of 240 pantries and shoes in the pie. How many pantries should be in a middle-flavor-flavor-flavor-flavor pie?

Free View Solution

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Hint

Detail

Your Free Hint

Your Free Solution

Your Free Solution

Solution: There are ratio of pantries to shoes is $\pi:2$, so there are π pantries and shoes in the pie. There are 240 pantries and shoes total, so there are $\frac{\pi}{\pi+2}\cdot240=70$ pantries and $240-70=170$ shoes. We wish to add pantries until the ratio of pantries to shoes is 3. At which point the number of pantries is $\frac{3}{2}\cdot170=255$ pantries and of shoes. We'll still have 170 shoes after adding pantries, so we need $255-70=75$ pantries total. This means we need $75-70=5$ more pantries.

7.2 Multi-way Ratios

A ratio is a handy gadget for comparing two quantities. But it's also useful for comparing more than two quantities. For example, suppose I pet store like 4 dogs, 5 cats, and 11 goldfish. We would say that the numbers of dogs, cats, and goldfish are in the ratio of $4:5:11$. Since 4, 5, and 11 have no common factors, we cannot simplify this ratio any further. If a different (more exotic) pet store had 9 geckos, 12 iguanas, and 21 snakes, then we would say that the numbers of geckos, iguanas, and snakes are in the ratio of $9:12:21$. But now, since 3 is a common factor of 9, 12, and 21, we can divide each term of the ratio by 3, to say that the numbers of geckos, iguanas, and snakes are in the ratio of

$$9:12:21 = \frac{9}{3}:\frac{12}{3}:\frac{21}{3} = 3:4:7.$$

Just as with a two-way ratio, a multi-way ratio only gives you information about the relative quantities of the items—it doesn't tell you anything about the total number of items. For example, if you know that a third pet store has hamsters, guinea pigs, and rabbits in the ratio $1:2:5$, all you know is that for every hamster, there are 2 guinea pigs and 5 rabbits. There might be 1 hamster, 2 guinea pigs, and 5 rabbits; or there might be 6 hamsters, 12 guinea pigs, and 30 rabbits; more generally, there are n hamsters, $2n$ guinea pigs, and $5n$ rabbits for some number n.

Although the multiple parts of a multi-way ratio may at first look confusing, they're really not that much different from a two-way ratio, as we will see in the following problems.

Problems

Problem 7.6 Jump to Solution

Simplify the following ratios:

(a) $5:15:10$

(b) $6:10:9$

(c) $8:32:4:8$

(d) $\left(\frac{1}{2}:\frac{1}{3}:\frac{2}{3}\right)$

(e) $\left(\frac{3}{4}:\frac{1}{4}:\frac{1}{2}:\frac{5}{4}\right)$

Problem 7.7 Jump to Solution

A bowling tournament pays out prizes to the top 3 players in the ratio $5:2:1$. If the total prize money is \$1,000, then how much does the first-place winner receive?

Problem 7.8 Jump to Solution

Sam wants to bake a cake that requires butter, flour, sugar, and milk in the ratio $1:6:2:1$. Sam has $\frac{1}{2}$ cup of sugar. How much of the other ingredients does he need?

Problem 7.9 Source: MATHCOUNTS Jump to Solution

Jamal needs three gallons of a mix that is two parts blue paint, three parts white paint, and one part red paint. How many gallons of red paint will he need?

Problem 7.10 Jump to Solution

I have blocks, bikets, and bills in a bag. The ratio of the number of blocks to the number of bikets is $3:4$. The ratio of the number of bikets to the number of bills is $5:6$. What is the ratio of the number of blocks to the number of bills?

Just as with a two-part ratio, to **simplify** a multi-way ratio means to write the ratio using integers with no common factor larger than 1. So, the ratio

$$6:4:8$$

is not simplified because each part is a multiple of 2, while the ratio

$$\frac{1}{2}:\frac{1}{3}:\frac{2}{3}$$

is not simplified because the parts are not integers.

Problem 7.6 Edit

Solution for Problem 7.6:

(a) The greatest common factor of 5, 15, and 10 is 5, so we divide each part of the ratio by 5:

$$5:15:10 = \frac{5}{5}:\frac{15}{5}:\frac{10}{5} = 1:3:2.$$

(b) It is a little more difficult to see what the greatest common factor of 6, 10, and 9 is. We can get a little more insight by factoring each number:

$$6 = 2\cdot 3,\ 10 = 2\cdot 5,\ 9 = 3\cdot 3.$$

Now we can see there is no factor greater than 1 that is common to all three numbers, so the ratio $6:10:9$ is already simplified.

(c) We notice that 4 divides all the terms of the ratio, so we have

$$8:32:4:8 = \frac{8}{4}:\frac{32}{4}:\frac{4}{4}:\frac{8}{4} = 2:8:1:2.$$

(d) We handle fractions a little bit differently: now we want to find a number we can multiply each term by so that each becomes a integer. This is usually the least common denominator of the fractions. In the ratio $\frac{1}{2}:\frac{1}{3}:\frac{2}{3}$ the common denominator is 6, so we multiply:

$$\frac{1}{2}:\frac{1}{3}:\frac{2}{3} = \left(\frac{1}{2}\cdot 6\right):\left(\frac{1}{3}\cdot 6\right):\left(\frac{2}{3}\cdot 6\right) = 3:2:4.$$

(e) Mixed numbers only look more complicated—they're really just the same as fractions. We normally find them easier to work with if we convert them to fractions:

$$3:\frac{1}{4}:\frac{1}{2}:\frac{5}{4} = \frac{12}{4}:\frac{1}{4}:\frac{2}{4}:\frac{5}{4} = \frac{26}{4}.$$

Then, as in part (d) above, we multiply by the least common denominator, which in this example is $3\cdot 4\cdot 5 = 60$, to convert the ratio to integers:

$$\frac{10}{3}:\frac{7}{4}:\frac{26}{4}:\frac{5}{4} = \left(\frac{10}{3}\cdot 60\right):\left(\frac{7}{4}\cdot 60\right):\left(\frac{26}{4}\cdot 60\right)$$
$$= 200:255:312.$$

□

Just as with a two-way ratio, we can think of a multi-way ratio in terms of "parts of the whole" as in the next problem.

Problem 7.7 Edit

A bowling tournament pays out prizes to the top 3 players in the ratio $5:2:1$. If the total prize money is \$1,000, then how much does the first-place winner receive?

Solution for Problem 7.7: The $5:2:1$ ratio means that for every \$5 the winner gets, the second-place player gets \$2 and the third-place player gets \$1. In other words, for every \$5 that the winner gets, a total of $\$5+\$2+\$1 = \$8$ is paid out. Therefore, the winner gets $\frac{5}{8}$ of the total money paid out. Since the total prize money is \$1,000, and the winner gets $\frac{5}{8}$ of the total, he gets $\frac{5}{8}\cdot\$1,000 = \$625$.

In Problem 7.7, we saw that a ratio of $5:2:1$ lead to the first quantity being $\frac{5}{5+2+1} = \frac{5}{8}$ of the whole. In the same way, the second-place winner gets $\frac{2}{5+2+1} = \frac{2}{8} = \frac{1}{4}$ of the prize money, and the third-place winner gets $\frac{1}{5+2+1} = \frac{1}{8}$ of the prize money.

> **Concept:** In a multi-part ratio, you can often think of each term as a "part of the whole." That is, you add all the terms in the ratio to get the "whole," and then each individual term makes up part of that whole.
>
> **WARNING!** Again, remember that ratio is a relative concept. The ratio only tells you the fraction of the whole that each part represents—it doesn't tell you anything about how much the total or each part is.

Going back to Problem 7.7, the ratio itself only tells us that the first-place winner gets $\frac{5}{8}$ of the money. The ratio doesn't tell us exactly how much the winner gets. It is only with the additional information about the total prize money that we can compute the actual money the winner receives.

Problem 7.8 Edit

Sam wants to bake a cake that requires butter, flour, sugar, and milk in the ratio $1:6:2:1$. Sam has $\frac{1}{2}$ cup of sugar. How much of the other ingredients does he need?

Solution for Problem 7.8: We'll present three different methods for solving this problem.

Method 1: Convert the ratio to match the given quantity. The given ratio of butter to flour to sugar to milk is $1:6:2:1$. But we only have $\frac{1}{2}$ cup of sugar, so we convert the ratio so that $\frac{1}{2}$ appears in the "sugar" position. Since the given ratio has a 2 in that position, we need to divide each term of the ratio by 4:

$$1:6:2:1 = \frac{1}{4}:\frac{6}{4}:\frac{2}{4}:\frac{1}{4} = \frac{1}{4}:\frac{3}{2}:\frac{1}{2}:\frac{1}{4}.$$

Now we can just read from the ratio the quantities of the other ingredients that correspond to $\frac{1}{2}$ cup of sugar: $\frac{1}{4}$ cup of butter, $\frac{3}{2}$ cup of flour, and $\frac{1}{4}$ cup of milk.

Method 2: Use separate two-part ratios: We can break up the multi-way ratio into several separate two-part ratios, where each ratio compares some ingredient to sugar. For example, the ratio of butter to sugar is $1:2$—this is part a ratio consisting of the butter and sugar terms from the original 4-part ratio.

> **Concept:** We can remove terms from a multi-way ratio to get a simpler ratio that only compares some of the quantities from the original ratio.

Since the butter and sugar are in ratio $1:2$ we know that there is half as much butter as sugar. We have $\frac{1}{2}$ cup of sugar, so the amount of butter is $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$ cup.

Similarly, the ratio of flour to sugar is $6:2 = 3:1$, so there is three times as much flour as sugar. Thus, since there is $\frac{1}{2}$ cup of sugar, there are $3\cdot\frac{1}{2} = \frac{3}{2}$ cups of flour. Finally, the ratio of sugar to milk is $2:1$, so there is $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$ cup of milk.

Method 3: Compute the total: We can use the "parts of the whole" way of thinking with our 4-part ratio. The sugar is $\frac{2}{1+6+2+1} = \frac{2}{10} = \frac{1}{5}$ of the entire ingredients, and we have $\frac{1}{2}$ cup of sugar. Thus the total quantity of ingredients is $\frac{1}{2}\div\frac{1}{5} = \frac{5}{2}$ cups.

Now we can use the "parts of the whole" to compute the other ingredient amounts. The butter is $\frac{1}{10}$ of the whole, so there is $\frac{1}{10}\cdot\frac{5}{2} = \frac{1}{4}$ cup of butter. Next, the flour is $\frac{6}{10}$ of the whole, so there is $\frac{6}{10}\cdot\frac{5}{2} = \frac{3}{2}$ cups of flour. Finally, the milk is $\frac{1}{10}$ of the whole, so there is $\frac{1}{10}\cdot\frac{5}{2} = \frac{1}{4}$ cup of milk.□

> **Concept:** None of the techniques from Problem 7.8 are "right" or "wrong." Ratios are best approached with flexible thinking. Ratios can be interpreted in many different ways, and you should use the method that you feel most comfortable with, or the method that appears to work best for the particular problem that you're working on.

Problem 7.9 Source: MATHCOUNTS

Jamal needs three gallons of a mix that is two parts blue paint, three parts white paint, and one part red paint. How many gallons of red paint will he need?

Solution for Problem 7.9: Since we are given the total amount and we want to find the amount of one of the parts, the "parts of the whole" method will probably work best. The given ratio is $2:3:1$ of blue : white : red, and the total is 3 gallons. So the red paint is $\frac{1}{2+3+1} = \frac{1}{6}$ of the whole, and thus Jamal needs $\frac{1}{6}\cdot 3 = \frac{1}{2}$ gallon of red paint. □

Problem 7.10 Edit

I have blocks, bikets, and bills in a bag. The ratio of the number of blocks to the number of bikets is $3:4$. The ratio of the number of bikets to the number of bills is $5:6$. What is the ratio of the number of blocks to the number of bills?

Solution for Problem 7.10: Let's make a little chart of the data we're given:

$$\text{blocks} : \text{bikets} \qquad \text{bikets} : \text{bills}$$
$$3 : 4 \qquad\qquad 5 : 6$$

We'd be able to compare bills and blocks if the number of bikets in the above two ratios were equal. So let's make them equal! We can do this by multiplying the parts of the first ratio by 5 and the parts of the second ratio by 4:

$$\text{blocks} : \text{bikets} \qquad \text{bikets} : \text{bills}$$
$$15 : 20 \qquad\qquad 20 : 24$$

Aha—now we can write it as a 3-way ratio of blocks to bikets to bills:

$$\text{blocks} : \text{bikets} : \text{bills}$$
$$15 : 20 : 24$$

The problem asked us for the relationship between blocks and bills, and we can read this information from our 3-way ratio. We see that blocks and bills are in the ratio $15:24$, which can be simplified as

$$\text{blocks} : \text{bills} = 15:24 = \frac{15}{3}:\frac{24}{3} = 5:8.$$

□

Multi-Part Ratios

Exercises

7.2.1: Edit

A log whose length is 60 inches is cut into three pieces in the ratio $1:3:5$. What is the number of inches in the length of the shortest piece?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The shortest piece is $\frac{1}{1+3+5} = \frac{1}{9}$ of the whole. So, the length of the shortest piece is

$$\frac{1}{9}\cdot 60 = \boxed{\text{7}}\ \text{inches} = \boxed{6}\ \text{inches}$$

7.2.2: Edit

Three numbers have ratio $1:2:3$, and their sum is 48. What is the greatest of these numbers?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The greatest of the numbers is $\frac{3}{1+2+3} = \frac{1}{2}$ of the sum of the numbers. Since the sum of the numbers is 48, the largest number is $\frac{1}{2}\cdot 48 = \boxed{24}$.

7.2.3: Source: MATHCOUNTS

Purple paint is made with a $16:3:1$ ratio of

$$\text{white paint} : \text{blue paint} : \text{red paint}.$$

How much white paint is needed in order to make one gallon of purple paint?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The white paint is $\frac{16}{16+3+1} = \frac{4}{5}$ of the mixture, so $\frac{4}{5}\cdot 1 = \boxed{\frac{4}{5}\text{ gallon}}$ is needed to make 1 gallon of purple paint.

7.2.4: Source: MATHCOUNTS

These friends, Alexa, Bruno, and Cameka, pooled their money to start a lemonade stand. Alexa contributed \$20, Bruno contributed \$30, and Cameka contributed \$50. After a month, their lemonade stand had earned \$2,000, and they went to distribute this money in the ratio as the money that was invested. How many dollars will Bruno receive?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The ratio of Alexa, Bruno, and Cameka's investments is $25:30:50$. Dividing all parts of this ratio by 5 gives $5:6:10$. Therefore, Bruno receives $\frac{6}{5+6+10} = \frac{6}{21}$ of the money, so he should receive $\frac{6}{21}$ of the earnings. Therefore, Bruno will receive $\frac{1}{3}\cdot 2000 = \boxed{\$800}$ dollars.

7.2.5: Edit

The top four winners in a golf tournament share the prize money in the ratio $9:5:2:1$. If the top prize winner receives \$45,000, then how much prize money is awarded in total?

You may type any additional notes you have here.

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Solution: The top winner receives $\frac{9}{9+5+2+1} = \frac{9}{17}$ of the total, so the ratio of the top winner to the total is $9:17$. Multiplying this ratio by 5000 to make the top winner equal to 45000 gives $45000:85000$, so the total prize pool is $\boxed{\$85,000}$.

7.2.6: Source: MATHCOUNTS

Alex owns three times as many brown shoes as red shoes, twice as many black shoes as brown shoes, and four times as many white shoes as red shoes. What is the ratio of the number of white shoes to the number of black shoes he owns?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: From the information in the problem, we have

$$\frac{\text{brown}}{\text{red}} = \frac{3}{1}, \quad \frac{\text{black}}{\text{brown}} = \frac{2}{1}, \quad \frac{\text{white}}{\text{red}} = \frac{4}{1}.$$

Conveniently, red's number is the same in the first and third ratios. So, we can combine these two ratios and write

$$\frac{\text{brown}}{\text{red}} = \frac{3}{1} = \frac{4}{1} = \frac{\text{white}}{\text{red}}.$$

To combine these two ratios, we need the number corresponding to brown shoes to be the same in both. So, we multiply both parts of the second ratio by 3:

$$\frac{\text{brown}}{\text{red}} = \frac{3}{1} = \frac{4}{1} = \frac{\text{white}}{\text{red}}, \quad \frac{\text{black}}{\text{brown}} = \frac{2}{1} = \frac{6}{3}.$$

Now, we can combine the ratios:

$$\frac{\text{black}}{\text{brown}} = \frac{6}{3} = \frac{4}{1} = \frac{\text{white}}{\text{red}}.$$

So, we see that the ratio of white shoes to black shoes is $4:6$, which equals $\boxed{2:3}$.

7.2.7★: Source: MATHCOUNTS

Three siblings have a gift of \$1,000 to split in the ratio $2:\frac{1}{2}:\frac{1}{4}$. What is the greatest number of dollars that any of the siblings will receive?

Hint: Can you write the ratio in an easier-to-use form?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: In order to make the ratio easier to work with, we start by simplifying it. The least common denominator of the fractions in the ratio is 12, so we multiply all parts of the ratio by 12. That gives

$$12:\frac{1}{2}\cdot 12:\frac{1}{4}\cdot 12 = 6:1:3.$$

Therefore, the sibling who gets the most money receives $\frac{6}{6+1+3} = \frac{6}{10}$ of the money. Since there is \$1,000 to divide, the one who gets the most money receives $\frac{6}{10}(\$1000) = \boxed{\$600}$.

7.3 Proportions

Whenever we have two ratios that are equal, we have a **proportion**. The most common usage of proportion is when we have two changing quantities that are related in such a way that their ratio doesn't change.

For example, suppose that Marisol's secret recipe for chocolate milk uses 8 ounces of milk and 2 ounces of chocolate syrup, and produces a 10-ounce glass of chocolate milk. The ratio of milk to chocolate syrup is $8:2$ or $4:1$. If Mario wants to make a big pitcher of chocolate milk for 6 people, then he will need $6 \cdot 8 = 48$ ounces of milk and $6 \cdot 2 = 12$ ounces of chocolate syrup, so the ratio of milk to chocolate syrup is $48:12$, which is still $4:1$. No matter what quantity of chocolate milk that we want, the ratio of milk to chocolate syrup will always be $4:1$. We say that the milk and chocolate syrup are **proportional** or in **proportion**.

Problems

Problem 7.11 Jump to Solution

Charlotte is planning a vacation to Europe. The exchange rate is 1 dollar equals 0.6 euros, or $\$1 = €0.60$. If Charlotte wants to have €300 for her trip, then how many dollars does she need to convert?

Problem 7.12 *Source: MATHCOUNTS* Jump to Solution

A recipe calls for $2\frac{1}{2}$ cups of flour and 4 eggs. If only 3 eggs are used, then how many cups of flour should be used?

Problem 7.13 Jump to Solution

Sadie is 3 feet tall and at 6 p.m. in Sunnytown, Sadie casts an 8-foot shadow. Nick is 5 feet tall. How long is his shadow at 6 p.m. in Sunnytown?

Problem 7.14 Jump to Solution

My map of upstate New York has the scale $\frac{1}{4}\text{ inch} = 5\text{ miles}$. If Buffalo and Albany are 13 inches apart on my map, then how far apart are the cities?

Problem 7.15 Jump to Solution

Sylvia is an architect designing a new building. The building will be 30 feet tall, and the windows will each be 8 feet high. Sylvia draws blueprints for the building on which the building is 8 inches tall. How tall are the windows on Sylvia's blueprints?

Problem 7.16 Jump to Solution

My wallet-size photo of my pet cat Snookums is 3 cm wide and 5 cm tall. If I want a larger photo to put on my wall, and I want the area of the photo to be 135 square centimeters, then how many centimeters wide should the larger photo be?

Problem 7.11

Charlotte is planning a vacation to Europe. The exchange rate is 1 dollar equals 0.6 euros, or $\$1 = €0.60$. If Charlotte wants to have €300 for her trip, then how many dollars does she need to convert?

Solution for Problem 7.11: Since $\$1$ equals €0.60, the ratio between an equal amount of dollars and euros is $1:0.6$, which simplifies to $5:3$. We let x be the number of dollars that equals €300, which is what we want to find. Then as a proportion we have

$$5:3 = x:300,$$

which gives $\frac{5}{3} = \frac{x}{300}$. We can solve for x as $x = 300 \cdot \frac{5}{3} = 500$. Hence, Charlotte must convert $500 in order to receive €300. □

Problem 7.12 *Source: MATHCOUNTS*

A recipe calls for $2\frac{1}{2}$ cups of flour and 4 eggs. If only 3 eggs are used, then how many cups of flour should be used?

Solution for Problem 7.12: There are two primary methods we can use.

Method 1: Set up a proportion by equating ratios. Because the recipe calls for $2\frac{1}{2}$ cups of flour and 4 eggs, we know that the ratio of flour to eggs should always equal $2\frac{1}{2} : 4$. If we only have 3 eggs, then we need an amount of flour so that the ratio of flour to eggs still equals $2\frac{1}{2} : 4$. This means that if we have x cups of flour to go with our 3 eggs, we must have

$$2\frac{1}{2} : 4 = x : 3.$$

We solve this by writing the equation with fractions:

$$\frac{2\frac{1}{2}}{4} = \frac{x}{3}.$$

Multiplying both sides of this equation by 12 gives $3 \cdot \left(2\frac{1}{2}\right) = 4x$. So $\frac{15}{2} = 4x$, and dividing by 4 gives $\frac{15}{8} = x$. Therefore we need $\frac{15}{8} = 1\frac{7}{8}$ cups of flour.

Method 2: Scale the quantities. Because we only have 3 eggs and the recipe calls for 4 eggs, we are only using $\frac{3}{4}$ of the recipe amount. In order for the ratio of flour to eggs to remain constant, we also need to use $\frac{3}{4}$ of the recipe amount of flour. Therefore, the amount of flour that we need is

$$\frac{3}{4} \cdot 2\frac{1}{2} = \frac{3}{4} \cdot \frac{5}{2} = \frac{15}{8} = 1\frac{7}{8}$$

cups. □

In the next problem, we use the geometric fact that at any given time, the length of an object is proportional to the length of its shadow. (This is based on the geometric concept of **similarity**, which you will learn when you take a geometry course.)

Problem 7.13

Sadie is 3 feet tall and at 6 p.m. in Sunnytown, Sadie casts an 8-foot shadow. Nick is 5 feet tall. How long is his shadow at 6 p.m. in Sunnytown?

Solution for Problem 7.13: The information about Sadie tells us that the length of an object (or person) and the length of its shadow are proportional in the ratio $3:8$. If Nick's shadow has length x, then the ratio $5:x$ must equal the ratio $3:8$. Therefore, $\frac{5}{x} = \frac{3}{8}$ and multiplying both sides by $8x$ gives $40 = 3x$, so $x = \frac{40}{3} = 13\frac{1}{3}$. Thus, Nick's shadow is $13\frac{1}{3}$ feet long. □

Problem 7.14

My map of upstate New York has the scale $\frac{1}{4}\text{ inch} = 5\text{ miles}$. If Buffalo and Albany are 13 inches apart on my map, then how far apart are the cities?

Solution for Problem 7.14: A map's scale is another example of a proportion—it tells us that the ratio of the distance on the map to the distance in the real world is constant. For my map, this ratio is

$$\frac{1}{4}\text{ inch} : 5\text{ miles}.$$

Multiplying by 4 simplifies this ratio to $1\text{ inch} : 20\text{ miles}$, so that 1 inch on the map corresponds to 20 miles in real life. Thus, the 13 inches on the map between Buffalo and Albany means that the cities are $13 \cdot 20 = 260$ miles apart. □

Problem 7.15

Sylvia is an architect designing a new building. The building will be 30 feet tall, and the windows will each be 8 feet high. Sylvia draws blueprints for the building on which the building is 8 inches tall. How tall are the windows on Sylvia's blueprints?

Solution for Problem 7.15: We start with the ratio

$$\text{building height} : \text{window height} = 30\text{ feet} : 8\text{ feet} = 30 : 8 = 15 : 4.$$

Thus, on the blueprints, we must have the same ratio between the heights of the building and the window. So if the window height on the blueprints is x inches, then we have

$$15 : 4 = 8 : x,$$

so that $\frac{15}{4} = \frac{8}{x}$. Multiplying both sides of this equation by $4x$ gives $15x = 32$, so $x = \frac{32}{15} = 2\frac{2}{15}$. Therefore, the windows are $2\frac{2}{15}$ inches tall on the blueprints. □

Problem 7.16

My wallet-size photo of my pet cat Snookums is 3 cm wide and 5 cm tall. If I want a larger photo to put on my wall, and I want the area of the photo to be 135 square centimeters, then how many centimeters wide should the larger photo be?

Solution for Problem 7.16: The assumption in this problem is that no matter what the size, the picture will always have the same shape. More precisely, this means that the ratio of width to height will remain constant. Since the wallet-size photo has width 3 cm and height 5 cm, the ratio of width to height will always be $3:5$. But we're given neither the width nor the height of the larger photo: we're only given the area. So how do we set up a proportion?

The proportion tells us that the width is $3x$ cm and the height is $5x$ cm for some number x. This means that the area is $(3x) \cdot (5x) = 15x^2$ square centimeters. If I want a larger photo with area 135 square centimeters, then we must have $15x^2 = 135$, or $x^2 = 135/15 = 9$. Therefore, $x = 3$. So my larger photo will be $3x = 3(3) = 9$ cm wide (and $5x = 5(3) = 15$ cm high). □

Exercises

7.3.1: *Source: AMC 5*

A ream of paper containing 500 sheets is 5 cm thick. How many sheets of this type of paper would there be in a stack 7.5 cm high?

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Let there be x sheets in the 7.5 cm stack, so we have $500 : 5 = x : 7.5$. Writing this using fractions, we have $\frac{500}{5} = \frac{x}{7.5}$, so $\frac{x}{7.5} = 100$. Multiplying both sides by 7.5 gives $x = \boxed{750}$.

7.3.2:

An American traveling in Japan wishes to exchange American money (dollars, symbol $) for Japanese money (yen, symbol ¥). If the exchange rate is $1 = ¥80, then how many dollars will the traveler need to purchase ¥10,000?

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Solution: Let x be the number of dollars the traveler needs. Since the ratio of dollars to yen is $1:80$, we have $1:80 = x:10000$. Writing this using fractions gives $\frac{1}{80} = \frac{x}{10000}$. Multiplying both sides by 10000 gives $x = \frac{10000}{80} = \frac{1000}{8} = \boxed{125\text{ dollars}}$.

7.3.3: *Source: MATHCOUNTS*

Alexia designed a logo 2 inches wide and 1.5 inches tall to be used on her school's website. The school wants the logo to appear on the website as 8 inches wide. How tall will the logo be on the website if it is enlarged proportionally?

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Your Submission: Solution

Solution: The width of the website logo is 4 times as large as Alexia's design, so the height of the website logo must also be 4 times as large as Alexia's design, or $4 \cdot 1.5 = \boxed{6\text{ inches}}$.

7.3.4:

A bank has two flagpoles next to each other. If the taller 30-foot pole (flying the U.S. flag) casts a shadow of 20 feet, and the shorter flagpole (flying the state flag) casts a shadow of 15 feet, then how tall is the shorter flagpole?

You may type any additional notes you have here.

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Solution: Considering the taller object, we have

$$\text{Object height} : \text{Shadow height} = 30 : 20 = 3 : 2.$$

So, we have

$$\text{Short pole height} : 15\text{ feet} = 3 : 2.$$

Multiplying both parts of the ratio on the right by 7.5, we have

$$\text{Short pole height} : 15\text{ feet} = 22.5 : 15.$$

So, the height of the short pole is $\boxed{22.5\text{ feet}}$.

We might also have reasoned as follows. The shadow of the shorter pole is $\frac{15}{20}$ of the shadow of the taller pole. So, the height of the shorter pole is $\frac{15}{20}$ of the height of the taller pole. Therefore, the height of the shorter pole is $\frac{15}{20} \cdot (30\text{ feet}) = \boxed{22.5\text{ feet}}$.

7.3.5:

If $\frac{1}{4}$ inch on a map represents 50 miles, then what is the number of miles represented by $2\frac{7}{8}$ inches?

Preview: Solution

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Your Submission: Solution

Solution: We have

$$\text{Map} : \text{Actual distance} = \frac{1}{4}\text{ inch} : 50\text{ miles}.$$

Multiplying by 4 makes this ratio $1\text{ inch} : 200\text{ miles}$. So, each inch on the map represents 200 miles Therefore, $2\frac{7}{8}$ inches represents $200 \cdot \left(2\frac{7}{8}\right) = \boxed{575\text{ miles}}$.

7.3.6: *Source: MATHCOUNTS*

A drafterperson makes a scale drawing of a 100 meter × 30 meter building, where 1 centimeter represents 2.5 meters. How many centimeters are in the smaller dimension of the drawing of the building?

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Your Submission: Solution

Solution: Let x cm be the smaller dimension in the drawing. Since we must have

$$\text{Drawing} : \text{Building} = 1\text{ cm} : 2.5\text{ m},$$

we have $x\text{ cm} : 30\text{ m} = 1\text{ cm} : 2.5\text{ m}$. Writing this as fractions, we have $\frac{x}{30} = \frac{1}{2.5}$. Multiplying by 30 gives $x = \frac{30}{2.5} = \frac{300}{25} = 12$. So, the smaller dimension of the drawing is $\boxed{12\text{ cm}}$.

7.3.7: *Source: AMC 5*

Twelve friends met for dinner at Oscar's Oversuffed Oyster House, and each ordered one meal. The portions were so large, there was enough food for 18 people. If they share, then how many meals should they have ordered to have just enough food for the 12 of them?

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Your Submission: Solution

Solution: The 12 meals they ordered were enough for 18 people, so we have the ratio

$$\text{Meals needed} : \text{People} = 12 : 18 = 2 : 3.$$

So, we need $\frac{2}{3}$ as many meals as we have people. Therefore, to feed 12 people, we need $\frac{2}{3} \cdot 12 = \boxed{8}$ meals.

7.3.8★: *Source: AMC 8*

For every 3° rise in temperature, the volume of a certain gas expands by 4 cubic centimeters. If the volume of the gas is 24 cubic centimeters when the temperature is 32°, then what was the volume of the gas when the temperature was 20°?

Preview: Solution

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Your Submission: Solution

Solution: As the temperature rose from 20° to 32°, it rose by 3° exactly 4 times. Each time the temperature rises 3°, the volume of the gas increases by 4 cubic centimeters. So, as the temperature rose 4 times by 3°, the volume expanded by $4 \cdot 4 = 16$ cubic centimeters. After this expansion, the volume of the gas is 24 cubic centimeters, so the volume of the gas before the expansion was $24 - 16 = \boxed{8\text{ cubic centimeters}}$.

7.4 Conversions

There is a special type of ratio that is useful for converting between different units of measurement. Let's illustrate how this works with a simple example.

We know that there are 12 inches in a foot and there are 3 feet in a yard. Suppose you want to use this information to compute the number of inches in a yard. Of course, you can ponder this information in your head, or you may even have the answer memorized. But let's carefully work through two methods that we can use to solve the problem. These methods will help us work through harder conversion problems, where the answer is not so obvious.

Method 1: Set up ratios. We'll use the a similar method to Problem 7.16 (the problem with the bills and blocks). We can write ratios to express the relationships between the units:

$$\frac{\text{inches}}{12} = \frac{1}{1}, \quad \frac{\text{feet}}{3} = \frac{1}{1}, \quad \frac{\text{yards}}{1} = \frac{1}{1}.$$

We'd like to combine this into a 3-way ratio relating three units. To do that, we need the "feet" amount in both 2-way ratios to match. The easiest way to do this is to multiply both parts of the first ratio by 3:

$$\frac{\text{inches}}{36} = \frac{3}{1}, \quad \frac{\text{feet}}{3} = \frac{1}{1}, \quad \frac{\text{yards}}{1} = \frac{1}{1}.$$

Now we can write it as a 3-way ratio:

$$\text{inches} : \text{feet} : \text{yards} := 36 : 3 : 1$$

Removing the "feet" gives us a ratio of inches : yards := 36 : 1, so there are 36 inches in a yard.

Method 2: Use conversion factors. If we write the ratios from Method 1 as fractions, then we have what are called **conversion factors**. To help us keep track of what's going on, we'll write the units as part of the fraction. So we'd write

$$\frac{12 \text{ inches}}{1 \text{ foot}} = \frac{1 \text{ foot}}{3 \text{ feet}} = \frac{3 \text{ feet}}{1 \text{ yards}}.$$

(For consistency, we usually write all units in plural, so we write the word-looking "1 foot" instead of "1 foot.") Multiplying the conversion factors together will cancel the "feet" and leave us with a conversion factor relating inches to yards:

$$\frac{12 \text{ inches}}{1 \text{ foot}} \cdot \frac{3 \text{ feet}}{1 \text{ yards}} = \frac{36 \text{ inches}}{1 \text{ yards}}.$$

So there are 36 inches in a yard.

Compare the two methods used above. They're really the same thing! The conversion factors from Method 2 are just a convenient way for us to keep track of the ratios from Method 1.

There are two key ideas to keep in mind when using conversion factors. First,

> **Concept:** Think of conversion factors as fractions that are equal to 1.

For example, we know that there are 12 inches in a foot, so we think of

$$12 \text{ inches} = 1 \text{ foot}.$$

In other words, the quantities "12 inches" and "1 foot" are equal quantities; so it makes sense to write them equal to each other in an equation. Going one step further, this then makes the fraction

$$\frac{12 \text{ inches}}{1 \text{ foot}} = 1.$$

The right way to think about this is as a fraction with equal numerator and denominator, so of course it is equal to 1. Also, we can just as easily write its reciprocal too:

$$\frac{1 \text{ foot}}{12 \text{ inches}} = 1.$$

Our second key idea about conversion factors is:

> **Concept:** We multiply conversion factors together to cancel units.

We see this concept if we revisit our earlier computation:

$$\frac{12 \text{ inches}}{1 \text{ foot}} \cdot \frac{3 \text{ feet}}{1 \text{ yards}} = \frac{36 \text{ inches}}{1 \text{ yards}}.$$

All we're doing is multiplying two fractions on the left that are each equal to 1, so naturally (since $1 \cdot 1 = 1$) the quantity on the right side of the equation is also equal to 1. But we also notice that the "feet" units cancel. You might see this more clearly if we put in some missing steps in the above calculation:

$$\begin{aligned}\frac{12 \text{ inches}}{1 \text{ foot}} \cdot \frac{3 \text{ feet}}{1 \text{ yards}} &= \frac{(12 \text{ inches}) \cdot (3 \text{ feet})}{(1 \text{ foot}) \cdot (1 \text{ yards})} \\ &= \frac{(12 \cdot 3)(\text{inches} \cdot \cancel{\text{feet}})}{(1 \cdot 1)(\cancel{\text{feet}} \cdot \text{yards})} \\ &= \frac{36 \text{ inches}}{1 \text{ yards}}.\end{aligned}$$

The beauty of this is that it helps us prevent mistakes. For example, if we incorrectly tried to combine conversion factors as:

$$\frac{12 \text{ inches}}{1 \text{ foot}} \cdot \frac{1 \text{ yards}}{3 \text{ feet}} = \frac{4(\text{inches} \cdot \text{yards})}{(\text{feet} \cdot \text{feet})}$$

we see that the units don't cancel properly, so we probably made a mistake somewhere.

Conversion factors make it easy to convert units. For example, suppose we want to convert the length "6 yards" into inches. We know that a conversion factor is just a fraction that equals 1, and multiplying "6 yards" by 1 doesn't change the length. Therefore, we have

$$6 \text{ yards} = (6 \text{ yards}) \cdot 1 = (6 \text{ yards}) \cdot \frac{36 \text{ inches}}{1 \text{ yards}}.$$

But now the units nicely cancel, and we can finish the computation:

$$6 \text{ yards} = (6 \cancel{\text{yards}}) \cdot \frac{36 \text{ inches}}{1 \cancel{\text{yards}}} = \frac{6 \cdot 36 \text{ inches}}{1} = 216 \text{ inches}.$$

Notice how the "yards" units cancelled, and we are left with just the "inches" units, as we want. Therefore, 6 yards is equal to 216 inches.

Conversion Factors Part 1

Conversion Factors Part 2

Problems

Problem 7.17 — Jump to Solution

How many yards equals 90 inches?

Problem 7.18 — Jump to Solution

A tablespoon is half of a fluid ounce, a cup is 8 fluid ounces, and a gallon is 16 cups. How many tablespoons are in a gallon?

Problem 7.19 — Jump to Solution

Will took $1,000 on his trip to Japan, where the exchange rate between dollars and yen is $1 = ¥90. He spent ¥45,000 on his hotel room and ¥11,250 on meals and souvenirs. How much money (in dollars) did he have remaining at the end of his trip?

Problem 7.20 — Jump to Solution

The density of water is approximately 8.3 pounds per gallon, and there are 4 quarts in a gallon. How much does 7 quarts of water weigh?

Problem 7.21 — Jump to Solution

An inch is approximately 2.5 centimeters. Approximately how many square centimeters are in a square inch?

Problem 7.17

How many yards equals 90 inches?

Solution for Problem 7.17: We discovered above that 1 yard is equal to 36 inches. To convert 90 inches into yards, we start with the quantity "90 inches" and multiply by the appropriate conversion factor until we get something with the units "yards." In this case, it's easy: we just need a number with inches in the denominator (to cancel the inches in our initial quantity) and yards in the numerator (so we'll be left with yards). Our calculation is:

$$90 \text{ inches} = 90 \text{ inches} \cdot \frac{1 \text{ yards}}{36 \text{ inches}} = \frac{90}{36} \text{ yards} = 2.5 \text{ yards}.$$

Thus, 90 inches is the same as 2.5 yards.

> **Concept:** The reason the calculation in Problem 7.17 works is that the conversion factor is equal to 1. That is, because 1 yards := 36 inches, the fraction $\frac{1 \text{ yards}}{36 \text{ inches}} = 1$ equals 1. Therefore, multiplying 90 inches by this fraction is the same as multiplying by 1, and hence the length does not change.

Problem 7.18

A tablespoon is half of a fluid ounce, a cup is 8 fluid ounces, and a gallon is 16 cups. How many tablespoons are in a gallon?

Solution for Problem 7.18: We want to convert "1 gallon" into some number of tablespoons. But we don't have a single gallons-to-tablespoons conversion factor. Instead, we have to use the multiple conversion factors that we are given in the problem statement. First, we show a step-by-step solution.

We first convert gallons to cups. We don't really need a "conversion factor" for this, since we can just read this data from the problem statement:

$$1 \text{ gallon} = 16 \text{ cups}.$$

Next, we convert cups to ounces, using the cups-to-ounces conversion:

$$\begin{aligned}1 \text{ gallon} &= 16 \text{ cups} \\ &= 16 \text{ cups} \cdot \frac{8 \text{ ounces}}{1 \text{ cups}} \\ &= (16 \cdot 8) \text{ ounces} \\ &= 128 \text{ ounces}.\end{aligned}$$

Finally, we convert ounces to tablespoons (abbreviated tbsp):

$$\begin{aligned}1 \text{ gallon} &= 128 \text{ ounces} \\ &= 128 \text{ ounces} \cdot \frac{1 \text{ tbsp}}{2 \text{ ounces}} \\ &= \frac{128}{2} \text{ tbsp} \\ &= 256 \text{ tbsp}.\end{aligned}$$

So there are 256 tablespoons in a gallon.

But conversion factors are nice in that we can use more than one of them at the same time. In particular, we could do the gallons-to-tablespoons conversion all at once:

$$\begin{aligned}1 \text{ gallon} &= 1 \text{ gallon} \cdot \frac{16 \text{ cups}}{1 \text{ gallon}} \cdot \frac{8 \text{ ounces}}{1 \text{ cups}} \cdot \frac{1 \text{ tbsp}}{2 \text{ ounces}} \\ &= \frac{16 \cdot 8}{2} \text{ tbsp} \\ &= 256 \text{ tbsp}.\end{aligned}$$

As long as we are sure that each conversion factor equals 1 (meaning that its numerator and denominator represent the same quantity), and that the units cancel properly, we can line up as many conversion factors as might be necessary to do a complicated conversion. :)

We can use conversion factors whenever we wish to compare two different units—they don't necessarily have to be typical "measurements" like length or volume. For example:

Problem 7.19

Will took $1,000 on his trip to Japan, where the exchange rate between dollars and yen is $1 = ¥90. He spent ¥45,000 on his hotel room and ¥11,250 on meals and souvenirs. How much money (in dollars) did he have remaining at the end of his trip?

Solution for Problem 7.19: Method 1: Convert and then convert back. When Will went to Japan, his dollars became:

$$\$1{,}000 = \frac{¥90}{\$1} \cdot \$1{,}000 = ¥90{,}000.$$

This is just like any other conversion: notice how the conversion factor $\frac{¥90}{\$1}$ equals 1 because ¥90 equals $1, and notice how the $ units cancel leaving us with the desired ¥ units. After his spending, he was left with:

$$¥90{,}000 - ¥45{,}000 - ¥11{,}250 = ¥33{,}750.$$

When Will got home, he converted his remaining yen back into dollars:

$$¥33{,}750 \cdot \frac{\$1}{¥90} = \frac{33{,}750}{90} = \$375.$$

So Will had $375 dollars remaining after his trip.

Method 2: Convert just the spending. Rather than convert Will's entire bankroll to yen and then convert it back, we can just figure out how much he spent in dollars. His total spending was ¥45,000 + ¥11,250 = ¥56,250, so in dollars this is

$$¥56{,}250 \cdot \frac{\$1}{¥90} = \frac{56{,}250}{90} = \$625.$$

Thus Will spent the equivalent of $625 on his trip, and had $1,000 − $625 = $375 remaining. :)

Problem 7.20

The density of water is approximately 8.3 pounds per gallon, and there are 4 quarts in a gallon. How much does 7 quarts of water weigh?

Solution for Problem 7.20: We want to convert from quarts to pounds. We can use two conversion factors: one for quarts to gallons, and one for gallons to pounds:

$$\begin{aligned}7 \text{ quarts} &= 7 \text{ quarts} \cdot \frac{1 \text{ gallons}}{4 \text{ quarts}} \cdot \frac{8.3 \text{ pounds}}{1 \text{ gallons}} \\ &= \frac{7 \cdot 8.3}{4} \text{ pounds} \\ &= 14.525 \text{ pounds}.\end{aligned}$$

:)

Problem 7.21

An inch is approximately 2.5 centimeters. Approximately how many square centimeters are in a square inch?

Solution for Problem 7.21: Method 1: Reason geometrically. A square inch is the area of a square that is 1 inch on each side. But this same square is 2.5 centimeters on each side, so its area is $(2.5\text{ cm})^2 = 6.25$ square centimeters.

Method 2: Use conversion factors. We wish to convert from square inches (written in^2) to square centimeters (written cm^2). So we need to cancel the inches units twice and be left with the centimeters units twice. Thus, we need to multiply by two conversion factors:

$$1 \text{ in}^2 = 1 \text{ in}^2 \cdot \frac{2.5 \text{ cm}}{1 \text{ in}} \cdot \frac{2.5 \text{ cm}}{1 \text{ in}} = (2.5)(2.5) \text{ cm}^2 = 6.25 \text{ cm}^2.$$

:)

Exercises

7.4.1:

Suppose that one US dollar is worth C$1.25 in Canadian dollars. If I want to buy a C$15 book in Canada, then how many US dollars do I need?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: I need 1 US dollar for every 1.25 Canadian dollars, so my 15 Canadian dollars convert to $15/1.25 = 12$ US dollars. We can also do this using a conversion factor:

$$\text{C\$15} = \text{C\$15} \cdot \frac{\text{US\$1}}{\text{C\$1.25}} = \text{US\$}\frac{15}{1.25} = \boxed{\text{US\$12}}.$$

7.4.2:

Basketball center Steve Tovolat is 7 feet 2 inches in height. What is Steve's height in inches?

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Each foot is 12 inches, so 7 feet is $7 \cdot 12 = 84$ inches. Adding the 2 extra inches gives $84 + 2 = \boxed{86}$ inches.

7.4.3:

Recall that 1 inch is approximately 2.5 centimeters. What is the area, in square centimeters, of a square that is $\frac{1}{2}$ foot long on each side?

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: First, since there are 12 inches in a foot, there are $12 \cdot \frac{1}{2} = 6$ inches in $\frac{1}{2}$ foot. Thus the square is 6 inches per side. We could first convert the side length to centimeters, giving

$$6 \text{ in} = 6 \cdot \frac{2.5 \text{ cm}}{1 \text{ in}} = (6 \cdot 2.5) \text{ cm} = 15 \text{ cm}.$$

Thus the area of the square is $15 \cdot 15 = \boxed{225}\ \text{cm}^2$.

Alternatively, we could use the fact (from the text) that $1 \text{ in}^2 = 6.25 \text{ cm}^2$, and convert the 36 square inches of area to square centimeters:

$$36 \text{ in}^2 = 36 \text{ in}^2 \cdot \frac{6.25 \text{ cm}^2}{1 \text{ in}^2} = (36 \cdot 6.25) \text{ cm}^2 = \boxed{225}\ \text{cm}^2.$$

7.4.4:

There are approximately 28.35 grams in an ounce, and 16 ounces in a pound. How many grams does a quarter-pound hamburger weigh? Round your answer to nearest whole number of grams.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We line up the conversion factors:

$$\frac{1}{4} \text{ pounds} \cdot \frac{16 \text{ ounces}}{1 \text{ pounds}} \cdot \frac{28.35 \text{ grams}}{1 \text{ ounces}} = \boxed{113 \text{ grams}}.$$

7.4.5:

Natalya's secret recipe for peanut butter cookies calls for $\frac{3}{2}$ cups of flour. Unfortunately, Natalya has lost all of her measuring equipment except for a teaspoon. There are 3 teaspoons in a tablespoon, $\frac{1}{2}$ ounce in a tablespoon, and 8 ounces in a cup. How many teaspoons of flour does Natalya need for her recipe?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: We line up the conversion factors:

$$\begin{aligned}2.5 \text{ cups} &= 2.5 \text{ cups} \cdot \frac{8 \text{ oz}}{1 \text{ cup}} \cdot \frac{0.5 \text{ tbsp}}{1 \text{ oz}} \cdot \frac{3 \text{ tsp}}{1 \text{ tbsp}} \\ &= \frac{2.5 \cdot 8 \cdot 3}{0.5} \text{ tsp} \\ &= 120 \text{ tsp}.\end{aligned}$$

Thus Natalya needs $\boxed{120}$ teaspoons for her recipe.

7.4.6:

1600 meters is 1 kilometer, and 100 hectares is one square kilometer. How many square meters are in 1 hectare?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We have (note ha is short for hectare):

$$1 \text{ ha} = 1 \text{ ha} \cdot \left(\frac{1 \text{ km}^2}{100 \text{ ha}}\right) \cdot \left(\frac{1000 \text{ m}}{1 \text{ km}}\right)^2 = \frac{1000^2}{100} \text{ m}^2 = \boxed{10000 \text{ m}^2}.$$

7.4.7:

On the planet Gobbob, the unit of currency is the Plokker, the unit of weight is the ekaw, and the unit of volume is the pinuak. The precious liquid vimey is worth 400 Plokkers per pinuak and has a density of 20 ekaw per pinuak. If Aaron has 500 pinuak of vimey, then how many Plokkers is her vimey worth?

Hint: Don't let the weird words fool you; this problem is just like many others you've solved in the section.

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Despite the foreign words, we can still use conversion factors!

$$\begin{aligned}500 \text{ pinuak} &= 500 \text{ pinuak} \cdot \frac{20 \text{ ekaw}}{1 \text{ pinuak}} \cdot \frac{400 \text{ Plokkers}}{1 \text{ ekaw}} \\ &= (500 \cdot 20 \cdot 400) \text{ Plokkers} \\ &= \boxed{4{,}000{,}000 \text{ Plokkers}}.\end{aligned}$$

7.5. Speed

Speed is probably always further with time **of** speed. Speed is a measure of how fast something is moving.

For example, suppose a car is travelling at a constant speed of 40 miles per hour. After 1 hour, it has travelled 40 miles; after 2 hours, it has travelled 80 miles; after 3 hours, it has travelled 120 miles; and so on. Notice that the ratio of distance travelled (in miles) to time travelled (in hours) is always 40:1.

$$40:1 \quad \text{or} \quad 80:2 \quad \text{or} \quad 120:3$$

In general, if we travel for a hours, at 5 miles per hour, we will have travelled 5b miles, for a ratio of distance to time of 5b : a or 5b : 1.

Concept: Speed is the ratio of distance to time.

We can write Speed in the ratio of distance to time" as the equation:

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

This equation can be rearranged as:

$$\text{distance} = \text{speed} \times \text{time}$$

and also as:

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

The units associated with speed hold, or remember these equations. For example, the speed of a car (in km/h) is usually given in "miles per hour" abbreviated "mph". The car **goes** essentially means "travels" so a car **goes** 40 miles in 1 hour means it takes distance (in miles) and time (in hours) and produces speed (in mph). Similarly, a train that travels 5 miles in 2 hours has a speed of 2.5 miles per hour.

$$\frac{5 \text{ miles}}{2 \text{ hours}} = 2.5 \text{ miles per hour} = 2.5 \text{ miles per hour}$$

Similarly, a car that travels at a speed of 70 miles per hour for 3 hours covers a distance of:

$$\text{distance} = \text{speed} \times \text{time} = 70 \text{ miles/hour} \times 3 \text{ hours} = 210 \text{ miles}$$

We can also think of speed as a type of conversion factor, which converts between hours and distance. To repeat our previous example, a car that travels at a speed of 70 miles per hour for 3 hours is essentially "converting" the 3 hours of time into a distance, as:

$$3 \text{ hours} \times \frac{70 \text{ miles}}{1 \text{ hour}} = 210 \text{ miles}$$

Problems

Problem 7.22 Jump to Solution

(a) How far does a car travelling at 75 miles per hour travel in 2.5 hours?

(b) If a truck travels at a constant speed and travels 230 miles in 3.5 hours, then at what speed do the truck travel?

(c) How long will a motorcycle travelling at 90 miles per hour travel to travel 450 miles?

Problem 7.23 Jump to Solution SOLVED! Jump to Solution

A freight train travels 1 mile in 1 minute 30 seconds. All three trains move towards either of the train tracks. Find the speed of each train.

Problem 7.24 Jump to Solution Source: CHALLENGE! Jump to Solution

If you could fly 60 minutes at a rate of 4 miles and then run for 30 minutes at a rate of 10 mph, then how many miles have you gone at the end of your run? Continuous? What was your average speed for the journey?

Problem 7.25 Jump to Solution

Oddly since the 90 miles from her home to her office at an average speed of 70 miles per hour. Running late, she encountered heavy traffic and travelled the first 40 miles at an average speed of 40 miles per hour. What was her average speed for the entire 90 miles?

Problem 7.26 Jump to Solution

Jump to Solution

Jump to Solution

(a) Jack travels for a hour at a rate of 5 miles per hour. Jill travels for 2 hours at a rate of 3 miles per hour. Who travels 100 miles last and who travels last known at 4 p.m., and biking at a constant speed of 15 miles per hour due east. At what time do they meet?

Problem 7.27 Jump to Solution

(a) On Monday, Yogi and Boo-Boo start at the same place and at the same time on a 400-mile circular track and run in opposite directions. Yogi runs at 5 miles per hour and Boo-Boo runs at 3 miles per hour. How many times do they meet (not including the start) after starting?

(b) On Tuesday, Yogi and Boo-Boo start at the same place and at the same time on a 400-mile circular track and run in the same direction. Yogi runs at 5 miles per second and Boo-Boo runs at 3 miles per second. In how many seconds will they first meet (not including the start)?

The train leaves Upper of speed problem gives you two of the three pieces of their speed, time, distance! and asks you to compute the third.

Problem 7.28 Jump to Solution

(a) How far does a car travelling at 75 miles per hour travel in 2.5 hours?

(b) If a truck travels at a constant speed and travels 230 miles in 3.5 hours, then at what speed do the truck travel?

(c) How long will a motorcycle travelling at 90 miles per hour travel to travel 450 miles?

Problem 7.29 Source: SOLVED! Jump to Solution

A freight train travels 1 mile in 1 minute 30 seconds. All three trains, how many miles will the train travel in 1 hour?

Problem 7.30 Source: CHALLENGE! Jump to Solution

If you could fly 60 minutes at a rate of 4 miles and then run for 30 minutes at a rate of 10 mph, then how many miles have you gone at the end of your run? Continuous? What was your average speed for the journey?

Problem 7.31 Jump to Solution

Oddly since the 90 miles from her home to her office at an average speed of 70 miles per hour. Running late, she encountered heavy traffic and travelled the first 40 miles at an average speed of 40 miles per hour. What was her average speed for the entire 90 miles?

Problem 7.32 Source: SOLVED! Jump to Solution

Jump to Solution

(a) Jack travels for a hour at a rate of 5 miles per hour. Jill travels for 2 hours at a rate of 3 miles per hour. Who travels 100 miles last and who travels last known at 4 p.m., and biking at a constant speed of 15 miles per hour due east. At what time do they meet?

Problem 7.33 Source: SOLVED!

(a) On Monday, Yogi and Boo-Boo start at the same place and at the same time on a 400-mile circular track and run in opposite directions. Yogi runs at 5 miles per hour and Boo-Boo runs at 3 miles per hour. How many times do they meet (not including the start) after starting?

(b) On Tuesday, Yogi and Boo-Boo start at the same place and at the same time on a 400-mile circular track and run in the same direction. Yogi runs at 5 miles per second and Boo-Boo runs at 3 miles per second. In how many seconds will they first meet (not including the start)?

Thanks! Nothing special about "miles" and "hours" in Problem 7.28. Speed can be measured in any units you want, as long as the units are consistent. Yogi and Boo-Boo, in which case, it should be expressed in "bears". Thus, we think the distance (in miles) by the time (in hours).

Problem 7.34 Source: SOLVED!

Yogi's train travels 1 mile 3 minutes 30 seconds. At this rate, how many miles will the train travel in 1 hour?

Solution for Problem 7.28: Give minutes 30 seconds is 3.5 minutes, so the train's speed is

$$\text{speed} = \frac{1 \text{ miles}}{3.5 \text{ minutes}} = \frac{2}{7} \text{ miles/minute}$$

There are 60 minutes an hour, so the distance travelled in 60 minutes is

$$\text{distance} = (60 \text{ minutes}) \cdot \frac{2}{7} \text{ miles/minute} = 17.14 \text{ miles}$$

Another way to solve Problem 7.28: It is reason as follows: the number of "7-minutes" in 3.5 minutes (3.5 seconds" intervals in a 60-minute hour is $\frac{60}{3.5} = 17.14$. The train covers a mile in each of these 60-minutes. Therefore, the train covers 60 miles an hour.

Problem 7.35 Source: SOLVED! Jump to Solution

If you walk 10 miles at a rate of 5 miles an hour and then for 30 minutes at a rate of 10 mph, then how many miles have you gone at the end of your run? Continuous? What was your average speed for the journey?

Solution for Problem 7.34: We can compute the walking distance and running distance separately. Here and there, You walk for 30 minutes, which means, as a rate of 5 mph, so the walking distance is $\frac{1}{2}$ hours $\times 5$ mph $= \frac{5}{2}$ miles. You run for 30 minutes, which is $\frac{1}{2}$ hour, at a rate of 10 mph, so the running distance is $\frac{1}{2}$ hours $\times 10$ mph $= 5$ miles. So the total distance you covered is $\frac{5}{2} + 5 = 7.5$ miles.

Now we can also compute the average speed for the journey. You covered 10 miles in $\frac{5}{2}$ hours, so your average speed was

$$\frac{10 \text{ miles}}{\frac{5}{2} \text{ hours}} = \frac{20}{5} \text{ miles/hour} = 4 \text{ mph}.$$

Problem 7.36 Source: SOLVED!

Oddly since the 90 miles from her home to her office at an average speed of 70 miles per hour. Running late, she encountered heavy traffic and travelled the first 40 miles at an average speed of 40 miles per hour. What was her average speed for the entire 90 miles?

Solution for Problem 7.35: You might think:

Bogus Solution: Oddly since half of her trip at 70 mph and the other half of her trip at 30 mph, her average speed is the average of 70 and 30, namely $\frac{70+30}{2} = 50$ mph.

This cannot average speeds in the same way, as we will see when we compute the amount of time the trip takes.

The trip from the home to the office was 50 miles at 70 mph, so it takes

$$\frac{50 \text{ miles}}{70 \text{ mph}} = \frac{5}{7} \text{ hours}$$

The trip from the office to the home was 50 miles at 30 mph, so it takes 1 hour. Thus, the entire 100 miles trip takes $\frac{5}{7}$ hours of travel time, and we can compute the average speed to be

$$\frac{100 \text{ miles}}{\frac{12}{7} \text{ hours}} = \frac{175}{3} \text{ mph} \neq 50 \text{ mph}.$$

Therefore the average speed of the round-trip journey is 50 miles per hour.

WARNING! Speeds do not usually "average" in the way that you might expect them to.

Exercise: There is a relationship between the speeds for the two portions of the journey in Problem 7.36 that will allow you to write a formula for the **harmonic mean** of the speeds for each half of the journey:

$$\frac{2}{\frac{1}{70} + \frac{1}{30}} = 42$$

Show generally the harmonic mean of two numbers a and b is

$$\frac{2}{\frac{1}{a} + \frac{1}{b}}$$

Harmonic mean is another kind of "average" that has many uses in advanced mathematics.

Problem 7.37 Source: SOLVED!

Suppose Jim travels for a 1 miles at a speed of 1 mph and then for 1 mile at a speed of 2 mph. Jim travels 100 miles this way and then for 100 miles at a speed of 100 mph. Jim travels 100 miles at a speed of 101 mph. Which trip did Jim take the most time?

Solution for Problem 7.36: Let's begin with the easy part. Jim travels 100 miles at 101 mph, so it takes $\frac{100 \text{ miles}}{101 \text{ mph}}$ hours of time.

Jim travels 100 miles at 100 mph, so it takes $\frac{100 \text{ miles}}{100 \text{ mph}} = 1$ hour of time.

Jim travels 1 miles at 1 mph and 1 miles at 2 mph, so it takes $\frac{1 \text{ miles}}{1 \text{ mph}} + \frac{1 \text{ miles}}{2 \text{ mph}} = 1.5$ hours of time.

Clearly, the 1 miles at 1 mph, and the 1 miles at 2 mph, Jim takes the most time. Why? Because Jim spends most of his time at 1 mph, so it takes him 1 hour to go 1 mile. Jim spends only a little time at 2 mph, so it takes him only 0.5 hours to go 1 mile. Jim spends most of his time at 100 mph, so it takes him only 1 hour to go 100 miles. Jim spends most of his time at 101 mph, so it takes him only 0.99 hours to go 100 miles.

Method 2: Think about how the average speed is each when. Jim takes 1 hour. Thus, the entire 100 miles trip takes 1 hours of travel time, and we can compute the average speed to be

$$\frac{100 \text{ miles}}{1 \text{ hour}} = 100 \text{ mph}.$$

Method 3: Jim travels 1 miles at a speed of 1 mph, so it takes 1 hour. Thus, the entire 100 miles trip takes 1 hours of travel time, and we can compute the average speed to be

$$\frac{100 \text{ miles}}{1 \text{ hour}} = 100 \text{ mph}.$$

Therefore the average speed of the round-trip journey is 50 miles per hour.

Concept: When there are two people or objects moving simultaneously, it is often easier to think about the distance between the objects, rather than trying to keep track of the objects separately.

Try this concept in the next problem.

Problem 7.38 Source: CHALLENGE!

(a) On Monday, Yogi and Boo-Boo start at the same place and at the same time on a 400-mile circular track and run in opposite directions. Yogi runs at 5 miles per hour and Boo-Boo runs at 3 miles per hour. How many times do they meet (not including the start) after starting?

(b) On Tuesday, Yogi and Boo-Boo start at the same place and at the same time on a 400-mile circular track and run in the same direction. Yogi runs at 5 miles per second and Boo-Boo runs at 3 miles per second. In how many seconds will they first meet (not including the start)?

(c) Where they met in the same direction, on both of the distances between them is zero. Yogi and Boo-Boo, in which case, it should be expressed in "bears". Thus, we have to think about the distance between them, as a rate of 2 ft / 5 ft = 0 miles per second. Do they will meet after 200 miles.

Source: CHALLENGE! Jump to Solution

Jump to Solution

Jump to Solution

7.6 Other Rates

Speed is just a special example of a **rate**. Whenever a quantity changes by a certain amount in a fixed unit of time, we have a rate. The idea of rate is very flexible and can be used in a lot of different situations.

Problems

Problem 7.28 Jump to Solution

Jason can type at a rate of 40 words per minute. How long will it take him to type a 2,000 word essay?

Problem 7.29 Jump to Solution

A hose fills a swimming pool at a rate of 0.5 gallons per second. If the pool's capacity is 9,000 gallons, then how many hours does it take for the hose to completely fill an empty pool?

Problem 7.30 Jump to Solution

Julie wants to give a 45-minute speech, and she speaks 120 words per minute. Her written notes contain 500 words per page. How many pages should she prepare?

Problem 7.31 Jump to Solution

Rajiv's car has tires that have circumference 75 inches. (The circumference of a tire is the distance around the outside of the tire.) How many revolutions will his tires make if Rajiv drives to a store that is $\frac{1}{4}$ mile away? (Assume that the drive is completely straight, and recall that 1 mile is 5,280 feet.)

Problem 7.32 Jump to Solution

If 5 woodchucks could chuck 50 cords of wood in 6 days, then how many cords of wood could 7 woodchucks chuck in 6 days?

Problem 7.33 Jump to Solution

Tom can paint Mr. Thatcher's fence in 6 hours, while Huck can paint it 8. Thatcher's fence in 5 hours. If they work together, then how long will it take them to paint the fence?

Just as with speed, the use of the word "per" is likely a signal that we are working with a rate. Also, just as with speed, we can use the units to our advantage when solving problems involving rates.

Here is a basic example:

Problem 7.28 7/7

Jason can type at a rate of 40 words per minute. How long will it take him to type a 2,000 word essay?

Solution for Problem 7.28: Method 1: Direct reasoning. Jason needs to type 2000 words, and for every 40 words that he will type, he will need 1 minute. Therefore, he needs $\frac{2000}{40} = 50$ minutes to type the entire 2000 words.

Method 2: Conversion factor. Jason's typing speed can be converted into "words" and vice versa, so we can use a conversion factor. We need to arrange the data so that the "words" units cancel and we are left with "minutes" units:

$$\text{time} = 2000 \text{ words} \cdot \frac{1 \text{ minute}}{40 \text{ words}} = \frac{2000}{40} \text{ minutes} = 50 \text{ minutes}.$$

So it takes Jason 50 minutes to type the essay. We can also easily check this answer: if he types for 50 minutes, and he types 40 words per minute, then he will type a total of

$$(50 \text{ minutes}) \cdot \left(40 \frac{\text{words}}{\text{minute}}\right) = (50 \cdot 40) \text{ words} = 2000 \text{ words},$$

as required. ◻

Concept: Use the units in your mathematical expressions to help you figure out how to use the information in the problem.

Problem 7.29 7/7

A hose fills a swimming pool at a rate of 0.5 gallons per second. If the pool's capacity is 9,000 gallons, then how many hours does it take for the hose to completely fill an empty pool?

Solution for Problem 7.29: First, we can compute how many seconds are necessary. This is just a basic conversion problem:

$$\begin{aligned}\text{time in seconds} &= 9000 \text{ gallons} \cdot \frac{1 \text{ second}}{0.5 \text{ gallons}} \\ &= \frac{9000}{0.5} \text{ seconds} \\ &= 18000 \text{ seconds}.\end{aligned}$$

We can also convert seconds to hours using conversion factors. This gives

$$18000 \text{ seconds} = (18000 \text{ seconds}) \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{18000}{60 \cdot 60} \text{ hours} = 5 \text{ hours}.$$

Thus it will take 5 hours to fill the pool. ◻

Concept: Unit conversions are just another type of rate. We can work with rates much like we do conversion factors. In particular, we them in mathematical expressions so that the units cancel in the way that you want.

Problem 7.30 7/7

Julie wants to give a 45-minute speech, and she speaks 120 words per minute. Her written notes contain 500 words per page. How many pages should she prepare?

Solution for Problem 7.30: We have a couple of different rates here: both the words per minute and the words per page. We could do this as a two-step problem, or can both rates at once.

Method 1: Count words first, then count pages. First, we compute how many words should be in the speech. Julie wants to talk for 45 minutes and she will use 120 words per minute, so she will speak $(45) \cdot (120) = 5400$ words. Then, we compute the number of pages: 5400 words at 500 words per page will need $\frac{5400}{500} = 10.8$ pages.

Method 2: Do all at once using conversion factors. We write an expression that cancels the units we don't want and leaves the unit we do want:

$$\text{number of pages} = (45 \text{ minutes}) \cdot \frac{120 \text{ words}}{1 \text{ minutes}} \cdot \frac{1 \text{ pages}}{500 \text{ words}} = \frac{45 \cdot 120}{500} \text{ pages} = 10.8 \text{ pages}.$$

◻

Problem 7.31 7/7

Rajiv's car has tires that have circumference 75 inches. (The circumference of a tire is the distance around the outside of the tire.) How many revolutions will his tires make if Rajiv drives to a store that is $\frac{1}{4}$ mile away? (Assume that the drive is completely straight, and recall that 1 mile is 5,280 feet.)

Solution for Problem 7.31: We can set this up as a product of conversion factors:

$$\begin{aligned}&\text{number of revolutions} \\ &= \left(\frac{1}{4} \text{ miles}\right) \cdot \frac{5280 \text{ feet}}{1 \text{ miles}} \cdot \frac{12 \text{ inches}}{1 \text{ feet}} \cdot \frac{1 \text{ revolutions}}{75 \text{ inches}} \\ &= \frac{5280 \cdot 12}{4 \cdot 75} \text{ revolutions} \\ &= 211.2 \text{ revolutions}.\end{aligned}$$

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Batman Part 1

Work problems are another particular type of rate problem. Here is a classic (if slightly confusing) example:

Problem 7.32 7/7

If 5 woodchucks could chuck 50 cords of wood in 6 days, then how many cords of wood could 7 woodchucks chuck in 6 days?

Solution for Problem 7.32: What information would be most useful? It would be helpful if we knew how much wood 1 woodchuck could chuck in 1 day. Fortunately, it is not too difficult to find this out:

If

5 woodchucks could chuck 50 cords of wood in 6 days,

and since 1 woodchuck chucks $\frac{1}{5}$ as much wood as 5 woodchucks, then we know that

1 woodchuck could chuck 10 cords of wood in 6 days.

Then, since a woodchuck can chuck $\frac{1}{6}$ as much wood in 1 day as she can chuck in 6 days, we conclude that

1 woodchuck could chuck 2.5 cords of wood in 1 day.

Now we use this to answer our problem. If

1 woodchuck could chuck 2.5 cords of wood in 1 day,

then

1 woodchuck could chuck (6 · 2.5) cords of wood in 6 days,

and thus

7 woodchucks could chuck (6 · 2.5 · 7) cords of wood in 6 days.

Thus the answer is 6 · 2.5 · 7 = 105 cords of wood. ◻

Problem 7.33 7/7

Tom can paint Mr. Thatcher's fence in 6 hours, while Huck can paint it 8. Thatcher's fence in 5 hours. If they work together, then how long will it take them to paint the fence?

Solution for Problem 7.33: Our plan is to determine their work rates, and then add them. In particular, Tom can paint $\frac{1}{6}$ of a fence per hour, and Huck can paint $\frac{1}{8}$ of a fence per hour. So, together they can paint $\frac{1}{6} + \frac{1}{8} = \frac{4}{24} + \frac{3}{24} = \frac{7}{24}$ of a fence per hour. Therefore, the time to paint the whole fence is

$$\begin{aligned}&\frac{1 \text{ fence}}{\text{hours per fence}} = \frac{30}{7} \text{ fence} = 2\frac{6}{7} \text{ hours},\end{aligned}$$

or a little over 2.7 hours. ◻

The key step in our solution to Problem 7.33 was considering how much work Tom and Huck each do per hour.

Concept: Work problems can often be solved by considering the amount of work each worker does per unit of time.

Problem Part 2 - Work Problems

Exercises

7.6.1: 7/7

Casey has to build a 100-foot-long fence. If it takes her 15 minutes to build 1 foot of the fence, then how many hours will it take her to complete the fence?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Solution Solution

Solution: Since it takes Casey 15 minutes to build each foot of the fence, it takes her $15 \cdot 100 = 1500$ minutes to build all 100 feet of the fence. We can then convert the minutes to hours:

$$1500 \text{ minutes} = \frac{1 \text{ hours}}{60 \text{ minutes}} \cdot \frac{1500}{60} \text{ hours} = \boxed{25 \text{ hours}}.$$

We also could have used conversion factors to do the whole problem from the beginning. Since she builds 1 foot per 15 minutes, we have

$$100 \text{ feet} \cdot \frac{15 \text{ minutes}}{1 \text{ feet}} \cdot \frac{1 \text{ hours}}{60 \text{ minutes}} = \frac{100 \cdot 15}{60} \text{ hours} = \boxed{25 \text{ hours}}.$$

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7.6.2: 7/7

Phil can type a page of his new novel in 20 minutes. If he writes for 8 hours, then how many pages will he type?

Preview Solution

You may type any additional notes you have here.

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Your Solution Solution

Solution: Phil produces 1 page every 20 minutes. So in 8 · 20 = 60 minutes (which is 1 hour), he produces 3 pages. Therefore, in 8 hours, he produces $3 \cdot 8 = \boxed{24 \text{ pages}}$.

Again, we can use conversion factors:

$$8 \text{ hours} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{1 \text{ page}}{20 \text{ minutes}} = \frac{8 \cdot 60}{20} \text{ pages} = \boxed{24 \text{ pages}}.$$

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7.6.3: Source: MGCMS 7/7

A kangaroo chases a rabbit that starts 150 feet ahead of the kangaroo. For every 12 foot hop of the kangaroo, the rabbit makes a 7 foot leap. How many leaps will the kangaroo have to make to catch up to the rabbit if the two animals always leap at the same time?

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Your Solution Solution

Solution: The kangaroo gains 5 feet on every leap. The kangaroo must gain 150 feet total, so it must jump $\frac{150}{5} = \boxed{30 \text{ times}}$ to catch the rabbit.

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7.6.4: Source: AMC 8 7/7

Maria buys computer disks at a price of 4 for $5 and sells them at a price of 3 for $5. How many computer disks must she sell in order to make a profit of $100?

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Your Solution Solution

Solution: We first determine how much money Maria makes per disk. Since she buys 4 disks for $5, the per-disk rate she pays is $\frac{5 \text{ dollars}}{4 \text{ disks}}$. She sells 3 disks for $5, so the per-disk rate she receives is $\frac{5 \text{ dollars}}{3 \text{ disks}}$. Therefore, her profit per disk is

$$\frac{5 \text{ dollars}}{3 \text{ disks}} - \frac{5 \text{ dollars}}{4 \text{ disks}} = \frac{5 \text{ dollars}}{12 \text{ disks}}.$$

Since this is the amount she makes for each disk, the number of disks she needs to make $100 is

$$\frac{100 \text{ dollars}}{\frac{5 \text{ dollars}}{12 \text{ disks}}} = \frac{100 \cdot 12}{5} \text{ disks} = \boxed{240 \text{ disks}}.$$

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7.6.5: Source: AMC 8 7/7

At the beginning of a trip, the mileage odometer read 56,299 miles. The driver filled the gas tank with 2 gallons of gasoline. During the trip, the driver filled the tank again with 12 gallons of gasoline when the odometer read 56,561. At the end of the trip, the driver filled the tank again with 20 gallons of gasoline. The odometer read 57,061. To the nearest tenth, what was the car's average miles-per-gallon for the entire trip?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Solution Solution

Solution: The entire trip was 57,061 − 56,299 = 762 miles. The tank was full after the driver added the 6 gallons of gas at the beginning. After that, the driver had to add a total of 12 + 20 = 32 gallons of gas to have the gas tank completely filled at the end. So, the car used 32 gallons of gas while traversing the 762 miles. That gives an average miles-per-gallon rate of

$$\frac{762 \text{ miles}}{32 \text{ gallons}} = \frac{381}{16} \frac{\text{miles}}{\text{gallon}} = \boxed{23.8 \frac{\text{miles}}{\text{gallon}}}.$$

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7.6.6: Source: MGCMS 7/7

A twelve-hour clock loses 1 minute every hour. Suppose it shows the correct time now. What is the least number of hours from now when it will again show the correct time?

Hide Solution | Reset

Your Solution Solution

Solution: The clock will again show the correct time when it has lost a total of 12 hours. Every hour, the clock loses 1 minute. We want the clock to lose 12 hours, which is 12 · 60 = 720 minutes. It will take 720 hours to lose 720 minutes, so it will take $\boxed{720 \text{ hours}}$ to lose 12 hours.

As with many of our other problems, we also could have used conversion factors to solve the problem:

$$1 \text{ hours time} \cdot \frac{1 \text{ minute lost}}{1 \text{ minute time}} \cdot \frac{12 \text{ hours lost}}{1 \text{ hours lost}} \cdot \frac{60 \text{ minutes lost}}{1 \text{ hours lost}} = 12 \cdot 60 \text{ hours time} = \boxed{720 \text{ hours}}.$$

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to peel the rest of the potatoes. Christen peels 5 potatoes in exactly four 4 minutes, so she peels $5 \cdot 4 = \boxed{20 \text{ potatoes}}$.

7.6.7: Source: AMC 8 7/7

Homer began peeling a pile of 44 potatoes at the rate of 3 potatoes per minute. Four minutes later, Christen joined him and peeled at the rate of 5 potatoes per minute. When they finished, how many potatoes had Christen peeled?

Hide Solution | Reset

Your Solution Solution

Solution: Homer peels 3 potatoes a minute, so in the four minutes he works alone, he peels 3 · 4 = 12 potatoes. This leaves 44 − 12 = 32 potatoes for them to peel together. Together, they peel 3 + 5 = 8 potatoes per minute, so they need

$$32 \text{ potatoes} \cdot \frac{4 \text{ minutes}}{8 \text{ potatoes}} = 4 \text{ minutes}$$

to peel the rest of the potatoes. Christen peels 5 potatoes a minute for 4 minutes, so she peels $5 \cdot 4 = \boxed{20 \text{ potatoes}}$.

7.6.8: Source: MGCMS 7/7

The cold-water faucet of a bath tub can fill the tub in 15 minutes. The drain, when opened, can empty the full tub in 30 minutes. Suppose the tub is empty and the faucet and drain are both opened at the same time. How long will it take to fill the tub?

Hide Solution | Reset

Your Solution Solution

Solution: Since the faucet can fill the tub in 15 minutes, it fills $\frac{1}{15}$ of the tub every minute. Similarly, the drain empties $\frac{1}{30}$ of the tub every minute. Therefore, the fraction of the tub that is filled each minute when the faucet and drain are both open is

$$\frac{1}{15} - \frac{1}{30} = \frac{2}{30} - \frac{1}{30} = \frac{1}{30}.$$

Since $\frac{1}{30}$ of the tub is filled each minute, it will take $\boxed{30 \text{ minutes}}$ for the tub to fill.

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7.6.9: 7/7

Roger can shovel his family's driveway in 1 hour. His older sister Alexis can shovel the driveway in $\frac{1}{2}$ hour. If they work together, then how long will it take them to shovel the driveway?

Hide Solution | Reset

Your Solution Solution

Solution: Roger can shovel the driveway in an hour, he can shovel $\frac{1}{1}$ of the driveway every minute. Similarly, Alexis can shovel the driveway in 30 minutes, so she shovels $\frac{1}{30}$ of the driveway each minute. Together, each minute they shovel

$$\frac{1}{1} + \frac{1}{30} = \frac{30}{30} + \frac{1}{30} = \frac{31}{30}$$

of the driveway. Therefore, they need $\boxed{30 \text{ minutes}}$ to shovel the entire driveway.

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7.6.10: Source: MGCMS 7/7

Three water pipes are used to fill a swimming pool. The first pipe alone takes 6 hours to fill the pool, the second pipe alone takes 12 hours to fill the pool, and the third pipe alone takes 24 hours to fill the pool. If all three pipes are opened at the same time, then how long will it take to fill the pool?

Hint: Problems involving rates and work can often be solved by thinking about how much work is done per unit of time.

Hint: What fraction of the pool is filled each hour?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Solution Solution

Solution: The first pipe fills $\frac{1}{6}$ of the pool in 1 hour, the second pipe fills $\frac{1}{12}$ of the pool in 1 hour, and the third pipe fills $\frac{1}{24}$ of the pool in 1 hour. So together the pipes fill

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{24} = \frac{4}{24} + \frac{2}{24} + \frac{1}{24} = \frac{7}{24}$$

of the pool in 1 hour. Therefore, the three pipes together will take $\boxed{7 \text{ hours}}$ to fill the pool.

7.7 Summary

A ratio is used to compare the *relative* quantities of two or more groups or items. However, a ratio only compares the quantities to each other–it doesn't tell us the actual values of the quantities. For example, suppose that you know that a certain history class has a ratio of girls to boys of $2:3$. All this tells you is that for every 2 girls, there are 3 boys. It doesn't tell you how many boys or girls there are.

Concept:	A ratio gives a *relative* comparison of two quantities. It doesn't tell you anything about the total amount of the quantities.

Definition:	To **simplify** a ratio means to write it as a ratio of integers with no common factor greater than 1.

Important:	Suppose we are using a ratio to compare two quantities that together make up a group (such as girls and boys in a class). If the two quantities are in the ratio $a:b$, then the first quantity makes up $\frac{a}{a+b}$ of the whole, and the second quantity makes up $\frac{b}{a+b}$ of the whole.

Whenever we have two ratios that are equal, we have a **proportion**. The most common usage of proportion is when we have two changing quantities that are related in such a way that their ratio doesn't change.

A commonly used ratio is **speed**, which is the ratio of distance to time. We can write this as an equation as

$$\text{speed} = \frac{\text{distance}}{\text{time}}.$$

This equation can be rearranged as

$$(\text{speed}) \cdot (\text{time}) = \text{distance}$$

and also as

$$\text{time} = \frac{\text{distance}}{\text{speed}}.$$

The units associated with speed help us remember these equations. The word **per** essentially means "divided by," so a speed in miles per hour means to take distance (in miles) and divide by time (in hours).

WARNING!!	Speeds do not usually "average" in the way that you might expect them to.

Speed is just a special example of a **rate**. Whenever a quantity changes by a certain amount in a fixed unit of time, we have a rate. Just as with speed, the use of the word "per" is often a signal that we are working with a rate. Also, just as with speed, we can use the units to our advantage when solving problems.

Concept:	Pay close attention to units in word problems! Use the units in your mathematical expressions to help you figure out how to use the information in the problem.

Unit conversions are just another type of rate. We can work with rates much like we work with conversion factors. In particular, we use them in mathematical expressions so that the units cancel in the way that we want.

Challenge Problems

7.58:

Source: MATHCOUNTS

A stack of 45 dimes is divided into three piles in the ratio $\frac{1}{6}:\frac{1}{3}:\frac{1}{4}$. How many dimes are in the pile with the least number of dimes?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The common denominator of the fractions is 12, so we can convert all the terms in the ratio to integers by multiplying them by 12:

$$\frac{1}{6}:\frac{1}{3}:\frac{1}{4}=12\cdot\frac{1}{6}:12\cdot\frac{1}{3}:12\cdot\frac{1}{4}=2:4:3.$$

The pile with the least number of dimes has $\frac{2}{2+4+3}=\frac{2}{9}$ of the dimes, so that pile has $\frac{2}{9}\cdot 45=\boxed{10}$ dimes.

7.59:

Five workers together can build a road in 20 days. Suppose every worker works at the same rate. If three workers work on the road for 10 days before eleven more workers join them, then how long total will it take to build the road?

Hint: Problems involving rates and work can often be solved by thinking about how much work is done per unit of time.

Hint: How long would it take one worker to build the whole road? What fraction of the road does one worker complete in one day?

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Because 5 workers can build a road in 20 days, each worker can build $\frac{1}{5}$ of a road in 20 days. Therefore, each worker can build 1 road working alone in $20\cdot 5=100$ days. So, each worker builds $\frac{1}{100}$ road each day. Therefore, three workers together build $\frac{3}{100}$ of a road each day, so in 10 days they build $10\cdot\frac{3}{100}=\frac{3}{10}$ of a road. At this point, they have $1-\frac{3}{10}=\frac{7}{10}$ of the road to finish.

After the 11 extra workers join in, the 14 workers build $\frac{14}{100}$ of the road each day. So, to finish the remaining $\frac{7}{10}$ of a road, the workers must work for

$$\frac{\frac{7}{10}\ \text{road}}{\frac{14}{100}\ \frac{\text{road}}{\text{day}}}=\frac{7}{10}\cdot\frac{100}{14}\ \text{days}=5\ \text{days}.$$

Therefore, the road is built in $10+5=\boxed{15\ \text{days}}$.

7.60:

Source: MOEMS

When Paul crossed the finish line of a 60-meter race, he was ahead of Robert by 10 meters and ahead of Sam by 20 meters. Suppose Robert and Sam continue to race to the finish line without changing their rates of speed. By how many meters will Robert beat Sam?

Hint: By how many meters would Robert beat Sam if the race were 50 meters? 100 meters? 150 meters?

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: When Paul finishes, Robert has run $60-10=50$ meters and Sam has run $60-20=40$ meters. Therefore, when Robert and Sam run for the same amount of time, Sam covers $\frac{40}{50}=\frac{4}{5}$ of the distance that Robert covers. So, while Robert runs the final 10 meters of the race, Sam runs $\frac{4}{5}\cdot 10=8$ meters. This means Robert's lead over Sam increases by 2 more meters, and he beats Sam by $10+2=\boxed{12}$ meters.

7.61:

Source: MATHCOUNTS

If $4:x^2=x:16$, then what is the value of x?

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Multiplying both parts of $4:x^2$ by x gives $4:x^2=4x:x^3$, and multiplying both parts of $x:16$ by 4 gives $x:16=4x:64$. Substituting these into $4:x^2=x:16$ gives $4x:x^3=4x:64$, so $x^3=64$, which gives $x=\boxed{4}$.

We also could have written the ratios in the given equation as fractions to get $\frac{4}{x^2}=\frac{x}{16}$. Multiplying both sides by x^2 and by 16 gets rid of the denominators and gives $4\cdot 16=x\cdot x^2$, so $64=x^3$ and $x=\boxed{4}$.

7.62:

Source: MATHCOUNTS

The Big Telescope Company sells circular mirrors. Their largest mirrors have radii of 5 meters and their smallest mirrors have radii of 1 meter. The cost of every mirror is proportional to the cube of the mirror's radius. What is the ratio of the total cost of 25 of the company's smallest mirrors to the cost of one of the company's largest mirrors?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Let the cost of one of the smallest mirrors be x, so the cost of 25 of these mirrors is $25x$. The radius of the largest mirror is 5 times the radius of the smallest mirror. The cost of the mirror is proportional to the cube of the radius, so the cost of the largest mirror is $5^3=125$ times the cost of the smallest mirror. Therefore, the cost of the largest mirror is $125x$, and the desired ratio is $25x:125x$. Dividing both parts of the ratio by $25x$ gives $\boxed{1:5}$.

7.63:

Source: MOEMS

Kim was elected class president. She received 3 votes for every 2 that Amy got. No one else ran. However, if 8 of the people who voted for Kim had voted for Amy instead, Kim would have received only 1 vote for every 2 that Amy would have gotten. How many people voted?

Hint: Assign a variable so you can write expressions and set up equations. What's often a useful way to assign a variable in a ratio problem?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The ratio of Kim's votes to Amy's votes is $3:2$, so Kim received $3x$ votes and Amy received $2x$ votes for some value of x. If 8 of the people had voted for Amy instead of Kim, then Kim would have $3x-8$ votes and Amy would have $2x+8$ votes. If this had happened, then Amy would have twice as many votes as Kim, so $2x+8=2(3x-8)$. Expanding the right-hand side gives $2x+8=6x-16$. Subtracting $2x$ from both sides and adding 16 to both sides gives $24=4x$, so $x=6$. Since $3x+2x=5x$ people voted, the number of voters is $5\cdot 6=\boxed{30}$.

7.64:

When the Slowpoke Marathon began, the ratio of runners to joggers was 2 to 19. If 4200 participants began the race (each either a runner or a jogger, but not both), and 500 joggers dropped out of the race but all the rest of the participants finished, then what was the ratio of runners to joggers among those who finished the race?

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: First, we determine how many joggers started the race. Since the ratio of runners to joggers was $2:19$ at the start of the race, the joggers are $\frac{19}{2+19}=\frac{19}{21}$ of the participants. There were 4200 participants at the start of the race, so $4200\cdot\frac{19}{21}=3800$ of them are joggers. The other $4200-3800=400$ are runners. Since 500 joggers dropped out, only $3800-500=3300$ finished. Therefore, the ratio of runners to joggers among those who finished is $400:3300=\boxed{4:33}$.

7.65:

Source: AMC 8

The students in Mrs. Reed's English class are reading the same 760-page novel. Three friends, Alice, Bob, and Chandra, are in the class. Alice reads a page in 20 seconds, Bob reads a page in 45 seconds, and Chandra reads a page in 30 seconds.

(a) If Bob and Chandra both read the whole book, then Bob will spend how many more seconds reading than Chandra?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: Bob spends 15 seconds more on each page, so he spends

$$760\ \text{pages}\cdot\frac{15\ \text{seconds}}{1\ \text{page}}=\boxed{11400\ \text{seconds}}$$

more than Chandra.

(b) Chandra and Bob, who each have a copy of the book, decide that they can save time by "team reading" the novel. In this scheme, Chandra will read from page 1 to a certain page and Bob will read from the next page through page 760, finishing the book. When they are through, they will tell each other about the parts they read. What is the last page that Chandra should read so that she and Bob spend the same amount of time reading the novel?

Hint: If we know what fraction of the book Chandra reads, then we can figure out how many pages she reads.

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: In 90 seconds, Bob reads 2 pages and Chandra reads 3 pages. So, for any fixed amount of time, the ratio of the amount Bob reads to the amount Chandra reads is $2:3$. Therefore, Chandra must read $\frac{3}{2+3}=\frac{3}{5}$ of the book, and during that time Bob will read the other $\frac{2}{5}$. This means Chandra should read $\frac{3}{5}\cdot 760=456$ pages, which means she should stop reading after completing page $\boxed{456}$.

(c) Before Chandra and Bob start reading, Alice says she would like to team read with them. If they divide the book into three sections so that each reads for the same length of time, then how many seconds will each have to read?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: In 90 seconds, Alice reads $\frac{90}{20}=4.5$ pages. So, in 90 seconds, the three together read $4.5+2+3=9.5$ pages. Therefore, the total amount of time they need to read is

$$760\ \text{pages}\cdot\frac{90\ \text{seconds}}{9.5\ \text{pages}}=\boxed{7200\ \text{seconds}}.$$

7.66★:

Source: AMC 8

Buses from Dallas to Houston leave every hour on the hour. Buses from Houston to Dallas leave every hour on the half hour. The trip from one city to the other takes 5 hours. Assuming the buses travel on the same highway, how many Dallas-bound buses does a Houston-bound bus pass on the highway (not in the station)?

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We count the number of Dallas-bound buses that a single Houston-bound bus passes. There are two groups of Dallas-bound buses that we have to consider. First, there are the buses that were on the way to Dallas at the time the Houston-bound bus leaves Dallas. Second, there are the buses that leave Houston after the Houston-bound bus starts but before it arrives at Houston.

In order for a bus to be on the way to Dallas when the Houston-bound bus leaves, it must have left Houston less than 5 hours before the Houston-bound bus leaves. There is one such bus each hour, for a total of 5 buses.

In order for a Dallas-bound bus to leave Houston after the Houston-bound bus leaves but before the Houston-bound bus arrives, the Dallas-bound must leave Houston within 5 hours after the Houston-bound bus leaves. Again, there is one such bus each hour, for a total of 5 more buses that the Houston-bound bus passes.

Therefore, the Houston-bound bus passes $5+5=\boxed{10}$ Dallas-bound buses.

Review Problems

7.34. ⏱ 4 ☆ 7

Jane's pet turtle is asleep at the ground at $3 - 2i$ where we fix units, then how many steps are left?

You may type any additional notes you have here.

Hide Solution | Email

Your Submission: Solution

Solution: Multiplying both parts of the ratio by 5 gives

$$\text{ratio} = \text{drop} = 15 - 2T,$$

so there are $\boxed{27}$ drops at the present.

7.35. ⏱ 4 ☆ 7

Suppose it takes 5 girls at a summer camp 4 hrs. If the total number of students at the camp is 140, then how many boys are at the camp?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Email

Your Submission: Solution

Solution: Since the ratio of boys to girls at the camp is $4 : 5$, the boys are $\frac{4}{4+5} = \frac{4}{9}$ of the students at the camp. There are 100 students, so there are $\frac{4}{9} \cdot 100 = \boxed{44}$ boys.

7.36. ⏱ 4 ☆ 7 Source: MATHCOUNTS

Consider one pound is where ounces, what is the ratio of 1 pound, 4 ounces to 5 pounds, 10 ounces?

Preview Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We express both weights in ounces, so that we can compare them. To pound, 4 ounces equals $1 \cdot 16 + 4 = 20$ ounces; 5 pounds, 10 ounces equals $5 \cdot 16 + 10 = 90$ ounces. So, the ratio of the given weights is $20 : 90$, which is $\boxed{2 : 9}$ in simplest form.

7.37. ⏱ 4 ☆ 7

A board that is 12 meters long is cut into 2 pieces whose lengths have a ratio of $7 : 5$. What is the length of the longer piece?

You may type any additional notes you have here.

Hide Solution | Email

Your Submission: Solution

Solution: The longer piece is $\frac{7}{7+5} = \frac{7}{12}$ of the whole board, so the length of the longer piece is $\frac{7}{12} \cdot (12 \text{ meters}) = \boxed{7 \text{ meters}}$.

7.38. ⏱ 4 ☆ 7

The ratio of boys to girls in an assembly is 4 to 3. How many students are present if there are 67 girls?

You may type any additional notes you have here.

Hide Solution | Email

Your Submission: Solution

Solution 1: The number of girls is $\frac{3}{3+4} = \frac{3}{7}$ of the total number of students, so the total number of students is $\frac{7}{3}$ times the number of girls. Therefore, there are $\frac{7}{3} \cdot 67 = \boxed{157}$ students.

Solution 2: Let x be the number of times a unit fraction is 67 and the number of girls is $3x$. Since there are 67 girls, we have $3x = 67$, so $x = \frac{67}{3}$. Therefore, the total number of students is $7x = 7x = 7(\frac{67}{3}) = \boxed{157}$.

7.39. ⏱ 4 ☆ 7 Source: MOEMS

For every $\$5$ Mariana spends, Anita spends $\$3$. Anita spends $\$120$ more than Mariana does. How many dollars does Anita spend?

You may type any additional notes you have here.

Hide Solution | Email

Your Submission: Solution

Solution: The ratio of the amount Mariana spends to the amount Anita spends is $5 : 3$. So, for some value of x, Mariana spends $5x$ dollars and Anita spends $3x$ dollars. Anita spends $\$120$ more than Mariana spends, so $3x - 5x = 120$. Simplifying gives $-2x = 120$, so $x = -60$. Therefore, Anita spends $3x = \boxed{-180}$ dollars.

7.40. ⏱ 4 ☆ 7

Originally, there are 20 fish in a tank and each fish is a guppy or an angelfish. The ratio of guppies to angelfish in the tank is $9 : 2$. Twenty more fish are added to the tank. Each new fish is either a guppy or an angelfish. The ratio of guppies to angelfish after the fish are added is $2 : 3$. How many guppies were added to the tank?

You may type any additional notes you have here.

Hide Solution | Email

Your Submission: Solution

Solution: Initially, the fraction of fish that are guppies is $\frac{9}{9+2} = \frac{9}{11}$, so there are $\frac{9}{11} \cdot 20 = 18$ guppies in the tank.

After adding the new fish, there are $20 + 20 = 40$ fish in the tank. Since the ratio of guppies to angelfish is now $2 : 3$, we know that $\frac{2}{2+3} = \frac{2}{5}$ of these fish are guppies. So, there are $\frac{2}{5} \cdot 40 = 16$ guppies in the tank, which means $16 - 12 = \boxed{4}$ guppies were added.

7.41. ⏱ 4 ☆ 7 Source: MATHCOUNTS

The first partner in a business decides to split the profits of their company in the ratio $7 : 8 : 5$. If the profit one year is $\$35,000$, then what is the largest amount of profit received by one of the four partners?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: The person who receives the largest amount gets $\frac{8}{7+8+5} = \frac{8}{20} = \frac{2}{5}$ of the total profit. So, this person receives $\frac{2}{5} \cdot (\$35,000) = \boxed{\$14,000}$.

7.42. ⏱ 4 ☆ 7

The statue of Abraham Lincoln in the Lincoln Memorial in Washington, D.C., is 6 meters tall. On the back of the \$5 bill, the statue of Abraham Lincoln is 2.5 cm tall. If a person is standing 25 cm tall on the back of the \$5 bill, and assuming that the distance from the person's head to the actual Memorial in Washington, D.C., is 1 km, how many meters tall is the person's head?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: The statue's height and the Memorial's height are in proportion. On the back of the \$5 bill, the Memorial is 25 mm and the statue is 5 mm, so the Memorial is 5 times as tall as the statue. Therefore, the real Memorial is 5 times as tall as the real statue, or $5 \cdot (6 \text{ meters}) = \boxed{30 \text{ meters}}$ tall.

7.43. ⏱ 4 ☆ 7

On a map, two mountains are $\frac{1}{2}$ inches apart. If $\frac{1}{4}$ of an inch on the map represents 80 miles, then how many miles apart are the two mountains?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution 1: Since $\frac{1}{4}$ inch represents 80 miles, we know that $\frac{1}{2} = 1$ inch represents $2 \cdot 80 = 160$ miles. Therefore, $\frac{1}{2}$ inches represents

$$\frac{1}{2} \cdot 160 = \frac{1}{2} \cdot 160 = 2^{-1} \cdot 20 = \boxed{80 \text{ miles}}.$$

Solution 2: We use conversion factors. Since $\frac{1}{4}$ inch corresponds to 80 miles, $\frac{1}{2}$ inches corresponds to

$$\left(\frac{1}{2} \text{ inches}\right) \cdot \frac{80 \text{ miles}}{\frac{1}{4} \text{ inches}} = \frac{1}{2} \cdot \frac{80}{\frac{1}{4}} \cdot 2 \text{ miles}$$
$$= \boxed{160 \text{ miles}}.$$

7.44. ⏱ 4 ☆ 7

Jane's pet turtle is 14 days, and a mile is 5 furlongs. If a desert caravan travels 15 miles per day, then how many furlongs does it travel in a fortnight?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: We set up conversion factors:

$$30 \frac{\text{miles}}{\text{day}} \cdot 5 \frac{\text{miles}}{\text{furlong}} \cdot 8 \frac{\text{furlongs}}{\text{fortnight}} \cdot 14 \frac{\text{days}}{\text{fortnight}}$$
$$= (30 \cdot 5 \cdot 8 \cdot 14) \frac{\text{miles}}{\text{day}} \cdot \frac{\text{furlongs}}{\text{miles}} \cdot \frac{\text{days}}{\text{fortnight}}$$
$$= (1120) \frac{\text{furlongs}}{\text{fortnight}}.$$

7.45. ⏱ 4 ☆ 7 Source: MATHCOUNTS

A trivial by trivial photo is enlarged proportionally such that its smaller dimension is now 1 foot 3 inches. How many inches are in the larger dimension?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: The ratio of the shorter dimension to the larger dimension is $3 : 5$. The smaller dimension of the enlarged picture is $12 + 3 = 15$ inches. Multiplying both parts of $3 : 5$ by 5 gives $15 : 25$, so the larger dimension of the picture is $\boxed{25 \text{ inches}}$.

7.46. ⏱ 4 ☆ 7

On my tourist map of Quebec, the distance between the state representative Montreal and Quebec City is 10 cm. If the scale of the map is $1 \text{ cm} = 5 \text{ km}$, then how many km are it from Montreal and Quebec City? (Note that $55 \text{ cm} = 1 \text{ m}$.)

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: The ratio of the distance on the map to the actual distance is $\frac{1}{5}$ cm : 1 km. Integers are easier to work with than fractions, so we multiply both parts of the ratio by 5 to give 1 cm : 5 km. Next, we notice that the distance on the map between Montreal and Quebec City is 10 cm, so we multiply both parts of the ratio by 10 to give 10 cm : 50 km. Finally, we note that there are 10 cm in every 1 m, so we multiply 10 cm : 50 km. Multiplying both parts of the ratio 5 km by 10 gives $\boxed{50 \text{ km}}$.

7.47. ⏱ 4 ☆ 7

Ducks weighed from 1 to 4 pounds per gallon, and the density of liquid D is 5 pounds per gallon. What quantity of liquid D weighs the same as 30 gallons of liquid B?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: Ducks weighed from 1 to 4 pounds per gallon, and the density of liquid D is 5 pounds per gallon. What quantity of liquid D weighs the same as 30 gallons of liquid B?

We could also set up two conversion factors as:

$$30 \text{ gal of } B \to 30 \text{ gal of } A \cdot \frac{4 \text{ lb of } A}{1 \text{ gal of } A} \cdot \frac{1 \text{ lb of } B}{1 \text{ lb of } A} \cdot \frac{1 \text{ gal of } D}{4 \text{ lb of } B}$$
$$= \frac{30 \cdot 4}{1 \cdot 1} \cdot \frac{1}{4} \text{ gal of } B$$
$$= \boxed{30 \text{ gal of } B}$$

7.48. ⏱ 4 ☆ 7 Source: MOEMS

Suppose a bag of liquid A weighs 20 pounds per gallon, and the density of liquid B is 4 pounds per gallon. What quantity of liquid B weighs the same as 30 gallons of liquid B?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: Bob's questionnaire ate his mastercard in broken. He is sighing at a constant speed. He knows himself and finds that it takes him 1 minute and 30 seconds to chin 1 mile. How fast is he sighing in miles per hour?

Preview Solution

You may type any additional notes you have here.

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Your Solution: Solution

Solution: Since 1 minute is 60 seconds, he sighs 1 mile in 90 seconds. We can use conversion factors to convert "miles per second" to "miles per hour":

$$\frac{1 \text{ mile}}{90 \text{ seconds}} \cdot \frac{60 \text{ seconds}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}}$$
$$= \frac{60 \cdot 60}{90} \text{ miles per hour}$$
$$= \boxed{40 \text{ miles per hour}}.$$

7.50. ⏱ 4 ☆ 7 Source: AMC 8

Elisa earns tips in the pool. When she first started, she completed 10 laps in 25 minutes. Now she can finish 12 laps in 24 minutes. By how many minutes has her time improved her tip factor?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: When Elisa started, it took her 25 minutes to cover 10 laps, so each lap took her $\frac{25}{10} = \frac{5}{2}$ minutes. Now, she swims 12 laps in 24 minutes, so each lap takes $\frac{24}{12} = 2$ minutes. Since $\frac{5}{2} = 2\frac{1}{2}$, she has improved her lap time by $\frac{1}{2}$ minutes.

7.51. ⏱ 4 ☆ 7

A northbound train from Miami to Jacksonville made the 324-mile journey at an average speed of 50 miles per hour. On its southbound return trip, the train covers the same distance at an average speed of 48 miles per hour. To the nearest tenth of a mile per hour, what was the train's average speed for the 648-mile round trip journey?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: The northbound train travels 324 miles at 50 miles per hour, so its trip takes

$$\frac{324 \text{ miles}}{50 \text{ miles per hour}} = \frac{324}{50} \text{ hours} = 6.48 \text{ hours}.$$

The southbound train travels 324 miles at 48 miles per hour, so its trip takes

$$\frac{324 \text{ miles}}{48 \text{ miles per hour}} = \frac{324}{48} \text{ hours} = 6.75 \text{ hours}.$$

Combining these, the train covers $324 + 324 = 648$ miles in $6.48 + 6.75 = 13.23$ hours, for an average speed of

$$\frac{648 \text{ miles}}{13.23 \text{ hours}} = \frac{648}{13.23} \text{ mph} = 48.\overline{9} \text{ mph} \approx \boxed{48.9 \text{ miles per hour}}.$$

7.52. ⏱ 4 ☆ 7 Source: MOEMS

Jane's train is 270 miles from New York City along a certain route. A car starts from Boston at 1:00 PM and travels along the route from Boston to New York at a steady rate of 60 mph. Another car starts from New York at 1:30 PM and travels along this route toward Boston at a steady rate of 90 mph. At what time do the cars pass each other?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: During the first half hour of its journey, the car from Boston goes

$$\text{dist} = \frac{\text{dist}}{\text{time}} \cdot \text{time} = 60 \cdot \frac{1}{2} = 30 \text{ miles}.$$

So, at 1:00 PM, the cars are $270 - 30 = 270$ miles apart. Therefore, the distance between the two cars decreases by $90 + 60 = 150$ miles per hour. Since they are 270 miles apart, the distance decreases by 90 miles per hour. It takes

$$\frac{\text{dist}}{\text{rate}} = \frac{90 \text{ miles}}{150 \text{ miles per hour}} = 3 \text{ hours}$$

for the distance to decrease to 0. Therefore, the cars meet at $\boxed{4:30 \text{ PM}}$.

7.53. ⏱ 4 ☆ 7

A swimming pool is one yard long, 100 gallons of water. If water evaporates at the rate of 10.5 gallons per day and no other water is added or removed, then how much water will be in the pond after 30 days?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: In 30 days, the amount of water that evaporates is

$$(30 \text{ days}) \cdot \left(10.5 \frac{\text{gallons}}{\text{day}}\right) = 315 \text{ gallons}.$$

Therefore, the amount of water left after 30 days is $1000 - 315 = \boxed{685 \text{ gallons}}$.

7.54. ⏱ 4 ☆ 7 Source: MOEMS

Vegas has brown cattle that are to provide light. Cassie, 8 cents for month. 12 minutes. Cassie 12 turns brown as fast as Leslie. 2 Leslie 2 turns three times as fast as cattle. 2! What is the greatest total number of minutes of light that all three cattle can provide?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: Since cassie 12 turns twice as fast as cattle, A, it takes half as long for cattle B to turn out as cattle A. Therefore, cattle B turns for $\frac{12}{2} = 6$ minutes. Similarly, cattle C turns for one third as long as cattle B, so cattle C turns for $\frac{6}{3} = 2$ minutes. Therefore, if we turn one cattle at a time, the cattle can give $12 + 6 + 2 = \boxed{20 \text{ minutes}}$ of light.

7.55. ⏱ 4 ☆ 7

Working alone, Jaime can mow her lawn in 75 minutes. If Bob helps her, then the two can mow the lawn in 30 minutes. How long does it take Bob to mow the lawn alone?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: Since Jaime can mow the lawn in 75 minutes, she can mow $\frac{1}{75}$ of her lawn each minute. Therefore, in 30 minutes, she mows $\frac{30}{75} = \frac{2}{5}$ of the lawn. This means that Bob mows the other $1 - \frac{2}{5} = \frac{3}{5}$ of the lawn in 30 minutes. There are several ways we can finish from here.

Method 1: Since Bob mows $\frac{3}{5}$ of the lawn in 30 minutes, he can mow $\frac{1}{5}$ of the lawn in $\frac{30}{3} = 10$ minutes, which means he can mow the lawn once in $\boxed{50 \text{ minutes}}$.

We could also use conversion factors to find rate of lawn mowing, the times $\frac{1}{5}$ of the lawn in 30 minutes, so his rate is

$$\frac{30 \text{ minutes}}{\frac{1}{5} \text{ lawn}} = \frac{30 \text{ minutes}}{\frac{1}{5} \text{ lawn}} = \boxed{150 \text{ minutes}} \text{ per lawn}.$$

7.56. ⏱ 4 ☆ 7

Cactus is going on vacation from Mexico to Canada, and a bond stop in New York. He forgot to exchange his pesos for British pounds, so he stops off in New York. He exchanges 2000 pesos for 100 British pounds for his trip. 12.5 Mexican pesos can be exchanged for 1 dollar, and 1 dollar can be exchanged for 0.62 pounds. To the nearest peso, how many pesos will Cactus have to exchange in order to get 2000 British pounds?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: Applying conversion factors for each of the exchanges, we have

$$2000 \text{ pounds} \cdot \frac{100 \text{ pounds}}{2000 \text{ pounds}} \cdot \frac{1 \text{ dollar}}{0.62 \text{ pounds}} \cdot \frac{12.5 \text{ pesos}}{1 \text{ dollar}}$$
$$= \frac{2000 \cdot 12.5}{0.62} \text{ pesos} = \boxed{50{,}000 \text{ pesos}}.$$

7.57. ⏱ 4 ☆ 7

Four short-order cooks can make 24 omelets in 10 minutes. If a diner gets a trip order for 90 omelets that needs to be ready in 15 minutes, then how many more cooks do they need to complete the order on time?

You may type any additional notes you have here.

Hide Solution | Email

Your Solution: Solution

Solution: We know that

4 short-order cooks can make 24 omelets in 10 minutes

1 cook can make $\frac{24}{4}$ omelets in 4 cooks can, so

1 short-order cook can make 6 omelets in 10 minutes.

If the cook has 15 minutes to make 90 omelets, then he has $\frac{15}{10} = \frac{3}{2}$ times as much time. So, he can make $\frac{3}{2}$ times as many omelets. Since $\frac{3}{2} \cdot 6 = 9$ we have

1 short-order cook can make 9 omelets in 15 minutes.

Since we need 90 omelets in these 15 minutes, and each cook makes 9 omelets in 15 minutes, we need $\frac{90}{9} = \boxed{10 \text{ cooks}}$.

8

8.1 What is a Person?

Question 1 Question 2 Question 3 Question 4 Question 5 Question 6 Question 7 Question 8 Question 9 Question 10 Question 11 Question 12 Question 13 Question 14 Question 15 Question 16 Question 17 Question 18 Question 19 Question 20 Question 21 Question 22 Question 23 Question 24 Question 25 Question 26 Question 27 Question 28 Question 29 Question 30 Question 31 Question 32 Question 33 Question 34 Question 35 Question 36 Question 37 Question 38 Question 39 Question 40 Question 41 Question 42 Question 43 Question 44 Question 45 Question 46 Question 47 Question 48 Question 49 Question 50 Question 51 Question 52 Question 53 Question 54 Question 55 Question 56 Question 57 Question 58 Question 59 Question 60 Question 61 Question 62 Question 63 Question 64 Question 65 Question 66 Question 67 Question 68 Question 69 Question 70 Question 71 Question 72 Question 73 Question 74 Question 75 Question 76 Question 77 Question 78 Question 79 Question 80 Question 81 Question 82 Question 83 Question 84 Question 85 Question 86 Question 87 Question 88 Question 89 Question 90 Question 91 Question 92 Question 93 Question 94 Question 95 Question 96 Question 97 Question 98 Question 99 Question 100

Question 1

Which of the following statements are true?

- (a) $\{1, 2, 3\} \subset \{1, 2, 3\}$
- (b) $\{1, 2, 3\} \subset \{1, 2, 3, \{1, 2, 3\}\}$
- (c) $\{1, 2, 3\} \in \{1, 2, 3, \{1, 2, 3\}\}$
- (d) $\{1, 2, 3\} \in \{1, 2, 3\}$

Correct Answer: (a), (b), (c), (d)

Question 2

Which of the following are equal sets?

- (a) $\{1, 2, 3\}$
- (b) $\{3, 2, 1\}$
- (c) $\{1, 1, 2, 2, 3, 3\}$
- (d) $\{1, 2, 3, 4\}$

Correct Answer: (a), (b), (c)

Question 3

- (a) $\{x \in \mathbb{R} : x^2 = 1\}$
- (b) $\{x \in \mathbb{R} : x^2 = 0\}$
- (c) $\{x \in \mathbb{R} : x^2 = -1\}$
- (d) $\{x \in \mathbb{R} : x^2 = 2\}$

Correct Answer: (a), (b), (c), (d)

Question 4

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 5

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 6

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 7

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 8

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 9

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 10

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 11

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 12

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 13

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 14

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 15

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 16

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 17

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 18

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 19

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 20

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 21

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 22

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 23

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 24

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 25

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 26

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 27

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 28

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 29

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 30

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 31

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 32

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 33

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 34

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 35

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 36

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 37

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 38

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 39

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 40

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 41

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 42

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 43

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 44

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 45

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 46

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 47

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 48

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 49

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 50

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 51

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 52

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 53

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 54

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 55

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 56

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 57

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 58

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 59

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 60

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 61

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 62

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 63

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 64

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 65

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 66

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 67

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 68

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 69

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 70

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 71

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 72

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 73

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 74

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 75

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 76

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 77

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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Correct Answer: (a), (b), (c), (d)

Question 78

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 79

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- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
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Correct Answer: (a), (b), (c), (d)

Question 80

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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Correct Answer: (a), (b), (c), (d)

Question 81

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 82

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 83

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 84

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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Correct Answer: (a), (b), (c), (d)

Question 85

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 86

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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Correct Answer: (a), (b), (c), (d)

Question 87

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 88

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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Correct Answer: (a), (b), (c), (d)

Question 89

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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Correct Answer: (a), (b), (c), (d)

Question 90

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 91

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 92

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 93

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
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- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 94

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 95

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
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- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 96

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 97

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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Correct Answer: (a), (b), (c), (d)

Question 98

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 99

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
- (b) $\{x \in \mathbb{R} : x^2 \leq 0\}$
- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (c), (d)

Question 100

- (a) $\{x \in \mathbb{R} : x^2 \geq 0\}$
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- (c) $\{x \in \mathbb{R} : x^2 \geq 1\}$
- (d) $\{x \in \mathbb{R} : x^2 \leq 1\}$

Correct Answer: (a), (b), (

8.2 Word Problems

Because percents are used frequently in the real world, they show up a lot in word problems. Word problems involving percents are really no different than any other sort of problem. The important steps to remember are:

Concept:
1. Read the problem carefully!
2. Convert the words into mathematics.
3. Solve the mathematical problem.
4. Write your answer in terms of the word problem, and make sure that it makes sense.

Here are several word problems involving percents:

Problems

Problem 8.0 ★ Jump to Solution

Stephanie bought a new computer that cost $900. Where Stephanie lives, the sales tax is 7%. How much sales tax does Stephanie have to pay for her computer?

Problem 8.9 ★ Jump to Solution

Raja has 4 blue shirts, 5 black shirts, and 6 white shirts. What percent of his shirts are white?

Problem 8.10 ★ Jump to Solution

If a school has 300 girls and 200 boys, what percent of the students are girls?

Problem 8.11 Source: AMC 8 ★ Jump to Solution

An athletic legend team will be 107% of the theoretical maximum team size. The maximum team size, in boxes per minute, is found by subtracting the athletic age, in years, from 330. What is the target team size of an athlete who is 25 years old? (Round to the nearest team size per minute.)

Problem 8.12 ★ Jump to Solution

All of the students in Mr. Sato's History class took an exam. Each student either passed or failed. 85% of the students passed and 3 students failed. How many students are in the class?

Problem 8.13 ★ Jump to Solution

Patti takes 200 flavors to her middle school to sell on Valentine's Day. She first sells 30% of the flavors to 6th grade students. She then sells 50% of the remaining flavors to 7th grade students. Finally, she sells 30% of the remaining flavors to 8th grade students. How many flavors does she have left?

Problem 8.14 ★ Jump to Solution

A 100-million bottle of salad dressing initially consists of 80% oil and 20% vinegar. 40% of the dressing is used, and then 4 millions of vinegar are added to the bottle. What percentage of the mixture is now vinegar?

Problem 8.0 ★ 🔒

Stephanie bought a new computer that cost $900. Where Stephanie lives, the sales tax rate is 7%. How much sales tax does Stephanie have to pay for her computer?

Solution for Problem 8.0: This is a very common "real world" use of percents. Most U.S. states have **sales tax** that gets added to many purchases. Sometimes, the price you see on the tag is the price you actually pay, but sometimes it is not. For example, at a store or restaurant, simply multiplies the sales tax percentage by the price to determine the amount of tax due. As an equation:

$$\text{Sales tax due} = (\text{Purchase price}) \cdot (\text{Sales tax rate}).$$

In our problem, the item being purchased—a computer—has a price of $900, and the sales tax rate is 7%. So we compute:

$$\begin{aligned}\text{Sales tax due} &= 900 \cdot \frac{7}{100} \\ &= \frac{6300}{100} \\ &= 63 \\ &= \$63.\end{aligned}$$

Thus, Stephanie must pay $63 in sales tax.

Sidenote: Another common percent that is used in restaurants is for tipping. In North America, it is typical for 15% of the bill to be used for waiter or waitress. For example, if her meal cost $100, then she would pay $100 as the meal cost, and an additional $100 · $\frac{15}{100}$ = $15 as a tip to their server. In fancier restaurants, the tip rate might be 18% or 20%.

Since 7% or 63% is a common number to use in the sales tax, we'll be easy way to compute approximately how much to tip to double the sales tax. Think about why this works!

Problem 8.9 ★ 🔒

Raja has 4 blue shirts, 5 black shirts, and 6 white shirts. What percent of his shirts are white?

Solution for Problem 8.9: Raja has 4 white shirts and 4 + 5 + 6 = 15 shirts total. So the percent of his shirts that are white is

$$\frac{4}{4+5+6} = \frac{4}{15} = \frac{80}{300} = 80\%.$$

Problem 8.10 ★ 🔒

If a school has 300 girls and 200 boys, what percent of the students are girls?

Solution for Problem 8.10: Avoid the following common mistake:

Bogus Solution: There are 300 girls and 200 boys, so the percent of girls is

$$\frac{300}{200} = \frac{3}{2} = 150\%.$$

This "answer" is clearly incorrect: the percent of girls must be between 0% (no girls) and 100% (all girls). The correct solution is that we must compare the girls to the total number of students.

The school has 300 + 200 = 500 total students, so the percent of the students who are girls is

$$\frac{300}{500} = \frac{3}{5} = \frac{60}{100} = 60\%.$$

Problem 8.11 Source: AMC 8 ★ 🔒

An athletic legend team will be 107% of the theoretical maximum team size. The maximum team size, in boxes per minute, is found by subtracting the athletic age, in years, from 330. What is the target team size of an athlete who is 25 years old? (Round to the nearest team size per minute.)

Solution for Problem 8.11: This is a 2-step problem. First, we have to find the athlete's maximum team size. Second, we have to find the target team size.

To find the maximum team size, we simply follow the directions in the problem: we subtract the athlete's age of 25 from 330, to get 330 − 25 = 304. This is the maximum team size for the athlete.

Then, we are told that the target team size is 80% of the maximum team size. So to find the target team size, we multiply the maximum team size by 80%:

$$80\% \cdot 194 = \frac{4}{5} \cdot 194 = \frac{776}{5}.$$

But that's not the answer—the problem asked us to round to the nearest team per minute. So we must round $\frac{776}{5}$ to the nearest integer. We have $\frac{776}{5} = 155\frac{1}{5}$, so we round up to 155 as 155.2 is 155, Alternatively, we can note that the nearest multiple of 5 to 776 is 775, so the answer is $\frac{775}{5} = 155$ boxes per minute.

Problem 8.12 ★ 🔒

All of the students in Mr. Sato's History class took an exam. Each student either passed or failed. 85% of the students passed and 3 students failed. How many students are in the class?

Solution for Problem 8.12: What bit of information do we need to answer this problem?

Concept: When deciding how to proceed with a problem, ask yourself: "What information do I need to be able to solve the problem?" Then see if you can find that information.

We know that 3 students failed, and it would be nice to know what percent of the total that is. But we do know that 3 are the entire class is 100% of the students, and 85% of the students passed, we know that 100% − 85% = 15% of the students failed. So 3 students equals 15% of the total number of students in the class.

Let's set n to be the number of students in the class, so that we can write an equation for n.

Concept: When trying to find an unknown quantity, it is often useful to assign a variable to that quantity, so that you can write an equation.

Since 3 students is 15% of the class, we have the equation

$$3 = \frac{15}{100} n = \frac{15n}{100}.$$

Multiplying by $\frac{100}{15}$, we find $n = 20$. Thus, and dividing by 15 gives $20 = n$. Therefore, there are 20 students in the class.

To check our answer, we note that if 3 students failed, then $20 - 3 = 17$ passed. This must be 85% of all students, and indeed 85% of 20 students total: $85\% \cdot 20 = \frac{17}{20} \cdot 20 = 17$ students. ✓

Problem 8.13 ★ 🔒

Patti takes 200 flavors to her middle school to sell on Valentine's Day. She first sells 30% of the flavors to 6th grade students. She then sells 50% of the remaining flavors to 7th grade students. Finally, she sells 30% of the remaining flavors to 8th grade students. How many flavors does she have left?

Solution for Problem 8.13: Our role in this problem is as an accountant. We need to keep track of her sales during the day.

First, she sells 30% of her original 200 flavors to her 6th grade students. She computes $30\% \cdot 200 = \frac{30}{100} \cdot 200 = 60$ so she sells 60 flavors. This means she has $200 - 60 = 140$ remaining.

Next, she sells 50% of her 140 remaining flavors to the 7th grade students. We compute $50\% \cdot 140 = \frac{1}{2} \cdot 140 = 50$, so she sells 50 flavors. This means she has $140 - 50 = 90$ remaining.

Next, she sells 30% of her 90 remaining flavors to the 8th grade students. We compute $30\% \cdot 90 = \frac{3}{10} \cdot 90 = 27$, so she sells 27 flavors. This means she has $90 - 27 = 63$ remaining.

So she ends the day with 63 flavors.

We could actually solve this problem in one calculation, by keeping track of how many flavors she sees at each day. We see that first she sees $200 \cdot \frac{70}{100}$ flavors after the 6th grade sales, and then $200 \cdot \frac{70}{100} \cdot \frac{50}{100}$ flavors after the 7th grade sales, and then $200 \cdot \frac{70}{100} \cdot \frac{50}{100} \cdot \frac{70}{100}$ flavors after the 8th grade sales, and then 30% of the remaining flavors after the 8th grade sales. So, at the end of the day, she keeps

$$30\% \text{ of } 40\% \text{ of } 50\% \text{ of } 200 = 30\% \cdot 40\% \cdot 50\% \cdot 200$$
$$= \frac{3}{10} \cdot \frac{2}{5} \cdot \frac{1}{2} \cdot 200$$
$$= \frac{3}{10} \cdot \frac{200}{10}$$
$$= 63$$

flavors.

Problem 8.14 ★ 🔒

A 100-million bottle of salad dressing initially consists of 80% oil and 20% vinegar. 40% of the dressing is used, and then 4 millions of vinegar are added to the bottle. What percentage of the mixture is now vinegar?

Solution for Problem 8.14: As in the previous problem, our best approach is to keep track of how the various quantities change.

We start by losing 40% of the original 100 ml of dressing, which means that 60% remains. That gives us 60ml of dressing left in the bottle. Since the composition of the dressing doesn't change (the amount that was changed), we still have 80% oil and 20% vinegar. This means that we have

$$80\% \cdot 60\text{ ml} = 48\text{ ml of oil} \quad \text{and} \quad 20\% \cdot 60\text{ ml} = 12\text{ ml of vinegar.}$$

Next, we add the additional 4 ml of vinegar. After this, we still have 48 ml of oil, but we have $12 + 4 = 16$ ml of vinegar, and the total amount of dressing is $48 + 16 = 64$ ml. Therefore, the percent of the dressing that is vinegar is

$$\frac{16}{64} = \frac{1}{4} = 25\%.$$

It is no surprise that the new vinegar percentage of 25% is higher than the original percentage of 20%: vinegar since we have added vinegar (but not oil), the percentage of vinegar in the final should be larger.

Percent Word Problems Part 1

Percent Word Problems Part 2

Exercises

8.2.1:

What percent of the 5 × 5 square is the right is shaded?

You may type any additional notes here. **Hide Solution** **Saved**

Your Submission Solution

Solution: There are 5 · 5 = 25 total 1 × 1 squares, and 2 · 2 + 3 · 3 = 4 + 9 = 13 of these are shaded. So $\frac{13}{25}$ of the small squares are shaded. Converting $\frac{13}{25}$ to a percent gives $\frac{13}{25} = \frac{13 \cdot 4}{25 \cdot 4} = \frac{52}{100}$ (as the percent of the 5 × 5 square that is shaded.

8.2.2: Source: MATHCOUNTS ★ 🔒

At a meeting there were 4 parents, 28 students, and 6 teachers. What percent of people at the meeting were students?

You may type any additional notes here. **Hide Solution** **Saved**

Your Submission Solution

Solution: There were $4 + 28 + 6 = 40$ people at the meeting, and 28 of them were students. So, the percent of people at the meeting who were students is $\frac{28}{40} = \frac{70}{100} = \boxed{70\%}$.

8.2.3: ★ 🔒

In the Arbonian State Senate, 20 of the 50 state senators are female. What percent of the state senators are male?

Preview Solution

You may type any additional notes here. **Hide Solution** **Saved**

Your Submission Solution

Solution: Since 20 of the 50 senators are female, the other $50 - 20 = 15$ are male. Therefore, the percent of senators who are male is $\frac{15}{50} = \frac{30}{100} = \boxed{30\%}$.

8.2.4: ★ 🔒

Prunus the Pacifist makes a 6% commission on each property she sells.

(a) If Rhonda sells a $270,000 house, how how much commission does she make?

Preview Solution

You may type any additional notes here. **Hide Solution** **Saved**

Your Submission Solution

Solution: Rhonda's commission is 6% of the sales price of $270,000. So, her commission is

$$6\% \text{ of } \$270{,}000 = \frac{6}{100} \cdot \$270{,}000$$
$$= \frac{1620000}{100}$$
$$= \$16{,}200$$

(b) If Rhonda sells a condo and receives a commission of $15,000 then how much did the condo sell for?

You may type any additional notes here. **Hide Solution** **Saved**

Your Submission Solution

Solution: Let x be the sales price of the condo in dollars. Since $15,000 is 6% of the sales price, we seek the number x such that 6% of x is $15,000. We have $6\% \cdot \frac{6}{100} \cdot x = \frac{6}{100} x$, so we must have

$$\frac{6}{100} x = \$15{,}000.$$

Multiplying both sides by $\frac{100}{6}$ gives

$$\frac{6}{100} x \cdot \frac{100}{6} = \$15{,}000 \cdot \frac{100}{6} = \frac{\$15{,}000}{1} \cdot \frac{100}{6} = \$150{,}000 \cdot \frac{100}{6} = \$250{,}000.$$

So, the sales price of the condo is $250,000.

8.2.5: Source: AMC 8 ★ 🔒

Ken's mathematics final had 75 problems: 10 arithmetic, 30 algebra, and 35 geometry problems. Although she answered 70% of the arithmetic, 40% of the algebra, and 60% of the geometry problems correctly, she did not pass the test because she got less than 60% of the problems right. How many more questions would she have needed to answer correctly to earn a 60% passing grade?

Preview Solution

You may type any additional notes here. **Hide Solution** **Saved**

Your Submission Solution

Solution: First, we determine how many questions Tori answered correctly: 70% of the arithmetic questions is $0.70 \cdot 10 = 7$ problems, 40% of the algebra problems is $0.40 \cdot 30 = 12$ problems, 60% of the geometry problems is $\frac{60}{100} \cdot 35 = \frac{6}{10} \cdot 35 = 21$ problems. This is a total of $7 + 12 + 21 = 40$ problems answered correctly.

Next, we find how many questions must be answered correctly to pass. There are $10 + 30 + 35 = 75$ questions on the test, the most answer 60% of these correctly to pass, which is a total of $60\% \cdot 75 = \frac{3}{5} \cdot 75 = 45$ questions.

Therefore, Tori needed to answer $45 - 40 = \boxed{5}$ more questions correctly to pass.

8.2.6: Source: AMC 8 ★ 🔒

A mixture of 30 liters of paint is 25% red tint, 30% yellow tint, and 45% water. Five liters of yellow tint are added to the original mixture. What is the percent of yellow tint in the new mixture?

You may type any additional notes here. **Hide Solution** **Saved**

Your Submission Solution

Solution: The new mixture has 30 + 5 = 35 liters of paint. To determine the percentage of yellow tint, we find the number of liters of yellow tint in the mixture. The original 30 liters have 30% yellow tint, so the amount of yellow tint in the original mixture is

$$30\% \cdot (30 \text{ liters}) = 0.3 \cdot (30 \text{ liters}) = 9 \text{ liters}.$$

The new mixture has 5 more liters of yellow tint, for a total of 14 liters. Therefore, the percentage of yellow tint in the new mixture is

$$\frac{14}{35} = \frac{2}{5} = \frac{40}{100} = \boxed{40\%}.$$

8.2.7: ★ 🔒

Wind-ex, Ltd's gas tank is 40% empty. It contains 3 gallons of gas. How many gallons of gas does it contain when it is 50% full?

You may type any additional notes here. **Hide Solution** **Saved**

Your Submission Solution

Solution: Since the tank is 3 gallons when it is 40% empty, these 3 gallons fill 100% − 40% = 30% of the tank. This means that the 3 gallons are 30% = $\frac{3}{10}$ of the whole tank. So, the whole tank holds $3 \cdot \frac{10}{3} = 10$ gallons. Therefore, when the tank is 50% full, it has

$$50\% \cdot 10 = \frac{50}{100} \cdot 10 = \frac{5}{10} \cdot 10 = \frac{50}{10} = 5$$

gallons of gas. (We also might have noticed that when it was 40% empty, the tank held 3 gallons, so 60% of the tank would hold $15 \cdot 3 = 5$ gallons.)

8.2.8: ★ 🔒

Jon Eng played Jon Fun against each other, among their friends Sam and Jessica. Emily won 60% of the time, and Fred won the other 1 games. (There were no ties.) How many games did Emily win?

You may type any additional notes here. **Hide Solution** **Saved**

Your Submission Solution

Solution: Since Emily won 60% of the games, Fred won the other 100% − 60% = 40%. Therefore, the ratio of the number of Emily's wins to the number of Fred's wins is 60 : 40. Dividing both parts of this ratio by 5 gives 12 : 8. Since Fred won 1 games, we conclude that Emily won $\boxed{12}$ games.

8.2.9: ★ 🔒

The Fighting Tomatoes, a micro-league baseball team, won 20 of their first 50 games. How many of the remaining 40 games must the Tomatoes win so that they finish with exactly 60% of their games at the end of the season?

You may type any additional notes here. **Hide Solution** **Saved**

Your Submission Solution

Solution: The season has 50 + 40 = 90 games, so 70% of the team's games at the end of the season will be $90 \cdot 0.7 = 63$ wins. The Fighting Tomatoes have already won 20 games, so they need to win $63 - 20 = \boxed{43}$ more games.

8.2.10a:

The table to the right gives the percent of students in each grade at West Pacifick and East Pacifick combined. Suppose that 30% of the students in West Pacifick and East Pacifick take 10th grade. In this section, combined, what percent of the students are in grade 9?

Grade	West	East
9th	35%	45%
10th	40%	35%
11th	25%	20%

You may type any additional notes here. **Hide Solution** **Saved**

Your Submission Solution

Solution: The two schools combined have 200 students. The table tells us that 30% of the 100 students in West Pacifick are 9th graders. Therefore, West Pacifick has $30\% \cdot 100 = \frac{30}{100} \cdot 100 = 30$ 9th graders. The table also tells us that 45% of East Pacifick's 200 students are 9th graders. So, there are $45\% \cdot 200 = \frac{45}{100} \cdot 200 = 90$ students in 9th grade at East Pacifick.

The two schools together have 30 + 90 = 120 students in grade 9 out of the 200 total students in the two schools. So, the percentage of students in the two schools combined who are in grade 9 is $\frac{120}{200} = \frac{3}{5} = \boxed{60\%}$.

8.2.11a: Source: MOEMS ★ 🔒

50% students at Euclid University wear a left-handed glove. 75% of the students passed the exam. Suppose instead only 10% of the students had failed the exam. How many more passing grades would there have been? (The exam is either passed or failed.)

Preview Solution

You may type any additional notes here. **Hide Solution** **Saved**

Your Submission Solution

Solution: If only 10% of the students had failed the exam, 100% − 10% = 90% would have passed the exam. This is an additional 90% − 75% = 15% more than the 75% who did actually pass the exam. Since there are 50 students total, the additional 15% who passed the exam consist of

$$15\% \cdot 500 = \frac{15}{100} \cdot 500 = 15 \cdot 5 = \boxed{75}$$

students.

9.3 Percent Increase and Decrease

Percent increase is the amount by which a quantity increases in terms of percent. For example, you can probably guess ask the "10% off" sales at your favorite store. Percent decrease is the amount by which a quantity decreases in terms of percent. For example, you can probably guess that a 10% decrease means that the quantity will be 10% less than the original.

To find the percent increase or decrease, we use the following formula:

$$\frac{\text{amount of change}}{\text{original amount}} = \frac{\text{new amount} - \text{original amount}}{\text{original amount}}$$

and then solving the increase in your case that the price of a DVD player is 10% more than the original price. Suppose the quantity is decreasing by 10%? Suppose the quantity is increasing by 10%? Suppose the quantity is increasing by 10% or decreasing by 10%? Suppose the quantity is increasing by 10% or decreasing by 10%? Suppose the quantity is increasing by 10% or decre

8.4 Summary

A **percent** is another way of writing a fraction. We can write

$$x\% = \frac{x}{100}$$

where x is any number. So percents are nothing really new:

Concept: A percent is just a number. It's a fraction with a hidden denominator of 100.

Sometimes the word "percentage" is used instead of "percent."

Concept: Remember, a percent is just a fraction, so all of the things that we can do with fractions, we can do with percents too.

Important: Many percents come up so often that you'll probably memorize them. The "quarter" percents are very common:

$$25\% = \frac{25}{100} = \frac{1}{4}, \qquad 50\% = \frac{50}{100} = \frac{1}{2}, \qquad 75\% = \frac{75}{100} = \frac{3}{4}.$$

You should also immediately recognize that $10\% = \frac{10}{100} = \frac{1}{10}$, so any percent that is a multiple of 10% is easy to compute:

$$20\% = 2 \cdot 10\% = \frac{2}{10} = \frac{1}{5}, \qquad 30\% = 3 \cdot 10\% = \frac{3}{10},$$

and so on.

One reason that percents are so commonly used is that they are easy to represent using decimals. As we've already seen, decimals are nice because they let us use our familiar base-10 number system to write fractions as well as integers. Since percents are based on 100, we see that 1 percent is equal to 1 one-hundredth, which we write as

$$1\% = \frac{1}{100} = 0.01.$$

Because percents are used frequently in the real world, they show up a lot in word problems. The key ideas for percent word problems are the same as for any type of word problem:

Concept:

- Read the problem carefully!
- Convert the words to mathematics.
- Solve the mathematical problem.
- Write your answer in terms of the word problem, and make sure that it makes sense.

Percents are commonly used to represent an increase or decrease to some quantity. In these situations, we take a percent times the *original* quantity and add or subtract it to get a new quantity.

For example, the quantity that is "a 30% increase from 400" is computed by first computing 30% of 400:

$$30\% \text{ of } 400 = 30\% \cdot 400 = \frac{3}{10} \cdot 400 = 120,$$

and then adding this increase to our original 400 to get $400 + 120 = 520$. We could also compute this quantity by realizing that

$$\text{a } 30\% \text{ increase from } 400$$

is the same quantity as

$$130\% \text{ of } 400,$$

and then

$$130\% \text{ of } 400 = 130\% \cdot 400 = \frac{13}{10} \cdot 400 = 13 \cdot 40 = 520.$$

Challenge Problems

8.39:

Source: MATHCOUNTS

If 25% of n is 18, then what is 125% of n?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $125\% = 5 \cdot 25\%$, we know that 125% of a number is 5 times 25% of that same number. Since 25% of n is 18, we know that 125% of n is $5 \cdot 18 = \boxed{90}$.

Alternatively, we could have found n. Since 25% of n is 18, we know that $\frac{1}{4}$ of n is 18. This means that $n = 18 \cdot 4 = 72$. We then compute that 125% of n is $\frac{125}{100} \cdot 72 = \frac{5}{4} \cdot 72 = \boxed{90}$, as before.

8.40:

Source: AMC 8

If 20% of a number is 12, then what is 30% of the same number?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The ratio of 30% of a number to 20% of that same number is $30 : 20 = 3 : 2$. So, 30% of a number is $\frac{3}{2}$ times 20% the same number. Since 20% of the number in the problem is 12, we know that 30% of the same number is $\frac{3}{2} \cdot 12 = \boxed{18}$.

Alternatively, we could have found the number. We are given that $20\% = \frac{1}{5}$ of the number is 12, so the number is $12 \cdot 5 = 60$. Therefore, 30% of the same number is $0.3 \cdot 60 = \boxed{18}$.

8.41:

Source: MATHCOUNTS

Find x if 20% of x equals 12% of $(x + 20)$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 20% of x equals $0.2x$ and 12% of $x + 20$ is

$$0.12(x + 20) = 0.12x + 0.12 \cdot 20 = 0.12x + 2.4.$$

Therefore, we must have $0.2x = 0.12x + 2.4$. Multiplying both sides by 100 gets rid of the decimals and leaves $20x = 12x + 240$. Subtracting $12x$ from both sides gives $8x = 240$, so $x = \boxed{30}$.

8.42:

Source: AMC 8

Three bags of jelly beans contain 26, 28, and 30 beans. In each bag, the percent of the beans that are yellow is 50%, 25%, and 20%, respectively. All three bags of candy are dumped into one bowl. What percent of the beans in the bowl are yellow? Round your answer to the nearest integer.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The numbers of yellow beans in the bags are $\frac{1}{2} \cdot 26 = 13$, $\frac{1}{4} \cdot 28 = 7$, and $\frac{1}{5} \cdot 30 = 6$, respectively. So, the total number of yellow beans is $13 + 7 + 6 = 26$. The total number of beans is $26 + 28 + 30 = 84$, so, to the nearest whole percent, the percentage of beans that are yellow is $\frac{26}{84} \approx 0.3095 \approx \boxed{31\%}$.

8.43:

Cody and Tyler were once the same height. Since then, Tyler has grown 20%, while Cody has grown half as much as Tyler. Cody's height is now 5 feet, 6 inches. How tall is Tyler now?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since there are 12 inches in a foot, Cody's height is $12 \cdot 5 + 6 = 66$ inches. Cody grew half as much as Tyler, and they started from the same height. So, while Tyler grew by 20%, Cody grew by 10%. Therefore, Cody's height is 66 inches after he grew by 10%. Since Cody is now 10% taller than he was, his current height is $100\% + 10\% = 110\%$ of his original height. So, if we let Cody's original height be x inches, we must have $1.1 \cdot x = 66$. Dividing both sides by 1.1 gives $x = 60$, so Cody was originally 60 inches tall.

Tyler was the same height as Cody, 60 inches, before growing by 20%. While growing, Tyler's height increased by $20\% \cdot 60 = 0.2 \cdot 60 = 12$ inches. So, he is now $\boxed{72 \text{ inches}}$, which is $\boxed{6 \text{ feet}}$.

8.44:

Source: AMC 8

Miki has a dozen oranges of the same size and a dozen pears of the same size. Miki can use her juicer to extract 8 ounces of pear juice from 3 pears and 8 ounces of orange juice from 2 oranges. She makes a pear-orange juice blend from an equal number of pears and oranges. What percent of the blend is pear juice?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since Miki gets 8 ounces of pear juice from 3 pears, she gets $\frac{8}{3}$ ounces of juice from 1 pear. Similarly, Miki gets 8 ounces of orange juice from 2 oranges, so she gets $\frac{8}{2} = 4$ ounces of juice from 1 orange. When Miki combines 1 pear and 1 orange, she gets $\frac{8}{3} + 4 = \frac{20}{3}$ ounces total. Of these $\frac{20}{3}$ ounces, $\frac{8}{3}$ ounces are pear juice. So, the pear juice is

$$\frac{8/3}{20/3} = \frac{8}{3} \cdot \frac{3}{20} = \frac{8}{20} = \frac{2}{5} = \boxed{40\%}$$

of the total.

8.45:

At some point in the season, the Unicorns had won 60% of their basketball games. After that point, they won 8 more games and lost 2, to finish the season having won 65% of their games. How many games did the Unicorns play during the season?

Hint: Assign a variable.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let x be the number of games the Unicorns played during the season. The Unicorns won 60% of the first $x - 10$ games, which is $60\% \cdot (x - 10)$ games. Combining this with the 8 games they won in the last 10 games gives a total of

$$60\% \cdot (x - 10) + 8 = 0.6(x - 10) + 8 = 0.6x - 6 + 8 = 0.6x + 2$$

wins. We also know that the Unicorns won 65% of their x games, which is a total of $65\% \cdot x = 0.65x$ wins. Setting this equal to our other expression for the total number of wins gives $0.65x = 0.6x + 2$, so $0.05x = 2$. Writing $0.05 = \frac{5}{100} = \frac{1}{20}$, we have $\frac{1}{20}x = 2$, so $x = \boxed{40}$.

8.46:

If 100 is decreased by $a\%$ and the result is decreased by $b\%$, then what is the total percent decrease (in terms of a and b)? (You can assume that both a and b are between 0 and 100.)

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: A decrease of $a\%$ is a decrease of 100 to $100 - a$. A decrease of this by $b\%$ is a decrease of

$$\frac{b}{100} \cdot (100 - a) = \frac{b}{100} \cdot 100 - \frac{b}{100} \cdot a = b - \frac{ab}{100}.$$

Therefore, the result of the $b\%$ decrease is

$$\begin{aligned} 100 - a - \left(b - \frac{ab}{100}\right) &= 100 - a - b + \frac{ab}{100} \\ &= 100 - \left(a + b - \frac{ab}{100}\right). \end{aligned}$$

So, the total decrease from 100 is $a + b - \frac{ab}{100}$. Since we started with 100, this is a $\boxed{a + b - \frac{ab}{100}}$ percent decrease.

8.47★:

Grapes are 80% water (by weight), and raisins are 20% water (by weight). If we start with 500 grams of grapes and remove enough water to turn them into raisins, then what is the weight of the raisins that result?

Hint: There's water, and there's other stuff.

Hint: How much other stuff?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Our original 500 grams of grapes consists of $(80\%) \cdot 500 = \frac{4}{5} \cdot 500 = 400$ grams of water and $500 - 400 = 100$ grams of pulp. After removing enough water to leave raisins, we still have 100 grams of pulp, which should be 80% of the weight of the raisins. Thus, the weight of the raisins is

$$100/80\% = 100/\frac{4}{5} = 100 \cdot \frac{5}{4} = \boxed{125 \text{ grams}}.$$

Review Problems

8.21: Source: MATHCOUNTS

Compute 40% of 20% of 10% of 80,000.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have

$$\begin{aligned} 40\% \text{ of } 20\% \text{ of } 10\% \text{ of } 80,000 &= 40\% \text{ of } 20\% \text{ of } (10\% \text{ of } 80,000) \\ &= 40\% \text{ of } 20\% \text{ of } (0.10 \cdot 80,000) \\ &= 40\% \text{ of } 20\% \text{ of } 8000 \\ &= 40\% \text{ of } (0.2 \cdot 8000) \\ &= 40\% \text{ of } 1600 \\ &= 0.4 \cdot 1600 \\ &= \boxed{640} \end{aligned}$$

8.22:

18% of 50 is what percent of 24?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 18% of 50 is $\frac{18}{100} \cdot 50 = \frac{9}{50} \cdot 50 = 9$. So, we want to know what percent of 24 is 9. Since $\frac{9}{24} = \frac{3}{8} = 0.375 = 37.5\%$, we know that 9 is $\boxed{37.5\%}$ of 24.

8.23:

If there are 240 boys in a school with a total of 960 students, then what percent of the students are girls?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since there are 240 boys, there are $960 - 240 = 720$ girls. Therefore, the percentage of students who are girls is $\frac{720}{960} = \frac{3}{4} = 0.75 = \boxed{75\%}$.

8.24:

The grading scale shown to the right is used at Jones Junior High. The fifteen scores in Mr. Freeman's class were:

89, 72, 54, 97, 77, 92, 85, 74, 75, 63, 84, 78, 71, 80, 90.

In Mr. Freeman's class, what percent of the students received a grade of C?

Score	Grade
90-100	A
80-89	B
70-79	C
60-69	D
0-69	F

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There are 15 scores in the list. There are 6 scores in the list that have a grade of C, so the percentage of students who receive a C is $\frac{6}{15} = \frac{2}{5} = \boxed{40\%}$.

8.25: Source: AMC 8

The glass gauge on a cylindrical coffee maker shows there are 45 cups left when the coffee maker is 36% full. How many cups of coffee does it hold when it is full?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 45 is 36% of the number of cups in the coffee maker. So, we must find the number x such that 36% of x is 45. Since $\frac{36}{100} = \frac{9}{25}$, we must have $\frac{9}{25} \cdot x = 45$. Multiplying both sides by $\frac{25}{9}$ gives $x = \frac{25}{9} \cdot 45 = 25 \cdot 5 = 125$. So, the coffee maker makes $\boxed{125}$ cups of coffee.

8.26: Source: AMC 8

During the softball season, Judy had 35 hits. Among her hits were 1 home run, 1 triple, and 5 doubles. The rest of her hits were singles. What percent of her hits were singles?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Judy had $1 + 1 + 5 = 7$ hits that were not singles, so the remaining $35 - 7 = 28$ hits were singles. Therefore, the percent of hits that were singles was $\frac{28}{35} = \frac{4}{5} = \frac{80}{100} = \boxed{80\%}$.

8.27:

Any quarter has a face value of \$0.25. An eccentric collector offers to buy state quarters for 600% of their face value. At that rate, how much will Larry receive for his collection of all 50 state quarters?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Larry's 50 state quarters have a face value of $50 \cdot \$0.25 = \$12.50$. 600% of a number is 6 times the number, so Larry receives $6 \cdot \$12.50 = \boxed{\$75}$ for his whole collection.

8.28: Source: AMC 8

Katie wants a fancy new music player that costs \$300. Katie can buy it in her home state and pay 6% sales tax, or she can drive to a neighboring state and pay only 3% sales tax. How much does Katie save on the player if she drives to the neighboring state?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Katie pays 3% less by driving to the neighboring state. So, she saves $3\% \cdot \$300 = 0.03 \cdot \$300 = \boxed{\$9}$.

8.29: Source: AMC 8

Sally is playing basketball. After she takes 20 shots, she has made 55% of her shots. After she takes 5 more shots, she raises her percentage of shots made to 56%. How many of the last 5 shots did she make?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since Sally made 55% of her first 20 shots, she made $\frac{55}{100} \cdot 20 = \frac{11}{20} \cdot 20 = 11$ of her first 20 shots. Since she made 56% of her first 25 shots, she made $\frac{56}{100} \cdot 25 = \frac{14}{25} \cdot 25 = 14$ of her first 25 shots. Therefore, she made $14 - 11 = \boxed{3}$ of her last 5 shots.

8.30:

200 students enrolled in the Percentages 101 class in 2010. A 20% increase in enrollment is expected each year. How many students are expected to enroll in the class in 2012?

Solution

Hide Solution

Reset

Your Submission: Solution

Solution: From 2010 to 2011, enrollment increases by 20% from 200, which is an increase of $0.2 \cdot 200 = 40$ students. So, enrollment is 240 students in 2011. From 2011 to 2012, enrollment increases by 20% from 240, which is an increase of $0.2 \cdot 240 = 48$ students. So, enrollment is $240 + 48 = \boxed{288}$ in 2012.

8.31:

The January price of a television was \$2200. This price was raised by 10% to produce the February price of the television. The February price was decreased by 15% to produce the March price of the television. What was the March price?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: A 10% increase of \$2200 is an increase of $\frac{1}{10} \cdot \$2200 = \$220$, to $\$2200 + \$220 = \$2420$. A decrease of 15% from \$2420 is a decrease of $0.15 \cdot \$2420 = \$363$, which makes the price $\$2420 - \$363 = \boxed{\$2057}$.

8.32: Source: AMC 8

Antoinette gets 70% on a 10-problem test, 80% on a 20-problem test, and 90% on a 30-problem test. If the three tests are combined into one 60-problem test, then what is her overall percent, rounded to the nearest whole percent?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: On the 10-problem test, she answers $0.7 \cdot 10 = 7$ questions correctly. On the 20-problem test, she answers $0.8 \cdot 20 = 16$ questions correctly. On the 30-problem test, she answers $0.9 \cdot 30 = 27$ questions correctly. So, she answers $7 + 16 + 27 = 50$ of the 60 total questions correctly, which means she answers $\frac{50}{60} = \frac{5}{6}$ of the questions correctly. Since $\frac{5}{6} = 0.83$, her percentage to the nearest whole percent is $\boxed{83\%}$.

8.33:

Suppose Paul receives a 6% raise every year. After four such raises, what is the total percentage increase to the nearest whole percent?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: After one such raise, Paul's new salary is 106% of his old salary, which means that his new salary is 1.06 times his old salary. Similarly, each of his raises multiplies his salary by 1.06. So, four such raises multiplies his salary by $1.06^4 \approx 1.262$, which is an increase of 0.262 times his original salary. Therefore, to the nearest percent, his salary has increased by $\boxed{26\%}$.

8.34: Source: AMC 8

Jack had a bag of 128 apples. He sold 25% of them to Jill. Next he sold 25% of those remaining to June. Of those apples still in his bag, he gave the most shiny one to his teacher. How many apples did Jack have then?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $25\% = \frac{1}{4}$, Jack sold $\frac{1}{4} \cdot 128 = 32$ apples to Jill, leaving him with $128 - 32 = 96$ apples. He then sold $\frac{1}{4} \cdot 96 = 24$ to June, leaving him with $96 - 24 = 72$. After giving his teacher the most shiny one of these, he was left with $\boxed{71}$ apples.

8.35: Source: AMC 8

A shopper buys a \$100 coat on sale for 20% off. An additional \$5 is taken off the sale price by using a discount coupon. A sales tax of 8% is paid on the final selling price. What is the total amount the shopper pays for the coat?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The 20% discount from \$100 reduces the price by \$20 to \$80. The discount coupon reduces the price to $\$80 - \$5 = \$75$. The tax of 8% of this sales price equals $\frac{8}{100} \cdot \$75 = \frac{2}{25} \cdot \$75 = \$6$, so the total amount the shopper pays is $\$75 + \$6 = \boxed{\$81}$.

8.36:

Polly the Penguin invested \$250 in the Antarctic stock market. During the first year her investment suffered a 15% loss, but during the second year the remaining investment showed a 20% gain. Over the two-year period, what was the percent loss or gain in Polly's investment?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Polly's 15% loss in the first year reduced her \$250 by

$$\frac{15}{100} \cdot \$250 = \frac{3}{20} \cdot \$250 = 3 \cdot \frac{\$250}{20} = 3 \cdot \$12.50 = \$37.50,$$

to \$212.50. Then, in the second year her \$212.50 is increased by 20%, which is an increase of $0.2 \cdot \$212.50 = \$42.50$. So, at the end of her second year, her investment is $\$212.50 + \$42.50 = \$255$. Therefore, her original investment of \$250 increased in value by \$5 total. Since a 1% increase in value is \$2.50, her \$5 increase represents a $\boxed{2\% \text{ gain}}$.

Alternatively, the loss in the first year multiplies Polly's investment by 0.85, and the gain in the second year multiplies the investment by 1.2. So over the two-year period, the investment is multiplied by $(0.85)(1.2) = 1.02$, which is a 2% gain.

8.37: Source: AMC 8

Sale prices at the Ajax Outlet Store are 50% below original prices. On Saturdays, an additional discount of 20% off the sale price is given. What is the Saturday price of a coat whose original price is \$180?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: A sale price of 50% off halves the price from \$180 to \$90. An additional discount of 20% off of this new price is a discount of $0.2 \cdot \$90 = \$18$. This makes the Saturday price $\$90 - \$18 = \boxed{\$72}$.

8.38:

Jack and Jill both work at the King's Ice Cream Shoppe. The King levies a 20% sales tax on all purchases. A customer comes in and orders an ice cream cone that costs 5 bonks (the bonk is a currency equal to 100 borklets). The customer also has a coupon for 10% off.

Jack says: "The total price will be highest if we first apply the 10% coupon to the price of the cone, and then compute the sales tax on the discounted price."

Jill says: "No–the total price will be highest if we first add the sales tax to the original price of the cone, and then apply the coupon."

Who's right?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since a bonk equals 100 borklets, the 5 bonk ice cream cone costs 500 borklets.

First, we compute the final price if we use Jack's strategy of applying the coupon first. The coupon gives a discount of $\frac{1}{10} \cdot 500 = 50$ borklets, making the price 450 borklets. Then, the 20% tax applied to this price increases the total by $\frac{1}{5} \cdot 450 = 90$ borklets, to 540 borklets.

Next, we compute the price with Jill's strategy of applying the tax first: A 20% tax on 500 borklets is an increase of $\frac{1}{5} \cdot 500 = 100$ borklets, to 600 borklets. A 10% discount from 600 borklets is a discount of $\frac{1}{10} \cdot 600 = 60$ borklets, to $600 - 60 = 540$ borklets.

So,

$$\boxed{\text{the two strategies result in the same final cost}}.$$

Is this just a coincidence? Try changing the original numbers in the problem, and see if the two strategies still produce the same final cost.

9.1 Square Roots

Back in Chapter 3, we learned about finding the square of a number. In this chapter, we investigate going the other direction: we start with the square, and figure out what number was squared to produce the square.

Throughout this chapter, you can assume that all numbers are positive.

9.1.1 From Squares to Square Roots

The square of 5 is 25. Going the other direction, we say that 5 is a square root of 25.

Definition: The **square root** of a nonnegative number x is the nonnegative number whose square is x.

We denote the square root of x by $\sqrt{x}$, read as "the square root of x", or "root x", or "radical x".

Both of the appearances of "nonnegative" in this definition are very important! It means some that it cannot be nonnegative in order to have a square root, and it means that the square root is nonnegative, even though there numbers that square to that number. For example, the square root of 25 is 5, not −5.

The nonnegative "nonnegative" is not arbitrary, but it is essential to negative. We define $\sqrt{x}$ to be the nonnegative number whose square is x, so we define $\sqrt{x}$ to be the nonnegative number whose square is x.

The length of the side of the square with area 25 is 5 inches.

The length of the side of the square with area x is $\sqrt{x}$ inches.

The first one is more simple. It would have applications, we only care about the nonnegative number whose square is x, so we define $\sqrt{x}$ to be the nonnegative number.

Obviously, simply saying "$\sqrt{x}$" is never quite clear enough, so we use "radicand." "radicand." "radicand."

9.2 Square Roots of Non-square Integers

In the previous section, we only worked with square roots of perfect squares. In this section, we investigate square roots of integers that are not perfect squares.

We'll start with $\sqrt{2}$. Does $\sqrt{2}$ even exist? Is there a number whose square is 2? We know that there isn't an integer that equals $\sqrt{2}$, since there is no integer whose square is 2. Is there a quotient of two integers (a fraction) whose square is 2? The ancient Greeks believed so for quite some time, and legend has it that the man who finally proved that no such quotient exists was drowned at sea for upsetting this belief. (You will have a chance to prove for yourself that no such quotient exists as a Challenge Problem. Don't worry, no one will drown you at sea if you succeed!)

So, there isn't an integer whose square is 2, and there isn't a fraction whose square is 2. But the number $\sqrt{2}$ does exist! It's a new kind of number that we call an **irrational number**.

> **Definition:** An **irrational number** is a number that cannot be expressed as the quotient of two integers.

We can't express $\sqrt{2}$ as an integer or a fraction, so we can't express it exactly as a decimal, either. However, we can approximate it using one important rule:

> **Important:** If a and b are nonnegative numbers such that $a > b$, then we have $\sqrt{a} > \sqrt{b}$. In other words, when comparing the square roots of two nonnegative numbers, the larger number has the larger square root. Similarly, if $\sqrt{a} > \sqrt{b}$, then $a > b$.

Since $1^2 = 1$ and $2^2 = 4$, we expect that the number whose square is 2 must be between 1 and 2. Our rule above tells us that this intuition is correct. In this section, we'll learn how to use this rule repeatedly to approximate $\sqrt{2}$ to as many decimal places as we like, but we will never be able to write a decimal expression that exactly equals $\sqrt{2}$. Moreover, a decimal approximation of $\sqrt{2}$ does not ever regularly repeat the way the repeating decimals we studied in Section 6.4 do.

Problems

Problem 9.8 — Jump to Solution

(a) Evaluate $\left(\sqrt{3}\right)^2$.

(b) Evaluate $\left(\sqrt{3}\right)^3$.

Problem 9.9 — Jump to Solution

(a) Explain why $\sqrt{2}$ must be less than 1.5.

(b) Estimate $\sqrt{2}$ to the nearest tenth.

Problem 9.10 — Jump to Solution

Which integers have a square root that is greater than 7 and less than 8?

Problem 9.11 — Jump to Solution

How many integers are between $\sqrt{13}$ and $\sqrt{131}$?

Problem 9.12 — Jump to Solution

Find the largest integer less than $\sqrt{80{,}999{,}999}$.

Problem 9.13 — Jump to Solution

Which is larger, $7 - \sqrt{11}$ or $6 - \sqrt{15}$?

Problem 9.8

(a) Evaluate $\left(\sqrt{3}\right)^2$.

(b) Evaluate $\left(\sqrt{3}\right)^3$.

Solution for Problem 9.8:

(a) By definition, $\sqrt{3}$ is the number whose square is 3. So, while we can't write a decimal or fraction that exactly equals $\sqrt{3}$, we do know that the square of $\sqrt{3}$ is simply 3.

(b) We know how to square $\sqrt{3}$, but here we want the sixth power. Once again, our exponent laws help us. Taking the sixth power of a number is the same as first squaring the number and then cubing the result:

$$\left(\sqrt{3}\right)^6 = \left(\sqrt{3}\right)^{2\cdot 3} = \left[\left(\sqrt{3}\right)^2\right]^3 = (3)^3 = 512.$$

□

Problem 9.9

(a) Explain why $\sqrt{2}$ must be less than 1.5.

(b) Estimate $\sqrt{2}$ to the nearest tenth.

Solution for Problem 9.9:

(a) Since $1.5^2 = 2.25$, any number that is at least 1.5 has a square that is at least 2.25. This means that no number that is at least 1.5 can be the square root of 2. So, the square root of 2 must be less than 1.5.

(b) So far, we know that $\sqrt{2}$ is between 1 and 1.5. Since 1.5^2 is closer to 2 than 1^2 is, we think that $\sqrt{2}$ is probably closer to 1.5 than to 1. Therefore, we try squaring 1.4 next. We have $1.4^2 = 1.96$, so we need to square a slightly larger number than 1.4 to get 2. We find that $1.41^2 = 1.9881$ and $1.42^2 = 2.0164$, so $\sqrt{2}$ is between 1.41 and 1.42. Therefore, $\sqrt{2}$ rounded to the nearest tenth is 1.4.

□

In a similar method as we used to estimate $\sqrt{2}$, we can estimate the square root of any non-square number. We square numbers that we think are close to the desired square root. We then compare the resulting squares to the number whose square root we are trying to find. For example, in Problem 9.9, we found $1.4^2 = 1.96$ and $1.5^2 = 2.25$. Since 2 is between 1.96 and 2.25, we know that $\sqrt{2}$ is between 1.4 and 1.5.

Problem 9.10

Which integers have a square root that is greater than 7 and less than 8?

Solution for Problem 9.10: Since 7 is the square root of 49 and 8 is the square root of 64, the square root of any integer between 49 and 64 is between 7 and 8. We can visualize this on the number line:

So, the following integers have square roots greater than 7 and less than 8:

$$50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63.$$

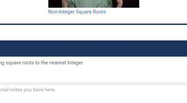
□

Just as we can find square roots that fall between integers, we can find integers that fall between square roots.

Problem 9.11

How many integers are between $\sqrt{13}$ and $\sqrt{131}$?

Solution for Problem 9.11: First, we figure out which two consecutive integers $\sqrt{13}$ is between. To do so, we find the two consecutive perfect squares that 13 falls between. Since 13 is between 9 and 16, we know that $\sqrt{13}$ is between 3 and 4. Similarly, since 131 is between 121 (which is 11^2) and 144 (which is 12^2), we know that $\sqrt{131}$ is between 11 and 12. Again, we can visualize these relationships with a number line:



So, the integers between $\sqrt{13}$ and $\sqrt{131}$ are the integers from 4 up to 11. There are 8 such integers. □

Problem 9.12

Find the largest integer less than $\sqrt{80{,}999{,}999}$.

Solution for Problem 9.12: We start by trying to find an integer near $\sqrt{80{,}999{,}999}$. Since $81 = 9^2$, we start by considering $\sqrt{81{,}000{,}000}$. We're in luck, since this is an integer:

$$\begin{aligned}\sqrt{81{,}000{,}000} &= \sqrt{81 \cdot 1{,}000{,}000}\\ &= \sqrt{9^2 \cdot 1000^2}\\ &= \sqrt{(9 \cdot 1000)^2}\\ &= 9000.\end{aligned}$$

So, $\sqrt{80{,}999{,}999}$ is less than 9000. The next perfect square smaller than 9000^2 is 8999^2. Rather than computing 8999^2 to compare it to 80,999,999, we remember what we learned about consecutive squares in Section 2.1 here. Since $9000^2 = 81{,}000{,}000$, we have

$$\begin{aligned}8999^2 &= 9000^2 - 9000 - 8999\\ &= 81{,}000{,}000 - 9000 - 8999\\ &= 80{,}991{,}000 - 8999,\end{aligned}$$

which is definitely less than 80,999,999. So, the largest integer less than $\sqrt{80{,}999{,}999}$ is 8999. □

Problem 9.13

Which is larger, $7 - \sqrt{11}$ or $6 - \sqrt{15}$?

Solution for Problem 9.13: We typically write the product of a number and a square root without using a multiplication symbol. So, we write $2\sqrt{3}$ to refer to the product of 2 and $\sqrt{3}$, and in this problem we are asked to compare $7\sqrt{11}$ and $6\sqrt{15}$.

We know how to compare two square roots. The order of the square roots is the same as the order of the squares of the two square roots. For example, we know $\sqrt{11} > \sqrt{10}$ because $11 > 10$. Similarly, the order of any two nonnegative numbers is the same as the order of their squares. Maybe comparing squares will work on this problem, too.

> **Concept:** When faced with a new problem, try strategies that you have used to solve similar problems.

Applying the exponent law $(ab)^2 = a^2b^2$, we have

$$\begin{aligned}\left(7\sqrt{11}\right)^2 &= (7)^2\left(\sqrt{11}\right)^2 = (49)(11) = 539,\\ \left(6\sqrt{15}\right)^2 &= (6)^2\left(\sqrt{15}\right)^2 = (36)(15) = 540.\end{aligned}$$

These two results are easy to compare. Since $6\sqrt{15}$ has a larger square than $7\sqrt{11}$ (and both numbers are nonnegative), we know that $6\sqrt{15}$ is larger than $7\sqrt{11}$. □

> **Concept:** We can often compare expressions involving square roots by comparing the squares of the expressions.

Non-Integer Square Roots

Exercises

9.2.1:

Round each of the following square roots to the nearest integer.

(a) $\sqrt{78}$

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: We have $8^2 = 64$ and $9^2 = 81$, so $\sqrt{78}$ is between 8 and 9. Since $8.5^2 = 72.25$, we have that $8.5 < \sqrt{78} < 9$, so the nearest integer to $\sqrt{78}$ is $\boxed{9}$.

(b) $\sqrt{200}$

You may type any additional notes you have here.

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Solution: We have $14^2 = 196$ and $15^2 = 225$, so $14 < \sqrt{200} < 15$, and we expect that $\sqrt{200}$ is closer to 14 than to 15. Since $14.5^2 = 210.25$, we have $14 < \sqrt{200} < 14.5$, so the closest integer to $\sqrt{200}$ is $\boxed{14}$.

(c) $\sqrt{40004}$

You may type any additional notes you have here.

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Solution: Since $60^2 = 3600$, we know that $\sqrt{40004}$ is greater than 60. Computing squares of numbers slightly larger than 60, we find that $62^2 = 3969$ and $64^2 = 4096$, so $63 < \sqrt{40004} < 64$. Since $63.5^2 = 4032.25$, we have $63 < \sqrt{40004} < 63.5$, and the integer closest to $\sqrt{40004}$ is $\boxed{63}$.

9.2.2:

How many integers are between $\sqrt{7}$ and $\sqrt{220}$?

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Solution: We have $2^2 < 7 < 3^2$, so $2 < \sqrt{7} < 3$. We also have $14^2 < 220 < 15^2$, so $14 < \sqrt{220} < 15$. Therefore, the integers between $\sqrt{7}$ and $\sqrt{220}$ are $3, 4, 5, 6, \ldots, 14$. There are $\boxed{12}$ numbers in this list.

9.2.3:

What is the largest integer that is less than $\sqrt{83} - \sqrt{35}$?

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Solution: Since $9^2 = 81$, we know that $\sqrt{83}$ is just a little bit more than 9. Since $6^2 = 36$, we know that $\sqrt{35}$ is just a little bit less than 6. When we subtract a number that is a little less than 6 from a number that is a little more than 9, we get a number that is a bit more than $\boxed{3}$. To make sure that $\sqrt{83} - \sqrt{35}$ isn't greater than 4, we can estimate both $\sqrt{83}$ and $\sqrt{35}$ more closely. We find that $9.1 < \sqrt{83} < 9.2$ and $5.9 < \sqrt{35} < 6$. Since $\sqrt{83}$ is less than 9.2 and $\sqrt{35}$ is greater than 5.9, their difference is less than $9.2 - 5.9 = 3.3$.

9.2.4:

Compute $\left(\sqrt{14}\right)^4$.

You may type any additional notes you have here.

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Solution:

$$\left(\sqrt{14}\right)^4 = \left(\sqrt{14}\right)^{2\cdot 2} = \left(\left(\sqrt{14}\right)^2\right)^2 = (14)^2 = \boxed{196}.$$

9.2.5:

What integer is closest to $4\sqrt{5}$?

You may type any additional notes you have here.

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Solution 1: Approximate $\sqrt{5}$. We approximate $\sqrt{5}$ and then multiply the result by 4. We have $2.2^2 = 4.84$ and $2.3^2 = 5.29$, so $2.2 < \sqrt{5} < 2.3$. Multiplying by 4, we have $8.8 < 4\sqrt{5} < 9.2$. Since $4\sqrt{5}$ is between 8.8 and 9.2, the integer closest to $4\sqrt{5}$ is $\boxed{9}$.

Solution 2: Guess $4\sqrt{5}$. We have $\left(4\sqrt{5}\right)^2 = 4^2\left(\sqrt{5}\right)^2 = 16(5) = 80$. Since $8^2 < \left(4\sqrt{5}\right)^2 < 9^2$, we know that $4\sqrt{5}$ is between 8 and 9. Furthermore, we have $8.5^2 = 72.25$, so $4\sqrt{5}$ is between 8.5 and 9. Therefore, the integer closest to $4\sqrt{5}$ is $\boxed{9}$.

9.2.6:

How many digits are in the square root of the perfect square 108,868,356?

You may type any additional notes you have here.

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Solution: We have $10{,}000^2 = 100{,}000{,}000$ and $100{,}000^2 = 10{,}000{,}000{,}000$, so the square root of 108,868,356 is between 10,000 and 100,000. Therefore, it has $\boxed{5}$ digits. (For the record, $\sqrt{108{,}868{,}356} = 10{,}434.$)

9.2.7: Source: MATHCOUNTS

Between what two consecutive integers on the number line is the sum $\sqrt{30} + \sqrt{50}$ located?

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Solution: Since $5^2 < 30 < 6^2$, we have $5 < \sqrt{30} < 6$. Since $7^2 < 50 < 8^2$, we have $7 < \sqrt{50} < 8$. Combining these, we know that $12 < \sqrt{30} + \sqrt{50} < 14$. But is $\sqrt{30} + \sqrt{50}$ less than or greater than 13? Since $\sqrt{50}$ is very close to 7 (because $7^2 = 49$) and $\sqrt{30}$ is between 5 and 6, we expect that $\sqrt{30} + \sqrt{50}$ is less than 13. To make sure, we notice that $5.5^2 = 30.25$, so $\sqrt{30} < 5.5$, and $7.1^2 = 50.41$, so $\sqrt{50} < 7.1$. Therefore, $\sqrt{30} + \sqrt{50} < 5.5 + 7.1 = 12.6$. So, $\sqrt{30} + \sqrt{50}$ is between $\boxed{12 \text{ and } 13}$.

9.2.8: Source: MATHCOUNTS

Which of the following numbers is largest: $\sqrt{75}$, $\frac{73}{9}$, or 50% of $\frac{68}{4}$?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: 50% of $\frac{68}{4}$ equals 17, we know that 50% of $\frac{68}{4}$ is 8. Next, dividing 9 into 75 gives 8.3. (We might also note $\frac{75}{9} = \frac{25}{3} = 8\frac{1}{3}$.) Finally, we note that $8.5^2 = 72.25$, so $8.5 < \sqrt{75}$. Therefore, $\boxed{\sqrt{75}}$ is the largest of the three numbers.

9.3 Arithmetic with Square Roots

Exercises 1–6 are intended to illustrate Order of Operations and square roots.

Problem 1 Mark this and return

Find the exact value of $\sqrt{4} + \sqrt{25} - \sqrt{9}$.

- $\sqrt{4} + \sqrt{25} - \sqrt{9} = 2 + 5 - 3$
- $\sqrt{4} + \sqrt{25} - \sqrt{9} = 2 + 5 - 3 = 4$
- $\sqrt{4} + \sqrt{25} - \sqrt{9} = 2 + 5 - 3 = 4$ Order of operations: Do all square roots first.
- $\sqrt{4} + \sqrt{25} - \sqrt{9} = 2 + 5 - 3 = 4$ Order of operations: Do $\sqrt{4}$ and $\sqrt{25}$ first, then $\sqrt{9}$.

Problem 2 Mark this and return

Compute and simplify the following.

$$\sqrt{4} \cdot \sqrt{25} \cdot \sqrt{9}$$

- $\sqrt{4} \cdot \sqrt{25} \cdot \sqrt{9}$
- $\sqrt{4} \cdot \sqrt{25} \cdot \sqrt{9} = 2 \cdot 5 \cdot 3$
- $\sqrt{4} \cdot \sqrt{25} \cdot \sqrt{9} = 2 \cdot 5 \cdot 3 = 30$

Problem 3 Mark this and return

Compute and simplify the following.

- $\frac{\sqrt{4}}{\sqrt{9}}$
- $\frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$
- $\frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$
- $\frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$

Problem 4 Mark this and return

Compute and simplify the following.

- $\sqrt{\frac{4}{9}}$
- $\sqrt{\frac{4}{9}} = \frac{2}{3}$
- $\sqrt{\frac{4}{9}} = \frac{2}{3}$
- $\sqrt{\frac{4}{9}} = \frac{2}{3}$

Problem 5 Mark this and return

- Compare $\sqrt{25}$
- Compare $\sqrt{9+16}$
- Compare $\sqrt{9} + \sqrt{16}$
- Which figure is greater, $\sqrt{25}$ or $\sqrt{9} + \sqrt{16}$?

Problem 6 Mark this and return

- Compare $\sqrt{25}$
- Compare $\sqrt{9+16}$
- Compare $\sqrt{9} + \sqrt{16}$
- Which figure is greater, $\sqrt{25}$ or $\sqrt{9} + \sqrt{16}$?

Problem 7 Mark this and return

Find the exact value of $\sqrt{9 \cdot 25}$.

- $\sqrt{9 \cdot 25} = \sqrt{225}$
- $\sqrt{9 \cdot 25} = \sqrt{225} = 15$
- $\sqrt{9 \cdot 25} = \sqrt{225} = 15$ Do the product first.
- $\sqrt{9 \cdot 25} = \sqrt{225} = 15$ Do the product first. Do $\sqrt{225}$ last.

Problem 8 Mark this and return

Find the exact value of $\sqrt{9} \cdot \sqrt{25}$.

- $\sqrt{9} \cdot \sqrt{25} = 3 \cdot 5$
- $\sqrt{9} \cdot \sqrt{25} = 3 \cdot 5 = 15$
- $\sqrt{9} \cdot \sqrt{25} = 3 \cdot 5 = 15$ Do $\sqrt{9}$ and $\sqrt{25}$ first.
- $\sqrt{9} \cdot \sqrt{25} = 3 \cdot 5 = 15$ Do $\sqrt{9}$ and $\sqrt{25}$ first. Do the product last.

Problem 9 Mark this and return

Find the exact value of $\sqrt{\frac{9}{25}}$.

- $\sqrt{\frac{9}{25}} = \frac{3}{5}$
- $\sqrt{\frac{9}{25}} = \frac{3}{5}$
- $\sqrt{\frac{9}{25}} = \frac{3}{5}$
- $\sqrt{\frac{9}{25}} = \frac{3}{5}$

Problem 10 Mark this and return

Find the exact value of $\frac{\sqrt{9}}{\sqrt{25}}$.

- $\frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}$
- $\frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}$
- $\frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}$
- $\frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}$

Problem 11 Mark this and return

Find the exact value of $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25}$.

- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = \sqrt{225} \cdot \sqrt{225}$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = \sqrt{225} \cdot \sqrt{225} = 15 \cdot 15$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = \sqrt{225} \cdot \sqrt{225} = 15 \cdot 15 = 225$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = \sqrt{225} \cdot \sqrt{225} = 15 \cdot 15 = 225$ Do the product last.

Problem 12 Mark this and return

Find the exact value of $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25}$ in Exercise 11 above.

- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last.
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last. Do $\sqrt{9}$ and $\sqrt{25}$ first.

Problem 13 Mark this and return

Find the exact value of $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25}$ in Exercise 11 above.

- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last.
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last. Do $\sqrt{9}$ and $\sqrt{25}$ first.

Problem 14 Mark this and return

Find the exact value of $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25}$ in Exercise 11 above.

- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last.
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last. Do $\sqrt{9}$ and $\sqrt{25}$ first.

Problem 15 Mark this and return

Find the exact value of $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25}$ in Exercise 11 above.

- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last.
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last. Do $\sqrt{9}$ and $\sqrt{25}$ first.

Problem 16 Mark this and return

Find the exact value of $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25}$ in Exercise 11 above.

- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last.
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last. Do $\sqrt{9}$ and $\sqrt{25}$ first.

Problem 17 Mark this and return

Find the exact value of $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25}$ in Exercise 11 above.

- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last.
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last. Do $\sqrt{9}$ and $\sqrt{25}$ first.

Problem 18 Mark this and return

Find the exact value of $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25}$ in Exercise 11 above.

- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last.
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last. Do $\sqrt{9}$ and $\sqrt{25}$ first.

Problem 19 Mark this and return

Find the exact value of $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25}$ in Exercise 11 above.

- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last.
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last. Do $\sqrt{9}$ and $\sqrt{25}$ first.

Problem 20 Mark this and return

Find the exact value of $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25}$ in Exercise 11 above.

- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last.
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last. Do $\sqrt{9}$ and $\sqrt{25}$ first.

Problem 21 Mark this and return

Find the exact value of $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25}$ in Exercise 11 above.

- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$
- $\sqrt{9 \cdot 25} \cdot \sqrt{9 \cdot 25} = 9 \cdot 25 = 225$ Do the product last.
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Problem 22 Mark this and return

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9.4 Summary

Definition: The **square root** of a nonnegative number n is the nonnegative number whose square is n. We express the square root as $\sqrt{n}$, where the $\sqrt{\ }$ symbol is called a **radical**.

WARNING!! ⊘ The square root of a nonnegative number is a nonnegative number by definition. For example, we have $\sqrt{16} = 4$. We *never* say that $\sqrt{16}$ is -4.

Square roots satisfy the following properties for all nonnegative numbers a and b:

- $\left(\sqrt{a}\right)^2 = a$
- $\sqrt{a^2} = a$
- $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$
- If $b \neq 0$, then $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$.
- If $a < b$, then $\sqrt{a} < \sqrt{b}$.
- If $\sqrt{a} > \sqrt{b}$, then $a > b$.

WARNING!! ⊘ If a and b are positive, then $\sqrt{a} + \sqrt{b}$ is **NEVER** equal to $\sqrt{a+b}$.

WARNING!! ⊘ If a and b are positive, then the value of $\sqrt{a^2 + b^2}$ is **NEVER** equal to $a + b$.

Challenge Problems

5.01

(a) For what positive numbers x is $x^2 > 2x$?

Choice Solution

There are infinitely many solutions.

Hide Solution | Hide Hint

Text Solution Solution

Suppose x is a number x^2 and $2x$ positive, we know that x is the square root of x^2, the more have

$$x = \sqrt{x^2} > \sqrt{2x} = \sqrt{2}\sqrt{x}$$

(b) For what positive number x is $x^2 = 2x$?

Choice Solution

There are infinitely many solutions.

Hide Solution | Hide Hint

Text Solution Solution

Suppose x is a number x^2 and $2x$ is equal, we know that $x^2 = 2x$ or $x^2 - 2x = 0$ or $x(x-2) = 0$ or $x = 0$ or $x = 2$

5.02

Find x if $\frac{1}{x} = 3$

Choice Solution

$x = \frac{1}{3}$

Hide Solution | Hide Hint

Text Solution Solution

Since $\frac{1}{x} = 3$, we know that $\frac{1}{x}$ and 3 are reciprocals. The reciprocal of 3 is $\frac{1}{3}$, so we have that $x = \frac{1}{3}$

Since $\frac{1}{\frac{1}{x}} = x$, we have another answer $\frac{1}{\frac{1}{x}} = \frac{1}{3}$

5.03

Find x if $\frac{1}{x} = \frac{2}{3}$

Choice Solution

$x = \frac{3}{2}$

Hide Solution | Hide Hint

Text Solution Solution

Since $\frac{1}{x} = \frac{2}{3}$, we know that $\frac{1}{x}$ and $\frac{2}{3}$ are reciprocals. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, so we have that $x = \frac{3}{2}$

Since $\frac{1}{\frac{1}{x}} = x$, we have another answer $\frac{1}{\frac{1}{x}} = \frac{3}{2}$

5.04 Score: MATH/SAT: 10

The **geometric mean** of two nonnegative numbers is the square root of their product.

(a) Find the geometric mean of 4 and 9.

Choice Solution

$\sqrt{36} = 6$

Hide Solution | Hide Hint

Text Solution Solution

Solution: The geometric mean of 4and 9 is the square root of their product $\sqrt{36}$. $\sqrt{36} = 6$

Since $\sqrt{36} = 6$ and $6 \times 6 = 36$, we know that $\sqrt{36}$ is a square root of 36. $\sqrt{36} = 6$ or -6.

(b) Is it possible for the geometric mean of two nonnegative to be an integer?

Choice Solution

Yes

Hide Solution | Hide Hint

Text Solution Solution

Solution: Since $\sqrt{36}$ is an integer, the geometric mean of 4and 9 is an integer.

(c) In another example, the geometric mean of $\frac{1}{2}$ and $\frac{1}{2}$ is $\sqrt{\frac{1}{2} \cdot \frac{1}{2}} = \frac{1}{2}$.

Choice Solution

Since $\sqrt{\frac{1}{2}}$ is not rational, the geometric mean of $\frac{1}{2}$ and $\frac{1}{2}$ is $\sqrt{\frac{1}{2}}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: Since $\sqrt{\frac{1}{2}}$ is not rational, the geometric mean of $\frac{1}{2}$ and $\frac{1}{2}$ is $\sqrt{\frac{1}{2}}$

(d) For what values of x does the geometric mean of x^2 equal x?

Choice Solution

$x = 0$ or $x = 1$

Hide Solution | Hide Hint

Text Solution Solution

Solution: For x to be equal to the geometric mean of x^2 and x, x must be the square root of the product of x^2 and x.

$\sqrt{x^2 \cdot x} = x$

Squaring both sides, we have $x^3 = x^2$, and we have $x^3 - x^2 = 0$

$x^2(x-1) = 0$

$x = 0$ or $x = 1$

5.05 Score: MATH/SAT: 10

Find a number whose square is equal to the square root of the number.

Choice Solution

$x = 1$

Hide Solution | Hide Hint

Text Solution Solution

Solution: Let x be the number. Then $x^2 = \sqrt{x}$ and we have the equation

$x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$ or $x^3 = 1$

$x = 0$ or $x = 1$

Solution: If x is equal to the square root of the number, we have $x^2 = \sqrt{x}$

$x^4 = x$

$x = 1$

5.06 Score: MATH/SAT: 10

Find x if $\frac{1}{x} = \frac{1}{2} + \frac{1}{3}$

Choice Solution

$x = \frac{6}{5}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: The sum of two reciprocals is equal to the reciprocal of their product.

$\frac{1}{x} = \frac{1}{2} + \frac{1}{3}$

$\frac{1}{x} = \frac{2+3}{2 \cdot 3}$

$\frac{1}{x} = \frac{5}{6}$

$x = \frac{6}{5}$

5.07

Find x if $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

Choice Solution

$x = \frac{12}{13}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$\frac{1}{x} = \frac{6+4+3}{12}$

$\frac{1}{x} = \frac{13}{12}$

$x = \frac{12}{13}$

5.08 Score: MATH/SAT: 10

Find x if $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

Choice Solution

$x = \frac{60}{47}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

$\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

$\frac{1}{x} = \frac{30+20+15+12}{60}$

$\frac{1}{x} = \frac{77}{60}$

$x = \frac{60}{47}$

5.09 Score: MATH/SAT: 10

Find a number whose square is equal to its square root.

Choice Solution

$x = 1$

Hide Solution | Hide Hint

Text Solution Solution

Solution: Let x be the number. Then $x^2 = \sqrt{x}$ and we have the equation

$x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$ or $x^3 = 1$

$x = 0$ or $x = 1$

Solution: If x is equal to the square root of the number, we have $x^2 = \sqrt{x}$

$x^4 = x$

$x = 1$

5.10 Score: MATH/SAT: 10

Find x if $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

Choice Solution

$x = \frac{60}{49}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

$\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

$\frac{1}{x} = \frac{30+20+15+12+10}{60}$

$\frac{1}{x} = \frac{87}{60}$

$x = \frac{60}{49}$

5.11

Find x if $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$

Choice Solution

$x = \frac{420}{419}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$

$\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$

$\frac{1}{x} = \frac{420+280+210+168+140+120}{840}$

$\frac{1}{x} = \frac{1337}{840}$

$x = \frac{420}{419}$

5.12 Score: MATH/SAT: 10

Find x if $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

Choice Solution

$x = \frac{840}{839}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$\frac{1}{x} = \frac{840+560+420+336+280+240+210}{1680}$

$\frac{1}{x} = \frac{2847}{1680}$

$x = \frac{840}{839}$

5.13

Find x if $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}$

Choice Solution

$x = \frac{1260}{1259}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}$

$\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}$

$\frac{1}{x} = \frac{1260+840+630+504+420+360+315+280}{2520}$

$\frac{1}{x} = \frac{3609}{2520}$

$x = \frac{1260}{1259}$

5.14

Find x if $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$

Choice Solution

$x = \frac{2520}{2519}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$

$\frac{1}{x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$

$\frac{1}{x} = \frac{2520+1680+1260+1008+840+720+630+560+504}{5040}$

$\frac{1}{x} = \frac{7929}{5040}$

$x = \frac{2520}{2519}$

5.15

Find the sum of the reciprocals of the first ten positive integers.

Choice Solution

$\frac{7381}{2520}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$

$\frac{2520+1260+840+630+504+420+360+315+280+252}{2520}$

$\frac{7381}{2520}$

5.16

Find the sum of the reciprocals of the first twelve positive integers.

Choice Solution

$\frac{86021}{27720}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12}$

$\frac{27720+13860+9240+6930+5544+4620+3960+3465+3080+2772+2520+2310}{27720}$

$\frac{86021}{27720}$

5.17

Find the sum of the reciprocals of the first twenty positive integers.

Choice Solution

$\frac{55835}{12936}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.18

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.19

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.20

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.21

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.22

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.23

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.24

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.25

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.26

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.27

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.28

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.29

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.30

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

Hide Solution | Hide Hint

Text Solution Solution

Solution: $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19} + \frac{1}{20}$

$\frac{12936+6468+4312+3234+2587+2156+1848+1617+1437+1293+1176+1078+995+924+862+808+760+718+680+646}{12936}$

$\frac{55835}{12936}$

5.31

Find the sum of the reciprocals of the first n positive integers.

Choice Solution

$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac

Review Problems

5.1.1. ▼ ⚑ ⚙

Evaluate each of the following:

(a) $\sqrt{\left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^2}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{\left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^2} &= \sqrt{3^2}\,\sqrt{4^2}\,\sqrt{2^2}\,\sqrt{3^2} = 3 \cdot 4 \cdot 2 \cdot 3 = \boxed{72}\end{aligned}$$

(b) $\sqrt{3^2}\,\sqrt{3^4}\,\sqrt{3^6}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{3^2}\,\sqrt{3^4}\,\sqrt{3^6} &= 3^1 \cdot 3^2 \cdot 3^3 \\ &= 3^{1+2+3} \\ &= 3^6 \\ &= \boxed{729}\end{aligned}$$

(c) $\sqrt{7}\,\sqrt{3}\,\sqrt{7^3}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{7}\,\sqrt{3}\,\sqrt{7^3} &= \sqrt{7}\,\sqrt{3}\,\sqrt{7^2}\,\sqrt{7^1} \\ &= \sqrt{3}\,7^{1+2} \\ &= \sqrt{3}\,7^3 \\ &= \boxed{7\sqrt{3}}\end{aligned}$$

(d) $\sqrt{3}\,\sqrt{6}\,\sqrt{2}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{3}\,\sqrt{6}\,\sqrt{2} &= \sqrt{3}\,\sqrt{3}\,\sqrt{2}\,\sqrt{2} \\ &= 3\,\sqrt{2}\,\sqrt{2} \\ &= 3\,(\sqrt{2})^2 \\ &= \boxed{6}\end{aligned}$$

(e) $\sqrt{5}\,\sqrt{6}\,\sqrt{5}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{5}\,\sqrt{6}\,\sqrt{5} &= \sqrt{5}\,\sqrt{5}\,\sqrt{6} \\ &= (\sqrt{5})^2\,\sqrt{6} \\ &= \boxed{5\sqrt{6}}\end{aligned}$$

(f) $\sqrt{3}\,\sqrt{6}\,\sqrt{2}\,\sqrt{3}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{3}\,\sqrt{6}\,\sqrt{2}\,\sqrt{3} &= \sqrt{3}\,\sqrt{3}\,\sqrt{6}\,\sqrt{2} \\ &= 3\,\sqrt{6}\,\sqrt{2} \\ &= 3\,(\sqrt{2})^2\,\sqrt{3} \\ &= \boxed{6\sqrt{3}}\end{aligned}$$

(g) $\sqrt{3}\,\sqrt{6}\,\sqrt{2}\,\sqrt{3}\,\sqrt{6}\,\sqrt{2}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{3}\,\sqrt{6}\,\sqrt{2}\,\sqrt{3}\,\sqrt{6}\,\sqrt{2} &= \sqrt{3}\,\sqrt{3}\,\sqrt{6}\,\sqrt{6}\,\sqrt{2}\,\sqrt{2} \\ &= 3\,\sqrt{6}\,\sqrt{6}\,\sqrt{2}\,\sqrt{2} \\ &= 3\,\sqrt{6}\,\sqrt{6}\,(\sqrt{2})^2 \\ &= 3\,\sqrt{6}\,\sqrt{6}\,2 \\ &= \boxed{36}\end{aligned}$$

(h) $\sqrt{3}\,\sqrt{6}\,\sqrt{2}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{3}\,\sqrt{6}\,\sqrt{2} &= \sqrt{3}\,\sqrt{3}\,\sqrt{2}\,\sqrt{2} \\ &= 3\,(\sqrt{2})^2 \\ &= \boxed{6}\end{aligned}$$

(i) $\sqrt{\frac{3}{2}}\,\sqrt{\frac{4}{3}}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{\frac{3}{2}}\,\sqrt{\frac{4}{3}} &= \sqrt{\frac{3}{2} \cdot \frac{4}{3}} \\ &= \sqrt{\frac{4}{2}} \\ &= \boxed{\sqrt{2}}\end{aligned}$$

(j) $\sqrt{\frac{3}{2}}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{\frac{3}{2}} &= \frac{\sqrt{3}}{\sqrt{2}} \\ &= \boxed{\frac{\sqrt{6}}{2}}\end{aligned}$$

(k) $\sqrt{\frac{3}{2}}\,\sqrt{\frac{4}{3}}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{\frac{3}{2}}\,\sqrt{\frac{4}{3}} &= \frac{\sqrt{3}}{\sqrt{2}}\,\frac{\sqrt{4}}{\sqrt{3}} \\ &= \frac{\sqrt{4}}{\sqrt{2}} \\ &= \boxed{\sqrt{2}}\end{aligned}$$

(l) $\sqrt{3}\,\sqrt{2}\,\sqrt{3}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{3}\,\sqrt{2}\,\sqrt{3} &= \sqrt{3}\,\sqrt{3}\,\sqrt{2} \\ &= 3\,\sqrt{2} \\ &= \boxed{3\sqrt{2}}\end{aligned}$$

(m) $\sqrt{3}\,\sqrt{2}\,\sqrt{3}\,\sqrt{2}\,\sqrt{3}$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{3}\,\sqrt{2}\,\sqrt{3}\,\sqrt{2}\,\sqrt{3} &= \sqrt{3}\,\sqrt{3}\,\sqrt{2}\,\sqrt{2} \\ &= 3\,\sqrt{2}\,\sqrt{2} \\ &= 3\,(\sqrt{2})^2 \\ &= \boxed{6}\end{aligned}$$

5.1.2. Source: MATHCONTEST. ▼ ⚑

Find x in the form of the expression $\sqrt{(3)^2 + 16}$ where $x = 5$ and $y = (3)^2$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\sqrt{(3)^2 + y} = \sqrt{(3)^2 + 5} = \sqrt{(3)^2 + (5)} = \boxed{\sqrt{14}}$$

5.1.3. ▼ ⚑ ⚙

Simplify $\sqrt{\frac{3}{2}}$ or $\sqrt{1.5}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{\frac{3}{2}} &= \frac{\sqrt{3}}{\sqrt{2}} \\ &= \boxed{\frac{\sqrt{6}}{2}}\end{aligned}$$

5.1.4. ▼ ⚑ ⚙

Simplify $\sqrt{\frac{3}{2}}$ or $\sqrt{1.5}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{\frac{3}{2}} &= \sqrt{\frac{3}{2} \cdot \frac{2}{2}} \\ &= \boxed{\frac{\sqrt{6}}{2}}\end{aligned}$$

5.1.5. ▼ ⚑ ⚙

Simplify $\sqrt{\frac{3}{2}}$ or $\sqrt{1.5}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{\frac{3}{2}} &= \sqrt{\frac{3}{2} \cdot \frac{2}{2}} \\ &= \boxed{\frac{\sqrt{6}}{2}}\end{aligned}$$

5.1.6. ▼ ⚑ ⚙

Which perfect cubes less than 100 have square roots that are integers?

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}1^3 &= 1 \\ 2^3 &= 8 \\ 3^3 &= 27 \\ 4^3 &= 64 \\ 5^3 &= 125 \\ 6^3 &= 216 \\ 7^3 &= 343 \\ 8^3 &= 512 \\ 9^3 &= 729 \\ 10^3 &= 1000\end{aligned}$$

Only 1^3 and 4^3 have square roots that are integers.

5.1.7. ▼ ⚑ ⚙

Find x in the form of a positive value of x

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = 1}$$

5.1.8. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.9. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.10. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.11. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.12. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.13. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.14. Source: MATHCONTEST. ▼ ⚑

Find x in the form of the expression $\sqrt{(3)^2}$ where $x = 3$ and $y = (3)^2$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\sqrt{(3)^2} = x = \sqrt{(3)^2} = x = \sqrt{(3)} = \boxed{\sqrt{3}}$$

5.1.15. ▼ ⚑ ⚙

Simplify $\sqrt{\frac{3}{2}}$ or $\sqrt{1.5}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{\frac{3}{2}} &= \frac{\sqrt{3}}{\sqrt{2}} \\ &= \boxed{\frac{\sqrt{6}}{2}}\end{aligned}$$

5.1.16. ▼ ⚑ ⚙

Which perfect cubes less than 100 have square roots that are integers?

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}1^3 &= 1 \\ 2^3 &= 8 \\ 3^3 &= 27 \\ 4^3 &= 64 \\ 5^3 &= 125 \\ 6^3 &= 216 \\ 7^3 &= 343 \\ 8^3 &= 512 \\ 9^3 &= 729 \\ 10^3 &= 1000\end{aligned}$$

Only 1^3 and 4^3 have square roots that are integers.

5.1.17. ▼ ⚑ ⚙

Find x in the form of a positive value of x

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = 1}$$

5.1.18. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.19. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.20. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.21. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.22. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.23. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.24. Source: MATHCONTEST. ▼ ⚑

Find x in the form of the expression $\sqrt{(3)^2}$ where $x = 3$ and $y = (3)^2$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\sqrt{(3)^2} = x = \sqrt{(3)^2} = x = \sqrt{(3)} = \boxed{\sqrt{3}}$$

5.1.25. ▼ ⚑ ⚙

Simplify $\sqrt{\frac{3}{2}}$ or $\sqrt{1.5}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{\frac{3}{2}} &= \frac{\sqrt{3}}{\sqrt{2}} \\ &= \boxed{\frac{\sqrt{6}}{2}}\end{aligned}$$

5.1.26. ▼ ⚑ ⚙

Which perfect cubes less than 100 have square roots that are integers?

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}1^3 &= 1 \\ 2^3 &= 8 \\ 3^3 &= 27 \\ 4^3 &= 64 \\ 5^3 &= 125 \\ 6^3 &= 216 \\ 7^3 &= 343 \\ 8^3 &= 512 \\ 9^3 &= 729 \\ 10^3 &= 1000\end{aligned}$$

Only 1^3 and 4^3 have square roots that are integers.

5.1.27. ▼ ⚑ ⚙

Find x in the form of a positive value of x

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = 1}$$

5.1.28. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.29. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.30. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.31. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.32. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.33. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.34. Source: MATHCONTEST. ▼ ⚑

Find x in the form of the expression $\sqrt{(3)^2}$ where $x = 3$ and $y = (3)^2$

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\sqrt{(3)^2} = x = \sqrt{(3)^2} = x = \sqrt{(3)} = \boxed{\sqrt{3}}$$

5.1.35. ▼ ⚑ ⚙

Simplify $\sqrt{\frac{3}{2}}$ or $\sqrt{1.5}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}\sqrt{\frac{3}{2}} &= \frac{\sqrt{3}}{\sqrt{2}} \\ &= \boxed{\frac{\sqrt{6}}{2}}\end{aligned}$$

5.1.36. ▼ ⚑ ⚙

Which perfect cubes less than 100 have square roots that are integers?

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\begin{aligned}1^3 &= 1 \\ 2^3 &= 8 \\ 3^3 &= 27 \\ 4^3 &= 64 \\ 5^3 &= 125 \\ 6^3 &= 216 \\ 7^3 &= 343 \\ 8^3 &= 512 \\ 9^3 &= 729 \\ 10^3 &= 1000\end{aligned}$$

Only 1^3 and 4^3 have square roots that are integers.

5.1.37. ▼ ⚑ ⚙

Find x in the form of a positive value of x

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = 1}$$

5.1.38. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.39. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.40. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.41. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.42. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.43. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.44. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.45. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.46. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.47. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.48. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.49. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.50. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.51. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.52. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.53. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.54. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.55. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.56. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.57. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.58. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.59. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.60. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.61. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.62. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.63. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.64. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.65. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.66. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.67. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.68. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.69. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.70. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.71. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.72. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.73. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.74. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.75. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.76. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.77. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.78. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.79. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

5.1.80. ▼ ⚑ ⚙

Find x, if $x = \sqrt{3}$.

Preview Solution

Hide Solution | Hide Hint | Hide Help

Your Hint:

Your Solution:

$$\boxed{x = \sqrt{3}}$$

10 Angles

Angles are important in many fields, and will cover a variety of topics in geometry. We will start with the basics, such as what is a **plane** (flat surface), what is a **line** (straight path), what is an **angle** (two lines joined at a point), and what is **parallel** (lines never meeting).

Angles are important because they are the glue between geometric figures, and **triangles** (3D shapes) have many terms throughout the course. Triangles are important because they are the building blocks for all other shapes, and we will see why later.

10.1 Measuring Angles

Angles are measured in degrees, and there are 360 degrees in a full circle, or a full turn. There could be no, or any amount, in between.

Figure 10.1.1 Angles

Now we can put a lower or a round limit (no point left) on a whole (which, in straight path, from one point to another is called a **line**, and if it goes on forever in both directions, it is called a **line segment**). Now we can measure the angle between the two points. We can measure the angle between two points, and we can measure the angle between two points, and we can measure the angle between two points.

If we continue a segment from point to degrees in both directions, we have a **line**.

Figure 10.1.2 Line

Line $\overleftrightarrow{AB}$ is shown in figure 10.1. We sometimes use a lowercase letter to identify a line, such as line ℓ in the figure. The lowercase letter is used when we don't care about the points that are on the line, or when there are too many points to name.

Line $\overleftrightarrow{AB}$ is also a **plane**, which is a flat surface that goes on forever in all directions. The smallest angle is the angle, or the amount of turn, that **angle** is given in the **degrees** (°) of the angle.

Figure 10.1.3 Angle

We refer to the two rays in figure 10.1 as the two rays that form the angle. The point where the two rays meet is called the **vertex**.

Figure 10.1.4 Measuring an Angle

Figure 10.1 shows that we can continue to the measure angle. We place the protractor on the model so that the vertex of the angle is at the center of the protractor, and the ray $\overrightarrow{AB}$ is on the 0° mark. The angle measure is the number of degrees between the 0° mark and the other ray, $\overrightarrow{AC}$. In figure 10.1.4, we see that the angle measure is 40°. We can also say that $m\angle ABC = 40°$ or $m\angle CBA = 40°$ because we don't care about the order of the points.

Problem 10.1.1 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.1.2 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.1.3 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.1.4 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.1.5 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.1.6 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.1.7 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.1.8 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

10.2 Angle Relationships

Angles are important because they are the glue between geometric figures and **triangles** (3D shapes) have many terms throughout the course. Triangles are important because they are the building blocks for all other shapes, and we will see why later.

10.2.1 Measuring Angles

Angles are measured in degrees, and there are 360 degrees in a full circle, or a full turn. There could be no, or any amount, in between. We can put a lower or a round limit (no point left) on a whole (which, in straight path, from one point to another is called **line**, and if it goes on forever in both directions, it is called a **line segment**). Now we can measure the angle between the two points. We can measure the angle between two points, and we can measure the angle between two points, and we can measure the angle between two points.

If we continue a segment from point to degrees in both directions, we have a **line**.

Figure 10.2.1 Line

Line $\overleftrightarrow{AB}$ is shown in figure 10.2. We sometimes use a lowercase letter to identify a line, such as line ℓ in the figure. The lowercase letter is used when we don't care about the points that are on the line, or when there are too many points to name.

Line $\overleftrightarrow{AB}$ is also a **plane**, which is a flat surface that goes on forever in all directions. The smallest angle is the angle, or the amount of turn, that **angle** is given in the **degrees** (°) of the angle.

Figure 10.2.2 Angle

We refer to the two rays in figure 10.2 as the two rays that form the angle. The point where the two rays meet is called the **vertex**.

Figure 10.2.3 Measuring an Angle

Figure 10.2 shows that we can continue to the measure angle. We place the protractor on the model so that the vertex of the angle is at the center of the protractor, and the ray $\overrightarrow{AB}$ is on the 0° mark. The angle measure is the number of degrees between the 0° mark and the other ray, $\overrightarrow{AC}$. In figure 10.2.3, we see that the angle measure is 40°. We can also say that $m\angle ABC = 40°$ or $m\angle CBA = 40°$ because we don't care about the order of the points.

Problem 10.2.1 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.2.2 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.2.3 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.2.4 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.2.5 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.2.6 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.2.7 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.2.8 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

10.3 Angle Relationships

Angles are important because they are the glue between geometric figures and **triangles** (3D shapes) have many terms throughout the course. Triangles are important because they are the building blocks for all other shapes, and we will see why later.

10.3.1 Measuring Angles

Angles are measured in degrees, and there are 360 degrees in a full circle, or a full turn. There could be no, or any amount, in between. We can put a lower or a round limit (no point left) on a whole (which, in straight path, from one point to another is called **line**, and if it goes on forever in both directions, it is called a **line segment**). Now we can measure the angle between the two points. We can measure the angle between two points, and we can measure the angle between two points, and we can measure the angle between two points.

If we continue a segment from point to degrees in both directions, we have a **line**.

Figure 10.3.1 Line

Line $\overleftrightarrow{AB}$ is shown in figure 10.3. We sometimes use a lowercase letter to identify a line, such as line ℓ in the figure. The lowercase letter is used when we don't care about the points that are on the line, or when there are too many points to name.

Line $\overleftrightarrow{AB}$ is also a **plane**, which is a flat surface that goes on forever in all directions. The smallest angle is the angle, or the amount of turn, that **angle** is given in the **degrees** (°) of the angle.

Figure 10.3.2 Angle

We refer to the two rays in figure 10.3 as the two rays that form the angle. The point where the two rays meet is called the **vertex**.

Figure 10.3.3 Measuring an Angle

Figure 10.3 shows that we can continue to the measure angle. We place the protractor on the model so that the vertex of the angle is at the center of the protractor, and the ray $\overrightarrow{AB}$ is on the 0° mark. The angle measure is the number of degrees between the 0° mark and the other ray, $\overrightarrow{AC}$. In figure 10.3.3, we see that the angle measure is 40°. We can also say that $m\angle ABC = 40°$ or $m\angle CBA = 40°$ because we don't care about the order of the points.

Problem 10.3.1 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.3.2 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.3.3 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.3.4 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.3.5 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.3.6 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.3.7 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.3.8 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

10.4 Angle Relationships

Angles are important because they are the glue between geometric figures and **triangles** (3D shapes) have many terms throughout the course. Triangles are important because they are the building blocks for all other shapes, and we will see why later.

10.4.1 Measuring Angles

Angles are measured in degrees, and there are 360 degrees in a full circle, or a full turn. There could be no, or any amount, in between. We can put a lower or a round limit (no point left) on a whole (which, in straight path, from one point to another is called **line**, and if it goes on forever in both directions, it is called a **line segment**). Now we can measure the angle between the two points. We can measure the angle between two points, and we can measure the angle between two points, and we can measure the angle between two points.

If we continue a segment from point to degrees in both directions, we have a **line**.

Figure 10.4.1 Line

Line $\overleftrightarrow{AB}$ is shown in figure 10.4. We sometimes use a lowercase letter to identify a line, such as line ℓ in the figure. The lowercase letter is used when we don't care about the points that are on the line, or when there are too many points to name.

Line $\overleftrightarrow{AB}$ is also a **plane**, which is a flat surface that goes on forever in all directions. The smallest angle is the angle, or the amount of turn, that **angle** is given in the **degrees** (°) of the angle.

Figure 10.4.2 Angle

We refer to the two rays in figure 10.4 as the two rays that form the angle. The point where the two rays meet is called the **vertex**.

Figure 10.4.3 Measuring an Angle

Figure 10.4 shows that we can continue to the measure angle. We place the protractor on the model so that the vertex of the angle is at the center of the protractor, and the ray $\overrightarrow{AB}$ is on the 0° mark. The angle measure is the number of degrees between the 0° mark and the other ray, $\overrightarrow{AC}$. In figure 10.4.3, we see that the angle measure is 40°. We can also say that $m\angle ABC = 40°$ or $m\angle CBA = 40°$ because we don't care about the order of the points.

Problem 10.4.1 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.4.2 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.4.3 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.4.4 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.4.5 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.4.6 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.4.7 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.4.8 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

10.5 Angle Relationships

Angles are important because they are the glue between geometric figures and **triangles** (3D shapes) have many terms throughout the course. Triangles are important because they are the building blocks for all other shapes, and we will see why later.

10.5.1 Measuring Angles

Angles are measured in degrees, and there are 360 degrees in a full circle, or a full turn. There could be no, or any amount, in between. We can put a lower or a round limit (no point left) on a whole (which, in straight path, from one point to another is called **line**, and if it goes on forever in both directions, it is called a **line segment**). Now we can measure the angle between the two points. We can measure the angle between two points, and we can measure the angle between two points, and we can measure the angle between two points.

If we continue a segment from point to degrees in both directions, we have a **line**.

Figure 10.5.1 Line

Line $\overleftrightarrow{AB}$ is shown in figure 10.5. We sometimes use a lowercase letter to identify a line, such as line ℓ in the figure. The lowercase letter is used when we don't care about the points that are on the line, or when there are too many points to name.

Line $\overleftrightarrow{AB}$ is also a **plane**, which is a flat surface that goes on forever in all directions. The smallest angle is the angle, or the amount of turn, that **angle** is given in the **degrees** (°) of the angle.

Figure 10.5.2 Angle

We refer to the two rays in figure 10.5 as the two rays that form the angle. The point where the two rays meet is called the **vertex**.

Figure 10.5.3 Measuring an Angle

Figure 10.5 shows that we can continue to the measure angle. We place the protractor on the model so that the vertex of the angle is at the center of the protractor, and the ray $\overrightarrow{AB}$ is on the 0° mark. The angle measure is the number of degrees between the 0° mark and the other ray, $\overrightarrow{AC}$. In figure 10.5.3, we see that the angle measure is 40°. We can also say that $m\angle ABC = 40°$ or $m\angle CBA = 40°$ because we don't care about the order of the points.

Problem 10.5.1 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.5.2 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.5.3 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.5.4 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.5.5 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.5.6 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.5.7 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.5.8 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

10.6 Angle Relationships

Angles are important because they are the glue between geometric figures and **triangles** (3D shapes) have many terms throughout the course. Triangles are important because they are the building blocks for all other shapes, and we will see why later.

10.6.1 Measuring Angles

Angles are measured in degrees, and there are 360 degrees in a full circle, or a full turn. There could be no, or any amount, in between. We can put a lower or a round limit (no point left) on a whole (which, in straight path, from one point to another is called **line**, and if it goes on forever in both directions, it is called a **line segment**). Now we can measure the angle between the two points. We can measure the angle between two points, and we can measure the angle between two points, and we can measure the angle between two points.

If we continue a segment from point to degrees in both directions, we have a **line**.

Figure 10.6.1 Line

Line $\overleftrightarrow{AB}$ is shown in figure 10.6. We sometimes use a lowercase letter to identify a line, such as line ℓ in the figure. The lowercase letter is used when we don't care about the points that are on the line, or when there are too many points to name.

Line $\overleftrightarrow{AB}$ is also a **plane**, which is a flat surface that goes on forever in all directions. The smallest angle is the angle, or the amount of turn, that **angle** is given in the **degrees** (°) of the angle.

Figure 10.6.2 Angle

We refer to the two rays in figure 10.6 as the two rays that form the angle. The point where the two rays meet is called the **vertex**.

Figure 10.6.3 Measuring an Angle

Figure 10.6 shows that we can continue to the measure angle. We place the protractor on the model so that the vertex of the angle is at the center of the protractor, and the ray $\overrightarrow{AB}$ is on the 0° mark. The angle measure is the number of degrees between the 0° mark and the other ray, $\overrightarrow{AC}$. In figure 10.6.3, we see that the angle measure is 40°. We can also say that $m\angle ABC = 40°$ or $m\angle CBA = 40°$ because we don't care about the order of the points.

Problem 10.6.1 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.6.2 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.6.3 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.6.4 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.6.5 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.6.6 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.6.7 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.6.8 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

10.7 Angle Relationships

Angles are important because they are the glue between geometric figures and **triangles** (3D shapes) have many terms throughout the course. Triangles are important because they are the building blocks for all other shapes, and we will see why later.

10.7.1 Measuring Angles

Angles are measured in degrees, and there are 360 degrees in a full circle, or a full turn. There could be no, or any amount, in between. We can put a lower or a round limit (no point left) on a whole (which, in straight path, from one point to another is called **line**, and if it goes on forever in both directions, it is called a **line segment**). Now we can measure the angle between the two points. We can measure the angle between two points, and we can measure the angle between two points, and we can measure the angle between two points.

If we continue a segment from point to degrees in both directions, we have a **line**.

Figure 10.7.1 Line

Line $\overleftrightarrow{AB}$ is shown in figure 10.7. We sometimes use a lowercase letter to identify a line, such as line ℓ in the figure. The lowercase letter is used when we don't care about the points that are on the line, or when there are too many points to name.

Line $\overleftrightarrow{AB}$ is also a **plane**, which is a flat surface that goes on forever in all directions. The smallest angle is the angle, or the amount of turn, that **angle** is given in the **degrees** (°) of the angle.

Figure 10.7.2 Angle

We refer to the two rays in figure 10.7 as the two rays that form the angle. The point where the two rays meet is called the **vertex**.

Figure 10.7.3 Measuring an Angle

Figure 10.7 shows that we can continue to the measure angle. We place the protractor on the model so that the vertex of the angle is at the center of the protractor, and the ray $\overrightarrow{AB}$ is on the 0° mark. The angle measure is the number of degrees between the 0° mark and the other ray, $\overrightarrow{AC}$. In figure 10.7.3, we see that the angle measure is 40°. We can also say that $m\angle ABC = 40°$ or $m\angle CBA = 40°$ because we don't care about the order of the points.

Problem 10.7.1 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.7.2 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.7.3 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.7.4 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.7.5 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.7.6 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.7.7 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.7.8 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

10.8 Angle Relationships

Angles are important because they are the glue between geometric figures and **triangles** (3D shapes) have many terms throughout the course. Triangles are important because they are the building blocks for all other shapes, and we will see why later.

10.8.1 Measuring Angles

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If we continue a segment from point to degrees in both directions, we have a **line**.

Figure 10.8.1 Line

Line $\overleftrightarrow{AB}$ is shown in figure 10.8. We sometimes use a lowercase letter to identify a line, such as line ℓ in the figure. The lowercase letter is used when we don't care about the points that are on the line, or when there are too many points to name.

Line $\overleftrightarrow{AB}$ is also a **plane**, which is a flat surface that goes on forever in all directions. The smallest angle is the angle, or the amount of turn, that **angle** is given in the **degrees** (°) of the angle.

Figure 10.8.2 Angle

We refer to the two rays in figure 10.8 as the two rays that form the angle. The point where the two rays meet is called the **vertex**.

Figure 10.8.3 Measuring an Angle

Figure 10.8 shows that we can continue to the measure angle. We place the protractor on the model so that the vertex of the angle is at the center of the protractor, and the ray $\overrightarrow{AB}$ is on the 0° mark. The angle measure is the number of degrees between the 0° mark and the other ray, $\overrightarrow{AC}$. In figure 10.8.3, we see that the angle measure is 40°. We can also say that $m\angle ABC = 40°$ or $m\angle CBA = 40°$ because we don't care about the order of the points.

Problem 10.8.1 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.8.2 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.8.3 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.8.4 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.8.5 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.8.6 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.8.7 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.8.8 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

10.9 Angle Relationships

Angles are important because they are the glue between geometric figures and **triangles** (3D shapes) have many terms throughout the course. Triangles are important because they are the building blocks for all other shapes, and we will see why later.

10.9.1 Measuring Angles

Angles are measured in degrees, and there are 360 degrees in a full circle, or a full turn. There could be no, or any amount, in between. We can put a lower or a round limit (no point left) on a whole (which, in straight path, from one point to another is called **line**, and if it goes on forever in both directions, it is called a **line segment**). Now we can measure the angle between the two points. We can measure the angle between two points, and we can measure the angle between two points, and we can measure the angle between two points.

If we continue a segment from point to degrees in both directions, we have a **line**.

Figure 10.9.1 Line

Line $\overleftrightarrow{AB}$ is shown in figure 10.9. We sometimes use a lowercase letter to identify a line, such as line ℓ in the figure. The lowercase letter is used when we don't care about the points that are on the line, or when there are too many points to name.

Line $\overleftrightarrow{AB}$ is also a **plane**, which is a flat surface that goes on forever in all directions. The smallest angle is the angle, or the amount of turn, that **angle** is given in the **degrees** (°) of the angle.

Figure 10.9.2 Angle

We refer to the two rays in figure 10.9 as the two rays that form the angle. The point where the two rays meet is called the **vertex**.

Figure 10.9.3 Measuring an Angle

Figure 10.9 shows that we can continue to the measure angle. We place the protractor on the model so that the vertex of the angle is at the center of the protractor, and the ray $\overrightarrow{AB}$ is on the 0° mark. The angle measure is the number of degrees between the 0° mark and the other ray, $\overrightarrow{AC}$. In figure 10.9.3, we see that the angle measure is 40°. We can also say that $m\angle ABC = 40°$ or $m\angle CBA = 40°$ because we don't care about the order of the points.

Problem 10.9.1 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.9.2 ◉ Hide My Solution

Find the measure of $\angle ABC$, $\angle CAB$, $\angle DAB$, and $\angle DAC$ (the double arc symbolizes the five problem).

Solution

Problem 10.9.3 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.9.4 ◉ Hide My Solution

In the figure, lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O, and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.9.5 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.9.6 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.9.7 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

Problem 10.9.8 ◉ Hide My Solution

Line $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ intersect at point O and $m\angle POR = 50°$. Find the measure of $\angle POS$.

Solution

10.10 Angle Relationships

Angles are important because they are the glue between geometric figures and **triangles** (3D shapes) have many terms throughout the course. Triangles are important because they are the building blocks for all other shapes, and we will see why later.

10.10.1 Measuring Angles

Angles are measured in degrees, and there are 360 degrees in a full circle, or a full turn. There could be no, or any amount, in between. We can put a lower or a round limit (no point left) on a whole (which, in straight path, from one point to another is called **line**, and if it goes on forever in both directions, it is called a **line segment**). Now we can measure the angle between the two points. We can measure the angle between two points, and we can measure the angle between two points, and we can measure the angle between two points.

If we continue a segment from point to degrees in both directions, we have a **line**.

Figure 10.10.1 Line

Line $\overleftrightarrow{AB}$ is shown in figure 10.10. We sometimes use a lowercase letter to identify a line, such as line ℓ in the figure. The lowercase letter is used when we don't care about the points that are on the line, or when there are too many points to name.

Line $\overleftrightarrow{AB}$ is also a **plane**, which is a flat surface that goes on forever in all directions. The smallest angle is the angle, or the amount of turn, that **angle** is given in the **degrees** (°) of the angle.

Figure 10.10.2 Angle

We refer to the two rays in figure 10.10 as the two rays that form the angle. The point where the two rays meet is called the **vertex**.

Figure 10.10.3 Measuring an Angle

Figure 10.10 shows that we can continue to the measure angle. We place the protractor on the model so that the vertex of the angle is at the center of the protractor, and the ray $\overrightarrow{AB}$ is on the 0° mark. The angle measure is the

10.2 Parallel Lines

Having learned about what happens when two lines meet, we should wonder about what happens if they don't. If two lines in a plane do not meet, we say that they are **parallel**. We can indicate that lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel by writing $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

As shown above, we can use little arrows to mark lines that are parallel. Those little arrows can really clutter up a diagram, so we won't always include them.

Problems

Problem 10.9 Jump to Solution

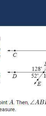
Draw a pair of parallel lines like those shown below. Then draw a line that crosses both of the parallel lines. With a protractor, measure all the angles formed between your line and both of the parallel lines. Write the angle measures in the angles you form. Try it again with a different pair of parallel lines. Do you notice anything interesting?

Problem 10.10 Jump to Solution

Lines m and n are parallel, and we are given the measure of one angle in the diagram as shown. Find the values of a, b, c, w, x, y, and z.

Problem 10.11 Jump to Solution

A chicken starts at point X on one side of a road. It starts walking across the road along a path that makes a 22° angle with the side of the road, as shown. Before making it to the other side, the chicken makes a sharp turn (at point Y) and starts along a new path, which makes a 71° angle with the old path, as shown. If the opposite sides of the road are parallel, what is the measure of the acute angle that the path of the chicken makes with the far side of the road at point Z?



We start studying parallel lines by taking a look at the angles formed when a line intersects a pair of parallel lines.

Problem 10.9

Draw a pair of parallel lines like those shown below. Then draw a line that crosses both of the parallel lines. With a protractor, measure all the angles formed between your line and both of the parallel lines. Write the angle measures in the angles you form. Try it again with a different pair of parallel lines. Do you notice anything interesting?

Solution for Problem 10.9: In the diagram to the right, we have parallel lines $\overleftrightarrow{CG}$ and $\overleftrightarrow{DF}$, and we have added line $\overleftrightarrow{EH}$, which meets $\overleftrightarrow{CG}$ and $\overleftrightarrow{DF}$ at A and B, respectively. We call a line that cuts across parallel lines a **transversal**. Measuring all 8 angles in the diagram, we find the measures shown. There are two groups of 4 equal angles!

We could have seen that some of these angles are equal without using a protractor. We must have $\angle BAC = \angle BAC$ and $\angle ABFE = \angle DBEC$ because these are pairs of vertical angles. But why are the acute angles at A equal to the acute angles at B?

One way to see why is to imagine sliding line $\overleftrightarrow{DF}$ on top of $\overleftrightarrow{CG}$ so that point B is on top of point A. Then, $\angle ABF$ would be right on top of $\angle BAC$. This isn't a proof, but it does give us some idea why these angles have the same measure.

Each obtuse angle in the diagram can be combined with one of the acute angles to form a straight line. So, each obtuse angle is supplementary to each acute angle. $\square$

> **Important:** The angles formed when a transversal intersects two parallel lines come in two groups of four equal angles as shown:
>
> $$\begin{array}{ccccccc} a & = & c & = & e & = & g \\ b & = & d & = & f & = & h \end{array}$$
>
> Each of the first set of angles is supplementary to each of the second set of angles. That is, the sum of any angle in the first group and any angle in the second group is 180°. So, either all 8 angles are right angles, or 4 of them are acute while the other 4 are obtuse.

Problem 10.10

Lines m and n are parallel, and we are given the measure of one angle in the diagram as shown. Find the values of a, b, c, w, x, y, and z.

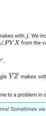
Solution for Problem 10.10: When a transversal intersects parallel lines, equal angles come in groups of four as we saw in Problem 10.9. Therefore, we know that $b = x = y = 55^\circ$. We also know that each angle in the other "group of four" has a measure that is supplementary to 55°:

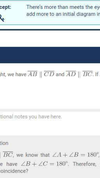
$$a = c = w = z = 180^\circ - 55^\circ = 145^\circ.$$

$\square$

We have seen that a transversal forms two sets of four equal angles when it intersects two parallel lines. We can use these relationships in reverse! That is, we can use the angle relationships we just learned to figure out when lines are parallel.

For example, in the diagram at the right, line ℓ intersects lines k and ℓ. If we can determine that $x = 4$ or $x = m$, then we know that $k \parallel \ell$. Similarly if we determine that $x + v = 180^\circ$ or $x + u = 180^\circ$, then we know that $k \parallel \ell$.





Angles and Parallel Lines

Now that we understand the relationships between angles when a transversal intersects parallel lines, let's try a more challenging problem.

Problem 10.11

A chicken starts at point X on one side of a road. It starts walking across the road along a path that makes a 22° angle with the side of the road, as shown. Before making it to the other side, the chicken makes a sharp turn (at point Y) and starts along a new path, which makes a 71° angle with the old path, as shown. If the opposite sides of the road are parallel, what is the measure of the acute angle that the path of the chicken makes with the far side of the road at point Z?

Solution for Problem 10.11: We'd like to see what we know about parallel lines and angles, but neither $\overleftrightarrow{XY}$ nor $\overleftrightarrow{YZ}$ intersects both sides of the road. So, we add a third line, through point Y and parallel to both sides of the road, as shown in the diagram below. We'll label this line ℓ, and let the sides of the road be j and k. Both $\overleftrightarrow{XY}$ and $\overleftrightarrow{YZ}$ are transversals that intersect a pair of parallel lines. Now we can use what we know about angles and parallel lines.

Since $j \parallel \ell$, we know that the acute angle that $\overleftrightarrow{XY}$ makes with ℓ equals the acute angle $\overleftrightarrow{XY}$ makes with j. We include this information in the diagram by writing 22° inside $\angle PYX$ in our diagram. We then find $\angle PYZ$ by subtracting $\angle PYX$ from the value of $\angle XYZ$ that we are given in the problem. We find that

$$\angle PYZ = \angle XYZ - \angle XYP = 71^\circ - 22^\circ = 49^\circ.$$

Finally, because $\ell \parallel k$, the acute angle that $\overleftrightarrow{YZ}$ makes with k is congruent to the acute angle $\overleftrightarrow{YZ}$ makes with ℓ. So the path of the chicken makes a 49° angle with the far side of the road. $\square$

As seen in Problem 10.11, parallel lines are so helpful that sometimes we add an extra parallel line to a problem in order to find a solution.

> **Concept:** There's more than meets the eye in many geometry problems! Sometimes we have to add more to an initial diagram in order to solve a problem.

Exercises

10.2.1:

In the diagram on the right, we have $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. If $\angle A = 73^\circ$, then what is the measure of $\angle C$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, we know that $\angle A + \angle B = 180^\circ$, so $\angle B = 180^\circ - \angle A = 107^\circ$. Similarly, since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, we have $\angle B + \angle C = 180^\circ$. Therefore, $\angle C = 180^\circ - \angle B = \boxed{73^\circ}$. Notice that $\angle A = \angle C$. Is that a coincidence?

10.2.2:

In the diagram below, we have $j \parallel k$. Find x.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Because $j \parallel k$, the acute angles in the diagram are supplementary to the obtuse angles in the diagram. Specifically, this means that the labeled angle measures must add to 180°:

$$x + x + 30^\circ = 180^\circ.$$

Therefore, we have $2x + 30^\circ = 180^\circ$, so $2x = 150^\circ$ and $x = \boxed{75^\circ}$

10.2.3:

In the diagram below, $\overleftrightarrow{AD}$ and $\overleftrightarrow{CF}$ intersect at point E, and $\overleftrightarrow{AD} \parallel \overleftrightarrow{FG}$. We also have $\angle CEB = 3\angle AEB$ and $\angle EFG = 2\angle AEB$, as shown. Find the measure of $\angle CED$ in degrees.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Since $\overleftrightarrow{AD} \parallel \overleftrightarrow{FG}$, we have $\angle CED = \angle CFG$, so $\angle CED = 2t$. The three angles $\angle AEB$, $\angle BEC$ and $\angle CEAD$ together form a straight angle, so they sum to 180°. Therefore, we have $t + 3t + 2t = 180^\circ$. Simplifying gives $6t = 180^\circ$, and dividing by 6 gives $t = 30^\circ$, which means $\angle CED = 2t = \boxed{60^\circ}$

10.2.4:

Lines j and k are parallel. If line ℓ is perpendicular to line j, then must line ℓ be perpendicular to line k?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: $\boxed{\text{Yes}}$. Since j and k are parallel, line ℓ makes the same angles with k that it makes with j. So, because ℓ makes a 90° angle with j, it also makes a 90° angle with k. This means that lines ℓ and k are perpendicular.

10.2.5:

If I draw 8 parallel lines on a piece of paper, into how many different non-overlapping regions will the lines divide the paper?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: There are 7 regions that only have one of the lines as a border and there are 7 regions between a pair of consecutive parallel lines, for a total of $7 + 7 = \boxed{14}$

10.2.6:

In the diagram, line d is perpendicular to line a, and line d is parallel to line c. Line b passes through the intersection of lines a and c. If the acute angle between lines a and b measures 47°, then what is the measure of the acute angle between lines b and d?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Since $a \parallel d$ and a is perpendicular to d, we know that a is perpendicular to c as well. We mark the angle between a and c as a right angle in our diagram on the right.

We next consider the three angles at point P, which is the intersection of lines a, b, and c. The measures of the three angles with vertex P that together make line a must have sum 180°. So, the acute angle between b and c must have measure $180^\circ - 90^\circ - 47^\circ = 43^\circ$. Since c and d are parallel, the acute angle between b and d has the same measure as the acute angle between b and c, which is $\boxed{43^\circ}$

10.2.7:

Line m intersects lines j and k forming angles with the measures shown at the right. Are lines j and k parallel?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: $\boxed{\text{No}}$. The other angle between m and k is $180^\circ - 80.09^\circ = 99.91^\circ$, which is not equal to the corresponding angle that m makes with j. If j and k were parallel, these two angles would have to be equal. So, lines j and k are not parallel.

10.2.8★:

Lines m and n are parallel. Two rays are drawn from point A, forming angles with m and n with the measures shown. What is the measure of the acute angle formed by these two rays?

Hint: What useful extra line might we add to the diagram?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: We draw line ℓ through A parallel to m and n. Since $\ell \parallel n$, we have $\angle BAD = 46^\circ$. Since $\ell \parallel m$, we have $\angle CAD = 34^\circ$. Therefore, we have

$$\angle BAC = \angle BAD - \angle CAD = \boxed{15^\circ}$$

10.2 Angles in Polygons

A polygon is a closed figure with straight sides. The vertices of the polygon, the "corner points", are those points where two sides meet.

The four segments that form the boundaries of a polygon can be called all the edges of the polygon. If we connect two vertices that are not adjacent in the polygon, we have a **diagonal**, such as diagonal BD in the diagram.

Polygons are classified according to the number of edges they have. Triangles have 3 edges, **quadrilaterals** have 4 edges, **pentagons** have 5 edges, **hexagons** have 6 edges, **heptagons** have 7 edges, **octagons** have 8 edges, **nonagons** have 9 edges, **decagons** have 10 edges, **dodecagons** have 12 edges, **pentadecagons** have 15 edges, **icosagons** have 20 edges, **triacontagons** have 30 edges, **tetracontagons** have 40 edges, **pentacontagons** have 50 edges, **hexacontagons** have 60 edges, **hecatontagons** have 100 edges, **chiliagons** have 1000 edges, **myriagons** have 10000 edges, **megagons** have 1000000 edges, **gigagons** have 1000000000 edges, **teragons** have 1000000000000 edges, **petagons** have 1000000000000000 edges, **exagons** have 1000000000000000000 edges, **zettagons** have 1000000000000000000000 edges, **yottagons** have 1000000000000000000000000 edges, **xenagons** have 1000000000000000000000000000 edges, **dekagons** have 1000000000000000000000000000000 edges, **megagons** have 1000000000000000000000000000000000 edges, **megagons** have 1000000000000000000000000000000000

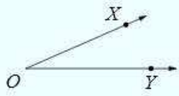
10.4 Summary

Definitions:

- A **point** is, well, a point. The great Greek mathematician Euclid called a point "that which has no part." We can't do much better than that vague description. We typically label points with capital letters.
- A straight path connecting two points is called a **segment**, and the original two points are the **endpoints** of the segment. We refer to a segment by its endpoints, such as $\overline{AB}$. We remove the bar to refer to the length of the segment: AB.
- If we start at a point, then head in one direction forever, we form a **ray**. Our starting point is the **vertex** of the ray, and we identify a ray as $\overrightarrow{AB}$, where A is the vertex of the ray and B is some other point on the ray.
- If we continue a line segment past its endpoints forever in both directions, we form a **line**, which we write as $\overleftrightarrow{AB}$.

Definitions:

Two rays that share an origin form an **angle**. The common origin of the rays is the **vertex** of the angle. We use the symbol $\angle$ to refer to an angle, and we use a point on each side and the vertex to identify the angle, such as $\angle XOY$ at the right. We sometimes use just the vertex to identify the angle when it is clear what angle we mean. For example, we can write $\angle O$ to refer to the angle at the right.



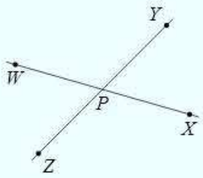
We can use a **protractor** to measure angles (see Section 10.1 here). The **semicircular** arc of the protractor is divided into 180 degrees, so that a whole circle is 360 degrees.

Definitions:

- An angle smaller than $90°$ is an **acute angle**.
- A $90°$ angle is a **right angle**. Lines, segments, or rays that form a right angle are said to be **perpendicular**.
- An angle between $90°$ and $180°$ is an **obtuse angle**.
- An angle that measures $180°$ is a **straight angle**.
- An angle that measures between $180°$ and $360°$ is a **reflex angle**.

Definitions:

- Two angles whose measures add to $180°$ are **supplementary angles**. Angles that together make up a straight angle form a particularly useful example of supplementary angles.
- Two angles whose measures add to $90°$ are **complementary angles**.
- Two angles that have the same measure are called **congruent angles**.
- When two lines intersect, they form two pairs of **vertical angles**, such as $\angle WPZ$ and $\angle YPX$ on the right. Vertical angles are congruent.



Definitions:

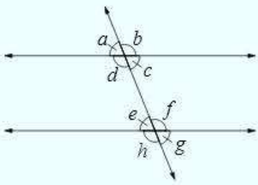
Two lines that do not intersect are **parallel**. A line that intersects multiple parallel lines is called a **transversal line**.

Important:

The angles formed when a transversal intersects two parallel lines come in two groups of four equal angles as shown:

$$\begin{aligned} a &= c = e = g \\ b &= d = f = h \end{aligned}$$

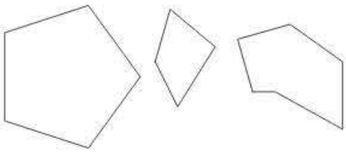
Each of the first set of angles is supplementary to each of the second set of angles.



Important:

The relationships described above when a transversal intersects two lines can also be used to show that two lines are parallel.

A **polygon** is a simple closed figure with line segments as boundaries. For example, all triangles are polygons. Several polygons are shown on the right. The line segments that form the boundaries of a polygon are the **sides** of the polygon. Each pair of consecutive sides of a polygon meet at a **vertex** of the polygon. Each angle inside a polygon formed by a pair of consecutive sides is called an **interior angle** of the polygon. If we connect two vertices that are not adjacent on the polygon, we form a **diagonal** of the polygon.



Important:

The sum of the interior angles in a triangle is $180°$, and the sum of the interior angles of a polygon with n sides is $180(n-2)$ degrees.

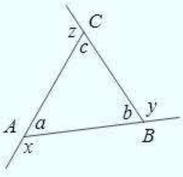
Definitions:

Triangles can be classified by their angles.

- A triangle with a right angle as an interior angle is a **right triangle**.
- A triangle with an obtuse angle as an interior angle is a **obtuse triangle**.
- A triangle in which all three angles are acute is an **acute triangle**.

Important:

When we extend the sides of a triangle past the vertices, we form **exterior angles** of the triangle. For example, the angles with measures x, y, and z on the right are exterior angles of $\triangle ABC$, while the angles with measures a, b, and c are interior angles of the triangle. The sum of the exterior angles of any triangle is $360°$.



Challenge Problems

10.37:

Source: AMC 12

Find the degree measure of an angle whose complement is 25% of its supplement.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let x be the measure of the angle. Its complement then has measure $90^\circ - x$ and its supplement has measure $180^\circ - x$. Since 25% of the supplement equals the complement, we have $\frac{1}{4}(180^\circ - x) = 90^\circ - x$. Multiplying both sides by 4 gives $180^\circ - x = 4(90^\circ - x)$. Expanding the right side gives $180^\circ - x = 360^\circ - 4x$. Subtracting 180° from both sides, and adding $4x$ to both sides, gives $3x = 180^\circ$, so $x = \boxed{60^\circ}$.

10.38:

Source: AMC 8

If $\angle A = 20^\circ$ and $\angle AFG = \angle AGF$ in the diagram at the right, then how many degrees is $\angle B + \angle D$?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: If we can find $\angle BFD$, we can use $\triangle BFD$ to find $\angle B + \angle D$. We also see that $\angle BFD + \angle AFG = 180^\circ$, so if we can find $\angle AFG$, we can solve the problem. In $\triangle AGF$, we have $\angle A + \angle AFG + \angle AGF = 180^\circ$, so

$$\angle AFG + \angle AGF = 180^\circ - \angle A = 160^\circ.$$

We also know that $\angle AFG = \angle AGF$. Combining this with $\angle AFG + \angle AGF = 160^\circ$, we have $\angle AFG = \angle AGF = 80^\circ$. Now, we have

$$\angle BFD = 180^\circ - \angle AFG = 180^\circ - 80^\circ = 100^\circ.$$

From triangle BFD, we have $\angle B + \angle D + \angle BFD = 180^\circ$, so

$$\angle B + \angle D = 180^\circ - \angle BFD = 180^\circ - 100^\circ = \boxed{80^\circ}.$$

10.39:

Source: MATHCOUNTS

Point A is on the edge of a circular disk. Every day at noon, the disk is rotated 150° in a counter-clockwise direction. What day of the week will it be the next time point A is at the same position that it was at 10 a.m. on Saturday?

You may type any additional notes you have here.

Hide Solution

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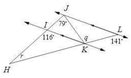
Your Submission: Solution

Solution: The point will be back in its original position when it has rotated by a multiple of 360°. We could list multiples of 150° until we hit a multiple of 360°, but instead, we compute what portion of a circle is 150°. We have $\frac{150^\circ}{360^\circ} = \frac{5 \cdot 30^\circ}{12 \cdot 30^\circ} = \frac{5}{12}$. So, each day, the disk is rotated $\frac{5}{12}$ of a circle. After n days, it has been rotated $\frac{5n}{12}$ of a circle. The smallest positive integer n for which $\frac{5n}{12}$ is an integer is $n = 12$. We have to be a bit careful–the disk is rotated on the first Saturday, so 12 spins of the disk is 11 days after Saturday, which brings us to $\boxed{\text{Wednesday}}$.

10.40:

Find angle measures q and r in the diagram below.



You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

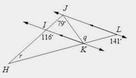
Solution: We have $\angle JLH = 180^\circ - 141^\circ = 39^\circ$. Since $\overleftrightarrow{IK} \parallel \overleftrightarrow{JL}$, we have $\angle HJL = \angle HIK = 116^\circ$. Next, we see that

$$\angle KJL = \angle HJL - \angle HJK = 116^\circ - 79^\circ = 37^\circ.$$

From $\triangle JKL$, we have

$$q = 180^\circ - \angle KJL - \angle JLK = 180^\circ - 37^\circ - 39^\circ = \boxed{104^\circ}.$$

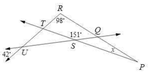
From triangle HJL, we have $r + \angle HJL + \angle JLH = 180^\circ$, so $r + 116^\circ + 39^\circ = 180^\circ$. This gives us $r + 155^\circ = 180^\circ$, so $r = \boxed{25^\circ}$.



10.41:

Find the value of x in the diagram at the right.

Hint: $RPSU$ is a quadrilateral!



Preview: Solution

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: Since vertical angles are congruent, we have $\angle TUS = 42^\circ$. We also have

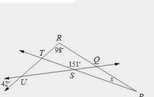
$$\angle TSU = 180^\circ - \angle TSQ = 180^\circ - 151^\circ = 29^\circ,$$

so $\angle PSQ = \angle TSU = 29^\circ$. The sum of the interior angles of $RPSU$ must be 360°, even though one of the angles is a reflex angle. The angle of $RPSU$ at vertex S has measure

$$\angle PSQ + \angle QST + \angle TSU = 209^\circ,$$

so

$$x = 360^\circ - 209^\circ - 42^\circ - 98^\circ = \boxed{11^\circ}.$$



10.42★:

Source: MATHCOUNTS

In quadrilateral $ABCD$ angle BAD and angle CDA are divided into three equal angles as shown. What is the measure of $\angle AFD$ in degrees?

Hint: What does quadrilateral $ABCD$ tell you about $x + y$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: From triangle AFD, we have

$$\begin{aligned}\angle AFD &= 180^\circ - \angle FAD - \angle FDA \\ &= 180^\circ - 2x - 2y \\ &= 180^\circ - 2(x + y).\end{aligned}$$

So, if we can find $x + y$, we can find $\angle AFD$. From quadrilateral $ABCD$, we have

$$\angle DAB + \angle B + \angle C + \angle CDA = 360^\circ,$$

so $3x + 110^\circ + 100^\circ + 3y = 360^\circ$. Simplifying the left side gives $3x + 3y + 210^\circ = 360^\circ$. Subtracting 210° from both sides gives $3x + 3y = 150^\circ$. Dividing by 3 gives $x + y = 50^\circ$. Substituting this into our expression for $\angle AFD$ gives

$$\angle AFD = 180^\circ - 2(50^\circ) = 180^\circ - 100^\circ = \boxed{80^\circ}.$$

10.43★:

Draw a five-pointed star like the one shown below. Find the sum of the measures of the angles at the five points of the star. Notice anything interesting? Test your observation for a few more stars, and then see if you can explain why it must be true.

Hint: To what other angles in the diagram can you relate angles of the star?

Hint: I see five little triangles. I also see five larger triangles!

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There is a pentagon in the middle of the star. If we extend the sides of any of the pentagon's interior angles, we reach two of the points of the star. In this way, we can build a triangle for each vertex of the pentagon. One of these triangles is shown in bold on the right. There are five such triangles, one for each vertex of the pentagon. If we add the measures of the angles in all five triangles (counting an angle twice if it appears in two triangles), we get $5 \cdot 180^\circ = 900^\circ$, because the angles of each triangle sum to 180°. When adding up all these angles, we include each angle of the pentagon once and each of the points of the star twice. The angles of the pentagon sum to 540°, leaving $900^\circ - 540^\circ = 360^\circ$. This must equal twice the sum of the angles at the points of the star (since these angles were included twice in the sum), so

$$\boxed{\text{the angles at the points of the star must sum to } 360^\circ/2 = 180^\circ}.$$

(There are many ways to solve this problem; see if you can find others.)

10.44★:

What is the greatest number of acute interior angles that a decagon can have? (Note: the decagon may be concave!)

Hint: Try the problem with a quadrilateral: can all of the angles of a quadrilateral be acute?

Hint: What do you know about the angles of a decagon? What does this tell you about the number of acute angles a decagon could have?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: A decagon has 10 sides, so the sum of its interior angles is $(10 - 2)(180^\circ) = 1440^\circ$. Suppose the decagon has n acute angles. The sum of these n angles is less than $90n$ degrees. The remaining $10 - n$ angles must each be less than 360° (the angles may be reflex angles). So, the sum of all of the angle measures must be less than

$$90n + 360(10 - n)$$

degrees. Expanding the product gives $90n + 3600 - 360n$ degrees, and simplifying gives $3600 - 270n$ degrees.

We now know that if there are n acute angles in the decagon, then the sum of the angles in the decagon must be less than $3600 - 270n$ degrees. We also know that the sum of all the interior angles must be 1440 degrees. So, it is only possible to make a decagon with n acute angles if $3600 - 270n > 1440$. Adding $270n$ to both sides of the inequality and subtracting 1440 from both sides gives $2160 > 270n$. Dividing by 270 gives $\frac{2160}{270} > n$. Simplifying the fraction gives $8 > n$. This tells us that it is impossible to have 8 or more acute angles. The diagram at the right shows that we can make a decagon with $\boxed{7}$ acute angles.

Review Problems

10.21:

Two lines intersect such that the measure of one of the angles formed by the lines is five times the measure of another of the angles formed by the lines. What are the measures of the angles formed by the lines?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Intersecting lines form two pairs of congruent angles. Let x be the measure of each smaller angle, so $5x$ is the measure of each larger angle. Each smaller angle can be combined with a larger angle to form a straight angle, so we must have $x + 5x = 180^\circ$. This gives us $6x = 180^\circ$, so $x = 30^\circ$. Therefore, the angles formed by the lines have measures $\boxed{30^\circ \text{ and } 150^\circ}$.

10.22:

Find $\angle BAE$ in the diagram below:

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The angles around a point must sum to 360°, so we have

$$\angle BAE = 360^\circ - 65^\circ - 132^\circ - 71^\circ = \boxed{92^\circ}.$$

10.23:

In the diagram below, we have $\overleftrightarrow{OW} \perp \overleftrightarrow{OP}$ and $\overleftrightarrow{OX} \perp \overleftrightarrow{OZ}$. If $\angle WOZ$ is five times $\angle XOY$, then what is the measure of $\angle XOY$?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

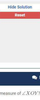
Solution: Let $x = \angle XOY$, so $\angle WOZ = 5x$. Since $\overleftrightarrow{OW} \perp \overleftrightarrow{OP}$, we have

$$\angle WOX = 90^\circ - \angle XOY = 90^\circ - x.$$

Since we also have $\overleftrightarrow{OX} \perp \overleftrightarrow{OZ}$, we have $\angle XOZ = 90^\circ$, and

$$\angle WOZ = \angle WOX + \angle XOZ = (90^\circ - x) + 90^\circ = 180^\circ - x.$$

We now have both $\angle WOZ = 5x$ and $\angle WOZ = 180^\circ - x$, so $5x = 180^\circ - x$. Adding x to both sides gives $6x = 180^\circ$, so $x = \boxed{30^\circ}$.



10.24:

What is the number of degrees in the smaller angle between the hour hand and the minute hand on a clock at 8:30?

Source: MATHCOUNTS

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: At 8:30, the hour hand points exactly between the 8 and the 9 on the face of the clock, as shown in the diagram on the right. There are $360/12 = 30$ degrees between adjacent numbers on the clock, so at 8:30 the hour hand is 15 degrees past the 8. The 8 is $2 \cdot 30 = 60$ degrees past the 6, which is where the minute hand is pointing. So, the smaller angle between the hour and minute hand at 8:30 is $15^\circ + 60^\circ = \boxed{75^\circ}$.

10.25:

Source: MATHCOUNTS

A revolving restaurant rotates one complete revolution every 56 minutes. In the 21 minutes it takes to eat the peaches jubilee dessert, through how many degrees does the restaurant revolve?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: In the 21 minutes it takes to eat dessert, the restaurant goes through $\frac{21}{56}$ of a rotation. Since a full rotation is 360°, the restaurant revolves during dessert by

$$\frac{21}{56} \cdot 360^\circ = \frac{3}{8} \cdot 360^\circ = 3 \cdot 45^\circ = \boxed{135^\circ}.$$

10.26:

In the diagram below, the measure of $\angle ACA$ is 50% greater than the measure of $\angle BCA$. Angle ACY is a right angle. What is the measure of $\angle ACY$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $\angle ACA$ is 50% greater than $\angle ACB$, we have $\angle ACA + \angle ACB = 180^\circ$. Since $\angle ACA$ is 50% greater than $\angle ACB$, we have $\angle ACA = 1.5\angle ACB$. So $1.5\angle ACB + \angle ACB = 180^\circ$. Therefore, we have $2.5\angle ACB = 180^\circ$, so $\angle ACB = \frac{180^\circ}{2.5} = 72^\circ$. We have $\angle BCY = 180^\circ - \angle ACY = 90^\circ$, so

$$\angle ACY = \angle BCY - \angle ACB = 90^\circ - 72^\circ = \boxed{18^\circ}.$$

10.27:

A line intersects parallel lines ℓ and m forming angles with measures $3x + 4^\circ$ and 131°, as shown in the diagram below. Find the value of x.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Because $\ell \parallel m$, the angles labeled in the diagram must add to 180°. Therefore, we have the equation $3x + 4^\circ + 131^\circ = 180^\circ$. Simplifying the left side gives $3x + 135^\circ = 180^\circ$, so $3x = 45^\circ$. Dividing by 3 gives $x = \boxed{15^\circ}$.

10.28:

If line a is perpendicular to two different lines b and c, then must we have $b \parallel c$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Yes. A diagram for this problem is shown below. If a is perpendicular to both b and c, then the corresponding angles indicated in the diagram are indeed equal. So b must be parallel to c.

10.29:

(a) Find x in the diagram at the right.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The angles at Q together make a straight angle, so $2x + (x + 12^\circ) = 180^\circ$. Simplifying the left side gives $3x + 12^\circ = 180^\circ$, so $3x = 168^\circ$ and $x = \boxed{56^\circ}$.

(b) Find $x'R$ in the diagram above.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have $x = 56^\circ$ from part (a), so $\angle P = 56^\circ$ and $\angle PQR = 2x = 112^\circ$. Now that we have the measures of two angles of $\triangle PQR$, we can find the third:

$$\angle R = 180^\circ - \angle P - \angle PQR = \boxed{12^\circ}.$$

Extra challenge: Could we have determined that $\angle R = 12^\circ$ without ever finding x?

10.30:

In the diagram on the right, we have $\overline{EG} \parallel \overline{DH}$.

(a) Find $\angle DEF$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Because $\overline{EF} \parallel \overline{DH}$, we must have $\angle D + \angle E = 180^\circ$. The box at $\angle D$ tells us that $\angle D = 90^\circ$, so $\angle E = 180^\circ - \angle D = \boxed{90^\circ}$.

(b) Find $\angle EFH$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Because $\overline{EF} \parallel \overline{DH}$, we have $\angle EFH = \angle FHD = 180^\circ$. Therefore, we have

$$\angle EFH = 180^\circ - \angle FHD = 180^\circ - 75^\circ = \boxed{105^\circ}.$$

(c) Find $\angle FHG$.

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: Because $\overline{EG} \parallel \overline{DH}$, we have $\angle HFG = \angle FHD = 75^\circ$. In $\triangle FGH$, we have

$$\angle HFG + \angle G + \angle FHG = 180^\circ,$$

so $75^\circ + 56^\circ + \angle FHG = 180^\circ$, which gives $\angle FHG = \boxed{47^\circ}$.

10.31:

One angle of a triangle has measure 20° greater than another angle of the triangle and half the measure of the third angle of the triangle. Find the measures of all three angles.

Hide Solution

Reset

Your Submission: Solution

Solution: Let the first angle mentioned have measure x, so the second angle has measure $x + 20^\circ$ and the third angle has measure $2x$. Since these three angles are the angles of a triangle, we must have $x + (x + 20^\circ) + 2x = 180^\circ$. Simplifying the left side gives $4x = 20^\circ = 180^\circ$. Adding 20° to both sides gives $4x = 200^\circ$, and dividing by 4 gives $x = 50^\circ$. Therefore, the first angle has measure $\boxed{50^\circ}$, the second has measure $x + 20^\circ = \boxed{70^\circ}$, and the third has measure $2x = \boxed{100^\circ}$.

10.32:

In the diagram at the right, side $\overline{BC}$ of $\triangle ABC$ is extended past B to a point D. Explain why $\angle ABD$ must equal $\angle A + \angle C$.

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: In triangle ABC, we must have $\angle A + \angle C + \angle ABC = 180^\circ$, so $\angle A + \angle C = 180^\circ - \angle ABC$. We also have $\angle ABC + \angle ABD = 180^\circ$, so $\angle ABD = 180^\circ - \angle ABC$. Since $\angle A + \angle C$ and $\angle ABD$ both equal $180^\circ - \angle ABC$, we have $\angle ABD = \angle A + \angle C$.

10.33:

Source: MATHCOUNTS

The four angles of a quadrilateral are in the ratio of $1 : 2 : 3 : 4$. What is the measure of the smallest angle?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let x be the measure of the smallest angle, so the angles have measures x, $2x$, $3x$, and $4x$. The sum of the angles in a quadrilateral is 360°, so we have $x + 2x + 3x + 4x = 360^\circ$, which means that $10x = 360^\circ$. Dividing by 10 gives $x = \boxed{36^\circ}$.

10.34:

Find angle measures x and y in the figure below. Note that the double arrows on $\overline{BE}$ and $\overline{DI}$ mean that $\overline{BE} \parallel \overline{DI}$.

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: Since $\overline{EH} \parallel \overline{FD}$, we have $y = \angle HED = \boxed{36^\circ}$. Since we have $\angle EFD + \angle GFD = 180^\circ$ and hence

$$\angle GFD = 180^\circ - 95^\circ = 85^\circ.$$

From $\triangle GFD$, we have

$$\angle FGD = 180^\circ - \angle GFD - \angle FDG = 180^\circ - 85^\circ - 36^\circ = 59^\circ.$$

Since $\overline{HI} \parallel \overline{EG}$, we have $x + \angle EGD = 180^\circ$, so

$$x = 180^\circ - \angle EGD = 180^\circ - 59^\circ = \boxed{121^\circ}.$$

10.35:

Find angle measures h and j in the diagram below.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: First, we have $\angle ADF = 180^\circ - 130^\circ = 50^\circ$. From $\triangle ADF$, we find

$$h = 180^\circ - \angle ADF - \angle AFD = 180^\circ - 50^\circ - 91^\circ = \boxed{39^\circ}.$$

Since $\overline{CE} \parallel \overline{AD}$, we have $\angle FCE = \angle FAD$, so $j = h = \boxed{39^\circ}$.

10.36:

What is the measure of each interior angle of a regular nonagon?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: A nonagon has 9 sides. The sum of the interior angles in a nonagon is $(9 - 2)(180^\circ) = 7(180^\circ)$. Each angle in a regular nonagon has the same measure, so each measures

$$\frac{7(180^\circ)}{9} = 7\left(\frac{180^\circ}{9}\right) = 7(20^\circ) = \boxed{140^\circ}.$$

Another way to do this problem is to use the fact that the exterior angles of any polygon add to 360°. (We showed this earlier in Exercise **10.33**.) So, each exterior angle of a regular nonagon has measure $\frac{360^\circ}{9} = 40^\circ$. Therefore, each interior angle has measure $180^\circ - 40^\circ = \boxed{140^\circ}$.

Solution: $(11 \times 11 - 1) \div 5$ | Solution: $(2 \times 2) \times 11 + 11$ | Solution: $11 \times 13 - 7 \times 17$ | Solution: $(13 \times 19 + 17) \div 11$

Problems to combat or mathematics has existed as much mystery, romanticism, misconception and human interest as the number pi. — William L. Schaaf

CHAPTER 11 Perimeter and Area

In this chapter, we discuss methods to measure the size of geometric objects. You've probably familiar with many of the formulas and concepts we will discuss in this chapter. The goals of this chapter are to give you better intuition for why these formulas work and to apply them to challenging problems.

11.1 Measuring Segments

Back in Section 10.1, we introduced the line segment.

Figure 11.1.A Segment

Recall that we use the endpoints to label the segment. For example, $\overline{AB}$ is the segment from A to B. To refer to the length of the segment, we omit the bar. For example, if you measure $\overline{AB}$ in Figure 11.1 with a ruler, you'll find that AB equals 2 inches. We often leave out the units in geometry problems. So, for example, we might write $AB = 2$.

One segment is a segment. To make sense of two segments, we need to know how they are related. We use the **congruent** symbol, $\cong$, to denote that two segments are **congruent** if they have the same length. If we have more than two segments, we use a different number of tick marks for each. For example, the pairs of tick marks on $\overline{ZX}$ and $\overline{WY}$ above indicate that $ZX = WY$, and that these lengths need not be the same as XM and YZ.

Figure 11.1.B Midpoint and Marking Segments of Equal Length

In Figure 11.2, M is the midpoint of $\overline{XY}$. In the diagram, we indicate that $XM = MY$ with the little tick marks along $\overline{XM}$ and $\overline{MY}$. We say that two segments are **congruent** if they have the same length. If we have more than two pairs of congruent segments, we use a different number of tick marks for each. For example, the pairs of tick marks on $\overline{ZX}$ and $\overline{WY}$ above indicate that $ZX = WY$, and that these lengths need not be the same as XM and YZ.

One way to measure a closed figure is by the total length of its boundary. We call this the **perimeter** of the figure. For example, the perimeter of a polygon is the sum of the lengths of its sides.

Problems

Problem 11.1.1 ☐ ✎ ☐

Farmer Fred wants to fence the oddly-shaped region shown in bold at the right. If each of the squares shown has sides that are 10 feet long, and fence costs $7 per foot, then how much will Fred's fence cost?

Solution for Problem 11.1.1: The total length of each bit of fence, and then add up all these lengths. But there's a more clever solution. Suppose the farmer started at the upper right corner, point P in the diagram on the right, and walked counterclockwise all the way around the region to fence the fence. His path will always be directly down, left, up, or right on our diagram. He'll walk downward along 5 squares and back up along 5 squares. Similarly, he'll walk 7 squares to the left and 7 squares to the right. So, in total, he walks a total of

$$5 + 5 + 7 + 7 = 24$$

sides of the squares. Each of the 24 side lengths is 10 feet, so he walks 240 feet. The fence costs $7 per foot, so Fred's fence costs $240 \cdot \$7 = \$1680$.

Another quick way to find the length of the fence is to rearrange the fence into a simpler shape. We do so by moving the parts of the fence that are inside the grid out to the boundary of the grid, as shown on the right. The old parts of the fence inside the grid are shown in gray. The new parts (which are slightly smaller) are shown in black. The top and the bottom of the rectangle are congruent, and the left and the right sides are congruent. As before, we find that the perimeter is a total of $2(5 + 7) = 24$ side lengths of the squares in the original diagram.

Important: Opposite sides of a rectangle are congruent. We often call the lengths of different sides of a rectangle the **length** and the **width** of the rectangle. So, if the length of a rectangle is l and the width is w, then the perimeter of the rectangle is $2l + 2w$.

Problem 11.2 Source: AMC 8 ☐ ✎ ☐

A triangle is called **equilateral** if all of its sides have the same length. In the diagram at the right, all three triangles are equilateral. Point F is the midpoint of $\overline{AC}$ and C is the midpoint of $\overline{AE}$. If $AB = 4$, then what is the perimeter of $ABCDEFG$?

Solution for Problem 11.2: The desired perimeter is

$$AB + BC + CD + DE + EF + FG + GA.$$

Since $\triangle ABC$ is equilateral and $AB = 4$, we know that $BC = AC = 4$ as well. Because D is the midpoint of $\overline{BC}$, we have $AD = CD = BC/2 = 4/2 = 2$. Since $\triangle ABC$ is equilateral and $AD = 2$, we have $AE = DE = AD = 2$. Then, because C is the midpoint of $\overline{AE}$, we have $AC = CE = AE/2 = 2/2 = 1$. And, because $\triangle ACD$ is equilateral, we have $CF = EF = CD = 1$.

Just whole paragraph is just a long way of saying "Long segments have length 4 medium segments have length 2, and short segments have length 1." The sides of $ABCDEFG$ consist of two long segments ($\overline{AB}$ and $\overline{BC}$), two medium segments ($\overline{CD}$ and $\overline{DE}$), and three short segments ($\overline{EF}$, $\overline{FG}$, and $\overline{GA}$), so the perimeter is $2(4) + 2(2) + 3(1) = 15$.

Notice that our final answer equals the sum of the perimeters of $\triangle ABC$ and $\triangle EFG$. We can see why with a clever rearrangement:

Problem 11.3 ☐ ✎ ☐

Points D and C are on segment $\overline{AB}$ such that $AC : CD = 3 : 1$ and D is the midpoint of $\overline{AC}$. If $BC = 5$, then what is AD?

Solution for Problem 11.3: We start with a quick sketch to help guide us. We must make sure to get the points in the right order. Since D and C are on $\overline{AB}$, we know that A and D are at the ends of the segment. Since D is the midpoint of $\overline{AC}$, we know that D is between A and C. The equation $AC : CD = 3 : 1$ tells us that AC is longer than CD, so C is closer to D than to A. We also write the given length $BC = 6$ on our diagram, as shown.

Concept: Diagrams help organize information in geometry problems. Quick sketches can help you avoid errors, and give you hints to find solutions.

Since D is the midpoint of $\overline{AC}$, we have $AB = BC$, so $AB = 6$, as well. We add this information to our diagram.

Concept: Update your diagram as you find more information.

All we have left to find is CD. We now have $BC = AB + BC = 12$, so the given ratio $AC : CD = 3 : 1$ tells us that $12 : CD = 3 : 1$. Multiplying the ratio on the right by 4 gives us $12 : CD = 12 : 4$, so $CD = 4$. Finally, we have $AD = AC + CD = 16$.

Problem 11.4 ☐ ✎ ☐

A triangle is called **isosceles** if two of its sides are congruent. The two congruent sides are called the **legs** of the triangle and the other side is called the **base** of the triangle. Suppose an isosceles triangle has perimeter 45 and length of each leg is twice the length of the base. What is the length of the base?

Solution for Problem 11.4: We assign a variable to the quantity we seek. Let b be the length of the base of the triangle. Since the length of each leg is twice the length of the base, each leg has length $2b$. We can display all this information in a diagram, as shown at the right. Since the perimeter of the triangle is 45, we have

$$b + 2b + 2b = 45.$$

Simplifying the left side gives $5b = 45$, and dividing by 5 gives $b = 9$. Therefore, the length of the base is 9. ☐

Concept: Assigning variables can be just as useful in geometry problems as in other types of problems.

We combined this strategy with our "organize geometric information with a diagram" strategy to solve the problem.

Problem 11.5 ☐ ✎ ☐

Problems 11.2 and 11.4 are intended to reinforce fundamental definitions and accessible triangles. An equilateral triangle is also an isosceles triangle. That is, "isosceles" does not mean that the base must have a different length than the legs. A triangle with no two sides congruent is called **scalene**. "Scalene" is just a fancy way to say "not isosceles."

Problem 11.5 ☐ ✎ ☐

(a) Mary leaves home and walks 7 miles in one direction before stopping at room for lunch. The same day, Jeff leaves from the same home and walks 4 miles before also stopping at room for lunch. In a straight line, how far apart are Mary and Jeff now? Explain.

(b) Is it possible for a triangle to have side lengths 4, 7, and 12? Why or why not?

(c) Suppose a triangle has side lengths a, b, and c. Is it possible for $a + b$ to be less than c? Is it possible for $a + b$ to equal c?

Solution for Problem 11.5:

(a) Mary and Jeff will be farthest apart at lunchtime if they walk in opposite directions. If they walk in opposite directions, they will be $7 + 4 = 11$ miles apart. Therefore, they cannot have lunch 12 miles apart.

(b) No.

We can view this as essentially the same as the previous part. The side of length 7 represents Mary's bike and the side of length 4 represents Jeff's bike. If we let the distance between Mary and Jeff at lunchtime be the length of the third side, then part (a) tells us that the third side length cannot be more than 11. Therefore, if the lengths of two sides of a triangle are 4 and 7, then the third side length cannot possibly be 12. Similarly, no triangle can have side lengths 4, 7 and 11.

We can think about this from the same way as we did in the previous part. Suppose Mary starts from home and bikes a miles while Jeff starts from the same home and bikes b miles. The farthest apart they can be at the end of their hikes is $a + b$ miles. So, if $c > a + b$, they can't possibly be at the same spot.

As in part (b), we can consider Mary's hike to be one side of a triangle, Jeff's hike to be another side of the triangle, and the segment connecting the endpoints of their hikes to be the third side of the triangle. So, the vertices of this triangle are the endpoints of the two hikes and the home where they both start. We know that the length of the third side of this triangle is at most $a + b$. So, the length of this triangle can't possibly be larger than the sum of the lengths of the other two sides. Similarly, it is impossible for a triangle to have side lengths a, b, and c if $c > a + b$.

Next, we consider whether or not we can have $c = a + b$. The only way Mary's and Jeff's hikes can end a + b miles apart is if they walk in opposite directions. This means that their home is on the segment connecting the endpoints of their hikes. In other words, the "triangle" is just a line segment, as shown in the diagram. So, we cannot make a triangle with side lengths a, b, c if $c = a + b$.

c.

In Problem 11.5, we discovered that if a, b, and c are the side lengths of a triangle, then $a + b > c$. This powerful relationship is called the **Triangle Inequality**. We also often write the Triangle Inequality as:

Important: For any three points A, B and C we have

$$AB + BC \geq AC.$$

We have $AB = BC = AC$ if and only if B is on $\overline{AC}$.

In other words, to get from point A to point C, it's shorter to go straight from A to C than to go first to some other point B and on $\overline{AC}$. The Triangle Inequality is just a fancy way of saying "The shortest distance between two points is a straight line."

Length and Perimeter | Triangle Inequality Introduction

Exercises

11.1.1: ☐ ✎ ☐

If each square in the diagram at the right has side length 1, then what is the perimeter of the figure traced in bold?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission:

Solution: Starting from the upper left corner and going clockwise, we add the side lengths and find a perimeter of

$$3 + 1 + 2 + 4 + 1 + 2 + 6 + 4 + 2 + 2 + 1 + 2 + 5 + 7 = \boxed{42}.$$

11.1.2: ☐ ✎ ☐

I am drawing a picture on a 12 inch by 16 inch rectangular piece of paper for my art class. My teacher tells me that I must leave a 1 inch margin on all sides of the paper, since that region will be covered by the frame. What is the perimeter of the region in which I can still draw?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission:

Solution: The diagram at the right shows the effect of including a margin. The dashed rectangle is the remaining area in which I can draw. Since the margin is 1 inch on all sides, the length and width of the dashed rectangle are 2 inches shorter than the length and width of the solid rectangle. This leaves a 14 inch by 10 inch rectangle in which I can still draw. The perimeter of this region is

$$2(14 + 10) = 2(24) = \boxed{48 \text{ inches}}.$$

11.1.3: ☐ ✎ ☐

Segment $\overline{AB}$ is 12 inches long. Points X and Y are selected on $\overline{AB}$ such that $\overline{AX}$ and $\overline{BY}$ are congruent, and the ratio of AX to AY is 1 : 4. What is the length of $\overline{XY}$?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission:

Solution: Since $AX : AY = 1 : 4$, we have $AY = 4AX$, which means $XY = 3AX$, and X is closer to A than Y is. Since $BY = AX$, we have

$$AB = AX + XY + BY = AX + 3AX + AX = 5AX.$$

Therefore, we have $5AX = 12$, so $AX = \frac{12}{5}$. We then have

$$BX = XY + BY = 3AX + AX = 4AX = 4\left(\frac{12}{5}\right) = \boxed{\frac{48}{5} \text{ inches}}.$$

Solution for Problem 11.1.3

11.1.4: Source: AMC 8 ☐ ✎ ☐

The sides of a triangle have lengths 6, 15, and x, where x is a positive integer. What is the smallest possible value of x?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission:

Solution: The Triangle Inequality tells us that $6.5 + x$ must be greater than 15. So, x is at least $\boxed{4}$.

11.1.5: ☐ ✎ ☐

One side of an isosceles triangle is three times as long as another side of the triangle. If the perimeter of the triangle is 140, then what is the length of the base of the triangle?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission:

Solution: Let x be the length of the first side of the triangle, so $3x$ is the length of the second side. Since the triangle is isosceles, the third side must have length x or $3x$. But the third side cannot have length x, since the triangle is x, x, $3x$ do not satisfy the Triangle Inequality. Therefore, the sides of the triangle have lengths x, $3x$, and $3x$. So, the perimeter of the triangle is $x + 3x + 3x = 7x$, so we have $7x = 140$, which means $x = 20$. The base of the triangle thus has length $\boxed{60}$.

11.1.6: ☐ ✎ ☐

Squares are constructed on each of the sides of a triangle as shown to the right. If the perimeter of the triangle is 17, then what is the perimeter of the nine-sided figure that is composed of the remaining three sides of each of the squares?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission:

Solution: Each solid side of a square has length equal to one of the sides of the triangle. There are three solid sides equal to each side of the triangle, so the perimeter of the nine-sided figure is three times the perimeter of the triangle. Therefore, the desired perimeter is $3(17) = \boxed{51}$.

11.1.7 a: Source: MATHCOUNTS ☐ ✎ ☐

Suzanne has determined that swimming the length of a rectangular pool 60 times is the same distance as swimming its perimeter 18 times. What is the ratio of the width to the length of this swimming pool?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission:

Solution: Let the length of the pool be l and the width be w. The perimeter of the pool is $2l + 2w$. Since 18 times the perimeter equals 60 lengths, we have $18(2l + 2w) = 60l$. Expanding the product on the left gives $(18)2l + (18)2w = 60l$, so $36l + 36w = 60l$. Subtracting $36l$ from both sides gives $36w = 24l$. Dividing both sides by 2 gives $\frac{36w}{2} = 24l$, and dividing both sides by 24 gives $\frac{w}{24} = \frac{l}{24}$. Therefore, the ratio of the width to the length is $\boxed{\frac{2}{3}}$.

11.1.8 a: ☐ ✎ ☐

An isosceles triangle has integer side lengths and perimeter 20. What are the possible values of the length of each leg?

Hint: How long would the base have to be if the legs each have length 11 to such a triangle possible?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission:

Solution: Let the base have length b and each leg have length a. Since the perimeter is 20, we must have $2a + b = 20$, so $b = 20 - 2a$. Since a is at least 1, we have $b \leq 18$. Expanding the product on the left gives $(18)2l + (18)2w = 60l$, so $36l + 36w = 60l$. Subtracting $36l$ from both sides gives $36w = 24l$. Dividing both sides by 2 gives $\frac{36w}{2} = 24l$, and dividing both sides by 24 gives $\frac{w}{24} = \frac{l}{24}$. Therefore, the ratio of the width to the length is $\boxed{\frac{2}{3}}$.

11.1.8 a: ☐ ✎ ☐

An isosceles triangle has integer side lengths and perimeter 20. What are the possible values of the length of each leg?

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11.2 Area

The area of a figure is a measure of how much it covers the surface of a closed figure, we use **area** to measure the space contained inside the figure.

We square all the right angles lies of length 1 inch. We say that the area of the square is "1 square inch" where "square inch" is a unit of area. We say that the area of a rectangle is "2 square inch" if the area of the square area "1 inch long, the two rectangles area should be "1 square inch" Rather than writing out "square inch" or "square foot" we sometimes abbreviate the units using the symbol "in²" for "square inch" and "ft²" for "square foot" and so on.

If we lower the units and we have a square with side length 1 foot, the area is "1 square unit" but we usually have the units "square feet" or "square inches" or "square cm" or "square km" depending on the units we use.

Of course, we'll find the area of the square with side length 1 inch and 1 ft, we find the area of figure as the number of unit squares that are needed to cover the figure. As you probably know, the area of a rectangle is the product of its length and width.

The area of a rectangle is $A = l \cdot w$

For example, to rectangle $WXYZ$ of the right, the length is 5 and the width is 3, So, the area of $WXYZ$ is $5 \cdot 3 = 15$ square units. The shape of the right prism is a great area why this formula for the area of a rectangle is so important. Suppose we want to find the area of the right triangle ABC in the right prism, the area is half the area of the rectangle $WXYZ$.

Finally, we sometimes use brackets to refer to area, so we write $[WXYZ] = 15$ to mean that the area of $WXYZ$ is 15 square units.

Problems

Problem 11.1 Jump to Solution

In rectangle $PQRS$, we have $PQ = 4$ and $QR = 9$. What is the area of $\triangle PQR$?

Problem 11.2 Jump to Solution

Rectangle $ABCD$ has length 6 and width 4. Point E is the midpoint of the diagonal AC. Point F is the midpoint of the side CD. What is the area of $\triangle ABE$ and $\triangle ACDF$?

Problem 11.3 Jump to Solution

Rectangle $ABCD$ has length 6 and width 4. Point E is the midpoint of the diagonal AC. Point F is the midpoint of the side CD. What is the area of $\triangle ABE$ and $\triangle ACDF$?

Problem 11.4 Jump to Solution

Two rectangles are drawn in the figure below. What is the area of the shaded region?

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Two rectangles are drawn in the figure below. What is the area of the shaded region?

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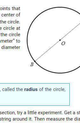
Two rectangles are drawn in the figure below

11.3 Circles

You probably recognize the bold curve at the right as a **circle**. This circle consists of all points that are 1 inch from point O. Point O is called the **center** of the circle. Any segment from the center of a circle to the circle itself is called a **radius** of the circle. So, for example, $\overline{OA}$ is a radius of the circle. We also use the term "radius" to refer to the length of such a segment. So, the radius of the circle at the right is 1 inch. A **diameter** of the circle is a line segment that connects two points on the circle and passes through the center of the circle. As with "radius," we can also use the word "diameter" to refer to the length of a diameter. The diameter of a circle is twice its radius, since each diameter consists of two radii ("radii" is the plural of radius).



Definition: A *circle* consists of all points that are a fixed distance, called the **radius** of the circle, from a given point, called the **center** of the circle.

The perimeter of a circle is called the circle's **circumference**. Before reading the rest of this section, try a little experiment. Get a string, and find numerous circular objects. Measure the distance around each object by wrapping the string around it. Then measure the diameter of the object. Finally, for each object find the quotient

$$\frac{\text{Distance Around the Object}}{\text{Diameter of Object}}$$

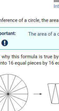
You should find that in each case the quotient is a little more than 3. (If you get anything much different, try measuring and dividing again!)

In every circle, the circumference divided by the diameter equals the same number. This special number is called **pi** and is almost always written as the Greek letter π, pronounced "pie." Just as with $\sqrt{2}$, we can't write a decimal number that equals π, but we can approximate π. To the nearest hundredth, π rounds to 3.14.

Sidenote: For some reason, memorizing huge portions of π is a bit of a sport among mathlovers. This should help you get started:

3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679

(Please don't tell your parents you got this π-memorizing idea from us.)



Introduction to Circles

In addition to the circumference of a circle, the area of a circle is also related to π:

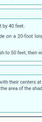
Important: The area of a circle with radius r is πr^2.

We can get a sense for why this formula is true by slicing up a circle and rearranging the pieces. On the left below, we have a circle with radius r that is divided into 16 equal pieces by 16 equally spaced radii. These pieces are called **sectors**.

Next, we rearrange the sectors as shown in the second figure above. Then, we take half of one of the end sectors and slide it to the other end as shown in the final figure on the right above. This final figure resembles a rectangle.

The circumference of the original circle is equally divided among the top and bottom of our "rectangle." So, the "length" of the "rectangle" is half the circumference of the circle, which is $\frac{1}{2}(2\pi r) = \pi r$. The "width" of the "rectangle" is simply the radius of the circle, which is r.

Since the area of a rectangle is its length times its width, the area of our "rectangle" is $(\pi r)(r) = \pi r^2$. The "rectangle" is composed of the exact same pieces as our original circle, so the area of the circle is the same as the area of the "rectangle." This isn't a completely accurate explanation for why the area of the original circle is πr^2, since the "rectangle" isn't exactly a rectangle. But, hopefully it gives you some intuition for why the formula is true.


I have a square that will slide a length 12 feet. Which of the quotes configurations below should I use?
Square
Square
Square

11.4 Summary

The **perimeter** of a closed figure is the total length of its boundary. For example, the perimeter of a rectangle with length l and width w is $2(l+w)$.

Important:

For any three points A, B, and C, we have

$$AB + BC \geq AC.$$

We have $AB + BC = AC$ if and only if B is on $\overline{AC}$. This relationship is called the **Triangle Inequality**.

The **area** of a closed figure is the number of unit squares (or pieces of squares) needed to exactly cover the figure. We sometimes use brackets to refer to area, so that $[ABC]$ means the area of $\triangle ABC$. The area of a rectangle equals the product of its length and width.

Important:

To find the area of a triangle, we select one side to be the "base" of the triangle. The perpendicular segment from the vertex opposite the base to the line containing the base is the **altitude** to that base. The area then is

$$\frac{\text{base length} \times \text{altitude length}}{2}.$$

A **circle** consists of all points that are a fixed distance, called the **radius** of the circle, from a given point, called the **center** of the circle. We also use the word "radius" to describe a segment from the center of the circle to a point on the circle. A **diameter** is a line segment that connects two points on the circle and passes through the circle's center. We also use the word "diameter" to mean the length of a diameter. The perimeter of a circle is called the circle's **circumference**.

Important:

If a circle has diameter d and radius r, then:

- $d = 2r$.
- The circumference of the circle is πd, or $2\pi r$, where π is a number that is approximately 3.14. (The symbol π is called "pi.")
- The area of the circle is πr^2.

Challenge Problems

11.39:

Source: MATHCOUNTS

Points A, B, C, and D lie on a line in alphabetical order. If $AB : BD = 5 : 7$ and $AC : CD = 13 : 11$, determine the ratio $AB : BC : CD$.

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Since $AB : BD = 5 : 7$, we know that there is some x such that $AB = 5x$ and $BD = 7x$. Similarly, since $AC : CD = 13 : 11$, there is some y such that $AC = 13y$ and $CD = 11y$. So, we have both $AD = AB + BD = 12x$ and $AD = AC + CD = 24y$, which means $12x = 24y$. This gives us $x = 2y$, so $AB = 5x = 10y$ and

$$BC = AD - AB - CD = 24y - 10y - 11y = 3y.$$

Therefore, we have

$$AB : BC : CD = 10y : 3y : 11y = \boxed{10 : 3 : 11}.$$

Another approach is to choose a length for $\overline{AD}$. A convenient choice is 24, since this will allow us to avoid fractions as segment lengths. Because $AC : CD = 13 : 11$ and $AC + CD = 24$, we have $AC = 13$ and $CD = 11$. Because $AB : BD = 5 : 7$ and $AB + BD = 24$, we have $AB = 10$ and $BD = 14$. We then have

$$BC = BD - CD = 14 - 11 = 3,$$

so $AB : BC : CD = \boxed{10 : 3 : 11}$ as before.

11.40:

In rectangle $ABCD$, point X is the midpoint of $\overline{AD}$ and Y is the midpoint of $\overline{CD}$. What fraction of the area of the rectangle is enclosed by $\triangle AXY$?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The area of the rectangle is $(AD)(DC)$. Since $\overline{YD}$ is the altitude from vertex Y of $\triangle AXY$, we have $[AXY] = (AX)(DY)/2$. Since X is the midpoint of $\overline{AD}$, we have $AX = AD/2$. Since Y is the midpoint of $\overline{DC}$, we have $DY = DC/2$. So, we have

$$[AXY] = \frac{(AX)(DY)}{2} = \frac{(AD/2)(DC/2)}{2} = \frac{(AD)(DC)}{8},$$

which means that the area of $\triangle AXY$ is $\boxed{\frac{1}{8}}$ the area of the rectangle.

11.41:

Point T is on side $\overline{QR}$ of $\triangle PQR$. Find the ratio QT/QR if the area of $\triangle PQT$ is 75 and the area of $\triangle PTR$ is 40.

Hint: We are given the areas of triangles. $\overline{QT}$ and $\overline{QR}$ are bases of triangles.

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Let h be the length of the altitude from P to $\overline{QR}$. This is the height from P for triangles PQT, PQR, and PTR. So, $[PQT] = (QT)(h)/2$ and $[PQR] = (QR)(h)/2$. When we divide the first equation by the second, we have

$$\frac{[PQT]}{[PQR]} = \frac{(QT)(h)/2}{(QR)(h)/2} = \frac{(QT)(h)}{(QR)(h)} = \frac{2}{(QR)(h)} = \frac{QT}{QR}.$$

In other words, if two triangles share an altitude, then the ratio of the triangles' areas equals the ratio of the lengths of the bases to which the altitude is drawn. Here, we have $[PQT] = 75$ and

$$[PQR] = [PQT] + [PTR] = 75 + 40 = 115,$$

so we have

$$\frac{QT}{QR} = \frac{[PQT]}{[PQR]} = \frac{75}{115} = \boxed{\frac{15}{23}}.$$

11.42:

In the diagram on the right, $WXYZ$ is a rectangle. The area of triangle ZXA is 36, and $ZA = 3AY$.

(a) If $XY = 12$, then what is the area of rectangle $WXYZ$?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We are given XY and seek for $[WXYZ]$. We have $[WXYZ] = (XY)(ZY)$, so we only need ZY to find $[WXYZ]$. We are also given $[ZXA]$. Since $\overline{XY}$ is the altitude from X to $\triangle ZXA$, we have $[ZXA] = (XY)(ZA)/2$. So, we can use the given value of $[ZXA]$ together with the value of XY to find ZA. Then, we use ZA to find ZY. Since $ZA = 3AY$, we have $AY = \frac{ZA}{3}$ so

$$ZY = ZA + AY = ZA + \frac{ZA}{3} = \frac{4ZA}{3}.$$

Since $[ZXA] = (XY)(ZA)/2$, we have $36 = 12(ZA)/2$, so $36 = 6ZA$. Therefore, we have $ZA = 6$, so $ZY = \frac{4ZA}{3} = \frac{4 \cdot 6}{3} = 8$, and $[WXYZ] = (ZY)(XY) = \boxed{96}$.

(b) If $XY = 9$, then what is the area of rectangle $WXYZ$?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Similar to part (a), since $[ZXA] = (XY)(ZA)/2$, we have $36 = 9(ZA)/2$, so $72 = 9ZA$. Therefore, we have $ZA = 8$, so $ZY = \frac{4ZA}{3} = \frac{4 \cdot 8}{3} = \frac{32}{3}$, and

$$[WXYZ] = (ZY)(XY) = \frac{32}{3} \cdot 9 = \boxed{96}.$$

(c) If $XY = 6$, then what is the area of rectangle $WXYZ$?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Similar to part (a), since $[ZXA] = (XY)(ZA)/2$, we have $36 = 6(ZA)/2$, so $36 = 3ZA$. Therefore, we have $ZA = 12$, so $ZY = \frac{4ZA}{3} = \frac{4 \cdot 12}{3} = 16$, and $[WXYZ] = (ZY)(XY) = \boxed{96}$.

(d) Do you notice a pattern in your answers to the first three parts? Will this pattern hold for other values of XY?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We found the same result for $[WXYZ]$ in all three parts. It's unlikely that this is a coincidence. Let's investigate. The area of $WXYZ$ is twice the area of $\triangle ZXY$. Triangles ZXY and ZXA have the same height from vertex X, and base $\overline{ZY}$ of $\triangle ZXY$ is $\frac{4}{3}$ times as long as base $\overline{ZA}$ of $\triangle ZXA$, so

$$[ZXY] = \frac{4}{3}[ZXA] = \frac{4}{3}(36) = 48.$$

Finally, we have $[WXYZ] = 2[ZXY] = 96$, no matter what XY is.

11.43:

What is the side length of a regular polygon whose interior angles each measure 168 degrees and whose perimeter is 120 cm?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The sum of the angles in a polygon with n sides is $(n-2)(180^\circ)$, so each angle in a regular polygon has measure $\frac{(n-2)(180^\circ)}{n}$. Since this must equal 168°, we have

$$\frac{(n-2)(180^\circ)}{n} = 168^\circ.$$

Multiplying both sides by n gives $(n-2)(180^\circ) = (n)(168^\circ)$. Expanding the product on the left gives $(n)(180^\circ) - 360^\circ = (n)(168^\circ)$. Adding 360° to both sides, and subtracting $(n)(168^\circ)$ from both sides, gives $(n)(12^\circ) = 360^\circ$. Dividing both sides by 12° gives $n = 30$. Since the polygon has 30 congruent sides and perimeter 120 cm, each side has length $120/30 = \boxed{4 \text{ cm}}$.

A much faster way to tackle this problem is to use the fact that the exterior angles of a polygon add to 360°. Since each interior angle is 168°, each exterior angle is $180^\circ - 168^\circ = 12^\circ$. Since all the exterior angles add to 360° and each measures 12°, there must be $\frac{360^\circ}{12^\circ} = 30$ of them. We then use the perimeter to determine that each side has length $\boxed{4 \text{ cm}}$ as before.

11.44:

In quadrilateral $ABCD$, each of the side lengths is an integer, and $AD = BC$. If we have $AB : AD = 2 : 5$ and $AD : CD = 3 : 4$, then what is the smallest possible perimeter of the quadrilateral?

Hint: AD is in both of the given ratio relationships.

Hint: We are given that $AB : AD = 2 : 5$, and that AB and AD are both integers. Could AD possibly be 47 Why or why not? Just from these given facts, what are the possible values of AD

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Since $AB : AD = 2 : 5$, we have $AB = 2x$ and $AD = 5x$ for some value of x. Since AB and AD must be integers, and the greatest common divisor of 2 and 5 is 1, the value of x must be an integer. Similarly, since $AD : CD = 3 : 4$, there is some integer y for which $AD = 3y$ and $CD = 4y$. Combining our expressions for AD, we have $5x = 3y$. The smallest positive integers x and y for which this equation holds are $x = 3$ and $y = 5$. So, we have $AB = 2x = 6$, $AD = 5x = 15$, $CD = 4y = 20$, and $BC = AD = 15$. Finally, the perimeter of $ABCD$ is $6 + 15 + 20 + 15 = \boxed{56}$.

11.45:

$ABCDEF$ is a regular hexagon in which diagonal $\overline{AD}$ has length 16. Find the perimeter of $ABCDEF$.

Hint: Draw all three long diagonals.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Drawing all three long diagonals of a regular hexagon splits the regular hexagon into 6 equilateral triangles, as shown on the right. $\overline{AD}$ consists of two sides of these triangles, so each equilateral triangle has side length $16/2 = 8$. The sides of the regular hexagon consist of six sides of these triangles, so the perimeter of the hexagon is $6 \cdot 8 = \boxed{48}$.

11.46:

Rebecca walks 100 feet in a straight line. She then turns 20 degrees to the left and walks another 100 feet, and then turns 20 degrees to the left again. She continues this pattern until she reaches the point where she started. How far did she walk?

Hint: What shape does Rebecca's path form? What angles of this shape do you know?

You may type any additional notes you have here.

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Your Submission: Solution

Solution: At each of her turns her path has a $180^\circ - 20^\circ = 160^\circ$ angle. Suppose she forms a regular polygon with her walk. (We don't know for sure that her angles will work out; we are just checking if it is possible that they do so.) Then, the 20° by which she turns is an exterior angle of the polygon. Since the exterior angles of the polygon sum to 360° and each has measure 20°, the polygon must have $\frac{360^\circ}{20^\circ} = 18$ angles. Checking, we see that each interior angle of a regular polygon with 18 sides is indeed $\frac{(18-2)(180^\circ)}{18} = 160^\circ$. Therefore, Rebecca's walk forms a regular polygon with 18 sides. Each side has length 100 feet, so she walks $18 \cdot 100 = \boxed{1800 \text{ feet}}$. (See the solution to 11.43 see how to find the number of sides of the polygon without considering the exterior angles.)

For a considerably more challenging problem, try to figure out how far Rebecca would walk if she turned 165° clockwise every 100 feet!

11.47:

Source: MATHCOUNTS

In the figure shown, points D, E, and F are the midpoints of semicircles ADB, BEC, and DFE, respectively. If the radius of each semicircle is 1, then what is the area of the shaded region?

Hint: We can find the areas of some complicated figures by rearranging them into figures whose areas we can find.

You may type any additional notes you have here.

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Your Submission: Solution

Solution: We break the upper semicircle into two quarter-circles and slide those two quarter-circular regions into the locations shown in the diagram at right. Now, the shaded region is a rectangle whose width is the radius of each semicircle and whose length is the diameter of each semicircle. Since the radius of each semicircle is 1, the area of the rectangle is $(1)(2) = \boxed{2}$.

11.48:

Teri has sticks of length 9 inches, 12 inches, and 14 inches. She cuts the same amount off of each stick. After cutting this amount off each stick, she is no longer able to make a triangle by attaching the sticks end to end. What is the smallest amount Teri could have cut off of each stick?

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The only way Teri cannot make a triangle is if the sum of the lengths of the two shortest sticks is not greater than the length of the longest stick. If she cuts x of each stick, then the sticks have lengths $9 - x$, $12 - x$, and $14 - x$. The sum of the two shortest sticks then is $(9 - x) + (12 - x)$, which equals $21 - 2x$. If she can't make a triangle, then this sum must be less than or equal to the length of the longest stick, so $21 - 2x \leq 14 - x$. Adding $2x$ to both sides and subtracting 14 from both sides gives $7 \leq x$. Therefore, the smallest amount Teri could have cut off each stick is $\boxed{7 \text{ inches}}$. If she cuts 7 inches off of each stick, she'll have sticks with lengths 2 inches, 5 inches, and 7 inches. Since $2 + 5$ is not greater than 7, she cannot make a triangle with these sticks.

11.49★:

Source: MATHCOUNTS

Suppose we have $AB = 8$, $BC = 10$, $CD = 16$, and $DA = 12$ in quadrilateral $ABCD$. How many different possible integer values are there for AC?

You may type any additional notes you have here.

Hide Solution

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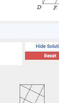
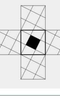
Your Submission: Solution

Solution: We apply the Triangle Inequality separately to $\triangle ABC$ and $\triangle ADC$. Applying it to $\triangle ABC$, we see that AC must be greater than 2 and less than 18. Applying it to $\triangle ADC$, we see that AC must be greater than 4 and less than 28. Combining these, we see that AC must be greater than 4 and less than 18, so the possible lengths are 5, 6, 7, ..., 17. There are $\boxed{13}$ numbers in this list.



11.50★:

Points E, F, G, and H are midpoints of sides of square $ABCD$ as shown. If $\overline{AB}$ is 20 units long, then what is the area of the shaded region?



Hint: It might help to think outside the box on this one.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We do a little "outside the box" thinking, as shown at the right. We extend the pattern beyond the given initial square past each side, as shown. We see that each of the triangle pieces inside the original square fits together with a quadrilateral piece inside the square to form a square that is the same size as the black square. Therefore, inside the original square, the 8 white pieces that are quadrilaterals or triangles fit together to make 4 squares that are the same size as the black square. This means that the black square's area is $\frac{1}{5}$ the area of $ABCD$. The area of $ABCD$ is $(20)(20) = 400$ square units, so the area of the black square is $\boxed{80 \text{ square units}}$.

11.17.

Segment EN has midpoint M and point N is the midpoint of segment EP. What is the ratio of MN to EP?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Let the length of EP be x. We start with a diagram:

M is the midpoint of EN, so MN is $\frac{1}{2}$ of EN. Similarly, N is the midpoint of EP, so EN is $\frac{1}{2}$ of EP. Combining these, we see that MN is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ of EP. Therefore, $MN : EP = \boxed{1:4}$.

11.18.

The length and width of a rectangle are each increased by 10%.

(a) What percent is the perimeter of the rectangle increased?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Let the original length be l and the original width be w. After increasing both by 10%, the length is $1.1l$ and the width is $1.1w$.

The original perimeter is $2(l + w)$. The perimeter of the new rectangle is

$$2(1.1l + 1.1w) = 2(1.1(l + w)) = 1.1 \cdot 2(l + w).$$

Therefore, the new perimeter is 1.1 times the old perimeter, which means it is $\boxed{10\%}$ greater than the old perimeter.

(b) At what percent is the area of the rectangle increased?

You may type any additional notes you have here.

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Your Submission Solution

Let the original length be l and the original width be w. After increasing both by 10%, the length is $1.1l$ and the width is $1.1w$.

The original area is lw. The area of the new rectangle is

$$(1.1l)(1.1w) = 1.21lw.$$

The new area is 1.21 times the old area, so it is $\boxed{21\%}$ greater than the old area.

11.19. Source: MATHCOUNTS ⧉

The perimeter of a hexagon is 305 cmts. One side of the hexagon is 45 cmts long and the lengths of the other sides are in the ratio $1 : 2 : 3 : 4 : 5$. Find the positive difference between the lengths of the longest and shortest sides.

You may type any additional notes you have here.

Hide Solution

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Your Submission Solution

Solution: Let the smallest of the five sides in the ratio $1 : 2 : 3 : 4 : 5$ have length x, so the five sides have lengths $x, 2x, 3x, 4x$, and $5x$. Then the perimeter of the hexagon is $x + 2x + 3x + 4x + 5x + 45 = 305$. Simplifying this gives a perimeter of $15x + 45$. We are given that the perimeter is 305, so $15x + 45 = 305$. Subtracting 45 from both sides gives $15x = 275$ and dividing by 15 gives $x = 17.\overline{3}$. So, the sides have lengths $17.\overline{3}$, $34.\overline{6}$, 52, $69.\overline{3}$, and $86.\overline{6}$. Therefore, the positive difference between the lengths of the longest and shortest sides is $86.\overline{6} - 17.\overline{3} = \boxed{69 \text{ mmts}}$.

11.20.

An architect draws a rectangular room to scale. The drawing is 12 cm long and 8 cm wide. The shorter dimension of the actual room is 20 feet. What is the longer actual room?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: The ratio of the length of the drawing to the width of the drawing is $12 : 8$. Simplifying this ratio, we have

$$\text{longer dimension} : \text{shorter dimension} = 3 : 2.$$

Since the shorter dimension of the room is 20 feet, we have

$$\text{longer dimension} : 20 \text{ feet} = 3 : 2.$$

Multiplying both parts of the ratio on the right by 10 gives

$$\text{longer dimension} : 20 \text{ feet} = 30 : 20,$$

so the longer dimension is 30 feet. (We also might have used the $3 : 2$ ratio to note that the longer dimension is $\frac{3}{2}$ times the shorter dimension. This means that the longer dimension of the room is $\frac{3}{2}(20) = 30$ feet.)

The perimeter of the room is $2(30 + 20) = \boxed{100 \text{ feet}}$.

11.21. Source: MODEL ⧉

Square tiles 9 inches on a side exactly cover the floor of a rectangular room. The border tiles are white and all other tiles are blue. The room measures 18 feet by 15 feet. How many tiles are white?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: We have to be careful about the corners of the room. There are four corner tiles. Along one width of the room, the corner tiles cover $2 \cdot 9 = 18$ inches already. The width of the room is $15 \cdot 12 = 180$ inches, so we must tile 162 inches more. This requires $\frac{162}{9} = 18$ more tiles. The other width of the room also requires 18 more tiles. As with each width, 18 inches of each length of the room are already covered by corner tiles. The length of the room is $18 \cdot 12 = 216$ inches, so $216 - 18 = 198$ inches remain to tiled. This requires $\frac{198}{9} = 22$ tiles for each length. Including all 4 corners, both widths and both lengths, we need $4 + 18 + 18 + 22 + 22 = \boxed{82 \text{ tiles}}$.

11.22. Source: MATHCOUNTS ⧉

If adjacent sides meet at right angles in the figure at the right, what is the number of centimeters in the perimeter of the figure?

2
1
1
1

12.1 The Pythagorean Theorem

In a right triangle, the side of the triangle opposite the right angle is called the **hypotenuse** and the other two sides are called the **legs** of the triangle. We use the letters a and b for the legs and c for the hypotenuse, as shown in the figure.

In this section, we explain one of the most famous math theorems, the **Pythagorean Theorem**, which is a powerful relationship among the lengths of the sides of a right triangle. It is called by a striking nickname, "the cow of many pounds of the Pythagorean Theorem" (pronounced "pythag-uhrem-ee").

Problems

Problem 12.1 Guide to Solution

Do you know right triangles with legs of lengths 3 and 4 and whose hypotenuse is the sides of square $WXYZ$ are shown, such that $PW = QX = RY = SZ = 1$ and $PQ = QR = RS = ST = 1$.

a) Explain why $PT = 1$ and $PS = 2$ and why $PQRS$ is a square.

b) What is the area of $PQRST$?

c) What is the area of $WXYZ$?

d) Find WX.

Problem 12.2 Guide to Solution

Do you know right triangles with legs of lengths 3 and 4 and whose hypotenuse is the sides of square $WXYZ$ are shown, such that $PW = QX = RY = SZ = 1$ and $PQ = QR = RS = ST = 1$.

a) Explain why $PT = 1$ and $PS = 2$ and why $PQRS$ is a square.

b) What is the area of $PQRST$?

c) Find the area of $WXYZ$.

d) Find WX.

Problem 12.3 Guide to Solution

Find the missing side lengths in each of the three triangles shown below.

Problem 12.4 Guide to Solution

Find the hypotenuse of a right triangle in the larger triangle of the triangle's legs are 1 and 2.

Problem 12.5 Guide to Solution

In Problems 12.1 and 12.3, we have seen two right triangles in which all three side lengths are integers. Can you find any more right triangles in which all three side lengths are integers?

Can you find a value of the hypotenuse that is an integer?

Problem 12.6 Guide to Solution

Find the hypotenuse of a right triangle whose legs are 3 - 4 and 4 - 6.

Find the hypotenuse of a right triangle whose legs are 3 - 5 and 4 - 6.

Find the hypotenuse of a right triangle whose legs are $\sqrt{2}$ and $\sqrt{3}$.

Find the hypotenuse of a right triangle whose legs are $\frac{1}{2}$ and $\frac{1}{3}$.

Problem 12.7 Guide to Solution

The length of one leg of a right triangle is 210 and the triangle's hypotenuse has length 750. What is the length of the other leg?

Problem 12.8 Guide to Solution

Consider right triangle with legs of lengths 3 and 4 as pictured to the side. Let $WXYZ$ be a square with side length 1 and let P be a point on WX with $PX = 1$. Let Q be a point on XY with $QX = 1$.

a) Show that $\angle PYX = \angle PWX$ and hence $\angle PWX = \angle QXY$.

b) Show that $\triangle PWX \cong \triangle QXY$.

c) Find the area of $WXYZ$.

d) Find WX.

Solution for Problem 12.1

a) Back in Problem 12.1, we have learned that the area of a right triangle is $\frac{1}{2}ab$. Therefore, in right triangle PWX we have

$$\triangle PWX \text{ has area } \frac{1}{2}(1)(2) = 1.$$

Since triangles WXZ and PWX are identical, we have $\triangle PWX \cong \triangle WXZ$. Substituting this into the equation above gives

$$\triangle WXZ \text{ has area } \frac{1}{2}(1)(2) = 1.$$

We are told that $WXYZ$ is a square, so $\angle WXZ = 45^\circ$. Therefore, we have

$$\angle PWX = \angle PWZ = \angle WXZ = \angle ZXY = 45^\circ.$$

$$\angle PWX + \angle WXY = \angle PWZ + \angle ZWX = 90^\circ.$$

$$\angle PWX + \angle WXY = \angle WXZ + \angle ZXY = 90^\circ.$$

Therefore, PWX is a straight angle. This means that P lies on side XY. Since $WXYZ$ is a square, we know that all sides of quadrilateral $PQRS$ lack side of $PQRST$ has length $1 = 1 - 1$, and each angle of $PQRS$ is the right angle of one of the triangles. So, we know $PQRS$ is a square, and so the area of the square $PQRS$ is the square with side length 1 which is 1 square unit.

b) Since $PQRS$ is a square with side length 1, we know $PQRS$ has area 1 square unit.

c) Each right triangle has area $\frac{1}{2}(1)(2) = 1$. Four triangles have right triangles from $PQRS$ leaves $WXYZ$ as we have

$$[WXYZ] = [PQRS] + 4[\triangle] = 1 + 4 = 5 = 25.$$

d) The area of $WXYZ$ is the square of its side length. Because the area of $WXYZ$ is 25, its side length must be $\sqrt{25}$, which equals 5.

Problem 12.2 Guide to Solution

Solution for Problem 12.2

a) Since PT is a side of a right triangle with legs of lengths 3 and 4, $PT = 1$.

Since $PQRS$ is a square, $PQRST$ is a square, and the vertices of $WXYZ$ are on the sides of $PQRST$. Each side of $PQRST$ has length $1 = 1$ in the sum of PQ and PQ is $1 + 1$ PT the sum of PQ is $1 + 1$ and PQ is 1 of PQ and PQ is 1 of PQ and PQ is 1 of $PQ

12.2 Some Special Triangles

In this section, we use the Pythagorean Theorem to investigate a few common special types of triangles.

Problem 12.0 ⇧ Jump to Solution

DEF is an isosceles triangle with $DE = EF = 26$ and $DF = 20$.

(a) Suppose M is on line DF such that $\overline{EM}$ is an altitude of the isosceles triangle. Why must M be the midpoint of $\overline{DF}$?

(b) Find the area of $\triangle DEF$.

Problem 12.1 ⇧ Jump to Solution

(a) Suppose $\triangle ABC$ is equilateral with side length 6. Find the length of an altitude of $\triangle ABC$.

(b) Find the area of $\triangle ABC$.

Problem 12.10 ⇧ Jump to Solution

(a) If one leg of an isosceles right triangle has length 6, then what are the lengths of the other leg and the hypotenuse of the triangle?

(b) If one leg of an isosceles right triangle has length a, then what are the lengths of the other leg and the hypotenuse of the triangle?

(c) If we know the length of one leg of an isosceles right triangle, then what's a fast way to find the length of the hypotenuse?

Problem 12.11 ⇧ Jump to Solution

A **30-60-90 triangle** is a triangle whose angle measures 30°, 60°, and 90°.

(a) Describe how to divide an equilateral triangle into two congruent 30-60-90 triangles.

(b) Suppose the leg opposite the 30° angle in a 30-60-90 triangle has length 1. Use your answer to part (a) to find the hypotenuse of the triangle. What is the length of the other leg?

(c) Suppose the leg opposite the 30° angle in a 30-60-90 triangle has length s. In terms of s, what are the lengths of the hypotenuse and the other leg of the triangle?

Problem 12.0.A ⇧ ⇩ ✓

DEF is an isosceles triangle with $DE = EF = 26$ and $DF = 20$.

Question: Suppose M is on line DF such that $\overline{EM}$ is an altitude of the isosceles triangle. Why must M be the midpoint of $\overline{DF}$?

Question: Find the area of $\triangle DEF$.

Solution for Problem 12.0

Triangles DEM and FEM are right triangles. Applying the Pythagorean Theorem to these triangles, we have

$$DM^2 + EM^2 = DE^2 = 26^2 = 676,$$
$$FM^2 + EM^2 = EF^2 = 26^2 = 676.$$

Since the right-hand sides of these equations are equal, the left-hand sides must also be equal. So, we have $DM^2 = FM^2$. Since DM and FM are lengths, this means that $DM = FM$. Therefore, we have $DM = FM$, which means that M is the midpoint of $\overline{DF}$.

Important: The altitude to the base of an isosceles triangle divides the base into two congruent segments.

(b)

Since M is the midpoint of $\overline{DF}$, we have $DM = DF/2 = 10$. Applying the Pythagorean Theorem to $\triangle DEM$ gives $DM^2 + EM^2 = DE^2$, so

$$10^2 + EM^2 = 26^2.$$

This gives us

$$EM^2 = 26^2 - 10^2 = 676 - 100 = 576.$$

We take the square root to find $EM = 24$. (Notice that we also could have found EM by noticing that $\triangle DEM$ has length 5 and triangle ABC has length 13. So, using the $(5, 12, 13)$ Pythagorean triple, we know that the other leg has length $12 \cdot 2 = 24$.) The area of $\triangle DEF$ then is $(DF)(EM)/2 = 240$ square units.

Concept: Building right triangles and applying the Pythagorean Theorem is one of the most powerful techniques in geometry and trigonometry.

In Problem 12.0, we discovered that we can think of an isosceles triangle as a pair of identical right triangles glued together along a common leg. This way of viewing an isosceles triangle reveals another important fact about isosceles triangles:

Important: In an isosceles triangle, the angles opposite the equal sides have the same measure. If two sides of a triangle are equal, then the angles opposite those sides are equal. In an isosceles triangle, the angle at the vertex opposite the base bisects the angle formed by the two sides opposite those angles are equal.

The equal angles of an isosceles triangle are called the **base angles** of the triangle, and the other angle is called the **vertex angle** of the triangle.

Problem 12.1 ⇧ ⇩ ✓

(a) Suppose $\triangle ABC$ is equilateral with side length 1. Find the length of an altitude of $\triangle ABC$.

(b) Find the area of $\triangle ABC$.

Solution for Problem 12.1

(a)

We start by drawing altitude $\overline{CD}$ of the triangle. Because $CA = CB$, we know that this altitude divides base $\overline{AB}$ into equal pieces. So, we have $AD = DB = 1$. Applying the Pythagorean Theorem to $\triangle ADC$ gives

$$AD^2 + DC^2 = AC^2.$$

Since $AD = 1$ and $AC = 6$, we have $\overline{\ }^2 + DC^2 = 6^2$, which means $DC^2 = 36 - 1 = 35$.

Taking the square root gives $DC = \sqrt{35} = \sqrt{5} \cdot \sqrt{7} = 5\sqrt{7}$.

(b) Now that we have the length of an altitude, we can find the area.

$$\{ABC\} = \frac{(AB)(CD)}{2} = \frac{(1)(\sqrt{35})}{2} = \frac{\sqrt{35}}{2}$$
$$= \frac{1}{2} \cdot 5\sqrt{7}$$
$$= 2 \cdot 5\sqrt{7}$$
$$= 5\sqrt{7} \text{ square units.}$$

Isosceles triangle Pythagorean Theorem

Problem 12.10 ⇧ ⇩ ✓

(a) If one leg of an isosceles right triangle has length 6, then what are the lengths of the other leg and the hypotenuse of the triangle?

(b) If one leg of an isosceles right triangle has length a, then what are the lengths of the other leg and the hypotenuse of the triangle?

(c) If we know the length of one leg of an isosceles right triangle, then what's a fast way to find the length of the hypotenuse?

Solution for Problem 12.10

(a)

Since the triangle is isosceles, two of the sides must have the same length. We know that the hypotenuse of a right triangle is the longest side of the triangle. So, it must be the leg that has the same length as an isosceles right triangle. This means the other leg of the triangle is also the problem has length 6. So, then we use the Pythagorean Theorem to find the hypotenuse. If we let c be the hypotenuse, then the Pythagorean Theorem tells us that

$$c^2 = 6^2 + 6^2 = 36 + 36 = 36.$$

Taking the square root gives us

$$c = \sqrt{36} = \sqrt{36 \cdot 2} = 6\sqrt{2}.$$

We conclude that each leg of the right triangle has length 3 and the hypotenuse has length $3\sqrt{2}$.

(b)

We follow essentially the same steps as in the previous part. Since the triangle is isosceles, the legs have the same length, so the second leg has length a. If we let c be the hypotenuse, then the Pythagorean Theorem tells us that

$$a^2 + a^2 = 6^2 = 6 \cdot 6 = 12a.$$

Taking the square root gives us

$$c = \sqrt{12a} = \sqrt{6 \cdot 2 \cdot a} = \sqrt{6a}.$$

We conclude that each leg of the right triangle has length a and the hypotenuse has length $\sqrt{6a}$.

(c)

So far we have been an isosceles right triangle with side lengths $5, 5, \sqrt{2}$ and one with side lengths $6, 6, 6\sqrt{2}$. It looks like there's a pattern. We know that the leg of an isosceles right triangle must always have the same length, but it always true that the hypotenuse is $\sqrt{2}$ times each leg?

We investigate by assigning a variable to the leg length. Suppose each leg has length x. Then, the sum of the squares of the legs is $x^2 + x^2$, which equals $2x^2$. So the Pythagorean Theorem tells us that the square of the hypotenuse equals $2x^2$, which means the length of the hypotenuse is $\sqrt{2x^2}$. We can simplify $\sqrt{2x^2}$ as follows:

$$\sqrt{2x^2} = \sqrt{x^2} \cdot \sqrt{2} = x\sqrt{2}.$$

We therefore see that the hypotenuse is $\sqrt{2}$ times as long as each leg. (As a side note, we usually write $x\sqrt{2}$ instead of $\sqrt{2}x$ to make it clear that the x is not inside the radical.)

Important: In an isosceles right triangle, the legs are congruent and the hypotenuse is $\sqrt{2}$ times as long as each leg.

Note that the two acute angles of an isosceles right triangle must be congruent, since they are opposite the congruent sides of the triangle. Because the acute angles of a right triangle must sum to 90°, each of these two angles has measure $90^\circ/2 = 45^\circ$. For this reason, isosceles right triangles are often referred to as **45-45-90 triangles**.

Isosceles Right Triangle

Problem 12.11 ⇧ ⇩ ✓

A **30-60-90 triangle** is a triangle whose angle measures 30°, 60°, and 90°.

(a) Describe how to divide an equilateral triangle into two congruent 30-60-90 triangles.

(b) Suppose the leg opposite the 30° angle in a 30-60-90 triangle has length 1. Use your answer to part (a) to find the hypotenuse of the triangle. What is the length of the other leg?

(c) Suppose the leg opposite the 30° angle in a 30-60-90 triangle has length s. In terms of s, what are the lengths of the hypotenuse and the other leg of the triangle?

Solution for Problem 12.11

(a) First, we note that all three angles of an equilateral triangle must be equal, because any two of the angles are opposite equal sides of the triangle. So, each angle of an equilateral triangle must measure $180^\circ/3 = 60^\circ$. Now, we can divide an equilateral triangle into two congruent triangles by drawing an altitude to one side of the triangle. Applying the Pythagorean Theorem to $\triangle ITS$ gives $TI^2 + IS^2 = TS^2$, so

$$TI^2 + (TS/2)^2 = TS^2 \Rightarrow TI^2 = 3/4 \cdot TS^2 \Rightarrow TI = \sqrt{3}/2 \cdot TS.$$

Therefore, we have $TI = \sqrt{3}/2 \cdot 12$. (We could also have used the Pythagorean triple $(12, 16, 20)$.) Since $\triangle ITS$ is isosceles with $IT = IS$, we know that the altitude $\overline{IT}$ divides base $\overline{TS}$ into two congruent segments. Therefore, we have $TS = 2 \cdot TI = 2 \cdot 6 = 12$.

We immediately have $IT = IS = \frac{12}{2} = 6$, and TI/IS are two triangles by drawing $\overline{PR}$, as shown on the right. Since $QP = QR$, we know that $\triangle QPR$ is isosceles with $\angle QPR = \angle QRP$. Since $\angle Q = 60^\circ$, these two base angles each are $\angle P = \angle R = (180^\circ - 60^\circ)/2 = 60^\circ$. Therefore, each measures 60°, which means $\triangle QPR$ is equilateral. Therefore, $PR = 14$. $\angle RPS = 120^\circ - \angle QPR = 60^\circ$, and $\angle PRS = 90^\circ - 60^\circ = 30^\circ$.

Turning to $\triangle PRS$, we have

$$CS = 180^\circ - \angle BPS - \angle PRB = 90^\circ.$$

So PRS is a 30-60-90 triangle with hypotenuse $PR = 14$. The leg opposite the 30° equals half the hypotenuse, so $PS = \frac{14}{2}$. The longer leg is $\sqrt{3}$ times the shorter leg, so $SR = \frac{7\sqrt{3}}{2}$.

30-60-90 Triangle

Exercises

12.2.1: ⇧ ⇩ ✓

Find the missing side lengths below.

A
B
C
D
E
F
G
H
I
J
K
L
M
N
O
P
Q
R
S
T
U
V
W
X
Y
Z
6
6
6

12.3.2 Types of Quadrilaterals

In the section on properties of parallelograms, **rectangle** is a parallelogram in which all the angles are right angles. The diagonals of a rectangle are equal in length. A rectangle is also a parallelogram, so it has all the properties of a parallelogram.

Figure 12.3 a Rectangle

Figure 12.3 b Square

In the section on properties of rhombuses, a rhombus is a parallelogram in which all the sides are equal in length. The diagonals of a rhombus are perpendicular to each other and bisect the angles of the rhombus.

Problems

Problem 12.10 ◀ Hide Hint ◀ Hide Solution

Prove that a square is a rhombus.

Problem 12.11 ◀ Hide Hint ◀ Hide Solution

Prove that a rectangle is a parallelogram.

Problem 12.12 ◀ Hide Hint ◀ Hide Solution

Prove that a square is a rectangle.

Problem 12.13 ◀ Hide Hint ◀ Hide Solution

Prove that a rectangle is a rhombus if and only if its diagonals are equal.

Problem 12.14 ◀ Hide Hint ◀ Hide Solution

Prove that a rhombus is a square if and only if its diagonals are equal.

Problem 12.15 ◀ Hide Hint ◀ Hide Solution

Prove that a rectangle is a square if and only if its diagonals are equal.

Problem 12.16 ◀ Hide Hint ◀ Hide Solution

A **trapezoid** is a quadrilateral in which two sides are parallel. The two parallel sides of a trapezoid are called the **bases** of the trapezoid. The perpendicular distance between the bases is the **height** of the trapezoid. The angles adjacent to the bases are called **base angles**. In the figure, the bases are $\overline{AB}$ and $\overline{CD}$. The bases are shown in red. The height is shown in blue.

Figure 12.4 Trapezoid

Prove that the sum of the interior angles of a trapezoid is 360°.

Problem 12.17 ◀ Hide Hint ◀ Hide Solution

Prove that the sum of the interior angles of a trapezoid is 360°.

Problem 12.18 ◀ Hide Hint ◀ Hide Solution

Prove that the sum of the interior angles of a trapezoid is 360°.

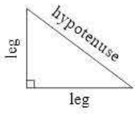
Problem 12.19 ◀ Hide Hint ◀ Hide Solution

Prove that the sum of the interior angles of a trapezoid is 360°.

**Problem

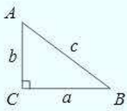
12.4 Summary

In a right triangle, the side of the triangle opposite the right angle is called the **hypotenuse** and the other two sides are called the **legs** of the triangle. We also often use the terms "legs" and "hypotenuse" to refer the lengths of the legs and hypotenuse of a right triangle.



Important: The **Pythagorean Theorem** tells us that in any right triangle, the sum of the squares of the legs equals the square of the hypotenuse. So, in the diagram to the right, we have

$$a^2 + b^2 = c^2.$$

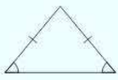


The Pythagorean Theorem also works in reverse: if the sum of the squares of two sides of a triangle equals the square of the third side, then the triangle is a right triangle.

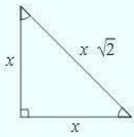
A **Pythagorean triple** is a group of three positive integers that satisfy the Pythagorean Theorem equation. So, for example, $\{3, 4, 5\}$ is a Pythagorean triple, as are $\{5, 12, 13\}$ and $\{8, 15, 17\}$. There are infinitely many Pythagorean triples.

Important: If we multiply all three side lengths of a right triangle by the same positive number, then the three new side lengths also satisfy the Pythagorean Theorem. In other words, if side lengths a, b, and c satisfy $a^2 + b^2 = c^2$, then $(na)^2 + (nb)^2 = (nc)^2$ for any number n.

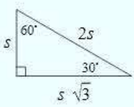
Important: In an isosceles triangle, the angles opposite the equal sides have the same measure. If two sides of a triangle are equal, then the angles opposite those sides are equal. Similarly, if two angles of a triangle are equal, then the sides opposite those angles are equal.



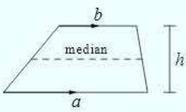
Important: In an isosceles right triangle, the legs are congruent and the hypotenuse is $\sqrt{2}$ times as long as each leg. An isosceles triangle is often called a **45-45-90 triangle**.



Important: In a right triangle with acute angles of $30°$ and $60°$, the side lengths are in the ratio $1 : \sqrt{3} : 2$ as shown to the right. Such a triangle is often called a **30-60-90 triangle**.



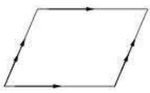
Definitions: A **trapezoid** is a quadrilateral with two parallel sides. The segment connecting the midpoints of the other two sides is the **median** of the trapezoid, and the distance between the two parallel sides is the **height** of the trapezoid.



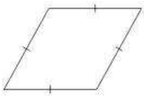
Important:
- The median of a trapezoid is parallel to the bases of the trapezoid, and equal in length to the average of the lengths of the bases.
- The area of a trapezoid equals the height of the trapezoid times the length of the median of the trapezoid. So, the area of the trapezoid above is $h(a + b)/2$.

Definition: A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel.

Important:
- The area of a parallelogram is the product of a side length (the base) and the distance between that side and the opposite side of the parallelogram. This distance between opposite sides is called a **height** of the parallelogram.
- In any parallelogram, the opposite sides are equal, and the opposite angles are equal.



A Parallelogram



A Rhombus

Definition: A quadrilateral is a **rhombus** if all of its sides are equal.

Important:
- Every rhombus is a parallelogram. Therefore, everything that is true about parallelograms is true about every rhombus.
- The diagonals of a rhombus are perpendicular. The area of a rhombus is half the product of its diagonals (and also equals its base times its height).

Definition: A quadrilateral in which all angles are equal is a **rectangle**.

Important:
- All rectangles are parallelograms, so all that is true of parallelograms is true of rectangles.
- Let two consecutive sides of a rectangle have lengths ℓ and w. The area of the rectangle is ℓw, and the diagonals of the rectangle both have length $\sqrt{\ell^2 + w^2}$.

A Rectangle

A Square

Definition: A quadrilateral in which all sides are equal and all angles are equal is a **square**.

Important:
- Each square is a parallelogram, a rectangle, and a rhombus, so all that is true of parallelograms, rectangles, or rhombuses is true of squares.
- If the side length of a square is s and its diagonal is d, then $d = s\sqrt{2}$ and the area of the square is s^2, or $d^2/2$.

WARNING!! Although every rhombus is a parallelogram, not every parallelogram is a rhombus. Therefore, if we prove a property that is true for every rhombus, this property is not necessarily true for every parallelogram. (The same warning holds for rectangles–rectangles are parallelograms, but not all parallelograms are rectangles, etc.)

12.37:

In the diagram at the left, CY is the center of the circle, $\angle CXOY$ is a rectangle, and the area of the circle is 196π. What is the length of diagonal $\overline{YP}$ of the rectangle?

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The diagonals of a rectangle are congruent, so $YP = OX$. $\overline{OXY}$ is a radius of the circle. Since the area of the circle is 196π, the circle radius is $\sqrt{196} = 14$. Therefore, $YP = \boxed{14}$.

12.38:

In the diagram at the right, $\overline{DF}$ ∥ $\overline{BC}$, and the area of $\triangle ABD$ is 13. Given that the area of $\triangle ABC$ is $\frac{4}{3}$ of the area of $\triangle ADF$, find $\angle ABD$.

You are given the equation involving $\triangle B$ and $\triangle D$. How are $\overline{AB}$ and $\overline{CD}$ related to the triangles mentioned in the equation?

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Solution: Let A be the height of the trapezoid. The height from D of $\triangle ABD$ is also h, as is the height from D of $\triangle BCD$. Since $[ABD] = 13$, we have $\frac{1}{2}AB(h) = 13 \Rightarrow 2.5(h) = 13$. Dividing both sides by h and multiplying both sides by 2 gives $AB = 2.5h$. Substituting this into $[ABC] = \frac{1}{2}(2.5h)(h) = 1.25h^2$. Since $[ABC] = \frac{4}{3}[ADF]$, we have $1.25h^2 = \frac{4}{3}[ADF]$. $[ADF] = \frac{1}{2}(h)(CD)$ and dividing by $\frac{1}{2}$ gives $(2.5h)(CD) = 2.5(h) = 2$. Therefore, $CD = \frac{2}{2.5} = 0.8$. Hence, $AB = 2.5h = \frac{2}{0.8} = 2.5$.

12.39:

Source: MATHCOUNTS ✓

$ABCD$ is a square, M and N are midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. What is the ratio of the area of $\triangle ADN$ to the area of $\triangle ADM$? Express your answer as a fraction in simplest form.

Hint: What do the two triangles have in common?

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Solution: Let $\overline{DM}$ ∥ $\overline{BN}$ = x, so the side length of the square is $2x$. Therefore, we have $[ADM] = x^2$ $\frac{1}{2}$ We also have $AM = CN = x$, so

$$[AMD] = [CMD] = (x)(2x)/2 = x^2.$$

Finally we have

$$\begin{aligned}[BND] &= [ABCD] - [NBD] - [AMD] - [CND] \\ &= 4x^2 - \frac{x^2}{2} - x^2 \\ &= \frac{5}{2}x^2.\end{aligned}$$

Therefore, we have

$$[MBND]/[ADM] = (x^2)/(2x^2/2) = \boxed{1/2}.$$

Perhaps a quicker way to see this is to note that $[AMD]$ and $[MBND]$ share a base, $\overline{MD}$. So, the ratio of the areas of the triangles equals the ratio of their heights to that common side. The altitudes to $\overline{MD}$ of these two triangles together form diagonal $\overline{AC}$. Since the DN DOM is twice, Dee if you can figure out why $\overline{MN}$ divides this diagonal into a 1 : 2 ratio.

12.40:

A 30-60-90 triangle is drawn on the exterior of equilateral triangle ABC as shown so that the hypotenuse of the right triangle is one side of the equilateral triangle. If the area of the right triangle is 6, what is the perimeter of ABC?

Hint: What length strategies do you know?

Hint: Is there a right triangle that has the desired segment as a side?

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Solution: Let $x = \overline{BC}$ and $\angle ABC = 60°$, so we have $\angle ABD = 30° + 60° = 90°$. Therefore, we can find AD by first finding $\overline{BD}$ and $\overline{AB}$. Let y be the length of $\overline{BD}$. Then, we can apply the Pythagorean Theorem to $\triangle ABD$ to find $\overline{AB}$ in terms of y: $\overline{AB} = \sqrt{x^2 - y^2}$. Since $\overline{BD} = \frac{1}{2}\overline{BC}$, $\overline{BC} = 2\overline{BD}$ or $x = 2y$. Hence, $\overline{AB}$ can be written as $\overline{AB} = \sqrt{3}y$. Applying the Pythagorean Theorem to $\triangle ABD$ gives

$$\begin{aligned}AD^2 &= AB^2 + BD^2 \\ &= (\sqrt{3}y)^2 \\ &= (\sqrt{3}y)^2 \\ &= 3y^2 \\ &= (\sqrt{3}y)^2 \\ &= 3y^2 \\ &= 3y^2 \\ &= 3y

12.21: Source: MATHCOUNTS

Two bicyclists start at the intersection of two perpendicular roads. One rides east at a rate of 9 miles per hour, while the other rides south at a rate of 17 miles per hour. How many miles are in the shortest distance between them at the end of 2 hours?

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Solution: The eastbound rider travels $9 \cdot 3 = 27$ miles in three hours and the southeast rider travels $17 \cdot 3 = 36$ miles. The shortest distance between them then is the hypotenuse of a right triangle with legs of lengths 27 and 36 miles. We can use the Pythagorean Theorem to find the hypotenuse. Or, we could notice that $27 = 3 \cdot 9$ and $36 = 4 \cdot 9$, so, applying the $\{3, 4, 5\}$ Pythagorean triple, we know the hypotenuse is $5 \cdot 9 = \boxed{45}$ miles.

12.22: Source: MATHCOUNTS

A right triangle has one leg of length 48 and hypotenuse with length 52. What is the length of the other leg? (Challenge: Try to do this problem in your head.)

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Solution: The leg has length $4 \cdot 12$ and the hypotenuse has length $4 \cdot 13$. Recalling the Pythagorean triple $\{5, 12, 13\}$, we notice that the other leg has length $4 \cdot 5 = \boxed{20}$.

12.23: Source: MATHCOUNTS

Pat knows that one leg of a certain right triangle is 300 cm and the hypotenuse is 400 cm. Pat notes that $300 = 3 \cdot 100$ and $400 = 4 \cdot 100$, and then uses the $\{3, 4, 5\}$ Pythagorean triple to determine that the other leg must be $5 \cdot 100 = 500$ cm. Explain why Pat's method doesn't work, and determine the correct length of the other leg of the triangle.

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Solution: In the $\{3, 4, 5\}$ Pythagorean triple, the 3 and the 4 are both leg lengths, but in the given problem, the side with length $4 \cdot 100$ is the hypotenuse, not a leg. We do have $400 = 5 \cdot 80$, but the given leg is not either 3 or 4 times 80, so we cannot use the $\{3, 4, 5\}$ Pythagorean triple to solve the problem. Instead, use the Pythagorean Theorem. Letting x be the length of the other leg, we have $x^2 + 300^2 = 400^2$, so

$$\begin{aligned} x^2 &= 400^2 - 300^2 \\ &= 4^2 \cdot 100^2 - 3^2 \cdot 100^2 \\ &= (16(100^2) - 9(100^2)) \\ &= 7(100^2), \end{aligned}$$

Taking the square root gives $x = \sqrt{7(100^2)} = \boxed{100\sqrt{7}\text{ cm}}$.

12.24:

(a) What is the area of a right triangle that has legs with lengths 7 inches and 24 inches?

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Solution: The area of a right triangle is half the product of the lengths of its legs, so the area of the given triangle is $(7)(24)/2 = \boxed{84\text{ in}^2}$.

(b) What is the length of the altitude to the hypotenuse of the triangle in part (a)?

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Solution: First, we find the hypotenuse length. From the Pythagorean Theorem, or from remembering the $\{7, 24, 25\}$ Pythagorean triple, we find that the hypotenuse of the triangle has length 25 inches. The area of the triangle is half the product of the hypotenuse and the desired altitude length. Let the desired altitude have length h. Using the area we found in part (a), we have $25h/2 = 84$. Multiplying both sides by 2 gives $25h = 168$, and dividing by 25 gives $h = \boxed{168/25\text{ inches}}$.

12.25:

Find the area of the quadrilateral on the right.

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Solution: We find the areas of the two right triangles separately. We start with the triangle on the lower left. Let the missing leg have length x. The Pythagorean Theorem gives $14^2 + x^2 = 28^2$, so $x^2 = 28^2 - 14^2$. We can simplify the expression on the right-hand side by noticing that $28 = 2 \cdot 14$. So taking the square root gives

$$\begin{aligned} x &= \sqrt{28^2 - 14^2} = \sqrt{(2 \cdot 14)^2 - 14^2} = \sqrt{2^2 \cdot 14^2 - 14^2} \\ &= \sqrt{4(14^2) - 14^2} = \sqrt{3 \cdot 14^2} = 14\sqrt{3}. \end{aligned}$$

(We might also have noticed that the given leg is half the hypotenuse, so the triangle is a 30-60-90 triangle.) Therefore, the right triangle on the lower left has area $(14)(14\sqrt{3})/2 = 98\sqrt{3}$ in^2.

Turning to the other right triangle, we let b be the length of the missing leg and again apply the Pythagorean Theorem. We find $26^2 + b^2 = 28^2$, so

$$b^2 = 28^2 - 26^2 = 784 - 676 = 108.$$

Taking the square root gives $b = \sqrt{108} = \sqrt{36 \cdot 3} = 6\sqrt{3}$. So, the area of the upper right triangle is $(26)(6\sqrt{3})/2 = 78\sqrt{3}$ in^2.

Combining the two triangles gives a total area of $98\sqrt{3} + 78\sqrt{3} = \boxed{176\sqrt{3}\text{ in}^2}$.

12.26: Source: MATHCOUNTS

A garden is laid out in the fashion shown in the diagram at the right. If only the shaded isosceles right triangles are used for planting, what is the total area, in square feet, that is to be used for planting?

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Solution: We present two methods for finding the total area.

Method 1: The Hard Way. Because the hypotenuse of an isosceles right triangle is $\sqrt{2}$ times a leg of the triangle, the leg length of each of the triangle sections is $\frac{32}{\sqrt{2}}$. Therefore, the area of each of the triangle sections is

$$\left(\frac{1}{2}\right)\left(\frac{32}{\sqrt{2}}\right)\left(\frac{32}{\sqrt{2}}\right) = \frac{1}{2} \cdot \frac{32^2}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2} \cdot \frac{256}{2} = 64\text{ ft}^2.$$

Combining all four sections gives a total area of $4(64) = \boxed{256\text{ ft}^2}$.

Method 2: The Easy Way. Pushing together all four right triangles forms a square with side length 16 ft. This square has area $(16)^2 = \boxed{256\text{ ft}^2}$.

12.27:

In $\triangle ABC$, we have $\angle A = 30^\circ$, $\angle B = 60^\circ$, and $AB = 8$. Find BC, AC, and the area of $\triangle ABC$.

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Solution: We have $\angle C = 180^\circ - 30^\circ - 60^\circ = 90^\circ$, so $\triangle ABC$ is a right triangle. Moreover, because the acute angles of the triangle are 30° and 60°, the triangle is a 30-60-90 triangle. Therefore, the short leg of the triangle is opposite the 30° angle and is half the hypotenuse, so $BC = 8/2 = \boxed{4}$. The longer leg is $\sqrt{3}$ times the shorter leg, so $AC = \boxed{4\sqrt{3}}$. The area is half the product of the legs, so $[ABC] = \boxed{8\sqrt{3}}$.

12.28:

(a) What is the greatest possible angle measure in an isosceles triangle that has an angle measuring 54 degrees?

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Solution: Either the 54° angle is one of the two congruent angles, or it isn't.

Case 1: The 54° angle is one of the two congruent angles. Then, other two angles are 54° and the third angle is $180^\circ - 54^\circ - 54^\circ = 72^\circ$.

Case 2: The 54° angle is not one of the two congruent angles. Then, the other two angles are congruent, and their measures must sum to $180^\circ - 54^\circ$, which equals 126°. Since the two congruent angles sum to 126°, they each measure $126^\circ/2 = 63^\circ$.

Considering the two cases above, the largest possible measure of an angle of the triangle is $\boxed{54^\circ}$.

(b) What is the least possible angle measure in an isosceles triangle that has an angle measuring 54 degrees?

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Solution: Considering the two cases in part (a), the smallest possible measure of an angle of the triangle is $\boxed{54^\circ}$.

12.29:

Find the perimeter and the area of the quadrilateral shown at the right.

You may type any additional notes you have here.

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Solution: Because $\angle A + \angle B = 180^\circ$, we know that $\overline{BC} \parallel \overline{AD}$, so $ABCD$ is a trapezoid. The bases of the trapezoid have lengths 5 and 15, and the height of the trapezoid is 8, so the area of the trapezoid is $\frac{5 + 15}{2} \cdot 8 = \boxed{80}$.

To find the perimeter of the trapezoid, we must find CD. We draw the altitude from C to $\overline{AD}$, forming the right triangle CED shown in the diagram. Since $BC = EA$ is a rectangle, we have $ED = AB - BC = 15 - 5 = 10$. We also have $CE = 8$. Recalling the $\{3, 4, 5\}$ Pythagorean Triple (which is 3 times the Pythagorean Triple in $\triangle CEA$ or recognizing the $\{6, 8, 10\}$ Pythagorean triple (which is 2 times the $\{3, 4, 5\}$ Pythagorean triple), we find that $CD = 10$, so the perimeter of $ABCD$ is $8 + 5 + 15 + 10 = \boxed{42}$.

12.30:

The lengths of the diagonals of a rhombus are 10 inches and 24 inches. What are the perimeter and the area of the rhombus?

You may type any additional notes you have here.

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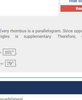
Solution: The area of a rhombus is half the product of its diagonals, so the area of the rhombus is $(10)(24)/2 = \boxed{120\text{ in}^2}$. The diagonals of a rhombus bisect each other, so drawing the diagonals of the rhombus splits the rhombus into four right triangles with legs of length 5 and 12 inches. Recalling the $\{5, 12, 13\}$ Pythagorean triple, or applying the Pythagorean Theorem to these triangles, we find that each side of the rhombus has length 13 inches. So, the perimeter of the rhombus is $4(13) = \boxed{52\text{ inches}}$.

12.31: Source: MATHCOUNTS

An element in the diagram on the right, a 25-foot ladder reaches 24 feet up the side of a building. Then the top of the ladder slides 4 feet down the side of the building and the bottom slides 15 feet from the wall. How many additional feet does the bottom of the ladder slide out from the base of the building?



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Solution: Connecting the foot of the ladder to the base of the building completes a right triangle in which part of the building wall is a leg and the ladder is the hypotenuse. From the Pythagorean Theorem, or from the $\{7, 24, 25\}$ Pythagorean triple, we see that the base of the ladder is initially 7 feet from the building. After the top slides down 4 feet, the top is 20 feet from the ground. The ladder is still 25 feet long, so from the Pythagorean Theorem or recognizing the $\{15, 20, 25\}$ Pythagorean triple (which is 5 times the $\{3, 4, 5\}$ Pythagorean triple), we see that the base of the ladder is 15 feet from the wall after sliding. Therefore, the base of the ladder slid $15 - 7 = \boxed{8\text{ feet}}$.

12.32: Source: MATHCOUNTS

A tessellation is composed of four regular hexagons and a rhombus as shown in the diagram on the right. How many degrees are in an acute angle of the rhombus?

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Solution: The sum of the angles of a hexagon is $(6 - 2)(180^\circ) = 720^\circ$, so each angle of a regular hexagon is $720^\circ/6 = 120^\circ$. We'll next compute the measure of an obtuse angle of the rhombus. Each obtuse angle of the rhombus shares its vertex with two angles of hexagons, so the sum of the obtuse angles must sum to 360°, so the measure of each obtuse angle of the rhombus is $360^\circ - 120^\circ - 120^\circ = 120^\circ$. The rhombus is also a parallelogram, which means its opposite sides are parallel. Therefore, consecutive angles in the rhombus must be supplementary, which means each acute angle has measure $180^\circ - 120^\circ = \boxed{60^\circ}$.

12.33:

Label each statement as true or false, and explain why your answer is correct.

(a) A quadrilateral can have exactly three right angles among its interior angles.

You may type any additional notes you have here.

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Solution: $\boxed{\text{True}}$. As explained in the text, every square is a rectangle and every rectangle is a parallelogram, so every square is a parallelogram.

(b) If both pairs of opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.

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Solution: $\boxed{\text{True}}$. Let the quadrilateral be $ABCD$, so $\angle A = \angle C$ and $\angle B = \angle D$. Therefore, we have $\angle A + \angle B = \angle C + \angle D = 180^\circ$. Since the sum of all four angles must be 360°, we must have $\angle A + \angle B = \angle C + \angle D = 180^\circ$. Since $\angle A + \angle B = 180^\circ$, we have $\overline{BC} \parallel \overline{AD}$. Similarly we can start with $\angle A = \angle C$ and $\angle B = \angle D$ to find $\angle A + \angle B = \angle B + \angle C$, which gives $\angle A = \angle D = 180^\circ$ so $\overline{AB} \parallel \overline{CD}$. Therefore, both pairs of opposite sides of $ABCD$ are parallel, so $ABCD$ is a parallelogram.

(c) The diagonals of a rhombus have the same length.

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Solution: $\boxed{\text{False}}$. For example, the rhombus at the right has form the tetrahedron has diagonals of different length.

12.34:

The measure of one interior angle of a rhombus is 79°. What are the measures of the other three interior angles?

You may type any additional notes you have here.

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Solution: Let the rhombus be $EFGH$, with $\angle E$ being the given 79° angle. Every rhombus is a parallelogram. Since opposite sides of the rhombus are parallel, each pair of consecutive angles is supplementary. Therefore, we have $\angle E + \angle F = \angle E + \angle H = 180^\circ$, which gives us

$$\angle F = \angle H = 180^\circ - \angle E = \boxed{101^\circ}.$$

Finally, the opposite angles of a parallelogram are congruent, so $\angle G = \angle E = \boxed{79^\circ}$.

12.35:

The midpoints of the sides of square $WXYZ$ are connected to form another quadrilateral.

(a) Explain why the new quadrilateral must also be a square.

You may type any additional notes you have here.

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Solution: The quadrilateral is formed by connecting the midpoints of the sides of $WXYZ$ as $ABCD$, as shown in the diagram at the right. First, we show that all four sides of $ABCD$ are congruent. Each side of $ABCD$ is the hypotenuse of an isosceles right triangle whose legs are half the side length of the square. Therefore, all four sides of $ABCD$ have lengths $\sqrt{2}$ times half the side length of the square. Next, we show that all of the angles of $ABCD$ are right angles. We see that

$$\angle DAB = 180^\circ - (\angle WAD - \angle ZAB).$$

Since $\angle WAD$ and $\angle ZAB$ are isosceles right triangles, we have $\angle WAD = \angle ZAB = 45^\circ$, so

$$\angle DAB = 180^\circ - 45^\circ - 45^\circ = 90^\circ.$$

Similarly, all four angles of $ABCD$ are right angles. Since all four angles of $ABCD$ are congruent and all four sides of $ABCD$ are congruent, $ABCD$ is a square.

(b) If the area of $WXYZ$ is 900, then what is the area of the new quadrilateral?

Preview Solution

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Solution: The side length of $WXYZ$ is $\sqrt{900} = 30$. As described in the previous part, each side of $ABCD$ has length equal to $\sqrt{2}$ times half the side length of $WXYZ$, so the side length of $ABCD$ is $15\sqrt{2}$, which means

$$[ABCD] = \left(15\sqrt{2}\right)^2 = 15^2\left(\sqrt{2}\right)^2 = 225(2) = \boxed{450}.$$

We also could have figured out that $[ABCD] = (WXYZ)/2$ by drawing the diagonals of $ABCD$. These diagonals divide $WXYZ$ into four smaller squares with area 225 each. Half of these squares lie inside $ABCD$, while the other half of each lie outside $ABCD$. So, the area of $ABCD$ is $\frac{1}{2} \cdot 225 \cdot 2 = \boxed{450}$.

12.36:

The measures of two angles in a parallelogram add to 204°. Find the measure of each of the other two angles.

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Solution: Because opposite sides of a parallelogram are parallel, each pair of consecutive angles adds to 180°. So, the two angles that add to 204° must be opposite each other. The opposite angles of a parallelogram are congruent, so each of these angles must be $204^\circ/2 = 102^\circ$. The other two angles are congruent, and each is supplementary to each of the 102° angles. Therefore, these other two angles have measure $180^\circ - 102^\circ = \boxed{78^\circ}$.

Section 13 — Data and Statistics

A set of your choosing, can be simply from 1 to 100 if you prefer or anything in between. Its size will be denoted by n. Its elements will be denoted by $x_1, x_2, x_3, \ldots, x_n$. Sometimes we will need to talk about the sum of the elements of the set. Instead of writing $x_1 + x_2 + x_3 + \ldots + x_n$ we can use the new symbol $\sum x$ (read as "the sum of x"). Sometimes we will need to talk about the sum of the squares of the elements of the set. Instead of writing $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2$ we can use the new symbol $\sum x^2$.

13.1 Basic Statistics

Very basic statistics. Given a set of numbers, it is convenient to have a single number to describe some feature of the numbers or the lot. Here are the standard ways to choose that representative:

Mean: The average of the numbers. Add up all the numbers and divide by the number of numbers to find the average. For example, the average of $\{1, 2, 3, 4, 5\}$ is $\frac{1+2+3+4+5}{5} = \frac{15}{5} = 3$. The average is also denoted by $\bar{x}$ (pronounced "x-bar").

We sometimes use the word "average" as a verb. For example, we may say that a student's physics average 70 points given some set of exam results. Or we may say that a student's physics average has improved from last year to this year. Or we may say that a student's physics average is better than his chemistry average.

Median: Put the set of numbers in order from lowest to highest, and choose the one in the middle. For example, the median of $\{1, 2, 3, 4, 5\}$ is 3.

If there are an even number of numbers in the group, then the median is the average of the two in the middle. For example, the median of the set

$$\{1, 2, 3, 4, 5, 6\}$$

is the average of 3 and 4, which is $\frac{3+4}{2} = \frac{7}{2} = 3.5$.

Mode: The number of a group of numbers to be counted the most times. For example, the mode of the set

$$\{1, 2, 3, 3, 4, 5\}$$

is 3, because 3 occurs more times than any other number.

Average, median, and mode are all measures of what is typical in a set of numbers. Sometimes one of them is more useful than the others. Sometimes it doesn't matter which one we use because they all give the same answer. Sometimes it doesn't matter which one we use because they all give different answers. Sometimes it doesn't matter which one we use because they all give different answers. Sometimes it doesn't matter which one we use because they all give different answers

13.2 Limits of Basic Statistics

Now that we know how to compute average, median, and mode, we will think about what sort of information they tell us. And, perhaps more importantly, we will think about what sort of information they don't tell us.

Problems

Problem 13.8 ⇧ Jump to Solution

You have to choose one of four people in your math club to be your partner on a math assignment. You'd like to choose someone who is very good at math! Here are the four students' scores on their previous seven assignments.

Anna:94, 93, 90, 93, 92, 91, 0
Bob:98, 94, 33, 33, 96, 97, 33
Carol:89, 89, 86, 88, 87, 84, 85
Doug:100, 14, 3, 198, 11, 2, 21

(a) Which student has the highest average score?

(b) Which student has the highest median score?

(c) Which student has the highest mode score?

(d) Which student would you choose as your partner?

Problem 13.9 ⇧ Jump to Solution

The average wealth of a person in Richville is $150,000 and the average wealth of a person in Poorville is $20,000.

(a) Suppose a person is called "filthy sticking rich" if the person has over one million dollars. Can we tell which town has more filthy sticking rich people?

(b) Can we tell which town has the higher median wealth?

(c) If both towns have the same number of people, can we tell which town has the higher total wealth?

(d) Suppose a person with $1,000,000 and 4 people who have $0 more into Richville. Will the average wealth of Richville go up or down?

Problem 13.10 ⇧ Jump to Solution

The median wealth of a person in Goldtown is $150,000 and the median wealth of a person in Tintown is $20,000. Each town has 9000 people.

(a) Suppose a town qualifies for special government subsidies if each of at least half of its residents has less than $60,000. Does either town necessarily qualify for the subsidies?

(b) Can we tell which town has the higher average wealth?

(c) Suppose a person with $1,000,000,000 and 4 people who have $0 move into Goldtown. Will the median wealth of Goldtown go up or down?

Problem 13.8

You have to choose one of four people in your math club to be your partner on a math assignment. You'd like to choose someone who is very good at math! Here are the four students' scores on their previous seven assignments.

Anna:94, 93, 90, 93, 92, 91, 0
Bob:98, 94, 33, 33, 96, 97, 33
Carol:89, 89, 86, 88, 87, 84, 85
Doug:100, 14, 3, 198, 11, 2, 21

(a) Which student has the highest average score?

(b) Which student has the highest median score?

(c) Which student has the highest mode score?

(d) Which student would you choose as your partner?

Solution for Problem 13.8

(a) Here are the averages:

$$\text{Anna's average} = \frac{94+93+90+93+92+91+0}{7} = \frac{553}{7} = 79,$$

$$\text{Bob's average} = \frac{98+94+33+33+96+97+33}{7} = \frac{474}{7} = 67\frac{5}{7},$$

$$\text{Carol's average} = \frac{89+89+86+88+87+84+85}{7} = \frac{607}{7} = 86\frac{5}{7},$$

$$\text{Doug's average} = \frac{100+14+3+198+11+2+21}{7} = \frac{351}{7} = 35\frac{6}{7}.$$

Carol has the highest average

(b) We find the students' median scores by putting each student's scores in order from least to greatest:

Anna: 0, 90, 91, 92, 93, 93, 94
Bob: 33, 33, 33, 94, 96, 97, 98
Carol: 84, 85, 86, 87, 88, 89, 89
Doug: 2, 3, 11, 14, 21, 100, 198

For each student, the corresponding median is the middle number in the list. Each student's median is underlined. Bob has the highest median.

(c) The mode is the most frequent score. Each student has one repeated score, so we can quickly find each student's mode score:

Anna: 93 Bob: 33 Carol: 89 Doug: 100

Doug has the highest mode.

(d) There's no clearly correct answer to this question. Carol has the highest average, Bob has the highest median, and Doug has the highest mode. But you might be tired of hearing about Bob.

Looking at all of the scores, it's pretty clear that you probably don't want to choose Doug the student with the highest model. Most of the time he does very poorly. Mode is usually not a very helpful statistic.

Even though Bob has the highest median, Bob appears to have considerably more likely than Anna or Carol to do badly. So choosing Bob seems to bring a higher risk of a low score.

Carol, with the highest average, seems to be more reliable than Anna, but Anna's single bad score of 0 suggests that there may be some other explanation for her poor performance on that assignment. Maybe she simply didn't turn it in.

If you can be sure that Anna will do well, then it looks like she has a good chance to do better than Carol, because all of Anna's nonzero scores are higher than all of Carol's scores.

The key point of Problem 13.8 is that all of the basic statistics we have studied so far are usually poor substitutes for considering all of the data. Of course, sometimes we don't have all of the data, or there may be too much data to consider all of it. As you study more statistics in the future, you'll learn methods for analyzing large batches of data.

For now, we'll take a closer look at the limitations of the basic tools we have studied so far. It's particularly important to understand the strengths and weaknesses of average and median, since these are the two statistics that people use (and misuse!) the most.

Problem 13.9

The average wealth of a person in Richville is $150,000 and the average wealth of a person in Poorville is $20,000.

(a) Suppose a person is called "filthy sticking rich" if the person has over one million dollars. Can we tell which town has more filthy sticking rich people?

(b) Can we tell which town has the higher median wealth?

(c) If both towns have the same number of people, can we tell which town has the higher total wealth?

(d) Suppose a person with $1,000,000 and 4 people who have $0 more into Richville. Will the average wealth of Richville go up or down?

Solution for Problem 13.9

(a) First notice that we aren't given any information about how many people are in each town.

WARNING!	Neither the average nor the median of a group of numbers tells us anything about how numbers are in the group.

On the one hand, it's possible that everyone in Richville has exactly $150,000, so there is no one in Richville who is filthy sticking rich. At the same time, Poorville might consist of 100 people: one person with $2,000,000, who is definitely filthy sticking rich, and 99 people with no money at all. Note that this does make the average wealth in Poorville equal to $20,000, as required. In this case, Poorville has more filthy sticking rich people.

On the other hand, it's possible that everyone in Poorville has exactly $20,000 (so that no one is filthy sticking rich), while Richville consists of one filthy sticking rich person with $15,000,000 and 99 people who each have nothing. In this case, Richville has more filthy sticking rich people.

So without any additional information, we can't tell which town has more filthy sticking rich people.

(b) On the one hand, suppose the towns' populations are:

Richville	Poorville
50 people with $150,000 each	1 person with $1,000,000
	49 people with 0 each

The average and median wealth in Richville are each $150,000, whereas the average wealth in Poorville is $1,000,000/50 = $20,000 and the median wealth in Poorville is $0. In this case, Richville has a higher median wealth than Poorville.

On the other hand, suppose the towns' populations are:

Richville	Poorville
1 person with $1,500,000	10 people with $20,000 each
1 people with $0	90 people with 0 each

The average and median wealth in Poorville are each $20,000, whereas the average wealth in Richville is $1,500,000/2 = $750,000 and the median wealth in Richville is $0. In this case, Poorville has a higher median wealth than Richville.

So without any additional information, we can't tell which town has the higher median wealth.

Important:	We cannot determine anything about the median of a set of data just by knowing the average of the data.

(c) For each town, the average wealth equals the total wealth divided by the number of people. So, the total wealth equals the product of the average wealth and the number of people. If the two towns have the same number of people, then Richville must have the higher total wealth. Specifically, if both towns have n people, then Richville has a total wealth of $150,000n, and Poorville has a total wealth of $20,000n.

Note that this analysis only worked because we were told that the number of people in each town was the same. If we weren't given this additional bit of information, we couldn't tell which town had the higher total wealth. For example, if the population of Richville was just 1 and the population of Poorville was 10, then the total wealth of Richville would be $150,000 · 1 = $150,000 whereas the total wealth of Poorville would be $20,000 · 10 = $200,000. In this example, the total wealth of Poorville would be greater.

Important:	Without knowing the size of the data sets, we cannot compare the sums of the data just by comparing averages.

(d) The average wealth equals the total wealth divided by the number of people. In other words, the average equals the amount each person would have if we divided the total wealth equally among the people. The five people who move into Richville have a total of $1,000,000. To keep the town average at $150,000, we need to give these five new people only need a total of $750,000, which is $250,000 less than the total wealth of Richville. So the average wealth of Richville will drop.

Problem 13.9 shows us some features and limitations of average. In particular, in our examples in parts (a), (b), and (d), we see that it is possible for a single number to have a huge effect on the average. For example, if an extremely rich person moves into your neighborhood, the average wealth of your neighborhood will rise a lot! But the other people in your neighborhood haven't gotten any richer; even though adding by the average wealth of the neighborhood, it looks like the neighborhood has become much richer.

Problem 13.10

The median wealth of a person in Goldtown is $150,000 and the median wealth of a person in Tintown is $20,000. Each town has 9000 people.

(a) Suppose a town qualifies for special government subsidies if each of at least half of its residents has less than $60,000. Does either town necessarily qualify for the subsidies?

(b) Can we tell which town has the higher average wealth?

(c) Suppose a person with $1,000,000,000 and 4 people who have $0 move into Goldtown. Will the median wealth of Goldtown go up or down?

Solution for Problem 13.10

(a) Suppose we live a pauper in Tintown based on how much wealth they have. The population of Tintown is even, so the median wealth, $20,000, is the average wealth of the two people in the middle. So, either the middle two people have $20,000 each, or the poorer of the two has less than $20,000 and the richer person has more than $20,000. If the richer person is the poorer of the middle, but the two median cannot be higher than the $20,000. Each of these people therefore has less than $60,000, so Tintown qualifies for the subsidies.

Over in Goldtown, the median wealth is $150,000. If the two people in the middle have equal wealth, then each person in the richer half of the line has at least $150,000. But all we know about the people in the poorer half of the line is that each person has no more than $150,000. It is possible for each person in the poorer half to have less than $60,000. For example, suppose each person in the poorer half of Goldtown has $50,000 and each person in the richer half of Goldtown has $250,000. Then, the median wealth of Goldtown is $150,000, but Goldtown still qualifies for the subsidies.

The median value only tells us how much the "middle" person has in each town, and that half the people in town have at least this much. But the richer people in town could have any amount of wealth, and the poorer half of the people can be anywhere between zero and the median.

Similarly, because half the people have much higher total wealth than the median, the median wealth can be much higher than the average. For example, if the population of Richville was just 1 and the population of Poorville was 10, then the total wealth of Richville would be $150,000 · 1 = $150,000 whereas the total wealth of Poorville would be $20,000 · 10 = $200,000. In this example, the total wealth of Poorville would be greater.

Important:	Without knowing the size of the data sets, we cannot compare the sums of the data just by comparing medians.

(b) The average wealth equals the total wealth divided by the number of people. In other words, the average equals the amount each person would have if we divided the total wealth equally among the people. The five people who move into Richville have a total of $1,000,000. To keep the town average at $150,000, we need to give these five new people only need a total of $750,000, which is $250,000 less than the total wealth of Poorville. So the average wealth of Poorville will drop.

Concept:	Only the middle value (or middle two values) in a group of numbers is used to determine the median. The other values (below the median, or decreasing the values below the median, does not affect the median.

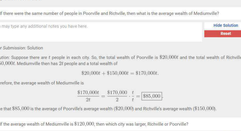
(c) Suppose that, before the five new people arrive, we line up all the people in Goldtown based on their wealth, with the richest first and the poorest last. Since there are 9000 people, the median is the average wealth of the middle two people, who are the 4000th and 4001st people in the line.

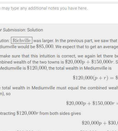
When the five new people arrive and find their places in the line, the four with $0 will join the poorer end, and the person with $1,000,000,000 will be someone in the rich half of the line. So, in this new line there will be 9005 people. The person who joined the rich end causes the people who were the 4000th and 4001st people to move up to the 4001st and 4002nd in the line.

In this new line of 9005 people, the middle person is 4003rd in line. Most notably, this person is better, and therefore not richer than, the two people whose average gave us the original median. So, the median wealth of Goldtown must have gone up.

We can't actually say that the new median must be lower than the old median, since we might have many people in the middle with the same wealth. So, it's possible that the new median is equal to the old median, but the new median cannot be higher than the old median.

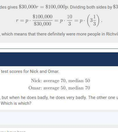
The interesting observation here is that the new people brought a total of one billion dollars to the town, but the median did not rise. That's because we added more poor people than rich people. Once again, we see that the actual values above and below the median value do not affect the median. The median only tells us what the middle is.





Limits of Basic Statistics Part 1

Limits of Basic Statistics Part 2


The median wealth of a town is the average wealth of the two people in the middle.
Median wealth of Rich Town is $150,000.
Median wealth of Poor Town is $20,000.
Rich Town
Poor Town
$0
$0
$0
$0
$1,000,000,000
$1,000,000,000
$1,000,000,000

13.3 Tables, Graphs, and Charts

In this section, you will use the number of students in ninth, seventh, and eighth grades at four middle schools to represent a collection of data. In this section, we explore a variety of ways to display data when that pool consisting of 4 of them.

Problems

Problem 13.1 ⏱ Estimate 5 minutes

Table 13.1 lists the number of students in ninth, seventh, and eighth grades at four middle schools. Use this information to answer the questions. Assume that the number of students in each grade at each school is the same as the number of students in each grade at the four schools, and the last column corresponds to the total number of students in each school.

Table 13.1 Students Attending Four Middle Schools

	Students in Ninth Grade	Students in Seventh Grade	Students in Eighth Grade	Total Students
East Middle School	200	210	190	600
West Middle School	150	160	140	450
North Middle School	250	240	260	750
South Middle School	300	290	310	900

(a) Complete the table by filling in the missing entries.

(b) How many more eighth graders are there in South Middle School than in East Middle School?

(c) Which grade has the most students? Which grade has the fewest students?

(d) Which school has the highest average number of students per grade?

(e) Which school has the highest percentage of students in the seventh grade?

Problem 13.2 ⏱ Estimate 5 minutes ✓ Grade by Solution

Suppose a group of 40 Fun Middle Schools students ranked their favorite school during history class. The results are shown in the **bar chart** on the right. Describe each of the following statements as true, false, or impossible. Explain your reasoning.

(a) More boys than girls chose the Little Volcanoes as their favorite school.

(b) Half the students chose either the Little Volcanoes or the Bee's Knees.

(c) Half the students think about what school their 9th and 8th Grade Love at Together.

(d) Half the students chose the Little Volcanoes.

(e) Most students spend more time thinking about 9th Grade about Middle School.

Problem 13.3 ⏱ Estimate 5 minutes ✓ Grade by Solution

There are two math teams at Mount Academy, The Little Mistakers and The Bees. At the end of each day, each team keeps grades that reflect the number of points they earned. The table below shows the **bar graph** for a week. Use this information to answer the questions.

Describe each of the following as true, false, or not enough information to determine from the graph.

(a) The Bees had fewer points than The Little Mistakers at the start of Monday, but more points than The Little Mistakers at the end of Friday.

(b) The Little Mistakers had more points than the Bees on Wednesday.

(c) The Little Mistakers had more points than the Bees on Friday.

(d) There was a tie between the Little Mistakers and the Bees.

(e) The Little Mistakers earned more than 20 points on Friday.

Problem 13.4 ⏱ Estimate 5 minutes ✓ Grade by Solution

Mount Academy's Fun Middle Schools Bee's Knees Band, Inc. is a band that has been around for a long time. The band's current lineup is 9th grade boys and 8th grade girls. The Bee's Knees band plan to hire a new lead singer

13.4 Summary

We studied three **statistics**, which are numbers used to give information about groups of numbers:

- **Average**: The average of a group of numbers is the sum of the numbers divided by the number of numbers. So, the average of 3, 5, 6, and 10 is $\frac{3+5+6+10}{4}$, which equals 6. The average is also called the **mean** or the **arithmetic mean**.
- **Median**: If we list a group of numbers from least to greatest, the median of the group is the number in the middle. So, the median of the numbers in the list

  $$4, 5, 7, 8, 11$$

  is 7. If there is an even number of numbers, then the median is the average of the middle two numbers.
- **Mode**: The mode of a group of numbers is the number that appears most frequently in the group. So, the mode of

  $$3, 3, 3, 3, 4, 5, 6, 7$$

  is 3. A group can have multiple modes if there are multiple numbers that appear the same number of times.

Average and median are used much more often than mode. There are significant limitations on what these statistics can tell us about a group of data. The median only tells us what the middle number is (or the average of the middle two numbers). It doesn't tell us anything about how far the numbers in the group are from the middle. Numbers that are much greater or much less than most of the other numbers in a group are called **outliers**. Outliers tend to have a very large effect on the average of the group, but very little effect on the median.

There are many ways to display data, such as tables (Section 13.3 **here**), bar charts (Section 13.3 **here**), line graphs (Section 13.3 **here**), stem-and-leaf plots (Section 13.3 **here**), and pie charts (Section 13.3 **here**).

Challenge Problems

13.27:

Source: AMC 8

There is a list of seven numbers. The average of the first four numbers is 5, and the average of the last four numbers is 8. If the average of all seven numbers is $6\frac{4}{7}$, then what number is common to both sets of four numbers?

Hint: Problems involving averages can often be solved by thinking about sums.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The sum of the first four numbers is $4 \cdot 5 = 20$. The sum of the last four numbers is $4 \cdot 8 = 32$. If we add these two sums, we find that $20 + 32 = 52$ is the sum of all seven numbers plus an extra copy of the middle number. The sum of all seven numbers is $7 \cdot \left(6\frac{4}{7}\right) = 46$, so the middle number must be $52 - 46 = \boxed{6}$.

13.28:

The mean, median, and mode of the five numbers 5, 7, 8, A, B are equal. (The list has a single mode.) Find all possible values of $A + B$.

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: In order for there to be a unique mode, one of the numbers must be repeated, but we can't have two numbers be repeated. It doesn't matter which is the repeated value, A or B, so we'll let A be the repeated value. There are four possibilities to consider:

Case 1: 5 is repeated. Our list then is 5, 5, 7, 8, B. Since the mode is 5, the median and mean must also be 5. Since 7 and 8 together are a total of $2 + 3 = 5$ greater than the mean, we know that B is 5 less than the mean. This tells us that $B = 5 - 5 = 0$, making our list 0, 5, 5, 7, 8. This list does indeed have mean, median, and mode equal to 5. In this case, $A + B = 5$.

Case 2: 7 is repeated. Our list then is 5, 7, 7, 8, B. The median and mean must also be 7. Since 8 is 1 greater than the mean and 5 is 2 less than the mean, the remaining number must also be 1 greater than the mean. That makes the list 5, 7, 7, 8, 8. But the mode of this list is not unique! So, it is impossible for 7 to be the repeated number.

Case 3: 8 is repeated. Our list then is 5, 7, 8, 8, B. The median and mean must also be 8. Since 5 is 3 less than the mean and 7 is 1 less than the mean, the remaining number must be $3 + 1 = 4$ greater than the mean. This makes our list 5, 7, 8, 8, 12. This list has mean, median, and mode equal to 8, and we have $A + B = 20$.

Case 4: A and B are equal. Our list then is 5, 7, 8, A, A. The mean and median must also be A. The mean of this list is $\frac{5 + 7 + 8 + A + A}{5}$. Simplifying this and setting it equal to A gives

$$\frac{20 + 2A}{5} = A.$$

Multiplying both sides by 5 gives $20 + 2A = 5A$. Subtracting $2A$ from both sides gives $20 = 3A$, so $A = \frac{20}{3} = 6\frac{2}{3}$. This makes our list 5, $6\frac{2}{3}$, $6\frac{2}{3}$, 7, 8, which has mean, median, and mode equal to $6\frac{2}{3}$. In this case, $A + B = 13\frac{1}{3}$.

Combining all of our cases, the possible values of $A + B$ are $\boxed{5, 13\frac{1}{3}, \text{ and } 20}$.

13.29:

The Back In My Day club made the histogram at right to represent the ages of its members.

(a) What is the least possible median age of the members?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: We start by noting that there are $2 + 4 + 8 + 10 + 7 = 31$ people total.

The number of people who are less than 80 years old is $2 + 4 + 8 = 14$. There are 7 people in their 90s and 10 in their 80s. So, less than half the people are below 80 years old and more than half are below 90 years old. This means that the median age is in the 80s. Therefore, the smallest it can be is $\boxed{80}$, which would indeed occur if all the people in their 80s are exactly 80.

(b) What is the greatest possible median age?

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: As explained in the first part, the median must be in the 80s. Therefore, the greatest the median age can be is $\boxed{89}$. For example, if all the people in their 80s were 89 years old, the median would be 89.

(c) What is the least possible average age?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: There are 31 people. To make the average of the ages as small as possible, we must make the sum of the ages as small as possible. The least possible sum occurs when each of the ages is as low as possible, which is the corresponding multiple of 10 for each age. So, the lowest possible sum is

$$\begin{aligned} &2 \cdot 50 + 4 \cdot 60 + 8 \cdot 70 + 10 \cdot 80 + 7 \cdot 90 \\ &= 10(2 \cdot 5 + 4 \cdot 6 + 8 \cdot 7 + 10 \cdot 8 + 7 \cdot 9) \\ &= 2330. \end{aligned}$$

Therefore, the least possible average is $\frac{2330}{31} = \boxed{75\frac{5}{31}} \approx 75.16$.

(d) What is the greatest possible average?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The average is greatest if all the people are as old as possible. This will occur if all the people in their 50s are 59, all the people in their 60s are 69, and so on. To sum these ages, we add the sum of the 50s and the sum of the 60s separately. We found the sum of these terms in part (c) as 2330. The sum of the ages in the 70s is $31 \cdot 9 = 279$, so the sum of all 31 ages is $2330 + 279 = 2609$, and the desired greatest possible average is $\frac{2609}{31} = \boxed{84\frac{25}{31}} \approx 84.16$. Seeing that this result is exactly 9 greater than the result for part (c), we note that adding 9 to each of the ages in part (c) results in adding 9 to the average, as expected.

13.30:

(a) Two lists of numbers that have the same average are combined to form a longer list. Must the average of the new list be the same as the average of each of the original shorter lists?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution $\boxed{\text{Yes}}$. The total amount by which the above-average numbers in the first list exceeds the average must equal the total amount by which the average exceeds the below-average numbers in the first list. Similarly, the total amount by which the above-average numbers in the second list exceeds the average must equal the total amount by which the average exceeds the below-average numbers in the second list. So, when we combine the two lists, the total amount by which numbers above the common average exceed the common average equals the total amount by which the common average exceeds the below-average numbers. This means that the combined list must have the same average as the common average of the two lists.

(b) Two lists of numbers that have the same median are combined to form a longer list. Must the median of the new list be the same as the median of each of the original shorter lists?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution $\boxed{\text{Yes}}$. Suppose we start with the first list, in order. The median must be the middle number of this list, or the average of the two middle numbers. When we combine the second list with this first list, there must be just as many new numbers above this median as below, since this median is the middle number (or between the two middle numbers) of the second list, too. So, the common median will be the median of the new list, too.

(c) Two lists of numbers that have the same mode are combined to form a longer list. Is the mode of the new list necessarily the same as the mode of each of the original shorter lists?

Preview: Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution $\boxed{\text{Yes}}$. The mode is the number that appears most often. If a number appears more than any other in the first list, and it appears more than any other in the second list, then it definitely will have more total appearances across both lists than any other number. So, the mode of the combined list is the same as the mode of each individual list.

(d) Two lists of numbers are combined to form a longer list. The mode of the new list is 3. Is it possible that neither of the original lists had 3 as its mode?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution $\boxed{\text{Yes}}$. Suppose the lists are 5,5,5,3,3 and 2,2,2,3,3. The mode of the first list is 5 and the mode of the second is 2. The combined list is 2,2,2,3,3,3,3,5,5,5. This list has mode 3.

13.31:

My four closest friends have weekly allowances of \$13, \$17, \$24, and \$30. What are the possible values of my weekly allowance if the median of our allowances equals the average of our allowances?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We start by noting that the total allowance of my friends is $\$13 + \$17 + \$24 + \$30 = \$84$. We then consider three cases:

Case 1: My weekly allowance is at most \$17. The median allowance then is \$17. In order for the average also to be \$17, the total of all 5 allowances must be $5 \cdot (\$17) = \$85$. So, my allowance must be $\$85 - \$84 = \$1$, which is indeed no more than \$17, and I need to have a conversation with my parents.

Case 2: My weekly allowance is at least \$24. The median allowance then is \$24. In order for the average also to be \$24, the total of all 5 allowances must be $5 \cdot (\$24) = \$120$. So, my allowance must be $\$120 - \$84 = \$36$, which is greater than \$24, as expected.

Case 3: My weekly allowance is between \$17 and \$24. The median allowance then is my allowance. Let my allowance be x dollars. The average allowance then is $\frac{84 + x}{5}$ dollars. Since this must equal the median, which is my allowance, we must have

$$\frac{84 + x}{5} = x.$$

Multiplying both sides by 5 gives $84 + x = 5x$, so $84 = 4x$ and $x = 21$, which is between 17 and 24 as desired. (My allowance is the average of my four friends' allowances in this case. Is this a coincidence?)

Combining the three cases, the possible values of my weekly allowance are $\boxed{\$1, \$21, \text{ and } \$36}$.

13.32:

We asked 124 people to rank four card games from favorite (1st) to least favorite (4th). The results are shown in the table at the right. Two of the entries in the table are wrong. Fix the errors by changing two of the numbers in the table.

	1st	2nd	3rd	4th
Magic	34	56	23	11
Pokemon	12	14	38	64
SET	57	41	23	3
Yu-Gi-Oh!	21	19	44	46

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since each card receives one vote per person, the four rows must each sum to 124. Similarly, each column must sum to 124. We include a row and a column to our table for these totals, and we have the table shown on the right. Looking at this table, we expect that the erroneous rows are those for Pokemon and Yu-Gi-Oh!, and the erroneous columns are those for 2nd and 3rd. The Pokemon row total is 4 too high, as is the total for the 3rd place column. So, we can fix both by subtracting 4 from the number of 3rd place votes received by Pokemon. Similarly, we can fix the Yu-Gi-Oh! row and the 2nd place column by subtracting 6 from the number of 2nd place votes Yu-Gi-Oh! received. The resulting table is shown below.

	1st	2nd	3rd	4th	Total
Magic	34	56	23	11	124
Pokemon	12	14	38	64	124
SET	57	41	23	3	124
Yu-Gi-Oh!	21	19	44	46	130
Total	124	130	128	124	

	1st	2nd	3rd	4th	Total
Magic	34	56	23	11	124
Pokemon	12	**8**	**34**	64	124
SET	57	41	23	3	124
Yu-Gi-Oh!	21	**13**	44	46	124
Total	124	124	124	124	

13.33★:

For a set of ten numbers, removing the largest number decreases the average by 1. Removing the smallest number increases the average by 2. What is the positive difference between the largest and the smallest of these ten numbers?

Hint: Sometimes thinking of the average of a list of numbers as "balancing" the list helps solve problems.

Hint: How far must the largest number be from the average of the original list?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The average of the smallest 9 numbers is 1 less than the average of all 10 numbers. So, on average, each of the 9 smallest numbers is 1 less than the average of all 10 numbers. This means that the 9 smallest numbers together are a total of $9 \cdot 1 = 9$ less than the average of all 10 numbers. So, the largest number must be 9 greater than the average of all 10 numbers.

Similarly, the average of the largest 9 numbers is 2 greater than the average of all 10 numbers. So, on average, each of the 9 largest numbers is 2 greater than the average of all 10 numbers. This means that the 9 largest numbers together are a total of $9 \cdot 2 = 18$ greater than the average of all 10 numbers. So, the smallest number must be 18 less than the average of all 10 numbers.

Since the largest number is 9 greater than the average of all 10 numbers, and the smallest number is 18 less than this average number, we know that the smallest and largest are $9 + 18 = \boxed{27}$ apart.

Review Problems

13.16:

I have four dogs whose average weight is 63 pounds. I also have three cats. The average weight of all seven of my animals is 41 pounds. What is the average weight of my cats?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The total weight of the dogs is $4 \cdot 63 = 252$ pounds. The total weight of all 7 animals is $7 \cdot 41 = 287$ pounds. So, the total weight of all three cats is $287 - 252 = 35$ pounds, which means the average weight of each cat is $\frac{35}{3} = \boxed{11\frac{2}{3}}$ pounds.

13.17:

The average age of the ten people on my basketball team was 13.5, but then a 15-year-old joined our team and an 11-year-old quit the team. What is the average age of my team now?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: When the 15-year-old joins and the 11-year-old quits, the sum of the ages of the people on the team increases by 4 years. Since there are 10 people on the team, this increases the average age by $\frac{4}{10} = 0.4$ years, to $13.5 + 0.4 = \boxed{13.9}$ years.

13.18:

Source: MOEMS

The average of five numbers is 18. Let the first number be increased by 1, the second number by 2, the third number by 3, the fourth by 4, and the fifth number by 5. What is the average of the list of increased numbers?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since the average of the five numbers is 18, their sum is $5 \cdot 18 = 90$. The total increase of the numbers is $1 + 2 + 3 + 4 + 5 = 15$, so the sum of the increased numbers is $90 + 15 = 105$. Therefore, the average of the increased numbers is $\frac{105}{5} = \boxed{21}$.

We also could have noticed that the average increase is the average of 1, 2, 3, 4, and 5, which is 3. Since the average increase is 3, the average of the increased numbers is $18 + 3 = 21$.

13.19:

Suppose you took eight math tests this semester. If your average score on your first six tests was 84 and your average score on all eight tests was 86, then what was the average of your last two test scores?

Hide Solution

Reset

Your Submission: Solution

Solution: Since the average of the five numbers is 18, their sum is $5 \cdot 18 = 90$. The total increase of the numbers is $1 + 2 + 3 + 4 + 5 = 15$, so the sum of the increased numbers is $90 + 15 = 105$. Therefore, the average of the increased numbers is $\frac{105}{5} = \boxed{21}$.

We also could have noticed that the first six tests are on average 2 points below the final average. So, the first six tests are $6 \cdot 2 = 12$ points below average. This means the first two tests must be a total of 12 points above average. That's $\frac{12\text{x}}{2} = 6$ points per test on average that these two final scores must exceed the overall average, which means the average on the final two tests must be $86 + 6 = \boxed{92}$.

13.20:

Must the median of a group of consecutive integers equal the average of the group?

Preview: Solution

Yes

Hide Solution

Reset

Your Submission: Solution

Solution: Yes. As we saw in **13.5.9**, the average of a group of consecutive numbers equals the middle number if there are an odd number of numbers, and the average of the group equals the average of the two middle terms if there are an even number of numbers. In both cases, the average equals the median.

13.21:

Larry writes a list of numbers that have average 14, median 21 and mode 11.

(a) Moe creates a list by adding 12 to each number in Larry's list. What are the average, median, and mode of Moe's list?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: *Average:* Suppose there are n numbers in the list. Adding 12 to each number adds $12n$ to the sum of the numbers. Since the sum is increased by $12n$ while the number of numbers stays the same, the average increases by $\frac{12n}{n} = 12$. So, the new average is $14 + 12 = \boxed{26}$.

Median: Suppose we place the numbers in Larry's list in increasing order. Adding 12 to each number in this list won't change the order; Moe's new list is in increasing order, too. So, the middle number of Moe's list is 12 more than the middle number of Larry's list. This means that Moe's list's median is $21 + 12 = \boxed{33}$. If there are an even number of numbers, then the median is the average of the middle two numbers. We can use the same argument as in our first paragraph to see that Moe's median is still 12 more than Larry's median.

Mode: If two numbers are different in Larry's list, then the corresponding numbers are different in Moe's list. If two numbers are the same in Larry's list, then the corresponding numbers are the same in Moe's list. So, for any number x in Larry's list, the number of times x appears in Larry's list equals the number of times that $x + 12$ appears in Moe's list. Therefore, the number that appears most frequently in Moe's list is 12 more than the number that appears most frequently in Larry's list. This means that the mode of Moe's list is $11 + 12 = \boxed{23}$.

(b) Curly creates a list by multiplying each number in Larry's list by 2. What are the average, median, and mode of Curly's list?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: *Average:* We follow essentially the same process as in the previous part. Doubling each number in a sum doubles the sum. So, the sum of the numbers in Curly's list is double the sum of the numbers in Larry's list. The lists have the same number of numbers, so Curly's average is double Larry's average. This means that Curly's list has average $2 \cdot 14 = \boxed{28}$.

Median: Suppose we place the numbers in Larry's list in increasing order. Multiplying each number in this list by 2 won't change the order; Curly's new list is in increasing order, too. So the middle number of Curly's list is 2 times the middle number of Larry's list. This means that Curly's list median is $2 \cdot 21 = \boxed{42}$. (If there are an even number of numbers, then the median is the average of the middle two numbers. We can use the same argument as in our first paragraph to see that Curly's median is still 2 times Larry's median.)

Mode: The number of times a certain number appears in Larry's list equals the number of times double that number appears in Curly's list. Therefore, the number that appears most frequently in Curly's list is 2 times the number that appears most frequently in Larry's list. This means that the mode of Curly's list is $2 \cdot 11 = \boxed{22}$.

13.22:

Find the average of the following numbers:

9420365988, 9420365994, 9420366003, 9420365981, 9420365991.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: All 5 of the numbers are very close to 9420365980. If we subtract 9420365980 from each number, our list becomes

$$8, 14, 23, 1, 11.$$

The average of the numbers in this list is

$$\frac{8 + 14 + 23 + 1 + 11}{5} = \frac{57}{5} = 11.4.$$

Each number in the original list is 9420365980 greater than the corresponding number in our list with average 11.4. Therefore, the average of the numbers in the original list is

$$11.4 + 9420365980 = \boxed{9420365991.4}.$$

13.23:

Each of 9 friends chooses her favorite positive integer.

(a) The median of the chosen numbers is 91. What is the smallest the average of the 9 chosen numbers could be?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: In order to make the average as small as possible, the sum of the chosen numbers must be as small as possible. Because the median of the numbers is 91, we know that five of the nine numbers are at least 91. To make the sum as small as possible, we set all five of these numbers equal to 91. The smallest the other four numbers can be is 1, so the smallest the sum of all nine numbers can be is $4 \cdot 1 + 5 \cdot 91 = 459$. This means that the smallest the average can be is $\frac{459}{9} = \boxed{51}$.

(b) The median of the chosen numbers is 91. Is there a limit to how large the average of the chosen numbers can be? If so, what is the largest the average can be?

Hide Solution

Reset

Your Submission: Solution

Solution: There is no maximum. The median only tells us that at least one of the numbers is 91, and four other numbers are at least 91. It doesn't place any limit on how large these four other numbers are, so they can be as large as we like. Therefore, there's no limit on how large the sum of all 9 numbers is, which means there is no limit to how large the average of the numbers can be.

(c) The average of the chosen numbers is 91. What is the smallest the median of the 9 chosen numbers could be?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since the average is 91, the sum of the 9 chosen numbers is $9 \cdot 91 = 819$. We might have one friend choose 811 and each of the other eight friends choose 1. The median of these nine choices is 1 and the sum is $811 + 8 \cdot 1 = 819$. Therefore, it is possible for 1 to be the median. Since the friends must choose positive integers, no one can choose a number smaller than 1. This means that $\boxed{1}$ is the smallest possible median.

(d)★ The average of the chosen numbers is 91. What is the largest the median of the chosen numbers could be?

Hint: Problems involving averages can often be solved by thinking about sums.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: As in the previous part, the sum of the chosen numbers is $9 \cdot 91 = 819$. Suppose the median is x. Since the sum of all nine numbers is 819, we must have

$$x + 819 - (\text{sum of the other eight numbers}).$$

Therefore, to make x as large as possible, we must make the sum of the other eight numbers as small as possible.

Since x is the median of the nine numbers, there are four other numbers that are no greater than x and four others that are no less than x. The smallest the numbers in the first group can be is 1, and the smallest the numbers in the second group can be is x. So, the median is as large as possible when four of the numbers are 1 and the other five are the same. If five of the numbers are 1, then the sum of the other five is $819 - 4 = 815$. Since these other five numbers must be the same to make the median as large as possible, the greatest the median can be is $\frac{815}{5} = \boxed{163}$.

As an extra challenge, try to figure out the greatest that the median can be if we have more friends choosing favorite positive integers, but the average of the integers chosen is still 91. What if there are 100 friends? 1000 friends? A million friends?

13.24:

Consider the two lists of numbers below:

List A: 34, 54, 161, 443, 87, 43, 76, 339, 38, 654, 75, 164, 870
List B: 5674784, 54, 65, 12, 654, 765, 12, 34, 98, 56, 72, 34, 86

(a) Is it easy to tell quickly which list has the higher average?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Yes. List B. The average of each list is the sum of the list divided by the number of numbers in the list. None of the numbers in List A is greater than 1000, but List B has a number that is greater than fifty million! All the numbers in both lists are positive, so the sum of List B is far greater than the sum of List A. Therefore, the average of List B is far greater than that of List A.

(b) Is it easy to tell quickly which list has the higher median?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Yes. We have to put the numbers in order to figure out what the median is in each list. We can't tell at a glance which list is more likely to have a larger middle number. (For those of you keeping score at home, the median of List A is 87 and the median of list B is 65.)

(c) What do the first two parts tell us about outliers?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: One extreme value, such as the large number in List B, can have a huge effect on the average, but not a very large impact on the median. That is, outliers affect the average more than they affect the median.

13.25:

Source: MATHCOUNTS

The chart at the right shows where Meadow Lark Lane Middle School students do their homework. The options are their bedroom (BR), the kitchen (K), the dining room (D), and the family living room (FLR). Using these data, how many of the 880 students at the school do their homework in the dining room?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The chart shows that 20% of the students do their homework in the dining room. Therefore, there are $0.2 \cdot 880 = \boxed{176}$ students who do their homework in the dining room.

13.26:

Five people live along a long, straight road. The table below gives the distance in miles between pairs of houses. For example, the distance between Adrian's house and Walter's house is 11 miles.

	Adrian	Dan	Laurie	Jon	Walter
Adrian	0	7	15	1	11
Dan	3	0	12	7	8
Laurie	15	12	0	5	4
Jon	10	7	5	0	6
Walter	11	8	4	1	0

(a) Fill in the missing entries.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The number in the upper left corner is 0 because the distance between Adrian's house and Adrian's house is obviously 0. Similarly, each number on the diagonal from upper left to lower right must be 0. This allows us to fill in two of the missing entries with 0 (corresponding to Laurie's house and Walter's house).

Next, we note that the entry in Adrian's row and Dan's column is the same as the entry in Dan's row and Adrian's column, since both of these entries represent the distance between Adrian's house and Dan's house. Similarly, the entries between each pair of houses appears twice in the table. This allows us to fill in all the rest of the missing entries once we know the corresponding to the distance between Walter's house and Jon's house.

Looking at the bottom two rows, we see that the distances from each person to Walter and to Jon differ by 1 mile, so we conclude that Walter's house and Jon's house are 1 mile apart. The completed table is shown below:

	Adrian	Dan	Laurie	Jon	Walter
Adrian	0	3	15	10	11
Dan	3	0	12	7	8
Laurie	15	12	0	5	4
Jon	10	7	5	0	1
Walter	11	8	4	1	0

(b) Which two people live the farthest apart? Which two people live the closest?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The largest number in the table is 15, which tells us that Laurie and Adrian are the farthest apart. The smallest number corresponding to a distance between two different people is 1, so Jon and Walter are closest.

(c) Draw an accurate map showing where the people live along the road.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since Laurie and Adrian are farthest apart, they are at opposite ends of the road. We can then use the distance between Adrian (or Laurie) and each of the others to place all 5 people along the road. Dan is closest to Adrian, so we place him first, 3 miles away. Next closest is Jon, who is 10 miles from Adrian. Jon is $10 - 3 = 7$ miles farther from Adrian than Dan, so Jon is 7 miles from Dan, as shown. Similarly, Walter is $11 - 10 = 1$ mile from Jon and Laurie is $15 - 11 = 4$ miles from Walter:

3 Adrian — 7 Dan — 1 Jon Walter — 4 Laurie

(We could also reverse the entire map, placing Adrian at the far right and Laurie at the far left.)

14.1 Counting with Addition and Subtraction

Problems

Problem 14.1 Using by Solution

How many numbers are in the list

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16?$$

Problem 14.2 Using by Solution

How many numbers are in the list

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27?$$

We could stop and ask the question: "How many numbers are there between 5 and 27 inclusive?" **Inclusive** means that we include the 5 and the 27 in our count.

Problem 14.3 Using by Solution

How many integers are in $\{-3, -2, -1, 0, 1, 2, 3\}$? Hint: Subtract 3 from each element of the set.

Problem 14.4 Using by Solution

How many multiples of 3 are between 3 and 27?

Problem 14.5 Using by Solution

How many multiples of 3 are between 12 and 27?

Problem 14.6 Using by Solution

How many multiples of 3 are between 12 and 27?

Problem 14.7 Using by Solution

How many multiples of 3 are between 12 and 27?

Problem 14.8 Using by Solution

How many multiples of 3 are between 12 and 27?

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27$$

Solution for Problem 14.1. Obviously, there are 16 numbers. ✓

That was pretty easy. The counting was already done for us. Many other counting problems can be reduced to this type of counting.

Problem 14.9 Using by Solution

$$9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27$$

We could also ask the question: "How many numbers are there between 9 and 27 inclusive?" **Inclusive** means that we include the 9 and the 27 in our count.

Solution for Problem 14.2. We could count them from left to right and find them are 16 numbers. However, a more clever way is to approach the problem as counting the number of numbers in the set. Obviously, we are subtracting 9 from every number in the set.

$$\begin{array}{rcl} 27 - 9 & = & 18 \\ \text{Add 1 to the answer} & & +1 \\ \text{Our answer} & & 19 \end{array}$$

We know how to count the new list! There are 19 items in the list. So, there are 19 items in our original list. ✓

Problem 14.3 Subtracting 3 from each element of the set, we get $\{-6, -5, -4, -3, -2, -1, 0\}$. There are 7 numbers in the set. ✓

Concept When presented with a complicated problem, try to turn it into a simpler problem that we know how to solve.

You may also notice that in Problem 14.2, the question asking what integers are between 5 and 27 is the same as asking what integers are $27 - 5 = 22$. This is one trick that has the number of items (16) in the list. Subtract the first number from the last number.

Inclusive means that we include the first and last number in the list.

Solution for Problem 14.4. We could count them from left to right and find there are 16 numbers. However, a more clever way is to approach the problem as counting the number of multiples of 3 in the set. Obviously, we are subtracting 3 from every number in the set.

$$\begin{array}{rcl} 27 - 3 & = & 24 \\ \text{Add 1 to the answer} & & +1 \\ \text{Our answer} & & 25 \end{array}$$

We know how to count the new list! There are 25 items in the list. So, there are 25 items in our original set. ✓

Problem 14.5 Subtracting 12 from each element of the set, we get $\{-9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. There are 15 numbers in the set. ✓

Concept Subtracting 1 from the last number and the first number in the list gives us the number of items in the list.

Problem 14.6 Using by Solution

Problem 14.7 Using by Solution

Problem 14.8 Using by Solution

Problem 14.9 Using by Solution

Problem 14.10 Using by Solution

Problem 14.11 Using by Solution

Problem 14.12 Using by Solution

Problem 14.13 Using by Solution

Problem 14.14 Using by Solution

Problem 14.15 Using by Solution

Problem 14.16 Using by Solution

Problem 14.17 Using by Solution

Problem 14.18 Using by Solution

Problem 14.19 Using by Solution

Problem 14.20 Using by Solution

Problem 14.21 Using by Solution

Problem 14.22 Using by Solution

Problem 14.23 Using by Solution

Problem 14.24 Using by Solution

Problem 14.25 Using by Solution

Problem 14.26 Using by Solution

Problem 14.27 Using by Solution

Problem 14.28 Using by Solution

Problem 14.29 Using by Solution

Problem 14.30 Using by Solution

Problem 14.31 Using by Solution

Problem 14.32 Using by Solution

Problem 14.33 Using by Solution

Problem 14.34 Using by Solution

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Problem 14.62 Using by Solution

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Problem 14.66 Using by Solution

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Problem 14.68 Using by Solution

Problem 14.69 Using by Solution

Problem 14.70 Using by Solution

Problem 14.71 Using by Solution

Problem 14.72 Using by Solution

Problem 14.73 Using by Solution

Problem 14.74 Using by Solution

Problem 14.75 Using by Solution

Problem 14.76 Using by Solution

Problem 14.77 Using by Solution

Problem 14.78 Using by Solution

Problem 14.79 Using by Solution

Problem 14.80 Using by Solution

Problem 14.81 Using by Solution

Problem 14.82 Using by Solution

Problem 14.83 Using by Solution

Problem 14.84 Using by Solution

Problem 14.85 Using by Solution

Problem 14.86 Using by Solution

Problem 14.87 Using by Solution

Problem 14.88 Using by Solution

Problem 14.89 Using by Solution

Problem 14.90 Using by Solution

Problem 14.91 Using by Solution

Problem 14.92 Using by Solution

Problem 14.93 Using by Solution

Problem 14.94 Using by Solution

Problem 14.95 Using by Solution

Problem 14.96 Using by Solution

Problem 14.97 Using by Solution

Problem 14.98 Using by Solution

Problem 14.99 Using by Solution

Problem 14.100 Using by Solution

Exercises

14.1.1. 🔊 ✎ ✓

How many numbers are in each of the following lists?

(a) $-10, -9, -8, \ldots, 90, 91$

Prompt: Solution

Your Solution: Hide Solution

Solution: We subtract 90 from each number in the list to get $1, 2, 3, \ldots, 99$ as these are $\boxed{99}$ numbers. We also count from zero and find -10. It is obvious to everyone that for how many numbers we between -9 and 0 inclusive. $10 - 0 = 10$. ✓

(b) $-37, -35, -33, \ldots, 33, 35$

Prompt: Solution

Your Solution: Hide Solution

Solution: Consecutive numbers in the list are 2 apart, and each number is 1 more than a multiple of 2. So, we subtract 1 from each number in the list to get $-36, -34, -32, \ldots, 32, 34$.

There are $\boxed{35}$ numbers in the list. ✓

Prompt: Solution

Your Solution: Hide Solution

Solution: Consecutive numbers in the list are 3 apart, and each number is 1 more than a multiple of 3. So, we subtract 1 from each number in the list to get $-4, -2, 0, 2, 4, \ldots, 32, 34$.

There are $\boxed{20}$ numbers in the list. ✓

Prompt: Solution

Your Solution: Hide Solution

Solution: Consecutive numbers in the list are 4 apart, and each number is 1 more than a multiple of 4. So, we subtract 1 from each number in the list to get $-5, -3, -1, 1, 3, 5, \ldots, 31, 33$.

There are $\boxed{20}$ numbers in the list. ✓

Prompt: Solution

Your Solution: Hide Solution

Solution: Consecutive numbers in the list are 5 apart, and each number is 1 more than a multiple of 5. So, we subtract 1 from each number in the list to get $-6, -4, -2, 0, 2, 4, \ldots, 30, 32$.

There are $\boxed{20}$ numbers in the list. ✓

14.1.2. 🔊 ✎ ✓

Fill in the blanks in the sentence: "Counting from 1 to 100 inclusive, we count $\boxed{100}$ numbers." Counting from 1 to 100, how many times do we write the digit 1?

Prompt: Solution

Your Solution: Hide Solution

Solution: Since 10^2 when counting from the first, there are 10^2 people in front of me. Since $10 \cdot 10^2$ when counting from the first, there are $10 \cdot 10^2$ people in front of me. Counting me, the people in front of me and the people behind me, there are $1 + 10^2 + 10 \cdot 10^2 = \boxed{1111}$ people.

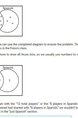
14.1.3. 🔊 ✎ ✓

Fill in the blanks: In my school, $\boxed{37}$ students like math, $\boxed{48}$ like of both math. Spanish, and $\boxed{25}$ of them both Spanish and math. How many students are in my school?

Prompt: Solution

Your Solution: Hide Solution

Solution: We start with a Venn diagram, as shown at the right. We place 25 in the overlap region for both math and Spanish. We place $37 - 25 = 12$ in the math only region. We place $48 - 25 = 23$ in the Spanish only region. We add $12 + 25 + 23 = \boxed{60}$ students in our school. Since we know the number of students in our school, we know the number of students in our Spanish only region. Yes, because $60 - 37 - 23 = 0$. ✓ Wow! We don't need to know the number of students in our school!



Venn Diagram with Math and Spanish

Exercises

14.1.5. 🔊 ✎ ✓

How many numbers are in each of the following lists?

(a) $-40, -39, -38, \ldots, 90, 91$

Prompt: Solution

Your Solution: Hide Solution

Solution: We subtract 90 from each number in the list to get $-1, 0, 1, \ldots, 99$ as these are $\boxed{100}$ numbers. We also count from zero and find -40. It is obvious to everyone that for how many numbers are between -39 and 0 inclusive. $0 - (-1) = 1$. ✓

(b) $-37, -35, -33, \ldots, 33, 35$

Prompt: Solution

Your Solution: Hide Solution

Solution: Consecutive numbers in the list are 2 apart, and each number is 1 more than a multiple of 2. So, we subtract 1 from each number in the list to get $-36, -34, -32, \ldots, 32, 34$.

There are $\boxed{35}$ numbers in the list. ✓

Prompt: Solution

Your Solution: Hide Solution

Solution: Consecutive numbers in the list are 3 apart, and each number is 1 more than a multiple of 3. So, we subtract 1 from each number in the list to get $-4, -2, 0, 2, 4, \ldots, 32, 34$.

There are $\boxed{20}$ numbers in the list. ✓

Prompt: Solution

Your Solution: Hide Solution

Solution: Consecutive numbers in the list are 4 apart, and each number is 1 more than a multiple of 4. So, we subtract 1 from each number in the list to get $-5, -3, -1, 1, 3, 5, \ldots, 31, 33$.

There are $\boxed{20}$ numbers in the list. ✓

Prompt: Solution

Your Solution: Hide Solution

Solution: Consecutive numbers in the list are 5 apart, and each number is 1 more than a multiple of 5. So, we subtract 1 from each number in the list to get $-6, -4, -2, 0, 2, 4, \ldots, 30, 32$.

There are $\boxed{20}$ numbers in the list. ✓

14.1.6. 🔊 ✎ ✓

Fill in the blanks in the sentence: "Counting from 1 to 100 inclusive, we count $\boxed{100}$ numbers." Counting from 1 to 100, how many times do we write the digit 1?

Prompt: Solution

Your Solution: Hide Solution

Solution: We place a 1 in front of the number. Now, we have 100 numbers and we have 100 numbers.

14.2 The Multiplication Principle

Problems

Problem 14.8 Jump to Solution

You have three shirts and four pairs of pants. How many outfits consisting of one shirt and one pair of pants can you make?

Problem 14.9 Jump to Solution

In how many ways can we form an international commission if we must choose one European country from among 6 European countries, one Asian country from among 4, one North American country from among 3, and one African country from among 7?

Problem 14.10 Jump to Solution

In how many ways can we form a license plate using only digits (0–9) and capital letters (other than O and I), given that each plate has 6 characters, the first of which is a digit, and the second of which is a letter?

Problem 14.11 Jump to Solution

In how many ways can I arrange four different books from left to right on a shelf?

Problem 14.12 Jump to Solution

Your math club has 16 members. In how many ways can it select a president, a vice-president, and a treasurer if no member can hold more than one office?

Often we'll be counting the number of outcomes of a series of events. Here's an example of this type of problem.

Problem 14.8

You have three shirts and four pairs of pants. How many outfits consisting of one shirt and one pair of pants can you make?

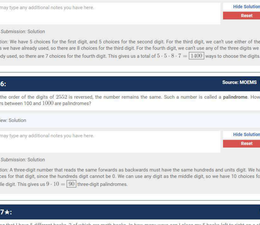
Solution for Problem 14.8: In this problem, the number of possibilities is so small that we can just list them.

If our shirts are labeled S_1, S_2, S_3 and our pants are labeled P_1, P_2, P_3, P_4, then we can list all of the possible outfits:

$$S_1P_1 \quad S_1P_2 \quad S_1P_3 \quad S_1P_4$$
$$S_2P_1 \quad S_2P_2 \quad S_2P_3 \quad S_2P_4$$
$$S_3P_1 \quad S_3P_2 \quad S_3P_3 \quad S_3P_4$$

So there are 12 outfits. ◻

Listing the outfits like this is somewhat annoying. We can better visualize the outfits by using a diagram. One sort of diagram is a **tree**, as shown below. We start at the dot at the top, and each arrow is a choice of one item: the first arrow (from the dot at the top) is the choice of pants, and the second arrow (from the choice of pants) is the choice of shirt. Each complete path of two arrows leads to a complete outfit.



Another type of diagram we can use for Problem 14.8 is a **grid**. A grid looks a bit like a multiplication table: we put the choices of pants along the top and the choices of shirts along the side. Each box in the grid is a complete outfit, as shown below:

	Pants			
	P_1	P_2	P_3	P_4
Shirts S_1	S_1P_1	S_1P_2	S_1P_3	S_1P_4
S_2	S_2P_1	S_2P_2	S_2P_3	S_2P_4
S_3	S_3P_1	S_3P_2	S_3P_3	S_3P_4

For both the tree and the grid, we can count outfits by using the following reasoning: we have four choices for the pants, and for each of these four choices of pants, we have three choices for the shirt. Therefore, there are $4 \cdot 3 = 12$ outfits.

> **Concept:** The point of drawing a tree or a grid is to keep our counting organized. Clear organization is very important for solving counting problems.

It's easy for us to draw a tree or a grid in Problem 14.8 because the numbers are small. Let's look at a more complicated version of the same type of problem.

Problem 14.9

In how many ways can we form an international commission if we must choose one European country from among 6 European countries, one Asian country from among 4, one North American country from among 3, and one African country from among 7?

Solution for Problem 14.9: Clearly drawing a grid or a tree is not going to be practical—there are too many choices! So we'll have to do this problem by reasoning out the answer.

We tackle a problem like this in steps.

Step 1: There are 6 ways to choose a European country.

Step 2: For each European country, we can choose an Asian country in 4 ways, for a total of $6 \cdot 4 = 24$ ways to choose both a European and an Asian country.

Step 3: For each pair of countries that we've chosen in Steps 1 and 2, we can choose a North American country in 3 ways. So there are $24 \cdot 3 = 72$ ways to choose 3 countries.

Step 4: For each triple of countries that we've chosen in Steps 1–3, we can choose an African country in 7 ways. So there are $72 \cdot 7 = 504$ ways to choose 4 countries.

So we see that there are $6 \cdot 4 \cdot 3 \cdot 7 = 504$ possibilities. ◻

We say that the choices in Problem 14.9 are **independent**, meaning that each decision does not depend on the others and does not affect the others. Specifically, we choose a European country, then an Asian country, then a North American country, then an African country. Each choice doesn't depend on or affect the other choices.

> **Concept:** We use multiplication to count the number of outcomes for a sequence of independent events.

Problem 14.10

In how many ways can we form a license plate using only digits (0–9) and capital letters (other than O and I), given that each plate has 6 characters, the first of which is a digit, and the second of which is a letter?

Solution for Problem 14.10: Since each character does not depend on any of the other characters, our choices are independent. There are 10 choices for the first character (any digit from 0 through 9), there are 24 choices for the second character (any letter A–Z except for O or I), and there are 34 choices for each of the other four characters (any digit 0–9 or any letter A–Z except O or I).

Therefore, since the choices are independent, we multiply the number of choices at each step, and we have

$$10 \cdot 24 \cdot 34 \cdot 34 \cdot 34 \cdot 34 = 10 \cdot 24 \cdot 34^4 = 320{,}720{,}640$$

ways to form our license plate. ◻

In some counting problems, we make a series of choices, but later choices will depend on some of the earlier choices. This is unlike the problems that we've done before. For example, in Problem 14.8, our choices of shirt didn't depend at all on our choice of pants. Things are a little bit different in the next problem.

Problem 14.11

In how many ways can I arrange four different books from left to right on a shelf?

Solution for Problem 14.11: We cannot just count $4 \cdot 4 \cdot 4 \cdot 4$, because once we place the first book on the left, we no longer have 4 choices for the second book. We can't reuse the first book, so we only have 3 choices remaining for the second book. Let's carefully count our choices step by step:

Step 1: We have 4 choices for which book to place on the left;

Step 2: Regardless of which book we chose in step 1, we have 3 books remaining, so we have 3 choices for the second book. Thus we have a total of $4 \cdot 3 = 12$ choices for the first two books on the left;

Step 3: We have 2 books remaining, so we have 2 choices for the third book. Thus we have a total of $12 \cdot 2 = 24$ choices for the first three books;

Step 4: We only have 1 book remaining, so we have 1 choice for the fourth book. Thus we have a total of $24 \cdot 1 = 24$ choices for all four books.

So the answer is 24. ◻

One important thing to note about Problem 14.11 is that although the choices themselves are not independent at each step, the number of choices at each step is independent of our previous choices. For example, no matter which book we choose in Step 1, we always have 3 remaining books to choose from in Step 2. Therefore, we can get our answer by multiplying the number of choices at each step, so that $4 \cdot 3 \cdot 2 \cdot 1 = 24$ is the answer.

If you're not yet convinced, we could represent our choices in this problem as a tree, where the books are labeled A, B, C, and D, as shown in the picture on the next page. We can see that there are 4 arrows from the starting dot for the first choice, then 3 arrows for the next choice, and so on, and indeed there are 24 possible arrangements.

Here's another example of a counting problem in which the choices are not independent.

Problem 14.12

Your math club has 16 members. In how many ways can it select a president, a vice-president, and a treasurer if no member can hold more than one office?

Solution for Problem 14.12: Once again, our choices are not independent. Once a student has been chosen president, she is not available to be chosen vice-president or treasurer. However, the number of choices we have for each position is the same no matter who is chosen.

We have 16 choices for president. Once we've chosen a president, then we have 15 people remaining to choose from for vice-president. Then, we have 14 people remaining to choose from for treasurer.

Therefore, there are

$$16 \cdot 15 \cdot 14 = 3360$$

ways to fill the three offices. ◻

The last two problems contain examples of **permutations**. A permutation occurs whenever we have to choose several different items, one at a time, from a large group of items.

Exercises

14.2.1:

Suppose I have 7 shirts and 4 ties. How many shirt-and-tie outfits can I make?

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: For each of the 7 choices I have for a shirt, I have 4 choices for a tie, giving me $7 \cdot 4 =$ 28 shirt-and-tie outfits.

14.2.2:

For each of 9 colors, I have one shirt and one tie of that color. How many shirt-and-tie outfits can I make if I refuse to wear a shirt and a tie of the same color?

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: Similar to the previous exercise, I have $9 \cdot 9 =$ 81 shirt-and-tie outfits, but I refuse to wear the 9 outfits that consist of a shirt and a tie of the same color. This leaves me $81 - 9 =$ 72 outfits. Another way to think of this is to suppose I choose my shirt first, and then my tie. I have 9 choices for the shirt, but then I only have 8 choices for the tie because I can't pick the one that is the same color as my shirt. So, I have $9 \cdot 8 =$ 72 possible outfits, as before.

14.2.3:

In how many ways can 5 people stand in a line?

You may type any additional notes you have here.

Hide Solution
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Your Submission: Solution

Solution: Similar to the previous exercise, I have 5 choices for who stands at the front of the line, then 4 people remaining to choose from for the next spot in line, then 3 people to choose from for the next spot in line, then 2 people for the next-to-last spot, and just 1 person remaining to put at the end of the line. This gives me a total of $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =$ 120 ways to arrange the people in line.

14.2.4:

A shoplooter sells house numbers. She has a large supply of the digits 1, 2, 3, and 8, but no other digits. How many different three-digit house numbers could be made using only the digits in her supply?

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: She has 4 choices for each digit, so she can make $4 \cdot 4 \cdot 4 = 4^3 =$ 64 different three-digit house numbers.

14.2.5: Source: AMC 8

How many four-digit positive integers are there such that the leftmost digit is odd, the second digit is even, and all four digits are different?

You may type any additional notes you have here.

Hide Solution
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Your Submission: Solution

Solution: We have 5 choices for the first digit, and 5 choices for the second digit. For the third digit, we can't use either of the two digits we have already used, so there are 8 choices for the third digit. For the fourth digit, we can't use any of the three digits we have already used, so there are 7 choices for the fourth digit. This gives us a total of $5 \cdot 5 \cdot 8 \cdot 7 =$ 1,400 ways to choose the digits.

14.2.6: Source: MOEMS

When the order of the digits of 2022 is reversed, the number remains the same. Such a number is called a **palindrome**. How many integers between 100 and 1000 are palindromes?

Preview: Solution

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: A three-digit number that reads the same forwards as backwards must have the same hundreds and units digit. We have 9 choices for that digit, since the hundreds digit cannot be 0. We can use any digit as the middle digit, so we have 10 choices for the middle digit. This gives us $9 \cdot 10 =$ 90 three-digit palindromes.

14.2.7★:

Suppose that I have 5 different books, 2 of which are math books. In how many ways can I place my 5 books left-to-right on a shelf if I want a math book on both ends?

Preview: Solution

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: At first, it might seem that the answer is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, since 5 books like I have 5 choices for the first book, 4 for the next, and so on. But we have two math books on the ends! So, we choose them first. We have 2 choices for which math book to put on the left end, and we have to save the other math book for the right end. That leaves 3 choices for which book to put to the right of the first math book, then 2 choices for the next non-math book, and 1 choice for the last non-math book. Then, we place our other math book on the right end of the shelf. This gives us a total of $2 \cdot 3 \cdot 2 \cdot 1 \cdot 1 =$ 12 ways to solve the books.

14.2.8★: Source: AMC 8

How many integers between 99 and 999 contain exactly one 0?

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: There are a number of ways to tackle this problem, but perhaps the most clever is to think about making a three-digit number with one 0 as follows:

Step 1: Place the 0. There are 2 places where we can put the 0: in the tens place or the units place.

Step 2: Place digits from 1-9 in the other two places. No matter where we place the 0, there are 9 choices for each of the other two digits in our three-digit number. Therefore, for each possible placement of the 0, there are $9 \cdot 9 =$ 81 ways to choose the other two digits to make a three-digit number with one zero.

So, there are 2 choices for where we can put the 0, and each gives us 81 three-digit numbers with one 0. This gives us a total of $2 \cdot 81 =$ 162 three-digit numbers with one 0.

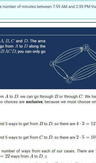
14.3 Casework

Many counting problems can be solved by considering different cases—that is, by dividing what we're trying to count into two or more separate categories. This approach is called **casework**. The key to solving problems using casework is to be organized, and to be careful that you don't miss any cases.

Problems

Problem 14.13 Jump to Solution

The figure to the right represents a road map between 6 villages, labeled A, B, C and D. The only roads connecting villages A and B are the roads from A to C and from C to B. In how many ways can you get from A to B along this map? (For example, one path is $ACCB$, and another is $ACDCB$.)



Problem 14.14 Jump to Solution

On the island of Abcdef, the alphabet has only 6 letters, and every word in their language has no more than 3 letters in it. How many words are possible? (A word can use a letter more than once, but 0 letters does not count as a word.)

Problem 14.15 Jump to Solution

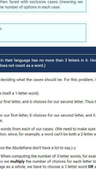
How many positive integer divisors (a, b) satisfy $a^2 + b < 20$?

Problem 14.16 Source: MOEMS Jump to Solution

A digital clock shows time in the form HH:MM. On a certain day, what is the number of minutes between 7:00 AM and 2:00 PM that HH is greater than MM?

Problem 14.17

The figure to the right represents a road map between 4 villages, labeled A, B, C and D. The only roads connecting villages A and B are the roads from A to C and from C to B. In how many ways can you get from A to B along this map? (For example, one path is $ACDCB$, another is $ACDACB$, and another is $ACDACDACB$.)



Solution for Problem 14.13. Broadly speaking, we have two ways to get from A to B: we can go through D or through C. We have to go through C either way, so we can't go through both. We say that these two choices are **exclusive**, because we must choose one or the other, but not both.

Our two exclusive choices will form our two cases.

Case 1: Go from A to D. There are 4 ways to get from A to D, and 2 ways to get from D to C; so there are $4 \cdot 2 = 12$ ways to get from A to D through D.

Case 2: Go from A to C. There are 2 ways to get from A to C, and 3 ways to get from C to D; so there are $2 \cdot 3 = 12$ ways to get from A to D through C.

To count the total number of ways to get from A to D, we now add the number of ways from each of our cases. There are 12 ways through D and another 12 ways through C; so there are a total of $12 + 12 = 22$ ways from A to D.

We can multiply in some of the steps of Solution 14.13 (for example, counting the paths from A to D through D), but in other steps (when we added the results from each case)?

Solution for Problem 14.14. On the island of Abcdef, the alphabet has only 6 letters, and every word in their language has no more than 3 letters in it. How many words are possible? (A word can use a letter more than once, but 0 letters does not count as a word.)

Solution for Problem 14.15. Often, the tricky part of casework problems is deciding what the cases should be. For this problem, it makes sense to use as our cases the number of letters in each word.

Case 1: one-letter words. There are 6 words because 6 of the 6 letters is itself a 1-letter word.

Case 2: two-letter words. To form a two-letter word, we have 6 choices for our first letter, and 6 choices for our second letter. Then there are $6 \cdot 6 = 36$ two-letter words.

Case 3: three-letter words. To form a three-letter word, we have 6 choices for our first letter, 6 choices for our second letter, and 6 choices for our third letter. There are $6 \cdot 6 \cdot 6 = 216$ three-letter words possible.

So to get the total number of words in the language, we add the number of words from each of our cases. (We need to make sure that the cases are exclusive, meaning that they don't overlap. But that's clear in this solution, since, for example, a word can't be both a 2-letter word and a 3-letter word at the same time.)

Therefore there are $6 + 36 + 216 = 258$ words possible on Abcdef. (I guess the Abcdefians don't have a lot to say.)

Again, in this problem, you have to know when to multiply and when to add. When computing the number of 3-letter words, for example, we have to choose the first letter **AND** the second letter **AND** the third letter, so we **multiply** the number of choices for each letter to get the number of 3-letter words. However, when choosing a word from the language as a whole, we have to choose a 1-letter word **OR** a 2-letter word **OR** a 3-letter word, so we **add** the number of choices for each length of word to get the number of total words.

Problem 14.18

How many pairs of positive integers (x, y) satisfy $x^2 + x < 24$?

Solution for Problem 14.18. It may not be obvious how to proceed with this problem, but a little experimentation might lead you to determine the possible values of x.

Since $0 < x^2 < 24$, we can see that x must be one of $1, 2, 3, 4$ or 5; let's see these as our cases.

Case 1: $x = 1$ Then $x^2 + x = 1 + 1 = 2$. So $x^2 + x < 24$ is true, so there are 2 possible choices for y when $x = 1$, since y can be any integer from 1 to 2.

Case 2: $x = 2$ Then $x^2 + x = 2 + 2 = 4$. So there are 19 possible choices for y when $x = 2$

Case 3: $x = 3$ Then $x^2 + x = 3 + 3 = 6$. So there are 17 possible choices for y when $x = 3$.

Case 4: $x = 4$ Then $x^2 + x = 4 + 4 = 8$ So there are 15 possible choices for y when $x = 4$.

Case 5: $x = 5$ Then $x^2 + x = 5 + 5 = 10$ So there are 13 possible choices for y when $x = 5$.

So to get the total number of pairs of positive integers satisfying the inequality, we add up all of our possible cases, and see that there are $2 + 19 + 14 + 11 + 8 = 54$ possible pairs.

We can see from this problem that sometimes it's not immediately clear how we want to set up our different cases. Breaking up a problem into cases, and deciding what the different cases are, is a skill that you'll get better at with practice. It is important to make sure that the cases are exclusive—that is, that they don't overlap—and also that every possible solution to the original problem is accounted for in one of the cases.

Problem 14.19 Source: MOEMS

A digital clock shows time in the form HH:MM. On a certain day, what is the number of minutes between 7:00 AM and 2:00 PM that HH is greater than MM?

Solution for Problem 14.19. We first notice that the hours in the time period described in the problem are 6, 7, 8, 9, 10, 11, 12, 1, and 2. We can then do casework in two ways.

Method 1: Use the hours as the cases. In the 6 o'clock hour, the minutes from 00 to 07 (inclusive) satisfy the condition that the hour is greater than the minutes. Thus, there are 8 minutes in the 6 o'clock hour that we count.

The same logic shows that in the 8 o'clock hour, there are 8 minutes that we need to count. So we just need to add up the hours:

$$8 + 9 + 10 + 11 + 12 + 1 + 2 = 53,$$

and our answer is that there are 53 minutes satisfying the condition of the problem.

Problem 14.20 Source: MOEMS

A digital clock shows time in the form HH:MM. On a certain day, what is the number of minutes between 7:00 AM and 2:00 PM that HH is greater than MM?

Solution for Problem 14.19. We first notice that the hours in the time period described in the problem are 6, 7, 8, 9, 10, 11, 12, 1, and 2. We can then do casework in two ways.

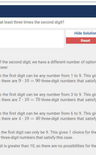
Method 1: Use the hours as the cases. In the 6 o'clock hour, the minutes from 00 to 07 (inclusive) satisfy the condition that the hour is greater than the minutes. Thus, there are 8 minutes in the 6 o'clock hour that we count.

The same logic shows that in the 8 o'clock hour, there are 8 minutes that we need to count. So we just need to add up the hours:

$$8 + 9 + 10 + 11 + 12 + 1 + 2 = 53,$$

and our answer is that there are 53 minutes satisfying the condition of the problem.

Casework Counting Part 1



Casework Counting Part 2

Exercises

14.3.1:

Write the integers from 1 to 150 inclusive. What is the total number of digits that must be written?

Hide Solution

Reset

Your Solution

Solution: The 9 single numbers contribute 1 digit each, for a total of 9 digits. The 90 two-digit numbers contribute 2 digits each, for a total of $90 \cdot 2 = 180$ digits. There are 51 three-digit numbers from 100 to 150. These contribute a total of $51 \cdot 3 = 153$ digits. Therefore, there are $9 + 180 + 153 = 342$ digits.

14.3.2:

I have two hats. In one hat are balls numbered 1 through 15. In the other hat are balls numbered 16 through 25. I first choose a hat, then from that hat, I select a ball, without replacing the balls between selections. How many different ordered selections of 2 balls are possible? (By "ordered selections," we mean that "ball 1 then ball 2" is considered different from "ball 2 then ball 1.")

Hide Solution

Reset

Your Solution

Solution: There are two possibilities for which hat I choose. We tackle these separately:

Case 1: I choose the hat with balls 1 through 15. I then have 15 choices for the first ball I choose, and for each of these choices I have 14 possibilities for the second ball. Therefore, there are a total of $15 \cdot 14 = 210$ possible selections in this case.

Case 2: I choose the hat with balls 16 through 25. There are $25 - 16 + 1 = 10$ balls in this hat. So, I have 10 choices for the first ball and 9 for the second, for a total of $10 \cdot 9 = 90$ selections.

Combining the two cases, we have $210 + 90 = \boxed{300}$ different ordered selections.

14.3.3:

(a) How many paths are there from A to D in the diagram shown below, if we can only travel in the direction of the arrows?

Preview Solution

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Your Solution

Solution: We go through either B or C to get to D. The number of paths going from A to B is $2 \cdot 1 = 2$, and the number of paths going from A to D through C is $2 \cdot 3$. Thus the total number of paths from A to D is $2 \cdot 1 + 2 \cdot 3 = \boxed{8}$.

(b) How many paths are there from D to E in the diagram shown below, if we can only travel in the direction of the arrows?

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Solution: To get from D to E, we go through one of E, F, or G. The three cases give a total of $(1 \cdot 1) + (2 \cdot 3) + (2 \cdot 1) = \boxed{13}$ paths from D to E.

(c) How many paths are there from A to E in the diagram shown below, if we can only travel in the direction of the arrows?

Preview Solution

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Reset

Your Solution

Solution: All paths must go through D. We saw in part (a) that there are 8 paths from A to D. We saw in part (b) that there are 13 paths from D to E. The choice of path from A to D is independent of the choice of path from D to E. Therefore we multiply the number of paths from A to D by the number of paths from D to E to get the number of paths from A to E. The answer is $8 \cdot 13 = \boxed{104}$.

14.3.4:

How many positive 3-digit numbers have the property that the first digit is at least three times the second digit?

Hide Solution

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Your Solution

Solution: We always have 10 options for the last digit. For each choice of the second digit, we have a different number of options for the first digit. So, we consider each possible second digit as a separate case.

Case 1: The second digit is 0. Then, three times the second digit is 0, so the first digit can be any number from 1 to 9. This gives 9 choices for the first digit, and there are 10 choices for the last digit, so there are $9 \cdot 10 = 90$ three-digit numbers that satisfy this case.

Case 2: The second digit is 1. Then, three times the second digit is 3, so the first digit can be any number from 3 to 9. This gives 7 choices for the first digit, and there are 10 choices for the last digit, so there are $7 \cdot 10 = 70$ three-digit numbers that satisfy this case.

Case 3: The second digit is 2. Then, three times the second digit is 6, so the first digit can be any number from 6 to 9. This gives 4 choices for the first digit, and there are 10 choices for the last digit, so there $4 \cdot 10 = 40$ three-digit numbers that satisfy this case.

Case 4: The second digit is 3. Then, three times the second digit is 9, so the first digit can only be 9. This gives 1 choice for the first digit, and there are 10 choices for the last digit, so there are $1 \cdot 10 = 10$ three-digit numbers that satisfy this case.

Case 5: The second digit is 4 or greater. In this case, the second digit is greater than 10, so there are no possibilities for the first digit. Therefore, no three-digit numbers satisfy this case.

Combining our cases, we have a total of $90 + 70 + 40 + 10 = \boxed{210}$ numbers that satisfy the problem.

14.3.5:

How many positive 4-digit numbers have the last digit equal to the sum of the first two digits?

Hide Solution

Reset

Your Solution

Solution: We list the possible last digits, and the possibilities for the first two digits for each choice of last digit:

Last Digit	First Two Digits
0	(0, 0)
1	(1, 0)
2	(1, 1), (2, 0)
3	(1, 2), (2, 1), (3, 0)
4	(1, 3), (2, 2), (3, 1), (4, 0)
5	(1, 4), (2, 3), (3, 2), (4, 1), (5, 0)
6	(1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 0)
7	(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), (7, 0)
8	(1, 7), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (7, 1), (8, 0)
9	(1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1), (9, 0)

The third digit can be any of the 10 digits. The answer is:

$$(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) \cdot 10 = \boxed{450}.$$

There is a quicker solution. Once we choose the last digit, we need the first digit to be between 1 and the chosen last digit, inclusive. If, for example, we choose 4 for the last digit, we can use 1, 2, 3, 4 for the first digit. The last digit is 2 where 2 is any digit from 0 to 9, then there are 2 choices for the first digit. The second digit is the last digit minus the first digit, so there is just one choice for the second digit. Thus, there are 2 choices. The third digit can be any of the 10 digits. Thus the answer is $(1 + 2 + \cdots + 9) \cdot 10 = \boxed{450}$.

14.3.6: Source: AMC 8

How many positive two-digit numbers have digits whose sum is a perfect square?

Hide Solution

Reset

Your Solution

Solution: The greatest the sum of the digits of a two-digit integer can be is $9 + 9 = 18$. Therefore, we have four possibilities to consider if the sum of the digits is a perfect square: $1, 4, 9$, or 16.

Case 1: The sum of the digits is 1. The only two-digit number that satisfies this is 10, so there is 1 possibility in this case.

Case 2: The sum of the digits is 4. For each tens digit from 1 to 4, there is one such number: 13, 22, 31, 40. So, there are 4 possibilities in this case.

Case 3: The sum of the digits is 9. For each tens digit from 1 to 9, there is one such number: 18, 27, 36, 45, 54, 63, 72, 81, 90. So, there are 9 possibilities in this case.

Case 4: The sum of the digits is 16. For each tens digit from 7 to 9, there is one such number: 79, 88, 97. So, there are 3 possibilities in this case.

Combining our cases, we have $1 + 4 + 9 + 3 = \boxed{17}$ numbers that satisfy the problem.

14.3.7a:

How many squares of any size can be formed by connecting dots in the grid shown to the right?

Hint: Don't forget squares whose sides aren't horizontal or vertical lines!

Preview Solution

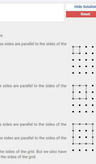
Hide Solution

Reset

Your Solution

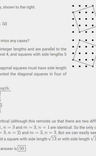
Solution: We can make each case be counting squares of a particular size.

Case 1: 1×1 "horizontal" squares There are sixteen 1×1 squares whose sides are parallel to the sides of the grid, as shown to the right.



Case 2: 2×2 "horizontal" squares There are nine 2×2 squares whose sides are parallel to the sides of the grid, as shown to the right.

Case 3: 3×3 "horizontal" squares There are four 3×3 squares whose sides are parallel to the sides of the grid, as shown to the right.



Case 4: 4×4 "horizontal" squares There is only one 4×4 square whose sides are parallel to the sides of the grid, namely the border of the entire grid.

This gives $16 + 9 + 4 + 1 = 30$ squares whose sides are parallel to the sides of the grid. But we also have squares which are "diagonal," meaning that their sides are not parallel to the sides of the grid.

Case 5: $\sqrt{2} \times \sqrt{2}$ "diagonal" squares There are 9 $\sqrt{2} \times \sqrt{2}$ squares, as shown to the right.

Case 6: $\sqrt{5} \times \sqrt{5}$ "diagonal" squares There are 8 $\sqrt{5} \times \sqrt{5}$ squares, as shown to the right. Note that these squares come in two different orientations.

Case 7: $\sqrt{10} \times \sqrt{10}$ "diagonal" squares There is only 1 $\sqrt{10} \times \sqrt{10}$ square—it is shown at right.

Case 8: $\sqrt{13} \times \sqrt{13}$ "diagonal" squares There are 2 $\sqrt{13} \times \sqrt{13}$ squares, shown to the right.

This gives a total of

$$16 + 9 + 4 + 1 + 9 + 8 + 1 + 2 = 50$$

squares in the grid.

This seems like all of the squares, but how can we be sure that we didn't miss any cases?

We can be certain that we have all of the squares, because we have listed every integer lengths and are parallel to the sides of the grid. We counted all the squares with side lengths 1, 2, 3, and 4, and squares with side lengths 5 and 10 are impossible, because the hypotenuse of a right triangle with two catheti one, two and five units long is $\sqrt{29}$ units long.

Two catheti one, two units long

We can also see that the Pythagorean Theorem, that all of the diagonal squares must have side length $\sqrt{n}$ for some n, and is one positive integer. We can also see that the diagonal squares in four of these cases:

m	n	Side Length
1	2	$\sqrt{2}$
1	5	$\sqrt{5}$
1	10	$\sqrt{10}$
2	3	$\sqrt{13}$

Note that the case $m = 1, n = 2$ and the case $m = 2, n = 1$ are identical (although this reminds us that there are two different orientations of squares with side length $\sqrt{5}$). Similarly the cases $m = 1, n = 3$ and $m = 3, n = 1$ are identical. So the only cases that we might have missed are $m = 2, n = 2$ (which would be a square with side length $\sqrt{8}$) and $m = 3, n = 3$. But we can easily see that such squares will not fit into the grid: there is no way to insert into the grid a square with side length $\sqrt{8}$ or with side length $\sqrt{18}$.

Thus we're accounted for all the cases, and counted all the squares; the answer is $\boxed{50}$.

14.4 Counting Pairs

Problems

Problem 14.17 Jump to Solution

A round-robin tennis tournament consists of each player playing every other player exactly once. How many matches will be held during an 8-person round-robin tennis tournament?

Problem 14.18 Jump to Solution

A round-robin tennis tournament consists of each player playing every other player exactly once. Find a formula for the number of matches that will be held during a n-person round-robin tennis tournament, where $n \geq 2$.

Problem 14.19 Jump to Solution

Five women and four men are at a party.

(a) If each person shakes hands with each other person, then how many total handshakes are there?

(b) If no two men shake hands, but each woman shakes hands with each other person, then how many total handshakes are there?

Problem 14.20 Jump to Solution

(a) How many diagonals does a triangle have?

(b) How many diagonals does a quadrilateral have?

(c) How many diagonals does a pentagon have?

(d) How many diagonals does a hexagon have?

(e) Find a formula for the number of diagonals in a polygon with n sides, where n is an integer greater than 2.

Problem 14.17

A round-robin tennis tournament consists of each player playing every other player exactly once. How many matches will be held during an 8-person round-robin tennis tournament?

Solution for Problem 14.17: Say you're one of the players. How many matches will you play?

Each player plays 7 matches, one against each of the other 7 players.

So what's wrong with the following reasoning?

> **Bogus Solution:** Each of the eight players plays 7 games, so there are $8 \cdot 7 = 56$ total games played.

Suppose two of the players are Alice and Bob. Among Alice's 7 matches is a match against Bob. Among Bob's 7 matches is a match against Alice. When we count the total number of matches as $8 \cdot 7$, the match between Alice and Bob is counted twice, once for Alice and once for Bob.

Therefore, since $8 \cdot 7 = 56$ counts each match twice, we must divide this total by 2 to get the total number of matches. Hence the number of matches in an 8-player round-robin tournament is $\frac{8 \cdot 7}{2} = 28$.

Here's another way to solve Problem 14.17. We'll go one player at a time, and keep a running count of the number of matches involving those players.

Let's start with Alice. We already know that Alice plays 7 matches.

Next, we move on to Bob. We also know that Bob plays 7 matches, but we've already counted the match that he plays with Alice. So Bob plays in 6 more matches that we haven't already counted. Therefore, our running total~the number of matches involving Alice or Bob~is $7 + 6 = 13$.

Then, we go to the next player, Carol. We know that Carol plays 7 matches, but we've already counted two of them: the one between Carol and Alice, and the one between Carol and Bob. So Carol plays in 5 more matches that we haven't already counted. Therefore, our running total~the number of matches involving Alice, Bob, or Carol~is $7 + 6 + 5 = 18$.

If we keep going in this manner, we'll eventually arrive at a total of

$$7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$$

matches.

Problem 14.17 is an example of counting pairs of objects~in this case, pairs of players in a tennis tournament. Now that we know what happens with 8 players, let's take a look at what happens when there are n players. This is an example of **generalizing** a problem: we have solved the problem for a specific number, 8, and now we're going to find a formula that will work for any positive integer number of players.

Problem 14.18

A round-robin tennis tournament consists of each player playing every other player exactly once. Find a formula for the number of matches that will be held during a n-person round-robin tennis tournament, where $n \geq 2$.

Solution for Problem 14.18: If we proceed as in the solution to Problem 14.17, we see that each of the n players must play every other player, so each player must play $n - 1$ matches. This leads to a preliminary count of $n(n-1)$ matches.

But this method counts each match twice, once for each player. Thus we must divide by 2, to get the answer $\frac{n(n-1)}{2}$.

Another way to approach the general tournament problem in Problem 14.18 is as follows:

Number the players from 1 through n.

Player 1 plays players $2, 3, \ldots, n$ for a total of $n - 1$ matches.

Player 2 plays players $1, 3, 4, \ldots, n$. But we've already counted the match between players 1 and 2. So we only need to add on the matches between player 2 and players $3, 4, \ldots, n$, for another $n - 2$ matches.

Similarly, for player 3, we've already counted the matches against players 1 and 2, so we need to add on the matches between player 3 and players $4, 5, \ldots, n$, for another $n - 3$ matches.

And so on.

So the total number of matches is the sum of our counts above, which is

$$(n-1) + (n-2) + (n-3) + \cdots + 2 + 1.$$

Since this must match the count that we had before, we've just shown that

$$(n-1) + (n-2) + (n-3) + \cdots + 2 + 1 = \frac{n(n-1)}{2}.$$

Our work above gives us a formula for the sum of the first k positive integers! If we let $n = k + 1$ in the equation we just found (and reverse the order of the sum so our sum goes from 1 to k), then we get

$$1 + 2 + \cdots + (k-2) + (k-1) + k = \frac{(k+1)(k)}{2}.$$

So, our counting explanations have given us an algebra formula!

> **Concept:** We can sometimes use counting explanations to find algebraic relationships.

We also could have used algebra to find a formula for the sum of the first k positive integers. We write

$$S = 1 + 2 + \cdots + (k-1) + k.$$

We can, of course, just as easily write S in reverse order:

$$S = k + (k-1) + \cdots + 2 + 1.$$

This may seems like a useless thing to do, but watch what happens when we add them together:

$$\begin{array}{rcccccccccc} S & = & 1 & + & 2 & + & \cdots & + & k-1 & + & k \\ + \ S & = & k & + & k-1 & + & \cdots & + & 2 & + & 1 \\ \hline 2S & = & k+1 & + & k+1 & + & \cdots & + & k+1 & + & k+1 \end{array}$$

The last line has k copies of $k + 1$, and hence

$$2S = k(k+1).$$

So, we divide by 2 to get $S = \frac{k(k+1)}{2}$.

> **Important:** For any positive integer k,
>
> $$1 + 2 + \cdots + (k-1) + k = \frac{k(k+1)}{2}.$$

> **Sidenote:**
>
> Legend has it that the famous German mathematician Carl Friedrich Gauss (1777~1855) once used the above formula to very quickly compute the sum of the integers from 1 to 100. What is remarkable about this is that Gauss was only 7 years old at the time! His schoolmaster gave his class, as an exercise, the assignment to calculate
>
> $$1 + 2 + 3 + \cdots + 99 + 100,$$
>
> figuring that it would take the students many minutes. The teacher was quite surprised when young Gauss came up with the answer almost immediately! Gauss, even at age 7, recognized that there was a quick way to add consecutive integers, and used the same process that we used to prove our formula.

In some problems, we don't necessarily want to count all pairs, but just some of them.

Problem 14.19

Five women and four men are at a party.

(a) If each person shakes hands with each other person, then how many total handshakes are there?

(b) If no two men shake hands, but each woman shakes hands with each other person, then how many total handshakes are there?

Solution for Problem 14.19:

(a) There are 9 people total, and each must shake hands with each other person. So, each person shakes hands 8 times. This gives us a preliminary count of $9 \cdot 8$ handshakes. But this counts each handshake twice, once for each person in the handshake. So, we must divide by 2 to get a total of $9 \cdot 8/2 = 36$ handshakes. (Notice that this problem is essentially the same as the round-robin tournament problem where we've replaced "matches" with "handshakes.")

(b) What's wrong with this solution:

> **Bogus Solution:** Each woman shakes hands with 8 other people, and there are 5 women, for a preliminary total of $8 \cdot 5 = 40$ handshakes. But this counts each handshake twice, once for each person in the handshake. So, we must divide by 2 to get a total of $40/2 = 20$ handshakes.

The problem here is that the count $8 \cdot 5$ does not count each handshake twice! A handshake between two women is counted twice, but a handshake between a man and a woman is counted only once, just for the woman.

> **WARNING!** When dividing by 2 in a counting problem to correct for counting each item twice, make sure you really were counting each item twice!

We have to handle the handshakes between two women separately from handshakes between a man and a woman. Each of the 5 women shakes hands with 4 other women, for a preliminary count of $5 \cdot 4$ handshakes. Suppose Alice and Bob are two of the women. Our count of 5 counts the handshake between Alice and Bob twice~once among Alice's 4 handshakes with other women, and once among Bob's 4 handshakes with other women. So, here we have to divide by 2, since $5 \cdot 4$ counts each handshake twice, once for each woman. This gives us $5 \cdot 4/2 = 10$ handshakes between two women.

Turning to handshakes between a man and a woman, each of the women shakes hands with 4 men, for a preliminary count of $5 \cdot 4 = 20$ handshakes. Suppose Alice is one of the women and Carl is one of the men. The handshake between Alice and Carl is only counted once, just among Alice's 4 handshakes with men. So, we don't divide the preliminary count by 2, because each handshake between a woman and a man is only counted once. There are 20 handshakes between a woman and a man.

Combining the handshakes between two women with those between a woman and a man, we have $10 + 20 = 30$ handshakes.

Problem 14.20

(a) How many diagonals does a triangle have?

(b) How many diagonals does a quadrilateral have?

(c) How many diagonals does a pentagon have?

(d) How many diagonals does a hexagon have?

(e) Find a formula for the number of diagonals in a polygon with n sides, where n is an integer greater than 2.

Solution for Problem 14.20:

(a) A triangle has 0 diagonals.

(b) A quadrilateral has 2 diagonals.

(c) As shown at the left below, a pentagon has 5 diagonals.

A Pentagon and Its Diagonals

A Hexagon and Its Diagonals

(d) A hexagon and its diagonals are shown at the right above. Rather than counting the diagonals directly, we count the diagonals by thinking about each vertex. Each vertex can connect to the other 5 vertices, but 2 of these connections are edges, not diagonals. So each vertex has $5 - 2 = 3$ diagonals. (This is a bit like the handshaking problem! Thinking of a diagonal as a handshake between two vertices, each vertex "shakes hands" with 3 other vertices.)

There are 6 vertices of a hexagon and each connects to 3 vertices by a diagonal, so it appears that there are $6 \cdot 3 = 18$ diagonals. However, this counts each diagonal twice, once for each vertex at its endpoints. To correct for this, we must divide by 2, giving $18/2 = 9$ diagonals.

(e) We can use our process from part (d) for any polygon. A polygon with n sides has n vertices. As in part (d), each vertex can connect to the other $n - 1$ vertices, but 2 of these connections are edges, not diagonals. So each vertex has $(n-1) - 2 = n - 3$ diagonals.

Since each of the n vertices has $n - 3$ diagonals, it appears that there are $n(n-3)$ diagonals. But again, this counts each diagonal twice, once for each vertex at its endpoints. So, we correct our count by dividing by 2, giving $n(n-3)/2$ diagonals for a polygon with n sides. Checking, we see that this formula does give us the values we found in the first four parts.

Counting Pairs

Exercises

14.4.1: Source: MOEMS

There are five girls in a tennis class. How many different doubles teams of two girls each can be formed from the students in the class?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There are 5 ways to choose the first girl for a team and 4 ways to choose the second girl. This gives us $5 \cdot 4$ ways to choose the two girls, but this counts each possible team twice, once for each order in which we choose the girls. Therefore, there are $(5 \cdot 4)/2 = \boxed{10}$ different doubles teams possible.

14.4.2: Source: MOEMS

A club has 12 members and needs to choose 2 members to be co-presidents. How many different pairs of co-presidents are possible?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: There are 12 ways to choose the first person to be a co-president, and then 11 ways to choose the second person. This gives us a total of $12 \cdot 11$ ways to choose the two people, but this counts each pair of co-presidents twice, once for each order of choosing them. Therefore, the possible number of pairs of co-presidents is $(12 \cdot 11)/2 = \boxed{66}$.

14.4.3: Source: MOEMS

Two students are needed to work in the school store during the lunch hour every day and four students volunteer for this work. What is the greatest number of days that can be arranged in which no pair of the four students work together more than once?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The greatest number of days for which no pair works together more than once occurs when each student works with each other student exactly once. So, we need to count the number of different pairs of students we can form from a group of 4 students. There are 4 students to choose as the first student and 3 left to choose as the second. This gives us a total of $(4 \cdot 3)/2 = \boxed{6}$ pairs. We divide by 2 because $4 \cdot 3$ counts each pair twice, once for each order in which we can pick the students in a given pair.

14.4.4: Source: MOEMS

Six people participated in a checkers tournament. Each participant played exactly three games with each of the other participants. How many games were played in all?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Each person plays 3 other people 3 times, so each of the 6 people plays $5 \cdot 3 = 15$ games. In multiplying $6 \cdot 15$, each game is counted twice, once for each player in the game, so we divide by two to get the answer of $(6 \cdot 15)/2 = \boxed{45}$ games.

14.4.5: Source: MOEMS

A sports conference has 12 teams in two divisions of 6. How many games are in a complete season for the conference if each team must play every other team in its own division twice and every team in the other division once?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Each team plays the other 5 teams in its division twice and each of the 6 teams in the other division once. This gives a total of $5 \cdot 2 + 6 \cdot 1 = 16$ games played by each team. There are 12 teams total, and each plays 16 games. In multiplying $12 \cdot 16$, each game is counted twice, once for each team in the game, so we divide by two to get the answer of $(12 \cdot 16)/2 = \boxed{96}$ games.

14.4.6a:

Find a formula for the sum of the first n even integers: $2 + 4 + 6 + \cdots + 2n$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We'll try the same strategy we used in the book to compute the sum of the first n positive integers. We let S equal the sum of the first n even integers, and then write the sum both forwards and backwards. We then add the two sums:

$$\begin{array}{rcccccccccc} S & = & 2 & + & 4 & + & \cdots & + & 2(n-1) & + & 2n \\ + \ S & = & 2n & + & 2(n-1) & + & \cdots & + & 4 & + & 2 \\ \hline 2S & = & 2(n+1) & + & 2(n+1) & + & \cdots & + & 2(n+1) & + & 2(n+1) \end{array}$$

The last line has n copies of $2(n+1)$, and hence $2S = n(2(n+1))$, so $2S = 2n(n+1)$. We divide by 2 to get $S = \boxed{n(n+1)}$.

Notice that our final result is two times our formula for the sum $1 + 2 + 3 + \cdots + n$. Watch what happens when we multiply this sum by 2:

$$2(1 + 2 + 3 + \cdots + n) = 2 + 4 + 6 + \cdots + 2n.$$

So, the sum of the first n positive even integers is indeed twice the sum of the first n positive integers.

14.4.7a:

How many interior diagonals does an icosahedron have? (An **icosahedron** is a 3-dimensional figure with 20 triangular faces and 12 vertices, with 5 faces meeting at each vertex. An **interior diagonal** is a segment connecting two vertices that do not lie on a common face.)

Hint: Consider a simpler problem; how did we solve the problem about polygon diagonals in the text?

Hint: Each vertex is connected to how many other vertices by an edge of the icosahedron?

Hint: Each vertex is connected to how many other vertices by an interior diagonal of the icosahedron?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There are 12 vertices in the icosahedron, so from each vertex there are potentially 11 other vertices to which we could extend a diagonal. However, 5 of these 11 points are connected to the original point by an edge of the icosahedron, so they are not connected by interior diagonals. So each vertex is connected to 6 other points by interior diagonals. This gives a preliminary count of $12 \cdot 6 = 72$ interior diagonals. However, we have counted each diagonal twice (once for each of its endpoints), so we must divide by 2 to correct for this overcounting, and the answer is $\frac{12 \cdot 6}{2} = \boxed{36}$ diagonals.

14.5 Probability

Now we turn from counting to **probability**. Unfortunately, while even most three-year-olds know what "counting" means, the concept of probability is a bit difficult to describe precisely.

Let's start with an example. Suppose we have a coin, some dice, a deck of cards, and a bag with some balls in it. We flip the coin over and over and over again, we expect that about half the time we'll flip heads and about half the time we'll flip tails. We express this mathematically by saying that the **probability** that a coin flip will turn up heads is $\frac{1}{2}$.

This is what we mean by probability: if an outcome O, the fraction of times that we expect an outcome to occur if we perform the experiment over and over. Notice that we can't say the exact fraction of times we have success. This is the difficulty in defining probability. Probability only makes sense if we are trying to measure an event that is not definite.

In basic examples, we'll compute probability by counting the number of equally likely outcomes, then counting how many of these outcomes are "successes". The probability of success is the ratio of the number of successful outcomes to the total number of possible outcomes. For instance, in our coin-flipping example above, there are 2 possible outcomes (heads and tails), and 1 successful outcome (heads), so the probability of flipping heads is $\frac{1}{2}$.

But we have a very important caveat:

Important: We must be sure that each possible outcome is equally likely.

Problems

Problem 14.21 Jump to Solution

The faces of a fair 6-sided die are numbered $\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6}$. ("Fair" means that each side is equally likely to be rolled.) What is the probability that when the die is rolled, a $\boxed{5}$ is facing up?

Problem 14.22 Jump to Solution

What is the probability that when a fair 6-sided die is rolled, an odd number faces up?

Problem 14.23 Jump to Solution

What is the greatest possible probability for any event? What is the least possible probability for any event?

Problem 14.24 Jump to Solution

A standard deck of cards has 52 cards divided into 4 suits, each of which has 13 cards. Two of the suits (♡ and ♢, called "hearts" and "diamonds") are red, and the other two (♠ and ♣, called "spades" and "clubs") are black. The cards in the deck are placed in random order (usually by a process called "shuffling"). What is the probability that the top two cards are both red?

Problem 14.25 Jump to Solution

Suppose we have 2 fair 6-sided dice numbered from $\boxed{1}$ to $\boxed{6}$. If we roll them both, then what is the probability that the two numbers shown sum to 7?

Let's start with a basic example.

Problem 14.21

The faces of a fair 6-sided die are numbered $\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6}$. ("Fair" means that each side is equally likely to be rolled.) What is the probability that when the die is rolled, a $\boxed{5}$ is facing up?

Solution for Problem 14.21: When we roll the die, there are 6 equally likely outcomes: $\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6}$, or $\boxed{6}$. In this problem, there is only one "successful" outcome, namely when a $\boxed{5}$ comes up. Therefore, the probability that a $\boxed{5}$ is rolled is the 1 successful outcome divided by the total 6 equally likely outcomes, or $\frac{1}{6}$.

As we discussed in the introduction to this section, the way to interpret this probability is that if we were to roll our die many, many times, we would expect that about $\frac{1}{6}$ of our rolls would come up $\boxed{5}$.

In Problem 14.21 we saw the basic way to calculate probabilities.

Concept: If all outcomes are **equally likely**, then the probability of success is

$$\frac{\text{Number of successful outcomes}}{\text{Number of possible outcomes}}$$

Here's another example with our trusty die:

Problem 14.32 Jump to Solution

What is the probability that when a fair 6-sided die is rolled, an odd number faces up?

Solution for Problem 14.22: In this problem, as in Problem 14.21, there are 6 equally likely outcomes. Three of those outcomes are successful: $\boxed{1}, \boxed{3}, \boxed{5}$. Therefore, the probability is $\frac{3}{6} = \frac{1}{2}$.

Now that we've done a couple of basic probability problems, let's ask some important general questions about probability.

Problem 14.33 Jump to Solution

What is the greatest possible probability for any event? What is the least possible probability for any event?

Solution for Problem 14.23: Since the probability of an event occurring is essentially the proportion of times that we expect the event to occur, the largest the probability could be is 1. This occurs when the event happens every time.

Another way to look at this is to recall that for any event, we have

$$\text{Probability of success} = \frac{\text{Number of successful outcomes}}{\text{Number of possible outcomes}}$$

Clearly the number of successful outcomes cannot be greater than the number of possible outcomes, so the probability must be no greater than 1. The smallest it can be is 0 (no successful outcomes) when the number of successful outcomes equals the number of possible outcomes, and the probability is 1. For example, if the event is "a single die roll is less than 10," then the probability is 1, since there are 6 successful outcomes out of 6 possible outcomes.

Similarly, the smallest the probability could ever be is 0. This occurs when the event cannot happen, so the number of successful outcomes is 0. For example, if the event is "a single die roll is 7," then the probability is 0, because there are 0 successful outcomes out of 6 possible outcomes.

Important: When working on a probability problem, if you ever get a probability that's less than 0 or greater than 1, then you've made a mistake! All probabilities are greater than or equal to 0, and less than or equal to 1.

Let's try a slightly more complicated problem.

Problem 14.34 Jump to Solution

Suppose we have 2 fair 6-sided dice numbered from $\boxed{1}$ to $\boxed{6}$. If we roll them both, then what is the probability the two numbers shown sum to 7?

Solution for Problem 14.25: We might incorrectly think like this:

Bogus Solution: Any number from 2 to 12 could occur, so there are 11 possibilities. 7 is one of those, so the probability is $\frac{1}{11}$.

The error here is that the 11 possibilities are **not equally likely**. For example, a sum of 7 is more likely than rolling a sum of 2. In order to correctly compute probability, we need to count equally likely outcomes, and then determine how many of those possibilities result in a sum of 7. But how can we count equally likely outcomes? Why is the sum of two dice?

There are 6 equally likely outcomes for each die, so there are $6 \cdot 6 = 36$ total equally likely outcomes for both die together. It may be easier to see this if you think of one of the die as being red and the other die as being white. There are 6 outcomes for the black die and 6 outcomes for the white die, for a total of $6 \cdot 6 = 36$ outcomes for the pair of dice. The outcomes that sum to 7 are $\boxed{1}\boxed{6}, \boxed{2}\boxed{5}, \boxed{3}\boxed{4}, \boxed{4}\boxed{3}, \boxed{5}\boxed{2}$, and $\boxed{6}\boxed{1}$, for a total of 6 equally likely successful outcomes. Therefore, the probability of rolling a sum of 7 is $\frac{6}{36} = \frac{1}{6}$.

We can see all of the possible equally likely outcomes for the two dice in the following table:

		Die #1					
		■	■	■	■	■	■
Die #2	□	1	2	3	4	5	6
	□	2	3	4	5	6	7
	□	3	4	5	6	7	8
	□	4	5	6	7	8	9
	□	5	6	7	8	9	10
	□	6	7	8	9	10	11
	□	7	8	9	10	11	12

Notice that 6 of the entries in the above table are 7, and there are 36 total entries in the table, so the probability of rolling a 7 is $\frac{6}{36} = \frac{1}{6}$.

Probability and Equally Likely Events

Exercises

14.5.1:

Suppose we roll a fair 6-sided die. What is the probability that

(a) $\boxed{4}$ is rolled?

Hide Solution

Edit

Your Solution

(b) an even number is rolled?

Hide Solution

Edit

Your Solution

(c) a perfect square is rolled?

Hide Solution

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Your Solution

14.5.2:

Suppose that we have an 8-sided die with 2 red sides, 3 yellow sides, and a blue side. What is the probability of rolling a yellow side?

Hide Solution

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Your Solution

14.5.3:

Consider a standard deck of 52 cards (as described in Problem 14.24). The deck is randomly arranged. What is the probability that

(a) the top card is ♠?

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Your Solution

(b) the top card is ♥?

Hide Solution

Edit

Your Solution

(c) the top card is a face card (a Jack, Queen, or King)?

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Your Solution

(d) the top card is a 3?

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Your Solution

(e) the top card is the 2 of ♡?

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Your Solution

(f) the top card is a red card?

Hide Solution

Edit

Your Solution

(g) the top card is a black card?

Hide Solution

Edit

Your Solution

14.5.4:

Suppose we flip four coins simultaneously: a penny, a nickel, a dime, and a quarter. What is the probability that

(a) they all come up heads?

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Your Solution

(b) the penny and nickel both come up heads?

Hide Solution

Edit

Your Solution

14.5.5:

Suppose that we roll two fair 6-sided dice. What is the probability that the two numbers rolled sum to 9?

Hide Solution

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Your Solution

14.5.6: Source: AIME I

Two dice are thrown. What is the probability that the product of the two numbers is a multiple of 5?

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Your Solution

14.5.7a: Source: AIME I

A pair of 6-sided dice has the sides of each die numbered 1 through 6. Each side has the same probability of landing face up. What is the probability that the product of the two numbers on the sides that land face up is greater than 20?

Hide Solution

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Your Solution

14.5.8a: Source: AIME I

At a party, there are only single women and married men with their wives. The probability that a randomly selected woman is single is $\frac{3}{7}$. What Fraction of the people in the room are married men?

Hide Solution

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Your Solution

14.6 Summary

In this chapter, we learned how to use arithmetic to solve counting problems.

> **Important:** Don't just blindly add, subtract, multiply, and divide–think about what you're doing!

Counting a list of numbers like $1, 2, 3, \ldots, 38$ is easy! Counting many other lists of numbers is easy too: use arithmetic to convert it into a list that looks like 1,2,3,. . .. The number of items in the list $a, a+1, \ldots, b-1, b$, where $a \leq b$ are integers, is $b - a + 1$.

> **Concept:** When presented with a complicated problem, try to reduce or simplify it to a simpler problem that you know how to solve.

Counting items which lie in one or more overlapping sets requires a thoughtful use of addition and subtraction. We can use a Venn diagram to help keep track of the thought process. When using a Venn diagram, we should initially try to fill in numbers that correspond exactly to a single region of the diagram. We may have to place variables in our diagram: if so, it's usually wise to choose our variable to represent the answer to the problem.

> **Concept:** When assigning a variable, it's usually best to let your variable represent the answer to the problem.

When counting multiple independent events, we multiply: the number of ways that A **and** B can occur is

$$\begin{aligned}&(\text{the number of ways } A \text{ can occur})\\ &\quad \textbf{times } (\text{the number of ways } B \text{ can occur}).\end{aligned}$$

Sometimes our later choices depend on our earlier choices, so our counting must take that into consideration.

Casework can often help us break a counting problem into more manageable pieces. We split the possibilities into exclusive cases, meaning that every outcome shows up in exactly one case. Then, to count the total, you simply add the counts for each case. Casework requires you to be organized and careful, and to make sure that you don't omit any outcomes in your cases.

> **Concept:** When faced with a series of independent choices, one after the other, we **multiply** the number of options at each step. When faced with exclusive options (meaning we can't choose more than one), we **add** the number of options.

To count pairs of items, we first count the number of ways to select 2 items from our group, and then divide by 2 because we've counted each pair twice (once in each order). In general, if we have n items, then there are $\dfrac{n(n-1)}{2}$ pairs of items.

> **Important:** For any positive integer n,
>
> $$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

If all outcomes are *equally likely*, then the probability of success is

$$\frac{\text{Number of successful outcomes}}{\text{Number of possible outcomes}}.$$

It is very important that all of the outcomes are equally likely!

When working on a probability problem, if you ever get a probability that's less than 0 or greater than 1, then you've made a mistake! All probabilities are greater than or equal to 0, and less than or equal to 1.

Most probability problems are just two counting problems: counting the number of total outcomes and counting the number of successful outcomes. So we can take advantage of all of the counting techniques that we've learned to solve probability problems as well.

Challenge Problems

14.43:

Which integers is satisfy $\frac{1}{2} > \frac{1}{x} > \frac{1}{10}$, and how much such integers are there?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We must have $\frac{1}{x} > \frac{1}{10}$ and $\frac{1}{x} < \frac{1}{2}$. We'll tackle $\frac{1}{x} > \frac{1}{10}$ first. Since is positive, we can multiply both sides by 2 and by x to get $x > 2$. Next, we tackle $\frac{1}{x} < \frac{1}{2}$. We multiply both sides by 100 and by x, and we have $100 > 5x$. Dividing by 5 gives $100/5 > x$. Combining $x > 2$ and $\frac{100}{5} > x$, we have $\frac{100}{5} > x > 2$. Since $\frac{100}{5} = 20\frac{1}{5}$, the integers is that satisfy $\frac{100}{5} > x > 2$ are $\{3, 4, 5, \ldots, 20\}$. There are $20 - 3 + 1 = \boxed{18}$ such integers.

14.44:

My classroom has 11 rows of chairs, with 11 chairs in each row. The chairs in each row are numbered from 1 to 11.

(a) How many chairs have odd numbers?

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Each row has odd-numbered chairs $1, 3, 5, 7, 9, 11$, for a total of 6 odd-numbered chairs in each row. Since there are 11 rows, there are a total of $11 \cdot 6 = \boxed{66}$ chairs with odd numbers.

(b)* Suppose we replaced all of the occurrences of "11" in this problem with n, where n is an odd number. Can you find a formula in terms of n for the number of chairs with odd numbers? Is the formula different if n is even?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: If n is odd, each row has odd-numbered chairs $1, 3, 5, \ldots, n$. n. Adding 1 to each number in this list and dividing each result by 2, we get $1, 2, 3, \ldots, \frac{n-1}{2}, \frac{n+1}{2}$. So there are $\frac{n+1}{2}$ odd-numbered chairs in each row and n rows, for a total of $\boxed{\frac{n(n+1)}{2}}$.

If n is even, each row has odd-numbered chairs $1, 3, 5, \ldots, n - 1, n$. Adding 1 to each number in this list (to make the numbers even) and then dividing each result by 2 gives $1, 2, 3, \ldots, \frac{n-2}{2}, \frac{n}{2}$. So there are $\frac{n}{2}$ odd-numbered chairs in each row and n rows for a total of $\boxed{\frac{n^2}{2}}$.

14.45:

There are 7 married couples at a party. At the start of the party, every person shakes hands once with every other person except his or her spouse. How many handshakes are there?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: All 14 people shake hands with 12 other people (everyone except themselves and their spouse). In multiplying $14 \cdot 12$ each handshake is counted twice, so we divide by two to get the answer of $\frac{14 \cdot 12}{2} = \boxed{84}$ handshakes.

14.46:

When writing the numbers from 1 to 500, how many times will you write the digit 3?

Hint: Can you split the problem into cases you can handle?

Hint: How many times does 3 appear as the tens digit?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We consider the units digit, the tens digit, and the hundreds digit separately. For simplicity in our computation below, let's assume that we write every number using three digits, including leading zeroes. (It's necessary (so that we'll write 22 as "022"). Doing this doesn't change the number of 3's that we write, so it doesn't affect our answer.

Units digit: We count the number 3 in the units digit. There are 5 choices for the hundreds digit (0, 1, 2, 3, 4), and 10 choices for the tens digit, for a total of $5 \cdot 10 = 50$ numbers with 3 as the units digit. Of the hundreds digit is 3, the resulting number is a one-digit or two-digit number). Therefore, there are 10 3's in the units place.

Alternatively, we could have noted that we write one 3 in the units place for each block of 10 numbers, for a total of $500/10 = 50$ 3's in the units place.

Tens digit: We count the numbers with 3 as the tens digit. After placing the 3 in the tens place, there are 5 ways to choose the hundreds digit and 10 ways to choose the units digit. So, there are $5 \cdot 10 = 50$ numbers with 3 in the tens place. Therefore, there are 50 3's in the tens place.

Alternatively, we might note that we write the 3 in the tens digit 10 times in each of the 30s, the 130s, the 230s, the 330s, and the 430s.

Hundreds digit: All 100 numbers from 300 to 399 have a 3 in the hundreds place, so there are 100 3's in the hundreds place.

Combining these cases gives a total of $50 + 50 + 100 = \boxed{200}$ 3's.

14.47:

In a sports league, there are 20 total teams, divided into 4 divisions of 5 teams each. Over the course of a season, each team plays each of the other teams in its own division 3 times, and each of the other teams in the other divisions twice. How many games does the league have in a complete season?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Each team plays the four other teams within its division three times, for a total of 12 intra-division games. Each team plays the other 15 teams twice for a total of 30 inter-division games. This gives a total of $12 + 30 = 42$ games per team. However, $20 \cdot 42$ counts every game twice (once for each of the opposing teams) so we must divide by two. The answer is $\frac{20 \cdot 42}{2} = \boxed{420}$.

14.48:

Source: AMC 8 ✎

Keiko tosses one penny and Ephraim tosses two pennies. What is the probability that Ephraim gets the same number of heads that Keiko gets?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Each flip has 2 possible outcomes, so there are $2 \cdot 2 \cdot 2 = 8$ different possible outcomes for all three flips. There are two possible cases in which they could have the same number of heads: they could each get 0 heads, or they could each get 1 head.

Case 1: Both get 0 heads. The only way this can happen is if all three flips are tails. There is only 1 way this can happen.

Case 2: Both get 1 head. Keiko's flip must be heads, and Ephraim's flips can be heads then tails, or tails then heads. So, there are 2 possibilities in this case.

Combining both cases, we see that there are 3 ways for the two to get the same number of heads, so the desired probability is $\boxed{\frac{3}{8}}$.

14.49:

Source: AMC 8 ✎

If two dice are tossed, what is the probability that the product of the numbers showing on the tops of the dice is greater than 10?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: There are $6 \cdot 6 = 36$ possible rolls of the two dice. It is impossible for the dice to have a product greater than 10 if either die is $\boxed{1}$. When one die is $\boxed{1}$, the only way the product is greater than 10 is if the other is $\boxed{2}$. Either the could be $\boxed{1}$, so there are 2 ways to do this.

Otherwise, if neither die is $\boxed{1}$ or $\boxed{2}$, then they each must show $\boxed{3}$, $\boxed{4}$, $\boxed{5}$, or $\boxed{6}$ and there are $4 \cdot 4 = 16$ ways for this to occur. Of these 16 possibilities, only $\boxed{4}$ results in a product greater than 10. So, there are 19 ways to produce a product greater than 10 when both dice are greater than $\boxed{2}$.

This gives us $2 + 16 = 17$ total rolls out of 36 in which the product of the dice is greater than 10, so the probability is $\boxed{\frac{17}{36}}$.

14.50:

Source: MATHCOUNTS ✎

Pace uses a spinner to select a number from 1 through 3 inclusive, each with equal probability. Mary uses a different spinner to select a number from 1 through 10 inclusive, each with equal probability. What is the probability that the product of their numbers is less than 10?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: If Pace's first number is 1 or 2, then any of the 10 numbers can appear on the second spinner, so this gives 10 successful spins. If Pace's first number is 3, then Mary's number must be less than $10/3 = 3\frac{1}{3}$. So, Mary's number must be 1, 2, or 3. If Pace's number is 3, then the second spinner must show 1 through 3 (inclusive), so there are 3 more successful spins. Finally, if Pace's first number is 1, then the second spinner must show 1 through 9 (inclusive). So there are a total of $10 + 10 + 9 + 3 + 3 = 41$ successful spins, and $3 \cdot 10 = 30$ possible spins, giving a probability of $\boxed{\frac{41}{30}}$.

14.51:

Source: MOEMS ✎

In Park School's 8th grade, 33 students like volleyball, 34 like soccer, 18 like basketball, 20 like volleyball and soccer, 13 like volleyball and basketball, 8 like soccer and basketball, 3 like all three sports, and 12 like none of these sports. How many students are in Park School's 8th grade?

Hint: This sure looks like a Venn diagram problem, but there are three categories!

Hint: So, you should start with three circles!

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We build a Venn diagram, but this time, we need three circles, one for each sport. We place them so that each pair overlaps, and so that there is a region in which all three overlap. As we often do with 2-circle Venn diagrams, we work from the inside out. Since there are 3 students who like all three sports, we place a 3 in the center.

Subtracting these 3 from the 10 who like volleyball and basketball gives us 7 who like volleyball and basketball but not soccer. We place 7 in the region inside the volleyball and basketball circles, but outside the soccer circle. Similarly, we place $20 - 3 = 17$ in the region where only volleyball and soccer overlap, and we place $8 - 3 = 5$ in the region where only soccer and basketball overlap.

Next, we turn to the students who only like basketball. We have 39 total who like basketball. We have 7, 3, and 5, accounting for the three regions basketball shares with other sports, so that leaves $39 - 7 - 3 - 5 = 24$ who only like basketball. We place 24 in the basketball-only region. Similarly, there are $34 - 5 - 3 - 17 = 9$ who only like soccer and $33 - 17 - 3 - 7 = 6$ who only like volleyball. Finally, we place 12 outside all 3 circles for the students who don't like any of the sports.

Adding all of the numbers in the diagram, we have

$$12 + 6 + 24 + 9 + 7 + 5 + 17 + 3 = \boxed{83}$$

students.

14.52*:

In my fencing finals, there are twice as many boys who are in high school as there are boys who are in middle school. Of the students in the role who are in middle school, the boys in middle school are twice as many girls as there are boys. Half of the girls who are in the club are in high school. If there are 72 people in my fencing club, how many are middle school boys?

Hint: Assign a variable.

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We draw a Venn diagram as shown at the right. We have one circle for boys and one for middle school. Let there be $2x$ boys in middle school, so we place x in the overlap region as shown.

Since there are twice as many boys in high school as in middle school, there are $2x$ boys in high school. We place $2x$ in the region inside the boys circle but not the middle school circle.

There are three times as many middle school girls as middle school boys, so there are $6x$ middle school girls. We place this in the region inside the middle school circle but not the boys circle.

Finally, half the girls are in high school, so there are just as many high school girls as middle school girls. So, there are $6x$ girls in high school, and we place $6x$ outside both circles.

Altogether, we see that there are $x + 2x + 2x + 36 = 9x$ students in the club, so we have $9x = 72$. This gives us $x = \boxed{8}$ middle school boys.

14.53*:

Source: AMC 8 ✎

There are 120 seats in a row. What is the fewest number of seats that must be occupied so that the next person to be seated must sit next to someone?

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: In any consecutive three seats, at least one of the seats must be occupied. Otherwise, the next person can safely sit in the middle of the three empty seats. We can break the 120 seats into 40 groups of 3 consecutive seats. We need at least one person in each of the 40 groups, so we need at least 40 people. If there are 40 people, then the middle chair in each of these groups of 3 consecutive seats, then the next person is forced to sit next to someone. Therefore, the lowest number of seats that must be occupied is indeed $\boxed{40}$.

14.54*:

In how many ways can you spell the word NOON in the grid below? You can start on any letter, then on each step you can move one letter in any direction (up, down, left, right, or diagonal). You cannot visit the same letter twice.

Hint: Experiment. Start from an N in the corner—how many options do you have? What if you start from an N that's not in a corner?

N	N	N	N
N	O	O	N
N	O	O	N
N	N	N	N

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Case 1: start at an N in a corner. If we start at one of the 4 N's in a corner, then we have only 1 choice for the first O, then 3 choices for the second O, then 5 choices for the second N, for a total of $4 \cdot 1 \cdot 3 \cdot 5 = 60$ ways to form NOON, starting from a corner.

Case 2: start at an N on a side. If we start at one of the 8 N's on a side, then we have 2 choices for the first O, then 3 choices for the second O. If we choose the second O adjacent to our original N, then we only have 4 choices for the final N; otherwise (for the other two choices of the second O) we have 5 choices for the final N. Thus we have $8 \cdot 2 \cdot (3 \cdot 5 + 1 \cdot 4) = 224$ ways to form NOON, starting from a side.

Adding our counts from the two cases gives a total of $60 + 224 = \boxed{284}$ ways to form NOON.

14.55*:

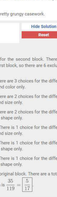
Two-thirds of the dogs at the dog pound are pit bulls. If half the dogs at the pound are female, and three-quarters of the male dogs at the pound are pit bulls, if there are 14 female pit bulls at the pound, how many dogs are at the pound?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We organize the information with the Venn diagram at the right, with a circle for pit bulls and a circle for males. We place 14 in the non-overlapping portion of the pit bull circle for the 14 female pit bulls. Let x be # male dogs that are not pit bulls, so we place an x in the non-overlapping portion of the male dog circle.



Since $\frac{2}{3}$ of the male dogs are pit bulls, there are 3 male pit bulls for every 1 male dog that is not a pit bull. There are 3 times as many male pit bulls as non-pit bull males. So, the number of male pit bulls is $3x$, which we place in the overlap region.

Since $\frac{1}{2}$ of the dogs at the pound are female, there are just as many females as males. There are $x + 3x = 4x$ males, so there are $4x$ females. We already have 14 of the females in the pit bulls, so the remaining $4x - 14$ are not female, which allows us to complete the Venn diagram shown at the upper right.

The only piece of information left to use is the fact that $\frac{3}{4}$ of the dogs are pit bulls. There are $3x + 14$ pit bulls at the pound, and a total of $8x$ dogs in the pound, so we must have $3x + 14 = \frac{3}{4}(8x)$. Multiplying both sides by 3 gives $3(3x + 14) = 2(8x)$, so $9x + 42 = 16x$. Therefore, we have $7x = 42$ and $x = 6$. Substituting this value of x into our Venn diagram above gives us the Venn diagram at the right, and we have $9x = \boxed{54}$ dogs total.

14.56*:

Source: AMC ✎

We connect dots with toothpicks in a grid as shown at right. If there are 10 horizontal toothpicks in each row and 20 vertical rows in each column, how many total toothpicks are there?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Since we use 20 vertical toothpicks in each column, there are 21 rows of dots; since we use 10 horizontal toothpicks, there are 11 columns of dots. Then, since there are 20 vertical toothpicks in each column and 11 columns, there are $20 \cdot 11 = 220$ vertical toothpicks. Similarly, since there are 10 horizontal toothpicks in each row and 21 rows, there are $21 \cdot 10 = 210$ vertical toothpicks. This gives a total of $220 + 210 = \boxed{430}$ toothpicks.

14.57*:

Source: AIME ✎

I have 120 blocks. Each block is one of 2 different materials, 3 different colors, 4 different sizes, and 5 different shapes. No two blocks share exactly the same set of four properties. I take two blocks at random. What is the probability the two blocks have exactly two of these four properties in common?

Hint: Not all problems have pretty solutions. Sometimes you have to get your hands dirty with some pretty gruesome casework.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We choose the first block at random, so there are 119 other blocks to choose from for the second block. There are $(4 \cdot 3)(2 \cdot 4) = 96$ different ways to choose the two properties that the blocks share with the first block, so there are 6 exclusive cases.

Case 1: Same material and color. The other block must have a different size and different shape. There are 3 choices for the different size and 4 choices for the different shape, for a total of $2 \cdot 3 \cdot 4 = 12$ blocks with the same material and size only.

Case 2: Same material and shape. The other block must have a different color and different shape. There are 2 choices for the different color and 4 choices for the different shape, for a total of $2 \cdot 2 \cdot 4 = 8$ blocks with the same material and size only.

Case 3: Same material and shape. The other block must have a different color and different size. There are 2 choices for the different color and 3 choices for the different size, for a total of $2 \cdot 2 \cdot 3 = 12$ blocks with the same material and shape only.

Case 4: Same color and size. The other block must have a different material and different shape. There is 1 choice for the different material and 4 choices for the different shape, for a total of $1 \cdot 4 = 4$ blocks with the same color and size only.

Case 5: Same color and shape. The other block must have a different material and different size. There is 1 choice for the different material and 3 choices for the different size, for a total of $1 \cdot 3 = 3$ blocks with the same color and shape only.

Case 6: Same size and shape. The other block must have a different material and different color. There is 1 choice for the different material and 2 choices for the different color, for a total of $1 \cdot 2 = 2$ blocks with the same size and shape only.

Summing all these cases, there are 35 blocks with exactly two characteristics in common with the original block. There are a total of 119 other blocks, so the probability that the two blocks have exactly two characteristics in common is $\frac{35}{119} = \boxed{\frac{5}{17}}$.

Review Problems

14.26:

How many numbers are in the following lists?

(a) $-7, -6, -5, \ldots, 23, 24$

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Adding 6 to each number gives $1, 2, 3, \ldots, 32$, so there are $\boxed{32}$ numbers in the list.

(b) $7, 14, 21, \ldots, 686$

You may type any additional notes you have here.

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Your Submission: Solution

Solution: Each number is a multiple of 7. Dividing each number by 7 gives $1, 2, 3, \ldots, 98$, so there are $\boxed{98}$ numbers in the list.

(c) $3.5, 6.5, 9.5, 12.5, \ldots, 87.5, 90.5$

Preview Solution

You may type any additional notes you have here.

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Your Submission: Solution

Solution: The list counts by 3's. We make the list start from 3 by subtracting 0.5 from each number, which gives $3, 6, 9, \ldots, 90$. Dividing each number in the new list by 3 gives $1, 2, 3, \ldots, 30$, so there are $\boxed{30}$ numbers in the list.

(d) $86, 82, 78, \ldots, 14, 10$

Preview Solution

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The last counts down by 4's. We make the list start from 3 by subtracting 6 from each number, which gives $80, 76, \ldots, 8, 4$. Dividing each number by 4 gives $20, 19, \ldots, 2, 1$. There are 21 numbers from 1 to 21, so our list (which is missing 13 has $\boxed{20}$ numbers.

14.27: Source: MOEMS

At a group of 25 girls, 8 joined the track team, 13 joined the math team, and 6 joined both teams. How many of the girls did not join either team?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: Since 6 joined both teams, there are $8 - 6 = 2$ who joined only the track team, and $13 - 6 = 7$ who joined only the math team. This gives us a total of $2 + 6 + 7 = 15$ who are on at least one of the teams, leaving $25 - 15 = \boxed{10}$ who are not on either team. We can represent this information with the Venn diagram at the right.

14.28:

There are 40 students in Mrs. Reusch's 5th grade class. If there are three times as many students with blond hair as with blue eyes, 3 students with blond hair and blue eyes, and 15 students with neither blond hair nor blue eyes, how many students have blue eyes?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: We use the Venn diagram shown at the right, with one circle for blond hair and one for blue eyes. We start with 15 students in the circles for students who have neither, and we place 3 in the overlap region for those who have both. We let x be the total number of students with blue eyes, so $x - 3$ have blue eyes but not blond hair. There are $3x$ total students with blond hair, so there are $3x - 3$ students with blond hair but not blue eyes. The sum of all four numbers in the diagram equals the total number of students, so

$$(3x - 3) + 3 + (x - 3) + 15 = 40.$$

Simplifying the left side gives $4x + 12 = 40$. Subtracting 12 and then dividing by 4 gives $x = 7$, so $\boxed{7}$ students have blue eyes.

14.29:

You have 4 shirts, 3 pairs of pants, and 6 hats. How many outfits can you make consisting of one shirt, one pair of pants, and one hat?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: There are 4 options for shirts, 3 for pants, and 6 for hats, and all of these choices are independent. Thus there are a total of $4 \cdot 3 \cdot 6 = \boxed{72}$ outfits.

14.30: Source: AMC 8

A haunted house has six windows. In how many ways can George the Ghost enter the house by one window and leave by a different window?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: George has 6 choices for which window to enter first. No matter which window he enters, he can't use the same window when leaving, so he has only 5 choices to leave through. Therefore, he has $6 \cdot 5 = \boxed{30}$ ways to enter by one window and leave by another.

14.31:

How many 3-letter combinations can be formed if the first letter must be a vowel (A, E, I, O, or U), and the second letter must be different from the first letter?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: We have 5 choices for the first letter. There are 26 choices for the third letter. The second letter can't be the same as the third. So, once the third letter is chosen, there are only 25 choices left for the second letter. Therefore, there are $5 \cdot 26 \cdot 25 = \boxed{3250}$ such 3-letter combinations.

14.32:

How many 5-digit numbers have the second digit odd and the fifth digit at least four times the second digit?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: We have 9 choices for the first digit (can't start with 0), 10 for the third digit, and 10 for the fourth digit. So, there are $9 \cdot 10 \cdot 10 = 900$ ways to choose these three digits.

Next, we have to count how many ways we can choose the second and fifth digits. We can't choose 2 or greater for the second digit, since then we will have no options for the fifth digit. Thus, since the second digit must be odd and less than 2, it must be 1. This means that the fifth digit must be at least 4, so we have 6 choices for the fifth digit (any digit from 4 to 9 inclusive).

For each of the 5 ways to choose the second and fifth digits, there are 900 ways to choose the other three digits, so there are $6 \cdot 900 = \boxed{5,400}$ ways to form the number.

14.33: Source: AMC 8

Pat Peele has plenty of 0's, 1's, 2's, 4's, 5's, 6's, 7's, 8's, and 9's, but he has only twenty-two 3's. How far can he number the pages of his scrapbook with these digits?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: Pat uses two 2's in the first 10 pages: one for 2 and one for 12. While numbering the pages 20 through 29, he uses 10 2's (at the tens digits and the ones digits of 22). So, he has up to 12 2's in the first 29 pages. He'll use one 2 for 32, and another 2 every 10 numbers after that. He only has 8 2's left after page 32, and he'll use one every 10 pages, so he'll use his last 2 on page 112. (Note that he won't use any 2's on tens digits between 32 and 112.) However, Pat can still number pages after page 112 until he hits a number with a 2 in it. So, Pat can number all the way through page $\boxed{119}$.

14.34: Source: MATHCOUNTS

How many positive, even three digit numbers exist such that the sum of the hundreds digit and the tens digit equals the units digit?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: The units digit must be 2, 4, 6, or 8 (it can't be 0 because then the number would have to be 200, which isn't allowed). If the units digit is 2, then the hundreds digits can be any digit from 1 to 8, and the middle digit is then necessarily a minus the hundreds digit. Therefore, there are a such numbers with units digit a. (2 ending with 2, 4 ending with 4, and so on.) So there are $2 + 4 + 6 + 8 = \boxed{20}$ such numbers.

14.35: Source: MOEMS

A baseball league has nine teams. During the season, each of the nine teams plays exactly three games with each of the other teams. What is the total number of games played?

Preview Solution

Solution

Hide Solution

Reset

Your Submission: Solution

Solution: Each team plays each of the other 8 teams 3 times, so each team plays $8 \cdot 3 = 24$ games. There are 9 teams total. In multiplying $9 \cdot 24$, each games is counted twice, once for each team in the game, so we divide by two to get the answer of $(9 \cdot 24)/2 = \boxed{108}$ games.

14.36: Source: AMC 8

Tyler has entered a buffet line in which he chooses one kind of meat, two different vegetables, and one dessert from the selections below. If the order of food items is not important, how many different meals might he choose?

Meat: beef, chicken, pork

Vegetables: baked beans, corn, potatoes, tomatoes

Dessert: brownie, chocolate cake, chocolate pudding, ice cream

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: Tyler has 3 choices of meat and 4 choices of dessert. There are 4 ways for him to choose his first vegetable and 3 ways to choose his second vegetable, for a preliminary count of $4 \cdot 3$ ways to choose two vegetables. But this counts each possible pair of vegetables twice, once for each order in which they're chosen. So, we have to divide by 2, for two 1 $(4 \cdot 3)/2 = 6$ ways to choose his vegetables.

He has 3 choices of meat, 4 choices of dessert, and 6 choices of pairs of vegetables, so he has $3 \cdot 4 \cdot 6 = \boxed{72}$ different choices for meals.

14.37: Source: AMC 8

A singles tournament had six players. Each player played every other player only once, with no ties. If Helen won 4 games, two won 3 games, Janet won 2 games, Kendra won 2 games, and Lara won 2 games, how many games did Monica win?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: We don't need to figure out exactly who beat whom—we just need to count. Each of the 6 players plays 5 games, one against each other player. This produces a total of $(5 \cdot 6)/2 = 15$ games. Since the other 5 players won a total of $4 + 3 + 2 + 2 + 2 = 13$ games, this leaves $15 - 13 = \boxed{2}$ games that Monica won.

14.38:

Two fair 6-sided dice are rolled. What is the probability

(a) that "doubles" are rolled (that is, the two dice show the same number)?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There are $6 \cdot 6 = 36$ possible rolls of the dice.

There are 6 different ways to roll doubles (⚀⚀, ⚁⚁, ⚂⚂, ⚃⚃, ⚄⚄, ⚅⚅), which means the probability of rolling doubles is $\frac{6}{36} = \boxed{\frac{1}{6}}$.

(b) that the sum rolled is greater than 3 but less than 7?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We count the number of acceptable outcomes with casework.

Case 1: the first die is a ⚁. This means the second die can be either ⚀, ⚁, or ⚂. So there are 3 ways.

Case 2: the first die is a ⚂. This means the second die can be either ⚀, ⚁, or ⚂. So there are 3 ways.

Case 3: the first die is a ⚃. This means the second die can be either ⚀ or ⚁. So there are 2 ways.

Case 4: the first die is a ⚄. This means the second die can be either ⚀ or ⚁. So there are 2 ways.

Case 5: the first die is a ⚅. This means the second die can only be ⚀. So there is 1 way.

Case 6: the first die is a ⚀. No value of the second die will give a total less than 7.

The number of ways to roll greater than 3 but less than 7 is $3 + 3 + 2 + 2 + 1 = 12$, and there are $6 \cdot 6 = 36$ possible rolls of the dice. This means the probability of rolling this is $\frac{12}{36} = \boxed{\frac{1}{3}}$.

(c) that at least one of the dice shows a ⚀?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There are $6 \cdot 6 = 36$ possible rolls of the dice.

Solution 1: There are 6 ways that the first roll can be a ⚀ (⚀⚀, ⚀⚁, ⚀⚂, ⚀⚃, ⚀⚄, ⚀⚅), and there are 6 ways that the second roll can be a ⚀ (⚀⚀, ⚁⚀, ⚂⚀, ⚃⚀, ⚄⚀, ⚅⚀). However, we counted twice when we say both ⚀⚀, so the number of ways that at least one of the dice is ⚀ is $6 + 6 - 1 = 11$. This means that the probability that at least one of the dice is ⚀ is $\boxed{\frac{11}{36}}$.

Solution 2: There are 5 ways in which the first roll is not ⚀, and 5 ways in which the second roll is not ⚀, so there are $5 \cdot 5 = 25$ ways in which neither die shows ⚀. Therefore there are $36 - 25 = 11$ ways in which one or both dice show ⚀. So the probability of this is $\frac{11}{36}$.

14.39: Source: MOEMS

A bag contains some marbles, all of the same size. Eight of them are black. The rest are red. The probability of drawing a red marble from the bag is $\frac{2}{3}$. Find the total number of red marbles in the bag.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since the probability of drawing a red marble is $\frac{2}{3}$, we know that $\frac{2}{3}$ of the marbles in the bag are red. This means that the other $1 - \frac{2}{3} = \frac{1}{3}$ of the marbles are black. Since the probability of drawing a red marble is double the probability of drawing a black marble, there are twice as many red marbles as black marbles. Therefore, there are $2 \cdot 8 = \boxed{16}$ red marbles.

14.40:

A box contains 5 white balls and 6 black balls.

(a) A ball is drawn out of the box at random. What is the probability that the ball is white?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There are 5 white balls out of 11 total balls, and each of the balls is equally likely to be drawn, so the probability that the ball is white is $\boxed{\frac{5}{11}}$.

(b) Two balls are drawn out of the box at random. What is the probability that they both are white?

You may type any additional notes you have here.

Hide Solution

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Your Submission: Solution

Solution: First we count the number of ways to draw a pair of balls from the box. There are 11 choices for the first ball and 10 for the second, for a total of $11 \cdot 10/2 = 55$ ways to choose the balls. We divide by 2 because $11 \cdot 10$ counts each possible pair of balls twice, once for each order in which we could draw them.

Next, we count the number of ways to choose two white balls. There are 5 choices for the first ball and 4 for the second, for a total of $(5) \cdot (4)/2 = 10$ ways to choose two white balls. Again, we divide by 2 because $5 \cdot 4$ counts each possible pair of white balls twice, once for each order in which we can choose the balls.

Combining these counts, the probability both balls are white is $\frac{10}{55} = \boxed{\frac{2}{11}}$.

14.41: Source: AMC 8

If one recipient the "1" certain times displayed on a digital watch are palindromes. These examples are 11:00, 11:41, and 11:11. How many times during a 12-hour period will be palindromes?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We consider three-digit times and four-digit times separately.

Case 1: Three-digit times. There are 5 choices for the hour digit. For each choice of hour digit, there are 6 choices for the middle digit (only the digits 0–5 may be used). The last digit must be the same as the hour digit, so it only has 1 choice. This gives us $5 \cdot 6 = 30$ three-digit palindromic times.

Case 2: Four-digit times. There are only 3 choices for the hour: 10, 11, or 12. For each choice of hour, the minutes digits must be the reverse of the hour digits: 10:01, 11:11, 12:21. So, there are 3 four-digit palindromic times.

Combining these cases gives us $6 + 3 = \boxed{33}$ palindromic times.

14.42: Source: AMC 8

Diana and Apollo each roll a standard die obtaining a number at random from 1 to 6. What is the probability that Diana's number is larger than Apollo's number?

You may type any additional notes you have here.

Hide Solution

Reset

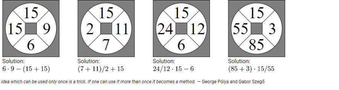
Your Submission: Solution

Solution 1: Casework. There are $6 \cdot 6 = 36$ possible pairs of rolls. We can count the outcomes in which Diana gets a higher roll with some casework based on Apollo's roll:

Apollo's roll	Number of higher rolls
⚀	5
⚁	4
⚂	3
⚃	2
⚄	1
⚅	0

Combining these gives $5 + 4 + 3 + 2 + 1 + 0 = 15$ ways Diana can have a higher roll than Apollo. So, the desired probability is $\frac{15}{36} = \boxed{\frac{5}{12}}$.

Solution 2: Some craftiness. Of the 36 possible rolls, there are exactly 6 ways the two can roll the same number. Of the other 30 rolls, half the time Apollo roll is higher and half the time Diana's roll is higher. So, there are 15 ways in which Diana can roll a higher number than Apollo. As before, this gives us a probability of $\frac{15}{36} = \boxed{\frac{5}{12}}$.



Solution: $6 \cdot 9 - (15 + 15)$ | Solution: $(7 + 11)/2 + 15$ | Solution: $24/12 \cdot 15 - 6$ | Solution: $(85 + 3) \cdot 15/55$

An idea which can be used only once is a trick. If one can use it more than once it becomes a method. — George Pólya and Gabor Szegö

CHAPTER 15 Problem-Solving Strategies

We finish the book with a chapter covering common general strategies for tackling problems. These strategies have a common goal:

> **Concept:** Simplify the problem.

15.1 Find a Pattern

Much of learning consists simply of observing patterns. Solving math problems is no different. An excellent problem solver is playful and experiments with problems. Often these experiments reveal patterns that lead to a solution.

Problems

Problem 15.1 Jump to Solution

What is the 2010th letter in the sequence below?

$$ABCDEDCBAABCDEDCBAABCDEDCBAABCDE\ldots$$

Problem 15.2 Jump to Solution

In this problem, we find the units digit (ones digit) of 2^{2011}.

(a) Find the units digit of each of the first 8 powers of 2, starting with 2^1.

(b) Find the units digit of 2^{2011}.

Problem 15.3 *Source: MOEMS* Jump to Solution

Michelle's Number Recycling Machine obeys two rules:

1. If an inserted number has exactly one digit, double the number.
2. If an inserted number has exactly two digits, compute the sum of the digits.

The first number Michelle inserts is 1. Then every answer she gets is inserted back into the machine until fifty numbers are inserted. What is the fiftieth number to be inserted?

Problem 15.4 Jump to Solution

In the figure below, each row of ×'s has two more ×'s than the row above it. Altogether, how many ×'s are contained in the first 30 rows?

Problem 15.1

What is the 2010th letter in the sequence below?

$$ABCDEDCBAABCDEDCBAABCDEDCBAABCDE\ldots$$

Solution for Problem 15.1: We certainly don't want to write out 2010 letters! Fortunately, the letters follow a clear pattern. The same 9-letter block repeats over and over:

$$ABCDEDCBA.$$

We'll have to write this block over and over many times before reaching the 2010th letter. To figure out how many times, we divide 2010 by 9, which has quotient 223 and remainder 3. So, in writing the first 2010 letters, we write this repeating block 223 times, and then write 3 more letters. Therefore, the 2010th letter is C. ☐

In Problem 15.1, the pattern is part of the problem statement. In many problems, the pattern isn't given in the problem, and we must discover it by experimenting.

Problem 15.2

What is the units digit (ones digit) of 2^{2011}?

Solution for Problem 15.2: Computing 2^{2011} by hand will take a while. Many calculators aren't able to output all the digits of such a large number, so even using a calculator may not help solve the problem. Instead, we'll have to play with the problem a bit by computing several powers of 2 and looking for a pattern. Starting from 2^1, the first eight powers of 2 are

$$2, 4, 8, 16, 32, 64, 128, 256.$$

At first, it may not be obvious that there's a pattern. But remember, the problem asks for the units digit of 2^{2011}, so let's look at just the units digits of the numbers in this list:

$$2, 4, 8, 6, 2, 4, 8, 6.$$

The same block of 4 digits repeats twice. Will this pattern continue forever?

> **Important:** Noticing a pattern isn't enough. We have to be sure that the pattern continues in order to use it to solve a problem. This usually means that we have to figure out why the pattern occurs.

The units digit of the product of two integers is the same as the units digit of the product of the units digits of the integers. For example, the units digit of $342 \cdot 486$ is the same as the units digit of $2 \cdot 6$. So, all we have to consider are the units digits of the powers of 2.

We start with 2, and multiplying any number that ends in 2 by 2 gives a number that ends in 4. This means that any 2 in the list of units digits of powers of 2 must be followed by a 4. Similarly, 4 must be followed by 8, and 8 must be followed by 6. Since 6 · 2 ends in 2, the next units digit is 2 again. We already know that after 2 comes 4, after 4 comes 8, and after 8 comes 6. Then, we've back to 2 yet again. So, we see that the units digits repeat the cycle 2, 4, 8, 6 over and over.

Now our goal is to figure out what digit is 2011th if we repeat the cycle 2, 4, 8, 6 until we have 2011 terms. 2011 divided by 4 has quotient 502 and remainder 3. So, we will write the cycle 502 times, and then write 3 more terms to get to the 2011th term. Therefore, the 2011th term is 8, so the units digit of 2^{2011} is 8.

Notice that we don't even have to find the quotient when 2011 is divided by 4. All we really care about is the remainder. Since the remainder is 3, we know that to get to the 2011th term, we have to write three more terms after writing the last complete cycle 2, 4, 8, 6. ☐

It's very easy to be off by one term when using patterns to solve a problem as we did in Problem 15.2. For example, suppose we wanted to find the units digit of 2^{2000}. Following our work in the previous problem, we find the remainder when we divide 2000 by 4. The remainder is 0, so do we take the first term in the cycle 2, 4, 8, 6, or do we take the last?

To get to the 2000th term, we write the cycle $2000/4 = 500$ times and then write *no more extra terms*. We write the 2000th term when we write the last term of the cycle for the 500th time. So, the units digit of 2^{2000} is 6.

> **WARNING!!** Be careful not to be off by one term when using patterns to solve a problem.

One way to check that you're not off by one is to test small cases. For example, to test our reasoning above for the units digit of 2^{2000}, we can consider what happens when 2 is raised to smaller exponents that are multiples of 4. Both 2^4 and 2^8 end in 6, which gives us more confidence that our answer for 2^{2000} is correct.

Problem 15.3 *Source: MOEMS*

Michelle's Number Recycling Machine obeys two rules:

1. If an inserted number has exactly one digit, double the number.
2. If an inserted number has exactly two digits, compute the sum of the digits.

The first number Michelle inserts is 1. Then every answer she gets is inserted back into the machine until fifty numbers are inserted. What is the fiftieth number to be inserted?

Solution for Problem 15.3: As with the previous problem, we experiment a bit and hope we find a pattern. Following the two given rules, the first several numbers Michelle's machine produces are

$$1, 2, 4, 8, 16, 7, 14, 5, 10, 1, 2, 4, 8.$$

Once the machine produces 1 for a second time, we know that the machine will repeat the first 9 numbers over and over. Since 50 divided by 9 leaves a remainder of 5, the 50th number in the list is the same as the 5th in this repeating block of numbers. This means that the 50th number in the list is 16. ☐

Problem 15.4

In the figure below, each row of ×'s has two more ×'s than the row above it. Altogether, how many ×'s are contained in the first 30 rows?

Solution for Problem 15.4: First, we simplify the problem by getting rid of the symbols, and instead write the number of ×'s in each row. The numbers of ×'s in the rows are

$$1, 3, 5, 7, 9, 11, \ldots.$$

So, to count all the ×'s in the first 30 rows, we could add the first 30 positive odd numbers. But instead, let's add up shorter lists and look for a pattern.

> **Concept:** Considering a simpler version of a problem can help us solve the original problem.

The first 5 such sums are

$$\begin{aligned} 1 &= 1, \\ 1 + 3 &= 4, \\ 1 + 3 + 5 &= 9, \\ 1 + 3 + 5 + 7 &= 16, \\ 1 + 3 + 5 + 7 + 9 &= 25. \end{aligned}$$

Each sum is a perfect square! The sum of the first 2 odd numbers is 2^2, the sum of the first 3 odd numbers is 3^2, the sum of the first 4 odd numbers is 4^2, and so on.

We now expect that the sum of the first 30 positive odd numbers is $30^2 = 900$. But we still have to figure out why summing the first n positive odd numbers results in n^2. Now that we know what to look for, there are a number of ways to see why it is true.

One way is to use what we learned back in Section 2.1 here about consecutive squares. There, we learned that

$$n^2 + 2n + 1 = (n + 1)^2.$$

The first square is just the first positive odd number: $1^2 = 1$. To get from 1^2 to 2^2, we add the odd number $2 \cdot 1 + 1$:

$$1^2 + 2 \cdot 1 + 1 = 2^2.$$

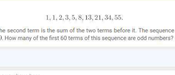
Then, to get from 2^2 to 3^2, we add the next odd number, $2 \cdot 2 + 1$:

$$2^2 + 2 \cdot 2 + 1 = 3^2.$$

And so on:

$$\begin{aligned} 3^2 + 2 \cdot 3 + 1 &= 4^2, \\ 4^2 + 2 \cdot 4 + 1 &= 5^2, \\ 5^2 + 2 \cdot 5 + 1 &= 6^2. \end{aligned}$$

We can also use some clever diagrams to explain why summing the first n positive odd numbers equals n^2:



Now, we can confidently state that there are 900 ×'s total in the first 30 rows. ☐

Find a Pattern

Exercises

15.1.1: Source: MATHCOUNTS

It takes exactly 74 colored beads on a string to make a necklace. The beads are strung in the following order: one red, one orange, two yellow, one green, and one blue. Then the pattern repeats, starting again with one red bead. If the first bead of the necklace is red, what is the color of the last bead used to make the necklace?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solve

Solution: The beads on the necklace repeat the same pattern of 6 bead colors over and over. Since 74 divided by 6 has quotient 12 and remainder 2, the 74th bead is 2 beads into this 6-bead pattern. The second bead of the pattern is orange.

15.1.2:

The **Fibonacci sequence** begins

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55.$$

Each term of the sequence after the second term is the sum of the two terms before it. The sequence continues forever. For example, the eleventh term is $34 + 55$, or 89. How many of the first 60 terms of this sequence are odd numbers?

Preview: Solve

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solve

Solution: In the first ten terms shown, it looks like there's a pattern of two odd numbers followed by an even number. To see why this pattern will continue, note that the sum of two odd numbers is even. So, two odd numbers must be followed by an even number. Similarly, the sum of an odd and even is odd. So, both "odd then even" and "even then odd" will be followed by an odd number. Putting these observations together, we see that the Fibonacci sequence consists of blocks of two odd numbers and an even number. In the first 60 terms, there are $60/3 = 20$ such blocks, for a total of $20 \cdot 2 = 40$ odd numbers.

15.1.3:

In the table of numbers below, what number is directly above 119?

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 2 & 3 & 4 & & & \\ & & 5 & 6 & 7 & 8 & 9 & & \\ & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \\ \iddots & & & & \vdots & & & & \ddots \end{array}$$

Preview: Solve

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solve

Solution: The last number in each row appears to be a perfect square. To see why this is the case, note that there is 1 number in the first row, 3 numbers in the second row, 5 numbers in the third row, and so on. In each row, we add the next odd number amount of numbers. So, the total number of numbers on or above each row is the sum of consecutive odd numbers starting from 1. As explained in the text, such a sum is always a perfect square. This means the last number in each row is a perfect square.

Since 119 is 2 less than the perfect square 121, we know that 119 is two numbers before the last number of the row that ends with 121. The row above 119's row ends in the previous square, 100, and this final number is directly above the next-to-last number in 119's row, namely 120. So, the number just to 100's left is directly above 119, which means that the number directly above 119 is 99.

15.1.4:

All of the even numbers from 2 through 288, except those ending in 0, are multiplied together. What is the units digit (ones digit) of the product?

Hint: We aren't multiplying the same units digit over and over again. But we are multiplying the same group of units digits over and over again.

Preview: Solve

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solve

Solution: We only care about the units digits of the numbers in the product. The first four numbers are $2 \cdot 4 \cdot 6 \cdot 8$. The product of the units digits of the next four numbers is also $2 \cdot 4 \cdot 6 \cdot 8$. Similarly, the product of the units digits in the rest of the product just repeats this block over and over.

We have $2 \cdot 4 \cdot 6 \cdot 8 = 8 \cdot 6 \cdot 8 = 48 \cdot 8$, which has units digit 4 (since $8 \cdot 8 = 64$). The rest of the product in the problem consists of 28 more blocks of $2 \cdot 4 \cdot 6 \cdot 8$. Each block has a product with units digit 4. The entire product consists of 29 such blocks, each with a product with units digit 4. Therefore, the product has the same units digit as 4^{29}.

4^1 ends in 4, 4^2 ends in 6, and the units digit of 4^3 is back to 4. So, the units digits of the powers of 4 cycle back and forth between 4 and 6, starting with $4^1 = 4$. Since 29 is odd, we see that 4^{29} has 4 as its units digit. Therefore, the original product has units digit 4.

15.2 Make a List

If a strategy is good enough to allow Santa Claus to distribute presents to all the children in the world, it ought to be good enough to help us solve some math problems.

Problems

Problem 15.5 *Source: MOEMS* Jump to Solution

Find a two-digit positive integer with all these properties:

- the tens digit is larger than the ones digit (units digit),
- the difference between the digits is greater than 3,
- the sum of the digits is greater than 10, and
- the integer is a multiple of 12.

Problem 15.6 Jump to Solution

Elizabeth spent exactly $7 for some 15-cent stamps and some 31-cent stamps. How many 15-cent stamps did she buy?

Problem 15.7 Jump to Solution

A **partition** of a positive integer n is a way of writing n as the sum of positive integers in which the order of the numbers in the sum doesn't matter. For example, the partitions of 3 are 3, $2+1$, and $1+1+1$. We consider $2+1$ and $1+2$ to be the same, so there are 3 partitions of 3. How many partitions of 6 are there?

Problem 15.5 *Source: MOEMS*

Find a two-digit positive integer with all these properties:

- the tens digit is larger than the ones digit (units digit),
- the difference between the digits is greater than 3,
- the sum of the digits is greater than 10, and
- the integer is a multiple of 12.

Solution for Problem 15.5: It would take way too long to test every single two-digit number. We narrow our search by first focusing on just one of the properties. But which one?

> **Concept:** When faced with several restrictions in a problem, it's often best to consider the most restrictive first.

Each of the first three properties is satisfied by quite a few two-digit numbers, but there aren't many two-digit multiples of 12. So, we start by listing the multiples of 12:

$$12, 24, 36, 48, 60, 72, 84, 96.$$

Now that we have a short list of possible solutions, we can quickly go through the other properties and eliminate possibilities from this list. Since the tens digit must be larger than the ones digit, we can eliminate the first four numbers in the list, leaving

$$\cancel{12}, \cancel{24}, \cancel{36}, \cancel{48}, 60, 72, 84, 96.$$

The next property, that the difference between the digits is greater than 3, eliminates 96:

$$\cancel{12}, \cancel{24}, \cancel{36}, \cancel{48}, 60, 72, 84, \cancel{96}.$$

Finally, since the sum of the digits must be greater than 10, we can eliminate 60 and 72, and we are left with 84 as the answer to the problem. □

Problem 15.6

Elizabeth spent exactly $7 for some 15-cent stamps and some 31-cent stamps. How many 15-cent stamps did she buy?

Solution for Problem 15.6: We often start word problems by assigning variables and writing equations. Let's try that here. First, we note that $7 is 700 cents. If we let a be the number of 15-cent stamps and b the number of 31-cent stamps, then we must have

$$15a + 31b = 700.$$

That's all we know about a and b, and this equation isn't terribly helpful. We could just guess values of a or b and hope we get lucky, but there are a lot of possibilities.

Instead, we take a more organized approach. After paying for all the 31-cent stamps, the amount remaining must be divisible by 15 cents. So, we imagine that Elizabeth starts with 700 cents and pays for the 31-cent stamps one at a time. We then list the number of cents she has remaining at each step, looking for a multiple of 15:

$$700, 669, 638, 607, 576, 545, 514, 483, 452, 421, 390.$$

Since 390 is divisible by both 3 and 5, it is divisible by 15. Elizabeth spent $700 - 390 = 310$ cents on 31-cent stamps, so she bought 10 31-cent stamps. With the other 390 cents, she bought $390/15 = 26$ 15-cent stamps.

You might have noticed that we could have saved a lot of work with a little bit of thinking. Instead of buying the 31-cent stamps one at a time, we might have noticed that the number of 31-cent stamps she bought must have been a multiple of 5. This is the only way the number of cents left for 15-cent stamps could be a multiple of 5. So, we can shorten our list by buying five 31-cent stamps at a time. Each block of five 31-cent stamps costs $5 \cdot 31 = 155$ cents, so our list is

$$700, 545, 390.$$

We reach the desired multiple of 15 much faster this way! Furthermore, it's easy with this method to see why there are no other combinations she could have bought. If we continue the list, we have

$$700, 545, 390, 235, 80.$$

The only multiple of 15 in this list is 390. So, the only way she can buy 31-cent stamps and 15-cent stamps is if she spends 390 cents on 15-cent stamps and 310 cents on 31-cent stamps. □

In the previous problem, we saw that doing a little bit of thinking before listing can sometimes shorten the amount of listing you have to do. In the next problem, we see that sometimes we need to do some careful thinking just to make sure we have a complete list.

Problem 15.7

A **partition** of a positive integer n is a way of writing n as the sum of positive integers in which the order of the numbers in the sum doesn't matter. For example, the partitions of 3 are 3, $2+1$, and $1+1+1$. We consider $2+1$ and $1+2$ to be the same, so there are 3 partitions of 3. How many partitions of 6 are there?

Solution for Problem 15.7: If we just start listing partitions of 6 as we think of them, it will be hard to know when we've found them all. So, we should take an organized approach.

> **Concept:** Organized lists are much more useful than disorganized lists.

One way to organize the partitions is to group them based on the largest number in the partition. Obviously, there's only one partition that has 6, namely

$$6.$$

Similarly, there's only one partition in which 5 is the largest number:

$$5+1.$$

The partitions with 4 as the greatest number are

$$4+2,$$
$$4+1+1.$$

The partitions with 3 as the greatest number are

$$3+3,$$
$$3+2+1,$$
$$3+1+1+1.$$

The partitions with 2 as the greatest number are

$$2+2+2,$$
$$2+2+1+1,$$
$$2+1+1+1+1.$$

The only partition with 1 as the greatest number is

$$1+1+1+1+1+1.$$

Altogether, we count 11 partitions of 6. □

Make a List

Exercises

15.2.1:

In Problem 15.6, why did we choose to imagine buying the 31-cent stamps one at a time, rather than starting with buying the 15-cent stamps?

You may type any additional notes you here.

Hide Solution

Reset

Your Submission: Solution

Solution: We tackled the problem by counting backwards from 700 by 31's while looking for a multiple of 15. This is much easier than counting backwards from 700 by 15's while looking for a multiple of 31 for two reasons. First, it is far easier to spot multiples of 15 than it is to spot multiples of 31. Only numbers that end in 0 or 5 can possibly be a multiple of 15. So, we don't have to do anything besides look at the last digit to discard most of the numbers we hit as we subtract 31's. Moreover, we cover a lot more ground as we subtract 31's than when we subtract 15's, so we have fewer numbers to check.

15.2.2:

I am less than 6 feet tall but more than 2 feet tall. My height in inches is a multiple of 7 and is also 1 more than a multiple of 6. What is my height?

Preview: Solution

You may type any additional notes you here.

Hide Solution

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Your Submission: Solution

Solution: My height is a number of inches that is greater than 24 (2 feet), less than 72 (6 feet), a multiple of 7, and 1 more than a multiple of 6. To find my height, I list the multiples of 7 greater than 24 and less than 72:

$$28, 35, 42, 49, 56, 63, 70.$$

The only number in this list that is 1 more than multiple of 6 is 49. So, I'm $\boxed{4 \text{ feet, } 1 \text{ inch}}$ tall.

15.2.3: Source: MOEMS

List all two-digit positive integers that satisfy both of the following:

1. The tens and ones digits are consecutive numbers, and
2. The integer is the product of two consecutive numbers.

Preview: Solution

You may type any additional notes you here.

Hide Solution

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Your Submission: Solution

Solution: It's easiest to start with the second condition, since there aren't many two-digit numbers that are the product of two consecutive numbers. (It's also much easier to multiply numbers than to find factorizations, which is what we'd have to do if we started by listing numbers whose digits are consecutive numbers.) We build a list of two-digit numbers that satisfy the second condition by multiplying pairs of consecutive numbers. The full list is

$$3 \cdot 4 = 12,$$
$$4 \cdot 5 = 20,$$
$$5 \cdot 6 = 30,$$
$$6 \cdot 7 = 42,$$
$$7 \cdot 8 = 56,$$
$$8 \cdot 9 = 72,$$
$$9 \cdot 10 = 90.$$

The only numbers in this list with consecutive numbers as digits are $\boxed{12 \text{ and } 56}$.

15.2.4:

If a number is divided by 3, the remainder is 0. If the number is divided by 5, the remainder is 4. If the number is divided by 11, the remainder is 7. The number has two digits. What is the number?

Preview: Solution

You may type any additional notes you here.

Hide Solution

Reset

Your Submission: Solution

Solution: We could start by listing all the two-digit multiples of 3, but that would be a pretty long list. Instead, we start by listing all the two-digit numbers that are 7 more than a multiple of 11. We can do this by starting with 7 and counting by 11's. The two-digit numbers we thus create are

$$18, 29, 40, 51, 62, 73, 84, 95.$$

Next, we note that because the number leaves a remainder of 4 when divided by 5, the number must end in 4 or 9. That leaves only 29 and 84 as possibilities. 29 is not divisible by 3, but $\boxed{84}$ is.

15.2.5★:

The three-digit number 114 has a **digit sum** of $1+1+4$, or 6. How many three-digit positive integers have a digit sum of 6?

Hint: What is a good way to organize the three-digit numbers whose digit sum is 6?

Hint: How many start with 6? How many start with 5?

You may type any additional notes you here.

Hide Solution

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Your Submission: Solution

Solution: We make an organized list, where we organize the numbers by the hundreds digit. Clearly, there are no numbers starting with 7 or greater that have a digit sum of 6. There is one such number starting with 6, namely 600. Continuing in this manner, we produce the following organized list:

Hundreds digit	Numbers with digit sum 6
6	600
5	510, 501
4	420, 411, 402
3	330, 321, 312, 303
2	240, 231, 222, 213, 204
1	150, 141, 132, 123, 114, 105

Notice how each row is nicely organized. This makes us confident that we didn't miss any, and didn't count any twice. It also reveals a nice pattern. We see that there are $1+2+3+4+5+6 = \boxed{21}$ three-digit numbers with digit sum 6.

15.3 Draw a Picture

The old adages "Seeing is believing" and "A picture is worth a thousand words" are applicable to many math problems. Naturally, this is often true of geometry problems, but it's also true of some problems that don't appear to be about pictures at all.

Problems

Problem 15.8 Jump to Solution

Wally the wandering walrus swims 6 miles north, then 3 miles east, then 3 miles north, then 15 miles west. How far is he from where he started? (You can assume Earth is flat for this problem.)

Problem 15.9 Jump to Solution

A frog is at the bottom of a 12-meter well. Each morning he climbs up 5 meters. Each night he slides down 3 meters. If he starts climbing on a Sunday, on which day will he reach the top of the well and escape?

Problem 15.10 Jump to Solution

Abel, Bernoulli, Cantor, and Descartes have a race. Bernoulli finished between Abel and Descartes. Cantor is happy that he didn't finish last, and Descartes bragged all day long about beating Abel. If no one finished between Cantor and Abel, then in what order did the participants finish?

Problem 15.11 Jump to Solution

Below are the pairs of cities connected by direct flights by GetUThere Airlines. For each pair of cities, there are flights in both directions. Is it possible to get from Boston to Philadelphia on a series of GetUThere flights?

Boston–New York	Baltimore–Washington	Buffalo–Newark
Buffalo–Albany	Philadelphia–Erie	New York–Washington
New York–Baltimore	Erie–Newark	Albany–Philadelphia
Newark–Albany	Baltimore–Boston	Newark–Philadelphia

We start with a problem in which drawing a picture is a pretty obvious first step.

Problem 15.8

Wally the wandering walrus swims 6 miles north, then 3 miles east, then 3 miles north, then 15 miles west. How far is he from where he started? (You can assume Earth is flat for this problem.)

Solution for Problem 15.8: At the right, the solid arrows show Wally's path. As shown in the dashed right triangle, Wally wanders 9 miles north and 12 miles west in total. Applying the Pythagorean Theorem, or recalling the $\{9, 12, 15\}$ Pythagorean triple, we find that Wally ends 15 miles from where he started. □

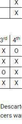
Problem 15.9

A frog is at the bottom of a 12-meter well. Each morning he climbs up 5 meters. Each night he slides down 3 meters. If he starts climbing on a Sunday, on which day will he reach the top of the well and escape?

Solution for Problem 15.9: What's wrong with this solution:

> **Bogus Solution:** Since he climbs up 5 meters each morning and slides back 3 meters each night, he moves upward 2 meters total each day of the week. Therefore, he will climb 12 meters in $12/2 = 6$ days. He starts climbing on a Sunday, so his sixth day of climbing is the following Friday.

We see the flaw in this reasoning when we draw a picture that shows both his climbs and his slides. Our picture is shown at the right. We draw up arrows for the climbs and down arrows for the slides, and note the total height climbed at each step. We see that his height first goes over 12 meters after the fifth climb, so he escapes the well on the fifth day of climbing. He starts climbing on a Sunday, so his fifth and last day of climbing is a Thursday. □



> **Concept:** A quick sketch can help prevent errors.

Problem 15.10

Abel, Bernoulli, Cantor, and Descartes have a race. Bernoulli finished between Abel and Descartes. Cantor is happy that he didn't finish last, and Descartes bragged all day long about beating Abel. If no one finished between Cantor and Abel, in what order did the participants finish?

Solution for Problem 15.10: We need a good way to organize the information in the problem. Each piece of information eliminates possibilities: For example, because Bernoulli finished between Abel and Descartes, we know that Bernoulli did not finish first or last. We visualize this information by making a table with a row for each racer and a column for each position. We place X's in the 1st and 4th columns of Bernoulli's row to indicate that Bernoulli didn't finish in either of these positions.

	1st	2nd	3rd	4th
Abel				
Bernoulli	X			X
Cantor				
Descartes				

Next, we place an X in Cantor's 4th column since Cantor wasn't last. Because Descartes beat Abel, we know that Abel wasn't first and Descartes wasn't last. Placing X's for these facts as well, we now have the table at the right:

	1st	2nd	3rd	4th
Abel	X			
Bernoulli	X			X
Cantor				X
Descartes				X

Now, it's clear that Abel must have been 4th, since none of the other runners could have been 4th. We then immediately know that Cantor was 3rd, since no one was between Cantor and Abel. We place O's in Abel's 4th column and Cantor's 3rd column to indicate that we know these placements. We then can place X's in the remainder of Abel's and Cantor's rows, and in the remainder of the 3rd column.

	1st	2nd	3rd	4th
Abel	X	X	X	O
Bernoulli	X		X	X
Cantor	X	X	O	X
Descartes			X	X

Now we have the full story. There's only one possibility left for Bernoulli, 2nd place, and only Descartes could have been in 1st place. The completed chart is shown at the left, and the order of the racers was

Descartes, Bernoulli, Cantor, Abel.

We can check our answer by making sure that this order agrees with all the information in the problem. In the finishing order we found above, Bernoulli is between Abel and Descartes. Cantor is not last, Descartes beats Abel, and no one is between Cantor and Abel. □

Problem 15.11

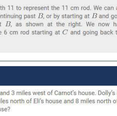
Below are the pairs of cities connected by direct flights by GetUThere Airlines. For each pair of cities, there are flights in both directions. Is it possible to get from Boston to Philadelphia on a series of GetUThere flights?

Boston–New York	Baltimore–Washington	Buffalo–Newark
Buffalo–Albany	Philadelphia–Erie	New York–Washington
New York–Baltimore	Erie–Newark	Albany–Philadelphia
Newark–Albany	Baltimore–Boston	Newark–Philadelphia

Solution for Problem 15.11: We know that Boston and Philadelphia are not connected by a direct flight, but we can't tell whether or not there's a sequence of flights that goes from Boston to Philadelphia. To help hunt for such a sequence, we draw a picture that includes all the flights. We make each city a point, and connect each pair of points that is connected by a direct flight. The result is shown below:

Now it's obvious that there is no way to get from Boston to Philadelphia. □

> **Sidenote:** **Graph theory** is a powerful field of mathematics that we can use to study connections between pairs of items. In Problem 15.11, we constructed a **graph** to represent connections between cities. The points in the graph for the cities are called **vertices** and the connections are **edges** of the graph.



Draw a Picture

Exercises

15.3.1:

A lumberjack can cut a log into five pieces in 20 minutes. How long would it take to cut a log of the same size and shape into seven pieces?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We might think that 5 pieces requires 5 cuts, but a quick sketch makes us realize that we only need 4 cuts. In the diagram at the right, the log is represented by a line segment, and the dots on the log are the cuts.

The lumberjack makes 4 cuts in 20 minutes, so it takes $20/4 = 5$ minutes to make each cut. Just as he needs 4 cuts to make 5 pieces, he needs 6 cuts to make 7 pieces. Each cut takes 5 minutes, so he needs $5 \cdot 6 =$ 30 minutes to cut the log into 7 pieces.

15.3.2:

The lengths of three rods are 6 cm, 10 cm, and 11 cm. How can you use these rods to measure a length of 15 cm?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We start by drawing $\overline{AB}$ with length 11 to represent the 11 cm rod. We can add the 10 cm rod to the end of this either by continuing past B, or by starting at B and going back towards A. Let's try extending past B, as shown at the right. We now have $AC = AB + BC = 21$. So, if we add the 6 cm rod starting at C and going back towards B, the other end of the rod will be $21 - 6 = 15$ cm from A, as desired.

15.3.3:

Albert's house is 5 miles east of Bella's house and 3 miles west of Carmot's house. Dolly's house is 6 miles east of Carmot's house, and 4 miles east of Ella's house. Frank's house is 5 miles north of Ella's house and 8 miles north of Greta's house. To the nearest tenth of a mile, how far apart are Belle's house and Greta's house?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We draw a diagram to figure out how the houses are situated. The resulting diagram is on the right. Each person's house is represented by a point labeled with the first letter of the person's name. Therefore, B is 5 miles to the left of A, and C is 3 miles to the right of A. Since Dolly is 6 miles east of Carmot and 4 miles east of Ell, we know that Ell is $6 - 4 = 2$ miles east of Carmot. Finally, since Frank is 5 miles north of Ell and Greta is 8 miles south of Frank, we know that Greta is 3 miles south of Ell.

As shown at the right, we find that Greta is 10 miles east and 3 miles south of Belle. Applying the Pythagorean Theorem to $\triangle BEG$, we have $BG^2 = 10^2 + 3^2$, so $BG = \sqrt{109} \approx 10.4$. Therefore, to the nearest tenth of a mile, Belle's house and Greta's house are 10.4 miles apart.

15.3.4:

A man wearing red pants has red shoes. A man with blue pants has blue shoes. A man with green pants has green shoes. They exchange shoes so that each man is wearing one shoe from each of the other two men. After they leave, you only remember that the man in green pants had a red shoe on his right foot. What color shoe is each man wearing on each foot?

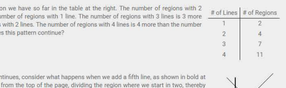
You may type any additional notes you have here.

Hide Solution

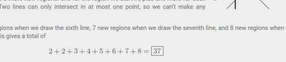
Reset

Your Submission: Solution

Solution: We start with a quick sketch of the shoes, and we name each person based on the color of their pants. (Yes, those are shoes in the diagram.) We label Green's right shoe "Red," since we know Green's right shoe is red.



We know that Green's shoes have different colors, and that neither is green. So, his left shoe must be blue. Also, we know that Blue's right shoe must be red or green, but the red right shoe is already taken by Green. So, Blue's right shoe is green.



Only the blue shoe remains for Red's right foot. Blue's left shoe can't be green or blue, so it must be Red. Red's left shoe can't be blue (Green has that shoe) and it can't be red, so it must be green. We now have the full arrangement of shoes:

15.3.5★:

One line divides a plane into two regions. Two lines can divide a plane into at most four regions. What is the maximum number of regions possible using eight lines?

Hint: Draw a picture. Keep adding lines. Count the regions you have each time. Do you see a pattern in your results?

You may type any additional notes you have here.

Hide Solution

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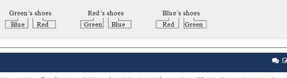
Your Submission: Solution

Solution: Below are diagrams for three lines and four lines.



We organize the information we have so far in the table at the right. The number of regions with 2 lines is 2 more than the number of regions with 1 line. The number of regions with 3 lines is 3 more than the number of regions with 2 lines. The number of regions with 4 lines is 4 more than the number of regions with 3 lines. Does this pattern continue?

# of Lines	# of Regions
1	2
2	4
3	7
4	11

To see why this pattern continues, consider what happens when we add a fifth line, as shown in bold at the right. We start the line from the top of the page, dividing the region where we start in two, thereby adding one more new region. Each time we cross one of the four old lines with our new line, we enter another region of the 4-line configuration. Then we cut that region into two pieces, again adding a new region. We therefore create 5 more new regions: one for the region we start in, plus one more for each of the 4 lines we cross. Two lines can only intersect in at most one point, so we can't make any additional new regions.

Similarly, we add 6 new regions when we draw the sixth line, 7 new regions when we draw the seventh line, and 8 new regions when we draw the eighth line. This gives a total of

$$2 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \boxed{37}$$

regions.

15.4 Work Backwards

In some math problems, we are given the situation at the end of a process and must figure out what happened to bring about that end. We can often solve these problems by working backwards from the end, reversing each step in the process described in the problem.

Problems

Problem 15.12 *Source: MOEMS* Jump to Solution

David buys a Beanie Baby. He later sells it to Jessica and loses $3 on the deal. Jessica makes a profit of $6 by selling it to Bryan for $25. How much did David pay for the Beanie Baby?

Problem 15.13 Jump to Solution

Danielle chooses a number. She multiplies it by 3, then adds 6, then divides by 3, and finally subtracts 6. Her end result is 4. What number did she choose?

Problem 15.14 *Source: MATHCOUNTS* Jump to Solution

The surface of a pond is being covered by oil leaking from a pipeline. The amount of surface area covered by the oil is doubling each day. At this rate, the surface of the pond will be completely covered with oil at the end of the twenty-fifth day. What fraction of the surface area of the pond will be covered by oil at the end of 20 days?

Problem 15.15 Jump to Solution

Alice gave Bob as many dollars as Bob had. Bob then gave Alice as many dollars as Alice then had. At this point, each had 24 dollars. How much did Alice have at the beginning?

Problem 15.16 *Source: MATHCOUNTS* Jump to Solution

In a sequence of five integers, the third integer is the sum of the previous two, the fourth integer is the sum of the previous three, and the fifth integer is the sum of the previous four. If the sum of the five integers is 248, what is the third integer in the sequence?

Problem 15.12

David buys a Beanie Baby. He later sells it to Jessica and loses $3 on the deal. Jessica makes a profit of $6 by selling it to Bryan for $25. How much did David pay for the Beanie Baby?

Solution for Problem 15.12: Rather than starting at the beginning by assigning a variable to the price David paid for the Beanie Baby, we start at the end. Jessica made a profit of $6 by selling the Beanie Baby for $25, so she must have bought it for $25 − $6 = $19. David lost $3 when he sold the Beanie Baby for $19, so he must have bought it for $19 + $3 = $22. □

Problem 15.13

Danielle chooses a number. She multiplies it by 3, then adds 6, then divides by 3, and finally subtracts 6. Her end result is 4. What number did she choose?

Solution for Problem 15.13: Just for fun, let's try working this problem "forwards." Let x be Danielle's number. We build an expression step by step for her final number:

Danielle's action	Resulting expression
Multiply by 3	$3x$
Add 6	$3x+6$
Divide by 3	$(3x+6)/3$
Subtract 6	$(3x+6)/3-6$

We know that this final number equal 4, so we have an equation:

$$\frac{3x+6}{3}-6=4.$$

It would be awfully easy to make a mistake while setting up or solving this equation. Let's try going backwards and see if that's easier.

We'll start with the end result, 4, and we'll go backwards through the steps, undoing each. For example, Danielle's last step was "subtract 6." Going backwards, we add 6 to see that Danielle's number was 10 before she subtracted 6. Here's what we find when we start with 4 and go backwards through the steps, undoing each:

Danielle's action	Undo action	Resulting number
Subtract 6	Add 6	10
Divide by 3	Multiply by 3	30
Add 6	Subtract 6	24
Multiply by 3	Divide by 3	8

Therefore, Danielle started with 8. (Note that $x=8$ satisfies our earlier equation.) □

Our "work backwards" solution to Problem 15.13 is closely related to solving the equation we wrote for the problem. Here's the equation we wrote for the problem:

$$\frac{3x+6}{3}-6=4.$$

Now, let's go through the "Undo actions" from our table, in order:

Undo action	Resulting equation
Add 6	$\frac{3x+6}{3}=10$
Multiply by 3	$3x+6=30$
Subtract 6	$3x=24$
Divide by 3	$x=8$

So, our "working backwards" is essentially just solving the equation without writing the equation in the first place!

Problem 15.14 *Source: MATHCOUNTS*

The surface of a pond is being covered by oil leaking from a pipeline. The amount of surface area covered by the oil is doubling each day. At this rate, the surface of the pond will be completely covered with oil at the end of the twenty-fifth day. What fraction of the surface area of the pond will be covered by oil at the end of 20 days?

Solution for Problem 15.14: What's wrong with this solution:

> **Bogus Solution:** Since 20 days is $\frac{20}{25}=\frac{4}{5}$ of the number of days it takes to cover the whole pond, $\frac{4}{5}$ of the pond must be covered at the end of 20 days.

This solution assumes that the amount the oil spreads each day is the same. But that's not the case! For example, on the fourth day the amount covered by oil doubles, which means that it increases by the same amount as was covered on all three previous days combined.

The portion of the pond covered doubles each day when going forwards in time. So, if we go backwards in time, the amount of the pond covered is halved each day. Since the pond is completely covered at the end of the twenty-fifth day, $\frac{1}{2}$ is covered at the end of the twenty-fourth day. Going back another day, the portion covered at the end of the twenty-third day is $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$. Similarly, we have:

End of Day	Portion Covered
25	1
24	$\frac{1}{2}$
23	$\frac{1}{4}$
22	$\frac{1}{8}$
21	$\frac{1}{16}$
20	$\frac{1}{32}$

The pond is $\frac{1}{32}$ covered with oil at the end of 20 days. □

Problem 15.15

Alice gave Bob as many dollars as Bob had. Bob then gave Alice as many dollars as Alice then had. At this point, each had 24 dollars. How much did Alice have at the beginning?

Solution for Problem 15.15: Solving this with algebra would require two variables and great care in setting up equations. Let's try going backwards. They both finish with $24. In the final step, Alice receives as much money as she has. So, she doubles her money to end up with $24. Therefore, she must have previously had only $12, and Bob gave her $12 more. Since Bob has $24 dollars after giving Alice $12, he must have had $36 before giving Alice any money.

Going back one more step, we know that Alice has $12 and Bob has $36 after Alice gives Bob money. Since Alice gave Bob as much money as Bob already had, Bob doubled his money to get to $36. Therefore, he originally had $18 before Alice gave him $18 more. Alice had $12 after giving the $18 to Bob, so she started with $12 + $18 = $30 dollars.

> **Important:** It's particularly important to check our answer when working backwards to solve a complicated problem. We can usually do so by using our answer and working forwards through the problem.

We check that Alice starting with $30 and Bob with $18 does indeed end with both of them having $24:

Action	Alice's amount	Bob's amount
Start	$30	$18
Alice gives Bob the amount Bob had	$12	$36
Bob gives Alice the amount Alice had	$24	$24

Our answer checks out, so Alice started with $30. □

All four of these problems have one key feature in common. Each tells a story in which we know the end, and are trying to figure out what the situation was sometime earlier. This is a huge clue to try working backwards from the known end situation.

> **Concept:** When we know a lot about the end state of a problem, and want to know where we started, we should try working backwards.

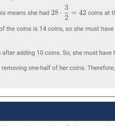
Problem 15.16 *Source: MATHCOUNTS*

In a sequence of five integers, the third integer is the sum of the previous two, the fourth integer is the sum of the previous three, and the fifth integer is the sum of the previous four. If the sum of the five integers is 248, what is the third integer in the sequence?

Solution for Problem 15.16: We know something about the end situation, so let's work backwards. The sum of all five integers is 248. But the sum of the first four integers equals the fifth integer. So, the sum of the first five integers is double the fifth integer. Therefore, the fifth integer is $248/2=124$ and the sum of the first four integers is 124.

Similarly, the fourth integer is the sum of the first three integers, so the sum of the first four integers is double the fourth integer. This makes the fourth integer $124/2=62$, and the sum of the first three integers is also 62.

We take one more step backwards. The third integer is the sum of the first two integers, so the sum of the first three integers is double the third integer. Therefore, the third integer is $62/2=31$. □



Exercises

15.4.1: *Source: AMC 8*

Granny Smith has $63. Elberta has $2 more than Anjou and Anjou has one-third as much as Granny Smith. How many dollars does Elberta have?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: Anjou has one-third as much as Granny Smith, so Anjou has $\frac{1}{3}(\$63)=\$21$. Elberta has $2 more than Anjou, so Elberta has $\$21+\$2=\boxed{\$23}$.

15.4.2:

Suppose you enter an elevator at a certain floor. Then the elevator moves up 6 floors, down 4 floors, and up 2 floors. You are then at floor 7. At which floor did you initially enter the elevator?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: You ended on floor 7 after going up 2 floors, so you were on floor 5 before going up 2 floors. You were on floor 5 after going down 4 floors, so you were on floor 9 before going down 4 floors. You were on floor 9 after going up 6 floors, so you were on floor $\boxed{3}$ before going up 6 floors.

15.4.3:

Early one morning, Joy took out one-half of the coins from her coin bank, and in the evening she put in 10 coins. The next morning, she took out one-third of the coins in the bank, and that evening she put in 4 coins. The next morning, she took out one-half the coins in the bank, leaving 16 coins. How many coins were in the bank to begin with?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: Working backwards, Joy had 16 coins after taking one-half of the coins out, so she had $2\cdot 16=32$ coins at the start of the last day.

The previous evening, she put 4 coins in the bank, so she had $32-4=28$ coins in the coin bank before that evening.

That morning, she had these 28 coins after taking one-third of the coins out of the coin bank. Therefore, these 28 coins are $\frac{2}{3}$ of the coins she had at the start of the morning. This means she had $28\cdot\frac{3}{2}=42$ coins at the start of the morning. (We also could have reasoned that if 28 is $\frac{2}{3}$ of the coins, then $\frac{1}{3}$ of the coins is 14 coins, so she must have had $28+14=42$ coins before removing a third of them.)

The previous evening, she had these 42 coins after adding 10 coins. So, she must have had 32 coins before that evening.

Finally, these 32 coins are what she had after removing one-half of her coins. Therefore, she started out with twice this amount, $\boxed{64}$ coins.

15.4.4: *Source: AMC 8*

A list of 8 numbers is formed by beginning with two given numbers. Each new number in the list is the product of the two previous numbers. Find the first number if the last three are shown:

$$\underline{\ ?\ },\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\ 16\ },\ \underline{\ 64\ },\ \underline{\ 1024\ }.$$

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: Working backwards, the fifth number times 16 is 64, so the fifth number is 4:

$$\underline{\ ?\ },\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\ 4\ },\ \underline{\ 16\ },\ \underline{\ 64\ },\ \underline{\ 1024\ }.$$

The fourth number times 4 is 16, so the fourth number is also 4:

$$\underline{\ ?\ },\ \underline{\quad},\ \underline{\quad},\ \underline{\ 4\ },\ \underline{\ 4\ },\ \underline{\ 16\ },\ \underline{\ 64\ },\ \underline{\ 1024\ }.$$

Continuing backwards, we find that the third number is 1, the second number is 4, and the first number is $\boxed{\frac{1}{4}}$:

$$\underline{\ \tfrac{1}{4}\ },\ \underline{\ 4\ },\ \underline{\ 1\ },\ \underline{\ 4\ },\ \underline{\ 4\ },\ \underline{\ 16\ },\ \underline{\ 64\ },\ \underline{\ 1024\ }.$$

15.4.5:

Serena and Visala had a combined total of $180. Serena then gave Visala $20, and then Visala gave Serena a quarter of the money Visala had. After this, they each had the same amount. How much money did Serena start with?

Preview: Solution

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: Since Serena and Visala together have $180 at the start of the problem, they will always have a total of $180 throughout the problem. At the end, they each have $90. Visala has this $90 after giving $\frac{1}{4}$ of her money to Serena. Therefore, $90 is $\frac{3}{4}$ of the money she had before giving any money to Serena. So, Visala had $\frac{4}{3}(\$90)=\$120$ before giving any money to Serena. Therefore, Visala had $120 after receiving $20 from Serena, which means Visala had $100 before receiving money from Serena. This means that Serena started with the other $180 − $100 = $\boxed{\$80}$.

15.5 Summary

In this chapter, we discussed four powerful problem-solving strategies. As you continue your math studies, you will use these strategies on all sorts of problems:

- Find a pattern
- Make a list
- Draw a picture
- Work backwards

All four of these strategies have a common goal:

Concept: Simplify the problem.

Challenge Problems

15.36:

Source: AMC 8 ✎ ✓

There are 24 four-digit positive integers that use each of the four digits 2, 4, 5, and 7 exactly once. Listed in numerical order from smallest to largest, what is the 15th integer in the list?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: For each choice of thousands digit, there are 3 choices remaining for the hundreds digit, then 2 choices remaining for the tens digit, and then finally 1 choice for the ones digit. Therefore, there are $3 \cdot 2 \cdot 1 = 6$ numbers with each thousands digit. So, there are 6 four-digit numbers that start with 2. This means that the 13th in the list is the fifth number that starts with 4. We list the six numbers that start with 4 in increasing order:

$$4247,\ 4257,\ 4527,\ 4572,\ \boxed{5224},\ 5242.$$

15.37:

The half of the water is poured out of a full container. Then one-third of the remainder is poured out. Continue the process: one-fourth of the remainder, then one-fifth of the remainder, then one-sixth of the remainder, and so on. After how many pourings does exactly one ninth of the original water remain?

Hint: What fraction of the water remains after 1 pour? After 2 pours? After 3 pours?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: After the first pour, there is $\frac{1}{2}$ of the water remaining. On the next pour, $\frac{1}{3}$ is removed, leaving $\frac{2}{3}$ of the container remaining. Therefore, the total remaining is $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$ of after two pours.

On the next pour, $\frac{1}{4}$ of the amount remaining is poured out, leaving $\frac{3}{4}$ of the container remaining. Therefore, the container is $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$ full after three pours.

We see a pattern: After the fourth pour, which is $\frac{1}{5}$ of the remaining water, there is $\frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}$ of the water remaining. Similarly, after the fifth pour, $\frac{1}{6}$ of the water remains. After the sixth pour, $\frac{1}{7}$ of the water remains. After the seventh pour, $\frac{1}{8}$ of the water remains.

And after pour $\boxed{9}$, exactly $\frac{1}{10}$ of the water remains.

15.38:

Consider all pairs of positive integers in which both numbers are less than 10. The two integers in each pair can be the same or be different. How many different products are possible if the two integers are multiplied?

Hint: We have $2 \cdot 3 = 6 = 3 \cdot 2$ and $4 \cdot 6 = 6 \cdot 4$. So, we have to be careful, since some products will appear multiple times. How can we be organized in writing all the possible products?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Clearly all the numbers from 1 to 9 can be achieved, by pairing 1 with the appropriate number. To count others in an organized manner, we make a table. We cross out all the single-digit numbers as well as any instance of a number after its first appearance:

	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	9	12	15	18	21	24	27	
4		16	20	24	28	32	36	
5			25	30	35	40	45	
6			36	42	48	54		
7				49	56	63		
8					64	72		
9						81		

We count 27 numbers in the table, and combining these with our 9 one-digit numbers, we have $\boxed{36}$ possible products.

15.39:

Source: AMC 8 ✎ ✓

A 3 by 3 square is divided into four 1 by 1 squares. The goal of this problem is to find all ways to color the square in grey or white so that exactly two of the four 1 by 1 squares are grey. How many ways are there? Hint: Rotations and reflections are considered different. Hint: Rotations are considered different. Hint: Rotations are considered different.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: We make an organized list of pictures. First, we consider the configurations in which the bottom left square is grey, since both of the restrictions on grey squares apply to this square. Neither the square above it nor the square to its right can be white, so these two squares must also be grey. The upper right corner is both above and to the right of a grey square, so it must be grey too. This gives us the one configuration shown to the right.

Now, we fix the bottom left as white and consider the cases for the upper right. It cannot be grey, too, but there's no restriction on the bottom right square. This gives us the two configurations shown on the right.

Having covered the cases in which at least one of the left squares is grey, we now consider those in which the bottom left square is white. Since the bottom right square is adjacent to the grey and the upper right square white, but the other three configurations with both squares on the left white are OK.

This gives us a total of $\boxed{5}$ possible configurations.

15.40:

Source: AMC 8 ✎ ✓

A certain calculator has only two keys [ ] ✓ and [ ] ✓ When you press one of the keys, the calculator automatically displays the result of applying the corresponding operation to the number on the screen. Initially the number 1 is on the screen. If you pressed [ ] ✓ it would display "2", then pressing [ ] ✓ would then display "4". What is the smallest number that can be displayed with exactly three "[ ] ✓"?

Hint: It's not immediately obvious what four first few presses should be if we want to get from 1 to 200 as fast as possible. If we just double 7 times, we'll get to 128. Then what? If it's not obvious how to proceed going forwards...

Hint: What is probably the best button we'll press if we get from 1 to 200 as fast as possible?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Working forwards from 1, it's not at all clear what a good strategy is to get to 200. But if we work backwards, it becomes more clear. So, instead going from 1 to 200 by adding 1 or doubling at each step, we start from 200 and work our way down to 1 by subtracting 1 or halving at each step.

If we simply halve whenever possible, we get

$$200, 100, 50, 25, 24, 12, 6, 3, 2, 1.$$

This gives us $\boxed{8}$ steps.

To see why we cannot get to 200 in 8 steps, we first note that pressing either [ ] ✓ or [ ] ✓ on the first step results in the new number being even. So, we must have pressed the button on the second step, because we need to get to an odd number. If we pressed, starting from 2 on the displayed number, if we press [ ] ✓ 3 times, we get 24. What if we instead start with 3 displayed, and press [ ] ✓ 4 times and [ ] ✓ once? If we first press [ ] ✓ and then [ ] ✓ 3 times, we get $3 \cdot 2^4 = 192$. Pressing [ ] ✓ on any other step results in 2^n or in the other ways results in even numbers, so, we can't get to 200 in 8 total steps. The highest we can reach in 7 steps is $2^7 = 128$, so we can't reach 200 in 7 or fewer steps, either.

15.41:

The product of three positive integers (not necessarily different) is 40. How many sets of 3 integers have this property if the order of the 3 integers is a set does not matter?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: We make an organized list. We list the numbers in each set from least to greatest. We'll start with the sets that begin with 1.

$$\{1, 1, 40\},\quad \{1, 2, 20\},\quad \{1, 4, 10\},\quad \{1, 5, 8\}.$$

Next, we move on to those that begin with 2 (so the other two numbers multiply to 20, and 1 is not among the numbers):

$$\{2, 2, 10\},\quad \{2, 4, 5\}.$$

There are no sets in which 4 is the smallest number, since such a set would have a product of at least $4 \cdot 4 \cdot 4 = 64$. Therefore, there are $\boxed{7}$ sets of numbers that satisfy the property in the problem.

15.42:

How many paths with length 9 units are there from A to B along the grid at the right? Notice that one segment in the path along the right-most grid line has length 2 units.

Hint: For the dots that are 1 step away from A, determine how many paths there are to B that only go upward or rightward at each step. What about the dots that are 2 steps away from B? Or the dots that are 3 steps away from B?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Our path can only be 1 unit away, and 2 and 3 are 9 unit steps apart along the grid. Therefore, each step we can go up or to the right. We start backwards to count the number of total paths from A to B. We'll label points in the grid with letters so we can refer to them. To the upper right of A point, we'll place the number of ways to get from that point to B along only upward or rightward steps. We start by putting 1's along the top and the right, since there's only one path to B from each of these points.

Then, we turn to point C. We can either go up one step from C or right one step from C. We know the number of ways to get from the point above C to B and the point to the right of C to B. So, we can add up ways to finish. The number of ways to finish from C is $1 + 3 = 4$.

We place a 2 at C in the grid, as shown below. Now, we can continue working backwards. From point D, we can either go up to C or to the right. We know there are 2 ways to go right, from which point we know there is 1 path. So, there are $2 + 1 = 3$ paths from D. From E, we can either go up to D from which there are 3 paths, or go right 1, from which there's one way to finish. So, there are $3 + 1 = 4$ paths from E.

Going down the next column, we see that from F, we can go right to C, from which there are 2 ways to finish, or we can go up 1, from which there is 1 way to finish. This gives $2 + 1 = 3$ paths from F. From G, we can go up to F, from which there are 3 paths, or we can go right 1, from which there are 4 paths. So, there are $3 + 4 = 7$ paths from G. From H, we can go up to G, from which there are 7 paths, or go right 1, from which there are 4 paths. So, there are $7 + 4 = 11$ paths from H.

Continuing in this manner, we can work our way backwards throughout the entire grid. Notice that we are careful to remember that we cannot go up from point X to point X in the grid. We can only go rightward from X, or the number of paths from X to B is the same as the number of paths from X to B. We find that there are $\boxed{53}$ paths from A to B.

15.43:

(a) Mario starts hiking on a path at noon, and stops at 8 p.m. He rests until noon the following day, and then starts hiking back towards his original starting point along the same path. He gets back to the starting point 9 hours after he starts walking. He walks at constant speed on the way out and on the way back. Show that there is a point on the path that Mario visits at the exact same time on both days.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: We make an organized list of time (vertical) versus distance from the starting point (horizontal). The bottom of the graph is noon and the top is 8 p.m. On the outbound trip, the graph must go from the bottom left corner (noon at the starting point) to the upper right corner (8 p.m. at the end of his first hike). One possible such graph is shown below. We know that on the way back, he walks at the same speed, so the slope of the second day's graph is point. So, the upper left (8 p.m. back at the original starting point). Clearly the graph for the second day must intersect the graph for the first day at some point. At that time, he is at the same point on both days. Hence, he is the same on both days, so Mario is at the same place on the trail at the same time.

Another way to think about this problem is to imagine that Mario took both trips on the same day. Then, Mario and his after ego must meet on the path somewhere!

(b) Suppose Mario only takes 4 hours on his return trip. Must there still be a point on the path that Mario visits at exactly the same time on both days?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: The situation is essentially the same as in part (a). The only difference here is that Mario's return path graph ends at point 4 p.m. instead of 8 p.m. Hence, at noon on 4 p.m., back at the original starting point. Points A and C will always be on opposite sides of Mario's first day graph (or C will be on the first day graph if Mario just stays at the start point for four hours on the first day), so the second day graph and the first must intersect.

15.44:

In the game of Tribal, each team has five players. In Tribal league, each player is on exactly two different teams, and no two players are teammates on both teams. How many possible rosters are there in Tribal league?

Hint: Draw a picture. Let each player be a point. What will teams be?

Hint: Each team is a triangle connecting three points. How many points do you need to satisfy the conditions in the problem?

You may type any additional notes you have here.

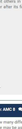
Hide Solution | Reset

Your Submission Solution

Solution: We start by drawing a picture. Each player is represented by a point, and each team is represented by a triangle connecting the points of the players on that team. Suppose the players on the first team are A, B, and C. We draw triangle ABC to represent this team. Player A must be on another team, but B and C cannot be on this team. So, we need two more players, D and E. We draw team ADE dashed to distinguish it from team ABC.

Next, D must be on another team. D cannot be on a team with both D and E, since D and E cannot be together on a second team. So, we need another player, F. We then form team DEF, which we represent with a dotted triangle. Finally, we form team CEF.

We therefore see that it is impossible to have more than 5 players, and the diagram at the right shows that it is possible to have a league with exactly $\boxed{15}$ players.



15.45:

Find the last two digits (the tens digit and the units digit of 2^{2011}.

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Just as the units digit of a product depends only on the units digits of the numbers being multiplied, the last two digits of a product depend only on the last two digits of the numbers being multiplied. So, rather than computing powers of 2, we just start with 2 and only keep track of the last two digits as we multiply by 2 over and over. These are the first few results:

$$07,\ 49,\ 43,\ 03,\ 07,\ 49.$$

We see that the final two digits of powers of 2 repeat in a cycle of 4 terms. Since 2011 divided by 4 has a remainder of 3, the last two digits of 2^{2011} are the third term in this cycle, which is $\boxed{43}$.

15.46:

Consider a 5 x 5 square with 25 cells. Each pair of cells is directly connected by either a train route or by an airplane route. Explain why there must be three towns such that all three are directly connected to each other by the same mode of transportation.

Hint: Let's start with a 5 x 5 grid.

Hint: Is it possible for one town to be connected to exactly two towns by train and exactly two towns by airplane?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Let J be one of the cities. Only J must be connected to at least three of the other cities by the same mode of transportation. To see why, consider what happens if J is not connected to at least three of the other cities by the same mode of transportation. Then J can be connected to at most 2 by train and at most 2 by airplane. That's a total of 4 connections, but J must be connected directly to all 24 other cities. So, J must be connected to at least 3 of the other cities by the same mode of transportation.

Suppose these three cities are Z_1, C, and Z_2. We illustrate that J is connected to them by the same mode of transportation with solid line segments in the diagram. The dashed lines represent the other mode of transportation. Actually, it could also be connected to Z_1 and/or Z_2 by solid lines, but all that matters is that Z_1 and Z_2 are also connected to Z_1 and Z_2.

Now, consider any other city X. If X is connected to Z_1 by the same mode as J is, then these two cities and J are directly connected to each other by the same mode of transportation. However, if X has of Z_1 and Z_2 and are connected by a solid line segment, then they must all be connected by dashed segment. Then cities Z_1, C, and Z_2 are directly connected to each other by the same mode of transportation.

Therefore, there must be three towns that are directly connected to each other by the same mode of transportation.

15.47:

A cryptarithm is a math puzzle in which the digits in a simple equation are replaced with letters. Each digit is represented by one letter, and each letter represents a different digit. So, for example, we might represent $53 + 50 = 101$ as $AB + BC = BCD$. In the cryptarithm $SEND + MORE = MONEY$, what digit does the letter Y represent?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: The sum of two four-digit numbers cannot be 20000 or greater. So, if the sum of two four-digit numbers is a five-digit number, then the first digit of the five-digit number is 1. This tells us that $M = 1$. We place that in the appropriate places:

Next, we note that the sum $S + 1$ in the thousands place, plus a carry from the hundreds place if there is one, must be 10 or greater. So, there must be a carry from the hundreds place to the thousands place. This tells us that $S \geq 9$.

We know that S is the same in the S place in the thousands place to the thousands in the sum, then $E + 10$ is 9 and we have a carry from the tens to the hundreds. Then, N would have to be 0, which is already taken. Therefore, there is no carry from the hundreds place to the thousands place. This tells us that $S = 9$.

Still looking at the hundreds place, we see that N is one more than E, and that there must be a carry from the tens to the hundreds place. Since N is one greater than E, the letters in the tens place will be a set that either E is N or N is one greater than E, so there is a carry in the units place. This is always taken, we know R is 8.

Now, we know that N is one greater than E, and that there must be a carry in the units place. The greatest E can be is 7, since 8 and 9 are taken. So, E cannot be 6, since this would make a carry in the units place impossible. We continue with guess-and-check, trying values of E to see which works. We only have to check 3, 4, 5, and 7, since $E = 2$ forces N to be 0, which is taken.

Case 1: E is 3. Then D must be 7, which forces Y to be 0. But 0 is already taken, so E cannot be 3.

Case 2: E is 4. Then D must be a 6. Y which forces Y to be 1. Again, this is impossible, so E cannot be 4.

Case 3: E is 5. Then D cannot be 6 (since this forces Y to be 1), but D can be 7. Then we can complete the cryptarithm as follows:

Case 4: E is 7. This is just to make sure there aren't two solutions. Then, N must be 1, so D can only be 4 or 5. But this forces Y to be 1. Once again, this is impossible, so E cannot be 7.

We conclude that the cryptarithm reads for 9567 + 1085 = 10652, so $Y = \boxed{2}$.

15.48:

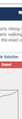
Five married couples get together at a party. At the start of the party each person shakes hands with everyone they didn't know before the party. After all the handshakes, Kyle, one of the husbands, asks everyone else how many hands they shook. He received each number from 0 to 8 as an answer once. How many hands did Kyle shake?

You may type any additional notes you have here.

Hide Solution | Reset

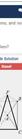
Your Submission Solution

Solution: We draw a picture. We represent each person with a point, and we draw a segment between two points if the two people shook hands. Since no one shakes hands with their spouse, for each person, the person who shook hands with 8 people must have shaken hands with everyone except his or her spouse. Let's call this person X, and let X_1 represent the spouse of X. We then connect X with all the other people as shown. Now, everyone except X has shaken hands with at least one person, so we can add in lines from X to each other person.

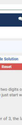


Next, we note that the person who shook 7 hands must have shaken hands with everyone except X and his or her own spouse. We call this person C, and connect this person to everyone except his spouse and X. Now everyone X and X's spouse has shaken hands at least twice, which means X's spouse must be the person who shook hands with just 1 person. We'll call this spouse D.

Continuing in the same manner, the person who shook hands with 5 others must have shaken hands with everyone not labeled so far (besides his or her spouse), and the person who shook hands with 2 people must be the person who shook hands with 3 others. We label these people E (5) and F (2).



Next, the person who shook hands with 3 others must have shaken hands with the two unlabeled people and their spouse. We call this person G, and connect this person to everyone except his spouse and G's spouse. Finally, the person who shook hands with 4 people must have shaken hands with everyone except himself, his spouse and G's spouse has shaken hands at least twice, which means G's spouse must be the person who shook hands with just 1 person. We'll call this spouse H.



Now, we have accounted for all people except X and X_1. We know that X shook hands with everyone except himself and his spouse, so we know that X_1 shook hands with everyone except himself and his spouse. Hence, X_1 shook hands with everyone else, so we can add in lines from X_1 to everyone else.

Now, we can see that X shook hands with 3 people, X_1 shook hands with 9 people, C shook hands with 7 people, D shook hands with 6 people, E shook hands with 5 people, F shook hands with 4 people, and G shook hands with 2 people. Hence, Kyle shook hands with 8 people.

15.17:

The last Monday of a particular month is on the 27th day of the month. What day of the week is the first day of the month?

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: Counting backwards by 7s, the 20th, 13th, and 6th are also Mondays. Counting backwards 3 more days, we see that the first day of the month is Wednesday.

15.18: Source: MCS95 ✓ ✓

Dan, Harry, and Ken each like a different sport among tennis, baseball, and soccer. Dan does not like baseball or soccer. Harry does not like baseball. Name the favorite sport of each person.

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: We start by making a grid with one row for each person and one column for each sport. We then include the given information by placing X's in the baseball column for Dan and Harry and in the soccer column for Dan. Since each person has a favorite sport, we can fill in Ken's tennis and that line is the only person who can possibly fill baseball.

	Tennis	Baseball	Soccer
Dan		X	X
Harry		X	
Ken	X		

We place X's in the grid for Dan's and Ken's sports, and X's in the tennis and soccer columns for Harry. Now only Harry for soccer, and we now have the completed grid at the right. Conclusion: Dan's favorite sport is soccer, and Ken's is baseball.

	Tennis	Baseball	Soccer
Dan		X	X
Harry	X	X	X
Ken	X	X	

15.19: Source: MCS95 ✓ ✓

The "Cartesian" sequence, each number after the first number is the sum of the preceding three numbers. For example, if the first three numbers are 5, 6, and 7, then the fourth number is 18 because $5 + 6 + 7 = 18$, and the fifth number is 31 because $6 + 7 + 18 = 31$. Write a program that prompts the user for the first three numbers and prints the sequence as $P, Q, R, 150$, and 251 in that order. What is the value of P?

You may type any additional notes here.

Hide Solution

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Your Submission Solution

Solution: Working backwards, the fifth number is the sum of the second, third, and fourth, so $Q + R = 150 - 251$. Simplifying the left side gives $Q + 2R = -101$, so $Q = R - 101$. The sum of the first three numbers equals the fourth, so $P + Q + R = 150$. Since $Q = R - 101$, we have $P + 150 = 150$, which means $P = \boxed{0}$.

15.20: Source: MCS95 ✓ ✓

In the table below, all the positive integers are arranged in columns. Under what letter will the number 100 appear?

	1	2	5	6
3	4	7	8	13
9	10	14	15	21
11	12	16	17	22
18	19	23	24	31
20	25	26	32	33
27	28	34	35	42
29	30	36	37	43
38	39	44	45	52
40	41	46	47	53
48	49	54	55	62
50	51	56	57	63
58	59	64	65	72
60	61	66	67	73
68	69	74	75	82
70	71	76	77	83
78	79	84	85	92
80	81	86	87	93
88	89	94	95	102
90	91	96	97	103
98	99	104	105	112
100	101	106	107	113
108	109	114	115	122
110	111	116	117	123
118	119	124	125	132
120	121	126	127	133
128	129	134	135	142
130	131	136	137	143
138	139	144	145	152
140	141	146	147	153
148	149	154	155	162
150	151	156	157	163
158	159	164	165	172
160	161	166	167	173
168	169	174	175	182
170	171	176	177	183
178	179	184	185	192
180	181	186	187	193
188	189	194	195	202
190	191	196	197	203
198	199	204	205	212
200	201	206	207	213

We get a little bigger than 100 in column $\boxed{F}$.

15.21:

Find the largest integer that is less than 1000, two more than a multiple of 3, two more than a multiple of 5, and odd.

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: 1000 is 2 more than a multiple of 3. We start from here and count down by 15 to generate a list of numbers that are 2 more than a multiple of 3. We rule out any of the numbers that are even and then check the remaining numbers to see if they are two more than a multiple of 5.

1000, 997, 982, 967, 952, 937, 922, 907, 892, 877, 862, ...

1000 could have counted down by 15 times, so we know the number must be odd. 997 is clearly a multiple of 5, so 997 is a number that is a multiple of 3. Since 997 is 5 less than 1000, we have that 967 is a multiple of 5. Finally, 937 is 2 more than 935, which is a multiple of 5. Therefore, the desired answer is $\boxed{937}$.

15.22: Source: AMC 8 ✓ ✓

Ann produces a sequence of positive integers by following the three rules below. She starts with a positive integer then applies the appropriate rule to the result, and repeats.

- **Rule 1:** If the integer is less than 10, multiply it by 9.
- **Rule 2:** If the integer is even and greater than 9, divide it by 2.
- **Rule 3:** If the integer is odd and greater than 9, subtract 5 from it.

Here is a sample sequence: 23, 18, 9, 81, 76, ..., 19. Find the 99th term of the sequence that begins 99, 45, ...

You may type any additional notes here.

Hide Solution

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Your Submission Solution

Solution: Applying the rules in the problem, we have

99, 45, 44, 22, 11, 6, 54, 27, 22, 11, 6, 54, ...

Once we hit 22 for a second time, we realize that the sequence is going to repeat the 5-term block

22, 11, 6, 54, 27

over and over. We have to be careful, though, because there are 3 terms in the sequence before we start the repeating block. Therefore, to get to the 99th term of the sequence, we write these first 3 terms and then write the repeating block of 5 terms 19 times. So, the 99th term of the sequence is the last term of the repeating block, which is $\boxed{27}$.

15.23: Source: MCS95 ✓ ✓

Jones began with a number. He doubled the number to 2, subtracted 5 from the quotient, built the square root of the difference, added to the square root, and took the square root of the sum. His final result was 3. What was Arthur's original number?

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: We work backwards, starting from the final result, 3, and reversing each action.

Action taken	Reverse action	Result
Take square root	Square	9
Add 1	Subtract 1	8
Take square root	Square	64
Subtract 6	Add 6	70
Divide by 2	Multiply by 2	140

Arthur's original number was $\boxed{140}$.

15.24: Source: AMC 8 ✓ ✓

In the table below, the integers from 100 down to 2 are arranged in columns P, Q, R, and S. Write the letter of the column that contains the number 25.

P	Q	R	S
	100	99	98
93	92	91	90
	85	86	87
81	80	79	78
	71	72	73
69	68	67	66
	57	58	59
55	54	53	52
	43	44	45
41	40	39	38
	29	30	31
27	26	25	24
	15	16	17
13	12	11	10
	3	4	5
2	1		

You may type any additional notes here.

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Your Submission Solution

Solution: The first number in column P is the sum of the first 0 numbers so put in the table. The next number in column P is the sum of the first 1 number, 93, so put in the table. The next number in column P is the sum of the first 2 numbers, so we put the sum of the numbers in column P by starting with 93 and counting downward by 93, since 93 is greater than a multiple of 3, and we form column P by counting downward from 93 by 93, since P has all the positive numbers less than 100 and that is more than a multiple of 3.

Similarly, each number in column S after the first 0 is less than the previous number in column S. The first number in column S is 98. 99 is 1 more than a multiple of 3, and we form column S by counting downward from 99 by 3, since S has all the positive numbers less than 100 that are 1 more than a multiple of 3. Since 25 is 1 more than a multiple of 4, we know that 25 is in column $\boxed{R}$.

15.25:

Jack is a slow eater who spent one half of her money and then $15 more. She then went to another store where she spent one third of her remaining money and then $15 more. She then had $4 left. How much did she have when she entered the first store?

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: We work backwards through Lu's actions in the table below:

Amount after action	Lu's action	Amount before action
$20	Spent $15 more	$35
$30	Spent one third of her money	$45
$90	Spent $15 more	$105
$90	Spent half her money	$180

So, she started with $$\boxed{\$180}$$.

15.26:

Find the units digit (zero digit) of 3^{4^5}.

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: We find the units digits of the first few powers of 3 and look for a pattern. The first few powers of 3 are $3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 177147, 531441, 1594323, 4782969, 14348907, 43046721$. It seems that the units digit of 3^n is 3, 9, 7, 1, and back. Since 4^5 is 256, we know that the units digit of 3^{4^5} is the same as the units digit of 3^{256}. So, the units digit of 3^{4^5} is the last number in this repeating 4-term block, which is $\boxed{1}$.

15.27: Source: AMC 8 ✓ ✓

Four generous friends, each with some cash, redistribute their money as follows: Ann gives enough money to Jan and Tip to double their amount of money that each has. Jan then gives enough to Ann and Tip to double their amounts. Finally, Tip gives Ann and Jan enough to double their amounts. If Tip has $56 when they begin and $56 when they end, what is the total amount that the four friends have?

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: We work backwards through Tip's actions in the table below:

Tip's $56	Tip's action	Tip's $56
$28	Gave Ann and Jan enough to double their amounts	$56
$14	Gave Ann and Jan enough to double their amounts	$28
$7	Gave Ann and Jan enough to double their amounts	$14
$56	Gave Ann and Jan enough to double their amounts	$7

Solution: Tip has $56 at the beginning then Ann doubles Tip's money in the first step to $72 and Jan doubles it to $112 in the next step. Tip has $56 at the end, so we know that Jan ended up with $112 − $56 = $56 in the last step. This means that Jan ended up with the amount of money Ann and Jan have, so Ann and Jan have a total of 2 · $56 = $112 in the end. Combining this with Tip's $56, the three together must have $$\boxed{\$168}$$.

15.28:

Two-thirds of the people in a room are seated in three-fourths of the chairs. The rest of the people are standing. If there are 5 empty chairs, how many people are in the room?

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: Three-fourths of the chairs are used, so one-fourth of the chairs are not used. Since 5 chairs are not used, there must be 20 chairs total and 15 people seated. Since two-thirds of the people are seated, the total number of people is $15 \div \frac{2}{3} = 15 \cdot \frac{3}{2} = \boxed{22}$ people total.

15.29: Source: AMC 8 ✓ ✓

A box contains gold coins. If the coins are equally divided among six people, four coins are left over. If the coins are equally divided among five people, three coins are left over. If the box holds the smallest number of coins that meets these two conditions, how many coins are left over when equally divided among seven people?

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: The number of coins must be 4 more than a multiple of 6, so we make a list of such numbers by starting with 4 and counting by 6's:

4, 10, 16, 22, 28, ...

The first number in this list that is 3 more than a multiple of 5 is 28. When these 28 coins are divided among 7 people, there are $\boxed{0}$ left over.

15.30: Source: AMC 8 ✓ ✓

Suppose there is a special key on a calculator that replaces the number a number currently displayed with the number given by the formula $\frac{1}{1-x}$. For example, if the number 2 is displayed then pushing the key will replace 2 with $\frac{1}{1-2} = -1$. Pushing the key again will replace -1 with $\frac{1}{1-(-1)} = \frac{1}{2}$. Now suppose the calculator is displaying 3. After the special key is pressed 100 times in a row, what number will the calculator display?

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: We make a list of the numbers that occur as we press the key over and over:

Press number	Calculation Sheet	Resulting number
1	$\frac{1}{1-3} = \frac{1}{-2} = -0.25$	-0.25
2	$\frac{1}{1-(-0.25)} = \frac{1}{1.25} = 0.8$	0.8
3	$\frac{1}{1-0.8} = \frac{1}{0.2} = 5$	5

Once we see the 3 repeat, we know that the same sequence will repeat over and over: $-0.25, 0.8$. We do have to be careful, the calculator already has the 3 displayed when we press special for the first time. So, -0.25 is what results after the first press, 0.8 is what results after the second press, and 3 is what results after the third press. So, after 100 presses, we have -0.25. Since 100 divided by 3 has a remainder of 1, the 100th number that results is the first in this repeating block, which is $\boxed{-0.25}$.

15.31:

You want to bring a wolf, a goat, and a cabbage across a river. You are the cover and don't get out of the boat. The wolf wants to eat the goat, and the goat wants to eat the cabbage. The wolf will not bite you if you bring as you are now. Besides yourself, there is room for only one item in the boat. How can you bring all three across the river?

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: When you take the first trip across the river, you will have to leave two of the items on shore. The only two you can leave together without an undesirable (for you) eating incident are the cabbage and the wolf. So, clearly you have to take the goat across first.

Thinking ahead, during your final trip across the river, you will have two items already on the destination shore while you are bringing the third across. So, you want to be sure that the two items you leave on the starting shore will not cause a problem. The goat and the cabbage in addition to being the last item brought across. That means you'll have to take the goat across the river in the "wrong" direction at some point.

We organize the information with a picture. We have two sides to our river and a letter for the cabbage (C), goat (G), and Wolf (W), and you (Y).

	LEFT	RIGHT
Y	C	W G
GRT	W	C
Y	C W	G
GRT	C	W
Y	C G	W
GRT		C W G

15.32: Source: AMC 8 ✓ ✓

How many positions of P between Q and R in Problem 18.7 in Section 15.2 have for the definition of a perfect $\sqrt{\ }$?

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: We organize the list of partitions of 7 by the largest number in the partition.

7

Largest number is 6:

6 1

Largest number is 5:

5 2, 5 1 1

Largest number is 4:

4 3, 4 2 1, 4 1 1 1

Largest number is 3:

3 3 1, 3 2 2, 3 2 1 1, 3 1 1 1 1

Largest number is 2:

2 2 2 1, 2 2 1 1 1, 2 1 1 1 1 1

Largest number is 1:

1 1 1 1 1 1 1

Counting them all, we find that we have $\boxed{15}$ partitions of 7.

15.33:

Four people come to a river in the night. There is a narrow bridge that can hold only two people at a time. Because it's night, a torch must be used when crossing the bridge. Unfortunately, they have only one torch. The four people need to get across the river as a group. You cannot throw the torch across, and it is dangerous to walk across the bridge at 1 minute. When two people cross the bridge together, they must move at the slower person's pace. The torch burns out in 15 minutes. Find a way for all four people to get across the bridge before the torch burns out.

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: The key insight is that if the 5-minute person and the 8-minute person (that is, the slow people) travel separately, then those two will take 13 minutes total. So, we want to have the 1-minute and 2-minute people cross first, and then one of them will come back with the original torch. The other slow person completes the journey. This is impossible to accomplish, so we look for a process in which the two slower people cross the bridge together.

Since the people can't be the last to cross, since there would be no way for them to get the torch from the faster people on the other side of the bridge before crossing. So, not only must the slow people cross together, but there must be a fast person on the side of the bridge to bring the torch back. This requires our solution send the two slow people across together, and have one person with the torch for the slow people. Then the slow people cross, and the fast person on the far side brings back the torch to fetch the last person. Now, everyone is across in 15 minutes.

We organize the information in the table shown at the right. Initially, all four start on the left side, where we use numbers to show how long each person takes to cross the bridge.

	Left	On-bridge	Right
	1 2 5 8		
	5 8	1 2	
	5 8	1	2
	5 8	2	1
	5	1 2	8
	5	1	2 8
		2	1 5 8
		1	2 5 8

The Estimate and the Estimate people must travel across together, but someone also needs to be on the far side to return the torch quickly. Therefore, we send the 1-minute and 2-minute people across first. The 1-minute person then brings the torch back, and the 5-minute and 8-minute people then cross. Finally, the 2-minute person brings the torch back, and the 1-minute and 2-minute people cross together at the end. (The solo trips of the 1-minute person and 2-minute person could be swapped.) Adding the underlined numbers, we see that the process takes 15 minutes, as required.

15.34: Source: AMC 8 ✓ ✓

Find six positive integers that have these properties:

1. the sum of the squares of their digits is 85; and
2. each digit is larger than the one to its left.

Find the largest integer with these properties.

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: We need a sum of distinct different positive squares that equals 50. We search for such a sum by first listing the squares that are less than 50:

1, 4, 9, 16, 25, 36, 49.

We quickly see one such sum which adds up to 85, namely $49 + 9 + 16 + 1 + 4 + 25 = 85$. Therefore, the number 123456 satisfies both properties. We next check 25, and we find $25 + 36 + 9 + 4 + 1 + 1 = 85$. So, the number 123456 is not the largest. We then try other squares in the list to less than 50, so there are no more possibilities. Therefore, the largest number that satisfies both properties is $\boxed{123456}$.

15.35:

Cindy walks at a constant rate of 2 miles per hour. She leaves home to walk to her friend Jenny's house at 6 am. When she is halfway from her house to Jenny's house, she sees her dog Jack for the first time. Jack runs at a constant rate of 6 miles per hour. Jack runs from Cindy's house to Jenny's house (there she turned around), she finds her phone. She turns back around to walk to Jenny's and arrives there at 10 am. How far apart are Cindy's house and Jenny's house?

Hint: Draw a diagram!

You may type any additional notes here.

Hide Solution

Email

Your Submission Solution

Solution: In the diagram, C is Cindy's house, J is Jenny's house, B is the halfway point where Cindy turns around, and P is where she finds the phone. Let d be the distance in miles between the two houses, so $CJ = d$. Since $CB = \frac{1}{2}CJ = \frac{d}{2}$, $BJ = \frac{1}{2}CJ = \frac{d}{2}$.

Since Cindy walks 2 miles per hour for 1 hour, she walks a total of 2 miles. Therefore, we have $\frac{d}{2} = 2$. Multiplying both sides by 2, we get $d = 4$.

$$\text{speed} = 2 = \left(\frac{1}{2}\right) \cdot \left(\frac{d}{2}\right) = \frac{d}{4} = \frac{d}{4} \cdot \frac{M}{H} = \frac{d}{4} \cdot \frac{6}{1}$$

Since Cindy walks 2 miles per hour for 1 hour, she walks a total of 2 miles. Therefore, we have $\frac{d}{2} = 2$. Multiplying both sides by 2, we get $d = 4$.