Acknowledgements

Contests

We would like to thank the following contests for allowing us to use a selection of their problems in this book:

- The **American Mathematics Competitions**, a series of contests for U.S. middle and high school students. The **AMC 8**, **AMC 10**, and **AMC 12** contests are multiple-choice tests that are taken by over 350,000 students every year. Top scorers on the AMC 10 and AMC 12 are invited to take the **American Invitational Mathematics Examination (AIME)**, which is a more difficult, short-answer contest. Approximately 10,000 students every year participate in the AIME. Then, based on the results of the AMC and AIME contests, about 500 students are invited to participate in the **USA Junior Mathematical Olympiad (USAJMO)** and **USA Mathematical Olympiad (USAMO)**, 2-day, 9-hour examinations in which each student must show all of his or her work. Results from the USA(J)MO are used to invite students to the Math Olympiad Summer Program, at which the U.S. team for the **International Mathematical Olympiad (IMO)** is trained. More information about the AMC contests can be found on the AMC website at http://www.maa.org/math-competitions.
- **MATHCOUNTS®**, the premier contest for U.S. middle school students. MATHCOUNTS is a national enrichment, coaching, and competition program that promotes middle school mathematics achievement through grassroots involvement in every U.S. state and territory, with over 160,000 students participating in 2013-14. President Barack Obama, and former Presidents Bush, Clinton, Bush and Reagan have all recognized MATHCOUNTS in White House ceremonies. The MATHCOUNTS program has also received two White House citations as an outstanding private sector initiative. More information is available at http://www.mathcounts.org.
- **Math Olympiads for Elementary and Middle Schools (MOEMS)**, an international program for students in grades 4-8. Created in 1977 by Dr. George Lenchner, MOEMS offers beginning problem solvers 5 contests each year. In 2013-14, nearly 150,000 students worldwide from 6,000 teams participated in MOEMS. More information is available at http://www.moems.org.

How We Wrote This Book

This book is a collaborative effort of the staff of the Art of Problem Solving. The three authors listed on the cover would like to particularly thank our colleagues listed below:

Jason Batterson, Jeremy Copeland, Larry Evans, and Shannon Rogers, for providing extensive proofreading, as well as valuable guidance on pedagogy and subject coverage;

Vanessa Rusczyk, for designing the cover and also contributing greatly to the interior design of the book;

Arun Alagappan, Founder and President of Advantage Testing (employer of one of the authors), for supporting this project and for providing a model of educational excellence for the last 25 years; and

Josh Zucker, whose comments about how he learned mathematics inspired the questions-before-the-lessons approach of the text.

The print version of this book was written using the LaTeX document processing system, and the diagrams were prepared using Metapost and Asymptote. We thank the authors of the various LaTeX packages that we used while preparing this book, and also the brilliant authors of *The LaTeX Companion* for writing a reference book that is not only thorough but also very readable.

The source files for the print book were initially converted to this online book using a script written in the Python programming language. Palmer Mebane managed the conversion of this book from print to online, and Paul Salerno also wrote tools used in the conversion. Jason Batterson and the Jacob Tyler Creative Group designed the look and feel of this online book. James Fung, Shelley Garg, Kyle Guillet, Tasha Moyer, David Patrick, Shannon Rogers, Amy Szczepanski, Deven Ware, Phyllis Xu, and Laura Zehender all helped to review and edit the online book content.

Dedication

R.B.–I dedicate this book to my mom and dad (Krishna Rao and Jaganmohan Rao), my wife Ranu, and my daughter Meena. Thank you for always believing in me.

D.P.–Personal thanks to my most influential mathematical mentors:
my parents (when I was very young),
Gerry Rising (when I was moderately young),
and Michael Artin (when I was not-so-young).

R.R.–For my students, in the hopes that they learn from me as much as I learn from them.

Acknowledgements

Contests

We would like to thank the following contests for allowing us to use a selection of their problems in this book:

- The **American Mathematics Competitions**, a series of contests for U.S. middle and high school students. The **AMC 8**, **AMC 10**, and **AMC 12** contests are multiple-choice tests that are taken by over 350,000 students every year. Top scorers on the AMC 10 and AMC 12 are invited to take the **American Invitational Mathematics Examination (AIME)**, which is a more difficult, short-answer contest. Approximately 10,000 students every year participate in the AIME. Then, based on the results of the AMC and AIME contests, about 500 students are invited to participate in the **USA Junior Mathematical Olympiad (USAJMO)** and **USA Mathematical Olympiad (USAMO)**, 2-day, 9-hour examinations in which each student must show all of his or her work. Results from the USA(J)MO are used to invite students to the Math Olympiad Summer Program, at which the U.S. team for the **International Mathematical Olympiad (IMO)** is trained. More information about the AMC contests can be found on the AMC website at http://www.maa.org/math-competitions.
- **MATHCOUNTS®**, the premier contest for U.S. middle school students. MATHCOUNTS is a national enrichment, coaching, and competition program that promotes middle school mathematics achievement through grassroots involvement in every U.S. state and territory, with over 160,000 students participating in 2013-14. President Barack Obama, and former Presidents Bush, Clinton, Bush and Reagan have all recognized MATHCOUNTS in White House ceremonies. The MATHCOUNTS program has also received two White House citations as an outstanding private sector initiative. More information is available at http://www.mathcounts.org.
- **Math Olympiads for Elementary and Middle Schools (MOEMS)**, an international program for students in grades 4-8. Created in 1977 by Dr. George Lenchner, MOEMS offers beginning problem solvers 5 contests each year. In 2013-14, nearly 150,000 students worldwide from 6,000 teams participated in MOEMS. More information is available at http://www.moems.org.

How We Wrote This Book

This book is a collaborative effort of the staff of the Art of Problem Solving. The three authors listed on the cover would like to particularly thank our colleagues listed below:

Jason Batterson, Jeremy Copeland, Larry Evans, and Shannon Rogers, for providing extensive proofreading, as well as valuable guidance on pedagogy and subject coverage;

Vanessa Rusczyk, for designing the cover and also contributing greatly to the interior design of the book;

Arun Alagappan, Founder and President of Advantage Testing (employer of one of the authors), for supporting this project and for providing a model of educational excellence for the last 25 years; and

Josh Zucker, whose comments about how he learned mathematics inspired the questions-before-the-lessons approach of the text.

The print version of this book was written using the LaTeX document processing system, and the diagrams were prepared using Metapost and Asymptote. We thank the authors of the various LaTeX packages that we used while preparing this book, and also the brilliant authors of *The LaTeX Companion* for writing a reference book that is not only thorough but also very readable.

The source files for the print book were initially converted to this online book using a script written in the Python programming language. Palmer Mebane managed the conversion of this book from print to online, and Paul Salerno also wrote tools used in the conversion. Jason Batterson and the Jacob Tyler Creative Group designed the look and feel of this online book. James Fung, Shelley Garg, Kyle Guillet, Tasha Moyer, David Patrick, Shannon Rogers, Amy Szczepanski, Deven Ware, Phyllis Xu, and Laura Zehender all helped to review and edit the online book content.

Dedication

R.B.–I dedicate this book to my mom and dad (Krishna Rao and Jaganmohan Rao), my wife Ranu, and my daughter Meena. Thank you for always believing in me.

D.P.–Personal thanks to my most influential mathematical mentors:
my parents (when I was very young),
Gerry Rising (when I was moderately young),
and Michael Artin (when I was not-so-young).

R.R.–For my students, in the hopes that they learn from me as much as I learn from them.

How to Use This Book

Learn by Solving Problems

This book is probably very different from most of the math books that you have read before. We believe that the best way to learn mathematics is by solving problems. Lots and lots of problems. In fact, we believe that the best way to learn mathematics is to try to solve problems that you don't know how to do. When you discover something on your own, you'll understand it much better than if someone just tells it to you.

Most of the sections of this book begin with several problems. The solutions to these problems will be covered in the text, but try to solve the problems *before* reading the section. If you can't solve some of the problems, that's OK, because they will all be fully solved as you read the section. Even if you solve all of the problems, it's still important to read the section, both to make sure that your solution is correct, and also because you may find that the book's solution is simpler or easier to understand than your own.

If you find that the problems are too easy, this means that you should try harder problems. Nobody learns very much by solving problems that are too easy for them.

Navigating This Book

From any page in the book, you can click on the image of the book's cover in the top-left corner to view the table of contents. On large-screen devices, you can also click anywhere on the left-side navigation bar to jump to the corresponding section of the book. You can use the left and right arrows at the top of the page to move to the previous or next chapter or section of the book.

Most sections begin with all of the problems that appear in the section, as explained in the "Learn by Solving Problems" subsection above. Clicking on the *Jump to Solution* link will jump forward in the section to where that problem and its solution appear in the text.

Hovering over a paragraph will create a small icon in the left margin. Clicking on that icon will pop up a window with a permanent link to that paragraph. You can cut-and-paste this link into an email message or community post.

Interactive Features

There are several interactive features built into the book. All online books are linked to the AoPS community, so that students using the online book can discuss the book with other students. Click on the Community icon next to a section or problem to view all of the discussions about that section or problem. Click the New Topic icon to start a new discussion topic about the section or problem.

This book is also linked to *Alcumus*, Art of Problem Solving's innovative online learning system. Clicking on the Alcumus link at the top of a section takes you to Alcumus. This gives you the opportunity to work on additional practice problems that reinforce the material in your current section of the book. (Not all sections are linked to Alcumus, so the icon may not appear at the top of some sections.) To learn more about Alcumus, click here.

This book contains embedded video lessons. Just click on the "play" button on a video to watch the video. You can also view the entire Prealgebra video library.

Explanation of Icons

Throughout the book, you will see various shaded boxes and icons.

Concept:	This will be a general problem-solving technique or strategy. These are the "keys" to becoming a better problem solver!
Important:	This will be something important that you should learn. It might be a formula, a solution technique, or a caution.
WARNING!!	Beware if you see this box! This will point out a common mistake or pitfall.
Sidenote:	This box will contain material which, although interesting, is not part of the main material of the text. It's OK to skip over these boxes, but if you read them, you might learn something interesting!
Bogus Solution:	Just like the impossible cube shown to the left, there's something wrong with any "solution" that appears in this box.
Extra!	This is an "Extra!" and might be a quote, some biographical or historical background, or perhaps an interesting idea to think about.

Exercises, Review Problems, and Challenge Problems

Most sections end with several **Exercises**. These will test your understanding of the material that was covered in the section that you just finished. You should try to solve *all* of the exercises. Exercises marked with a ★ are more difficult.

Most chapters have a section containing **Review Problems**. These are problems which test your understanding of the material covered in the chapter. Your goal should be to solve most or all of the Review Problems for every chapter — if you're unable to do this, it means that you haven't yet mastered the material, and you should probably go back and read the chapter again.

All of the chapters end with a section containing **Challenge Problems**. These problems are generally more difficult than the other problems in the book, and will really test your mastery of the material. Some of them are very, very hard — the hardest ones are marked with a ★. Don't necessarily expect to be able to solve all of the Challenge Problems on your first try — these are difficult problems even for experienced problem solvers. If you are able to solve a large number of Challenge Problems, then congratulations, you are on your way to becoming an expert problem solver!

You can type your solution or notes for any Exercise, Review Problem, or Challenge Problem directly into the book. Your work will automatically be saved. You won't be able to view the solution to a problem until you type something in the solution box.

Many problems come with one or more **hints**. You can view any available hint by clicking on the *Hint* link after the problem statement. You can then hide the hint by again clicking on the *Hint* link. It is very important that you first try to solve the problem without peeking at the hints. Only after you've seriously thought about a problem and are stuck should you look at a hint. Also, for problems which have multiple hints, use the hints one at a time; don't go to the second hint until you've thought about the first one.

The **solutions** to all of the Exercises, Review Problems, and Challenge Problems are built into the book. Clicking the *Show Solution* button will display the solution, but you won't be able to view the solution until you've made an attempt to solve the problem and typed something into the solution box. Once you've viewed the solution, you can add notes to the solution in a separate box. You can also click the *Reset* button to clear your solution and notes and start fresh.

Here are some very important things to keep in mind about the solutions:

1. Make sure that you make a serious attempt at the problem before looking at the solution. You should think *hard* about a problem before deciding to give up and look at the solution. Remember, once you view a solution, you can't change what you typed for your solution.
2. After you solve a problem, it's usually a good idea to read the solution, even if you think you know how to solve the problem. Our solution might show you a quicker or more concise way to solve the problem, or it might have a completely different solution method that you might not have thought of.
3. If you have to look at the solution in order to solve a problem, make sure that you make a note of that problem. You can then come back to the problem in a week or two to make sure that you are able to solve it on your own without resorting to the solution.

Resources

Here are some other good resources for you to further pursue your study of mathematics:

- Art of Problem Solving has a complete library of books (both print and online) specifically designed for avid math students:
  - The *Introduction* series: *Introduction to Algebra*, *Introduction to Counting & Probability*, *Introduction to Number Theory*, and *Introduction to Geometry*, designed for students in grades 6-10.
  - The *Intermediate* series: *Intermediate Algebra*, *Intermediate Counting & Probability*, *Precalculus*, and *Calculus*, designed for students in grades 9-12.
  - The *Problem Solving* series: designed for students preparing for math competitions. In addition to our classics *the Art of Problem Solving, Volume 1: the Basics* (for students in grades 7-10 preparing for MATHCOUNTS and the AMC 8/10/12 contests) and *the Art of Problem Solving, Volume 2: and Beyond* (for students in grades 9-12 preparing for advanced contests such as the AIME), we also have *Competition Math for Middle School*.
  - *Beast Academy*: a full, rigorous, entertaining curriculum for aspiring math beasts in grades 2-5.
- The Art of Problem Solving website contains many other resources for students.
  - The AoPS Community has tens of thousands of members (if you are reading this online book, you're a member too!) and millions of posts on a variety of math, problem solving, and other fun topics.
  - Alcumus, our free adaptive online learning system containing over 13,000 practice problems.
  - For the Win!, our free interactive online game inspired by the MATHCOUNTS Countdown Round.
  - Our vast video library contains hundreds of videos featuring AoPS founder Richard Rusczyk.
  - Learn LaTeX, the mathematical typesetting system used by most professional mathematicians and scientists, from our widely-used LaTeX guide, and practice your LaTeX skills with the TeXeR.
  - AoPS Community members collaborate to build the AoPSWiki.
  - We have a collection of articles on a variety of problem-solving topics.
- You can hone your problem solving skills (and perhaps win prizes!) by participating in various math contests. Please see the **Acknowledgements** section of this book for more information.

A Note to Teachers and Parents

We believe that students learn best when they are challenged with hard problems that at first they may not know how to do. This is the motivating philosophy behind this book.

Rather than first introducing new material and then giving students exercises, we present problems at the start of each section that students should try to solve *before* the new material is presented. The goal is to get students to discover the new material on their own. Often, complicated problems are broken into smaller parts, so that students can discover new techniques one piece at a time. Then the new material is formally presented in the text, and full solutions to each problem are explained, along with problem-solving strategies.

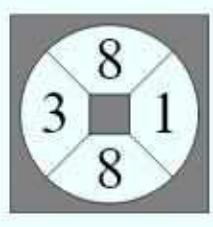
We hope that teachers will find that many students will discover most of the material in this book on their own by working through the problems. Other students may learn better from a more traditional approach of first seeing the new material, then working the problems. Teachers have the flexibility to use either approach when teaching from this book.

The book is linear in coverage. Generally, students and teachers should progress straight through the book in order, without skipping chapters. In general, chapters are not equal in length, so different chapters may take different amounts of classroom time.

After completing this book, students should be ready to continue with any book in the Art of Problem Solving's *Introduction* series of textbooks. The books in the *Introduction* series can be used in any order, although we generally recommend that *Introduction to Geometry* be used last.

Game: **The 24 Game**

See the four numbers on the card below?



Can you combine these four numbers to make the number 24? You may add, subtract, multiply, or divide. You may also use parentheses. You can't use any other symbols. You have to use each of the four numbers exactly once.

In the example above, one solution is $(3+1)\times 8-8$. Another solution is $(3-1)\times 8+8$. Yet another solution is $(8+1)\times 8\div 3$.

This game is called the **24® game**. For more information about it, visit http://www.24game.com.

For fun, this book has a few 24 cards at the top of each chapter. Give them a try! Some of them are hard, but each has at least one solution. If you get stuck, the button right below each card shows the solution.

Copyright © 2011, 2013, 2014, 2015, 2016, 2017, 2019 AoPS Incorporated. All Rights Reserved.

Reproduction of any portion of this book without the written permission of AoPS Incorporated is strictly prohibited, except for "fair use" or other noncommercial uses as defined in Sections 107 and 108 of the U.S. Copyright Act.

This online book may be used solely under the terms of AoPS Incorporated's Online Book License Agreement, available HERE.

Published by:
AoPS Incorporated
10865 Rancho Bernardo Rd Ste 100
San Diego, CA 92127-2102
(858) 675-4555
books@artofproblemsolving.com

Print Version ISBN: 978-1-934124-21-5 (text), 978-1-934124-22-2 (solutions)

24® is a registered trademark of Suntex International Inc.
MATHCOUNTS® is a registered trademark of the MATHCOUNTS Foundation.

Cover image designed by Vanessa Rusczyk using KaleidoTile software. Cover includes the original oil painting *Bush Poppy, El Cajon Mountain*, © 2011 Vanessa Rusczyk.

Table of Contents

Contents and Introduction

Table of Contents
Legal Information
How to Use This Book
Acknowledgements

1 Properties of Arithmetic

1.1 Why Start with Arithmetic?
1.2 Addition
1.3 Multiplication
1.4 Negation
1.5 Subtraction
1.6 Reciprocals
1.7 Division
1.8 Summary
Review Problems
Challenge Problems

2 Exponents

2.1 Squares
2.2 Higher Exponents
2.3 Zero as an Exponent
2.4 Negative Exponents
2.5 Summary
Review Problems
Challenge Problems

3 Number Theory

3.1 Multiples
3.2 Divisibility Tests
3.3 Prime Numbers
3.4 Prime Factorization
3.5 Least Common Multiple
3.6 Divisors
3.7 Greatest Common Divisor
3.8 Summary
Review Problems
Challenge Problems

4 Fractions

4.1 What is a Fraction?
4.2 Multiplying Fractions
4.3 Dividing by a Fraction
4.4 Raising Fractions to Powers
4.5 Simplest Form of a Fraction
4.6 Comparing Fractions
4.7 Adding and Subtracting Fractions
4.8 Mixed Numbers
4.9 Summary
Review Problems
Challenge Problems

5 Equations and Inequalities

5.1 Expressions
5.2 Solving Linear Equations I
5.3 Solving Linear Equations II
5.4 Word Problems
5.5 Inequalities
5.6 Summary
Review Problems
Challenge Problems

6 Decimals

6.1 Arithmetic with Decimals
6.2 Rounding
6.3 Decimals and Fractions
6.4 Repeating Decimals
6.5 Summary
Review Problems
Challenge Problems

7 Ratios, Conversions, and Rates

7.1 What is a Ratio?
7.2 Multi-way Ratios
7.3 Proportions
7.4 Conversions
7.5 Speed
7.6 Other Rates
7.7 Summary
Review Problems
Challenge Problems

8 Percents

8.1 What is a Percent?
8.2 Word Problems
8.3 Percent Increase and Decrease
8.4 Summary
Review Problems
Challenge Problems

9 Square Roots

9.1 From Squares to Square Roots
9.2 Square Roots of Non-square Integers
9.3 Arithmetic with Square Roots
9.4 Summary
Review Problems
Challenge Problems

10 Angles

10.1 Measuring Angles
10.2 Parallel Lines
10.3 Angles in Polygons
10.4 Summary
Review Problems
Challenge Problems

11 Perimeter and Area

11.1 Measuring Segments
11.2 Area
11.3 Circles
11.4 Summary
Review Problems
Challenge Problems

12 Right Triangles and Quadrilaterals

12.1 The Pythagorean Theorem
12.2 Some Special Triangles
12.3 Types of Quadrilaterals
12.4 Summary
Review Problems
Challenge Problems

13 Data and Statistics

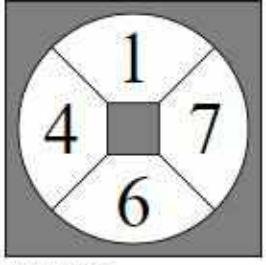
13.1 Basic Statistics
13.2 Limits of Basic Statistics
13.3 Tables, Graphs, and Charts
13.4 Summary
Review Problems
Challenge Problems

14 Counting

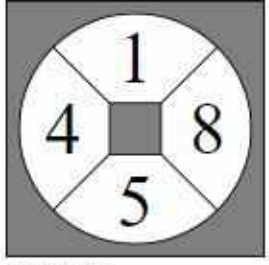
14.1 Counting with Addition and Subtraction
14.2 The Multiplication Principle
14.3 Casework
14.4 Counting Pairs
14.5 Probability
14.6 Summary
Review Problems
Challenge Problems

15 Problem-Solving Strategies

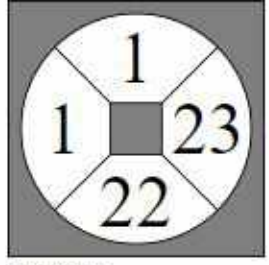
15.1 Find a Pattern
15.2 Make a List
15.3 Draw a Picture
15.4 Work Backwards
15.5 Summary
Review Problems
Challenge Problems



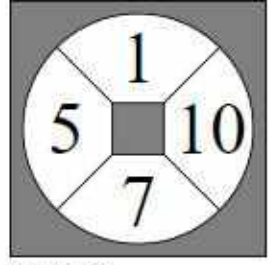
Solution:
$(7+1-4)\times 6$



Solution:
$(5+1)\times(8-4)$ or
$(5-1)\times 4+8$



Solution:
$(1+1)\times 23-22$



Solution:
$5\times 7-10-1$

Arithmetic is being able to count up to twenty without taking off your shoes. — Mickey Mouse

CHAPTER 1

Properties of Arithmetic

1.1 Why Start with Arithmetic?

You know how to add, subtract, multiply, and divide. In fact, you may already know how to solve many of the problems in this chapter. So why do we start this book with an entire chapter on arithmetic?

To answer this question, go back to the title of this book: ***Prealgebra***. What is prealgebra? Not everybody agrees on what "prealgebra" means, but we (the writers of this book) like to think of prealgebra as the bridge between *arithmetic* and *algebra*.

Arithmetic refers to the basics of adding, subtracting, multiplying, dividing, and (maybe) more exotic things like squares and square roots. You probably learned most of these basics already. The hardest thing that you usually do in arithmetic is "word problems" like "If Jenny has 5 apples and Timmy has 7 apples, then how many apples do they have together?" As you get older, the numbers get bigger, but the problems don't really get much harder. Arithmetic is great when trying to solve simple problems like counting apples. But when the problems get more complicated–like trying to compute a rocket's trajectory, or trying to analyze a financial market, or trying to count the number of ways a text message can be routed through a cellular phone network–we need a more advanced toolbox.

That toolbox is *algebra*. Algebra is the language of all advanced mathematics. Algebra gives us tools to take our concepts from arithmetic and make them *general*, meaning that we can use the concepts not just for arithmetic problems, but for other sorts of problems, too.

To take a simple example, you can use arithmetic to show that

$$2\times(3+5)=(2\times 3)+(2\times 5),$$

because the left side equals 2×8, which is 16, and the right side equals $6+10$, which is also 16. But algebra gives us the much more general tool that

$$a\times(b+c)=(a\times b)+(a\times c),$$

no matter what numbers a, b, and c are. And in higher mathematics, a, b, and c might not even be numbers as you recognize them now, but might be more complicated mathematical objects. (Even more generally, "+" and "×" might not mean addition and multiplication as you think of them now, but instead might represent more complicated mathematical operations. But we're getting ahead of ourselves a little bit!)

So our initial goal, in Chapter 1, is to carefully lay down the rules of arithmetic, and to give you some ideas as to *why* these rules are true. Once you know the rules, you'll be ready to start thinking algebraically.

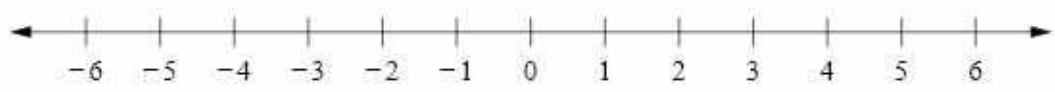
Also, by the time you start reading this book, you are mathematically mature enough to start thinking about not just *how* to perform various calculations, but *why* the techniques used in those calculations work. Understanding why mathematics works is the key to solving harder problems. If you only understand how techniques work but not why they work, you'll have a lot more difficulty modifying those techniques to solve more complicated problems. So, throughout this book, we will rarely just tell you how something works–we'll usually show you why it works.

By the end of this chapter, you should be able to explain *why* the following computations are true:

- $(-5)\times(-7)=35$ (and not -35)
- $(1990\times 1991)-(1989\times 1990)=3980$ (and be able to compute this in your head!)
- $8\div\frac{1}{7}=56$
- $(4\times 10\times 49)\div(2\times 5\times 7)=28$ (again, in your head!)

You'll know all these things not because you've blindly applied some calculation, or because you've memorized some formula–instead, you'll understand the mathematics behind all these expressions.

Unfortunately, different mathematicians and different textbooks may use slightly different words for the same concept, in the same way that what an American calls a "truck" is called a "lorry" by people in Great Britain. So, before we go any farther, we want to make sure that we all agree on some of the words that we're going to use.



The **number line** is shown above. It goes on forever in both directions. Every number that we will consider in this book is somewhere on the number line. The tick marks on the number line above indicate the **integers**. An integer is a number without a fractional part:

$$\ldots,-7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7,\ldots$$

(The symbol ... at either end of the above list means that the list goes on forever in that direction. The ... symbol is called an **ellipsis**.)

However, as you know, there are many numbers on the number line other than integers. For instance, most fractions such as $\frac{1}{2}$ are not integers. (But some fractions are integers–we'll explore this further in Chapter 4.)

A number is called **positive** if it is to the right of 0 on the number line. In other words, a number is positive if it is greater than 0. A number is called **negative** if it is to the left of 0 on the number line. That is, a number is negative if it is less than 0. For example, 2 is positive, while -2 is negative. Note that 0 itself is neither positive nor negative, and that every number is either positive, negative, or 0.

A number is called **nonnegative** if it is not negative. In other words, a nonnegative number is positive or 0. Similarly, a number is called **nonpositive** if it is not positive. Finally, a number is called **nonzero** if it is not equal to 0. Note that 0 is nonnegative and nonpositive.

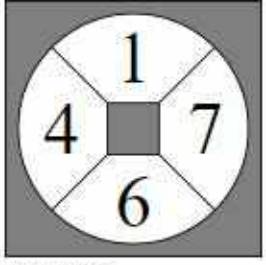
Sidenote: ♫

A lot of people use the term **whole number** to mean a nonnegative integer. In other words, a whole number is one of the numbers $0, 1, 2, \ldots$. These same people use the term **natural number** to mean a positive integer. In other words, a natural number is one of the numbers $1, 2, 3, \ldots$.

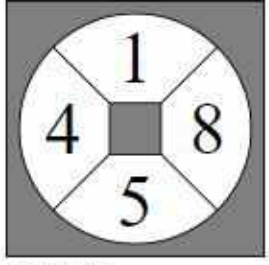
However–and this is the really irritating part–a lot of people use the term **natural number** to mean a nonnegative integer. In other words, a natural number is one of the numbers $0, 1, 2, \ldots$. These same people use the term **whole number** to mean a positive integer. In other words, a whole number is one of the numbers $1, 2, 3, \ldots$.

And some of both of these groups of people might also use the term **counting number** to mean either whole number or natural number.

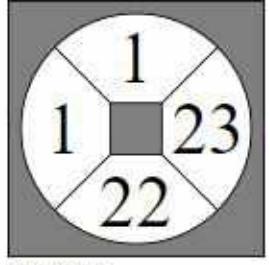
These people have been arguing for centuries, and they will likely never agree. Since we don't like to argue, we will stick with the very clear terms **positive integers** for the numbers $1, 2, 3, \ldots$, and **nonnegative integers** for the numbers $0, 1, 2, \ldots$.



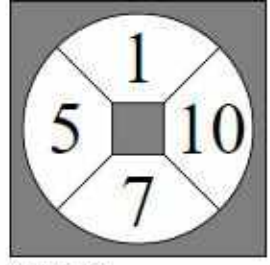
Solution:
$(7+1-4)\times 6$



Solution:
$(5+1)\times(8-4)$ or
$(5-1)\times 4+8$



Solution:
$(1+1)\times 23-22$



Solution:
$5\times 7-10-1$

Arithmetic is being able to count up to twenty without taking off your shoes. — Mickey Mouse

CHAPTER 1

Properties of Arithmetic

1.1 Why Start with Arithmetic?

You know how to add, subtract, multiply, and divide. In fact, you may already know how to solve many of the problems in this chapter. So why do we start this book with an entire chapter on arithmetic?

To answer this question, go back to the title of this book: ***Prealgebra***. What is prealgebra? Not everybody agrees on what "prealgebra" means, but we (the writers of this book) like to think of prealgebra as the bridge between *arithmetic* and *algebra*.

Arithmetic refers to the basics of adding, subtracting, multiplying, dividing, and (maybe) more exotic things like squares and square roots. You probably learned most of these basics already. The hardest thing that you usually do in arithmetic is "word problems" like "If Jenny has 5 apples and Timmy has 7 apples, then how many apples do they have together?" As you get older, the numbers get bigger, but the problems don't really get much harder. Arithmetic is great when trying to solve simple problems like counting apples. But when the problems get more complicated–like trying to compute a rocket's trajectory, or trying to analyze a financial market, or trying to count the number of ways a text message can be routed through a cellular phone network–we need a more advanced toolbox.

That toolbox is *algebra*. Algebra is the language of all advanced mathematics. Algebra gives us tools to take our concepts from arithmetic and make them *general*, meaning that we can use the concepts not just for arithmetic problems, but for other sorts of problems, too.

To take a simple example, you can use arithmetic to show that

$$2\times(3+5)=(2\times 3)+(2\times 5),$$

because the left side equals 2×8, which is 16, and the right side equals $6+10$, which is also 16. But algebra gives us the much more general tool that

$$a\times(b+c)=(a\times b)+(a\times c),$$

no matter what numbers a, b, and c are. And in higher mathematics, a, b, and c might not even be numbers as you recognize them now, but might be more complicated mathematical objects. (Even more generally, "+" and "×" might not mean addition and multiplication as you think of them now, but instead might represent more complicated mathematical operations. But we're getting ahead of ourselves a little bit!)

So our initial goal, in Chapter 1, is to carefully lay down the rules of arithmetic, and to give you some ideas as to *why* these rules are true. Once you know the rules, you'll be ready to start thinking algebraically.

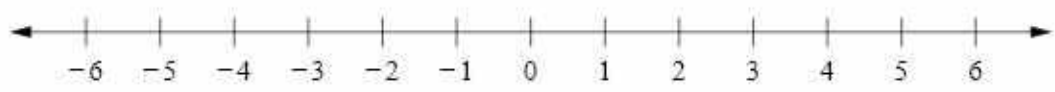
Also, by the time you start reading this book, you are mathematically mature enough to start thinking about not just *how* to perform various calculations, but *why* the techniques used in those calculations work. Understanding why mathematics works is the key to solving harder problems. If you only understand how techniques work but not why they work, you'll have a lot more difficulty modifying those techniques to solve more complicated problems. So, throughout this book, we will rarely just tell you how something works–we'll usually show you why it works.

By the end of this chapter, you should be able to explain *why* the following computations are true:

- $(-5)\times(-7)=35$ (and not -35)
- $(1990\times 1991)-(1989\times 1990)=3980$ (and be able to compute this in your head!)
- $8\div\frac{1}{7}=56$
- $(4\times 10\times 49)\div(2\times 5\times 7)=28$ (again, in your head!)

You'll know all these things not because you've blindly applied some calculation, or because you've memorized some formula–instead, you'll understand the mathematics behind all these expressions.

Unfortunately, different mathematicians and different textbooks may use slightly different words for the same concept, in the same way that what an American calls a "truck" is called a "lorry" by people in Great Britain. So, before we go any farther, we want to make sure that we all agree on some of the words that we're going to use.



The **number line** is shown above. It goes on forever in both directions. Every number that we will consider in this book is somewhere on the number line. The tick marks on the number line above indicate the **integers**. An integer is a number without a fractional part:

$$\ldots,-7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7,\ldots$$

(The symbol ... at either end of the above list means that the list goes on forever in that direction. The ... symbol is called an **ellipsis**.)

However, as you know, there are many numbers on the number line other than integers. For instance, most fractions such as $\frac{1}{2}$ are not integers. (But some fractions are integers–we'll explore this further in Chapter 4.)

A number is called **positive** if it is to the right of 0 on the number line. In other words, a number is positive if it is greater than 0. A number is called **negative** if it is to the left of 0 on the number line. That is, a number is negative if it is less than 0. For example, 2 is positive, while -2 is negative. Note that 0 itself is neither positive nor negative, and that every number is either positive, negative, or 0.

A number is called **nonnegative** if it is not negative. In other words, a nonnegative number is positive or 0. Similarly, a number is called **nonpositive** if it is not positive. Finally, a number is called **nonzero** if it is not equal to 0. Note that 0 is nonnegative and nonpositive.

Sidenote: ♫

A lot of people use the term **whole number** to mean a nonnegative integer. In other words, a whole number is one of the numbers $0, 1, 2, \ldots$. These same people use the term **natural number** to mean a positive integer. In other words, a natural number is one of the numbers $1, 2, 3, \ldots$.

However–and this is the really irritating part–a lot of people use the term **natural number** to mean a nonnegative integer. In other words, a natural number is one of the numbers $0, 1, 2, \ldots$. These same people use the term **whole number** to mean a positive integer. In other words, a whole number is one of the numbers $1, 2, 3, \ldots$.

And some of both of these groups of people might also use the term **counting number** to mean either whole number or natural number.

These people have been arguing for centuries, and they will likely never agree. Since we don't like to argue, we will stick with the very clear terms **positive integers** for the numbers $1, 2, 3, \ldots$, and **nonnegative integers** for the numbers $0, 1, 2, \ldots$.

1.2 Addition

We'll start by exploring the simplest arithmetic operation **addition**. There's not a whole lot to explore, but as we'll see, a solid knowledge of the basic properties of addition makes complicated-looking calculations easy.

> **Important:** **How to use this book:** Most sections will begin with problems, like those shown below. You should first try to solve the problems. Then, continue reading the section, and compare your solutions to the solutions presented in the book.

Problems

Problem 1.1 Jump to Solution

Using the two pictures below, explain why $2 + 3 = 3 + 2$.

Problem 1.2 Jump to Solution

Using the two pictures below, explain why $(2 + 3) + 4 = 2 + (3 + 4)$.

Reminder: The parentheses tell you what to compute first. For example, $(3 + 4) \times 5$ equals 7×5, whereas $3 + (4 \times 5)$ equals $3 + 20$.

Problem 1.3 Source: (b) MATHCOUNTS Jump to Solution

(a) Using the properties from Problems 1.1 and 1.2, explain why

$$472 + (219 + 28) = (472 + 28) + 219.$$

(b) Compute the sum $472 + (219 + 28)$.

Problem 1.4 Source: MATHCOUNTS Jump to Solution

Compute

$$(2 + 12 + 22 + 32) + (8 + 18 + 28 + 38).$$

Problem 1.5 Jump to Solution

Find the sum $1 + 2 + 3 + \cdots + 19 + 20$. Reminder: The ellipsis $\cdots$ means that we should include all the numbers in the pattern. So we are adding the positive integers from 1 to 20.

Problem 1.6 Jump to Solution

Using the picture below, explain why $2 + 0 = 2$.

Problem 1.1

Using the two pictures below, explain why $2 + 3 = 3 + 2$.

Solution for Problem 1.1: First let's look at the picture on the left. The first row has 2 squares; the second row has 3 squares. So in total, there are $2 + 3$ squares.

Now let's look at the picture on the right. The first row has 3 squares; the second row has 2 squares. In total, there are $3 + 2$ squares.

The picture on the right, however, is just an upside-down version of the picture on the left. Flipping a picture upside down doesn't change the number of squares. So we conclude that $2 + 3 = 3 + 2$. □

Whenever we add two numbers, the order of the numbers does not matter. For example, $5 + 17 = 17 + 5$ and $32 + 999 = 999 + 32$. There are an infinite number of such examples. Of course, we can't write down an infinite number of examples. Instead, we can write the equation

$$\text{first number} + \text{second number} = \text{second number} + \text{first number}.$$

This is a long equation to write. We can shorten it by letting a represent the first number and b represent the second number. Then our equation becomes

$$a + b = b + a.$$

Here a and b stand for any numbers (and possibly the same number). For instance, if we let $a = 2$ and $b = 3$, then we have our original example $2 + 3 = 3 + 2$. If we let $a = 999$ and $b = 200$, then we get $999 + 200 = 200 + 999$. Note that a has the same value throughout the equation; as does b; both, however, may change from one problem to the next. Letters such as a and b that represent numbers are called **variables**.

The rule that $a + b = b + a$ for all numbers a and b is called the **commutative property** of addition.

> **Important:** **Addition is commutative:** Let a and b be numbers. Then
>
> $$a + b = b + a.$$

Commutative Property of Addition

In Problem 1.1, we explained why $a + b = b + a$ is true for one particular example, when $a = 2$ and $b = 3$. But one example doesn't prove that $a + b = b + a$ for all a and b. We don't have the tools in this book to explain why this must hold for any two numbers, but Problem 1.1 should give you good intuition for why it is true for positive integers.

Throughout the rest of this chapter, we will explore many more arithmetic rules. We will use examples and pictures to give intuition for why these rules work. These examples and pictures are not proofs, but they should give you a feel for why these rules must be true.

The commutative property is concerned with adding two numbers. What if we add three numbers?

Problem 1.2

Using the two pictures below, explain why $(2 + 3) + 4 = 2 + (3 + 4)$.

Reminder: The parentheses tell you what to compute first. For example, $(3 + 4) \times 5$ equals 7×5, whereas $3 + (4 \times 5)$ equals $3 + 20$.

Solution for Problem 1.2: For each picture, we will first count the number of light squares, then count the number of dark squares, and finally add the two counts.

Let's start with the picture on the left. It has $(2 + 3)$ light squares and 4 dark squares. So altogether it has $(2 + 3) + 4$ squares.

Next, let's look at the picture on the right. It has 2 light squares and $(3 + 4)$ dark squares. So altogether it has $2 + (3 + 4)$ squares.

What's the difference between the two pictures? The only difference is the color of the middle row. Changing the color doesn't change the number of squares. So we conclude that $(2 + 3) + 4 = 2 + (3 + 4)$. □

We get a similar equation for any three numbers: $(a + b) + c = a + (b + c)$. In other words, first adding a and b and then adding c is the same as adding a to $b + c$. This property is called the **associative property** of addition.

> **Important:** **Addition is associative:** Let a, b, and c be numbers. Then
>
> $$(a + b) + c = a + (b + c).$$

Associative Property of Addition

> **WARNING!!** Students sometimes mix up the names "commutative" and "associative." In the commutative property, the numbers are moved around ("commuted") on the two sides of the equation. In the associative property, the numbers stay in the same place, but are grouped ("associated") differently.

Together, the commutative and associative properties are sneakily powerful, as they let us add a list of numbers in any order. The next problem is an illustration of this **any-order principle**.

Problem 1.3 Source: (b) MATHCOUNTS

(a) Using the properties from Problems 1.1 and 1.2, explain why

$$472 + (219 + 28) = (472 + 28) + 219.$$

(b) Compute the sum $472 + (219 + 28)$.

Solution for Problem 1.3:

(a) Let's start with the left side of the equation:

$$472 + (219 + 28).$$

We will try to make it look like the right side. To do that, we need to switch the order of the 219 and the 28. We can do so by the commutative property. In other words, we replace $472 + (219 + 28)$ with the equal quantity

$$472 + (28 + 219).$$

This expression is close to what we want: $(472 + 28) + 219$ in fact, these two expressions are equal by the associative property.

Let's combine the pieces of our explanation into a nice chain of equations:

$$\begin{aligned} 472 &+ (219 + 28) \\ &= 472 + (28 + 219) && \text{commutative property} \\ &= (472 + 28) + 219. && \text{associative property} \end{aligned}$$

(b) By part (a), the quantities $472 + (219 + 28)$ and $(472 + 28) + 219$ are equal, so we can compute $(472 + 28) + 219$ instead. But this is easy to compute: the sum $472 + 28$ is 500, so we are left with $500 + 219$. The answer is 719.

□

The point of Problem 1.3 is that we can rearrange the numbers in our addition to make the addition easier to compute. It's easier to first compute $472 + 28$, and then compute $500 + 219$ than it would have been to start with $219 + 28$ and then add that sum to 472.

In a similar way, any addition problem can be rearranged without changing the sum. Usually we won't bother to write all the individual steps of the rearrangement, like we did in Problem 1.3. Instead, we'll use our knowledge of the commutative and associative properties to just go ahead and rearrange a sum in whatever way is best. Let's apply that principle to solve the next problem.

Problem 1.4 Source: MATHCOUNTS

Compute

$$(2 + 12 + 22 + 32) + (8 + 18 + 28 + 38).$$

Solution for Problem 1.4: We could start with 2, then add 12, then add 22, and so on, but that's too much work. Instead, let's try to rearrange the sum in a useful way. Let's pair up the numbers so that each pair has the same sum. Specifically, let's pair each number in the first group with a number in the second group:

$$(2 + 38) + (12 + 28) + (22 + 18) + (32 + 8).$$

The first pair of numbers adds up to 40, so does the second pair, the third pair, and the fourth pair. So our sum becomes

$$40 + 40 + 40 + 40.$$

The answer is 160. □

Let's use the any-order principle to compute a longer sum.

Problem 1.5

Find the sum $1 + 2 + 3 + \cdots + 19 + 20$. Reminder: The ellipsis $\cdots$ means that we should include all the numbers in the pattern. So we are adding the positive integers from 1 to 20.

Solution for Problem 1.5: We definitely don't want to add the 20 numbers one at a time. Instead, let's try again to rearrange the numbers into pairs, so that each pair has the same sum. We pair the smallest number with the largest, the second-smallest with the second-largest, and so on:

$$(1 + 20) + (2 + 19) + (3 + 18) + \cdots + (10 + 11).$$

We have grouped the 20 numbers into 10 pairs. Each pair adds up to 21. So our sum becomes

$$21 + 21 + 21 + 21 + 21 + 21 + 21 + 21 + 21 + 21.$$

Adding 10 copies of 21 is the same as multiplying 10 and 21. So the answer is 210. □

Sum the Numbers from 1 to 100

Finally, let's look at one more property of addition. What happens when we add zero to a number?

Problem 1.6

Using the picture below, explain why $2 + 0 = 2$.

Solution for Problem 1.6: On one hand, there are 2 squares. On the other hand, we can say there are 2 light squares and 0 dark squares. So we get the equation $2 + 0 = 2$. □

Adding zero to any number doesn't change the number.

> **Important:** **Adding zero:** Let a be a number. Then
>
> $$a + 0 = a.$$

Exercises

1.2.1:

Compute $99 + 99 + 99 + 101 + 101 + 101$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We can group each 99 with a 101:

$$\begin{aligned} &99 + 99 + 99 + 101 + 101 + 101 \\ &= (99 + 101) + (99 + 101) + (99 + 101) \\ &= 200 + 200 + 200 \\ &= \boxed{600} \end{aligned}$$

1.2.2:

Compute

$$1999 + 2001 + 1999 + 2001 + 1999 + 2001 + 1999 + 2001.$$

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We can group the numbers into pairs without moving them:

$$\begin{aligned} &1999 + 2001 + 1999 + 2001 + 1999 + 2001 + 1999 + 2001 \\ &= (1999 + 2001) + (1999 + 2001) \\ &\quad + (1999 + 2001) + (1999 + 2001) \\ &= 4000 + 4000 + 4000 + 4000 = \boxed{16{,}000} \end{aligned}$$

1.2.3:

Compute

$$(3 + 13 + 23 + 33 + 43) + (7 + 17 + 27 + 37 + 47).$$

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We can pair each number in the first group with a number in the second group:

$$\begin{aligned} &(3 + 13 + 23 + 33 + 43) + (7 + 17 + 27 + 37 + 47) \\ &= (3 + 47) + (13 + 37) + (23 + 27) + (33 + 17) + (43 + 7) \\ &= 50 + 50 + 50 + 50 + 50 = \boxed{250} \end{aligned}$$

1.2.4:

Compute

$$(1 + 2 + 3 + \cdots + 49 + 50) + (99 + 98 + 97 + \cdots + 51 + 50).$$

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Again let's match each number in the first group with a number in the second:

$$\begin{aligned} &(1 + 2 + 3 + \cdots + 49 + 50) + (99 + 98 + 97 + \cdots + 51 + 50) \\ &= (1 + 99) + (2 + 98) + (3 + 97) + \cdots + (49 + 51) + (50 + 50) \\ &= \underbrace{100 + 100 + 100 + \cdots + 100 + 100}_{50 \text{ numbers}} = 50 \cdot 100 = \boxed{5000} \end{aligned}$$

1.3 Multiplication

Problems

Problem 1.1 ⏱ 10 ⚙ by Solution

Using the two pictures below, explain why $2 \times 3 = 3 \times 2$.

Problem 1.2 ⏱ 10 ⚙ by Solution

By counting the dots in the picture below in two different ways, explain why

$$(1 + 2) \times (1 + 2 + 3) = (1 + 2) \times (6).$$

(a) Compute $(1 + 2) \times (6)$.

(b) Compute $(1 + 2) \times (1 + 2 + 3)$.

Problem 1.3 ⏱ 10 ⚙ by Solution

Using the pictures below, explain why $(1 + 2) \times (1 + 2 + 3) = (1 + 2) \times (6)$

Problem 1.4 ⏱ 10 ⚙ by Solution

Compute $(1 + 9) \times (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)$.

Problem 1.5 ⏱ 10 ⚙ by Solution

What is the value of $(1^2 + 1)(1^2 + 2)(1^2 + 3)(1^2 + 4)(1^2 + 5)$?

Problem 1.6 ⏱ 10 ⚙ by Solution

Consider the multiplication facts below:

$$\begin{aligned} 1 \times 7 &= 7 \\ 2 \times 7 &= 14 \\ 3 \times 7 &= 21 \\ 4 \times 7 &= 28 \\ 5 \times 7 &= 35 \\ 6 \times 7 &= ____ \end{aligned}$$

(a) Find a pattern in the multiplication answers.

(b) Assuming that the pattern continues, predict the answer to the last multiplication.

Problem 1.7 ⏱ 20 ⚙ by Solution

Using the two pictures below, explain why $7 \times 3 = 3 \times 7$.

Solution for Problem 1.7 The picture on the left has 3 rows. Each row has 7 squares. So the total number of squares is $7 + 7 + 7$, which is 7×3.

The picture on the right has 3 rows but now has 7 squares. So the total number of squares is $3 + 3 + 3$ which is 3×7.

Because the picture on the left and the picture on the right, both have 21 squares, we know that 7×3 and 3×7 are both equal to 21, so we conclude that $7 \times 3 = 3 \times 3$.

Note that the two pictures in fact are reverse the numbers in multiplication just as we use in addition. In other words, multiplication is commutative, just like addition. Note that we can not conclude that 7×3 is equal to 3×7 for all numbers, but we know that we can do it just fine with these two numbers. We have $a \times b = b \times a$ for all positive numbers a and b. We call this **the commutative property of multiplication**.

Note that the picture on the left shows 7×3 as 3 groups of 7 and the picture on the right shows 7×3 as 7 groups of 3. Another reason is that the cause-and-effect interpretation of the value a which is cause and b is the cause and we have $a \times b$ is the effect and $b \times a$ is the effect.

We will often go a step further writing $a \times b$ as ab. This notation is even quicker to write. We can't always leave out the dot though. For example, to express 3×5 we can't write 35, because that number is thirty-five in such cases, we will write either $3 \cdot 5$ or $3(5)$ for multiplication.

Do you see the commutative property of multiplication?

Important:	**Multiplication is commutative** Let a and b be numbers. Then
	$ab = ba.$

is multiplication associative? Yes! The next example is slightly trickier

Problem 1.8 ⏱ 20 ⚙ by Solution

By counting the dots in the picture below in two different ways, explain why

$$(1 + 2) \times (1 + 2 + 3) = (1 + 2) \times (6).$$

Solution for Problem 1.8 On one hand we have $(1 + 2) \times (6)$ from such, so let's reorder the numbers to simplify our task. Let's look for easy pairs such as $7 + 3$ that lead to nice round numbers. For example, let's try

$$(1 + 2) \times (6) = (1 + 2)(3)$$

The product of 3 and $(1 + 2)$. With the product of 6 and $(1 + 2)$. Both ways we will arrive at the same value.

The answer is 60,480,000.

We've already seen that $(1 + 2)$ is a special number for addition: adding zero does nothing, so there is similar number for multiplication?

Problem 1.9 ⏱ 20 ⚙ by Solution

Using the picture below, explain why $7 \times (1 + 2) = (7 \times 1) + (7 \times 2)$.

Solution for Problem 1.9 On one hand there are 3 rows. Each row has 7 squares. So the total is $7 + 7 + 7$ squares. On the other hand, there are 3 groups. So we conclude that $3 \times 7 = 21$.

The example suggests that multiplying by 1 does not change the number, and this leads to another rule:

Important:	**Multiplying by 1** Let a be a number. Then
	$1a = a.$

defines we continue with properties of arithmetic, we need to understand how to combine different operations. We know that if we are just just working on addition, we know that we can reorder the numbers and the sum is the same. Similarly, we know that if we are just working only on multiplication, we know that we can reorder the product of numbers in the same calculation. For instance, what does $7 \times 9 \times 3$ mean?

Problem 1.10 ⏱ 10 ⚙ by Solution

(a) Compute $(5 \times 6) \times 7$.

(b) Compute $5 \times (6 \times 7)$.

Solution for Problem 1.10 On one hand there are 5 rows. Each row has 7 squares. So the total is $7 + 7 + 7 + 7 + 7$. On the other hand, there are 5 groups. So we conclude that $5 \times 7 = 35$.

This example suggests that multiplying by 5 does not change the number, and this leads to another rule:

Important:	**Multiplying by 5** Let a be a number. Then
	$5a = a.$

defines we continue with properties of arithmetic, we need to understand how to combine different operations. We know that if we are just just working on addition, we know that we can reorder the numbers and the sum is the same. Similarly, we know that if we are just just working only on multiplication, we know that we can reorder the product of numbers in the same calculation. For instance, what does $7 \times 9 \times 3$ mean?

Problem 1.11 ⏱ 10 ⚙ by Solution

(a) Compute $(5 \times 6) \times 7$.

(b) Compute $5 \times (6 \times 7)$.

Solution for Problem 1.11 On one hand there are 5 rows. Each row has 7 squares. So the total is $7 + 7 + 7 + 7 + 7$. On the other hand, there are 5 groups. So we conclude that $5 \times 7 = 35$.

This example suggests that multiplying by 5 does not change the number, and this leads to another rule:

Important:	**Order of operations** Perform the operations in an expression in the following order.
	1. Perform all operations inside parentheses first.
	2. Compute powers. (We cover powers in chapter 1).
	3. Multiply and divide from left to right.
	4. Add and subtract from left to right.

So to compute $5 + 6 \times 7$ we must do the multiplication first:

$$5 + 6 \times 7 = 5 + 42 = 47.$$

Note, we can fix a powerful rule that connects addition and multiplication. So usual, we'll motivate the rule with a simple example.

Problem 1.12 ⏱ 20 ⚙ by Solution

Using the picture below, explain why $7 \times (9 + 3) = (7 \times 9) + (7 \times 3)$.

Solution for Problem 1.12 On one hand there are 3 rows. Each row has $(9 + 3)$ squares. So the total is $7 \times (9 + 3)$ squares. On the other hand, there are $(9 + 3)$ dark squares, and there are (3×7) light squares. So altogether there are $(7 \times 3) + (3 \times 9)$ squares.

We have counted the same squares in two different ways. So we conclude that

$$7 \times (9 + 3) = (7 \times 9) + (7 \times 3).$$

Problem 1.12 is an example of a very useful rule that relates multiplication and addition. For any three numbers, a, b and c we have

$$a \times (b + c) = (a \times b) + (a \times c).$$

The multiplication is **distributed** (handed out) to the two parts of the addition. For that reason, this property is called **the distributive property of multiplication over addition**. In the **distributive property** of multiplication.

Because multiplication is commutative, we can reverse each of the products above, and write

$$(b + c) \times a = (b \times a) + (c \times a).$$

In other words, the distributive property works when the sum is first $(b + c)$ or a is on back a. $(b + c)$.

To reduce clutter, we write the distributive property in the following way:

Important:	**Multiplication distributes over addition** Let a, b and c be numbers. Then
	$a(b + c) = ab + ac \quad \text{and} \quad (b + c)a = ba + ca.$

The equation $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ are really the same equation, we include them both to highlight the fact that the distributive property can be used when the sum is first or second.

Distributive Property

Let's explore the distributive property further. What can this property do for us?

Problem 1.13 ⏱ 20 ⚙ by Solution

Compute $(7 \times 9) + (7 \times 3)$.

Solution for Problem 1.13 Instead of separately computing 7×9 and 7×3, we can use the distributive property.

$$(7 \times 9) + (7 \times 3) = 7(9 + 3)$$

On the right side, the sum $9 + 3$ is 3 times as large as 9. So we have a square in 3 7×3.

Instead of computing 7×9, we can use the distributive property to make the multiplication simpler. Writing (7×9) as 7×3 we can rewrite the sum as

$$(7 \times 3) + (3 \times 3) = (7 + 3) \times 3 = 10 \times 3 = 30.$$

The advantage of using the distributive property as we just did the sum of 7×9 and 7×3 - 49 we both easy to compute! Our answer is

$$56 - 49 = 7.$$

As we see in Problem 1.13, we can use the distributive property "in other direction" Using the distributive property to rewrite $a(b + c)$ as $ab + ac$ is called **expanding**. Using the distributive property to rewrite $ab + ac$ as $a(b + c)$ is called **factoring**.

For example, in the first step of the solution to Problem 1.13, we factored $(7 \times 9) + (7 \times 3)$ to write it as the simpler $7(9 + 3)$ in the second step, we expanded $7(9 + 3)$ to write it as the more complicated $(7 \times 9) + (7 \times 3)$.

Concept:	**Factoring** means using the distributive property to rewrite something of the form $ab + ac$ as the simpler expression $a(b + c)$.
	In this example, the number 37 is called a **common factor** of both products as the number 37 is a factor of both products. We can factor out 37 to write $37 \times 10 + 37 \times 5$ as $37(10 + 5)$. We might also say that we factored a common factor of 37.

So, we have used the distributive property with some of the numbers. What about larger sums?

Problem 1.14 ⏱ 20 ⚙ by Solution

Compute $(7 \times 9) + (7 \times 3) + (7 \times 2)$.

Solution for Problem 1.14 We know that we can use the distributive property if we have two products with a common factor. Like ab and ac which have the common factor a. So we can factor out a from the products 7×9 and 7×3.

But now the products (7×9), (7×3) and (7×2) have a common factor of 7. So we can use the distributive property again:

$$(7 \times 9) + (7 \times 3) + (7 \times 2) = 7(9 + 3 + 2) = 7(14).$$

By the associative property of addition, we can add the numbers in any order, so we can quickly to just $(7 \times 9 + 3)$ - 63 but $(7 \times 3 + 2)$ - 23 so 7×14 - 98 equals 98, so we know it without reading them.

Suppose we have $(7 \times 9) + (7 \times 3) + (7 \times 2) + (7 \times 1)$. Now, we can factor out a common factor of 7 from the four products (7×9), (7×3), (7×2) and (7×1) and we can do it in the exactly the same way as we did in the solution to Problem 1.14. We

1.4 Negation

Take a look at the number line above. Note that -2 and 2 are the same distance from 0, but are on opposite sides of 0. What happens when we add -2 + 2? Of course, you know that we get 0. We can think of the sum -2 + 2 on the number line as starting at the position -2 and moving 2 units to the right, so that we end up at 0.

This idea of two numbers that sum to 0 is the key concept behind **negation**:

> **Definition:** The **negation** of a number x, written $-x$, is the number that we add to x to get 0. That is,
>
> $$-x + x = 0.$$
>
> The negation of x is also called the **opposite** of x or the **additive inverse** of x. We also sometimes pronounce $-x$ as "minus x" or "negative x."

For example, -1 is the negation of 1, since -1 + 1 = 0, and -2 is the negation of 2, since -2 + 2 = 0. The number line shows us that there is clearly a negation for any positive number. For example, -286 is the negation of 286, since -286 + 286 = 0.

You might ask: why do we bother with a new word "negation" when we already have negative numbers? The reason is that although so far we have mentioned the negations of positive numbers, we have not yet discussed the negations of zero and negative numbers. For example, the negation of 0 is -0, whatever that is. (We will find out soon enough.) The negation of -6 is $-(-6)$, whatever that is. (Again, we will soon find out.) Every number—positive, negative, or zero—has a negation.

Problems

Problem 1.14 Jump to Solution

Consider the negation facts below:

$$\begin{aligned}\text{The negation of 4 is } &-4.\\ \text{The negation of 3 is } &-3.\\ \text{The negation of 2 is } &-2.\\ \text{The negation of 1 is } &-1.\\ \text{The negation of 0 is } &____.\\ \text{The negation of } -1 \text{ is } &____.\\ \text{The negation of } -2 \text{ is } &____.\end{aligned}$$

(a) Consider where the numbers at the ends of the facts above are on the number line. After the first number, how is each number related to the number above it?

(b) Based on the pattern from part (a), what is the negation of 0?

(c) Based on the pattern from part (a), what is the negation of -1?

(d) Based on the pattern from part (a), what is the negation of -2?

Problem 1.17 Jump to Solution

Consider the multiplication facts below:

$$\begin{aligned}4 \cdot 3 &= 12\\ 3 \cdot 3 &= 9\\ 2 \cdot 3 &= 6\\ 1 \cdot 3 &= 3\\ 0 \cdot 3 &= 0\\ (-1) \cdot 3 &= ____.\end{aligned}$$

(a) Consider where the numbers at the ends of the facts above are on the number line. After the first number, how is each number related to the number above it?

(b) Based on the pattern from part (a), what should $(-1) \cdot (-3)$ be?

Every number has a negation. What's the negation of zero? What's the negation of a negative number? Let's find out.

Problem 1.16

Consider the negation facts below:

$$\begin{aligned}\text{The negation of 4 is } &-4.\\ \text{The negation of 3 is } &-3.\\ \text{The negation of 2 is } &-2.\\ \text{The negation of 1 is } &-1.\\ \text{The negation of 0 is } &____.\\ \text{The negation of } -1 \text{ is } &____.\\ \text{The negation of } -2 \text{ is } &____.\end{aligned}$$

(a) Consider where the numbers at the ends of the facts above are on the number line. After the first number, how is each number related to the number above it?

(b) Based on the pattern from part (a), what is the negation of 0?

(c) Based on the pattern from part (a), what is the negation of -1?

(d) Based on the pattern from part (a), what is the negation of -2?

Solution for Problem 1.16:

(a) The numbers at the ends of the facts in the list are -4, -3, -2 and -1. After the first number in the list, each number is one unit to the right of the number before it.

(b) The pattern from part (a) suggests that the answer is 1 unit to the right of -1. That number is 0. So -0, the negation of 0, should be 0.

We can also see this from our definition of negation. The negation of 0 was defined as the number that we can add to 0 to get 0. That is, by our definition, the number -0 satisfies

$$-0 + 0 = 0.$$

But adding 0 to a number does nothing, so adding 0 to -0 gives -0:

$$-0 + 0 = -0.$$

Since -0 + 0 equals both -0 and 0, we know that $-0 = 0$.

(c) Starting from 0 and moving 1 unit to the right, we get 1. So $-(-1)$ the negation of -1, is 1.

Again, we can use the definition of negation. We know that 1 and -1 sum to 0:

$$1 + (-1) = 0.$$

But we also know that $-(-1)$ defined as the number whose sum with -1 is 0:

$$-(-1) + (-1) = 0.$$

Comparing the above two equations indicates that $-(-1) = 1$.

(d) Starting from 1 and moving 1 unit to the right, we get to 2. So $-(-2)$, the negation of -2, is 2. We also know that $2 + (-2) = 0$, and we must have $-(-2) + (-2) = 0$. So we conclude that $-(-2) = 2$.

$\Diamond$

In Problem 1.16, we saw that the negation of -1 is 1, and the negation of -2 is 2. Similarly, for any number x, the negation of $-x$ is the original number x. A double negation seems to "cancel out," giving us the original number back again. So we suspect that $-(-x) = x$ for all numbers x, and we proved it in Problem 1.16 in the special case of positive numbers. But a pattern is not enough to be sure that $-(-x) = x$ for all numbers x. How do we know that the pattern continues forever?

We can **prove** that $-(-x) = x$ for all numbers x, using a very clever idea. Consider the sum

$$x + (-x) + (-(-x)).$$

That is, we are adding x, its negation $-x$, and the negation of $-x$. By the associative property of addition, we can add these three in any order. If we start by adding the first two, we have $x + (-x) = 0$, so

$$x + (-x) + (-(-x)) = 0 + (-(-x)) = -(-x).$$

However, suppose we start by adding $(-x) + (-(-x))$ first, since $-(-x)$ is the negation of $-x$, we have $(-x) + (-(-x)) = 0$, so find

$$x + (-x) + (-(-x)) = x + 0 = x.$$

We just showed that $x + (-x) + (-(-x))$ equals both $-(-x)$ and x, so we must have

$$-(-x) = x.$$

> **Important:** **Negation of negation:** Let x be any number. Then
>
> $$-(-x) = x.$$
>
> **WARNING!!** Even though $-x$ is sometimes spoken as "negative x," it does not have to be a negative number. For example, when $x = -1$, the value of $-x$ is the positive number 1. The negation of any negative number is positive.

Problem 1.17

Consider the multiplication facts below:

$$\begin{aligned}4 \cdot 3&= 12\\ 3 \cdot 3 &= 9\\ 2 \cdot 3 &= 6\\ 1 \cdot 3 &= 3\\ 0 \cdot 3 &= 0\\ (-1) \cdot 3 &= ____.\end{aligned}$$

(a) Consider where the numbers at the ends of the facts above are on the number line. After the first number, how is each number related to the number above it?

(b) Based on the pattern from part (a), what should $(-1) \cdot (-3)$ be?

Solution for Problem 1.17:

(a) The numbers at the ends of the facts in the list are 12, 9, 6, 3, and 0. On a number line, each number after the first is 3 units to the left of the number before it.

(b) The pattern from part (a) suggests that the answer is 3 units to the left of 0. That number is -3. So $(-1) \cdot 3 = -3$.

$\Diamond$

In words, multiplying a number by -1 appears to be the same as negating the number. This rule is handy. We can replace multiplication by -1 with negation if it helps us, we can also go the other way, replacing negation with multiplication—whichever way helps us solve a problem.

> **Important:** **Multiplying by -1:** Let x be a number. Then
>
> $$(-1)x = -x.$$

But just as before, a pattern is not a proof. So let's prove that $(-1)x$ is the negation of x. We can do this if we can show that

$$(-1)x + x = 0,$$

because if a number added to x gives 0, then our definition of negation says that the number is the negation of x. The clever idea here is to use that fact that $x = 1x$ (since multiplying by 1 doesn't change a number. Then, we have the following computation:

$$\begin{aligned}(-1)x + x &\\ = (-1)x + (1)x &\quad \text{replacing } x \text{ by } (1)x\\ = ((-1) + 1)x &\quad \text{using the distributive property to factor } x\\ = (0)x &\quad \text{definition of the negation } -1\\ = 0. &\quad \text{multiplication by zero.}\end{aligned}$$

So indeed, $(-1)x$ is the negation of x, and therefore

$$(-1)x = -x.$$

We can use the fact that $-x = (-1)x$ to understand how to multiply by negative numbers.

Problems

Problem 1.18 Jump to Solution

Using the fact that $-x = (-1)x$, explain why

$$(-2) \cdot 3 = -(2 \cdot 3).$$

Problem 1.19 Jump to Solution

Using Problem 1.18, explain why

$$-2(-3) = 2 \cdot 3.$$

Problem 1.20 Jump to Solution

Using the fact that $-x = (-1)x$, explain why

$$-(4 + 5) = (-4) + (-5).$$

Problem 1.18

Using the fact that $-x = (-1)x$, explain why

$$(-2) \cdot 3 = -(2 \cdot 3).$$

Solution for Problem 1.18: We know that negation is the same as multiplying by -1, so let's use that fact twice:

$$\begin{aligned}(-2) \cdot 3 &= ((-1) \cdot 2) \cdot 3 && \text{multiplying by } -1\\ &= (-1) \cdot (2 \cdot 3) && \text{associative property}\\ &= -(2 \cdot 3). && \text{multiplying by } -1\end{aligned}$$

In particular, since $(-2) \cdot 3 = -(2 \cdot 3)$ we know that $(-2) \cdot 3 = -6$. $\square$

There was nothing particularly special about the numbers "2" or "3" in Problem 1.18. In the same way, we can show that $(-x)y = -(xy)$ for any numbers x and y.

> **Important:** **Multiplying by negation:** Let x and y be numbers. Then
>
> $$(-x)y = -(xy),$$
> $$x(-y) = -(xy).$$

When x and y are positive, the equation $(-x)y = -(xy)$ says that a negative number times a positive number is negative, or "negative times positive is negative." By the commutative property, we can say the rule the other way around too: "positive times negative is negative."

How do we multiply a negative number by a negative number? Let's try an example.

Problem 1.19

Using Problem 1.18, explain why

$$-2(-3) = 2 \cdot 3.$$

Solution for Problem 1.19: We know how to multiply by a negative, so let's do that twice:

$$\begin{aligned}-2(-3) &= -(2(-3)) && \text{multiplying by negation}\\ &= -(-(2 \cdot 3)) && \text{multiplying by negation}\\ &= 2 \cdot 3. && \text{negation of negation}\end{aligned}$$

In particular, $-2(-3)$ is the positive number 6. $\square$

In the same way, this works for any numbers x and y:

> **Important:** **Negation times negation:** Let x and y be numbers. Then
>
> $$(-x)(-y) = xy.$$

When x and y are positive, this equation says that a negative number times a negative number is positive. Students are often taught the chant "negative times negative is positive." Now you know why the chant is true!

How do we add negatives? Let's find out in the next problem.

Problem 1.20

Using the fact that $-x = (-1)x$, explain why

$$-(4 + 5) = (-4) + (-5).$$

Solution for Problem 1.20: Let's use the fact that negation is the same as multiplying by -1:

$$\begin{aligned}-(4 + 5) &= (-1)(4 + 5) && \text{multiplying by } -1\\ &= (-1)(4) + (-1)(5) && \text{distributive property}\\ &= (-4) + (-5). && \text{multiplying by } -1\end{aligned}$$

In particular, our work above tells us that $(-4) + (-5) = -9$. $\square$

Similarly, the negation of $x + y$ equals $(-x) + (-y)$ in fancier words, negation distributes over addition.

> **Important:** **Negation of sum:** Let x and y be numbers. Then
>
> $$-(x + y) = (-x) + (-y).$$

Negation also distributes over longer sums. For example,

$$-(x + y + z) = (-x) + (-y) + (-z).$$

Exercises

1.4.1:

Compute $-631 + (114 + 631)$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution:

Solution: By rearranging,

$$\begin{aligned}-631 + (114 + 631) &= (-631 + 631) + 114 = 0 + 114 = \boxed{114}\end{aligned}$$

1.4.2:

What is the sum of all of the negative integers that are greater than -5?

Preview: Solution:

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The negative integers greater than -5 are -4, -3, -2, and -1. Their sum is

$$\begin{aligned}-4 + (-3) + (-2) + (-1) &= (-4 + 3 + 2 + 1) = \boxed{-10}\end{aligned}$$

1.4.3:

What is the sum

$$-10 + (-9) + (-8) + \cdots + 9 + 10 + 11 + 12?$$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let's group numbers with their negations:

$$\begin{aligned}&-10 + (-9) + (-8) + \cdots + 9 + 10 + 11 + 12\\ &= (-10 + 10) + (-9 + 9) + \cdots + (-1 + 1) + 0 + 11 + 12\\ &= \underbrace{0 + 0 + \cdots + 0}_{10 \text{ zeros}} + 0 + 11 + 12\\ &= 0 + 0 + 11 + 12 = 11 + 12 = \boxed{23}\end{aligned}$$

1.4.4: Source: MATHCOUNTS

What is the value of $210 \cdot 5 + 105 \cdot (-9)$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution:

$$\begin{aligned}210 \cdot 5 + 105 \cdot (-9) &= 105 \cdot 2 \cdot 5 + 105 \cdot (-9)\\ &= 105 \cdot 10 + 105 \cdot (-9)\\ &= 105(10 + (-9))\\ &= 105 \cdot 1\\ &= \boxed{105}\end{aligned}$$

1.4.5: Source: MATHCOUNTS

What is

$$9342 + (-438)719 + (-9340) = (-438)(-719)?$$

Preview: Solution:

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution:

Solution: The two terms with 438 and 719 add up to zero:

$$\begin{aligned}9342 &= (-438)719 + (-9340) = (-438)(-719)\\ &= 9342 + (-438 \cdot 719) + (-9340) + 438 \cdot 719\\ &= (9342 + (-9340)) + (-438 \cdot 719) + 438 \cdot 719\\ &= (9342 - 9340) + 0 = 9342 - 9340 = \boxed{2}\end{aligned}$$

1.5 Subtraction

Also so as now subtraction can be such – 29 of course, you know that 9 – 3 = 7. But in order to generalize subtraction to written more complicated numbers such as negatives or fractions, it will be useful to define subtraction as a combination of addition and negation. Thus, the notion of finding the inverse of things is a very important one in mathematics and underlies the chapter for addition and negation.

Definition:	Let a and b be any numbers. Then the difference $a - b$ (pronounced "a minus b") is defined as: $a + (-b)$
	The subtraction $a - b$ is sometimes called the **difference** $a - b$.

Subtraction is nothing further than a special case of addition.

For instance, 3 – 2 equals 3 + (–2) using our new definition. But if this definition is going to make any sense, we had better have this equal 1. So we check that 3 + (–2) = 3 – 2 = 1. Thus using our shiny subtraction rule for addition and negation:

$$\begin{aligned} 8 - 3 &= 8 + (-3) && \text{definition of subtraction} \\ &= (-3) + 8 && \text{commutative property} \\ &= 3 + (-3) + 8 && \text{associative property} \\ &= 3 + 8 + (-3) && \text{commutative property} \\ &= 11 + (-3) && \text{adding rules} \end{aligned}$$

Hooray! Our new definition of subtraction matches up with what we already know about subtracting positive integers! Specifically, using the definition that 3 – 2 = 1. So we were able to prove that 8 – 3 = 11 in similar style. 3 – 2 is the number that fits the blank in the addition problem 2 + ___ = 3.

$$2 + ___ = 3$$

So we can think of subtraction as the reverse of addition. This is very important to think about subtraction.

Important:	Let a, b, and c be numbers. If $a + b = c$, then $c - a = b$ and $c - b = a$.

Our goal is to explain how subtraction works with zero and with negative numbers, and to explain some of the other properties of subtraction.

EXERCISE!	Negative and subtraction tools like some, but are different operations. Negative takes one number and changes it into its opposite. Subtraction takes two numbers and finds their difference. Even though negative and subtraction use the same symbol (the minus sign), you should distinguish in your mind between negative and subtraction.

Problems

Problem 1.21 Using its Solution

Use the definition of subtraction to answer the following questions:

(a) What is 5 – 3?
(b) What is 27 – 29?
(c) What is 3 – 3?

Problem 1.22 Using its Solution

Use the definition of subtraction to explain the following equations:

(a) 11 – 2 = 11 + (–2) = 13
(b) 3 – 5 = 3 + (–5) = –2
(c) 11 – 11 = 11 + (–11) = 0

Problem 1.23 Using its Solution

Find numbers a and b such that $a - b$ is not equal to $b - a$.

Problem 1.24 Using its Solution

Find numbers a, b, and c such that $(a - b) - c$ is not equal to $a - (b - c)$.

Problem 1.25 Using its Solution

What is the value of 2500 – 2996 – 2500?

Problem 1.26 Using its Solution

What is the value of

$$2011 - 2375 + 6179 - 2011 - 2375 - 6179?$$

Problem 1.27 Using its Solution

Find numbers a, b, and c such that $a - b - c$ is not equal to $a - (b - c)$.

(a) Explain why $(29 - 35) + 35 = 29 - 35 + 35$.
(b) Compute $(29 - 35) + 35$.
(c) $29 - 35 + 35 = 29$.

Source: 2010 AMC 10 Problem 1

Use the fact that $\frac{1}{999} = 0.\overline{001}$ to calculate $999\overline{001}\overline{001}$.

Use the problem confirmed our intuition about how subtraction between with 0.

Problem 1.28 Using its Solution

Use the definition of subtraction to answer the following questions:

(a) What is 1 – 1?
(b) What is 2 – 2?
(c) What is 3 – 3?

Solution for Problem 1.28:

(a) Subtraction is defined as addition of a negative:

$$\begin{aligned} 9 - 2^2 &= 9 + (-2^2) && \text{definition of subtraction} \\ &= -(-9) \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} 2^3 - 3^2 &= 2^3 + (-3^2) && \text{definition of subtraction} \\ &= 3^2 + (-2^3) && \text{commutative property} \end{aligned}$$

(c) Again in a similar way,

$$\begin{aligned} 2^3 - 4 &= 2^3 + (-4) && \text{definition of subtraction} \\ &= 4^2 + (-8) && \text{negative of zero} \\ &= 8 && \text{adding rules} \end{aligned}$$

Hooray! Now we're talking special! Wow! Very special! Now we can see these subtraction properties in number other than 0.

Important:	Let a be any number. Subtract the following properties: **Subtracting the zero:** $a - 0 = a$. **Self subtraction:** $a - a = 0$. **Negative of zero:** $-0 = 0$.

Next, how do we subtract when some of the numbers involved are negative? You may already know some rules about negative numbers and subtraction. Let's see if we can prove them using our new definition.

Problem 1.29 Using its Solution

Use the definition of subtraction to explain the following equations:

(a) 11 – (–3) = 11 + (–(–3)) = 14.
(b) 3 – (–5) = 3 + (–(–5)) = 8.
(c) 11 – (–3) = 11 + (–(–3)).

Solution for Problem 1.29:

(a) Let's change the subtraction to an addition:

$$\begin{aligned} 11 - (-3) &= 11 + (-(-3)) && \text{definition of subtraction} \\ &= 11 + 3 && \text{negative of negative} \end{aligned}$$

(b) In a similar way,

$$\begin{aligned} 11 - (-3) &= 11 + (-(-3)) && \text{definition of subtraction} \\ &= 11 + 3 && \text{negative of zero} \end{aligned}$$

(c) This part takes a few more steps, but we can do it! $11 - (-3) = 11 + (-(-3))$ because the right-hand side is more complicated than the left-hand side and we want $(11 - 3)$ to easily be since that's equal $11 + 3$.

Example:	Now we know that two subtractions are equal. It's other name is: (and) from the more complicated expression

Hot Take:

$$\begin{aligned} -(11 - 3) &= -(11 + (-3)) && \text{definition of subtraction} \\ &= -(11 + (-(-3))) && \text{definition of subtraction} \\ &= -(11 - (-3)) && \text{definition of subtraction} \end{aligned}$$

So $11 - (3) = (11 - (3))$.

Hence $-(11 - 3) = -(11 - (-3))$.

This means that we will begin subtraction as we did with addition or multiplication. For example, $(5)(3) = (5)(3)$ equal to

$$(5)(3) = (5)(3) = (5)(3).$$

because we subtract from left to right. But we can change the order of subtraction.

$$(5)(3) = (5)(3) = (5)(3).$$

There is good news, though. Remember that we can change the order of addition and negatives, and addition is commutative and associative. So we can use the following shiny subtraction rules:

Concept:	**Its same subtraction problems:** 1. Change all subtractions to addition. 2. Rearrange the addition using the commutative and associative properties 3. Optional Change some of the addition back to subtractions.

Let's use this subtraction to addition strategy at work.

Problem 1.30 Using its Solution

What is the value of $2011 - 2996 - 2500$?

Problem 1.31 Using its Solution

What is the value of

$$2011 - 2375 + 6179 - 2011 - 2375 - 6179?$$

Source: 2010 AMC 10 Problem 1

Solution for Problem 1.31: First, let's convert the two subtractions to additions:

$$2011 + (-2996) + (-2500)$$

Now we can group similar terms together:

$$\begin{aligned} &(2011 + (-2011)) + (-2996) + (-2500) \\ &= (2011 + (-2011)) + (-2996 + (-2500)) \end{aligned}$$

The sum $(-2011 + (2011))$ is zero, so we have:

$$-2996 + (-5)$$

Therefore, the answer is $-2996 + (-5)$.

Problem 1.32 Using its Solution

What is the value of $2011 - 2996 - 2500$

Problem 1.33 Using its Solution

What is the value of

$$2011 - 2375 + 6179 - 2011 - 2375 - 6179$$

Source: 2010 AMC 10 Problem 1

Solution for Problem 1.33: Remember that some differences are computed from left to right. We could keep the subtracts in this given order, but the calculations get big. It's more important to use the commutative property to rearrange the subtracts.

First, we can rewrite the subtracts as adds:

$$2011 + (-2375) + 6179 + (-2011) + (-2375) + (-6179)$$

Second, we bring similar subtracts together:

$$(2011 + (-2011)) + (-2375 + (-2375)) + (6179 + (-6179))$$

Third, we convert some of the same back to differences:

$$(2011 - 2011) + (-2375 - (2375)) + (6179 - 6179)$$

Each of these differences have value of zero:

$$0 + 0 + 0$$

So the answer is **0**!

With practice, you'll be able to go from the original expression

$$2011 - 2375 + 6179 - 2011 - 2375 - 6179$$

to the rearrangement

$$(2011 - 2011) + (2375 - (2375)) + (6179 - 6179)$$

without from converting the subtractions to adds. Some of the numbers involved in this problem were originally being added (246, 1475, and 6179) and still being added after the rearrangement. Some of the numbers that were originally being subtracted (246, 1475, and 6179) are still being subtracted after the rearrangement.

Having discussed the commutative and associative properties, we turn to the distributive property. Remember that multiplication distributes over addition. Does multiplication distribute over subtraction too? The real problem is a test case.

Problem 1.34 Using its Solution

(a) Explain why $(29 - 35) + 35 = 29 - 35 + 35$.
(b) Compute $(29 - 35) + 35$.

Source: 2010 AMC 10 Problem 1

Solution for Problem 1.34: As proposed by the problem, we can write $(29 - 35) + 35$ as

$$29 - 35 + 35 = (29 - 35) + 35$$

As the distributive property now substitutes, we get:

$$29 \cdot 35 + 35 \cdot (29 - 35)$$

Now our difference is just $35(29 - 35)$, which is $35(-6)$.

Even better, multiplication distributes over any combination of additions and subtractions. For example, we can expand $4(9 - 7 + 4)$ into $(4 \cdot 9) - (4 \cdot 7) + 4$.

The last problem has a surprising use of the distributive property.

Problem 1.35 Using its Solution

Compute $999 \cdot 999 + 999$.

Source: 2010 AMC 10 Problem 1

Solution for Problem 1.35: As proposed by the problem, we can write $999 \cdot 999 + 999$ as

$$999 \cdot 999 + 999 = (999 \cdot 999) + (999 \cdot 1)$$

Now, we can factor out 999:

$$999 \cdot (999 + 1)$$

Finally, we get $999 \cdot 1000 = 999000$.

Exercises

1.5.1 Source: MATH PROBLEMS

What is the value of $99{,}987 - 1{,}000 - 99{,}987$?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution:

Solution: Subtracting a number from itself gives zero, so

$$99{,}987 - 1{,}000 - 99{,}987 = 99{,}987 - 99{,}987 - 1{,}000 = 0 - 1{,}000 = -1{,}000.$$

1.5.2 Source: MATH PROBLEMS

Compute $2000 - 1999 + 1998 - 1997$.

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution:

Solution: Rearranging to suit your given sum,

$$(1998 + 2000) - (1997 + 1999) = 3998 - 3996 = 2.$$

1.5.3 Source: MATH PROBLEMS

The city of Jacksonville had a high temperature of 70° and a low temperature of 52°. At the same day, Bismarck, North Dakota, had a high temperature of 63° and a low temperature of 27°. What was the difference between the two cities' temperature ranges?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution:

Solution: The first set of high-minus-degrees-Fahrenheit-low temperatures, we compute the high minus the low:

$$70^\circ - 52^\circ = 18^\circ \text{ and } 63^\circ - 27^\circ = 36^\circ.$$

1.5.4 Source: MATH PROBLEMS

Evaluate $5 - (-10) - (-5)$.

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution:

Solution:

$$\begin{aligned} 5 - (-10) - (-5) &= 5 + (-(-10)) + (-(-5)) \\ &= 5 + (-(-10)) \\ &= 5 + 10 \\ &= 15 \\ &= 5 - (-5) \\ &= 5 + 5 \\ &= 10. \end{aligned}$$

1.5.5 Source: MATH PROBLEMS

If x and y are positive such that $y - x = 2$ what is the value of $y^2 - x^2$?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution:

Solution: $y^2 - x^2 = (y - x)(y + x) = 2(y + x)$.

1.5.6 Source: MATH PROBLEMS

$$2011 - 2375 + 6179 - 2011 - 2375 - 6179$$

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution:

Solution: We will place the added parts first followed by the subtracted parts.

$$\begin{aligned} &(2011 - 2011) + (6179 - 2375) + (-2375 - 6179) \\ &= (2011 + 6179) + (-2375 - 2375 - 6179) \\ &= 8190 - (2375 + 2375 + 6179) = 8190 - 10{,}929 = -2{,}739. \end{aligned}$$

1.5.7 Source: MATH 8

$$1000 - 1993 + 1992 - 1991 + 1990 - 1989 + \cdots + 1000 - 999$$

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution:

Solution: Let's compute the sum of each number in the first sum with a number in the second sum that has the same final two digits.

$$\begin{aligned} &(1000 - 1993) + (1992 - 1991) + (1990 - 1989) + \cdots + (1000 - 999) \\ &= (1000 - 1993) + (1992 - 1991) + (1990 - 1989) + \cdots + (1000 - 999) \\ &= -993 + 1 + 1 + \cdots + 1 = -993 + 500 \cdot 1 = -493. \end{aligned}$$

(0 matches)

1.5.8

$$20 + 24 + 27 + 30 + 33 + 41 + 71 + 92 + 73$$

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution:

Solution: Let's rewrite this as follows:

$$\begin{aligned} &(20 + 24 + 27 + 30 + 33 + 41 + 71 + 92 + 73) \\ &= (20 + 24 + 27 + 30 + 33 + 41 + 71 + 92 + 73) \\ &= 1{,}000 - 1{,}000 = 0. \end{aligned}$$

1.5.9 Source: MATH 8

$$1000 - 1001 + 1000 - 1000 + 1000 - 1000 + \cdots + 1000 - 1000$$

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution:

Solution: Let's rewrite this as follows:

$$\begin{aligned} &1000 - 1001 + 1000 - 1000 + 1000 - 1000 + \cdots + 1000 - 1000 \\ &=

1.6 Reciprocals

We learned back in Section 1.2 here that adding 0 to a number doesn't change the number:

$$0 + x = x$$

for any x. For this reason, we call 0 the **identity** for addition. In Section 1.4 we also learned that for any number x, there is a number called the negation of x (written $-x$) whose sum with x is this identity, 0. That is, $-x$ is the number that goes in the blank in the equation

$$___ + x = 0.$$

Does multiplication have an identity? It isn't 0, since multiplying any number by 0 gives 0, not the original number. Of course, 1 is the number we seek:

$$1 \cdot x = x$$

for any number x, so 1 is the identity for multiplication.

But the question we really want to ask is: given any number x, what number can we put in the blank to solve the equation

$$___ \cdot x = 17$$

The answer to this question is called the **reciprocal** of x.

> **Definition:** For any number x, the **reciprocal** of x, written as $\frac{1}{x}$, is the number such that
>
> $$\frac{1}{x} \cdot x = 1.$$
>
> This number is also called the **multiplicative inverse** of x, and we can say $\frac{1}{x}$ as "1 over x."

Of course, you've seen this sort of number before: $\frac{1}{2}$ looks like a fraction. If $x = 2$, then $\frac{1}{2}$ is the fraction one-half; if $x = 3$, then $\frac{1}{3}$ is the fraction one-third, and so on. Later, in Chapter 4, we'll see that reciprocals of positive integers are exactly the same thing as the fractions that you already know. However, for now, we'll just treat a reciprocal as a new kind of number—a number that "magically" multiplies with another number to give a product of 1. In this section, we figure out some properties of reciprocals, and then in Chapter 4, we will use these properties to learn about fractions.

We're going to assume that every number, except for 0, has a reciprocal. However, the number 0 cannot have a reciprocal. We will see why in our first problem.

Problems

Problem 1.29 Jump to Solution

(a) What is the product of 0 and any number?

(b) Using part (a), explain why 0 doesn't have a reciprocal.

Problem 1.30 Jump to Solution

Explain why the reciprocal of 1 is 1.

Problem 1.31 Jump to Solution

(a) What is the product of $\frac{1}{2}$ and 2?

(b) Using part (a), explain why the reciprocal of $\frac{1}{2}$ is 2.

Problem 1.32 Jump to Solution

(a) What is the product of $5 \cdot 7$ and $\frac{1}{5} \cdot \frac{1}{7}$?

(b) Using part (a), explain why the reciprocal of $5 \cdot 7$ is $\frac{1}{5} \cdot \frac{1}{7}$.

Problem 1.33 Jump to Solution

(a) What is the product of -8 and $-\frac{1}{8}$?

(b) Using part (a), explain why the reciprocal of -8 is $-\frac{1}{8}$.

We'd like to assume that every number has a reciprocal. But we can't do this, because the number 0 is special:

Problem 1.29

(a) What is the product of 0 and any number?

(b) Using part (a), explain why 0 doesn't have a reciprocal.

Solution for Problem 1.29:

(a) We know that 0 times any number is 0. (If you've forgotten why this is true, go back and review Section 1.3 here.)

(b) If the number $\frac{1}{0}$ existed, then $\frac{1}{0} \cdot 0$ would have to equal 1. But recall part (a): the product of any number and 0 must be 0. So $\frac{1}{0}$ cannot exist! Therefore, 0 can't have a reciprocal.

□

Problem 1.29 tells us that we have to be careful with reciprocals. A number must be nonzero in order to have a reciprocal.

> **WARNING!!** The reciprocal of 0 is undefined.

Every nonzero number does in fact have a reciprocal. Let's now look at some properties of reciprocals. The "nicest" number for multiplication is 1, so it makes sense to first look at $\frac{1}{1}$.

Problem 1.30

Explain why the reciprocal of 1 is 1.

Solution for Problem 1.30: By definition, the reciprocal of 1 is the number that goes in the blank to solve the equation

$$___ \cdot 1 = 1.$$

But we know that $1 \cdot 1 = 1$. So the reciprocal of 1 is 1. □

Problem 1.30 confirms the "obvious" fact that $\frac{1}{1} = 1$. We also saw a general strategy for dealing with reciprocals:

> **Concept:** **Reciprocal strategy:** To show that two numbers are reciprocals of each other, multiply them and check that their product is 1.

Remember that the negation of the negation of a number is the original number. Is there a similar result for reciprocals?

Problem 1.31

(a) What is the product of $\frac{1}{2}$ and 2?

(b) Using part (a), explain why the reciprocal of $\frac{1}{2}$ is 2.

Solution for Problem 1.31:

(a) By the definition of reciprocal, the product of $\frac{1}{2}$ and 2 is 1.

(b) From part (a), we see that the number that fills in the blank to solve

$$___ \cdot \frac{1}{2} = 1$$

is 2. But that number, by definition, is the reciprocal of $\frac{1}{2}$. So the reciprocal of $\frac{1}{2}$ is 2.

□

We just showed that the reciprocal of the reciprocal of 2 is 2. In the same way, for every nonzero number x, we can show that the reciprocal of $\frac{1}{x}$ is x: We can write that property as

$$\frac{1}{\left(\frac{1}{x}\right)} = x.$$

> **Important:** **Reciprocal of reciprocal:** Let x be a nonzero number. Then $\frac{1}{x}$ is nonzero and its reciprocal is x.

How do we multiply reciprocals? The next problem shows how.

Problem 1.32

(a) What is the product of $5 \cdot 7$ and $\frac{1}{5} \cdot \frac{1}{7}$?

(b) Using part (a), explain why the reciprocal of $5 \cdot 7$ is $\frac{1}{5} \cdot \frac{1}{7}$.

Solution for Problem 1.32:

(a) Let's multiply:

$$\begin{aligned} &(5 \cdot 7)\left(\frac{1}{5} \cdot \frac{1}{7}\right) && \text{commutative and associative properties} \\ &= \left(\frac{1}{5} \cdot 5\right)\left(\frac{1}{7} \cdot 7\right) && \\ &= 1 \cdot 1 && \text{definition of reciprocal (twice)} \\ &= 1. && \text{multiplying by 1} \end{aligned}$$

So the product is 1.

(b) From part (a), we see that $5 \cdot 7$ and $\frac{1}{5} \cdot \frac{1}{7}$ are reciprocals of each other because their product is 1. So the reciprocal of $5 \cdot 7$ is $\frac{1}{5} \cdot \frac{1}{7}$. As an equation, we have

$$\frac{1}{5} \cdot \frac{1}{7} = \frac{1}{5 \cdot 7}.$$

□

There was nothing special about the numbers 5 and 7 in Problem 1.32—the same result holds for any two nonzero numbers. We conclude that the reciprocal of a product is the product of reciprocals.

> **Important:** **Reciprocal of product:** Let x and y be nonzero numbers. Then xy is nonzero and its reciprocal is $\frac{1}{x} \cdot \frac{1}{y}$. That is,
>
> $$\frac{1}{x} \cdot \frac{1}{y} = \frac{1}{xy}.$$

A similar result holds for longer products. For example,

$$\frac{1}{3} \cdot \frac{1}{5} \cdot \frac{1}{7} = \frac{1}{3 \cdot 5 \cdot 7} = \frac{1}{105}.$$

How do we find the reciprocal of a negative number? The next problem shows the way.

Problem 1.33

(a) What is the product of -8 and $-\frac{1}{8}$?

(b) Using part (a), explain why the reciprocal of -8 is $-\frac{1}{8}$.

Solution for Problem 1.33:

(a) Let's multiply the two numbers:

$$\begin{aligned} \left(\frac{-1}{8}\right) \cdot (-8) &= \frac{1}{8} \cdot 8 && \text{negation times negation} \\ &= 1. && \text{reciprocal property} \end{aligned}$$

So the product is 1.

(b) From part (a), -8 and $-\frac{1}{8}$ are reciprocals of each other. So the reciprocal of -8 is $-\frac{1}{8}$. As an equation,

$$\frac{1}{-8} = -\frac{1}{8}.$$

□

In a similar way, we can take the reciprocal of any negative. The reciprocal of a negation is the negation of the reciprocal.

> **Important:** **Reciprocal of negation:** Let x be a nonzero number. Then
>
> $$\frac{1}{-x} = -\frac{1}{x}.$$

What's a Reciprocal?

Exercises

1.6.1:

What is the reciprocal of -2?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: By the "reciprocal of negation" property, and because the reciprocal of 1 is 1, we have

$$\frac{1}{-1} = -\frac{1}{1} = \boxed{-1}.$$

1.6.2:

What number is *not* the reciprocal of any number?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The only number that is not a reciprocal is $\boxed{0}$. Zero times any number is 0, not 1, so zero can't be the reciprocal of a number.

In contrast, any number besides zero is a reciprocal. Namely, if x is a nonzero number, then x is the reciprocal of $\frac{1}{x}$.

1.6.3:

What is the product of any nonzero number and twice its reciprocal?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let x be the nonzero number. Twice its reciprocal is $2 \cdot \frac{1}{x}$. So the product of x and twice its reciprocal is

$$x\left(2 \cdot \frac{1}{x}\right) = 2\left(\frac{1}{x} \cdot x\right) = 2 \cdot 1 = \boxed{2}.$$

1.6.4:

Multiply the negation of a positive number by the reciprocal of that same positive number. What is the product?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let x be the positive number. When we multiply the reciprocal of x by the negation of x, we get

$$\frac{1}{x}(-x) = -\left(\frac{1}{x} \cdot x\right) = \boxed{-1}.$$

1.6.5: Source: AMC 8

Compute $(2 \cdot 3 \cdot 4)\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let's use the distributive law:

$$\begin{aligned} &(2 \cdot 3 \cdot 4)\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) \\ &= (2 \cdot 3 \cdot 4)\frac{1}{2} + (2 \cdot 3 \cdot 4)\frac{1}{3} + (2 \cdot 3 \cdot 4)\frac{1}{4} \\ &= \left(\frac{1}{2} \cdot 2\right)(3 \cdot 4) + \left(\frac{1}{3} \cdot 3\right)(2 \cdot 4) + \left(\frac{1}{4} \cdot 4\right)(2 \cdot 3) \\ &= 1(3 \cdot 4) + 1(2 \cdot 4) + 1(2 \cdot 3) \\ &= 3 \cdot 4 + 2 \cdot 4 + 2 \cdot 3 \\ &= 12 + 8 + 6 \\ &= \boxed{26}. \end{aligned}$$

1.7 Division

Yesterday already learn how to do simple division, such as $12 \div 3$. You might think of this as "I have 12 apples, if I share them out to 3 friends, each friend get 4 apples." Today we will learn more about division, and how it is related to other operations.

Division is the inverse of multiplication, just like subtraction is the inverse of addition. Division is also the inverse of multiplication, because it undoes what multiplication does.

In a similar way, we will define division as a combination of multiplication and subtraction.

Definition: Let a and b be numbers such that b is not zero. Then the quotient $a \div b$ (pronounced "a divided by b") is defined as

$$a \div b = a \times \frac{1}{b}$$

In other words, dividing by a number means multiplying by its reciprocal. For instance, $10 \div 2$ equals $10 \times \frac{1}{2}$. But for the definition to be well-defined, we need to make sure that b is not zero, because the reciprocal of zero is undefined, just like we can't divide by zero.

Since $a \div b$ is defined as $a \times \frac{1}{b}$, it can be proved that the inverse of multiplication is also true for division.

IMPORTANT: You can't divide by zero. Zero is a problem! Before dividing by a number, be sure that the number is zero.

Problems

Problem 1.a Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Using the definition of division and the fact that $\frac{1}{2} \div \frac{1}{3}$ equals $\frac{1}{2} \times \frac{3}{1}$.

Problem 1.b Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try to cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.c Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try to cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.d Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try to cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.e Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try to cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.f Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try to cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.g Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try to cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.h Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try to cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.i Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try to cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.j Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try to cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.k Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.l Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.m Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.n Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.o Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.p Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.q Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.r Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.s Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.t Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.u Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.v Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.w Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Problem 1.x Using the definition of division, find the quotient $\frac{1}{2} \div \frac{1}{3}$.

Let's try cross number. Use the definition of division to answer the following questions.

(a) $\frac{1}{2} \div \frac{1}{3} =$

(b) $\frac{1}{2} \times \frac{3}{1} =$

(c) $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} =$

Exercises

1.7.1. Calculate

$$(1 \times 1) \div (1 \times 2) + (2 \times 3) \div (3 \times 4) + (4 \times 5) \div (5 \times 6) =$$

Hide Solution

Show Solution

1.7.2. Calculate

$$(1 \times 1) \div (1 \times 2) + (2 \times 3) \div (3 \times 4) + (4 \times 5) \div (5 \times 6) =$$

Hide Solution

Show Solution

1.7.3. Calculate

$$(1 \times 1) \div (1 \times 2) + (2 \times 3) \div (3 \times 4) + (4 \times 5) \div (5 \times 6) =$$

Hide Solution

Show Solution

1.7.4. Calculate

$$(1 \times 1) \div (1 \times 2) + (2 \times 3) \div (3 \times 4) + (4 \times 5) \div (5 \times 6) =$$

Hide Solution

Show Solution

1.7.5. What is $\frac{1}{2} \div \frac{1}{3}$?

Hide Solution

Show Solution

1.7.6. What is $\frac{1}{2} \div \frac{1}{3}$?

Hide Solution

Show Solution

1.7.7. What is the value of $\frac{1}{2} \div \frac{1}{3}$?

Hide Solution

Show Solution

1.7.8. Calculate

$$(2 \div 3) \div (3 \div 4) + (4 \div 5) \div (5 \div 6) =$$

Hide Solution

Show Solution

1.7.9. Calculate

$$(2 \div 3) \div (3 \div 4) + (4 \div 5) \div (5 \div 6) =$$

Hide Solution

Show Solution

1.7.10. Calculate

$$(2 \div 3) \div (3 \div 4) + (4 \div 5) \div (5 \div 6) =$$

Hide Solution

Show Solution

1.7.11. Calculate

$$(2 \div 3) \div (3 \div 4) + (4 \div 5) \div (5 \div 6) =$$

Hide Solution

Show Solution

1.8 Summary

In this chapter, we explored the basic arithmetic operations of addition, negation, subtraction, multiplication, reciprocation, and division. Our goal was to derive as many properties about these operations as possible.

Important:

Order of operations: Perform the operations in an expression in the following order.

1. Evaluate expressions inside parentheses first.
2. Compute powers. (We cover powers in Chapter 2.)
3. Multiply and divide from left to right.
4. Add and subtract from left to right.

We described that the following properties of arithmetic hold for all numbers:

Important:

Let a, b, and c be numbers.

Addition is commutative: $a + b = b + a$.
Addition is associative: $(a + b) + c = a + (b + c)$.
Adding zero: $a + 0 = a$.
Multiplication is commutative: $ab = ba$.
Multiplication is associative: $(ab)c = a(bc)$.
Multiplying by 1: $1a = a$.
Multiplication distributes over addition: $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.
Negation property: $-a + a = 0$.
Reciprocal property: If a is nonzero, then $\frac{1}{a} \cdot a = 1$.

These nine rules are the only assumptions that we need! Starting from these few properties, we can figure out a surprising number of other useful facts.

We defined subtraction and division in terms of addition and multiplication.

Definitions:

Let a and b be numbers.

Subtraction: $a - b = a + (-b)$.

Division: If b is nonzero, then $a \div b = a \cdot \frac{1}{b}$.

From these few basic properties and definitions, we proved dozens of other important properties of arithmetic. You shouldn't have to memorize these properties. Instead, you should be comfortable with *why* each of the following rules of arithmetic are true. If you aren't, go back and review the appropriate section of the chapter.

Important:

Let a, b, and c be numbers.

Negation of negation: $-(-a) = a$.
Negation of sum: $-(a + b) = (-a) + (-b)$.
Multiplying by zero: $0a = 0$.
Multiplying by -1: $(-1)a = -a$.
Multiplying by negation: $(-a)b = -(ab)$ and $a(-b) = -(ab)$.
Negation times negation: $(-a)(-b) = ab$.
Subtracting from zero: $0 - a = -a$.
Self subtraction: $a - a = 0$.
Subtracting zero: $a - 0 = a$.
Subtraction of negation: $a - (-b) = a + b$.
Subtraction from negation: $-a - b = -(a + b)$.
Negation of subtraction: $-(a - b) = b - a$.
Multiplication distributes over subtraction: $a(b - c) = ab - ac$ and $(b - c)a = ba - ca$.
Reciprocal of reciprocal: If a is nonzero, then the reciprocal of $\frac{1}{a}$ is a.
Reciprocal of negation: If a is nonzero, then $\frac{1}{-a} = -\frac{1}{a}$.
Reciprocal of product: If a and b are nonzero, then $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$.
Dividing into zero: If a is nonzero, then $0 \div a = 0$.
Self division: If a is nonzero, then $a \div a = 1$.
Dividing by 1: $a \div 1 = a$.
Dividing into 1: If x is nonzero, then $1 \div x = \frac{1}{x}$.
Dividing by reciprocal: If b is nonzero, then $a \div \frac{1}{b} = ab$.
Dividing into negation: If b is nonzero, then $(-a) \div b = -(a \div b)$.
Dividing by negation: If b is nonzero, then $a \div (-b) = -(a \div b)$.
Negation divided by negation: If b is nonzero, then $(-a) \div (-b) = a \div b$.
Cancel common factor: If a and c are nonzero, then $(ab) \div (ac) = b \div c$.
Division by a number distributes over addition: If c is nonzero, then $(a + b) \div c = a \div c + b \div c$.
Division by a number distributes over subtraction: If c is nonzero, then $(a - b) \div c = a \div c - b \div c$.

WARNING!!

Keep these arithmetic warnings in mind:

- Subtraction is neither commutative nor associative. That is, $a - b$ and $b - a$ are **not** necessarily equal, and neither are $a - (b - c)$ and $(a - b) - c$.
- You can't divide by zero. Division by zero is undefined. Before dividing by a number, be sure that the number is nonzero.

WARNING!!

Watch out for these, too:

- Division is neither commutative nor associative. That is, $a \div b$ and $b \div a$ are **not** necessarily equal, and neither are $a \div (b \div c)$ and $(a \div b) \div c$.
- We can use the distributive property to divide a sum by a number, but we can't use the distributive property to divide a number by a sum. In other words, $a \div (b + c)$ is **not** necessarily equal to $a \div b + a \div c$. Similarly, $a \div (b - c)$ is **not** the same as $a \div b - a \div c$.

Challenge Problems

1.62:

What is the sum of the first sixty-one positive integers?

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We know that we can sum an even number of consecutive integers by grouping the integers into pairs. So, we'll first add the first 60 positive integers, and then add 61 to the resulting sum. Grouping the first 60 integers into 30 pairs gives

$$\begin{aligned} 1+2+\cdots+59+60 &= (1+60)+(2+59)+\cdots+(30+31) \\ &= 61+61+\cdots+61 = 30\cdot 61 = 1830. \end{aligned}$$

(M Qualifier)

So, the sum of the first 61 positive integers is $1830+61 = \boxed{1891}$.

1.64:

What is the value of the sum $5+10+15+\cdots+95+100$?

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Let's group the 20 numbers into 10 pairs:

$$\begin{aligned} &5+10+15+\cdots+95+100 \\ &= (5+100)+(10+95)+(15+90)+\cdots+(50+55) \\ &= 105+105+105+\cdots+105 \\ &= 10\cdot 105 = \boxed{1050}. \end{aligned}$$

(J Qualifier)

1.65:

What is the sum

$$-100+(-99)+(-98)+\cdots+97+98?$$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Assume

$$\begin{aligned} &-100+(-99)+(-98)+\cdots+97+98 \\ &= -200+(-99+98)+(-98+97)+(-97+97)+\cdots \\ &+(-1+1)+0 \\ &= -200+(-99)+0+0+\cdots+0+0 \\ &= -200 = 99 = \boxed{-199}. \end{aligned}$$

1.66:

Source: MATHCOUNTS

Find the sum

$$(-39)+(-37)+(-35)+\cdots+(-1).$$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: After factoring out the negative, we will group the 20 numbers into 10 pairs:

$$\begin{aligned} &-39+(-37)+(-35)+\cdots+(-1) \\ &= -(39+37+35+\cdots+1) \\ &= -((39+1)+(37+3)+(35+5)+\cdots+(21+19)) \\ &= -(40+40+40+\cdots+40) \\ &= -(20\cdot 40) \end{aligned}$$

(M Qualifier)

$$= \boxed{-800}.$$

1.67:

Source: MOEMS

Find the value of

$$100-98+96-94+92-90+\cdots+8-6+4-2.$$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We will group the 50 numbers into 25 pairs:

$$\begin{aligned} &100-98+96-94+92-90+\cdots+8-6+4-2 \\ &= (100-98)+(96-94)+(92-90)+\cdots+(8-6)+(4-2) \\ &= \underbrace{2+2+2+\cdots+2+2}_{25 \text{ times}} \\ &= 25\cdot 2 \\ &= \boxed{50}. \end{aligned}$$

1.68:

Source: MATHCOUNTS

What is the product

$$\begin{aligned} &(49+(-50))(36+(-9))(32+(-8))\cdots(-32+8) \\ &(-36+9)(-49+50), \end{aligned}$$

where the first number in each factor is decreasing by 4, and the second number in each factor is increasing by 1?

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: The factors of this product are equal to 30, 27, 24, and so on until -30. So in between, one of the factors equals zero, namely the factor $(9+8)$. Because one of the factors equals zero, and zero times anything is zero, the entire product is $\boxed{0}$.

1.69:

Source: MATHCOUNTS

"Schwing" a one-digit number to make it a two-digit number (for example, making 3 into 323) is equivalent to multiplying by eleven. Schwing a two-digit number to make it a four-digit number (for example, making 23 into 2323) is equivalent to multiplying the two-digit number by what value?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Let's express 2323 in terms of 23:

$$2323 = 2300+23 = 23\cdot 100+23\cdot 1 = 23(100+1) = 23\cdot 101,$$

So 2323 is 101 times 23. The same method works for any two-digit starting number. For example, 7474 is 101 times 74. So the answer is $\boxed{101}$.

1.70:

Source: MATHCOUNTS

The number 222,222 is equal to the product 37,037 · 6. What is the product of 37,037 and 27?

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Because 6 times 37,037 is 222,222, we know that 3 times 37,037 is 111,111. So we have

$$27\cdot 37{,}037 = 9\cdot 3\cdot 37{,}037 = 9\cdot 111{,}111 = \boxed{999{,}999}.$$

1.71:

Given numbers a and b, let $a \oplus b$ equal $2a+2b$. For example, $3 \oplus 4$ equals 14.

(a) Show that $\oplus$ is commutative.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Let a and b be any numbers. Using the definition of the operation $\oplus$, we have

$$a \oplus b = 2a+2b.$$

Also

$$b \oplus a = 2b+2a = 2a+2b.$$

The two expressions are equal. So $\oplus$ is commutative.

(b) Show that $\oplus$ is not associative.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: There are many examples that show that $\oplus$ is not associative. For instance, choose $a=1$, $b=0$, and $c=0$. On one hand,

$$\begin{aligned} (a \oplus b) \oplus c &= (1 \oplus 0) \oplus 0 \\ &= (2\cdot 1+2\cdot 0) \oplus 0 \\ &= 2 \oplus 0 \\ &= 2\cdot 2+2\cdot 0 \\ &= 4. \end{aligned}$$

On the other hand,

$$\begin{aligned} a \oplus (b \oplus c) &= 1 \oplus (0 \oplus 0) \\ &= 1 \oplus (2\cdot 0+2\cdot 0) \\ &= 1 \oplus 0 \\ &= 2\cdot 1+2\cdot 0 \\ &= 2. \end{aligned}$$

The two expressions are not equal. So $\oplus$ is not associative.

From both parts, we see that an operation can be commutative without being associative.

1.72:

Given numbers a and b, let $a \oplus b$ equal a. For example, $3 \oplus 4$ equals 3.

(a) Show that $\oplus$ is associative.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Let a, b, and c be any numbers. Using the definition of the operation $\oplus$, we have

$$(a \oplus b) \oplus c = a \oplus c = a.$$

Also,

$$a \oplus (b \oplus c) = a \oplus b = a.$$

The two expressions are equal. So $\oplus$ is associative.

(b) Show that $\oplus$ is not commutative.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: There are many examples that show that $\oplus$ is not commutative. For instance, choose $a=0$ and $b=1$. On one hand,

$$a \oplus b = 0 \oplus 1 = 0.$$

On the other hand,

$$b \oplus a = 1 \oplus 0 = 1.$$

The two expressions are not equal. So $\oplus$ is not commutative.

From both parts, we see that an operation can be associative without being commutative.

1.73:

Let a, b, x, and y be numbers. Show that $(a+b)(x+y)$ equals $ax+ay+bx+by$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: The key is to use the distributive property in two steps:

$$\begin{aligned} (a+b)(x+y) &= a(x+y)+b(x+y) \\ &= (ax+ay)+(bx+by) \\ &= ax+ay+bx+by. \end{aligned}$$

1.74:

Source: MATHCOUNTS

Let a, b, and c be numbers. Simplify the expression $\{a-[b-(c)]-[(a-b)-c]\}$.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Because negation distributes over addition and subtraction, we have

$$\begin{aligned} &\{a-[b-(c)]-[(a-b)-c]\} \\ &= \{a-[b+c]-[a-b-c]\} \\ &= \{a-b-c]-a-[-b]-[-c]\} \\ &= \{a-b-c]-a+b+c \\ &= \{a-a]+\{-b+b]+\{+c+c\} = 0+0+2c = \boxed{2c}. \end{aligned}$$

1.75a:

Source: AMC 8

Find the sum of the digits in the answer to

$$\underbrace{9999\ldots99}_{99 \text{ digits}} \times \underbrace{4444\ldots44}_{44 \text{ digits}}.$$

The first number has 99 digits, each of which is a 9. The second number also has 99 digits, each of which is a 4.

Hint: Simplify the problem: How many hundreds 999 · 44, 999 · 444, or 9999 · 4444?

Hint: Multiplying $\underbrace{9999\ldots99}_{99 \text{ digits}}$ by $\underbrace{4444\ldots44}_{44 \text{ digits}}$ looks hard. Is there a really long number that's easy to multiply $\underbrace{4444\ldots44}_{44 \text{ digits}}$ by?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: A string of nines is 1 less than a power of ten, so the product is

$$\begin{aligned} &\underbrace{9999\ldots99}_{99 \text{ digits}} \times \underbrace{4444\ldots44}_{44 \text{ digits}} \\ &= (\underbrace{10000\ldots00}_{100 \text{ digits}} - 1)\underbrace{4444\ldots44}_{44 \text{ digits}} \\ &= \underbrace{40000\ldots00}_{144 \text{ digits}} - \underbrace{4444\ldots44}_{44 \text{ digits}} = \underbrace{4444\ldots44}_{44 \text{ digits}}\underbrace{0000\ldots00}_{100 \text{ digits}} - \underbrace{4444\ldots44}_{44 \text{ digits}} \\ &= \underbrace{4444\ldots4443}_{43 \text{ digits}}\underbrace{5555\ldots5556}_{56 \text{ digits}}. \end{aligned}$$

The product has 10 fours, 1 three, 43 fives, and 1 six. So the sum of the digits is

$$\begin{aligned} &10\times 4+1\times 3+43\times 5+1\times 6 = 10+4+5+(3+6) \\ &= 10+9+9 \\ &= 19+19 \\ &= 34+9 \\ &= \boxed{43}. \end{aligned}$$

1.76a:

Given numbers a and b, define $a \Delta b$ to be $a+ab+b$. For example, $2 \Delta 3 = 2+3(2)+3 = 11$.

(a) Show that the operation Δ is commutative.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We must show that $a \Delta b = b \Delta a$ for all numbers a and b. We compute:

$$a \Delta b = a+ab+b$$

and

$$b \Delta a = b+ba = a.$$

But

$$a+ab+b = b+ab+a = b+ba = a$$

by the commutativity of addition and multiplication. So $a \Delta b = b \Delta a$, and therefore Δ is commutative.

(b) Show that Δ is associative.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We must show that $(a \Delta b) \Delta c = a \Delta (b \Delta c)$. We compute:

$$\begin{aligned} (a \Delta b) \Delta c &= (a+ab+b) \Delta c \\ &= (a+ab+b)+(a+ab+b)c+c \\ &= a+ab+b+ac+abc+bc+c \\ &= a+b+c+ab+ac+bc+abc, \end{aligned}$$

and

$$\begin{aligned} a \Delta (b \Delta c) &= a \Delta (b+bc+c) \\ &= a+a(b+bc+c)+(b+bc+c) \\ &= a+ab+abc+ac+b+bc+c \\ &= a+b+c+ab+ac+bc+abc. \end{aligned}$$

These are equal, so Δ is associative.

(c) What number is the identity of Δ? That is, what is the number f such that $a \Delta f = a$ for all values of a?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We need to find a number f such that $a \Delta f = a$ for all numbers a. So we compute

$$a \Delta f = a+af+f.$$

For the right-hand side above to always equal a, we need $af+f$ to equal 0 for all a. Factoring f out of $af+f$ gives $(a+1)f$, which will be zero for all a if $f = \boxed{0}$.

(d) What number is the inverse of Δ with respect to 0? That is, what number goes in the blank to solve $___ \Delta 1 = 2$, where f is the number that you found in part (c)?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We want to find the number x so that $x \Delta 1 = 0$. But

$$x \Delta 1 = x+(x)(1)+1 = 2x+1.$$

So we would like $2x+1 = 0$. This means that $2x = -1$, so our answer is

$$x = (-1)\div 2 = -(1\div 2) = -\left(\frac{1}{1}\div\frac{2}{1}\right) = \boxed{-\frac{1}{2}}.$$

Review Problems

1.42: Source: AMC 8

Compute

$$90 + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99.$$

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: We pair the smallest number with the largest, the second smallest with the second largest, and so on:

$$\begin{aligned} &90 + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 \\ &= (90 + 99) + (91 + 98) + (92 + 97) + (93 + 96) + (94 + 95) \\ &= 189 + 189 + 189 + 189 + 189 = 5 \cdot 189 = \boxed{945}. \end{aligned}$$

We also might view each number after the first in the sum as 10 plus a 1-digit number, so

$$\begin{aligned} &90 + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 \\ &= 90 + (90 + 1) + (90 + 2) + (90 + 3) + \cdots + (90 + 9) \\ &= (90 + 90 + 90 + 90 + 90 + 90 + 90 + 90 + 90 + 90) \\ &\quad + (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) \\ &= 10 \cdot 90 + 45 = 900 + 45 = \boxed{945}. \end{aligned}$$

1.43:

Compute the product $25 \cdot (12 \cdot 8)$ in your head.

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: This can be done in your head if you mentally rearrange the numbers using the commutative and associative properties of multiplication:

$$25 \cdot (12 \cdot 8) = (12 \cdot 8) \cdot 25 = 12 \cdot 200 = \boxed{2400}.$$

1.44:

Compute

$$3(101 + 103 + 105 + 107 + 109 + 111 + 113 + 115 + 117 + 119).$$

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: We pair the smallest number with the largest, the second smallest with the second largest, and so on:

$$\begin{aligned} &3(101 + 103 + 105 + 107 + 109 + 111 + 113 + 115 + 117 + 119) \\ &= 3(101 + 119) + (103 + 117) + (105 + 115) + (107 + 113) \\ &\quad + (109 + 111)) \\ &= 3(220 + 220 + 220 + 220 + 220) \\ &= 3 \cdot 5 \cdot 220 = 3 \cdot 1100 = \boxed{3300}. \end{aligned}$$

1.45: Source: MATHCOUNTS

Simplify the expression

$$(1 \cdot 2) + (3 \cdot 4) - (5 \cdot 6) + (7 \cdot 8) - (9 \cdot 0).$$

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: The first expression is

$$\begin{aligned} &42 + 7 - 6 \cdot 4 + 2 \cdot (-1) \cdot 0 = 42 + 7 - 36 + 0 \\ &\qquad = 42 + 7 - 36 \\ &\qquad = 49 - 36 \\ &\qquad = 13. \end{aligned}$$

Anything times zero is zero, so the second expression is

$$(42 + 7 - 6 \cdot 4 + 2 \cdot (-1)) \cdot 0 = 0.$$

The difference between the two expressions is $13 - 0 = \boxed{13}$.

1.46: Source: MATHCOUNTS

What is $42 + 7 - 6 \cdot 4 + 2 \cdot (-1) \cdot 0$ minus $(42 + 7 - 6 \cdot 4 + 2 \cdot (-1) \cdot 0)$?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: The first expression is

$$\begin{aligned} &42 + 7 - 6 \cdot 4 + 2 \cdot (-1) \cdot 0 = 42 + 7 - 36 + 0 \\ &\qquad = 42 + 7 - 36 \\ &\qquad = 49 - 36 \\ &\qquad = 13. \end{aligned}$$

Anything times zero is zero, so the second expression is

$$(42 + 7 - 6 \cdot 4 + 2 \cdot (-1)) \cdot 0 = 0.$$

The difference between the two expressions is $13 - 0 = \boxed{13}$.

1.47:

What is the value of

$$(185 + 378 + 579) - (85 + 178 + 279)?$$

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution

$$\begin{aligned} &(185 + 378 + 579) - (85 + 178 + 279) \\ &= ((185 - 85) + (378 + 178) + (579 - 279) \\ &= 100 + 500 + 300 \\ &= \boxed{900}. \end{aligned}$$

1.48:

Calculate

$$11 + (-15) + 11 - (-15) + 11 - 15 - (11 + 15).$$

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution:

$$\begin{aligned} &11 + (-15) + 11 - (-15) + 11 - 15 - (11 + 15) \\ &= 11 - 15 + 11 + 15 + 11 - 15 - 11 - 15 \\ &= 11 + 11 + 11 - 15 - 15 - 15 - 15 \\ &= 11 + 11 + 11 - 15 \\ &= 22 - 15 = 7 = \boxed{-8}. \end{aligned}$$

1.49: Source: MATHCOUNTS

Express in simplest form $6\left(2(8 - 98) - (19 - 96)\right)$

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: Subtraction is addition of a negative, so

$$\begin{aligned} &6\left(2(8 - 98) - (19 - 96)\right) = 6\left(2(8 - 98) + (-(19 - 96))\right) \\ &\qquad = 6\left(2(8 - 98) + (98 - 19)\right) \\ &\qquad = 6\left(2(8 - 19) + (98 - 98)\right) \\ &\qquad = 6(8 + 0) \\ &\qquad = 6 \cdot 8 \\ &\qquad = \boxed{48}. \end{aligned}$$

1.50:

Compute

$$1 - 3 + 5 - 7 + 9 - 11 + 13 - 15 + 17 - 19 + 21 - 23 + 25.$$

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: Let's pair up the first and second numbers, the third and fourth numbers, and so on:

$$\begin{aligned} &1 - 3 + 5 - 7 + 9 - 11 + 13 - 15 + 17 - 19 + 21 - 23 + 25 \\ &= (1 - 3) + (5 - 7) + (9 - 11) + (13 - 15) + (17 - 19) \\ &\quad + (21 - 23) + 25 \\ &= (-2) + (-2) + (-2) + (-2) + (-2) + (-2) + 25 \\ &= 6(-2) + 25 = -12 + 25 = \boxed{13}. \end{aligned}$$

1.51:

Evaluate $699 \cdot 1587 - 692 \cdot 1587$.

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: We can factor:

$$699 \cdot 1587 - 692 \cdot 1587 = (699 - 692)1587 = 7 \cdot 1587 = \boxed{11107}.$$

1.52: Source: MATHCOUNTS

Express in simplest form $(-39)\left((-35) + (-75) + (-6)(5)\right)$

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: Let's use our rules for multiplying and subtracting negatives.

$$\begin{aligned} &(-39)\left((-35) + (-75) + (-6)(5)\right) \\ &= (-39)\left(-35 - (-75 \cdot 5)\right) \\ &= (-39)(-35 + 375) \\ &= (-39)(-35 + 18) \\ &= (-39) \cdot 43 \\ &= (-39) \cdot 43 \\ &= \boxed{-1677}. \end{aligned}$$

1.53: Source: AMC 8

Compute $(1299) + 3(299) + 2(299) + 294$.

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: Let's rewrite 294 as $299 - 1$, and then factor out 299:

$$\begin{aligned} &(1299) + 3(299) + 2(299) + 294 \\ &= (1299) + 3(299) + 2(299) + 299 - 1 \\ &= (1299) + 3(299) + 2(299) + 1(299) - 1 \\ &= (1 + 3 + 2 + 1)299 - 1 \\ &= 10 \cdot 299 - 1 \\ &= 2990 - 1 \\ &= \boxed{2989}. \end{aligned}$$

1.54:

Evaluate

$$40 \div \frac{1}{2} + 40 \div \frac{1}{3} + 40 \div \frac{1}{4} + 40 \div \frac{1}{5}.$$

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: This is probably easiest if we convert the operations involving reciprocals to operations involving integers using the definition of division:

$$\begin{aligned} &40 \div \frac{1}{2} + 40 \div \frac{1}{3} + 40 \div \frac{1}{4} + 40 \div \frac{1}{5} \\ &= 40 \cdot 2 + 40 \cdot 3 + 40 \cdot 4 + 40 \cdot 5 \\ &= 5 \times 320 = 5 \times 200 \\ &= \boxed{1600}. \end{aligned}$$

1.55: Source: MATHCOUNTS

Express in simplest form $(8 + (-3))(4 - 12)$.

Solution

Your Submission Solution

Solution:

$$\begin{aligned} &(8 + (-3))(4 - 12) = (-8 + 3)(4 - 12) \\ &\qquad = (-2)(-8) \\ &\qquad = 2 \cdot 8 \\ &\qquad = \boxed{16}. \end{aligned}$$

1.56:

Simplify

$$(-13) + (-14) + (-13) + (-15) - (-13).$$

Solution

Your Submission Solution

Solution

$$\begin{aligned} &(-13) + (-14) + (-13) + (-15) - (-13) \\ &= (-13) + (-13) + (-13) - (-13) \\ &= (-13) + (-13) - (-13) \\ &= (-13) + (-13) \\ &= (-13) + 0 \\ &= \boxed{-13}. \end{aligned}$$

1.57: Source: MATHCOUNTS

What is the value of 123,123 divided by 1001?

Solution

Your Submission Solution

Solution: Rewriting 123,123 as 123,000 + 123 allows us to factor out 123:

$$\begin{aligned} &123,123 = 123,000 + 123 \\ &\qquad = 123 \cdot 1000 + 123 \cdot 1 \\ &\qquad = 123(1000 + 1) \\ &\qquad = 123 \cdot 1001. \end{aligned}$$

So, we have

$$123,123 \div 1001 = (123 \cdot 1001) \div (1 \cdot 1001) = 123 \div 1 = \boxed{123}.$$

1.58:

What is the reciprocal of $2 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{9}$?

Solution

Your Submission Solution

Solution: The reciprocal of a product is the product of reciprocals, so the reciprocal of $2 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{9}$ is $\frac{1}{2} \cdot \frac{3}{1} \cdot \frac{2}{1} \cdot \frac{9}{1}$. This simplifies as

$$\frac{1}{2} \cdot \frac{3}{1} \cdot \frac{2}{1} \cdot \frac{9}{1} = \left(\frac{1}{2} \cdot \frac{2}{1}\right) \cdot \left(\frac{3}{1} \cdot \frac{9}{1}\right) = (1 \cdot 1) \cdot (3 \cdot 9) = 3 \cdot 9 = \boxed{27}.$$

1.59:

Compute $\frac{1}{2} \div \frac{1}{6}$.

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: The reciprocal of $\frac{1}{6}$ is 6, so

$$\frac{1}{2} \div \frac{1}{6} = \frac{1}{2} \cdot 6 = 6 \cdot \frac{1}{2} = 6 \div 2 = \boxed{3}.$$

1.60:

Compute $(3 \cdot 4) \div \left(\frac{1}{5} \cdot \frac{1}{6}\right)$.

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution

$$(3 \cdot 4) \div \left(\frac{1}{5} \cdot \frac{1}{6}\right) = 12 \div \frac{1}{30} = 12 \cdot 30 = \boxed{360}.$$

1.61: Source: MATHCOUNTS

Jack sets up all the even integers from 2 to 100, inclusive. Jill sets up all the integers from 1 to 100, inclusive. What is Sean's sum divided by Jule's sum?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: Sean's sum is

$$2 + 4 + 6 + \cdots + 98 = 2(1 + 2(2) + 3(3) + \cdots + 2(50))$$
$$= 2(1 + 2 + 3 + \cdots + 250).$$

Jule's sum is

$$1 + 2 + 3 + \cdots + 250.$$

So Sean's sum is twice Jule's sum. Therefore, Sean's sum divided by Jule's sum is $\boxed{2}$.

1.62:

Gary wanted to compute $200 \div 10 \div 2$ and his reasoning was: "test, $10 \div 2$ is 5, so $200 \div 10 \div 2$ is the same as $200 \div 5$, which is 40, so the answer is 40." Is Gary correct? Why or why not?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reveal

Your Submission Solution

Solution: Division is not associative, so Gary is incorrect. We must compute the divisions from left to right. The correct answer is $200 \div 10 \div 2 = 20 \div 2 = \boxed{10}$.

$\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$

$\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$ $\sqrt{-1}$

2.2.6:

Express each of the following numbers as a power of 2.

(a) $(2^2)^4$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: By the power of a power rule, we have $(2^2)^4 = 2^{2\cdot 4} = \boxed{2^{12}}$.

(b) $4\cdot 4\cdot 4\cdot 4\cdot 4$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We can express repeated multiplication as a power:

$$4\cdot 4\cdot 4\cdot 4\cdot 4 = 4^5 = (2^2)^5 = (2^2)^5 = \boxed{2^{10}}.$$

We might also think of this as, "Each 4 is a product of 2 2's, so the product of 5 4's is a product of $2\cdot 5 = 10$ 2's."

(c) $2^{20}\cdot 2^{13}\div 2^6$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Applying the product and quotient of powers (same base) rules, we have

$$2^{20}\cdot 2^{13}\div 2^6 = 2^{20+13}\div 2^6 = 2^{20+13-6} = \boxed{2^{27}}.$$

(d) $2^{25}\cdot 4^{20}\cdot 8^{30}$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: By power of a power,

$$\begin{aligned}2^{25}\cdot 4^{20}\cdot 8^{30} &= 2^{25}\cdot (2^2)^{20}\cdot (2^3)^{30}\\ &= 2^{25}\cdot 2^{40}\cdot 2^{90}\\ &= 2^{25+40+90}\\ &= \boxed{2^{155}}\end{aligned}$$

(e) $4^4\cdot 8^8\cdot 16^{16}$

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: By power of a power,

$$\begin{aligned}4^4\cdot 8^8\cdot 16^{16} &= (2^2)^4\cdot (2^3)^8\cdot (2^4)^{16}\\ &= 2^8\cdot 2^{24}\cdot 2^{64}\\ &= 2^{8+24+64}\\ &= \boxed{2^{96}}\end{aligned}$$

(f) $4^7\div 2^2$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: By power of a power and quotient of powers,

$$\begin{aligned}4^7\div 2^2 &= (2^2)^7\div 2^2\\ &= 2^{7}\div 2^2\\ &= 2^{7-2}\\ &= \boxed{2^5}.\end{aligned}$$

(g) $\frac{1}{2}\cdot 8^{50}$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: By power of a power and quotient of powers,

$$\frac{1}{2}\cdot 8^{50} = 8^{50}\div 2 = (2^3)^{50}\div 2^1 = 2^{150}\div 2^1 = 2^{150-1} = \boxed{2^{149}}.$$

(h) 256

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $256 = 16^2$, we have $256 = (2^4)^2 = 2^{4\cdot 2} = \boxed{2^8}$.

(i) $2^{20}+2^{20}+2^{20}+2^{20}$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Because repeated addition is multiplication,

$$2^{20}+2^{20}+2^{20}+2^{20} = 4\cdot 2^{20} = 2^2\cdot 2^{20} = 2^{2+20} = \boxed{2^{22}}.$$

(j)★ $4^4+4(4^4)+6(4^4)+4(4^4)+4^4$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: By factoring out 4^4, and then using the power of a power and product of powers rules, we have

$$\begin{aligned}4^4+4(4^4)+6(4^4)+4(4^4)+4^4 &= 1(4^4)+4(4^4)+6(4^4)+4(4^4)+1(4^4)\\ &= (1+4+6+4+1)\,4^4\\ &= 16\cdot 4^4\\ &= 2^4\cdot (2^2)^4\\ &= 2^4\cdot 2^8\\ &= 2^{4+8}\\ &= \boxed{2^{12}}.\end{aligned}$$

(k)★ $4^{15}\div (4^3)^5$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: By power of a power and quotient of powers,

$$4^{15}\div (4^3)^5 = 4^{15}\div 4^5 = 4^{15-5} = 4^{10} = (2^2)^{10} = \boxed{2^{20}}.$$

(l)★ $2^{20}-2^{19}$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let's express 2^{20} in terms of 2^{19}, and then factor:

$$\begin{aligned}2^{20}-2^{19} &= 2^{1+19}-2^{19}\\ &= 2^1\cdot 2^{19}-2^{19}\\ &= 2\cdot 2^{19}-2^{19}\\ &= 2\cdot 2^{19}-1\cdot 2^{19}\\ &= (2-1)2^{19}\\ &= 1\cdot 2^{19}\\ &= \boxed{2^{19}}.\end{aligned}$$

2.2.7:

Source: MATHCOUNTS

How many digits are in the product of 91 and 10^{17}?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: When we expand 10^{17}, we get

$$10^{17} = \underbrace{10\times 10\times\cdots\times 10\times 10}_{17\text{ times}} = 10\underbrace{00\ldots00}_{17\text{ zeros}}.$$

So the product of 91 and 10^{17} is

$$91\times 10^{17} = 91\times 10\underbrace{00\ldots00}_{17\text{ zeros}} = 91\underbrace{00\ldots00}_{17\text{ zeros}}.$$

The final number has 2 nonzero digits followed by 17 zeros. So the number of digits is $2+17 = \boxed{19}$.

2.2.8:

Determine the number of digits in the value of $2^{36}\cdot 5^{17}$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let's pair each factor of 5 with a factor of 2 to form factors of 10, since it's easy to multiply by 10:

$$2^{36}\cdot 5^{17} = 2^{3}\cdot 2^{33}\cdot 5^{17} = 2^{3}\cdot (2\cdot 5)^{17} = 2^{3}\cdot 10^{17} = 8\cdot 10^{17}.$$

Since 10^{17} multiplied out is a 1 followed by 13 zeros, we know that $8\cdot 10^{17}$ multiplied out is an 8 followed by 13 zeros, for a total of $\boxed{14}$ digits.

2.2.9:

Source: MATHCOUNTS

Which of the numbers 2^{100}, 3^{75}, and 5^{50} is the largest?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: All three exponents are 25 times something. So we can write all three numbers as 25th powers:

$$\begin{aligned}2^{100} &= 2^{4\cdot 25} = (2^4)^{25} = 16^{25},\\ 3^{75} &= 3^{3\cdot 25} = (3^3)^{25} = 27^{25},\\ 5^{50} &= 5^{2\cdot 25} = (5^2)^{25} = 25^{25}.\end{aligned}$$

Of the three bases 16, 27, and 25, the largest is 27. So the largest of the numbers on the right is 27^{25}. Therefore, $\boxed{3^{75}}$ is the largest of the numbers on the left.

2.2.10★:

Source: AMC 8

When $10^{93}-93$ is expressed as a single number, what is the sum of its digits?

Hint: What are the digits of the number?

Hint: Simplify the problem. What if we replaced 10^{93} with just 10^3? 10^4? 10^5?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 10^{93} is a 1 followed by 93 zeros, so it has 94 digits. Subtracting 93 produces a 93-digit number in which the first 91 digits are nines and the last two digits are 07.

$$\underbrace{100\ldots00}_{93\text{ zeros}} - 93 = \underbrace{99\ldots99}_{91\text{ nines}}07.$$

The sum of the digits of this number is

$$\begin{aligned}\underbrace{9+9+\cdots+9+9}_{91\text{ nines}} + 0 + 7 &= 91\cdot 9 + 0 + 7\\ &= 819 + 0 + 7\\ &= \boxed{826}.\end{aligned}$$

2.2.11★:

Source: MATHCOUNTS

Which of the numbers

$$2^{27},\quad 3^{18},\quad 3^{9},\quad 4^{27},\quad 4^{9}$$

has the greatest value?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let's first simplify the numbers and then convert the powers of 4 to powers of 2:

$$\begin{aligned}2^{27} &= 2^{27},\\ 3^{18} &= 3^{18},\\ 3^{9} &= 3^{9},\\ 4^{27} &= 4^{3} = (2^2)^3 = 2^{6},\\ 4^{9} &= 4^{3} = (2^2)^3 = 2^{6}.\end{aligned}$$

All the numbers except the third one are powers of 2. Of the powers of 2, the largest is 2^{27}. The third number is 3^{9}, which is less than $4^{18} = (2^2)^{18} = 2^{2\cdot 18} = 2^{36}$. So the largest number is 2^{27}, which came from $\boxed{2^{27}}$.

2.2 Higher Exponents

Let's start by thinking about what it means to "square" a number raised to the 3rd power in Lesson 2 Quiz. The Lesson 2 of Integer 2 also called **power** and the 3rd is called "cubed" and 3th equals 3^3. 3^3 also and it's more special name the higher powers. We can 3^3 is called "three cubed" or "the third power of three" or "three to the third power". Notice that 3^3 means $3 \cdot 3 \cdot 3$ and 3^3 equals 27. We read this number as "three cubed" or "the third power of three." So, we can write $3^3 = 27$. We read this as "three to the third power is 27." In a power with the 2nd and power is the third power, 3^3 is a power of 3 and 3^3 is the third power of 3.

Of course, there are 2nd powers and 3rd powers, and so on. For example, 2^4 means

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 4 \cdot 2 \cdot 2 = 8 \cdot 2 \cdot 2 = 16 \cdot 2 = 32$$

2^4 is read as "to the third power." So, we can state that the right-hand side as follows:

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

The phrase "Whatever 2^4 will be is one hundred sixty-two." 2^4 is read as "two to the fourth power." The power of 2nd is called "two to the 4th is equal to sixteen." 2^4 is read as "two to the fourth power." 2^4 is read as "two to the fourth power." 2^4 is read as "two to the fourth power

2.3 Zero as an Exponent

In the last two sections, we defined a^1, a^2, a^3, a^4, and so on. What about a^{0}? We will define it in this section.

Problems

Problem 2.21 Jump to Solution

Consider the exponent facts below:

$$\begin{aligned} 2^4 &= 16 \\ 2^3 &= 8 \\ 2^2 &= 4 \\ 2^1 &= 2 \\ 2^0 &= ____. \end{aligned}$$

(a) In these equations, what pattern do you see in the numbers on the right?

(b) Assuming that your pattern continues, predict the value of 2^0.

Problem 2.22 Jump to Solution

Let m and n be positive integers such that m is greater than n. In the last section, we introduced the quotient of powers (same base) rule:

$$2^{m-n} = 2^m \div 2^n.$$

Suppose that this quotient rule is true even when m is equal to n. What is 2^0?

In our first problem, we will predict the value of 2^0.

Problem 2.21

Consider the exponent facts below:

$$\begin{aligned} 2^4 &= 16 \\ 2^3 &= 8 \\ 2^2 &= 4 \\ 2^1 &= 2 \\ 2^0 &= ____. \end{aligned}$$

(a) In these equations, what pattern do you see in the numbers on the right?

(b) Assuming that your pattern continues, predict the value of 2^0.

Solution for Problem 2.21:

(a) Each number on the right is the number above it divided by 2. For instance, 16 divided by 2 is 8, and 8 divided by 2 is 4.

(b) According to the pattern, the missing number should be 2 divided by 2, which is 1. So we predict 2^0 is 1.

□

Let's take a look at another reason we might expect 2^0 to equal 1.

Problem 2.22

Let m and n be positive integers such that m is greater than n. In the last section, we introduced the quotient of powers (same base) rule:

$$2^{m-n} = 2^m \div 2^n.$$

Suppose that this quotient rule is true even when m is equal to n. What is 2^0?

Solution for Problem 2.22: Let's choose $m = 1$ and $n = 1$. Then the equation becomes

$$2^{1-1} = 2^1 \div 2^1.$$

Since $1 - 1 = 0$, the left-hand side is 2^0. The right-hand side equals 1, since any nonzero number divided by itself is 1. So, the equation above becomes

$$2^0 = 1.$$

□

Problems 2.21 and 2.22 give us two reasons why it is convenient to define $2^0 = 1$. Similarly, we can see why we define $a^0 = 1$ for any number a.

> **Definition:** Let a be any number. Then a^0 is defined to be 1.

Our definition includes $0^0 = 1$, even though we can't use our explanations in Problems 2.21 and 2.22 to see why we should define 0^0 this way. You won't see the expression 0^0 often; we define it to be 1 in part to avoid having to write "except when the base is 0" in statements like $a^0 \cdot a^n = a^n$.

Zero as an Exponent

Now, let's solve a few problems that contain 0th powers.

Problems

Problem 2.23 Jump to Solution

Evaluate $6^0 + 6^1 + 6^2$.

Problem 2.24 Jump to Solution

Let a be any number. Simplify $4a^0(4a)^0$.

Problem 2.25 *Source: MATHCOUNTS* Jump to Solution

Let $P = (2 - 3 - 4 + 7)^{2347}$ and $Q = (-2 + 3 + 4 - 7)^{2347}$. What is the value of

$$(2 + 3 + 4 + 7)^{P+Q}?$$

Problem 2.23

Evaluate $6^0 + 6^1 + 6^2$.

Solution for Problem 2.23: Because $6^0 = 1$, we have

$$6^0 + 6^1 + 6^2 = 1 + 6 + 36 = 43.$$

□

Problem 2.24

Let a be any number. Simplify $4a^0(4a)^0$.

Solution for Problem 2.24: Both 0th powers are equal to 1, so $4a^0(4a)^0 = 4 \cdot 1 \cdot 1 = 4$. □

The next problem doesn't involve 0th powers . . . or does it?

Problem 2.25 *Source: MATHCOUNTS*

Let $P = (2 - 3 - 4 + 7)^{2347}$ and $Q = (-2 + 3 + 4 - 7)^{2347}$. What is the value of

$$(2 + 3 + 4 + 7)^{P+Q}?$$

Solution for Problem 2.25: The exponent 2347 is scary. Let's first try to simplify P and Q. The value of P is

$$P = (2 - 3 - 4 + 7)^{2347} = 2^{2347}.$$

Because 2347 is odd, the value of Q is

$$Q = (-2 + 3 + 4 - 7)^{2347} = (-2)^{2347} = -2^{2347}.$$

Aha! The values of P and Q are negations of each other. In other words, $P + Q$ is zero. So the expression we want is

$$(2 + 3 + 4 + 7)^{P+Q} = (2 + 3 + 4 + 7)^0 = 1.$$

□

Exercises

2.3.1:

Evaluate the following expressions.

(a) $56 \div 4 \div 3 \cdot 2^0$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We evaluate the exponent first:

$$56 \div 4 \div 3 \cdot 2^0 = 56 \div 4 \div 3 \cdot 1 = 14 \div 3 \cdot 1 = 14 \div 3 = \boxed{17}.$$

(b) $7^4(8 - 2^3) + 11^{6(8)-32}$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We evaluate inside the parentheses first:

$$\begin{aligned} 7^4(8 - 2^3) + 11^{6(8)-32} &= 7^4(8 - 8) + 11^{6(8)-32} \\ &= 7^4 \cdot 0 + 11^{6(8)-32}. \end{aligned}$$

We don't have to compute 7^4; the product $7^4 \cdot 0$ is 0 no matter what 7^4 is. So, we have

$$\begin{aligned} 7^4 \cdot 0 + 11^{6(8)-32} &= 0 + 11^{6(8)-32} \\ &= 11^{6(8)-32} \\ &= 11^{52-32} \\ &= 11^9 \\ &= \boxed{1}. \end{aligned}$$

(c) $7^0 + 3^2 \cdot 4 - 2(14 - 8 \div 2)$

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We evaluate inside the parentheses first, and then the exponents:

$$\begin{aligned} 7^0 + 3^2 \cdot 4 - 2(14 - 8 \div 2) &= 7^0 + 3^2 \cdot 4 - 2(14 - 4) \\ &= 7^0 + 3^2 \cdot 4 - 2 \cdot 10 \\ &= 1 + 9 \cdot 4 - 2 \cdot 10 \\ &= 1 + 36 - 20 \\ &= 37 - 20 \\ &= \boxed{17}. \end{aligned}$$

2.3.2: Source: MATHCOUNTS

Consecutive powers of 3 are added to form this sequence: $3^0, 3^1 + 3^2, 3^3 + 3^4 + 3^5$, and so on. What is the value of the fourth term of this sequence?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The fourth term is $3^9 + 3^{10} + 3^{11}$; its value is

$$\begin{aligned} 3^9 + 3^{10} + 3^{11} &= 1 + 3 + 9 + 27 \\ &= \boxed{40}. \end{aligned}$$

2.3.3:

When $x = 2$ and $y = -2$, what is the value of $x^{x+y} + y^{x-y}$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let's replace each x with 2 and each y with -2:

$$\begin{aligned} x^{x+y} + y^{x-y} &= 2^{2+(-2)} + (-2)^{2-(-2)} \\ &= 2^0 + (-2)^4 \\ &= 1 + 16 \\ &= \boxed{17}. \end{aligned}$$

2.3.4:

Let n be a number. Evaluate $3n^0 \cdot (7n)^0$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Anything raised to the 0th power is 1, so $3n^0 \cdot (7n)^0 = 3 \cdot 1 \cdot 1 = \boxed{3}$.

2.3.5: Source: MATHCOUNTS

Let x be a number. Simplify $6^0x^2 + 6x^2$. Express your answer as a number times a power of x.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: By replacing 6^0 with 1 and then factoring out x^2, we get

$$6^0x^2 + 6x^2 = 1x^2 + 6x^2 = (1 + 6)x^2 = \boxed{7x^2}.$$

2.4 Negative Exponents

Assume that $b^0 = 1$ for any nonzero real number b and that the exponent laws hold for all integer exponents such that $b^m \cdot b^n = b^{m+n}$ and that the rule for the quotient of powers $\frac{b^m}{b^n} = b^{m-n}$ holds for $m > n$.

Problems

Problem 1.1 [0] Answer a Problem

Evaluate the exponent for the following:

$\frac{2^5}{2^2} = 2^{\square}$

$\frac{3^7}{3^4} = 3^{\square}$

$\frac{4^9}{4^6} = 4^{\square}$

$\frac{5^{11}}{5^8} = 5^{\square}$

(a) In these expressions, what pattern does you see for the exponents on the right?

(b) Using the pattern you observed, what do you think the exponent should be for $\frac{2^3}{2^5}$?

(c) Write the fraction $\frac{2^3}{2^5}$ as a fraction with 2 raised to the power of 3 in the numerator and 2 raised to the power of 5 in the denominator.

(d) Write the fraction $\frac{2^3}{2^5}$ as a fraction with 2 raised to the power of 3 in the numerator and 2 raised to the power of 5 in the denominator.

Problem 1.2 [0] Answer a Problem

(a) Each number on the right is the number on the left raised to the power of 2. Explain why.

(b) According the Pythagorean Theorem, the square of the hypotenuse of a right triangle is the sum of the squares of the two legs. Write the value of $\sqrt{\square}$ $\sqrt{\square}$ $\sqrt{\square}$

2.5 Summary

In this chapter, we introduced the concept of exponents.

Definition: Let a be any number. Let n be an integer. If n is positive, then

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ copies of } a}.$$

For instance, $a^4 = a \cdot a \cdot a \cdot a$. If n is zero, then $a^n = 1$. If a is nonzero and n is positive, then

$$a^{-n} = \frac{1}{a^n}.$$

The expression a^n is called a **power**, with **base** a and **exponent** n.

Important: Let a and b be numbers. Let m and n be integers.

Product of powers (same base): $a^m a^n = a^{m+n}$.
Product of powers (same exponent): $a^n b^n = (ab)^n$.
Quotient of powers (same base): $a^m \div a^n = a^{m-n}$.
Quotient of powers (same exponent): $a^n \div b^n = (a \div b)^n$.
Power of power: $(a^m)^n = a^{mn}$.
Power of reciprocal: $\left(\frac{1}{a}\right)^n = \frac{1}{a^n}$.
Negation in exponent: $a^{-n} = \frac{1}{a^n}$.
Power of negation: If n is even, then $(-a)^n = a^n$. If n is odd, then $(-a)^n = -a^n$.

Before using each property, make sure that the powers in it are defined. For example, in the product rule $a^m a^n = a^{m+n}$, we can't choose $a = 0$ and $m = -1$, because 0^{-1} is undefined.

Challenge Problems

2.57

How many of two consecutive positive integers differ by 42? What is the sum(s) of the two integers?

Hint: How do you get from one perfect square to the next perfect square?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We could find squares until we find two that differ by 42, but we want to the best given we a faster approach. We can get from the perfect square x^2 to the next perfect square $(x+1)^2$ by adding a term $2x+1$. So, the squares x^2 and $(x+1)^2$ differ by $2x+1 = 42$. In other words, the squares of two consecutive integers differ by the sum of the integers. This means we are looking for two consecutive integers that add to 42. So, we can find x and $x+1$ by finding x and $x+1$ as the consecutive integers whose square differs by 42.

2.58 Source: MATHCOUNTS ★

Find the sum of all positive integers between Fahrenheit (1947-1969) have fewer than one four different positive integers A, B, C, and D such that $A \times B \times C \times D = 1536$ and $A + B + C + D = 1536$ where A is the sum $A + B + C + D$?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The first five positive cubes are 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, and 1728. We can square 1728 as: $1728^2 = 2985984$. We can find the sum of the digits of 2985984 as: $2+9+8+5+9+8+4 = 45$. The sum of the digits of positive perfect cubes add up to 1728. So, A, B, C, and D are 1, 2, 3, and 4 in some order. Therefore, then sum $\boxed{10}$.

2.59

Express $x^2 - 2^2$ as a power of x.

Hint: Let's find express the number as a power of x, and then convert it to a power of x.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let's first express the number as a power of x, and then convert it to a power of x.

$$\begin{aligned} x^2 - 2^2 &= (x^2 - 2^2)^{\frac{1}{2}} \\ &= x^{\frac{2}{2}} - 2^{\frac{2}{2}} \\ &= x^1 - 2^1 \\ &= x^1 - 2^1 \\ &= x^1 - 2^1 \\ &= x^{\frac{1}{2}} - 2^{\frac{1}{2}} \end{aligned}$$

2.60

What five-digit positive integer with an 8 in the thousands place is the cube of an integer?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We are looking for a cube between 80,000 and 90,000. The cube of 40 is 64,000 and the cube of 50 is 125,000. In the middle, the cube of 45 is 91,125. A little lower, the cube of 44 is $\boxed{85,184}$.

2.61

Express $\frac{3^{10}}{9^5}$ as a power of $\frac{1}{3}$.

Hint: Can you write 9 as a power of 3?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Because $3^2 = 9$, we have

$$3^{10} = 3^{10} = (3^2)^5 = 9^5.$$

Because $9^5 = 9^5$, we have

$$3^{10} = 9^5 \Rightarrow 3^{10} = (3^{-1})^{-5} = \boxed{\left(\frac{1}{3}\right)^{-5}}$$

2.62 Source: MATHCOUNTS ★

Express $x^2 + 2^2 + 10^2 + x^2 + \ldots + 2004^2$ as a power of 2.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: By connecting the product of squares to the square of a product, and then expressing each number as a power of 2, we get:

$$\begin{aligned} &2^2 + 4^2 + 10^2 + \ldots + 2004^2 \\ &= 2^2 + 2^2 \times 2^2 + 2^2 \times 5^2 + \ldots + 2^2 \times 1002^2 \\ &= 2^2 \times (1^2 + 2^2 + 5^2 + \ldots + 1002^2) \\ &= 2^2 \times (1^2 + 1^2 + 2

Review Problems

2.24 ✎ 2

Evaluate the following expressions:

(a) $-4(2^3-10^2)-4(-8)$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We evaluate inside the parenthesis first:

$$-4(2^3-10^2)-4(-8) = -4(8-100)-4(-8)$$
$$= -4(-92)-4(-8)$$
$$= 4\cdot 92+4\cdot 8$$
$$= 368+32$$
$$= \boxed{400}$$

(b) $(5-3)^2+(2-5)^2$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We evaluate inside parenthesis first:

$$(5-3)^2+(2-5)^2 = 2^2+(-3)^2 = 4+9 = \boxed{13}$$

(c) $5\cdot 2^2-(5\cdot 2)^2$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We evaluate inside parenthesis first:

$$5\cdot 2^2-(5\cdot 2)^2 = 5\cdot 4-(10)^2 = 20-100 = \boxed{-80}$$

(d) $5+(-8)^2+(2^3)^2$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We evaluate 2^3 inside the parenthesis first:

$$5+(-8)^2+(2^3)^2 = 5+(-8)^2+(8)^2 = 5+64+64 = \boxed{133}$$

Then, we compute

$$(-8)^2+(-8)^2 = 64+64 = 128,$$

so

$$5+(-8)^2+(8^2)^2 = 5+128 = \boxed{133}.$$

2.26 ✎ 2

Which week does P occur?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The first number P is $\boxed{2}$. The second number is $2^2 = \boxed{4}$, the first number exceeds the second number by $2(4-1) = \boxed{6}$.

2.29 ✎ 2 Source: MATHCOUNTS ✓

What is the value of the sum

$$\sum_{n=1}^{100} \sqrt{n+1} - \sum_{n=1}^{100} \sqrt{n}?$$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Denote $\sum_{n=1}^{100} \sqrt{n+1} - \sum_{n=1}^{100} \sqrt{n}$ by $\sum_{n=1}^{100} \sqrt{n+1} - \sum_{n=1}^{100} \sqrt{n}$. Because 3000 is odd, we have $(-1)^{3000} = 1$. Since any power of -1 is either 1 or -1, we have $(-1)^{3000} = 1$. So we find that

$$\sum_{n=1}^{100} \sqrt{n+1} - \sum_{n=1}^{100} \sqrt{n} = \sum_{n=1}^{100} \sqrt{n+1} - \sum_{n=1}^{100} \sqrt{n}$$
$$= 1 - 1$$
$$= \boxed{0}$$

2.40 ✎ 2 Source: MATHCOUNTS ✓

A rock produces a large crater C.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since C is the "equation backwards", $\boxed{\text{P.S.}}$ Because x is a positive integer, the only solution is $x = \boxed{5}$.

We also can solve the problem with the power of zero, by noting that $2^x = 2^5 \Rightarrow x^0 = 5^0$ or $x = \boxed{5}$.

2.41 ✎ 2

What pre-to the 20th century (the years 1901 through 2000) was a perfect square?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The square of 44 is 1936 and the square of 45 is 2025. Since 1936^2 is less than and 1936^2, to be high, the square we seek (and 1936^2 is the perfect square 1936^2 in the middle, the square of 44 in 1936, and to a little too high, or little lower the square of $\boxed{1936}$.

2.42 ✎ 2

What is the value of x^2 - $2x$ when $x = 3$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Substituting 3 for x we get

$$x^2 - 2x = 3^2 - 2\cdot 3 = 9 - 6 = \boxed{3}.$$

2.43 ✎ 2

What is the value of $3x^2 + 2x$ when $x = 3$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Substituting 3 for x we get

$$3x^2 + 2x = 3\cdot 3^2 + 2\cdot 3 = 3\cdot 9 + 6 = \boxed{33}.$$

2.44 ✎ 2

What is the value of $x^2 + 2x + 1$ when $x = 3$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Substituting 3 for x we get

$$x^2 + 2x + 1 = 3^2 + 2\cdot 3 + 1 = 9 + 6 + 1 = \boxed{16}.$$

2.45 ✎ 2 Source: MATHCOUNTS ✓

Let x be the value of $\sqrt{x}$ - x. What is the value of x^2?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The value of x is $1/4$, which is less than one, and the value of x^2 is $1/16$, which is too high. In the middle, the value of x is $1/2$. A little higher, the value of x is $\boxed{1/4}$. So $x^2 = \boxed{1/16}$.

2.46 ✎ 2 Source: MATHCOUNTS ✓

Suppose you have a rock that weighs x pounds. What is the value of x^2?

You may type any additional notes you have here.

Hide Solution

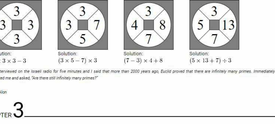
Reset

Your Submission: Solution

Solution: The value of x is $1/4$, which is less than one, and the value of x^2 is $1/16$, which is too high. In the middle, the value of x is $1/2$. A little higher, the value of x is $\boxed{1/4}$. So $x^2 = \boxed{1/16}$.

2.47 ✎ 2 Source: MATHCOUNTS ✓

Suppose you have a rock that weighs x pounds. What is the value of x^2? Hint:



Solution: $3 \times 3 \times 3 - 3$

Solution: $(3 \times 5^{-1}) \times 3$

Solution: $(7 - 3) \times 4 + 8$

Solution: $(5 \times 13 + 7) \div 3$

I was interested on the Israeli radio for five minutes and I said that more than 2000 years ago, Euclid proved that there are infinitely many primes. Immediately the host interrupted me and asked, "Are there still infinitely many primes?"

– Noga Alon

CHAPTER 3 Number Theory

Number theory is the study of integers.

3.1 Multiples

We know that 12 equals 3 times 4. In other words, 12 equals some integer times 4. For that reason, we say that 12 is a **multiple** of 4.

> **Definition:** Let a and b be numbers. We say that a is a **multiple** of b if a equals b times some integer. In other words, a is a multiple of b if there is an integer n such that $a = bn$.

For instance, 7 is not a multiple of 4, because we cannot write 7 as the product of 4 and an integer. Note that -12 is a multiple of 4, because -12 equals -3 times 4. Similarly, 0 is a multiple of 4, because 0 equals 0 times 4.

In this chapter, we'll talk about division differently than we did in Chapter 1. We'll use the "quotient and remainder" concept of division, which you probably used when you first learned about division. As an example, when we divide 13 by 4, the quotient is 3 and the remainder is 1.

Using this view of division, we can say that an integer a is a multiple of an integer $b \neq 0$ divided by b has remainder 0. So, for example, 12 is a multiple of 4 because 12 divided by 4 has remainder 0, while 13 is not a multiple of 4 because 13 divided by 4 has remainder 1.

Problems

Problem 3.1 — Jump to Solution

(a) Both 147 and 357 are multiples of 7. Is $147 + 357$ a multiple of 7?

(b) Suppose k is a multiple of 7. Must $k + 7$ be a multiple of 7?

(c) Suppose r and s are multiples of 7. Must $r + s$ be a multiple of 7?

(d) Suppose k is a multiple of 7. Is it possible for $k \div 23$ to be a multiple of 7?

Problem 3.2 — *Source: MODMO* — Jump to Solution

What number between 100 and 200 is both a perfect square and a multiple of 7?

Problem 3.3 — *Source: MATHCOUNTS* — Jump to Solution

What is the greatest three-digit number that is a multiple of 13?

Problem 3.4 — Jump to Solution

(a) How many integers between 2 and 1001 are multiples of 5?

(b) How many integers between 150 and 300 are multiples of 9?

Problem 3.5 — Jump to Solution

(a) Must every multiple of 15 also be a multiple of 3?

(b) Must every multiple of 3 also be a multiple of 15?

Problem 3.1

(a) Both 147 and 357 are multiples of 7. Is $147 + 357$ a multiple of 7?

(b) Suppose k is a multiple of 7. Must $k + 7$ be a multiple of 7?

(c) Suppose r and s are multiples of 7. Must $r + s$ be a multiple of 7?

(d) Suppose k is a multiple of 7. Is it possible for $k \div 23$ to be a multiple of 7?

Solution for Problem 3.1:

(a) We have $147 = 7 \cdot 21$ and $357 = 7 \cdot 51$. We can add these two using the distributive property:

$$147 + 357 = 7 \cdot 21 + 7 \cdot 51 = 7 \cdot (21 + 51) = 7 \cdot 72,$$

So, $147 + 357$ can be written as 7 times an integer, which means that $147 + 357$ is a multiple of 7.

(b) Because k is a multiple of 7, there is an integer a for which $k = 7a$. So, adding 7 to k gives $k + 7 = 7a + 7$. We can then factor out 7 to write

$$k + 7 = 7a + 7 = 7 \cdot a + 7 \cdot 1 = 7(a + 1).$$

Since a is an integer, so is $a + 1$. Therefore, we can write $k + 7$ as 7 times some integer, which means that $k + 7$ is indeed a multiple of 7. Moreover, we can also see that adding 7 to k gives us the next multiple of 7.

(c) We can use our solution to part (b) of as a guide. Because r is a multiple of 7, there is an integer m such that $r = m \cdot 7$. Similarly, because s is a multiple of 7, there is an integer n such that $s = n \cdot 7$. We then have $r + s = 7m + 7n$, and factoring out 7 gives

$$r + s = 7m + 7n = 7(m + n).$$

Because m and n are integers, $m + n$ is an integer. So the equation $r + s = 7(m + n)$ says that $r + s$ is 7 times some integer. In other words, $r + s$ is a multiple of 7.

(d) Part (c) tells us that the sum of any two multiples of 7 is a multiple of 7. So, if k is a multiple of 7, then $k + 21$ and $k + 28$ are both multiples of 7. Moreover, because $k + 21$ and $k + 28$ are 7 apart, they are consecutive multiples of 7. So, there are no multiples of 7 between $k + 21$ and $k + 28$. Specifically, there are no multiples of 7 between k and $k + 23$.

We might also have thought about division to see why $k + 23$ is not a multiple of 7. Since $k + 21$ is a multiple of 7 and $k + 23$ is 2 greater than $k + 21$, we know that $k + 23$ divided by 7 has a remainder of 2. So, $k + 23$ is not a multiple of 7.

Part (b) of Problem 3.1 tells us that if we start at any multiple of 7 and count by 7s, we'll generate multiples of 7 from our starting point onward. Similarly, we can also generate multiples of 7 by starting from those multiples of 7 and subtracting 7 over and over. For example, the following are all multiples of 7:

$$700, 693, 686, 679, 672, 665, 658, 651, \ldots.$$

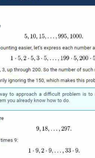
> **Important:** If we start from any multiple of n and count either upward or downward by n's, all the numbers we hit will be multiples of n.

Part (c) of Problem 3.1 is an example of a neat property of multiples. If we add two multiples of a number, we get another multiple of the same number. Similarly, the difference between any two multiples of a number is another multiple of that same number.

> **Important:** If a and b are multiples of c, then both $a + b$ and $a - b$ are multiples of c.

Finally, part (d) of Problem 3.1 is an example of why we know that we don't skip any multiples of 7 when we count by 7s starting from a multiple of 7. We can go through basically the same steps to see that if we add a multiple of 7 and a number that is not a multiple of 7, then the resulting sum is not a multiple of 7.

> **Important:** If we start with a multiple of n and add (or subtract) a number that is not a multiple of n, then the resulting sum (or difference) is not a multiple of n.



Problem 3.2 — *Source: MODMO*

What number between 100 and 200 is both a perfect square and a multiple of 7?

> **WARNING!** In this book, when we say "between 100 and 200," we don't include 100 and 200.

Solution for Problem 3.2: The perfect squares between 100 and 200 are 121, 144, 169, and 196. When we divide each of these squares by 7, we find that only 196 is a multiple of 7. To be specific, 196 is 28 times 7. So the answer is 196.

Here's another way to see that 196 is a multiple of 7. Note that 196 is 14 times 14, and 14 is a multiple of 7. Specifically, we have $14 = 2 \cdot 7$, so

$$196 = 14 \cdot 14 = 14 \cdot 2 \cdot 7 = 28 \cdot 7.$$

Therefore, 196 is a multiple of 7. $\square$

Problem 3.3 — *Source: MATHCOUNTS*

What is the greatest three-digit number that is a multiple of 13?

Solution for Problem 3.3: The three-digit numbers are the integers from 100 to 999. We are looking for the greatest such integer that is a multiple of 13. So let's divide 999 by 13. We get a quotient of 76 and a remainder of 11. In other words, 999 is 11 more than a multiple of 13. So the multiple of 13 we want is $999 - 11$, which is 988.

We also can solve this problem using our insights about multiples from Problem 3.1. We find a multiple of 13 near 1000, and then add or subtract multiples of 13 to find the largest three-digit multiple of 13. For example, we know that 1000 and 100 are multiples of 13, so $1300 - 100 = 1170$ is a multiple of 13, as is $1170 - 100 = 1070$. Therefore,

$$1040 = 2(13) = 1040 - 20 = 1001$$

is a multiple of 13, which means that the largest multiple of 13 less than 1000 is $1001 - 13 = 988$. $\square$

Let's try some counting problems involving multiples.

Problem 3.4 — *Source: MATHCOUNTS*

(a) How many integers between 2 and 1001 are multiples of 5?

(b) How many integers between 150 and 300 are multiples of 9?

Solution for Problem 3.4:

(a) The multiples of 5 between 2 and 1004 are

$$5, 10, 15, \ldots, 995, 1000.$$

How many numbers is that? To make the counting easier, let's express each number as an integer times 5:

$$1 \cdot 5, 2 \cdot 5, 3 \cdot 5, \ldots, 199 \cdot 5, 200 \cdot 5.$$

In other words, we are multiplying 5 by $1, 2, 3$, up through 200. So the number of such multiples is 200.

(b) Let's simplify the problem a bit by temporarily ignoring the 150, which makes this problem look like part (a).

> **Concept:** One way to approach a difficult problem is to relate the problem to an easier problem you already know how to do.

The positive multiples of 9 less than 300 are

$$9, 18, \ldots, 297.$$

Let's express these numbers as an integer times 9:

$$1 \cdot 9, 2 \cdot 9, \ldots, 33 \cdot 9.$$

So there are 33 such multiples.

We need to remove the numbers less than 150. The positive multiples of 9 less than 150 are

$$9, 18, \ldots, 144.$$

We can express each as an integer times 9:

$$1 \cdot 9, 2 \cdot 9, \ldots, 16 \cdot 9.$$

So there are 16 such multiples.

If we remove these 16 multiples of 9 less than 150 from the 33 multiples of 9 less than 300, we are left with $33 - 16 = 17$ multiples between 150 and 300. $\square$

Problem 3.5 — *Source: MATHCOUNTS*

(a) Must every multiple of 15 also be a multiple of 3?

(b) Must every multiple of 3 also be a multiple of 15?

Solution for Problem 3.5:

(a) Suppose that m is a multiple of 15, so we have $m = 15n$ for some integer n. Since $15 = 3 \cdot 5$, we can write $m = 3 \cdot 5n$, which means m is 3 times the integer $5n$. Therefore, m is a multiple of 3, as well.

(b) Not every multiple of 3 is a multiple of 15. For example, 3, 6, 9, 12, and 18 are all multiples of 3, but none of these numbers is a multiple of 15.

$\square$

Part (a) of Problem 3.5 tells us that because 15 is a multiple of 3, every multiple of 15 is also a multiple of 3. This is an example of another neat property of multiples:

> **Important:** If a is a multiple of b, then every multiple of a is also a multiple of b.

Exercises

3.1.1:

What are the 20 smallest positive multiples of 6?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We can list positive multiples of 6 by starting from 6 and counting by 6's:

$$\boxed{6, 12, 18, 24, 30, 36, 42, 48, 54, 60}$$

3.1.2: — **Source: AMC 8**

There are many positive two-digit multiples of 7, but only two of these multiples have a digit sum of 10. (The **digit sum** of an integer is the sum of its digits.) What are these two multiples of 7?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We find the positive multiples of 7 less than 100 by starting from 7 and counting by 7's. We compute the sum of the digits of each, and find that the two with digit sum 10 are $\boxed{28 \text{ and } 91}$.

3.1.3:

What is the sum of all positive integers less than 100 that are multiples of 13?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The desired multiples of 13 are $13, 26, 39, 52, 65, 78$, and 91. These have sum

$$13 \cdot 1 + 13 \cdot 2 + 13 \cdot 3 + 13 \cdot 4 + 13 \cdot 5 + 13 \cdot 6 + 13 \cdot 7$$

$$= 13(1 + 2 + 3 + 4 + 5 + 6 + 7) = 13(28) = \boxed{364}.$$

3.1.4:

What is the least positive number that can be added to 173 so that the result is a multiple of 20?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The sum of the desired number and 173 must be the smallest multiple of 20 that is greater than 173, which is 180. So, the desired number is $180 - 173 = \boxed{7}$.

3.1.5:

What is the smallest positive four-digit multiple of 11?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since 1000 divided by 11 leaves remainder 10, we know that 1000 is 1 more than a multiple of 11. Therefore, $1000 - 1 = 999$ is a multiple of 11, and $999 \div 11 = \boxed{1003}$ is the smallest four-digit multiple of 11. (We also might have noticed that since 1000 divided by 11 leaves a remainder of 10, we know that 1000 is 3 less than a multiple of 11.)

3.1.6:

What is the greatest four-digit multiple of 18?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The greatest four-digit number is 9999. Dividing 9999 by 18 leaves a remainder of 9, so $9999 - 9 = \boxed{9990}$ is the largest four-digit multiple of 18.

Here's a quick way to see that 9999 divided by 18 has remainder 9. The number 9999 is clearly a multiple of 9, since $9999 = 9 \cdot 1111$. But 9999 is not a multiple of 2. The multiples of 9 alternate between even numbers that are multiples of 18 (like 0, 18, 36, 54) and odd numbers that are 9 more than multiples of 18 (like 9, 27, 45). So, 9999 must be 9 more than a multiple of 18.

3.1.7: — **Source: MATHCOUNTS**

What is the greatest three-digit multiple of 33 that can be written using three different digits?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: First, we find the greatest multiple of 33. Since 999 divided by 33 has remainder 0, we know that 990 is the largest three-digit multiple of 33. But the digits of 990 are not all different. So, we consider the next smaller multiple of 33, which is $990 - 33 = 957$. The digits of 957 are different, so $\boxed{957}$ is the greatest three-digit multiple of 33 that can be written with three different digits.

3.1.8: — **Source: MATHCOUNTS**

How many integers between 17 and 2678 are multiples of 11?

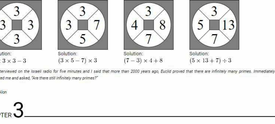
You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The smallest multiple of 11 between 17 and 2678 is $2 \cdot 11 = 22$. But what is the largest? Dividing 2678 by 11 gives quotient 243 and remainder 5, so the largest multiple of 11 between 17 and 2678 is $243 \cdot 11$. Therefore, the integers we must count are $2 \cdot 11, 3 \cdot 11, 4 \cdot 11, \ldots, 243 \cdot 11$. This list has all of the first 243 positive multiples of 11 except 11 itself, so there are $\boxed{242}$ integers between 17 and 2678 that are multiples of 11.



Solution: $3 \times 3 \times 3 - 3$

Solution: $(3 \times 5^{-1}) \times 3$

Solution: $(7 - 3) \times 4 + 8$

Solution: $(5 \times 13 + 7) \div 3$

I was interested on the Israeli radio for five minutes and I said that more than 2000 years ago, Euclid proved that there are infinitely many primes. Immediately the host interrupted me and asked, "Are there still infinitely many primes?"

– Noga Alon

CHAPTER 3 Number Theory

Number theory is the study of integers.

3.1 Multiples

We know that 12 equals 3 times 4. In other words, 12 equals some integer times 4. For that reason, we say that 12 is a **multiple** of 4.

> **Definition:** Let a and b be numbers. We say that a is a **multiple** of b if a equals b times some integer. In other words, a is a multiple of b if there is an integer n such that $a = bn$.

For instance, 7 is not a multiple of 4, because we cannot write 7 as the product of 4 and an integer. Note that -12 is a multiple of 4, because -12 equals -3 times 4. Similarly, 0 is a multiple of 4, because 0 equals 0 times 4.

In this chapter, we'll talk about division differently than we did in Chapter 1. We'll use the "quotient and remainder" concept of division, which you probably used when you first learned about division. As an example, when we divide 13 by 4, the quotient is 3 and the remainder is 1.

Using this view of division, we can say that an integer a is a multiple of an integer $b \neq 0$ divided by b has remainder 0. So, for example, 12 is a multiple of 4 because 12 divided by 4 has remainder 0, while 13 is not a multiple of 4 because 13 divided by 4 has remainder 1.

Problems

Problem 3.1 — Jump to Solution

(a) Both 147 and 357 are multiples of 7. Is $147 + 357$ a multiple of 7?

(b) Suppose k is a multiple of 7. Must $k + 7$ be a multiple of 7?

(c) Suppose r and s are multiples of 7. Must $r + s$ be a multiple of 7?

(d) Suppose k is a multiple of 7. Is it possible for $k \div 23$ to be a multiple of 7?

Problem 3.2 — *Source: MODMO* — Jump to Solution

What number between 100 and 200 is both a perfect square and a multiple of 7?

Problem 3.3 — *Source: MATHCOUNTS* — Jump to Solution

What is the greatest three-digit number that is a multiple of 13?

Problem 3.4 — Jump to Solution

(a) How many integers between 2 and 1001 are multiples of 5?

(b) How many integers between 150 and 300 are multiples of 9?

Problem 3.5 — Jump to Solution

(a) Must every multiple of 15 also be a multiple of 3?

(b) Must every multiple of 3 also be a multiple of 15?

Problem 3.1

(a) Both 147 and 357 are multiples of 7. Is $147 + 357$ a multiple of 7?

(b) Suppose k is a multiple of 7. Must $k + 7$ be a multiple of 7?

(c) Suppose r and s are multiples of 7. Must $r + s$ be a multiple of 7?

(d) Suppose k is a multiple of 7. Is it possible for $k \div 23$ to be a multiple of 7?

Solution for Problem 3.1:

(a) We have $147 = 7 \cdot 21$ and $357 = 7 \cdot 51$. We can add these two using the distributive property:

$$147 + 357 = 7 \cdot 21 + 7 \cdot 51 = 7 \cdot (21 + 51) = 7 \cdot 72,$$

So, $147 + 357$ can be written as 7 times an integer, which means that $147 + 357$ is a multiple of 7.

(b) Because k is a multiple of 7, there is an integer a for which $k = 7a$. So, adding 7 to k gives $k + 7 = 7a + 7$. We can then factor out 7 to write

$$k + 7 = 7a + 7 = 7 \cdot a + 7 \cdot 1 = 7(a + 1).$$

Since a is an integer, so is $a + 1$. Therefore, we can write $k + 7$ as 7 times some integer, which means that $k + 7$ is indeed a multiple of 7. Moreover, we can also see that adding 7 to k gives us the next multiple of 7.

(c) We can use our solution to part (b) of as a guide. Because r is a multiple of 7, there is an integer m such that $r = m \cdot 7$. Similarly, because s is a multiple of 7, there is an integer n such that $s = n \cdot 7$. We then have $r + s = 7m + 7n$, and factoring out 7 gives

$$r + s = 7m + 7n = 7(m + n).$$

Because m and n are integers, $m + n$ is an integer. So the equation $r + s = 7(m + n)$ says that $r + s$ is 7 times some integer. In other words, $r + s$ is a multiple of 7.

(d) Part (c) tells us that the sum of any two multiples of 7 is a multiple of 7. So, if k is a multiple of 7, then $k + 21$ and $k + 28$ are both multiples of 7. Moreover, because $k + 21$ and $k + 28$ are 7 apart, they are consecutive multiples of 7. So, there are no multiples of 7 between $k + 21$ and $k + 28$. Specifically, there are no multiples of 7 between k and $k + 23$.

We might also have thought about division to see why $k + 23$ is not a multiple of 7. Since $k + 21$ is a multiple of 7 and $k + 23$ is 2 greater than $k + 21$, we know that $k + 23$ divided by 7 has a remainder of 2. So, $k + 23$ is not a multiple of 7.

Part (b) of Problem 3.1 tells us that if we start at any multiple of 7 and count by 7s, we'll generate multiples of 7 from our starting point onward. Similarly, we can also generate multiples of 7 by starting from those multiples of 7 and subtracting 7 over and over. For example, the following are all multiples of 7:

$$700, 693, 686, 679, 672, 665, 658, 651, \ldots.$$

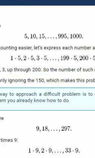
> **Important:** If we start from any multiple of n and count either upward or downward by n's, all the numbers we hit will be multiples of n.

Part (c) of Problem 3.1 is an example of a neat property of multiples. If we add two multiples of a number, we get another multiple of the same number. Similarly, the difference between any two multiples of a number is another multiple of that same number.

> **Important:** If a and b are multiples of c, then both $a + b$ and $a - b$ are multiples of c.

Finally, part (d) of Problem 3.1 is an example of why we know that we don't skip any multiples of 7 when we count by 7s starting from a multiple of 7. We can go through basically the same steps to see that if we add a multiple of 7 and a number that is not a multiple of 7, then the resulting sum is not a multiple of 7.

> **Important:** If we start with a multiple of n and add (or subtract) a number that is not a multiple of n, then the resulting sum (or difference) is not a multiple of n.



Problem 3.2 — *Source: MODMO*

What number between 100 and 200 is both a perfect square and a multiple of 7?

> **WARNING!** In this book, when we say "between 100 and 200," we don't include 100 and 200.

Solution for Problem 3.2: The perfect squares between 100 and 200 are 121, 144, 169, and 196. When we divide each of these squares by 7, we find that only 196 is a multiple of 7. To be specific, 196 is 28 times 7. So the answer is 196.

Here's another way to see that 196 is a multiple of 7. Note that 196 is 14 times 14, and 14 is a multiple of 7. Specifically, we have $14 = 2 \cdot 7$, so

$$196 = 14 \cdot 14 = 14 \cdot 2 \cdot 7 = 28 \cdot 7.$$

Therefore, 196 is a multiple of 7. $\square$

Problem 3.3 — *Source: MATHCOUNTS*

What is the greatest three-digit number that is a multiple of 13?

Solution for Problem 3.3: The three-digit numbers are the integers from 100 to 999. We are looking for the greatest such integer that is a multiple of 13. So let's divide 999 by 13. We get a quotient of 76 and a remainder of 11. In other words, 999 is 11 more than a multiple of 13. So the multiple of 13 we want is $999 - 11$, which is 988.

We also can solve this problem using our insights about multiples from Problem 3.1. We find a multiple of 13 near 1000, and then add or subtract multiples of 13 to find the largest three-digit multiple of 13. For example, we know that 1000 and 100 are multiples of 13, so $1300 - 100 = 1170$ is a multiple of 13, as is $1170 - 100 = 1070$. Therefore,

$$1040 = 2(13) = 1040 - 20 = 1001$$

is a multiple of 13, which means that the largest multiple of 13 less than 1000 is $1001 - 13 = 988$. $\square$

Let's try some counting problems involving multiples.

Problem 3.4 — *Source: MATHCOUNTS*

(a) How many integers between 2 and 1001 are multiples of 5?

(b) How many integers between 150 and 300 are multiples of 9?

Solution for Problem 3.4:

(a) The multiples of 5 between 2 and 1004 are

$$5, 10, 15, \ldots, 995, 1000.$$

How many numbers is that? To make the counting easier, let's express each number as an integer times 5:

$$1 \cdot 5, 2 \cdot 5, 3 \cdot 5, \ldots, 199 \cdot 5, 200 \cdot 5.$$

In other words, we are multiplying 5 by $1, 2, 3$, up through 200. So the number of such multiples is 200.

(b) Let's simplify the problem a bit by temporarily ignoring the 150, which makes this problem look like part (a).

> **Concept:** One way to approach a difficult problem is to relate the problem to an easier problem you already know how to do.

The positive multiples of 9 less than 300 are

$$9, 18, \ldots, 297.$$

Let's express these numbers as an integer times 9:

$$1 \cdot 9, 2 \cdot 9, \ldots, 33 \cdot 9.$$

So there are 33 such multiples.

We need to remove the numbers less than 150. The positive multiples of 9 less than 150 are

$$9, 18, \ldots, 144.$$

We can express each as an integer times 9:

$$1 \cdot 9, 2 \cdot 9, \ldots, 16 \cdot 9.$$

So there are 16 such multiples.

If we remove these 16 multiples of 9 less than 150 from the 33 multiples of 9 less than 300, we are left with $33 - 16 = 17$ multiples between 150 and 300. $\square$

Problem 3.5 — *Source: MATHCOUNTS*

(a) Must every multiple of 15 also be a multiple of 3?

(b) Must every multiple of 3 also be a multiple of 15?

Solution for Problem 3.5:

(a) Suppose that m is a multiple of 15, so we have $m = 15n$ for some integer n. Since $15 = 3 \cdot 5$, we can write $m = 3 \cdot 5n$, which means m is 3 times the integer $5n$. Therefore, m is a multiple of 3, as well.

(b) Not every multiple of 3 is a multiple of 15. For example, 3, 6, 9, 12, and 18 are all multiples of 3, but none of these numbers is a multiple of 15.

$\square$

Part (a) of Problem 3.5 tells us that because 15 is a multiple of 3, every multiple of 15 is also a multiple of 3. This is an example of another neat property of multiples:

> **Important:** If a is a multiple of b, then every multiple of a is also a multiple of b.

Exercises

3.1.1:

What are the 20 smallest positive multiples of 6?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We can list positive multiples of 6 by starting from 6 and counting by 6's:

$$\boxed{6, 12, 18, 24, 30, 36, 42, 48, 54, 60}$$

3.1.2: — **Source: AMC 8**

There are many positive two-digit multiples of 7, but only two of these multiples have a digit sum of 10. (The **digit sum** of an integer is the sum of its digits.) What are these two multiples of 7?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We find the positive multiples of 7 less than 100 by starting from 7 and counting by 7's. We compute the sum of the digits of each, and find that the two with digit sum 10 are $\boxed{28 \text{ and } 91}$.

3.1.3:

What is the sum of all positive integers less than 100 that are multiples of 13?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The desired multiples of 13 are $13, 26, 39, 52, 65, 78$, and 91. These have sum

$$13 \cdot 1 + 13 \cdot 2 + 13 \cdot 3 + 13 \cdot 4 + 13 \cdot 5 + 13 \cdot 6 + 13 \cdot 7$$

$$= 13(1 + 2 + 3 + 4 + 5 + 6 + 7) = 13(28) = \boxed{364}.$$

3.1.4:

What is the least positive number that can be added to 173 so that the result is a multiple of 20?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The sum of the desired number and 173 must be the smallest multiple of 20 that is greater than 173, which is 180. So, the desired number is $180 - 173 = \boxed{7}$.

3.1.5:

What is the smallest positive four-digit multiple of 11?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since 1000 divided by 11 leaves remainder 10, we know that 1000 is 1 more than a multiple of 11. Therefore, $1000 - 1 = 999$ is a multiple of 11, and $999 \div 11 = \boxed{1003}$ is the smallest four-digit multiple of 11. (We also might have noticed that since 1000 divided by 11 leaves a remainder of 10, we know that 1000 is 3 less than a multiple of 11.)

3.1.6:

What is the greatest four-digit multiple of 18?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The greatest four-digit number is 9999. Dividing 9999 by 18 leaves a remainder of 9, so $9999 - 9 = \boxed{9990}$ is the largest four-digit multiple of 18.

Here's a quick way to see that 9999 divided by 18 has remainder 9. The number 9999 is clearly a multiple of 9, since $9999 = 9 \cdot 1111$. But 9999 is not a multiple of 2. The multiples of 9 alternate between even numbers that are multiples of 18 (like 0, 18, 36, 54) and odd numbers that are 9 more than multiples of 18 (like 9, 27, 45). So, 9999 must be 9 more than a multiple of 18.

3.1.7: — **Source: MATHCOUNTS**

What is the greatest three-digit multiple of 33 that can be written using three different digits?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: First, we find the greatest multiple of 33. Since 999 divided by 33 has remainder 0, we know that 990 is the largest three-digit multiple of 33. But the digits of 990 are not all different. So, we consider the next smaller multiple of 33, which is $990 - 33 = 957$. The digits of 957 are different, so $\boxed{957}$ is the greatest three-digit multiple of 33 that can be written with three different digits.

3.1.8: — **Source: MATHCOUNTS**

How many integers between 17 and 2678 are multiples of 11?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The smallest multiple of 11 between 17 and 2678 is $2 \cdot 11 = 22$. But what is the largest? Dividing 2678 by 11 gives quotient 243 and remainder 5, so the largest multiple of 11 between 17 and 2678 is $243 \cdot 11$. Therefore, the integers we must count are $2 \cdot 11, 3 \cdot 11, 4 \cdot 11, \ldots, 243 \cdot 11$. This list has all of the first 243 positive multiples of 11 except 11 itself, so there are $\boxed{242}$ integers between 17 and 2678 that are multiples of 11.

3.2 Divisibility Tests

Another way to say that 12 is a multiple of 4 is to say that 12 is **divisible** by 4.

Definition: Let a be a number, and let b be a non-zero number. We say that a is **divisible** by b if a is a multiple of b.

In this section, we explore some shortcuts for determining whether or not one number is divisible by another. You've probably familiar with several of these shortcuts already but do the exercises, just in case, you don't also understand why these shortcuts work.

Problem 3.1 ⟳ Jump to Solution

Consider the following fact:

(a) Note that

$$123123 - 123000 = 123 \cdot 1001 = 123 \cdot 7 \cdot 11 \cdot 13$$

How can we use this fact to tell that 123123 is divisible by 7?

(b) How can we quickly tell that 303027 is not divisible by 7?

Problem 3.2 ⟳ Jump to Solution

How can we tell at a glance whether or not a number is a multiple of 10?

Problem 3.3 ⟳ Jump to Solution

(a) Is every multiple of 10 also a multiple of 5?

(b) How can we tell at a glance whether or not a number is a multiple of 5?

(c) How can we tell at a glance whether or not a number is a multiple of 2?

Problem 3.4 ⟳ Jump to Solution

(a) Which of the following numbers are divisible by 2?

$$12, \quad 312, \quad 512, \quad 2512, \quad 43427$$

(b) Is every multiple of 100 also a multiple of 2?

(c) Use your answer to part (b) to explain why 5,437,512,000 is divisible by 4.

(d) Use your answer to part (b) to explain why 4,000,512 is not divisible by 4.

Problem 3.5 ⟳ Jump to Solution

In the problem, we discover a method for determining whether or not a number is divisible by 3. Notice that

$$765 = 7 \cdot 100 + 6 \cdot 10 + 5$$
$$= 7 \cdot (99 + 1) + 6 \cdot (9 + 1) + 5$$
$$= 7 \cdot 99 + 7 + 6 \cdot 9 + 6 + 5$$
$$= 7 \cdot 99 + 6 \cdot 9 + (7 + 6 + 5)$$

(a) As shown above, 765 equals $7 \cdot 99 + 6 \cdot 9 + (7 + 6 + 5)$. Notice that $7 + 6 + 5$ is divisible by 3. Explain why this tells us that 765 is divisible by 3.

(b) Notice that $9 = 5 \cdot 2 + 1 = 18$ (double why this tells us that 9014 is divisible by 3.

(c) Determine whether or not 1561 is divisible by 3 without dividing it into 1561.

Problem 3.6 ⟳ Jump to Solution

In the previous problem, we noticed that

$$765 = 7 \cdot 99 + 6 \cdot 9 + (7 + 6 + 5).$$

(a) Are both 99 and 9 divisible by 3?

(b) Is 765 divisible by 3?

(c) Is 765 divisible by 9?

(d) Explain why 9 765 is not divisible by 9 without dividing it into 9.

Problem 3.7 ⟳ Jump to Solution

(a) Is the sum digit in the four-digit number 2,653, 9 2,653,9 is divisible by 3 and by 9, then what are the possible values of ?

Problem 3.8 ⟳ Jump to Solution

(a) Note that $123123 = 123000 + 123$. How can we use this fact to tell that 123123 is divisible by 11?

(b) How can we quickly tell that 303027 is not divisible by 11?

Problem 3.9 ⟳ Jump to Solution

(a) Since 15 is a multiple of 3, we know that 123000, 123000, and 123 are all multiples of 3. From our work in Section 3.1, we know that the sum of numbers that are multiples of 3 is also a multiple of 3. So, we know 123000 + 123000 + 123 is a multiple of 3. Which means that 123123 is divisible by 3.

Instead of using the previous part, we know that 303000 + 300000 + 3000 + 30 is a multiple of 3 because 3 is a multiple of 3. Since 303027 is 1 more than a multiple of 3, we know that dividing 303027 by 3 will give a remainder of 1. Therefore, 303027 is not divisible by 3.

Our work in Problem 3.6 gives us two strategies for testing whether a number is divisible by some other number b:

- Write out a sum of numbers that are divisible by b. We did this when we wrote 123123 as 123000 + 123000 + 123.
- Compare a number to another number that is divisible by b. We did this when we noted that 303027 is greater than 303000.

For the rest of this section, we will use these two tactics to develop strategies for testing for divisibility by specific numbers.

Problem 3.10 ⟳ Jump to Solution

How can we tell at a glance whether or not a number is a multiple of 10?

Solution for Problem 3.2: We compute the product of 10 and an integer by placing a 0 at the end of the integer. For example, $10 \cdot 23 = 230$. Conversely, we can tell that 230 is a multiple of 10 because it ends in a 0. For example, we can tell that 235 is not a multiple of 10 because it ends in a 5. Since 235 is 5 more than a multiple of 10, we know that 235 is not a multiple of 10.

Problem 3.11 ⟳ Jump to Solution

(a) Is every multiple of 10 also a multiple of 5?

(b) How can we tell at a glance whether or not a number is a multiple of 5?

(c) How can we tell at a glance whether or not a number is a multiple of 2?

Solution for Problem 3.3:

(a) Since 15 is a multiple of 5, we know that 123000, 123000, and 123 are all multiples of 5. From our work in Section 3.1, we know that the sum of numbers that are multiples of 5 is also a multiple of 5. So, we know 123000 + 123000 + 123 is a multiple of 5. Which means that 123123 is divisible by 5.

Instead of using the previous part, we know that 303030 + 300000 + 3000 + 30 is a multiple of 5 because 5 is a multiple of 5. Since 303027 is 1 more than a multiple of 5, we know that dividing 303027 by 5 will give a remainder of 1. Therefore, 303027 is not divisible by 5.

(b) Our work in Problem 3.6 gives us two strategies for testing whether a number is divisible by some other number b:

- Write out a sum of numbers that are divisible by b. We did this when we wrote 123123 as 123000 + 123000 + 123.
- Compare a number to another number that is divisible by b. We did this when we noted that 303027 is greater than 303000.

For the rest of this section, we will use these two tactics to develop strategies for testing for divisibility by specific numbers.

Problem 3.12 ⟳ Jump to Solution

How can we tell at a glance whether or not a number is a multiple of 10?

Solution for Problem 3.2: We compute the product of 10 and an integer by placing a 0 at the end of the integer. For example, $10 \cdot 23 = 230$. Conversely, we can tell that 230 is a multiple of 10 because it ends in a 0. For example, we can tell that 235 is not a multiple of 10 because it ends in a 5. Since 235 is 5 more than a multiple of 10, we know that 235 is not a multiple of 10.

Problem 3.13 ⟳ Jump to Solution

(a) Is every multiple of 10 also a multiple of 5?

(b) How can we tell at a glance whether or not a number is a multiple of 5?

(c) How can we tell at a glance whether or not a number is a multiple of 2?

Solution for Problem 3.4:

(a) Yes, 10 is a multiple of 5.

(b) Since 15 is a multiple of 5, we know that 123000, 123000, and 123 are all multiples of 5. From our work in Section 3.1, we know that the sum of numbers that are multiples of 5 is also a multiple of 5. So, we know 123000 + 123000 + 123 is a multiple of 5. Which means that 123123 is divisible by 5.

Instead of using the previous part, we know that 303030 + 300000 + 3000 + 30 is a multiple of 5 because 5 is a multiple of 5. Since 303027 is 1 more than a multiple of 5, we know that dividing 303027 by 5 will give a remainder of 1. Therefore, 303027 is not divisible by 5.

(c) Our work in Problem 3.6 gives us two strategies for testing whether a number is divisible by some other number b:

- Write out a sum of numbers that are divisible by b. We did this when we wrote 123123 as 123000 + 123000 + 123.
- Compare a number to another number that is divisible by b. We did this when we noted that 303027 is greater than 303000.

For the rest of this section, we will use these two tactics to develop strategies for testing for divisibility by specific numbers.

Problem 3.14 ⟳ Jump to Solution

How can we tell at a glance whether or not a number is a multiple of 10?

Solution for Problem 3.2: We compute the product of 10 and an integer by placing a 0 at the end of the integer. For example, $10 \cdot 23 = 230$. Conversely, we can tell that 230 is a multiple of 10 because it ends in a 0. For example, we can tell that 235 is not a multiple of 10 because it ends in a 5. Since 235 is 5 more than a multiple of 10, we know that 235 is not a multiple of 10.

Problem 3.15 ⟳ Jump to Solution

(a) Is every multiple of 10 also a multiple of 5?

(b) How can we tell at a glance whether or not a number is a multiple of 5?

(c) How can we tell at a glance whether or not a number is a multiple of 2?

Solution for Problem 3.8

(a) Because 10 is a multiple of 5, every multiple of 10 is also a multiple of 5.

(b) The multiples of 10 are the numbers that end in 0. So, we can tell if a number is a multiple of 5 by seeing if it ends in 0 or 5. For example, we know that 1,000,000,000

3.3 Prime Numbers

Every positive integer is divisible by 1 and by itself. Some numbers are not divisible by any other positive integers. For example, 5 is divisible by 1 and 5, but not by any other positive integer. Mathematicians have given such numbers a special name.

Definition: **A prime number** is a positive integer that is divisible by exactly two positive integers: 1 and the number itself. A **composite number** is a positive integer that is divisible by more than two positive integers. This means that a number is composite if it is divisible by some positive integer besides 1 and the number itself.

For example, 2 is divisible by 1 and by 2, but it is not divisible by any other positive number. Therefore, 2 is prime. Meanwhile, 12 is divisible by 1, by 2, by 3, by 4, by 6, by 12 and by 12 itself. Therefore, 12 is the only positive integer that is neither prime nor composite. Each integer greater than 1 is either prime or composite.

We often use the word "prime" as a noun, as in, "2 is a prime."

Problems

Problem 3.14 ⟲ Jump to Solution

List every prime that is less than 20.

Problem 3.15 ⟲ Jump to Solution

Meena is trying to determine whether or not 113 is prime. She's doing so by checking 113 for divisibility by each positive integer starting with 2; she sees that 113 is not divisible by 2, but she stops there. She then checks 113 is divisible by 3. Since $1 + 1 + 3 = 5$ is not divisible by 3, she knows that 113 is not divisible by 3. Then she checks 113 is divisible by 4. Since $113 \div 4 = 28\frac{1}{4}$ is not divisible by 4, she knows that 113 is not divisible by 4. Then she checks 113 is divisible by 5. Since $113 \div 5 = 22\frac{3}{5}$ is not divisible by 5, she knows that 113 is not divisible by 5.

(a) Meena says, "Since 113 is not divisible by 2, I know that 113 isn't divisible by 4." Is she right? Why or why not?

(b) How does Meena know at a glance that 113 is not divisible by 5?

(c) She then says, "Since 113 is not divisible by 3, I know that it isn't divisible by 6." Is she right? Why or why not?

(d) Meena divides 113 by 7 and finds a remainder of 1, so she knows that 113 is not divisible by 7. She then says, "I don't have to test 8, 9, or 10. I know that 113 isn't divisible by any of those!" How does she know without testing?

(e) Meena then says, "Since 11^2 is 121, I don't have to test any numbers higher than 10. I know know that 113 is prime." Why doesn't Meena have to test any more numbers?

Problem 3.16 ⟲ Jump to Solution

Describe each of the following numbers as prime or composite:

(a) 61

(b) 91

(c) 143

(d) 157

Problem 3.17 ⟲ Jump to Solution

Find the largest two-digit composite number in which both digits are prime.

Problem 3.18 ⟲ Jump to Solution

Find all pairs of primes whose sum is 61.

Problem 3.19 ⟲ Jump to Solution

Start with the grid of numbers on the right, and perform the following process:

	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99

(a) Circle the smallest number that is not already circled or crossed out.

(b) Cross out all the multiples of the number you circled in Step 1, except for the circled number itself.

(c) If every number is either circled or crossed out, then stop. Otherwise, go back to Step 1.

When you finish, do you notice anything interesting about the circled numbers?

Problem 3.14

List every prime that is less than 20.

Solution for Problem 3.14: In the introduction, we noted that 1 is not prime and 2 is prime.

The number 3 is divisible by 1 and by 3, but not by 2. A positive number cannot be divisible by a number greater than itself, so we don't have to check if 3 is divisible by 4 or by any other number greater than 3. Therefore, we know that 3 is prime.

Since 4 is divisible by 2, we know that 4 is composite. Similarly, 6, 8, 10, 12, 14, 16, and 18 are all divisible by 2 and therefore are composite. Of course, you probably recognize these as even numbers. Any number that is divisible by 2 is even, and any integer that is not divisible by 2 is an odd number. Since every even number is divisible by 2, the only even prime is 2. So, we don't have to check any more even numbers.

Continuing in this manner, we find that 9 is divisible by 3, so it is composite. 5, 7, 11, 13, 17, and 19 are prime. The only other numbers we see are prime, so note that 1 is not prime by definition, all the other even numbers are divisible by 2, and both 9 and 15 are multiples of 3.

Problem 3.15

Meena is trying to determine whether or not 113 is prime. She's doing so by checking 113 for divisibility by each positive integer starting with 2; she sees that 113 is not divisible by 2, but she stops there. She then checks 113 is divisible by 3. Since $1 + 1 + 3 = 5$ is not divisible by 3, she knows that 113 is not divisible by 3.

(a) Meena says, "Since 113 is not divisible by 2, I know that 113 isn't divisible by 4." Is she right? Why or why not?

(b) How does Meena know at a glance that 113 is not divisible by 5?

(c) She then says, "Since 113 is not divisible by 3, I know that it isn't divisible by 6." Is she right? Why or why not?

(d) Meena divides 113 by 7 and finds a remainder of 1, so she knows that 113 is not divisible by 7. She then says, "I don't have to test 8, 9, or 10. I know that 113 isn't divisible by any of those!" How does she know without testing?

(e) Meena then says, "Since 11^2 is 121, I don't have to test any numbers higher than 10. I know know that 113 is prime." Why doesn't Meena have to test any more numbers?

Solution for Problem 3.15

(a) Yes, she is right. Every multiple of 4 is also a multiple of 2, since 4 is a multiple of 2. But Meena already knows that 113 is not a multiple of 2, so she knows that 113 cannot possibly be a multiple of 4.

(b) The last digit of 113 is not 0 or 5, so she knows that 113 is not divisible by 5.

(c) Yes, she is right. As in part (a), every multiple of 6 is also a multiple of 2, since 6 is a multiple of 2. Meena knows that 113 is not a multiple of 2, so she knows that 113 cannot be a multiple of 6.

(d) She doesn't have to test 8 or 10 because these are both multiples of 2. Since 113 is not a multiple of 2, Meena knows that 113 is not divisible by any even number.

Every multiple of 9 is also a multiple of 3. Since Meena knows that 113 is not a multiple of 3, she knows that 113 cannot be a multiple of 9, either. Therefore, Meena doesn't have to test 9. 9, or 10.

(e) Meena knows that 113 is not a multiple of any integer from 2 through 10. So, if 113 is the product of two integers, and neither integer is 1, then both of the integers must be greater than 10.

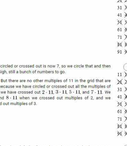
The smallest possible product of two integers greater than 10 is $11 \cdot 11 = 121$, which is larger than 113. Any other product of two integers greater than 10 will be even larger. So, 113 cannot be written as the product of two integers that are greater than 1. This means that 113 is divisible only by 1 and by itself, so 113 is prime.

Our work in Problem 3.15 revealed two very important points about testing whether or not a number is prime.

Important: If a number is not divisible by a prime, then it is not divisible by any multiple of that prime. So, in testing whether or not a number is prime, we only need to test if that number is divisible by prime numbers.

Important: To check if a number is prime, we only need to test if it is divisible by primes whose squares are less than or equal to that number.

For example, to test whether or not 59 is prime, we only need to check if it is divisible by 2, 3, 5, or 7. Since 11^2 is greater than 59, we don't need to test any other primes.



How to Tell if a Number is Prime

Problem 3.16

Describe each of the following numbers as prime or composite:

(a) 61

(b) 91

(c) 143

(d) 157

Solution for Problem 3.16:

(a) Since $11^2 = 121$ is greater than 61, we only need to test for divisibility by 2, 3, 5, and 7. Since 61 is not divisible by any of these primes, we know that 61 is prime.

(b) $91 = 7 \cdot 13$, so 91 is composite.

(c) $143 = 11 \cdot 13$, so 143 is composite.

(d) Since $13^2 = 169$'s greater than 157, we only need to test for divisibility by 2, 3, 5, 7, and 11. Since 157 is not divisible by any of these primes, we know that 157 is prime.

Problem 3.17

Find the largest two-digit composite number in which both digits are prime.

Solution for Problem 3.17: The one-digit primes are 2, 3, and 7. The largest two-digit number we can form with these digits is 77, which is indeed composite because $77 = 7 \cdot 11$.

Problem 3.18

Find all pairs of primes whose sum is 61.

Solution for Problem 3.18: If two integers sum to 61, then one of the integers must be even and the other odd. The only even prime is 2, so one of the primes must be 2. We have $61 - 2 = 59$, and 59 is prime, so 2 and 59 is the only pair of primes whose sum is 61. ◻

Problem 3.19

Start with the grid of numbers on the right, and perform the following process:

	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99

(a) Circle the smallest number that is not already circled or crossed out.

(b) Cross out all the multiples of the number you circled in Step 1, except for the circled number itself.

(c) If every number is either circled or crossed out, then stop. Otherwise, go back to Step 1.

When you finish, do you notice anything interesting about the circled numbers?

Solution for Problem 3.19: The smallest number in the grid is 2, so we start by circling 2 in Step 1. Step 2 then tells us to cross out all the multiples of 2 except for 2, so we cross out all the even numbers except 2. Moving on to Step 1, we still have plenty of numbers that are not circled or crossed out, so we go back to Step 1.

	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99

The smallest number that was not circled or crossed out already is 3, so we circle 3. We then cross out all the other multiples of 3. As we go, we notice that some of the numbers we crossed out in Step 2 are already crossed out—these are the numbers that are also multiples of 2. After crossing out all the multiples of 3 besides 3, we still have many numbers that are not circled or crossed out. Back to Step 1 we go.

	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99

Now the smallest number that was not circled or crossed out already is 5, so we circle 5. We then cross out all the other multiples of 5. There are still plenty of numbers to go. Back to Step 1.

	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99

The smallest number that was not circled or crossed out is now 7, so we circle that and then cross out the other multiples of 7, still a bunch of numbers to go.

	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99

The next number we'll circle is 11. But there are no other multiples of 11 in the grid that are not circled or crossed out. This is because we have circled or crossed out all the multiples of 2, 3, 5, and 7 already. That means we have crossed out $2 \cdot 11$, $3 \cdot 11$, $5 \cdot 11$, and $7 \cdot 11$. We circle 11 and cross out $11 \cdot 11$. We now have all the composite numbers crossed out and all the composite number will get crossed out in the step after we circle the smallest prime that evenly divides the composite number. So, we must have all primes in the grid circled and all composite numbers crossed out when we finish.

	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99

This process is called the **Sieve of Eratosthenes** after the ancient Greek mathematician who developed it. There's nothing special about 100 in our example above of the Sieve of Eratosthenes. We could have started with a much larger grid to find many more primes.

Sidenote: The Sieve of Eratosthenes is a very efficient way to generate long lists of prime numbers. For example, if you wanted to write a computer program to list all of the primes between 1 and 1,000,000, the Sieve of Eratosthenes would be a good way to do it. Eratosthenes rather famously excluded number one at time with the process we used in Problems 3.15.

Exercises

3.3.1:

What is the sum of all the prime numbers between 60 and 90?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We can read the primes between 60 and 90 off of the Sieve of Eratosthenes in the text. Or, we can test each odd number between 60 and 90 to see if it's prime. (The even numbers are obviously not prime.) 61 and 67 are divisible by 3, and 65 is divisible by 5. Both 63 and 69 are prime, and their sum is $\boxed{132}$.

3.3.2: Source: MATHCOUNTS

Find every number between 70 and 80 that is not prime and is not a multiple of 2, 3, or 5.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: All of the even numbers are multiples of 2, and 75 is a multiple of 5. That leaves 71, 73, 77, and 79. We find that $77 = 7 \cdot 11$ is not prime and is not a multiple of 2, 3, or 5. Meanwhile, 71, 73, and 79 are all prime. Therefore, $\boxed{77}$ is the only number between 70 and 80 that is not prime and is not a multiple of 2, or 5.

3.3.3:

What is the largest prime number jouch that it strikes pick less than 1000?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 1000 divided by 4 is 125. So, we seek the largest prime number that is less than 125. Even numbers greater than 2 aren't prime, so we only have to check odd numbers. We have $123 = 3 \cdot 41$, $121 = 11^2$, $119 = 7 \cdot 17$, $117 = 3 \cdot 39$, and $115 = 5 \cdot 23$. So these are all composite. Since 113 isn't divisible by 2, 3, 5, or 7, and 11^2 is greater than 113, we know that $\boxed{113}$ is the desired prime.

3.3.4:

How many pairs of primes are there such that sum of the pair is 99?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: If two integers add to 99, then either both are odd or both are even. We don't have to check the even pairs, since at least one of two even numbers that add to 99 is a composite multiple of 2. So, we check all the pairs of odd numbers that add to 99:

$$99 = 1 + 98 = 3 + 97 = 5 + 94 = 7 + 92 = 9 + 90 = 11 + 88 = 13 + 86 = 15 + 84 = 17 + 82 = 19 + 80 = 21 + 78 = 23 + 76 = 25 + 74 = 27 + 72 = 29 + 70 = 31 + 68 = 33 + 66 = 35 + 64 = 37 + 62 = 39 + 60 = 41 + 58 = 43 + 56 = 45 + 54 = 47 + 52 = 49 + 50 = 51 + 48 = 53 + 46 = 55 + 44 = 57 + 42 = 59 + 40 = 61 + 38 = 63 + 36 = 65 + 34 = 67 + 32 = 69 + 30 = 71 + 28 = 73 + 26 = 75 + 24 = 77 + 22 = 79 + 20 = 81 + 18 = 83 + 16 = 85 + 14 = 87 + 12 = 89 + 10 = 91 + 8 = 93 + 6 = 95 + 4 = 97 + 2$$

The only sums that consist of two prime numbers are $3 + 97$, $11 = 29$ and $17 + 23$. So, there are $\boxed{3}$ pairs of prime numbers that add to 40.

3.3.5:

(a) The product of all prime numbers between 1 and 80 is divided by 10. What is the remainder?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The product of all prime numbers between 1 and 80 includes 2 and 5. Multiplying these gives 10, which means the product of all the primes between 1 and 80 is a multiple of 10. Therefore, the remainder is $\boxed{0}$ when this product is divided by 10.

(b) ■ The product of all prime numbers between 1 and 80 is divided by 4. What is the remainder?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The product of all prime numbers between 1 and 80 includes exactly one 2, and has no other even numbers. So, we know that the product is even, but is not a multiple of 4 (which is 2 times 2). When we add numbers divisible by 4, the remainder is 0, and when an even number is divided by 4, the remainder is 0 or 2. The remainder is only 0 if the number is a multiple of 4. We know that the product in the problem is even but not a multiple of 4, so the remainder $\boxed{2}$ when the product is divided by 4.

3.3.6:

How many groups of three prime numbers add to 20? Note: (2, 3, 5) is considered the same group of three primes as (3, 2, 5).

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The sum of three integers is even only if the three integers are even or if two of the integers are odd and the other even. The only even prime is 2, so the only integer that is the sum of three primes is even is $2 + 2 + 2 = 6$. So, we must have 2 and two odd primes that add to 20. The pairs of odd primes that add to 20 are

$$20 = 1 + 19 = 3 + 17 = 5 + 15 = 7 + 13 = 9 + 11.$$

Two of these sums consist of two prime integers $(3 + 17$ and $7 + 13)$, so there are $\boxed{2}$ groups of three primes whose sum is 22.

3.3.7+: Source: MOEMS

Suppose P and Q both represent prime numbers and

$$5P + 7Q = 101.$$

Find the value of the prime P.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The sum of two odd numbers is even, so $5P$ and $7Q$ cannot both be odd. Therefore, either $5P$ or $7Q$ is even, which means P or Q is 2.

If P is 2, then the equation becomes $10 + 7Q = 109$. This means that $7Q$ must be 99. But 99 is not a multiple of 7, so there is no integer Q for which $7Q$ is 99.

If Q is 2, then the equation becomes $5P + 14 = 109$. This means that $5P$ is 95, so P must be $95 \div 5 = \boxed{19}$.

Extra: Finding large prime numbers is a thrill for some mathematicians and computer scientists. As of this writing, the largest known prime is

$$2^{74,207,281} - 1,$$

which has 22,338,618 digits! This prime was discovered in 2016 by the **Great Internet Mersenne Prime Search (GIMPS)**, which is a collaborative internet-based search for large primes. Visit http://www.mersenne.org/ to join the search!

3.4 Prime Factorization

Every whole number greater than 1 is a product of integers. So, if a number is composite, then we can factor it by writing it as a product of two smaller integers. For example, we can factor 15 as $15 = 3 \cdot 5$. Since 3 is composite, we also can factor it into a product of smaller integers, $3 = 1 \cdot 3$.

$$15 = 3 \cdot 5 = 1 \cdot 3 \cdot 5$$

Since 2 and 5 are prime, we cannot write either of them as the product of two smaller positive integers. We have expressed 15 as a product of prime numbers. We say that $1 \cdot 3 \cdot 5$ is the prime factorization of 15. Notice that the factors 1, 3, and 5 are all prime numbers.

We usually write prime factorizations using powers. For example, the prime factorization of 15 is $3^1 \cdot 5^1$. Notice that we write the primes in ascending order. For example, we will often write the prime factorization of 1 as 2^0 instead of 3^0 or 5^0.

It's easy to keep track of your work when finding the prime factorization of a small number like 15. When finding the prime factorization of a larger number, it helps to use a tree diagram to keep track of your work.

Suppose we want to find the prime factorization of 72. We start by writing 72. We can factor 72 as $8 \cdot 9$. We write 8 and 9 above this to factor branches, leaving 72 as a root. Since 8 is a composite, we can also decompose it in the tree, so we write the factors 2 and 4 below 8. Since 9 is a composite, we can also decompose it in the tree, so we write the factors 3 and 3 below 9.

Since 2 and 3 are prime, we can factor 4 into the product of smaller numbers, 2 and 2. Each of these are prime, so we can write 2 below 4. Since 3 is prime, we can factor 3 into 1 and 3. Since 1 is not prime, we have our prime factorization. We can now read off the prime factorization of 72 from our factor tree.

$$72 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$$

Problems

Problem 3.20 ✎ Copy to Solution

Find the prime factorization of each of the following numbers:

(a) 18
(b) 30
(c) 60
(d) 90

Problem 3.21 ✎ Copy to Solution

Find the prime factorization of each of the following powers:

(a) 2^4
(b) 3^4
(c) 5^4
(d) 7^4

Problem 3.22 ✎ Copy to Solution

Why can we use the prime factorization of a number to tell whether or not the number is a perfect square?

Problem 3.23 ✎ Copy to Solution

Find the smallest integer squared equals 90 - 400.

Problem 3.24 ✎ Copy to Solution

Find multiples two positive integers and gets a product of 1600. If another of Kate's integers ends in 5, then what is the sum of Kate's integers?

Problem 3.25 ✎ Copy to Solution

(a) Find the product of the 14 smallest prime numbers. Explain why the prime factorization of $2^{14} - 1$ must include a prime that is larger than 29. Hint: $2^{14} - 1 = (2^7 - 1)(2^7 + 1)$.

(b) Let N be the product of the 100 smallest prime numbers. Explain why the prime factorization of $N^2 - 1$ must include a prime that is larger than 541. Hint: $N^2 - 1 = (N - 1)(N + 1)$.

Problem 3.26 ✎ ✓

Find the prime factorization of each of the following numbers:

(a) 18
(b) 30
(c) 60
(d) 90

Solution for Problem 3.26

(a) We start with $18 = 2 \cdot 9$, and then $9 = 3 \cdot 3$, which gives us the factor tree shown at the right. We then have

$$18 = 2^1 \cdot 3^2$$

as the prime factorization of 18.

Of course, this isn't the only factor tree we could have built for 18. We could have started with $18 = 3 \cdot 6$, and produced the factor tree above at the left. This factor tree also produces the prime factorization

$$18 = 2^1 \cdot 3^2$$

(b) We start with $90 = 2 \cdot 45$ and then $45 = 5 \cdot 9$ and $9 = 3 \cdot 3$. This produces the factor tree on the right. Reading the primes from the factor tree gives the prime factorization

$$90 = 2^1 \cdot 3^2 \cdot 5^1.$$

We also could have used our result from part (a), that we found that $18 = 2^1 \cdot 3^2$, to write

$$90 = 2 \cdot 18 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2 \cdot 5^1.$$

(c) We can easily divide 2160 by 2, so we start with $2160 = 2 \cdot 1080$. We then have $1080 = 2 \cdot 540$ and $540 = 2 \cdot 270$. We can then factor 270 as $270 = 5 \cdot 54$ and factor 54 as $54 = 3 \cdot 18$. Reading the primes from the factor tree gives

$$2160 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 = 2^4 \cdot 3^3 \cdot 5^1.$$

We also could have started with 2160 as a product of smaller numbers. For example, we could have factored 2160 as $2160 = 5 \cdot 432$. We then have $432 = 2 \cdot 216$ and $216 = 3 \cdot 72$. We can factor 72 as $72 = 2 \cdot 36$ and factor 36 as $36 = 3 \cdot 12$. We can factor 12 as $12 = 2 \cdot 6$ and factor 6 as $6 = 2 \cdot 3$. Reading the primes from the factor tree gives the same prime factorization of positive integers, without having to include "orange" in every line.

(d) We can easily divide 90 by 2, so we start with $90 = 2 \cdot 45$. We recognize 45 as a multiple of 5, so we can factor 45 as $45 = 5 \cdot 9$. We can factor 9 as $9 = 3 \cdot 3$. Reading the primes from the factor tree gives

$$90 = 2 \cdot 5 \cdot 3 \cdot 3 = 2^1 \cdot 3^2 \cdot 5^1.$$

We also might have seen that 90 is easily divisible by 9, and started our factor tree with $90 = 9 \cdot 10$. We can recognize 10 as 10 times 1, so we have the factor tree shown at the left. We could continue with the factor tree, but recognizing 9 as the product of primes, we once again can compute our prime factor

$$90 = 9 \cdot 10 = 3^2 \cdot 2^1 \cdot 5^1 = 2^1 \cdot 3^2 \cdot 5^1.$$

Every positive integer has a prime factorization. We can see why by considering the primes, the composites, and 1 separately.

- The primes have the prime factorization of a number is just the number itself. For example, $5 = 5^1$.
- The composites have a product of primes, so we can write composite number has a prime factorization. Any composite number can be written as the product of two smaller numbers, then then continue our factorizing process until those smaller positive numbers. For example, we can factor 15 as $15 = 3 \cdot 5$. Since 3 and 5 are prime, we can write the prime factorization of 15 as $3^1 \cdot 5^1$. Notice that we write the primes in ascending order. For example, we will often write the prime factorization of 1 as 2^0 instead of 3^0 or 5^0.
- 1 is a special case. It is neither prime nor composite. We can write the prime factorization of 1 as 2^0 or 3^0 or 5^0.

We can see that the prime factorization of a number is all whether or not the number is a perfect square?

Navigation: Prime factorization of perfect square numbers, Factoring the prime factorization of a perfect square.

We have the prime factorization of a number, we can determine if it is a perfect square or not. For example, we can factor 15 as $15 = 3 \cdot 5$. Since 3 and 5 are prime, we can write the prime factorization of 15 as $3^1 \cdot 5^1$. Since the exponents are not all even, we can see that 15 is not a perfect square.

We have the prime factorization of a number, we can determine if it is a perfect square or not. For example, we can factor 15 as $15 = 3 \cdot 5$. Since 3 and 5 are prime, we can write the prime factorization of 15 as $3^1 \cdot 5^1$. Since the exponents are not all even, we can see that 15 is not a perfect square.

Important: Fundamental Theorem of Arithmetic: Every positive integer has a unique prime factorization.

Do you become more experienced with primes and prime factorizations, you probably would find you have trying to find prime factorizations of numbers. For example, you might find you have trying to find the prime factorization of 1000. Instead of trying to find the factors of 1000, you might instead try to factor 1000 as

$$10 \cdot 10 = 5 \cdot 2 \cdot 5 \cdot 2$$

and have the exponent to find the prime factorization of 1000.

Knowing the prime factorization of a number can be a powerful tool in solving problems about properties of whole numbers.

Example: Find the prime factorization of 1000 by using the factor trees of integers. Because every integer has a unique prime factorization, we can be particularly useful in problems involving products of integers.

Problem 3.27 ✎ ✓

Find the prime factorization of each of the following perfect squares:

(a) 36
(b) 100
(c) 144
(d) 196

(e) Find two of the prime factorization of a number to tell whether or not the number is a perfect square?

Solution for Problem 3.27:

(a) $36 = 6^2 = (2^1 \cdot 3^1)^2 = 2^2 \cdot 3^2$
(b) $36 = 6^2 = (2^1 \cdot 5^1)^2 = 2^2 \cdot 5^2$
(c) $144 = 12^2 = (2^2 \cdot 3^1)^2 = 2^4 \cdot 3^2$
(d) $144 = 12^2 = (2^2 \cdot 3^1)^2 = 2^4 \cdot 3^2$

(e) Knowing the prime factorization of a number can tell us whether or not it is a perfect square. For example, we can factor 15 as $15 = 3 \cdot 5$. Since 3 and 5 are prime, we can write the prime factorization of 15 as $3^1 \cdot 5^1$. Since the exponents are not all even, we can see that 15 is not a perfect square.

We can see that the prime factorization of a number is all whether or not the number is a perfect square?

Navigation: Prime factorization of perfect square numbers, Factoring the prime factorization of a perfect square.

Important: All of the exponents in the prime factorization of a perfect square are even. Any prime factorization of a number with all of the exponents are even is the prime factorization of a perfect square.

Problem 3.28 ✎ ✓

What positive integer squared equals 90 - 400?

Solution for Problem 3.28: We can factor the prime factorization of the whole.

$$90 = 2 \cdot 3 \cdot 3 \cdot 5 = 2^1 \cdot 3^2 \cdot 5^1$$
$$400 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 = 2^4 \cdot 5^2$$

Next, we multiply these prime factorizations.

$$90 \cdot 400 = (2^1 \cdot 3^2 \cdot 5^1) \cdot (2^4 \cdot 5^2)$$
$$= 2^1 \cdot 2^4 \cdot 3^2 \cdot 5^1 \cdot 5^2$$
$$= 2^5 \cdot 3^2 \cdot 5^3$$

Both of the exponents are even, so $2^5 \cdot 3^2$ is a perfect square. Notice that

$$2^5 \cdot 3^2 = (2^2 \cdot 3^1)^2 = (12)^2$$

So the number we seek is 12.

Problem 3.29 ✎ ✓

Find multiples two positive integers and gets a product of 1600. If another of Kate's integers ends in 5, then what is the sum of Kate's integers?

Solution for Problem 3.29: Once again we have a problem about a product of integers, so we'll think about prime factorizations. The prime factorization of 1600 is

$$1600 = 16 \cdot 100$$
$$= (2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 5 \cdot 5)$$
$$= 2^4 \cdot 2^2 \cdot 5^2$$
$$= 2^6 \cdot 5^2$$

We therefore seek two numbers whose product is $2^6 \cdot 5^2$ and such that another number ends in 5. Any multiple of 10 ends in 0, and any number ends with 5 is in the prime factorization of 5. Since 5 is a factor of 1600, we know that one of Kate's integers must be a factor of 5. We know that one of Kate's integers must be a factor of 5. We know that one of Kate's integers must be a factor of 5. We

3.5 Least Common Multiple

Suppose a and b are two positive integers. A number is a multiple of a and b if we can find a number that can be written as **common multiple** of a and b. The smallest common multiple of a and b is called the **least common multiple** of a and b. For example, the multiples of 6 and 9, the multiples of 9 and 12, the common multiples of 6 and 9, and the least common multiple of 6 and 9 are:

Multiples of 6: $6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, 102, 108, 114, 120, \ldots$

Multiples of 9: $9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, 117, 126, \ldots$

Common multiples of 6 and 9: $18, 36, 54, 72, \ldots$

The least common multiple of 6 and 9: 18

Problems

Problem 3.35 ★ Level 1 Topic: Multiples

Let S be the smallest positive multiple of 15.

Let T be the smallest positive multiple of 20.

Let U be the smallest positive multiple of 35.

What is the least common multiple of S and T? What is the least common multiple of S and U? What is the least common multiple of T and U? What is the least common multiple of S, T and U?

Problem 3.36 ★ Level 1 Topic: Multiples

Let S be the smallest positive multiple of 15.

Let T be the smallest positive multiple of 20.

Let U be the smallest positive multiple of 35.

What is the least common multiple of S and T? What is the least common multiple of S and U? What is the least common multiple of T and U? What is the least common multiple of S, T and U?

Problem 3.37 ★ Level 1 Topic: Multiples

Find the prime factorizations of the least common multiple of 6 and 9. Compare these prime factorizations to the prime factorizations of 6 and 9. What do you notice?

Problem 3.38 ★ Level 1 Topic: Multiples

Find the prime factorizations of 75×2^3. Find the prime factorizations of each number below to determine if this number is a multiple of 75×2^3.

(a) $900 = 2^2 \cdot 3^2$

(b) $1350 = 2 \cdot 3^3 \cdot 5^2$

(c) $3600 = 2^4 \cdot 3^2 \cdot 5^2$

Suppose a number N has prime factorization of two positive integers a and b. How can we use these prime factorizations to tell if a is a multiple of b?

Problem 3.39 ★ Level 1 Topic: MATHCOUNTS

How can we use the prime factorization of 12 and 18 to find lcm$[12, 18]$?

Problem 3.40 ★ Level 1 Topic: Multiples

Consider each of the following:

(a) lcm$[20, 100]$

(b) lcm$[20, 35]$

(c) lcm$[2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3]$

(d) lcm$[6, 12, 36]$

Problem 3.41 ★ Level 1 Topic: Multiples

(a) lcm$[x, y] = 100$. 100 and x are 20. 100 and y are 25. What is x?

(b) lcm$[20, 35] = 140$. 140 and x are 35. 140 and y are 20. What is x?

(c) Let x and y be positive integers. Given that lcm$[x, 100] = 20$ and lcm$[y, 100] = 20$, what is lcm$[x, y]$?

Problem 3.42 ★ Level 1 Topic: Multiples

Consider each of the following:

(a) lcm$[20, 100]$

(b) lcm$[20, 35]$

(c) lcm$[2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3]$

(d) lcm$[6, 12, 36]$

Problem 3.43 ★ Level 1 Topic: MATHCOUNTS

(a) lcm$[x, 20] = 100$. 100 and x are 20. What is x?

(b) lcm$[x, 20] = 140$. 140 and x are 20. What is x?

(c) Let x and y be positive integers. Given that lcm$[x, 100] = 20$ and lcm$[y, 100] = 20$, what is lcm$[x, y]$?

Problem 3.44 ★ Level 1 Topic: MATHCOUNTS

(a) lcm$[x, 20] = 100$. 100 and x are 20. What is x?

(b) lcm$[x, 20] = 140$. 140 and x are 20. What is x?

(c) Let x and y be positive integers. Given that lcm$[x, 100] = 20$ and lcm$[y, 100] = 20$, what is lcm$[x, y]$?

Exercises

3.5.1 Source: MATHCOUNTS

Consider each of the following:

(a) lcm$[20, 25]$

(b) lcm$[2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3]$

(c) lcm$[6, 12, 36]$

Hide Solution

Your Solution

Your Feedback

Your Self-Assessment

Solution: Find the prime factorizations of the lcm. The smallest two positive multiples of 25 are 25 and 50. Since 50 is divisible by 5, we have lcm$[20, 25] = 100$.

(a) lcm$[20, 25]$

Your Solution

Your Feedback

Your Self-Assessment

Solution: Find the prime factorizations of the lcm. The smallest two positive multiples of $2^3 \cdot 3^2 \cdot 5^4$ are $2^3 \cdot 3^2 \cdot 5^4$ and $2^4 \cdot 3^2 \cdot 5^4$. Since $2^4 \cdot 3^2 \cdot 5^4$ is divisible by $2^2 \cdot 3^3 \cdot 5^3$, we have lcm$[2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3] = 2^4 \cdot 3^2 \cdot 5^4$.

(b) lcm$[2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3]$

Your Solution

Your Feedback

Your Self-Assessment

Solution: Find the prime factorizations of the lcm. The smallest two positive multiples of 6 are 6 and 12. Since 12 is divisible by 12, we have lcm$[6, 12] = 12$.

(c) lcm$[6, 12]$

Your Solution

Your Feedback

Your Self-Assessment

3.5.2 Source: MATHCOUNTS

Consider each of the following:

(a) lcm$[20, 25]$

(b) lcm$[2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3]$

(c) lcm$[6, 12, 36]$

Hide Solution

Your Solution

Your Feedback

Your Self-Assessment

Solution: Find the prime factorizations of the lcm. The smallest two positive multiples of 25 are 25 and 50. Since 50 is divisible by 5, we have lcm$[20, 25] = 100$.

(a) lcm$[20, 25]$

Your Solution

Your Feedback

Your Self-Assessment

Solution: Find the prime factorizations of the lcm. The smallest two positive multiples of $2^3 \cdot 3^2 \cdot 5^4$ are $2^3 \cdot 3^2 \cdot 5^4$ and $2^4 \cdot 3^2 \cdot 5^4$. Since $2^4 \cdot 3^2 \cdot 5^4$ is divisible by $2^2 \cdot 3^3 \cdot 5^3$, we have lcm$[2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3] = 2^4 \cdot 3^2 \cdot 5^4$.

(b) lcm$[2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3]$

Your Solution

Your Feedback

Your Self-Assessment

Solution: Find the prime factorizations of the lcm. The smallest two positive multiples of 6 are 6 and 12. Since 12 is divisible by 12, we have lcm$[6, 12] = 12$.

(c) lcm$[6, 12]$

Your Solution

Your Feedback

Your Self-Assessment

3.5.3 Source: MATHCOUNTS

Consider each of the following:

(a) lcm$[20, 25]$

(b) lcm$[2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3]$

(c) lcm$[6, 12, 36]$

Hide Solution

Your Solution

Your Feedback

Your Self-Assessment

Solution: Find the prime factorizations of the lcm. The smallest two positive multiples of 25 are 25 and 50. Since 50 is divisible by 5, we have lcm$[20, 25] = 100$.

(a) lcm$[20, 25]$

Your Solution

Your Feedback

Your Self-Assessment

Solution: Find the prime factorizations of the lcm. The smallest two positive multiples of $2^3 \cdot 3^2 \cdot 5^4$ are $2^3 \cdot 3^2 \cdot 5^4$ and $2^4 \cdot 3^2 \cdot 5^4$. Since $2^4 \cdot 3^2 \cdot 5^4$ is divisible by $2^2 \cdot 3^3 \cdot 5^3$, we have lcm$[2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3] = 2^4 \cdot 3^2 \cdot 5^4$.

(b) lcm$[2^3 \cdot 3^2 \cdot 5^4, 2^2 \cdot 3^3 \cdot 5^3]$

Your Solution

Your Feedback

Your Self-Assessment

Solution: Find the prime factorizations of the lcm. The smallest two positive multiples of 6 are 6 and 12. Since 12 is divisible by 12, we have lcm$[6, 12] = 12$.

(c) lcm$[6, 12]$

Your Solution

Your Feedback

Your Self-Assessment

3.5.4 Source: MATHCOUNTS

What is the largest common multiple of 6 and 12 that is less than 100?

Hide Solution

Your Solution

Your Feedback

Your Self-Assessment

Solution: The common multiples of 6 and 12 are the factors of 6 and 12. Since lcm$[6, 12] = 12$, we can list the common multiples of 6 and 12 by listing the multiples of 12. The largest common multiple of 6 and 12 that is less than 100 is $12 \times 8 = 96$.

3.5.5 Source: MATHCOUNTS

Consider the prime factorizations of the positive integers a and b. How can we use these prime factorizations to tell if a is a multiple of b?

Hide Solution

Your Solution

Your Feedback

Your Self-Assessment

Solution: Suppose a and b are two positive integers. Let $a = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_n^{a_n}$ and $b = p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_n^{b_n}$. Then a is a multiple of b if and only if $a_1 \geq b_1$, $a_2 \geq b_2$, $\ldots$, $a_n \geq b_n$.

3.5.6 Source: MATHCOUNTS

Let x and y be positive integers. Given that lcm$[x, 100] = 20$ and lcm$[y, 100] = 20$, what is lcm$[x, y]$?

Hide Solution

Your Solution

Your Feedback

Your Self-Assessment

Solution: Since lcm$[x, 100] = 20$, x is a multiple of 20. Since lcm$[y, 100] = 20$, y is a multiple of 20. Since x and y are multiples of 20, lcm$[x, y]$ is a multiple of 20. Since lcm$[x, 100] = 20$, lcm$[x, y]$ is a multiple of 100. Since lcm$[x, y]$ is a multiple of 20 and lcm$[x, y]$ is a multiple of 100, lcm$[x, y] = 100$.

3.5.7 Source: MATHCOUNTS

Let x and y be positive integers. Given that lcm$[x, 100] = 20$ and lcm$[y, 100] = 20$, what is lcm$[x, y]$?

Hide Solution

Your Solution

Your Feedback

Your Self-Assessment

Solution: Since lcm$[x, 100] = 20$, x is a multiple of 20. Since lcm$[y, 100] = 20$, y is a multiple of 20. Since x and y are multiples of 20, lcm$[x, y]$ is a multiple of 20. Since lcm$[x, 100] = 20$, lcm$[x, y]$ is a multiple of 100. Since lcm$[x, y]$ is a multiple of 20 and lcm$[x, y]$ is a multiple of 100, lcm$[x, y] = 100$.

3.5.8 Source: MATHCOUNTS

Let x and y be positive integers. Given that lcm$[x, 100] = 20$ and lcm$[y, 100] = 20$, what is lcm$[x, y]$?

Hide Solution

Your Solution

Your Feedback

Your Self-Assessment

Solution: Since lcm$[x, 100] = 20$, x is a multiple of 20. Since lcm$[y, 100] = 20$, y is a multiple of 20. Since x and y are multiples of 20, lcm$[x, y]$ is a multiple of 20. Since lcm$[x, 100] = 20$, lcm$[x, y]$ is a multiple of 100. Since lcm$[x, y]$ is a multiple of 20 and lcm$[x, y]$ is a multiple of 100, lcm$[x, y] = 100$.

3.6 Divisors

In Section 3.1, we learned that 12 is a multiple of 4, because 12 equals 4 times an integer. We can also express this fact by saying, "4 is a **divisor** of 12."

> **Definition:** Let a be a nonzero integer and b be an integer. We say that a is a **divisor** of b if b is a multiple of a. In other words, b is a times some integer.

We can also say that 4 is a **factor** of 12. The nouns "divisor" and "factor" mean the same thing. For example, the positive factors of 6 are 1, 2, 3, and 6. The negative factors of 6 are $-1, -2, -3$, and -6.

One way we sometimes think of divisors is as "is a divisor of b if dividing b by a leaves no remainder." So, because 60 divided by 6 has no remainder, 6 is a divisor of 60. Since 72 divided by 5 has remainder 2, we know that 5 is not a divisor of 72.

> **Important:** If a and b are nonzero integers, then all of the following statements mean the same thing:
>
> - a is a divisor of b.
> - a is a factor of b.
> - b is a multiple of a.
> - b is divisible by a.

Problems

Problem 3.34 Jump to Solution

Find all of the positive divisors of 84.

Problem 3.35 Jump to Solution

(a) Count the positive divisors of each integer from 8 to 18. Which of these integers have an odd number of positive divisors?

(b) There is one integer between 20 and 30 that has an odd number of positive divisors. Which one?

(c) Which positive integers have an odd number of positive divisors?

Problem 3.36 Jump to Solution

Note that 20 is a divisor of 60.

(a) Must every divisor of 20 also be a divisor of 60?

(b) Must every divisor of 60 also be a divisor of 20?

Problem 3.37 Jump to Solution

(a) Suppose 3 is a divisor of k; is 3 a divisor of $k+3$?

(b) Suppose 3 is a divisor both of b and of c. Must 3 be a divisor of $b+c$?

(c) Suppose 3 is a divisor of $b+c$. Must 3 be a divisor both of b and of c?

Problem 3.34

Find all of the positive divisors of 84.

Solution for Problem 3.34: We start off testing each positive integer, starting with 1.

Is 1 a divisor of 84? Yes, because $84 = 1 \cdot 84$. This equation shows that both 1 and 84 are divisors of 84.

Is 2 a divisor of 84? Yes, because $84 = 2 \cdot 42$. Both 2 and 42 are divisors of 84.

In the same way, we can find other pairs that multiply to 84:

$$3 \cdot 28 = 84, \qquad 4 \cdot 21 = 84, \qquad 6 \cdot 14 = 84, \qquad 7 \cdot 12 = 84.$$

None of 5, 8, or 9 is a divisor of 84. Rather than continuing to test the numbers from 10 up to 84 to see if they are divisors of 84, we notice that $10 \cdot 10$ is greater than 84. So, it's impossible to find two numbers greater than 9 whose product is 84. We've already tested all the numbers from 1 to 9, so we have found all the pairs of integers whose product is 84.

In summary, 84 has 12 positive divisors: 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, and 84. ◻

As we saw in Problem 3.34, when hunting for pairs of positive integers that multiply to n, we only have to test numbers up to the first integer whose square is greater than n.

Problem 3.35

(a) Count the positive divisors of each integer from 8 to 18. Which of these integers have an odd number of positive divisors?

(b) There is one integer between 20 and 30 that has an odd number of positive divisors. Which one?

(c) Which positive integers have an odd number of positive divisors?

Solution for Problem 3.35:

(a) Below is a table of the positive divisors of each integer from 8 to 18, as well as a count of the positive divisors of each integer:

Number	Divisors	Count	Number	Divisors	Count
8	1, 2, 4, 8	4	14	1, 2, 7, 14	4
9	1, 3, 9	3	15	1, 3, 5, 15	4
10	1, 2, 5, 10	4	16	1, 2, 4, 8, 16	5
11	1, 11	2	17	1, 17	2
12	1, 2, 3, 4, 6, 12	6	18	1, 2, 3, 6, 9, 18	6
13	1, 13	2			

The only integers from 8 to 18 that have an odd number of positive divisors are 9 and 16:

(b) The perfect squares 9 and 16 are the only numbers in part (a) that have an odd number of positive divisors. So, we guess that the perfect square 25 is the number between 20 and 30 that has an odd number of positive divisors. Checking, we find that the only positive divisors of 25 are 1, 5, and 25. Indeed, 25 is the integer between 20 and 30 that has an odd number of positive divisors.

(c) From the previous two parts, we suspect that perfect squares have an odd number of positive divisors, and the rest of the positive integers have an even number of positive divisors.

The positive divisors of an integer come in pairs of numbers whose product is that integer. If an integer is not a perfect square, the two numbers in a such a pair cannot ever be the same. When we count the positive divisors of a non-square integer, they come in pairs, so the count is even:

$$12 = 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4.$$

If an integer is a perfect square, then one pair has two numbers that are the same. All the rest of the positive divisors of the integer come in pairs of different numbers:

$$36 = 1 \cdot 36 = 2 \cdot 18 = 3 \cdot 12 = 4 \cdot 9 = 6 \cdot 6.$$

So, when we count the positive divisors of a perfect square, we count by twos for the pairs with different numbers, and then count only one divisor for the pair that has two numbers that are the same. This final divisor makes the count of positive divisors odd.

◻

Problem 3.36

Note that 20 is a divisor of 60.

(a) Must every divisor of 20 also be a divisor of 60?

(b) Must every divisor of 60 also be a divisor of 20?

Solution for Problem 3.36:

(a) Yes. If an integer d is a divisor of 20, then 20 is a multiple of d. This means that every multiple of 20 is also a multiple of d. Specifically, 60 is a multiple of 20, so 60 is a multiple of d. Therefore, d is a divisor of 60.

(b) No. For example, 30 is a divisor of 60, but 30 is not a divisor of 20.

◻

> **Important:** If a is a divisor of b, then every divisor of a is also a divisor of b.

Problem 3.37

(a) Suppose 3 is a divisor of k; is 3 a divisor of $k+3$?

(b) Suppose 3 is a divisor both of b and of c. Must 3 be a divisor of $b+c$?

(c) Suppose 3 is a divisor of $b+c$. Must 3 be a divisor both of b and of c?

Solution for Problem 3.37:

(a) 3 is a divisor of k, so k is a multiple of 3. Adding 3 to a multiple of 3 gives another multiple of 3, so $k+3$ is a multiple of 3. Therefore, 3 is a divisor of $k+3$.

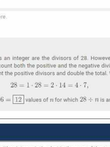
(b) Since 3 is a divisor of both b and c, we know that both b and c are multiples of 3. The sum of two multiples of 3 must also be a multiple of 3. (See Problem 3.1 if you don't remember why.) Therefore, $b+c$ is a multiple of 3, which means that 3 is a divisor of $b+c$.

(c) No. Suppose $b = 2$ and $c = 1$. Then, 3 is a divisor of $b+c$, but 3 is not a divisor of b or c.

◻

The principles we explored in Problem 3.37 are the same as the rules we learned about multiples in Problem 3.1. Here are the rules written in terms of "divisors" instead of "multiples":

> **Important:** Let a, b, and c be integers. If c is a divisor of a and of b, then c is a divisor of $a+b$ and of $a-b$.



A Challenging Divisor Problem

Exercises

3.6.1:

What is the product of all positive factors of 6?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have $6 = 1 \cdot 6 = 2 \cdot 3$, so the product of the positive factors is $1 \cdot 6 \cdot 2 \cdot 3 = 6 \cdot 6 = 6^2 = \boxed{36}$.

3.6.2:

What is the sum of the positive divisors of 18?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have $18 = 1 \cdot 18 = 2 \cdot 9 = 3 \cdot 6$, so the sum of the positive divisors of 18 is $1 + 2 + 3 + 6 + 9 + 18 = \boxed{39}$.

3.6.3:

How many positive factors does 32 have?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have $32 = 1 \cdot 32 = 2 \cdot 16 = 4 \cdot 8$, so 32 has $\boxed{6}$ divisors.

3.6.4:

For how many integers n is $28 \div n$ an integer?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The values of n such that $28 \div n$ is an integer are the divisors of 28. However, we must be careful; the problem didn't specify that n must be positive. So, we must count both the positive and the negative divisors. The negative divisors are simply the negations of the positive divisors, so we'll count the positive divisors and double the total. We have

$$28 = 1 \cdot 28 = 2 \cdot 14 = 4 \cdot 7,$$

so 28 has 6 positive divisors. This gives us $2 \cdot 6 = \boxed{12}$ values of n for which $28 \div n$ is an integer.

3.6.5: Source: MATHCOUNTS

The product of two positive integers is 2005. If neither integer is 1, what is the sum of the two integers?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have $2005 = 5 \cdot 401$. Since 401 is not divisible by any of the primes less than 20, and 23^2 is greater than 401, we know that 401 is prime. So, the only possibility for the desired sum is $5 + 401 = \boxed{406}$.

3.6.6: Source: MOEMS

The product of the three-digit number ABC and the single-digit number D is 1673. If A, B, C, and D represent different digits, then what three-digit number does ABC represent?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: D is a one-digit divisor of 1673. We know that D isn't 1 because D times a three-digit number equals 1673. So, we seek a one-digit divisor of 1673 besides 1. Since 1673 is odd, we know that D can't be even. We can also use divisibility tests to quickly eliminate 3, 5, and 9 as candidates for D. That leaves 7. We have $1673 \div 7 = 239$, so the desired ABC is $\boxed{239}$.

3.6.7: Source: AMC 8

A lucky year is one in which at least one date, when written in the form month/day/year, has the following property: the product of the month times the day equals the last two digits of the year. For example, 1956 is a lucky year because it has the date 7/8/56 and $7 \cdot 8 = 56$. Which of the following is NOT a lucky year: 1990, 1991, 1992, 1993, or 1994?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let A be the units digit in the year 199A. In order for 199A to be lucky, the two-digit number 9A must be the product of two integers in which one integer is from 1 to 12 (the month) and the other integer is no greater than the number of days in the corresponding month.

We have $90 = 3 \cdot 30$, so 1990 is lucky. (3/30/90)

We have $91 = 7 \cdot 13$, so 1991 is lucky. (7/13/91)

We have $92 = 4 \cdot 23$, so 1992 is lucky. (4/23/92)

We have $93 = 3 \cdot 31$, so 1993 is lucky. (3/31/93)

That leaves 1994 as the year that is not lucky. To see that 1994 is not lucky, note that the only pairs of positive divisors of 94 that multiply to 94 are $94 = 1 \cdot 94 = 2 \cdot 47$. Neither of these pairs leads to a valid date.

3.6.8:

What is the greatest positive integer less than 100 that has an odd number of positive divisors?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Perfect squares have an odd number of positive divisors, and integers that are not perfect squares have an even number of positive divisors. So, we seek the largest square that is less than 100, which is $\boxed{81}$.

3.6.9: Source: MATHCOUNTS

What is the greatest integer less than 10000 that is a factor of $11000 + 1100 + 11$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The sum is 12,111. The largest divisor of 12,111 is 12,111 itself, but that's larger than 10,000. The largest divisor of 12,111 and the smallest positive divisor have product 12,111. This gives us the pair of divisors 1,111 and 1. Similarly, the product of the next largest divisor and the next smallest positive divisor is 12,111. So, we can find the next largest divisor by finding the next smallest divisor.

Since 12,111 is odd, 2 is not a divisor. But the sum of the digits of 12,111 is a multiple of 3, so we know that 3 is a divisor of 12,111. We have $12{,}111 = 3 \cdot 4{,}037$, so the greatest factor that is less than 10,000 is $\boxed{4{,}037}$.

3.7 Greatest Common Divisor

Let a and b be two integers. Then the number d is the number **a common divisor** of a and b, common divisors are also referred to as **common factors**.

Let a be the divisors of 18 and b = 1, 2, 3, 6, 9, 18, and d put together, then d is the divisors of a and b = 1, 2, 3, 6, 9, 18, 18, 36, 54, 108, 216. The number 18 is the number d = 1, 2, 3, 6, 9, 18 is the greatest common divisor of a and b = 18.

The greatest positive integer that is a divisor of both a and b is called the **greatest common divisor** of a and b, meaning of the common divisors of a and b, the largest is d, the greatest common divisor is **greatest common factor**, d and b is d, common divisors are also referred to as **common factors**.

Problems

Problem 3.1 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.2 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.3 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.4 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.5 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.6 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.7 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.8 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.9 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.10 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.11 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.12 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.13 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.14 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.15 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.16 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.17 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.18 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.19 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.20 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.21 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.22 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.23 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.24 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.25 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.26 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.27 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.28 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Problem 3.29 ▶ Jump to Solution

Let a = 2 and b = 3.

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Problem 3.30 ▶ Jump to Solution

Find the positive factorials of each of the positive divisors of 24. Compute these prime factorizations to the prime factorization of the factorial of the positive divisors of 24.

Solutions

Problem 3.1

(a) Find the positive divisors of a.

(b) Find the positive divisors of b.

(c) Find the positive common divisors of a and b.

(d) What is the greatest common divisor of a and b?

Solution For Problem 3.1 (a)

Let a = 2.

Then the positive divisors of a are

$$1, 2, 3, 4, 6, 12.$$

(b) The divisors of b are

$$1, 2, 3, 4, 6, 12.$$

(c) The divisors that appear in both a and b are the positive common divisors of a and b. Thus a = 2, 3, and b = 2, 3.

(d) The greatest common divisor of a and b is d = 2, 3 and b = 2, 3 and d = 2, 3 and b = 2, 3 and d = 2, 3 and b = 2, 3 and

3.8 Summary

Definition: A **prime number** is a positive integer that is divisible by exactly 2 positive integers: 1 and the number itself. A **composite number** is a positive integer that is divisible by some positive integer besides 1 and the number itself.

Important: **Fundamental Theorem of Arithmetic**. Every integer greater than 1 can be written as the product of one or more primes in exactly one way (disregarding the order of the primes in the product).

Definition: Let a and b be integers. We say that a is a **multiple** of b if a equals b times some integer. In other words, a is a multiple of b if there is an integer n such that $a = bn$. If a is a multiple of b and b is nonzero, then we say that b is a **divisor**, or **factor**, of a, and that a is **divisible** by b.

Important: Let a, b, and c be numbers.

- If a and b are multiples of c, then both $a + b$ and $a - b$ are multiples of c.
- If a is a multiple of b, then every multiple of a is also a multiple of b.

We developed several useful divisibility tests:

Number	Condition under which n is divisible by the number
2	Units digit of n is 0, 2, 4, 6, or 8
3	Sum of the digits of n is a multiple of 3
4	Number formed by last two digits of n is a multiple of 4
5	Units digit of n is 0 or 5
6	Divisible by 2 and by 3
9	Sum of the digits of n is a multiple of 9
10	Units digit of n is 0

If a number is a multiple of both a and b, then we say that the number is a **common multiple** of a and b. The smallest positive integer that is a multiple of both a and b is called the **least common multiple** of a and b. We refer to the least common multiple of a and b as $\operatorname{lcm}[a, b]$. Every multiple of $\operatorname{lcm}[a, b]$ is a common multiple of a and b, and every common multiple of a and b is a multiple of $\operatorname{lcm}[a, b]$.

Important: To find the prime factorization of the least common multiple of a group of numbers, we first find the prime factorization of each of the numbers. The prime factorization of the least common multiple is the product of the highest power of each prime factor that appears in the prime factorizations of the numbers.

If a number is a divisor of both a and b, then we say that the number is a **common divisor** of a and b. The greatest positive integer that is a divisor of both a and b is called the **greatest common divisor** of a and b. We refer to the greatest common divisor of a and b as $\gcd(a, b)$. Every divisor of $\gcd(a, b)$ is a common divisor of a and b, and every common divisor of a and b is a divisor of $\gcd(a, b)$. We say that two integers are **relatively prime** if their greatest common divisor is 1.

Important: We can find the greatest common divisor of a group of numbers with the following process:

Step 1: Find the prime factorization of each number.

Step 2: Identify the primes that appear in *all* of the prime factorizations. If no primes appear in all of the prime factorizations, then the greatest common divisor of the numbers is 1.

Step 3: Among the prime factorizations, find the smallest power of each prime from Step 2.

Step 4: Multiply the powers of primes found in Step 3.

Important: For any positive integers a, b, and n, we have

$$\operatorname{lcm}[na, nb] = n\operatorname{lcm}[a, b],$$
$$\gcd(na, nb) = n\gcd(a, b).$$

Challenge Problems

3.72: Source: MATHCOUNTS ⚑ 27

Find the single digit number $\overline{5CB4A7}$ is divisible by 8, the remainders is 4. List both possible values of the digit $\overline{B}$.

Hint: What's the nearest multiple of 8?

You may type any additional notes you have here.

Hide Solution

Send

Your Submission: Solution

Solution: Since $\overline{5CB4A7}$ is 4 more than a multiple of 8, we know that $\overline{5CB4A7} - 4 = \overline{5CB4A3}$ is a multiple of 8 since $\overline{5CB4A7} = \overline{5CB4A3} + 4$. Since $\overline{5CB4A3}$ is a multiple of 8, the number $\overline{5CB4A3}$ is a multiple of 4 only when the hundred digit number is a multiple of 4. The only two multiple of 4 with 1 as the ones digit are 12 and 32, so the possible values of $\overline{B}$ are 1 and 3. Double that to get possible values for $\overline{A}$ and it turns out that the number is divisible by 8 if the number formed by the last two digits is. This fact only applies to numbers of the form $\overline{5CB4A3}$ divisible by 8, in general a number is divisible by 8 if the number formed by its last three digits is divisible by 8.

3.73: Source: MATHCOUNTS ⚑ 27

What is the largest integer that is greater than 2 and is least common multiple of $\overline{3E}$, what are the possible values of the larger of the two integers?

You may type any additional notes you have here.

Hide Solution

Send

Your Submission: Solution

Solution: If two positive integers are relatively prime, then the product of the integers equals their least common multiple. So, the product of the two numbers is $\overline{3E}$. Since $\overline{E}$ is the ones digit of the product of 3 $\overline{B}$ there are only two pairs of positive integers whose product is $\overline{E}$. These are 1 and 3 and 2 and 6. So the two possible values of the larger number are $\overline{B}$ and $2\overline{B}$.

3.74: Source: MATHCOUNTS ⚑ 27

What is the largest multiple of 13 that can be written using each digit 1, 3, 5, ..., 9 exactly once?

You may type any additional notes you have here.

Hide Solution

Send

Your Submission: Solution

Solution: The multiples of 13 are the numbers that are multiples of both 3 and 4. A number is a multiple of 13 if the sum of its digits is a multiple of 3. Since

$$0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

is a multiple of 3, any number using each digit once is a multiple of 3. Since $100 \equiv 4 \pmod{4}$, we only care how to construct the largest such number that is divisible by 4. This means we must make the final two digits of the number form a multiple of 4. We want the largest such number, so we want the digits to be as large as possible. The largest multiple of 4 that can be formed using two digits is 96, so we want the final two digits to be the largest multiple of 4 we can form. Digits need that this to be further to the left in the number, such as 9876543210. We can't put both 7 and 9 in the final two digits, since 10 is not a multiple of 4. We can put 9 and 2 in the final two digits, $\overline{9876543210}$ is a multiple of 13.

3.75: Source: MATHCOUNTS ⚑ 27

Find the single number $\overline{AEGD}$ is divisible by. What is the sum of $\overline{A}$ and $\overline{D}$?

You may type any additional notes you have here.

Hide Solution

Send

Your Submission: Solution

Solution: We have $\overline{3E} = \overline{G}$, $\overline{A}$ and $\overline{E}$ are relatively prime, so we can test for divisibility by 3 through testing for divisibility by $\overline{G}$. In fact, we need $\overline{AEGD}$ to be a multiple of $\overline{G}$ for $\overline{AEGD}$ to be a multiple of 3. So $\overline{AEGD}$ is a multiple of 3 if and only if $\overline{AEGD}$ is a multiple of $\overline{G}$. The only two multiples of 3 with 1 as the ones digit are 21 and 51. So, our number is $\overline{AE21}$ or $\overline{AE51}$.

In order for the number to be a multiple of 4, the sum of the digits must be a multiple of 4. Therefore, if our number is $\overline{AE21}$, then $\overline{A} + \overline{E} + 5 = \overline{E} + \overline{A} + 12$ must be a multiple of 4. The only digit for which this is true is $\overline{A} = 6$, which gives us $\overline{E} = 1$. If our number is $\overline{AE51}$, then $\overline{A} + \overline{E} + 8$ is a multiple of 4. So $\overline{A} + \overline{E} + 8 = 4$ or $\overline{A} + \overline{E} = 12$. The only two digits for which this is true is $\overline{A} = 6$, which gives us $\overline{E} = 6$. So we have $\overline{A} = 6$ and $\overline{E} = 1$ or $\overline{A} = 6$ and $\overline{E} = 6$. For this case as well, the only possible sum of $\overline{A}$ and $\overline{D}$ is 12.

3.76: Source: MATHCOUNTS ⚑ 27

The number $\overline{AEGD}$ is a six-digit integer in which $\overline{A}$ and $\overline{D}$ are digits, and the number is divisible by 72. Find the value of $\overline{A}$ and the value of $\overline{D}$.

You may type any additional notes you have here.

Hide Solution

Send

Your Submission: Solution

Solution: We have $\overline{3E} = \overline{G}$, $\overline{A}$ and $\overline{E}$ are relatively prime, so we can test for divisibility by 3 through testing for divisibility by $\overline{G}$. In fact, we need $\overline{AEGD}$ to be a multiple of $\overline{G}$ for $\overline{AEGD}$ to be a multiple of 3. So $\overline{AEGD}$ is a multiple of 3 if and only if $\overline{AEGD}$ is a multiple of $\overline{G}$. The only two multiples of 3 with 1 as the ones digit are 21 and 51. So, our number is $\overline{AE21}$ or $\overline{AE51}$.

In order for the number to be a multiple of 4, the sum of the digits must be a multiple of 4. Therefore, if our number is $\overline{AE21}$, then $\overline{A} + \overline{E} + 5 = \overline{E} + \overline{A} + 12$ must be a multiple of 4. The only digit for which this is true is $\overline{A} = 6$, which gives us $\overline{E} = 1$. If our number is $\overline{AE51}$, then $\overline{A} + \overline{E} + 8$ is a multiple of 4. So $\overline{A} + \overline{E} + 8 = 4$ or $\overline{A} + \overline{E} = 12$. The only two digits for which this is true is $\overline{A} = 6$, which gives us $\overline{E} = 6$. So we have $\overline{A} = 6$ and $\overline{E} = 1$ or $\overline{A} = 6$ and $\overline{E} = 6$. For this case as well, the only possible sum of $\overline{A}$ and $\overline{D}$ is 12.

3.77: Source: MATHCOUNTS ⚑ 27

Find the largest factor of 2320 that is not divisible by 5.

Hint: How much the prime factorization of 2320 be related to the prime factorization of 2020?

Hint: If a number is not a multiple of 5, then what must be true about its prime factorization?

You may type any additional notes you have here.

Hide Solution

Send

Your Submission: Solution

Solution: We think about the prime factorization of 2320 and its divisors, since we can use prime factorization to tell if one number is a divisor of another. The prime factorization of 2320 is

$$2320 = 2^4 \cdot 5 \cdot 29$$

In order for the number to be a multiple of 5, the exponent of 5 must be at least 1. Therefore, if our number is 2020, then $2020 = 2^2 \cdot 5 \cdot 101$ is a divisor of another. The prime factorization of 2020 is

$$2020 = 2^2 \cdot 5 \cdot 101$$

We want the divisor to be as large as possible without being divisible by 5, including 5^1 and 5^0 in the prime factorization of the divisor doesn't affect whether or not the divisor is divisible by 5, so we include both. Since the divisor cannot be a multiple of 101, the divisor can't be a multiple of both 2 and 5. This means the divisor can't have both 2 and 5 in the prime factorization. So, when we include the largest power of 2 that we can without including both 2 and 5 in the prime factorization, we include the largest 2^3 and 2^0. Since $2^3 = 8$ is larger than $2^0 = 1$, we include the power of 8 in the divisor.

Therefore, the largest divisor that is not a multiple of 5 is $2^3 \cdot 5^0 \cdot 101 = \boxed{808}$.

3.78: Source: MATHCOUNTS ⚑ 27

Find both the product of three different prime numbers. Is it possible for the sum of the digits of each product to be 10? Why or why not?

Hint: What do you know about a number whose digits sum to 10?

You may type any additional notes you have here.

Hide Solution

Send

Your Submission: Solution

Solution: No, it is impossible. A digits sum to 10, then the number must be a multiple of 9, which is 3^2. A product of different primes can only have factor of 3, so each product can't be a multiple of 9. Therefore, the sum of the digits of each product can't be 10.

3.79: Source: MATHCOUNTS ⚑ 27

The least common multiple of 12, 16, 24, and is 400. What is the least possible value of the positive integer n?

Hint: How do you find the least common multiple of a group of numbers? What must be true about n in order for $\text{lcm}[12, 16, 24, n] = 400$?

You may type any additional notes you have here.

Hide Solution

Send

Your Submission: Solution

Solution: We can find the least common multiple of a group of numbers by finding the prime factorizations of those numbers, and taking the largest power of each prime. We can find the prime factorizations of the given numbers. Unfortunately, we don't know the prime factorization of the first three numbers, and of the least common multiple we are given.

$$12 = 2^2 \cdot 3^1,$$
$$16 = 2^4 \cdot 3^0,$$
$$24 = 2^3 \cdot 3^1,$$
$$400 = 2^4 \cdot 5^2 \cdot 5^1,$$

We know that

$$\text{lcm}[12, 16, 24] = 400 = 2^4 \cdot 3^1 \cdot 5^2 \cdot 5^1,$$

In forming the least common multiple, we can get the 2^4 and the 5^2 from the prime factorizations of 12, and the 5^1 from the prime factorization of 16. We can get the 3^1 and 5^1 from any of the 12, 16, or 24. So, it must come from n. Therefore, n must a positive multiple of 5, which means the smallest possible value of n is $\boxed{5}$.

3.80: Source: MATHCOUNTS ⚑ 27

For each of the following pairs of numbers, find the product of the numbers, and then find the product of the greatest common divisor and the least common multiple of the numbers.

(a) 18, 24

Hide Solution

Send

Your Submission: Solution

Solution: We have $18 = 2^1 \cdot 3^2$ and $24 = 2^3 \cdot 3^1$. So $\gcd(18, 24) = 2^1 \cdot 3^1 = 6$ and $\text{lcm}[18, 24] = 2^3 \cdot 3^2 = 72$. Therefore, we have

$$18 \cdot 24 = (2^1 \cdot 3^2) \cdot (2^3 \cdot 3^1)$$
$$= (2^1 \cdot 3^1) \cdot (2^3 \cdot 3^2)$$
$$= 2^4 \cdot 3^3,$$
$$\gcd(18, 24) \cdot \text{lcm}[18, 24] = (2^1 \cdot 3^1) \cdot (2^3 \cdot 3^2)$$
$$= (2^1 \cdot 3^1) \cdot (2^3 \cdot 3^2)$$
$$= 2^4 \cdot 3^3.$$

Both products equal $2^4 \cdot 3^3 = 432$.

(b) 35, 42

Preview Solution

Hide Solution

Send

Your Submission: Solution

Solution: We have $35 = 5^1 \cdot 7^1$ and $42 = 2^1 \cdot 3^1 \cdot 7^1$ so $\gcd(35, 42) = 7^1 = 7$ and

$$\text{lcm}[35, 42] = 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1 = 210.$$

Therefore, we have

$$35 \cdot 42 = (5^1 \cdot 7^1) \cdot (2^1 \cdot 3^1 \cdot 7^1)$$
$$= 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^2,$$
$$\gcd(35, 42) \cdot \text{lcm}[35, 42] = (5^1 \cdot 7^1) \cdot (2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1)$$
$$= 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^2.$$

Both products equal $2^1 \cdot 3^1 \cdot 5^1 \cdot 7^2 = 1470$.

(c) 84, 84

Preview Solution

Hide Solution

Send

Your Submission: Solution

Solution: We have $84 = 2^2 \cdot 3^1 \cdot 7^1$ and $\gcd(84, 84) = 2^2 \cdot 3^1 \cdot 7^1 = 84$ and $\text{lcm}[84, 84] = 2^2 \cdot 3^1 \cdot 7^1 = 84$.

Therefore, we have

$$\begin{aligned} 84 \cdot 84&= (2^2 \cdot 3^1 \cdot 7^1) \cdot (2^2 \cdot 3^1 \cdot 7^1) \\ &= (2^2 \cdot 3^1 \cdot 7^1) \cdot (2^2 \cdot 3^1 \cdot 7^1) \\ \gcd(84, 84) \cdot \text{lcm}[84, 84] &= (2^2 \cdot 3^1 \cdot 7^1) \cdot (2^2 \cdot 3^1 \cdot 7^1) \\ &= (2^2 \cdot 3^1 \cdot 7^1) \cdot (2^2 \cdot 3^1 \cdot 7^1). \end{aligned}$$

Both products equal $2^4 \cdot 3^2 \cdot 7^2 = 7056$.

(d) Do you see an interesting pattern in your answers? Will that pattern hold for any two positive integers you start with? If so, why?

Preview Solution

Hide Solution

Send

Your Submission: Solution

Solution: In each part, we started with two positive integers, a and b, and found that $ab = \gcd(a, b) \cdot \text{lcm}[a, b]$. We saw that these products were the same by seeing that the prime factorizations of ab and $\gcd(a, b) \cdot \text{lcm}[a, b]$ were the same. To see why this must always happen for any a and b, let's think about a single prime factor. There are three possibilities for each prime factor. 1. The prime factor appears in the prime factorization of a and b 2. The prime factor is in the prime factorization of a but it does not show up in the prime factorization of b 3. The prime factor is in the prime factorization of b but it does not show up in the prime factorization of a. Suppose p is the prime factor, and p^x is the power of p in the prime factorization of a and p^y is the power of p in the prime factorization of b. Case 1. The prime appears in the prime factorization of both a and b. Suppose p^x is the power of p in the prime factorization of a, and p^y is the power of p in the prime factorization of b. Then, the prime factorization of ab includes the product of p^x and p^y, and the prime factorization of $\gcd(a, b) \cdot \text{lcm}[a, b]$ has the larger of p^x and p^y. Since $p^x \cdot p^y = p^{\max(x,y)} \cdot p^{\min(x,y)}$, the product of the prime factors in ab and $\gcd(a, b) \cdot \text{lcm}[a, b]$ are the same. Case 2. The prime appears in the prime factorization of a but not in the prime factorization of b. Then, the prime factorization of ab includes p^x, and $\gcd(a, b) \cdot \text{lcm}[a, b]$ includes p^x. So, the prime factors of ab and $\gcd(a, b) \cdot \text{lcm}[a, b]$ are the same, which means that $ab = \gcd(a, b) \cdot \text{lcm}[a, b]$. Case 3. The prime appears in the prime factorization of b but not in the prime factorization of a. Then, the prime factorization of ab includes p^y, and $\gcd(a, b) \cdot \text{lcm}[a, b]$ includes p^y. So, the prime factors of ab and $\gcd(a, b) \cdot \text{lcm}[a, b]$ are the same, which means that $ab = \gcd(a, b) \cdot \text{lcm}[a, b]$. In these three cases, we saw that $ab = \gcd(a, b) \cdot \text{lcm}[a, b]$. So, we know that $ab = \gcd(a, b) \cdot \text{lcm}[a, b]$ for any two positive integers a and b. To see why, we saw that $ab = \gcd(a, b) \cdot \text{lcm}[a, b]$ for any two positive integers a and b. To see why, we saw that the prime factorizations of ab and $\gcd(a, b) \cdot \text{lcm}[a, b]$ were the same. So, we know that $ab = \gcd(a, b) \cdot \text{lcm}[a, b]$ for any two positive integers a and b.

3.81: Source: MATHCOUNTS ⚑ 27

How many terminal zeros does the integer equal to $90^{10} - 10^{90}$ have?

Hint: Factoring an integer by 10 means you will be left with a factor of the integer. How do you factor of 10?

Hint: How might the prime factorization of 10 be helpful?

You may type any additional notes you have here.

Hide Solution

Send

Your Submission: Solution

Solution: We multiply a number by 10 by adding a zero to the end of the number. Similarly, we multiply by five 10s, or 10^5, by adding five zeros to the end of the number. So, to figure out how many zeros there are at the end of $90^{10} - 10^{90}$, we write the number as the product of a power of 10 times some number. We can factor out a power of 10.

We get factors of 10 when we combine factors of 2 and factors of 5. So, we'll start by writing the prime factorization of $90^{10} - 10^{90}$ to see how many factors of each we have. Since $90^{10} = 10 \cdot 9^{10}$, $2 \cdot 5 = 10$, $90 = 2 \cdot 3 \cdot 5 = 3 \cdot 10$ we have

$$90^{10} - 10^{90} = (3 \cdot 10)^{10} - 10^{90}$$
$$= (3^{10} \cdot 10^{10}) - 10^{90}$$
$$= 3^{10} \cdot 10^{10} - 10^{90}$$
$$= 10^{10} \cdot (3^{10} - 10^{80})$$

We can pull each of the 10 s with a factor of 2 to make a factor of 10.

$$2^{10} \cdot 3^{10} \cdot 5^{10} - 2^{90} \cdot 5^{90}$$
$$= 2^{10} \cdot 3^{10} \cdot 5^{10}$$
$$= 2^{10} \cdot 3^{10} \cdot 5^{10}$$
$$= 2^{10} \cdot 3^{10} \cdot 5^{10}$$

So, there are 10 terminal zeros.

3.82: Source: MATHCOUNTS ⚑ 27

The number 56 is both a perfect cube and a perfect square, since $7^3 = 56$ and $8^2 = 56$. What is the least larger number that is both a perfect square and a perfect cube?

Hint: Use the prime factorization of a perfect square must be even. Let's see if there is similar fact for the exponents in the prime factorization of a perfect cube.

Hint: Use the prime factorization of a perfect square must be even. What similar fact is true about the exponents in the prime factorization of a perfect cube?

Hide Solution

Send

Your Submission: Solution

Solution: The exponents in the prime factorization of a perfect square must be even. Let's see if there is similar fact for the exponents in the prime factorization of a perfect cube. We know that the prime factorization of 56 is the fact that five cubes to see if we find anything interesting.

$$7^3, \quad 2^3 \cdot 7^3, \quad (2^2)^3 \cdot 7^3 = 2^6 \cdot 7^3, \quad 2^3 \cdot (2 \cdot 3)^3 = 2^3 \cdot 6^3.$$

It looks like all exponents in the prime factorization of a perfect cube must be multiples of 3. The exemplar of 7^3 and 6^3 above show why this occurs. To get the prime factorization of a perfect cube, 6^3 and 7^3 are the prime factors of the cube. Where we take each a prime factorization, we multiply the exponents in the prime factorization by 3. For example, the cube of 7^3 is

$$(7^3)^3 = 7^{3 \cdot 3} = 7^9.$$

Therefore, each of the exponents in the prime factorization of a cube must be a multiple of 3.

Back of the exponents in the prime factorization of a square must be a multiple of 2, and each of the exponents in the prime factorization of a cube must be a multiple of 3. Therefore, if a number is a square and a cube, then each of the exponents in its prime factorization must be a multiple of 6. So, we want to find the least number that is greater than 56 and has all of its exponents in its prime factorization equal to 6. The next number is $2^6 \cdot 7^6$. The next number is $2^6 \cdot 3^6 \cdot 7^6$ and 7^6 is greater than 56. So, the next number is $2^6 \cdot 3^6 \cdot 7^6 = \boxed{1176490}$.

3.83a: Source: MATHCOUNTS ⚑ 27

Suppose that n and $n+$ are positive integers with n greater than 5, and with $\text{lcm}[n, 5] = 60$ and $\gcd[n, 5] = 10$ Find the largest prime factor of n and $n+$ prime factor n?

Hint: How do you usually find the least common multiple and the greatest common divisor of two numbers? Can you use that process here?

Hint: What do the prime factorizations of $\text{lcm}[n, 5]$ and $\gcd[n, 5]$ tell you about the prime factorizations of n and $n+$?

You may type any additional notes you have here.

Hide Solution

Send

Your Submission: Solution

Solution: We know that we can find the prime factorizations of $\text{lcm}[n, 5]$ and $\gcd[n, 5]$ from the prime factorizations of n and 5. So, we find the prime factorizations of the given values of $\text{lcm}[n, 5]$ and $\gcd[n, 5]$, hoping these will help us learn about the prime factorizations of n and 5.

$$\text{lcm}[n, 5] = 60 = 2^2 \cdot 3^1 \cdot 5^1, \quad \gcd[n, 5] = 10 = 2^1 \cdot 5^1.$$

Since 2^1 is the prime factorization of $\gcd[n, 5]$ and $\gcd[n, 5]$ is a factor of n, we know that 2^1 is both the largest and the smallest power of 2 that appears in the prime factorization of n and 5. Therefore, 2^1 appears in the prime factorizations of both n and 5. Similarly, 5^1 is the smallest power of 5 that appears in $\gcd[n, 5]$ and $\gcd[n, 5]$ is a factor of n, so 5^1 appears in the prime factorizations of n and 5.

Since 2^2 is a factor of $\text{lcm}[n, 5]$ and $\text{lcm}[n, 5]$, we know that 2^2 is the prime factorization of one of n and 5, but not the other. Similarly, 3^1 is a factor of $\text{lcm}[n, 5]$ and $\text{lcm}[n, 5]$, so 3^1 is the prime factorization of one of n and 5.

So, the prime factorization of n and 5 both include 5^1. 5^1 is the prime factorization of exactly one of n and 5 and instead 3^1, also the prime factorization of exactly one of n and 5 includes 2^2 possibly the same one that includes 5^1. Neither has any other prime factors besides these two since no other primes appear in the prime factorization of $\text{lcm}[n, 5]$. Since n is greater than 5, it must be n that has 2^2 in its prime factorization. Therefore, $\boxed{3}$ is the largest prime that is a factor of n but not of 5.

3.84a: Source: MATHCOUNTS ⚑ 27

What is the smallest positive multiple of 63 that ends in 0 and 1 in its digits?

Hint: We don't know a divisibility rule for 63, but we do know divisibility rules for some of the divisors of 63.

Preview Solution

Hide Solution

Send

Your Submission: Solution

Solution: We have $\gcd[n, 5] = 10$, so n is the multiples of 10 are the numbers that are multiples of both 5 and 2. Therefore, we seek the smallest positive integer that is divisible by both 5 and 2 such that the integer has only 0s and 1s as digits.

A number is divisible by 5 if it ends in 0 or 5. Our desired number can only have 0s and 1s as digits, so it must end in 0.

Since our desired number must be divisible by 2, it must be even. So, it must end in 0. The only nonzero digits in our desired number are 1s. So, our desired number must have 1 1s in order for the sum of its digits to be a multiple of 5.

Combining these two conditions, $\boxed{1111111110}$ is the smallest positive multiple of 63 that only has 0s and 1s as digits.

3.85a: Source: MATHCOUNTS ⚑ 27

What is the sum of the digits of the number $(10^{99} - 9)^2$?

Hint: Find pattern involving digits by. What key insight did you make and how is the last to help you learn how to test for divisibility by 11?

Hide Solution

Send

Your Submission: Solution

Solution: We will express $10^{99} - 9$ as the sum of two numbers that are easy to divide by 9.

$$\begin{aligned} (10^{99} - 9) &= 9 \cdot (10^{98} - 1) \\ &= 9 \cdot (9 \cdot 1111111111 \ldots 1111111111) \\ &= (9 \cdot 9) \cdot 1111111111 \ldots 1111111111 \\ &= 81 \cdot 1111111111 \ldots 1111111111 \\ &= (80 + 1) \cdot 1111111111 \ldots 1111111111 \\ &= 80 \cdot 1111111111 \ldots 1111111111 + 1 \cdot 1111111111 \ldots 1111111111 \end{aligned}$$

So the sum of the digits is $(1 + 2 + \ldots + 9) = \boxed{45}$.

3.86a: Source: MATHCOUNTS ⚑ 27

How many positive integers less than 1000 are divisible by 2 and 3 but not 5?

Hint: Simplify the problem first. Count the numbers less than 1000 that are divisible by 2 and 3.

Hint: Then count the ones that are divisible by 5 and subtract the number of multiples of 5. How many?

You may type any additional notes you have here.

Hide Solution

Send

Your Submission: Solution

Solution: The numbers that are divisible by both 2 and 3 are the multiples of 6. So, we want to count the positive numbers less than 1000 that are multiples of 6, but not of 5. Since 1000 divides by 6 is a multiple of 5 between 1 and this naturally 1, 2, 3, 4, 5, ..., 166. When we multiply 6 by each of these, we get a multiple of 6 less than 1000 that is also a multiple of 5. Excluding these 166 from our count leaves count of 166 leaves 166 − 166 = $\boxed{0}$ numbers less than 1000 that are multiples of 6 but not multiples of 5.

3.87a: Source: MATHCOUNTS ⚑ 27

Students in city school are numbered in order from 1 to 1000. Initially, they all are closed. Then 1000 students in city school. The 1st student goes through the school and opens every locker. The 2nd student goes through the school and for every 2nd locker, if the locker is closed, the student closes it, and if the locker is open, the student opens it. The 3rd student goes through the school and for every 3rd locker, if the locker is closed, the student closes it, and if the locker is open, the student opens it. This continues until the 1000th student goes through the school and for every 1000th locker, the student closes it, and if the locker is open, the student opens it. How many lockers are open after all of the students have finished, how many lockers are open?

Hint: Let's say that a student "touches" a locker if the either opens or closes the locker. Which students touch locker 10? Which students touch locker 7? Locker 6? Locker 9?

You may type any additional notes you have here.

Hide Solution

Send

Your Submission: Solution

Solution: Let's say that a student "touches" a locker if the either opens or closes the locker. Which students touch locker 10? Which students touch locker 7? Locker 6? Locker 9?

Students touch locker 10 are the 1st student, the 2nd student, the 5th student, the 10th student touches every 10th locker starting with locker 1, the 2nd student touches every locker that is a multiple of 2, the 5th student touches every 5th locker starting with locker 5, the 10th student touches every locker that is a multiple of 10.

Therefore, locker number n is touched by every student whose number evenly divides n. So, the number of times locker n gets touched is the number of divisors of n. Lockers that are touched an odd number of times are closed, and lockers that are touched an even number of times are open. Perfect squares are the only integers with an odd number of divisors. So, the number of lockers that are open is the number of perfect squares less than 1000.

The largest perfect square less than 1000 is $31^2 = 961$ so there are 31 perfect squares less than 1000.

Review Problems

3.47.

Find the prime factorization of each of the following numbers:

(a) 252

Preview Solution

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have

$$630 = 2 \cdot 315 = 2 \cdot 3 \cdot 105 = \underline{2 \cdot 3 \cdot 5} \cdot 21$$

We also might have noticed that

$$630 = 700 - 7 = 5(140 - 1) = 5(140 = 5(2^2 \cdot 35) = 2^2 \cdot 5^2 \cdot 7^1.$$

(b) 5400

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We can use divisibility tests to see quickly that 5400 is not divisible by 2, 3, or 5. Dividing 5400 by 7 leaves a remainder of 2. Dividing 5400 by 11 leaves a remainder of 10. Dividing 5400 by 13 leaves a remainder of 12. Next, we try 17, and find $5400 = 17 \cdot 318$. Since 318 is divisible by 3, we can continue:

$$5400 = 17 \cdot 318 = \underline{17 \cdot 3} \cdot 106$$

(c) 30100

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution:

$$\begin{aligned} 30,100 &= 301 \cdot 100 \\ &= 19 \cdot 301 \cdot 10^2 \\ &= 19 \cdot 7 \cdot 43 \cdot 2^2 \cdot 5^2 \\ &= \underline{2^2 \cdot 5^2 \cdot 7^1 \cdot 19^1 \cdot 43^1}. \end{aligned}$$

3.48.

Under the modulus-7 system, $1000_{(7)}(7)$ is divisible by 9?

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Since 9 is a multiple of 3, we in $1000_{(7)}(7)$. Therefore, the remainder is $\boxed{0}$ when we divide $1000_{(7)}(7)$ by 9.

3.49.

What is the sum of the two smallest multiples of 9 that are greater than 100?

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Since 100 divided by 10 has remainder 1, we know that $\boxed{101}$ is a multiple of 9. So, the next two multiples of 9 are $109 = 9 \cdot 109$ and $118 = 9 \cdot 13$. Thus our sum is $109 + 118 = \boxed{227}$.

3.50.

What is the largest multiple of 7 that is less than 2000?

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Since 2000 divided by 7 has remainder 5, we know that 2000 is 5 more than a multiple of 7. Therefore, the largest multiple of 7 less than 2000 is $2000 - 5 = \boxed{1995}$.

We might also have started by checking the multiples of 7 near 2000. Since $7 \cdot 2 = \boxed{14}$, we have $7 \cdot 20 = \boxed{140}$, $7 \cdot 200 = \boxed{1400}$, $7 \cdot 2000 = \boxed{14000}$, $7 \cdot 1000 = \boxed{7000}$, $7 \cdot 2000 = \boxed{14000}$, $7 \cdot 1500 = \boxed{10500}$, $7 \cdot 1800 = \boxed{12600}$, $7 \cdot 1900 = \boxed{13300}$ are all multiples of 7 as well as the largest multiple of 7 less than 2000 $\boxed{1995}$.

3.51.

Which of the following numbers is divisible by 7

45,624, 346,556, 90,230,490 9,011,017

Preview Solution

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: A number is divisible by 7 if the sum of its digits is divisible by 7. Otherwise, the number is not divisible by 7. Summing the digits of each of the numbers in the problem gives:

$$\begin{aligned} 45,624 &= 4 + 5 + 6 + 2 + 4 = 21, \\ 346,556 &= 3 + 4 + 6 + 5 + 5 + 6 = 29, \\ 90,230,490 &= 9 + 0 + 2 + 3 + 0 + 4 + 9 = 27, \\ 9,011,017 &= 9 + 0 + 1 + 1 + 0 + 1 = 12. \end{aligned}$$

The only number whose digits sum to a multiple of 7 is 45,624 (21). So, the only number divisible by 7 is $\boxed{45,624}$.

3.52.

Which one of the following four-digit numbers is not divisible by 4: 2444, 2564, 2684, 2744, or 2884?

Preview Solution

Solution: A number is multiple of 4 if its final two digits form a multiple of 4. Therefore, 2444, 2564, 2672, and 2376 are multiples of 4. Since 84 is not a multiple of 4, the number 2884 is not divisible by 4. Note the original number is not a multiple of 4. Since 2a is not a multiple of 4, we know that $\boxed{2444}$ is not a multiple of 4.

3.53.

Any base10 that is number is divisible by 5 if its units digit is 0 or 5. What is a similar rule that we can use for base 7 a number is divisible by 5?

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We get a similar rule by taking the first several positive multiples of 5:

$$25, 50, 75, 100, 125, 150, 175, 200, 225, 250, 275, 300.$$

It looks like every multiple of 5 ends in 25, 50, or 75. Therefore, we can tell if a base 7 number is divisible by 5 by looking at the number's last two digits. Specifically, we can tell if a base 7 number is divisible by 5 if the sum of its last two digits is 7, 14, or 21.

Since any multiple of 100 is a multiple of 25, we can tell if a base 7 number is a multiple of 25 by the two-digit number is also a multiple of 25. The only numbers we can form with two digits that are multiples of 25 are 00, 25, and 50. So, if a number ends in 00, 25, 50, or 75, then the number is divisible by 25. Without, we can tell if a number is divisible by 5 by looking at the last two digits. Specifically, we can tell if a base 7 number is divisible by 5 if the sum of its last two digits is 7, 14, or 21.

3.54. Source: MAS05 **TF**

Amanda arranges the digits 1, 2, 3, and 7 to write a four-digit number. The 7 is not in the 1 but not the 3. The 3 is not in the 7 but not in the 1. The 1 is not in the 3 but not in the 7. The digit sum is divisible by 7. What is Amanda's four-digit number?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We are looking for a number that is divisible by 7 and only has 1, 2, 3, and 7 as digits. Except for the last digit, the tricks that the 7 and the 3 are not adjacent. So, we can't have the digits 37 in the middle. Therefore, the number ends in the two digits 73. The 7 is not in the 1, so 73 is not a number ends 1173. Therefore, the number is $\boxed{2173}$.

3.55. Source: MAS06 **TF**

Compute sum of the following:

(a) $\text{lcm}[20, 60]$

Preview Solution

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $20 = \underline{2^2} \cdot \underline{5^1}$ and $60 = \underline{2^2} \cdot \underline{3^1} \cdot \underline{5^1}$, so

$$\text{lcm}[20, 60] = 2^2 \cdot 3^1 \cdot 5^1 = \boxed{120}.$$

We also might have noted that

$$\text{lcm}[20, 60] = \text{lcm}[20 \cdot 3, 1 \cdot 3] = 3 \text{lcm}[20, 1] = 3 \cdot 40 = 120.$$

(b) $\text{lcm}[90, 75]$

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $90 = \underline{2^1} \cdot \underline{3^2}$ and $75 = \underline{3^1} \cdot \underline{5^2}$, so

$$\text{lcm}[75, 90] = 2^1 \cdot 3^2 \cdot 5^2 = \boxed{450}.$$

We also might have noted that

$$\text{lcm}[75, 90] = \text{lcm}[25 \cdot 3, 5 \cdot 3] = 3 \text{lcm}[25, 5] = 3 \cdot 50 = 150.$$

(c) $\text{lcm}[24, 25, 28]$

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $24 = \underline{2^3} \cdot \underline{3^1}$, $25 = \underline{5^2}$, and $28 = 1 \cdot 7 = 2^2 \cdot 7^1$, so

$$\text{lcm}[24, 25, 28] = 2^3 \cdot 3^1 \cdot 5^2 \cdot 7^1 = \boxed{2100}.$$

(d) $\text{lcm}[18, 45, 75]$

Preview Solution

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $18 = \underline{2^1} \cdot \underline{3^2}$, $45 = \underline{3^2} \cdot \underline{5^1}$, and $75 = 3 \cdot 5^2$, so

$$\text{lcm}[75, 18, 45] = 2^1 \cdot 3^2 \cdot 5^2 = \boxed{450}.$$

3.56.

Compute sum of the following:

(a) $\text{gcd}[18, 45]$

Preview Solution

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $18 = \underline{2^1} \cdot \underline{3^2}$ and $75 = 3 \cdot \underline{5^2}$, so $\text{gcd}[18, 75] = 3^1 = \boxed{3}$.

We also might have noted that

$$\text{gcd}[18, 75] = \text{gcd}[18 \cdot 3, 3 \cdot 5] = 3 \text{gcd}[18, 5] = 3 \cdot 1 = 3.$$

(b) $\text{gcd}[150, 385]$

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have $150 = 15 \cdot 10 = 3^1 \cdot 5^2 \cdot 2^1$ and

$$385 = 55 \cdot 7 = 5^1 \cdot 11^1 \cdot 7^1 = 5^1 \cdot 7^1 \cdot 11^1,$$

so $\text{gcd}[150, 385] = 5^1 = \boxed{5}$.

(c) $\text{gcd}[120, 385]$

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have

$$120 = 3 \cdot 10 = \underline{3^1} \cdot \underline{5^1} \cdot \underline{2^3}$$

and

$$385 = 5 \cdot 10 = 5^1 \cdot 11^1 \cdot 7^1,$$

so

$$\text{gcd}[120, 385] = 5^1 \cdot 11^1 = \boxed{55}.$$

(d) $\text{gcd}[120, 385, 1000]$

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We have

$$\begin{aligned} 120 &= 3 \cdot 10 = 3^1 \cdot \underline{5^1} \cdot 2^3, \\ 385 &= 5 \cdot 10 = 5^1 \cdot 11^1 \cdot 7^1, \\ 1000 &= 10 \cdot 100 = 5^1 \cdot 2^3 \cdot 5^2, \end{aligned}$$

so

$$\text{gcd}[120, 385, 1000] = 5^1 = \boxed{5}.$$

3.57.

If the $\text{gcd}[\text{number}, 7]$ is 0 divisible by, then what is the largest possible value of the digit d?

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: The largest number d can be the numbers that are multiples of both 0 and 7. The number 70 divides by 7, so 0 is a multiple of 7. In order for a number to be a multiple of 7, the last digit must be a multiple of 7. Thus, we can rule out the digits of 7 (0d0) 7 7 7

$\frac{4}{4}$ $\frac{4}{4}$ $\frac{9}{9}$ $\frac{1}{1}$ $\frac{4}{4}$ $\frac{10}{10}$ $\frac{9}{9}$ $\frac{16}{16}$

$4 \times 4 = 4 \times 4$ $\quad 9 \times 1 = 9 \times 1$ $\quad 4 \times 10 = 4 \times 10$ $\quad 9 \times 16 = 9 \times 16$

For a set of two people to create with fractions — Green Grade

Section 4 **Fractions**

4.1. What is a Fraction?

In Section 1.1, we discussed division. For example, $12 \div 3 = \frac{12}{3} = 4$. We also saw that $\frac{1}{3} \div 2 = \frac{1}{6}$ but what is $2 \div 3$? It is a number too, though not an integer. We often write this number equal to $2 \div 3$ as $\frac{2}{3}$.

Definition: If a is a number and b is a non-zero number, then the **fraction** $\frac{a}{b}$ equals $a \div b$.

Because every fraction is a division, we can find fractions of decimals as learned in Chapter 1 to work with fractions. Here are some properties we've seen in this section, written with fraction notation:

Important:

- **Dividing into zero:** If a is not zero, then $\frac{0}{a} = 0$.
- **Self division:** If a is not zero, then $\frac{a}{a} = 1$.
- **Dividing by 1:** $\frac{a}{1} = a$.
- **Dividing into a negative:** If b is not zero, then $\frac{a}{-b} = -\frac{a}{b}$.
- **Dividing by a negative:** If a is not zero, then $\frac{-a}{b} = -\frac{a}{b}$.
- **Negative divided by negative:** If b is not zero, then $\frac{-a}{-b} = \frac{a}{b}$.

Of course, you've already seen fraction notation in this book, when we talked about reciprocals. When a is nonzero, the reciprocal $\frac{1}{a}$ is simply $\frac{1}{a} = a^{-1}$.

Because every fraction is a division, we can find fractions of decimals as learned in Chapter 1 to work with fractions. Here are some properties we've seen in this section, written with fraction notation:

above, we repeatedly used that denominators of fractions cannot be 0. This is because $\frac{a}{0}$ or $\frac{0}{0}$ and division by 0 is not defined.

Problems

Problem 4.1 ▾ jump to Solution

Simplify each of the following fractions.

(a) $\frac{3}{1}$ = ___

(b) $\frac{1}{3}$ = ___

(c) $\frac{3}{3}$ = ___

(d) $\frac{0}{3}$ = ___

(e) $\frac{3}{0}$ = ___

(f) $\frac{0}{0}$ = ___

Problem 4.2 ▾ jump to Solution

For what values of x between 1 and 40 is $\frac{x}{40}$ an integer?

Problem 4.3 ▾ jump to Solution

Find a number between two consecutive integers on the number line below. Is divided into five equal pieces. For each labeled dot, find a number that corresponds to that point on the number line.

Problem 4.4 ▾ jump to Solution

(a) Use the number line to explain why $\frac{1}{2} = \frac{2}{4}$.

(b) Between what two consecutive integers is $\frac{7}{4}$?

Problem 4.5 ▾ jump to Solution

Use the number line to explain why $\frac{1}{2} = \frac{2}{4}$.

Problem 4.6 ▾ jump to Solution

Find two fractions that are equal to $\frac{1}{2}$.

Problem 4.7 ▾ 🔒 ⚙

Simplify each of the following fractions.

(a) $\frac{6}{1}$ = ___

(b) $\frac{1}{6}$ = ___

(c) $\frac{6}{6}$ = ___

(d) $\frac{0}{6}$ = ___

(e) $\frac{6}{0}$ = ___

(f) $\frac{0}{0}$ = ___

Solution for Problem 4.7

Problem 4.8 ▾ 🔒 ⚙

For what values of x between 1 and 40 is $\frac{x}{40}$ an integer?

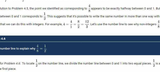
Solution for Problem 4.8 Since $\frac{x}{40} = x \div 40$, the number x must be a multiple of 1 in order for $\frac{x}{40}$ to be an integer. The only multiples of 1 between 1 and 40 are 1, 2, 3, ..., 38, and 40.

Problem 4.9 ▾ 🔒 ⚙

Find a number between two consecutive integers on the number line below. Is divided into five equal pieces. For each labeled dot, find a number that corresponds to that point on the number line.

Solution for Problem 4.9 Because the dots divide the number line between 4 and 5 into 5 equal pieces, the right end of the first of these pieces is $\frac{1}{5}$ the right end of the second piece is $\frac{2}{5}$ the third piece is $\frac{3}{5}$ and so on.

Doing a little one-step division from 5, we find that $\frac{1}{5}$ corresponds to negative numbers, by starting from 4 and going leftward, the left end of the first piece is $-\frac{1}{5}$. The left end of the second piece is $-\frac{2}{5}$ the third piece is $-\frac{3}{5}$ and so on. Continuing in this way, we can label the number line as shown below.



We find that $\frac{8}{5} = \frac{4}{5} + \frac{4}{5} = \frac{4}{5} + 1$ and $\frac{17}{5} = 3 + \frac{2}{5}$.

Problem 4.10 ▾ 🔒 ⚙

(a) Use the number line to explain why $\frac{1}{2} = \frac{2}{4}$.

(b) Between what two consecutive integers is $\frac{7}{4}$?

Solution for Problem 4.10

(a) We locate the number $\frac{1}{2}$ on the number line by dividing the segment between 0 and 1 into 2 equal pieces. $\frac{1}{2}$ is the point at the right end of the first of these pieces.

Since $\frac{2}{4}$ is the right end of the seventh of these pieces, we see that $\frac{1}{2}$ is between $\frac{1}{4}$ and $\frac{3}{4}$ is between 0 and 1.

Similarly, if the numerator and denominator of a fraction are positive, we can quickly compare the fraction to 1.

Important:

Suppose a and b are positive.

- a is less than 1 if a is less than b.
- a equals 1 if $a = b$.
- a is greater than 1 if a is greater than b.

(b) We can write a fraction $\frac{a}{b}$ as an integer if a is a multiple of b. So, to determine which consecutive integers $\frac{7}{4}$ is between, we must find the consecutive multiples of 5 that sit in between. Since 4 is between 0 and 5, we know that $\frac{7}{4}$ is between $\frac{0}{5} = 0$ and $\frac{5}{5} = 1$.

Therefore, $\frac{7}{4}$ is between 0 and 1.

Problem 4.11 ▾ 🔒 ⚙

(a) Use the number line to explain why $\frac{1}{2} = \frac{2}{4}$.

(b) Between what two consecutive integers is $\frac{7}{4}$?

The points A, B, C, and D on the number line below correspond to the numbers $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, and $\frac{7}{2}$ in some order. Match each point to the correct number.

Solution for Problem 4.11 Point A is the only labeled point to the left of 0, so it must correspond to the only negative number in the list, which is $-\frac{1}{2}$.

Point B is between 0 and 1. Of the three positive numbers in our list, only $\frac{1}{2}$ is a number that is less than its denominator. Therefore $\frac{1}{2}$ is the only positive number in the list that is less than 1. So, point B corresponds to $\frac{1}{2}$.

Since $\frac{3}{2} = \frac{1}{2} + 1$ Point C must be $\frac{3}{2}$. That leaves point D for $\frac{7}{2}$. As a quick check, we note that $\frac{7}{2}$ is between 3 and 4, as $\frac{7}{2} = \frac{6}{2} + \frac{1}{2} = 3 + \frac{1}{2}$ and $\frac{7}{2}$ is between 2 and 3 on the number line.

In our solution to Problem 4.3, the point we identified as $\frac{7}{4}$ appears to be exactly halfway between 3 and 4. Are the point D and $\frac{7}{4}$ the same dot? Unfortunately, no. This suggests that it's possible to write the same number in more than one way with fractions. It's clear that we can do this with integers. For example, $3 = \frac{6}{2} = \frac{9}{3}$ and $-2 = \frac{-4}{2}$ and $\frac{-6}{3}$ and $\frac{-8}{4}$ and so on. We can do the same thing with fractions. For example, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$ and $\frac{-1}{2} = \frac{-2}{4} = \frac{-3}{6}$ and $\frac{-4}{-8}$ and so on.

Problem 4.12 ▾ 🔒 ⚙

Use the number line to explain why $\frac{1}{2} = \frac{2}{4}$.

Solution for Problem 4.12 To locate $\frac{1}{2}$ on the number line, we divide the number line between 0 and 1 into two equal pieces. $\frac{1}{2}$ is the right end of the first piece.

To locate $\frac{2}{4}$ on the number line, we divide the number line between 0 and 1 into eight equal pieces. We can do so by dividing each of the two pieces we already have between 0 and 1 into four equal pieces.

So, the right end of the fourth of these eight small pieces between 0 and 1 is the same as the right end of the first of the two equal pieces we started with. Since $\frac{1}{2}$ is the right end of the first piece, we see that $\frac{1}{2} = \frac{2}{4}$.

Problem 4.13 ▾ 🔒 ⚙

Find two fractions that are equal to $\frac{1}{2}$.

Solution for Problem 4.13 We can look back of these three equal pieces between 0 and 1 into two equal pieces, so that we now have six equal pieces between 0 and 1. The point corresponding to $\frac{1}{2}$ is the right end of the 3rd piece, so we see that $\frac{1}{2} = \frac{3}{6}$.

Similarly, we could look back around all our original three equal pieces into three equal pieces, so that there are 9 equal pieces between 0 and 1. The point corresponding to $\frac{1}{2}$ is the right end of the 4th of these 9 pieces, so we see that $\frac{1}{2} = \frac{4}{9}$.

Therefore, both $\frac{3}{6}$ and $\frac{4}{9}$ are fractions equal to $\frac{1}{2}$. These are not the only fractions that equal $\frac{1}{2}$. For example, $\frac{5}{10}$, $\frac{6}{12}$, $\frac{7}{14}$, $\frac{8}{16}$, $\frac{9}{18}$, $\frac{10}{20}$ and $\frac{11}{22}$ and so on. Isn't fun? Why do fractions $\frac{1}{2}$ and $\frac{2}{4}$ equal?

Exercises

4.1.1. ▾ 🔒 ⚙

Find the value of each of the following:

(a) $\frac{-5}{1}$ =

You may type any additional notes you have here.

Hide Solution

Reset

Solution: Applying our rule for dividing into negative, we have $\frac{-5}{1} = -(5 \div 1) = \boxed{-5}$.

(b) $\frac{0}{-3}$ =

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Solution: $\frac{0}{-3} = (0 \div 3) = \boxed{0}$

(c) $\frac{-10}{-10}$ =

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution: Any nonzero number divided by itself equals 1, so $\frac{-10}{-10} = \boxed{1}$.

(d) $\frac{0}{0}$ =

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution: It is undefined by any nonzero number in it, so $\frac{0}{0} = \boxed{\text{DNE}}$.

4.1.2. ▾ 🔒 ⚙

Compute $\frac{10 \times 5}{10 \times 2}$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution: $\frac{10 \times 5}{10 \times 2} = \frac{10}{10} \times \frac{5}{2} = \boxed{\frac{25}{2}}$.

4.1.3. Source: MATHCOUNTS ▾ 🔒 ⚙

Compute $\frac{1 \times 2 \times 3 \times 4}{2 \times 3 \times 4 \times 5}$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution:

$$\frac{1 \times 2 \times 3 \times 4}{2 \times 3 \times 4 \times 5} = \frac{1}{5} \times \frac{2}{2} \times \frac{3}{3} \times \frac{4}{4} = \boxed{\frac{1}{5}}$$

4.1.4. ▾ 🔒 ⚙

Compute $\frac{1}{2} \div \frac{1}{3}$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution: We first find the multiple of 3 that sit in between. We have $\frac{0}{3} = 0$ and $\frac{3}{3} = 1$. So $\frac{1}{3}$ is between $\frac{0}{3}$ and $\frac{3}{3}$. We now zoom in to the form on the number line. We show $\frac{1}{3}$ below.

Another way to think about this problem is to consider the location of $\frac{1}{3}$ on the number line. Suppose we break the number line between 0 and 1 into equal pieces like in Example 1 above. The fraction $\frac{1}{3}$ which equals 1, is the right end of the $\frac{1}{3}$rd piece, or the right end of 3. The fraction $\frac{1}{2}$ is the right end of two pieces, so it is closer to $\frac{1}{3}$ than is any other integer.

4.1.5. ▾ 🔒 ⚙

$\frac{0}{0} = ?$ Clearly $\frac{0}{0} = 0$ but this value of $\frac{0}{0}$ ...

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution: Since $(-10) \div 0 = 0$, we have

$$(x) \div \frac{0}{0} = (-10) \div \frac{0}{0} = (-10) \div 0 = \boxed{-10}.$$

4.1.6. ▾ 🔒 ⚙

The points D, E, C, and B on the number line below correspond to the numbers $\frac{-3}{2}$, $\frac{-1}{2}$, $\frac{1}{2}$, and $\frac{3}{2}$ in some order. Match each point to the correct number.

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution: Step 1: Find the $\frac{1}{2}$ unit.

Solution: We locate the $\frac{1}{2}$ on the number line by dividing the number line between 0 and 1 into 2 equal pieces. $\frac{1}{2}$ is the right end of the 1st of these pieces.

Step 2: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 3: Find the $\frac{-1}{2}$ unit.

Solution: We locate $\frac{-1}{2}$ on the number line by dividing the number line between 0 and 1 into 2 equal pieces. $\frac{-1}{2}$ is the right end of the 1st of these pieces.

Step 4: Find the $\frac{-3}{2}$ unit.

Solution: We locate $\frac{-3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{-3}{2}$ is the right end of the 3rd of these pieces.

Step 5: Find the $\frac{-3}{2}$ unit.

Solution: We locate $\frac{-3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{-3}{2}$ is the right end of the 3rd of these pieces.

Step 6: Find the $\frac{-1}{2}$ unit.

Solution: We locate $\frac{-1}{2}$ on the number line by dividing the number line between 0 and 1 into 2 equal pieces. $\frac{-1}{2}$ is the right end of the 1st of these pieces.

Step 7: Find the $\frac{1}{2}$ unit.

Solution: We locate $\frac{1}{2}$ on the number line by dividing the number line between 0 and 1 into 2 equal pieces. $\frac{1}{2}$ is the right end of the 1st of these pieces.

Step 8: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 9: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 10: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 11: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 12: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 13: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 14: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 15: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 16: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 17: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 18: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 19: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 20: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 21: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 22: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 23: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 24: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 25: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 26: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 27: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 28: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 29: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 30: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 31: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 32: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 33: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 34: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 35: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 36: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 37: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 38: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 39: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 40: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 41: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 42: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 43: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 44: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 45: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 46: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 47: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 48: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 49: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 50: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 51: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 52: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 53: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 54: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 55: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 56: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 57: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 58: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 59: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 60: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 61: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 62: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 63: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 64: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 65: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 66: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 67: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 68: Find the $\

$\frac{4}{4}$ $\frac{4}{4}$ $\frac{9}{9}$ $\frac{1}{1}$ $\frac{4}{4}$ $\frac{10}{10}$ $\frac{9}{9}$ $\frac{16}{16}$

$4 \times 4 = 4 \times 4$ $\quad 9 \times 1 = 9 \times 1$ $\quad 4 \times 10 = 4 \times 10$ $\quad 9 \times 16 = 9 \times 16$

For a set of two people to create with fractions — Green Grade

Section 4 **Fractions**

4.1. What is a Fraction?

In Section 1.1, we discussed division. For example, $12 \div 3 = \frac{12}{3} = 4$. We also saw that $\frac{1}{3} \div 2 = \frac{1}{6}$ but what is $2 \div 3$? It is a number too, though not an integer. We often write this number equal to $2 \div 3$ as $\frac{2}{3}$.

Definition: If a is a number and b is a non-zero number, then the **fraction** $\frac{a}{b}$ equals $a \div b$.

Because every fraction is a division, we can find fractions of decimals as learned in Chapter 1 to work with fractions. Here are some properties we've seen in this section, written with fraction notation:

Important:

- **Dividing into zero:** If a is not zero, then $\frac{0}{a} = 0$.
- **Self division:** If a is not zero, then $\frac{a}{a} = 1$.
- **Dividing by 1:** $\frac{a}{1} = a$.
- **Dividing into a negative:** If b is not zero, then $\frac{a}{-b} = -\frac{a}{b}$.
- **Dividing by a negative:** If a is not zero, then $\frac{-a}{b} = -\frac{a}{b}$.
- **Negative divided by negative:** If b is not zero, then $\frac{-a}{-b} = \frac{a}{b}$.

Of course, you've already seen fraction notation in this book, when we talked about reciprocals. When a is nonzero, the reciprocal $\frac{1}{a}$ is simply $\frac{1}{a} = a^{-1}$.

Because every fraction is a division, we can find fractions of decimals as learned in Chapter 1 to work with fractions. Here are some properties we've seen in this section, written with fraction notation:

above, we repeatedly used that denominators of fractions cannot be 0. This is because $\frac{a}{0}$ or $\frac{0}{0}$ and division by 0 is not defined.

Problems

Problem 4.1 ▾ jump to Solution

Simplify each of the following fractions.

(a) $\frac{3}{1}$ = ___

(b) $\frac{1}{3}$ = ___

(c) $\frac{3}{3}$ = ___

(d) $\frac{0}{3}$ = ___

(e) $\frac{3}{0}$ = ___

(f) $\frac{0}{0}$ = ___

Problem 4.2 ▾ jump to Solution

For what values of x between 1 and 40 is $\frac{x}{40}$ an integer?

Problem 4.3 ▾ jump to Solution

Find a number between two consecutive integers on the number line below. Is divided into five equal pieces. For each labeled dot, find a number that corresponds to that point on the number line.

Problem 4.4 ▾ jump to Solution

(a) Use the number line to explain why $\frac{1}{2} = \frac{2}{4}$.

(b) Between what two consecutive integers is $\frac{7}{4}$?

Problem 4.5 ▾ jump to Solution

Use the number line to explain why $\frac{1}{2} = \frac{2}{4}$.

Problem 4.6 ▾ jump to Solution

Find two fractions that are equal to $\frac{1}{2}$.

Problem 4.7 ▾ 🔒 ⚙

Simplify each of the following fractions.

(a) $\frac{6}{1}$ = ___

(b) $\frac{1}{6}$ = ___

(c) $\frac{6}{6}$ = ___

(d) $\frac{0}{6}$ = ___

(e) $\frac{6}{0}$ = ___

(f) $\frac{0}{0}$ = ___

Solution for Problem 4.7

Problem 4.8 ▾ 🔒 ⚙

For what values of x between 1 and 40 is $\frac{x}{40}$ an integer?

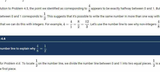
Solution for Problem 4.8 Since $\frac{x}{40} = x \div 40$, the number x must be a multiple of 1 in order for $\frac{x}{40}$ to be an integer. The only multiples of 1 between 1 and 40 are 1, 2, 3, ..., 38, and 40.

Problem 4.9 ▾ 🔒 ⚙

Find a number between two consecutive integers on the number line below. Is divided into five equal pieces. For each labeled dot, find a number that corresponds to that point on the number line.

Solution for Problem 4.9 Because the dots divide the number line between 4 and 5 into 5 equal pieces, the right end of the first of these pieces is $\frac{1}{5}$ the right end of the second piece is $\frac{2}{5}$ the third piece is $\frac{3}{5}$ and so on.

Doing a little one-step division from 5, we find that $\frac{1}{5}$ corresponds to negative numbers, by starting from 4 and going leftward, the left end of the first piece is $-\frac{1}{5}$. The left end of the second piece is $-\frac{2}{5}$ the third piece is $-\frac{3}{5}$ and so on. Continuing in this way, we can label the number line as shown below.



We find that $\frac{8}{5} = \frac{4}{5} + \frac{4}{5} = \frac{4}{5} + 1$ and $\frac{17}{5} = 3 + \frac{2}{5}$.

Problem 4.10 ▾ 🔒 ⚙

(a) Use the number line to explain why $\frac{1}{2} = \frac{2}{4}$.

(b) Between what two consecutive integers is $\frac{7}{4}$?

Solution for Problem 4.10

(a) We locate the number $\frac{1}{2}$ on the number line by dividing the segment between 0 and 1 into 2 equal pieces. $\frac{1}{2}$ is the point at the right end of the first of these pieces.

Since $\frac{2}{4}$ is the right end of the seventh of these pieces, we see that $\frac{1}{2}$ is between $\frac{1}{4}$ and $\frac{3}{4}$ is between 0 and 1.

Similarly, if the numerator and denominator of a fraction are positive, we can quickly compare the fraction to 1.

Important:

Suppose a and b are positive.

- a is less than 1 if a is less than b.
- a equals 1 if $a = b$.
- a is greater than 1 if a is greater than b.

(b) We can write a fraction $\frac{a}{b}$ as an integer if a is a multiple of b. So, to determine which consecutive integers $\frac{7}{4}$ is between, we must find the consecutive multiples of 5 that sit in between. Since 4 is between 0 and 5, we know that $\frac{7}{4}$ is between $\frac{0}{5} = 0$ and $\frac{5}{5} = 1$.

Therefore, $\frac{7}{4}$ is between 0 and 1.

Problem 4.11 ▾ 🔒 ⚙

(a) Use the number line to explain why $\frac{1}{2} = \frac{2}{4}$.

(b) Between what two consecutive integers is $\frac{7}{4}$?

The points A, B, C, and D on the number line below correspond to the numbers $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, and $\frac{7}{2}$ in some order. Match each point to the correct number.

Solution for Problem 4.11 Point A is the only labeled point to the left of 0, so it must correspond to the only negative number in the list, which is $-\frac{1}{2}$.

Point B is between 0 and 1. Of the three positive numbers in our list, only $\frac{1}{2}$ is a number that is less than its denominator. Therefore $\frac{1}{2}$ is the only positive number in the list that is less than 1. So, point B corresponds to $\frac{1}{2}$.

Since $\frac{3}{2} = \frac{1}{2} + 1$ Point C must be $\frac{3}{2}$. That leaves point D for $\frac{7}{2}$. As a quick check, we note that $\frac{7}{2}$ is between 3 and 4, as $\frac{7}{2} = \frac{6}{2} + \frac{1}{2} = 3 + \frac{1}{2}$ and $\frac{7}{2}$ is between 2 and 3 on the number line.

In our solution to Problem 4.3, the point we identified as $\frac{7}{4}$ appears to be exactly halfway between 3 and 4. Are the point D and $\frac{7}{4}$ the same dot? Unfortunately, no. This suggests that it's possible to write the same number in more than one way with fractions. It's clear that we can do this with integers. For example, $3 = \frac{6}{2} = \frac{9}{3}$ and $-2 = \frac{-4}{2}$ and $\frac{-6}{3}$ and $\frac{-8}{4}$ and so on. We can do the same thing with fractions. For example, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$ and $\frac{-1}{2} = \frac{-2}{4} = \frac{-3}{6}$ and $\frac{-4}{-8}$ and so on.

Problem 4.12 ▾ 🔒 ⚙

Use the number line to explain why $\frac{1}{2} = \frac{2}{4}$.

Solution for Problem 4.12 To locate $\frac{1}{2}$ on the number line, we divide the number line between 0 and 1 into two equal pieces. $\frac{1}{2}$ is the right end of the first piece.

To locate $\frac{2}{4}$ on the number line, we divide the number line between 0 and 1 into eight equal pieces. We can do so by dividing each of the two pieces we already have between 0 and 1 into four equal pieces.

So, the right end of the fourth of these eight small pieces between 0 and 1 is the same as the right end of the first of the two equal pieces we started with. Since $\frac{1}{2}$ is the right end of the first piece, we see that $\frac{1}{2} = \frac{2}{4}$.

Problem 4.13 ▾ 🔒 ⚙

Find two fractions that are equal to $\frac{1}{2}$.

Solution for Problem 4.13 We can look back of these three equal pieces between 0 and 1 into two equal pieces, so that we now have six equal pieces between 0 and 1. The point corresponding to $\frac{1}{2}$ is the right end of the 3rd piece, so we see that $\frac{1}{2} = \frac{3}{6}$.

Similarly, we could look back around all our original three equal pieces into three equal pieces, so that there are 9 equal pieces between 0 and 1. The point corresponding to $\frac{1}{2}$ is the right end of the 4th of these 9 pieces, so we see that $\frac{1}{2} = \frac{4}{9}$.

Therefore, both $\frac{3}{6}$ and $\frac{4}{9}$ are fractions equal to $\frac{1}{2}$. These are not the only fractions that equal $\frac{1}{2}$. For example, $\frac{5}{10}$, $\frac{6}{12}$, $\frac{7}{14}$, $\frac{8}{16}$, $\frac{9}{18}$, $\frac{10}{20}$ and $\frac{11}{22}$ and so on. Isn't fun? Why do fractions $\frac{1}{2}$ and $\frac{2}{4}$ equal?

Exercises

4.1.1. ▾ 🔒 ⚙

Find the value of each of the following:

(a) $\frac{-5}{1}$ =

You may type any additional notes you have here.

Hide Solution

Reset

Solution: Applying our rule for dividing into negative, we have $\frac{-5}{1} = -(5 \div 1) = \boxed{-5}$.

(b) $\frac{0}{-3}$ =

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Solution: $\frac{0}{-3} = (0 \div 3) = \boxed{0}$

(c) $\frac{-10}{-10}$ =

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution: Any nonzero number divided by itself equals 1, so $\frac{-10}{-10} = \boxed{1}$.

(d) $\frac{0}{0}$ =

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution: It is undefined by any nonzero number in it, so $\frac{0}{0} = \boxed{\text{DNE}}$.

4.1.2. ▾ 🔒 ⚙

Compute $\frac{10 \times 5}{10 \times 2}$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution: $\frac{10 \times 5}{10 \times 2} = \frac{10}{10} \times \frac{5}{2} = \boxed{\frac{25}{2}}$.

4.1.3. Source: MATHCOUNTS ▾ 🔒 ⚙

Compute $\frac{1 \times 2 \times 3 \times 4}{2 \times 3 \times 4 \times 5}$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution:

$$\frac{1 \times 2 \times 3 \times 4}{2 \times 3 \times 4 \times 5} = \frac{1}{5} \times \frac{2}{2} \times \frac{3}{3} \times \frac{4}{4} = \boxed{\frac{1}{5}}$$

4.1.4. ▾ 🔒 ⚙

Compute $\frac{1}{2} \div \frac{1}{3}$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution: We first find the multiple of 3 that sit in between. We have $\frac{0}{3} = 0$ and $\frac{3}{3} = 1$. So $\frac{1}{3}$ is between $\frac{0}{3}$ and $\frac{3}{3}$. We now zoom in to the form on the number line. We show $\frac{1}{3}$ below.

Another way to think about this problem is to consider the location of $\frac{1}{3}$ on the number line. Suppose we break the number line between 0 and 1 into equal pieces like in Example 1 above. The fraction $\frac{1}{3}$ which equals 1, is the right end of the $\frac{1}{3}$rd piece, or the right end of 3. The fraction $\frac{1}{2}$ is the right end of two pieces, so it is closer to $\frac{1}{3}$ than is any other integer.

4.1.5. ▾ 🔒 ⚙

$\frac{0}{0} = ?$ Clearly $\frac{0}{0} = 0$ but this value of $\frac{0}{0}$ ...

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution: Since $(-10) \div 0 = 0$, we have

$$(x) \div \frac{0}{0} = (-10) \div \frac{0}{0} = (-10) \div 0 = \boxed{-10}.$$

4.1.6. ▾ 🔒 ⚙

The points D, E, C, and B on the number line below correspond to the numbers $\frac{-3}{2}$, $\frac{-1}{2}$, $\frac{1}{2}$, and $\frac{3}{2}$ in some order. Match each point to the correct number.

You may type any additional notes you have here.

Hide Solution

Reset

Your Assessment: Solution

Solution: Step 1: Find the $\frac{1}{2}$ unit.

Solution: We locate the $\frac{1}{2}$ on the number line by dividing the number line between 0 and 1 into 2 equal pieces. $\frac{1}{2}$ is the right end of the 1st of these pieces.

Step 2: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 3: Find the $\frac{-1}{2}$ unit.

Solution: We locate $\frac{-1}{2}$ on the number line by dividing the number line between 0 and 1 into 2 equal pieces. $\frac{-1}{2}$ is the right end of the 1st of these pieces.

Step 4: Find the $\frac{-3}{2}$ unit.

Solution: We locate $\frac{-3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{-3}{2}$ is the right end of the 3rd of these pieces.

Step 5: Find the $\frac{-3}{2}$ unit.

Solution: We locate $\frac{-3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{-3}{2}$ is the right end of the 3rd of these pieces.

Step 6: Find the $\frac{-1}{2}$ unit.

Solution: We locate $\frac{-1}{2}$ on the number line by dividing the number line between 0 and 1 into 2 equal pieces. $\frac{-1}{2}$ is the right end of the 1st of these pieces.

Step 7: Find the $\frac{1}{2}$ unit.

Solution: We locate $\frac{1}{2}$ on the number line by dividing the number line between 0 and 1 into 2 equal pieces. $\frac{1}{2}$ is the right end of the 1st of these pieces.

Step 8: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 9: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 10: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 11: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 12: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 13: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 14: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 15: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 16: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 17: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 18: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 19: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 20: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 21: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 22: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 23: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 24: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 25: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 26: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 27: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 28: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 29: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 30: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 31: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 32: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 33: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 34: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 35: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 36: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 37: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 38: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 39: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 40: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 41: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 42: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 43: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 44: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 45: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 46: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 47: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 48: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 49: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 50: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 51: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 52: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 53: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 54: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 55: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 56: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 57: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 58: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 59: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 60: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 61: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 62: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 63: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 64: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 65: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 66: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 67: Find the $\frac{3}{2}$ unit.

Solution: We locate $\frac{3}{2}$ on the number line by dividing the number line between 0 and 1 into 6 equal pieces. $\frac{3}{2}$ is the right end of the 3rd of these pieces.

Step 68: Find the $\

4.3 Multiplying Fractions

We often use the word "of" to mean multiplication. For example, suppose there are 12 apples in each apple basket in the grocery store, and you count four many total apples there are in 2 of these baskets, we compute $2 \cdot 12 = 24$. We often avoid using "of" with integers, saying "2 baskets" rather than "2 of the baskets." But with fractions, "half of a basket" instead of "$\frac{1}{2}$ of a basket" makes sense.

Suppose we want to find $\frac{1}{2}$ of $\frac{3}{4}$ of a basket. If there are 12 apples in each basket? Since "half" means multiply, there are $\frac{1}{2}$ apples. So compute $\frac{1}{2} \cdot \frac{3}{4}$. We call this multiplying fractions.

Here are a couple of properties we learned in Chapter 1 that will be helpful in learning how to multiply fractions.

Definition of division: If b is not zero then $\frac{a}{b} = a \div b$	
Reciprocal of product: If a and b are not zero then $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$	

Problems

Problem 4.3

Jump to Solution

Compute $\frac{1}{2} \cdot \frac{3}{4}$.

Problem 4.4

Jump to Solution

In this problem, we compute $\frac{1}{3} \cdot \frac{5}{7}$.

(a) Use our definition of division to write $\frac{1}{3}$ as the product of an integer and the reciprocal of an integer.

(b) Use the associative property of multiplication and your result from part (a) to compute $\frac{1}{3} \cdot \frac{5}{7}$. Hint: To "compute," we mean find a number that equals $\frac{1}{3} \cdot \frac{5}{7}$.

Problem 4.5

Jump to Solution

In this problem, we compute $\frac{5}{6} \cdot \frac{3}{4}$.

(a) Write $\frac{5}{6}$ as $\frac{5}{2} \cdot \frac{1}{3}$.

(b) Explain why $\frac{5}{2} \cdot \frac{1}{3} = \frac{5 \cdot 1}{2 \cdot 3}$.

(c) Compute $\frac{5}{6} \cdot \frac{3}{4}$.

Problem 4.6

Jump to Solution

Compute $\frac{1}{2} \cdot \frac{3}{4}$.

Problem 4.7

Jump to Solution

In this problem, we compute $\frac{1}{3} \cdot \frac{5}{7}$.

(a) Use the definition of division to write $\frac{1}{3}$ as the product of an integer and the reciprocal of an integer.

(b) Use the associative property of multiplication and your result from part (a) to compute $\frac{1}{3} \cdot \frac{5}{7}$.

Problem 4.8

Jump to Solution

In this problem, we compute $\frac{5}{6} \cdot \frac{3}{4}$.

(a) Write $\frac{5}{6}$ as $\frac{5}{2} \cdot \frac{1}{3}$.

(b) Explain why $\frac{5}{2} \cdot \frac{1}{3} = \frac{5 \cdot 1}{2 \cdot 3}$.

(c) Compute $\frac{5}{6} \cdot \frac{3}{4}$.

Problem 4.9

Jump to Solution

Compute $\frac{11}{12} \cdot \frac{27}{35}$.

(a) What is $\frac{11}{12}$ of $\frac{27}{35}$?

(b) What is $\frac{27}{35}$ of $\frac{11}{12}$?

Problem 4.10

Jump to Solution

Many starts with 100 pennies. One **gives** $\frac{1}{4}$ of the pennies to her brother Mitch. Mitch then gives $\frac{2}{3}$ of the pennies he receives to his sister Mary. How many pennies does Mary give to her mother?

Problem 4.11 Video Solution

Video 4.11

Solution for Problem 4.11 Our definition of division tells us that $\frac{1}{2} \div 3 = \frac{1}{2} \cdot \frac{1}{3}$ mapping this to $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

We can also see that $\frac{1}{2} \div 3 = \frac{1}{6}$ the number line. We begin the number line by dividing the number line between 0 and 1 into three equal pieces. $\frac{1}{3}$ is the right end of the first piece. $\frac{1}{2}$ is now to be third $\frac{1}{2}$. $\frac{1}{6}$ is the right end of the second of these three equal pieces between 0 and 1.

Problem 4.12 Video Solution

Video 4.12

Solution for Problem 4.12 In the previous problem, we saw how to multiply an integer and the reciprocal of an integer using the definition of division. So, we identify nothing $\frac{1}{2}$ and multiplying

$$3 \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right).$$

Applying the associative property of multiplication

$$3 \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right) = \left(3 \cdot \frac{1}{2}\right) \cdot \frac{1}{3}.$$

Finally, applying the definition of division gives

$$\left(3 \cdot \frac{1}{2}\right) \cdot \frac{1}{3} = \frac{3}{2} \cdot \frac{1}{3}.$$

$$\frac{3}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

We can also verify that $\frac{1}{2} \div 3 = \frac{1}{6}$ the number line. We begin the number line by dividing the number line between 0 and 1 into three equal pieces. $\frac{1}{3}$ is the right end of the first piece. $\frac{1}{2}$ is now to be third $\frac{1}{2}$. $\frac{1}{6}$ is the right end of the second of these three equal pieces between 0 and 1.

Problem 4.13 Video Solution

Video 4.13

Solution for Problem 4.13 We know the same strategy we used to compute $\frac{1}{2} \cdot \frac{1}{3}$ mapping this to $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ and $\frac{1}{2}$ we can see. Applying this to $\frac{1}{2}$ gives

$$\frac{1}{2} \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right).$$

Once again, we interpret this result on the number line. We start with the location of $\frac{1}{2}$ dividing the number line between 0 and 1 into 3 equal pieces.

The point that corresponds to $\frac{1}{2} \cdot \frac{1}{3}$ we can find this $\frac{1}{2}$ $\frac{1}{6}$ $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

Problem 4.14 Video Solution

Video 4.14

Solution for Problem 4.14 We follow the same strategy we used to compute $\frac{1}{2} \cdot \frac{1}{3}$ we use the definition of division to write each $\frac{1}{2}$ as the product of an integer and a reciprocal

$$\frac{1}{2} \cdot \left(\frac{1}{2} \cdot \left(\frac{1}{2} \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right)\right)\right).$$

We then use the associative and commutative properties of multiplication to group the integers and the reciprocals

$$\left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right) = 16 \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right).$$

Finally, applying the reciprocal of a product gives rise to the definition of division we have

$$\left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right) = 16 \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right) = \frac{16}{2} \cdot \frac{1}{3}.$$

Once again, we can interpret this result on the number line. We start by locating $\frac{1}{2}$ on the number line.

Then, we find $\frac{1}{3}$ $\frac{1}{2}$ on the number line by dividing the number line between 0 and $\frac{1}{2}$ into three equal pieces. The product $\frac{1}{2} \cdot \frac{1}{3}$ is the right end of the second of these pieces.

To see that $\frac{1}{2}$ is also at the right end of the second of these pieces, we divide each of our initial five equal pieces between 0 and 1 into 2 equal pieces each. This gives us $\frac{10}{2}$ $\frac{1}{2} = \frac{10}{2} \cdot \frac{1}{3}$ pieces.

Between 0 and 1 there are $5 \cdot 2 = 10$ of these pieces. Since $\frac{1}{2}$ is the $5 \cdot 1 = 5$ th piece, the number $\frac{1}{2} \cdot \frac{1}{3} = \frac{5}{10}$. Pictures, the number equal to the right endpoint of the 5th of these 10 equal pieces between 0 and 1.

We can similarly handle any product $\frac{1}{2} \cdot \frac{1}{3}$ where a and b are nonzero

$$\begin{aligned} \frac{1}{a} \cdot \left(\frac{1}{b} \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right)\right) &= \frac{1}{a} \cdot \left(\frac{1}{b} \cdot \frac{1}{2} \cdot \frac{1}{3}\right) && \text{definition of division} \\ &= \left(\frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{2}\right) \cdot \frac{1}{3} && \text{associative property} \\ &= \left(\frac{1}{ab} \cdot \frac{1}{2}\right) \cdot \frac{1}{3} && \text{definition of division} \end{aligned}$$

Important! If a is nonzero, then $\frac{1}{a} \cdot \frac{1}{a} = \frac{1}{a^2}$.

Since $\frac{1}{a^2} = \frac{1}{a} \cdot \frac{1}{a}$ we have $\frac{1}{a^2} = \frac{1}{a} \cdot \frac{1}{a}$ as well.

Problem 4.15 Video Solution

Video 4.15

Solution for Problem 4.15 We know that $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ and the corresponding fraction $\frac{1}{6}$ is quite a crime, instead, we see that crime is clearly a multiple of 3. Using the rule $\frac{1}{a^2} = \frac{1}{a} \cdot \frac{1}{a}$ that we just learned, we have

$$\frac{1}{6^2} = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

Since $\frac{1}{6^2} = \frac{1}{6} \cdot \frac{1}{6}$ we have $\frac{1}{6^2} = \frac{1}{6} \cdot \frac{1}{6}$

4.3 Dividing by a Fraction

Recall that we define division in terms of multiplication:

$$a \div b = a \cdot \frac{1}{b}$$

In this section, we combine this definition with the rule for multiplying fractions to learn how to divide by fractions.

Problems

Problem 4.16 Jump to Solution

What is $\frac{1}{2} \div \frac{3}{4}$?

Problem 4.17 Jump to Solution

(a) What is the reciprocal of $\frac{3}{5}$?

(b) If a and b are nonzero, then what is the reciprocal of $\frac{a}{b}$?

Problem 4.18 Jump to Solution

What is $3 \div \frac{5}{7}$?

Problem 4.19 Jump to Solution

What is $\frac{1}{2} \div \frac{1}{3}$?

Problem 4.20 Jump to Solution

What is $\frac{12}{5} \div \frac{3}{10}$?

Problem 4.21 Jump to Solution

(a) $\frac{3}{4}$ of what number?

Problem 4.22 Jump to Solution

Each panel of fencing material is $\frac{3}{8}$ feet long. How many panels do I need to build a 60-foot fence?

Problem 4.16 Tag: Fractions

What is $\frac{1}{2} \div \frac{3}{4}$?

Solution for Problem 4.16: Since $a \div b = a \cdot \frac{1}{b}$, we have

$$\frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \cdot \frac{1}{\frac{3}{4}} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.$$

Q

What about dividing by a fraction? Our definition of division tells us that to divide by a fraction, we multiply by the reciprocal of the fraction. So, we'd love to figure out how to find the reciprocal of a fraction.

Problem 4.17 Tag: Fractions

(a) What is the reciprocal of $\frac{3}{5}$?

(b) If a and b are nonzero, then what is the reciprocal of $\frac{a}{b}$?

Solution for Problem 4.17:

(a) We seek a number such that the product of $\frac{3}{5}$ and the number is 1. Suppose that this number is a fraction $\frac{m}{n}$. Applying the rule for multiplying fractions, we have

$$\frac{3}{5} \cdot \frac{m}{n} = \frac{3m}{5n}.$$

The fraction $\frac{3m}{5n}$ equals 1 if the numerator and denominator are equal. One easy way to accomplish this is by letting $m = 5$ and $n = 3$:

$$\frac{3m}{5n} = \frac{3 \cdot 5}{5 \cdot 3} = \frac{3 \cdot 5}{3 \cdot 5} = 1.$$

So, $\frac{5}{3}$ is the reciprocal of $\frac{3}{5}$.

We also could have found the answer by applying the definition of division together with the rule for the product of reciprocals:

$$\frac{1}{\frac{3}{5}} = \frac{1}{3} \div \frac{1}{5} = \frac{1}{3} \cdot \frac{5}{1}.$$

Since the reciprocal of $\frac{3}{5}$ is 1, we have

$$\frac{1}{\frac{3}{5}} = \frac{1}{3} \cdot \frac{5}{1} = \frac{5}{3}.$$

(b) In part (a), we found that the reciprocal of $\frac{3}{5}$ is formed by swapping the numerator and denominator to get $\frac{5}{3}$. Maybe we can find the reciprocal of any fraction the same way! Let's check if the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$:

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = 1.$$

Since $\frac{b}{a} \cdot \frac{a}{b} = 1$, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

As in part (a), we could have applied the definition of division together with the rule for the product of reciprocals:

$$\frac{1}{\frac{a}{b}} = \frac{1}{a} \div \frac{1}{b} = \frac{1}{a} \cdot \frac{b}{1}.$$

Since the reciprocal of $\frac{a}{b}$ is 1, we have

$$\frac{1}{\frac{a}{b}} = \frac{1}{a} \cdot \frac{b}{1} = \frac{b}{a}.$$

Q

Important: If a and b are nonzero, then the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. In other words,

$$\frac{1}{\frac{a}{b}} = \frac{b}{a}.$$

Problem 4.18 Tag: Fractions

What is $3 \div \frac{5}{7}$?

Solution for Problem 4.18: The definition of division tells us that $3 \div \frac{5}{7}$ equals 3 times the reciprocal of $\frac{5}{7}$. The reciprocal of $\frac{5}{7}$ is $\frac{7}{5}$, so we have

$$3 \div \frac{5}{7} = \frac{3}{1} \cdot \frac{7}{5} = \frac{3 \cdot 7}{1 \cdot 5} = \frac{21}{5}.$$

Usually, we leave out the intermediate step $\frac{3}{1} \cdot \frac{7}{5}$ above and just write $3 \div \frac{5}{7} = \frac{3}{1} \cdot \frac{7}{5}$.

Problem 4.19 Tag: Fractions

What is $\frac{1}{2} \div \frac{1}{3}$?

Solution for Problem 4.19: Since the reciprocal of $\frac{1}{3}$ is $\frac{3}{1}$, we have

$$\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \cdot \frac{3}{1} = \frac{1 \cdot 3}{2 \cdot 1} = \frac{3}{2}.$$

Q

Similarly, we can now write a rule for dividing by a fraction:

Important: If b, c, and d are nonzero, then

$$\frac{b}{c} \div \frac{d}{e} = \frac{b}{c} \cdot \frac{e}{d}.$$

Fractions Dividing

Problem 4.20 Tag: Fractions

What is $\frac{12}{5} \div \frac{3}{10}$?

Solution for Problem 4.20: Here we have a fraction in which the numerator and the denominator are both fractions, all this means is that we are dividing the fraction in the numerator, $\frac{12}{5}$, by the fraction in the denominator, $\frac{3}{10}$. In order to divide by $\frac{3}{10}$, we need to find the reciprocal of a negative fraction. Fortunately, in section 1.4 here we learned the rule for the reciprocal of a negative:

$$\frac{1}{-\frac{a}{b}} = -\frac{b}{a}.$$

So, we have $\frac{1}{-\frac{3}{10}} = -\frac{10}{3}$. Now, we can perform our division:

$$\frac{12/5}{-3/10} = \frac{12}{5} \cdot \left(-\frac{10}{3}\right) = \frac{12}{5} \cdot \left(-\frac{10}{3}\right) = -\frac{12 \cdot 10}{5 \cdot 3} = -\frac{120}{15} = -8.$$

Since 14 is a multiple of 2 and 5 is a multiple of 3, we can simplify the final computation:

$$-\frac{12 \cdot 10}{5 \cdot 3} = -\frac{12 \cdot 2}{1 \cdot 3} = -\frac{12}{3} = -4 \cdot 3 = -12.$$

Q

Problem 4.21 Tag: Fractions

(a) $\frac{3}{4}$ of what number?

Solution for Problem 4.21: Since "$\frac{3}{4}$" means "multiply" the question asks, "(x) equals $\frac{3}{4}$ times what number?" This means we seek the number that fills the box in the equation

$$32 = \frac{3}{4} \cdot \boxed{}$$

If we still don't know what to do to fill in the box here, we can use a very powerful problem-solving strategy:

Concept: If you don't know how to solve a problem at first, try solving a simpler version of the problem.

It's probably the fraction that makes this problem difficult, so we think about how to solve a similar problem without a fraction. Imagine instead that the problem were to find the number that fills in the equation

$$32 = 4 \cdot \boxed{}$$

This is more familiar. We learned back in section 1.7 that we can think of division as the reverse of multiplication, and the number that fills this box is $32 \div 4 = 8$.

Let's return to the equation with the fraction:

$$32 = \frac{3}{4} \cdot \boxed{}$$

We also could have used some number sense to solve this problem. Since $\frac{3}{4}$ of a number is 32, we know that $\frac{1}{4}$ of that same number is half of 32, or 16. Since $\frac{1}{4}$ equals 16, the whole equals 4 times 16, or 64. We notice that in this solution, we divided 32 by 3 and then multiplied the result by 4. Compare these two steps to the steps we used to compute $32 \div \frac{3}{4}$ above—they're the same steps!

Problem 4.22 Tag: Fractions

Each panel of fencing material is $\frac{3}{8}$ feet long. How many panels do I need to build a 60-foot fence?

Solution for Problem 4.22: Once again, we can start by simplifying the problem to get a better handle on it. Suppose the problem were instead:

Each panel of fencing material is 1 feet long. How many panels do I need to build a 60-foot fence?

Now it's more clear that we have to divide. Each panel is 1 feet, so I need $60 \div 1 = 12$ panels to have all feet of fence.

Returning to our original problem, we see that if each panel is $\frac{3}{8}$ feet, then the number of panels needed for a 60-foot fence is

$$60 \div \frac{3}{8} = \frac{60}{1} \cdot \frac{8}{3} = \frac{60 \cdot 8}{1 \cdot 3} = \frac{480}{3} = 160 \text{ feet of fence } \frac{1}{2}$$

As a check, we confirm that 3 panels do indeed provide $3 \cdot \frac{3}{8} = \frac{3 \cdot 3}{1 \cdot 8} = \frac{9}{8} = 1\frac{1}{8}$ feet of fence. $\frac{1}{2}$

Exercises

4.3.1:

Evaluate each of the following:

(a) $\frac{1}{2} \div \frac{3}{2}$.

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\frac{1}{2} \div \frac{3}{2} = \frac{1}{2} \cdot \frac{2}{3} = \boxed{\frac{1}{3}}$

(b) $7 \div \frac{7}{8}$.

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $7 \div \frac{7}{8} = 7 \cdot \frac{8}{7} = \frac{7 \cdot 8}{7} = 8 = \boxed{8}$

(c) $\frac{11/3}{5/3}$.

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\frac{11/3}{5/3} = \frac{11}{3} \div \frac{5}{3} = \frac{11}{3} \cdot \frac{3}{5} = \frac{11}{3 \cdot 5} = \boxed{\frac{11}{5}}$

(d) $\left(-\frac{3}{4}\right) \div \left(-\frac{7}{2}\right)$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The quotient of two negative numbers is positive, so we have

$$\left(-\frac{3}{4}\right) \div \left(-\frac{7}{2}\right) = \frac{3}{4} \div \frac{7}{2} = \frac{3}{4} \cdot \frac{2}{7} = \frac{3 \cdot 2}{4 \cdot 7} = \boxed{\frac{3}{14}}$$

4.3.2:

What is the quotient when $\frac{3}{4}$ is divided by $\frac{7}{8}$?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: When $\frac{3}{4}$ is divided by $\frac{7}{8}$ the result is $\frac{3}{4} \div \frac{7}{8} = \frac{3}{4} \cdot \frac{8}{7} = \boxed{\frac{6}{7}}$

4.3.3:

If $x = \frac{7}{8}$ what is the value of $\frac{36}{x}$?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\frac{36}{x} = 36 \div x = 36 \div \frac{7}{8} = 36 \cdot \frac{8}{7} = \frac{36 \cdot 8}{7} = 4 \cdot 12 = 4 \cdot \boxed{12}$

4.3.4:

(a) $40 \div \frac{5}{8}$ of what number?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We must determine what number is in the blank in $\frac{5}{8} \cdot \boxed{} = 40$, so we divide: $40 \div \frac{5}{8} = 40 \cdot \frac{8}{5} = 8 \cdot 8 = \boxed{64}$

(b) $\frac{3}{4} \div \frac{9}{8}$ of what number?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We must determine what number is in the blank in $\frac{9}{8} \cdot \boxed{} = \frac{3}{4}$, so we divide: $\frac{3}{4} \div \frac{9}{8} = \frac{3}{4} \cdot \frac{8}{9} = \boxed{\frac{2}{3}}$

4.3.5:

(a) Dividing $\frac{1}{2}$ by what number gives $\frac{3}{4}$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: To get a feel for these questions, we first think about the problem without fractions. Suppose we are asked, "Dividing 6 by what number gives 2?" Then, the answer is 3, which is $6 \div 2$. Similarly, the answer to "Dividing 40 by what number is 8?" is 5, which is $40 \div 8$. So, it seems like the answer to the question,

"Dividing a by what number equals b?"

is $a \div b$. So, we expect that $x \div \frac{1}{2} = \frac{3}{4}$ equals 3. To see that this is correct, we write $x \div \frac{1}{2} = \frac{x}{1} \cdot \frac{2}{1}$ and we have

$$x \div \left(\frac{1}{2}\right) = x \cdot \frac{2}{1} = \frac{x}{1} \cdot \frac{2}{1} = \frac{x \cdot 2}{1 \cdot 1} = 2x.$$

So, indeed, $x \div \frac{1}{2}$ is the number that we must divide $\frac{1}{2}$ by to get $\frac{3}{4}$.

Using what we just learned, we must divide $\frac{1}{2} \div \frac{3}{4}$ to get x: $\frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}$. $\boxed{\frac{2}{3}}$ Checking our answer, we find that $\frac{2}{3} \div \frac{1}{2} = \frac{2}{3} \cdot \frac{2}{1} = \frac{4}{3}$ as expected!

(b) Dividing $\frac{3}{4}$ by what number gives $\frac{9}{8}$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Looking at the solution in part (a), and again using what we just learned, we must divide $\frac{3}{4} \div \frac{9}{8} = \frac{3}{4} \cdot \frac{8}{9} = \frac{3 \cdot 8}{4 \cdot 9} = \frac{3 \cdot 2}{1 \cdot 3} = \frac{6}{3} = 2$. Checking our answer, we find $\frac{3}{4} \div \frac{9}{8} = \frac{3}{4} \cdot \frac{8}{9} = \frac{3 \cdot 8}{4 \cdot 9} = \frac{3 \cdot 2}{1 \cdot 3} = \frac{6}{3} = 2$, as required.

(c) Dividing $\frac{3}{4}$ by what number gives $\frac{9}{8}$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We must divide $\frac{3}{4} \div \frac{9}{8} = \frac{3}{4} \cdot \frac{8}{9} = \frac{3 \cdot 8}{4 \cdot 9} = \frac{3 \cdot 2}{1 \cdot 3} = \boxed{\frac{2}{3}}$

4.3.6:

Dividing $\frac{1}{2}$ into what number gives a quotient of 20?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Once again, we get a handle on the problem by trying a problem without fractions. Suppose the problem were:

"Dividing 2 into what number gives a quotient of 20?"

Now, it's more clear what the answer is: we divide 2 into 40 to get a quotient of 20. Notice that $40 \div \frac{1}{2} = \frac{40}{1} \cdot \frac{2}{1}$. So, it looks like the answer to the question, "Dividing a into what number gives a quotient of b?" is simply ab. We can quickly check this by simplifying $(a \cdot b) \div a$:

$$(a \cdot b) \div a = (a \cdot b) \cdot \frac{1}{a} = \frac{a}{a} \cdot b.$$

This makes sense: if we divide the product of a and b by a, we get b. That's how division works, it undoes multiplication!

Returning to the original problem, to get a quotient of 20, we divide $\frac{1}{2}$: $20 \div \frac{1}{2} = \frac{20}{1} \cdot \frac{2}{1} = \boxed{40}$. Checking our answer, we have $12 \div \frac{1}{2} = \frac{12}{1} \cdot \frac{2}{1} = 6 \cdot 2 = 20$, as required.

4.3.7: Source: AMC 8

Multiplying a number by $\frac{3}{4}$ and then dividing the result by $\frac{3}{4}$ has the same effect as multiplying the original number by what number?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Suppose our original number is x. Multiplying x by $\frac{3}{4}$ gives $\left(x \cdot \frac{3}{4}\right)$. Dividing this number by $\frac{3}{4}$ gives

$$\left(x \cdot \frac{3}{4}\right) \div \frac{3}{4} = \left(x \cdot \frac{3}{4}\right) \cdot \frac{4}{3} = x \cdot \left(\frac{3}{4} \cdot \frac{4}{3}\right) = x \cdot \left(\frac{3 \cdot 4}{4 \cdot 3}\right) = x \cdot \left(\frac{12}{12}\right) = x.$$

So, the result is the same as multiplying the original number by $\boxed{1}$.

4.3.8:

I have a scoop that holds $\frac{1}{3}$ cup of flour. If my recipe calls for 5 cups of flour, how many scoops do I need?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: If we multiply the number of scoops by the size of the scoop, we get the total amount of flour:

$$(\text{Number of scoops}) \cdot (\text{Size of scoop}) = (\text{Total amount of flour}).$$

So, to find the number of scoops, we divide the total amount of flour by the size of each scoop. Therefore, I need $5 \div \frac{1}{3} = \frac{5}{1} \cdot \frac{3}{1} = \frac{5 \cdot 3}{1 \cdot 1} = \boxed{15}$ scoops.

4.4 Raising Fractions to Powers

Problems

Problem 4.23 — Jump to Solution

(a) Compute $\left(\frac{2}{5}\right)^3$.

(b) Let b be nonzero and let n be a positive integer. Explain why $\left(\frac{b}{1}\right)^n = \frac{b^n}{1^n}$.

Problem 4.24 — Jump to Solution

(a) Evaluate $\left(\frac{3}{5}\right)^{-2}$.

(b) Evaluate $\left(\frac{b}{1}\right)^{-n}$.

Problem 4.25 — Jump to Solution

Suppose a and b are nonzero, and n is a positive integer. Explain why

$$\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}.$$

Problem 4.26 — Jump to Solution

Compute $\frac{(25/31)^2(31/21)^2}{(21/31)^{-2}}$.

Problem 4.23

(a) Compute $\left(\frac{2}{5}\right)^3$.

(b) Let b be nonzero and let n be a positive integer. Explain why $\left(\frac{b}{1}\right)^n = \frac{b^n}{1^n}$.

Solution for Problem 4.23:

(a) We apply the rule for fraction multiplication:

$$\left(\frac{2}{5}\right)^3 = \frac{2}{5}\cdot\frac{2}{5}\cdot\frac{2}{5} = \frac{2\cdot2\cdot2}{5\cdot5\cdot5} = \frac{2\cdot2\cdot2}{5\cdot5\cdot5} = \frac{2^3}{5^3} = \frac{8}{125}.$$

(b) If n is a positive integer, then $\left(\frac{b}{1}\right)^n$ is a product of n copies of $\frac{b}{1}$. Applying the rule for multiplying fractions tells us that such a product equals $\frac{b^n}{1^n}$:

$$\left(\frac{b}{1}\right)^n = \underbrace{\frac{b}{1}\times\frac{b}{1}\times\cdots\times\frac{b}{1}}_{n \text{ copies}} = \frac{\overbrace{b\times b\times\cdots\times b}^{n \text{ copies}}}{\underbrace{1\times1\times\cdots\times1}_{n \text{ copies}}} = \frac{b^n}{1^n}.$$

□

> **Important:** If n is a positive integer and b is not 0, then
>
> $$\left(\frac{b}{1}\right)^n = \frac{b^n}{1^n}.$$

This isn't something new. It's just the quotient of powers rule ($x \div 1^n = x^n$ = x^n written using fractions.

Problem 4.24

(a) Evaluate $\left(\frac{3}{5}\right)^{-2}$.

(b) Evaluate $\left(\frac{b}{1}\right)^{-n}$.

Solution for Problem 4.24:

(a) Back in Section 3.4, we defined a^{-n} for any nonzero a to be the reciprocal of a. Since the reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$, we have

$$\left(\frac{3}{5}\right)^{-2} = \frac{5}{3}.$$

Similarly, we have:

> **Important:** If a and b are nonzero, then
>
> $$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}.$$

This is just another way to write our earlier rule for finding the reciprocal of a fraction.

(b) In Section 3.4, we also defined a^{-n} for any nonzero a and any positive integer n as follows:

$$a^{-n} = \frac{1}{a^{\underline{n}}}.$$

So, $\left(\frac{b}{1}\right)^{-n}$ equals the reciprocal of $\left(\frac{b}{1}\right)^n$. Applying the rule we discovered in this section here for raising a fraction to a positive integer power gives

$$\left(\frac{b}{1}\right)^{-n} = \frac{1}{(1/b)^n} = \frac{1}{b^n/1^n} = \frac{1^n}{b^n} = \frac{1^n}{b^n},$$

which we learned in Section 2.4 here. Applying this rule to $\left(\frac{b}{1}\right)^{-n}$, together with the rule for finding the reciprocal of a fraction, we have

$$\left(\frac{b}{1}\right)^{-n} = \left(\left(\frac{b}{1}\right)^{-1}\right)^n = \left(\frac{1}{b}\right)^n = \frac{1^n}{b^n} = \frac{27}{64}.$$

□

Our second solution to part (b) is the way we usually compute negative powers of fractions.

Problem 4.25

Suppose a and b are nonzero, and n is a positive integer. Explain why

$$\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}.$$

Solution for Problem 4.25: We have

$$\begin{aligned}\left(\frac{a}{b}\right)^{-n} &= \left(\left(\frac{a}{b}\right)^{-1}\right)^n && \text{power of a power}\\ &= \left(\frac{b}{a}\right)^n && \text{reciprocal of a fraction}\\ &= \frac{b^n}{a^n}, && \text{power of a fraction.}\end{aligned}$$

□

> **Important:** If a and b are nonzero and n is a positive integer, then
>
> $$\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}.$$

Rather than memorizing this as a separate formula, we usually think of this with the intermediate step

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}.$$

As an Exercise, you'll use $\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$ to explain why we can apply the same rule to negative powers of fractions that we use for positive powers of fractions:

> **Important:** If a and b are nonzero and n is any integer, then
>
> $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Problem 4.26

Compute $\frac{(25/31)^2(31/21)^2}{(21/31)^{-2}}$.

Solution for Problem 4.26: We could compute this the long way, multiplying out each of the powers of fractions. But instead, we notice that $\frac{31}{21}$ appears in both the numerator and the denominator. We can use an exponent law to simplify:

$$\begin{aligned}\frac{(25/31)^2(31/21)^2}{(21/31)^{-2}} &= \frac{(25/31)^2(31/21)^2}{(31/21)^2} && (21/31)^{-2}\\ &= (25/31)^2\cdot(31/21)^{2-2} && \text{(exponent law)}\\ &= (25/31)^2\cdot(31/21)^0.\end{aligned}$$

Here are two solutions from this point:

Solution 1: Use the rule for raising fractions to powers. We have

$$\left(\frac{25}{31}\right)^2\cdot\left(\frac{31}{21}\right)^0 = \frac{25^2}{31^2}\cdot\frac{31^0}{21^0} = \frac{25^2\cdot31^0}{31^2\cdot21^0}.$$

The numerator and denominator of the final fraction are equal, so the fraction equals 1.

Solution 2: Notice that the powers are the same. Since both fractions in $(25/31)^2\cdot(31/21)^0$ are raised to the same power, we apply an exponent law:

$$\left(\frac{25}{31}\right)^2\cdot\left(\frac{31}{21}\right)^0 = \left(\frac{25}{31}\cdot\frac{31}{21}\right)^2.$$

Since $\frac{31}{21}$ is the reciprocal of $\frac{21}{31}$, the product $\frac{25}{31}\cdot\frac{31}{21}$ equals 1. This means our expression equals 1^2, which is 1. □

Exercises

4.4.1:

Compute each of the following:

(a) $\left(\frac{2}{3}\right)^3$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: $\left(\frac{2}{3}\right) = \frac{2^3}{3^3} = \boxed{\frac{8}{27}}$

(b) $\left(-\frac{3}{7}\right)^2$.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Any number raised to the power 0 equals 1, so $\left(-\frac{3}{7}\right)^0 = \boxed{1}$.

(c) $\left(\frac{5}{8}\right)^{-3}$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution:

$$\left(\frac{5}{8}\right)^{-3} = \left(\left(\frac{5}{8}\right)^{-1}\right)^3 = \left(\frac{8}{5}\right)^3 = \frac{8^3}{5^3} = \boxed{\frac{512}{125}}$$

(d) $\left(-\frac{3}{7}\right)^{-2}$.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution:

$$\left(-\frac{3}{7}\right)^{-2} = \frac{(-3)^{-2}}{7^{-2}} = \frac{-243}{32} = \boxed{-\frac{243}{32}}$$

(e) $\left(\frac{1}{1.25}\right)^3$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Since $\left(\frac{1}{1}\right) = \frac{1^3}{1^3} = \frac{1}{1^3} = \frac{1}{1}$, we have $\frac{1}{1.25} = \frac{1}{1.125} = \frac{125}{1} = \boxed{125}$.

(f) $\frac{(3/8)^2}{(5/3)^2}$.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution:

$$\frac{(3/8)^2}{(5/3)^2} = \frac{3^2/8^2}{5^2/3^2} = \frac{9/64}{625/36} = \frac{9}{64}\cdot\frac{36}{625} = \frac{9}{64}\cdot\frac{36}{625} = \boxed{\frac{81}{1600}}$$

4.4.2:

(a) $\frac{3}{2}$ raised to what power equals $\frac{27}{64}$?

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We might recognize 27 as 3^3 and 64 as 2^6, but if we don't, we can simply compute a few powers of $\frac{3}{2}$ and hope we get lucky. We have $\left(\frac{3}{2}\right)^1 = \frac{3}{2}$, $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$, and $\left(\frac{3}{2}\right)^3 = \frac{27}{8}$. So, we raise $\frac{3}{2}$ to the power $\boxed{3}$ to get $\frac{27}{64}$.

(b) $\frac{5}{3}$ raised to what power equals $\frac{16}{9}$?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We see that 9 is 3^2 and 16 is 4^2, so $\frac{16}{9} = \frac{4^2}{3^2} = \left(\frac{4}{3}\right)^2$. But we need $\frac{16}{9}$ as a power of $\frac{5}{3}$ not as a power of $\frac{4}{3}$. Since $\frac{4}{3}$ is the reciprocal of $\frac{3}{4}$ we can use negative exponents:

$$\left(\frac{4}{3}\right)^2 = \left(\left(\frac{3}{4}\right)^{-1}\right)^2 = \left(\frac{3}{4}\right)^{-2}.$$

Therefore, we raise $\frac{5}{3}$ to the power $\boxed{-2}$ to get $\frac{16}{9}$.

4.4.3:

Compute $\frac{(2/16411)^{-2}}{(3/16411)^{-2}}$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution 1: Raise both fractions to the 4th power. It's a little scary to raise 16411 to the 4th power, but maybe we'll get lucky and not have to. We have

$$\frac{(2/16411)^{-2}}{(3/16411)^{-2}} = \frac{2^{-2}/16411^{-2}}{3^{-2}/16411^{-2}} = \frac{2^{-2}}{3^{-2}} = \frac{(16411^4)^{-2}}{(16411^4)^{-2}} = \frac{2^{-2}}{3^{-2}}.$$

Power Since the numerator and denominator $\frac{16411^4}{16411^4}$ are the same, we can apply the exponent laws $\frac{16411^4}{16411^4} = 1$. So, we have $\frac{2^{-2}}{3^{-2}} = \frac{16411^4}{16411^4} = \frac{2^{-2}}{3^{-2}} = \boxed{\frac{9}{4}}$.

Solution 2: Use the fact that $\frac{a^{-n}}{b^{-n}} = \left(\frac{b}{a}\right)^n$. The hard here is that a and b themselves are fractions! We have

$$\frac{(2/16411)^{-2}}{(3/16411)^{-2}} = \left(\frac{16411/2}{16411/3}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}.$$

Conveniently, once again the 16411's cancel out and we have $\left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \boxed{\frac{9}{4}}$.

4.4.4:

Compute $\frac{(5/8)^{-3}(5/3)^2}{(5/3)^{-1}}$.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: All three fractions are the same, so we can apply the exponent laws $a^{-n} \cdot a^{-m} = a^{-m-n}$ and $a^n/a^m = a^{n-m}$:

$$\frac{(5/8)^{-3}(5/3)^2}{(5/3)^{-1}} = \frac{(5/8)^{-3}\cdot(5/3)^2}{(5/3)^{-1}} = \frac{(5/8)^{-3}}{(5/3)^{-3}} = (5/8)^{-3}\cdot(5/3)^3 = (5/8)^3 = \boxed{\left(\frac{35}{8}\right)^3}$$

4.4.5:

Compute $\left(\frac{3}{4}\right)^{-1}\left(\frac{4}{3}\right)^{-1}\left(\frac{3}{4}\right)^{-1}$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: One of the fractions is not the same as the other two, so it looks like we can't use the same strategy as the previous problem. However, $\frac{3}{4}$ is the reciprocal of $\frac{4}{3}$ so we can use negative exponents! Since $\frac{3}{4} = \left(\frac{4}{3}\right)^{-1}$, we have $\left(\frac{3}{4}\right)^{-1} = \left(\left(\frac{4}{3}\right)^{-1}\right)^{-1} = \left(\frac{4}{3}\right)^{-1}$. Then, we have

$$\begin{aligned}\left(\frac{3}{4}\right)^{-1}\left(\frac{4}{3}\right)^{-1}\left(\frac{3}{4}\right)^{-1} &= \left(\frac{4}{3}\right)^{-1}\left(\frac{4}{3}\right)^{-1}\left(\frac{4}{3}\right)^{-1}\\ &= \left(\frac{4}{3}\right)^{-3} && \text{(exp. law)}\\ &= \left(\frac{3}{4}\right)^{3}\\ &= \boxed{\frac{27}{64}}\end{aligned}$$

4.4.6★:

Suppose a and b are nonzero, and n is a positive integer. In the text, we showed that $\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$. Explain why

$$\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}}$$

by showing that $\frac{a^{-n}}{b^{-n}}$ also equals $\frac{b^n}{a^n}$.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We have

$$\begin{aligned}\frac{a^{-n}}{b^{-n}} &= a^{-n}\cdot b^{-n} && \text{definition of fraction}\\ &= \frac{1}{a^n}\cdot\frac{1}{b^n} && \text{negative in exponent}\\ &= \frac{1}{a^n}\cdot b^n && \text{division by a fraction}\\ &= \frac{b^n}{a^n}. && \text{product of fractions}\end{aligned}$$

Since $\left(\frac{a}{b}\right)^{-n}$ and $\frac{a^{-n}}{b^{-n}}$ both equal $\frac{b^n}{a^n}$, we must have $\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}}$.

4.5 Simplest Form of a Fraction

A fraction $\frac{a}{b}$ is in **simplest form** if its numerator and denominator have no positive common divisor besides 1. For example, the numerator and the denominator of $\frac{5}{8}$ have 2 as a common divisor, so $\frac{5}{8}$ is not in simplest form. The only positive common divisor of the numerator and denominator of $\frac{3}{8}$ is 1, so $\frac{3}{8}$ is in simplest form. We saw back in Problem 4.4 that the fractions $\frac{3}{8}$ and $\frac{5}{8}$ are equal. So, we can write $\frac{5}{8}$ in simplest form as $\frac{3}{8}$.

Problems

Problem 4.27 ↩ Jump to Solution

Write $\frac{12}{18}$ in simplest form.

Problem 4.28 ↩ Jump to Solution

One of the properties of division that we learned in Section 1.7 was the "cancel common factor" property: if b and c are not zero, then

$$(ma) \div (mb) = a \div b.$$

Use fraction multiplication to explain why the cancel common factor property works.

Problem 4.29 ↩ Jump to Solution

Write $\frac{105}{210}$ in simplest form.

Problem 4.30 ↩ Jump to Solution

Compute each of the following in simplest form:

(a) $\frac{6}{9} \cdot \frac{12}{8}$

(b) $\left(\frac{2}{3}\right) \cdot \left(\frac{9}{10}\right)$

(c) $\left(\frac{2}{3}\right) \div \left(\frac{10}{9}\right)$

(d) $\frac{56}{21} \div \frac{14}{15}$

(e) $\frac{56}{21} \div \frac{15}{14}$

Problem 4.31 ↩ Jump to Solution

What fraction of 96 is 64? Answer as a fraction in simplest form.

Problem 4.32 ↩ Jump to Solution

(a) Simplify $\frac{2}{3} \cdot \frac{3}{2}$.

(b) Simplify $\frac{2}{3} \div \frac{3}{2}$.

(c) Suppose x is nonzero. Simplify $\frac{2x^2}{3x^2}$.

Problem 4.33 ↩ Jump to Solution

If x and y are not zero, then simplify $\frac{49x^3y^2}{7x^2y^5}$.

Problem 4.27 🔖 ✓

Write $\frac{12}{18}$ in simplest form.

Solution for Problem 4.27: Since 12 and 18 have 6 as a common divisor, we have

$$\frac{12}{18} = \frac{2 \cdot 6}{3 \cdot 6} = \frac{2}{3}.$$

2 and 3 have only 1 as a positive common divisor, so $\frac{2}{3}$ is in simplest form. ▪

Fraction Simplification

Problem 4.28 🔖 ✓

One of the properties of division that we learned in Section 1.7 was the "cancel common factor" property: if b and c are not zero, then

$$(ma) \div (mb) = a \div b.$$

Use fraction multiplication to explain why the cancel common factor property works.

Solution for Problem 4.28: We start by writing $(ma) \div (bc)$ in the fraction $\frac{ma}{bc}$. Even our discussion of multiplying fractions, we know that $\frac{ma}{bc} = \frac{m}{b} \cdot \frac{a}{c}$. Since $\frac{m}{m} = 1$ we have $\frac{m}{b} = \frac{m}{b} \cdot \frac{m}{m} = \frac{m^2}{bm}$. So we have $(ma) \div (bc) = m \cdot a \div b \cdot c$.

When the numerator and the denominator of a fraction have a positive common factor besides 1, we can use the fact that $\frac{mx}{my} = \frac{x}{y}$ to simplify the fraction.

Problem 4.29 🔖 ✓

Write $\frac{105}{210}$ in simplest form.

Solution for Problem 4.29: 5 is a common divisor of 105 and 140, and we have $\frac{105}{210} = \frac{21 \cdot 5}{42 \cdot 5} = \frac{21}{42}$. So $\frac{105}{210}$ is not in simplest form because 40 and 108 are both divisible by 9:

$$\frac{21}{42} = \frac{3 \cdot 7}{6 \cdot 7} = \frac{3}{6}.$$

The only positive common divisor of 3 and 12 is 1, so $\frac{3}{6}$ is in simplest form.

We also could have used the prime factorizations of 105 and 140 to help us find common divisors. Since $105 = 3^1 \cdot 5^1$ and $140 = 2^2 \cdot 7^1$, 5 is a common divisor. We can write the prime factors of 21 and 42 as factors of 5 in common, so

$$\frac{105}{210} = \frac{3^1 \cdot 5^1 \cdot 7^1}{2^1 \cdot 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1} = \frac{3^1 \cdot 7^1}{2^1 \cdot 2^1 \cdot 3^1} = \frac{3^1}{2^1 \cdot 2^1} = \frac{3}{4}.$$

▪

Concept: We can use the prime factorizations of the numerator and denominator of a fraction to simplify the fraction.

Fraction Calculations

Problem 4.30 🔖 ✓

Compute each of the following in simplest form:

(a) $\frac{6}{9} \cdot \frac{12}{8}$

(b) $\left(\frac{2}{3}\right) \cdot \left(\frac{9}{10}\right)$

(c) $\left(\frac{2}{3}\right) \div \left(\frac{10}{9}\right)$

(d) $\frac{56}{21} \div \frac{14}{15}$

(e) $\frac{56}{21} \div \frac{15}{14}$

Solution for Problem 4.30:

(a) We have

$$\frac{6}{9} \cdot \frac{12}{8} = \frac{2 \cdot 3}{3 \cdot 3} \cdot \frac{3 \cdot 4}{2 \cdot 4}.$$

We also could have simplified $\frac{6}{9}$ and $\frac{12}{8}$ before multiplying. Since $\frac{6}{9} = \frac{2 \cdot 3}{3 \cdot 3} = \frac{2}{3}$ and $\frac{12}{8} = \frac{3 \cdot 4}{2 \cdot 4} = \frac{3}{2}$, we have

$$\frac{6}{9} \cdot \frac{12}{8} = \frac{2}{3} \cdot \frac{3}{2}.$$

(b) The product of two negative numbers is positive, we have

$$\left(\frac{2}{3}\right) \cdot \left(\frac{9}{10}\right) = \frac{2 \cdot 9}{3 \cdot 10} = \frac{2 \cdot 3 \cdot 3}{3 \cdot 2 \cdot 5}.$$

We still have to compute $\frac{2 \cdot 9}{3 \cdot 10}$. We could multiply out $2 \cdot 9$ and $3 \cdot 10$, and then look for common factors. However, we can save a lot of time by simplifying the two fractions before multiplying. We have $\frac{2}{3} = \frac{2 \cdot 3}{3 \cdot 1}$ and $\frac{9}{10} = \frac{3 \cdot 3}{2 \cdot 5}$, so

$$\frac{2 \cdot 9}{3 \cdot 10} = \frac{2 \cdot 3 \cdot 3}{3 \cdot 2 \cdot 5}.$$

We say that we **cancel** a common divisor of 4 in the final step above. We sometimes express this cancellation with a slash through the numbers being cancelled:

$$\frac{2 \cdot 3 \cdot 3}{3 \cdot 2 \cdot 5} = \frac{\cancel{2} \cdot \cancel{3} \cdot 3}{\cancel{3} \cdot \cancel{2} \cdot 5}.$$

(c) The product of a positive number and a negative number is negative, we have

$$\left(\frac{2}{3}\right) \div \left(\frac{10}{9}\right) = \frac{2}{3} \cdot \left(-\frac{9}{10}\right) = -\frac{2 \cdot 9}{3 \cdot 10}.$$

We still must compute $\frac{2 \cdot 9}{3 \cdot 10}$. We can't simplify either fraction. However, the numerator of each fraction has divisors in common with the denominator of the other fraction. We can perform a clever manipulation to allow us to take advantage of these common factors:

$$-\left(\frac{2 \cdot 9}{3 \cdot 10}\right) = -\left(\frac{2 \cdot 9}{3 \cdot 1}\right) \cdot \left(\frac{1}{10}\right) = -\left(\frac{2 \cdot 3 \cdot 3}{3 \cdot 1}\right) \cdot \left(\frac{1}{2 \cdot 5}\right)$$

$$= -\left(\frac{\cancel{2} \cdot \cancel{3} \cdot 3}{\cancel{3} \cdot 1}\right) \cdot \left(\frac{1}{\cancel{2} \cdot 5}\right) = -\left(\frac{3}{1}\right) \cdot \left(\frac{1}{5}\right) = -\frac{3}{5}.$$

Since $\frac{10}{10} = \frac{10}{10} = \frac{10}{10} = 1$, we have

$$-\left(\frac{2 \cdot 9}{3 \cdot 10}\right) \cdot \left(\frac{10}{10}\right) = -

4.6 Comparing Fractions

In this section, we reverse our fraction simplification strategies from the previous section to compare values of fractions.

Problems

Problem 4.34 Jump to Solution

Which is greater, $\frac{3}{5}$ or $\frac{4}{5}$?

Problem 4.35 Jump to Solution

In this problem, we determine which is greater, $\frac{3}{5}$ or $\frac{7}{10}$.

(a) What fraction with denominator 10 is equal to $\frac{3}{5}$?

(b) Which is greater, $\frac{3}{5}$ or $\frac{7}{10}$?

Problem 4.36 Jump to Solution

(a) Which is greater, $\frac{5}{11}$ or $\frac{8}{11}$?

(b) Which is greater, $\frac{5}{6}$ or $\frac{7}{9}$?

Problem 4.37 Jump to Solution

Which is greater, $\frac{2}{7}$ or $\frac{3}{7}$?

Problem 4.38 Jump to Solution

Which is greater, $-\frac{17}{14}$ or $-\frac{41}{35}$?

Problem 4.39 Jump to Solution

Which is greater, $\frac{541}{539}$ or $\frac{399}{401}$?

Problem 4.34

Which is greater, $\frac{4}{5}$ or $\frac{3}{5}$?

Solution for Problem 4.34: We can see that $\frac{4}{5}$ is greater than $\frac{3}{5}$ by considering the number line. To locate both fractions on the number line, we divide the number line between 0 and 1 into 5 equal pieces. Since $\frac{3}{5}$ is at the right end of the 3rd piece and $\frac{4}{5}$ is at the right end of the 4th piece, we conclude that $\frac{4}{5}$ is to the right of $\frac{3}{5}$.

Therefore, $\frac{4}{5}$ is greater than $\frac{3}{5}$.

In general, if two fractions have the same positive denominator, then the fraction with the greater numerator is the greater of the two fractions.

Problem 4.35

Which is greater, $\frac{3}{5}$ or $\frac{7}{10}$?

Solution for Problem 4.35: We know how to compare two fractions that have the same denominator, so it would be convenient if we could express the two fractions with the same denominator. To do so, we use our fraction simplification strategy from Section 4.3 in reverse. Multiplying $\frac{3}{5}$ by 1 doesn't change the value of the fraction, so we multiply $\frac{3}{5}$ by $\frac{2}{2}$ which equals 1, to express $\frac{3}{5}$ with 10 as the denominator:

$$\frac{3}{5} = \frac{3}{5} \cdot \frac{2}{2} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10}.$$

Since 2 is greater than 6, we know that $\frac{7}{10}$ is greater than $\frac{6}{10}$, so $\frac{7}{10}$ is greater than $\frac{3}{5}$.

We often leave out the "multiplying by 1" step when writing a fraction with a new denominator. Instead, we go straight to multiplying the numerator and denominator by the same factor:

$$\frac{3}{5} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10}.$$

When we write multiple fractions with the same denominator, we say that we write the fractions with a **common denominator**.

Problem 4.36

(a) Which is greater, $\frac{5}{11}$ or $\frac{8}{11}$?

(b) Which is greater, $\frac{5}{6}$ or $\frac{7}{9}$?

Solution for Problem 4.36:

(a) We know how to compare fractions that have the same denominator. Unfortunately, neither of our denominators in this part is a multiple of the other. So, we change both denominators to make them the same. When writing two fractions with a common denominator, we can always use the product of the fractions' denominators as the common denominator. Writing each fraction with $7 \cdot 11$ as the denominator gives

$$\frac{5}{7} = \frac{5 \cdot 11}{7 \cdot 11} = \frac{55}{77}, \qquad \frac{8}{11} = \frac{8 \cdot 7}{11 \cdot 7} = \frac{56}{77}.$$

Since $\frac{56}{77}$ is greater than $\frac{55}{77}$, we know that $\frac{8}{11}$ is greater than $\frac{5}{7}$.

(b) Once again, we can use the product of the denominators as our common denominator. Writing each fraction with $6 \cdot 9$ as the denominator gives

$$\frac{5}{6} = \frac{5 \cdot 9}{6 \cdot 9} = \frac{45}{54}, \qquad \frac{7}{9} = \frac{7 \cdot 6}{9 \cdot 6} = \frac{42}{54}.$$

Since $\frac{45}{54}$ is greater than $\frac{42}{54}$, we know that $\frac{5}{6}$ is greater than $\frac{7}{9}$.

Thinking back to Section 3.5, we realize that we could have used a smaller common denominator. The common denominator must be a multiple of both 6 and 9. The least common multiple of 6 and 9 is 18, so we can write both $\frac{5}{6}$ and $\frac{7}{9}$ with a denominator of 18:

$$\frac{5}{6} = \frac{5 \cdot 3}{6 \cdot 3} = \frac{15}{18}, \qquad \frac{7}{9} = \frac{7 \cdot 2}{9 \cdot 2} = \frac{14}{18}.$$

Once again, we see that $\frac{5}{6}$ is greater than $\frac{7}{9}$.

+

The **least common denominator** of two or more fractions is the least common multiple of their denominators. In part (a) of Problem 4.36, the least common denominator is 77, while in part (b), the least common denominator is 18.

Problem 4.37

Which is greater, $\frac{2}{7}$ or $\frac{3}{7}$?

Solution for Problem 4.37: The least common denominator of $\frac{2}{7}$ and $\frac{3}{7}$ is 35. Writing both fractions with this denominator gives

$$\frac{2}{7} = \frac{2 \cdot 5}{7 \cdot 5} = \frac{10}{35}, \qquad \frac{3}{5} = \frac{2 \cdot 7}{5 \cdot 7} = \frac{14}{35},$$

so we see that $\frac{3}{7}$ is greater than $\frac{2}{7}$.

We also could have considered the number line. To locate $\frac{2}{7}$ on the number line, we split the number line between 0 and 1 into 7 equal pieces, and $\frac{2}{7}$ is at the right end of the 2nd piece. Similarly, to locate $\frac{3}{7}$ on the number line, we split the number line between 0 and 1 into 5 equal pieces, and $\frac{3}{7}$ is at the right end of the 2nd piece. We make larger pieces when we make 5 pieces than we do when we make 7 pieces, so $\frac{3}{7}$ is farther from 0 than $\frac{2}{7}$ is from 0. Therefore, $\frac{3}{7}$ is greater than $\frac{2}{7}$.

In general, if two positive fractions have the same numerator, then the fraction with the smaller denominator is the greater fraction.

Comparing Fractions Part 1

Problem 4.38

Which is greater, $-\frac{17}{14}$ or $-\frac{41}{35}$?

Solution for Problem 4.38: Before we even start thinking about common denominators, we note that the numbers in this problem are negative. So, we first have to think about how comparing negative numbers is different from comparing positive numbers. When comparing two positive numbers, the number that is farthest from 0 is the greater number. When comparing two negative numbers, this is reversed! The number closest to 0 is the greater number. For example, -2 is greater than -10.

Now, we're ready to compare the fractions by finding a common denominator. The least common multiple of 14 and 35 is 70, so we write each fraction with 70 as the denominator:

$$-\frac{17}{14} = \frac{-17 \cdot 5}{14 \cdot 5} = \frac{-85}{70}, \qquad -\frac{41}{35} = \frac{-41 \cdot 2}{35 \cdot 2} = \frac{-82}{70}.$$

Since $-\frac{82}{70}$ is closer to 0, it is the greater of these two negative numbers. Therefore, $-\frac{41}{35}$ is greater than $-\frac{17}{14}$.

Problem 4.39

Which is greater, $\frac{541}{539}$ or $\frac{399}{401}$?

Solution for Problem 4.39: Writing these fractions with a common denominator looks like a pain. Fortunately, we don't have to! Since the numerator of $\frac{541}{539}$ is greater than its denominator, the fraction $\frac{541}{539}$ is greater than 1. Meanwhile, the numerator of $\frac{399}{401}$ is less than its denominator, so $\frac{399}{401}$ is less than 1. Putting these together, $\frac{541}{539}$ must be greater than $\frac{399}{401}$.

Comparing Fractions Part 2

Exercises

4.6.1:

In each of the following pairs of numbers, which number is smaller?

(a) $\frac{3}{4}, \frac{7}{5}$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Writing the two fractions with a common denominator gives us $\frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{15}{20}$ and $\frac{7}{5} = \frac{7 \cdot 4}{5 \cdot 4} = \frac{14}{20}$. Since 14 is smaller than 15, we know that $\frac{14}{20}$ is smaller than $\frac{15}{20}$. This means $\boxed{\frac{7}{5}}$ is the smaller fraction.

(b) $\frac{2}{3}, \frac{3}{4}$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Every negative number is less than 0, and 0 is less than every positive number. So, every negative number is less than every positive number, which means that $\boxed{-\frac{2}{3}}$ is the smaller number.

(c) $-\frac{2}{3}, -\frac{3}{5}$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Since the numbers are negative, the number that is farther from 0 on the number line is the lesser number. To locate $-\frac{2}{3}$ on the number line, we go two steps of length $\frac{1}{3}$ to the left of 0. To get to $-\frac{3}{5}$, we must take one more step leftward of length $\frac{1}{5}$. So, $-\frac{2}{3}$ is to the left of $-\frac{3}{5}$ on the number line, which means that $\boxed{-\frac{3}{5}}$ is the smaller number.

4.6.2:

For each of the following lists of numbers, arrange the numbers in decreasing order (from largest to smallest).

(a) $\frac{1}{2}, \frac{3}{4}, \frac{5}{12}$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We write all three fractions with a common denominator. The least common multiple of 2, 4, and 12 is 12. We have $\frac{1}{2} = \frac{1 \cdot 6}{2 \cdot 6} = \frac{6}{12}$, $\frac{3}{4} = \frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12}$, and $\frac{5}{12} = \frac{5}{12}$. So the numbers from largest to smallest are $\boxed{\frac{3}{4}, \frac{1}{2}, \frac{5}{12}}$.

(b) $\frac{3}{2}, \frac{5}{4}, \frac{5}{3}$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: The least common denominator of these three fractions is 24. We have $\frac{3}{2} = \frac{3 \cdot 12}{2 \cdot 12} = \frac{36}{24}$, $\frac{5}{4} = \frac{5 \cdot 6}{4 \cdot 6} = \frac{30}{24}$, and $\frac{5}{3} = \frac{5 \cdot 8}{3 \cdot 8} = \frac{40}{24}$, so the numbers from largest to smallest are $\boxed{\frac{5}{3}, \frac{3}{2}, \frac{5}{4}}$.

(c) $-\frac{5}{4}, -\frac{5}{2}, -\frac{13}{3}$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Since $\frac{5}{2}$ is the only positive number in the list, it is the largest number. To compare $-\frac{5}{4}$ and $-\frac{13}{3}$, we can think about what pair of consecutive integers each is between. Since 5 is between 4 and 8, we know that $\frac{5}{4}$ is between $\frac{4}{4} = 1$ and $\frac{8}{4} = 2$. So, $-\frac{5}{4}$ is between -2 and -1, which means $-\frac{5}{4}$ is greater than -3. Similarly, since 13 is between 12 and 15, we know that $\frac{13}{3}$ is between $\frac{12}{3} = 4$ and $\frac{15}{3} = 5$. This means that $-\frac{13}{3}$ is between -5 and -4, which means $-\frac{13}{3}$ is less than -3. Therefore, from largest to smallest, the numbers are $\boxed{-\frac{5}{4}, -3, -\frac{13}{3}}$.

4.6.3: Source: AMC 8

Which one of these numbers is less than its reciprocal?

$$-2, \quad -1, \quad 0, \quad 1, \quad 2$$

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: The reciprocal of -2 is $-\frac{1}{2}$, which equals $-\frac{1}{2}$. Since -2 is less than -1 and $-\frac{1}{2}$ is greater than -1 (because $-\frac{1}{2}$ is between -1 and 0), we know that $\boxed{-2}$ is less than $-\frac{1}{2}$.

The reciprocal of -1 is $\frac{1}{-1} = -1$, which is not greater than -1. The number 0 does not have a reciprocal. The reciprocal of 1 is 1, which is not greater than 1. The reciprocal of 2 is $\frac{1}{2}$, which is less than 2. So, none of the other numbers in the list is less than its reciprocal.

4.6.4:

Which number is greater, $\frac{3}{2011}$ or $\frac{3}{2012}$?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Writing the two fractions with a common denominator gives $\frac{3}{2011} = \frac{3 \cdot 2012}{2011 \cdot 2012}$ and $\frac{3}{2012} = \frac{3 \cdot 2011}{2011 \cdot 2012}$. Since $3 \cdot 2012$ is greater than $3 \cdot 2011$, we see that $\boxed{\frac{3}{2011}}$ is greater than $\frac{3}{2012}$.

We might also have thought about the number line. To locate $\frac{3}{2011}$ on the number line, we split the number line between 0 and 1 into 2011 equal pieces, and $\frac{3}{2011}$ is at the right end of the 3rd piece. Similarly, to locate $\frac{3}{2012}$ we split the number line between 0 and 1 into 2012 equal pieces, and $\frac{3}{2012}$ is at the right end of the 3rd piece. We make smaller pieces when we make 2012 pieces than we do when we make 2011 pieces. Therefore, $\frac{3}{2012}$ is not as far from 0 as $\frac{3}{2011}$, which means $\boxed{\frac{3}{2011}}$ is greater than $\frac{3}{2012}$.

4.6.5:

Which number is greater, $\frac{19}{30}$ or $\frac{22}{35}$?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Since the least common multiple of 30 and 35 is 210, the least common denominator of $\frac{19}{30}$ and $\frac{22}{35}$ is 210. We have $\frac{19}{30} = \frac{19 \cdot 7}{30 \cdot 7} = \frac{133}{210}$ and $\frac{22}{35} = \frac{22 \cdot 6}{35 \cdot 6} = \frac{132}{210}$. So $\boxed{\frac{19}{30}}$ is the greater fraction.

4.6.6:

Which number is greater, $\frac{506}{101}$ or $\frac{509}{102}$?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We see that the numerators of the fractions are close to 5 times the denominators. So, we compare each fraction to 5. Since $\frac{505}{101} = 5$, we see that $\frac{506}{101}$ is greater than 5. Since $\frac{510}{102} = 5$, we see that $\frac{509}{102}$ is less than 5. Combining these, we know that $\boxed{\frac{506}{101}}$ is greater than $\frac{509}{102}$.

4.7 Adding and Subtracting Fractions

Sometimes we need to add or subtract fractions that are not in simplest form. We need them to possess a common denominator to work with them. Suppose we want to add $\frac{1}{2}+\frac{1}{3}$. We can change the fraction $\frac{1}{2}$ to an equivalent fraction with denominator 6 by multiplying both the numerator and denominator by 3. We can change the fraction $\frac{1}{3}$ to an equivalent fraction with denominator 6 by multiplying both the numerator and denominator by 2. In one step, this is:

$$\frac{1}{2}+\frac{1}{3}=\frac{1\cdot 3}{2\cdot 3}+\frac{1\cdot 2}{3\cdot 2}$$

Writing these intermediate steps in a fraction, we have the following:

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{3+2}{6}=\frac{5}{6}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator.

Add each of the following:

Problem 1

Find the sum of the following in simplest form:

$$\frac{1}{4}+\frac{2}{4}$$

$$\frac{1}{4}+\frac{2}{4}=\frac{1+2}{4}=\frac{3}{4}$$

Problem 2

Find the sum of the following in simplest form:

$$\frac{1}{5}+\frac{3}{5}$$

$$\frac{1}{5}+\frac{3}{5}=\frac{1+3}{5}=\frac{4}{5}$$

Problem 3

Find the sum of the following in simplest form:

$$\frac{1}{6}+\frac{4}{6}$$

$$\frac{1}{6}+\frac{4}{6}=\frac{1+4}{6}=\frac{5}{6}$$

Problem 4

Find the sum of the following in simplest form:

$$\frac{1}{7}+\frac{5}{7}$$

$$\frac{1}{7}+\frac{5}{7}=\frac{1+5}{7}=\frac{6}{7}$$

Problem 5

Find the sum of the following in simplest form:

$$\frac{1}{8}+\frac{6}{8}$$

$$\frac{1}{8}+\frac{6}{8}=\frac{1+6}{8}=\frac{7}{8}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 8 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 8 is the common denominator.

Problem 6

Find the sum of the following in simplest form:

$$\frac{1}{9}+\frac{2}{9}$$

$$\frac{1}{9}+\frac{2}{9}=\frac{1+2}{9}=\frac{3}{9}=\frac{1}{3}$$

Problem 7

Find the sum of the following in simplest form:

$$\frac{1}{10}+\frac{3}{10}$$

$$\frac{1}{10}+\frac{3}{10}=\frac{1+3}{10}=\frac{4}{10}=\frac{2}{5}$$

Problem 8

Find the sum of the following in simplest form:

$$\frac{1}{11}+\frac{4}{11}$$

$$\frac{1}{11}+\frac{4}{11}=\frac{1+4}{11}=\frac{5}{11}$$

Problem 9

Find the sum of the following in simplest form:

$$\frac{1}{12}+\frac{5}{12}$$

$$\frac{1}{12}+\frac{5}{12}=\frac{1+5}{12}=\frac{6}{12}=\frac{1}{2}$$

Problem 10

Find the sum of the following in simplest form:

$$\frac{1}{13}+\frac{6}{13}$$

$$\frac{1}{13}+\frac{6}{13}=\frac{1+6}{13}=\frac{7}{13}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 13 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 13 is the common denominator.

Problem 11

Find the sum of the following in simplest form:

$$\frac{1}{14}+\frac{3}{14}$$

$$\frac{1}{14}+\frac{3}{14}=\frac{1+3}{14}=\frac{4}{14}=\frac{2}{7}$$

Problem 12

Find the sum of the following in simplest form:

$$\frac{1}{15}+\frac{4}{15}$$

$$\frac{1}{15}+\frac{4}{15}=\frac{1+4}{15}=\frac{5}{15}=\frac{1}{3}$$

Problem 13

Find the sum of the following in simplest form:

$$\frac{1}{16}+\frac{5}{16}$$

$$\frac{1}{16}+\frac{5}{16}=\frac{1+5}{16}=\frac{6}{16}=\frac{3}{8}$$

Problem 14

Find the sum of the following in simplest form:

$$\frac{1}{17}+\frac{6}{17}$$

$$\frac{1}{17}+\frac{6}{17}=\frac{1+6}{17}=\frac{7}{17}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 17 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 17 is the common denominator.

Problem 15

Find the sum of the following in simplest form:

$$\frac{1}{18}+\frac{3}{18}$$

$$\frac{1}{18}+\frac{3}{18}=\frac{1+3}{18}=\frac{4}{18}=\frac{2}{9}$$

Problem 16

Find the sum of the following in simplest form:

$$\frac{1}{19}+\frac{4}{19}$$

$$\frac{1}{19}+\frac{4}{19}=\frac{1+4}{19}=\frac{5}{19}$$

Problem 17

Find the sum of the following in simplest form:

$$\frac{1}{20}+\frac{5}{20}$$

$$\frac{1}{20}+\frac{5}{20}=\frac{1+5}{20}=\frac{6}{20}=\frac{3}{10}$$

Problem 18

Find the sum of the following in simplest form:

$$\frac{1}{21}+\frac{6}{21}$$

$$\frac{1}{21}+\frac{6}{21}=\frac{1+6}{21}=\frac{7}{21}=\frac{1}{3}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 21 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 21 is the common denominator.

Problem 19

Find the sum of the following in simplest form:

$$\frac{1}{22}+\frac{3}{22}$$

$$\frac{1}{22}+\frac{3}{22}=\frac{1+3}{22}=\frac{4}{22}=\frac{2}{11}$$

Problem 20

Find the sum of the following in simplest form:

$$\frac{1}{23}+\frac{4}{23}$$

$$\frac{1}{23}+\frac{4}{23}=\frac{1+4}{23}=\frac{5}{23}$$

Problem 21

Find the sum of the following in simplest form:

$$\frac{1}{24}+\frac{5}{24}$$

$$\frac{1}{24}+\frac{5}{24}=\frac{1+5}{24}=\frac{6}{24}=\frac{1}{4}$$

Problem 22

Find the sum of the following in simplest form:

$$\frac{1}{25}+\frac{6}{25}$$

$$\frac{1}{25}+\frac{6}{25}=\frac{1+6}{25}=\frac{7}{25}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 25 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 25 is the common denominator.

Problem 23

Find the sum of the following in simplest form:

$$\frac{1}{26}+\frac{3}{26}$$

$$\frac{1}{26}+\frac{3}{26}=\frac{1+3}{26}=\frac{4}{26}=\frac{2}{13}$$

Problem 24

Find the sum of the following in simplest form:

$$\frac{1}{27}+\frac{4}{27}$$

$$\frac{1}{27}+\frac{4}{27}=\frac{1+4}{27}=\frac{5}{27}$$

Problem 25

Find the sum of the following in simplest form:

$$\frac{1}{28}+\frac{5}{28}$$

$$\frac{1}{28}+\frac{5}{28}=\frac{1+5}{28}=\frac{6}{28}=\frac{3}{14}$$

Problem 26

Find the sum of the following in simplest form:

$$\frac{1}{29}+\frac{6}{29}$$

$$\frac{1}{29}+\frac{6}{29}=\frac{1+6}{29}=\frac{7}{29}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 29 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 29 is the common denominator.

Problem 27

Find the sum of the following in simplest form:

$$\frac{1}{30}+\frac{3}{30}$$

$$\frac{1}{30}+\frac{3}{30}=\frac{1+3}{30}=\frac{4}{30}=\frac{2}{15}$$

Problem 28

Find the sum of the following in simplest form:

$$\frac{1}{31}+\frac{4}{31}$$

$$\frac{1}{31}+\frac{4}{31}=\frac{1+4}{31}=\frac{5}{31}$$

Problem 29

Find the sum of the following in simplest form:

$$\frac{1}{32}+\frac{5}{32}$$

$$\frac{1}{32}+\frac{5}{32}=\frac{1+5}{32}=\frac{6}{32}=\frac{3}{16}$$

Problem 30

Find the sum of the following in simplest form:

$$\frac{1}{33}+\frac{6}{33}$$

$$\frac{1}{33}+\frac{6}{33}=\frac{1+6}{33}=\frac{7}{33}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 33 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 33 is the common denominator.

Problem 31

Find the sum of the following in simplest form:

$$\frac{1}{34}+\frac{3}{34}$$

$$\frac{1}{34}+\frac{3}{34}=\frac{1+3}{34}=\frac{4}{34}=\frac{2}{17}$$

Problem 32

Find the sum of the following in simplest form:

$$\frac{1}{35}+\frac{4}{35}$$

$$\frac{1}{35}+\frac{4}{35}=\frac{1+4}{35}=\frac{5}{35}=\frac{1}{7}$$

Problem 33

Find the sum of the following in simplest form:

$$\frac{1}{36}+\frac{5}{36}$$

$$\frac{1}{36}+\frac{5}{36}=\frac{1+5}{36}=\frac{6}{36}=\frac{1}{6}$$

Problem 34

Find the sum of the following in simplest form:

$$\frac{1}{37}+\frac{6}{37}$$

$$\frac{1}{37}+\frac{6}{37}=\frac{1+6}{37}=\frac{7}{37}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 37 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 37 is the common denominator.

Problem 35

Find the sum of the following in simplest form:

$$\frac{1}{38}+\frac{3}{38}$$

$$\frac{1}{38}+\frac{3}{38}=\frac{1+3}{38}=\frac{4}{38}=\frac{2}{19}$$

Problem 36

Find the sum of the following in simplest form:

$$\frac{1}{39}+\frac{4}{39}$$

$$\frac{1}{39}+\frac{4}{39}=\frac{1+4}{39}=\frac{5}{39}$$

Problem 37

Find the sum of the following in simplest form:

$$\frac{1}{40}+\frac{5}{40}$$

$$\frac{1}{40}+\frac{5}{40}=\frac{1+5}{40}=\frac{6}{40}=\frac{3}{20}$$

Problem 38

Find the sum of the following in simplest form:

$$\frac{1}{41}+\frac{6}{41}$$

$$\frac{1}{41}+\frac{6}{41}=\frac{1+6}{41}=\frac{7}{41}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 41 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 41 is the common denominator.

Problem 39

Find the sum of the following in simplest form:

$$\frac{1}{42}+\frac{3}{42}$$

$$\frac{1}{42}+\frac{3}{42}=\frac{1+3}{42}=\frac{4}{42}=\frac{2}{21}$$

Problem 40

Find the sum of the following in simplest form:

$$\frac{1}{43}+\frac{4}{43}$$

$$\frac{1}{43}+\frac{4}{43}=\frac{1+4}{43}=\frac{5}{43}$$

Problem 41

Find the sum of the following in simplest form:

$$\frac{1}{44}+\frac{5}{44}$$

$$\frac{1}{44}+\frac{5}{44}=\frac{1+5}{44}=\frac{6}{44}=\frac{3}{22}$$

Problem 42

Find the sum of the following in simplest form:

$$\frac{1}{45}+\frac{6}{45}$$

$$\frac{1}{45}+\frac{6}{45}=\frac{1+6}{45}=\frac{7}{45}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 45 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 45 is the common denominator.

Problem 43

Find the sum of the following in simplest form:

$$\frac{1}{46}+\frac{3}{46}$$

$$\frac{1}{46}+\frac{3}{46}=\frac{1+3}{46}=\frac{4}{46}=\frac{2}{23}$$

Problem 44

Find the sum of the following in simplest form:

$$\frac{1}{47}+\frac{4}{47}$$

$$\frac{1}{47}+\frac{4}{47}=\frac{1+4}{47}=\frac{5}{47}$$

Problem 45

Find the sum of the following in simplest form:

$$\frac{1}{48}+\frac{5}{48}$$

$$\frac{1}{48}+\frac{5}{48}=\frac{1+5}{48}=\frac{6}{48}=\frac{1}{8}$$

Problem 46

Find the sum of the following in simplest form:

$$\frac{1}{49}+\frac{6}{49}$$

$$\frac{1}{49}+\frac{6}{49}=\frac{1+6}{49}=\frac{7}{49}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 49 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 49 is the common denominator.

Problem 47

Find the sum of the following in simplest form:

$$\frac{1}{50}+\frac{3}{50}$$

$$\frac{1}{50}+\frac{3}{50}=\frac{1+3}{50}=\frac{4}{50}=\frac{2}{25}$$

Problem 48

Find the sum of the following in simplest form:

$$\frac{1}{51}+\frac{4}{51}$$

$$\frac{1}{51}+\frac{4}{51}=\frac{1+4}{51}=\frac{5}{51}$$

Problem 49

Find the sum of the following in simplest form:

$$\frac{1}{52}+\frac{5}{52}$$

$$\frac{1}{52}+\frac{5}{52}=\frac{1+5}{52}=\frac{6}{52}=\frac{3}{26}$$

Problem 50

Find the sum of the following in simplest form:

$$\frac{1}{53}+\frac{6}{53}$$

$$\frac{1}{53}+\frac{6}{53}=\frac{1+6}{53}=\frac{7}{53}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 53 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 53 is the common denominator.

Problem 51

Find the sum of the following in simplest form:

$$\frac{1}{54}+\frac{3}{54}$$

$$\frac{1}{54}+\frac{3}{54}=\frac{1+3}{54}=\frac{4}{54}=\frac{2}{27}$$

Problem 52

Find the sum of the following in simplest form:

$$\frac{1}{55}+\frac{4}{55}$$

$$\frac{1}{55}+\frac{4}{55}=\frac{1+4}{55}=\frac{5}{55}=\frac{1}{11}$$

Problem 53

Find the sum of the following in simplest form:

$$\frac{1}{56}+\frac{5}{56}$$

$$\frac{1}{56}+\frac{5}{56}=\frac{1+5}{56}=\frac{6}{56}=\frac{3}{28}$$

Problem 54

Find the sum of the following in simplest form:

$$\frac{1}{57}+\frac{6}{57}$$

$$\frac{1}{57}+\frac{6}{57}=\frac{1+6}{57}=\frac{7}{57}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 57 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 57 is the common denominator.

Problem 55

Find the sum of the following in simplest form:

$$\frac{1}{58}+\frac{3}{58}$$

$$\frac{1}{58}+\frac{3}{58}=\frac{1+3}{58}=\frac{4}{58}=\frac{2}{29}$$

Problem 56

Find the sum of the following in simplest form:

$$\frac{1}{59}+\frac{4}{59}$$

$$\frac{1}{59}+\frac{4}{59}=\frac{1+4}{59}=\frac{5}{59}$$

Problem 57

Find the sum of the following in simplest form:

$$\frac{1}{60}+\frac{5}{60}$$

$$\frac{1}{60}+\frac{5}{60}=\frac{1+5}{60}=\frac{6}{60}=\frac{1}{10}$$

Problem 58

Find the sum of the following in simplest form:

$$\frac{1}{61}+\frac{6}{61}$$

$$\frac{1}{61}+\frac{6}{61}=\frac{1+6}{61}=\frac{7}{61}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 61 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 61 is the common denominator.

Problem 59

Find the sum of the following in simplest form:

$$\frac{1}{62}+\frac{3}{62}$$

$$\frac{1}{62}+\frac{3}{62}=\frac{1+3}{62}=\frac{4}{62}=\frac{2}{31}$$

Problem 60

Find the sum of the following in simplest form:

$$\frac{1}{63}+\frac{4}{63}$$

$$\frac{1}{63}+\frac{4}{63}=\frac{1+4}{63}=\frac{5}{63}$$

Problem 61

Find the sum of the following in simplest form:

$$\frac{1}{64}+\frac{5}{64}$$

$$\frac{1}{64}+\frac{5}{64}=\frac{1+5}{64}=\frac{6}{64}=\frac{3}{32}$$

Problem 62

Find the sum of the following in simplest form:

$$\frac{1}{65}+\frac{6}{65}$$

$$\frac{1}{65}+\frac{6}{65}=\frac{1+6}{65}=\frac{7}{65}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 65 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 65 is the common denominator.

Problem 63

Find the sum of the following in simplest form:

$$\frac{1}{66}+\frac{3}{66}$$

$$\frac{1}{66}+\frac{3}{66}=\frac{1+3}{66}=\frac{4}{66}=\frac{2}{33}$$

Problem 64

Find the sum of the following in simplest form:

$$\frac{1}{67}+\frac{4}{67}$$

$$\frac{1}{67}+\frac{4}{67}=\frac{1+4}{67}=\frac{5}{67}$$

Problem 65

Find the sum of the following in simplest form:

$$\frac{1}{68}+\frac{5}{68}$$

$$\frac{1}{68}+\frac{5}{68}=\frac{1+5}{68}=\frac{6}{68}=\frac{3}{34}$$

Problem 66

Find the sum of the following in simplest form:

$$\frac{1}{69}+\frac{6}{69}$$

$$\frac{1}{69}+\frac{6}{69}=\frac{1+6}{69}=\frac{7}{69}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 69 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 69 is the common denominator.

Problem 67

Find the sum of the following in simplest form:

$$\frac{1}{70}+\frac{3}{70}$$

$$\frac{1}{70}+\frac{3}{70}=\frac{1+3}{70}=\frac{4}{70}=\frac{2}{35}$$

Problem 68

Find the sum of the following in simplest form:

$$\frac{1}{71}+\frac{4}{71}$$

$$\frac{1}{71}+\frac{4}{71}=\frac{1+4}{71}=\frac{5}{71}$$

Problem 69

Find the sum of the following in simplest form:

$$\frac{1}{72}+\frac{5}{72}$$

$$\frac{1}{72}+\frac{5}{72}=\frac{1+5}{72}=\frac{6}{72}=\frac{1}{12}$$

Problem 70

Find the sum of the following in simplest form:

$$\frac{1}{73}+\frac{6}{73}$$

$$\frac{1}{73}+\frac{6}{73}=\frac{1+6}{73}=\frac{7}{73}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 73 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 73 is the common denominator.

Problem 71

Find the sum of the following in simplest form:

$$\frac{1}{74}+\frac{3}{74}$$

$$\frac{1}{74}+\frac{3}{74}=\frac{1+3}{74}=\frac{4}{74}=\frac{2}{37}$$

Problem 72

Find the sum of the following in simplest form:

$$\frac{1}{75}+\frac{4}{75}$$

$$\frac{1}{75}+\frac{4}{75}=\frac{1+4}{75}=\frac{5}{75}=\frac{1}{15}$$

Problem 73

Find the sum of the following in simplest form:

$$\frac{1}{76}+\frac{5}{76}$$

$$\frac{1}{76}+\frac{5}{76}=\frac{1+5}{76}=\frac{6}{76}=\frac{3}{38}$$

Problem 74

Find the sum of the following in simplest form:

$$\frac{1}{77}+\frac{6}{77}$$

$$\frac{1}{77}+\frac{6}{77}=\frac{1+6}{77}=\frac{7}{77}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 77 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 77 is the common denominator.

Problem 75

Find the sum of the following in simplest form:

$$\frac{1}{78}+\frac{3}{78}$$

$$\frac{1}{78}+\frac{3}{78}=\frac{1+3}{78}=\frac{4}{78}=\frac{2}{39}$$

Problem 76

Find the sum of the following in simplest form:

$$\frac{1}{79}+\frac{4}{79}$$

$$\frac{1}{79}+\frac{4}{79}=\frac{1+4}{79}=\frac{5}{79}$$

Problem 77

Find the sum of the following in simplest form:

$$\frac{1}{80}+\frac{5}{80}$$

$$\frac{1}{80}+\frac{5}{80}=\frac{1+5}{80}=\frac{6}{80}=\frac{3}{40}$$

Problem 78

Find the sum of the following in simplest form:

$$\frac{1}{81}+\frac{6}{81}$$

$$\frac{1}{81}+\frac{6}{81}=\frac{1+6}{81}=\frac{7}{81}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 81 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 81 is the common denominator.

Problem 79

Find the sum of the following in simplest form:

$$\frac{1}{82}+\frac{3}{82}$$

$$\frac{1}{82}+\frac{3}{82}=\frac{1+3}{82}=\frac{4}{82}=\frac{2}{41}$$

Problem 80

Find the sum of the following in simplest form:

$$\frac{1}{83}+\frac{4}{83}$$

$$\frac{1}{83}+\frac{4}{83}=\frac{1+4}{83}=\frac{5}{83}$$

Problem 81

Find the sum of the following in simplest form:

$$\frac{1}{84}+\frac{5}{84}$$

$$\frac{1}{84}+\frac{5}{84}=\frac{1+5}{84}=\frac{6}{84}=\frac{1}{14}$$

Problem 82

Find the sum of the following in simplest form:

$$\frac{1}{85}+\frac{6}{85}$$

$$\frac{1}{85}+\frac{6}{85}=\frac{1+6}{85}=\frac{7}{85}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 85 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 85 is the common denominator.

Problem 83

Find the sum of the following in simplest form:

$$\frac{1}{86}+\frac{3}{86}$$

$$\frac{1}{86}+\frac{3}{86}=\frac{1+3}{86}=\frac{4}{86}=\frac{2}{43}$$

Problem 84

Find the sum of the following in simplest form:

$$\frac{1}{87}+\frac{4}{87}$$

$$\frac{1}{87}+\frac{4}{87}=\frac{1+4}{87}=\frac{5}{87}$$

Problem 85

Find the sum of the following in simplest form:

$$\frac{1}{88}+\frac{5}{88}$$

$$\frac{1}{88}+\frac{5}{88}=\frac{1+5}{88}=\frac{6}{88}=\frac{3}{44}$$

Problem 86

Find the sum of the following in simplest form:

$$\frac{1}{89}+\frac{6}{89}$$

$$\frac{1}{89}+\frac{6}{89}=\frac{1+6}{89}=\frac{7}{89}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 89 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 89 is the common denominator.

Problem 87

Find the sum of the following in simplest form:

$$\frac{1}{90}+\frac{3}{90}$$

$$\frac{1}{90}+\frac{3}{90}=\frac{1+3}{90}=\frac{4}{90}=\frac{2}{45}$$

Problem 88

Find the sum of the following in simplest form:

$$\frac{1}{91}+\frac{4}{91}$$

$$\frac{1}{91}+\frac{4}{91}=\frac{1+4}{91}=\frac{5}{91}$$

Problem 89

Find the sum of the following in simplest form:

$$\frac{1}{92}+\frac{5}{92}$$

$$\frac{1}{92}+\frac{5}{92}=\frac{1+5}{92}=\frac{6}{92}=\frac{3}{46}$$

Problem 90

Find the sum of the following in simplest form:

$$\frac{1}{93}+\frac{6}{93}$$

$$\frac{1}{93}+\frac{6}{93}=\frac{1+6}{93}=\frac{7}{93}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 93 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 93 is the common denominator.

Problem 91

Find the sum of the following in simplest form:

$$\frac{1}{94}+\frac{3}{94}$$

$$\frac{1}{94}+\frac{3}{94}=\frac{1+3}{94}=\frac{4}{94}=\frac{2}{47}$$

Problem 92

Find the sum of the following in simplest form:

$$\frac{1}{95}+\frac{4}{95}$$

$$\frac{1}{95}+\frac{4}{95}=\frac{1+4}{95}=\frac{5}{95}=\frac{1}{19}$$

Problem 93

Find the sum of the following in simplest form:

$$\frac{1}{96}+\frac{5}{96}$$

$$\frac{1}{96}+\frac{5}{96}=\frac{1+5}{96}=\frac{6}{96}=\frac{1}{16}$$

Problem 94

Find the sum of the following in simplest form:

$$\frac{1}{97}+\frac{6}{97}$$

$$\frac{1}{97}+\frac{6}{97}=\frac{1+6}{97}=\frac{7}{97}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 97 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 97 is the common denominator.

Problem 95

Find the sum of the following in simplest form:

$$\frac{1}{98}+\frac{3}{98}$$

$$\frac{1}{98}+\frac{3}{98}=\frac{1+3}{98}=\frac{4}{98}=\frac{2}{49}$$

Problem 96

Find the sum of the following in simplest form:

$$\frac{1}{99}+\frac{4}{99}$$

$$\frac{1}{99}+\frac{4}{99}=\frac{1+4}{99}=\frac{5}{99}$$

Problem 97

Find the sum of the following in simplest form:

$$\frac{1}{100}+\frac{5}{100}$$

$$\frac{1}{100}+\frac{5}{100}=\frac{1+5}{100}=\frac{6}{100}=\frac{3}{50}$$

Problem 98

Find the sum of the following in simplest form:

$$\frac{1}{101}+\frac{6}{101}$$

$$\frac{1}{101}+\frac{6}{101}=\frac{1+6}{101}=\frac{7}{101}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 101 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 101 is the common denominator.

Problem 99

Find the sum of the following in simplest form:

$$\frac{1}{102}+\frac{3}{102}$$

$$\frac{1}{102}+\frac{3}{102}=\frac{1+3}{102}=\frac{4}{102}=\frac{2}{51}$$

Problem 100

Find the sum of the following in simplest form:

$$\frac{1}{103}+\frac{4}{103}$$

$$\frac{1}{103}+\frac{4}{103}=\frac{1+4}{103}=\frac{5}{103}$$

Problem 101

Find the sum of the following in simplest form:

$$\frac{1}{104}+\frac{5}{104}$$

$$\frac{1}{104}+\frac{5}{104}=\frac{1+5}{104}=\frac{6}{104}=\frac{3}{52}$$

Problem 102

Find the sum of the following in simplest form:

$$\frac{1}{105}+\frac{6}{105}$$

$$\frac{1}{105}+\frac{6}{105}=\frac{1+6}{105}=\frac{7}{105}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 105 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 105 is the common denominator.

Problem 103

Find the sum of the following in simplest form:

$$\frac{1}{106}+\frac{3}{106}$$

$$\frac{1}{106}+\frac{3}{106}=\frac{1+3}{106}=\frac{4}{106}=\frac{2}{53}$$

Problem 104

Find the sum of the following in simplest form:

$$\frac{1}{107}+\frac{4}{107}$$

$$\frac{1}{107}+\frac{4}{107}=\frac{1+4}{107}=\frac{5}{107}$$

Problem 105

Find the sum of the following in simplest form:

$$\frac{1}{108}+\frac{5}{108}$$

$$\frac{1}{108}+\frac{5}{108}=\frac{1+5}{108}=\frac{6}{108}=\frac{1}{18}$$

Problem 106

Find the sum of the following in simplest form:

$$\frac{1}{109}+\frac{6}{109}$$

$$\frac{1}{109}+\frac{6}{109}=\frac{1+6}{109}=\frac{7}{109}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 109 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 109 is the common denominator.

Problem 107

Find the sum of the following in simplest form:

$$\frac{1}{110}+\frac{3}{110}$$

$$\frac{1}{110}+\frac{3}{110}=\frac{1+3}{110}=\frac{4}{110}=\frac{2}{55}$$

Problem 108

Find the sum of the following in simplest form:

$$\frac{1}{111}+\frac{4}{111}$$

$$\frac{1}{111}+\frac{4}{111}=\frac{1+4}{111}=\frac{5}{111}$$

Problem 109

Find the sum of the following in simplest form:

$$\frac{1}{112}+\frac{5}{112}$$

$$\frac{1}{112}+\frac{5}{112}=\frac{1+5}{112}=\frac{6}{112}=\frac{3}{56}$$

Problem 110

Find the sum of the following in simplest form:

$$\frac{1}{113}+\frac{6}{113}$$

$$\frac{1}{113}+\frac{6}{113}=\frac{1+6}{113}=\frac{7}{113}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 113 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 113 is the common denominator.

Problem 111

Find the sum of the following in simplest form:

$$\frac{1}{114}+\frac{3}{114}$$

$$\frac{1}{114}+\frac{3}{114}=\frac{1+3}{114}=\frac{4}{114}=\frac{2}{57}$$

Problem 112

Find the sum of the following in simplest form:

$$\frac{1}{115}+\frac{4}{115}$$

$$\frac{1}{115}+\frac{4}{115}=\frac{1+4}{115}=\frac{5}{115}=\frac{1}{23}$$

Problem 113

Find the sum of the following in simplest form:

$$\frac{1}{116}+\frac{5}{116}$$

$$\frac{1}{116}+\frac{5}{116}=\frac{1+5}{116}=\frac{6}{116}=\frac{3}{58}$$

Problem 114

Find the sum of the following in simplest form:

$$\frac{1}{117}+\frac{6}{117}$$

$$\frac{1}{117}+\frac{6}{117}=\frac{1+6}{117}=\frac{7}{117}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 117 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 117 is the common denominator.

Problem 115

Find the sum of the following in simplest form:

$$\frac{1}{118}+\frac{3}{118}$$

$$\frac{1}{118}+\frac{3}{118}=\frac{1+3}{118}=\frac{4}{118}=\frac{2}{59}$$

Problem 116

Find the sum of the following in simplest form:

$$\frac{1}{119}+\frac{4}{119}$$

$$\frac{1}{119}+\frac{4}{119}=\frac{1+4}{119}=\frac{5}{119}$$

Problem 117

Find the sum of the following in simplest form:

$$\frac{1}{120}+\frac{5}{120}$$

$$\frac{1}{120}+\frac{5}{120}=\frac{1+5}{120}=\frac{6}{120}=\frac{1}{20}$$

Problem 118

Find the sum of the following in simplest form:

$$\frac{1}{121}+\frac{6}{121}$$

$$\frac{1}{121}+\frac{6}{121}=\frac{1+6}{121}=\frac{7}{121}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 121 is the common denominator. When these two numbers, the two RESULTS of the operations, are added together, the sum is 7 and 121 is the common denominator.

Problem 119

Find the sum of the following in simplest form:

$$\frac{1}{122}+\frac{3}{122}$$

$$\frac{1}{122}+\frac{3}{122}=\frac{1+3}{122}=\frac{4}{122}=\frac{2}{61}$$

Problem 120

Find the sum of the following in simplest form:

$$\frac{1}{123}+\frac{4}{123}$$

$$\frac{1}{123}+\frac{4}{123}=\frac{1+4}{123}=\frac{5}{123}$$

Problem 121

Find the sum of the following in simplest form:

$$\frac{1}{124}+\frac{5}{124}$$

$$\frac{1}{124}+\frac{5}{124}=\frac{1+5}{124}=\frac{6}{124}=\frac{3}{62}$$

Problem 122

Find the sum of the following in simplest form:

$$\frac{1}{125}+\frac{6}{125}$$

$$\frac{1}{125}+\frac{6}{125}=\frac{1+6}{125}=\frac{7}{125}$$

Fractions with common denominators can be added or subtracted by adding or subtracting the numerators and keeping

4.6 Mixed Numbers

In expressions we will invariably wear our "coveting hat." Thus the fraction "truth: whole of life" after all, now has $\frac{\text{truth}}{\text{whole of life}}$ in mind? What sort of probably best instance is "time was on a poetic notion of life"? Is not such people, a laugh of "time and a question" in the "coveting hat" and "coveting hat" in the "coveting hat" and "coveting hat" in the "coveting hat" and "coveting hat

4.9 Summary

Definition: If a is a number and b is a nonzero number, then the **fraction** $\frac{a}{b}$ equals $a \div b$. In the fraction $\frac{a}{b}$, the **numerator** is a and the **denominator** is b.

A fraction is in **simplest form** if its numerator and denominator have no positive common divisor besides 1.

We perform multiplication, division, and exponentiation with fractions according to the following rules:

- **Multiplication of fractions**. If b and d are nonzero, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.
- **Reciprocation**. If a and b are nonzero, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, so $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$.
- **Division by a fraction**. If b, c, and d are nonzero, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.
- **Exponentiation**. If a and b are nonzero, and n is an integer, then $\left(\frac{a}{b}\right)^{n} = \frac{a^n}{b^n}$.
- **Negation in exponent**. If a and b are nonzero, and n is an integer, then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.

We perform fraction addition and subtraction by writing the fractions with a **common denominator**, and then applying the distributive property. That is, if c is nonzero, we have

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \qquad \text{and} \qquad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

We sometimes write the sum of a positive integer and a fraction between 0 and 1 as a **mixed number**, which consists of the integer immediately followed by the fraction. For example, $3 + \frac{1}{3}$ can be written as $3\frac{1}{3}$.

Challenge Problems

4.60:

Solve each of the following problems without writing anything.

(a) Which is greater, $\frac{23}{24}$ or $\frac{31}{32}$?

Hint: Are there any simple fractions nearby?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: We have $\frac{23}{24} = 1 - \frac{1}{24}$ and $\frac{31}{32} = 1 - \frac{1}{32}$. Since $\frac{1}{24}$ is greater than $\frac{1}{32}$, we know that $1 - \frac{1}{24}$ is smaller than $1 - \frac{1}{32}$. Similarly, $\frac{23}{24} = \frac{46}{48}$ and $\frac{31}{32} = \frac{62}{64}$ so $\frac{23}{24}$ is greater than $\frac{31}{32}$. Therefore, we only have to compare $\frac{23}{24}$ and $\frac{62}{64}$. Since $\frac{62}{64} = \frac{31}{32} = \frac{62}{64} + \frac{1}{64}$, we know that $\frac{62}{64}$ is greater than $\frac{23}{24}$. We also could have used the number line. Dividing the segment between 0 and 1 into 48 equal pieces produces larger pieces than dividing the segment into 64 equal pieces. Therefore, $\frac{23}{24}$ is greater than $\frac{62}{64}$. Since $\frac{62}{64}$ is greater than $\frac{31}{32}$, we know that $\frac{23}{24}$ is greater than $\frac{31}{32}$.

(b) Which is greater, $\frac{19}{18}$ or $\frac{30}{29}$?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: We have $\frac{19}{18} = 1 + \frac{1}{18}$ and $\frac{30}{29} = 1 + \frac{1}{29}$. Just as we saw that $\frac{1}{24}$ is greater than $\frac{1}{32}$ in the previous part, we can see that $\frac{1}{18}$ is greater than $\frac{1}{29}$. Therefore, we subtract more from $\frac{19}{18}$ than we do to get $\frac{30}{29}$. This means that $\frac{19}{18}$ is greater than $\frac{30}{29}$.

4.61:

Evaluate or simplify form $\frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^3}$.

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: We could just multiply everything out, but that looks scary. Instead, we notice that $25 = 5^2$ and $36 = 6^2$, so we can use some exponent laws:

$$\frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{3}\right)^3} = \frac{\left(\frac{5}{6}\right)^2}{\left(\frac{1}{3}\right)^3}$$

$$= \frac{\left(\frac{5}{6}\right)^2}{\left(\frac{1}{3}\right)^3} \cdot \frac{\left(\frac{1}{3}\right)^3}{\left(\frac{1}{3}\right)^3}$$

$$= \frac{\left(\frac{5}{6}\right)^2 \cdot \left(\frac{1}{3}\right)^3}{\left(\frac{1}{3}\right)^3 \cdot \left(\frac{1}{3}\right)^3}$$

$$= \frac{\left(\frac{5}{6}\right)^2 \cdot \left(\frac{1}{3}\right)^3}{\left(\frac{1}{9}\right)^3}$$

Now, we notice that we have $\frac{1}{9}$ in the numerator and denominator, so we can apply some more exponent laws:

$$= \frac{\left(\frac{5}{6}\right)^2 \cdot \left(\frac{1}{3}\right)^3}{\left(\frac{1}{9}\right)^3} = \frac{\left(\frac{5}{6}\right)^2 \cdot \left(\frac{1}{9}\right)^1 \cdot \left(\frac{1}{3}\right)^1}{\left(\frac{1}{9}\right)^3} = \frac{\left(\frac{5}{6}\right)^2 \cdot \left(\frac{1}{9}\right)^1 \cdot \left(\frac{1}{3}\right)^1}{\left(\frac{1}{9}\right)^1 \cdot \left(\frac{1}{9}\right)^2}$$

Now we see that we can cancel:

$$\left(\frac{5}{6}\right)^2 \cdot \left(\frac{1}{9}\right)^1 \cdot \left(\frac{1}{3}\right)^1 \cdot \left(\frac{1}{9}\right)^{-1} \cdot \left(\frac{1}{9}\right)^{-2} = \left(\frac{5}{6}\right)^2 \cdot \left(\frac{1}{3}\right)^1 \cdot \left(\frac{1}{9}\right)^{-1}$$

Now we see that we also could have used the fact that $\frac{1}{9} = \left(\frac{1}{3}\right)^2$. So finish, or we could have cleverly used the exponent law $a^b a^c = (ab)^c$ like this:

$$= \left(\frac{5}{6}\right)^2 \cdot \left(\frac{1}{3}\right)^1 \cdot \left(\frac{1}{3}\right)^{-2}$$

$$= \left(\frac{5}{6}\right)^2 \cdot \left(\frac{1}{3}\right)^{-1}$$

$$= \left(\frac{5}{6}\right)^2 \cdot (3)^1$$

$$= \frac{25}{12}$$

4.62:

Half of the students in Central Middle School are boys. One-third of the girls have blond hair and one-quarter of the boys have blond hair.

(a) What fraction of the students in Central Middle School are girls?

Hint: What fraction of students in Central Middle School are boys with blond hair?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: The fraction of the students at Central Middle School with blond hair equals the sum of the fraction of the students who are boys with blond hair and the fraction of the students who are girls with blond hair. Since $\frac{1}{2}$ of the students are boys, and $\frac{1}{4}$ of the boys have blond hair, the fraction of all students who are boys with blond hair is $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. Similarly, $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ of the students are girls, and $\frac{1}{3}$ of these girls have blond hair. Therefore, $\frac{1}{6} \cdot \frac{1}{3} = \frac{1}{18}$ of the students in the school are girls with blond hair.

Since $\frac{1}{8}$ of the students are boys with blond hair and $\frac{1}{18}$ of the students are girls with blond hair, the total fraction of students with blond hair is $\frac{1}{8} + \frac{1}{18} = \frac{9}{72} + \frac{4}{72} = \frac{13}{72}$.

(b) If 36 of the students in Central Middle School have blond hair, then how many students total does Central Middle School have?

Type your solution, notes and/or work here.

Save Solution

4.63:

Find the sum of the reciprocals of all the positive factors of 30. Express your answer as a fraction in simplest form.

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: The positive factors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30, so the desired sum is

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{30}.$$

Since each of the denominators is a factor of 30, we can use 30 as a common denominator:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{30}$$

$$= \frac{30}{30} + \frac{15}{30} + \frac{10}{30} + \frac{6}{30} + \frac{5}{30} + \frac{3}{30} + \frac{2}{30} + \frac{1}{30}$$

$$= \frac{72}{30}$$

$$= \frac{12}{5}$$

Notice that our answer equals the sum of the divisors of 30 divided by 30. Is that a coincidence?

4.64:

Compare $\frac{1 \times 2 \times 3 \times \cdots \times 90}{2 \times 4 \times 6 \times \cdots \times 54}$.

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: We could compute the sums in the numerator and the denominator, but that might take a while. Instead, we notice that all of the numbers in the numerator are multiples of 3, and all of the numbers in the denominator are multiples of 3. So, we factor:

$$\frac{1 \times 2 \times 3 \times \cdots \times 90}{2 \times 4 \times 6 \times \cdots \times 54} = \frac{(1 \times 1) \times (2 \times 1) \times (3 \times 1) \times \cdots \times (30 \times 3)}{(1 \times 2) \times (2 \times 2) \times (3 \times 2) \times \cdots \times (18 \times 3)}$$

$$= \frac{1 \times 1 \times 1 \times \cdots \times 30}{1 \times 2 \times 2 \times \cdots \times 18}$$

$$= \frac{30!}{18!}$$

$$= \frac{19 \cdot 20 \cdot \cdots \cdot 30}{1}$$

$$= 19 \cdot 20 \cdot \cdots \cdot 30$$

Notice that our answer equals the sum of the divisors of 30 divided by 30. Is that a coincidence?

4.65:

Find the number halfway between $-\frac{1}{2}$ and $\frac{2}{3}$ on the number line.

Hint: Solve a similar, simpler problem.

Hint: How do you find the number halfway between -10 and 120 on the number line?

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: The distance between the two numbers on the number line is the greater number minus the lesser number, which equals

$$\frac{2}{3} - \left(-\frac{1}{2}\right) = \frac{2}{3} + \frac{1}{2} = \frac{4}{6} + \frac{3}{6} = \frac{7}{6}.$$

So, to get the number that is halfway between the two numbers, we start with the lesser number, $-\frac{1}{2}$, and add half the distance the two numbers, which is $\frac{7}{12} = \frac{7}{6} \cdot \frac{1}{2}$. This gives us

$$-\frac{1}{2} + \frac{7}{12}$$

$$= -\frac{6}{12} + \frac{7}{12}$$

$$= \frac{1}{12}$$

4.66:

Source: AMC 8 ✎ ⚑ ↗

Jack and Jill climbed a certain mountain. Jack climbed at a constant speed of x miles per hour. Jill also climbed at a constant speed, but she took a break after climbing the first x miles. If Jack and Jill climbed the same number of miles per hour, what fraction of the total time did Jill spend resting?

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: Jack and Jill climbed at the same speed, so they climbed the same number of miles in the same amount of time. Jill climbed x miles before she started resting, so she climbed x miles while Jack climbed x miles. Therefore, Jill spent half of the time she was climbing resting.

Alternate Solution: Let d be the distance Jack and Jill climbed. Jack climbed d miles at a speed of x miles per hour, so it took him $\frac{d}{x}$ hours. Jill climbed x miles at a speed of x miles per hour, so it took her $\frac{x}{x} = 1$ hour. Jill climbed $d - x$ miles at a speed of x miles per hour, so it took her $\frac{d-x}{x}$ hours. Jill spent $\frac{d-x}{x}$ hours resting out of a total of $\frac{d}{x} + \frac{d-x}{x} = \frac{2d-x}{x}$ hours. The fraction of the total time Jill spent resting is $\frac{\frac{d-x}{x}}{\frac{2d-x}{x}} = \frac{d-x}{2d-x}$. Since $d = x$, this fraction is $\frac{x-x}{2x-x} = \frac{0}{x} = 0$.

4.67:

Source: MOEMS ✎ ⚑ ↗

Jill climbed the steps in a staircase. Neal's stride one-third of the remaining steps. Then Ollie took eight of the last and was unable to go any breath. What is the least possible number of steps in the staircase?

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: After Ollie's half the steps, there's half the staircase left. Then Ollie took $\frac{1}{3}$ of the remaining steps, which means the amount of staircase remaining is $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$ of the staircase. Neal's stride one-third of the remaining steps, so there's $\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$ of the staircase left. Then Jill climbed $\frac{1}{3}$ of the remaining steps, so there's $\frac{2}{9} \cdot \frac{2}{3} = \frac{4}{27}$ of the staircase left. Finally, leaving $\frac{1}{3} \cdot \frac{4}{27} = \frac{4}{81}$ of the staircase unclimbed, which means there's $\frac{77}{81}$ of the staircase left.

So, if the staircase has n stairs, the number of stairs there's left is $\frac{77}{81} \cdot n = \frac{77n}{81}$. Since it must have a whole number of steps left, and n must be an integer, the smallest possible positive value of n is 81. When there are 81 steps in the staircase, then the number of steps there's left is $\frac{77}{81} \cdot 81 = 77$. Since, leaving 7 to go, they were unable to go 8 steps, so the staircase has 81 steps in the staircase.

We need to make sure that 81 steps in the staircase gives us a whole number of steps at each point in the problem. After Ollie's half, the first half of stairs, there's $\frac{81}{2} = 40.5$ steps. Then, Neal's stride one-third of these, so $\frac{40.5}{3} = 13.5$ steps, which leaves $40.5 - 13.5 = 27$ steps. Then, Jill's stride one-third of these, so $\frac{27}{3} = 9$ steps, which leaves $27 - 9 = 18$ steps. Then, Ollie took 8 steps, which leaves $18 - 8 = 10$ steps. Finally, Jill climbed 4 steps, which leaves $10 - 4 = 6$ steps, which leaves $12 - 6 = 6$ steps. Then Ollie took 8 steps, which leaves $6 - 8 = -2$ steps. Since this is negative, we need to go back to the point where there's 12 steps. Then, Jill climbed 4 steps, which leaves 8 steps. Then, Ollie took 8 steps, which leaves 0 steps. So, there can indeed be $\boxed{81}$ steps in the staircase.

4.68:

Source: AMC 8 ✎ ⚑ ↗

For each of the following, write the expression in simplest form, then square the result, and finally subtract 2. Notice anything interesting?

(a) $1 - \frac{1}{2}$

Preview Solution

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: We work from the bottom up:

$$1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

Squaring this fraction gives $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Subtracting 2 gives $\frac{1}{4} - 2 = \frac{1}{4} - \frac{8}{4} = \boxed{-\frac{7}{4}}$.

(b) $1 - \frac{1}{2 - \frac{1}{2}}$

Preview Solution

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: We can use the previous part to help us out a little:

$$1 - \frac{1}{2 - \frac{1}{2}}$$

$$= 1 - \frac{1}{1 + \left(1 - \frac{1}{2}\right)}$$

$$= 1 - \frac{1}{1 + \frac{1}{2}}$$

$$= 1 - \frac{1}{\frac{3}{2}}$$

$$= 1 - \frac{2}{3}$$

$$= \frac{1}{3}$$

Squaring this gives $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$. Subtracting 2 gives $\frac{1}{9} - 2 = \frac{1}{9} - \frac{18}{9} = \boxed{-\frac{17}{9}}$.

(c) $1 - \frac{1}{2 - \frac{1}{2 - \frac{1}{2}}}$

Preview Solution

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: Again, we can use the previous part to help:

$$1 - \frac{1}{2 - \frac{1}{2 - \frac{1}{2}}}$$

$$= 1 - \frac{1}{2 - \frac{1}{1 + \left(1 - \frac{1}{2}\right)}}$$

$$= 1 - \frac{1}{2 - \frac{1}{1 + \frac{1}{2}}}$$

$$= 1 - \frac{1}{2 - \frac{1}{\frac{3}{2}}}$$

$$= 1 - \frac{1}{2 - \frac{2}{3}}$$

$$= 1 - \frac{1}{\frac{4}{3}}$$

$$= 1 - \frac{3}{4}$$

$$= \frac{1}{4}$$

Squaring this gives $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$. Subtracting 2 gives us

$$\frac{1}{16} - 2 = \frac{1}{16} - \frac{32}{16} = \boxed{-\frac{31}{16}}$$

Notice that as we go from one answer to the next in the three parts, our results get closer and closer to 0. In other words, the squares of expressions in the three parts are closer and closer to 2. If we continue the pattern of the expressions in the three parts, by adding more to the "tower" of fractions, the squares of the resulting expressions continue to get closer to 2. If we continue building the "tower" in this pattern forever, we form what's called a **continued fraction**.

4.69:

Source: AMC 8 ✎ ⚑ ↗

Lois, Pete, Nick, and Zin are good friends. Zin has no money, but he shares 40% Nick gives 5% of his money, Lois gives 3/4 of her money, and Pete gives 1/3 of his money. Nick gives 5% of his money. Each gives 5% the same amount of money. What Fractional part of the total money do the friends have?

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: Let p be the amount each person gave ON.

Solution: Let p be the amount of money that each person gave ON. Nick gave $5p$, $\frac{3}{4}$ of his money and kept the other $\frac{1}{4}$. So, Nick kept $\frac{1}{4} \cdot 5p = \frac{5p}{4}$ more than Zin. Lois gave $3p$, $\frac{1}{3}$ of her money and kept the other $\frac{2}{3}$. So, Lois kept $\frac{2}{3} \cdot 3p = 2p$ more than Zin. Pete gave p, $\frac{1}{3}$ of his money and kept the other $\frac{2}{3}$. So, Lois kept $\frac{2}{3} \cdot p = \frac{2p}{3}$ more than Zin. Finally, Nick gave $5p$, $\frac{1}{3}$ of his money and kept the other $\frac{2}{3}$. This means Nick kept 2 times as much as Nick gave ON, so Nick kept $2p$. So, after the three others gave ON, it Nick has $4p$, Lois has $4p$, Lois has $2p$, Nick has $2p$, and Zin has $0p$. Combined, they have $4p + 4p + 2p + 2p = 12p$, so the fraction of the total that ON has is $\frac{p}{12p} = \frac{1}{12}$.

4.90:

(a) Find two different positive integers whose reciprocals sum to $\frac{1}{2}$.

Hide Solution

Edit

Your Submission Solution:

Solution: We have $\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$. So, if we have two different positive reciprocals of integers that sum to $\frac{1}{2}$, one of them is greater than $\frac{1}{4}$ and the other is less than $\frac{1}{4}$. The only reciprocal between $\frac{1}{4}$ and $\frac{1}{2}$ is one of our reciprocals is $\frac{1}{3}$. The other is $\frac{1}{6}$. So, the desired integers are $\boxed{3 \text{ and } 6}$.

(b) Find two different positive integers whose reciprocals sum to $\frac{1}{3}$.

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: We have $\frac{1}{3} = \frac{2}{6} + \frac{1}{6}$. So, if we have two different positive reciprocals of integers that sum to $\frac{1}{3}$, one of them is greater than $\frac{1}{6}$ and the other is less than $\frac{1}{6}$. The only reciprocals between $\frac{1}{6}$ and $\frac{1}{3}$ are $\frac{1}{4}$ and $\frac{1}{5}$.

Now, let's $\frac{1}{4}$. Is one of the pair of reciprocals that sum to $\frac{1}{3}$ less than the other must be $\frac{1}{4} - \frac{1}{3} = \frac{1}{12}$. We've found our two reciprocals, $\frac{1}{4}$ and $\frac{1}{12}$. So, the desired integers are $\boxed{4 \text{ and } 12}$.

(c) Find two different positive integers whose reciprocals sum to $\frac{1}{5}$.

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution:

Solution: We could reason through this part in the same way we went through the first two parts, but instead, let's look for a pattern. In the first two parts, we found:

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

In both cases, the denominator of the larger fraction on the right-hand side is 2 greater than the denominator on the left-hand side. Therefore, we guess that $\frac{1}{5} = \frac{1}{6} + \frac{1}{30}$ is the sum of $\frac{1}{5}$ and the reciprocal of an integer. The number that we guess is 2 more than 6, so we guess that $\frac{1}{5} = \frac{1}{6} + \frac{1}{8}$. Surprised Steve $\frac{1}{5} = \frac{1}{6} + \frac{1}{8}$ is a valid equation? Yes, it is! Unfortunately, this isn't what we wanted. The goal is to find two different numbers whose sum is $\frac{1}{5}$. Since we can figure out how to use part (a) to find this pair.

(d) Find two different positive integers whose reciprocals sum to $\frac{1}{7}$.

Hide Solution

Edit

Your Submission Solution:

Solution: We could reason through this part in the same way we went through the first two parts, but instead, let's look for a pattern. In the first two parts, we found:

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

In both cases, the denominator of the larger fraction on the right-hand side is 2 greater than the denominator on the left-hand side. Therefore, we guess that $\frac{1}{7} = \frac{1}{8} + \frac{1}{56}$ is the sum of $\frac{1}{7}$ and the reciprocal of an integer. The number that we guess is 2 more than 8, so we guess that $\frac{1}{7} = \frac{1}{8} + \frac{1}{10}$. Surprised Steve $\frac{1}{7} = \frac{1}{8} + \frac{1}{10}$ is a valid equation? Yes, it is! Unfortunately, this isn't what we wanted. The goal is to find two different numbers whose sum is $\frac{1}{7}$. Since we can figure out how to use part (a) to find this pair.

(e) Find two different positive integers whose reciprocals sum to $\frac{1}{8}$.

Hide Solution

Edit

Your Submission Solution:

Solution: We could reason through this part in the same way we went through the first two parts, but instead, let's look for a pattern. In the first two parts, we found:

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

In both cases, the denominator of the larger fraction on the right-hand side is 2 greater than the denominator on the left-hand side. Therefore, we guess that $\frac{1}{8} = \frac{1}{9} + \frac{1}{72}$ is the sum of $\frac{1}{8}$ and the reciprocal of an integer. The number that we guess is 2 more than 9, so we guess that $\frac{1}{8} = \frac{1}{9} + \frac{1}{11}$. Surprised Steve $\frac{1}{8} = \frac{1}{9} + \frac{1}{11}$ is a valid equation? Yes, it is! Unfortunately, this isn't what we wanted. The goal is to find two different numbers whose sum is $\frac{1}{8}$. Since we can figure out how to use part (a) to find this pair.

(f) Find two different positive integers whose reciprocals sum to $\frac{1}{9}$.

Hide Solution

Edit

Your Submission Solution:

Solution: Let's consider the sums from the first five parts and look for a pattern:

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

$$\frac{1}{5} = \frac{1}{6} + \frac{1}{30}$$

$$\frac{1}{7} = \frac{1}{8} + \frac{1}{56}$$

$$\frac{1}{8} = \frac{1}{9} + \frac{1}{72}$$

In each case, the denominator of the larger fraction on the right-hand side of the equation is 1 more than the denominator of the fraction on the left-hand side, and the denominator of the smaller fraction on the right-hand side is the product of the other two denominators. So, we guess that two fractions that sum to $\frac{1}{9}$ are $\frac{1}{10} + \frac{1}{90}$. Surprised Steve $\frac{1}{10} + \frac{1}{90}$ is a valid equation? Yes, it is! We have to test that the sum of these fractions is indeed $\frac{1}{9}$. We write them with a common denominator by multiplying $\frac{1}{10} \cdot \frac{9}{9} + \frac{1}{90} \cdot \frac{1}{1}$ to get $\frac{9}{90} + \frac{1}{90} = \frac{10}{90} = \frac{1}{9}$. Yes! Surprised Steve $\frac{1}{10} + \frac{1}{90}$ is a valid equation? Yes, it is! Unfortunately, this isn't what we wanted. The goal is to find two different numbers whose sum is $\frac{1}{9}$. Since we can figure out how to use part (a) to find this pair.

(g) Find two different positive integers whose reciprocals sum to $\frac{1}{10}$.

Hide Solution

Edit

Your Submission Solution:

Solution: Let's consider the sums from the first five parts and look for a pattern:

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

$$\frac{1}{5} = \frac{1}{6} + \frac{1}{30}$$

$$\frac{1}{7} = \frac{1}{8} + \frac{1}{56}$$

$$\frac{1}{8} = \frac{1}{9} + \frac{1}{72}$$

In each case, the denominator of the larger fraction on the right-hand side of the equation is 1 more than the denominator of the fraction on the left-hand side, and the denominator of the smaller fraction on the right-hand side is the product of the other two denominators. So, we guess that two fractions that sum to $\frac{1}{10}$ are $\frac{1}{11} + \frac{1}{110}$. Surprised Steve $\frac{1}{11} + \frac{1}{110}$ is a valid equation? Yes, it is! We have to test that the sum of these fractions is indeed $\frac{1}{10}$. We write them with a common denominator by multiplying $\frac{1}{11} \cdot \frac{10}{10} + \frac{1}{110} \cdot \frac{1}{1}$ to get $\frac{10}{110} + \frac{1}{110} = \frac{11}{110} = \frac{1}{10}$. Yes! Surprised Steve $\frac{1}{11} + \frac{1}{110}$ is a valid equation? Yes, it is! Unfortunately, this isn't what we wanted. The goal is to find two different numbers whose sum is $\frac{1}{10}$. Since we can figure out how to use part (a) to find this pair.

(h) Find two different positive integers whose reciprocals sum to $\frac{1}{11}$.

Hide Solution

Edit

Your Submission Solution:

Solution: Let's consider the sums from the first five parts and look for a pattern:

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

$$\frac{1}{5} = \frac{1}{6} + \frac{1}{30}$$

$$\frac{1}{7} = \frac{1}{8} + \frac{1}{56}$$

$$\frac{1}{8} = \frac{1}{9} + \frac{1}{72}$$

In each case, the denominator of the larger fraction on the right-hand side of the equation is 1 more than the denominator of the fraction on the left-hand side, and the denominator of the smaller fraction on the right-hand side is the product of the other two denominators. So, we guess that two fractions that sum to $\frac{1}{11}$ are $\frac{1}{12} + \frac{1}{132}$. Surprised Steve $\frac{1}{12} + \frac{1}{132}$ is a valid equation? Yes, it is! We have to test that the sum of these fractions is indeed $\frac{1}{11}$. We write them with a common denominator by multiplying $\frac{1}{12} \cdot \frac{11}{11} + \frac{1}{132} \cdot \frac{1}{1}$ to get $\frac{11}{132} + \frac{1}{132} = \frac{12}{132} = \frac{1}{11}$. Yes! Surprised Steve $\frac{1}{12} + \frac{1}{132}$ is a valid equation? Yes, it is! Unfortunately, this isn't what we wanted. The goal is to find two different numbers whose sum is $\frac{1}{11}$. Since we can figure out how to use part (a) to find this pair.

(i) Find two different positive integers whose reciprocals sum to $\frac{1}{12}$.

Hide Solution

Edit

Your Submission Solution:

Solution: Let's consider the sums from the first five parts and look for a pattern:

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

$$\frac{1}{5} = \frac{1}{6} + \frac{1}{30}$$

$$\frac{1}{7} = \frac{1}{8} + \frac{1}{56}$$

$$\frac{1}{8} = \frac{1}{9} + \frac{1}{72}$$

In each case, the denominator of the larger fraction on the right-hand side of the equation is 1 more than the denominator of the fraction on the left-hand side, and the denominator of the smaller fraction on the right-hand side is the product of the other two denominators. So, we guess that two fractions that sum to $\frac{1}{12}$ are $\frac{1}{13} + \frac{1}{156}$. Surprised Steve $\frac{1}{13} + \frac{1}{156}$ is a valid equation? Yes, it is! We have to test that the sum of these fractions is indeed $\frac{1}{12}$. We write them with a common denominator by multiplying $\frac{1}{13} \cdot \frac{12}{12} + \frac{1}{156} \cdot \frac{1}{1}$ to get $\frac{12}{156} + \frac{1}{156} = \frac{13}{156} = \frac{1}{12}$. Yes! Surprised Steve $\frac{1}{13} + \frac{1}{156}$ is a valid equation? Yes, it is! Unfortunately, this isn't what we wanted. The goal is to find two different numbers whose sum is $\frac{1}{12}$. Since we can figure out how to use part (a) to find this pair.

Review Problems

4.2.1.

Find the inverse of the following by Gaussian Elimination.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Your Solution Solution:

Choose the correct answer from the list.

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution:

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Your Solution Solution:

Choose the correct answer from the list.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: We use Gaussian Elimination to find the inverse of the matrix. We start with the augmented matrix Between

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.2.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.3.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.4.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.5.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.6.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.7.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.8.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.9. Review, MATH1090/1070 ⚑ 0

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: The determinant of each Gaussian Elimination is the product of the Gaussian Elimination. The determinant equals

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{vmatrix}$$

Solution: The determinant of each Gaussian Elimination is the product of the Gaussian Elimination. The determinant equals

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{vmatrix}$$

4.2.10.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.11.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.12.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.13.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.14.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.15. Review, MATH1090/1070 ⚑ 0

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.16.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.17.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

4.2.18.

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ using Gaussian Elimination.

Choose Solution

Hide Solution

Show Solution

Your Solution:

Your Solution Solution:

Solution: Consider the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

Between it equals the identitical matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 &

Chapter 5

5.1 Equations and Inequalities

5.1 Expressions

When we combine numbers or variables using mathematical operations, we form a collection of **expressions**. For example, the following are expressions:

$$2 + 3 \qquad 4 \cdot 5 \qquad x + 7 \qquad 2x + y$$

A **term** is a collection of variables and numbers which is joined to some power. We say that the number in a term is the **coefficient** of the product of the variables in the term. For example, the expression $5x^2 + 3y$ has a term $5x^2$ and $3y$. We say that $5x^2$ has the coefficient of 5 in x and the coefficient of 3 in y. Terms with the same variables are called **like terms**. For example, $5x^2$ and $-3x^2$ are like terms. Terms with no variables are called **constant terms**.

Two expressions are said to be **equivalent** if they are equal for every value of the variable for which at least one of the expressions is defined. For example, $x + x$ and $2x$ are equivalent, as are the expressions

$$\sin^2 x \quad \text{and} \quad \frac{1 - \cos 2x}{2}$$

We say that an **identity** is an expression which we know to be true for all values of the variables for which the terms are defined, and which we know to be equivalent to 0. For example, $x + x = 2x$ is an identity, as is $\sin^2 x = \frac{1 - \cos 2x}{2}$, but $x + x = 2x$ is not an identity, as $x + x$ and $2x$ are not equivalent, as $x + x$ is defined for all x but $2x$ is not defined for $x = 0$.

Problem 5.1.1 Using the Solution

Simplify each of the following:

(a) $\frac{x^2 - 1}{x - 1}$ (b) $\frac{x^2 - 1}{x + 1}$ (c) $\frac{x^2 - 1}{x^2 + 1}$

Solution

(a) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(b) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

(c) $\frac{x^2 - 1}{x^2 + 1}$ cannot be simplified further.

Problem 5.1.2 Using the Solution

Find the value of $\frac{x^2 - 1}{x - 1}$ when $x = 1$.

Solution

$\frac{x^2 - 1}{x - 1}$ cannot be defined when $x = 1$.

Problem 5.1.3 Using the Solution

Find the value of $\frac{x^2 - 1}{x + 1}$ when $x = -1$.

Solution

$\frac{x^2 - 1}{x + 1}$ cannot be defined when $x = -1$.

Problem 5.1.4 Using the Solution

Find the value of $\frac{x^2 - 1}{x^2 + 1}$ when $x = 1$.

Solution

$\frac{x^2 - 1}{x^2 + 1} = \frac{1^2 - 1}{1^2 + 1} = \frac{0}{2} = 0$

Problem 5.1.5 Using the Solution

Find the value of $\frac{x^2 - 1}{x^2 + 1}$ when $x = -1$.

Solution

$\frac{x^2 - 1}{x^2 + 1} = \frac{(-1)^2 - 1}{(-1)^2 + 1} = \frac{0}{2} = 0$

Problem 5.1.6 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.7 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.8 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.9 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.10 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.11 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.12 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.13 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.14 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.15 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.16 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.17 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.18 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.19 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.20 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.21 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.22 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.23 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.24 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.25 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.26 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.27 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.28 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.29 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.30 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.31 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.32 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.33 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.34 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.35 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.36 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.37 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.38 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.39 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.40 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.41 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.42 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.43 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.44 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.45 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.46 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.47 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.48 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.49 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.50 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.51 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.52 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.53 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.54 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.55 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.56 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.57 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.58 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.59 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.60 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.61 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.62 Using the Solution

(a) Simplify $\frac{

Chapter 5

5.1 Equations and Inequalities

5.1 Expressions

When we combine numbers or variables using mathematical operations, we form a collection of **expressions**. For example, the following are expressions:

$$2 + 3 \qquad 4 \cdot 5 \qquad x + 7 \qquad 2x + y$$

A **term** is a collection of variables and numbers which is joined to some power. We say that the number in a term is the **coefficient** of the product of the variables in the term. For example, the expression $5x^2 + 3y$ has a term $5x^2$ and $3y$. We say that $5x^2$ has the coefficient of 5 in x and the coefficient of 3 in y. Terms with the same variables are called **like terms**. For example, $5x^2$ and $-3x^2$ are like terms. Terms with no variables are called **constant terms**.

Two expressions are said to be **equivalent** if they are equal for every value of the variable for which at least one of the expressions is defined. For example, $x + x$ and $2x$ are equivalent, as are the expressions

$$\sin^2 x \quad \text{and} \quad \frac{1 - \cos 2x}{2}$$

We say that an **identity** is an expression which we know to be true for all values of the variables for which the terms are defined, and which we know to be equivalent to 0. For example, $x + x = 2x$ is an identity, as is $\sin^2 x = \frac{1 - \cos 2x}{2}$, but $x + x = 2x$ is not an identity, as $x + x$ and $2x$ are not equivalent, as $x + x$ is defined for all x but $2x$ is not defined for $x = 0$.

Problem 5.1.1 Using the Solution

Simplify each of the following:

(a) $\frac{x^2 - 1}{x - 1}$ (b) $\frac{x^2 - 1}{x + 1}$ (c) $\frac{x^2 - 1}{x^2 + 1}$

Solution

(a) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(b) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

(c) $\frac{x^2 - 1}{x^2 + 1}$ cannot be simplified further.

Problem 5.1.2 Using the Solution

Find the value of $\frac{x^2 - 1}{x - 1}$ when $x = 1$.

Solution

$\frac{x^2 - 1}{x - 1}$ cannot be defined when $x = 1$.

Problem 5.1.3 Using the Solution

Find the value of $\frac{x^2 - 1}{x + 1}$ when $x = -1$.

Solution

$\frac{x^2 - 1}{x + 1}$ cannot be defined when $x = -1$.

Problem 5.1.4 Using the Solution

Find the value of $\frac{x^2 - 1}{x^2 + 1}$ when $x = 1$.

Solution

$\frac{x^2 - 1}{x^2 + 1} = \frac{1^2 - 1}{1^2 + 1} = \frac{0}{2} = 0$

Problem 5.1.5 Using the Solution

Find the value of $\frac{x^2 - 1}{x^2 + 1}$ when $x = -1$.

Solution

$\frac{x^2 - 1}{x^2 + 1} = \frac{(-1)^2 - 1}{(-1)^2 + 1} = \frac{0}{2} = 0$

Problem 5.1.6 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.7 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.8 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.9 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.10 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.11 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.12 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.13 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.14 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.15 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.16 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.17 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.18 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.19 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.20 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.21 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.22 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.23 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.24 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.25 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.26 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.27 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.28 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.29 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.30 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.31 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.32 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.33 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.34 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.35 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.36 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.37 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.38 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.39 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.40 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.41 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.42 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.43 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.44 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.45 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.46 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.47 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.48 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.49 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.50 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.51 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.52 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.53 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.54 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.55 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.56 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.57 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.58 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.59 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.60 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.61 Using the Solution

(a) Simplify $\frac{x^2 - 1}{x^2 + 1}$ (b) $\frac{x^2 - 1}{x - 1}$ (c) $\frac{x^2 - 1}{x + 1}$

Solution

(a) $\frac{x^2 - 1}{x^2 + 1}$ cannot be further simplified.

(b) $\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$ provided $x \neq 1$

(c) $\frac{x^2 - 1}{x + 1} = \frac{(x+1)(x-1)}{x+1} = x - 1$ provided $x \neq -1$

Problem 5.1.62 Using the Solution

(a) Simplify $\frac{

5.2 Solving Linear Equations I

An **equation** states that two quantities are equal. The most basic type of equation comes from arithmetic. For example,

$$2 + 6 = 8 + 0.$$

You've already seen many examples of this sort of equation.

So far in this book, nearly every equation with variables has been used to say that two expressions are equivalent, such as

$$a + b = b + a.$$

In this section, we introduce equations with a variable such that the equation is true for only some values of the variable. Unfortunately, we don't have the symbol "$=$" to mean that two expressions are equivalent and to write equations that are only true for some values of a variable.

For example, the equation $x + 3 = 9$ does not tell us that $x = 3$ or $x = 6$ for all values of x. If $x = 3$, then $x + 3 = 6$, not 9, so the equation $x + 3 = 9$ is not true when $x = 3$. However, if $x = 6$, then $x + 3 = 9$, so the equation $x + 3 = 9$ is true when $x = 6$. The **solution** to an equation are the values of the variable that make the equation true. So $x = 6$ is a solution to $x + 3 = 9$.

We can also write an equation where the left and right sides are both variable expressions. For example, the equation

$$x + 3 = 4x.$$

1. We can replace any expression with an equivalent expression. For example, in the equation

$$5x - 6x + 3 = 4x$$

we can simplify the left-hand side to $x + 3$ to get the equivalent equation

$$x + 3 = 4x.$$

2. We can perform the same mathematical operation to both sides of the equation. For example, starting with the equation $x + 3 = 4x$, we can subtract 3 from both sides of the equation to get

$$x + 3 - 3 = 4x - 3.$$

Simplifying both sides of the equation then gives $x = 4x$ and we have found the solution to the equation. Looking back to the original equation, $5x - 6x + 3 = 4x$ we see that when we have $x = 1$ we get $5 \cdot 1 - 6 \cdot 1 + 3 = 4 \cdot 1$ ($5 + 3 = 4$) which is indeed a true equation.

Important: If you add, subtract, multiply or divide the expression on one side of the equation by something, then you have to do the same to the expression on the other side of the equation.

We often solve equations with one variable by performing operations on both sides of the equation and simplifying expressions until the variable is alone on one side of the equation. When we do this, we say that we **isolate** the variable.

In this section, we focus on solving **linear equations**. An equation is a linear equation if every term in the equation is a constant term or is a constant times the first power of the variable. So,

$$x + 4x + 5 = 3 - 6x \qquad \text{and} \qquad 2y + 7 = 3 - 2y$$

are linear equations. The equation

$$x^2 = 36 \qquad \text{and} \qquad \frac{3}{y - 1} = 19$$

are not linear equations.

Problems

Problem 5.0 Jump to Solution

Consider the equation $x - 12 = 299$. We will solve this equation in several different ways.

(a) Use your understanding of numbers to find a value of x that makes the equation true.

(b) Use the number line to find a value of x that makes the equation true.

(c) What number can be added to both sides of the equation to give an equation in which x is alone on the left side?

(d) Use part (c) to solve the equation.

Problem 5.9 Jump to Solution

Solve the following equations:

(a) $x - 4 = \frac{3}{2}$

(b) $4 - \frac{3}{2} = 2x + 3 - x + \frac{1}{2}$

Problem 5.10 Jump to Solution

Consider the equation $31x = 713$.

(a) By what number can we divide both sides of the equation to give an equation in which x is alone on the left side?

(b) Solve the equation.

Problem 5.11 Jump to Solution

Solve the following equations:

(a) $5x = -15$

(b) $24 = -75y$

(c) $\frac{3}{y} = \frac{9}{11}$

(d) $\frac{2x}{3} = \frac{9}{2} - \frac{15}{6}$

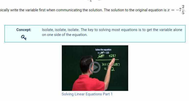
Problem 5.0

Solve the equation $x - 12 = 299$.

Solution for Problem 5.0 We present three different solutions.

Approach: The equation means that 12 less than x equals 299. Since 299 is 12 less than x, we know that x must be 12 more than 299. Therefore, x equals 299 + 12, which is 311.

Number Line: If we draw a number line for the equation $x - 12 = 299$ with x so that 299 is 12 steps to the left of x. This means that x is 12 steps to the right of 299, so x is 299 + 12 = 311.



Algebra: To solve the equation, we manipulate it until it reads $x =$ (a base condition). Therefore, we must get x alone on one side of the equation. To do so, we eliminate the -12 on the left side by adding 12 to both sides of the equation:

$$\begin{aligned} x - 12 &= 299 \\ + 12 &= +12 \\ x &= 311 \end{aligned}$$

We have therefore isolated x on the left side of the equation. We can now see that the solution to the equation $x - 12 = 299$ is $x = 311$.

Whichever method we use to find the solution, we can check our answer by substituting our solution, $x = 311$, back in to the original equation, $x - 12 = 299$, to get $311 - 12 = 299$. This equation is true, so our solution works. $\blacksquare$

Perhaps you noticed that each of our three solution approaches comes down to the same two step, adding 12 to 299 to get our answer. The first uses words, the second uses pictures, the third uses algebra. Whole logic and pictures are sometimes helpful in solving equations, algebraic manipulations are by far the most generally useful tools to solve equations. Try using algebra to solve the following equations.

Problem 5.9

Solve the following equations:

(a) $x - \frac{3}{2} = \frac{9}{2}$

(b) $4 - \frac{3}{2} = 2x + 3 - x + \frac{1}{2}$

Solution for Problem 5.9

(a) We isolate x by adding $\frac{3}{2}$ to both sides:

$$\begin{aligned} x - \tfrac{3}{2} &= \tfrac{9}{2} \\ + \tfrac{3}{2} &= + \tfrac{3}{2} \\ x &= \tfrac{12}{2} = 6 \end{aligned}$$

We finish by adding the mixed numbers on the right side.

We have therefore isolated x on the left side of the equation. We can now see that the solution to the equation $x - 12 = 299$ is $x = 301$.

Whichever method we use to find the solution, we can check our answer by substituting our solution, $x = 301$, back in to the original equation, $x - 12 = 299$, to get $301 - 12 = 299$. This equation is true, so our solution works. $\blacksquare$

Perhaps you noticed that each of our three solution approaches comes down to the same two step, adding 12 to 299 to get our answer. The first uses words, the second uses pictures, the third uses algebra. Whole logic and pictures are sometimes helpful in solving equations, algebraic manipulations are by far the most generally useful tools to solve equations. Try using algebra to solve the following equations.

Problem 5.9

Solve the following equations:

(a) $x - \frac{3}{2} = \frac{9}{2}$

(b) $4 - \frac{3}{2} = 2x + 3 - x + \frac{1}{2}$

Solution for Problem 5.9

(a) We isolate x by adding $\frac{3}{2}$ to both sides:

$$\begin{aligned} x - \tfrac{3}{2} &= \tfrac{9}{2} \\ + \tfrac{3}{2} &= + \tfrac{3}{2} \\ x &= \tfrac{12}{2} = 6 \end{aligned}$$

We finish by adding the mixed numbers on the right side.

We have therefore isolated x on the left side of the equation. We can now see that the solution to the equation $x - 12 = 299$ is $x = 301$.

Whichever method we use to find the solution, we can check our answer by substituting our solution, $x = 301$, back in to the original equation, $x - 12 = 299$, to get $301 - 12 = 299$. This equation is true, so our solution works. $\blacksquare$

Perhaps you noticed that each of our three solution approaches comes down to the same two step, adding 12 to 299 to get our answer. The first uses words, the second uses pictures, the third uses algebra. Whole logic and pictures are sometimes helpful in solving equations, algebraic manipulations are by far the most generally useful tools to solve equations. Try using algebra to solve the following equations.

Concept: Isolate, isolate, isolate. The key to solving most equations is to get the variable alone on one side of the equation.

Addition and subtraction are not the only tools we can use to solve linear equations.

Problem 5.10

Solve the equation $31x = 713$.

Solution for Problem 5.10 We can divide both sides of the equation by 31. This leaves x alone on the left:

$$\frac{31x}{31} = \frac{713}{31}.$$

Since $31x/31 = x$ and $713/31 = 23$, we have $x = 23$. $\blacksquare$

In this solution we used division to change the coefficient of x from 31 to 1. We could also have viewed this as multiplying both sides of the equation by the reciprocal of the coefficient of $31x$ to give $\frac{1}{31} \cdot 31x = \frac{1}{31} \cdot 713$. The $\frac{1}{31}$ and 31 cancel on the left, and we have $x = \frac{713}{31} = 23$.

Problem 5.11

Solve the following equations:

(a) $5x = -15$

(b) $24 = -75y$

(c) $\frac{3}{y} = \frac{9}{11}$

(d) $\frac{2x}{3} = \frac{9}{2} - \frac{15}{6}$

Solution for Problem 5.11

(a) We isolate $5x$ dividing both sides of the equation by 5:

$$\frac{5x}{5} = \frac{-15}{5}.$$

Since $\frac{5x}{5}$ simplifies to x, we have $x = \frac{-15}{5}$ as our solution.

(b) We divide both sides by -75:

$$\frac{24}{-75} = \frac{-75y}{-75}.$$

$\frac{24}{-75} = \frac{-8}{25}$. We usually write the variable first, so we can write this equation as

$$y = \frac{-8}{25}.$$

We finish by simplifying the right-hand side:

$$y = \frac{-8}{25} = \frac{-8 \cdot 4}{25 \cdot 4} = \frac{-32}{100}.$$

Therefore, the solution is $y = -\frac{8}{25}$.

We can check our answer by substituting $y = -\frac{8}{25}$ in the original equation. We see that $-75 \cdot \left(-\frac{8}{25}\right)$ does equal 24, so our answer is correct.

Important: When solving an equation, we can check our answer by substituting our answer back into the original equation. If the original equation is not satisfied by our answer, then we probably made a mistake and should solve the equation again.

(c) To get rid of the 3 in the denominator on the left side, we multiply both sides by 3:

$$3 \cdot \left(\frac{3}{y}\right) = 3 \cdot \left(\frac{9}{11}\right).$$

We have $3 \cdot \left(\frac{3}{y}\right) = 3 \cdot \frac{3}{y} = \frac{9}{y}$ and $3 \cdot \left(\frac{9}{11}\right) = \frac{27}{11}$ so the equation above simplifies to $\frac{9}{y} = \frac{27}{11}$.

(d) At first, it might look like we can't isolate x with one step. But if we write $-\frac{15}{6}$ as $\left(-\frac{5}{2}\right) \cdot 3$, we have

$$\left(\frac{2}{3}\right) \cdot x = \frac{9}{2}.$$

Now, we can isolate x by multiplying both sides of the equation by the reciprocal of the coefficient of x. The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$ and multiplying both sides of the equation by $\frac{3}{2}$ gives

$$\left(\frac{3}{2}\right) \cdot \left(\frac{2}{3}\right) \cdot x = \left(\frac{3}{2}\right) \cdot \frac{9}{2}.$$

The product of a number and its reciprocal is 1, so the left side simplifies to x, as planned. We therefore have

$$x = \left(\frac{3}{2}\right) \cdot \frac{9}{2} = \frac{3 \cdot 9}{2 \cdot 2} = \frac{27}{4}.$$

Checking our work, we find that when $x = \frac{27}{4}$, we have

$$\frac{2x}{3} = \frac{2 \cdot \frac{27}{4}}{3} = \frac{\cancel{54}^{\,27}}{\cancel{12}_{\,4}} = \frac{27}{4}, \qquad \frac{9}{2} - \frac{15}{6} = \frac{27}{6} - \frac{15}{6} = \frac{12}{6} = 2.$$

So, the equation is indeed satisfied when $x = \frac{27}{4}$. $\blacksquare$

Exercises

5.2.1:

Solve each of the following equations:

(a) $x + 235 = 137$

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Subtracting 235 from both sides gives $x = 137 - 235 = \boxed{-98}$.

(b) $x + \frac{1}{2} = \frac{-7}{2}$

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Subtracting $\frac{1}{2}$ from both sides gives $x = -\frac{7}{2} - \frac{1}{2} = \frac{-8}{2} = \boxed{-4}$.

(c) $-\frac{3}{4}x = +11 + x$

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Adding 14 to both sides gives $14 + \frac{3}{4}x = x$, so $x = \boxed{56}$.

(d) $-2y + \frac{3}{2} = 3y + \frac{1}{2}$

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Simplifying the left side gives $y + \frac{3}{2} = \frac{1}{2}$. Subtracting $\frac{3}{2}$ from both sides gives

$$y = \frac{1}{2} - \frac{3}{2} = \frac{-2}{2} = \boxed{-1}. \quad \frac{12}{3} - \frac{17}{3} = \frac{-5}{3}, \quad \frac{28}{7} - \frac{35}{7} = \frac{-7}{7}.$$

5.2.2:

Solve each of the following equations:

(a) $-7y = 343$

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Dividing both sides by -7 gives $y = 343/(-7) = \boxed{-49}$.

(b) $16x = 3\frac{1}{2}$

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We first write the right side as a fraction instead of a mixed number:

$$16x = \frac{7}{2}.$$

Multiplying both sides by $\frac{1}{16}$ (which is the same as dividing both sides by 16) gives

$$x = \frac{1}{16} \cdot \frac{7}{2} = \boxed{\frac{7}{32}}.$$

(c) $\frac{y}{-9} = \frac{6}{15}$

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Multiplying both sides by 5 cancels the 3 in the denominator on the left and gives $x = \frac{6}{5} = \boxed{\frac{12}{5}}$.

(d) $\frac{3y}{-2} = \frac{-15}{10}$

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We can write the left side as $-\frac{3y}{2}$. We can eliminate the coefficient of y in the equation by multiplying both sides by the reciprocal of the coefficient:

$$\left(-\frac{2}{3}\right) \cdot \left(\frac{3}{2}\right) \cdot y = \left(-\frac{2}{3}\right) \cdot \left(\frac{-15}{10}\right).$$

The product of the fractions on the left is 1, and we have $y = \left(-\frac{2}{3}\right) \cdot \left(\frac{-15}{10}\right) = \boxed{\frac{3}{2}}$.

5.2.3:

Solve the equation $\frac{3}{2} + y = 19\frac{1}{2}$.

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Subtracting $\frac{3}{2}$ from both sides gives $-y = 19\frac{1}{2} - 1\frac{1}{2}$. Since $-y = 18\frac{1}{2}$, we have $y = \boxed{-18\frac{1}{2}}$.

5.2.4:

Solve the equation $\frac{x - 2}{-2} = 3$.

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: We don't have a single step that will get us x. However, since $x - 2$ divided by 2 equals 2, and 14 is the number we divide by 7 to get 2, we know that $x - 2$ equals 14. Therefore, $x = \boxed{16}$.

5.2.5:

Solve the equation $3x - 7\frac{1}{2} = 24$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: As in the last problem, we can't isolate x in one step. However, since 3 times $x - 7$ equals 24, and 8 is the number we multiply by 3 to get 24, we know that $x - 7$ equals 8. Therefore, $x = \boxed{15}$.

5.2.6:

Find the value of x such that $x - 2$ is a solution to the equation $\frac{x}{2} = 5$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Save

Your Submission: Solution

Solution: Just as we can multiply both sides of the equation $\frac{x}{2} = 5$ by 2 to get $x = 2 \cdot 5$, we can multiply both sides of $\frac{x - 2}{3} = 3$ by $x - 2$ to get $x - 2 = 3x$. When $x = 2$ this equation becomes $2 = 3x$, and dividing both sides by 3 gives $x = \boxed{\frac{2}{3}}$.

We also could have substituted $x = \frac{2}{3}$ into the equation $\frac{x - 2}{3} = 3$ right away. Since $x - 2$ is a solution to $\frac{x}{2} = 5$, we have $\frac{x - 2}{2} = 5$. Multiplying both sides by x gives $x - 2x$, and dividing by 2 gives $x = \boxed{\frac{2}{3}}$.

5.3 Solving Linear Equations in 1 Variable

Problems

Problem 5.3.1 ▶ Jump to Solution

In this problem, we will solve the equation $9 + 8 = 45$.

Solve the equation by subtracting the appropriate constant from both sides.

Solve the equation by multiplying both sides.

Problem 5.3.2 ▶ Jump to Solution

In this problem, we will solve the equation $\frac{x}{3} + 5 = 8$ by $\frac{x}{3} = 3$, $x = 9$.

Solve the equation by subtracting the appropriate constant from both sides.

Solve the equation by multiplying both sides.

Solve the equation by dividing both sides.

Problem 5.3.3 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} - \frac{2x}{3} = \frac{x}{6} - 1$

(b) $\frac{x}{2} - \frac{x}{3} = \frac{x}{6} + 1$

(c) $\frac{x}{2} - \frac{x}{3} = \frac{x}{6} - 1$

(d) $\frac{x}{2} - \frac{x}{3} = \frac{x}{6} + 1$

(e) $\frac{x}{2} - \frac{x}{3} = \frac{x}{6} - 1$

(f) $\frac{x}{2} - \frac{x}{3} = \frac{x}{6} + 1$

Problem 5.3.4 ▶ Jump to Solution

Solve the following equations:

(a) First all values of x that satisfy $3x + 2 = 2x + 4$, $x + 2 = 2x - 2$.

(b) First all values of x that satisfy $2x + 3 = 3x + 2$, $3x + 2 = 2x + 3$.

Problem 5.3.5 ▶ Jump to Solution

First solve $x + 2 = 4x$ the equations $2x + 3 = 7$ and $x + 2 = 3$. Compare the same solution for x.

In the last section, we used addition and subtraction to solve some equations, and used multiplication and division to solve others. To solve more complicated equations, however, we have to be a little more careful.

Problem 5.3.6 ▶ Jump to Solution

Solve the equation $2 + 3 = 15$.

Solution Problem 5.3.6 This equation doesn't look exactly like any of the equations we already know how to solve. It may not be obvious immediately how to isolate x. However, we can isolate $3x$ by subtracting 2 from both sides:

$$\begin{aligned} 2 + 3 &= 15 \\ 2 + 3 - 2 &= 15 - 2 \\ 3 &= 13 \end{aligned}$$

Now we have an equation we know how to solve! We divide both sides by 3 to get $x = \frac{13}{3}$.

We can check our work by substituting this value back into the original equation: $2 + 3 = 15$. Indeed, $2 + 3 = 15$, so our answer works. We didn't have to add or subtract to solve this equation. We only used multiplication.

We can then distribute on the left side. Since

$$\begin{aligned} &\text{distributing} \\ &\text{gives} \end{aligned}$$

we have

$$\begin{aligned} &\text{distributing} \\ &\text{gives} \end{aligned}$$

We then subtract $\frac{x}{2}$ from both sides of the equation to get $\frac{x}{3} = \frac{x}{2} + 1$ and then $\frac{x}{3} = \frac{x}{2} + 1$ as before. x.

The equation $\frac{x}{3} = \frac{x}{2} + 1$ is the same as the equation we started the problem with, so we can solve it the same way.

Concept: Solving an equation that isn't exactly like an equation you have solved before. Sometimes it helps to look at the problem in a new way. Sometimes it helps to add a new step to the problem.

Now we have an equation we know how to solve! We add 6 to both sides to get $\frac{x}{3} = 7$. We then divide by $\frac{1}{3}$ to find $x = \frac{21}{2}$.

We can check our work by substituting this value back into the original equation: $\frac{x}{3} = \frac{x}{2} + 1$. Indeed, $\frac{x}{3} = \frac{x}{2} + 1$, so our answer works.

Concept: Solving an equation that isn't exactly like an equation you have solved before. Sometimes it helps to add a new step to the problem.

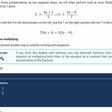
Solution Problem 5.3.6 This equation doesn't look exactly like any of the equations we already know how to solve. It may not be obvious immediately how to isolate x. However, we can isolate $\frac{x}{3}$ by subtracting $\frac{x}{2}$ from both sides:

$$\begin{aligned} \frac{x}{3} &= \frac{x}{2} + 1 \\ \frac{x}{3} - \frac{x}{2} &= \frac{x}{2} + 1 - \frac{x}{2} \\ \frac{x}{3} - \frac{x}{2} &= 1 \end{aligned}$$

Now we have an equation we know how to solve! We add 6 to both sides to get $\frac{x}{3} - \frac{x}{2} = 6$. We then divide by $\frac{1}{3}$ to find $x = \frac{21}{2}$.

We can check our work by substituting this value back into the original equation: $\frac{x}{3} = \frac{x}{2} + 1$. Indeed, $\frac{x}{3} = \frac{x}{2} + 1$, so our answer works.

Concept: Solving an equation that isn't exactly like an equation you have solved before. Sometimes it helps to add a new step to the problem.



Jack and Jill Tumbling Down the Hill

Here's a little more practice.

Problem 5.3.7 ▶ Jump to Solution

Solve the following equations:

(a) $\frac{x}{3} - \frac{2x}{3} = \frac{x}{6} - 1$

(b) $\frac{x}{2} - \frac{x}{3} = \frac{x}{6} + 1$

(c) $\frac{x}{2} - \frac{x}{3} = \frac{x}{6} - 1$

(d) $\frac{x}{2} - \frac{x}{3} = \frac{x}{6} + 1$

(e) $\frac{x}{2} - \frac{x}{3} = \frac{x}{6} - 1$

(f) $\frac{x}{2} - \frac{x}{3} = \frac{x}{6} + 1$

Solution for Problem 5.3.7

(a) Adding $\frac{x}{3}$ to both sides leaves the variable term on the left side, putting all the constant terms on the right:

$$\begin{aligned} \frac{x}{3} - \frac{2x}{3} &= \frac{x}{6} - 1 \\ \frac{x}{3} - \frac{2x}{3} + \frac{x}{3} &= \frac{x}{6} - 1 + \frac{x}{3} \\ \frac{x}{6} &= \frac{x}{6} - 1 + \frac{x}{3} \end{aligned}$$

Simplifying the right-hand side gives

$$\begin{aligned} \frac{x}{6} &= \frac{x}{6} - 1 + \frac{x}{3} \\ &= \frac{x}{6} - 1 + \frac{2x}{6} \\ &= \frac{x}{6} - 1 + \frac{x}{3} \end{aligned}$$

so we now have

$$\begin{aligned} \frac{x}{6} &= \frac{x}{6} - 1 + \frac{x}{3} \\ \frac{x}{6} &= \frac{x}{6} - 1 + \frac{x}{3} \end{aligned}$$

Multiplying both sides by $\frac{1}{6}$ gives the same as dividing both sides by $\frac{1}{6}$ gives

$$\begin{aligned} \frac{x}{6} &= \frac{x}{6} - 1 + \frac{x}{3} \\ \frac{x}{6} &= \frac{x}{6} - 1 + \frac{x}{3} \\ \frac{x}{6} &= \frac{x}{6} - 1 + \frac{x}{3} \\ \frac{x}{6} &= \frac{x}{6} - 1 + \frac{x}{3} \end{aligned}$$

(b) First, we use the distribution property to expand both sides:

$$\begin{aligned} &\text{distributing} \\ &\text{gives} \end{aligned}$$

Simplifying both sides gives

$$\begin{aligned} &\text{distributing} \\ &\text{gives} \end{aligned}$$

Which, we get all the terms with x on one side of the equation and all the constants on the other side. Subtracting $\frac{x}{3}$ from both sides gives $-\frac{2x}{3} + \frac{x}{3} = \frac{x}{6} - 1$ Subtracting $\frac{x}{3}$ from both sides gives $-\frac{2x}{3} + \frac{x}{3} = \frac{x}{6} - 1$ Finally, dividing both sides by $\frac{1}{6}$ gives $x = -2$.

Now, we have $\frac{x}{2} - \frac{x}{3} = \frac{x}{6} + 1$. Subtracting $\frac{x}{2}$ from both sides gives $-\frac{x}{3} = \frac{x}{6} + 1 - \frac{x}{2}$. Adding $\frac{x}{3}$ to both sides gives $0 = \frac{x}{6} + 1 - \frac{x}{2} + \frac{x}{3}$. Finally, we add a common factor $\frac{1}{6}$ to the right side to get $0 = \frac{1}{6}(x + 6 - 3x + 2x)

5.4 Word Problems

Most word problems can be solved using the following general method:

1. Read the problem carefully. Well, I don't actually know what you mean.

2. Convert the words to math.
3. Solve the math.
4. Convert your answer back to words.
5. Check your answer against the original question. (Make sure you solve the question that was asked).

Problems

Problem 5.18 Jump to Solution

Seven more than twice what number equals thirty-five?

Problem 5.19 Jump to Solution

Six plus half of a number equals four plus one-third of the same number. What is the number?

Problem 5.20 *Source: MATHCOUNTS* Jump to Solution

When you add 13 to a number and then divide the sum by 13, you get the same result as when you subtract 13 from the number and then divide the difference by 13. What is the number?

Problem 5.21 Jump to Solution

My sister and I are buying a television for our room. Because I am older, I will pay $45 more than my sister. If the television costs $200, then how much does my sister have to pay?

Problem 5.22 Jump to Solution

I bought a new comic book at the Comic Book Shoppe and paid entirely using quarters. If I had instead paid using only dimes, I would have needed 9 more coins. How much did the comic book cost?

Problem 5.23 Jump to Solution

A garage has 17 cars and motorcycles. Altogether, there are 56 wheels. How many of each type of vehicle are there?

Problem 5.24 Jump to Solution

Three years ago, I was two-thirds as old as I will be eight years from now. How old am I now?

Problem 5.25 Jump to Solution

In softball, a folds is worth 2 points and a glidds is worth 3 points. Kumquam and the Wazzits recently played for the Intergalactic Sheffield Championship. During the game, Kumquam scored eight more folds than the Wazzits, but scored five fewer glidds than the Wazzits. Together the two teams scored 70 points total. What was the final score?

Problem 5.18 ⧉ ↗

Seven more than twice what number equals thirty-five?

Solution for Problem 5.18 The first step is turning many word problems into math is assigning a variable to an unknown quantity. Here, we let x be the unknown number. Now, we can rewrite the problem as

Seven more than twice a number equals thirty-five.

We can write "Seven more than twice x" as $7 + 2x$, so we can rewrite the problem as

$$7 + 2x \text{ equals thirty-five.}$$

Now, it's clear how to write this as an equation:

$$7 + 2x = 35.$$

Subtracting 7 from both sides gives $2x = 28$, and dividing both sides by 2 gives $x = 14$.

Checking our answer, we see that seven more than twice 14 does indeed equal 35. Therefore, the desired number is 14. ◻

Problem 5.19 ⧉ ↗

Six plus half of a number equals four plus one-third of the same number. What is the number?

Solution for Problem 5.19 We again start by assigning a variable, x, to the unknown number. This makes our problem:

Six plus half of a equals four plus one-third of x.

Converting this sentence into an equation gives

$$6 + \frac{1}{2}x = 4 + \frac{1}{3}x.$$

Subtracting $\frac{1}{2}x$ from both sides gives

$$6 + \frac{1}{2}x - \frac{1}{2}x = 4 + \frac{1}{3}x - \frac{1}{2}x = 4,$$

so $6 + \frac{1}{2}x = 4$. Subtracting 6 from both sides gives $\frac{1}{2}x = -2$. Multiplying both sides by 6 gives $x = -12$.

We finish by checking our answer: Six plus half of -12 equals $6 + \frac{1}{2}(-12) = 6 - 6 = 0$ and $4 + \frac{1}{3}(-12)$ also equals 0. So, the number is -12. ◻

Problem 5.20 *Source: MATHCOUNTS* ⧉ ↗

When you add 13 to a number and then divide the sum by 13, you get the same result as when you subtract 13 from the number and then divide the difference by 13. What is the number?

Solution for Problem 5.20 Let's mix it up a little bit. We'll use n for the number this time. We'll also go straight from the words in the problem to the equation:

$$\frac{12 + n}{13} = \frac{n - 13}{13}.$$

Concept: When you write an equation to represent a problem, take a moment to check that your equation does correctly represent the problem before solving the equation.

Multiplying both sides by 12 and by 13 to cancel the denominators gives

$$13(12 + n) = 12(n - 13).$$

Expanding both sides gives

$$144 + 12n = 12n - 169.$$

Subtracting $12n$ from both sides gives $144 = -169$. Adding 169 to both sides gives $n = 313$.

Checking our answer, we see that adding 12 to 313 and dividing the sum by 13 gives 25. Subtracting 13 from 313 and then dividing the difference by 13 also gives 25. So the desired number is indeed 313. ◻

Word Problems Part 1

Of course, you won't often be confronted with problems written in terms of "a number" you must find. Instead, you'll usually be seeking a more meaningful unknown quantity.

Problem 5.21 ⧉ ↗

My sister and I are buying a television for our room. Because I am older, I will pay $45 more than my sister. If the television costs $200, then how much does my sister have to pay?

Solution for Problem 5.21 We don't know how much I pay or how much my sister pays. To which of these quantities should we assign a variable?

Concept: When assigning a variable in a problem with multiple unknown quantities, we usually assign the variable to the unknown quantity we care most about.

We wish to know how much my sister pays, so let s be the number of dollars she pays. We then express how much I pay in terms of my sister's variable.

Concept: We can often express multiple unknown quantities in terms of the same variable.

Since I pay $45 more than my sister does, I must pay $s + 45$ dollars. Together, we spend $200, so we must have

$$(s + 45) + s = 200.$$

Simplifying the left side gives $2s + 45 = 200$. Subtracting 45 from both sides gives $2s = 254$. Dividing both sides by 2 gives $s = 127$. Therefore, my sister pays $127.

Checking our answer, I must pay $127 + $45 = $172. Combining this with the $127 my sister pays gives $172 + $127 = $299, as expected. ◻

Word Problems Part 2

Sometimes it isn't immediately obvious what quantity the variable should represent or a problem.

Problem 5.22 ⧉ ↗

I bought a new comic book at the Comic Book Shoppe and paid entirely using quarters. If I had instead paid using only dimes, I would have needed 9 more coins. How much did the comic book cost?

Solution for Problem 5.22 At first, it looks like we should assign a variable to the cost of the comic book. But it's not immediately clear how we'd relate that to the information about the numbers of quarters and dimes.

Concept: Sometimes you might not find a way to use your first choice for assigning a variable to make an equation. If assigning a variable to the quantity you seek doesn't seem to work, try assigning a variable to a quantity you have information about. This is especially true when you can relate this quantity to what you seek.

We know something about the number of quarters I paid. Also, if we find the number of quarters I paid, then we can figure out how much the comic book costs. So, let q be the number of quarters I paid. In order to pay for the comic book, I would have needed $q + 9$ dimes. Both quarters and $q + 9$ dimes must equal the price of the comic book. Since a quarter is 25¢ cents, and $q + 9$ dimes is $10(q + 9)$ cents, we must have

$$25q = 10(q + 9).$$

Expanding the right side gives $25q = 10q + 90$. Subtracting $10q$ from both sides gives $15q = 90$, and dividing by 15 gives $q = 6$. Therefore, I paid 6 quarters for the comic book, which means the comic book cost $1.50. To check our answer, we note that $6 + 9 = 15 dimes is also $1.50. ◻

Word Problems Part 3

Problem 5.23 ⧉ ↗

A garage has 17 cars and motorcycles. Altogether, there are 56 wheels. How many of each type of vehicle are there?

Solution for Problem 5.23 Let c be the number of cars. Since there are 17 cars and motorcycles total, there are $17 - c$ motorcycles. Since each car has 4 wheels and each motorcycle has 2 wheels, the total number of wheels is $4c + 2(17 - c)$. Therefore, we must have

$$4c + 2(17 - c) = 56.$$

Expanding the left side gives $4c + 34 - 2c = 56$, and simplifying gives $2c + 34 = 56$. Subtracting 34 from both sides gives $2c = 22$ so $c = 11$. This means that there are 11 cars and $17 - 11 = 6$ motorcycles. Checking, we find that 11 cars and 6 motorcycles together have $11 \cdot 4 + 6 \cdot 2 = 56$ wheels. ◻

We also might have solved this problem with a little clever insight. If all 17 vehicles were motorcycles, then there are $17 \cdot 2 = 34$ wheels total. That's $56 - 34 = 22$ wheels too fast. Each time we replace a motorcycle with a car, the number of wheels increases by 2. So, if we start with 17 motorcycles, which together have 22 wheels too few, then we need to replace $22/2 = 11$ motorcycles with 11 cars in order to have 56 wheels total. ◻

Problem 5.24 ⧉ ↗

Three years ago, I was two-thirds as old as I will be eight years from now. How old am I now?

Solution for Problem 5.24 Let my age now be n. Three years ago, my age was $n - 3$, and eight years from now, my age will be $n + 8$. What's wrong with this next step:

Begin Solution: Converting the words in the problem to an equation gives

$$\frac{2}{3}(n - 3) = n + 8.$$

We set the equation up incorrectly. The problem tells us that

My age three years ago — two-thirds my age eight years from now.

Since my age three years ago is $n - 3$, and my age eight years from now is $n + 8$, we have the equation

$$n - 3 = \frac{2}{3}(n + 8).$$

Multiplying both sides by 3 gives $3(n - 3) = 2(n + 8)$. Expanding both sides gives $3n - 9 = 2n + 16$. Subtracting $2n$ from both sides gives $n - 9 = 16$, and adding 9 to both sides gives $n = 25$. Therefore, I'm 25 years old.

Checking, we see that three years ago I was 22, and eight years from now I'll be 33. Since $\frac{2}{3}(33) = 22$, our answer is correct. ◻

In our Bogus Solution to Problem 5.24, we started with the equation $\frac{2}{3}(n - 3) = n + 8$. Suppose we hadn't realized that we wrote the wrong equation, and proceeded to solve the equation. Multiplying both sides by 3 to get rid of the fraction gives

$$3 \cdot \frac{2}{3}(n - 3) = 3(n + 8),$$

so $2(n - 3) = 3(n + 8)$. Expanding both sides gives $2n - 6 = 3n + 24$. Subtracting $2n$ from both sides gives $-6 = n + 24$, and subtracting 24 from both sides gives $-30 = n$. Clearly this is ridiculous. My age can't be negative! This is a strong clue that we made an error somewhere and need to check our work.

WARNING! Always stop a moment to consider whether or not your final answer makes sense.

Problem 5.25 ⧉ ↗

In softball, a folds is worth 2 points and a glidds is worth 3 points. Kumquam and the Wazzits recently played for the Intergalactic Sheffield Championship. During the game, Kumquam scored eight more folds than the Wazzits, but scored five fewer glidds than the Wazzits. Together the two teams scored 70 points total. What was the final score?

Solution for Problem 5.25 Once again, it isn't obvious what quantity a variable should represent. We do know that the two teams together scored 70 points. So, if we let f represent the number of folds scored by the Wazzits' score was $70 - f$. Will that be useful?

What else do we know about the scores of the teams? Kumquam scored eight more folds, which is $f + 8$. Since glidds are worth 3 points, the Kumquam scored five fewer glidds, which is $70 - 5$ = 65 points, than the Wazzits. So, altogether, Kumquam scored 5 fewer points than twice the Wazzits' score. Twice the Wazzits' score is $2f$. We now have two expressions for the same quantity, the Wazzits' score, so we can write the equation

$$f + 5 = 2f - 5.$$

Adding 5 and subtracting f from both sides of $f + 5 = 2f - 5$ gives $10 = f$. Subtracting 10 from both sides gives $0 = f$. Multiplying both sides by 2 gives $0 = 2f$. Adding 5 to both sides gives $5 = 2f + 5$. Substituting $2f + 5$ for $f + 8$ gives $f + 8 = 2f + 5$. Multiplying both sides of $f + 8 = 2f + 5$ by 6 gives $6 = 2f$. $\boxed{3}$

Exercises

5.4.1: ⧉ ↗

Eddie thinks of a number, then doubles the number, and then multiplies the result by 3. If her final number is 55 more than her original number, then what was her original number?

You may type any additional notes you have here. Hide Solution / Reset

Your Solution

Solution: Let x be Eddie's number. Double her number is $2x$, and triple this is $6x$. Since this is 55 greater than her original number, we have $6x = x + 55$. Subtracting x from both sides gives $5x = 55$, and dividing by 5 gives $x = \boxed{11}$.

5.4.2: *Source: MOEMS* ⧉ ↗

If I add $\frac{1}{3}$ of a number, the result is $\frac{2}{3}$ of the number. What is the number?

Preview Solution

You may type any additional notes you have here. Hide Solution / Reset

Your Solution

Solution: Let x be the number. Adding $\frac{1}{3}$ to the number gives $\frac{1}{3} + x$, so we must have $\frac{1}{3} + x = \frac{2}{3}x$. Subtracting $\frac{1}{3}x$ from both sides gives $\frac{1}{3} + \frac{2}{3}x - \frac{1}{3}x = \frac{2}{3}x - \frac{1}{3}x$. Multiplying both sides of $\frac{1}{3} + \frac{1}{3}x = \frac{1}{3}x$ by 6 gives $2 + x = \boxed{2x}$.

5.4.3: ⧉ ↗

One of my dogs is 25 pounds heavier than the other and the two together weigh 137 pounds. How much does the heavier dog weigh?

You may type any additional notes you have here. Hide Solution / Reset

Your Submission

Solution: Let x be the weight of the heavier dog in pounds, so the lighter dog weighs $x - 25$ pounds. Since the two dogs together weigh 137 pounds, we have $x + (x - 25) = 137$. Adding 25 to both sides gives $2x = 162$. Dividing both sides gives $2x = 162$. Dividing both sides by 2 gives $x = 81$, so the heavier dog weighs $\boxed{81}$ pounds.

5.4.4: *Source: MATHCOUNTS* ⧉ ↗

What integer is triplet when one is added to three-fourths of it?

You may type any additional notes you have here. Hide Solution / Reset

Your Submission

Solution: Let n be the integer. Triple the integer is $3n$. The sum of $\frac{3}{4}$ and three-fourths of the integer is $\frac{3}{4} + \frac{3}{4}n$, so we have $3n = \frac{3}{4} + \frac{3}{4}n$. Subtracting $\frac{3}{4}n$ from both sides gives $3n - \frac{3}{4}n = \frac{3}{4}$, so $\frac{9}{4}n = \frac{3}{4}$. Multiplying both sides by $\frac{4}{9}$ gives $n = \boxed{\frac{1}{3}}$.

5.4.5: *Source: MOEMS* ⧉ ↗

The sum of the ages of three children is 32. The age of the oldest is twice the age of the youngest. The two older children differ by three years. What is the age of the youngest child?

You may type any additional notes you have here. Hide Solution / Reset

Your Submission

Solution: Let the age of the youngest child be y. The oldest is twice as old as the youngest, so the oldest is $2y$ years old. The two older children differ by three years, so the middle child is $2y - 3$ years old. The sum of their ages is 32, so $y + 2y + (2y - 3) = 32$. Simplifying the left side gives $5y - 3 = 32$, so $5y = 35$, which means $y = 7$. Therefore, the youngest child is $\boxed{7 \text{ years old}}$.

5.4.6: ⧉ ↗

If the sum of six consecutive even integers is 282, then what is the largest of the integers?

You may type any additional notes you have here. Hide Solution / Reset

Your Submission

Solution: Let the number of two-wheel bikes they have. Since they have 47 bikes total, they must have $47 - 2$ three-wheel bikes. The two-wheel bikes need 2! tires total per bike, and the three-wheel bikes need 3!/2! = 6 tires total, so we must have $2b + 3(47 - b) = 112$. Expanding the product on the left gives $2b + 141 - 3b = 112$. Simplifying the left side gives $141 - b = 112$. Subtracting 141 from both sides gives $-b = -29$, so $b = \boxed{29}$.

We also could have noted that if each of the 47 bikes has 2 wheels, then there are 94 wheels total. That's $112 - 94 = 18$ wheels too few. Each time we replace a two-wheel bike with a three-wheel bike, we gain one wheel. So, if we start with 47 two-wheel bikes, which have too few wheels, and then replace 18 of them with three-wheel bikes, we'll have $94 + 18 = 112$ wheels total and $47 - 18 = \boxed{29}$ two-wheel bikes. ◻

5.4.7: ⧉ ↗

Bobby's Bike Shack orders tires each week for its two-wheel bikes and three-wheel bikes. They order tires for all 47 of their bikes this week. If they ordered 112 tires, how many two-wheel bikes does the Bike Shack have?

You may type any additional notes you have here. Hide Solution / Reset

Your Submission

Solution: Let t be the number of two-wheel bikes they have. Since they have 47 bikes total, they must have $47 - t$ three-wheel bikes. The two-wheel bikes need 2! tires total per bike, and the three-wheel bikes need 3!/2! = 6 tires total, so we must have $2t + 3(47 - t) = 112$. Expanding the product on the left gives $2t + 141 - 3t = 112$. Simplifying the left side gives $141 - t = 112$. Subtracting 141 from both sides gives $-t = -29$, so $t = \boxed{29}$.

We also could have noted that if each of the 47 bikes has 2 wheels, then there are 94 wheels total. That's $112 - 94 = 18$ wheels too few. Each time we replace a two-wheel bike with a three-wheel bike, we gain one wheel. So, if we start with 47 two-wheel bikes, which have too few wheels, and then replace 18 of them with three-wheel bikes, we'll have $94 + 18 = 112$ wheels total and $47 - 18 = \boxed{29}$ two-wheel bikes. ◻

5.4.8: ⧉ ↗

In Problem 5.22, we solved the following problem:

I bought a new comic book at the Comic Book Shoppe and paid entirely using quarters. If I had instead paid using only dimes, I would have needed 9 more coins. How much did the comic book cost?

We first considered assigning a variable to the cost of the comic book, but instead found a solution by assigning a variable to the number of quarters I used to buy the comic book. However, it's possible to find the solution by assigning a variable to the cost of the comic book. How?

Write d is the cost of the comic book in cents, then what other quantities in the problem can you express in terms of d?

You may type any additional notes you have here. Hide Solution / Reset

Your Submission

Solution: If we let c be the number of cents in the cost of the comic book, then $\frac{c}{25}$ is the number of quarters I use to buy the comic book and $\frac{c}{10}$ is the number of dimes I would have used to buy the comic book. Since I would need 9 more dimes than quarters, we have

$$\frac{c}{10} = \frac{c}{25} + 9.$$

Multiplying both sides by 50 gives

$$50\left(\frac{c}{10}\right) = 50\left(\frac{c}{25}\right) + 50 \cdot 9,$$

so $2c = 450 + 5c$. Solving this equation gives $c = 150$, which means the comic book cost $$1.50. ◻

5.4.9: ⧉ ↗

The Phillies won 3 of their first 21 games. How many games in a row after these 21 games do the Phillies have to win in order to have won exactly $\frac{2}{3}$ of the games they have played?

You may type any additional notes you have here. Hide Solution / Reset

Your Submission

Solution: Let g be the number of games in a row the Phillies win. After these games, they have won $3 + g$ games total out of their $21 + g$ games. Since they must win $\frac{2}{3}$ of their games, we must have $3 + g = \frac{2}{3}(21 + g)$. Multiplying both sides by 3 gives $3(3 + g) = 2(21 + g)$. Expanding both sides gives $9 + 3g = 63 + 2g$. Subtracting $2g$ from both sides gives $15 + 2g = 63$, so $2g = 48$, which gives $g = \boxed{24}$.

5.5 Inequalities

Inequality is a mathematical expression that one equal. In this section, you will solve inequalities that are not equal. If we have two numbers a and b, we write $a > b$ if a is greater than b, and $a < b$ if a is less than b.

The $>$ symbol means "greater than", so $5 > 3$ means "5 is greater than 3". The $<$ symbol means "less than", so $2 < 10$ means "2 is less than 10".

Also, $a \geq b$ means "a is greater or equal to b". Similarly, $a \leq b$ means "a is less or equal to b".

Let's look at some examples:

- $5 > 2$ means "5 is greater than 2".
- $3 < 7$ means "3 is less than 7".
- $x \geq 5$ means "x is greater or equal to 5". Similarly, $x \leq 5$ means "x is less or equal to 5".

Tells us that x is greater than 5. For example, x could be 6 or 7 or $\frac{11}{2}$ and so on. x $\geq$ 5 also includes 5. We can graph the values of x that satisfy the inequality on the number line, as shown below:

We draw an open circle at 5 on the number line to indicate that $x > 5$ but it does not include 5. For $x \geq 5$, we draw a closed circle at 5 to indicate that x can be equal to 5. We can also use the number line to find the values of x that satisfy two inequalities at the same time. For example, $x \geq 5$ and $x < 8$ means that x is greater or equal to 5 but less than 8. We can also express the two inequalities as $5 \leq x < 8$. We can graph the values of x that satisfy the two inequalities on the number line:

We draw an open circle at 8 to indicate that $x < 8$ but it does not include 8. We draw a closed circle at 5 to indicate that $x \geq 5$ and x can be equal to 5.

Problems

Problem 1.20 ★ Video by Sofiaan

(a) Write down the inequality $x \geq 3$ as a single number.

(b) Write down the inequality $x \leq 3$ as a single number.

(c) Write down the inequality $x \geq 3$ and $x \leq 3$ as a single number.

Problem 1.21 ★ Video by Sofiaan

(a) Write down the inequality $x > 3$ as a single number.

(b) Write down the inequality $x < 3$ as a single number.

(c) Write down the inequality $x > 3$ and $x < 3$ as a single number.

Problem 1.22 ★ Video by Sofiaan

(a) Suppose $x \geq 2$ and $x \leq 5$. Write down the inequality $2 \leq x \leq 5$ as a single number.

(b) Write down the inequality $2 \leq x \leq 5$ as a single number.

(c) Write down the inequality $x \geq 2$ and $x \leq 5$ as a single number.

Problem 1.23 ★ Video by Sofiaan

(a) Suppose $x > 2$ and $x < 5$. Write down the inequality $2 < x < 5$ as a single number.

(b) Write down the inequality $2 < x < 5$ as a single number.

(c) Write down the inequality $x > 2$ and $x < 5$ as a single number.

Problem 1.24 ★ Video by Sofiaan

(a) Write down the inequality $x \geq 2$ as a single number.

(b) Write down the inequality $x \leq 2$ as a single number.

(c) Write down the inequality $x > 2$ as a single number.

Problem 1.25 ★ Video by Sofiaan

(a) Write down the inequality $x \leq 2$ as a single number.

(b) Write down the inequality $x < 2$ as a single number.

(c) Write down the inequality $x \geq 2$ and $x \leq 5$ as a single number.

Problem 1.26 ★ Video by Sofiaan

(a) Write down the inequality $x < 2$ as a single number.

(b) Write down the inequality $x \geq 2$ as a single number.

(c) Write down the inequality $x \leq 2$ and $x < 5$ as a single number.

Problem 1.27 ★ Video by Sofiaan

(a) Write down the inequality $x \leq 2$ and $x \geq 5$ as a single number.

(b) Write down the inequality $x < 2$ and $x \geq 5$ as a single number.

(c) Write down the inequality $x \leq 2$ and $x < 5$ as a single number.

Problem 1.28 ★ Video by Sofiaan

(a) Write down the inequality $x < 2$ and $x \geq 5$ as a single number.

(b) Write down the inequality $x \leq 2$ and $x \geq 5$ as a single number.

(c) Write down the inequality $x < 2$ and $x < 5$ as a single number.

Problem 1.29 ★ Video by Sofiaan

In this problem, we consider single inequalities where one multiply both sides of an inequality by a positive number. Suppose that $x > y$.

(a) What will happen to $\frac{x}{2}$ and $\frac{y}{2}$?

(b) What will happen to $\frac{x}{3}$ and $\frac{y}{3}$?

(c) What will happen to $\frac{x}{4}$ and $\frac{y}{4}$? Write a general rule for multiplying both sides of an inequality by a positive number. Suppose that $x > y$.

(d) What is $\frac{x}{10}$ and $\frac{y}{10}$?

(e) What is $\frac{x}{100}$ and $\frac{y}{100}$?

Problem 1.30 ★ Video by Sofiaan

In this problem, we consider single inequalities where one multiply both sides of an inequality by a negative number. Suppose that $x > y$.

(a) What is $-x$ and $-y$?

(b) What is $-2x$ and $-2y$?

(c) What is $-3x$ and $-3y$?

(d) What is $-4x$ and $-4y$? Write a general rule for multiplying both sides of an inequality by a negative number. Suppose that $x > y$.

(e) What is $-10x$ and $-10y$?

(f) What is $-100x$ and $-100y$?

Problem 1.31 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 1 > y + ___$

(b) $x - 1 > y - ___$

(c) $x + 10 > y + ___$

(d) $x - 10 > y - ___$

(e) $x + 100 > y + ___$

(f) $x - 100 > y - ___$

Problem 1.32 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + y > y + ___$

(b) $x - y > y - ___$

(c) $x + y > x + ___$

(d) $x - y > x - ___$

(e) $x + y > ___ + y$

(f) $x - y > ___ - y$

Problem 1.33 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2y > y + ___$

(b) $x - 2y > y - ___$

(c) $x + 2y > x + ___$

(d) $x - 2y > x - ___$

(e) $x + 2y > ___ + y$

(f) $x - 2y > ___ - y$

Problem 1.34 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3y > y + ___$

(b) $x - 3y > y - ___$

(c) $x + 3y > x + ___$

(d) $x - 3y > x - ___$

(e) $x + 3y > ___ + y$

(f) $x - 3y > ___ - y$

Problem 1.35 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4y > y + ___$

(b) $x - 4y > y - ___$

(c) $x + 4y > x + ___$

(d) $x - 4y > x - ___$

(e) $x + 4y > ___ + y$

(f) $x - 4y > ___ - y$

Problem 1.36 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10y > y + ___$

(b) $x - 10y > y - ___$

(c) $x + 10y > x + ___$

(d) $x - 10y > x - ___$

(e) $x + 10y > ___ + y$

(f) $x - 10y > ___ - y$

Problem 1.37 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100y > y + ___$

(b) $x - 100y > y - ___$

(c) $x + 100y > x + ___$

(d) $x - 100y > x - ___$

(e) $x + 100y > ___ + y$

(f) $x - 100y > ___ - y$

Problem 1.38 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2x > y + ___$

(b) $x - 2x > y - ___$

(c) $x + 2x > x + ___$

(d) $x - 2x > x - ___$

(e) $x + 2x > ___ + y$

(f) $x - 2x > ___ - y$

Problem 1.39 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3x > y + ___$

(b) $x - 3x > y - ___$

(c) $x + 3x > x + ___$

(d) $x - 3x > x - ___$

(e) $x + 3x > ___ + y$

(f) $x - 3x > ___ - y$

Problem 1.40 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4x > y + ___$

(b) $x - 4x > y - ___$

(c) $x + 4x > x + ___$

(d) $x - 4x > x - ___$

(e) $x + 4x > ___ + y$

(f) $x - 4x > ___ - y$

Problem 1.41 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10x > y + ___$

(b) $x - 10x > y - ___$

(c) $x + 10x > x + ___$

(d) $x - 10x > x - ___$

(e) $x + 10x > ___ + y$

(f) $x - 10x > ___ - y$

Problem 1.42 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100x > y + ___$

(b) $x - 100x > y - ___$

(c) $x + 100x > x + ___$

(d) $x - 100x > x - ___$

(e) $x + 100x > ___ + y$

(f) $x - 100x > ___ - y$

Problem 1.43 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + y > y + ___$

(b) $x - y > y - ___$

(c) $x + y > x + ___$

(d) $x - y > x - ___$

(e) $x + y > ___ + y$

(f) $x - y > ___ - y$

Problem 1.44 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2y > y + ___$

(b) $x - 2y > y - ___$

(c) $x + 2y > x + ___$

(d) $x - 2y > x - ___$

(e) $x + 2y > ___ + y$

(f) $x - 2y > ___ - y$

Problem 1.45 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3y > y + ___$

(b) $x - 3y > y - ___$

(c) $x + 3y > x + ___$

(d) $x - 3y > x - ___$

(e) $x + 3y > ___ + y$

(f) $x - 3y > ___ - y$

Problem 1.46 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4y > y + ___$

(b) $x - 4y > y - ___$

(c) $x + 4y > x + ___$

(d) $x - 4y > x - ___$

(e) $x + 4y > ___ + y$

(f) $x - 4y > ___ - y$

Problem 1.47 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10y > y + ___$

(b) $x - 10y > y - ___$

(c) $x + 10y > x + ___$

(d) $x - 10y > x - ___$

(e) $x + 10y > ___ + y$

(f) $x - 10y > ___ - y$

Problem 1.48 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100y > y + ___$

(b) $x - 100y > y - ___$

(c) $x + 100y > x + ___$

(d) $x - 100y > x - ___$

(e) $x + 100y > ___ + y$

(f) $x - 100y > ___ - y$

Problem 1.49 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2x > y + ___$

(b) $x - 2x > y - ___$

(c) $x + 2x > x + ___$

(d) $x - 2x > x - ___$

(e) $x + 2x > ___ + y$

(f) $x - 2x > ___ - y$

Problem 1.50 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3x > y + ___$

(b) $x - 3x > y - ___$

(c) $x + 3x > x + ___$

(d) $x - 3x > x - ___$

(e) $x + 3x > ___ + y$

(f) $x - 3x > ___ - y$

Problem 1.51 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4x > y + ___$

(b) $x - 4x > y - ___$

(c) $x + 4x > x + ___$

(d) $x - 4x > x - ___$

(e) $x + 4x > ___ + y$

(f) $x - 4x > ___ - y$

Problem 1.52 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10x > y + ___$

(b) $x - 10x > y - ___$

(c) $x + 10x > x + ___$

(d) $x - 10x > x - ___$

(e) $x + 10x > ___ + y$

(f) $x - 10x > ___ - y$

Problem 1.53 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100x > y + ___$

(b) $x - 100x > y - ___$

(c) $x + 100x > x + ___$

(d) $x - 100x > x - ___$

(e) $x + 100x > ___ + y$

(f) $x - 100x > ___ - y$

Problem 1.54 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + y > y + ___$

(b) $x - y > y - ___$

(c) $x + y > x + ___$

(d) $x - y > x - ___$

(e) $x + y > ___ + y$

(f) $x - y > ___ - y$

Problem 1.55 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2y > y + ___$

(b) $x - 2y > y - ___$

(c) $x + 2y > x + ___$

(d) $x - 2y > x - ___$

(e) $x + 2y > ___ + y$

(f) $x - 2y > ___ - y$

Problem 1.56 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3y > y + ___$

(b) $x - 3y > y - ___$

(c) $x + 3y > x + ___$

(d) $x - 3y > x - ___$

(e) $x + 3y > ___ + y$

(f) $x - 3y > ___ - y$

Problem 1.57 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4y > y + ___$

(b) $x - 4y > y - ___$

(c) $x + 4y > x + ___$

(d) $x - 4y > x - ___$

(e) $x + 4y > ___ + y$

(f) $x - 4y > ___ - y$

Problem 1.58 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10y > y + ___$

(b) $x - 10y > y - ___$

(c) $x + 10y > x + ___$

(d) $x - 10y > x - ___$

(e) $x + 10y > ___ + y$

(f) $x - 10y > ___ - y$

Problem 1.59 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100y > y + ___$

(b) $x - 100y > y - ___$

(c) $x + 100y > x + ___$

(d) $x - 100y > x - ___$

(e) $x + 100y > ___ + y$

(f) $x - 100y > ___ - y$

Problem 1.60 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2x > y + ___$

(b) $x - 2x > y - ___$

(c) $x + 2x > x + ___$

(d) $x - 2x > x - ___$

(e) $x + 2x > ___ + y$

(f) $x - 2x > ___ - y$

Problem 1.61 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3x > y + ___$

(b) $x - 3x > y - ___$

(c) $x + 3x > x + ___$

(d) $x - 3x > x - ___$

(e) $x + 3x > ___ + y$

(f) $x - 3x > ___ - y$

Problem 1.62 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4x > y + ___$

(b) $x - 4x > y - ___$

(c) $x + 4x > x + ___$

(d) $x - 4x > x - ___$

(e) $x + 4x > ___ + y$

(f) $x - 4x > ___ - y$

Problem 1.63 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10x > y + ___$

(b) $x - 10x > y - ___$

(c) $x + 10x > x + ___$

(d) $x - 10x > x - ___$

(e) $x + 10x > ___ + y$

(f) $x - 10x > ___ - y$

Problem 1.64 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100x > y + ___$

(b) $x - 100x > y - ___$

(c) $x + 100x > x + ___$

(d) $x - 100x > x - ___$

(e) $x + 100x > ___ + y$

(f) $x - 100x > ___ - y$

Problem 1.65 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + y > y + ___$

(b) $x - y > y - ___$

(c) $x + y > x + ___$

(d) $x - y > x - ___$

(e) $x + y > ___ + y$

(f) $x - y > ___ - y$

Problem 1.66 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2y > y + ___$

(b) $x - 2y > y - ___$

(c) $x + 2y > x + ___$

(d) $x - 2y > x - ___$

(e) $x + 2y > ___ + y$

(f) $x - 2y > ___ - y$

Problem 1.67 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3y > y + ___$

(b) $x - 3y > y - ___$

(c) $x + 3y > x + ___$

(d) $x - 3y > x - ___$

(e) $x + 3y > ___ + y$

(f) $x - 3y > ___ - y$

Problem 1.68 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4y > y + ___$

(b) $x - 4y > y - ___$

(c) $x + 4y > x + ___$

(d) $x - 4y > x - ___$

(e) $x + 4y > ___ + y$

(f) $x - 4y > ___ - y$

Problem 1.69 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10y > y + ___$

(b) $x - 10y > y - ___$

(c) $x + 10y > x + ___$

(d) $x - 10y > x - ___$

(e) $x + 10y > ___ + y$

(f) $x - 10y > ___ - y$

Problem 1.70 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100y > y + ___$

(b) $x - 100y > y - ___$

(c) $x + 100y > x + ___$

(d) $x - 100y > x - ___$

(e) $x + 100y > ___ + y$

(f) $x - 100y > ___ - y$

Problem 1.71 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2x > y + ___$

(b) $x - 2x > y - ___$

(c) $x + 2x > x + ___$

(d) $x - 2x > x - ___$

(e) $x + 2x > ___ + y$

(f) $x - 2x > ___ - y$

Problem 1.72 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3x > y + ___$

(b) $x - 3x > y - ___$

(c) $x + 3x > x + ___$

(d) $x - 3x > x - ___$

(e) $x + 3x > ___ + y$

(f) $x - 3x > ___ - y$

Problem 1.73 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4x > y + ___$

(b) $x - 4x > y - ___$

(c) $x + 4x > x + ___$

(d) $x - 4x > x - ___$

(e) $x + 4x > ___ + y$

(f) $x - 4x > ___ - y$

Problem 1.74 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10x > y + ___$

(b) $x - 10x > y - ___$

(c) $x + 10x > x + ___$

(d) $x - 10x > x - ___$

(e) $x + 10x > ___ + y$

(f) $x - 10x > ___ - y$

Problem 1.75 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100x > y + ___$

(b) $x - 100x > y - ___$

(c) $x + 100x > x + ___$

(d) $x - 100x > x - ___$

(e) $x + 100x > ___ + y$

(f) $x - 100x > ___ - y$

Problem 1.76 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + y > y + ___$

(b) $x - y > y - ___$

(c) $x + y > x + ___$

(d) $x - y > x - ___$

(e) $x + y > ___ + y$

(f) $x - y > ___ - y$

Problem 1.77 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2y > y + ___$

(b) $x - 2y > y - ___$

(c) $x + 2y > x + ___$

(d) $x - 2y > x - ___$

(e) $x + 2y > ___ + y$

(f) $x - 2y > ___ - y$

Problem 1.78 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3y > y + ___$

(b) $x - 3y > y - ___$

(c) $x + 3y > x + ___$

(d) $x - 3y > x - ___$

(e) $x + 3y > ___ + y$

(f) $x - 3y > ___ - y$

Problem 1.79 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4y > y + ___$

(b) $x - 4y > y - ___$

(c) $x + 4y > x + ___$

(d) $x - 4y > x - ___$

(e) $x + 4y > ___ + y$

(f) $x - 4y > ___ - y$

Problem 1.80 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10y > y + ___$

(b) $x - 10y > y - ___$

(c) $x + 10y > x + ___$

(d) $x - 10y > x - ___$

(e) $x + 10y > ___ + y$

(f) $x - 10y > ___ - y$

Problem 1.81 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100y > y + ___$

(b) $x - 100y > y - ___$

(c) $x + 100y > x + ___$

(d) $x - 100y > x - ___$

(e) $x + 100y > ___ + y$

(f) $x - 100y > ___ - y$

Problem 1.82 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2x > y + ___$

(b) $x - 2x > y - ___$

(c) $x + 2x > x + ___$

(d) $x - 2x > x - ___$

(e) $x + 2x > ___ + y$

(f) $x - 2x > ___ - y$

Problem 1.83 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3x > y + ___$

(b) $x - 3x > y - ___$

(c) $x + 3x > x + ___$

(d) $x - 3x > x - ___$

(e) $x + 3x > ___ + y$

(f) $x - 3x > ___ - y$

Problem 1.84 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4x > y + ___$

(b) $x - 4x > y - ___$

(c) $x + 4x > x + ___$

(d) $x - 4x > x - ___$

(e) $x + 4x > ___ + y$

(f) $x - 4x > ___ - y$

Problem 1.85 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10x > y + ___$

(b) $x - 10x > y - ___$

(c) $x + 10x > x + ___$

(d) $x - 10x > x - ___$

(e) $x + 10x > ___ + y$

(f) $x - 10x > ___ - y$

Problem 1.86 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100x > y + ___$

(b) $x - 100x > y - ___$

(c) $x + 100x > x + ___$

(d) $x - 100x > x - ___$

(e) $x + 100x > ___ + y$

(f) $x - 100x > ___ - y$

Problem 1.87 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + y > y + ___$

(b) $x - y > y - ___$

(c) $x + y > x + ___$

(d) $x - y > x - ___$

(e) $x + y > ___ + y$

(f) $x - y > ___ - y$

Problem 1.88 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2y > y + ___$

(b) $x - 2y > y - ___$

(c) $x + 2y > x + ___$

(d) $x - 2y > x - ___$

(e) $x + 2y > ___ + y$

(f) $x - 2y > ___ - y$

Problem 1.89 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3y > y + ___$

(b) $x - 3y > y - ___$

(c) $x + 3y > x + ___$

(d) $x - 3y > x - ___$

(e) $x + 3y > ___ + y$

(f) $x - 3y > ___ - y$

Problem 1.90 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4y > y + ___$

(b) $x - 4y > y - ___$

(c) $x + 4y > x + ___$

(d) $x - 4y > x - ___$

(e) $x + 4y > ___ + y$

(f) $x - 4y > ___ - y$

Problem 1.91 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10y > y + ___$

(b) $x - 10y > y - ___$

(c) $x + 10y > x + ___$

(d) $x - 10y > x - ___$

(e) $x + 10y > ___ + y$

(f) $x - 10y > ___ - y$

Problem 1.92 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100y > y + ___$

(b) $x - 100y > y - ___$

(c) $x + 100y > x + ___$

(d) $x - 100y > x - ___$

(e) $x + 100y > ___ + y$

(f) $x - 100y > ___ - y$

Problem 1.93 ★ Video by Sofiaan

In this problem, we consider single inequalities where one multiply both sides of an inequality by a positive number. Suppose that $x > y$.

(a) What will happen to $\frac{x}{2}$ and $\frac{y}{2}$?

(b) What will happen to $\frac{x}{3}$ and $\frac{y}{3}$?

(c) What will happen to $\frac{x}{4}$ and $\frac{y}{4}$? Write a general rule for multiplying both sides of an inequality by a positive number. Suppose that $x > y$.

(d) What is $\frac{x}{10}$ and $\frac{y}{10}$?

(e) What is $\frac{x}{100}$ and $\frac{y}{100}$?

Problem 1.94 ★ Video by Sofiaan

In this problem, we consider single inequalities where one multiply both sides of an inequality by a negative number. Suppose that $x > y$.

(a) What is $-x$ and $-y$?

(b) What is $-2x$ and $-2y$?

(c) What is $-3x$ and $-3y$?

(d) What is $-4x$ and $-4y$? Write a general rule for multiplying both sides of an inequality by a negative number. Suppose that $x > y$.

(e) What is $-10x$ and $-10y$?

(f) What is $-100x$ and $-100y$?

Problem 1.95 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 1 > y + ___$

(b) $x - 1 > y - ___$

(c) $x + 10 > y + ___$

(d) $x - 10 > y - ___$

(e) $x + 100 > y + ___$

(f) $x - 100 > y - ___$

Problem 1.96 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + y > y + ___$

(b) $x - y > y - ___$

(c) $x + y > x + ___$

(d) $x - y > x - ___$

(e) $x + y > ___ + y$

(f) $x - y > ___ - y$

Problem 1.97 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2y > y + ___$

(b) $x - 2y > y - ___$

(c) $x + 2y > x + ___$

(d) $x - 2y > x - ___$

(e) $x + 2y > ___ + y$

(f) $x - 2y > ___ - y$

Problem 1.98 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3y > y + ___$

(b) $x - 3y > y - ___$

(c) $x + 3y > x + ___$

(d) $x - 3y > x - ___$

(e) $x + 3y > ___ + y$

(f) $x - 3y > ___ - y$

Problem 1.99 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4y > y + ___$

(b) $x - 4y > y - ___$

(c) $x + 4y > x + ___$

(d) $x - 4y > x - ___$

(e) $x + 4y > ___ + y$

(f) $x - 4y > ___ - y$

Problem 1.100 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10y > y + ___$

(b) $x - 10y > y - ___$

(c) $x + 10y > x + ___$

(d) $x - 10y > x - ___$

(e) $x + 10y > ___ + y$

(f) $x - 10y > ___ - y$

Problem 1.101 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100y > y + ___$

(b) $x - 100y > y - ___$

(c) $x + 100y > x + ___$

(d) $x - 100y > x - ___$

(e) $x + 100y > ___ + y$

(f) $x - 100y > ___ - y$

Problem 1.102 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2x > y + ___$

(b) $x - 2x > y - ___$

(c) $x + 2x > x + ___$

(d) $x - 2x > x - ___$

(e) $x + 2x > ___ + y$

(f) $x - 2x > ___ - y$

Problem 1.103 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3x > y + ___$

(b) $x - 3x > y - ___$

(c) $x + 3x > x + ___$

(d) $x - 3x > x - ___$

(e) $x + 3x > ___ + y$

(f) $x - 3x > ___ - y$

Problem 1.104 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4x > y + ___$

(b) $x - 4x > y - ___$

(c) $x + 4x > x + ___$

(d) $x - 4x > x - ___$

(e) $x + 4x > ___ + y$

(f) $x - 4x > ___ - y$

Problem 1.105 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10x > y + ___$

(b) $x - 10x > y - ___$

(c) $x + 10x > x + ___$

(d) $x - 10x > x - ___$

(e) $x + 10x > ___ + y$

(f) $x - 10x > ___ - y$

Problem 1.106 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100x > y + ___$

(b) $x - 100x > y - ___$

(c) $x + 100x > x + ___$

(d) $x - 100x > x - ___$

(e) $x + 100x > ___ + y$

(f) $x - 100x > ___ - y$

Problem 1.107 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + y > y + ___$

(b) $x - y > y - ___$

(c) $x + y > x + ___$

(d) $x - y > x - ___$

(e) $x + y > ___ + y$

(f) $x - y > ___ - y$

Problem 1.108 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2y > y + ___$

(b) $x - 2y > y - ___$

(c) $x + 2y > x + ___$

(d) $x - 2y > x - ___$

(e) $x + 2y > ___ + y$

(f) $x - 2y > ___ - y$

Problem 1.109 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 3y > y + ___$

(b) $x - 3y > y - ___$

(c) $x + 3y > x + ___$

(d) $x - 3y > x - ___$

(e) $x + 3y > ___ + y$

(f) $x - 3y > ___ - y$

Problem 1.110 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 4y > y + ___$

(b) $x - 4y > y - ___$

(c) $x + 4y > x + ___$

(d) $x - 4y > x - ___$

(e) $x + 4y > ___ + y$

(f) $x - 4y > ___ - y$

Problem 1.111 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 10y > y + ___$

(b) $x - 10y > y - ___$

(c) $x + 10y > x + ___$

(d) $x - 10y > x - ___$

(e) $x + 10y > ___ + y$

(f) $x - 10y > ___ - y$

Problem 1.112 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 100y > y + ___$

(b) $x - 100y > y - ___$

(c) $x + 100y > x + ___$

(d) $x - 100y > x - ___$

(e) $x + 100y > ___ + y$

(f) $x - 100y > ___ - y$

Problem 1.113 ★ Video by Sofiaan

Suppose $x > y$. Fill in the blanks:

(a) $x + 2x > y + ___$

(b) $x - 2x > y - ___$

(c) $x + 2x > x + ___$

(d) $x - 2x > x - ___$

(e) $x + 2x > ___ + y$

(f) $x - 2x > ___ - y$

Problem 1.114 ★ Video by Sofiaan

Suppose $x

5.6 Summary

In this chapter, we learned how to solve **one-variable linear equations**. By "solving" an equation, we mean finding all values of the variable for which the equation is true. The "one-variable" in "one-variable linear equation" means that only one variable appears in the equation, though it may appear multiple times. The "linear" means that every term in the equation is a constant term or is a constant times the first power of the variable.

We say that we **solve** an equation when we find all values of the variable that make the equation true. The two most important tactics we use to solve equations are:

1. *We can replace any expression with an equivalent expression*. For example, in the equation

$$5x - 4x + 3 = 14,$$

we can simplify the left-hand side to $x + 3$, so the equation becomes

$$x + 3 = 14.$$

2. *We can perform the same mathematical operation to both sides of the equation*. For example, starting with the equation $x + 3 = 14$, we can subtract 3 from both sides of the equation to get

$$x + 3 - 3 = 14 - 3.$$

Simplifying both sides of the equation then gives $x = 11$, and we have found the solution to the equation. Looking back to the original equation, $5x - 4x + 3 = 14$, we see that when we have $x = 11$, we get $5 \cdot 11 - 4 \cdot 11 + 3 = 14$, which is indeed a true equation.

Important:	When solving an equation, we can check our answer by substituting it back into the original equation. If the original equation is not satisfied by our answer, then we made a mistake.

Important:	If a linear equation can be manipulated into an equation that is never true (such as $-1 = -5$), then there are no solutions to the equation. Similarly, if a linear equation can be manipulated into an equation that is always true (such as $4r + 5 = 4r + 5$), then all possible values of the variable are solutions to the original equation.

We can often solve word problems by turning them into linear equations.

Important:	The key to solving word problems is converting the words into the language of mathematics. To do so, assign a variable to be a quantity you seek. Then, try to build an equation to solve for that variable.

WARNING!!	When solving a word problem, define your variable clearly and use it exactly as you've defined it.

The statement $x > y$ means that x is greater than y. Similarly, $x < y$ means that x is less than y. Both $x > y$ and $x < y$ are **inequalities**. More specifically, they are strict inequalities, because in both cases we cannot have $x = y$.

We can also write nonstrict inequalities, such as $x \geq y$, which means that x is greater than or equal to y. Similarly, $x \leq y$ means that x is less than or equal to y.

Important:	Here are several useful rules regarding ways in which we can manipulate inequalities: ▪ If $a > b$ and $b > c$, then $a > c$. ▪ If $x > y$, then $x + c > y + c$ for any number c. If we also have $a > b$, then $x + a > y + b$. ▪ If $x > y$ and $a > 0$, then $xa > ya$.

Important:	If we multiply or divide an inequality by a negative number, we must reverse the direction of the inequality. For example, if $x > y$ and $a < 0$, then $xa < ya$.

Similar rules hold for nonstrict inequalities. For example, if $a \geq b$ and $b \geq c$, then $a \geq c$.

WARNING!!	Inequality rules that work when all the variables are positive don't always work when some of the variables are negative! Be careful when dealing with negative numbers (or expressions that can be negative) in inequalities.

Challenge Problems

5.56:

Source: MATHCOUNTS

Solve for x: $\frac{x}{2} = \frac{11}{12}$.

Hint: Consider a simpler problem. How would you handle the problem if you replaced the mixed numbers with integers?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: First, we write the mixed numbers as fractions, which gives

$$\frac{x}{2} = \frac{11}{12}.$$

Therefore, we have $\frac{2x}{2} = \frac{3}{2} \cdot \frac{11}{12} = \frac{11}{8}$. Multiplying both sides by 2x gives $\frac{2}{2} \cdot x = \frac{2x}{2} \cdot x = \frac{11}{8} \cdot 2x$, so $x = \frac{11}{8} \cdot 2x$ thus gives us $2x = 3$, and dividing by 3 gives $x = \boxed{\frac{11}{12}}$.

5.57:

What values of x satisfy the inequality $\frac{1}{x} \geq 0$?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We begin by temporarily multiply both sides by x and write $\frac{1}{x} \geq 0x$, so $\frac{1}{x} \geq 0$. However, we have to be careful. First, x cannot be 0, since we cannot divide by 0. Second, if x is negative, then we have to reverse the direction of the inequality symbol. But looking back at the original inequality, we can see that x cannot be negative, since this would make the greater side of $\frac{1}{x} \geq 0$ negative while the other side is positive. Therefore, x must be positive, which means we can indeed multiply by x to produce $\frac{1}{x} \geq 0x$. Dividing by 0 gives $\frac{1}{0} \geq x$, but we must remember that x has to be positive. So, the full solution is $0 < x \leq \frac{1}{0}$ or

$$\boxed{\text{all positive numbers (less than or equal to } \tfrac{1}{0}\text{)}}$$

5.58:

Kayla adds the same number to both the numerator and denominator of the fraction $\frac{1}{10}$. Her resulting fraction equals $\frac{1}{3}$. What number did she add to both the numerator and denominator of her original fraction?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let x be Kayla's number, so we have equal $\frac{1+x}{10+x} = \frac{1}{3}$. Multiplying both sides by 3 and by $10 + x$ gives

$$3 \cdot (10 + x) \cdot \frac{1+x}{10+x} = 3 \cdot \frac{1}{3} \cdot (10 + x).$$

The right-hand side is simply $2(10 + x)$ while the left side simplifies as

$$3 \cdot (10 + x) \cdot \frac{1+x}{10+x} = \frac{(10+x)(1+x)}{10+x}$$
$$= \frac{10(1+x)}{10+x}$$
$$= 2(1 + x).$$

So, our equation is now $2(1 + x) = 2(10 + x)$. Expanding both sides gives $2 + 2x = 20 + 2x$. Subtracting $2x$ and 2 from both sides gives $x = \boxed{18}$.

5.59:

Douglas writes down his favorite number, which is a two-digit positive integer. He then turns the number into a three-digit number by writing a 1 at the end of the number. This new number is 355 more than Douglas's favorite number. What is Douglas's favorite number?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let x be Douglas's favorite number. Next, we have to figure out how to represent the number Douglas makes when he writes a 1 at the end of his favorite number. We know his number is a two-digit number that results when we write a 0 at the end of a number. That's just 10 times the original number. So Douglas put a 0 at the end of his favorite number, he would make the number $10x$. The number he makes when he puts a 1 at the end of his favorite number is 1 greater than the number he would make by putting a 0 at the end, so it is $10x + 1$. We know that this number is 355 more than Douglas's favorite number, so we have that $10x + 1 = x + 355$. Subtracting x and 1 from both sides gives $9x = 354$, and dividing by 9 gives $x = \boxed{39}$.

5.60:

Source: MATHCOUNTS

Solve for y: $\frac{y}{36} = \frac{m+n}{6} + \frac{p-m}{120}$.

Hint: One step at a time. What variable can you solve for immediately?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We start with the first two fractions. $\frac{y}{36}$. Multiplying both sides by 72 gives $y = \frac{m+n}{6} \cdot 72 = 60$. Substituting this value in for y gives

$$\frac{5}{6} = \frac{60}{36} = \frac{m+60}{6} + \frac{p-m}{120}.$$

So, now we have $\frac{5}{6} = \frac{m+60}{6}$. Multiplying both sides by 84 gives $m + 60 = \frac{5}{6} \cdot 84 = 70$, so $m = 10$. Substituting this value in for m gives

$$\frac{5}{6} = \frac{70}{6} = \frac{10+60}{6} + \frac{p-10}{120}.$$

So, now we have $\frac{5}{6} = \frac{70}{120}$. Multiplying both sides by 120 gives $p - 10 = \frac{5}{6} \cdot 120 = 100$, so $p = \boxed{110}$.

5.61:

Suppose that x is nonzero.

(a) Express $\frac{x}{x} + \frac{x}{x}$ as a single fraction.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The denominators of the fractions are the same, so we can apply the distributive property

$$\frac{1}{x} + \frac{x}{x} = \frac{1+x}{x} = \boxed{\frac{x+1}{x}}$$

(b) Express $\frac{5}{x} + \frac{3}{2x}$ as a single fraction.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: If we write the denominators the same, we can apply the distributive property. The second denominator is twice the first, so we multiply the numerator and denominator of the second fraction by 2:

$$\frac{5}{x} + \frac{3}{2x} = \frac{5}{x} + \frac{3 \cdot 2}{2x \cdot 2} = \frac{5}{x} + \frac{6}{4x} = \frac{5 \cdot 4}{4x} + \frac{6}{4x} = \boxed{\frac{26}{4x}}$$

(c) Express $\frac{7}{16x} - \frac{3}{10x}$ as a single fraction.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The least common multiple of 16 and 10 is 80, and we have

$$\frac{7}{16x} - \frac{3}{10x} = \frac{7}{16x} \cdot \frac{5}{5} - \frac{3}{10x} \cdot \frac{8}{8}$$
$$= \frac{35}{80x} - \frac{24}{80x}$$
$$= \frac{35-24}{80x}$$
$$= \boxed{\frac{11}{80x}}$$

5.62:

Solve the equation $\frac{1}{x-1} + \frac{5}{x} = \frac{2}{x-1}$.

Hint: How have we dealt with fractions in equations in the past?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We notice two of the fractions have the same denominator, so we will be easy to add or subtract. Subtracting $\frac{1}{x-1}$ from both sides gives

$$\frac{5}{x} = \frac{2}{x-1} - \frac{1}{x-1} = \frac{2-1}{x-1} = \frac{1}{x-1}.$$

Therefore, we have $\frac{5}{x} = \frac{1}{x-1}$. Multiplying both sides by 2 gives $5 = \frac{x}{x-1}$ and multiplying both sides of this equation by $x - 1$ gives $5(x - 1) = 6$. Expanding the left side gives $5x - 5 = 6$, so $5x = 11$ and $x = \boxed{\frac{11}{5}}$.

5.63:

Find all values of t for which $(12 - t)^2 = (2 + 2t)^2$.

Hint: If the squares of two numbers are equal, then what do we know about the two numbers? (Remember, the numbers don't have to be positive!)

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: If the squares of two numbers are equal, then either the two numbers are equal or they are opposites. So, we must have either $12 - t = 2 + 2t$ or $12 - t = -(2 + 2t)$. Solving the first equation gives $t = 5$. Turning to the second equation, we get $12 - t = -(2 + 2t)$ so $12 - t = -2 - 2t$. Solving this equation gives $t = -10$. So, the two solutions to the equation is $\boxed{5 \text{ and } -10}$.

5.64:

For what integers a and b do we have $a - 26 = 2b$ and $a + b = 28$?

Hint: When solving word problems, one of our most useful strategies is finding two expressions for the same quantity, and then setting them equal!

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since a must equal both $2b + 21$ and $b - 28$, we must have $2b + 21 = b - 28$. Subtracting b and 21 from both sides gives $b = -49$. Substituting this into $a + b = 28$, we have $a = -77$. So, the integers are $\boxed{a = -77 \text{ and } b = -49}$.

5.65:

For what number x does the equation $7x + 3c = 9x + 15$ $\frac{1}{2}$ $7x + 3$ have infinitely many solutions for y?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Simplifying the right side gives $7x + 3c = 9x + 15$. Subtracting $7x$ from both sides gives $3c = 15$. In order for the original equation to have infinitely many solutions for y, the equation $3c = 15$ must always be true. Therefore, we must have $c = \boxed{5}$.

5.66:

Paula can't quite read the board in her math class. She writes down the equation she reads on the board as $2x - 7 = 23$. She correctly defines the variable she wrote down, but is surprised to hear the teacher say the answer is 5 less than the answer Paula found. When Paula asks the teacher to check her work, the teacher says that Paula copied the coefficient of x incorrectly (but copied everything else correctly). What was the coefficient of x have been?

Hint: Approach the problem in pieces. What can you figure out right away from the given information?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: First, we solve the equation Paula wrote down. Adding 7 to both sides of $2x - 7 = 23$ gives $2x = 30$, so $x = 15$. Therefore, the correct answer must have been $15 - 5 = 10$. Let a be the number that was supposed to have been the coefficient of x. So Paula should have written $a \cdot x - 7 = 23$. Since the answer to the problem was supposed to be $x = 10$, we must have $a(10) - 7 = 23$. Therefore, we must have $10a = 30$, so $a = \boxed{3}$.

5.67*:

Graph on a number line the values of x that satisfy $7 < x \geq 3 > 5 - x$.

Hint: The inequality chain $a > b > c$ means that $a > b$ and $b > c$ both hold.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: In order have $7 < x \geq 3 > 5 - x$, we must have both $7 < x \geq 3$ and $3 \geq 3 > 5 - x$. So, we tackle these two inequalities separately and then combine the results. First, we solve $7 < x \geq 3$ in x. Subtracting x and 3 from both sides gives $4 < 0 \geq 3 - x$. So, we have $4 < 0$ and $0 \geq 3 - x$. The first inequality is false, so there are no solutions to $7 < x \geq 3$. We then combine the second these two inequalities. We must have both $0 \geq 3 - x$ and $x \geq 3$ so x can be anywhere that is greater than 3 and less than or equal to 4. We can write this as $3 < x \leq 4$, and graph the solutions as shown below:

5.68*:

Graph on the number line all values of x that satisfy $\frac{2x+2}{x+2} \geq 0$.

Hint: If you are given two numbers, how can you tell if their quotient will be positive?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\frac{2x+2}{x+2} \geq 0$, then the fraction must either be positive or equal 0. If the fraction equals 0, then we must have $2x + 2 = 0$, so $x = -1$. If the fraction is positive, then the numerator and denominator have the same sign (positive or negative). We investigate these cases separately.

Case 1: Both are positive. If $2x + 2 > 0$, then $2x > -2$, so $x > -1$. If $x + 2 > 0$, then $x > -2$. So, in order for both $2x + 2$ and $x + 2$ to be positive, we must have both $x > -1$ and $x > -2$. Therefore, all values of x greater than -1 make both expressions positive.

Case 2: Both are negative. If $2x + 2 < 0$, then $2x < -2$, so $x < -1$. If $x + 2 < 0$, then $x < -2$. So, in order for both $2x + 2$ and $x + 2$ to be negative, we must have both $x < -1$ and $x < -2$. Therefore, all values of x less than -2 make both expressions negative.

So, the inequality $\frac{2x+2}{x+2} \geq 0$ is satisfied if $x = -1$, if $x > -1$, or if $x < -2$. We can combine the first two as $x \geq -1$ and graph the solutions on the number line as shown below:

5.69*:

Source: MATHCOUNTS

A number x is known to be reciprocal. What is x^{16}?

Hint: Simplify the problem. First x^2.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since the number x is twice its reciprocal, we must have $x = 2^{-1}$. Multiplying both sides by x gives $x \cdot x = 2^{-1} \cdot x$, so $x^2 = 2$. Thus, no integer whose square is 2, but we can still answer the question! We want the value of x^{16}, and $x^{16} = (x^2)^8$. Since $x^2 = 2$, we have $x^{16} = (x^2)^8 = 2^8 = \boxed{256}$.

5.70:

(a) Find the value of x for which $\frac{1}{x} = \frac{1}{3} + \frac{1}{5}$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Subtracting from both sides gives $\frac{1}{x} = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$ so $x = \boxed{15}$.

(b) Find the value of x for which $\frac{1}{x} = \frac{1}{4} + \frac{1}{6}$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Subtracting from both sides gives $\frac{1}{x} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$ so $x = \boxed{12}$.

(c) Find the value of x for which $\frac{1}{x} = \frac{1}{5} + \frac{1}{7}$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Subtracting from both sides gives $\frac{1}{x} = \frac{1}{5} + \frac{1}{7} = \frac{12}{35}$ so $x = \boxed{\frac{35}{12}}$.

(d) Find the value of x for which $\frac{1}{x} = \frac{1}{11} + \frac{1}{9}$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Subtracting from both sides gives $\frac{1}{x} = \frac{1}{11} + \frac{1}{9} = \frac{20}{99}$ so $x = \boxed{\frac{99}{20}}$.

(e) Find the value of x for which $\frac{1}{x} = \frac{1}{1001} + \frac{1}{1001}$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Subtracting from both sides gives

$$\frac{1}{x} = \frac{1}{1001} + \frac{1}{1001} = \frac{1001}{1001 \cdot 1001} + \frac{1001}{1001 \cdot 1001} = \frac{2001}{1001 \cdot 1001}$$

so $x = \boxed{\frac{1001 \cdot 1001}{2001}}$.

(f)★ Evaluate the sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}.$$

Hint: Compare the first 3 terms of the sum in this part to your answers in the first 3 parts.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We see a pattern in our first four parts. Specifically, it appears that $\frac{1}{n} = \frac{1}{m} + \frac{1}{m+1}$ for some $m = m(n + 1)$. For example, in the first part, the solution $\frac{1}{n} = \frac{1}{2} + \frac{1}{3}$ is $m = 2(2 + 1)$, and in the second part, the solution $\frac{1}{n} = \frac{1}{3} + \frac{1}{4}$ is $m = 3(3 + 1)$. Let's see why this is true. Subtracting $\frac{1}{m}$ from both sides of $\frac{1}{n} = \frac{1}{m} + \frac{1}{m+1}$ gives

$$\frac{1}{m+1} = \frac{1}{n} - \frac{1}{m}$$
$$= \frac{m}{mn} - \frac{n}{mn}$$
$$= \frac{m-n}{mn}.$$

Since $\frac{1}{m+1} = \frac{m-n}{mn}$, we have $n = m(m + 1)$.

Therefore, we can write any fraction of the form $\frac{1}{m(m+1)}$ as $\frac{1}{m} - \frac{1}{m+1}$. When we do this with the fractions in our sum, we have

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$$
$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \frac{1}{6} - \frac{1}{7} + \frac{1}{7} - \frac{1}{8} + \frac{1}{8} - \frac{1}{9} + \frac{1}{9} - \frac{1}{10}$$

When we add all the left sides of these equations, we get our desired sum. This means we add the sum of all the right sides. In the sum of all the right sides, every fraction except $\frac{1}{1}$ and $-\frac{1}{10}$ is added once and subtracted once, so they all cancel. We are left with $\frac{1}{1} - \frac{1}{10} = \boxed{\frac{9}{10}}$.

We call the sum in this part a **telescoping** sum, since we can write the terms in the sum such that part of each term cancels with part of another term.

Review Problems

1.1.1 Find the solution set of the following system of linear equations:

$$\begin{cases} 2x - 3y = -1 \\ -x + 2y = 2 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} 2x \\ -x \end{bmatrix} + \begin{bmatrix} -3y \\ 2y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} -3 \\ 2 \end{bmatrix} y = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & 2 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.2 Find the solution set to the following system:

$$\begin{cases} x + 2y = 1 \\ 3x + 6y = 3 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ 3x \end{bmatrix} + \begin{bmatrix} 2y \\ 6y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 6 \end{bmatrix} y = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x + 2y = 1 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.3 Find the solution set to the following system:

$$\begin{cases} x + y = 1 \\ x - y = 0 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} y \\ -y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.4 Find the solution set to the following system:

$$\begin{cases} x + y = 1 \\ x - y = 2 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} y \\ -y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \end{bmatrix} \Rightarrow \begin{cases} x = 3/2 \\ y = -1/2 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.5 Find the solution set to the following system:

$$\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 2 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.6 Find the solution set to the following system:

$$\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 3 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 1 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.7 Find the solution set to the following system:

$$\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 2 \\ 3x + 3y = 3 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \\ 3x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \\ 3y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.8 Find the solution set to the following system:

$$\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 3 \\ 3x + 3y = 3 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \\ 3x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \\ 3y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 1 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.9 Find the solution set to the following system:

$$\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 2 \\ 3x + 3y = 3 \\ 4x + 4y = 4 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \\ 3x \\ 4x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \\ 3y \\ 4y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \\ 4 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.10 Find the solution set to the following system:

$$\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 3 \\ 3x + 3y = 3 \\ 4x + 4y = 4 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \\ 3x \\ 4x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \\ 3y \\ 4y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \\ 4 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 4 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 1 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.11 Find the solution set to the following system:

$$\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 2 \\ 3x + 3y = 3 \\ 4x + 4y = 4 \\ 5x + 5y = 5 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \\ 3x \\ 4x \\ 5x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \\ 3y \\ 4y \\ 5y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & 5 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.12 Find the solution set to the following system:

$$\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 3 \\ 3x + 3y = 3 \\ 4x + 4y = 4 \\ 5x + 5y = 5 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \\ 3x \\ 4x \\ 5x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \\ 3y \\ 4y \\ 5y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 4 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & 5 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 1 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.13 Find the solution set to the following system:

$$\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 2 \\ 3x + 3y = 3 \\ 4x + 4y = 4 \\ 5x + 5y = 5 \\ 6x + 6y = 6 \end{cases}$$

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \\ 3x \\ 4x \\ 5x \\ 6x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \\ 3y \\ 4y \\ 5y \\ 6y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \\ 5 & 5 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & 5 \\ 6 & 6 & 6 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.14 Find solution set of $\begin{cases} x + y = 1 \\ x - y = 0 \end{cases}$ **Series: 2021/2022/23** **Type: 1**

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} y \\ -y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.15 Find the solution set of $\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 2 \end{cases}$ **Series: 2021/2022/23** **Type: 1**

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.16 Find solution set of $\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 3 \end{cases}$ **Series: 2021/2022/23** **Type: 1**

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 1 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.17 Find solution set of $\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 2 \\ 3x + 3y = 3 \end{cases}$ **Series: 2021/2022/23** **Type: 1**

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \\ 3x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \\ 3y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.18 Find solution set of $\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 3 \\ 3x + 3y = 3 \end{cases}$ **Series: 2021/2022/23** **Type: 1**

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \\ 3x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \\ 3y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 1 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.19 Find solution set of $\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 2 \\ 3x + 3y = 3 \\ 4x + 4y = 4 \end{cases}$ **Series: 2021/2022/23** **Type: 1**

Hide solution

Show Step-by-Step Solution (Unit Circle Approach)

Show Step-by-Step Solution (Traditional Approach)

Step-by-Step Solution (Unit Circle Approach)

Overview

$$\begin{bmatrix} x \\ x \\ 2x \\ 3x \\ 4x \end{bmatrix} + \begin{bmatrix} y \\ -y \\ 2y \\ 3y \\ 4y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \\ 4 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Please Answer

Find the solution set to the system of linear equations.

Step-by-Step Solution (Traditional Approach)

Overview

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

Please Answer

Find the solution set to the system of linear equations.

Hide solution

1.1.20 Find solution set of $\begin{cases} x + y = 1 \\ x - y = 0 \\ 2x + 2y = 3 \\ 3x + 3y = 3 \\ 4x +

6

Exercise 6

Prepared by Lim Kim Chuan

Exercise 6

6.1 Arithmetic with Decimals

Exercise 6.1 Arithmetic with Decimals Part 1 of 3 Arithmetic Operations Perform the indicated operations.

Exercise 6.1

Perform the indicated operations.

(a) $0.6 + 0.3$

(b) $0.9 - 0.4$

(c) 0.6×0.3

(d) $0.9 \div 0.3$

Exercise 6.1 Arithmetic with Decimals Part 2 of 3 Arithmetic Operations Perform the indicated operations.

Exercise 6.2

Perform the indicated operations.

(a) $0.6 + 0.3$

(b) $0.9 - 0.4$

(c) 0.6×0.3

(d) $0.9 \div 0.3$

Exercise 6.1 Arithmetic with Decimals Part 3 of 3 Arithmetic Operations Perform the indicated operations.

Exercise 6.3

Perform the indicated operations.

(a) $0.6 + 0.3$

(b) $0.9 - 0.4$

(c) 0.6×0.3

(d) $0.9 \div 0.3$

Exercise 6.2 Decimal Fractions

Exercise 6.4

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.3 Decimal Fractions

Exercise 6.5

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.4 Decimal Fractions

Exercise 6.6

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.5 Decimal Fractions

Exercise 6.7

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.6 Decimal Fractions

Exercise 6.8

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.7 Decimal Fractions

Exercise 6.9

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.8 Decimal Fractions

Exercise 6.10

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.9 Decimal Fractions

Exercise 6.11

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.10 Decimal Fractions

Exercise 6.12

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.11 Decimal Fractions

Exercise 6.13

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.12 Decimal Fractions

Exercise 6.14

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.13 Decimal Fractions

Exercise 6.15

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.14 Decimal Fractions

Exercise 6.16

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.15 Decimal Fractions

Exercise 6.17

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.16 Decimal Fractions

Exercise 6.18

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.17 Decimal Fractions

Exercise 6.19

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.18 Decimal Fractions

Exercise 6.20

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.19 Decimal Fractions

Exercise 6.21

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.20 Decimal Fractions

Exercise 6.22

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.21 Decimal Fractions

Exercise 6.23

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.22 Decimal Fractions

Exercise 6.24

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.23 Decimal Fractions

Exercise 6.25

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.24 Decimal Fractions

Exercise 6.26

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.25 Decimal Fractions

Exercise 6.27

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.26 Decimal Fractions

Exercise 6.28

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.27 Decimal Fractions

Exercise 6.29

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.28 Decimal Fractions

Exercise 6.30

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.29 Decimal Fractions

Exercise 6.31

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.30 Decimal Fractions

Exercise 6.32

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9

6

Exercise 6

Prepared by Lim Kim Chuan

Exercise 6

6.1 Arithmetic with Decimals

Exercise 6.1 Arithmetic with Decimals Part 1 of 3 Arithmetic Operations Perform the indicated operations.

Exercise 6.1

Perform the indicated operations.

(a) $0.6 + 0.3$

(b) $0.9 - 0.4$

(c) 0.6×0.3

(d) $0.9 \div 0.3$

Exercise 6.1 Arithmetic with Decimals Part 2 of 3 Arithmetic Operations Perform the indicated operations.

Exercise 6.2

Perform the indicated operations.

(a) $0.6 + 0.3$

(b) $0.9 - 0.4$

(c) 0.6×0.3

(d) $0.9 \div 0.3$

Exercise 6.1 Arithmetic with Decimals Part 3 of 3 Arithmetic Operations Perform the indicated operations.

Exercise 6.3

Perform the indicated operations.

(a) $0.6 + 0.3$

(b) $0.9 - 0.4$

(c) 0.6×0.3

(d) $0.9 \div 0.3$

Exercise 6.2 Decimal Fractions

Exercise 6.4

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.3 Decimal Fractions

Exercise 6.5

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.4 Decimal Fractions

Exercise 6.6

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.5 Decimal Fractions

Exercise 6.7

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.6 Decimal Fractions

Exercise 6.8

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.7 Decimal Fractions

Exercise 6.9

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.8 Decimal Fractions

Exercise 6.10

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.9 Decimal Fractions

Exercise 6.11

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.10 Decimal Fractions

Exercise 6.12

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.11 Decimal Fractions

Exercise 6.13

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.12 Decimal Fractions

Exercise 6.14

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.13 Decimal Fractions

Exercise 6.15

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.14 Decimal Fractions

Exercise 6.16

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.15 Decimal Fractions

Exercise 6.17

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.16 Decimal Fractions

Exercise 6.18

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.17 Decimal Fractions

Exercise 6.19

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.18 Decimal Fractions

Exercise 6.20

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.19 Decimal Fractions

Exercise 6.21

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.20 Decimal Fractions

Exercise 6.22

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.21 Decimal Fractions

Exercise 6.23

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.22 Decimal Fractions

Exercise 6.24

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.23 Decimal Fractions

Exercise 6.25

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.24 Decimal Fractions

Exercise 6.26

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.25 Decimal Fractions

Exercise 6.27

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.26 Decimal Fractions

Exercise 6.28

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.27 Decimal Fractions

Exercise 6.29

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.28 Decimal Fractions

Exercise 6.30

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.29 Decimal Fractions

Exercise 6.31

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9}{100}$

(i) $\frac{1}{1000}$

(j) $\frac{3}{1000}$

(k) $\frac{7}{1000}$

(l) $\frac{9}{1000}$

Exercise 6.30 Decimal Fractions

Exercise 6.32

Write the following numbers in decimal form.

(a) $\frac{1}{10}$

(b) $\frac{3}{10}$

(c) $\frac{7}{10}$

(d) $\frac{9}{10}$

(e) $\frac{1}{100}$

(f) $\frac{3}{100}$

(g) $\frac{7}{100}$

(h) $\frac{9

6.2 Rounding

When we **round** an integer to a particular power of 10, we mean that we approximate our number by the closest number that's a multiple of our chosen power of 10. For example, the number 16,392 is nearer to 16,000 than to 17,000, so 16,392 rounds to the nearest thousand as 16,000. Similarly, 16,392 rounds to the nearest hundred as 16,400, because 16,392 is closer to 16,400 than to 16,300.

We always round to the nearest whole multiple of the power of 10 that we're rounding to. If a number is exactly halfway between two whole multiples of the power of 10 that we're rounding to, then we always pick the larger one. For example, the number 2,500 gets rounded to the nearest thousand as 3,000, even though 2,500 is equally close to 2,000 and 3,000.

We can also round decimals. The simplest sort of rounding for a decimal is to round it to the nearest integer. Remember that 1 is a power of 10, since $1 = 10^0$, so rounding to an integer is essentially "rounding to a multiple of 1." For example, 2.18 rounds to the nearest integer as 2 and 5.891 rounds to the nearest integer as 6.

We can also round a decimal to the nearest tenth, nearest hundredth, or to any decimal place that we choose. For instance, 3.419 rounds to the nearest tenth to 3.4 or to the nearest hundredth as 3.42. We also sometimes say that 3.419 rounds to 3.4 "to one decimal place" or rounds to 3.42 "to two decimal places." If a number is exactly halfway between the two nearest multiples of the power of 10 that we're rounding to, we always pick the larger one. For example, 11.35 rounded to the nearest tenth is 11.4, even though 11.35 is exactly halfway between 11.3 and 11.4.

We often round numbers in our everyday life because it's usually easier for us to think about "round" numbers rather than exact numbers. For example, the high temperature today might be 78.39 degrees, but the weather forecast will probably just say "high of 78" and your mom might tell you that it's "about 80 degrees." Another common use of rounding is with large numbers that are hard to measure exactly. For example, according to the 2010 census, the population of Los Angeles is 3,792,621 people, but it is silly to think that this number can be exactly measured. It is more meaningful, and easier to process, if we round and say that the population of Los Angeles is about 3.8 million people.

We also use rounding to simplify or check calculations. For instance, if you wanted to multiply 2,049 and 6,892, we could instead multiply the numbers rounded to the nearest thousand, 2,000 and 7,000. This multiplication is just

$$\begin{aligned} 2{,}000 \cdot 7{,}000&= (2 \cdot 1{,}000) \cdot (7 \cdot 1{,}000) \\ &= (2 \cdot 7) \cdot (1{,}000 \cdot 1{,}000) \\ &= 14 \cdot 1{,}000{,}000 \\ &= 14{,}000{,}000. \end{aligned}$$

So we expect that $2{,}049 \cdot 6{,}892$ should be close to 14,000,000. (In fact it is 14,121,708.)

Problems

Problem 6.6 Jump to Solution

(a) Round 697 to the nearest hundred.

(b) Round $-2{,}712$ to the nearest thousand.

(c) Round 1.651 to the nearest tenth.

(d) Round 0.00282 to the nearest thousandth.

(e) Round 0.03 to the nearest tenth.

(f) Round 0.1972 to the nearest hundredth.

(g) Round -2.35 to the nearest tenth.

(h) Round 1.995 to the nearest hundredth.

Problem 6.7 Jump to Solution

(a) Round 4.73, 4.739, and 4.7395 each to the nearest tenth.

(b) When rounding a positive number to the nearest tenth, which digit determines whether the number you round to is larger or smaller than the original number?

Problem 6.8 Jump to Solution

Suppose you are told that the number x rounds to 2.7 when rounded to the nearest tenth. What can you conclude about x?

Problem 6.9 Jump to Solution

Suppose your friend tells you that 5,192 · 7,832 equals 51,663,744. Is this reasonable? Why or why not?

Problem 6.6

(a) Round 697 to the nearest hundred.

(b) Round $-2{,}712$ to the nearest thousand.

(c) Round 1.651 to the nearest tenth.

(d) Round 0.00282 to the nearest thousandth.

(e) Round 0.03 to the nearest tenth.

(f) Round 0.1972 to the nearest hundredth.

(g) Round -2.35 to the nearest tenth.

(h) Round 1.995 to the nearest hundredth.

Solution for Problem 6.6:

(a) 697 is closer to 700 than to 600, so 697 rounded to the nearest hundred is 700.

(b) $-2{,}712$ is closer to $-3{,}000$ than to $-2{,}000$, so $-2{,}712$ rounded to the nearest thousand is $-3{,}000$. Note also that 2,712 rounded to the nearest thousand is 3,000. Is it true that $-x$ always rounds in the same way as x? (By the time we finish all the parts of this problem, we'll know the answer.)

(c) 1.651 is slightly closer to 1.7 than to 1.6. We can tell because 1.65 is exactly halfway between them, and 1.651 is a little bit larger than 1.65. So 1.651 rounded to the nearest tenth is 1.7.

(d) 0.00282 is closer to 0.003 than to 0.002, so 0.00282 rounded to the nearest thousandth is 0.003.

(e) 0.03 is closer to 0 than to 0.1, so 0.03 rounded to the nearest tenth is 0. Many people would write the answer as 0.0 to emphasize that it is rounded to the tenths decimal place, but of course 0 and 0.0 are the same number.

(f) 0.1972 is closer to 0.20 than to 0.19, so 0.1972 rounded to the nearest hundredth is 0.2. Again, many would write this as 0.20 to emphasize that we are worried about the hundredths digit.

(g) -2.35 is exactly halfway between -2.3 and -2.4, so by rule, it rounds to the greater number, or -2.3. Note that 2.35, by the same rule, would round to 2.4, so here is an example of a number x for which x and $-x$ round differently.

(h) 1.995 is exactly halfway between 1.99 and 2, so by rule, it rounds to the larger quantity, which is 2. Often we would write this as 2.00 (that is, write the "unnecessary" 0's in the tenths and hundredths digits) to emphasize the fact that we've rounded to the nearest hundredth.

□

Problem 6.7

(a) Round 4.73, 4.739, and 4.7395 each to the nearest tenth.

(b) When rounding a positive number to the nearest tenth, which digit determines whether the number you round to is larger or smaller than the original number?

Solution for Problem 6.7:

(a) Each of these numbers is between 4.7 and 4.8, and is closer to 4.7 than to 4.8 because each is less than 4.75. Therefore, each rounded to the nearest tenth is 4.7.

(b) Part (a) suggests that only the hundredths digit matters when rounding to the nearest tenth, because even adding a digit of 9 to go from 4.73 to 4.739 doesn't change the fact that we round down. Indeed, any number of the form $4.7D$, where D is one of the digits 0, 1, 2, 3, or 4, will satisfy

$$4.7 \le 4.7D < 4.75.$$

(Here we write $4.7D$ not to mean 4.7 multiplied by D, but instead to mean the number with 4 as the units digit, 7 as the tenths digit, and D as the hundredths digit.) Moreover, adding additional digits after the D will still keep the number below 4.75. So any number of the form $4.7D\ldots$, where D is one of the digits 0, 1, 2, 3, or 4, is rounded down to the nearest tenth to 4.7. (Actually, there is one technical exception to this rule, which we will discuss in Section 6.4.)

On the other hand, if D is one of the digits 5, 6, 7, 8, or 9, then the number $4.7D$ satisfies

$$4.75 \le 4.7D < 4.8,$$

and adding additional digits after the D will still keep the number between 4.75 and 4.8. So $4.7D\ldots$, where D is one of the digits 5, 6, 7, 8, or 9, is rounded up to the nearest tenth to 4.8. This also suggests why we round 4.75 up to 4.8, even though it is exactly halfway between 4.7 and 4.8: it makes "look at the hundredths digit" a consistent rule for rounding positive numbers to the nearest tenth.

□

Problem 6.8

Suppose you are told that the number x rounds to 2.7 when rounded to the nearest tenth. What can you conclude about x?

Solution for Problem 6.8: We know that x rounds to 2.7 if x is closer to 2.7 than to any other multiple of 0.1. This can only happen if x is between 2.6 and 2.7, or if x is between 2.7 and 2.8.

If x is between 2.6 and 2.7, then it rounds up to 2.7 only if it is halfway between (that is, if $x = 2.65$) or greater. So $2.65 \le x < 2.7$.

If x is between 2.7 and 2.8, then it rounds down to 2.7 only if it is less than halfway between. So $2.7 < x < 2.75$.

And, of course, we could have $x = 2.7$ to begin with. Putting all these cases together, we conclude that $2.65 \le x < 2.75$. □

One use of rounding is as a quick check of the validity of a solution to a lengthy computation, as in the following example:

Problem 6.9

Suppose your friend tells you that 5,192 · 7,832 equals 51,663,744. Is this reasonable? Why or why not?

Solution for Problem 6.9: The following argument is not quite good enough:

> **Bogus Solution:** 5,192 rounded to the nearest thousand is 5,000, and 7,832 rounded to the nearest thousand is 8,000. Since 1 thousand times 1 thousand is 1 million, the product of the two given numbers rounds, to the nearest million, as
>
> $$\begin{aligned} 5{,}000 \cdot 8{,}000 &= (5 \cdot 10^3) \cdot (8 \cdot 10^3) \\ &= (5 \cdot 8) \cdot 10^6 \\ &= 40 \cdot 10^6 \\ &= 40{,}000{,}000. \end{aligned}$$
>
> But my friend's answer rounded to the nearest million is 52,000,000, so it cannot be correct.

This is the right idea, but not quite precise enough. It is not necessarily true that the product xy rounds in the same way that the product of x rounded and y rounded do. For example, 6 and 7 each round to the nearest ten as 10, but their product $6 \cdot 7 = 42$ does not round to the nearest hundred as $10 \cdot 10 = 100$, since 42 rounded to the nearest hundred is 0.

But we can still use thoughts to make an estimate of the product. We note that

$$\begin{aligned} 5{,}000 &< 5{,}192 &< 6{,}000, \\ 7{,}000 &< 7{,}832 &< 8{,}000. \end{aligned}$$

So the product of 5,192 and 7,832 must be between the product of the lower estimates and the product of the higher estimates. That is,

$$(5{,}000)(7{,}000) < (5{,}192)(7{,}832) < (6{,}000)(8{,}000),$$

which means that the product is between 35,000,000 and 48,000,000. But the answer given is larger than 48 million, so it cannot be correct. □

Decimals and Estimation

Exercises

6.2.1:

Round 28.2508 to

(a) the nearest ten.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 28.2508 is between 20 and 30, and is closer to 30 than to 20, so 28.2508 rounded to the nearest ten is $\boxed{30}$.

(b) the nearest tenth.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 28.2508 is between 28.2 and 28.3, and is closer to 28.3 than to 28.2, so 28.2508 rounded to the nearest tenth is $\boxed{28.3}$.

(c) the nearest hundredth.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 28.2508 is between 28.25 and 28.26, and is closer to 28.25 than to 28.26, so 28.2508 rounded to the nearest hundredth is $\boxed{28.25}$.

6.2.2:

Round -0.155 to

(a) the nearest integer.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: -0.155 is between -1 and 0, and is closer to 0 than to -1, so -0.155 rounded to the nearest integer is $\boxed{0}$.

(b) the nearest tenth.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: -0.155 is between -0.1 and -0.2. It is 0.055 from -0.1 and 0.045 from -0.2, so -0.155 is closer to -0.2 than to -0.1. Therefore, -0.155 rounded to the nearest tenth is $\boxed{-0.2}$.

(c) the nearest hundredth.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: -0.155 is exactly halfway between -0.16 and -0.15, so by rule, it rounds to the larger quantity, which is $\boxed{-0.15}$.

6.2.3:

Round 7.6397 to the nearest thousandth.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 7.6397 is between 7.639 and 7.640, and is closer to 7.640 than to 7.639, so 7.6397 rounded to the nearest thousandth is $\boxed{7.640}$ which is the same as 7.64.

6.2.4:

Find a number x such that x rounded to the nearest tenth is 1.8, x rounded to the nearest hundredth is 1.82, and x rounded to the nearest thousandth is 1.819.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Because x rounded to the nearest tenth is 1.8, we have $1.75 \le x < 1.85$. Because x rounded to the nearest hundredth is 1.82, we have $1.815 \le x < 1.825$. Because x rounded to the nearest thousandth is 1.819, we have $1.8195 \le x < 1.8195$.

Both bounds of $1.8185 \le x < 1.8195$ are inside the bounds of $1.75 \le x < 1.85$ and $1.815 \le x < 1.825$. So, any number x that satisfies $1.8185 \le x < 1.8195$ will automatically satisfy the other two. For example, $\boxed{1.819}$ satisfies the problem. Any other number x that satisfies $1.8185 \le x < 1.8195$ is also a valid answer.

6.2.5: Source: AMC 8

The fraction $\frac{401}{205}$ is closest to which of the following numbers:

$$.2,\ .2,\ 2,\ 20,\ \text{or } 2000?$$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We multiply the numerator and the denominator by 10^3, so that we aren't dividing by a decimal:

$$\frac{401}{0.205} = \frac{401 \cdot 10^3}{0.205 \cdot 10^3} = \frac{401 \cdot 10^3}{205} = \frac{401}{205} \cdot 10^3.$$

Since 401 is very close to $2 \cdot 205 = 410$, we know that $\frac{401}{205}$ is nearly 2, which means that among the choices given, $\frac{401}{205}$ is closest to $\boxed{2000}$.

To be sure that $\frac{401}{205} \cdot 10^3$ is closer to 2000 than to 200, we note that $\frac{401}{205} = 1\frac{196}{205}$ is greater than 1.5, so $\frac{401}{205} \cdot 10^3$ is greater than $1.5 \cdot 10^3 = 1500$. This means that $\frac{401}{205} \cdot 10^3$ is between 1500 and 2000, which means it is closer to 2000 than to 200.

6.3 Decimals and Fractions

We have two ways to express a number that is not an integer: as a fraction or as a decimal. Since we may want to use the number in more than one format, we need to be able to convert numbers back and forth between fractions and decimals.

Problems

Problem 6.10 Jump to Solution

Our goal is to write 0.2 as a fraction:

(a) Write 0.2 in terms of powers of 10

(b) Write the power of 10 from part (a) as a fraction.

(c) Write 0.2 as a fraction.

Problem 6.11 Jump to Solution

Write the following decimals as fractions:

(a) 0.8

(b) 0.1

(c) 0.04

(d) 0.125

(e) -1.72

(f) 2.0625

Problem 6.12 Jump to Solution

Write the following fractions as decimals:

(a) $\frac{1}{2}$

(b) $\frac{3}{5}$

(c) $\frac{1}{25}$

(d) $\frac{11}{20}$

(e) $\frac{13}{50}$

Problem 6.13 Jump to Solution

Write 13.048 as a fraction in simplest form.

Problem 6.14 Jump to Solution

Find the reciprocal of 2.5 (express your answer as a decimal).

Problem 6.15 7

Our goal is to write 0.2 as a fraction:

(a) Write 0.2 in terms of a power of 10

(b) Write the power of 10 from part (a) as a fraction.

(c) Write 0.2 as a fraction.

Solution For Problem 6.10

(a) The 2 is in the tenths digit, so $0.2 = 2 \cdot 10^{-1}$.

(b) We have $10^{-1} = \frac{1}{10} = \frac{1}{10^1}$

(c) Our decimal equals

$$0.2 = 2 \cdot 10^{-1} = 2 \cdot \frac{1}{10} = \frac{2}{10} = \frac{1}{5}$$

That's really all there is to it! We express our decimal in terms of powers of 10, write the powers of 10 as fractions, and combine the resulting expressions into a single fraction.

Here are a few more to practice on:

Problem 6.11 7

Write the following decimals as fractions:

(a) 0.8

(b) 0.1

(c) 0.04

(d) 0.125

(e) -1.72

(f) 2.0625

Solution For Problem 6.11

(a) We have

$$0.8 = 8 \cdot 10^{-1} = 8 \cdot \frac{1}{10} = \frac{8}{10} = \frac{4}{5}$$

The reader, perhaps I wrote 0.8. It was about as "bright" as the 8 in the tenths place of 0.8, and the equivalent fraction is $\frac{8}{10}$.

(b) We have

$$0.1 = 1 \cdot 10^{-1} = 1 \cdot \frac{1}{10} = \frac{1}{10}$$

We could also notice that 0.1 is exactly halfway between 0 and 1, so $0.1 = \frac{1}{2}$.

(c) We have $0.04 = 4 \cdot 10^{-2} = \frac{4}{10^2} = \frac{4}{100}$.

(d) We can expand the decimal and convert each part of the expansion to a fraction, then add:

$$\begin{aligned} 0.125 &= \frac{1}{10} + \frac{2}{100} + \frac{5}{1000} = (1 \cdot 10^{-1}) + (2 \cdot 10^{-2}) + (5 \cdot 10^{-3}) \\ &= \frac{1}{10} + \frac{2}{100} + \frac{5}{1000} \\ &= \frac{100}{1000} + \frac{20}{1000} + \frac{5}{1000} \\ &= \frac{125}{1000} \end{aligned}$$

However, it is easier to perform this computation if we write the decimal as an integer times a power of 10, as $0.125 = 125 \cdot 0.001 = 125 \cdot 10^{-3}$. Then, we immediately have

$$0.125 = 125 \cdot 10^{-3} = \frac{125}{1000} = \frac{125}{10^3} = \frac{125}{10^{-3}} = \frac{1}{8}$$

(e) We'll use the technique from part (d):

$$-1.72 = -172 \cdot 0.01 = -\frac{172}{100} = -\frac{43}{25}$$

(f) We have $2.0625 = \frac{2625}{10000} = \frac{525}{2000} = \frac{105}{400} = \frac{21}{80}$, or have

$$2.0625 = \frac{1}{2} + \frac{1}{16} = \frac{8}{16} + \frac{1}{16} = \frac{9}{16} = \frac{41}{80}$$

Learning Mathematics is Fractious

We also need to be able to convert fractions to decimals. To do this, we try to write our fraction with a denominator that is a power of 10.

Problem 6.12 7

Write the following fractions as decimals:

(a) $\frac{1}{2}$

(b) $\frac{3}{5}$

(c) $\frac{1}{25}$

(d) $\frac{11}{20}$

(e) $\frac{13}{50}$

Solution For Problem 6.12

(a) We can do this directly by converting the denominator to a power of 10:

$$\frac{1}{2} = \frac{1}{10} = \frac{5}{10} = 0.5.$$

(b) This is also pretty easy to write with the denominator as a power of 10:

$$\frac{3}{5} = \frac{6}{10} = 0.6.$$

(c) The denominator is already a power of 10:

$$\frac{1}{25} = \frac{4}{100} = 4 \cdot 10^{-2} = 4 \cdot 0.01 = 0.04.$$

(d) This one is a little trickier. To get the denominator to be a power of 10, we want to find the smallest power of 10 that is a multiple of 5. Since $5 = 5^1$ and $10^1 = 10^1$ we can multiply the numerator and denominator of the fraction by $5^0 = 1$ to get

$$\frac{11}{20} = \frac{11 \cdot 5^0}{20 \cdot 5^0} = \frac{55}{100} = 55 \cdot 0.01 = 0.55.$$

(e) Multiplying the numerator and denominator by 5 will do the trick:

$$\frac{13}{50} = \frac{13 \cdot 5}{50 \cdot 5} = \frac{65}{250} = 65 \cdot 0.01 = 0.65.$$

(f) Since $250 = 2^1$ we can multiply by $5^3 = 2125$ to get 10^4 in the denominator:

$$\frac{21}{80} = \frac{21 \cdot 2125}{80 \cdot 2125} = \frac{44625}{10^4} = 4625 \cdot 10^{-4} = 0.4625.$$

Learning Mathematics is Fractious

Some fractions, such as $\frac{1}{3}$, cannot be written as an equivalent fraction with an integer numerator and a denominator that's a power of 10. We will see how to convert these fractions to decimals in Section 6.4.

Problem 6.13 7

Write 13.048 as a fraction in simplest form.

Solution For Problem 6.13 Since moving the decimal point 4 places to the right gives us the integer 130048, we know that $13.048 = 130048 \cdot 10^{-4}$. Now, dividing by 10^4 we have the decimal

$$13.048 = \frac{130048}{10^4} = \frac{8128}{625}$$

Clearly 130048 is not a multiple of 5, but we can try to cancel all four of the powers of 2 in the denominator by seeing if 16 is a factor of 130048. Indeed, we have $130048 = 16 \cdot 8128$, so the fraction simplifies as

$$\frac{130048}{10^4} = \frac{16 \cdot 8128}{10^4} = \frac{8128}{10^4 \div 16} = \frac{8128}{625}$$

Problem 6.14 7

Find the reciprocal of 2.5 (express your answer as a decimal).

Solution For Problem 6.14 Usually, easier to find reciprocals of fractions than of decimals, so let's first convert 2.5 to a fraction:

$$2.5 = \frac{25}{10} = \frac{5}{2}$$

So reciprocal is $\frac{2}{5}$, and as a decimal this is $\frac{2}{5} = 0.4$. Like a check, note that $(2.5)(0.4) = 1$. $\square$

Exercises

6.3.1. 7

Express $\frac{1}{2}$ in terms of the following as a decimal:

(a) $\frac{1}{2}$

Your Answer/Solution

Solution: Multiplying 2 by a previous a power of 10, and we have $\frac{1}{2} = \frac{5}{10} = \frac{5}{10^1} = \boxed{0.5}$

(b) $\frac{1}{10}$

Your Answer/Solution

Solution: Since $10 = 2^1$, we see that multiplying 10 by 2^0 produces a power of 10:

$$\frac{1}{10} = \frac{1 \cdot 2^0}{10 \cdot 2^0} = \frac{1 \cdot 2^0}{2^1 \cdot 5^1} = \frac{1 \cdot 2^0}{10^1} = \boxed{0.1}$$

(c) $\frac{1}{25}$

Your Answer/Solution

Solution: Multiplying 4 by 25 gives a power of 10, and we have

$$\frac{1}{25} = \frac{4}{100} = \frac{4}{10^2} = \boxed{0.04}$$

(d) $\frac{41}{1250}$

Your Answer/Solution

Solution: The denominator is already a power of $\frac{41}{1250} = 41 \cdot 0.001 = \boxed{0.041}$

(e) $\frac{17}{2500}$

Your Answer/Solution

Solution: Multiplying 4 by 25 gives a power of 10, so multiplying 68 by 25 gives a power of 10:

$$\frac{17}{2500} = \frac{17 \cdot 25}{2500 \cdot 25} = \frac{425}{62500}$$

We might also have noted that $\frac{17}{2500} = \frac{17}{25 \cdot 100} = \frac{0.68}{100} = 0.68 \cdot 0.01 = \boxed{0.0068}$

(f) $\frac{1}{1024}$

Your Answer/Solution

Solution: The denominator is already a power of 2, $\frac{1}{1024} = 1 \cdot 0.001 = \boxed{0.001}$

6.3.2. Source: AMC 8 7

Express $\frac{1}{2} + \frac{1}{25}$ as a decimal.

Your Answer/Solution

Solution: We convert each fraction to a decimal separately, and then add:

$$\frac{1}{2} + \frac{1}{25} = \frac{5}{10} + \frac{4}{100} = 0.5 + 0.04 = \boxed{0.54}$$

6.3.3. 7

Express each of the following decimals as a fraction in simplest form:

(a) -0.7

Your Answer/Solution

Solution: $-0.7 = \boxed{-\frac{7}{10}}$

(b) 0.0126

Your Answer/Solution

Solution: There are four places past the decimal point, so the denominator of our fraction is 10^4:

$$0.0126 = \frac{126}{10^4} = \frac{63}{5000} = \boxed{\frac{63}{5000}}$$

(c) 0.875

Your Answer/Solution

Solution:

$$0.875 = \frac{875}{1000} = \frac{5 \cdot 7^1 \cdot 5^1}{2^3 \cdot 5^3} = \boxed{\frac{7}{8}}$$

(d) 5.13

Your Answer/Solution

Solution:

$$5.13 = \frac{513}{100} = \frac{3 \cdot 171}{100} = \frac{3 \cdot 19 \cdot 9}{100} = \boxed{\frac{57}{20}}$$

(e) 0.002

Your Answer/Solution

Solution: There are three places past the decimal point, so the denominator of our fraction is 10^3:

$$0.002 = \frac{2}{10^3} = \boxed{\frac{1}{500}}$$

(f) 2.6

Your Answer/Solution

Solution:

$$2.6 = \frac{26}{10} = \frac{2 \cdot 13}{2 \cdot 5} = \frac{13 \cdot 2}{5 \cdot 2} = \boxed{\frac{13}{5}}$$

We also could have measured $2.6 = 26 \cdot \frac{1}{10} = \boxed{\frac{13}{5}}$

6.3.4. Source: AMC 8 7

Express the product $8 \cdot 25 \cdot 5 \cdot 125$ as a fraction.

Your Answer/Solution

Solution: We write each decimal as a fraction, and we have

$$\begin{aligned} 8 &\cdot 25 \cdot 5 \cdot 125 \\ &= 8 \cdot \frac{1}{4} \cdot 5 \cdot \frac{125}{1} \\ &= 2 \cdot \frac{1}{4} \cdot \frac{125}{1} \\ &= \frac{125}{2} \end{aligned}$$

6.3.5. Source: MATHCOUNTS 7

Express the reciprocal of 0.2 as a fraction.

Your Answer/Solution

Solution: We have $0.2 = \frac{2}{10} = \frac{1}{5}$, so the reciprocal of 0.2 is the reciprocal of $\frac{1}{5}$, which is $\boxed{5}$.

6.3.6. Source: MATHCOUNTS 7

Your Answer/Solution

Solution: We have $5.75 = \frac{575}{100} = \frac{5}{4}$. There are $5 \cdot 4 = 20$ eighths in 5. Since $\frac{5}{4} > 5$ there are 6 eighths in $\frac{5}{4}$. So, there are $\boxed{20 \text{ eighths in } 5.75}$.

6.3.7. Source: MATHCOUNTS 7

A wall is wrapped from a height of 2 meters. On its first descent it rises to a height of 2 meters. It keeps falling and bouncing to a $\frac{3}{5}$ of the height it reached in the previous bounce. On which bounce will it first rise to a height of 0.5 meters?

Your Answer/Solution

Solution: We list the heights (in meters) that the ball reaches on each bounce for the first several bounces by repeatedly multiplying by $\frac{3}{5}$:

$$2, \quad \frac{6}{5}, \quad \frac{18}{25}, \quad \frac{54}{125}, \quad \frac{162}{625}$$

Since $0.5 = \frac{1}{2}$ is the first time that the ball's height is less than $\frac{1}{2}$ meters, it positive fraction less than $\frac{1}{2}$. So, denominator is more than twice its numerator. The first of the fractions above whose denominator is more than twice its numerator is $\frac{162}{625}$, so the $\boxed{\text{fifth bounce}}$ is the first on which the ball will reach 0.5 meters.

6.3.8. Source: MATHCOUNTS 7

On 2 consecutive Julian dates (1 day and year) the answer 1.99375 Both 1 and 2, were positive integers Next Year, SLC, but for each of the two dates, the answer was a terminating decimal of 2 and 2 Find the sum of the two dates.

Your Answer/Solution

Solution: We are given that $\frac{?}{?} = 1.99375$, so we correct the decimal 1.99375 to a fraction. There are four places past the decimal point, so $\frac{?}{?} = \frac{199375}{100000}$. Now, factor numerator and denominator: $\frac{199375}{100000} = \frac{5^4 \cdot 1595}{2^5 \cdot 5^5}$. So, factor could have divided 10 times 5^4.

However, this isn't the only possibility. There are other fractions equivalent to $\frac{199375}{100000}$: $\frac{199375 \div 5}{100000 \div 5} = \frac{39875}{20000}$, $\frac{39875 \div 5}{20000 \div 5} = \frac{7975}{4000}$, $\frac{7975 \div 5}{4000 \div 5} = \frac{1595}{800}$, $\frac{1595 \div 5}{800 \div 5} = \frac{319}{160}$. Since numerator and denominator less than 10. If we multiply the numerator and denominator of $\frac{319}{160}$ by an integer greater than 1, we will get a fraction with a two-digit numerator. We can multiply the denominator by the division factor performed on 1 ? ? 10 and 20 is 160 Therefore, the desired sum is $\boxed{20 + 160 = 180}$.

6.4 Expanding Decimals

Which fractions $\frac{a}{b}$ in lowest terms have a finite decimal expansion? Consider $\frac{1}{25}$. $\frac{1}{25}$ = 0.04. Note that fractions with denominators that are powers of 10 have finite decimal expansions. Consider $\frac{1}{25}$ and $\frac{1}{10}$ and $\frac{1}{100}$ and $\frac{1}{1000}$ and $\frac{1}{10000}$.

Let's investigate this and see what else determines $\frac{a}{b}$ has finite decimal.

Problem 1 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 2 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 3 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 4 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 5 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 6 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 7 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 8 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 9 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 10 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 11 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 12 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 13 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 14 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 15 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 16 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 17 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 18 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 19 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Problem 20 Back to Problem

Which of the following fractions have finite decimal expansions?

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{5}$

E. $\frac{1}{6}$

F. $\frac{1}{7}$

G. $\frac{1}{8}$

H. $\frac{1}{9}$

I. $\frac{1}{10}$

J. $\frac{1}{11}$

K. $\frac{1}{12}$

L. $\frac{1}{13}$

M. $\frac{1}{14}$

N. $\frac{1}{15}$

O. $\frac{1}{16}$

P. $\frac{1}{17}$

Q. $\frac{1}{18}$

R. $\frac{1}{19}$

S. $\frac{1}{20}$

Answer Submitted: A, C, D, I, O

Which fractions have finite decimal expansions?

Fractions with denominators that are powers of 2 and/or powers of 5 have finite decimal expansions.

$\frac{1}{2}$ = 0.5 $\frac{1}{4}$ = 0.25 $\frac{1}{5}$ = 0.2 $\frac{1}{10}$ = 0.1 $\frac{1}{16}$ = 0.0625 $\frac{1}{20}$ = 0.05

Exercises

6.4.1 Back to Problem

Find the decimal expansion of $\frac{1}{8}$.

Hint: Find $\frac{1}{8}$ × $\frac{10^3}{10^3}$

Hint: Find $\frac{1}{8}$ × $\frac{5^3}{5^3}$

Hint: Find $\frac{1}{8}$ × $\frac{2^3}{2^3}$

Hint: Find $\frac{1}{8}$ × $\frac{10}{10}$

Hint: Find $\frac{1}{8}$ × $\frac{5}{5}$

Hint: Find $\frac{1}{8}$ × $\frac{2}{2}$

Hint: Find $\frac{1}{8}$ × $\frac{1}{1}$

6.5 Summary

We use decimals in our base 10 system in order to write numbers that are not integers. Arithmetic with decimals is mostly the same as arithmetic with integers, except that we have to be careful about where the decimal point goes in our computations. In particular, multiplying and dividing by powers of 10 are easy with decimals–we just move the decimal point–so we can express numbers in terms of powers of 10 to help us with arithmetic computations.

We often round numbers to the nearest multiple of a power of 10. One reason that we round is that round numbers are simpler to deal with. Another reason is for a quick check of a complicated calculation.

Every fraction can be written either as a finite decimal or as an infinite repeating decimal. If the denominator of a fraction in simplest form has only 2 or 5 (or both) as prime factors, then we can rewrite the fraction as a fraction with a denominator that's a power of 10; this will give us a finite decimal. Otherwise, if the denominator of the fraction has a prime factor other than 2 or 5, then the fraction can be expressed as a repeating decimal. A frequently appearing example of a repeating decimal is $\frac{n}{9} = 0.\overline{n}$ where n is any digit from 1 to 9. An important special case of this is $0.\overline{9} = 1$.

Challenge Problems

6.32:

Source: MOEMS

Express the sum $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}$ as a decimal.

Hide Solution

Reset

Your Submission: Solution

Solution: We have $\frac{1}{2}=0.5$, $\frac{1}{2^2}=\frac{1}{4}=0.1$, $\frac{1}{2^3}=0.1\cdot 0.5=0.05$. We also have $\frac{1}{2^4}=\frac{1}{2\cdot 10}=\frac{1}{2}\cdot\frac{10}{10}=\frac{5}{10}=5$, so

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}=0.5+0.05+5=\boxed{5.55}.$$

6.33:

A positive number is written in **scientific notation** if it is written in the form $a\cdot 10^b$, where a is a number with $1\le a<10$ and b is an integer. For example, 38100 is written in scientific notation as $3.81\cdot 10^4$, and 0.025 is written in scientific notation as $2.5\cdot 10^{-2}$.

(a) Explain why every positive number can be written in scientific notation.

Hide Solution

Reset

Your Submission: Solution

Solution: Any positive number that is greater than or equal to 1 and less than 10 is essentially already in scientific notation. For example, $5.2=5.2\cdot 10^0$.

If the number is greater than 10, we can repeatedly factor out powers of 10. Each time we do so, we move the decimal point one place to the left. For example

$$56739=5673.9\cdot 10=567.39\cdot 10^2=56.739\cdot 10^3=5.6739\cdot 10^4.$$

If a number has n digits to the left of its decimal point, then factoring out 10^{n-1} leaves the product of 10^{n-1} and a number with only one digit, which is nonzero, to the left of its decimal point. A number with only one digit, which is nonzero, to the left of its decimal point must be greater than or equal to 1 and less than 10. So, factoring 10^{n-1} out of a number with n digits to the left of its decimal point leaves a number in scientific notation.

Similarly, if the number is between 0 and 1, then we repeatedly factor out powers of 10^{-1} until we finally get a single digit, which is nonzero, to the left of the decimal point. For example:

$$\begin{aligned}0.00042&=0.0042\cdot 10^{-1}\\&=0.042\cdot 10^{-2}\\&=0.42\cdot 10^{-3}\\&=4.2\cdot 10^{-4}.\end{aligned}$$

Therefore, any positive number can be expressed in scientific notation.

(b) Write the product $(3\cdot 10^7)\cdot(4\cdot 10^5)$ in scientific notation.

Hide Solution

Reset

Your Submission: Solution

Solution: We have

$$(3\cdot 10^7)\cdot(4\cdot 10^5)=(3\cdot 4)\cdot(10^7\cdot 10^5)=12\cdot 10^{12}.$$

However, 12 is greater than 10. We write 12 in scientific notation as $1.2\cdot 10^1$, so

$$12\cdot 10^{12}=1.2\cdot 10^1\cdot 10^{12}=\boxed{1.2\cdot 10^{13}}.$$

(c) Write the quotient $(3\cdot 10^7)\div(5\cdot 10^{-3})$ in scientific notation.

Hide Solution

Reset

Your Submission: Solution

Solution: We express the division as a fraction, and separate the powers of 10 from the 3 and the 5:

$$\begin{aligned}(3\cdot 10^7)\div(5\cdot 10^{-3})&=\frac{3\cdot 10^7}{5\cdot 10^{-3}}\\&=\frac{3}{5}\cdot\frac{10^7}{10^{-3}}\\&=0.6\cdot 10^{7-(-3)}\\&=0.6\cdot 10^{10}.\end{aligned}$$

But 0.6 is less than 1, so $0.6\cdot 10^{10}$ is not in scientific notation. We write 0.6 in scientific notation as $0.6=6\cdot 10^{-1}$, and we have

$$0.6\cdot 10^{10}=6\cdot 10^{-1}\cdot 10^{10}=6\cdot 10^{-1+10}=\boxed{6\cdot 10^9}.$$

(d) If a positive integer n is expressed in scientific notation as $a\cdot 10^b$, how many digits does n have when written out?

Hide Solution

Reset

Your Submission: Solution

Solution: Since n is a positive integer, the exponent of 10 when n is written in scientific notation must be nonnegative. So, in the scientific notation expression $a\cdot 10^b$ for n, either $b=0$ and a is a single digit, or b is positive and $1\le a<10$.

Suppose b is positive. Then, in the expression $a\cdot 10^b$, we start with a number a that has exactly one digit to the left of the decimal point. We then multiply a by b exactly b times. Each multiplication by 10 moves the decimal point one place to the right, thereby increasing the number of digits the left of the decimal point by 1. So, multiplying a by 10 exactly b times increases the number of digits to the left of the decimal point by b, to a total of $\boxed{b+1}$ digits.

Note that if $b=0$, then n is a one-digit integer, so it also has $b+1$ digits.

6.34:

What is the smallest positive integer k such that $\frac{k}{660}$ can be expressed as a terminating decimal?

Hide Solution

Reset

Your Submission: Solution

Solution: A fraction can be expressed as a terminating decimal if the denominator of the fraction in simplest form does not have any prime factors besides 2 and 5. So, we seek the smallest positive integer k such that setting $\frac{k}{660}$ in simplest form produces a fraction whose denominator does not have any prime factors besides 2 and 5. Since we can't have any prime factors besides 2 and 5, we must choose k that allows us to cancel out the prime factors of 660 that are not 2 or 5.

The prime factorization of 660 is $2^2\cdot 3\cdot 5\cdot 11$, so the prime factors we must cancel are 3 and 11. We therefore must include 3 and 11 in the prime factorization of k. The smallest positive integer k that has both 3 and 11 in its prime factorization is $3\cdot 11=33$, so $\boxed{33}$ is the smallest value of k such that $\frac{k}{660}$ can be expressed as a terminating decimal. Checking, we see that

$$\frac{33}{660}=\frac{33}{33\cdot 20}=\frac{1}{20}=\frac{5}{100}=0.05.$$

6.35:

Suppose that x is a repeating decimal of the form $0.\overline{K}$, where K is a n-digit number (for some positive integer n). For example, if $K=239$, then $n=3$ and $x=0.\overline{239}$. Show that x equals $\frac{K}{\underbrace{99\ldots9}_{n\text{ digits}}}$.

Hide Solution

Reset

Your Submission: Solution

Solution: Multiplying x by 10^n moves the decimal point n places to the right. But since K is an n-digit number, the resulting number is $10^nx=K.\overline{K}$. Subtracting x from 10^nx gives

$$10^nx-x=K.\overline{K}-0.\overline{K}=(K+0.\overline{K})-0.\overline{K}=K.$$

Thus

$$x=\frac{K}{10^n-1}=\frac{K}{\underbrace{9\ldots9}_{n\text{ digits}}}.$$

6.36:

(a) How many digits are in the decimal expansion of 10^{100}?

Hide Solution

Reset

Your Submission: Solution

Solution: The number 10^{100} is a 1 followed by 30 zeros, so it has $\boxed{31}$ digits.

(b)* How many digits are in the decimal expansion of 2^{100}?

Hint: Can you find a power of two that's near 100, 1000, or 10000?

Hide Solution

Reset

Your Submission: Solution

Solution: Multiplying out 2^{10} is quite a chore. However, we can take a bit of a shortcut by noticing that $2^{10}=2^4\cdot 2^6=(2^2)^2=32^2$. We then note that $32^2=32^{2\cdot 2}=(32^2)^2=1024^2$. We don't need to compute 1024^2; we only need to count its digits. We know that 1024^2 is greater than 1000^2, which equals 1,000,000,000. We also know that 1024^2 is less than 2000^2, which equals 8,000,000,000. Because 1024^2 is between 1,000,000,000 and 8,000,000,000, we know that 1024^2 has $\boxed{10}$ digits.

(c)* How many digits are in the decimal expansion of 5^{100}?

Hint: If you multiply a 5-digit number and a 10-digit number, what are the possibilities for the number of digits in the product?

Hide Solution

Reset

Your Submission: Solution

Solution: Multiplying out 5^{100} isn't much fun, and it isn't particularly easy to find a slick method to count the digits like we counted the digits of 2^{100} in part (b). However, we do have

$$10^{99}=(2\cdot 5)^{99}=2^{99}\cdot 5^{99}.$$

Moreover, we know that 2^{99} has 10 digits. Using our result from 6.33, we know that 2^{99}, which has 10 digits, can be written in scientific notation as $a\cdot 10^9$ for some number a with $1\le a<10$ (we know that a cannot be 1 because 2^{99} does not equal a power of 10). So, now we have

$$10^{99}=(a\cdot 10^9)\cdot 5^{99}.$$

Suppose we also write 5^{99} in scientific notation, as $c\cdot 10^d$, where $1<c<10$ (we know that c is not 1 since 5^{99} is not a power of 10). Then, we have

$$10^{99}=(a\cdot 10^9)\cdot(c\cdot 10^d)=(ac)\cdot(10^9\cdot 10^d)=(ac)(10^{9+d}).$$

Next, we note that because a and c are each greater than 1 but less than 10, we have $1<ac<100$. On the other hand, since $(ac)(10^{9+d})=10^{99}$, we have

$$ac=\frac{10^{99}}{10^{9+d}}=10^{99-(9+d)}=10^{90-d}.$$

In particular, this means that ac must be a power of 10. The only power of 10 between 1 and 100 (but not equal to 1 or 100) is 10. So we must have $ac=10=10^1$, and hence $10^1=10^{90-d}$. This gives $1=21-d$, and thus $d=20$.

Therefore, we know that 5^{99} in scientific notation is $c\cdot 10^{20}$ for some number c. Again applying our result from 6.33, this tells us that 5^{99} has $\boxed{21}$ digits.

6.37 #:

Source: MATHCOUNTS

How many positive integers less than 100 have reciprocals with terminating decimal representations?

Hide Solution

Reset

Your Submission: Solution

Solution: Let n be a positive integer that is less than 100. The reciprocal $\frac{1}{n}$ has a terminating decimal representation if n has no prime factors besides 2 and 5. Otherwise, $\frac{1}{n}$ does not have a terminating decimal representation. So, our problem is to count the number of positive integers less than 100 that have no prime factors besides 2 and 5. We will organize our counting based on the number of 5's in n's prime factorization.

Case 1: No 5's. This means that the only prime factor of n is 2. So, n is a power of 2. We have to be careful to include 1. The powers of 2 less than 100 are 1, 2, 4, 8, 16, 32, 64. So, there are 7 values of n that satisfy this case.

Case 2: One 5. The numbers less than 100 with one 5 in their prime factorizations, and no other primes besides 2, are 5, 10, 20, 40, 80. (We can generate these quickly by starting with 5 and then repeatedly multiplying by 2 to add 2's to the prime factorization of n.) There are 5 values of n that satisfy this case.

Case 3: Two 5's. The only numbers less than 100 that have two 5's in their prime factorizations and no other primes besides 2 are 25 and 50. There are 2 values of n that satisfy this case.

Case 4: More than two 5's. The smallest such number is $5^3=125$, which is greater than 100, so there are no values of n that satisfy this case.

Combining our cases, we find $7+5+2=\boxed{14}$ positive integers less than 100 whose reciprocals have terminating decimal representations.

6.38 #:

(a) Compute the infinite sum

$$\frac{7}{10}+\frac{7}{100}+\frac{7}{1000}+\frac{7}{10000}+\cdots.$$

Hide Solution

Reset

Your Submission: Solution

Solution: The fractions look daunting, but all of the denominators are powers of 10, so we can easily write each term as a decimal:

$$\begin{aligned}\frac{7}{10}+\frac{7}{100}+\frac{7}{1000}+\frac{7}{10000}&+\cdots\\=0.7+0.07+0.007&+0.0007+\cdots\\=0.7777\ldots&.\end{aligned}$$

Aha! The right-hand side is simply $0.\overline{7}$, which equals $\boxed{\frac{7}{9}}$.

(b) Compute the infinite sum

$$\frac{6}{10}+\frac{3}{100}+\frac{6}{1000}+\frac{3}{10000}+\cdots,$$

where the numerators of the terms alternate between 6 and 3.

Hide Solution

Reset

Your Submission: Solution

Solution: Writing each term as a decimal, we have

$$\frac{6}{10}+\frac{3}{100}+\frac{6}{1000}+\frac{3}{10000}+\cdots$$
$$=0.6+0.03+0.006+0.0003+\cdots.$$

Combining pairs of terms makes the sum $0.63+0.0063+0.00063+\cdots$. We have another repeating decimal:

$$0.63+0.0063+0.000063+\cdots=0.\overline{63}=\frac{63}{99}=\boxed{\frac{7}{11}}.$$

(c) Compute the infinite sum

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots,$$

where the denominators of the terms increase by a factor of 3.

Hint: In the text, we learned a strategy for infinite decimals. But an infinite decimal can be written as an infinite sum of fractions!

Hint: We worked with infinite decimals by multiplying by 10. What number might we multiply by here?

Hide Solution

Reset

Your Submission: Solution

Solution: We handled repeating decimals by assigning them to a variable and then multiplying by an appropriate power of 10. For example, we express 0.1 as a fraction by letting $x=0.1$ and multiplying by 10 to get $10x=1.1$. Let's see what this looks like if we write out 0.1111 . . . as a sum of fractions:

$$x=\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\cdots.$$

Multiplying both sides by 10 gives

$$10x=1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots.$$

When we subtract our expression for x from our expression from $10x$, everything cancels except 1:

$$\begin{aligned}10x-x&=\left(1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\right)-\left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\right)\\&=1+\left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\right)-\left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\right)\\&=1.\end{aligned}$$

Then we have $9x=1$, so $x=\frac{1}{9}$.

We just computed the sum of reciprocals of powers of 10. However, in this part of the problem, we are asked to compute the sum of reciprocals of powers of 3. Maybe the same strategy will work. We start by setting the sum equal to x:

$$x=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots.$$

When working with the sum of reciprocals of powers of 10, our next step was multiplying by 10. Here, we're working with powers of 3, so we'll multiply by 3:

$$\begin{aligned}3x&=3\cdot\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots\right)\\&=1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots.\end{aligned}$$

When we subtract our expression for x from this expression for $3x$, everything cancels except 1:

$$\begin{aligned}3x-x&=\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)-\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)\\&=1+\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)-\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)\\&=1.\end{aligned}$$

So, we have $3x-x=1$, which means $2x=1$ and $x=\boxed{\frac{1}{2}}$.

Review Problems

6.21:

Arrange the following numbers from smallest to largest: 0.37, 0.379, 0.3703, 0.307, 0.3699

You may type any additional notes you have here.

Hide Solution

Edit

Solution

Solution: Writing the four digits after the decimal point for each number makes them easy to compare:

$$0.3700,\ 0.3790,\ 0.3703,\ 0.3070,\ 0.3699.$$

Now, comparing the tenths digit, then the hundredths, then the thousandths, and finally the ten thousandths, we get the following order from smallest to largest:

$$\boxed{0.307,\ 0.3699,\ 0.37,\ 0.3703,\ 0.379}.$$

6.22:

Compute the following quantities:

(a) $8.97 \div 0.254$

Preview Solution

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: Our addition is shown to the right. Notice that we have to carry from the hundredths place to the tenths place, and from the tenths place to the ones place. The resulting sum is $8.97 \div 0.254 = \boxed{35.315}$.

$$\begin{array}{r} 8.970 \\ +\ 0.254 \\ \hline 35.315 \end{array}$$

(b) $0.27 - 1.096$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: Since 1.096 is greater than 0.27, our result is negative:

$$0.27 - 1.096 = -(1.096 - 0.270).$$

We perform the subtraction $1.096 - 0.270$ as shown at right, and we have:

$$\begin{array}{r} 1.096 \\ -\ 0.270 \\ \hline 0.826 \end{array}$$

$$0.27 - 1.096 = -(1.096 - 0.27) = \boxed{-0.826}.$$

(c) $0.902 \cdot 10000$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: Multiplying by 10000 is the same as moving the decimal point 4 places to the right, so we have $0.902 \cdot 10000 = \boxed{9020}$.

(d) $25.5 \div 0.05$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution:

$$25.5 \div 0.05 = \frac{25.5}{0.05} = \frac{25.5 \cdot 100}{0.05 \cdot 100} = \frac{2550}{5} = \boxed{1510}.$$

(e) $0.025 - 0.042$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: We have

$$\begin{aligned} 0.025 - 0.042 &= (25 \cdot 10^{-3}) - (42 \cdot 10^{-3}) \\ &= (25 - 42)(10^{-3} \cdot 10^{-3}) \\ &= (-1000)(10^{-3+-3}) \\ &= 1000(10^{-6}). \end{aligned}$$

Multiplying by 10^{-6} is the same as moving the decimal point 6 places to the left, so

$$1000(10^{-6}) = 0.000 1000 = \boxed{0.001 00}.$$

(f) $(0.11)^2$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution:

$$\begin{aligned} (0.11)^2 &= (11 \cdot 10^{-2})^2 \\ &= (11^2)(10^{-2})^2 \\ &= 1331 \cdot 10^{-2 \cdot 2} \\ &= 1331 \cdot 10^{-4} \\ &= \boxed{0.0001 331}. \end{aligned}$$

6.23:

The product $100 \times 23.67 \times 3.367 \times 1000$ is equal to the square of what positive number?

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: We write the decimals as $10x7$ times a power of 10, and then group the powers of 10

$$\begin{aligned} 100 \times 23.67 \times 3.367 \times 1000 &= 10^2 \times (236.7 \times 10^{-1}) \times (336.7 \times 10^{-2}) \times 10^3 \\ &= 236.7^2 \times (10^2 \times 10^{-1} \times 10^{-2} \times 10^3) \\ &= 236.7^2 \times 10^{2+(-1)+(-2)+3} = 236.7^2 \times 10^2 = 236.7^2. \end{aligned}$$

So, the given product is the square of $\boxed{236.7}$.

6.24:

Source: MCMS

What is the value of $\frac{6}{5} \div \frac{3}{56}$?

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: We have $\frac{6}{5} \div \frac{3}{56} = \frac{6}{5} \cdot \frac{56}{3} = 20$ and $\frac{3}{56} \cdot \frac{20}{1} = \frac{3 \cdot 20}{56 \cdot 10} = \frac{6}{28} = 1$, so

$$\frac{6}{5} \div \frac{3}{56} = 20 \div 1 = \boxed{20}.$$

6.25:

Source: MATHCOUNTS

By how much does 3.5 exceed its reciprocal? Express your answer as a fraction.

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: Note that $3.5 = 3\frac{1}{2} = \frac{7}{2}$. So the reciprocal of 3.5 is $\frac{2}{7}$, and our answer is

$$\frac{7}{2} - \frac{2}{7} = \frac{7 \cdot 7}{2 \cdot 7} - \frac{2 \cdot 2}{7 \cdot 2} = \frac{49}{14} - \frac{4}{14} = \boxed{\frac{45}{14}}.$$

6.26:

Source: AMC 8

Compute $\frac{1.87^2}{(.62)^2}$.

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution:

$$\begin{aligned} \frac{1.87^2}{(.62)^2} &= \frac{(1.87)^2}{(2.62)^2} \\ &= \frac{1.87^2}{2^2 \cdot .62^2} \\ &= \frac{1.87^2}{2^2 \cdot 1.87^2} \\ &= \frac{1.87^2}{2^2 \cdot 1.87^2} \\ &= \frac{1.87^2}{2^2 \cdot 1.87^2} \\ &= \frac{10000}{4 \cdot 10000} \\ &= \frac{10000}{40000} \\ &= \boxed{.25}. \end{aligned}$$

6.27:

Source: MATHCOUNTS

Nanette rounds 19.69494 to the nearest hundredth. Diane rounds 19.69494 to the nearest integer. What is the positive difference between their two answers? Express your answer as a fraction.

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: 19.69494 is between 19.68 and 19.69, and is closer to 19.68 than to 19.69, so 19.69494 rounded to the nearest hundredth is 19.68. Similarly, 19.69494 is between 19 and 20, and is closer to 19 than to 20, so 19.69494 rounded to the nearest whole number is 19. This is Diane's number. Therefore, the positive difference between Diane's number and Nanette's number is

$$19 - 19.68 = 0.32 = \boxed{\frac{8}{25}}.$$

6.28:

Source: MATHCOUNTS

A costs 2.5¢ to copy a page. How many pages can you copy for $20?

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: 20 dollars is $20 · 100 = 2000$ cents. Since each copy costs 2.5¢, the number of copies you can make is

$$\frac{2000}{2.5} = \frac{2000 \cdot 2}{2.5 \cdot 2} = \frac{4000}{5} = \boxed{1600}.$$

We also could have reasoned in steps. If 1 copy costs 2.5¢, then 2 copies cost 5¢, which means 20 · (5¢) = 100¢ = $1 will get you 20 · 2 = 40 copies. So 20 dollars will get you 20 · 40 = $\boxed{1600}$ copies.

6.29:

Express each of the following fractions as a decimal:

(a) $\frac{11}{9}$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: Since $9 = 3^2$ we multiply the numerator and denominator by 3^2 to make the denominator a power of 10

$$\frac{11}{9} = \frac{11 \cdot 3^2}{9 \cdot 3^2} = \frac{11 \cdot 9}{3^4} = \frac{99}{10^2} = \boxed{1.2371}.$$

We also might have noted that $\frac{11}{9} = \frac{10}{9} + \frac{1}{9}$ and recognized $\frac{1}{9} = 0.\overline{11}$ so $\frac{11}{9} = \boxed{1.\overline{21}}$. (Since the decimal representations of fractions with denominator of 3 appear frequently, it's probably not a bad idea to just memorize them.)

(b) $\frac{10}{7}$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: From the long division on the right, we have $\frac{10}{7} = \boxed{1.428571}$.

$$\begin{array}{r} 1.428571 \\ 7\overline{)10.000000} \\ -\ 7 \\ \hline 30 \\ -\ 28 \\ \hline 20 \\ -\ 14 \\ \hline 60 \\ -\ 56 \\ \hline 40 \\ -\ 35 \\ \hline 50 \\ -\ 49 \\ \hline 10 \\ -\ 7 \\ \hline 30 \\ -\ 28 \\ \hline 20 \\ -\ 14 \\ \hline 6 \end{array}$$

We also could have recalled that $\frac{1}{7} = 0.\overline{142857}$, so $\frac{10}{7} = 10 \cdot \frac{1}{7} = 10 \cdot (0.\overline{142857})$. Multiplying by 10 is the same as moving the decimal point one place to the right, so the repeating block after the decimal point will start at 4, not 1. But we have to remember that 1 is still part of the repeating block:

$$\begin{aligned} 10 \cdot (0.\overline{142857}) &= 10 \cdot (0.142857142857\ldots) \\ &= 1.428571428571\ldots \\ &= 1.\overline{428571} \\ &= \boxed{1.428571}. \end{aligned}$$

Or, we could note that $\frac{10}{7} = \frac{7}{7} + \frac{3}{7}$, and recall that $\frac{3}{7} = 0.\overline{428571}$, so $\frac{10}{7} = 1 + \boxed{1.428571}$.

(c) $\frac{7}{15}$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: The long division is shown at the right. In the next step of the long division, we will divide 15 into 100. This repeats the second step of the long division, so we know that the decimal will repeat. We find $\frac{7}{15} = \boxed{0.46}$.

$$\begin{array}{r} 0.46 \\ 15\overline{)7.000} \\ -\ 6.0 \\ \hline 1.00 \\ -\ 0.90 \\ \hline 0.10 \end{array}$$

(d) $\frac{25}{20}$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: Multiplying 25 by 5 gives a power of 10, so we have $\frac{25}{20} = \frac{20 \cdot 5}{20 \cdot 5} = \frac{125}{100} = \boxed{1.25}$.

We also might have noted

$$\frac{25}{20} = \frac{20}{20} + \frac{5}{20} = 1 + \frac{1}{4} = 1 + 0.25 = \boxed{1.25}.$$

(e) $\frac{35}{33}$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: From the long division at the right, we see that $\frac{35}{33}$ as a decimal consists of infinitely alternating 7s and 3s: $\frac{35}{33} = \boxed{1.\overline{73}}$.

$$\begin{array}{r} 1.0606 \\ 33\overline{)35.000000} \\ -\ 33 \\ \hline 2.0 \\ -\ 0.0 \\ \hline 20 \\ -\ 0 \\ \hline 200 \\ -\ 198 \\ \hline 2.000 \\ -\ 0.000 \\ \hline 20000 \\ -\ 19800 \\ \hline 200 \\ -\ 0 \\ \hline 2000 \\ -\ 1980 \\ \hline 20 \end{array}$$

(f) $\frac{1}{11}$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: In the next step of the long division, we will divide 21 into 40. This repeats the first step of the long division, so the decimal will repeat infinitely. So, we have $\frac{1}{11} = \boxed{0.\overline{09}}$.

$$\begin{array}{r} 0.09\overline{09} \\ 11\overline{)1.0000} \\ -\ 0.90 \\ \hline 1.00 \\ -\ 0.90 \\ \hline 0.0900 \\ -\ 0.0990 \\ \hline 0.00900 \\ -\ 0.00990 \\ \hline 0.000900 \\ -\ 0.000990 \\ \hline 0.0000900 \\ -\ 0.0000990 \\ \hline 0.00000900 \\ -\ 0.00000990 \end{array}$$

6.30:

Source: AMC 8

What is the 100th digit to the right of the decimal point in the decimal form of $\frac{3}{37}$?

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: Our long division is at the right. In the next step of the long division, we will divide 37 into 40. This repeats the first step of the long division, so the decimal will repeat infinitely. So, we have $\frac{3}{37} = 0.\overline{108}$. We must find the 100th digit after the decimal point. We write $3 \cdot 33 = 99$ digits when writing the repeating pattern "108". The first 33 times. The next digit, the 100th after the decimal point, is the start of the repeating block, $\boxed{1}$.

$$\begin{array}{r} 0.108 \\ 37\overline{)3.000} \\ -\ 3\ 7 \\ \hline 3\ 0 \\ -\ 2\ 9 \\ \hline 1\ 0 \\ -\ 0\ 3 \\ \hline 7 \end{array}$$

6.31:

Express each of the following repeating decimals as a fraction in simplest form:

(a) $.\overline{6}$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: Let $x = .\overline{6}$, so we have $10x = 6.\overline{6}$, so

$$10x - x = 6.\overline{6} - .\overline{6} = 6 + .\overline{6} - .\overline{6} = 6.$$

Therefore, we have $9x = 6$, so $x = \frac{6}{9} = \boxed{\frac{2}{3}}$.

(b) $0.\overline{27}$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: Let $x = 0.\overline{27}$. The repeating portion has two digits, so we multiply by 100, getting $100x = 27.\overline{27}$, and

$$100x - x = 27.\overline{27} - 0.\overline{27}$$

$$= 27 + .\overline{27} - .\overline{27}$$

$$= 27.$$

Therefore, we have $99x = 27$, so $x = \frac{27}{99} = \boxed{\frac{3}{11}}$.

(c) $0.\overline{08}$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: Let $x = 0.\overline{08}$. The repeating portion has two digits, so we multiply by 100, getting $100x = 8.\overline{08}$, and

$$100x - x = 8.\overline{08} - 0.\overline{08}$$

$$= 8 + .\overline{08} - .\overline{08}$$

$$= 8.$$

Therefore, we have $99x = 8$, so $x = \frac{8}{99} = \boxed{\frac{8}{99}}$.

(d) $0.\overline{32}$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: Let $x = 0.\overline{32}$. The repeating portion has two digits, so we multiply by 100, getting $100x = 32.\overline{32}$, and

$$100x - x = 32.\overline{32} - 0.\overline{32}$$

$$= 32 + .\overline{32} + 0.\overline{32} - 0.\overline{32}$$

$$= 32 + 0 - 0.\overline{32}$$

$$= 32.$$

Therefore, we have $99x = 32$. Multiplying both sides by 10 gives $990x = 320$, and dividing both sides by 990 gives $x = \frac{320}{990} = \frac{32}{99}$, so $\boxed{\frac{32}{99}}$.

(e) $0.\overline{40}$

You may type any additional notes you have here.

Hide Solution

Edit

Your Submission Solution

Solution: We recognize $0.\overline{40}$ as 1 $\overline{40}$s:

$$0.\overline{40} = 0.01 \cdot 0.\overline{40} = 0.01 \cdot 1 = 0.01.$$

This means

$$0.\overline{40} = 0.40 + 0.0\overline{40} = 0.40 + 0.01 = 0.41 = \boxed{\frac{41}{100}}.$$

$$\begin{array}{r} 7\,6\,3 \\ +\,1\,2\,4 \\ \hline 9\,0\,7 \end{array} \qquad \begin{array}{r} 7\,6\,3 \\ +\,1\,2\,4 \\ \hline 8\,8\,7 \end{array} \qquad \begin{array}{r} 7\,6\,3 \\ +\,1\,2\,4 \\ \hline 8\,9\,7 \end{array} \qquad \begin{array}{r} 7\,6\,3 \\ +\,1\,2\,4 \\ \hline 8\,9\,0 \end{array}$$

$$763 + 124 = 890 \qquad 763 + 124 = 887 \qquad 763 + 124 = 897 \qquad 763 + 124 = 890$$

I continued to do arithmetic with his fellow people through Nicolas's decrees. I remained aware of the point where so many cows are so much grass, and were that with cows in so many cows in Nicolas's grass.

— Agatha Christie

CHAPTER 7

Ratios, Conversions, and Rates

In this chapter we will discuss:

- **ratios** that compare two or more quantities,
- **conversions** between units of measurement measurements from one unit to another (for example, from inches to yards), and
- **rates** that measure how a quantity changes over time.

There are also ratios and rates that have two conversions and rates that are a ratio of two ratios.

7.1 What is a Ratio?

A **ratio** is used to compare the relative quantities of (usually) two groups or items of data.

A ratio is written as two numbers separated by a colon (:) or written in words. For example, a ratio of 2 to 3 means that for every two things there are three things. For example, there are 2 girls and 3 boys in the class, or there might be 10 girls and 15 boys in the class, or there might be 2 girls and 3 boys in the class, or there might be 2 girls and 3 boys in the class, or there might be 10 girls and 15 boys in the class, or there might be 2 girls and 3 boys in the class (in a very large classroom), (if you have to ask that if there are 2 girls, there have are the boys, but you should know that).

Concept
A ratio compares two quantities.

$$10 : 15 \qquad \text{10 to 15} \qquad \text{10/15}$$

Ratios behave a lot like fractions. They are usually written in **simplest form** as a ratio of two positive integers with no common factor larger than 1. To change our example, suppose that our ratio was something in mind there with 10 girls and 15 boys. We could write that the ratio of girls to boys is 10 to 15 or 10 : 15. However, we could also divide the students into 5 identical groups, where each group has 2 girls and 3 boys. We could write that the ratio of girls to boys is 2 : 3. It doesn't group, but some of the groups are the same. We could write of girls to boys is 2 : 3. Simplest form is 2 : 3.

We typically reduce to simplest form by dividing the greatest common factor from each part of the ratio. So to continue our example, since the greatest common factor of 10 and 15 is 5, we divide 10 by 5 to get 2 and 15 by 5 to get 3. This we make sense if we can factor.

$$15/5 = 3/1.$$

In words this means that for every 2 girls there is 1 boy. This process of reducing the ratio to simplest form is also called **simplifying** the ratio.

Definition: To **simplify** a ratio means to write it as a ratio of integers with no common factor larger than 1.

Problems

Problem 7.1 ⏱ 2'

Simplify the following ratios:

(a) $2 : 10$

(b) $9 : 6$

(c) $\frac{1}{2} : \frac{1}{3}$

(d) $\frac{1}{3} : \frac{1}{4}$

(e) $1.4 : 2.8$

Solution for Problem 7.1

(a) The greatest common factor of 2 and 10 is 2. We divide both parts of the ratio by 2, and we get

$$2 : 10 = \frac{2}{2} : \frac{10}{2} = 1 : 5.$$

Therefore the simplified ratio is $1 : 5$. We can also say the ratio greater than 1 is 5 to 1.

(b) The greatest common factor of 9 and 6 is 3, so we divide both parts of the ratio by 3, and we get

$$9 : 6 = \frac{9}{3} : \frac{6}{3} = 3 : 2.$$

As fractions, this is the same as $\frac{9}{6} = \frac{3}{2}$.

(c) To write $\frac{1}{2} : \frac{1}{3}$ as a ratio of positive integers, we need to multiply both parts of the ratio by some number that will cancel both denominators. The number we need is the least common multiple of the two denominators, which is 6. This gives us

$$\frac{1}{2} : \frac{1}{3} = \frac{1}{2} \cdot 6 : \frac{1}{3} \cdot 6 = 3 : 2.$$

As a fraction, this simplification is equivalent to

$$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\frac{1}{2} \cdot 6}{\frac{1}{3} \cdot 6} = \frac{3}{2}.$$

(d) It is usually easier to work with mixed numbers to find converting them to fractions. So we start by writing $1.4 : 2.8$ as $1\frac{4}{10} : 2\frac{8}{10}$ making our ratio

$$\frac{1\frac{4}{10}}{2\frac{8}{10}} = \frac{\frac{14}{10}}{\frac{28}{10}} = \frac{14}{28} = \frac{1}{2}.$$

Then, multiplying by 6 we remove the denominators, giving us

$$\frac{1}{3} : \frac{1}{4} = \frac{1}{3} \cdot 6 : \frac{1}{4} \cdot 6 = 2 : 3 = 21 : 15.$$

This is a ratio of two positive integers with no common factor (other than 1), so we are done.

(e) We can find write the ratio as a ratio of integers by multiplying by 10.

$$1.4 : 2.8 = 14 : 28.$$

Then, 14 and 28 have greatest common factor 2, so we divide by 2 to finish the simplification:

$$1.4 : 2.8 = 14 : 28 = \frac{14}{2} : \frac{28}{2} = 7 : 12.$$

◻

Problem 7.2 ⏱ 2'

The ratio of ratio to dogs in a pet shop is $1 : 3$. If there are 24 dogs in the shop, how many cats are there?

Solution for Problem 7.2 The ratio $2 : 3$ means that for every 2 cats, there are 3 dogs. Naturally, this also means that for every 4 dogs, there are 6 cats. So if there are 24 dogs, there are 6 times as many dogs as there are cats. So there are 6 times as many cats and so there are 6 times as many cats as there are dogs.

There are 24 dogs in the shop. We can think of this as 6 groups of 4 dogs each. Each group of 4 dogs has a corresponding group of 2 cats. So there are 6 groups of 2 cats, so there are $6 \cdot 2 = 12$ cats in the shop.

Another way to think about this problem is that, for the number of cats, we have

Concept
A ratio helps to assign a variable to an unknown quantity. Also, pick your variable wisely (big you remember what they represent, like c for "cats".

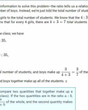
Since the ratio of cats to dogs in the shop is $2 : 3$, we have the equation

$$c : 24 = 2 : 3.$$

We can make the second ratio look like the first one by multiplying the parts of the second ratio by 3, giving

$$c : 24 = 2 : 3 = (2 \cdot 3) : (3 \cdot 3) = 6 : 18 = 1 : 3.$$

Comparing the first and last ratios above tells us that $c = 6$. So there are 6 cats in the shop. Indeed, we can check that $18 : 24 = 2 : 3$.



Setting two ratios equal, such as $c : 24 = 2 : 3$ in Problem 7.2, is an example of a **proportion**. We will cover proportions in more detail in Section 7.2.

Problem 7.3 ⏱ 2'

Billy's class has a ratio of girls to boys of $1 : 3$. If there are 36 students in the class, then how many of them are girls?

Solution for Problem 7.3 At first, it seems like we may not have enough information to solve this problem. We know ratio is a relationship between two quantities, but we don't know the number of girls or the number of boys. Instead, we will introduce the total number of students.

However, we can use the given ratio to construct a new ratio: the ratio of girls to the total number of students. We know that the $1 : 3$ girls to boys means there are 1 girls for every 4 boys. So there are 4 boys for every 1 girls. There are $1 + 3 = 4$ total students (girls and boys), and thus the ratio of girls to all students is $1 : 4$.

Let g be the number of girls in the class. Since there are 36 students in the class, we have

$$g : 4 = 1 : 36.$$

Multiplying the parts of the first ratio by 9 gives

$$36 : 36 = g : 36,$$

so $g = 36$ and there are 36 girls.

Another way to think of this is that girls make up $\frac{1}{4}$ of the total number of students, and boys make up $\frac{3}{4}$ of the total number of students. As a check, notice that $\frac{1}{4} \cdot 36 = 9$ is a girl and boys together make up all of the students.

Important: Suppose we are using a ratio to compare two quantities that together make up a group (such as girls and boys in a class). If the ratio of the first quantity to the second quantity is $m : n$, then the first quantity makes up $\frac{m}{m+n}$ of the whole, and the second quantity makes up $\frac{n}{m+n}$ of the whole.

The using this method in the next problem.

Problem 7.4 ⏱ 2'

Source: BATHOUSE7 ▶ Jump to Solution

A 1 foot length of rope is cut into two pieces whose lengths are in the ratio $1 : 4$. What is the length of the longer piece?

Solution for Problem 7.4 After we cut the rope, it will consist of two pieces that together make up the whole rope. Since the pieces have lengths in ratio $1 : 4$, the shorter piece of rope is $\frac{1}{1+4} = \frac{1}{5}$ of the original length, and the longer piece will make up $\frac{4}{1+4} = \frac{4}{5}$ of the original length. The longer piece is $\frac{4}{5}$ of a foot rope, so its length is

$$\frac{4}{5} \cdot (\text{1 foot}) = \left(\frac{4}{5}\right) \cdot (\text{1 foot}) = \text{foot}.$$

Problem 7.5 ⏱ 2'

Source: BATHOUSE7 ▶ Jump to Solution

My wife's family can top 30 pieces of candy. She only has Butterscotch and jelly beans, and the ratio of butterscotch to jelly beans is $7 : 3$. My wife eats some, and I want to split some jelly beans so that the ratio of butterscotch to jelly beans is $1 : 1$. How many jelly beans should I add?

Solution for Problem 7.5 We start by figuring out how many pieces of each type of candy were in the jar. Per practice, we'll show the two methods.

Method 1. Use a variable. Since we are using a ratio to compare two quantities that together make up a group (such as girls and boys in a class), the ratio of the first quantity to the second quantity is $7 : 3$, then the first quantity makes up $\frac{7}{7+3}$ of the whole, and the second quantity makes up $\frac{3}{7+3}$ of the whole.

Let b be the number of butterscotch. Since there are 30 pieces of candy, we have b of the butterscotch, and that each shape after we eat jelly beans, so we want the new total amount 2 jelly beans to be such that $\#\ \text{jelly beans} = 2b$. You can probably see right away that we need $b = 10$ and $\#\ \text{jelly beans} = 20$. If not, we can always compute by multiplying both parts of the $7 : 3$ ratio by 10 so that the first part of the ratio equals 30, so that

$$7 : 3 = 70 : 30.$$

This must equal b : b, so we must have $b = 30$.

We conclude that I need to add the jar to have 20 jelly beans. It starts with 10 jelly beans, so I need to add $20 - 10 = 8$ jelly beans to the jar.

Exercises

7.1.1: ⏱ 2'

Simplify the following ratios:

(a) $20 : 15$

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: The greatest common factor of 20 and 15 is 5. Dividing out this factor, we have $\frac{20}{5} : \frac{15}{5} = 4 : 3$.

(b) $6^2 : 6^4$

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Since $6^2 = (2 \cdot 3)^2 = 2^2 \cdot 3^2$ and $6^4 = (2 \cdot 3)^4 = 2^4 \cdot 3^4$, the greatest common factor of 6^2 and 6^4 is $2^2 \cdot 3^2$. Dividing by this common factor, we have

$$6^2 : 6^4 = \frac{2^2 \cdot 3^2}{2^2 \cdot 3^2} : \frac{2^4 \cdot 3^4}{2^2 \cdot 3^2} = 1 : 2^2 \cdot 3^2 = 1 : 36$$

(c) $\frac{1}{2} : \frac{1}{3}$

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Multiplying both fractions by 12 turns each into an integer:

$$\frac{1}{2} : \frac{1}{3} = \frac{1}{2} \cdot 12 : \frac{1}{3} \cdot 12 = 6 : 4$$

(d) $100 : 500$

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Dividing both 100 and 500 by 100 gives $100 : 500 = \frac{100}{100} : \frac{500}{100} = 1 : 5$.

7.1.2: ⏱ 2'

Simplify the following ratios:

(a) $20 : 15$

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: The greatest common factor of 20 and 15 is 5. Dividing out this factor, we have $\frac{20}{5} : \frac{15}{5} = 4 : 3$.

(b) $6^2 : 6^4$

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Since $6^2 = (2 \cdot 3)^2 = 2^2 \cdot 3^2$ and $6^4 = (2 \cdot 3)^4 = 2^4 \cdot 3^4$, the greatest common factor of 6^2 and 6^4 is $2^2 \cdot 3^2$. Dividing by this common factor, we have

$$6^2 : 6^4 = \frac{2^2 \cdot 3^2}{2^2 \cdot 3^2} : \frac{2^4 \cdot 3^4}{2^2 \cdot 3^2} = 1 : 2^2 \cdot 3^2 = 1 : 36$$

(c) $\frac{1}{2} : \frac{1}{3}$

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Multiplying both fractions by 12 turns each into an integer:

$$\frac{1}{2} : \frac{1}{3} = \frac{1}{2} \cdot 12 : \frac{1}{3} \cdot 12 = 6 : 4$$

(d) $100 : 500$

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Dividing both 100 and 500 by 100 gives $100 : 500 = \frac{100}{100} : \frac{500}{100} = 1 : 5$.

7.1.3: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

Solve: A radius 3 rectangular for some 3 revolutions (per 2 minutes. If pass 2 radius 3 revolutions in 1 minute, then how many revolutions does pass 2 radius 3 rotation?

Solution for Problem 7.1.3 We can think about the problem in two ways.

Method 1. Use a variable. Since the ratio of revolutions to minutes is $3 : 2$, the multiplying both parts of the ratio by 10 because 2 minutes 10 revolutions.

$$3 : 2 = (3 \cdot 5) : (2 \cdot 5) = 15 : 10.$$

So, 10 minutes $\frac{15}{10}$ revolutions when 2 radius 3 revolutions.

We also could have used the $7 : 2$ ratio to note that 10 minutes are many revolutions as 3, so, when 2 radius 10 revolutions, 10 minutes $\frac{15}{10} = \frac{3}{2}$ revolutions.

7.1.4: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

The ratio of girls to cows participating in rehearsal subtotal is 2 subtotal division listed is $7 : 4$. There are 45 girls in the program. What is the total subtotal of the program?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Method 1. Use a variable. Since the ratio of girls to cows is $7 : 4$, the multiplying both parts of the ratio by 10 because 2 subtotal 10 girls cows $7 : 4 = 7 \cdot 10 : 4 \cdot 10 = 70 : 40$.

So, 10 subtotal $\frac{70}{10}$ revolutions when 2 subtotal 10 revolutions.

We also could have used the $7 : 1$ ratio to note that the total number of students is $\frac{7}{7+4}$ times the number of girls. Therefore, there are $45 \cdot \frac{7+4}{7} = \frac{495}{7}$ total students.

7.1.5: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

Two numbers are in the ratio $3 : 7$. Their sum is 44. What is the greater of the two numbers?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Since the ratio of the smaller number to the larger number is $3 : 7$, the numbers are $3x$ and $7x$ for some value of x. Therefore, we must have $3x + 7x = 44$, so $11x = 44$. Dividing by 11 gives $x = 4$, so the larger number is $7 \cdot 4 = 28$.

7.1.6: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

An 8-inch-long subtraction sandwich is cut into two pieces whose lengths are in the ratio of $7 : 9$. How long is the shorter piece?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Since the ratio of the longer piece to the shorter piece is $7 : 9$, the shorter piece is $\frac{9}{7+9} = \frac{9}{16}$ of the whole sandwich. Since the sandwich is 8 inches long, the length of the shorter piece is $\frac{9}{16} \cdot (8 \text{ inches}) = \frac{9}{2}$ inches.

7.1.7: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

A father lifts 200 stacks of meat to be divided among his sons of and that is the ratio $5 : 3$, respectively. How many sons should he divide?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: All recreation $\frac{5}{8}$ of the land, so he gets $\frac{5}{8} \cdot 200 = \frac{1000}{8}$ stacks.

7.1.8: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

Two positive numbers are in the ratio of $1 : 9$. Their difference is 16. What is the sum of the two numbers?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Since the numbers are in the ratio $1 : 9$, the numbers are x and $9x$ for some value of x. Therefore, we have $9x - x = 16$ so $8x = 16$ so $x = 2$. Dividing by 5 gives $x = 4$, so the sum of the numbers is $4 + 9x = 13x = (13)(4) = 52$.

7.1.9: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

The ratio of teachers to students in a particular school is $1 : 14$. The ratio of female students to the total number of students is $4 : 5$. If there are 280 female students, then how many teachers are there?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: First, we find the total number of students. Since the ratio of female students to total number of students is $4 : 5$, the total number of students is $\frac{5}{4}$ the number of female students. Therefore, the total number of students is $\frac{5}{4} \cdot 280$. Reducing that $280 = 400$. It allows us to compute the total number of students.

$$\frac{5}{4} \cdot 280 = \frac{5}{4} \cdot 400 = 5 \cdot 100 = 500.$$

Next, we count the teachers. Since the ratio of teachers to students is $1 : 14$, the number of teachers is $\frac{1}{14}$ the number of students. There are 500 students, so there are $\frac{1}{14} \cdot 500 = \frac{250}{7}$ teachers.

7.1.10: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

The ratio of boxes to wins for Kyle's team is $3 : 5$. If the team had played the same number of games, but had won twice as many of its games, then what would the ratio of boxes to wins be?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: The ratio of boxes to wins for Kyle's team is $3 : 5$, which means that the team won 3 out of every 5 games. If it had won twice as many games, but played the same number of games, then it would have won 6 out of every 5 games. This leaves $7 - 5 = 2$ losses. So, the ratio of losses to wins is $2 : 6$, or $1 : 3$.

7.1.11: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

The ratio of pennies to dimes is as 2 to 3. $\frac{1}{2}$ and there are a total of 240 pennies and dimes in the jar. How many pennies should be added to make the ratio equal to 3 to 4?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Since the ratio of pennies to dimes is $2 : 3$, the pennies are $2x$ and the dimes are $3x$ for some value of x. Since we add pennies and dimes total, we have $x = \frac{240}{5} = 70$ pennies and $3x = 210$ dimes. We wish to add pennies until the ratio of pennies to dimes is $3 : 4$ which point the number of pennies is $\frac{3}{4} \cdot 210 = \frac{630}{4} = 157.5$ pennies and dimes. We'll still have 175 dimes after adding pennies, so we need $\frac{630}{4} - 175 = 70$ pennies total. This means we need $70 - 70 = \frac{70}{2}$ pennies.

$$\begin{array}{r} 7\,6\,3 \\ +\,1\,2\,4 \\ \hline 9\,0\,7 \end{array} \qquad \begin{array}{r} 7\,6\,3 \\ +\,1\,2\,4 \\ \hline 8\,8\,7 \end{array} \qquad \begin{array}{r} 7\,6\,3 \\ +\,1\,2\,4 \\ \hline 8\,9\,7 \end{array} \qquad \begin{array}{r} 7\,6\,3 \\ +\,1\,2\,4 \\ \hline 8\,9\,0 \end{array}$$

$$763 + 124 = 890 \qquad 763 + 124 = 887 \qquad 763 + 124 = 897 \qquad 763 + 124 = 890$$

I continued to do arithmetic with his fellow people through Nicolas's decrees. I remained aware of the point where so many cows are so much grass, and were that with cows in so many cows in Nicolas's grass.

— Agatha Christie

CHAPTER 7

Ratios, Conversions, and Rates

In this chapter we will discuss:

- **ratios** that compare two or more quantities,
- **conversions** between units of measurement measurements from one unit to another (for example, from inches to yards), and
- **rates** that measure how a quantity changes over time.

There are also ratios and rates that have two conversions and rates that are a ratio of two ratios.

7.1 What is a Ratio?

A **ratio** is used to compare the relative quantities of (usually) two groups or items of data.

A ratio is written as two numbers separated by a colon (:) or written in words. For example, a ratio of 2 to 3 means that for every two things there are three things. For example, there are 2 girls and 3 boys in the class, or there might be 10 girls and 15 boys in the class, or there might be 2 girls and 3 boys in the class, or there might be 2 girls and 3 boys in the class, or there might be 10 girls and 15 boys in the class, or there might be 2 girls and 3 boys in the class (in a very large classroom), (if you have to ask that if there are 2 girls, there have are the boys, but you should know that).

Concept
A ratio compares two quantities.

$$10 : 15 \qquad \text{10 to 15} \qquad \text{10/15}$$

Ratios behave a lot like fractions. They are usually written in **simplest form** as a ratio of two positive integers with no common factor larger than 1. To change our example, suppose that our ratio was something in mind there with 10 girls and 15 boys. We could write that the ratio of girls to boys is 10 to 15 or 10 : 15. However, we could also divide the students into 5 identical groups, where each group has 2 girls and 3 boys. We could write that the ratio of girls to boys is 2 : 3. It doesn't group, but some of the groups are the same. We could write of girls to boys is 2 : 3. Simplest form is 2 : 3.

We typically reduce to simplest form by dividing the greatest common factor from each part of the ratio. So to continue our example, since the greatest common factor of 10 and 15 is 5, we divide 10 by 5 to get 2 and 15 by 5 to get 3. This we make sense if we can factor.

$$15/5 = 3/1.$$

In words this means that for every 2 girls there is 1 boy. This process of reducing the ratio to simplest form is also called **simplifying** the ratio.

Definition: To **simplify** a ratio means to write it as a ratio of integers with no common factor larger than 1.

Problems

Problem 7.1 ⏱ 2'

Simplify the following ratios:

(a) $2 : 10$

(b) $9 : 6$

(c) $\frac{1}{2} : \frac{1}{3}$

(d) $\frac{1}{3} : \frac{1}{4}$

(e) $1.4 : 2.8$

Solution for Problem 7.1

(a) The greatest common factor of 2 and 10 is 2. We divide both parts of the ratio by 2, and we get

$$2 : 10 = \frac{2}{2} : \frac{10}{2} = 1 : 5.$$

Therefore the simplified ratio is $1 : 5$. We can also say the ratio greater than 1 is 5 to 1.

(b) The greatest common factor of 9 and 6 is 3, so we divide both parts of the ratio by 3, and we get

$$9 : 6 = \frac{9}{3} : \frac{6}{3} = 3 : 2.$$

As fractions, this is the same as $\frac{9}{6} = \frac{3}{2}$.

(c) To write $\frac{1}{2} : \frac{1}{3}$ as a ratio of positive integers, we need to multiply both parts of the ratio by some number that will cancel both denominators. The number we need is the least common multiple of the two denominators, which is 6. This gives us

$$\frac{1}{2} : \frac{1}{3} = \frac{1}{2} \cdot 6 : \frac{1}{3} \cdot 6 = 3 : 2.$$

As a fraction, this simplification is equivalent to

$$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\frac{1}{2} \cdot 6}{\frac{1}{3} \cdot 6} = \frac{3}{2}.$$

(d) It is usually easier to work with mixed numbers to find converting them to fractions. So we start by writing $1.4 : 2.8$ as $1\frac{4}{10} : 2\frac{8}{10}$ making our ratio

$$\frac{1\frac{4}{10}}{2\frac{8}{10}} = \frac{\frac{14}{10}}{\frac{28}{10}} = \frac{14}{28} = \frac{1}{2}.$$

Then, multiplying by 6 we remove the denominators, giving us

$$\frac{1}{3} : \frac{1}{4} = \frac{1}{3} \cdot 6 : \frac{1}{4} \cdot 6 = 2 : 3 = 21 : 15.$$

This is a ratio of two positive integers with no common factor (other than 1), so we are done.

(e) We can find write the ratio as a ratio of integers by multiplying by 10.

$$1.4 : 2.8 = 14 : 28.$$

Then, 14 and 28 have greatest common factor 2, so we divide by 2 to finish the simplification:

$$1.4 : 2.8 = 14 : 28 = \frac{14}{2} : \frac{28}{2} = 7 : 12.$$

◻

Problem 7.2 ⏱ 2'

The ratio of ratio to dogs in a pet shop is $1 : 3$. If there are 24 dogs in the shop, how many cats are there?

Solution for Problem 7.2 The ratio $2 : 3$ means that for every 2 cats, there are 3 dogs. Naturally, this also means that for every 4 dogs, there are 6 cats. So if there are 24 dogs, there are 6 times as many dogs as there are cats. So there are 6 times as many cats and so there are 6 times as many cats as there are dogs.

There are 24 dogs in the shop. We can think of this as 6 groups of 4 dogs each. Each group of 4 dogs has a corresponding group of 2 cats. So there are 6 groups of 2 cats, so there are $6 \cdot 2 = 12$ cats in the shop.

Another way to think about this problem is that, for the number of cats, we have

Concept
A ratio helps to assign a variable to an unknown quantity. Also, pick your variable wisely (big you remember what they represent, like c for "cats".

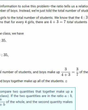
Since the ratio of cats to dogs in the shop is $2 : 3$, we have the equation

$$c : 24 = 2 : 3.$$

We can make the second ratio look like the first one by multiplying the parts of the second ratio by 3, giving

$$c : 24 = 2 : 3 = (2 \cdot 3) : (3 \cdot 3) = 6 : 18 = 1 : 3.$$

Comparing the first and last ratios above tells us that $c = 6$. So there are 6 cats in the shop. Indeed, we can check that $18 : 24 = 2 : 3$.



Setting two ratios equal, such as $c : 24 = 2 : 3$ in Problem 7.2, is an example of a **proportion**. We will cover proportions in more detail in Section 7.2.

Problem 7.3 ⏱ 2'

Billy's class has a ratio of girls to boys of $1 : 3$. If there are 36 students in the class, then how many of them are girls?

Solution for Problem 7.3 At first, it seems like we may not have enough information to solve this problem. We know ratio is a relationship between two quantities, but we don't know the number of girls or the number of boys. Instead, we will introduce the total number of students.

However, we can use the given ratio to construct a new ratio: the ratio of girls to the total number of students. We know that the $1 : 3$ girls to boys means there are 1 girls for every 4 boys. So there are 4 boys for every 1 girls. There are $1 + 3 = 4$ total students (girls and boys), and thus the ratio of girls to all students is $1 : 4$.

Let g be the number of girls in the class. Since there are 36 students in the class, we have

$$g : 4 = 1 : 36.$$

Multiplying the parts of the first ratio by 9 gives

$$36 : 36 = g : 36,$$

so $g = 36$ and there are 36 girls.

Another way to think of this is that girls make up $\frac{1}{4}$ of the total number of students, and boys make up $\frac{3}{4}$ of the total number of students. As a check, notice that $\frac{1}{4} \cdot 36 = 9$ is a girl and boys together make up all of the students.

Important: Suppose we are using a ratio to compare two quantities that together make up a group (such as girls and boys in a class). If the ratio of the first quantity to the second quantity is $m : n$, then the first quantity makes up $\frac{m}{m+n}$ of the whole, and the second quantity makes up $\frac{n}{m+n}$ of the whole.

The using this method in the next problem.

Problem 7.4 ⏱ 2'

Source: BATHOUSE7 ▶ Jump to Solution

A 1 foot length of rope is cut into two pieces whose lengths are in the ratio $1 : 4$. What is the length of the longer piece?

Solution for Problem 7.4 After we cut the rope, it will consist of two pieces that together make up the whole rope. Since the pieces have lengths in ratio $1 : 4$, the shorter piece of rope is $\frac{1}{1+4} = \frac{1}{5}$ of the original length, and the longer piece will make up $\frac{4}{1+4} = \frac{4}{5}$ of the original length. The longer piece is $\frac{4}{5}$ of a foot rope, so its length is

$$\frac{4}{5} \cdot (\text{1 foot}) = \left(\frac{4}{5}\right) \cdot (\text{1 foot}) = \text{foot}.$$

Problem 7.5 ⏱ 2'

Source: BATHOUSE7 ▶ Jump to Solution

My wife's family can top 30 pieces of candy. She only has Butterscotch and jelly beans, and the ratio of butterscotch to jelly beans is $7 : 3$. My wife eats some, and I want to split some jelly beans so that the ratio of butterscotch to jelly beans is $1 : 1$. How many jelly beans should I add?

Solution for Problem 7.5 We start by figuring out how many pieces of each type of candy were in the jar. Per practice, we'll show the two methods.

Method 1. Use a variable. Since we are using a ratio to compare two quantities that together make up a group (such as girls and boys in a class), the ratio of the first quantity to the second quantity is $7 : 3$, then the first quantity makes up $\frac{7}{7+3}$ of the whole, and the second quantity makes up $\frac{3}{7+3}$ of the whole.

Let b be the number of butterscotch. Since there are 30 pieces of candy, we have b of the butterscotch, and that each shape after we eat jelly beans, so we want the new total amount 2 jelly beans to be such that $\#\ \text{jelly beans} = 2b$. You can probably see right away that we need $b = 10$ and $\#\ \text{jelly beans} = 20$. If not, we can always compute by multiplying both parts of the $7 : 3$ ratio by 10 so that the first part of the ratio equals 30, so that

$$7 : 3 = 70 : 30.$$

This must equal b : b, so we must have $b = 30$.

We conclude that I need to add the jar to have 20 jelly beans. It starts with 10 jelly beans, so I need to add $20 - 10 = 8$ jelly beans to the jar.

Exercises

7.1.1: ⏱ 2'

Simplify the following ratios:

(a) $20 : 15$

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: The greatest common factor of 20 and 15 is 5. Dividing out this factor, we have $\frac{20}{5} : \frac{15}{5} = 4 : 3$.

(b) $6^2 : 6^4$

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Since $6^2 = (2 \cdot 3)^2 = 2^2 \cdot 3^2$ and $6^4 = (2 \cdot 3)^4 = 2^4 \cdot 3^4$, the greatest common factor of 6^2 and 6^4 is $2^2 \cdot 3^2$. Dividing by this common factor, we have

$$6^2 : 6^4 = \frac{2^2 \cdot 3^2}{2^2 \cdot 3^2} : \frac{2^4 \cdot 3^4}{2^2 \cdot 3^2} = 1 : 2^2 \cdot 3^2 = 1 : 36$$

(c) $\frac{1}{2} : \frac{1}{3}$

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Multiplying both fractions by 12 turns each into an integer:

$$\frac{1}{2} : \frac{1}{3} = \frac{1}{2} \cdot 12 : \frac{1}{3} \cdot 12 = 6 : 4$$

(d) $100 : 500$

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Dividing both 100 and 500 by 100 gives $100 : 500 = \frac{100}{100} : \frac{500}{100} = 1 : 5$.

7.1.2: ⏱ 2'

Simplify the following ratios:

(a) $20 : 15$

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: The greatest common factor of 20 and 15 is 5. Dividing out this factor, we have $\frac{20}{5} : \frac{15}{5} = 4 : 3$.

(b) $6^2 : 6^4$

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Since $6^2 = (2 \cdot 3)^2 = 2^2 \cdot 3^2$ and $6^4 = (2 \cdot 3)^4 = 2^4 \cdot 3^4$, the greatest common factor of 6^2 and 6^4 is $2^2 \cdot 3^2$. Dividing by this common factor, we have

$$6^2 : 6^4 = \frac{2^2 \cdot 3^2}{2^2 \cdot 3^2} : \frac{2^4 \cdot 3^4}{2^2 \cdot 3^2} = 1 : 2^2 \cdot 3^2 = 1 : 36$$

(c) $\frac{1}{2} : \frac{1}{3}$

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Multiplying both fractions by 12 turns each into an integer:

$$\frac{1}{2} : \frac{1}{3} = \frac{1}{2} \cdot 12 : \frac{1}{3} \cdot 12 = 6 : 4$$

(d) $100 : 500$

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Dividing both 100 and 500 by 100 gives $100 : 500 = \frac{100}{100} : \frac{500}{100} = 1 : 5$.

7.1.3: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

Solve: A radius 3 rectangular for some 3 revolutions (per 2 minutes. If pass 2 radius 3 revolutions in 1 minute, then how many revolutions does pass 2 radius 3 rotation?

Solution for Problem 7.1.3 We can think about the problem in two ways.

Method 1. Use a variable. Since the ratio of revolutions to minutes is $3 : 2$, the multiplying both parts of the ratio by 10 because 2 minutes 10 revolutions.

$$3 : 2 = (3 \cdot 5) : (2 \cdot 5) = 15 : 10.$$

So, 10 minutes $\frac{15}{10}$ revolutions when 2 radius 3 revolutions.

We also could have used the $7 : 2$ ratio to note that 10 minutes are many revolutions as 3, so, when 2 radius 10 revolutions, 10 minutes $\frac{15}{10} = \frac{3}{2}$ revolutions.

7.1.4: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

The ratio of girls to cows participating in rehearsal subtotal is 2 subtotal division listed is $7 : 4$. There are 45 girls in the program. What is the total subtotal of the program?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Method 1. Use a variable. Since the ratio of girls to cows is $7 : 4$, the multiplying both parts of the ratio by 10 because 2 subtotal 10 girls cows $7 : 4 = 7 \cdot 10 : 4 \cdot 10 = 70 : 40$.

So, 10 subtotal $\frac{70}{10}$ revolutions when 2 subtotal 10 revolutions.

We also could have used the $7 : 1$ ratio to note that the total number of students is $\frac{7}{7+4}$ times the number of girls. Therefore, there are $45 \cdot \frac{7+4}{7} = \frac{495}{7}$ total students.

7.1.5: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

Two numbers are in the ratio $3 : 7$. Their sum is 44. What is the greater of the two numbers?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Since the ratio of the smaller number to the larger number is $3 : 7$, the numbers are $3x$ and $7x$ for some value of x. Therefore, we must have $3x + 7x = 44$, so $11x = 44$. Dividing by 11 gives $x = 4$, so the larger number is $7 \cdot 4 = 28$.

7.1.6: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

An 8-inch-long subtraction sandwich is cut into two pieces whose lengths are in the ratio of $7 : 9$. How long is the shorter piece?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Since the ratio of the longer piece to the shorter piece is $7 : 9$, the shorter piece is $\frac{9}{7+9} = \frac{9}{16}$ of the whole sandwich. Since the sandwich is 8 inches long, the length of the shorter piece is $\frac{9}{16} \cdot (8 \text{ inches}) = \frac{9}{2}$ inches.

7.1.7: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

A father lifts 200 stacks of meat to be divided among his sons of and that is the ratio $5 : 3$, respectively. How many sons should he divide?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: All recreation $\frac{5}{8}$ of the land, so he gets $\frac{5}{8} \cdot 200 = \frac{1000}{8}$ stacks.

7.1.8: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

Two positive numbers are in the ratio of $1 : 9$. Their difference is 16. What is the sum of the two numbers?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Since the numbers are in the ratio $1 : 9$, the numbers are x and $9x$ for some value of x. Therefore, we have $9x - x = 16$ so $8x = 16$ so $x = 2$. Dividing by 5 gives $x = 4$, so the sum of the numbers is $4 + 9x = 13x = (13)(4) = 52$.

7.1.9: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

The ratio of teachers to students in a particular school is $1 : 14$. The ratio of female students to the total number of students is $4 : 5$. If there are 280 female students, then how many teachers are there?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: First, we find the total number of students. Since the ratio of female students to total number of students is $4 : 5$, the total number of students is $\frac{5}{4}$ the number of female students. Therefore, the total number of students is $\frac{5}{4} \cdot 280$. Reducing that $280 = 400$. It allows us to compute the total number of students.

$$\frac{5}{4} \cdot 280 = \frac{5}{4} \cdot 400 = 5 \cdot 100 = 500.$$

Next, we count the teachers. Since the ratio of teachers to students is $1 : 14$, the number of teachers is $\frac{1}{14}$ the number of students. There are 500 students, so there are $\frac{1}{14} \cdot 500 = \frac{250}{7}$ teachers.

7.1.10: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

The ratio of boxes to wins for Kyle's team is $3 : 5$. If the team had played the same number of games, but had won twice as many of its games, then what would the ratio of boxes to wins be?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: The ratio of boxes to wins for Kyle's team is $3 : 5$, which means that the team won 3 out of every 5 games. If it had won twice as many games, but played the same number of games, then it would have won 6 out of every 5 games. This leaves $7 - 5 = 2$ losses. So, the ratio of losses to wins is $2 : 6$, or $1 : 3$.

7.1.11: ⏱ 2'

Source: BATHOUSE7 ▶ Hide Solution

The ratio of pennies to dimes is as 2 to 3. $\frac{1}{2}$ and there are a total of 240 pennies and dimes in the jar. How many pennies should be added to make the ratio equal to 3 to 4?

Preview Solution

Your Step-by-step additional notes here plus here

Hint Solution

Your Submission: Solution

Explain: Since the ratio of pennies to dimes is $2 : 3$, the pennies are $2x$ and the dimes are $3x$ for some value of x. Since we add pennies and dimes total, we have $x = \frac{240}{5} = 70$ pennies and $3x = 210$ dimes. We wish to add pennies until the ratio of pennies to dimes is $3 : 4$ which point the number of pennies is $\frac{3}{4} \cdot 210 = \frac{630}{4} = 157.5$ pennies and dimes. We'll still have 175 dimes after adding pennies, so we need $\frac{630}{4} - 175 = 70$ pennies total. This means we need $70 - 70 = \frac{70}{2}$ pennies.

7.2 Multi-way Ratios

A ratio is a handy gadget for comparing two quantities. But it's also useful for comparing more than two quantities. For example, suppose a pet store has 4 dogs, 5 cats, and 11 goldfish. We would say that the numbers of dogs, cats, and goldfish are in the ratio of $4 : 5 : 11$. Since 4, 5, and 11 have no common factors, we cannot simplify this ratio any further. If a different (more exotic) pet store had 9 dogs, 12 cats, and 21 snakes, then we would say that the numbers of dogs, cats, and snakes are in the ratio of $9 : 12 : 21$. But now, since 3 is a common factor of 9, 12, and 21, we can divide each term of the ratio by 3, to say that the numbers of geckos, iguanas, and snakes are in the ratio of

$$9 : 12 : 21 = \frac{9}{3} : \frac{12}{3} : \frac{21}{3} = 3 : 4 : 7.$$

Just as with a two-way ratio, a multi-way ratio only gives you information about the relative quantities of the items—it doesn't tell you anything about the total number of items. For example, if you know that a third pet store has hamsters, guinea pigs, and rabbits in the ratio $1 : 2 : 5$, all you know is that for every hamster, there are 2 guinea pigs and 5 rabbits. There might be 1 hamster, 2 guinea pigs, and 5 rabbits; or there might be 6 hamsters, 12 guinea pigs, and 30 rabbits; more generally, there are n hamsters, $2n$ guinea pigs, and $5n$ rabbits for some number n.

Although the multiple parts of a multi-way ratio may at first look confusing, they're really not that much different from a two-way ratio, as we will see in the following problems.

Problems

Problem 7.6 Jump to Solution

Simplify the following ratios:

(a) $5 : 15 : 10$

(b) $6 : 10 : 9$

(c) $8 : 32 : 4 : 8$

(d) $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$

(e) $\left(3\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{5}{12}\right)$

Problem 7.7 Jump to Solution

A bowling tournament pays out prizes to the top 3 players in the ratio $5 : 2 : 1$. If the total prize money is \$1,000, then how much does the first-place winner receive?

Problem 7.8 Jump to Solution

Sam wants to bake a cake that requires butter, flour, sugar, and milk in the ratio $1 : 6 : 2 : 1$. Sam has $\frac{1}{2}$ cup of sugar. How much of the other ingredients does he need?

Problem 7.9 *Source: MATHCOUNTS* Jump to Solution

Jamal needs three gallons of a mix that is two parts blue paint, three parts white paint, and one part red paint. How many gallons of red paint will he need?

Problem 7.10 Jump to Solution

I have blocks, bikets, and bills in a bag. The ratio of the number of blokts to the number of bikets is $3 : 4$. The ratio of the number of bikets to the number of bills is $5 : 6$. What is the ratio of the number of blokts to the number of bills?

Just as with a two-part ratio, to **simplify** a multi-way ratio means to write the ratio using integers with no common factor larger than 1. So, the ratio

$$6 : 4 : 8$$

is not simplified because each part is a multiple of 2, while the ratio

$$\frac{1}{2} : \frac{1}{3} : \frac{2}{3}$$

is not simplified because the parts are not integers.

Problem 7.6 Edit

Solution for Problem 7.6:

(a) The greatest common factor of 5, 15, and 10 is 5, so we divide each part of the ratio by 5:

$$5 : 15 : 10 = \frac{5}{5} : \frac{15}{5} : \frac{10}{5} = 1 : 3 : 2.$$

(b) It is a little more difficult to see what the greatest common factor of 6, 10, and 9 is. We can get a little more insight by factoring each number:

$$6 = 2 \cdot 3,\ 10 = 2 \cdot 5,\ 9 = 3 \cdot 3.$$

Now we can see that there is no factor greater than 1 that is common to all three numbers, so the ratio $6 : 10 : 9$ is already simplified.

(c) We notice that 4 divides all the terms of the ratio, so we have

$$8 : 32 : 4 : 8 = \frac{8}{4} : \frac{32}{4} : \frac{4}{4} : \frac{8}{4} = 2 : 8 : 1 : 2.$$

(d) We handle fractions a little bit differently: now we want to find a number we can multiply each term by so that each becomes a integer. This is usually the least common denominator of the fractions. In the ratio $\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\right)$ the common denominator is 6, so we multiply:

$$\frac{1}{2} : \frac{1}{3} : \frac{2}{3} = \left(\frac{1}{2} \cdot 6\right) : \left(\frac{1}{3} \cdot 6\right) : \left(\frac{2}{3} \cdot 6\right) = 3 : 2 : 4.$$

(e) Mixed numbers only look more complicated—they're really just the same as fractions. We normally find them easier to work with if we convert them to fractions:

$$3\frac{1}{4} : \frac{1}{4} : \frac{1}{2} : \frac{5}{12} = \frac{13}{4} : \frac{1}{4} : \frac{1}{2} : \frac{5}{12}.$$

Then, as in part (d) above, we multiply by the least common denominator, which in this example is $3 \cdot 4 \cdot 5 = 60$, to convert the ratio to integers:

$$\frac{13}{4} : \frac{1}{4} : \frac{1}{2} : \frac{5}{12} = \left(\frac{13}{4} \cdot 60\right) : \left(\frac{1}{4} \cdot 60\right) : \left(\frac{1}{2} \cdot 60\right) : \left(\frac{5}{12} \cdot 60\right)$$
$$= 390 : 255 : 312.$$

□

Just as with a two-way ratio, we can think of a multi-way ratio in terms of "parts of the whole" as in the next problem.

Problem 7.7 Edit

A bowling tournament pays out prizes to the top 3 players in the ratio $5 : 2 : 1$. If the total prize money is \$1,000, then how much does the first-place winner receive?

Solution for Problem 7.7: The $5 : 2 : 1$ ratio means that for every \$5 the winner gets, the second-place player gets \$2 and the third-place player gets \$1. In other words, for every \$5 that the winner gets, a total of $\$5 + \$2 + \$1 = \$8$ is paid out. Therefore, the winner gets $\frac{5}{8}$ of the total money paid out. Since the total prize money is \$1,000, and the winner gets $\frac{5}{8}$ of the total, he gets $\frac{5}{8} \cdot \$1,000 = \$625$.

In Problem 7.7, we saw that a ratio of $5 : 2 : 1$ lead to the first quantity being $\frac{5}{5+2+1} = \frac{5}{8}$ of the whole. In the same way, the second-place winner gets $\frac{2}{5+2+1} = \frac{2}{8} = \frac{1}{4}$ of the prize money, and the third-place winner gets $\frac{1}{5+2+1} = \frac{1}{8}$ of the prize money.

> **Concept:** In a multi-part ratio, you can often think of each term as a "part of the whole." That is, you add all the terms in the ratio to get the "whole," and then each individual term makes up part of that whole.
>
> **WARNING!** Again, remember that ratio is a relative concept. The ratio only tells you the fraction of the whole that each part represents—it doesn't tell you anything about how much the total or each part is.

Going back to Problem 7.7, the ratio itself only tells us that the first-place winner gets $\frac{5}{8}$ of the money. The ratio doesn't tell us exactly how much the winner gets. It is only with the additional information about the total prize money that we can compute the actual money the winner receives.

Problem 7.8 Edit

Sam wants to bake a cake that requires butter, flour, sugar, and milk in the ratio $1 : 6 : 2 : 1$. Sam has $\frac{1}{2}$ cup of sugar. How much of the other ingredients does he need?

Solution for Problem 7.8: We'll present three different methods for solving this problem.

Method 1: Convert the ratio to match the given quantity. The given ratio of butter to flour to sugar to milk is $1 : 6 : 2 : 1$. But we only have $\frac{1}{2}$ cup of sugar, so we convert the ratio so the $\frac{1}{2}$ appears in the "sugar" position. Since the given ratio has a 2 in that position, we need to divide each term of the ratio by 4:

$$1 : 6 : 2 : 1 = \frac{1}{4} : \frac{6}{4} : \frac{2}{4} : \frac{1}{4} = \frac{1}{4} : \frac{3}{2} : \frac{1}{2} : \frac{1}{4}.$$

Now we can just read from the ratio the quantities of the other ingredients that correspond to $\frac{1}{2}$ cup of sugar: $\frac{1}{4}$ cup of butter, $\frac{3}{2}$ cup of flour, and $\frac{1}{4}$ cup of milk.

Method 2: Use separate two-part ratios: We can break up the multi-way ratio into several separate two-part ratios, where each ratio compares some ingredient to sugar. For example, the ratio of butter to sugar is $1 : 2$—this is part a ratio consisting of the butter and sugar terms from the original 4-part ratio.

> **Concept:** We can remove terms from a multi-way ratio to get a simpler ratio that only compares some of the quantities from the original ratio.

Since the butter and sugar are in ratio $1 : 2$ we know that there is half as much butter as sugar. We have $\frac{1}{2}$ cup of sugar, so the amount of butter is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ cup.

Similarly, the ratio of flour to sugar is $6 : 2 = 3 : 1$, so there is three times as much flour as sugar. Thus, since there is $\frac{1}{2}$ cup of sugar, there are $3 \cdot \frac{1}{2} = \frac{3}{2}$ cups of flour. Finally, the ratio of sugar to milk is $2 : 1$, so there is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ cup of milk.

Method 3: Compute the total. We can use the "parts of the whole" way of thinking with our 4-part ratio. The sugar is $\frac{1}{1+6+2+1} = \frac{2}{10} = \frac{1}{5}$ of the entire ingredients, and we have $\frac{1}{2}$ cup of sugar. Thus the total quantity of ingredients is $\frac{1}{2} \div \frac{1}{5} = \frac{5}{2}$ cups.

Now we can use the "parts of the whole" to compute the other ingredient amounts. The butter is $\frac{1}{10}$ of the whole, so there is $\frac{1}{10} \cdot \frac{5}{2} = \frac{1}{4}$ cup of butter. Next, the flour is $\frac{6}{10}$ of the whole, so there is $\frac{6}{10} \cdot \frac{5}{2} = \frac{3}{2}$ cups of flour. Finally, the milk is $\frac{1}{10}$ of the whole, so there is $\frac{1}{10} \cdot \frac{5}{2} = \frac{1}{4}$ cup of milk. □

> **Concept:** None of the techniques from Problem 7.8 are "right" or "wrong." Ratios are best approached with flexible thinking. Ratios can be interpreted in many different ways, and you should use the method that you feel most comfortable with, or the method that appears to work best for the particular problem that you're working on.

Problem 7.9 *Source: MATHCOUNTS*

Jamal needs three gallons of a mix that is two parts blue paint, three parts white paint, and one part red paint. How many gallons of red paint will he need?

Solution for Problem 7.9: Since we are given the total amount and we want to find the amount of one of the parts, the "parts of the whole" method will probably work best. The given ratio is $2 : 3 : 1$ of blue : white : red, and the total is 3 gallons. So the red paint is $\frac{1}{2+3+1} = \frac{1}{6}$ of the whole, and thus Jamal needs $\frac{1}{6} \cdot 3 = \frac{1}{2}$ gallon of red paint. □

Problem 7.10 Edit

I have blokts, bikets, and bills in a bag. The ratio of the number of blokts to the number of bikets is $3 : 4$. The ratio of the number of bikets to the number of bills is $5 : 6$. What is the ratio of the number of blokts to the number of bills?

Solution for Problem 7.10: Let's make a little chart of the data we're given:

$$\text{blokts} : \text{bikets} \qquad \text{bikets} : \text{bills}$$
$$3 : 4 \qquad\qquad 5 : 6$$

We'd be able to compare bills and blokts if the number of bikets in the above two ratios were equal. So let's make them equal! We can do this by multiplying the parts of the first ratio by 5 and the parts of the second ratio by 4:

$$\text{blokts} : \text{bikets} \qquad \text{bikets} : \text{bills}$$
$$15 : 20 \qquad\qquad 20 : 24$$

Aha—now we can write it as a 3-way ratio of blokts to bikets to bills:

$$\text{blokts} : \text{bikets} : \text{bills}$$
$$15 : 20 : 24$$

The problem asked us for the relationship between blokts and bills, and we can read this information from our 3-way ratio. We see that blokts and bills are in the ratio $15 : 24$, which can be simplified as

$$\text{blokts} : \text{bills} = 15 : 24 = \frac{15}{3} : \frac{24}{3} = 5 : 8.$$

□

Multi-Part Ratios

Exercises

7.2.1:

A log whose length is 60 inches is cut into three pieces in the ratio $1 : 3 : 5$. What is the number of inches in the length of the shortest piece?

Hide Solution

Reset

Your Submission: Solution

Solution: The shortest piece is $\frac{1}{1+3+5} = \frac{1}{9}$ *of the whole. So, the length of the shortest piece is*

$$\frac{1}{9} \cdot 60 = \boxed{\frac{20}{3} \text{ inches}} = \boxed{6\tfrac{2}{3} \text{ inches}}$$

7.2.2:

Three numbers have ratio $1 : 2 : 3$, and their sum is 48. What is the greatest of these three numbers?

Hide Solution

Reset

Your Submission: Solution

Solution: The greatest of the numbers is $\frac{3}{1+2+3} = \frac{1}{2}$ *of the sum of the numbers. Since the sum of the numbers is 48, the largest number is* $\frac{1}{2} \cdot 48 = \boxed{24}$.

7.2.3: *Source: MATHCOUNTS*

Purple paint is made with a $16 : 3 : 1$ ratio of

$$\text{white paint} : \text{blue paint} : \text{red paint}.$$

How much white paint is needed in order to make one gallon of purple paint?

Hide Solution

Reset

Your Submission: Solution

Solution: The white paint is $\frac{16}{16+3+1} = \frac{4}{5}$ *of the mixture, so* $\frac{4}{5} \cdot 1 = \boxed{\frac{4}{5} \text{ gallon}}$ *is needed to make 1 gallon of purple paint.*

7.2.4: *Source: MATHCOUNTS*

Three friends, Alexa, Bruno, and Carmela, pooled their money to start a lemonade stand. Alexa contributed \$20, Bruno contributed \$30, and Carmela contributed \$35. After a month, their lemonade stand had earned \$2,000, and they want to distribute this money in the same ratio as the money that was invested. How many dollars will Bruno receive?

Hide Solution

Reset

Your Submission: Solution

Solution: The ratio of Alexa, Bruno, and Carmela's investments is 25 : 30 : 35. Dividing all parts of this ratio by 5 gives 5 : 6 : 7. Therefore, Bruno invested $\frac{6}{5+6+7} = \frac{3}{16}$ *of the money, so he should receive* $\frac{3}{16}$ *of the earnings. Therefore, Bruno will receive* $\frac{3}{16} \cdot 2000 = \boxed{\$375}$ *dollars.*

7.2.5:

The top four winners in a golf tournament share the prize money in the ratio $9 : 5 : 2 : 1$. If the top prize winner receives \$45,000, then how much prize money is awarded in total?

Hide Solution

Reset

Your Submission: Solution

Solution: The top winner receives $\frac{9}{9+5+2+1} = \frac{9}{17}$ *of the total, so the ratio of the top winner to the total is* $9 : 17$. *Multiplying this ratio by 5000 to make the top winner equal to 45000 gives* $45000 : 85000$, *so the total prize pool is* $\boxed{\$85,000}$.

7.2.6: *Source: MATHCOUNTS*

Alex owns three times as many brown shoes as red shoes, twice as many black shoes as brown shoes, and four times as many white shoes as red shoes. What is the ratio of the number of white shoes to the number of black shoes he owns?

Hide Solution

Reset

Your Submission: Solution

Solution: From the information in the problem, we have

$$\frac{\text{brown}}{\text{red}} = \frac{3}{1}, \quad \frac{\text{black}}{\text{brown}} = \frac{2}{1}, \quad \frac{\text{white}}{\text{red}} = \frac{4}{1}.$$

Conveniently, red's number is the same in the first and third ratios. So, we can combine these two ratios and write

$$\frac{\text{brown}}{\text{red}} = \frac{3}{1} : \frac{4}{1}, \quad \frac{\text{white}}{\text{red}} = \frac{4}{1} : \frac{\text{brown}}{\text{red}} = \frac{3}{1} : 1.$$

To combine these two ratios, we need the number corresponding to brown shoes to be the same in both. So, we multiply both parts of the second ratio by 3:

$$\frac{\text{brown}}{\text{red}} = \frac{3}{1} : 1, \quad \frac{\text{white}}{\text{red}} = \frac{\text{brown}}{\text{red}} = \frac{3}{1} : 3.$$

Now, we can combine the ratios:

$$\frac{\text{black}}{\text{brown}} = \frac{2}{1} : 1, \quad \frac{\text{white}}{\text{brown}} = \frac{3}{1} : 3.$$

So, we see that the ratio of white shoes to black shoes is $6 : 4$, *which equals* $\boxed{3 : 2}$.

7.2.7★: *Source: MATHCOUNTS*

Three siblings have a gift of \$1,000 to split in the ratio $\frac{1}{2} : \frac{1}{3} : \frac{1}{4}$. What is the greatest number of dollars that any of the siblings will receive?

Hint: Can you write the ratio in an easier-to-use form?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: In order to make the ratio easier to work with, we start by simplifying it. The least common denominator of the fractions in the ratio is 12, so we multiply all parts of the ratio by 12. That gives

$$12 \cdot \frac{1}{2} : 12 \cdot \frac{1}{3} : 12 \cdot \frac{1}{4} = 6 : 4 : 3.$$

Therefore, the sibling who gets the most money receives $\frac{6}{6+4+3} = \frac{6}{13}$ *of the money. Since there is \$1,000 to divide, the one who gets the most money receives* $\frac{6}{13} \cdot \$1000 = \boxed{\$\frac{6000}{13}}$.

7.3 Proportions

Whenever we have two ratios that are equal, we have a **proportion**. The most common usage of proportion is when we have two changing quantities that are related in such a way that their ratio doesn't change.

For example, suppose that Marios secret recipe for chocolate milk uses 8 ounces of milk and 2 ounces of chocolate syrup, and produces a 10-ounce glass of chocolate milk. The ratio of milk to chocolate syrup is $8:2$, or $4:1$. If Mario wants to make a big pitcher of chocolate milk for 6 people, then he will need $6 \cdot 8 = 48$ ounces of milk and $6 \cdot 2 = 12$ ounces of chocolate syrup, so the ratio of milk to chocolate syrup is $48:12$, which is still $4:1$. No matter what quantity of chocolate milk that we want, the ratio of milk to chocolate syrup will always be $4:1$. We say that the milk and chocolate syrup are **proportional** or in **proportion**.

Problems

Problem 7.11 Jump to Solution

Charlotte is planning a vacation to Europe. The exchange rate is 1 dollar equals 0.6 euros, or $\$1 = €0.60$. If Charlotte wants to have €300 for her trip, then how many dollars does she need to convert?

Problem 7.12 *Source: MATHCOUNTS* Jump to Solution

A recipe calls for $2\frac{1}{2}$ cups of flour and 4 eggs. If only 3 eggs are used, then how many cups of flour should be used?

Problem 7.13 Jump to Solution

Sadie is 3 feet tall and at 6 p.m. in Sunnytown, Sadie casts an 8-foot shadow. Nick is 5 feet tall. How long is his shadow at 6 p.m. in Sunnytown?

Problem 7.14 Jump to Solution

My map of upstate New York has the scale $\frac{1}{4}\text{ inch} = 5\text{ miles}$. If Buffalo and Albany are 13 inches apart on my map, then how far apart are the cities?

Problem 7.15 Jump to Solution

Sylvia is an architect designing a new building. The building will be 30 feet tall, and the windows will each be 8 feet high. Sylvia draws blueprints for the building on which the building is 8 inches tall. How tall are the windows on Sylvia's blueprints?

Problem 7.16 Jump to Solution

My wallet-size photo of my pet cat Snookums is 3 cm wide and 5 cm tall. If I want a larger photo to put on my wall, and I want the area of the photo to be 135 square centimeters, then how many centimeters wide should the larger photo be?

Problem 7.11

Charlotte is planning a vacation to Europe. The exchange rate is 1 dollar equals 0.6 euros, or $\$1 = €0.60$. If Charlotte wants to have €300 for her trip, then how many dollars does she need to convert?

Solution for Problem 7.11: Since $\$1$ equals €0.60, the ratio between an equal amount of dollars and euros is $1:0.6$, which simplifies to $5:3$. We let x be the number of dollars that equals €300, which is what we want to find. Then as a proportion we have

$$5:3 = x:300,$$

which gives $\frac{5}{3} = \frac{x}{300}$. We can solve for x as $x = 300 \cdot \frac{5}{3} = 500$. Hence, Charlotte must convert $500 in order to receive €300. □

Problem 7.12 *Source: MATHCOUNTS*

A recipe calls for $2\frac{1}{2}$ cups of flour and 4 eggs. If only 3 eggs are used, then how many cups of flour should be used?

Solution for Problem 7.12: There are two primary methods we can use.

Method 1: Set up a proportion by equating ratios. Because the recipe calls for $2\frac{1}{2}$ cups of flour and 4 eggs, we know that the ratio of flour to eggs should always equal $2\frac{1}{2} : 4$. If we only have 3 eggs, then we need an amount of flour so that the ratio of flour to eggs still equals $2\frac{1}{2} : 4$. This means that if we have x cups of flour to go with our 3 eggs, we must have

$$2\frac{1}{2} : 4 = x : 3.$$

We solve this by writing the equation with fractions:

$$\frac{2\frac{1}{2}}{4} = \frac{x}{3}.$$

Multiplying both sides of this equation by 12 gives $3 \cdot \left(2\frac{1}{2}\right) = 4x$. So $\frac{15}{2} = 4x$, and dividing by 4 gives $\frac{15}{8} = x$. Therefore we need $\frac{15}{8} = 1\frac{7}{8}$ cups of flour.

Method 2: Scale the quantities. Because we only have 3 eggs and the recipe calls for 4 eggs, we are only using $\frac{3}{4}$ of the recipe amount. In order for the ratio of flour to eggs to remain constant, we also need to use $\frac{3}{4}$ of the recipe amount of flour. Therefore, the amount of flour that we need is

$$\frac{3}{4} \cdot 2\frac{1}{2} = \frac{3}{4} \cdot \frac{5}{2} = \frac{15}{8} = 1\frac{7}{8}$$

cups. □

In the next problem, we use the geometric fact that at any given time, the length of an object is proportional to the length of its shadow. (This is based on the geometric concept of **similarity**, which you will learn when you take a geometry course.)

Problem 7.13

Sadie is 3 feet tall and at 6 p.m. in Sunnytown, Sadie casts an 8-foot shadow. Nick is 5 feet tall. How long is his shadow at 6 p.m. in Sunnytown?

Solution for Problem 7.13: The information about Sadie tells us that the length of an object (or person) and the length of its shadow are proportional in the ratio $3:8$. If Nick's shadow has length x, then the ratio $5:x$ must equal the ratio $3:8$. Therefore, $\frac{5}{x} = \frac{3}{8}$ and multiplying both sides by $8x$ gives $40 = 3x$, so $x = \frac{40}{3} = 13\frac{1}{3}$. Thus, Nick's shadow is $13\frac{1}{3}$ feet long. □

Problem 7.14

My map of upstate New York has the scale $\frac{1}{4}\text{ inch} = 5\text{ miles}$. If Buffalo and Albany are 13 inches apart on my map, then how far apart are the cities?

Solution for Problem 7.14: A map's scale is another example of a proportion—it tells us that the ratio of the distance on the map to the distance in the real world is constant. For my map, this ratio is

$$\frac{1}{4}\text{ inch} : 5\text{ miles}.$$

Multiplying by 4 simplifies this ratio to $1\text{ inch} : 20\text{ miles}$, so that 1 inch on the map corresponds to 20 miles in real life. Thus, the 13 inches on the map between Buffalo and Albany means that the cities are $13 \cdot 20 = 260$ miles apart. □

Problem 7.15

Sylvia is an architect designing a new building. The building will be 30 feet tall, and the windows will each be 8 feet high. Sylvia draws blueprints for the building on which the building is 8 inches tall. How tall are the windows on Sylvia's blueprints?

Solution for Problem 7.15: We start with the ratio

$$\text{building height} : \text{window height} = 30\text{ feet} : 8\text{ feet} = 30 : 8 = 15 : 4.$$

Thus, on the blueprints, we must have the same ratio between the heights of the building and the window. So if the window height on the blueprints is x inches, then we have

$$15 : 4 = 8 : x,$$

so that $\frac{15}{4} = \frac{8}{x}$. Multiplying both sides of this equation by $4x$ gives $15x = 32$, so $x = \frac{32}{15} = 2\frac{2}{15}$. Therefore, the windows are $2\frac{2}{15}$ inches tall on the blueprints. □

Problem 7.16

My wallet-size photo of my pet cat Snookums is 3 cm wide and 5 cm tall. If I want a larger photo to put on my wall, and I want the area of the photo to be 135 square centimeters, then how many centimeters wide should the larger photo be?

Solution for Problem 7.16: The assumption in this problem is that no matter what the size, the picture will always have the same shape. More precisely, this means that the ratio of width to height will remain constant. Since the wallet-size photo has width 3 cm and height 5 cm, the ratio of width to height will always be $3:5$. But we're given neither the width nor the height of the larger photo: we're only given the area. So how do we set up a proportion?

The proportion tells us that the width is $3x$ cm and the height is $5x$ cm for some number x. This means that the area is $(3x) \cdot (5x) = 15x^2$ square centimeters. If I want a larger photo with area 135 square centimeters, then we must have $15x^2 = 135$, or $x^2 = 135/15 = 9$. Therefore, $x = 3$. So my larger photo will be $3x = 3(3) = 9$ cm wide (and $5x = 5(3) = 15$ cm high). □

Exercises

7.3.1: *Source: AMC 5*

A ream of paper containing 500 sheets is 5 cm thick. How many sheets of this type of paper would there be in a stack 7.5 cm high?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let there be x sheets in the 7.5 cm stack, so we have $500 : 5 = x : 7.5$. Writing this using fractions, we have $\frac{500}{5} = \frac{x}{7.5}$, so $\frac{x}{7.5} = 100$. Multiplying both sides by 7.5 gives $x = \boxed{750}$.

7.3.2:

An American traveling in Japan wishes to exchange American money (dollars, symbol $) for Japanese money (yen, symbol ¥). If the exchange rate is $1 = ¥80, then how many dollars will the traveler need to purchase ¥10,000?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let x be the number of dollars the traveler needs. Since the ratio of dollars to yen is $1:80$, we have $1:80 = x:10000$. Writing this using fractions gives $\frac{1}{80} = \frac{x}{10000}$. Multiplying both sides by 10000 gives $x = \frac{10000}{80} = \frac{1000}{8} = \boxed{125\text{ dollars}}$.

7.3.3: *Source: MATHCOUNTS*

Alexia designed a logo 2 inches wide and 1.5 inches tall to be used on her school's website. The school wants the logo to appear on the website as 8 inches wide. How tall will the logo be on the website if it is enlarged proportionally?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The width of the website logo is 4 times as large as Alexia's design, so the height of the website logo must also be 4 times as large as Alexia's design, or $4 \cdot 1.5 = \boxed{6\text{ inches}}$.

7.3.4:

A bank has two flagpoles next to each other. If the taller 30-foot pole (flying the U.S. flag) casts a shadow of 20 feet, and the shorter flagpole (flying the state flag) casts a shadow of 15 feet, then how tall is the shorter flagpole?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Considering the taller object, we have

$$\text{Object height} : \text{Shadow height} = 30 : 20 = 3 : 2.$$

So, we have

$$\text{Short pole height} : 15\text{ feet} = 3 : 2.$$

Multiplying both parts of the ratio on the right by 7.5, we have

$$\text{Short pole height} : 15\text{ feet} = 22.5 : 15.$$

So, the height of the short pole is $\boxed{22.5\text{ feet}}$.

We might also have reasoned as follows. The shadow of the shorter pole is $\frac{15}{20}$ of the shadow of the taller pole. So, the height of the shorter pole is $\frac{15}{20}$ of the height of the taller pole. Therefore, the height of the shorter pole is $\frac{15}{20} \cdot (30\text{ feet}) = \boxed{22.5\text{ feet}}$.

7.3.5:

If $\frac{1}{4}$ inch on a map represents 50 miles, then what is the number of miles represented by $2\frac{7}{8}$ inches?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have

$$\text{Map} : \text{Actual distance} = \frac{1}{4}\text{ inch} : 50\text{ miles}.$$

Multiplying by 4 makes this ratio $1\text{ inch} : 200\text{ miles}$. So, each inch on the map represents 200 miles Therefore, $2\frac{7}{8}$ inches represents $200 \cdot \left(2\frac{7}{8}\right) = 200\left(\frac{23}{8}\right) = \boxed{575\text{ miles}}$.

7.3.6: *Source: MATHCOUNTS*

A drafterperson makes a scale drawing of a 100 meter × 30 meter building, where 1 centimeter represents 2.5 meters. How many centimeters are in the smaller dimension of the drawing of the building?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let x cm be the smaller dimension in the drawing. Since we must have

$$\text{Drawing} : \text{Building} = 1\text{ cm} : 2.5\text{ m},$$

we have $x\text{ cm} : 30\text{ m} = 1\text{ cm} : 2.5\text{ m}$. Writing this as fractions, we have $\frac{x}{30} = \frac{1}{2.5}$. Multiplying by 30 gives $x = \frac{30}{2.5} = \frac{300}{25} = 12$. So, the smaller dimension of the drawing is $\boxed{12\text{ cm}}$.

7.3.7: *Source: AMC 8*

Twelve friends met for dinner at Oscar's Oversuffed Oyster House, and each ordered one meal. The portions were so large, there was enough food for 18 people. If they share, then how many meals should they have ordered to have just enough food for the 12 of them?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The 12 meals they ordered were enough for 18 people, so we have the ratio

$$\text{Meals needed} : \text{People} = 12 : 18 = 2 : 3.$$

So, we need $\frac{2}{3}$ as many meals as we have people. Therefore, to feed 12 people, we need $\frac{2}{3} \cdot 12 = \boxed{8}$ meals.

7.3.8★: *Source: AMC 8*

For every 3° rise in temperature, the volume of a certain gas expands by 4 cubic centimeters. If the volume of the gas is 24 cubic centimeters when the temperature is 32°, then what was the volume of the gas when the temperature was 20°?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: As the temperature rose from 20° to 32°, it rose by 3° exactly 4 times. Each time the temperature rises 3°, the volume of the gas increases by 4 cubic centimeters. So, as the temperature rose 4 times by 3°, the volume expanded by $4 \cdot 4 = 16$ cubic centimeters. After this expansion, the volume of the gas is 24 cubic centimeters, so the volume of the gas before the expansion was $24 - 16 = \boxed{8\text{ cubic centimeters}}$.

7.4 Conversions

There is a special type of ratio that is useful for converting between different units of measurement. Let's illustrate how this works with a simple example.

We know that there are 12 inches in a foot and there are 3 feet in a yard. Suppose you want to use this information to compute the number of inches in a yard. Of course, you can probably do this in your head, or you may even have the answer memorized. But let's carefully work through two methods that we can use to solve the problem. These methods will help us work through harder conversion problems, where the answer is not so obvious.

Method 1: Set up ratios. We'll use the ratio of a measurement similar to Problem 7.16 (the problem with the bills and blocks) and bills). We can write ratios to express the relationships between the units:

$$\frac{\text{inches}}{12}=\frac{1}{1}\quad\text{and}\quad\frac{\text{feet}}{3}=\frac{1}{1}\quad\text{and}\quad\frac{\text{yards}}{3}=\frac{1}{1}.$$

Well like to combine this into a 3-way ratio relating all three units. To do that, we need the "feet" amount in both 2-way ratios to match. The easiest way to do this is to multiply both parts of the first ratio by 3:

$$\frac{\text{inches}}{36}=\frac{3\ \text{feet}}{3}=\frac{1}{1}\quad\text{and}\quad\frac{\text{feet}}{3}=\frac{1}{1}\quad\text{and}\quad\frac{\text{yards}}{3}=\frac{1}{1}$$

Now we can write it as a 3-way ratio:

$$\text{inches}:\text{feet}:\text{yards}=36:3:1$$

Removing the "feet" gives us a ratio of inches : yards $=36:1$, so there are 36 inches in a yard.

Method 2: Use conversion factors. If we write the ratios from Method 1 as fractions, then we have what are called **conversion factors**. To help us keep track of what's going on, we'll write the units as part of the fraction. So we'd write

$$\frac{12\ \text{inches}}{1\ \text{foot}}\quad\text{and}\quad\frac{3\ \text{feet}}{1\ \text{yards}}.$$

(For consistency, we usually write all units in plural, so we write the word-looking "1 feet" instead of "1 foot.") Multiplying the conversion factors together will cancel the "feet" and leave us with a conversion factor relating inches to yards:

$$\frac{12\ \text{inches}}{1\ \cancel{\text{feet}}}\times\frac{3\ \cancel{\text{feet}}}{1\ \text{yards}}=\frac{36\ \text{inches}}{1\ \text{yards}}.$$

So there are 36 inches in a yard.

Compare the two methods used above. They're really the same thing! The conversion factors from Method 2 are just a convenient way for us to keep track of the ratios from Method 1.

There are two key ideas to keep in mind when using conversion factors: First,

> **Concept:** Think of conversion factors as fractions that are equal to 1.

Second,

> **Concept:** We multiply conversion factors together to cancel units.

We see this concept if we revisit our earlier computation:

$$\frac{12\ \text{inches}}{1\ \text{foot}}\cdot\frac{3\ \text{feet}}{1\ \text{yards}}=\frac{36\ \text{inches}}{1\ \text{yards}}.$$

All we're doing is multiplying two fractions on the left that are each equal to 1, so naturally (since $1\cdot1=1$) the quantity on the right side of the equation is also equal to 1. But we also notice that the "feet" units cancel. You might see this more clearly if we put in some missing steps in the above calculation:

$$\begin{aligned}\frac{12\ \text{inches}}{1\ \text{foot}}\cdot\frac{3\ \text{feet}}{1\ \text{yards}}&=\frac{(12\ \text{inches})\cdot(3\ \text{feet})}{(1\ \text{foot})\cdot(1\ \text{yards})}\\&=\frac{(12\cdot3)(\text{inches}\cdot\text{feet})}{(3\cdot1)(\text{feet}\cdot\text{yards})}\\&=\frac{36\ \text{inches}}{1\ \text{yards}}.\end{aligned}$$

The beauty of this is that it helps us prevent mistakes. For example, if we incorrectly tried to combine conversion factors as:

$$\frac{12\ \text{inches}}{1\ \text{foot}}\cdot\frac{1\ \text{yards}}{3\ \text{feet}}=\frac{4(\text{inches}\cdot\text{yards})}{(\text{feet}\cdot\text{feet})}$$

we see that the units don't cancel properly, so we probably made a mistake somewhere.

Conversion factors make it easy to convert units. For example, suppose we want to convert the length "6 yards" into inches. We know that a conversion factor is just a fraction that equals 1, and multiplying "6 yards" by 1 doesn't change the length. Therefore, we have

$$6\ \text{yards}=(6\ \text{yards})\cdot1=(6\ \text{yards})\cdot\frac{36\ \text{inches}}{1\ \text{yards}}.$$

But now the units nicely cancel, and we can finish the computation:

$$6\ \text{yards}=(6\ \cancel{\text{yards}})\cdot\frac{36\ \text{inches}}{1\ \cancel{\text{yards}}}=\frac{6\cdot36\ \text{inches}}{1}=216\ \text{inches}.$$

Notice how the "yards" units cancelled, and we are left with just the "inches" units, as we want. Therefore, 6 yards is equal to 216 inches.

Conversion Factors Part 1

Conversion Factors Part 2

Problems

Problem 7.17 Jump to Solution

How many yards equals 90 inches?

Problem 7.18 Jump to Solution

A tablespoon is half of a fluid ounce, a cup is 8 fluid ounces, and a gallon is 16 cups. How many tablespoons are in a gallon?

Problem 7.19 Jump to Solution

Will took $1,000 on his trip to Japan, where the exchange rate between dollars and yen is $1 = ¥90. He spent ¥15,000 on his hotel room and ¥11,250 on meals and souvenirs. How much money (in dollars) did he have remaining at the end of his trip?

Problem 7.20 Jump to Solution

The density of water is approximately 8.3 pounds per gallon, and there are 4 quarts in a gallon. How much does 7 quarts of water weigh?

Problem 7.21 Jump to Solution

An inch is approximately 2.5 centimeters. Approximately how many square centimeters are in a square inch?

Problem 7.17

How many yards equals 90 inches?

Solution for Problem 7.17: We discovered above that 1 yard is equal to 36 inches. To convert 90 inches into yards, we start with the quantity "90 inches" and multiply by the appropriate conversion factors until we get something with the units "yards." In this case, it's easy: we just need a number with 90 inches in the denominator (to cancel the inches in our initial quantity) and yards in the numerator (so we'll be left with yards). Our calculation is:

$$90\ \text{inches}=90\ \text{inches}\cdot\frac{1\ \text{yards}}{36\ \text{inches}}=\frac{90}{36}\ \text{yards}=2.5\ \text{yards}.$$

Thus, 90 inches is the same as 2.5 yards. ◻

> **Concept:** The reason the calculation in Problem 7.17 works is that the conversion factor is equal to 1. That is, because $1\ \text{yards}=36\ \text{inches}$, the fraction $\frac{1\ \text{yards}}{36\ \text{inches}}=1$ equals 1. Therefore, multiplying 90 inches by this fraction is the same as multiplying by 1, and hence the length does not change.

Problem 7.18

A tablespoon is half of a fluid ounce, a cup is 8 fluid ounces, and a gallon is 16 cups. How many tablespoons are in a gallon?

Solution for Problem 7.18: We want to convert "1 gallon" into some number of tablespoons. But we don't have a single gallons-to-tablespoons conversion factor. Instead, we have to use the multiple conversion factors that we are given in the problem statement. First, we show a step-by-step solution.

We first convert gallons to cups. We don't really need a "conversion factor" for this, since we can just read this data from the problem statement:

$$1\ \text{gallon}=16\ \text{cups}.$$

Next, we convert cups to ounces, using the cups-to-ounces conversion:

$$\begin{aligned}1\ \text{gallon}&=16\ \text{cups}\\&=16\ \text{cups}\cdot\frac{8\ \text{ounces}}{1\ \text{cups}}\\&=(16\cdot8)\ \text{ounces}\\&=128\ \text{ounces}.\end{aligned}$$

Finally, we convert ounces to tablespoons (abbreviated tbsp):

$$\begin{aligned}1\ \text{gallon}&=128\ \text{ounces}\\&=128\ \text{ounces}\cdot\frac{1\ \text{tbsp}}{2\ \text{ounces}}\\&=\frac{128}{2}\ \text{tbsp}\\&=256\ \text{tbsp}.\end{aligned}$$

So there are 256 tablespoons in a gallon.

But conversion factors are nice in that we can use more than one of them at the same time. In particular, we could do the gallons-to-tablespoons conversion all at once:

$$1\ \text{gallon}=1\ \text{gallon}\cdot\frac{16\ \text{cups}}{1\ \text{gallon}}\cdot\frac{8\ \text{ounces}}{1\ \text{cup}}\cdot\frac{1\ \text{tbsp}}{2\ \text{ounces}}$$
$$=\frac{16\cdot8}{2}\ \text{tbsp}$$
$$=256\ \text{tbsp}.$$

As long as we sure that each conversion factor equals 1 (meaning that its numerator and denominator represent the same quantity), and that the units cancel properly, we can line up as many conversion factors as might be necessary to do a complicated conversion. ◻

We can use conversion factors whenever we wish to compare two different units—they don't necessarily have to be typical "measurements" like length or volume. For example:

Problem 7.19

Will took $1,000 on his trip to Japan, where the exchange rate between dollars and yen is $1 = ¥90. He spent ¥15,000 on his hotel room and ¥11,250 on meals and souvenirs. How much money (in dollars) did he have remaining at the end of his trip?

Solution for Problem 7.19: Method 1: Convert and then convert back. When Will went to Japan, his dollars became:

$$\$1{,}000\cdot\frac{¥90}{\$1}=¥90{,}000.$$

This is just like any other conversion: notice how the conversion factor $\frac{¥90}{\$1}$ equals 1 because ¥90 equals $1, and notice how the $ units cancel leaving us with the desired ¥ units. After his spending, he was left with:

$$¥90{,}000-¥(15{,}000+11{,}250)=¥63{,}750.$$

When Will got home, he converted his remaining yen back into dollars:

$$¥63{,}750\cdot\frac{\$1}{¥90}=\$\frac{63{,}750}{90}=\$375.$$

So Will had $375 dollars remaining after his trip.

Method 2: Convert just the spending. Rather than convert Will's entire bankroll to yen and then convert it back, we can just figure out how much he spent in dollars. His total spending was ¥(15,000 + 11,250) = ¥26,250, so in dollars this is

$$¥26{,}250\cdot\frac{\$1}{¥90}=\$\frac{26{,}250}{90}=\$625.$$

Thus Will spent the equivalent of $625 on his trip, and $1,000 − $625 = $375 remaining. ◻

Problem 7.20

The density of water is approximately 8.3 pounds per gallon, and there are 4 quarts in a gallon. How much does 7 quarts of water weigh?

Solution for Problem 7.20: We want to convert from quarts to pounds. We can use two conversion factors: one for quarts to gallons, and one for gallons to pounds:

$$\begin{aligned}7\ \text{quarts}&=7\ \text{quarts}\cdot\frac{1\ \text{gallon}}{4\ \text{quarts}}\cdot\frac{8.3\ \text{pounds}}{1\ \text{gallon}}\\&=\frac{7\cdot8.3}{4}\ \text{pounds}\\&=14.525\ \text{pounds}.\end{aligned}$$

◻

Problem 7.21

An inch is approximately 2.5 centimeters. Approximately how many square centimeters are in a square inch?

Solution for Problem 7.21: Method 1: Reason geometrically. A square inch is the area of a square that is 1 inch on each side. But this same square is 2.5 centimeters on each side, so its area is $(2.5\ \text{cm})^2=6.25$ square centimeters.

Method 2: Use conversion factors. We wish to convert from square inches (written in^2) to square centimeters (written cm^2). So we need to cancel the inches units twice and be left with the centimeters units twice. Thus, we need to multiply by two conversion factors:

$$1\ \text{in}^2=1\ \text{in}^2\cdot\frac{2.5\ \text{cm}}{1\ \text{in}}\cdot\frac{2.5\ \text{cm}}{1\ \text{in}}=(2.5)(2.5)\ \text{cm}^2=6.25\ \text{cm}^2.$$

◻

Exercises

7.4.1:

Suppose that one US dollar is worth C$1.25 in Canadian dollars. If I want to buy a C$15 book in Canada, then how many US dollars do I need?

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: I need 1 US dollar for every 1.25 Canadian dollars, so my 15 Canadian dollars convert to $15/1.25 = 12$ US dollars. We can also do this using a conversion factor:

$$\text{C\$15}=\text{C\$15}\cdot\frac{\text{US\$1}}{\text{C\$1.25}}=\text{US\$}\frac{15}{1.25}=\boxed{\text{US\$12}}$$

7.4.2:

Basketball center Steve Tisdall is 7 feet 2 inches in height. What is Steve's height in inches?

Preview: Solution

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: Each foot is 12 inches, so 7 feet is $7\cdot12=84$ inches. Adding the 2 extra inches gives $84+2=\boxed{86}$ inches.

7.4.3:

Recall that 1 inch is approximately 2.5 centimeters. What is the area, in square centimeters, of a square that is $\frac{1}{2}$ feet long on each side?

Preview: Solution

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: First, since there are 12 inches in a foot, there are $12\cdot\frac{1}{2}=6$ inches in $\frac{1}{2}$ foot. Thus the square is 6 inches per side.

We could first convert the side length to centimeters, giving

$$6\ \text{in}=6\ \text{in}\cdot\frac{2.5\ \text{cm}}{1\ \text{in}}=(6\cdot2.5)\ \text{cm}=15\ \text{cm}.$$

Thus the area of the square is $15\cdot15=\boxed{225}\ \text{cm}^2$.

Alternatively, we could use the fact (from the text) that $1\ \text{in}^2=6.25\ \text{cm}^2$, and convert the 36 square inches of area to square centimeters:

$$36\ \text{in}^2=36\ \text{in}^2\cdot\frac{6.25\ \text{cm}^2}{1\ \text{in}^2}=(36\cdot6.25)\ \text{cm}^2=\boxed{225}\ \text{cm}^2.$$

7.4.4:

There are approximately 28.35 grams in an ounce, and 16 ounces in a pound. How many grams does a quarter-pound hamburger weigh? Round your answer to nearest whole number of grams.

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: We line up the conversion factors:

$$\frac{1}{4}\ \text{pounds}\cdot\frac{16\ \text{ounces}}{1\ \text{pounds}}\cdot\frac{28.35\ \text{grams}}{1\ \text{ounces}}=\boxed{113\ \text{grams}}$$

7.4.5:

Natalya's secret recipe for peanut butter cookies calls for $\frac{3}{2}$ cups of flour. Unfortunately, Natalya has lost all of her measuring equipment except for a teaspoon. There are 3 teaspoons in a tablespoon, $\frac{1}{2}$ ounce in a tablespoon, and 8 ounces in a cup. How many teaspoons of flour does Natalya need for her recipe?

Preview: Solution

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: We line up the conversion factors:

$$\begin{aligned}2.5\ \text{cups}&=2.5\ \text{cups}\cdot\frac{8\ \text{oz}}{1\ \text{cup}}\cdot\frac{1\ \text{tbsp}}{0.5\ \text{oz}}\cdot\frac{3\ \text{tsp}}{1\ \text{tbsp}}\\&=\frac{2.5\cdot8\cdot3}{0.5}\ \text{tsp}\\&=120\ \text{tsp}.\end{aligned}$$

Thus Natalya needs $\boxed{120}$ teaspoons for her recipe.

7.4.6:

1000 meters is 1 kilometer, and 100 hectares is one square kilometer. How many square meters are in 1 hectare?

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: We have (note ha is short for hectare):

$$1\ \text{ha}=1\ \text{ha}\cdot\frac{1\ \text{km}^2}{100\ \text{ha}}\cdot\left(\frac{1000\ \text{m}}{1\ \text{km}}\right)^2=\frac{1000^2}{100}\ \text{m}^2=\boxed{10000\ \text{m}^2}.$$

7.4.7:

On the planet Gobbob, the unit of currency is the Plokker, the unit of weight is the ekam, and the unit of volume is the pinpak. The precious liquid vimey is worth 400 Plokkers per ekam and has a density of 20 ekam per pinpak. If Aarne has 500 pinpak of vimey, then how many Plokkers is her vimey worth?

Hint: Don't let the weird words fool you; this problem is just like many others you've solved in the section.

Preview: Solution

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: Despite the foreign words, we can still use conversion factors!

$$\begin{aligned}500\ \text{pinpak}&=500\ \text{pinpak}\cdot\frac{20\ \text{ekam}}{1\ \text{pinpak}}\cdot\frac{400\ \text{Plokkers}}{1\ \text{ekam}}\\&=(500\cdot20\cdot400)\ \text{Plokkers}\\&=\boxed{4{,}000{,}000\ \text{Plokkers}}.\end{aligned}$$

7.5. Speed

Speed is probably already familiar with the idea of **speed**. Speed is a measure of how fast something is moving.

For example, suppose a car is travelling at a constant speed of 40 miles per hour. After 1 hour, it has travelled 40 miles; after 2 hours, it has travelled 80 miles; after 3 hours, it has travelled 120 miles; and so on. Notice that the ratio of distance travelled (in miles) to time travelled (in hours) is always the same:

$$\frac{40}{1} = \frac{80}{2} = \frac{120}{3} = \frac{240}{6}.$$

In general, if we travel for a time t, and we have travelled d miles, then, for a ratio of distance to time of $\frac{d}{t}$ miles per hour:

Concept: Speed is the ratio of distance to time.

We can write Speed in the ratio of distance to time" as the equation:

$$\text{speed} = \frac{\text{distance}}{\text{time}}.$$

This equation can be rearranged as:

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

and also as:

$$\text{distance} = \text{speed} \times \text{time}.$$

The units associated with speed depends on whether these equations. For example, the speed of a car (in the U.S.) is usually given in "miles per hour" abbreviated "mph". The car **can** be expressed in metres. Similarly, the time is usually given in hours but it can be given in minutes or seconds. The distance is usually given in miles but it can be given in kilometres or metres.

Example: $\frac{\text{miles}}{\text{hour}}$ miles per hour = 60 miles per hour; $\frac{\text{km}}{\text{s}}$ km per second = 5 km per second.

Similarly, a car that travels at a speed of 70 miles per hour for 2 hours covers a distance of $70 \times 2 = 140$ miles.

We can also think of speed as a way of conversions from one unit of measure to another between time and distance. To repeat our previous example, a car that travels at a speed of 70 miles per hour for 2 hours is essentially "converting" the 2 hours of time into a distance, as:

$$\text{distance} = \text{speed} \times \text{time} = \frac{70 \text{ miles}}{1 \text{ hour}} \times 2 \text{ hours} = 140 \text{ miles}.$$

Problems

Problem 7.22 Jump to Solution

(a) How far does a car travelling at 75 miles per hour travel in 2.5 hours?

(b) If a truck travels at a constant speed and travels 230 miles in 3.5 hours, then at what speed does the truck travel?

(c) How long will a motorcycle travelling at 55 miles per hour travel to travel 165 miles?

Problem 7.23 Jump to Solution SOLUTION: (a) Jump to Solution

A freight train travels 7 miles in 7 minutes. How many miles does it travel in 1 hour, if it travels at this speed?

Problem 7.24 Jump to Solution SOLUTION: (a) Jump to Solution

If you walk for 40 minutes at a rate of 4 mph and then run for 20 minutes at a rate of 10 mph, then how many miles have you gone at the end of your run? Continuous? What was your average speed for the journey?

Problem 7.25 Jump to Solution

Shelly drove the 165 miles from her home to her office at an average speed of 75 miles per hour. Coming home, she encountered heavy traffic and her average speed for the return trip was 55 miles per hour. What was her average speed for the entire 330 miles?

Problem 7.26 Jump to Solution

Jessica has to travel 4.7 km, and drives at a constant speed of 15 miles per hour. Jessica drives 150 miles due east of Alex. Jessica has known at 4 p.m., and drives at a constant speed of 15 miles per hour due west. At what time do they meet?

Problem 7.27 Jump to Solution

(a) On Monday, Yogi and Boo-Boo drive at the same place and at the same time on a 400-mile circular track and run in the same direction. Yogi runs at 5 miles per second and Boo-Boo runs at 3 miles per second. How many seconds will they first meet after starting?

(b) On Tuesday, Yogi and Boo-Boo drive at the same place and at the same time on a 400-mile circular track and run in the same direction. Yogi runs at 5 miles per second and Boo-Boo runs at 3 miles per second. How many seconds will they first meet after starting?

The next frame (type of speed problem) gives you two of the three pieces of data: speed, time, distance' and asks you to compute the third.

Problem 7.28 Jump to Solution

(a) How far does a car travelling at 75 miles per hour travel in 2.5 hours?

(b) If a truck travels at a constant speed and travels 230 miles in 3.5 hours, then at what speed does the truck travel?

(c) How long will a motorcycle travelling at 55 miles per hour travel to travel 165 miles?

Problem 7.29 Jump to Solution

The car travels 75 miles during each hour that it travels, so it travels a total of $75 \times 2.5 = 187.5$ miles.

We need think about the units to see why we should multiply by time. The time 2.5 hours to get the distance travelled in miles. Multiplying these two allows us to cancel out the "hours":

$$\text{distance} = \text{speed} \times \text{time}$$
$$= \frac{75 \text{ miles}}{1 \text{ hour}} \times 2.5 \text{ hours}$$
$$= 75 \times 2.5 \text{ miles}$$
$$= 187.5 \text{ miles}.$$

Again, we need to think about the units to see why we should multiply by time. The time 3.5 hours to get the distance travelled in miles. Multiplying these allows us to cancel out the "hours":

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$
$$= \frac{230 \text{ miles}}{3.5 \text{ hours}}$$
$$= 230 \div 3.5 \text{ miles per hour}$$
$$= 65.714\ldots \text{ miles per hour}.$$

It is a motorcycle process 55 miles in each hour, then it will cover 165 miles in $\frac{165}{55} = 3.25$ hours.

We use the first two parts, finding second units, we help us see what units we need to use. We need to convert 165 miles into hours. The time 3.25 hours to get the distance travelled in miles. Multiplying these allows us to cancel out the "hours":

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$
$$= \frac{165 \text{ miles}}{55 \text{ miles per hour}}$$
$$= 3 \text{ hours}$$

This may be easier to see (and remember) if we use the conversion factor which more like a conversion factor:

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$
$$= \frac{165 \text{ miles}}{55 \text{ miles per hour}} \times \frac{1 \text{ hour}}{1 \text{ hour}}$$
$$= 3 \text{ hours}.$$

Problem 7.30 Jump to Solution

Yogi's friend Jellystone Jones travels 1 mile in 1 minute 30 seconds. At this rate, how many miles will he travel in 1 hour?

Solution for Problem 7.30. One minute 30 seconds is $\frac{3}{2}$ minutes, so the hourly speed is

$$\text{speed} = \frac{1 \text{ mile}}{\frac{3}{2} \text{ minutes}} = \frac{2}{3} \text{ miles per minute}.$$

There are 60 minutes in an hour, so the distance travelled in 60 minutes is

$$\text{distance} = (60 \text{ minutes}) \times \left(\frac{2}{3} \text{ miles per minute}\right) = 40 \text{ miles}.$$

Another way to solve Problem 7.30 is to reason as follows: the number of "1 minutes 30 seconds" intervals in a 60 minute hour is $60 \div \frac{3}{2} = 40$. The train covers a mile in each of these 40 intervals. Therefore the train covers 40 miles an hour.

Problem 7.31 Jump to Solution SOLUTION: (a) Jump to Solution

If you walk for 40 minutes at a rate of 4 mph and then run for 20 minutes at a rate of 10 mph, then how many miles have you gone at the end of your run? Continuous? What was your average speed for the journey?

Problem 7.32 Jump to Solution

Shelly drove the 165 miles from her home to her office at an average speed of 75 miles per hour. Coming home, she encountered heavy traffic and her average speed for the return trip was 55 miles per hour. What was her average speed for the entire 330 miles?

Solution for Problem 7.32. You might think:

Bogus Solution: Shelly drove half of her trip at 75 mph and the other half of her trip at 55 mph. Therefore her average speed for the entire trip was the average of 75 and 55, namely 65 mph.

$$\frac{75 + 55}{2} = 65 \text{ mph}.$$

We cannot average speeds in this way, as we will see when we compute the amount of time the trip takes.

The trip from the home to the office was 165 miles at 75 mph, so it takes

$$\frac{165 \text{ miles}}{75 \text{ miles per hour}} = 2.2 \text{ hours}.$$

The trip from the office to the home was 165 miles at 55 mph, so it takes 3 hour. This, the entire 165 miles trip takes $\frac{165}{55}$ hours of travel time, and we can compute the average speed to be

$$\frac{330 \text{ miles}}{\frac{165}{55} \text{ hours}} = 66 \text{ mph}.$$

Therefore the average speed of the round trip journey is 66 miles per hour.

WARNING! Speeds do not usually "average" in the way that you might expect them to.

Advice: There is a relationship between the speeds for the two portions of the journey in the problem. If the speed for the first portion of the journey is s_1 and the speed for each half of the trip is s_2, then

$$\frac{s_1 + s_2}{2} = \frac{330}{\frac{165}{s_1} + \frac{165}{s_2}}.$$

Harmonic mean is another kind of "average" that has many uses in advanced mathematics.

Problem 7.33 Jump to Solution

Jacksonville, the house of a 1-mile circular track and a speed of 10 miles per hour and. Alex drives 150 miles due east of Alex. Alex has known at 4 p.m., and drives at a constant speed of 15 miles per hour due west. At what time do they meet?

Solution for Problem 7.33. There are a couple of different ways to solve this problem.

Method 1. Set up an equation. Let the time in hours it takes Jessica to make her meeting Alex's car variable since this is what we're trying to find. Jessica drives at a constant speed of 15 miles per hour, so she drives $15t$ miles in t hours. Alex's car drives at a constant speed of 15 miles per hour, so Alex's car drives $150 - 15t$ miles in t hours. Alex's starting at 7 a.m., and he moves right (along our imaginary number line) at 15 miles per hour. So his position at time t is $15t$. Jessica's starting at 7 a.m., and she moves left (along our imaginary number line) at 15 miles per hour. So her position at time t is $150 - 15t$. So when Jessica and Alex meet, they are at the same place, so we need to solve the equation:

$$15t = 150 - 15t.$$

They meet when their positions are equal, so at the time that they meet we have

$$15t = 150 - 15t$$
$$30t = 150$$
$$t = 5.$$

Expanding the equation gives $15t = 150 - 15t$ = 90 and simplifying gives $30t = 150$, $5t = 150/30 = 15$ 5. Therefore, we conclude that they meet at 5 hours after 7 a.m., or at 12 noon. Jessica's car has travelled a distance of $15 \times 5 = 75$ miles. Alex's car has travelled a distance of $150 - 15 \times 5 = 75$ miles.

Method 2. Think about how the people would actually behave in real life. Jessica's car has drive 150 miles between 7 and 8 a.m.; she has to drive 150 miles between 4 and 5 p.m. Alex's car has drive 150 miles between 7 and 8 a.m.; he has to drive 150 miles between 4 and 5 p.m.

Alex's car and Jessica's car are moving in the same direction, so they have to drive the distance between them at the speed difference, namely 20 miles per hour. They will meet when this distance is reduced all the way to 0. Since they start with 60 miles between them at 4 a.m., and they need to reduce this distance at a speed of 20 miles per hour, it takes 3 hours for them to meet. Jessica's car has travelled a distance of $15 \times 5 = 75$ miles. Alex's car has travelled a distance of $150 - 15 \times 5 = 75$ miles. Jessica's car is moving at a speed of 15 miles per hour, so it takes Jessica's car 5 hours to drive 75 miles. Alex's car has travelled a distance of $150 - 15 \times 5 = 75$ miles. Since Alex's car has travelled 75 miles and Jessica's car has travelled 75 miles, they meet at 12 noon.

Problem 7.34 Jump to Solution

(a) We can think of Yogi and Boo-Boo as being 400 miles apart at the start because they have to run a combined 400 miles until they meet each again. They reduce the distance between them at a rate of $5 + 3 = 8$ miles per second. So they will meet after $\frac{400}{8} = 50$ seconds.

(b) When they run in the same direction, we think of the distance between them as increasing. The rate at which the distance is increasing is the difference of their speeds, namely $5 - 3 = 2$ miles per second. They meet when Yogi "laps" Boo-Boo, meaning that the distance between them increases all the way to 400 miles. So they will meet after $\frac{400}{2} = 200$ seconds. Since Yogi's lead increases by 2 miles per second, it takes Yogi 200 seconds to lap Boo-Boo.

Problem 7.35 Jump to Solution

(a) When there are two people or objects moving simultaneously, it is often easier to think about the distance between the objects, rather than trying to keep track of the objects separately.

Consider the two people or objects moving in the same direction. Suppose the faster person is s_1 miles per second and the slower person is s_2 miles per second. The faster person is gaining on the slower person at a speed of $s_1 - s_2$ miles per second. Suppose the faster person is initially d miles behind the slower person. The faster person will catch up to the slower person after $\frac{d}{s_1 - s_2}$ seconds.

(b) Consider the two people or objects moving in opposite directions. Suppose the faster person is s_1 miles per second and the slower person is s_2 miles per second. The faster person is moving away from the slower person at a speed of $s_1 + s_2$ miles per second. Suppose the faster person is initially d miles behind the slower person. The faster person will be d miles ahead of the slower person after $\frac{d}{s_1 + s_2}$ seconds.

Problem 7.36 Jump to Solution

(a) Jessica drives from her home at 15 miles per hour to a shopping mall that is 20 miles from her home. On the return trip, Jessica drives heavy traffic and average 12 miles per hour. To her interest with her plan, what is the average speed for the round trip to and from the mall?

Solution for Problem 7.36. Jessica's round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{15 \text{ miles per hour}} = \frac{8}{3} \text{ hours}$$

to get to the mall. On the way back, the round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{12 \text{ miles per hour}} = \frac{10}{3} \text{ hours}$$

to get back home. So the round trip takes her $\frac{8}{3} + \frac{10}{3} = 6$ hours. Since the round trip covers 40 miles, it means that her average speed for the round trip to and from the mall is

$$\frac{40 \text{ miles}}{6 \text{ hours}} = \frac{20}{3} \text{ miles per hour}.$$

Problem 7.37 Jump to Solution

(a) Jessica drives from her home at 15 miles per hour to a shopping mall that is 20 miles from her home. On the return trip, Jessica drives at 12 miles per hour. To her interest with her plan, what is the average speed for the round trip to and from the mall?

Solution. Jessica's round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{15 \text{ miles per hour}} = \frac{8}{3} \text{ hours}$$

to get to the mall. On the way back, the round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{12 \text{ miles per hour}} = \frac{10}{3} \text{ hours}$$

to get back home. So the round trip takes her $\frac{8}{3} + \frac{10}{3} = 6$ hours. Since the round trip covers 40 miles, it means that her average speed for the round trip to and from the mall is

$$\frac{40 \text{ miles}}{6 \text{ hours}} = \frac{20}{3} \text{ miles per hour}.$$

Problem 7.38 Jump to Solution

(a) Jessica drives from her home at 15 miles per hour to a shopping mall that is 20 miles from her home. On the return trip, Jessica drives at 12 miles per hour. To her interest with her plan, what is the average speed for the round trip to and from the mall?

Solution. Jessica's round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{15 \text{ miles per hour}} = \frac{8}{3} \text{ hours}$$

to get to the mall. On the way back, the round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{12 \text{ miles per hour}} = \frac{10}{3} \text{ hours}$$

to get back home. So the round trip takes her $\frac{8}{3} + \frac{10}{3} = 6$ hours. Since the round trip covers 40 miles, it means that her average speed for the round trip to and from the mall is

$$\frac{40 \text{ miles}}{6 \text{ hours}} = \frac{20}{3} \text{ miles per hour}.$$

Problem 7.39 Jump to Solution

(a) Jessica drives from her home at 15 miles per hour to a shopping mall that is 20 miles from her home. On the return trip, Jessica drives at 12 miles per hour. To her interest with her plan, what is the average speed for the round trip to and from the mall?

Solution. Jessica's round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{15 \text{ miles per hour}} = \frac{8}{3} \text{ hours}$$

to get to the mall. On the way back, the round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{12 \text{ miles per hour}} = \frac{10}{3} \text{ hours}$$

to get back home. So the round trip takes her $\frac{8}{3} + \frac{10}{3} = 6$ hours. Since the round trip covers 40 miles, it means that her average speed for the round trip to and from the mall is

$$\frac{40 \text{ miles}}{6 \text{ hours}} = \frac{20}{3} \text{ miles per hour}.$$

Problem 7.40 Jump to Solution

(a) Jessica drives from her home at 15 miles per hour to a shopping mall that is 20 miles from her home. On the return trip, Jessica drives at 12 miles per hour. To her interest with her plan, what is the average speed for the round trip to and from the mall?

Solution. Jessica's round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{15 \text{ miles per hour}} = \frac{8}{3} \text{ hours}$$

to get to the mall. On the way back, the round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{12 \text{ miles per hour}} = \frac{10}{3} \text{ hours}$$

to get back home. So the round trip takes her $\frac{8}{3} + \frac{10}{3} = 6$ hours. Since the round trip covers 40 miles, it means that her average speed for the round trip to and from the mall is

$$\frac{40 \text{ miles}}{6 \text{ hours}} = \frac{20}{3} \text{ miles per hour}.$$

Exercises

7.5.1. Jump to Solution

(a) At 100 miles per hour, how far drives a car travel in 2 hours?

Please Solution

You may type any additional notes you have here.

Save

Your Submission Solution

Solution. A car that travels 50 miles in each hour for 2 hours goes

$$50 \left[\frac{\text{miles}}{\text{hour}}\right] \times 2 \left[\text{hours}\right] = 100 \text{ miles}.$$

Hide Solution

Save

(b) At 100 miles per hour, how long does a car to travel 300 miles?

You may type any additional notes you have here.

Save

Your Submission Solution

Solution. A car travels at a speed of 100 miles per hour, so it takes 3 hours to travel 300 miles.

$$300 \text{ miles} \times \frac{100 \text{ miles}}{1 \text{ hour}} = 300 \text{ miles} \times \frac{1 \text{ hour}}{100 \text{ miles}} = \frac{300}{100} \text{ hours} = \frac{3}{2} \text{ hours}$$

$$\frac{3}{2} \text{ hours} = \frac{3}{2} \text{ of an hour} = \frac{3}{2} \times 60 \text{ minutes}, \text{ a better way to write an answer is } \frac{3}{2} \text{ hours and } 30 \text{ minutes}.$$

Hide Solution

Save

(c) How fast drives a car have to travel to go 200 miles in 5 hours?

Please Solution

You may type any additional notes you have here.

Save

Your Submission Solution

Solution. The speed of the car in the distance it travels divided by the time it travels. If the car travel over 200 miles at a constant speed in 5 hours, it must travel

$$\frac{200 \text{ miles}}{5 \text{ hours}} = 40 \text{ miles each hour}. \text{ Therefore, its speed is}$$

$$\frac{200 \text{ miles}}{5 \text{ hours}} = 40 \left[\frac{\text{miles}}{\text{hour}}\right] = 40 \text{ miles per hour}.$$

Hide Solution

Save

7.5.2. Jump to Solution

(a) Yogi and Boo-Boo drive at the same place and at the same time on a 400-mile circular track and run in opposite directions. Yogi runs at 5 miles per second and Boo-Boo runs at 3 miles per second. How many seconds will they first meet after starting?

(b) On Tuesday, Yogi and Boo-Boo start at the same place and at the same time on a 400-mile circular track and run in the same direction. Yogi runs at 5 miles per second and Boo-Boo runs at 3 miles per second. How many seconds will they first meet after starting?

Source: AMC 8 Jump to Solution

Bill is lazy, and drinks 64 ounces in an hour and a half. Ken was assigned to traffic for 30 minutes, and then traveled 100 miles during the remaining time. What was the traffic speed during the 30 minutes?

Please Solution

You may type any additional notes you have here.

Save

Your Submission Solution

Solution. Altogether, the car travelled $90 + 100 = 190$ miles in 90 minutes, so its average speed is

$$\frac{190 \text{ miles}}{90 \text{ minutes}} = \frac{190 \text{ miles}}{\frac{3}{2} \text{ hours}} = \frac{190}{\frac{3}{2}} \text{ miles per hour} = \frac{380}{3} \text{ miles per hour}.$$

Hide Solution

Save

7.5.3. Jump to Solution SOLUTION: Jump to Solution

Peter had a 12:00 noon appointment that was 40 miles from his home. He drove from his home at an average rate of 40 miles per hour and arrived at the appointment at 1:00 p.m. How fast did he drive on the return trip from the appointment?

You may type any additional notes you have here.

Save

Your Submission Solution

Solution. Peter drove 80 miles at 40 miles per hour. So, the amount of time he drove was

$$\frac{80 \text{ miles}}{40 \text{ miles per hour}} = 2 \text{ hours}.$$

Since Peter arrived 12 minutes late, he arrived at 12:12 p.m. Therefore, since he drove $\frac{1}{5}$ hours, he left his house $\frac{1}{5}$ hours before 12:12 p.m., at 10:12 a.m.

Hide Solution

Save

7.5.4. Jump to Solution

(a) Jessica and Jeremy work together at a gas station. Jessica drives 15 miles away from the factory and drives at 60 miles per hour. Jeremy drives 15 miles away from the factory and drives at 45 miles per hour. If they leave their houses at the same time, then who arrives at the factory first and how long is it and the other arrives?

Please Solution

You may type any additional notes you have here.

Save

Your Submission Solution

Solution. Non-negative separable time lying 1 takes each person to travel to the factory. Jessica drives 60 miles per hour for 20 miles, so the trip takes him

$$\frac{20 \text{ miles}}{60 \text{ miles per hour}} = \frac{1}{3} \text{ hours}.$$

Jeremy drives 15 miles per hour for 20 miles, so his trip takes him

$$\frac{20 \text{ miles}}{45 \text{ miles per hour}} = \frac{4}{9} \text{ hours}.$$

Hence $\frac{1}{3} = \frac{3}{9} < \frac{4}{9}$ hours, so Jessica arrives first. Jeremy arrives $\frac{4}{9} - \frac{3}{9} = \frac{1}{9}$ hours later. Jessica arrives 1 hour, so Jeremy arrives 60 minutes, or Jeremy $\frac{1}{9}$ hours after Jessica.

$$\frac{1}{9} \text{ hours} = \frac{1}{9} \times 60 \text{ minutes} = \frac{60}{9} \text{ minutes} = 20 \text{ minutes}.$$

So, Jeremy arrives $20 - 20 = \frac{0}{0}$ minutes after Jessica.

Hide Solution

Save

7.5.5. Jump to Solution

(a) Jessica drives from her home at 30 miles per hour to a shopping mall that is 20 miles from her home. On the return trip, Jessica drives at 60 miles per hour. To her interest with her plan, what is the average speed for the round trip to and from the mall?

Please Solution

You may type any additional notes you have here.

Save

Your Submission Solution

Solution. Jessica's round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{30 \text{ miles per hour}} = \frac{4}{3} \text{ hours}$$

to get to the mall. On the way back, the round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{60 \text{ miles per hour}} = \frac{2}{3} \text{ hours}$$

to get back home. So the round trip takes her $\frac{4}{3} + \frac{2}{3} = 2$ hours. Since the round trip covers 40 miles, it means that her average speed for the round trip to and from the mall is

$$\frac{40 \text{ miles}}{2 \text{ hours}} = 20 \text{ miles per hour}.$$

Hide Solution

Save

7.5.6. Jump to Solution SOLUTION: Jump to Solution

A train is travelling 1 mile every 30 seconds. If the train continues at this rate, then how far will it travel in 1 hour?

Please Solution

You may type any additional notes you have here.

Save

Your Submission Solution

Solution 1. Divide the travelling time. First, we convert the 75 seconds to minutes. We have

$$75 \text{ seconds} = \frac{75}{60} \text{ minutes}$$
$$= \frac{5}{4} \text{ minutes}$$
$$= 1\frac{1}{4} \text{ minutes}.$$

So, the train goes 1 mile every $\frac{5}{4}$ minutes. Multiplying by 4, we see that the train goes 4 miles in 5 minutes. Multiplying by 12, the train goes 48 miles in 60 minutes. So, we conclude that the train travels 48 miles in 1 hour.

Solution 2. Multiply the travel speed by the time it travels. The train travels at a speed of 1 mile per 75 seconds. That is, the speed is $\frac{1 \text{ mile}}{75 \text{ seconds}}$. Multiplying by the time, we see that the train travels $\frac{1 \text{ mile}}{75 \text{ seconds}} \times 3600 \text{ seconds} = \frac{3600}{75} \text{ miles} = 48 \text{ miles}$ in 1 hour.

Hide Solution

Save

7.5.7. Jump to Solution

(a) Yogi and Boo-Boo drive at the same place and at the same time on a 400-mile circular track and run in the same direction. Yogi runs at 5 miles per second and Boo-Boo runs at 3 miles per second. How many seconds will they first meet after starting?

(b) On Tuesday, Yogi and Boo-Boo start at the same place and at the same time on a 400-mile circular track and run in the same direction. Yogi runs at 5 miles per second and Boo-Boo runs at 3 miles per second. How many seconds will they first meet after starting?

Source: AMC 8 Jump to Solution

Bill is lazy, and drinks 64 ounces in an hour and a half. Ken was assigned to traffic for 30 minutes, and then traveled 100 miles during the remaining time. What was the traffic speed during the 30 minutes?

Please Solution

You may type any additional notes you have here.

Save

Your Submission Solution

Solution. Altogether, the car travelled $90 + 100 = 190$ miles in 90 minutes, so its average speed is

$$\frac{190 \text{ miles}}{90 \text{ minutes}} = \frac{190 \text{ miles}}{\frac{3}{2} \text{ hours}} = \frac{190}{\frac{3}{2}} \text{ miles per hour} = \frac{380}{3} \text{ miles per hour}.$$

Hide Solution

Save

7.5.8. Jump to Solution SOLUTION: Jump to Solution

Peter had a 12:00 noon appointment that was 40 miles from his home. He drove from his home at an average rate of 40 miles per hour and arrived at the appointment at 1:00 p.m. How fast did he drive on the return trip from the appointment?

You may type any additional notes you have here.

Save

Your Submission Solution

Solution. Peter drove 80 miles at 40 miles per hour. So, the amount of time he drove was

$$\frac{80 \text{ miles}}{40 \text{ miles per hour}} = 2 \text{ hours}.$$

Since Peter arrived 12 minutes late, he arrived at 12:12 p.m. Therefore, since he drove $\frac{1}{5}$ hours, he left his house $\frac{1}{5}$ hours before 12:12 p.m., at 10:12 a.m.

Hide Solution

Save

7.5.9. Jump to Solution

(a) Jessica and Jeremy work together at a gas station. Jessica drives 15 miles away from the factory and drives at 60 miles per hour. Jeremy drives 15 miles away from the factory and drives at 45 miles per hour. If they leave their houses at the same time, then who arrives at the factory first and how long is it and the other arrives?

Please Solution

You may type any additional notes you have here.

Save

Your Submission Solution

Solution. Non-negative separable time lying 1 takes each person to travel to the factory. Jessica drives 60 miles per hour for 20 miles, so the trip takes him

$$\frac{20 \text{ miles}}{60 \text{ miles per hour}} = \frac{1}{3} \text{ hours}.$$

Jeremy drives 15 miles per hour for 20 miles, so his trip takes him

$$\frac{20 \text{ miles}}{45 \text{ miles per hour}} = \frac{4}{9} \text{ hours}.$$

Hence $\frac{1}{3} = \frac{3}{9} < \frac{4}{9}$ hours, so Jessica arrives first. Jeremy arrives $\frac{4}{9} - \frac{3}{9} = \frac{1}{9}$ hours later. Jessica arrives 1 hour, so Jeremy arrives 60 minutes, or Jeremy $\frac{1}{9}$ hours after Jessica.

$$\frac{1}{9} \text{ hours} = \frac{1}{9} \times 60 \text{ minutes} = \frac{60}{9} \text{ minutes} = 20 \text{ minutes}.$$

So, Jeremy arrives $20 - 20 = \frac{0}{0}$ minutes after Jessica.

Hide Solution

Save

7.5.10. Jump to Solution SOLUTION: Jump to Solution

(a) Jessica drives from her home at 30 miles per hour to a shopping mall that is 20 miles from her home. On the return trip, Jessica drives at 60 miles per hour. To her interest with her plan, what is the average speed for the round trip to and from the mall?

Please Solution

You may type any additional notes you have here.

Save

Your Submission Solution

Solution. Jessica's round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{30 \text{ miles per hour}} = \frac{4}{3} \text{ hours}$$

to get to the mall. On the way back, the round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{60 \text{ miles per hour}} = \frac{2}{3} \text{ hours}$$

to get back home. So the round trip takes her $\frac{4}{3} + \frac{2}{3} = 2$ hours. Since the round trip covers 40 miles, it means that her average speed for the round trip to and from the mall is

$$\frac{40 \text{ miles}}{2 \text{ hours}} = 20 \text{ miles per hour}.$$

Hide Solution

Save

7.5.11. Jump to Solution

(a) Jessica drives from her home at 15 miles per hour to a shopping mall that is 20 miles from her home. On the return trip, Jessica drives at 12 miles per hour. To her interest with her plan, what is the average speed for the round trip to and from the mall?

Solution. Jessica's round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{15 \text{ miles per hour}} = \frac{8}{3} \text{ hours}$$

to get to the mall. On the way back, the round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{12 \text{ miles per hour}} = \frac{10}{3} \text{ hours}$$

to get back home. So the round trip takes her $\frac{8}{3} + \frac{10}{3} = 6$ hours. Since the round trip covers 40 miles, it means that her average speed for the round trip to and from the mall is

$$\frac{40 \text{ miles}}{6 \text{ hours}} = \frac{20}{3} \text{ miles per hour}.$$

Hide Solution

Save

7.5.12. Jump to Solution

(a) Jessica drives from her home at 15 miles per hour to a shopping mall that is 20 miles from her home. On the return trip, Jessica drives at 12 miles per hour. To her interest with her plan, what is the average speed for the round trip to and from the mall?

Solution. Jessica's round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{15 \text{ miles per hour}} = \frac{8}{3} \text{ hours}$$

to get to the mall. On the way back, the round trip is 40 miles, so it takes her

$$\frac{40 \text{ miles}}{12 \text{ miles per hour}} = \frac{10}{3} \text{ hours}$$

to get back home. So the round trip takes her $\frac{8}{3} + \frac{10}{3} = 6$ hours. Since the round trip covers 40 miles, it means that her average speed for the round trip to and from the mall is

$$\frac{40 \text{ miles}}{6 \text{ hours}} = \frac{20}{3} \text{ miles per hour}.$$

Hide Solution

Save

7.6 Other Rates

Speed is just a special example of a **rate**. Whenever a quantity changes by a certain amount in a fixed unit of time, we have a rate. The idea of rate is very flexible and can be used in a lot of different situations.

Problems

Problem 7.28 Jump to Solution

Jason can type at a rate of 40 words per minute. How long will it take him to type a 2,000 word essay?

Problem 7.29 Jump to Solution

A hose fills a swimming pool at a rate of 0.5 gallons per second. If the pool's capacity is 9,000 gallons, then how many hours does it take for the hose to completely fill an empty pool?

Problem 7.30 Jump to Solution

Julie wants to give a 45-minute speech, and she speaks 120 words per minute. Her written notes contain 500 words per page. How many pages should she prepare?

Problem 7.31 Jump to Solution

Rajiv's car has tires that have circumference 75 inches. (The circumference of a tire is the distance around the outside of the tire.) How many revolutions will his tires make if Rajiv drives to a store that is $\frac{1}{4}$ mile away? (Assume that the drive is completely straight, and recall that 1 mile is 5,280 feet.)

Problem 7.32 Jump to Solution

If 5 woodchucks could chuck 50 cords of wood in 4 days, then how many cords of wood could 7 woodchucks chuck in 6 days?

Problem 7.33 Jump to Solution

Tom can paint Mr. Thatcher's fence in 6 hours, while Huck can paint it in 8 hours. Mr. Thatcher's fence is 3 hours. If they work together, then how long will it take them to paint the fence?

Just as with speed, the use of the word "per" is likely a signal that we are working with a rate. Also, just as with speed, we can use the units to our advantage when solving problems involving rates.

Here is a basic example:

Problem 7.28 7/7

Jason can type at a rate of 40 words per minute. How long will it take him to type a 2,000 word essay?

Solution for Problem 7.28: Method 1: Direct reasoning. Jason needs to type 2000 words, and for every 40 words that he will type, he will need 1 minute. Therefore, he needs $\frac{2000}{40} = 50$ minutes to type the entire 2000 words.

Method 2: Conversion factor. Jason's typing speed converts "minutes" into "words" and vice versa, so we can use a conversion factor. We need to arrange the data so that the "words" units cancel and we are left with "minutes" units:

$$\text{time} = 2000 \text{ words} \cdot \frac{1 \text{ minute}}{40 \text{ words}} = \frac{2000}{40} \text{ minutes} = 50 \text{ minutes}.$$

So it takes Jason 50 minutes to type the entire 2000 words. The conversion factor is the ratio of the times for 50 minutes, and the types 40 words per minute, then he will type a total of

$$(50 \text{ minutes}) \cdot \left(\frac{40 \text{ words}}{1 \text{ minute}}\right) = (50 - 40) \text{ words} = 2000 \text{ words},$$

as required. ◻

Concept: $\mathcal{A}_2$ Use the units in your mathematical expressions to help you figure out how to use the information in the problem.

Problem 7.29 7/7

A hose fills a swimming pool at a rate of 0.5 gallons per second. If the pool's capacity is 9,000 gallons, then how many hours does it take for the hose to completely fill an empty pool?

Solution for Problem 7.29: First, we can compute how many seconds are necessary. This is just a basic conversion problem:

$$\begin{aligned}\text{time in seconds} &= 9000 \text{ gallons} \cdot \frac{1 \text{ second}}{0.5 \text{ gallons}} \\ &= \frac{9000}{0.5} \\ &= 18000 \text{ seconds}.\end{aligned}$$

We can also convert seconds to hours using conversion factors. This gives

$$\begin{aligned}18000 \text{ seconds} &= (18000 \text{ seconds}) \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \\ &= \frac{18000}{60} \cdot \frac{1}{60} \text{ hours} \\ &= 5 \text{ hours}.\end{aligned}$$

Thus it will take 5 hours to fill the pool. ◻

Concept: $\mathcal{A}_2$ Unit conversions are just another type of rate. We can work with rates much like we do conversion factors, in particular, we them in mathematical expressions so that the units cancel in the way that you want.

Problem 7.30 7/7

Julie wants to give a 45-minute speech, and she speaks 120 words per minute. Her written notes contain 500 words per page. How many pages should she prepare?

Solution for Problem 7.30: We have a couple of different rates here both the words per minute and the words per page. We could do this as a two-step problem, or can both rates at once.

Method 1: Count words first, then count pages. First, we compute how many words should be in the speech. Julie wants to talk for 45 minutes and she will use 120 words per minute, so she will speak $(45) \cdot (120) = 5400$ words. Then, we compute the number of pages: 5400 words at 500 words per page will take $\frac{5400}{500} = 10.8$ pages.

Method 2: Do all at once using conversion factors. We write an expression that cancels the units we don't want and leaves the unit we do want:

$$\begin{aligned}\text{number of pages} &= (45 \text{ minutes}) \cdot \frac{120 \text{ words}}{1 \text{ minutes}} \cdot \frac{1 \text{ pages}}{500 \text{ words}} \\ &= \frac{45 \cdot 120}{500} \text{ pages} \\ &= 10.8 \text{ pages}.\end{aligned}$$

◻

Problem 7.31 7/7

Rajiv's car has tires that have circumference 75 inches. (The circumference of a tire is the distance around the outside of the tire.) How many revolutions will his tires make if Rajiv drives to a store that is $\frac{1}{4}$ mile away? (Assume that the drive is completely straight, and recall that 1 mile is 5,280 feet.)

Solution for Problem 7.31: We can set this up as a product of conversion factors:

$$\begin{aligned}&\text{number of revolutions} \\ &= \left(\frac{1}{4} \text{ miles}\right) \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} \cdot \frac{12 \text{ inches}}{1 \text{ feet}} \cdot \frac{1 \text{ revolutions}}{75 \text{ inches}} \\ &= \frac{5280 \cdot 12}{4 \cdot 75} \text{ revolutions} \\ &= 211.2 \text{ revolutions}.\end{aligned}$$

◻

Panel Part 1

Work problems are another particular type of rate problem. Here is a classic (if slightly confusing) example:

Problem 7.32 7/7

If 5 woodchucks could chuck 50 cords of wood in 4 days, then how many cords of wood could 7 woodchucks chuck in 6 days?

Solution for Problem 7.32: What information would be most useful? It would be helpful if we knew how much wood 1 woodchuck could chuck in 1 day. Fortunately it is not too difficult to find this out:

If

$$\text{5 woodchucks could chuck 50 cords of wood in 4 days,}$$

and since 1 woodchuck chucks $\frac{1}{5}$ as much wood as 5 woodchucks, then we know that

$$\text{1 woodchuck could chuck 10 cords of wood in 4 days.}$$

Then, since a woodchuck can chuck $\frac{1}{4}$ as much wood in 1 day as she can chuck in 4 days, we conclude that

$$\text{1 woodchuck could chuck 2.5 cords of wood in 1 day.}$$

Now we use this to answer our problem: if

$$\text{1 woodchuck could chuck 2.5 cords of wood in 1 day,}$$

then

$$\text{1 woodchuck could chuck (6 $\cdot$ 2.5) cords of wood in 6 days,}$$

and thus

$$\text{7 woodchucks could chuck (6 $\cdot$ 2.5 $\cdot$ 7) cords of wood in 6 days.}$$

Thus the answer is $6 \cdot 2.5 \cdot 7 = 105$ cords of wood. ◻

Problem 7.33 7/7

Tom can paint Mr. Thatcher's fence in 6 hours, while Huck can paint it in 8 hours. Mr. Thatcher's fence is 3 hours. If they work together, then how long will it take them to paint the fence?

Solution for Problem 7.33: Our plan is to determine their work rates, and then add them. In particular, Tom can paint $\frac{1}{6}$ of a fence per hour, and Huck can paint $\frac{1}{8}$ of a fence per hour. So, together they can paint $\frac{1}{6} + \frac{1}{8} = \frac{12}{48} + \frac{6}{48} = \frac{18}{48}$ of a fence per hour. Therefore, the time to paint the whole fence is

$$\frac{1 \text{ fence}}{\frac{18}{48} \text{ fences per hour}} = \frac{36}{11} \text{ hours} = 3\frac{2}{11} \text{ hours},$$

or a little over 2.7 hours. ◻

The key step in our solution to Problem 7.33 was considering how much work Tom and Huck each do per hour.

Concept: $\mathcal{A}_3$ Work problems can often be solved by considering the amount of work each worker does per unit of time.

Panel Part 2 - Work Problems

Exercises

7.6.1: 7/7

Carey has to build a 150-foot long fence. If it takes her 15 minutes to build 1 foot of the fence, then how many hours will it take her to complete the fence?

Preview Solution

You may type any additional notes you have here. Hide Solution Email

Your Solution Solution:

Solution: Since it takes Carey 15 minutes to build each foot of the fence, it takes her $15 \cdot 100 = 1500$ minutes to build all 100 feet of the fence. We can then convert the minutes to hours:

$$1500 \text{ minutes} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{1500}{60} \text{ hours} = \boxed{25 \text{ hours}}.$$

We also could have used conversion factors to do the whole problem from the beginning. Since she builds 1 foot per 15 minutes, we have

$$100 \text{ feet} \cdot \frac{15 \text{ minutes}}{1 \text{ feet}} \cdot \frac{1 \text{ hours}}{60 \text{ minutes}} = \frac{100 \cdot 15}{60} \text{ hours} = \boxed{25 \text{ hours}}.$$

7.6.2: 7/7

Phil can type a page of his new novel in 20 minutes. If he writes for 8 hours, then how many pages will he type?

Preview Solution

You may type any additional notes you have here. Hide Solution Email

Your Solution Solution:

Solution: Phil produces 1 page every 20 minutes. So in 3 · 20 = 60 minutes (which is 1 hour), he produces 3 pages. Therefore, in 8 hours, he produces $3 \cdot 8 = \boxed{24 \text{ pages}}$.

Again, we can use conversion factors:

$$8 \text{ hours} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{1 \text{ page}}{20 \text{ minutes}} = \frac{8 \cdot 60}{20} \text{ pages} = \boxed{24 \text{ pages}}.$$

7.6.3: Source: MGCMS 7/7

A kangaroo chases a rabbit that starts 150 feet ahead of the kangaroo. For every 12 foot hop of the kangaroo, the rabbit makes a 7 foot hop. How many hops will the kangaroo have to make to catch up to the rabbit if the two animals always leap at the same time?

You may type any additional notes you have here. Hide Solution Email

Your Solution Solution:

Solution: The kangaroo gains 5 feet on every hop. The kangaroo must gain 150 feet total, so it must jump $\frac{150}{5} = \boxed{30 \text{ times}}$ to catch the rabbit.

7.6.4: Source: AMC 8 7/7

Maria buys computer disks at a price of 4 for $5, and sells them at a price of 3 for $5. How many computer disks must she sell in order to make a profit of $100?

You may type any additional notes you have here. Hide Solution Email

Your Solution Solution:

Solution: We first determine how much money Maria makes per disk. Since she buys 4 disks for $5, the per-disk rate she pays is $\frac{\$5}{4 \text{ disks}}$. Similarly, she sells 3 disks for $5, so the per-disk rate she receives is $\frac{\$5}{3 \text{ disks}}$. Therefore, her profit per disk is

$$\frac{\$5}{3 \text{ disks}} - \frac{\$5}{4 \text{ disk}} = \frac{\$5}{12 \text{ disks}} - \frac{\$5}{12 \text{ disks}} = \frac{\$5}{12 \text{ disks}}.$$

Since this is the amount she makes for each disk, the number of disks she needs to make $100 is

$$\frac{100 \text{ dollars}}{\frac{\$5}{12 \text{ disks}}} = \frac{100 \cdot 12}{5} \text{ disks} = \boxed{240 \text{ disks}}.$$

7.6.5: Source: AMC 8 7/7

At the beginning of a trip, the mileage odometer read 56,200 miles. The driver filled the gas tank with 2 gallons of gasoline. During the trip, the driver filled his tank again with 12 gallons of gasoline when the odometer read 56,560. At the end of the trip, the driver filled the tank again with 20 gallons of gasoline. The odometer read 57,000. To the nearest hundred, what was the car's average miles-per-gallon for the entire trip?

Preview Solution

You may type any additional notes you have here. Hide Solution Email

Your Solution Solution:

Solution: The entire trip was 57,000 − 56,200 = 800 miles. The tank was full after the driver added the 6 gallons of gas at the beginning. After that, the driver had to add a total of 12 + 20 = 32 gallons of gas to have the gas tank completely filled at the end. So, the car used 32 gallons of gas while traveling the 800 miles. That gives an average miles-per-gallon rate of

$$\frac{800 \text{ miles}}{32 \text{ gallons}} = \frac{25}{1} \frac{\text{miles}}{\text{gallon}} = \boxed{25.0 \frac{\text{miles}}{\text{gallon}}}.$$

7.6.6: Source: MGCMS 7/7

A twelve-hour clock loses 1 minute every hour. Suppose it shows the correct time now. What is the least number of hours from now when it will again show the correct time?

You may type any additional notes you have here. Hide Solution Email

Your Solution Solution:

Solution: The clock will again show the correct time when it has lost a total of 12 hours. Every hour, the clock loses 1 minute. We want the clock to lose 12 hours, which is 12 · 60 = 720 minutes. It will take 720 hours to lose 720 minutes, so it will take $\boxed{720 \text{ hours}}$ to lose 12 hours.

As with many of our other problems, we also could have used conversion factors to solve the problem:

$$\begin{aligned}&1 \text{ hours time} \cdot \frac{1 \text{ minute lost}}{60 \text{ minutes time}} \cdot \frac{60 \text{ minutes lost}}{1 \text{ hours lost}} \\ &= 12 \cdot 60 \text{ hours time} \\ &= \boxed{720 \text{ hours}}.\end{aligned}$$

7.6.7: Source: AMC 8 7/7

Homer began peeling a pile of 44 potatoes at the rate of 3 potatoes per minute. Four minutes later, Christen joined him and peeled at the rate of 5 potatoes per minute. When they finished, how many potatoes had Christen peeled?

You may type any additional notes you have here. Hide Solution Email

Your Solution Solution:

Solution: Homer peels 3 potatoes a minute, so in the four minutes he works alone, he peels 3 · 4 = 12 potatoes. This leaves 44 − 12 = 32 potatoes for them to peel together. Together, they peel 3 + 5 = 8 potatoes per minute, so they need

$$\frac{32 \text{ potatoes}}{8 \frac{\text{potatoes}}{\text{minute}}} = 4 \text{ minutes}$$

to peel the rest of the potatoes. Christen peels 5 potatoes a minute for 4 minutes, so she peels 5 · 4 = $\boxed{20 \text{ potatoes}}$.

7.6.8: Source: MGCMS 7/7

The cold-water faucet of a bath tub can fill the tub in 15 minutes. The drain, when opened, can empty the full tub in 30 minutes. Suppose the tub is empty and the faucet and drain are both opened at the same time. How long will it take to fill the tub?

You may type any additional notes you have here. Hide Solution Email

Your Solution Solution:

Solution: Since the faucet can fill the tub in 15 minutes, it fills $\frac{1}{15}$ of the tub every minute. Similarly, the drain empties $\frac{1}{30}$ of the tub every minute. Therefore, the fraction of the tub that is filled each minute when the faucet and drain are both open is

$$\frac{1}{15} - \frac{1}{30} = \frac{2}{30} - \frac{1}{30} = \frac{1}{30}.$$

Since $\frac{1}{30}$ of the tub is filled each minute, it will take $\boxed{30 \text{ minutes}}$ for the tub to fill.

7.6.9: 7/7

Roger can shovel his family's driveway in 1 hour. His older sister Alexis can shovel the driveway in $\frac{1}{2}$ hour. If they work together, then how long will it take them to shovel the driveway?

You may type any additional notes you have here. Hide Solution Email

Your Solution Solution:

Solution: Roger can shovel the driveway in an hour, he can shovel $\frac{1}{60}$ of the driveway every minute. Similarly, Alexis can shovel the driveway in 30 minutes, so she shovels $\frac{1}{30}$ of the driveway each minute. Together, each minute they shovel

$$\frac{1}{60} + \frac{1}{30} = \frac{1}{60} + \frac{2}{60} = \frac{3}{60}$$

of the driveway. Therefore, they need $\boxed{20 \text{ minutes}}$ to shovel the entire driveway.

7.6.10: Source: MGCMS 7/7

Three water pipes are used to fill a swimming pool. The first pipe alone takes 6 hours to fill the pool, the second pipe alone takes 12 hours to fill the pool, and the third pipe alone takes 24 hours to fill the pool. If all three pipes are opened at the same time, then how long will it take to fill the pool?

Hint: Problems involving rates and work can often be solved by thinking about how much work is done per unit of time.

Hint: What fraction of the pool is filled each hour?

Preview Solution

You may type any additional notes you have here. Hide Solution Email

Your Solution Solution:

Solution: The first pipe fills $\frac{1}{6}$ of the pool in 1 hour, the second pipe fills $\frac{1}{12}$ of the pool in 1 hour, and the third pipe fills $\frac{1}{24}$ of the pool in 1 hour. So together the pipes fill

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{24} = \frac{4}{24} + \frac{2}{24} + \frac{1}{24} = \frac{7}{24}$$

of the pool in 1 hour. Therefore, the three pipes together will take $\boxed{\frac{24}{7} \text{ hours}}$ to fill the pool.

7.7 Summary

A ratio is used to compare the *relative* quantities of two or more groups or items. However, a ratio only compares the quantities to each other–it doesn't tell us the actual values of the quantities. For example, suppose that you know that a certain history class has a ratio of girls to boys of $2:3$. All this tells you is that for every 2 girls, there are 3 boys. It doesn't tell you how many boys or girls there are.

Concept:	A ratio gives a *relative* comparison of two quantities. It doesn't tell you anything about the total amount of the quantities.

Definition:	To **simplify** a ratio means to write it as a ratio of integers with no common factor greater than 1.

Important:	Suppose we are using a ratio to compare two quantities that together make up a group (such as girls and boys in a class). If the two quantities are in the ratio $a:b$, then the first quantity makes up $\frac{a}{a+b}$ of the whole, and the second quantity makes up $\frac{b}{a+b}$ of the whole.

Whenever we have two ratios that are equal, we have a **proportion**. The most common usage of proportion is when we have two changing quantities that are related in such a way that their ratio doesn't change.

A commonly used ratio is **speed**, which is the ratio of distance to time. We can write this as an equation as

$$\text{speed} = \frac{\text{distance}}{\text{time}}.$$

This equation can be rearranged as

$$(\text{speed}) \cdot (\text{time}) = \text{distance}$$

and also as

$$\text{time} = \frac{\text{distance}}{\text{speed}}.$$

The units associated with speed help us remember these equations. The word **per** essentially means "divided by," so a speed in miles per hour means to take distance (in miles) and divide by time (in hours).

WARNING!!	Speeds do not usually "average" in the way that you might expect them to.

Speed is just a special example of a **rate**. Whenever a quantity changes by a certain amount in a fixed unit of time, we have a rate. Just as with speed, the use of the word "per" is often a signal that we are working with a rate. Also, just as with speed, we can use the units to our advantage when solving problems.

Concept:	Pay close attention to units in word problems! Use the units in your mathematical expressions to help you figure out how to use the information in the problem.

Unit conversions are just another type of rate. We can work with rates much like we work with conversion factors. In particular, we use them in mathematical expressions so that the units cancel in the way that we want.

Challenge Problems

7.58:

Source: MATHCOUNTS

A stack of 45 dimes is divided into three piles in the ratio $\frac{1}{6}:\frac{1}{3}:\frac{1}{4}$. How many dimes are in the pile with the least number of dimes?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The common denominator of the fractions is 12, so we can convert all the terms in the ratio to integers by multiplying them by 12:

$$\frac{1}{6}:\frac{1}{3}:\frac{1}{4}=12\cdot\frac{1}{6}:12\cdot\frac{1}{3}:12\cdot\frac{1}{4}=2:4:3.$$

The pile with the least number of dimes has $\frac{2}{2+4+3}=\frac{2}{9}$ of the dimes, so that pile has $\frac{2}{9}\cdot 45=\boxed{10}$ dimes.

7.59:

Five workers together can build a road in 20 days. Suppose every worker works at the same rate. If three workers work on the road for 10 days before eleven more workers join them, then how long total will it take to build the road?

Hint: Problems involving rates and work can often be solved by thinking about how much work is done per unit of time.

Hint: How long would it take one worker to build the whole road? What fraction of the road does one worker complete in one day?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Because 5 workers can build a road in 20 days, each worker can build $\frac{1}{5}$ of a road in 20 days. Therefore, each worker can build 1 road working alone in $20\cdot 5=100$ days. So, each worker builds $\frac{1}{100}$ road each day. Therefore, three workers together build $\frac{3}{100}$ of a road each day, so in 10 days they build $10\cdot\frac{3}{100}=\frac{3}{10}$ of a road. At this point, they have $1-\frac{3}{10}=\frac{7}{10}$ of the road to finish.

After the 11 extra workers join in, the 14 workers build $\frac{14}{100}$ of the road each day. So, to finish the remaining $\frac{7}{10}$ of a road, the workers must work for

$$\frac{\frac{7}{10}\ \text{road}}{\frac{14}{100}\ \frac{\text{road}}{\text{day}}}=\frac{7}{10}\cdot\frac{100}{14}\ \text{days}=5\ \text{days}.$$

Therefore, the road is built in $10+5=\boxed{15\ \text{days}}$.

7.60:

Source: MOEMS

When Paul crossed the finish line of a 60-meter race, he was ahead of Robert by 10 meters and ahead of Sam by 20 meters. Suppose Robert and Sam continue to race to the finish line without changing their rates of speed. By how many meters will Robert beat Sam?

Hint: By how many meters would Robert beat Sam if the race were 50 meters? 100 meters? 150 meters?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: When Paul finishes, Robert has run $60-10=50$ meters and Sam has run $60-20=40$ meters. Therefore, when Robert and Sam run for the same amount of time, Sam covers $\frac{40}{50}=\frac{4}{5}$ of the distance that Robert covers. So, while Robert runs the final 10 meters of the race, Sam runs $\frac{4}{5}\cdot 10=8$ meters. This means Robert's lead over Sam increases by 2 more meters, and he beats Sam by $10+2=\boxed{12}$ meters.

7.61:

Source: MATHCOUNTS

If $4:x^2=x:16$, then what is the value of x?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Multiplying both parts of $4:x^2$ by x gives $4:x^2=4x:x^3$, and multiplying both parts of $x:16$ by 4 gives $x:16=4x:64$. Substituting these into $4:x^2=x:16$ gives $4x:x^3=4x:64$, so $x^3=64$, which gives $x=\boxed{4}$.

We also could have written the ratios in the given equation as fractions to get $\frac{4}{x^2}=\frac{x}{16}$. Multiplying both sides by x^2 and by 16 gets rid of the denominators and gives $4\cdot 16=x\cdot x^2$, so $64=x^3$ and $x=\boxed{4}$.

7.62:

Source: MATHCOUNTS

The Big Telescope Company sells circular mirrors. Their largest mirrors have radii of 5 meters and their smallest mirrors have radii of 1 meter. The cost of every mirror is proportional to the cube of the mirror's radius. What is the ratio of the total cost of 25 of the company's smallest mirrors to the cost of one of the company's largest mirrors?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let the cost of one of the smallest mirrors be x, so the cost of 25 of these mirrors is $25x$. The radius of the largest mirror is 5 times the radius of the smallest mirror. The cost of the mirror is proportional to the cube of the radius, so the cost of the largest mirror is $5^3=125$ times the cost of the smallest mirror. Therefore, the cost of the largest mirror is $125x$, and the desired ratio is $25x:125x$. Dividing both parts of the ratio by $25x$ gives $\boxed{1:5}$.

7.63:

Source: MOEMS

Kim was elected class president. She received 3 votes for every 2 that Amy got. No one else ran. However, if 8 of the people who voted for Kim had voted for Amy instead, Kim would have received only 1 vote for every 2 that Amy would have gotten. How many people voted?

Hint: Assign a variable so you can write expressions and set up equations. What's often a useful way to assign a variable in a ratio problem?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The ratio of Kim's votes to Amy's votes is $3:2$, so Kim received $3x$ votes and Amy received $2x$ votes for some value of x. If 8 of the people had voted for Amy instead of Kim, then Kim would have $3x-8$ votes and Amy would have $2x+8$ votes. If this had happened, then Amy would have twice as many votes as Kim, so $2x+8=2(3x-8)$. Expanding the right-hand side gives $2x+8=6x-16$. Subtracting $2x$ from both sides and adding 16 to both sides gives $24=4x$, so $x=6$. Since $3x+2x=5x$ people voted, the number of voters is $5\cdot 6=\boxed{30}$.

7.64:

When the Slowpoke Marathon began, the ratio of runners to joggers was 2 to 19. If 4200 participants began the race (each either a runner or a jogger, but not both), and 500 joggers dropped out of the race but all the rest of the participants finished, then what was the ratio of runners to joggers among those who finished the race?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: First, we determine how many joggers started the race. Since the ratio of runners to joggers was $2:19$ at the start of the race, the joggers are $\frac{19}{2+19}=\frac{19}{21}$ of the participants. There were 4200 participants at the start of the race, so $4200\cdot\frac{19}{21}=3800$ of them are joggers. The other $4200-3800=400$ are runners. Since 500 joggers dropped out, only $3800-500=3300$ finished. Therefore, the ratio of runners to joggers among those who finished is $400:3300=\boxed{4:33}$.

7.65:

Source: AMC 8

The students in Mrs. Reed's English class are reading the same 760-page novel. Three friends, Alice, Bob, and Chandra, are in the class. Alice reads a page in 20 seconds, Bob reads a page in 45 seconds, and Chandra reads a page in 30 seconds.

(a) If Bob and Chandra both read the whole book, then Bob will spend how many more seconds reading than Chandra?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Bob spends 15 seconds more on each page, so he spends

$$760\ \text{pages}\cdot\frac{15\ \text{seconds}}{1\ \text{page}}=\boxed{11400\ \text{seconds}}$$

more than Chandra.

(b) Chandra and Bob, who each have a copy of the book, decide that they can save time by "team reading" the novel. In this scheme, Chandra will read from page 1 to a certain page and Bob will read from the next page through page 760, finishing the book. When they are through, they will tell each other about the parts they read. What is the last page that Chandra should read so that she and Bob spend the same amount of time reading the novel?

Hint: If we know what fraction of the book Chandra reads, then we can figure out how many pages she reads.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: In 90 seconds, Bob reads 2 pages and Chandra reads 3 pages. So, for any fixed amount of time, the ratio of the amount Bob reads to the amount Chandra reads is $2:3$. Therefore, Chandra must read $\frac{3}{2+3}=\frac{3}{5}$ of the book, and during that time Bob will read the other $\frac{2}{5}$. This means Chandra should read $\frac{3}{5}\cdot 760=456$ pages, which means she should stop reading after completing page $\boxed{456}$.

(c) Before Chandra and Bob start reading, Alice says she would like to team read with them. If they divide the book into three sections so that each reads for the same length of time, then how many seconds will each have to read?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: In 90 seconds, Alice reads $\frac{90}{20}=4.5$ pages. So, in 90 seconds, the three together read $4.5+2+3=9.5$ pages. Therefore, the total amount of time they need to read is

$$760\ \text{pages}\cdot\frac{90\ \text{seconds}}{9.5\ \text{pages}}=\boxed{7200\ \text{seconds}}.$$

7.66★:

Source: AMC 8

Buses from Dallas to Houston leave every hour on the hour. Buses from Houston to Dallas leave every hour on the half hour. The trip from one city to the other takes 5 hours. Assuming the buses travel on the same highway, how many Dallas-bound buses does a Houston-bound bus pass on the highway (not in the station)?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We count the number of Dallas-bound buses that a single Houston-bound bus passes. There are two groups of Dallas-bound buses that we have to consider. First, there are the buses that were on the way to Dallas at the time the Houston-bound bus leaves Dallas. Second, there are the buses that leave Houston after the Houston-bound bus starts but before it arrives at Houston.

In order for a bus to be on the way to Dallas when the Houston-bound bus leaves, it must have left Houston less than 5 hours before the Houston-bound bus leaves. There is one such bus each hour, for a total of 5 buses.

In order for a Dallas-bound bus to leave Houston after the Houston-bound bus leaves but before the Houston-bound bus arrives, the Dallas-bound must leave Houston within 5 hours after the Houston-bound bus leaves. Again, there is one such bus each hour, for a total of 5 more buses that the Houston-bound bus passes.

Therefore, the Houston-bound bus passes $5+5=\boxed{10}$ Dallas-bound buses.

8

8.1 What is a Person?

Question 1 Mock Test 8 Mock Test 8 Mock Test 8

Consider the following statements in regard to Kantian ethics. Two of them are correct. Which are the correct statements? Choose the correct option.

(1) Kant's ethics is based on the idea that we should treat people as means to our own ends, as long as we do not treat them as means only.

(2) Kant's ethics is based on the idea that we should treat people as means to our own ends, as long as we do not treat them as means only.

(3) Kant's ethics is based on the idea that we should treat people as means to our own ends, as long as we do not treat them as means only.

(4) Kant's ethics is based on the idea that we should treat people as means to our own ends, as long as we do not treat them as means only.

Which of the following statements are correct?

Correct Answer: (1) and (3) only

Explanation

The reading of Kant's ethics in the reading Philosophy, Ethics and Religion, Kant's ethics is based on the idea that we should treat people as means to our own ends, as long as we do not treat them as means only.

8

8.1 What is a Person?

Question 1 Mock Test 8 Mock Test 8 Mock Test 8

Consider the following statements in regard to Kantian ethics. Two of them are correct. Which are the correct statements? Choose the correct option.

(1) Kant's ethics is based on the idea that we should treat people as means to our own ends, as long as we do not treat them as means only.

(2) Kant's ethics is based on the idea that we should treat people as means to our own ends, as long as we do not treat them as means only.

(3) Kant's ethics is based on the idea that we should treat people as means to our own ends, as long as we do not treat them as means only.

(4) Kant's ethics is based on the idea that we should treat people as means to our own ends, as long as we do not treat them as means only.

Which of the following statements are correct?

Correct Answer: (1) and (3) only

Explanation

The reading of Kant's ethics in the reading Philosophy, Ethics and Religion, Kant's ethics is based on the idea that we should treat people as means to our own ends, as long as we do not treat them as means only.

8.2 Word Problems

Because percents are used frequently in the real world, they show up a lot in word problems. Word problems involving percents are really no different than any other sort of problem. The important steps in order are:

Concept:
- Read the problem carefully!
- Convert the words into mathematics.
- Solve the mathematical problem.
- Write your answer in terms of the word problem, and make sure that it makes sense.

Here are several word problems involving percents:

Problems

Problem 8.0 ▶ Jump to Solution

Stephanie bought a new computer that cost $900. Where Stephanie lives, the sales tax is 7%. How much sales tax does Stephanie have to pay for her computer?

Problem 8.9 ▶ Jump to Solution

Rajiv has 4 blue shirts, 5 black shirts, and 6 white shirts. What percent of his shirts are white?

Problem 8.10 ▶ Jump to Solution

If a school has 200 girls and 200 boys, what percent of the students are girls?

Problem 8.11 Source: AMC 8 ▶ Jump to Solution

An athletic target team with 10 members shot 100 times. The maximum heart rate in beats per minute, is found by subtracting the athlete's age, in years, from 220. What is the target heart rate of an athlete who is 25 years old? (Round to the nearest beat per minute.)

Problem 8.12 ▶ Jump to Solution

All of the students in Mr. Sato's History class took an exam. Each student either passed or failed. 85% of the students passed and 3 students failed. How many students are in the class?

Problem 8.13 ▶ Jump to Solution

Patti takes 200 flavors to her middle school to sell on Valentine's Day. She first sells 30% of the flavors to 6th grade students. She then sells 50% of the remaining flavors to 7th grade students. Finally, she sells 30% of the remaining flavors to 8th grade students. How many flavors does she have left?

Problem 8.14 ▶ Jump to Solution

A 100-milliliter bottle of salad dressing initially consists of 30% oil and 20% vinegar. 40% of the dressing is used, and then 4 milliliters of vinegar are added to the bottle. What percentage of the mixture is now vinegar?

Problem 8.0 ✓

Stephanie bought a new computer that cost $900. Where Stephanie lives, the sales tax rate is 7%. How much sales tax does Stephanie have to pay for her computer?

Solution for Problem 8.0. This is a very common "real world" use of percents. Most U.S. states have **sales tax** that gets added to many purchases. Suppose that you go to a store and purchase something for $100. Depending on where you live (such as a store or restaurant) simply multiplies the sales tax by the purchase price to determine the amount of tax due. As an equation:

$$\text{Sales tax due} = (\text{Purchase price}) \cdot (\text{Sales tax rate}).$$

In our problem, the item being purchased= a computer has a price of $900, and the sales tax sales is 7%. So we compute:

$$\begin{aligned}\text{Sales tax due} &= 900 \cdot \frac{7}{100} \\ &= \frac{6300}{100} \\ &= 63 \\ &= \$63.\end{aligned}$$

Thus, Stephanie must pay $63 in sales tax.

Sidenote: Another common percent that is used in restaurants is for tipping. In North America, it is typical to tip 15% of the total bill to waiters or waitress. For example, if two people eat a meal that costs $80, then we must pay a tip equal to $80 · 0.15 = $12. Because $80 · $12 = $80 · $\frac{12}{100}$ = $80 · 0.15 = $12, the tip is then carried. In Iranian restaurants, the tip rate might be 18% or even 20%.

Since 7% or 0%, is a common sales tax rate in the U.S., we may wrap to compute approximately sales tax to tip to double the sales tax. Think about why this works!

Problem 8.9 ✓

Rajiv has 4 blue shirts, 5 black shirts, and 6 white shirts. What percent of his shirts are white?

Solution for Problem 8.9. Rajiv has 5 white shirts and a total of 4 + 5 + 6 = 15 shirts total. So the percent of his shirts that are white is

$$\frac{6}{15} = \frac{2}{5} = \frac{40}{100} = 40\%.$$

Problem 8.10 ✓

If a school has 200 girls and 200 boys, what percent of the students are girls?

Solution for Problem 8.10. Avoid the following common mistake:

Bogus Solution: There are 200 girls and 200 boys, so the percent of girls is

$$\frac{200}{200} = \frac{100}{100} = 100\%.$$

This "answer" is clearly incorrect: the percent of girls must be between 0% (no girls) and 100% (all girls). The correct solution is that we must compute the percentage of girls from both boys and girls:

The school has 200 + 200 = 500 total students, so the percent of the students who are girls is

$$\frac{200}{500} = \frac{40}{100} = 40\%.$$

Problem 8.11 Source: AMC 8

An athletic target team with 10 members shot 100 times. The maximum heart rate in beats per minute, is found by subtracting the athlete's age, in years, from 220. What is the target heart rate of an athlete who is 25 years old? (Round to the nearest beat per minute.)

Solution for Problem 8.11. This is a 2-step problem. First, we have to find the athlete's maximum heart rate. Second, we have to find the target heart rate.

To find the maximum heart rate, we simply follow the directions in the problem: we subtract the athlete's age of 25 from 220, to get 220 − 25 = 194. This is the maximum heart rate of the athlete.

Then, we are told that the target heart rate is 80% of the maximum heart rate. So to find the target heart rate, we multiply the maximum heart rate by 80%:

$$80\% \cdot 194 = \frac{4}{5} \cdot 194 = \frac{776}{5}.$$

But that's not the answer~the problem asked us to round to the nearest beat per minute. So we must round $\frac{776}{5}$ to the nearest integer. We find $\frac{776}{5} = 155.2$, so the closest integer to 155.2 is 155. Alternatively, we can note that the nearest multiple of 5 to 776 is 775, so the closest to $\frac{776}{5} = 155$ beats per minute.

Problem 8.12 ✓

All of the students in Mr. Sato's History class took an exam. Each student either passed or failed. 85% of the students passed and 3 students failed. How many students are in the class?

Solution for Problem 8.12. What bit of information do we need to answer this problem?

Concept: When deciding how to proceed with a problem, ask yourself: "What information do I need to be able to solve the problem?" Then see if you can find that information.

We know that 3 students failed, and it would be nice to know what percent of the total that is. But we do know that since the entire class is 100% of the students, and 85% of the students passed, we know that 100% − 85% = 15% of the students failed. So 3 students equals 15% of the total number of students in the class.

Let's set n to be the number of students in the class, so that we can write an equation for n.

Concept: When trying to find an unknown quantity, it is often useful to assign a variable to that quantity, so that you can write an equation.

Since 3 students is 15% of the class, we have the equation

$$3 = \frac{15}{100} n = \frac{15n}{100}.$$

Multiplying $3 = \frac{15n}{100}$ by 100 gives 300 = 15n, and dividing by 15 gives 20 = n. Therefore, there are 20 students in the class.

To check our answer, we note that if 3 students failed, then 20 − 3 = 17 passed. This must be 85% of all students, and indeed 85% of 20 students total: $85\% \cdot 20 = \frac{85}{100} \cdot 20 = \frac{17}{5} = 17$ students. ✓

Problem 8.13 ✓

Patti takes 200 flavors to her middle school to sell on Valentine's Day. She first sells 30% of the flavors to 6th grade students. She then sells 50% of the remaining flavors to 7th grade students. Finally, she sells 30% of the remaining flavors to 8th grade students. How many flavors does she have left?

Solution for Problem 8.13. Our role in this problem is as an accountant. We need to keep track of her sales during the day.

First, she sells 30% of her original 200 flavors to her 6th grade students. She computes $30\% \cdot 200 = \frac{30}{100} \cdot 200 = 60$, so she sells 60 flavors. This means she has 200 − 60 = 140 remaining.

Next, she sells 50% of her 140 remaining flavors to the 7th grade students. We compute $50\% \cdot 140 = \frac{1}{2} \cdot 140 = 50$, so she sells 50 flavors. This means she has 140 − 50 = 90 remaining.

Next, she sells 30% of her 90 remaining flavors to the 8th grade students. We compute $30\% \cdot 90 = \frac{3}{10} \cdot 90 = 27$, so she sells 27 flavors. This means she has 90 − 27 = 63 remaining.

So she ends the day with 63 flavors.

We could actually solve this problem in one calculation, by keeping track of how many flavors she sees at each step. We see that first she sells 30% of the flavors, so she keeps 70% of the flavors. Then she sells 50% of the remaining flavors, so she keeps 50% of the flavors. Finally, she sells 30% of the remaining flavors, so she keeps 70% of the remaining flavors after the 8th grade sales. So, at the end of the day, she keeps

$$70\% \text{ of } 50\% \text{ of } 70\% \text{ of } 200 = 70\% \cdot 50\% \cdot 70\% \cdot 200$$
$$= \frac{7}{10} \cdot \frac{1}{2} \cdot \frac{7}{10} \cdot 200$$
$$= \frac{49}{100} \cdot 200$$
$$= 49 \text{ flavors.}$$

Problem 8.14 ✓

A 100-milliliter bottle of salad dressing initially consists of 30% oil and 20% vinegar. 40% of the dressing is used, and then 4 milliliters of vinegar are added to the bottle. What percentage of the mixture is now vinegar?

Solution for Problem 8.14. As in the previous problem, our best approach is to keep track of how the various quantities change.

We start by using 40% of the original 100 ml of dressing, which means that 60% remains. That gives us 60ml of dressing left in the bottle. Since the composition of the dressing doesn't change (only the amount has changed), we still have 60% oil and 20% vinegar. This means that we have

$$60\% \cdot 100\text{ml} = 60\text{ml of oil} \quad \text{and} \quad 20\% \cdot 100\text{ml} = 12\text{ml of vinegar.}$$

Next, we add the additional 4 ml of vinegar. After this, we still have 60ml of oil, but we have 12 + 4 = 16ml of vinegar, and the total amount of dressing is 60 + 16 = 76ml. Therefore, the percent of the dressing that is vinegar is

$$\frac{16}{76} = \frac{4}{19} = 21\%.$$

It is no surprise that the new vinegar percentage of 21% is higher than the original percentage of 20%: vinegar since we have added vinegar (but not oil), the percentage of vinegar in the final should be larger.

Exercises

8.2.1: ✓

What percent of the 5 × 5 square is the right is shaded?

You may type any additional text here. | Hide Solution | Saved

Your Submission: Solution

Solution: There are 5 · 5 = 25 total 1 × 1 squares, and 2 · 2 + 3 · 3 = 4 + 9 = 13 of these are shaded. So $\frac{13}{25}$ of the small squares are shaded. Converting $\frac{13}{25}$ to a percent gives $\frac{13}{25} = \frac{13 \cdot 4}{25 \cdot 4} = \frac{52}{100} = 52\%$ of the percent of the 5 × 5 square that is shaded.

8.2.2: Source: MATHCOUNTS ✓

At a meeting there were 4 parents, 28 students, and 6 teachers. What percent of people at the meeting were students?

You may type any additional text here. | Hide Solution | Saved

Your Submission: Solution

Solution: There are 4 + 28 + 6 = 38 people at the meeting, and 28 of them were students. So, the percent of people at the meeting who were students is $\frac{28}{38} = \frac{14}{19} = \boxed{\frac{700}{19}\%}$.

8.2.3: ✓

In the Arkansas State Senate, 35 of the 50 state senators are female. What percent of the state senators are male?

Preview: Solution

You may type any additional text here. | Hide Solution | Saved

Your Submission: Solution

Solution: Since 35 of the 50 senators are female, the other 50 − 35 = 15 are male. Therefore, the percent of senators who are male is $\frac{15}{50} = \frac{3}{10} = \frac{30}{100} = \boxed{30\%}$.

8.2.4: ✓

Fred is the Realtor making a 6% commission on each property she sells.

(a) If Fred sells a $270,000 house, then how much commission does she make?

Preview: Solution

You may type any additional text here. | Hide Solution | Saved

Your Submission: Solution

Solution: Fred's commission is 6% of the sales price of $270,000. So, her commission is

$$\begin{aligned}6\% \text{ of } \$270{,}000&= \$270{,}000 \cdot \frac{6}{100} \\ &= \frac{\$270{,}000 \cdot 6}{100} \\ &= \$\,27{,}000 \\ &= \boxed{\$16{,}200}.\end{aligned}$$

(b) If Rhonda sells a condo and receives a commission of $15,000, then how much did the condo sell for?

You may type any additional text here. | Hide Solution | Saved

Your Submission: Solution

Solution: Let the sales price of the condo is dollars. Since $15,000 is 6% of the sales price, we seek the number x such that 6% of $x = $15,000. We have $6\% = \frac{6}{100} = \frac{3}{50}$, so we must have

$$\frac{3}{50} \cdot x = \$15{,}000.$$

Multiplying both sides by $\frac{50}{3}$ gives

$$x = \frac{50}{3} \cdot (\$15{,}000) = 50 \cdot \frac{\$15{,}000}{3} = 50 \cdot \$5{,}000 = \$250{,}000.$$

So, the sales price of the condo is $250,000.

8.2.5: Source: AMC 8 ✓

Ted's mathematics final had 75 problems: 10 arithmetic, 30 algebra, and 35 geometry problems. Although she answered 70% of the arithmetic, 40% of the algebra, and 60% of the geometry problems correctly, she did not pass the test because she got less than 60% of the problems right. How many more questions would she have needed to answer correctly to earn a 60% passing grade?

Preview: Solution

You may type any additional text here. | Hide Solution | Saved

Your Submission: Solution

Solution: First, we determine how many questions Ted answered correctly: 70% of the arithmetic problems is $0.70 \cdot 10 = 7$ problems, 40% of the algebra problems is $0.40 \cdot 30 = 12$ problems, 60% of the geometry problems is $\frac{60}{100} \cdot 35 = \frac{3}{5} \cdot 35 = 21$ problems. This is a total of $7 + 12 + 21 = 40$ problems answered correctly.

Next, we find how many questions must be answered correctly to pass. There are $70 + 30 + 35 = 75$ questions on the test. We must answer 60% of these correctly to pass, which is a total of $60\% \cdot 75 = \frac{3}{5} \cdot 75 = 45$ questions.

Therefore, Ted needed to answer $45 - 40 = \boxed{5}$ more questions correctly to pass.

8.2.6: Source: AMC 8 ✓

A mixture of 30 liters of paint is 25% red tint, 30% yellow tint, and 45% water. Five liters of yellow tint are added to the original mixture. What is the percent of yellow tint in the new mixture?

You may type any additional text here. | Hide Solution | Saved

Your Submission: Solution

Solution: The new mixture has 30 + 5 = 35 liters of paint. To determine the percentage of yellow tint, we find the number of liters of yellow tint in the mixture. The original mixture has 30% yellow tint, so the amount of yellow tint in the original mixture is

$$30\% \cdot (30 \text{ liters}) = 0.3 \cdot (30 \text{ liters}) = 9 \text{ liters}.$$

The new mixture has 5 more liters of yellow tint, for a total of 14 liters. Therefore, the percentage of yellow tint in the new mixture is:

$$\frac{14}{35} = \frac{2}{5} = \frac{40}{100} = \boxed{40\%}.$$

8.2.7: ✓

When we can't pay bills in 60% steps, it costs me 3 gallons of gas. How many gallons of gas does it contain when it is 50% full?

You may type any additional text here. | Hide Solution | Saved

Your Submission: Solution

Solution: Since the tank is 3 gallons when it is 60% empty, these 3 gallons fill 100% − 60% = 30% of the tank. This means that the 3 gallons are 30% = $\frac{30}{100} = \frac{3}{10}$ of the whole tank. So, the whole tank holds $3 \cdot 10 = 30$ gallons. Therefore, when the tank is 50% full, it has

$$50\% \cdot 30 = \frac{50}{100} \cdot 30 = \frac{1}{2} \cdot 30 = 15$$

gallons of gas. (We also might have noticed that when it was 60% empty, the tank held 3 gallons, so 40% of the tank held 30 − 3 = 27 gallons.)

8.2.8: ✓

Jon Eyeb played Jon Fyeb against each other many times last January. Eyeb won 65% of the time, and Fyeb won the other 7 games. (There were no ties.) How many games did Eyeb win?

You may type any additional text here. | Hide Solution | Saved

Your Submission: Solution

Solution: Since Eyeb won 65% of the games, Fyeb won the other 100% − 65% = 35%. Therefore, the ratio of the number of Eyeb's wins to the number of Fyeb's wins is 65 : 35. Dividing both parts of this ratio by 5 gives 13 : 7. Since Fyeb won 7 games, we conclude that Eyeb won $\frac{13}{7} \cdot 7 = \boxed{13}$ games.

8.2.9: ✓

The Fighting Tomatoes, a minor-league baseball team, won 20 of their first 30 games. How many of the remaining 40 games must the Tomatoes win so that they finish with exactly 70% of their games at home and 70% of their games on the road?

You may type any additional text here. | Hide Solution | Saved

Your Submission: Solution

Solution: The season has 50 = 30 + 40 games, so 70% of the games is a total of $70\% \cdot 50 = 0.7 \cdot 50 = 35$ games. The Fighting Tomatoes have already won 20 games, so they need to win $35 - 20 = \boxed{15}$ more games.

8.2.10: ✓

The table to the right gives the percent of students in each grade at West Pacifistic and East Pacifistic Elementary Schools. For example, 40% of the students at East Pacifistic are 5th students. In the two schools combined, what percent of the students are in grade 5?

Grade	West	East
5th	30%	40%
6th	40%	35%
7th	30%	25%

You may type any additional text here. | Hide Solution | Saved

Your Submission: Solution

Solution: The two schools combined have 200 students. The table tells us that 30% of the 100 students in West Pacifistic are 6th graders. Therefore, West Pacifistic has $30\% \cdot 100 = 30$ 6th graders. The table also tells us that 40% of East Pacifistic's 200 students are 6th graders. So, there are $40\% \cdot 200 = 80$ students in 6th grade at East Pacifistic.

The two schools together have 30 + 80 = 110 students in grade 6 out of the 200 total students in the two schools. So, the percentage of students in the two schools combined who are in grade 6 is $\frac{110}{200} = 55 = \boxed{55\%}$.

8.2.11: Source: MOEMS ✓

50% students at Euclid University took a math exam. 70% of the students passed the exam. Suppose instead only 10% of the students had failed the exam. How many more passing grades would there have been? (The exam is either passed or failed.)

Preview: Solution

You may type any additional text here. | Hide Solution | Saved

Your Submission: Solution

Solution: If only 10% of the students had failed the exam, 100% − 10% = 90% would have passed the exam. This is an additional 90% − 70% = 20% more than the 70% who did actually pass the exam. Since there are 50 students total, the additional 10% who passed the exam consist of

$$20\% \cdot 50 = 10 = \boxed{5}$$

students.

8.1 Percent Increase and Decrease

Percent increase and decrease are used to describe an increase or some quantity. For example, you can probably state with the "10% off" sales tag on a pair of jeans that the price of the jeans is now less than the original price. The original quantity and each subtract the amount by which it is greater than the original quantity.

A 10% increase means that the original is 100% increase from 80° is compared by first computing 10% of 80:

$$10\% \text{ of } 80 = 10\% \cdot 80 = \frac{10}{100} \cdot 80 = \frac{1}{10} \cdot 80 = 8$$

and then adding the increase to your original 80 to get 80 + 8 = 88°. With the sales tag, you compute the quantity by subtracting that 10% from the original 80 to get 80 − 8 = 72°. Percent increase and decrease are used to describe a change in a quantity and are used in calculations that involve a given percentage.

A 90% increase means that

$$\frac{90}{100} = \frac{9}{10}$$

That can be computed

$$10\% \text{ of } 80 = \frac{1}{10} \cdot 80 = 8$$

$$\text{(10% of 80)} + \text{(10% of 80)} = \frac{1}{10} \cdot 80 + \frac{1}{10} \cdot 80 = 8 + 8 = 16$$

Problems

Problem 1 ◯ Hide Solution

Compute the following quantities:

(a) 125% increase from 40

(b) 125% increase from 100

(c) 125% increase from 120

(d) 125% increase from 0

(e) 125% increase from 5

Problem 2 ◯ Hide Solution

Describe the percent increase or decrease given by the following changing quantities:

(a) an increase from 100 to 125

(b) a change from 50 to 25

(c) a change from 100 to 150

(d) a change from 100 to 100

(e) a change from 0 to 5

(f) a change from 5 to 0

(g) a change from 0 to 0

Problem 3 ◯ Hide Solution

Compute the following quantities:

(a) 125% increase from 40

(b) 125% increase from 100

(c) 125% increase from 120

(d) 125% increase from 0

Problem 4 ◯ Hide Solution

Describe the percent increase or decrease given by the following changing quantities:

(a) an increase from 100 to 125

(b) a change from 50 to 25

(c) a change from 100 to 150

(d) a decrease from 100 to 100

(e) a change from 0 to 5

(f) a change from 5 to 0

(g) a change from 0 to 0

(h) a change from 5 to 5

Problem 5 ◯ Hide Solution

Describe the percent increase or decrease given by the following changing quantities:

(a) an increase from 100 to 125

(b) a change from 50 to 25

(c) a change from 100 to 150

(d) a decrease from 100 to 100

(e) a change from 0 to 5

(f) a change from 5 to 0

(g) a change from 0 to 0

(h) a change from 5 to 5

(i) a change from 5 to 0.5

Problem 6 ◯ Hide Solution

A 125% increase Grade by Lemon Computer Co.'s stock price of $100 is $125. What is the percent increase in the stock price?

Problem 7 ◯ Hide Solution

A 125% increase Grade by Lemon Computer Co.'s stock price of $100 is $125. What is the percent increase in the stock price?

Problem 8 ◯ Hide Solution

A 125% increase Grade by Lemon Computer Co.'s stock price of $100 is $125. What is the percent increase in the stock price?

Problem 9 ◯ Hide Solution

A 125% increase Grade by Lemon Computer Co.'s stock price of $100 is $125. What is the percent increase in the stock price?

Problem 10 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 11 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 12 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 13 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 14 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 15 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 16 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 17 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 18 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 19 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 20 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 21 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 22 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 23 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 24 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 25 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 26 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 27 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 28 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 29 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 30 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 31 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 32 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 33 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 34 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 35 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 36 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 37 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 38 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 39 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 40 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 41 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 42 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 43 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 44 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 45 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 46 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 47 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 48 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 49 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 50 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 51 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 52 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 53 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 54 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 55 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 56 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 57 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 58 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 59 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 60 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 61 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 62 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 63 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 64 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 65 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 66 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 67 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 68 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 69 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 70 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 71 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 72 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 73 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 74 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 75 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 76 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 77 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 78 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 79 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 80 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 81 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 82 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 83 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 84 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 85 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 86 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 87 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 88 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 89 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 90 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 91 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 92 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 93 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 94 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 95 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 96 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 97 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 98 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 99 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Problem 100 ◯ Hide Solution

Percent increase in Grade by Lemon Computer Co.'s stock price of $100 is 125%. What is the percent increase in the stock price?

Exercises

8.1.1 Hide Solution

(a) What number is 20% more than 100?

Your Solution

Solution: 20% of 100 is $\frac{20}{100} \cdot 100 = \frac{1}{5} \cdot 100 = 20$ 20% more than 100 is $100 + 20 = 120$.

(b) What number is 50% less than 80?

Your Solution

Solution: 50% of 80 is $\frac{50}{100} \cdot 80 = \frac{1}{2} \cdot 80 = 40$ 50% less than 80 is $80 - 40 = 40$.

8.1.2 Hide Solution

(a) What number is 10% more than 40?

Your Solution

Solution: 10% of 40 is $\frac{10}{100} \cdot 40 = \frac{1}{10} \cdot 40 = 4$ 10% more than 40 is $40 + 10 = 50$.

(b) What number is 10% less than 40?

Your Solution

Solution: 10% of 40 is $\frac{10}{100} \cdot 40 = \frac{1}{10} \cdot 40 = 4$ 10% less than 40 is $40 - 10 = 30$.

8.1.3 Hide Solution

(a) What number is 100% more than 40?

Your Solution

Solution: 100% of 40 is $\frac{100}{100} \cdot 40 = 1 \cdot 40 = 40$ 100% more than 40 is $40 + 40 = 80$.

(b) What number is 100% less than 40?

Your Solution

Solution: 100% of 40 is $\frac{100}{100} \cdot 40 = 1 \cdot 40 = 40$ 100% less than 40 is $40 - 40 = 0$.

Alternatively, one might note that 0 is one 40 (100%) smaller than 0 and so 0 is 100% less than 40.

(c) What number is 200% more than 40?

Your Solution

Solution: 200% of 40 is $\frac{200}{100} \cdot 40 = 2 \cdot 40 = 80$ 200% more than 40 is $40 + 80 = 120$.

(d) What number is 200% less than 40?

Your Solution

Solution: 200% of 40 is $\frac{200}{100} \cdot 40 = 2 \cdot 40 = 80$ 200% less than 40 is $40 - 80 = -40$.

Alternatively, one might note that 0 and −40 are 40 (100%) apart from 0 and −40 are 40 (100%) apart from 0 and −40 are 40 (100%) apart from

8.4 Summary

A **percent** is another way of writing a fraction. We can write

$$x\% = \frac{x}{100}$$

where x is any number. So percents are nothing really new:

> **Concept:** A percent is just a number. It's a fraction with a hidden denominator of 100.

Sometimes the word "percentage" is used instead of "percent."

> **Concept:** Remember, a percent is just a fraction, so all of the things that we can do with fractions, we can do with percents too.

> **Important:** Many percents come up so often that you'll probably memorize them. The "quarter" percents are very common:
>
> $$25\% = \frac{25}{100} = \frac{1}{4}, \qquad 50\% = \frac{50}{100} = \frac{1}{2}, \qquad 75\% = \frac{75}{100} = \frac{3}{4}.$$
>
> You should also immediately recognize that $10\% = \frac{10}{100} = \frac{1}{10}$, so any percent that is a multiple of 10% is easy to compute:
>
> $$20\% = 2 \cdot 10\% = \frac{2}{10} = \frac{1}{5}, \qquad 30\% = 3 \cdot 10\% = \frac{3}{10},$$
>
> and so on.

One reason that percents are so commonly used is that they are easy to represent using decimals. As we've already seen, decimals are nice because they let us use our familiar base-10 number system to write fractions as well as integers. Since percents are based on 100, we see that 1 percent is equal to 1 one-hundredth, which we write as

$$1\% = \frac{1}{100} = 0.01.$$

Because percents are used frequently in the real world, they show up a lot in word problems. The key ideas for percent word problems are the same as for any type of word problem:

> **Concept:**
>
> - Read the problem carefully!
> - Convert the words to mathematics.
> - Solve the mathematical problem.
> - Write your answer in terms of the word problem, and make sure that it makes sense.

Percents are commonly used to represent an increase or decrease to some quantity. In these situations, we take a percent times the *original* quantity and add or subtract it to get a new quantity.

For example, the quantity that is "a 30% increase from 400" is computed by first computing 30% of 400:

$$30\% \text{ of } 400 = 30\% \cdot 400 = \frac{3}{10} \cdot 400 = 120,$$

and then adding this increase to our original 400 to get $400 + 120 = 520$. We could also compute this quantity by realizing that

$$\text{a } 30\% \text{ increase from } 400$$

is the same quantity as

$$130\% \text{ of } 400,$$

and then

$$130\% \text{ of } 400 = 130\% \cdot 400 = \frac{13}{10} \cdot 400 = 13 \cdot 40 = 520.$$

Challenge Problems

8.39:

Source: MATHCOUNTS

If 25% of n is 18, then what is 125% of n?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $125\% = 5 \cdot 25\%$, we know that 125% of a number is 5 times 25% of that same number. Since 25% of n is 18, we know that 125% of n is $5 \cdot 18 = \boxed{90}$.

Alternatively, we could have found n. Since 25% of n is 18, we know that $\frac{1}{4}$ of n is 18. This means that $n = 18 \cdot 4 = 72$. We then compute that 125% of n is $\frac{125}{100} \cdot 72 = \frac{5}{4} \cdot 72 = \boxed{90}$, as before.

8.40:

Source: AMC 8

If 20% of a number is 12, then what is 30% of the same number?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The ratio of 30% of a number to 20% of that same number is $30 : 20 = 3 : 2$. So, 30% of a number is $\frac{3}{2}$ times 20% the same number. Since 20% of the number in the problem is 12, we know that 30% of the same number is $\frac{3}{2} \cdot 12 = \boxed{18}$.

Alternatively, we could have found the number. We are given that $20\% = \frac{1}{5}$ of the number is 12, so the number is $12 \cdot 5 = 60$. Therefore, 30% of the same number is $0.3 \cdot 60 = \boxed{18}$.

8.41:

Source: MATHCOUNTS

Find x if 20% of x equals 12% of $(x + 20)$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 20% of x equals $0.2x$ and 12% of $x + 20$ is

$$0.12(x + 20) = 0.12x + 0.12 \cdot 20 = 0.12x + 2.4.$$

Therefore, we must have $0.2x = 0.12x + 2.4$. Multiplying both sides by 100 gets rid of the decimals and leaves $20x = 12x + 240$. Subtracting $12x$ from both sides gives $8x = 240$, so $x = \boxed{30}$.

8.42:

Source: AMC 8

Three bags of jelly beans contain 26, 28, and 30 beans. In each bag, the percent of the beans that are yellow is 50%, 25%, and 20%, respectively. All three bags of candy are dumped into one bowl. What percent of the beans in the bowl are yellow? Round your answer to the nearest integer.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The numbers of yellow beans in the bags are $\frac{1}{2} \cdot 26 = 13$, $\frac{1}{4} \cdot 28 = 7$, and $\frac{1}{5} \cdot 30 = 6$, respectively. So, the total number of yellow beans is $13 + 7 + 6 = 26$. The total number of beans is $26 + 28 + 30 = 84$, so, to the nearest whole percent, the percentage of beans that are yellow is $\frac{26}{84} \approx 0.3095 \approx \boxed{31\%}$.

8.43:

Cody and Tyler were once the same height. Since then, Tyler has grown 20%, while Cody has grown half as much as Tyler. Cody's height is now 5 feet, 6 inches. How tall is Tyler now?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since there are 12 inches in a foot, Cody's height is $12 \cdot 5 + 6 = 66$ inches. Cody grew half as much as Tyler, and they started from the same height. So, while Tyler grew by 20%, Cody grew by 10%. Therefore, Cody's height is 66 inches after he grew by 10%. Since Cody is now 10% taller than he was, his current height is $100\% + 10\% = 110\%$ of his original height. So, if we let Cody's original height be x inches, we must have $1.1 \cdot x = 66$. Dividing both sides by 1.1 gives $x = 60$, so Cody was originally 60 inches tall.

Tyler was the same height as Cody, 60 inches, before growing by 20%. While growing, Tyler's height increased by $20\% \cdot 60 = 0.2 \cdot 60 = 12$ inches. So, he is now $\boxed{72 \text{ inches}}$, which is $\boxed{6 \text{ feet}}$.

8.44:

Source: AMC 8

Miki has a dozen oranges of the same size and a dozen pears of the same size. Miki can use her juicer to extract 8 ounces of pear juice from 3 pears and 8 ounces of orange juice from 2 oranges. She makes a pear-orange juice blend from an equal number of pears and oranges. What percent of the blend is pear juice?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since Miki gets 8 ounces of pear juice from 3 pears, she gets $\frac{8}{3}$ ounces of juice from 1 pear. Similarly, Miki gets 8 ounces of orange juice from 2 oranges, so she gets $\frac{8}{2} = 4$ ounces of juice from 1 orange. When Miki combines 1 pear and 1 orange, she gets $\frac{8}{3} + 4 = \frac{20}{3}$ ounces total. Of these $\frac{20}{3}$ ounces, $\frac{8}{3}$ ounces are pear juice. So, the pear juice is

$$\frac{8/3}{20/3} = \frac{8}{3} \cdot \frac{3}{20} = \frac{8}{20} = \frac{2}{5} = \boxed{40\%}$$

of the total.

8.45:

At some point in the season, the Unicorns had won 60% of their basketball games. After that point, they won 8 more games and lost 2, to finish the season having won 65% of their games. How many games did the Unicorns play during the season?

Hint: Assign a variable.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let x be the number of games the Unicorns played during the season. The Unicorns won 60% of the first $x - 10$ games, which is $60\% \cdot (x - 10)$ games. Combining this with the 8 games they won in the last 10 games gives a total of

$$60\% \cdot (x - 10) + 8 = 0.6(x - 10) + 8 = 0.6x - 6 + 8 = 0.6x + 2$$

wins. We also know that the Unicorns won 65% of their x games, which is a total of $65\% \cdot x = 0.65x$ wins. Setting this equal to our other expression for the total number of wins gives $0.65x = 0.6x + 2$, so $0.05x = 2$. Writing $0.05 = \frac{5}{100} = \frac{1}{20}$, we have $\frac{1}{20}x = 2$, so $x = \boxed{40}$.

8.46:

If 100 is decreased by $a\%$ and the result is decreased by $b\%$, then what is the total percent decrease (in terms of a and b)? (You can assume that both a and b are between 0 and 100.)

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: A decrease of $a\%$ is a decrease of 100 to $100 - a$. A decrease of this by $b\%$ is a decrease of

$$\frac{b}{100} \cdot (100 - a) = \frac{b}{100} \cdot 100 - \frac{b}{100} \cdot a = b - \frac{ab}{100}.$$

Therefore, the result of the $b\%$ decrease is

$$\begin{aligned} 100 - a - \left(b - \frac{ab}{100}\right) &= 100 - a - b + \frac{ab}{100} \\ &= 100 - \left(a + b - \frac{ab}{100}\right). \end{aligned}$$

So, the total decrease from 100 is $a + b - \frac{ab}{100}$. Since we started with 100, this is a $\boxed{a + b - \frac{ab}{100}}$ percent decrease.

8.47★:

Grapes are 80% water (by weight), and raisins are 20% water (by weight). If we start with 500 grams of grapes and remove enough water to turn them into raisins, then what is the weight of the raisins that result?

Hint: There's water, and there's other stuff.

Hint: How much other stuff?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Our original 500 grams of grapes consists of $(80\%) \cdot 500 = \frac{4}{5} \cdot 500 = 400$ grams of water and $500 - 400 = 100$ grams of pulp. After removing enough water to leave raisins, we still have 100 grams of pulp, which should be 80% of the weight of the raisins. Thus, the weight of the raisins is

$$100/80\% = 100/\frac{4}{5} = 100 \cdot \frac{5}{4} = \boxed{125 \text{ grams}}.$$

Review Problems

8.21: Source: MATHCOUNTS

Compute 40% of 20% of 10% of 80,000.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have

$$\begin{aligned} 40\% \text{ of } 20\% \text{ of } 10\% \text{ of } 80,000 &= 40\% \text{ of } 20\% \text{ of } (10\% \text{ of } 80,000) \\ &= 40\% \text{ of } 20\% \text{ of } (0.10 \cdot 80,000) \\ &= 40\% \text{ of } 20\% \text{ of } 8000 \\ &= 40\% \text{ of } (0.2 \cdot 8000) \\ &= 40\% \text{ of } 1600 \\ &= 0.4 \cdot 1600 \\ &= \boxed{640}. \end{aligned}$$

8.22:

18% of 50 is what percent of 24?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 18% of 50 is $\frac{18}{100} \cdot 50 = \frac{9}{50} \cdot 50 = 9$. So, we want to know what percent of 24 is 9. Since $\frac{9}{24} = \frac{3}{8} = 0.375 = 37.5\%$, we know that 9 is $\boxed{37.5\%}$ of 24.

8.23:

If there are 240 boys in a school with a total of 960 students, then what percent of the students are girls?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since there are 240 boys, there are $960 - 240 = 720$ girls. Therefore, the percentage of students who are girls is $\frac{720}{960} = \frac{3}{4} = 0.75 = \boxed{75\%}$.

8.24:

The grading scale shown to the right is used at Jones Junior High. The fifteen scores in Mr. Freeman's class were:

89, 72, 54, 97, 71, 92, 85, 74, 75, 63, 84, 78, 71, 80, 90.

In Mr. Freeman's class, what percent of the students received a grade of C?

Score	Grade
90-100	A
80-89	B
70-79	C
60-69	D
0-69	F

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There are 15 scores in the list. There are 6 scores in the list that have a grade of C, so the percentage of students who receive a C is $\frac{6}{15} = \frac{2}{5} = \boxed{40\%}$.

8.25: Source: AMC 8

The glass gauge on a cylindrical coffee maker shows there are 45 cups left when the coffee maker is 36% full. How many cups of coffee does it hold when it is full?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: 45 is 36% of the number of cups in the coffee maker. So, we must find the number x such that 36% of x is 45. Since $\frac{36}{100} = \frac{9}{25}$, we must have $\frac{9}{25} \cdot x = 45$. Multiplying both sides by $\frac{25}{9}$ gives $x = \frac{25}{9} \cdot 45 = 25 \cdot 5 = 125$. So, the coffee maker makes $\boxed{125}$ cups of coffee.

8.26: Source: AMC 8

During the softball season, Judy had 35 hits. Among her hits were 1 home run, 1 triple, and 5 doubles. The rest of her hits were singles. What percent of her hits were singles?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Judy had $1 + 1 + 5 = 7$ hits that were not singles, so the remaining $35 - 7 = 28$ hits were singles. Therefore, the percent of hits that were singles was $\frac{28}{35} = \frac{4}{5} = \frac{80}{100} = \boxed{80\%}$.

8.27:

Any quarter has a face value of 80.25. An eccentric collector offers to buy state quarters for 600% of their face value. At that rate, how much will Larry receive for his collection of all 50 state quarters?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Larry's 50 state quarters have a face value of $50 \cdot 80.25 = \$12.50$. 600% of a number is 6 times the number, so Larry receives $6 \cdot \$12.50 = \boxed{\$75}$ for his whole collection.

8.28:

Katie wants a fancy new music player that costs $300. Katie can buy it in her home state and pay 8% sales tax, or she can drive to a neighboring state and pay only 3% sales tax. How much does Katie save on the player if she drives to the neighboring state?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Katie pays 3% less by driving to the neighboring state. So, she saves $3\% \cdot \$300 = 0.03 \cdot \$300 = \boxed{\$9}$.

8.29: Source: AMC 8

Sally is playing basketball. After she takes 20 shots, she has made 55% of her shots. After she takes 5 more shots, she raises her percentage of shots made to 56%. How many of the last 5 shots did she make?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since Sally made 55% of her first 20 shots, she made $\frac{55}{100} \cdot 20 = \frac{11}{20} \cdot 20 = 11$ of her first 20 shots. Since she made 56% of her first 25 shots, she made $\frac{56}{100} \cdot 25 = \frac{14}{25} \cdot 25 = 14$ of her first 25 shots. Therefore, she made $14 - 11 = \boxed{3}$ of her last 5 shots.

8.30:

200 students enrolled in the Percentages 101 class in 2010. A 20% increase in enrollment is expected each year. How many students are expected to enroll in the class in 2012?

Solution

Hide Solution

Reset

Your Submission: Solution

Solution: From 2010 to 2011, enrollment increases by 20% from 200, which is an increase of $0.2 \cdot 200 = 40$ students. So, enrollment is 240 students in 2011. From 2011 to 2012, enrollment increases by 20% from 240, which is an increase of $0.2 \cdot 240 = 48$ students. So, enrollment is $240 + 48 = \boxed{288}$ in 2012.

8.31:

The January price of a television was $2200. This price was raised by 10% to produce the February price of the television. The February price was decreased by 15% to produce the March price of the television. What was the March price?

You may type any additional notes you have here.

Hide Solution

Reset

Solution: A 10% increase of $2200 is an increase of $\frac{1}{10} \cdot \$2200 = \$220$, to $\$2200 + \$220 = \$2420$. A decrease of 15% from $2420 is a decrease of $0.15 \cdot \$2420 = \$363$, which makes the price $\$2420 - \$363 = \boxed{\$2057}$.

8.32: Source: AMC 8

Antoinette gets 70% on a 10-problem test, 80% on a 20-problem test, and 90% on a 30-problem test. If the three tests are combined into one 60-problem test, then what is her overall percent, rounded to the nearest whole percent?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: On the 10-problem test, she answers $0.7 \cdot 10 = 7$ questions correctly. On the 20-problem test, she answers $0.8 \cdot 20 = 16$ questions correctly. On the 30-problem test, she answers $0.9 \cdot 30 = 27$ questions correctly. So, she answers $7 + 16 + 27 = 50$ of the 60 total questions correctly, which means she answers $\frac{50}{60} = \frac{5}{6}$ of the questions correctly. Since $\frac{5}{6} = 0.8\overline{3}$, her percentage to the nearest whole percent is $\boxed{83\%}$.

8.33:

Suppose Paul receives a 6% raise every year. After four such raises, what is the total percentage increase to the nearest whole percent?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: After one such raise, Paul's new salary is 106% of his old salary, which means that his new salary is 1.06 times his old salary. Similarly, each of his raises multiplies his salary by 1.06. So, four such raises multiplies his salary by $1.06^4 \approx 1.262$, which is an increase of 0.262 times his original salary. Therefore, to the nearest percent, his salary has increased by $\boxed{26\%}$.

8.34: Source: AMC 8

Jack had a bag of 128 apples. He sold 25% of them to Jill. Next he sold 25% of those remaining to June. Of those apples still in his bag, he gave the most shiny one to his teacher. How many apples did Jack have then?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $25\% = \frac{1}{4}$, Jack sold $\frac{1}{4} \cdot 128 = 32$ apples to Jill, leaving him with $128 - 32 = 96$ apples. He then sold $\frac{1}{4} \cdot 96 = 24$ to June, leaving him with $96 - 24 = 72$. After giving his teacher the most shiny one of these, he was left with $\boxed{71}$ apples.

8.35: Source: AMC 8

A shopper buys a $100 coat on sale for 20% off. An additional $5 is taken off the sale price by using a discount coupon. A sales tax of 8% is paid on the final selling price. What is the total amount the shopper pays for the coat?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The 20% discount from $100 reduces the price by $20 to $80. The discount coupon reduces the price to $80 − $5 = $75. The tax of 8% of this sales price equals $\frac{8}{100} \cdot \$75 = \frac{2}{25} \cdot \$75 = \$6$, so the total amount the shopper pays is $75 + $6 = $\boxed{\$81}$.

8.36:

Polly the Penguin invested $250 in the Antarctic stock market. During the first year her investment suffered a 15% loss, but during the second year the remaining investment showed a 20% gain. Over the two-year period, what was the percent loss or gain in Polly's investment?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Polly's 15% loss in the first year reduced her $250 by

$$\frac{15}{100} \cdot \$250 = \frac{3}{20} \cdot \$250 = 3 \cdot \frac{\$250}{20} = 3 \cdot \$12.50 = \$37.50,$$

to $212.50. Then, in the second year her $212.50 is increased by 20%, which is an increase of $0.2 \cdot \$212.50 = \$42.50$. So, at the end of her second year, her investment is $212.50 + $42.50 = $255. Therefore, her original investment of $250 increased in value by $5 total. Since a 1% increase in value is $2.50, her $5 increase represents a $\boxed{2\%\text{ gain}}$.

Alternatively, the loss in the first year multiplies Polly's investment by 0.85, and the gain in the second year multiplies the investment by 1.2. So over the two-year period, the investment is multiplied by $(0.85)(1.2) = 1.02$, which is a 2% gain.

8.37: Source: AMC 8

Sale prices at the Ajax Outlet Store are 50% below original prices. On Saturdays, an additional discount of 20% off the sale price is given. What is the Saturday price of a coat whose original price is $180?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: A sale price of 50% off halves the price from $180 to $90. An additional discount of 20% off of this new price is a discount of $0.2 \cdot \$90 = \$18$. This makes the Saturday price $90 − $18 = $\boxed{\$72}$.

8.38:

Jack and Jill both work at the King's Ice Cream Shoppe. The King levies a 25% sales tax on all purchases. A customer comes in and orders an ice cream cone that costs 5 bonks (the bonk is a currency equal to 100 borklets). The customer also has a coupon for 10% off.

Jack says: "The total price will be highest if we first apply the 10% coupon to the price of the cone, and then compute the sales tax on the discounted price."

Jill says: "No–the total price will be highest if we first add the sales tax to the original price of the cone, and then apply the coupon."

Who's right?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since a bonk equals 100 borklets, the 5 bonk ice cream cone costs 500 borklets.

First, we compute the final price if we use Jack's strategy of applying the coupon first. The coupon gives a discount of $\frac{10}{100} = \frac{1}{10}$ of 500 borklets, making the price 450 borklets. Then, the 20% tax applied to this price increases the total by $\frac{1}{5} \cdot 450 = 90$ borklets, to 540 borklets.

Next, we compute the price with Jill's strategy of applying the tax first. A 20% tax on 500 borklets is an increase of $\frac{1}{5} \cdot 500 = 100$ borklets, to 600 borklets. A 10% discount from 600 borklets is a discount of $\frac{1}{10} \cdot 600 = 60$ borklets, to $600 − 60 = 540$ borklets.

So,

$$\boxed{\text{the two strategies result in the same final cost}}.$$

Is this just a coincidence? Try changing the original numbers in the problem, and see if the two strategies still produce the same final cost.

9.1 Square Roots

Back in Chapter 5, we learned about finding the square root of a number. In this chapter, we investigate going in the other direction: we start with the square, and figure out what number was squared to produce the square.

Throughout this chapter, you will be expected to know the square roots of the perfect squares.

9.1 From Squares to Square Roots

The square root of 9 is 3 because when we square 3, we get 9: $3^2 = 9$. We write this with symbols as:

$$\sqrt{9} = 3$$

You might wonder why we'd ever use that $\sqrt{\ }$ sign. It's almost the square of – it's close to it. The answer is that we already had a name for the number we get when we square 3: we call it the square of 3. We'd like to be able to use the corresponding name for the number we get when we square 9. But we can't, because there is more than one number that we can square to get 9.

Definition: The **square root** of a nonnegative number x is the nonnegative number whose square is x, and we write it as $\sqrt{x}$. We read the symbol $\sqrt{\ }$ as "the square root of," or simply "root."

Both of the appearances of "nonnegative" in this definition are very important! It makes sense that a root is nonnegative in order to be unique. It makes sense that we can't take the square root of a negative number, as nothing multiplied by itself has anything positive in it, and so there is no such number.

The symbol "nonnegative" is not written with a sign, but it is a perfectly nonsense. We usually write a nonnegative number whose square is x, so we define $\sqrt{x}$ to be the nonnegative square root of x. Nonnegative means "not negative."

$$\text{The square of the side of the square is } \sqrt{\text{The area of the square}}$$

$$\text{The height of the tree will be the square root of the shadow's length.}$$

The first one is mostly simple. It would make sense in most applications, we only care about the nonnegative number whose square is x, so we define $\sqrt{x}$ to be the nonnegative number.

Obviously, simply saying "if screen-square-then-sqrt(x^2)" then we "evaluate" "evaluate" is "simplify" is square root expression, we by the nonnegative square root of x is x itself, so we have the following:

$$\sqrt{x^2} = x \quad \text{and} \quad \sqrt{x^2} = x^2.$$

Problems

Problem 1 Solving for Solutions 1

Why is the following two problems have different answers?

1. Find all values of x for which $x^2 = 9$.
2. Find all values of x for which $x = \sqrt{9}$.

Problem 2 Solving for Solutions 1

Evaluate each of the following square roots:

1. $\sqrt{49}$
2. $\sqrt{144}$
3. $\sqrt{225}$
4. $\sqrt{400}$

Problem 3 Solving for Solutions 1

Evaluate each of the following square roots:

1. $\sqrt{\frac{1}{4}}$
2. $\sqrt{\frac{9}{16}}$
3. $\sqrt{\frac{25}{36}}$
4. $\sqrt{\frac{100}{121}}$

Problem 4 Solving for Solutions 1

Evaluate each of the following:

1. $\sqrt{0.01}$
2. $\sqrt{0.09}$
3. $\sqrt{0.25}$
4. $\sqrt{0.36}$

Problem 5 Solving for Solutions 1

Simplify each of the following:

1. $\sqrt{(-1)^2}$
2. $\sqrt{(-2)^2}$
3. $\sqrt{(-3)^2}$
4. $\sqrt{(-4)^2}$

Problem 6 Solving for Solutions 1

Simplify each of the following:

1. $\sqrt{(-1)^4}$
2. $\sqrt{(-2)^4}$
3. $\sqrt{(-3)^4}$
4. $\sqrt{(-4)^4}$

Problem 7 Solving for Solutions 1

Evaluate each of the following:

1. $\sqrt{\sqrt{1}}$
2. $\sqrt{\sqrt{\sqrt{1}}}$
3. $\sqrt{\sqrt{\sqrt{\sqrt{1}}}}$
4. $\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{1}}}}}$

Problem 8 Solving for Solutions 1

Why is the following two problems have different answers?

1. Find all values of x for which $x^2 = 9$.
2. Find all values of x for which $x = \sqrt{9}$.

Solution for Problem 8.1

1. We know that $3^2 = 9$, so $x = 3$ is one solution. We know that $(-3)^2 = 9$, so $x = -3$ is a solution. But for the equation $x = \sqrt{9}$, only $x = 3$ is a solution. We don't care about -3 in this case, because $\sqrt{9}$ is a number, and we know it is a number whose square equals 9, so we define the square root $x = \sqrt{9}$ to be only $x = 3$.

Problem 9 Solving for Solutions 1

Evaluate each of the following square roots:

1. $\sqrt{36}$
2. $\sqrt{64}$
3. $\sqrt{100}$
4. $\sqrt{169}$

Solution for Problem 9.2

1. Since $6^2 = 36$, we know $\sqrt{36} = 6$.
2. Since $8^2 = 64$, we know $\sqrt{64} = 8$.
3. Since $10^2 = 100$, we know $\sqrt{100} = 10$. We can't mention what number 325 is the square root. There was not my squaring different numbers, hoping to find a number whose square is 325. We notice that $20^2 = 400$ which is more than 325, so we need to consider numbers smaller than 20. We know $15^2 = 225$ which is less than 325, so we know the square root is between 15 and 20. We know $16^2 = 256$ and $17^2 = 289$ and $18^2 = 324$, so $19^2 = 361$ and $20^2 = 400$. We know that the square root is between 18 and 19, so we know that it is not an integer. We can't find the exact square root, so we leave it as $\sqrt{325}$.
4. Notice that we cannot continue in all of this by thinking of all square sequence before trying to square any numbers at all.

Problem 10 Solving for Solutions 1

Evaluate each of the following:

1. $\sqrt{\frac{1}{4}}$
2. $\sqrt{\frac{9}{16}}$
3. $\sqrt{\frac{25}{36}}$
4. $\sqrt{\frac{100}{121}}$

Problem 11 Solving for Solutions 1

Evaluate each of the following:

1. $\sqrt{0.01}$
2. $\sqrt{0.09}$
3. $\sqrt{0.25}$
4. $\sqrt{0.36}$

Problem 12 Solving for Solutions 1

Simplify each of the following:

1. $\sqrt{(-1)^2}$
2. $\sqrt{(-2)^2}$
3. $\sqrt{(-3)^2}$
4. $\sqrt{(-4)^2}$

Problem 13 Solving for Solutions 1

Simplify each of the following:

1. $\sqrt{(-1)^4}$
2. $\sqrt{(-2)^4}$
3. $\sqrt{(-3)^4}$
4. $\sqrt{(-4)^4}$

Problem 14 Solving for Solutions 1

Evaluate each of the following:

1. $\sqrt{\sqrt{1}}$
2. $\sqrt{\sqrt{\sqrt{1}}}$
3. $\sqrt{\sqrt{\sqrt{\sqrt{1}}}}$
4. $\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{1}}}}}$

Solution for Problem 14.2

1. We know that $1^2 = 1$, so $\sqrt{1} = 1$.
2. We know that $1^2 = 1$, so $\sqrt{1} = 1$. We know $\sqrt{1} = 1$, so $\sqrt{\sqrt{1}} = \sqrt{1} = 1$.
3. We know that $1^2 = 1$, so $\sqrt{1} = 1$. We know $\sqrt{1} = 1$, so $\sqrt{\sqrt{1}} = \sqrt{1} = 1$. We know $\sqrt{\sqrt{1}} = 1$, so $\sqrt{\sqrt{\sqrt{1}}} = \sqrt{1} = 1$.
4. We know that $1^2 = 1$, so $\sqrt{1} = 1$. We know $\sqrt{1} = 1$, so $\sqrt{\sqrt{1}} = \sqrt{1} = 1$. We know $\sqrt{\sqrt{1}} = 1$, so $\sqrt{\sqrt{\sqrt{1}}} = \sqrt{1} = 1$. We know $\sqrt{\sqrt{\sqrt{1}}} = 1$, so $\sqrt{\sqrt{\sqrt{\sqrt{1}}}} = \sqrt{1} = 1$.

Problem 15 Solving for Solutions 1

Evaluate each of the following square roots:

1. $\sqrt{36}$
2. $\sqrt{64}$
3. $\sqrt{100}$
4. $\sqrt{169}$

Solution for Problem 15.2

1. We know that $6^2 = 36$, so $\sqrt{36} = 6$.
2. We know that $8^2 = 64$, so $\sqrt{64} = 8$.
3. We know that $10^2 = 100$, so $\sqrt{100} = 10$.
4. We know that $13^2 = 169$, so $\sqrt{169} = 13$.

Problem 16 Solving for Solutions 1

Evaluate each of the following square roots:

1. $\sqrt{\frac{1}{4}}$
2. $\sqrt{\frac{9}{16}}$
3. $\sqrt{\frac{25}{36}}$
4. $\sqrt{\frac{100}{121}}$

Problem 17 Solving for Solutions 1

Find all x for $\sqrt{x^2} = -5$.

Problem 18 Solving for Solutions 1

Find all x for $\sqrt{x^2} = -5$.

Problem 19 Solving for Solutions 1

Find all x for $\sqrt{x^2} = 5$.

Problem 20 Solving for Solutions 1

Find all x for $\sqrt{x^2} = 5$.

Problem 21 Solving for Solutions 1

Find all x for $\sqrt{x^2} = x$.

Problem 22 Solving for Solutions 1

Find all x for $\sqrt{x^2} = x$.

Problem 23 Solving for Solutions 1

Find all x for $\sqrt{x^2} = x^2$.

Problem 24 Solving for Solutions 1

Find all x for $\sqrt{x^2} = x^2$.

Problem 25 Solving for Solutions 1

Find all x for $\sqrt{x^2} = -x$.

Problem 26 Solving for Solutions 1

Find all x for $\sqrt{x^2} = -x$.

Problem 27 Solving for Solutions 1

Find all x for $\sqrt{x^2} = -x^2$.

Problem 28 Solving for Solutions 1

Find all x for $\sqrt{x^2} = -x^2$.

Problem 29 Solving for Solutions 1

Find all x for $\sqrt{x^2} = 1 - x$.

Problem 30 Solving for Solutions 1

Find all x for $\sqrt{x^2} = 1 - x$.

Exercises

9.1.1. Solving for Solutions 1

Evaluate the following square roots, for extra challenge, try computing them without using anything.

1. $\sqrt{49}$

Your Solution(s) Hide Solution

Your Solution(s) Hide Solution

Your Solution(s) Hide Solution

9.1 Square Roots

Back in Chapter 5, we learned about finding the square root of a number. In this chapter, we investigate going in the other direction: we start with the square, and figure out what number was squared to produce the square.

Throughout this chapter, you will be expected to know the square roots of the perfect squares.

9.1 From Squares to Square Roots

The square root of 9 is 3 because when we square 3, we get 9: $3^2 = 9$. We write this with symbols as:

$$\sqrt{9} = 3$$

You might wonder why we'd ever use that $\sqrt{\ }$ sign. It's almost the square of – it's close to it. The answer is that we already had a name for the number we get when we square 3: we call it the square of 3. We'd like to be able to use the corresponding name for the number we get when we square 9. But we can't, because there is more than one number that we can square to get 9.

Definition: The **square root** of a nonnegative number x is the nonnegative number whose square is x, and we write it as $\sqrt{x}$. We read the symbol $\sqrt{\ }$ as "the square root of," or simply "root."

Both of the appearances of "nonnegative" in this definition are very important! It makes sense that a root is nonnegative in order to be unique. It makes sense that we can't take the square root of a negative number, as nothing multiplied by itself has anything positive in it, and so there is no such number.

The symbol "nonnegative" is not written with a sign, but it is a perfectly nonsense. We usually write a nonnegative number whose square is x, so we define $\sqrt{x}$ to be the nonnegative square root of x. Nonnegative means "not negative."

$$\text{The square of the side of the square is } \sqrt{\text{The area of the square}}$$

$$\text{The height of the tree will be the square root of the shadow's length.}$$

The first one is mostly simple. It would make sense in most applications, we only care about the nonnegative number whose square is x, so we define $\sqrt{x}$ to be the nonnegative number.

Obviously, simply saying "if screen-square-then-sqrt(x^2)" then we "evaluate" "evaluate" is "simplify" is square root expression, we by the nonnegative square root of x is x itself, so we have the following:

$$\sqrt{x^2} = x \quad \text{and} \quad \sqrt{x^2} = x^2.$$

Problems

Problem 1 Solving for Solutions 1

Why is the following two problems have different answers?

1. Find all values of x for which $x^2 = 9$.
2. Find all values of x for which $x = \sqrt{9}$.

Problem 2 Solving for Solutions 1

Evaluate each of the following square roots:

1. $\sqrt{49}$
2. $\sqrt{144}$
3. $\sqrt{225}$
4. $\sqrt{400}$

Problem 3 Solving for Solutions 1

Evaluate each of the following square roots:

1. $\sqrt{\frac{1}{4}}$
2. $\sqrt{\frac{9}{16}}$
3. $\sqrt{\frac{25}{36}}$
4. $\sqrt{\frac{100}{121}}$

Problem 4 Solving for Solutions 1

Evaluate each of the following:

1. $\sqrt{0.01}$
2. $\sqrt{0.09}$
3. $\sqrt{0.25}$
4. $\sqrt{0.36}$

Problem 5 Solving for Solutions 1

Simplify each of the following:

1. $\sqrt{(-1)^2}$
2. $\sqrt{(-2)^2}$
3. $\sqrt{(-3)^2}$
4. $\sqrt{(-4)^2}$

Problem 6 Solving for Solutions 1

Simplify each of the following:

1. $\sqrt{(-1)^4}$
2. $\sqrt{(-2)^4}$
3. $\sqrt{(-3)^4}$
4. $\sqrt{(-4)^4}$

Problem 7 Solving for Solutions 1

Evaluate each of the following:

1. $\sqrt{\sqrt{1}}$
2. $\sqrt{\sqrt{\sqrt{1}}}$
3. $\sqrt{\sqrt{\sqrt{\sqrt{1}}}}$
4. $\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{1}}}}}$

Problem 8 Solving for Solutions 1

Why is the following two problems have different answers?

1. Find all values of x for which $x^2 = 9$.
2. Find all values of x for which $x = \sqrt{9}$.

Solution for Problem 8.1

1. We know that $3^2 = 9$, so $x = 3$ is one solution. We know that $(-3)^2 = 9$, so $x = -3$ is a solution. But for the equation $x = \sqrt{9}$, only $x = 3$ is a solution. We don't care about -3 in this case, because $\sqrt{9}$ is a number, and we know it is a number whose square equals 9, so we define the square root $x = \sqrt{9}$ to be only $x = 3$.

Problem 9 Solving for Solutions 1

Evaluate each of the following square roots:

1. $\sqrt{36}$
2. $\sqrt{64}$
3. $\sqrt{100}$
4. $\sqrt{169}$

Solution for Problem 9.2

1. Since $6^2 = 36$, we know $\sqrt{36} = 6$.
2. Since $8^2 = 64$, we know $\sqrt{64} = 8$.
3. Since $10^2 = 100$, we know $\sqrt{100} = 10$. We can't mention what number 325 is the square root. There was not my squaring different numbers, hoping to find a number whose square is 325. We notice that $20^2 = 400$ which is more than 325, so we need to consider numbers smaller than 20. We know $15^2 = 225$ which is less than 325, so we know the square root is between 15 and 20. We know $16^2 = 256$ and $17^2 = 289$ and $18^2 = 324$, so $19^2 = 361$ and $20^2 = 400$. We know that the square root is between 18 and 19, so we know that it is not an integer. We can't find the exact square root, so we leave it as $\sqrt{325}$.
4. Notice that we cannot continue in all of this by thinking of all square sequence before trying to square any numbers at all.

Problem 10 Solving for Solutions 1

Evaluate each of the following:

1. $\sqrt{\frac{1}{4}}$
2. $\sqrt{\frac{9}{16}}$
3. $\sqrt{\frac{25}{36}}$
4. $\sqrt{\frac{100}{121}}$

Problem 11 Solving for Solutions 1

Evaluate each of the following:

1. $\sqrt{0.01}$
2. $\sqrt{0.09}$
3. $\sqrt{0.25}$
4. $\sqrt{0.36}$

Problem 12 Solving for Solutions 1

Simplify each of the following:

1. $\sqrt{(-1)^2}$
2. $\sqrt{(-2)^2}$
3. $\sqrt{(-3)^2}$
4. $\sqrt{(-4)^2}$

Problem 13 Solving for Solutions 1

Simplify each of the following:

1. $\sqrt{(-1)^4}$
2. $\sqrt{(-2)^4}$
3. $\sqrt{(-3)^4}$
4. $\sqrt{(-4)^4}$

Problem 14 Solving for Solutions 1

Evaluate each of the following:

1. $\sqrt{\sqrt{1}}$
2. $\sqrt{\sqrt{\sqrt{1}}}$
3. $\sqrt{\sqrt{\sqrt{\sqrt{1}}}}$
4. $\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{1}}}}}$

Solution for Problem 14.2

1. We know that $1^2 = 1$, so $\sqrt{1} = 1$.
2. We know that $1^2 = 1$, so $\sqrt{1} = 1$. We know $\sqrt{1} = 1$, so $\sqrt{\sqrt{1}} = \sqrt{1} = 1$.
3. We know that $1^2 = 1$, so $\sqrt{1} = 1$. We know $\sqrt{1} = 1$, so $\sqrt{\sqrt{1}} = \sqrt{1} = 1$. We know $\sqrt{\sqrt{1}} = 1$, so $\sqrt{\sqrt{\sqrt{1}}} = \sqrt{1} = 1$.
4. We know that $1^2 = 1$, so $\sqrt{1} = 1$. We know $\sqrt{1} = 1$, so $\sqrt{\sqrt{1}} = \sqrt{1} = 1$. We know $\sqrt{\sqrt{1}} = 1$, so $\sqrt{\sqrt{\sqrt{1}}} = \sqrt{1} = 1$. We know $\sqrt{\sqrt{\sqrt{1}}} = 1$, so $\sqrt{\sqrt{\sqrt{\sqrt{1}}}} = \sqrt{1} = 1$.

Problem 15 Solving for Solutions 1

Evaluate each of the following square roots:

1. $\sqrt{36}$
2. $\sqrt{64}$
3. $\sqrt{100}$
4. $\sqrt{169}$

Solution for Problem 15.2

1. We know that $6^2 = 36$, so $\sqrt{36} = 6$.
2. We know that $8^2 = 64$, so $\sqrt{64} = 8$.
3. We know that $10^2 = 100$, so $\sqrt{100} = 10$.
4. We know that $13^2 = 169$, so $\sqrt{169} = 13$.

Problem 16 Solving for Solutions 1

Evaluate each of the following square roots:

1. $\sqrt{\frac{1}{4}}$
2. $\sqrt{\frac{9}{16}}$
3. $\sqrt{\frac{25}{36}}$
4. $\sqrt{\frac{100}{121}}$

Problem 17 Solving for Solutions 1

Find all x for $\sqrt{x^2} = -5$.

Problem 18 Solving for Solutions 1

Find all x for $\sqrt{x^2} = -5$.

Problem 19 Solving for Solutions 1

Find all x for $\sqrt{x^2} = 5$.

Problem 20 Solving for Solutions 1

Find all x for $\sqrt{x^2} = 5$.

Problem 21 Solving for Solutions 1

Find all x for $\sqrt{x^2} = x$.

Problem 22 Solving for Solutions 1

Find all x for $\sqrt{x^2} = x$.

Problem 23 Solving for Solutions 1

Find all x for $\sqrt{x^2} = x^2$.

Problem 24 Solving for Solutions 1

Find all x for $\sqrt{x^2} = x^2$.

Problem 25 Solving for Solutions 1

Find all x for $\sqrt{x^2} = -x$.

Problem 26 Solving for Solutions 1

Find all x for $\sqrt{x^2} = -x$.

Problem 27 Solving for Solutions 1

Find all x for $\sqrt{x^2} = -x^2$.

Problem 28 Solving for Solutions 1

Find all x for $\sqrt{x^2} = -x^2$.

Problem 29 Solving for Solutions 1

Find all x for $\sqrt{x^2} = 1 - x$.

Problem 30 Solving for Solutions 1

Find all x for $\sqrt{x^2} = 1 - x$.

Exercises

9.1.1. Solving for Solutions 1

Evaluate the following square roots, for extra challenge, try computing them without using anything.

1. $\sqrt{49}$

Your Solution(s) Hide Solution

Your Solution(s) Hide Solution

Your Solution(s) Hide Solution

9.2 Square Roots of Non-square Integers

In the previous section, we only worked with square roots of perfect squares. In this section, we investigate square roots of integers that are not perfect squares.

We'll start with $\sqrt{2}$. Does $\sqrt{2}$ even exist? Is there a number whose square is 2? We know that there isn't an integer that equals $\sqrt{2}$, since there is no integer whose square is 2. Is there a quotient of two integers (a fraction) whose square is 2? The ancient Greeks believed so for quite some time, and legend has it that the man who finally proved that no such quotient exists was drowned at sea for upsetting this belief. (You will have a chance to prove for yourself that no such quotient exists as a Challenge Problem. Don't worry, no one will drown you at sea if you succeed!)

So, there isn't an integer whose square is 2, and there isn't a fraction whose square is 2. But the number $\sqrt{2}$ does exist! It's a new kind of number that we call an **irrational number**.

> **Definition:** An **irrational number** is a number that cannot be expressed as the quotient of two integers.

We can't express $\sqrt{2}$ as an integer or a fraction, so we can't express it exactly as a decimal, either. However, we can approximate it using one important rule:

> **Important:** If a and b are nonnegative numbers such that $a > b$, then we have $\sqrt{a} > \sqrt{b}$. In other words, when comparing the square roots of two nonnegative numbers, the larger number has the larger square root. Similarly, if $\sqrt{a} > \sqrt{b}$, then $a > b$.

Since $1^2 = 1$ and $2^2 = 4$, we expect that the number whose square is 2 must be between 1 and 2. Our rule above tells us that this intuition is correct. In this section, we'll learn how to use this rule repeatedly to approximate $\sqrt{2}$ to as many decimal places as we like, but we will never be able to write a decimal expression that exactly equals $\sqrt{2}$. Moreover, a decimal approximation of $\sqrt{2}$ does not ever regularly repeat the way the repeating decimals we studied in Section 6.4 do.

Problems

Problem 9.8 — Jump to Solution

(a) Evaluate $\left(\sqrt{3}\right)^3$.

(b) Evaluate $\left(\sqrt{3}\right)^6$.

Problem 9.9 — Jump to Solution

(a) Explain why $\sqrt{2}$ must be less than 1.5.

(b) Estimate $\sqrt{2}$ to the nearest tenth.

Problem 9.10 — Jump to Solution

Which integers have a square root that is greater than 7 and less than 8?

Problem 9.11 — Jump to Solution

How many integers are between $\sqrt{13}$ and $\sqrt{131}$?

Problem 9.12 — Jump to Solution

Find the largest integer less than $\sqrt{80{,}999{,}999}$.

Problem 9.13 — Jump to Solution

Which is larger, $7 \cdot \sqrt{11}$ or $6 \cdot \sqrt{15}$?

Problem 9.8

(a) Evaluate $\left(\sqrt{3}\right)^3$.

(b) Evaluate $\left(\sqrt{3}\right)^6$.

Solution for Problem 9.8:

(a) By definition, $\sqrt{3}$ is the number whose square is 5. So, while we can't write a decimal or fraction that exactly equals $\sqrt{3}$, we do know that the square of $\sqrt{3}$ is simply 3.

(b) We know how to square $\sqrt{3}$ but we've seen the sixth power. Once again, our exponent laws help us. Taking the sixth power of a number is the same as first squaring the number and then cubing the result:

$$\left(\sqrt{3}\right)^6 = \left(\sqrt{3}\right)^{2\cdot 3} = \left[\left(\sqrt{3}\right)^2\right]^3 = (3)^3 = 512.$$

□

Problem 9.9

(a) Explain why $\sqrt{2}$ must be less than 1.5.

(b) Estimate $\sqrt{2}$ to the nearest tenth.

Solution for Problem 9.9:

(a) Since $1.5^2 = 2.25$, any number that is at least 1.5 has a square that is at least 2.25. This means that no number that is at least 1.5 can be the square root of 2. So, the square root of 2 must be less than 1.5.

(b) So far, we know that $\sqrt{2}$ is between 1 and 1.5. Since 1.5^2 is closer to 2 than 1^2 is, we think that $\sqrt{2}$ is probably closer to 1.5 than to 1. Therefore, we try squaring 1.4 next. We have $1.4^2 = 1.96$, so we need to square a slightly larger number than 1.4 to get 2. We find that $1.41^2 = 1.9881$ and $1.42^2 = 2.0164$, so $\sqrt{2}$ is between 1.41 and 1.42. Therefore, $\sqrt{2}$ rounded to the nearest tenth is 1.4.

□

In a similar method as we used to estimate $\sqrt{2}$, we can estimate the square root of any non-square number. We square numbers that we think are close to the desired square root. We then compare the resulting squares to the number whose square root we are trying to find. For example, in Problem 9.9, we found $1.4^2 = 1.96$ and $1.5^2 = 2.25$. Since 2 is between 1.96 and 2.25, we know that $\sqrt{2}$ is between 1.4 and 1.5.

Problem 9.10

Which integers have a square root that is greater than 7 and less than 8?

Solution for Problem 9.10: Since 7 is the square root of 49 and 8 is the square root of 64, the square root of any integer between 49 and 64 is between 7 and 8. We can visualize this on the number line:

$$\underset{4}{\underset{\bullet}{\sqrt{16}}} \quad \underset{5}{\underset{\bullet}{\sqrt{25}}} \quad \underset{6}{\underset{\bullet}{\sqrt{36}}} \quad \underset{7}{\underset{\bullet}{\sqrt{49}}} \quad \underset{8}{\underset{\bullet}{\sqrt{64}}} \quad \underset{9}{\underset{\bullet}{\sqrt{81}}} \quad \underset{10}{\underset{\bullet}{\sqrt{100}}}$$

So, the following integers have square roots greater than 7 and less than 8:

$$50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63.$$

□

Just as we can find square roots that fall between integers, we can find integers that fall between square roots.

Problem 9.11

How many integers are between $\sqrt{13}$ and $\sqrt{131}$?

Solution for Problem 9.11: First, we figure out which two consecutive integers $\sqrt{13}$ is between. To do so, we find the two consecutive perfect squares that 13 falls between. Since 13 is between 9 and 16, we know that $\sqrt{13}$ is between 3 and 4. Similarly, since 131 is between 121 (which is 11^2) and 144 (which is 12^2), we know that $\sqrt{131}$ is between 11 and 12. Again, we can visualize these relationships with a number line:

$$\underset{3}{\underset{\bullet}{\sqrt{9}}} \quad \underset{3.6}{\sqrt{13}} \quad \underset{4}{\underset{\bullet}{\sqrt{16}}} \quad \underset{5}{\underset{\bullet}{\sqrt{25}}} \quad \underset{6}{\underset{\bullet}{\sqrt{36}}} \quad \underset{7}{\underset{\bullet}{\sqrt{49}}} \quad \underset{8}{\underset{\bullet}{\sqrt{64}}} \quad \underset{9}{\underset{\bullet}{\sqrt{81}}} \quad \underset{10}{\underset{\bullet}{\sqrt{100}}} \quad \underset{11}{\underset{\bullet}{\sqrt{121}}} \quad \underset{11.9}{\sqrt{131}} \quad \underset{12}{\underset{\bullet}{\sqrt{144}}}$$

So, the integers between $\sqrt{13}$ and $\sqrt{131}$ are the integers from 4 up to 11. There are 8 such integers. □

Problem 9.12

Find the largest integer less than $\sqrt{80{,}999{,}999}$.

Solution for Problem 9.12: We start by trying to find an integer near $\sqrt{80{,}999{,}999}$. Since $81 = 9^2$, we start by considering $\sqrt{81{,}000{,}000}$. We're in luck, since this is an integer:

$$\begin{aligned}\sqrt{81{,}000{,}000} &= \sqrt{81 \cdot 1{,}000{,}000}\\ &= \sqrt{9^2 \cdot 1000^2}\\ &= \sqrt{(9 \cdot 1000)^2}\\ &= 9000.\end{aligned}$$

So, $\sqrt{80{,}999{,}999}$ is less than 9000. The next perfect square smaller than 9000^2 is 8999^2. Rather than computing 8999^2 to compare it to 80,999,999, we remember what we learned about consecutive squares in Section 2.1 here. Since $9000^2 = 81{,}000{,}000$, we have

$$\begin{aligned}8999^2 &= 9000^2 - 9000 - 8999\\ &= 81{,}000{,}000 - 9000 - 8999\\ &= 80{,}991{,}000 - 8999,\end{aligned}$$

which is definitely less than 80,999,999. So, the largest integer less than $\sqrt{80{,}999{,}999}$ is 8999. □

Problem 9.13

Which is larger, $7 \cdot \sqrt{11}$ or $6 \cdot \sqrt{15}$?

Solution for Problem 9.13: We typically write the product of a number and a square root without using a multiplication symbol. So, we write $2\sqrt{3}$ to refer to the product of 2 and $\sqrt{3}$, and in this problem we are asked to compare $7\sqrt{11}$ and $6\sqrt{15}$.

We know how to compare two square roots. The order of two square roots is the same as the order of the squares of the two square roots. For example, we know $\sqrt{11} > \sqrt{10}$ because $11 > 10$. Similarly, the order of any two nonnegative numbers is the same as the order of their squares. Maybe comparing squares will work on this problem, too.

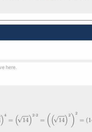
> **Concept:** When faced with a new problem, try strategies that you have used to solve similar problems.

Applying the exponent law $(ab)^2 = a^2b^2$, we have

$$\begin{aligned}\left(7\sqrt{11}\right)^2 &= (7)^2\left(\sqrt{11}\right)^2 = (49)(11) = 539,\\ \left(6\sqrt{15}\right)^2 &= (6)^2\left(\sqrt{15}\right)^2 = (36)(15) = 540.\end{aligned}$$

These two results are easy to compare. Since $6\sqrt{15}$ has a larger square than $7\sqrt{11}$ (and both numbers are nonnegative), we know that $6\sqrt{15}$ is larger than $7\sqrt{11}$. □

> **Concept:** We can often compare expressions involving square roots by comparing the squares of the expressions.



Non-Integer Square Roots

Exercises

9.2.1:

Round each of the following square roots to the nearest integer.

(a) $\sqrt{28}$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have $5^2 = 64$ and $9^2 = 81$, so $\sqrt{28}$ is between 8 and 9. Since $8.5^2 = 72.25$, we know that $8.5 < \sqrt{28} < 9$, so the nearest integer to $\sqrt{28}$ is $\boxed{9}$.

(b) $\sqrt{200}$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have $14^2 = 196$ and $15^2 = 225$, so $14 < \sqrt{200} < 15$, and we expect that $\sqrt{200}$ is closer to 14 than to 15. Since $14.5^2 = 210.25$, we have $14 < \sqrt{200} < 14.5$, so the closest integer to $\sqrt{200}$ is $\boxed{14}$.

(c) $\sqrt{4004}$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $60^2 = 3600$, we know that $\sqrt{4004}$ is greater than 60. Computing squares of numbers slightly larger than 60, we find that $62^2 = 3969$ and $64^2 = 4096$, so $63 < \sqrt{4004} < 64$. Since $63.5^2 = 4032.25$, we have $63 < \sqrt{4004} < 63.5$, and the integer closest to $\sqrt{4004}$ is $\boxed{63}$.

9.2.2:

How many integers are between $\sqrt{7}$ and $\sqrt{220}$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have $2^2 < 7 < 3^2$, so $2 < \sqrt{7} < 3$. We also have $14^2 < 220 < 15^2$, so $14 < \sqrt{220} < 15$. Therefore, the integers between $\sqrt{7}$ and $\sqrt{220}$ are 3, 4, 5, 6, . . . , 14. There are $\boxed{12}$ numbers in this list.

9.2.3:

What is the largest integer that is less than $\sqrt{83} - \sqrt{35}$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $9^2 = 81$, we know that $\sqrt{83}$ is just a little bit more than 9. Since $6^2 = 36$, we know that $\sqrt{35}$ is just a little bit less than 6. When we subtract a number that is a little less than 6 from a number that is a little more than 9, we get a number that is a lot more than $\boxed{3}$. To make sure that $\sqrt{83} - \sqrt{35}$ isn't greater than 4, we can estimate both $\sqrt{83}$ and $\sqrt{35}$ more closely. We find that $9.1 < \sqrt{83} < 9.2$ and $5.9 < \sqrt{35} < 6$. Since $\sqrt{83}$ is less than 9.2 and $\sqrt{35}$ is greater than 5.9, their difference is less than $9.2 - 5.9 = 3.3$.

9.2.4:

Compute $\left(\sqrt{14}\right)^4$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution:

$$\left(\sqrt{14}\right)^4 = \left(\sqrt{14}\right)^{2\cdot 2} = \left(\left(\sqrt{14}\right)^2\right)^2 = (14)^2 = \boxed{196}.$$

9.2.5:

What integer is closest to $4\sqrt{5}$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution 1: Approximate $\sqrt{5}$. We approximate $\sqrt{5}$ and then multiply the result by 4. We have $2.2^2 = 4.84$ and $2.3^2 = 5.29$, so $2.2 < \sqrt{5} < 2.3$. Multiplying by 4, we have $8.8 < 4\sqrt{5} < 9.2$. Since $4\sqrt{5}$ is between 8.8 and 9.2, the integer closest to $4\sqrt{5}$ is $\boxed{9}$.

Solution 2: Square $4\sqrt{5}$. We have $\left(4\sqrt{5}\right)^2 = 4^2\left(\sqrt{5}\right)^2 = 16(5) = 80$. Since $8^2 < \left(4\sqrt{5}\right)^2 < 9^2$, we know that $4\sqrt{5}$ is between 8 and 9. Furthermore, we have $8.5^2 = 72.25$, so $4\sqrt{5}$ is between 8.5 and 9. Therefore, the integer closest to $4\sqrt{5}$ is $\boxed{9}$.

9.2.6:

How many digits are in the square root of the perfect square 108,868,356?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have $10{,}000^2 = 100{,}000{,}000$ and $100{,}000^2 = 10{,}000{,}000{,}000$, so the square root of 108,868,356 is between 10,000 and 100,000. Therefore, it has $\boxed{5}$ digits. (For the record, $\sqrt{108{,}868{,}356} = 10{,}434$.)

9.2.7: Source: MATHCOUNTS

Between what two consecutive integers on the number line is the sum $\sqrt{30} + \sqrt{50}$ located?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $5^2 < 30 < 6^2$, we have $5 < \sqrt{30} < 6$. Since $7^2 < 50 < 8^2$, we have $7 < \sqrt{50} < 8$. Combining these, we know that $12 < \sqrt{30} + \sqrt{50} < 14$. But is $\sqrt{30} + \sqrt{50}$ less than or greater than 13? Since $\sqrt{50}$ is very close to 7 (because $7^2 = 49$ and $\sqrt{50}$ is between 9 and 8, we expect that $\sqrt{30} + \sqrt{50}$ is less than 13. To make sure, we notice that $5.5^2 = 30.25$, so $\sqrt{30} < 5.5$, and $7.1^2 = 50.41$, so $\sqrt{50} < 7.1$. Therefore, $\sqrt{30} + \sqrt{50} < 5.5 + 7.1 = 12.6$. So, $\sqrt{30} + \sqrt{50}$ is between $\boxed{12 \text{ and } 13}$.

9.2.8: Source: MATHCOUNTS

Which of the following numbers is largest: $\sqrt{75}$, $\frac{75}{5}$, or 50% of $\frac{68}{4}$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We have $\frac{68}{4} = 17$. Next, dividing 75 into 75 gives 8.3. (We might also note $\frac{75}{9} = \frac{25}{3} = 8\frac{1}{3}$.) Finally, we note that $8.5^2 = 72.25$, so $8.5 < \sqrt{75}$. Therefore, $\boxed{\sqrt{75}}$ is the largest of the three numbers.

9.3 Arithmetic with Square Roots

Exercises marked with a number in brackets (like this) are in the current review exercises.

Problem 1 Show Solution

Simplify:

(a) $\sqrt{\text{Two and twenty}}$ $\sqrt{x+3}$ $\sqrt{x^2}$

(b) $\sqrt{\text{Twenty-five}}$ $\sqrt{x^2+6x+9}$

(c) $\sqrt{\text{Twenty-five and twenty-one}}$ $\sqrt{x^2+6x+9+21}$ (Hint: x^2+6x+9 is a perfect square.)

Problem 2 Show Solution

Compute and simplify the following:

(a) $\sqrt{\text{Twenty-five}}$

(b) $\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}$

(c) $\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}$

Problem 3 Show Solution

Compute and simplify the following:

(a) $\sqrt{\frac{\text{Twenty-five}}{\text{Twenty-five}}}$

(b) $\sqrt{\frac{\text{Twenty-five}}{\text{Twenty-five}}}\sqrt{\text{Twenty-five}}$

(c) $\sqrt{\frac{\text{Twenty-five}}{\text{Twenty-five}}}\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}$

Problem 4 Show Solution

(a) $\sqrt{\text{Twenty-five}}$

(b) $\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}$

(c) $\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}$

Problem 5 Show Solution

Simplify $\sqrt{\text{Twenty-five}}$.

Problem 6 Show Solution

Simplify $\sqrt{\text{Twenty-five}}$.

Problem 7 Show Solution

(a) $\sqrt{\text{Twenty-five}}$

(b) $\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}$

(c) $\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}$

Problem 8 Show Solution

Simplify $\sqrt{\text{Twenty-five}}$.

Problem 9 Show Solution

Simplify $\sqrt{\text{Twenty-five}}$.

Problem 10 Show Solution

Simplify $\sqrt{\text{Twenty-five}}$.

Problem 11 Show Solution

(a) $\sqrt{\text{Twenty-five}}$

(b) $\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}$

(c) $\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}$

Problem 12 Show Solution

Simplify $\sqrt{\text{Twenty-five}}$.

Problem 13 Show Solution

Simplify $\sqrt{\text{Twenty-five}}$.

Problem 14 Show Solution

Simplify $\sqrt{\text{Twenty-five}}$.

Problem 15 Show Solution

(a) $\sqrt{\text{Twenty-five}}$

(b) $\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}$

(c) $\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}\sqrt{\text{Twenty-five}}$

Problem 16 Show Solution

Simplify $\sqrt{\text{Twenty-five}}$.

**Problem

9.4 Summary

Definition:	The **square root** of a nonnegative number n is the nonnegative number whose square is n. We express the square root as $\sqrt{n}$, where the $\sqrt{\ }$ symbol is called a **radical**.

WARNING!! ⊘	The square root of a nonnegative number is a nonnegative number by definition. For example, we have $\sqrt{16} = 4$. We *never* say that $\sqrt{16}$ is -4.

Square roots satisfy the following properties for all nonnegative numbers a and b:

- $\left(\sqrt{a}\right)^2 = a$
- $\sqrt{a^2} = a$
- $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$
- If $b \neq 0$, then $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$.
- If $a < b$, then $\sqrt{a} < \sqrt{b}$.
- If $\sqrt{a} > \sqrt{b}$, then $a > b$.

WARNING!! ⊘	If a and b are positive, then $\sqrt{a} + \sqrt{b}$ is **NEVER** equal to $\sqrt{a+b}$.

WARNING!! ⊘	If a and b are positive, then the value of $\sqrt{a^2 + b^2}$ is **NEVER** equal to $a + b$.

Challenge Problems

5.01 ■ 1

(a) For what positive numbers a and b is $\sqrt{a^2+b^2} = a+b$?

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: Square both sides of the equation, and we know that it is the square root of a^2, we have

$$a^2 + 2ab + b^2 = a^2 + b^2$$

(a) For what positive numbers a and b is $\sqrt{a^2+b^2} = a+b$?

5.01 ■ 2

Find x if $\sqrt{x} = \frac{1}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: Square both sides of the equation, we have $x = \frac{1}{4}$.

5.01 ■ 3

Find x if $\sqrt{x^2} = 3$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: Square both sides of the equation, we have $x^2 = 9$. Therefore $x = \pm 3$.

5.01 ■ 4

Find x if $\sqrt{x^2} = 3$ and x is a positive integer.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: Since $\sqrt{x^2} = 3$, we have $x = \pm 3$. Since x is a positive integer, $x = 3$.

5.01 ■ 5 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}}$ and $\sqrt{\frac{1}{3}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ and $\sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.

(a) Find $\sqrt{\frac{1}{2}}$ and $\sqrt{\frac{1}{3}}$.

5.01 ■ 6

Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$.

5.01 ■ 7

Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ to three decimal places.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\frac{3\sqrt{2}+2\sqrt{3}}{6} \approx 0.9471$ correct to three decimal places.

5.01 ■ 8 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ correct to three decimal places.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: The geometric mean of these two is equal to $\sqrt{\frac{1}{2} \cdot \frac{1}{3}} = \frac{1}{\sqrt{6}}$.

(a) Is it possible for the geometric mean of these two numbers to be an integer?

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\frac{1}{\sqrt{6}} \approx 0.4$, so it is not an integer.

5.01 ■ 9 Review: MATH2040F 15

(a) Find the arithmetic mean of $\sqrt{\frac{1}{2}}$ and $\sqrt{\frac{1}{3}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: The arithmetic mean of these two is equal to $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

(a) Is it possible for the arithmetic mean of these two numbers to be an integer?

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\frac{3\sqrt{2}+2\sqrt{3}}{12} \approx 0.4$, so it is not an integer.

5.01 ■ 10 Review: MATH2040F 15

(a) Find the harmonic mean of $\sqrt{\frac{1}{2}}$ and $\sqrt{\frac{1}{3}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: Since $\frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2}$ and $\frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{3}$, the harmonic mean is equal to $\frac{2}{\sqrt{2}+\sqrt{3}} = \sqrt{2}-\sqrt{3}$.

(a) Is it possible for the harmonic mean of these two numbers to be an integer?

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{2}-\sqrt{3} \approx -0.3$, so it is not an integer.

5.01 ■ 11 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\sqrt{\frac{1}{2} \cdot \frac{1}{3}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\sqrt{\frac{1}{2} \cdot \frac{1}{3}} = \frac{1}{\sqrt{6}}$.

(a) Find $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 12 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 13

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 14

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 15 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 16

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 17

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 18 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 19

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 20

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 21 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 22

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 23

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 24 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 25

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 26

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 27 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 28

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 29

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 30 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 31

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 32

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 33 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 34

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 35

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 36 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 37

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 38

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 39 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 40

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 41

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 42 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 43

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 44

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 45 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 46

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 47

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 48 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 49

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 50

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 51 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 52

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 53

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 54 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 55

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 56

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 57 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 58

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 59

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 60 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2}}} + \frac{1}{\sqrt{\frac{1}{3}}} = \sqrt{2}-\sqrt{3}$.

5.01 ■ 61

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{1}{\sqrt{\frac{1}{2} \cdot \frac{1}{3}}} = \sqrt{6}$.

5.01 ■ 62

(a) Find $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2}$.

Choose Solution

Hide Solution

Hide Answer

Your Solution

Solution: $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} = \frac{3\sqrt{2}+2\sqrt{3}}{6}$ and $\frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{3}}}{2} = \frac{3\sqrt{2}+2\sqrt{3}}{12}$.

5.01 ■ 63 Review: MATH2040F 15

(a) Find $\sqrt{\frac{1}{2}}

Review Problems

5.2.6.

Evaluate each of the following:

(a) $\sqrt{\sqrt{\sqrt{1}}}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt{\sqrt{\sqrt{1}}} = \sqrt{\sqrt{1^{1/2}}} = \sqrt{1^{1/4}} = 1^{1/8} = \boxed{1}$$

(b) $\sqrt{3} \cdot \sqrt{3} \cdot \sqrt{3}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt{3} \cdot \sqrt{3} \cdot \sqrt{3} = 3^{1/2} \cdot 3^{1/2} \cdot 3^{1/2} = 3^{3/2} = \boxed{3\sqrt{3}}$$

(c) $\sqrt{7} \cdot \sqrt{3} \cdot \sqrt{7}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt{7} \cdot \sqrt{3} \cdot \sqrt{7} = 7^{1/2} \cdot 3^{1/2} \cdot 7^{1/2} = 7^{1/2} \cdot 7^{1/2} \cdot 3^{1/2} = 7^{1/2+1/2} \cdot 3^{1/2} = 7^1 \cdot 3^{1/2} = \boxed{7\sqrt{3}}$$

(d) $\sqrt{3} \cdot \sqrt{6}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt{3} \cdot \sqrt{6} = \sqrt{3 \cdot 6} = \sqrt{3 \cdot 3 \cdot 2} = \sqrt{3^2 \cdot 2} = \boxed{3\sqrt{2}}$$

(e) $\sqrt{3} \cdot \sqrt{5}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt{3} \cdot \sqrt{5} = \sqrt{3 \cdot 5} = \sqrt{15} = \boxed{\sqrt{15}}$$

(f) $\sqrt{3} \cdot \sqrt{3} \cdot \sqrt{3}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt{3} \cdot \sqrt{3} \cdot \sqrt{3} = 3^{1/2} \cdot 3^{1/2} \cdot 3^{1/2} = 3^{1/2+1/2+1/2} = 3^{3/2} = \boxed{3\sqrt{3}}$$

No, you could have combined the three original numbers under a single radical in our first step:

$$\sqrt{3} \cdot \sqrt{3} \cdot \sqrt{3} = \sqrt{3 \cdot 3 \cdot 3} = \sqrt{3 \cdot 3 \cdot 3 \cdot 1} = \sqrt{3^2 \cdot 3 \cdot 1} = \boxed{3\sqrt{3}}$$

Furthermore, we could have noticed that $3 = 3 \cdot 1$. Or, taken the computation easier:

$$\sqrt{3} \cdot \sqrt{3} \cdot \sqrt{3} = \sqrt{3 \cdot 3} \cdot \sqrt{3} = \sqrt{3 \cdot 3 \cdot 1} \cdot \sqrt{3} = \sqrt{3 \cdot 3 \cdot 1 \cdot 3} = \sqrt{3^2 \cdot 3} = \boxed{3\sqrt{3}}$$

(g) $\sqrt[3]{\frac{1}{2}}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt[3]{\frac{1}{2}} = \sqrt[3]{\frac{1 \cdot 2^2}{2 \cdot 2^2}} = \sqrt[3]{\frac{2^2}{2^3}} = \boxed{\frac{\sqrt[3]{2^2}}{2}}$$

(h) $\sqrt[3]{\frac{3}{2}}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt[3]{\frac{3}{2}} = \sqrt[3]{\frac{3 \cdot 2^2}{2 \cdot 2^2}} = \sqrt[3]{\frac{3 \cdot 2^2}{2^3}} = \boxed{\frac{\sqrt[3]{3 \cdot 2^2}}{2}}$$

(i) $\sqrt{36}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt{36} = \sqrt{6 \cdot 6} = \sqrt{6^2} = \boxed{6}$$

(j) $\frac{\sqrt{3}}{\sqrt{2}}$, $\frac{\sqrt{6}}{2}$, $\frac{\sqrt{3}}{\sqrt{2}}$, $\frac{\sqrt{6}}{2}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{3 \cdot 2}}{\sqrt{2^2}} = \boxed{\frac{\sqrt{6}}{2}}$$

$$\frac{\sqrt{6}}{2} = \boxed{\frac{\sqrt{6}}{2}}$$

$$\frac{\sqrt{3}}{\sqrt{2}} = \boxed{\frac{\sqrt{6}}{2}}$$

$$\frac{\sqrt{6}}{2} = \boxed{\frac{\sqrt{6}}{2}}$$

(k) $\sqrt{\sqrt{3}}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt{\sqrt{3}} = \sqrt{3^{1/2}} = \boxed{3^{1/4}}$$

(l) $\sqrt{3^2 \cdot 2^3 \cdot 5^5}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt{3^2 \cdot 2^3 \cdot 5^5} = \sqrt{3^2 \cdot 2^2 \cdot 2 \cdot 5^4 \cdot 5} = \boxed{3 \cdot 2 \cdot 5^2 \sqrt{2 \cdot 5}}$$

(m) $\sqrt{3^2 \cdot 2^3 \cdot 5^5 \cdot 3^3 \cdot 2^5 \cdot 5^7}$

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt{3^2 \cdot 2^3 \cdot 5^5 \cdot 3^3 \cdot 2^5 \cdot 5^7} = \sqrt{3^2 \cdot 3^3 \cdot 2^3 \cdot 2^5 \cdot 5^5 \cdot 5^7} = \sqrt{3^{2+3} \cdot 2^{3+5} \cdot 5^{5+7}} = \sqrt{3^5 \cdot 2^8 \cdot 5^{12}} = \boxed{3^2 \cdot 2^4 \cdot 5^6 \sqrt{3 \cdot 5}}$$

5.2.8. Source: MATHCONTEST. ✎ ✓ ?

(a) Is the value of the expression $\sqrt{3^{-2}} \cdot 3^2$ rational or irrational?

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution:

$$\sqrt{3^{-2}} \cdot 3^2 = \sqrt{3^{-2}} \cdot 3^2 = \sqrt{3^{-2} \cdot 3^2} = \boxed{1}$$

5.2.9. Source: MATHCONTEST. ✎ ✓ ?

Simplify $\sqrt{\frac{3}{2}}$ or $\sqrt{1.5}$.

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution: First, we evaluate $\sqrt{1.5}$ by finding his prime factorization of 1.5 lb. He has four 1.5 lb. $\sqrt{1.5} = 3^{1/2} \cdot 5^{1/2}$, so

$$\sqrt{1.5} = 3^{1/2} \cdot 5^{1/2} = \sqrt{3^1 \cdot 5^1} = \sqrt{3 \cdot 5} = \boxed{\sqrt{15}}$$

Do we have

$$\sqrt{3} \cdot \sqrt{1.5} = \sqrt{3} \cdot \sqrt{3 \cdot 5} = \sqrt{3 \cdot 3 \cdot 5} = \boxed{3\sqrt{5}}$$

5.2.10. Source: MATHCONTEST. ✎ ✓ ?

Which perfect cubes less than 100 have square roots that are integers?

Hide Solution | Hide Hint

Your Hint:

Your Solution:

Your Evaluation: Correct | Incorrect

Solution: The nonnegative perfect cubes less than 100 are $0^3 = 0$, $1^3 = 1$, $2^3 = 8$, $3^3 = 27$, and $4^3 = 64$. One (last bullet) $\sqrt{0} = 0$, $\sqrt{1} = 1$, $\sqrt{8} = 2\sqrt{2}$, $\sqrt{27} = 3\sqrt{3}$, and $\sqrt{64} = 8$. Only 0, 1, and 64 have integer square roots. These three numbers are perfect cubes (powers, in fact, of 2). Thus, the three

10 Angles

Angles are formed when two lines, or rays, or line segments meet at a common point. In this chapter we will learn about **angle** measurement, the different types of angles, and the properties of angles formed when two lines intersect.

Angles are important because they are used in many fields such as geometry, physics and **engineering**. Well-trained engineers need to know the properties of angles in order to design and construct buildings, machines, and other structures that are safe and efficient.

10.1 Measuring Angles

Angles are measured in units called **degrees**. The symbol for a degree is a small circle, written like this: °. There are 360 degrees in a full turn, and 90 degrees in a quarter turn, or right angle.

Figure 10.1: 1 turn

Figure 10.2: 1/4 turn

In order to tell how much turn there is between two lines, we imagine a circle with its centre at the point where the two lines meet. There could be no, one or two arcs, as shown below:

Figure 10.3: 0 turn

Figure 10.4: 1 turn

Figure 10.5: 2 turns

We refer to the turn as an angle. If we have an angle $\angle BAC$ we can imagine a ray starting at A and rotating from AB to AC:

Figure 10.6: 0 turn

Figure 10.7: 1/4 turn

Figure 10.8: 1 turn

Figure 10.9: 2 turns

Notice that when we measure the angle, $\angle BAC$ is the same regardless of the length of the arms AB and AC. We can imagine that the arms are infinitely long when we measure the angle.

Figure 10.10: Measuring an Angle

Figure 10.10 shows that we can continue to rotate the ray AC. We define the position in the result so that the vertex of the angle is at A and the arm AB is in the position of the original ray. We can continue to rotate the ray AC until it is back in its original position. We can continue to rotate the ray AC until it is again back in its original position, and we can continue to rotate the ray AC as many times as we want. We can define $\angle BAC$ to be the amount that the ray AC has rotated.

Problem

Find the measure of $\angle ABC$, $\angle PQR$, $\angle STUV$, and $\angle WXYZ$ (the results are symbolic for the problem):

Solution for Problem 10.1 ▶ Hide Solution

Problem 10.2

Find the measure of $\angle ABC$, $\angle PQR$, $\angle STUV$, and $\angle WXYZ$ (the results are symbolic for the problem):

Solution for Problem 10.2 ▶ Hide Solution

Problem 10.3

Find the size of the angle between vertical and horizontal:

Solution for Problem 10.3 ▶ Hide Solution

Problem 10.4

Find the size of the angle between vertical and horizontal:

Solution for Problem 10.4 ▶ Hide Solution

Problem 10.5

Find the size of the angle between vertical and horizontal:

Solution for Problem 10.5 ▶ Hide Solution

Problem 10.6

Find the size of the angle between vertical and horizontal:

Solution for Problem 10.6 ▶ Hide Solution

Problem 10.7

Find the size of the angle between vertical and horizontal:

Solution for Problem 10.7 ▶ Hide Solution

Problem 10.8

Find the size of the angle between vertical and horizontal:

Solution for Problem 10.8 ▶ Hide Solution

Figure 10.11 shows that we can rotate a ray in two ways. We can rotate the ray in the normal way with the vertex at the centre of the circle and the arm AB in the position of the positive x-axis. We can also rotate the ray in the opposite way. We define the positive direction to be the direction that we rotate the ray AB to get to the ray AC. We define the negative direction to be the other direction.

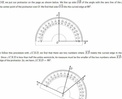
Figure 10.11: Positive and Negative Angles

Figure 10.12 shows that we can rotate the ray AC in the positive direction to the ray AB and then we can rotate the ray AC in the negative direction to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AB to get to the ray AC. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB and then we rotate the ray AB back to its original position.

Figure 10.12: Positive and Negative Angles

Problem

Find the size of $\angle ABC$ and $\angle PQR$:



Solution for Problem 10.9 ▶ Hide Solution

Problem 10.10

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.10 ▶ Hide Solution

Problem 10.11

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.11 ▶ Hide Solution

Problem 10.12

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.12 ▶ Hide Solution

When we follow the procedure with $\angle CAB$, we find that there are two numbers where $\overrightarrow{AB}$ lies on the circle in the normal way with the positive direction. We can define $\angle CAB$ to be the smaller of the two numbers. We can define $\angle CAB$ to be the number of the other number multiplied by -1.

Figure 10.13: $\angle CAB = -\angle BAC$

We can also use the diagram above to find the measure of $\angle CAB$. Once again, we single out a point on the normal circle with the positive direction. We can define $\angle CAB$ to be the number of the other number multiplied by -1. We can define $\angle CAB$ to be the number of $\angle BAC$ minus $\angle CAD$. We can also use the placement of the protractor to measure $\angle CAD$. Now $\overrightarrow{AD}$ lies on the normal ray of the protractor. We can define $\angle CAB$ to be the number of the other number multiplied by -1. We can define $\angle CAB$ to be $\angle CAD - \angle BAD$.

Figure 10.14: $\angle CAB = \angle CAD - \angle BAD$

Figure 10.15 shows $\angle PQR$. We can find the complement angle of $\angle PQR$ by drawing a line perpendicular to $\overline{QR}$ at Q. We can find the supplement angle of $\angle PQR$ by drawing a line perpendicular to $\overline{QR}$ at R. We can find the supplement angle of $\angle PQR$ by drawing a line perpendicular to $\overline{QR}$ at Q and then rotate the ray QR in the positive direction to the ray QP. We can find the supplement angle of $\angle PQR$ by drawing a line perpendicular to $\overline{QR}$ at R and then rotate the ray RQ in the positive direction to the ray RP.

Figure 10.15: Complement and Supplement of $\angle PQR$

Problem

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.13 ▶ Hide Solution

Problem 10.14

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.14 ▶ Hide Solution

Problem 10.15

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.15 ▶ Hide Solution

Problem 10.16

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.16 ▶ Hide Solution

Problem 10.17

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.17 ▶ Hide Solution

Problem 10.18

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.18 ▶ Hide Solution

Figure 10.16 shows that we can rotate a ray in two ways. We can rotate the ray AB in the normal way with the vertex at the centre of the circle and the arm AB in the position of the positive x-axis. We can also rotate the ray in the opposite way. We define the positive direction to be the direction that we rotate the ray AB to get to the ray AC. We define the negative direction to be the other direction.

Figure 10.16: Positive and Negative Angles

Figure 10.17 shows that we can rotate the ray AC in the positive direction to the ray AB and then we can rotate the ray AC in the negative direction to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AB to get to the ray AC. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB and then we rotate the ray AB back to its original position.

Figure 10.17: Positive and Negative Angles

Problem

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.19 ▶ Hide Solution

Problem 10.20

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.20 ▶ Hide Solution

Problem 10.21

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.21 ▶ Hide Solution

Problem 10.22

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.22 ▶ Hide Solution

We can use the protractor to find the angle between two lines. We can place the protractor so that the centre of the protractor is at the point where the two lines meet and the arm AB is in the position of the positive x-axis. We can find the angle $\angle BAC$ by reading the number where the line AC meets the protractor. We can also use the protractor to find the angle $\angle CAB$. We can find the angle $\angle CAB$ by reading the number where the line AC meets the protractor and then we multiply the number by -1. We can find the angle $\angle CAB$ by reading the number where the line AC meets the protractor and then we subtract the number of the line AB from the number of the line AC.

Figure 10.18: Using a Protractor

Figure 10.19: Using a Protractor

Problem

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.23 ▶ Hide Solution

Problem 10.24

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.24 ▶ Hide Solution

Problem 10.25

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.25 ▶ Hide Solution

Problem 10.26

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.26 ▶ Hide Solution

Figure 10.20 shows that we can rotate a ray in two ways. We can rotate the ray AB in the normal way with the vertex at the centre of the circle and the arm AB in the position of the positive x-axis. We can also rotate the ray in the opposite way. We define the positive direction to be the direction that we rotate the ray AB to get to the ray AC. We define the negative direction to be the other direction.

Figure 10.20: Positive and Negative Angles

Figure 10.21 shows that we can rotate the ray AC in the positive direction to the ray AB and then we can rotate the ray AC in the negative direction to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AB to get to the ray AC. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB and then we rotate the ray AB back to its original position.

Figure 10.21: Positive and Negative Angles

Problem

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.27 ▶ Hide Solution

Problem 10.28

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.28 ▶ Hide Solution

Problem 10.29

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.29 ▶ Hide Solution

Problem 10.30

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.30 ▶ Hide Solution

Figure 10.22 shows that we can rotate a ray in two ways. We can rotate the ray AB in the normal way with the vertex at the centre of the circle and the arm AB in the position of the positive x-axis. We can also rotate the ray in the opposite way. We define the positive direction to be the direction that we rotate the ray AB to get to the ray AC. We define the negative direction to be the other direction.

Figure 10.22: Positive and Negative Angles

Figure 10.23 shows that we can rotate the ray AC in the positive direction to the ray AB and then we can rotate the ray AC in the negative direction to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AB to get to the ray AC. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB and then we rotate the ray AB back to its original position.

Figure 10.23: Positive and Negative Angles

Problem

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.31 ▶ Hide Solution

Problem 10.32

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.32 ▶ Hide Solution

Problem 10.33

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.33 ▶ Hide Solution

Problem 10.34

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.34 ▶ Hide Solution

Figure 10.24 shows that we can rotate a ray in two ways. We can rotate the ray AB in the normal way with the vertex at the centre of the circle and the arm AB in the position of the positive x-axis. We can also rotate the ray in the opposite way. We define the positive direction to be the direction that we rotate the ray AB to get to the ray AC. We define the negative direction to be the other direction.

Figure 10.24: Positive and Negative Angles

Figure 10.25 shows that we can rotate the ray AC in the positive direction to the ray AB and then we can rotate the ray AC in the negative direction to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AB to get to the ray AC. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB and then we rotate the ray AB back to its original position.

Figure 10.25: Positive and Negative Angles

Problem

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.35 ▶ Hide Solution

Problem 10.36

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.36 ▶ Hide Solution

Problem 10.37

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.37 ▶ Hide Solution

Problem 10.38

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.38 ▶ Hide Solution

Problem 10.39

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.39 ▶ Hide Solution

Problem 10.40

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.40 ▶ Hide Solution

10.2 Types of Angles

Angles are classified according to their size. We have seen that there are 360 degrees in a full turn, 90 degrees in a quarter turn, or right angle. We define an **acute angle** to be an angle that is less than a right angle. We define an **obtuse angle** to be an angle that is more than a right angle but less than a half turn. We define a **reflex angle** to be an angle that is more than a half turn but less than a full turn.

Figure 10.26: Acute Angle

Figure 10.27: Obtuse Angle

Figure 10.28: Reflex Angle

Problem

Find the size of $\angle ABC$, $\angle PQR$, $\angle STUV$, and $\angle WXYZ$:

Solution for Problem 10.41 ▶ Hide Solution

Problem 10.42

Find the size of $\angle ABC$, $\angle PQR$, $\angle STUV$, and $\angle WXYZ$:

Solution for Problem 10.42 ▶ Hide Solution

Problem 10.43

Find the size of $\angle ABC$, $\angle PQR$, $\angle STUV$, and $\angle WXYZ$:

Solution for Problem 10.43 ▶ Hide Solution

Problem 10.44

Find the size of $\angle ABC$, $\angle PQR$, $\angle STUV$, and $\angle WXYZ$:

Solution for Problem 10.44 ▶ Hide Solution

10.3 Complementary and Supplementary Angles

Two angles are **complementary** if the sum of the angles is a right angle. Two angles are **supplementary** if the sum of the angles is a straight angle. Two angles are **complimentary** if they are very nice angles.

Problem

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.45 ▶ Hide Solution

Problem 10.46

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.46 ▶ Hide Solution

Problem 10.47

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.47 ▶ Hide Solution

Problem 10.48

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.48 ▶ Hide Solution

Problem 10.49

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.49 ▶ Hide Solution

Problem 10.50

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.50 ▶ Hide Solution

10.4 Vertical Angles

When two lines intersect, the angles opposite each other are called **vertical angles**. Figure 10.29 shows that $\angle AOB$ and $\angle COD$ are vertical angles. Figure 10.29 shows that $\angle AOC$ and $\angle BOD$ are vertical angles.

Figure 10.29: Vertical Angles

Problem

Find the size of $\angle AOB$, $\angle COD$, $\angle AOC$, and $\angle BOD$:

Solution for Problem 10.51 ▶ Hide Solution

Problem 10.52

Find the size of $\angle AOB$, $\angle COD$, $\angle AOC$, and $\angle BOD$:

Solution for Problem 10.52 ▶ Hide Solution

Problem 10.53

Find the size of $\angle AOB$, $\angle COD$, $\angle AOC$, and $\angle BOD$:

Solution for Problem 10.53 ▶ Hide Solution

Problem 10.54

Find the size of $\angle AOB$, $\angle COD$, $\angle AOC$, and $\angle BOD$:

Solution for Problem 10.54 ▶ Hide Solution

10.5 Angles Formed by Parallel Lines

When a line cuts two parallel lines, the angles on the same side of the line are called **corresponding angles**. Figure 10.30 shows that $\angle ABC$ and $\angle DEF$ are corresponding angles. Figure 10.30 shows that $\angle ABC$ and $\angle GHI$ are corresponding angles.

Figure 10.30: Corresponding Angles

Problem

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.55 ▶ Hide Solution

Problem 10.56

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.56 ▶ Hide Solution

Problem 10.57

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.57 ▶ Hide Solution

Problem 10.58

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.58 ▶ Hide Solution

When a line cuts two parallel lines, the angles on the opposite sides of the line are called **alternate angles**. Figure 10.31 shows that $\angle ABC$ and $\angle DEF$ are alternate angles. Figure 10.31 shows that $\angle ABC$ and $\angle GHI$ are alternate angles.

Figure 10.31: Alternate Angles

Problem

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.59 ▶ Hide Solution

Problem 10.60

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.60 ▶ Hide Solution

Problem 10.61

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.61 ▶ Hide Solution

Problem 10.62

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.62 ▶ Hide Solution

When a line cuts two parallel lines, the angles on the same side of the line are called **interior angles**. Figure 10.32 shows that $\angle ABC$ and $\angle DEF$ are interior angles. Figure 10.32 shows that $\angle ABC$ and $\angle GHI$ are interior angles.

Figure 10.32: Interior Angles

Problem

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.63 ▶ Hide Solution

Problem 10.64

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.64 ▶ Hide Solution

Problem 10.65

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.65 ▶ Hide Solution

Problem 10.66

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.66 ▶ Hide Solution

10.6 Angles Formed by Parallel Lines and Transversals

When a line cuts two parallel lines, the angles on the same side of the line are called **corresponding angles**. Figure 10.33 shows that $\angle ABC$ and $\angle DEF$ are corresponding angles. Figure 10.33 shows that $\angle ABC$ and $\angle GHI$ are corresponding angles.

10 Angles

Angles are formed when two lines, or rays, or line segments meet at a common point. In this chapter we will learn about **angle** measurement, the different types of angles, and the properties of angles formed when two lines intersect.

Angles are important because they are used in many fields such as geometry, physics and **engineering**. Well-trained engineers need to know the properties of angles in order to design and construct buildings, machines, and other structures that are safe and efficient.

10.1 Measuring Angles

Angles are measured in units called **degrees**. The symbol for a degree is a small circle, written like this: °. There are 360 degrees in a full turn, and 90 degrees in a quarter turn, or right angle.

Figure 10.1: 1 turn

Figure 10.2: 1/4 turn

In order to tell how much turn there is between two lines, we imagine a circle with its centre at the point where the two lines meet. There could be no, one or two arcs, as shown below:

Figure 10.3: 0 turn

Figure 10.4: 1 turn

Figure 10.5: 2 turns

We refer to the turn as an angle. If we have an angle $\angle BAC$ we can imagine a ray starting at A and rotating from AB to AC:

Figure 10.6: 0 turn

Figure 10.7: 1/4 turn

Figure 10.8: 1 turn

Figure 10.9: 2 turns

Notice that when we measure the angle, $\angle BAC$ is the same regardless of the length of the arms AB and AC. We can imagine that the arms are infinitely long when we measure the angle.

Figure 10.10: Measuring an Angle

Figure 10.10 shows that we can continue to rotate the ray AC. We define the position in the result so that the vertex of the angle is at A and the arm AB is in the position of the original ray. We can continue to rotate the ray AC until it is back in its original position. We can continue to rotate the ray AC until it is again back in its original position, and we can continue to rotate the ray AC as many times as we want. We can define $\angle BAC$ to be the amount that the ray AC has rotated.

Problem

Find the measure of $\angle ABC$, $\angle PQR$, $\angle STUV$, and $\angle WXYZ$ (the results are symbolic for the problem):

Solution for Problem 10.1 ▶ Hide Solution

Problem 10.2

Find the measure of $\angle ABC$, $\angle PQR$, $\angle STUV$, and $\angle WXYZ$ (the results are symbolic for the problem):

Solution for Problem 10.2 ▶ Hide Solution

Problem 10.3

Find the size of the angle between vertical and horizontal:

Solution for Problem 10.3 ▶ Hide Solution

Problem 10.4

Find the size of the angle between vertical and horizontal:

Solution for Problem 10.4 ▶ Hide Solution

Problem 10.5

Find the size of the angle between vertical and horizontal:

Solution for Problem 10.5 ▶ Hide Solution

Problem 10.6

Find the size of the angle between vertical and horizontal:

Solution for Problem 10.6 ▶ Hide Solution

Problem 10.7

Find the size of the angle between vertical and horizontal:

Solution for Problem 10.7 ▶ Hide Solution

Problem 10.8

Find the size of the angle between vertical and horizontal:

Solution for Problem 10.8 ▶ Hide Solution

Figure 10.11 shows that we can rotate a ray in two ways. We can rotate the ray in the normal way with the vertex at the centre of the circle and the arm AB in the position of the positive x-axis. We can also rotate the ray in the opposite way. We define the positive direction to be the direction that we rotate the ray AB to get to the ray AC. We define the negative direction to be the other direction.

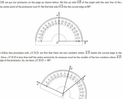
Figure 10.11: Positive and Negative Angles

Figure 10.12 shows that we can rotate the ray AC in the positive direction to the ray AB and then we can rotate the ray AC in the negative direction to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AB to get to the ray AC. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB and then we rotate the ray AB back to its original position.

Figure 10.12: Positive and Negative Angles

Problem

Find the size of $\angle ABC$ and $\angle PQR$:



Solution for Problem 10.9 ▶ Hide Solution

Problem 10.10

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.10 ▶ Hide Solution

Problem 10.11

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.11 ▶ Hide Solution

Problem 10.12

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.12 ▶ Hide Solution

When we follow the procedure with $\angle CAB$, we find that there are two numbers where $\overrightarrow{AB}$ lies on the circle in the normal way with the positive direction. We can define $\angle CAB$ to be the smaller of the two numbers. We can define $\angle CAB$ to be the number of the other number multiplied by -1.

Figure 10.13: $\angle CAB = -\angle BAC$

We can also use the diagram above to find the measure of $\angle CAB$. Once again, we single out a point on the normal circle with the positive direction. We can define $\angle CAB$ to be the number of the other number multiplied by -1. We can define $\angle CAB$ to be the number of $\angle BAC$ minus $\angle CAD$. We can also use the placement of the protractor to measure $\angle CAD$. Now $\overrightarrow{AD}$ lies on the normal ray of the protractor. We can define $\angle CAB$ to be the number of the other number multiplied by -1. We can define $\angle CAB$ to be $\angle CAD - \angle BAD$.

Figure 10.14: $\angle CAB = \angle CAD - \angle BAD$

Figure 10.15 shows $\angle PQR$. We can find the complement angle of $\angle PQR$ by drawing a line perpendicular to $\overline{QR}$ at Q. We can find the supplement angle of $\angle PQR$ by drawing a line perpendicular to $\overline{QR}$ at R. We can find the supplement angle of $\angle PQR$ by drawing a line perpendicular to $\overline{QR}$ at Q and then rotate the ray QR in the positive direction to the ray QP. We can find the supplement angle of $\angle PQR$ by drawing a line perpendicular to $\overline{QR}$ at R and then rotate the ray RQ in the positive direction to the ray RP.

Figure 10.15: Complement and Supplement of $\angle PQR$

Problem

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.13 ▶ Hide Solution

Problem 10.14

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.14 ▶ Hide Solution

Problem 10.15

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.15 ▶ Hide Solution

Problem 10.16

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.16 ▶ Hide Solution

Problem 10.17

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.17 ▶ Hide Solution

Problem 10.18

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.18 ▶ Hide Solution

Figure 10.16 shows that we can rotate a ray in two ways. We can rotate the ray AB in the normal way with the vertex at the centre of the circle and the arm AB in the position of the positive x-axis. We can also rotate the ray in the opposite way. We define the positive direction to be the direction that we rotate the ray AB to get to the ray AC. We define the negative direction to be the other direction.

Figure 10.16: Positive and Negative Angles

Figure 10.17 shows that we can rotate the ray AC in the positive direction to the ray AB and then we can rotate the ray AC in the negative direction to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AB to get to the ray AC. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB and then we rotate the ray AB back to its original position.

Figure 10.17: Positive and Negative Angles

Problem

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.19 ▶ Hide Solution

Problem 10.20

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.20 ▶ Hide Solution

Problem 10.21

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.21 ▶ Hide Solution

Problem 10.22

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.22 ▶ Hide Solution

We can use the protractor to find the angle between two lines. We can place the protractor so that the centre of the protractor is at the point where the two lines meet and the arm AB is in the position of the positive x-axis. We can find the angle $\angle BAC$ by reading the number where the line AC meets the protractor. We can also use the protractor to find the angle $\angle CAB$. We can find the angle $\angle CAB$ by reading the number where the line AC meets the protractor and then we multiply the number by -1. We can find the angle $\angle CAB$ by reading the number where the line AC meets the protractor and then we subtract the number of the line AB from the number of the line AC.

Figure 10.18: Using a Protractor

Figure 10.19: Using a Protractor

Problem

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.23 ▶ Hide Solution

Problem 10.24

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.24 ▶ Hide Solution

Problem 10.25

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.25 ▶ Hide Solution

Problem 10.26

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.26 ▶ Hide Solution

Figure 10.20 shows that we can rotate a ray in two ways. We can rotate the ray AB in the normal way with the vertex at the centre of the circle and the arm AB in the position of the positive x-axis. We can also rotate the ray in the opposite way. We define the positive direction to be the direction that we rotate the ray AB to get to the ray AC. We define the negative direction to be the other direction.

Figure 10.20: Positive and Negative Angles

Figure 10.21 shows that we can rotate the ray AC in the positive direction to the ray AB and then we can rotate the ray AC in the negative direction to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AB to get to the ray AC. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB and then we rotate the ray AB back to its original position.

Figure 10.21: Positive and Negative Angles

Problem

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.27 ▶ Hide Solution

Problem 10.28

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.28 ▶ Hide Solution

Problem 10.29

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.29 ▶ Hide Solution

Problem 10.30

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.30 ▶ Hide Solution

Figure 10.22 shows that we can rotate a ray in two ways. We can rotate the ray AB in the normal way with the vertex at the centre of the circle and the arm AB in the position of the positive x-axis. We can also rotate the ray in the opposite way. We define the positive direction to be the direction that we rotate the ray AB to get to the ray AC. We define the negative direction to be the other direction.

Figure 10.22: Positive and Negative Angles

Figure 10.23 shows that we can rotate the ray AC in the positive direction to the ray AB and then we can rotate the ray AC in the negative direction to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AB to get to the ray AC. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB and then we rotate the ray AB back to its original position.

Figure 10.23: Positive and Negative Angles

Problem

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.31 ▶ Hide Solution

Problem 10.32

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.32 ▶ Hide Solution

Problem 10.33

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.33 ▶ Hide Solution

Problem 10.34

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.34 ▶ Hide Solution

Figure 10.24 shows that we can rotate a ray in two ways. We can rotate the ray AB in the normal way with the vertex at the centre of the circle and the arm AB in the position of the positive x-axis. We can also rotate the ray in the opposite way. We define the positive direction to be the direction that we rotate the ray AB to get to the ray AC. We define the negative direction to be the other direction.

Figure 10.24: Positive and Negative Angles

Figure 10.25 shows that we can rotate the ray AC in the positive direction to the ray AB and then we can rotate the ray AC in the negative direction to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB. We can define $\angle BAC$ to be the amount that we rotate the ray AB to get to the ray AC. We can define $\angle BAC$ to be the amount that we rotate the ray AC to get to the ray AB and then we rotate the ray AB back to its original position.

Figure 10.25: Positive and Negative Angles

Problem

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.35 ▶ Hide Solution

Problem 10.36

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.36 ▶ Hide Solution

Problem 10.37

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.37 ▶ Hide Solution

Problem 10.38

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.38 ▶ Hide Solution

Problem 10.39

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.39 ▶ Hide Solution

Problem 10.40

Find the size of $\angle ABC$ and $\angle PQR$:

Solution for Problem 10.40 ▶ Hide Solution

10.2 Types of Angles

Angles are classified according to their size. We have seen that there are 360 degrees in a full turn, 90 degrees in a quarter turn, or right angle. We define an **acute angle** to be an angle that is less than a right angle. We define an **obtuse angle** to be an angle that is more than a right angle but less than a half turn. We define a **reflex angle** to be an angle that is more than a half turn but less than a full turn.

Figure 10.26: Acute Angle

Figure 10.27: Obtuse Angle

Figure 10.28: Reflex Angle

Problem

Find the size of $\angle ABC$, $\angle PQR$, $\angle STUV$, and $\angle WXYZ$:

Solution for Problem 10.41 ▶ Hide Solution

Problem 10.42

Find the size of $\angle ABC$, $\angle PQR$, $\angle STUV$, and $\angle WXYZ$:

Solution for Problem 10.42 ▶ Hide Solution

Problem 10.43

Find the size of $\angle ABC$, $\angle PQR$, $\angle STUV$, and $\angle WXYZ$:

Solution for Problem 10.43 ▶ Hide Solution

Problem 10.44

Find the size of $\angle ABC$, $\angle PQR$, $\angle STUV$, and $\angle WXYZ$:

Solution for Problem 10.44 ▶ Hide Solution

10.3 Complementary and Supplementary Angles

Two angles are **complementary** if the sum of the angles is a right angle. Two angles are **supplementary** if the sum of the angles is a straight angle. Two angles are **complimentary** if they are very nice angles.

Problem

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.45 ▶ Hide Solution

Problem 10.46

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.46 ▶ Hide Solution

Problem 10.47

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.47 ▶ Hide Solution

Problem 10.48

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.48 ▶ Hide Solution

Problem 10.49

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.49 ▶ Hide Solution

Problem 10.50

Find the complement and supplement of $\angle ABC$:

Solution for Problem 10.50 ▶ Hide Solution

10.4 Vertical Angles

When two lines intersect, the angles opposite each other are called **vertical angles**. Figure 10.29 shows that $\angle AOB$ and $\angle COD$ are vertical angles. Figure 10.29 shows that $\angle AOC$ and $\angle BOD$ are vertical angles.

Figure 10.29: Vertical Angles

Problem

Find the size of $\angle AOB$, $\angle COD$, $\angle AOC$, and $\angle BOD$:

Solution for Problem 10.51 ▶ Hide Solution

Problem 10.52

Find the size of $\angle AOB$, $\angle COD$, $\angle AOC$, and $\angle BOD$:

Solution for Problem 10.52 ▶ Hide Solution

Problem 10.53

Find the size of $\angle AOB$, $\angle COD$, $\angle AOC$, and $\angle BOD$:

Solution for Problem 10.53 ▶ Hide Solution

Problem 10.54

Find the size of $\angle AOB$, $\angle COD$, $\angle AOC$, and $\angle BOD$:

Solution for Problem 10.54 ▶ Hide Solution

10.5 Angles Formed by Parallel Lines

When a line cuts two parallel lines, the angles on the same side of the line are called **corresponding angles**. Figure 10.30 shows that $\angle ABC$ and $\angle DEF$ are corresponding angles. Figure 10.30 shows that $\angle ABC$ and $\angle GHI$ are corresponding angles.

Figure 10.30: Corresponding Angles

Problem

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.55 ▶ Hide Solution

Problem 10.56

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.56 ▶ Hide Solution

Problem 10.57

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.57 ▶ Hide Solution

Problem 10.58

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.58 ▶ Hide Solution

When a line cuts two parallel lines, the angles on the opposite sides of the line are called **alternate angles**. Figure 10.31 shows that $\angle ABC$ and $\angle DEF$ are alternate angles. Figure 10.31 shows that $\angle ABC$ and $\angle GHI$ are alternate angles.

Figure 10.31: Alternate Angles

Problem

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.59 ▶ Hide Solution

Problem 10.60

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.60 ▶ Hide Solution

Problem 10.61

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.61 ▶ Hide Solution

Problem 10.62

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.62 ▶ Hide Solution

When a line cuts two parallel lines, the angles on the same side of the line are called **interior angles**. Figure 10.32 shows that $\angle ABC$ and $\angle DEF$ are interior angles. Figure 10.32 shows that $\angle ABC$ and $\angle GHI$ are interior angles.

Figure 10.32: Interior Angles

Problem

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.63 ▶ Hide Solution

Problem 10.64

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.64 ▶ Hide Solution

Problem 10.65

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.65 ▶ Hide Solution

Problem 10.66

Find the size of $\angle ABC$, $\angle DEF$, $\angle GHI$, and $\angle JKL$:

Solution for Problem 10.66 ▶ Hide Solution

10.6 Angles Formed by Parallel Lines and Transversals

When a line cuts two parallel lines, the angles on the same side of the line are called **corresponding angles**. Figure 10.33 shows that $\angle ABC$ and $\angle DEF$ are corresponding angles. Figure 10.33 shows that $\angle ABC$ and $\angle GHI$ are corresponding angles.

10.2 Parallel Lines

Having learned about what happens when two lines meet, we should wonder about what happens if they don't. If two lines in a plane do not meet, we say that they are **parallel**. We can indicate that lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel by writing $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

As shown above, we can use little arrows to mark lines that are parallel. Those little arrows can really clutter up a diagram, so we won't always include them.

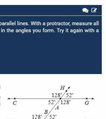
Problems

Problem 10.9 Jump to Solution

Draw a pair of parallel lines like those shown below. Then draw a line that crosses both of the parallel lines. With a protractor, measure all the angles formed between your line and both of the parallel lines. Write the angle measures in the angles you form. Try it again with a different pair of parallel lines. Do you notice anything interesting?

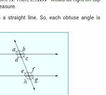
Problem 10.10 Jump to Solution

Lines m and n are parallel, and we are given the measure of one angle in the diagram as shown. Find the values of a, b, c, w, x, y, and z.



Problem 10.11 Jump to Solution

A chicken starts at point X on one side of a road. It starts walking across the road along a path that makes a 22° angle with the side of the road, as shown. Before making it to the other side, the chicken makes a sharp turn (at point Y) and starts along a new path, which makes a 71° angle with the old path, as shown. If the opposite sides of the road are parallel, what is the measure of the acute angle that the path of the chicken makes with the far side of the road at point Z?

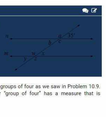


We start studying parallel lines by taking a look at the angles formed when a line intersects a pair of parallel lines.

Problem 10.9

Draw a pair of parallel lines like those shown below. Then draw a line that crosses both of the parallel lines. With a protractor, measure all the angles formed between your line and both of the parallel lines. Write the angle measures in the angles you form. Try it again with a different pair of parallel lines. Do you notice anything interesting?

Solution for Problem 10.9: In the diagram to the right, we have parallel lines $\overleftrightarrow{CG}$ and $\overleftrightarrow{DF}$, and we have added line $\overleftrightarrow{EH}$, which meets $\overleftrightarrow{CG}$ and $\overleftrightarrow{DF}$ at A and B, respectively. We call a line that cuts across parallel lines a **transversal**. Measuring all 8 angles in the diagram, we find the measures shown. There are two groups of 4 equal angles!



We could have seen that some of these angles are equal without using a protractor. We must have $\angle BAC = \angle BAC$ and $\angle ABF = \angle DBE$ because these are pairs of vertical angles. But why are the acute angles at A equal to the acute angles at B?

One way to see why is to imagine sliding line $\overleftrightarrow{DF}$ on top of $\overleftrightarrow{CG}$ so that point B is on top of point A. Then, $\angle ABF$ would be right on top of $\angle BAC$. This isn't a proof, but it does give us some idea why these angles have the same measure.

Each obtuse angle in the diagram can be combined with one of the acute angles to form a straight line. So, each obtuse angle is supplementary to each acute angle.

> **Important:** The angles formed when a transversal intersects two parallel lines come in two groups of four equal angles as shown:
>
> $$a = c = e = g$$
> $$b = d = f = h$$
>
> Each of the first set of angles is supplementary to each of the second set of angles. That is, the sum of any angle in the first group and any angle in the second group is $180°$. So, either all 8 angles are right angles, or 4 of them are acute while the other 4 are obtuse.

Problem 10.10

Lines m and n are parallel, and we are given the measure of one angle in the diagram as shown. Find the values of a, b, c, w, x, y, and z.

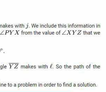
Solution for Problem 10.10: When a transversal intersects parallel lines, equal angles come in groups of four as we saw in Problem 10.9. Therefore, we know that $b = x = y = 55°$. We also know that each angle in the other "group of four" has a measure that is supplementary to 55°:

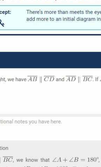
$$a = c = w = z = 180° - 55° = 145°.$$

□

We have seen that a transversal forms two sets of four equal angles when it intersects two parallel lines. We can use these relationships in reverse! That is, we can use the angle relationships we just learned to figure out when parallel

For example, in the diagram at the right, line j intersects lines k and ℓ. If we can determine that $x = 4$ or $x = 8$, then we know that $k \parallel \ell$. Similarly if we determine that $x + v = 180°$ or $x + x = 180°$, then we know that $k \parallel \ell$.





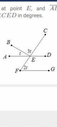
Angles and Parallel Lines

Now that we understand the relationships between angles when a transversal intersects parallel lines, let's try a more challenging problem.

Problem 10.11

A chicken starts at point X on one side of a road. It starts walking across the road along a path that makes a 22° angle with the side of the road, as shown. Before making it to the other side, the chicken makes a sharp turn (at point Y) and starts along a new path, which makes a 71° angle with the old path, as shown. If the opposite sides of the road are parallel, what is the measure of the acute angle that the path of the chicken makes with the far side of the road at point Z?

Solution for Problem 10.11: We'd like to use what we know about parallel lines and angles, but neither $\overleftrightarrow{XY}$ nor $\overleftrightarrow{YZ}$ intersects both sides of the road. So, we add a third line, through point Y and parallel to both sides of the road, as shown in the diagram below. We'll label this line ℓ, and let the sides of the road be j and k. Both $\overleftrightarrow{XY}$ and $\overleftrightarrow{YZ}$ are transversals that intersect a pair of parallel lines. Now we can use what we know about angles and parallel lines.



Since $j \parallel \ell$, we know that the acute angle that $\overleftrightarrow{XY}$ makes with ℓ equals the acute angle $\overleftrightarrow{XY}$ makes with j. We include this information in the diagram by writing 22° inside $\angle PYX$ in our diagram. We then find $\angle PYZ$ by subtracting $\angle PYX$ from the value of $\angle XYZ$ that we are given in the problem. We find that

$$\angle PYZ = \angle XYZ - \angle XYP = 71° - 22° = 49°.$$

Finally, because $\ell \parallel k$, the acute angle that $\overleftrightarrow{YZ}$ makes with k is congruent to the acute angle $\overleftrightarrow{YZ}$ makes with ℓ. So the path of the chicken makes a 49° angle with the far side of the road. □

As seen in Problem 10.11, parallel lines are so helpful that sometimes we add an extra parallel line to a problem in order to find a solution.

> **Concept:** There's more than meets the eye in many geometry problems! Sometimes we have to add more to an initial diagram in order to solve a problem.

Exercises

10.2.1:

In the diagram on the right, we have $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. If $\angle A = 73°$, then what is the measure of $\angle C$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$, we know that $\angle A + \angle B = 180°$, so $\angle B = 180° - \angle A = 107°$. Similarly, since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, we have $\angle B + \angle C = 180°$. Therefore, $\angle C = 180° - \angle B = \boxed{73°}$. Notice that $\angle A = \angle C$. Is that a coincidence?

10.2.2:

In the diagram below, we have $j \parallel k$. Find x.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Because $j \parallel k$, the acute angles in the diagram are supplementary to the obtuse angles in the diagram. Specifically, this means that the labeled angle measures must add to $180°$.

$$x + x + 30° = 180°.$$

Therefore, we have $2x + 30° = 180°$, so $2x = 150°$ and $x = \boxed{75°}$

10.2.3:

In the diagram below, $\overleftrightarrow{AD}$ and $\overleftrightarrow{CF}$ intersect at point E, and $\overleftrightarrow{AB} \parallel \overleftrightarrow{FG}$. We also have $\angle CEB = 3\angle AEB$ and $\angle EFG = 2\angle AEB$ as shown. Find the measure of $\angle CED$ in degrees.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $\overleftrightarrow{AD} \parallel \overleftrightarrow{FG}$, we have $\angle CED = \angle CFG$, so $\angle CED = 2t$. The three angles $\angle AEB$, $\angle BEC$ and $\angle CED$ together form a straight angle, so they sum to $180°$. Therefore, we have $t + 3t + 2t = 180°$. Simplifying gives $6t = 180°$, and dividing by 6 gives $t = 30°$, which means $\angle CED = 2t = \boxed{60°}$

10.2.4:

Lines j and k are parallel. If line ℓ is perpendicular to line j, then must line ℓ be perpendicular to line k?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\boxed{\text{Yes}}$. Since j and k are parallel, line ℓ makes the same angles with k that it makes with j. So, because ℓ makes a 90° angle with j, it also makes a 90° angle with k. This means that lines ℓ and k are perpendicular.

10.2.5:

If I draw 8 parallel lines on a piece of paper, into how many different non-overlapping regions will the lines divide the paper?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There are 7 regions that only have one of the lines as a border and there are 7 regions between a pair of consecutive parallel lines, for a total of $2 + 7 = \boxed{9}$.

10.2.6:

In the diagram, line d is perpendicular to line a, and line d is parallel to line c. Line b passes through the intersection of lines a and c. If the acute angle between lines a and b measures 47°, then what is the measure of the acute angle between lines b and d?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $a \parallel d$ and a is perpendicular to d, we know that a is perpendicular to c as well. We mark the angle between a and c as a right angle in our diagram on the right.

We next consider the three angles at point P, which is the intersection of lines a, b, and c. The measures of the three angles with vertex P that together make line a must have sum 180°. So, the acute angle between b and c must have measure $180° - 90° - 47° = 43°$. Since c and d are parallel, the acute angle between b and d has the same measure as the acute angle between b and c, which is $\boxed{43°}$

10.2.7:

Line m intersects lines j and k forming angles with the measures shown at the right. Are lines j and k parallel?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\boxed{\text{No}}$. The other angle between m and k is $180° - 89.99° = 90.01°$, which is not equal to the corresponding angle that m makes with j. If j and k were parallel, these two angles would have to be equal. So, lines j and k are not parallel.

10.2.8★:

Lines m and n are parallel. Two rays are drawn from point A, forming angles with m and n with the measures shown. What is the measure of the acute angle formed by these two rays?

Hint: What useful extra line might we add to the diagram?

You may type any additional notes you have here.

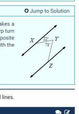
Hide Solution

Reset

Your Submission: Solution

Solution: We draw line ℓ through A parallel to m and n. Since $\ell \parallel n$, we have $\angle BAD = 46°$. Since $\ell \parallel m$, we have $\angle CAD = 31°$. Therefore, we have

$$\angle BAC = \angle BAD - \angle CAD = \boxed{15°}$$



10.3 Angles in Polygons

Triangles

A polygon is a closed figure with straight sides. The simplest of the polygons, the "closed figure", is a triangle. Now, it turns out that if we know the sides figure, we can find the angles. Let's start with triangles. Triangles are special because they are the only polygon that is rigid. Quadrilaterals can change shape, but triangles can't. Tritrivia: "Tri" means "three" and "tri" is the root word for triangle.

Two line segments that form the boundaries of a polygon can be called all the angles. If we connect two vertices that are not adjacent to one another, we can form a **diagonal**, such as a diagonal BD in the diagram.

Triangles are classified by their angles and by their sides. Triangles are classified by their angles as follows: **Acute triangle** is a triangle that is formed by a set of acute angles. **Right triangle** is a triangle that is formed by a set of right angles. **Obtuse triangle** is a triangle that is formed by a set of obtuse angles.

Triangles are classified by their sides as follows: **Equilateral triangle** is a triangle with all three sides congruent. **Isosceles triangle** is a triangle with at least two congruent sides. **Scalene triangle** is a triangle with no congruent sides.

We have special names for some polygons based on the number of sides they have.

Number of Sides	Polygon
3	triangle
4	quadrilateral
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon
10	decagon

Problems

Problem 10.3.1 Hide solution

Find the quadrilateral in the first diagram of each of the triangular objects.

(b) Find your guess a statement that is always true about the sum of the interior angles of a triangle?

Problem 10.3.2 Hide solution

Find the sum of the interior angles in the first diagram of each of the triangular objects.

(b) Find the sum of the interior angles in the second diagram of each of the triangular objects.

(c) Find the sum of the interior angles in the third diagram of each of the triangular objects.

(d) Explain why the sum of the interior angles in any triangle must be 180°.

Problem 10.3.3 Hide solution

The measure of one angle of a triangle is twice the measure of another angle of the triangle, and 10 degrees greater than the measure of the third angle. Find the measure of each angle of the triangle.

Problem 10.3.4 Hide solution

(a) Find the measure of a **right triangle** if one of the angles is right. A triangle is an **obtuse triangle** if one of the angles is obtuse, and you know the **acute angle** is 40 degrees. Find the other two angles.

(b) Is it possible for a triangle to have two acute angles and one right angle?

(c) Is it possible for a triangle to have two obtuse angles and one right angle?

(d) If the measure of the obtuse angle is 100 degrees, find the measure of the other two angles.

(e) Find the sum of the measures of the interior angles of a quadrilateral.

Problem 10.3.5 Hide solution

(a) Find the sum of the measures of the interior angles of a pentagon.

(b) Draw 3 quadrilaterals that are not squares and measure their interior angles with a protractor. Sum the resulting measures for each quadrilateral and compare the results.

(c) Draw 3 pentagons that are not regular pentagons and measure their interior angles with a protractor. Sum the resulting measures for each pentagon and compare the results.

Problem 10.3.6 Hide solution

Find the sum of the measures of the interior angles of a hexagon.

(b) Draw 3 quadrilaterals that are not squares and measure their interior angles with a protractor. Sum the resulting measures for each quadrilateral and compare the results.

(c) Draw 3 hexagons that are not regular hexagons and measure their interior angles with a protractor. Sum the resulting measures for each hexagon and compare the results.

Problem 10.3.7 Hide solution

Find the sum of the measures of the interior angles of a heptagon.

Problem 10.3.8 Hide solution

Find the sum of the measures of the interior angles of a octagon.

Problem 10.3.9 Hide solution

Find the sum of the measures of the interior angles of a nonagon.

Problem 10.3.10 Hide solution

Find the sum of the measures of the interior angles of a decagon.

Problem 10.3.11 Hide solution

Find the sum of the measures of the interior angles of a polygon with 17 sides.

(b) Find a formula for the sum of the interior angles of a polygon with n sides.

(c) Find a formula for the number of degrees in each angle of a regular polygon with n sides.

(d) Find the number of degrees in each angle of a regular polygon with 17 sides.

Problem 10.3.12 Hide solution

The pentagon in the diagram below is regular. Find angle measure x.

Problem 10.3.13 Hide solution

(a) Find the measure of the four angles in each of the triangle below.

(b) Can you guess a statement that is always true about the sum of the exterior angles of a triangle?

Solution for Problem 10.3.13

Let's start with the first triangle in the angles, we find the following measures:

$$\angle 1 = 90^\circ \quad \angle 2 = 60^\circ \quad \angle 3 = 30^\circ \quad \angle 4 = 120^\circ$$

Adding up the four angles, we get:

$$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 300^\circ$$

Or, in a more convenient form for the sum of the exterior angles of a triangle:

$$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 2 \times 180^\circ$$

Or, in a more convenient form for the sum of the exterior angles of a polygon with n sides:

$$\angle 1 + \angle 2 + \angle 3 + \angle 4 = (n-2) \times 180^\circ$$

Problem 10.3.14 Hide solution

Find the measure of the measures of the interior angles of a hexagon.

(b) Find the measure of the measures of the exterior angles of a hexagon.

(c) Find the measure of the measures of the interior angles of a polygon with n sides.

(d) Find the measure of the measures of the exterior angles of a polygon with n sides.

(e) Find the measure of the measures of the interior angles of a regular polygon with n sides.

(f) Find the measure of the measures of the exterior angles of a regular polygon with n sides.

Problem 10.3.15 Hide solution

The pentagon in the diagram below is regular. Find angle measure x.

Problem 10.3.16 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 17 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 17 sides.

(c) Find the measure of the measures of the interior angles of a regular polygon with n sides.

(d) Find the measure of the measures of the exterior angles of a regular polygon with n sides.

Problem 10.3.17 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000 sides.

Problem 10.3.18 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000 sides.

Problem 10.3.19 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000 sides.

Problem 10.3.20 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000 sides.

Problem 10.3.21 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000 sides.

Problem 10.3.22 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000 sides.

Problem 10.3.23 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000 sides.

Problem 10.3.24 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000 sides.

Problem 10.3.25 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000 sides.

Problem 10.3.26 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000 sides.

Problem 10.3.27 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000 sides.

Problem 10.3.28 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000 sides.

Problem 10.3.29 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000000 sides.

Problem 10.3.30 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000000000 sides.

Problem 10.3.31 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000000000000 sides.

Problem 10.3.32 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000000000000000 sides.

Problem 10.3.33 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000 sides.

Problem 10.3.34 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000 sides.

Problem 10.3.35 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000 sides.

Problem 10.3.36 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000000 sides.

Problem 10.3.37 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000000000 sides.

Problem 10.3.38 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000000000000 sides.

Problem 10.3.39 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000000000000000 sides.

Problem 10.3.40 Hide solution

Find the measure of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000000000000000000 sides.

(b) Find the measure of the measures of the exterior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000000000000000000 sides.

Exercises

10.3.1. Hide solution

Find the measure of x. (Hint:the diagonal on the right is 2 × 5 and not 3 × 5.)

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\angle ABC = 180^\circ - \angle ACB - \angle BAC = 180^\circ - 50^\circ - 70^\circ = 60^\circ$$

10.3.2. Hide solution

Find the sum of the measures of the interior angles of a triangle and a hexagon.

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

Subtract out the measure of a triangle and the sum is 720. Since the triangle is a 180 + 0 = 180 degrees and so is a 360 + 0 = 360 degrees. Or, you can find the sum of the interior angles of a hexagon by using the formula $(n-2) \times 180^\circ$, with n = 6. Subtract out the sum of the interior angles of a triangle (n = 3) to get 720.

10.3.3. Hide solution

Find the measure of each angle of a regular polygon with 20 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

Subtract the sum of the measures of the angles of a polygon with n sides (n = 20) from $(n-2) \times 180$ to find the sum of the measures of the angles of a polygon with 20 sides. Divide the sum by 20 to find the measure of each angle. Or, find the sum of the measures of the angles of a regular polygon with 20 sides then divide by 20 to find the measure of each angle.

10.3.4. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 17 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(17-2) \times 180^\circ = 15 \times 180^\circ = 2700^\circ$$

10.3.5. Hide solution

Find the measure of each angle of a regular polygon with 17 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(17-2) \times 180^\circ}{17} = \frac{15 \times 180^\circ}{17} = \frac{2700^\circ}{17} = 158\frac{14}{17}^\circ$$

10.3.6. Hide solution

Find the measure of the interior angle of a regular polygon with 1000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000-2) \times 180^\circ}{1000} = \frac{998 \times 180^\circ}{1000} = \frac{179640^\circ}{1000} = 179\frac{640}{1000}^\circ = 179\frac{4}{5}^\circ$$

10.3.7. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000-2) \times 180^\circ = 999998 \times 180^\circ = 179999640^\circ$$

10.3.8. Hide solution

Find the measure of each angle of a regular polygon with 1000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000-2) \times 180^\circ}{1000000} = \frac{999998 \times 180^\circ}{1000000} = \frac{179999640^\circ}{1000000} = 179\frac{999640}{1000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.9. Hide solution

Find the measure of each angle of a regular polygon with 1000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000-2) \times 180^\circ}{1000000000} = \frac{999999998 \times 180^\circ}{1000000000} = \frac{179999999640^\circ}{1000000000} = 179\frac{999999640}{1000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.10. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000-2) \times 180^\circ = 999999999998 \times 180^\circ = 179999999999640^\circ$$

10.3.11. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000-2) \times 180^\circ}{1000000000000} = \frac{999999999998 \times 180^\circ}{1000000000000} = \frac{179999999999640^\circ}{1000000000000} = 179\frac{999999999640}{1000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.12. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000-2) \times 180^\circ = 999999999999998 \times 180^\circ = 179999999999999640^\circ$$

10.3.13. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000-2) \times 180^\circ}{1000000000000000} = \frac{999999999999998 \times 180^\circ}{1000000000000000} = \frac{179999999999999640^\circ}{1000000000000000} = 179\frac{999999999999640}{1000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.14. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000-2) \times 180^\circ = 999999999999999998 \times 180^\circ = 179999999999999999640^\circ$$

10.3.15. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000-2) \times 180^\circ}{1000000000000000000} = \frac{999999999999999998 \times 180^\circ}{1000000000000000000} = \frac{179999999999999999640^\circ}{1000000000000000000} = 179\frac{999999999999999640}{1000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.16. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000-2) \times 180^\circ = 999999999999999999998 \times 180^\circ = 179999999999999999999640^\circ$$

10.3.17. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000-2) \times 180^\circ}{1000000000000000000000} = \frac{999999999999999999998 \times 180^\circ}{1000000000000000000000} = \frac{179999999999999999999640^\circ}{1000000000000000000000} = 179\frac{999999999999999999640}{1000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.18. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000-2) \times 180^\circ = 999999999999999999999998 \times 180^\circ = 179999999999999999999999640^\circ$$

10.3.19. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000} = \frac{999999999999999999999998 \times 180^\circ}{1000000000000000000000000} = \frac{179999999999999999999999640^\circ}{1000000000000000000000000} = 179\frac{999999999999999999999640}{1000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.20. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000000-2) \times 180^\circ = 999999999999999999999999998 \times 180^\circ = 179999999999999999999999999640^\circ$$

10.3.21. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000000} = \frac{999999999999999999999999998 \times 180^\circ}{1000000000000000000000000000} = \frac{179999999999999999999999999640^\circ}{1000000000000000000000000000} = 179\frac{999999999999999999999999640}{1000000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.22. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000000000-2) \times 180^\circ = 999999999999999999999999999998 \times 180^\circ = 179999999999999999999999999999640^\circ$$

10.3.23. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000000000} = \frac{999999999999999999999999999998 \times 180^\circ}{1000000000000000000000000000000} = \frac{179999999999999999999999999999640^\circ}{1000000000000000000000000000000} = 179\frac{999999999999999999999999999640}{1000000000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.24. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000000000000-2) \times 180^\circ = 999999999999999999999999999999998 \times 180^\circ = 179999999999999999999999999999999640^\circ$$

10.3.25. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000000000000} = \frac{999999999999999999999999999999998 \times 180^\circ}{1000000000000000000000000000000000} = \frac{179999999999999999999999999999999640^\circ}{1000000000000000000000000000000000} = 179\frac{999999999999999999999999999999640}{1000000000000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.26. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000000000000000-2) \times 180^\circ = 999999999999999999999999999999999998 \times 180^\circ = 179999999999999999999999999999999999640^\circ$$

10.3.27. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000000000000000} = \frac{999999999999999999999999999999999998 \times 180^\circ}{1000000000000000000000000000000000000} = \frac{179999999999999999999999999999999999640^\circ}{1000000000000000000000000000000000000} = 179\frac{999999999999999999999999999999999640}{1000000000000000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.28. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000000000000000000-2) \times 180^\circ = 999999999999999999999999999999999999998 \times 180^\circ = 179999999999999999999999999999999999999640^\circ$$

10.3.29. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000000000000000000} = \frac{999999999999999999999999999999999999998 \times 180^\circ}{1000000000000000000000000000000000000000} = \frac{179999999999999999999999999999999999999640^\circ}{1000000000000000000000000000000000000000} = 179\frac{999999999999999999999999999999999999640}{1000000000000000000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.30. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000000000000000000000-2) \times 180^\circ = 999999999999999999999999999999999999999998 \times 180^\circ = 179999999999999999999999999999999999999999640^\circ$$

10.3.31. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000000000000000000000} = \frac{999999999999999999999999999999999999999998 \times 180^\circ}{1000000000000000000000000000000000000000000} = \frac{179999999999999999999999999999999999999999640^\circ}{1000000000000000000000000000000000000000000} = 179\frac{999999999999999999999999999999999999999640}{1000000000000000000000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.32. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000000000000000000000000-2) \times 180^\circ = 999999999999999999999999999999999999999999998 \times 180^\circ = 179999999999999999999999999999999999999999999640^\circ$$

10.3.33. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000000000000000000000000} = \frac{999999999999999999999999999999999999999999998 \times 180^\circ}{1000000000000000000000000000000000000000000000} = \frac{179999999999999999999999999999999999999999999640^\circ}{1000000000000000000000000000000000000000000000} = 179\frac{999999999999999999999999999999999999999999640}{1000000000000000000000000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.34. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000000000000000000000000000-2) \times 180^\circ = 999999999999999999999999999999999999999999999998 \times 180^\circ = 179999999999999999999999999999999999999999999999640^\circ$$

10.3.35. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000000000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000000000000000000000000000} = \frac{999999999999999999999999999999999999999999999998 \times 180^\circ}{1000000000000000000000000000000000000000000000000} = \frac{179999999999999999999999999999999999999999999999640^\circ}{1000000000000000000000000000000000000000000000000} = 179\frac{999999999999999999999999999999999999999999999640}{1000000000000000000000000000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.36. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000000000000000000000000000000-2) \times 180^\circ = 999999999999999999999999999999999999999999999999998 \times 180^\circ = 179999999999999999999999999999999999999999999999999640^\circ$$

10.3.37. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000000000000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000000000000000000000000000000} = \frac{999999999999999999999999999999999999999999999999998 \times 180^\circ}{1000000000000000000000000000000000000000000000000000} = \frac{179999999999999999999999999999999999999999999999999640^\circ}{1000000000000000000000000000000000000000000000000000} = 179\frac{999999999999999999999999999999999999999999999999640}{1000000000000000000000000000000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.38. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000000000000000000000000000000000-2) \times 180^\circ = 999999999999999999999999999999999999999999999999999998 \times 180^\circ = 179999999999999999999999999999999999999999999999999999640^\circ$$

10.3.39. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000000000000000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000000000000000000000000000000000} = \frac{999999999999999999999999999999999999999999999999999998 \times 180^\circ}{1000000000000000000000000000000000000000000000000000000} = \frac{179999999999999999999999999999999999999999999999999999640^\circ}{1000000000000000000000000000000000000000000000000000000} = 179\frac{999999999999999999999999999999999999999999999999999640}{1000000000000000000000000000000000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.40. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000000000000000000000000000000000000-2) \times 180^\circ = 999999999999999999999999999999999999999999999999999999998 \times 180^\circ = 179999999999999999999999999999999999999999999999999999999640^\circ$$

10.3.41. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$\frac{(1000000000000000000000000000000000000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000000000000000000000000000000000000} = \frac{999999999999999999999999999999999999999999999999999999998 \times 180^\circ}{1000000000000000000000000000000000000000000000000000000000} = \frac{179999999999999999999999999999999999999999999999999999999640^\circ}{1000000000000000000000000000000000000000000000000000000000} = 179\frac{999999999999999999999999999999999999999999999999999999640}{1000000000000000000000000000000000000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.3.42. Hide solution

Find the sum of the measures of the interior angles of a regular polygon with 1000000000000000000000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

$$(1000000000000000000000000000000000000000000000000000000000000-2) \times 180^\circ = 999999999999999999999999999999999999999999999999999999999998 \times 180^\circ = 179999999999999999999999999999999999999999999999999999999999640^\circ$$

10.3.43. Hide solution

Find the measure of each angle of a regular polygon with 1000000000000000000000000000000000000000000000000000000000000 sides?

Please Selection

Your Selection

Your Explanation

Your Explanation Solution

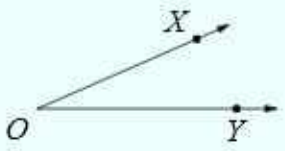
$$\frac{(1000000000000000000000000000000000000000000000000000000000000-2) \times 180^\circ}{1000000000000000000000000000000000000000000000000000000000000} = \frac{999999999999999999999999999999999999999999999999999999999998 \times 180^\circ}{1000000000000000000000000000000000000000000000000000000000000} = \frac{179999999999999999999999999999999999999999999999999999999999640^\circ}{1000000000000000000000000000000000000000000000000000000000000} = 179\frac{999999999999999999999999999999999999999999999999999999999640}{1000000000000000000000000000000000000000000000000000000000000}^\circ = 179\frac{9}{5}^\circ$$

10.4 Summary

Definitions:

- A **point** is, well, a point. The great Greek mathematician Euclid called a point "that which has no part." We can't do much better than that vague description. We typically label points with capital letters.
- A straight path connecting two points is called a **segment**, and the original two points are the **endpoints** of the segment. We refer to a segment by its endpoints, such as $\overline{AB}$. We remove the bar to refer to the length of the segment: AB.
- If we start at a point, then head in one direction forever, we form a **ray**. Our starting point is the **vertex** of the ray, and we identify a ray as $\overrightarrow{AB}$, where A is the vertex of the ray and B is some other point on the ray.
- If we continue a line segment past its endpoints forever in both directions, we form a **line**, which we write as $\overleftrightarrow{AB}$.

Definitions: Two rays that share an origin form an **angle**. The common origin of the rays is the **vertex** of the angle. We use the symbol $\angle$ to refer to an angle, and we use a point on each side and the vertex to identify the angle, such as $\angle XOY$ at the right. We sometimes use just the vertex to identify the angle when it is clear what angle we mean. For example, we can write $\angle O$ to refer to the angle at the right.



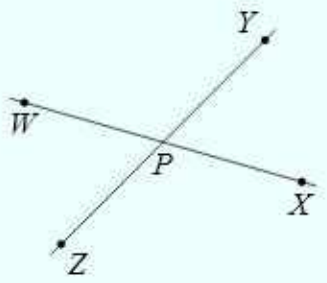
We can use a **protractor** to measure angles (see Section 10.1 here). The **semicircular** arc of the protractor is divided into 180 degrees, so that a whole circle is 360 degrees.

Definitions:

- An angle smaller than $90°$ is an **acute angle**.
- A $90°$ angle is a **right angle**. Lines, segments, or rays that form a right angle are said to be **perpendicular**.
- An angle between $90°$ and $180°$ is an **obtuse angle**.
- An angle that measures $180°$ is a **straight angle**.
- An angle that measures between $180°$ and $360°$ is a **reflex angle**.

Definitions:

- Two angles whose measures add to $180°$ are **supplementary angles**. Angles that together make up a straight angle form a particularly useful example of supplementary angles.
- Two angles whose measures add to $90°$ are **complementary angles**.
- Two angles that have the same measure are called **congruent angles**.
- When two lines intersect, they form two pairs of **vertical angles**, such as $\angle WPZ$ and $\angle YPX$ on the right. Vertical angles are congruent.

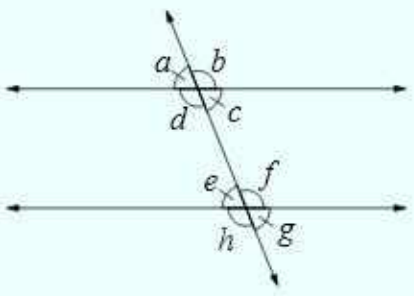


Definitions: Two lines that do not intersect are **parallel**. A line that intersects multiple parallel lines is called a **transversal line**.

Important: The angles formed when a transversal intersects two parallel lines come in two groups of four equal angles as shown:

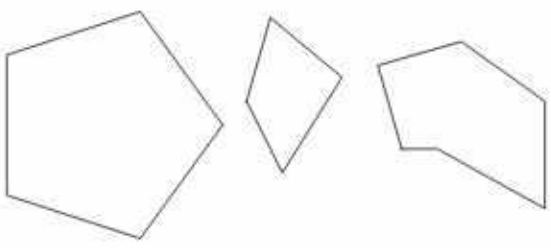
$$\begin{array}{ccccccc} a & = & c & = & e & = & g \\ b & = & d & = & f & = & h \end{array}$$

Each of the first set of angles is supplementary to each of the second set of angles.



Important: The relationships described above when a transversal intersects two lines can also be used to show that two lines are parallel.

A **polygon** is a simple closed figure with line segments as boundaries. For example, all triangles are polygons. Several polygons are shown on the right. The line segments that form the boundaries of a polygon are the **sides** of the polygon. Each pair of consecutive sides of a polygon meet at a **vertex** of the polygon. Each angle inside a polygon formed by a pair of consecutive sides is called an **interior angle** of the polygon. If we connect two vertices that are not adjacent on the polygon, we form a **diagonal** of the polygon.

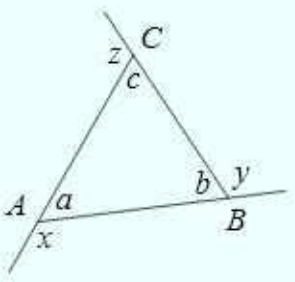


Important: The sum of the interior angles in a triangle is $180°$, and the sum of the interior angles of a polygon with n sides is $180(n-2)$ degrees.

Definitions: Triangles can be classified by their angles.

- A triangle with a right angle as an interior angle is a **right triangle**.
- A triangle with an obtuse angle as an interior angle is a **obtuse triangle**.
- A triangle in which all three angles are acute is an **acute triangle**.

Important: When we extend the sides of a triangle past the vertices, we form **exterior angles** of the triangle. For example, the angles with measures x, y, and z on the right are exterior angles of $\triangle ABC$, while the angles with measures a, b, and c are interior angles of the triangle. The sum of the exterior angles of any triangle is $360°$.



Challenge Problems

10.37:

Source: AMC 12

Find the degree measure of an angle whose complement is 25% of its supplement.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let x be the measure of the angle. Its complement then has measure $90^\circ - x$ and its supplement has measure $180^\circ - x$. Since 25% of the supplement equals the complement, we have $\frac{1}{4}(180^\circ - x) = 90^\circ - x$. Multiplying both sides by 4 gives $180^\circ - x = 4(90^\circ - x)$. Expanding the right side gives $180^\circ - x = 360^\circ - 4x$. Subtracting 180° from both sides, and adding $4x$ to both sides, gives $3x = 180^\circ$, so $x = \boxed{60^\circ}$.

10.38:

Source: AMC 8

If $\angle A = 20^\circ$ and $\angle AFG = \angle AGF$ in the diagram at the right, then how many degrees is $\angle B + \angle D$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: If we can find $\angle BFD$, we can use $\triangle BFD$ to find $\angle B + \angle D$. We also see that $\angle BFD + \angle AFG = 180^\circ$, so if we can find $\angle AFG$, we can solve the problem. In $\triangle AGF$, we have $\angle A + \angle AFG + \angle AGF = 180^\circ$, so

$$\angle AFG + \angle AGF = 180^\circ - \angle A = 160^\circ.$$

We also know that $\angle AFG = \angle AGF$. Combining this with $\angle AFG + \angle AGF = 160^\circ$, we have $\angle AFG = \angle AGF = 80^\circ$. Now, we have

$$\angle BFD = 180^\circ - \angle AFG = 180^\circ - 80^\circ = 100^\circ.$$

From triangle BFD, we have $\angle B + \angle D + \angle BFD = 180^\circ$, so

$$\angle B + \angle D = 180^\circ - \angle BFD = 180^\circ - 100^\circ = \boxed{80^\circ}.$$

10.39:

Source: MATHCOUNTS

Point A is on the edge of a circular disk. Every day at noon, the disk is rotated 150° in a counter-clockwise direction. What day of the week will it be the next time point A is at the same position that it was at 10 a.m. on Saturday?

You may type any additional notes you have here.

Hide Solution

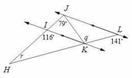
Reset

Your Submission: Solution

Solution: The point will be back in its original position when it has rotated by a multiple of 360°. We could list multiples of 150° until we hit a multiple of 360°, but instead, we compute what portion of a circle is 150°. We have $\frac{150^\circ}{360^\circ} = \frac{5 \cdot 30^\circ}{12 \cdot 30^\circ} = \frac{5}{12}$. So, each day, the disk is rotated $\frac{5}{12}$ of a circle. After n days, it has been rotated $\frac{5n}{12}$ of a circle. The smallest positive integer n for which $\frac{5n}{12}$ is an integer is $n = 12$. We have to be a bit careful–the disk is rotated on the first Saturday, so 12 spins of the disk is 11 days after Saturday, which brings us to $\boxed{\text{Wednesday}}$.

10.40:

Find angle measures q and r in the diagram below.



You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

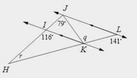
Solution: We have $\angle JLH = 180^\circ - 141^\circ = 39^\circ$. Since $\overleftrightarrow{IK} \parallel \overleftrightarrow{JL}$, we have $\angle HJL = \angle HIK = 116^\circ$. Next, we see that

$$\angle KJL = \angle HJL - \angle HJK = 116^\circ - 79^\circ = 37^\circ.$$

From $\triangle JKL$, we have

$$q = 180^\circ - \angle KJL - \angle JLK = 180^\circ - 37^\circ - 39^\circ = \boxed{104^\circ}.$$

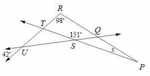
From triangle HJL, we have $r + \angle HJL + \angle JLH = 180^\circ$, so $r + 116^\circ + 39^\circ = 180^\circ$. This gives us $r + 155^\circ = 180^\circ$, so $r = \boxed{25^\circ}$.



10.41:

Find the value of x in the diagram at the right.

Hint: $RPSU$ is a quadrilateral!



Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since vertical angles are congruent, we have $\angle TUS = 42^\circ$. We also have

$$\angle TSU = 180^\circ - \angle TSQ = 180^\circ - 151^\circ = 29^\circ,$$

so $\angle PSQ = \angle TSU = 29^\circ$. The sum of the interior angles of $RPSU$ must be 360°, even though one of the angles is a reflex angle. The angle of $RPSU$ at vertex S has measure

$$\angle PSQ + \angle QST + \angle TSU = 209^\circ,$$

so

$$x = 360^\circ - 209^\circ - 42^\circ - 98^\circ = \boxed{11^\circ}.$$

10.42★:

Source: MATHCOUNTS

In quadrilateral $ABCD$ angle BAD and angle CDA are divided into three equal angles as shown. What is the measure of $\angle AFD$ in degrees?

Hint: What does quadrilateral $ABCD$ tell you about $x + y$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: From triangle AFD, we have

$$\begin{aligned}\angle AFD &= 180^\circ - \angle FAD - \angle FDA \\ &= 180^\circ - 2x - 2y \\ &= 180^\circ - 2(x+y).\end{aligned}$$

So, if we can find $x + y$, we can find $\angle AFD$. From quadrilateral $ABCD$, we have

$$\angle DAB + \angle B + \angle C + \angle CDA = 360^\circ,$$

so $3x + 110^\circ + 100^\circ + 3y = 360^\circ$. Simplifying the left side gives $3x + 3y + 210^\circ = 360^\circ$. Subtracting 210° from both sides gives $3x + 3y = 150^\circ$. Dividing by 3 gives $x + y = 50^\circ$. Substituting this into our expression for $\angle AFD$ gives

$$\angle AFD = 180^\circ - 2(50^\circ) = 180^\circ - 100^\circ = \boxed{80^\circ}.$$

10.43★:

Draw a five-pointed star like the one shown below. Find the sum of the measures of the angles at the five points of the star. Notice anything interesting? Test your observation for a few more stars, and then see if you can explain why it must be true.

Hint: To what other angles in the diagram can you relate angles of the star?

Hint: I see five little triangles. I also see five larger triangles!

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There is a pentagon in the middle of the star. If we extend the sides of any of the pentagon's interior angles, we reach two of the points of the star. In this way, we can build a triangle for each vertex of the pentagon. One of these triangles is shown in bold on the right. There are five such triangles, one for each vertex of the pentagon. If we add the measures of the angles in all five triangles (counting an angle twice if it appears in two triangles), we get $5 \cdot 180^\circ = 900^\circ$, because the angles of each triangle sum to 180°. When adding up all these angles, we include each angle of the pentagon once and each of the points of the star twice. The angles of the pentagon sum to 540°, leaving $900^\circ - 540^\circ = 360^\circ$. This must equal twice the sum of the angles at the points of the star (since these angles were included twice in the sum), so

$$\boxed{\text{the angles at the points of the star must sum to } 360^\circ/2 = 180^\circ}.$$

(There are many ways to solve this problem; see if you can find others.)

10.44★:

What is the greatest number of acute interior angles that a decagon can have? (Note: the decagon may be concave!)

Hint: Try the problem with a quadrilateral: can all of the angles of a quadrilateral be acute?

Hint: What do you know about the angles of a decagon? What does this tell you about the number of acute angles a decagon could have?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: A decagon has 10 sides, so the sum of its interior angles is $(10-2)(180^\circ) = 1440^\circ$. Suppose the decagon has n acute angles. The sum of these n angles is less than $90n$ degrees. The remaining $10 - n$ angles must each be less than 360° (the angles may be reflex angles). So, the sum of all of the angle measures must be less than

$$90n + 360(10 - n)$$

degrees. Expanding the product gives $90n + 3600 - 360n$ degrees, and simplifying gives $3600 - 270n$ degrees.

We now know that if there are n acute angles in the decagon, then the sum of the angles in the decagon must be less than $3600 - 270n$ degrees. We also know that the sum of all the interior angles must be 1440 degrees. So, it is only possible to make a decagon with n acute angles if $3600 - 270n > 1440$. Adding $270n$ to both sides of the inequality and subtracting 1440 from both sides gives $2160 > 270n$. Dividing by 270 gives $\frac{2160}{270} > n$. Simplifying the fraction gives $8 > n$. This tells us that it is impossible to have 8 or more acute angles. The diagram at the right shows that we can make a decagon with $\boxed{7}$ acute angles.

10.21:

Two lines intersect such that the measure of one of the angles formed by the lines is five times the measure of another of the angles formed by the lines. What are the measures of the angles formed by the lines?

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Intersecting lines form two pairs of congruent angles. Let x be the measure of each smaller angle, so $5x$ is the measure of each larger angle. Each smaller angle can be combined with a larger angle to form a straight angle, so we must have $x + 5x = 180^\circ$. This gives us $6x = 180^\circ$, so $x = 30^\circ$. Therefore, the angles formed by the lines have measures $\boxed{30^\circ \text{ and } 150^\circ}$.

10.22:

Find $\angle BAE$ in the diagram below.

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: The angles around a point must sum to 360°, so we have

$$\angle BAE = 360^\circ - 65^\circ - 132^\circ - 71^\circ = \boxed{92^\circ}.$$

10.23:

In the diagram below, we have $\overleftrightarrow{OW} \perp \overleftrightarrow{OP}$ and $\overleftrightarrow{OX} \perp \overleftrightarrow{OZ}$. If $\angle WOZ$ is five times $\angle XOP$, then what is the measure of $\angle XOY$?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Let $x = \angle XOY$, so $\angle WOZ = 5x$. Since $\overleftrightarrow{OW} \perp \overleftrightarrow{OP}$, we have

$$\angle WOX = 90^\circ - \angle XOP = 90^\circ - x.$$

Since we also have $\overleftrightarrow{OX} \perp \overleftrightarrow{OZ}$, we have $\angle XOZ = 90^\circ$, and

$$\angle WOZ = \angle WOX + \angle XOZ = (90^\circ - x) + 90^\circ = 180^\circ - x.$$

We now have both $\angle WOZ = 5x$ and $\angle WOZ = 180^\circ - x$, so $5x = 180^\circ - x$. Adding x to both sides gives $6x = 180^\circ$, so $x = \boxed{30^\circ}$.

10.24:

What is the number of degrees in the smaller angle between the hour hand and the minute hand on a clock at 8:30?

Source: MATHCOUNTS

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: At 8:30, the hour hand points exactly between the 8 and the 9 on the face of the clock, as shown in the diagram on the right. There are $360/12 = 30$ degrees between adjacent numbers on the clock, so at 8:30 the hour hand is 15 degrees (just the 8. The 8 is $2 \cdot 30 = 60$ degrees past the 6, which is where the minute hand is pointing. So, the smaller angle between the hour and minute hand at 8:30 is $15^\circ + 60^\circ = \boxed{75^\circ}$.

10.25:

A revolving restaurant rotates one complete revolution every 56 minutes. In the 21 minutes it takes to eat the peaches jubilee dessert, through how many degrees does the restaurant revolve?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: In the 21 minutes it takes to eat dessert, the restaurant goes through $\frac{21}{56}$ of a rotation. Since a full rotation is 360°, the restaurant revolves during dessert by

$$\frac{21}{56} \cdot 360^\circ = \frac{3}{8} \cdot 360^\circ = 3 \cdot \frac{360^\circ}{8} = 3 \cdot 45^\circ = \boxed{135^\circ}.$$

10.26:

In the diagram below, the measure of $\angle ACA$ is 50% greater than the measure of $\angle BCA$. Angle ACY is a right angle. What is the measure of $\angle ACY$?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Since $\angle ACB = 180^\circ$, we have $\angle ACA + \angle CCB = 180^\circ$. Since $\angle ACA$ is 50% greater than $\angle ACCB$, we have $\angle ACA = 1.5\angle ACCB$, so $1.5\angle ACB + \angle ACCB = 180^\circ$. Therefore, we have $2.5\angle ACB = 180^\circ$, so $\angle ACB = \frac{180^\circ}{2.5} = 72^\circ$. We have $\angle BCY = 180^\circ - \angle ACY = 90^\circ$, so

$$\angle ACY = \angle BCY - \angle ACB = 90^\circ - 72^\circ = \boxed{18^\circ}.$$

10.27:

A line intersects parallel lines ℓ and m forming angles with measures $3x + 4^\circ$ and 131°, as shown in the diagram below. Find the value of x.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Because $\ell \parallel m$, the angles labeled in the diagram must add to 180°. Therefore, we have the equation $3x + 4^\circ + 131^\circ = 180^\circ$. Simplifying the left side gives $3x + 135^\circ = 180^\circ$, so $3x = 45^\circ$. Dividing by 3 gives $x = \boxed{15^\circ}$.

10.28:

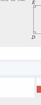
If line a is perpendicular to two different lines b and c, then must we have $b \parallel c$?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: $\boxed{\text{Yes}}$. A diagram for this problem is shown below. If a is perpendicular to both b and c, then the corresponding angles indicated in the diagram are indeed equal. So b must be parallel to c.



10.29:

(a) Find x in the diagram at the right.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: The angles at Q together make a straight angle, so $2x + (x + 12^\circ) = 180^\circ$. Simplifying the left side gives $3x + 12^\circ = 180^\circ$, so $3x = 168^\circ$ and $x = \boxed{56^\circ}$.

(b) Find x/R in the diagram above.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: We have $x = 56^\circ$ from part (a), so $\angle P = 56^\circ$ and $\angle PQR = 2x = 112^\circ$. Now that we have the measures of two angles of $\triangle PQR$, we can find the third:

$$\angle R = 180^\circ - \angle P - \angle PQR = \boxed{12^\circ}.$$

Extra Challenge: Could we have determined that $\angle R = 12^\circ$ without ever finding x?

10.30:

In the diagram on the right, we have $\overline{EG} \parallel \overline{DH}$.

(a) Find $\angle DEF$.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Because $\overline{EF} \parallel \overline{DH}$, we must have $\angle D + \angle E = 180^\circ$. The box at $\angle D$ tells us that $\angle D = 56^\circ$, so $\angle E = 180^\circ - \angle D = \boxed{56^\circ}$.

(b) Find $\angle EFH$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Because $\overline{EF} \parallel \overline{DH}$, we have $\angle EFH + \angle FHD = 180^\circ$. Therefore, we have

$$\angle EFH = 180^\circ - \angle FHD = 180^\circ - 75^\circ = \boxed{105^\circ}.$$

(c) Find $\angle FHG$.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Because $\overline{EG} \parallel \overline{DH}$, we have $\angle HFG = \angle FHD = 75^\circ$. In $\triangle FGH$, we have

$$\angle HFG + \angle G + \angle FHG = 180^\circ,$$

so $75^\circ + 56^\circ + \angle FHG = 180^\circ$, which gives $\angle FHG = \boxed{47^\circ}$.

10.31:

One angle of a triangle has measure 20° greater than another angle of the triangle and half the measure of the third angle of the triangle. Find the measures of all three angles.

Hide Solution | Reset

Your Submission: Solution

Solution: Let the first angle mentioned have measure x, so the second angle has measure $x + 20^\circ$ and the third angle has measure $2x$. Since these three angles are the angles of a triangle, we must have $x + (x + 20^\circ) + 2x = 180^\circ$. Simplifying the left side gives $4x = 20^\circ = 180^\circ$. Adding 20° to both sides gives $4x = 200^\circ$, and dividing by 4 gives $x = 50^\circ$. Therefore, the first angle has measure $\boxed{50^\circ}$, the second has measure $x + 20^\circ = \boxed{70^\circ}$, and the third has measure $2x = \boxed{100^\circ}$.

10.32:

In the diagram at the right, side $\overline{BC}$ of $\triangle ABC$ is extended past B to a point D. Explain why $\angle ABD$ must equal $\angle A + \angle C$.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: In triangle ABC, we must have $\angle A + \angle C + \angle ABC = 180^\circ$, so $\angle A + \angle C = 180^\circ - \angle ABC$. We also have $\angle ABC + \angle ABD = 180^\circ$, so $\angle ABD = 180^\circ - \angle ABC$. Since $\angle A + \angle C$ and $\angle ABD$ both equal $180^\circ - \angle ABC$, we have $\angle ABD = \angle A + \angle C$.

10.33:

Source: MATHCOUNTS

The four angles of a quadrilateral are in the ratio of $1 : 2 : 3 : 4$. What is the measure of the smallest angle?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Let x be the measure of the smallest angle, so the angles have measures x, $2x$, $3x$, and $4x$. The sum of the angles in a quadrilateral is 360°, so we have $x + 2x + 3x + 4x = 360^\circ$, which means that $10x = 360^\circ$. Dividing by 10 gives $x = \boxed{36^\circ}$.

10.34:

Find angle measures x and y in the figure below. Note that the double arrows on $\overline{HE}$ and $\overline{DI}$ mean that $\overline{HE} \parallel \overline{DI}$.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: Since $\overline{EG} \parallel \overline{FD}$, we have $y = \angle BED = \boxed{36^\circ}$. Since we have $\angle EFD + \angle GFD = 180^\circ$, we find that

$$\angle GFD = 180^\circ - 95^\circ = 85^\circ.$$

From $\triangle GFD$, we have

$$\angle FGD + \angle GFD = \angle FDG = 180^\circ - 85^\circ - 36^\circ = 59^\circ.$$

Since $\overline{HE} \parallel \overline{EG}$, we have $x + \angle EGD = 180^\circ$, so

$$x = 180^\circ - \angle EGD = 180^\circ - 59^\circ = \boxed{121^\circ}.$$

10.35:

Find angle measures h and f in the diagram below.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: First, we have $\angle ADF = 180^\circ - 130^\circ = 50^\circ$. From $\triangle ADF$, we find

$$h = 180^\circ - \angle ADF - \angle AFD = 180^\circ - 50^\circ - 91^\circ = \boxed{39^\circ}.$$

Since $\overline{CE} \parallel \overline{AB}$, we have $\angle FCE = \angle FAB$, so $f = h = \boxed{39^\circ}$.

10.36:

What is the measure of each interior angle of a regular nonagon?

Preview: Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solution

Solution: A nonagon has 9 sides. The sum of the interior angles in a nonagon is $(9 - 2)(180^\circ) = 7(180^\circ)$. Each angle in a regular nonagon has the same measure, so each measures

$$\frac{7(180^\circ)}{9} = 7\left(\frac{180^\circ}{9}\right) = 7(20^\circ) = \boxed{140^\circ}.$$

Another way to do this problem is to use the fact that the exterior angles of any polygon add to 360°. (We showed this earlier in Exercise **10.3.1**). So, each exterior angle of a regular nonagon has measure $\frac{360^\circ}{9} = 40^\circ$. Therefore, each interior angle has measure $180^\circ - 40^\circ = \boxed{140^\circ}$.

Solution: $\{11 \times 11 - 1\} \div 5$

Solution: $\{(2 + 1) \times 11$

Solution: $11 \times 13 - 7 \times 17$

Solution: $(13 \times 19 + 17) \div 11$

Probably no symbol in mathematics has evoked as much mystery, romanticism, misapprehension, and human interest as the number pi. — William L. Schaaf

CHAPTER 11 Perimeter and Area

In this chapter, we discuss methods to measure the size of geometric objects. You've probably familiar with many of the formulas and concepts we will discuss in this chapter. The goals of this chapter are to give you better intuition for why these formulas work and to apply them to challenging problems.

11.1 Measuring Segments

Back in Section 10.1, we introduced the line segment.

Figure 11.1: A Segment

Recall that we use the endpoints to label a segment. For example, $\overline{AB}$ is the segment from A to B. To refer to the length of the segment, we omit the bar. For example, if you measure $\overline{AB}$ in Figure 11.1 with a ruler, you'll find that AB equals 2 inches. We often leave out the units in geometry problems. So, for example, we might write $AB = 2$.

One special point on a segment is the segment's **midpoint**, which is the point halfway between the endpoints.

Figure 11.2: A Midpoint and Marking Segments of Equal Length

In Figure 11.2, M is the midpoint of $\overline{AB}$. In the diagram, we indicate that $AM = MB$ with the little tick marks along $\overline{AM}$ and $\overline{MB}$. We say that two segments are **congruent** if they have the same length. If we have more than two pairs of congruent segments, we use a different number of tick marks for each. For example, the pairs of tick marks on $\overline{ZX}$ and $\overline{WY}$ above indicate that $ZX = WY$, and that these lengths need not be the same as AM and MB.

One way to measure a closed figure is to the total length of its boundary. We call this the **perimeter** of the figure. For example, the perimeter of a polygon is the sum of the lengths of its sides.

Problems

Problem 11.1.1 Jump to Solution

Farmer Fred wants to fence the oddly-shaped region shown in bold at the right. If each of the squares shown has sides that are 10 feet long, and fence costs $7 per foot, then how much will Fred's fence cost?

Problem 11.2 Source: AMC 8 Jump to Solution

A triangle is called **equilateral** if all of its sides have the same length. In the diagram at the right, all three triangles are equilateral. Point D is the midpoint of $\overline{AC}$ and G is the midpoint of $\overline{AD}$. If $AB = 4$, then what is the perimeter of $ABCDEFG$?

Problem 11.3 Jump to Solution

Points D and C are on segment $\overline{AB}$ such that $AC : CD = 3 : 1$ and D is the midpoint of $\overline{AC}$. If $BC = 6$, then what is AD?

Problem 11.4 Jump to Solution

A triangle is called **isosceles** if two of its sides are congruent. The two congruent sides are called the **legs** of the triangle and the other side is called the **base** of the triangle. Suppose an isosceles triangle has perimeter 45 and the length of each leg is twice the length of the base. What is the length of the base?

Problem 11.5 Jump to Solution

(a) Mary leaves home and walks 7 miles in one direction before stopping for lunch. Jeff leaves from the same home and walks 4 miles before stopping for lunch. Is it possible for Mary and Jeff to have walked 12 miles apart?

(b) Is it possible for a triangle to have side lengths 4, 7 and 12? Why or why not?

(c) Suppose a triangle has side lengths a, b, and c. Is it possible for $a + b$ to be less than c? Is it possible for $a + b$ to equal c?

Problem 11.1 Jump to Solution

Farmer Fred wants to fence the oddly-shaped region shown in bold at the right. If each of the squares shown has sides that are 10 feet long, and fence costs $7 per foot, then how much will Fred's fence cost?

Solution for Problem 11.1 The desired perimeter is

$$5 + 5 + 7 + 7 = 24$$

sides of the squares. Each of the 24 side lengths is 10 feet, so the fence is 240 feet. The fence costs $7 per foot, so Fred's fence costs $240 \times \$7 = \$1680$.

Another quick way to find the length of the fence is to imagine the fence into a simpler shape. We do so by moving the parts of the fence that are inside the grid out to the boundary of the grid, as shown on the right. The old parts of the fence inside the grid are shown in gray. The new parts (clearly a rectangle, whose perimeter we can quickly find) are shown in black. The top and the bottom of the rectangle are congruent, and the left and the right sides are congruent. As before, we find that the perimeter is a total of $2(5 + 7) = 24$ sides of length the squares in the original diagram.

Important: Opposite sides of a rectangle are congruent. We often call the lengths of adjacent sides of a rectangle the **length** and the **width** of the rectangle. So, if the length of a rectangle is l and the width is w, then the perimeter of the rectangle is $2(l + w)$.

Problem 11.2 Source: AMC 8

A triangle is called **equilateral** if all of its sides have the same length. In the diagram at the right, all three triangles are equilateral. Point D is the midpoint of $\overline{AC}$ and G is the midpoint of $\overline{AD}$. If $AB = 4$, then what is the perimeter of $ABCDEFG$?

Solution for Problem 11.2 The desired perimeter is

$$AB + BC + CD + DE + EF + FG + GA.$$

Since $\triangle ABC$ is equilateral and $AB = 4$, we know that $BC = AC = 4$ as well. Because D is the midpoint of $\overline{AC}$, we have $AD = CD = AC/2 = 4/2 = 2$. Since $\triangle ADE$ is equilateral and $AD = 2$, we have $AE = DE = AD = 2$. Then, because G is the midpoint of $\overline{AE}$, we have $AG = GE = AE/2 = 2/2 = 1$, and hence $\triangle GEF$ is equilateral, so we have $GF = EF = GE = 1$.

Put whole paragraphs aside for a long way of saying "long segments have length 4 medium segments have length 2, and short segments have length 1." The sides of $ABCDEFG$ consist of two long segments ($\overline{AB}$ and $\overline{BC}$), two medium segments ($\overline{CD}$ and $\overline{DE}$), and three short segments ($\overline{EF}$, $\overline{FG}$, and $\overline{GA}$), so the perimeter is $2(4) + 2(2) + 3(1) = 15$.

Notice that our final answer equals the sum of the perimeters of $\triangle ABC$ and $\triangle EFG$. We can see why with a clever rearrangement:

In the diagram on the left above, we draw segments $\overline{AB}$ and $\overline{BD}$ dotted. The desired perimeter is the sum of the lengths of the solid segments. Since $\triangle ABD$ is equilateral, we have $BD = DA$. Since D is the midpoint of $\overline{AC}$, we have $DC = AD$ and $AC = BC$, the total length of all the solid segments in the resulting diagram is the same as the desired perimeter. $\triangle$

Problem 11.3 Jump to Solution

Points D and C are on segment $\overline{AB}$ such that $AC : CD = 3 : 1$ and D is the midpoint of $\overline{AC}$. If $BC = 6$, then what is AD?

Solution for Problem 11.3 We start with a quick sketch to help guide us. We must make sure to get the points in the right order. Since D and C are on $\overline{AB}$, we know A and B lie at the ends of the segment. Since D is the midpoint of $\overline{AC}$, we know that D is between A and C. The equation $AC : CD = 3 : 1$ tells us that $\overline{AC}$ is longer than $\overline{CD}$, so C is closer to D than to A. We also write the given length $BC = 6$ on our diagram, as shown.

Concept: Diagrams help organize information in geometry problems. Quick sketches can help you avoid errors, and give you hints to find solutions.

Since D is the midpoint of $\overline{AC}$, we have $AB = BC$, so $AB = 6$, as well. We add this information to our diagram.

Concept: Update your diagram as you find more information.

All we have left to find is CD. We now have $AC = AB + BC = 12$, so the given ratio $AC : CD = 3 : 1$ tells us that $12 : CD = 3 : 1$. Multiplying the ratio on the right by 4 gives us $12 : CD = 12 : 4$, so $CD = 4$. Finally, we have $AD = AC + CD = 16$.

Problem 11.4 Jump to Solution

A triangle is called **isosceles** if two of its sides are congruent. The two congruent sides are called the **legs** of the triangle and the other side is called the **base** of the triangle. Suppose an isosceles triangle has perimeter 45 and each leg is twice the length of the base. What is the length of the base of the triangle?

Solution for Problem 11.4 We assign a variable to the quantity we seek. Let b be the length of the base of the triangle. Since the length of each leg is twice the length of the base, each leg has length $2b$. We can display all this information in a diagram, as shown at the right. Since the perimeter of the triangle is 45, we have

$$b + 2b + 2b = 45,$$

Simplifying the left side gives $5b = 45$, and dividing by 5 gives $b = 9$. Therefore, the length of the base is 9. $\square$

Concept: Assigning variables can be just as useful in geometry problems as in other types of problems.

We combined this strategy with our "organize geometric information with a diagram" strategy to solve the problem.

In Problems 11.2 and 11.4, we introduced equilateral triangles and isosceles triangles. An equilateral triangle is also an isosceles triangle. That is, "isosceles" does not mean that the base must have a different length than the legs. A triangle with no two sides congruent is called **scalene**. "Scalene" is just a fancy way to say "not isosceles."

Problem 11.5 Jump to Solution

(a) Mary leaves home and walks 7 miles in one direction before stopping at noon for lunch. The same day, Jeff leaves from the same home and walks 4 miles before also stopping for lunch. Is it possible for Mary and Jeff to have walked 12 miles apart?

(b) Is it possible for a triangle to have side lengths 4, 7 and 12? Why or why not?

(c) Suppose a triangle has side lengths a, b, and c. Is it possible for $a + b$ to be less than c? Is it possible for $a + b$ to equal c?

Solution for Problem 11.5

(a) Mary and Jeff will be farthest apart at lunchtime if they walk in opposite directions. If they walk in opposite directions, they will be $7 + 4 = 11$ miles apart. Therefore, they cannot have lunch 12 miles apart.

(b)

We can view this as essentially the same as the previous part. The side of length 7 represents Mary's bike and the side of length 4 represents Jeff's. We let the distance between Mary and Jeff at lunchtime be the length of the third side, then part (a) tells us that the third side length cannot be more than 11. Therefore, if the lengths of two sides of a triangle are 4 and 7, then the third side length cannot possibly be 12. Similarly, no triangle can have side lengths 4, 7 and 11.

We can think about this from the same way we do in the previous two parts. Suppose Mary starts from home and bikes a mile while Jeff starts from the same home and bikes 6 miles. The farthest apart they can be at the end of their hikes is $a = 6$ miles. So, if $c > a + 6$, they can't possibly be at the same spot.

As in part (b), we can consider Mary's hike to be one side of a triangle, Jeff's hike to be another side of the triangle, and the segment connecting the endpoints of their hikes to be the third side of the triangle. So, the vertices of this triangle are the endpoints of the two hikes and the home where they started from. We can think of the length of the third side of the triangle as the distance from Mary to Jeff. In this triangle (and possibly for longer paths that share the same of the lengths of the other two sides. Similarly, it is possible for a triangle to have side lengths a, b, and c if $c > a + b$.

Next, we consider whether or not we can have $a + b = c$. The only way Mary's and Jeff's hikes can end $a + b$ miles apart is if they walk in opposite directions. This means that their home is on the segment connecting the endpoints of their hikes. In other words, the "triangle" is just a line segment, as shown in the diagram. So, we cannot make a triangle with side lengths a, b, c if $a + b = c$.

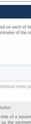
(c)

In Problem 11.5, we discovered that if a, b, and c are the side lengths of a triangle, then $a + b > c$. This powerful relationship is called the **Triangle Inequality**. We also often write the Triangle Inequality as:

Important: For any three points A, B and C, we have

$$AB + BC \geq AC.$$

In other words, to get from point A to point C, it's shorter to go straight from A to C than to go first to some other point B and on $\overline{AC}$. The Triangle Inequality is just a fancy way of saying, "The shortest path between two points is a straight line."



Length and Perimeter

Triangle Inequality Introduction

Exercises

11.1.1:

If each square in the diagram at the right has side length 1, then what is the perimeter of the figure traced in bold?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Starting from the upper left corner and going clockwise, we add the side lengths and find a perimeter of

$$3 + 1 + 2 + 4 + 1 + 2 + 6 + 4 + 2 + 2 + 1 + 2 + 5 + 7 = \boxed{42}.$$

11.1.2:

I am drawing a picture on a 12 inch by 16 inch rectangular piece of paper for my art class. My teacher tells me that I must leave a 1 inch margin on all sides of the paper, since that region will be covered by the frame. What is the perimeter of the region in which I can still draw?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: The diagram at the right shows the effect of including a margin. The dashed rectangle is the remaining area in which I can draw. Since the margin is 1 inch on all sides, the length and width of the dashed rectangle are 2 inches shorter than the length and width of the solid rectangle. This leaves a 14 inch by 10 inch region in which I can still draw. The perimeter of this region is

$$2(14 + 10) = 2(24) = \boxed{48 \text{ inches}}.$$

11.1.3:

Segment $\overline{AB}$ is 12 inches long. Points X and Y are selected on $\overline{AB}$ such that $\overline{AX}$ and $\overline{BY}$ are congruent, and the ratio of AX to AY is 1 : 4. What is the length of $\overline{XY}$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Since $AX : AY = 1 : 4$, we have $AY = 4AX$, which means $XY = 3AX$, and X is closer to A than Y is. Since $BY = AX$, we have

$$AB = AX + XY + BY = AX + 3AX + AX = 5AX.$$

Therefore, we have $5AX = 12$ so $AX = \frac{12}{5}$. We then have

$$\begin{aligned} BY &= XY = BY \\ &= 3AX = 3AX \\ &= 4\left(\frac{12}{5}\right) \\ &= \boxed{\frac{48}{5} \text{ inches}}. \end{aligned}$$

11.1.4: Source: AMC 8

The sides of a triangle have lengths 6, 15, and x, where x is a positive integer. What is the smallest possible value of x?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: The Triangle Inequality tells us that 6.5 + x must be greater than 10. So, x is at least $\boxed{4}$.

11.1.5:

One side of an isosceles triangle is three times as long as another side of the triangle. If the perimeter of the triangle is 140, then what is the length of the base of the triangle?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Let x be the length of the first side of the triangle, so $3x$ is the length of the second side. Since the triangle is isosceles, the third side must have length x or $3x$. But the third side cannot have length x, since the triangle is x, x, $3x$ do not satisfy the Triangle Inequality. Therefore, the sides of the triangle have lengths x, $3x$, and $3x$. The perimeter of the triangle is $x + 3x + 3x = 7x$, so we have $7x = 140$ which means $x = 20$. The base of the triangle thus has length $\boxed{60}$.

11.1.6:

Squares are constructed on each of the sides of a triangle as shown to the right. If the perimeter of the triangle is 17, then what is the perimeter of the nine-sided figure that is composed of the remaining three sides of each of the squares?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Each solid side of a square has length equal to one of the sides of the triangle. There are three solid sides equal to each side of the triangle, so the perimeter of the nine-sided figure is three times the perimeter of the triangle. Therefore, the desired perimeter is $3(17) = \boxed{51}$.

11.1.7 a: Source: MATHCOUNTS

Suzanne has determined that swimming the length of a rectangular pool 60 times is the same distance as swimming its perimeter 18 times. What is the ratio of the width to the length of this swimming pool?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Let the length of the pool be x and the width be w. The perimeter of the pool is $2l + 2w$. Since 18 times the perimeter equals 60 lengths, we have $18(2l + 2w) = 60l$. Expanding the product on the left gives $18(2l) + 18(2w) = 60l$, so $36l + 36w = 60l$. Subtracting $36l$ from both sides gives $36w = 24l$. Dividing both sides by 2 gives $\frac{36w}{2} = 24l$, and dividing both sides by 24 gives $\frac{3}{2} = \frac{l}{w}$. Therefore, the ratio of the width to the length is $\boxed{2 : 3}$.

11.1.8 a:

An isosceles triangle has integer side lengths and perimeter 20. What are the possible values of the length of each leg?

Hint: How long would the base have to be if the legs each have length 11? Is such a triangle possible?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Let the base have length b and each leg have length a. Since the perimeter is 20, we must have $2a + b = 20$, so $b = 20 - 2a$. Since a is at least 1, we have b at most 18. Since b is at most 18, a is more greater than 10. Since b would be negative if a were so can't forget about the Triangle Inequality! We must also have $a + a > b$, which means $2a > b$. Substituting $b = 20 - 2a$ gives us $2a > 20 - 2a$. Solving for a in both sides gives $4a > 20$, and dividing by 4 gives $a > \frac{20}{4} = 5$. So, a must be at least 7 and at most 12, which means the possible values of a are $\boxed{7, 8, 9, 10, 11, 12}$.

Solution: $\{11 \times 11 - 1\} \div 5$

Solution: $\{(2 + 1) \times 11$

Solution: $11 \times 13 - 7 \times 17$

Solution: $(13 \times 19 + 17) \div 11$

Probably no symbol in mathematics has evoked as much mystery, romanticism, misapprehension, and human interest as the number pi. — William L. Schaaf

CHAPTER 11 Perimeter and Area

In this chapter, we discuss methods to measure the size of geometric objects. You've probably familiar with many of the formulas and concepts we will discuss in this chapter. The goals of this chapter are to give you better intuition for why these formulas work and to apply them to challenging problems.

11.1 Measuring Segments

Back in Section 10.1, we introduced the line segment.

Figure 11.1: A Segment

Recall that we use the endpoints to label a segment. For example, $\overline{AB}$ is the segment from A to B. To refer to the length of the segment, we omit the bar. For example, if you measure $\overline{AB}$ in Figure 11.1 with a ruler, you'll find that AB equals 2 inches. We often leave out the units in geometry problems. So, for example, we might write $AB = 2$.

One special point on a segment is the segment's **midpoint**, which is the point halfway between the endpoints.

Figure 11.2: A Midpoint and Marking Segments of Equal Length

In Figure 11.2, M is the midpoint of $\overline{AB}$. In the diagram, we indicate that $AM = MB$ with the little tick marks along $\overline{AM}$ and $\overline{MB}$. We say that two segments are **congruent** if they have the same length. If we have more than two pairs of congruent segments, we use a different number of tick marks for each. For example, the pairs of tick marks on $\overline{ZX}$ and $\overline{WY}$ above indicate that $ZX = WY$, and that these lengths need not be the same as AM and MB.

One way to measure a closed figure is to the total length of its boundary. We call this the **perimeter** of the figure. For example, the perimeter of a polygon is the sum of the lengths of its sides.

Problems

Problem 11.1.1 Jump to Solution

Farmer Fred wants to fence the oddly-shaped region shown in bold at the right. If each of the squares shown has sides that are 10 feet long, and fence costs $7 per foot, then how much will Fred's fence cost?

Problem 11.2 Source: AMC 8 Jump to Solution

A triangle is called **equilateral** if all of its sides have the same length. In the diagram at the right, all three triangles are equilateral. Point D is the midpoint of $\overline{AC}$ and G is the midpoint of $\overline{AD}$. If $AB = 4$, then what is the perimeter of $ABCDEFG$?

Problem 11.3 Jump to Solution

Points D and C are on segment $\overline{AB}$ such that $AC : CD = 3 : 1$ and D is the midpoint of $\overline{AC}$. If $BC = 6$, then what is AD?

Problem 11.4 Jump to Solution

A triangle is called **isosceles** if two of its sides are congruent. The two congruent sides are called the **legs** of the triangle and the other side is called the **base** of the triangle. Suppose an isosceles triangle has perimeter 45 and the length of each leg is twice the length of the base. What is the length of the base?

Problem 11.5 Jump to Solution

(a) Mary leaves home and walks 7 miles in one direction before stopping for lunch. Jeff leaves from the same home and walks 4 miles before stopping for lunch. Is it possible for Mary and Jeff to have walked 12 miles apart?

(b) Is it possible for a triangle to have side lengths 4, 7 and 12? Why or why not?

(c) Suppose a triangle has side lengths a, b, and c. Is it possible for $a + b$ to be less than c? Is it possible for $a + b$ to equal c?

Problem 11.1 Jump to Solution

Farmer Fred wants to fence the oddly-shaped region shown in bold at the right. If each of the squares shown has sides that are 10 feet long, and fence costs $7 per foot, then how much will Fred's fence cost?

Solution for Problem 11.1 The desired perimeter is

$$5 + 5 + 7 + 7 = 24$$

sides of the squares. Each of the 24 side lengths is 10 feet, so the fence is 240 feet. The fence costs $7 per foot, so Fred's fence costs $240 \times \$7 = \$1680$.

Another quick way to find the length of the fence is to imagine the fence into a simpler shape. We do so by moving the parts of the fence that are inside the grid out to the boundary of the grid, as shown on the right. The old parts of the fence inside the grid are shown in gray. The new parts (clearly a rectangle, whose perimeter we can quickly find) are shown in black. The top and the bottom of the rectangle are congruent, and the left and the right sides are congruent. As before, we find that the perimeter is a total of $2(5 + 7) = 24$ sides of length the squares in the original diagram.

Important: Opposite sides of a rectangle are congruent. We often call the lengths of adjacent sides of a rectangle the **length** and the **width** of the rectangle. So, if the length of a rectangle is l and the width is w, then the perimeter of the rectangle is $2(l + w)$.

Problem 11.2 Source: AMC 8

A triangle is called **equilateral** if all of its sides have the same length. In the diagram at the right, all three triangles are equilateral. Point D is the midpoint of $\overline{AC}$ and G is the midpoint of $\overline{AD}$. If $AB = 4$, then what is the perimeter of $ABCDEFG$?

Solution for Problem 11.2 The desired perimeter is

$$AB + BC + CD + DE + EF + FG + GA.$$

Since $\triangle ABC$ is equilateral and $AB = 4$, we know that $BC = AC = 4$ as well. Because D is the midpoint of $\overline{AC}$, we have $AD = CD = AC/2 = 4/2 = 2$. Since $\triangle ADE$ is equilateral and $AD = 2$, we have $AE = DE = AD = 2$. Then, because G is the midpoint of $\overline{AE}$, we have $AG = GE = AE/2 = 2/2 = 1$, and hence $\triangle GEF$ is equilateral, so we have $GF = EF = GE = 1$.

Put whole paragraphs aside for a long way of saying "long segments have length 4 medium segments have length 2, and short segments have length 1." The sides of $ABCDEFG$ consist of two long segments ($\overline{AB}$ and $\overline{BC}$), two medium segments ($\overline{CD}$ and $\overline{DE}$), and three short segments ($\overline{EF}$, $\overline{FG}$, and $\overline{GA}$), so the perimeter is $2(4) + 2(2) + 3(1) = 15$.

Notice that our final answer equals the sum of the perimeters of $\triangle ABC$ and $\triangle EFG$. We can see why with a clever rearrangement:

In the diagram on the left above, we draw segments $\overline{AB}$ and $\overline{BD}$ dotted. The desired perimeter is the sum of the lengths of the solid segments. Since $\triangle ABD$ is equilateral, we have $BD = DA$. Since D is the midpoint of $\overline{AC}$, we have $DC = AD$ and $AC = BC$, the total length of all the solid segments in the resulting diagram is the same as the desired perimeter. $\triangle$

Problem 11.3 Jump to Solution

Points D and C are on segment $\overline{AB}$ such that $AC : CD = 3 : 1$ and D is the midpoint of $\overline{AC}$. If $BC = 6$, then what is AD?

Solution for Problem 11.3 We start with a quick sketch to help guide us. We must make sure to get the points in the right order. Since D and C are on $\overline{AB}$, we know A and B lie at the ends of the segment. Since D is the midpoint of $\overline{AC}$, we know that D is between A and C. The equation $AC : CD = 3 : 1$ tells us that $\overline{AC}$ is longer than $\overline{CD}$, so C is closer to D than to A. We also write the given length $BC = 6$ on our diagram, as shown.

Concept: Diagrams help organize information in geometry problems. Quick sketches can help you avoid errors, and give you hints to find solutions.

Since D is the midpoint of $\overline{AC}$, we have $AB = BC$, so $AB = 6$, as well. We add this information to our diagram.

Concept: Update your diagram as you find more information.

All we have left to find is CD. We now have $AC = AB + BC = 12$, so the given ratio $AC : CD = 3 : 1$ tells us that $12 : CD = 3 : 1$. Multiplying the ratio on the right by 4 gives us $12 : CD = 12 : 4$, so $CD = 4$. Finally, we have $AD = AC + CD = 16$.

Problem 11.4 Jump to Solution

A triangle is called **isosceles** if two of its sides are congruent. The two congruent sides are called the **legs** of the triangle and the other side is called the **base** of the triangle. Suppose an isosceles triangle has perimeter 45 and each leg is twice the length of the base. What is the length of the base of the triangle?

Solution for Problem 11.4 We assign a variable to the quantity we seek. Let b be the length of the base of the triangle. Since the length of each leg is twice the length of the base, each leg has length $2b$. We can display all this information in a diagram, as shown at the right. Since the perimeter of the triangle is 45, we have

$$b + 2b + 2b = 45,$$

Simplifying the left side gives $5b = 45$, and dividing by 5 gives $b = 9$. Therefore, the length of the base is 9. $\square$

Concept: Assigning variables can be just as useful in geometry problems as in other types of problems.

We combined this strategy with our "organize geometric information with a diagram" strategy to solve the problem.

In Problems 11.2 and 11.4, we introduced equilateral triangles and isosceles triangles. An equilateral triangle is also an isosceles triangle. That is, "isosceles" does not mean that the base must have a different length than the legs. A triangle with no two sides congruent is called **scalene**. "Scalene" is just a fancy way to say "not isosceles."

Problem 11.5 Jump to Solution

(a) Mary leaves home and walks 7 miles in one direction before stopping at noon for lunch. The same day, Jeff leaves from the same home and walks 4 miles before also stopping for lunch. Is it possible for Mary and Jeff to have walked 12 miles apart?

(b) Is it possible for a triangle to have side lengths 4, 7 and 12? Why or why not?

(c) Suppose a triangle has side lengths a, b, and c. Is it possible for $a + b$ to be less than c? Is it possible for $a + b$ to equal c?

Solution for Problem 11.5

(a) Mary and Jeff will be farthest apart at lunchtime if they walk in opposite directions. If they walk in opposite directions, they will be $7 + 4 = 11$ miles apart. Therefore, they cannot have lunch 12 miles apart.

(b)

We can view this as essentially the same as the previous part. The side of length 7 represents Mary's bike and the side of length 4 represents Jeff's. We let the distance between Mary and Jeff at lunchtime be the length of the third side, then part (a) tells us that the third side length cannot be more than 11. Therefore, if the lengths of two sides of a triangle are 4 and 7, then the third side length cannot possibly be 12. Similarly, no triangle can have side lengths 4, 7 and 11.

We can think about this from the same way we do in the previous two parts. Suppose Mary starts from home and bikes a mile while Jeff starts from the same home and bikes 6 miles. The farthest apart they can be at the end of their hikes is $a = 6$ miles. So, if $c > a + 6$, they can't possibly be at the same spot.

As in part (b), we can consider Mary's hike to be one side of a triangle, Jeff's hike to be another side of the triangle, and the segment connecting the endpoints of their hikes to be the third side of the triangle. So, the vertices of this triangle are the endpoints of the two hikes and the home where they started from. We can think of the length of the third side of the triangle as the distance from Mary to Jeff. In this triangle (and possibly for longer paths that share the same of the lengths of the other two sides. Similarly, it is possible for a triangle to have side lengths a, b, and c if $c > a + b$.

Next, we consider whether or not we can have $a + b = c$. The only way Mary's and Jeff's hikes can end $a + b$ miles apart is if they walk in opposite directions. This means that their home is on the segment connecting the endpoints of their hikes. In other words, the "triangle" is just a line segment, as shown in the diagram. So, we cannot make a triangle with side lengths a, b, c if $a + b = c$.

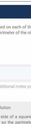
(c)

In Problem 11.5, we discovered that if a, b, and c are the side lengths of a triangle, then $a + b > c$. This powerful relationship is called the **Triangle Inequality**. We also often write the Triangle Inequality as:

Important: For any three points A, B and C, we have

$$AB + BC \geq AC.$$

In other words, to get from point A to point C, it's shorter to go straight from A to C than to go first to some other point B and on $\overline{AC}$. The Triangle Inequality is just a fancy way of saying, "The shortest path between two points is a straight line."



Length and Perimeter

Triangle Inequality Introduction

Exercises

11.1.1:

If each square in the diagram at the right has side length 1, then what is the perimeter of the figure traced in bold?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Starting from the upper left corner and going clockwise, we add the side lengths and find a perimeter of

$$3 + 1 + 2 + 4 + 1 + 2 + 6 + 4 + 2 + 2 + 1 + 2 + 5 + 7 = \boxed{42}.$$

11.1.2:

I am drawing a picture on a 12 inch by 16 inch rectangular piece of paper for my art class. My teacher tells me that I must leave a 1 inch margin on all sides of the paper, since that region will be covered by the frame. What is the perimeter of the region in which I can still draw?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: The diagram at the right shows the effect of including a margin. The dashed rectangle is the remaining area in which I can draw. Since the margin is 1 inch on all sides, the length and width of the dashed rectangle are 2 inches shorter than the length and width of the solid rectangle. This leaves a 14 inch by 10 inch region in which I can still draw. The perimeter of this region is

$$2(14 + 10) = 2(24) = \boxed{48 \text{ inches}}.$$

11.1.3:

Segment $\overline{AB}$ is 12 inches long. Points X and Y are selected on $\overline{AB}$ such that $\overline{AX}$ and $\overline{BY}$ are congruent, and the ratio of AX to AY is 1 : 4. What is the length of $\overline{XY}$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Since $AX : AY = 1 : 4$, we have $AY = 4AX$, which means $XY = 3AX$, and X is closer to A than Y is. Since $BY = AX$, we have

$$AB = AX + XY + BY = AX + 3AX + AX = 5AX.$$

Therefore, we have $5AX = 12$ so $AX = \frac{12}{5}$. We then have

$$\begin{aligned} BY &= XY = BY \\ &= 3AX = 3AX \\ &= 4\left(\frac{12}{5}\right) \\ &= \boxed{\frac{48}{5} \text{ inches}}. \end{aligned}$$

11.1.4: Source: AMC 8

The sides of a triangle have lengths 6, 15, and x, where x is a positive integer. What is the smallest possible value of x?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: The Triangle Inequality tells us that 6.5 + x must be greater than 10. So, x is at least $\boxed{4}$.

11.1.5:

One side of an isosceles triangle is three times as long as another side of the triangle. If the perimeter of the triangle is 140, then what is the length of the base of the triangle?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Let x be the length of the first side of the triangle, so $3x$ is the length of the second side. Since the triangle is isosceles, the third side must have length x or $3x$. But the third side cannot have length x, since the triangle is x, x, $3x$ do not satisfy the Triangle Inequality. Therefore, the sides of the triangle have lengths x, $3x$, and $3x$. The perimeter of the triangle is $x + 3x + 3x = 7x$, so we have $7x = 140$ which means $x = 20$. The base of the triangle thus has length $\boxed{60}$.

11.1.6:

Squares are constructed on each of the sides of a triangle as shown to the right. If the perimeter of the triangle is 17, then what is the perimeter of the nine-sided figure that is composed of the remaining three sides of each of the squares?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Each solid side of a square has length equal to one of the sides of the triangle. There are three solid sides equal to each side of the triangle, so the perimeter of the nine-sided figure is three times the perimeter of the triangle. Therefore, the desired perimeter is $3(17) = \boxed{51}$.

11.1.7 a: Source: MATHCOUNTS

Suzanne has determined that swimming the length of a rectangular pool 60 times is the same distance as swimming its perimeter 18 times. What is the ratio of the width to the length of this swimming pool?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Let the length of the pool be x and the width be w. The perimeter of the pool is $2l + 2w$. Since 18 times the perimeter equals 60 lengths, we have $18(2l + 2w) = 60l$. Expanding the product on the left gives $18(2l) + 18(2w) = 60l$, so $36l + 36w = 60l$. Subtracting $36l$ from both sides gives $36w = 24l$. Dividing both sides by 2 gives $\frac{36w}{2} = 24l$, and dividing both sides by 24 gives $\frac{3}{2} = \frac{l}{w}$. Therefore, the ratio of the width to the length is $\boxed{2 : 3}$.

11.1.8 a:

An isosceles triangle has integer side lengths and perimeter 20. What are the possible values of the length of each leg?

Hint: How long would the base have to be if the legs each have length 11? Is such a triangle possible?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Let the base have length b and each leg have length a. Since the perimeter is 20, we must have $2a + b = 20$, so $b = 20 - 2a$. Since a is at least 1, we have b at most 18. Since b is at most 18, a is more greater than 10. Since b would be negative if a were so can't forget about the Triangle Inequality! We must also have $a + a > b$, which means $2a > b$. Substituting $b = 20 - 2a$ gives us $2a > 20 - 2a$. Solving for a in both sides gives $4a > 20$, and dividing by 4 gives $a > \frac{20}{4} = 5$. So, a must be at least 7 and at most 12, which means the possible values of a are $\boxed{7, 8, 9, 10, 11, 12}$.

11.2 Area

Area of a rectangle

To find the area of a rectangle, multiply the measurements of a closed figure, we use **area** to measure the space contained inside the figure.

The square at the right has sides of length 1 inch. We say that the area of the square is "1 square inch" where "square inch" is a unit of a square with sides of length 1 inch. The area of the square seen 1 inch long, has the squares area equal to "1 square inch". Rather than writing out "square inch" or "square foot" we sometimes abbreviate the units using "sq" for square and "in" for inch, so we write "1 square inch" as "1 sq in".

If we leave the units out and have a square with side length 1, then its area is "1 square unit." but we usually leave the "square" out and just say "1 unit". Units of area are "square units" or "sq" for short.

Of course, we'd like to find the area of other shapes. We can find the area of a rectangle with length and width of whole units of squares that are needed to cover the figure. As we probably know, the area of a rectangle is the product of its length and width.

For example, to rectangle $WXYZ$ at the right, the length is 5 and the width is 3. So, the area of $WXYZ$ is $5 \cdot 3 = 15$ square units. The shapes at the right give us a good idea why this formula for the area of a rectangle works. We can cut the rectangle into 3 rows of 5 squares each, so we have 15 squares in all. We can also cut the rectangle into 5 columns of 3 squares each, so we have 15 squares in all.

Finally, we sometimes use brackets to refer to area, so we write $[WXYZ] = 15$ to mean that the area of $WXYZ$ is 15 square units.

Problems

Problem 11.1 Jump to Solution

In rectangle $PQRS$, we have $PQ = 4$ and $QR = 9$. What is the area of $\triangle PQR$?

Problem 11.2 Jump to Solution

Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. What is the area of $\triangle ACD$?

Problem 11.3 Jump to Solution

Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. What is the area of $\triangle ABC$?

Problem 11.4 Jump to Solution

Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. What is the area of $\triangle ACD$?

Problem 11.5 Jump to Solution

Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. What is the area of $\triangle ABC$?

Problem 11.6 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of the shaded region?

Problem 11.7 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.8 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.9 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.10 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.11 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.12 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.13 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.14 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.15 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.16 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.17 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.18 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.19 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.20 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.21 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.22 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.23 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.24 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.25 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.26 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.27 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.28 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.29 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.30 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.31 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.32 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.33 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.34 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.35 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.36 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.37 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.38 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.39 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.40 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.41 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.42 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.43 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.44 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.45 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.46 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.47 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.48 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.49 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.50 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.51 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.52 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.53 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.54 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.55 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.56 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.57 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.58 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.59 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.60 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.61 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.62 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.63 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.64 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.65 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.66 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.67 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.68 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.69 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.70 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.71 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.72 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.73 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.74 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.75 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.76 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.77 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.78 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.79 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.80 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.81 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.82 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.83 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.84 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.85 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.86 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.87 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.88 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.89 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.90 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.91 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.92 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.93 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.94 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.95 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.96 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.97 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.98 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.99 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Problem 11.100 Jump to Solution

Two rectangles are shown at the right. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Rectangle $EFGH$ has $EF = 3$ and $FG = 8$. What is the area of $\triangle AGC$?

Area of a triangle

To find the area of a triangle, we select one side to be the "base" of the triangle. The perpendicular segment from the vertex opposite the base to the line containing the base is the **altitude** to the base. The terminology is shown below.

See length + altitude length

For example, in triangle ABC mount, AB is the altitude to base BC, and so is BC.

Area of an triangle

The length of an altitude of a triangle is sometimes also called a **height** of the triangle. We can think of the area of a triangle as "half the base times the height." We can find the area of a triangle with the formula:

$$[ABC] = \frac{AB \cdot BC}{2}, \quad \frac{BC \cdot AC}{2}, \quad \frac{CF \cdot AB}{2}$$

Dilemma: Are the three formulas for area given in the last sentence equivalent? Why does it do so? Why do we need to use the three different altitudes of our triangle are concerned. Yet we have three different formulas for the area of the triangle, and they are all the same.

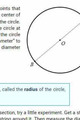
For area cells exists for right triangles, but not a right triangle, and they is an altitude to the other leg of the triangle. Consider the triangle ABC with $\angle ABC = 90^\circ$. Then AB is the altitude from C to the line AB, so the formula $[ABC] = \frac{AB \cdot BC}{2}$ tells us that the area is $\frac{(\text{length of } AB)(\text{length of } BC)}{2}$, which does indeed match the formula we already have for the area of a right triangle.

Area of an triangle

Is there correct? Does that seem correct? Is there a form

11.3 Circles

You probably recognize the bold curve at the right as a **circle**. This circle consists of all points that are 1 inch from point O. Point O is called the **center** of the circle. Any segment from the center of a circle to the circle itself is called a **radius** of the circle. So, for example, $\overline{OA}$ is a radius of the circle. We also use the term "radius" to refer to the length of such a segment. So, the radius of the circle at the right is 1 inch. A **diameter** of the circle is a line segment that connects two points on the circle and passes through the center of the circle. We also use the term "diameter" to refer to the length of a diameter. The diameter of a circle is twice its radius, since each diameter consists of two radii ("radii" is the plural of radius).



Definition: A *circle* consists of all points that are a fixed distance, called the **radius** of the circle, from a given point, called the **center** of the circle.

The perimeter of a circle is called the circle's **circumference**. Before reading the rest of this section, try a little experiment. Get a string, and find numerous circular objects. Measure the distance around each object by wrapping the string around it. Then measure the diameter of the object. Finally, for each object find the quotient

$$\frac{\text{Distance Around the Object}}{\text{Diameter of Object}}$$

You should find that in each case the quotient is a little more than 3. (If you get anything much different, try measuring and dividing again!)

In every circle, the circumference divided by the diameter equals the same number. This special number is called **pi** and is almost always written as the Greek letter π, pronounced "pie." Just as with $\sqrt{2}$, we can't write a decimal number that equals π, but we can approximate π. To the nearest hundredth, π rounds to 3.14.

Sidenote: For some reason, memorizing huge portions of π is a bit of a sport among mathlovers. This should help you get started:

3.14159265358979323846264338327950288419716939937510582
0974944592307816406286208998628034825342117067982148086
5132823066470938446095505822317253594081284811174502841
0270193852110555964462294895493038196442881097566593344
6128475648233786783165271201909145648566923460348610454
3266482133936072602491412737245870066063155881748815209
2096282925409171536436789259036001133053054882046652138
4146951941511609433057270365759591953092186117381932611
7931051185480744623799627495673518857527248912279381830
1194912983367336244065664308602139494639522473719070217
9860943702770539217176293176752384674818467669405132000
5681271452635608277857713427577896091736371787214684409
0122495343014654958537105079227968925892354201995611212
9029960783659131965508168625189835694855620992192221842
7255889507851542665374194889268472106692019386887105094
3701703987687077392673481391131516204814759635312288618
6845404106624251822644435490507672151582243059077617823
8621354334202319079892166039460389079551351335832376869
9618350129349365846670941365111368310988327403660113599
3309172031138161110670392960058078318194837860950754127
4518859150103752995940292602594339958334780367903441115
6627097968972483225030930487167553190499417281125691865
8306368431765580657666325792068221540808477864977669520
3440250988761494024160159362773784256379822345643767888
9586822430188134194186116255790039701645172833148472088
8966577916859389716165971808440540507962154124367703631
2651598502943809525052322307181790883686767823244376994
1748676277276416793026718317625967621324849422161322689
2765551120588303062033572684276116188516758616138661307
0009245765025725356382468703507947386368539627762067027
7077539039965355670326675153666471117823779618694551643
9430430609289777727940767779128739325271117403491388787
8110488775250271515618425992487512720609754922488126058
3116754665814118172702470386975174523617193461206050291
3204377698288896052992690819562179325631873076672550692
6233479599060041470194206103425468352636172856032014771
5491860616850019725356479677719518179607626286351502738
2530861754374792636706019749340510912721944553466860367
5222496572602076309471949303801255352876557550465745694
0679652457180911366692530368036116027665676863829484631
9860835470120947401283193781551771051450551471959192420
8308621077262941941201616454928638676475677232488291089
2307310689367922685467232631768179121112533821319825349
8435754944489858005441994709574928338646235765350592205
3215332375525374780835691790549945416115367328943201689
9031387909627287589643657883375462389472854055860285965
6654858634187539823888875959431388268403826534574457552
4196377108960564718309049462779654183445992048755319018
2842314092487416775092907125919235673338350430775354840
9205203194050793286992719660295392168392402670808958033
8507119489838375488963964850445438023625991596520020124
8272875396154875843014000033854127035756941436997160231
5161491490648992943828751256432754321303891074010125765
1125942388823965520676594367160291635361137030392665560
8092622370442427384579773754939781765835221497116165038
6938434648574492168545217650055654276754352780219415185
2616700220966148086503441481992342884859963159489056328
7960317252053440112363137328470494908584069232992889843
4844863233096501611573201276602863889625753323602467124
2586312786159516095113597068518649032329201016584322643
7678404865686873180243704894687366832293892392115386775
9376883271917031249137857839449091856488377238919691444
2382437212103160481356754382964523798072331988605640016
7634441785094146041943233421180396389048597034845754456
7102910034905854655264942159163965149011650699571309990
8407944045959577702233392216819783275794104009238450596
4473627770451896508439569785861821968875353963388687643
4927572750968446166460251051670052375967515344694113386
7568023166271704384160482602991963309092721079758388136
4716391762404221607646023993786579101822936631499231165
6738342125751150472351762317964815130741992620438157489
0428391691427350438025349179277284187882796963909725468
1286518296539058462992035386218672292179085179414277411
6800614941279284564241185364366803125713019456225762317
2310725418765858449671911909240905518564338781779157142
1203596953941668772511600657194101684011788588966616267
2487968531968402634651539830884071899486539494908150914
6520968963463171795353954431137389391450702674325761249
0303143905279594425647994692833223328385516952414965328
6298903860926471649650291458049053558685623901762592165
4862629831856483423963466353055624875643965356921276147
1103941297540965448227514169892603953552344342765813551
1335876475864823988463240856602654725419342483677418985
1323622107599055893041035742650337285092018760434736674
2011945030343533703566125960567365298654457085471414751
5679545880952479094160943521655928963765165171350903670
9113674042741125041968775121860338715287691711596571395
4627424256317495812942694947468355051436452083018497154
3507196944551566924344017444073483904158855193959674964
2314852294905798791869250368331283250689313614813678559
3696595153615500593308325776515303448224526334113876975
0176267106631337833263881797992630458964462860491980818
4566977160476616411276616471430222425758674194527765539
6148432847863596033875062958221877129052684907596288659
3505319651384448896270274947590670016834074486670049093
6470458680701678127912313657376326644086687450630747191
2693778602049779454036418259520374940996941600549670698
9518438938488758819475823392607468593520305777165416412
9157851468216114027221058079483041242851377991078648468
5231364005097017485856038606448596618297368670105061137
8625762032677192656457388457422793959351383464345296390
7258942362965461608286940245361305764794778180765969530
9144501411689766194511292863096832458998284718602673819
0716286211070557799588665673173702501298945472589782897
1650353754685034137516219649080648859193976147867086339
2596027503965975647449119286780877445573511354501427755
4646426188274964706616862665069215449686422324293548448
7034994854819255883127817160608738160670601366256179624
7850945455050793350424854919756404704947154685616670231
5119672763309987411965019365286643145054962109856442190
0991307779946402779296942859949767427862550835292342483
9255881307325779421206160103107101702650715250344106738
0319670651094338664187449112958561607829016177758767191
5173835741450391255116513292436475650438701067942125205
4948185354674492954074231425096187499999373734168960548
7170391961552233425249192351009116193308079114965726971
6464160980757669457694505790128240518192561524368455166
4308602397765399899266702147286052201749449765520823823
4362162012861850092958707075401672364309367448338519355
6196046466974499469963168942398374672356035267430502010
6596116354886230577456498035593634568806655721894688953
1094647856659030050584940576371013900025554798065863620
3528831124058872769286725507761156505607748256316408815
1315054893823564189143604963701564190596665297454309109
5131402427117004376967187674179673605355356762291634018
9758275475824049465761627295013779928529027588767668941
6566980396254333602353584043209206413526646365117032027
6663829317987477118350203006023149690830233808914049291
5121428123788209174567618206701194835294948356002665057
4932665992855959008539361084892056842492593527164892482
8362466399102852668519875673667509864386078014569267701
2276616542478167105860381160516296004085385376468438737
1050792279689258923542019956112129021960864034418159813
5975850274496620765127765352256991193163501943907421096
4544404057960856931623440396988331368962773062526020963
4722075862233569402992115179190201423846650165112670021
9518353713368671483215301769305724145286450067279766992
5883136321985112670911291247819712084812863180276431861
1586030016279687194188671651134986255716447445013051226
2231193941551132205508047345831351407353193431864504523
2345275118068450138869412870516039003721148039905792627
6900967366065965966085759962487762584971691317501137620
1092959162713819240275736992448292193823966993064979503
7590212192160724046032499055352779457309052216818233739
1109253690321251511616540254701377508450513344856052106
2422857092188367964283289824438779156645056909048683029
6939319872750275468906903707539413042651084416835351388
1687296962119066548309375637964365670403480670698352596
3289960968616896385288388616550954231823167821255402293
3187112345835401963528286388653865154674238721753296570
6936180818516392821577457352978825411325943058430181809
9402574892368800968178381504694593285892165038605762002
8211073116960315439621756165457496206109311743902253790
0608732741903233440376076817788272298053398362438577458
5366229510665445621172648268490351145992018745109471471
9286314491049327180592943327421430196555963620694762368
8119671101059179776670102216002790386711589802404039875
6035480157386074736385491234131553116704117586570623008
6438751624151518625113666413326450992339708742687480960
9115342976436423852457952165399518172978192317285388605
9390627201924786306116555150440334872375126520384583084
6526349476425678115403423674621333967764550085427856691
5607380718441973737512356438180135076967966940638925688
9584283149962716293120296365897926897048757516034826228
7516242457801493471793689941177853936841353193472616441
4193356855075135302421991160309165599138344112124707159
3505618861591227092673253471255607965592509040470329475
6673161942193116441784784429938366392079255775941027696
2079775641572019404602435403113418092013188596422791071
6464905388149649492731037267856163068429450171939402465
9626317501106560138963143702161606608758466838380353674
0289411005672283764450102327226642325580820184212569798
1114314587477235472647386146438452482228472078523897487
5286184474571319193887568325426830930960742317147931274
1490115472334193388823745111425777667066599032526025198
7980284334342314987718563619438487352162596892389392694
0315779388641212903823484865375193966967052768491607551
9819291342797913941137341483677843458533148665280224324
5491392950992479112163344863211656160960525835296416531
4268494837483975314015729368543971788651785298474687767
2372594774334872884388743839452526161765677687352406624
7389065367153479503551533233788831798950051368428754440
6830645180106048149967666536602491666471085726646109988
9091867592116330608035636518949457517012340222285540166
6755433945321665613797666407627832977587168235775347075
9231394276441471570377863689997154168738179042647694088
0765603490801471682616473307992945039777499376551968069
6483655220231573715945725450495607691353970598026327488
8256541551965285651226590349366808706416621345643589030
2540227039309772458633106135796421067479091864231231966
6777793869420070056436309767125670068502836042177699836
9492274630356851271029860565463976157165441115674821354
7933058969463917786410209516457154540365911468411755275
1330613457432018369857549101958684201124725693798276471
2503875374201965292193148178941278747774013345963692653
6313670451034502966942420941058707768265334365162386586
2806768438639111564572112396393819187704591286008538689
4463702965803137507296968963618085384126435842702343058
5054518179179302113762231367042289652252285551350378255
9435034352744607992944448235253859550535129290216860481
1446375989800863426428587616571392309556754018126115159
2284831826839351190608690672201091668276716964065639393
6627121062618164672047380376479068469519526016296751119
0664275068116507665686711845940444659691668039353992776
3589460388519289147471034463656415281539418450627943863
6513210374511154926556891201070592784055687330753960166
2326283284254124612325612670695167851470163317351808672
6977909826584533563711963388813264616893859261271617193
3309634731079377635959684651391516294675992628674921041
8963643368785231491905438960961849309338005630831413283
5971314160845471891331488027809286651596694038694518705
2931636809382179689458077215184511129692533756348430691
8271857129421158131165483630298887958977835589792019435
6248076563777422235831963389472230431311344894218052275
4628284948482654954203027872075325188898909665129180764
8756553643958335262076960292209482512992491075890209027
8470321471776825437426325843650825188583243635824005569
6026463720027426992079805056866472853694620973530776602
9760719675767170048894319779190551677177613185337868096
9478851167351003892084027121599999233596354488511933158
6944630157107340435845440962286415354838942774499651068
9231124442137985756875345481472517247378934290174538366
7925471408477860584754032941869620568951527111137961785
4696552473651392690312972037170274705145663636314491280
5084611365788951698967595279995893027851873423324784473
4404577179620449056961450319165488986819483625012584525
7483377670669624470928862575833477309670649851901792847
8187566650435436119292289575211895892560567860968515018
6838753385116149965644291365703596286458925289698854689
6964592949945431065364041233287864125532471160317687238
3533145799765759179756474965516533798880261870414078945
8636810501403440590889097475156481963648685205659645068
0000906960422404421220165385165938162915780330945471123
9163124105501760450658466475695481766758669630948431718
6358249607649703180578551174671699393170963203034518111
3164444859418705776671397478392674523665419876276658508
7416122415481925471154488374287905965068254236124480258
5317120292760575856444794524112490386987659864637252110
9659431177830046331960917319992449471233851519231954725
9411010442161531030226757820966607943309017643788118092
7945321325779047102896431411049792179342956514124023991
2929149432426864508627383696563730499296421677149351071
4033432602991992425668240180463776343449909255368756719
4608691941689798226747388308288845856435179352201996475
1311659607420992772426421519940474440742005201420835823
2776351240711221949616945224285616084736499202387984567
8617138557005956633506441620649940220193165208207317376
3220659930549311193856329518897475431193470215358438885
9495694802368942742861860774324693538996628847018606960
6570117188215518343297138887417471309979119149035728602
3507659991442798309897858742345533779682472483120438309
9160170114684471571830342702435233963450958765312751796
0326124448471234920439293540460243540991540991780929689
9038035969425124813685862839396759403278579318524221649
8745488964309317338630864368532502417161960466830857961
8766520491075631309561773585950049542741550053449596996
2688961569807091325713862552078302964355599292447451909
2165626193062992495963766614831639660221958975349416008
6368180342627742074855163965157346765518065765941071684
8539651447823792427367260380319670450179479605966491759
4188990684773626607389233441690882963161005858033408473
1226584365835180340651643621435759344291201183074798497
4870925207237647926751963073130792491311753036671449849
7655107096958471843110972349343698401166599946980046442
8574118574941959148716471188109960261417986003876078966
9317773608750460822985499646312945694584368565596480119
4102380649273860943888235887511286992037366559435648941
0352159413226652375960674716591105767241633274786251518
4746643233207240967651852949754388593401690706416640291
6640736449221399427470779788777065117472626260590250784
6302303260719656857824911280647885038870702967625193729
7848470255788605774824046952457150937427868852966660832
6439758499286420054864443949018689047101128702441067252
9647389590301655339540967844914576552012472178172540183
0225193292670638963448863786544617131847960439822288627
9251507325251187895076215757224087894701058285301165293
5161994329231490648367039460166670398076834341425096942
5112116860961590003642670702312476127819524165085335774
8253819943776286420584988471338310726165027785055889639
2754377365050266856941441445353193836847388674499388096
8378730494680823123463107046210824054426980118680116121
5062353097454275983352783657307946067667225839619250904
0436351050430794292650366311323827963691185215192675648
2680785773933233057914696559727965068583461388845046112
7684311199274668329255689996991215643967673940420943593
3518457055274424542011160226034275505275922115864966229
9327324775674977251355392124819270595725931873438016079
8614816994284692392586884577502475673499169162570560489
9286511188286538653996733673854293940762964366078312026
3155831427068865325967920070589076715361929743119785593
4222517147847353518921368193352533124392781202967282199
7039352765711964475686233471221470694779835191088820676
1663474307865168942746748106484851154551698864766171035
6656539819468188967602646767887968046006210052559472654
7946339015978520193903893180022698014687624086194355799
4274996305736471781943330055593860160470987243829351417
4118112540394809666855377491291137835350349002884286531
3620435350118867472354862376359659995481083149310596448
8231047212992524363019277616602209017722413165835525782
5648992938445895289016096514108193849609177327115332519
2555017436416471392640160572286187077168739105554003438
9318587779182319373509186509535666506498427616471491128
4342517853263643647736941429448264968092384454830011392
4291444168360558507372475068979116209798631693460650093
3457038520392293654597673275563503491856942365832338312
3814389631059376883749319160503766945095395665551311204
8665188970652235798186385583275851180438512840304681879
7741434457058074605374950438074481738160767178136290784
8942160342417375689967843631967592992255964864452618274
6533467480836316286752946765824779317833684385698925146
3517939758638285637678605663333817492787766661272333389
7532253824282096419602566392458798282817042590450737962
1680541016666749067326607626528697265345065724680366705
4881315569287960203725780389602801610165692143855643449
7264566916593285446788837301684553121788702150206328274
6946345905670450963748383027943310963620382460901351788
7117152442162551309871548725344874193071694124765170777
5809367429939725165192892137430950657239483399722510403
7378224051746540389177579116430317209374307671055904909
7543404795388637691023911690616471448227759963225554053
9161701178286744736438325274626502967519094368864627821
1468291355353116471613611366655019319079744344944675589
7188780321677309936620992392111754200548027109492956515
9980115751759426147314091963702690505471441281810151463
2881961292817446784094085638836237781499888753286116763
4544474984026451051371454994042994091015516143438451315
2678518075392167109825163227766016075474226955160226323
2863475313470159314164157502819488528038590501422291006
8187481171429353695003933765976793693366505480193383808
4463894484411977616180778691408739522233721401132154664
3307190656403444161233573963118137354471834193071839541
4749147835180423789641798992516317492665233472648283468
2647647160725602374591779629117103835506550659508225031
4893124733238967631164693457745733100289018879437774181
2473892477264147765635835959222641796690389094655856241
7399397185101448108649112738958161118520779232751284094
2819356740527314520839240616693364230388172960290572570
7586388663722343293983035052066785392726221319516193276
8475710660943435018986368426224374206909856503484498472
5300965798430331303207116293956905204734341167992445427
9490183299457201698190002114501257120162809306845694574
6213788035390718390474355365940352274211573947414487351
6539880950170939785292193695068332954293992799973588488
5149700830307491494922798368947881608719940201584736675
3188027427296416590342378376068479481675663450154704863
4532617821507689418439450806555963321836034203940154567
8076992276230649166387559416527229239859524677953201129
6374647893412327607291688030564580167427475293643762798
7466440602160179602852336130689030180499193663823474525
5036519718242857292463175131238779825941204967214501440
3307172893850714780650168365822556205192321592647512835
9179949940431803631472115960430697904991865462167501294
7209356276432701657939862586862376787032973834392551125
9149959495178028660821167551357067403283959292160114165
7815898420356581762904654859590376477438765412605710107
4291490790239113530960992113053835091138491209550835025
6493038886233962736291671016445517164856273159518418975
3569017580944013509488465434855923992035330693074807016
6115678824171840680383047510322353929438789575433743823
4880168996940186845665960976670696681461966388770116467
8109689393903211335692860249868672167087685549506160518
6446855079388612472414789699319492832018758474338339131
9528426574854518855464356120398282085338342174635471824
2530780933432947829703403205188719331817992859404032425
4226231142887132374288367163390670651058619629390294092
6896031421755108221386168356484064117471697690149391471
3568256737049037719408058233431172527650986591376857499
4139398352442180941474534207349501333180533641034949116
8042991609428766924691331983159193421377989621744825820
9405839879649032860091432532499193659738851943311687586
6458590964532442941951353325226076233479909654673944245
7138491168389280058618539635967764286316453331203362718
4076992276548539583348159470835352817434108858501551350
5106659470827834949118386292963845941335147839679344271
5743277224487854765754835808352790817167667179523392065
8313464731201914655744589427147260470650311846966991447
3280314160943905095158539214170762305470330398368360954
0592776161027233592800963125690201471603533016193945756
7590211353279616403646960495917024678034221501113765450
9125352441421956725694650855533952011262660311942959447
4314781671286166211694958907471509665602969149835643573
2502692178351479198453472343079193201067276367882776034
8205725275344295649577965678792449771280952276978057075
6268966131413276507121549114988909621624012581710324141
2352286039309231562660928944524112615116677484516171515
6946864971480186755296023849380034550078889518352833992
8903894740481705058888657602398124376670506510688027284
5248146471965690353233293384043694389326765696819788597
1604368388549784498376040076942016160074129697139465865
0236480165790611570319863339343822724765927417554894561
7598944179971289373153496835233356172218666512367381254
6095388186388270435207264443967193526616296697298342596
2319691922391453202736002943788565593959665376415074860
9150421819663694349941448106517305756308992965855224639
1382625638739831836351146701335544471046574766464344286
9436960258972417073781445699309759148282470540637221053
7468106233830475275677886971687824230013771490609670050
3541122903221471643675599678353399411005925365651496462
9314076821292774671105864693953516519526805057835588967
7948431165149351114337903134884380149941388876543909551
2813526616352477194550779235973417637539686663616820292
5309091353031404343362849460618442292256607411481392572
4768179625766135592321574189926428394846860436067892659
3674518179479688704913835201275471264581112633388067624
8352990354650172195602256391324193471910442959284334376
5188509815616943570067904862907686040837966790975157598
4678905165650294721852479235765764790911387381306766477
2606649947626479423309921673903951483488501461753940724
6433650992858458438602194319660012946074078054558772286
5200909962468180842461226388757421286710692949247438116
6496284306835034964202880702457157638162051318899748441
3379967697774226941966967835333573457475461446792068659
2250825548702782471026644365690602359943201986202150414
5395225322283438267386125480890642458179321403518306256
7207603439677628003225690561124962650618448751471876888
7176193475630699119949518671430235438116560381159987616
5646432441435950947050026586956473035425686233869337114
8351650591892441845960627963146294149182897491472453611
4737958065362215960835591508566278706396315788136650074
6315430774137869201212360337421276847404487034376967555
9648687309860911013965613936540127831011704941292684763
6056810891768309894776550201151835643111756374850579757
9157319618860231789128833036465756967266311866967033057
6395226058259286975572403101416128342015650778033183476
6319286116435996967992334875193479286909665315823205374
1856762186480439943292640317077167968201550187689338964
5750835253860930884462847417741797444388428388160987225
6034857953383275355819314438944535651147635018748794322
8138520780358241251023092822153710713783671444186838721
9181193884064275227054473909965094276649031723796042390
7351790128362895831184368888722147118475404207489965602
7495034800572343966618860992965351723446042475488677358
0849314864593861367469831751929673835260097992253425831
6491356163863303305160780809435472474670742041767535736
4212670403963601311672682677835965540511552860798329798
4156471289429274154041220947845411963751590860665959709
8362751210479724368944086617847823966745836992574115902
6403602818015967446819756038893956562178989714450967411
7539416113681186833991074785352983526942698906339025754
5176404343904431487792963781473837764488094674941675329
1028392185389388693546262618428649198724018550492599186
9608166374641432388533640967094437936838743168601862608
6747116055979466458835163994856216517630457408856488416
5375427426914149541091441342855026748686048470584121079
8837479392845691703854533821023425052319193800444968367
6222832860779273860232912721940762670960625889407430836
7253314071250702946501445034181067488460286609405853062
6069352451054577487436338234542179976091426412719367643
8811686194960771551509440609120202540331936333333333333
8888888888888888888888888888888888888888888888888888888
8888888888888888888888888888888888888888888888888888888
8888888888888888888888888888888888888888888888888888888

11.4 Summary

The **perimeter** of a closed figure is the total length of its boundary. For example, the perimeter of a rectangle with length l and width w is $2(l+w)$.

Important: For any three points A, B, and C, we have

$$AB + BC \geq AC.$$

We have $AB + BC = AC$ if and only if B is on $\overline{AC}$. This relationship is called the **Triangle Inequality**.

The **area** of a closed figure is the number of unit squares (or pieces of squares) needed to exactly cover the figure. We sometimes use brackets to refer to area, so that $[ABC]$ means the area of $\triangle ABC$. The area of a rectangle equals the product of its length and width.

Important: To find the area of a triangle, we select one side to be the "base" of the triangle. The perpendicular segment from the vertex opposite the base to the line containing the base is the **altitude** to that base. The area then is

$$\frac{\text{base length} \times \text{altitude length}}{2}.$$

A **circle** consists of all points that are a fixed distance, called the **radius** of the circle, from a given point, called the **center** of the circle. We also use the word "radius" to describe a segment from the center of the circle to a point on the circle. A **diameter** is a line segment that connects two points on the circle and passes through the circle's center. We also use the word "diameter" to mean the length of a diameter. The perimeter of a circle is called the circle's **circumference**.

Important: If a circle has diameter d and radius r, then:

- $d = 2r$.
- The circumference of the circle is πd, or $2\pi r$, where π is a number that is approximately 3.14. (The symbol π is called "pi.")
- The area of the circle is πr^2.

Challenge Problems

11.39:

Source: MATHCOUNTS

Points A, B, C, and D lie on a line in alphabetical order. If $AB : BD = 5 : 7$ and $AC : CD = 13 : 11$, determine the ratio $AB : BC : CD$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $AB : BD = 5 : 7$, we know that there is some x such that $AB = 5x$ and $BD = 7x$. Similarly, since $AC : CD = 13 : 11$, there is some y such that $AC = 13y$ and $CD = 11y$. So, we have both $AD = AB + BD = 12x$ and $AD = AC + CD = 24y$, which means $12x = 24y$. This gives us $x = 2y$, so $AB = 5x = 10y$ and

$$BC = AD - AB - CD = 24y - 10y - 11y = 3y.$$

Therefore, we have

$$AB : BC : CD = 10y : 3y : 11y = \boxed{10 : 3 : 11}.$$

Another approach is to choose a length for $\overline{AD}$. A convenient choice is 24, since this will allow us to avoid fractions as segment lengths. Because $AC : CD = 13 : 11$ and AC is a CD or AC, we have $AC = 13$ and $CD = 11$. Because $AB : BD = 5 : 7$ and $AB + BD = 24$, we have $AB = 10$ and $BD = 14$. We then have

$$BC = BD - CD = 14 - 11 = 3,$$

so $AB : BC : CD = \boxed{10 : 3 : 11}$ as before.

11.40:

In rectangle $ABCD$, point X is the midpoint of $\overline{AD}$ and Y is the midpoint of $\overline{CD}$. What fraction of the area of the rectangle is enclosed by $\triangle AXY$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The area of the rectangle is $(AD)(DC)$. Since $\overline{YD}$ is the altitude from vertex Y of $\triangle AXY$, we have $[AXY] = (AX)(DY)/2$. Since X is the midpoint of $\overline{AD}$, we have $AX = AD/2$. Since Y is the midpoint of $\overline{DC}$, we have $DY = DC/2$. So, we have

$$[AXY] = \frac{(AX)(DY)}{2} = \frac{(AD/2)(DC/2)}{2} = \frac{(AD)(DC)}{8},$$

which means that the area of $\triangle AXY$ is $\boxed{\frac{1}{8}}$ the area of the rectangle.

11.41:

Point T is on side $\overline{QR}$ of $\triangle PQR$. Find the ratio QT/QR if the area of $\triangle PQT$ is 75 and the area of $\triangle PTR$ is 40.

Hint: We are given the areas of triangles $\overline{QT}$ and $\overline{QR}$ are bases of triangles.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Let b be the length of the altitude from P to $\overline{QR}$. This is the height from P for triangles PQT, PQR, and PTR. So, $[PQT] = (QT)(b)/2$ and $[PQR] = (QR)(b)/2$. When we divide the first equation by the second, we have

$$\frac{[PQT]}{[PQR]} = \frac{(QT)(b)/2}{(QR)(b)/2} = \frac{(QT)(b)}{2} \cdot \frac{2}{(QR)(b)} = \frac{QT}{QR}.$$

In other words, if two triangles share an altitude, then the ratio of the triangles' areas equals the ratio of the lengths of the bases to which the altitude is drawn. Here, we have $[PQT] = 75$ and

$$[PQR] = [PQT] + [PTR] = 75 + 40 = 115,$$

so we have

$$\frac{QT}{QR} = \frac{[PQT]}{[PQR]} = \frac{75}{115} = \boxed{\frac{15}{23}}.$$

11.42:

In the diagram on the right, $WXYZ$ is a rectangle. The area of triangle ZXA is 36, and $ZA = 3AY$.

(a) If $XY = 12$, then what is the area of rectangle $WXYZ$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We are given XY and seek for $[WXYZ]$. We have $[WXYZ] = (XY)(ZY)$, so we only need ZY to find $[WXYZ]$. We are also given $[ZXA]$. Since $\overline{XY}$ is the altitude from X to $\triangle ZXA$, we have $[ZXA] = (XY)(ZA)/2$. So, we can use the given value of $[ZXA]$ together with the value of XY to find ZA. Then, we use ZA to find ZY. Since $ZA = 3AY$, we have $AY = \frac{ZA}{3}$.

$$ZY = ZA + AY = ZA + \frac{ZA}{3} = \frac{4ZA}{3}.$$

Since $[ZXA] = (XY)(ZA)/2$, we have $36 = 12(ZA)/2$, so $36 = 6ZA$. Therefore, we have $ZA = 6$, so $ZY = \frac{4ZA}{3} = \frac{4 \cdot 6}{3} = 8$, and $[WXYZ] = (ZY)(XY) = \boxed{96}$.

(b) If $XY = 9$, then what is the area of rectangle $WXYZ$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Similar to part (a), since $[ZXA] = (XY)(ZA)/2$, we have $36 = 9(ZA)/2$, so $72 = 9ZA$. Therefore, we have $ZA = 8$, so $ZY = \frac{4ZA}{3} = \frac{4 \cdot 8}{3} = \frac{32}{3}$, and

$$[WXYZ] = (ZY)(XY) = \frac{32}{3} \cdot 9 = \boxed{96}.$$

(c) If $XY = 6$, then what is the area of rectangle $WXYZ$?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Similar to part (a), since $[ZXA] = (XY)(ZA)/2$, we have $36 = 6(ZA)/2$, so $36 = 3ZA$. Therefore, we have $ZA = 12$, so $ZY = \frac{4ZA}{3} = \frac{4 \cdot 12}{3} = 16$, and $[WXYZ] = (ZY)(XY) = \boxed{96}$.

(d) Do you notice a pattern in your answers to the first three parts? Will this pattern hold for other values of XY?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We found the same result for $[WXYZ]$ in all three parts. It's unlikely that this is a coincidence. Let's investigate. The area of $WXYZ$ is twice the area of $\triangle ZXY$. Triangles ZXY and ZXA have the same height from vertex X, and base $\overline{ZY}$ of $\triangle ZXY$ is $\frac{4}{3}$ times as long as base $\overline{ZA}$ of $\triangle ZXA$, so

$$[ZXY] = \frac{4}{3}[ZXA] = \frac{4}{3}(36) = 48.$$

Finally, we have $[WXYZ] = 2[ZXY] = 96$, no matter what XY is.

11.43:

What is the side length of a regular polygon whose interior angles each measure 168 degrees and whose perimeter is 120 cm?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The sum of the angles in a polygon with n sides is $(n-2)(180^\circ)$, so each angle in a regular polygon has measure $\frac{(n-2)(180^\circ)}{n}$. Since this must equal 168°, we have

$$\frac{(n-2)(180^\circ)}{n} = 168^\circ.$$

Multiplying both sides by n gives $(n-2)(180^\circ) = (n)(168^\circ)$. Expanding the product on the left gives $(n)(180^\circ) - 360^\circ = (n)(168^\circ)$. Adding 360° to both sides, and subtracting $(n)(168^\circ)$ from both sides, gives $(n)(12^\circ) = 360^\circ$. Dividing both sides by 12° gives $n = 30$. Since the polygon has 30 congruent sides and perimeter 120 cm, each side has length $120/30 = \boxed{4 \text{ cm}}$.

A much faster way to tackle this problem is to use the fact that the exterior angles of a polygon add to 360°. Since each interior angle is 168°, each exterior angle is $180^\circ - 168^\circ = 12^\circ$. Since all the exterior angles add to 360° and each measures 12°, there must be $\frac{360^\circ}{12^\circ} = 30$ of them. We then use the perimeter to determine that each side has length $\boxed{4 \text{ cm}}$, as before.

11.44:

In quadrilateral $ABCD$, each of the side lengths is an integer, and $AD = BC$. If we have $AB : AD = 2 : 5$ and $AD : CD = 3 : 4$, then what is the smallest possible perimeter of the quadrilateral?

Hint: AB is in both of the given ratio relationships.

Hint: We are given that $AB : AD = 2 : 5$, and that AB and AD are both integers. Could AD possibly be 47 Why or why not? Just from these given facts, what are the possible values of AB?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since $AB : AD = 2 : 5$, we have $AB = 2x$ and $AD = 5x$ for some value of x. Since AB and AD must be integers, and the greatest common divisor of 2 and 5 is 1, the value of x must be an integer. Similarly, since $AD : CD = 3 : 4$, there is some integer y for which $AD = 3y$ and $CD = 4y$. Combining our expressions for AD, we have $5x = 3y$. The smallest positive integers x and y for which this equation holds are $x = 3$ and $y = 5$. So, we have $AB = 2x = 6$, $AD = 5x = 15$, $CD = 4y = 20$, and $BC = AD = 15$. Finally, the perimeter of $ABCD$ is $6 + 15 + 20 + 15 = \boxed{56}$.

11.45:

$ABCDEF$ is a regular hexagon in which diagonal $\overline{AD}$ has length 16. Find the perimeter of $ABCDEF$.

Hint: Draw all three long diagonals.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Drawing all three long diagonals of a regular hexagon splits the regular hexagon into 6 equilateral triangles, as shown on the right. $\overline{AD}$ consists of two sides of these triangles, so each equilateral triangle has side length $16/2 = 8$. The sides of the regular hexagon consist of six sides of these triangles, so the perimeter of the hexagon is $6 \cdot 8 = \boxed{48}$.

11.46:

Rebecca walks 100 feet in a straight line. She then turns 20 degrees to the left and walks another 100 feet, and then turns 20 degrees to the left again. She continues this pattern until she reaches the point where she started. How far did she walk?

Hint: What shape does Rebecca's path form? What angles of this shape do you know?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: At each of her turns her path has a $180^\circ - 20^\circ = 160^\circ$ angle. Suppose she forms a regular polygon with her walk. (We don't know for sure that her angles will work out; we are just checking if it is possible that they do so.) Then, the 20° by which she turns is an exterior angle of the polygon. Since the exterior angles of the polygon sum to 360° and each has measure 20°, the polygon must have $\frac{360^\circ}{20^\circ} = 18$ angles. Checking, we see that each interior angle of a regular polygon with 18 sides is indeed $\frac{(18-2)(180^\circ)}{18} = 160^\circ$. Therefore, Rebecca's walk forms a regular polygon with 18 sides. Each side has length 100 feet, so she walks $18 \cdot 100 = \boxed{1800 \text{ feet}}$. (See the solution to 11.43 see how to find the number of sides of the polygon without considering the exterior angles.)

For a considerably more challenging problem, try to figure out how far Rebecca would walk if she turned 165° clockwise every 100 feet!

11.47:

Source: MATHCOUNTS

In the figure shown, points D, E, and F are the midpoints of semicircles ADB, BEC, and DFE, respectively. If the radius of each semicircle is 1, then what is the area of the shaded region?

Hint: We can find the areas of some complicated figures by rearranging them into figures whose areas we can find.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We break the upper semicircle into two quarter-circles and slide those two quarter-circular regions into the bottom, shown in the diagram at right. Now, the shaded region is a rectangle whose width is the radius of each semicircle and whose length is the diameter of each semicircle. Since the radius of each semicircle is 1, the area of the rectangle is $(1)(2) = \boxed{2}$.

11.48:

Teri has sticks of length 9 inches, 12 inches, and 14 inches. She cuts the same amount off of each stick. After cutting this amount off each stick, she is no longer able to make a triangle by attaching the sticks end to end. What is the smallest amount Teri could have cut off of each stick?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The only way Teri cannot make a triangle is if the sum of the lengths of the two shortest sticks is not greater than the length of the longest stick. If she cuts x of each stick, then the sticks have lengths $9 - x$, $12 - x$, and $14 - x$. The sum of the two shortest sticks then is $(9 - x) + (12 - x)$ which equals $21 - 2x$. If she can't make a triangle, then this sum must be less than or equal to the length of the longest stick, so $21 - 2x \leq 14 - x$. Adding $2x$ to both sides and subtracting 14 from both sides gives $7 \leq x$. Therefore, the smallest amount Teri could have cut off each stick is $\boxed{7 \text{ inches}}$. If she cuts 7 inches off of each stick, she'll have sticks with lengths 2 inches, 5 inches, and 7 inches. Since $2 + 5$ is not greater than 7, she cannot make a triangle with these sticks.

11.49★:

Source: MATHCOUNTS

Suppose we have $AB = 8$, $BC = 10$, $CD = 16$, and $DA = 12$ in quadrilateral $ABCD$. How many different possible integer values are there for AC?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We apply the Triangle Inequality separately to $\triangle ABC$ and $\triangle ADC$. Applying it to $\triangle ABC$, we see that AC must be greater than 2 and less than 18. Applying it to $\triangle ADC$, we see that AC must be greater than 4 and less than 28. Combining these, we see that AC must be greater than 4 and less than 18, so the possible lengths are 5, 6, 7, ..., 17. There are $\boxed{13}$ numbers in this list.

11.50★:

Points E, F, G, and H are midpoints of sides of square $ABCD$ as shown. If $\overline{AB}$ is 20 units long, then what is the area of the shaded region?

Hint: It might help to think outside the box on this one.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We do a little "outside the box" thinking, as shown at the right. We extend the pattern beyond the given initial square part each side, as shown. We see that each of the triangle pieces inside the original square fits together with a quadrilateral piece inside the square to form a square that is the same size as the black square. Therefore, inside the original square, the 8 white pieces that are quadrilaterals or triangles fit together to make 4 squares that are the same size as the black square. This means that the black square's area is $\frac{1}{5}$ the area of $ABCD$. The area of $ABCD$ is $(20)(20) = 400$ square units, so the area of the black square is $\boxed{80 \text{ square units}}$.

11.17.

Segment EN has midpoint M and point N is the midpoint of segment EP. What is the ratio of MN to EP?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Let the length of EN be x. Start with a diagram:

M is the midpoint of EN, so MN is $\frac{1}{2}x$ of EN. Similarly, N is the midpoint of EP so EN is $\frac{1}{2}x$ of EP. Combining these, we see that MN is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ of EP. Therefore, $MN : EP = \boxed{1:4}$.

11.18.

The length and width of a rectangle are each increased by 10%.

(a) By what percent is the perimeter of the rectangle increased?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Let the original length be l and the original width be w. After increasing both by 10%, the length is $1.1l$ and the width is $1.1w$.

The original perimeter is $2(l + w)$. The perimeter of the new rectangle is

$$2(1.1l + 1.1w) = 2(1.1)(l + w) = 1.1 \cdot 2(l + w).$$

Therefore, the new perimeter is 1.1 times the old perimeter, which means it is $\boxed{10\%}$ greater than the old perimeter.

(b) By what percent is the area of the rectangle increased?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Let the original length be l and the original width be w. After increasing both by 10%, the length is $1.1l$ and the width is $1.1w$.

The original area is lw. The area of the new rectangle is

$$(1.1l)(1.1w) = 1.21lw.$$

The new area is 1.21 times the old area, so it is $\boxed{21\%}$ greater than the old area.

11.19. Source: MATHCOUNTS

The perimeter of a hexagon is 300 cmm. One side of the hexagon is 45 cmm long and the lengths of the other sides are in the ratio $3 : 2 : 5 : 1 : 4$. Find the positive difference between the lengths of the longest and shortest sides.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: Let the smallest of the five sides in the ratio $1 : 2 : 3 : 4 : 5$ have length x, so the five sides have lengths $x, 2x, 3x, 4x$ and $5x$. Then the perimeter of the hexagon is $x + 2x + 3x + 4x + 5x = 15x$. Simplifying this gives a perimeter of $15x = 45$. We are given that the perimeter is 300, so $15x = 45 = 300$. Subtracting 45 from both sides gives $15x = 255$ and dividing by 15 gives $x = 17$. So, the sides have lengths $17, 34, 51, 68, 85$, and 85. Therefore, the positive difference between the lengths of the longest and shortest sides is $85 - 17 = \boxed{68 \text{ mm}}$.

11.20.

An architect draws a rectangular room to scale. The drawing is 12 cm long and 8 cm wide. The shorter dimension of the actual room is 20 feet. What is the longer dimension?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: The ratio of the length of the drawing to the width of the drawing is $12 : 8$. Simplifying this ratio, we have

$$\text{longer dimension} : \text{shorter dimension} = 3 : 2.$$

Since the shorter dimension of the room is 20 feet, we have

$$\text{longer dimension} : 20 \text{ feet} = 3 : 2.$$

Multiplying both parts of the ratio on the right by 10, we have

$$\text{longer dimension} : 20 \text{ feet} = 30 : 20,$$

so the longer dimension is 30 feet. (We also might have used the $3 : 2$ ratio to note that the longer dimension is $\frac{3}{2}$ times the shorter dimension. This means that the longer dimension of the room is $\frac{3}{2}(20) = 30$ feet.)

The perimeter of the room is $2(30 + 20) = \boxed{100 \text{ feet}}$.

11.21. Source: MODMU

Square tiles 9 inches on a side exactly cover the floor of a rectangular room. The border tiles are white and all other tiles are blue. The room measures 15 feet by 15 feet. How many tiles are white?

Preview Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission Solution

Solution: We have to be careful about the corners of the room. There are four corner tiles. Along one width of the room, the corner tiles cover $2 \cdot 9 = 18$ inches already. The width of the room is $15 \cdot 12 = 180$ inches, so we must tile 162 inches more. This requires $\frac{162}{9} = 18$ more tiles. The other width of the room also requires 18 more tiles. As with each width, 18 inches of each length of the room are already covered by corner tiles. The length of the room is $15 \cdot 12 = 216$ inches, so $216 - 18 = 198$ inches remain to tiled. This requires $\frac{198}{9} = 22$ tiles for each length. Including all 4 corners, both widths and both lengths, we need $4 + 18 + 18 + 22 + 22 = \boxed{82 \text{ tiles}}$.

11.22. Source: MATHCOUNTS

If adjacent sides meet at right angles in the figure at the right, what is the number of centimeters in the perimeter of the figure?

2
1
1
1

Choose always the way with the lowest length (it has the cashier and soon needed it way and separate. → Pythagoras

CHAPTER 12

Right Triangles and Quadrilaterals

12.1 The Pythagorean Theorem

In a right triangle, the side of the triangle opposite the right angle is called the **hypotenuse** and the other two sides are called the **legs** of the triangle. We also refer to the terms "leg" and "hypotenuse" when we are dealing with right triangles.

In this section, we explain one of the most famous math theorems, the **Pythagorean Theorem**, which is a geometrical relationship among the three sides of a right triangle, and is called the "Pythagorean Theorem" after the name of the mathematician Pythagoras (Πυθαγόρας in pronounced "puθaɣoˈɾas").

Problems

Problem 12.1 Apply to Solution

In an isosceles right triangles with legs of lengths 3 and 4 are attached to the sides of square $WXYZ$ as shown, such that $PW = QX = RY = SZ = 1$ and $PQ = QR = RS = ST = 1$.

- Question one: $PW = 1$ and PQ and QR are equal to 1.
- (a) What is the area of $PQRST$?
- (b) What is the area of $WXYZ$?
- (c) Find W, X.

Problem 12.2 Apply to Solution

In this problem, we follow in the steps of the previous problem to prove the Pythagorean Theorem. We start again with four copies of a right triangle with legs of lengths a and b and hypotenuse of length c. We arrange the four triangles in a square with side length $a + b$, as shown, and let the hypotenuse of each right triangle have length c.

- (a) Find the area of $WXYZ$ in terms of a, b.
- (b) Find the area of $PQRST$ in terms of c and b.
- (c) Find the area of $WXYZ$ in terms of a and b.
- (d) Show that $a^2 + b^2 = c^2$.

Problem 12.3 Apply to Solution

Find the missing side lengths in each of the three triangles shown below.

Problem 12.4 Apply to Solution

Find the hypotenuse of a right triangle from the longest side of the triangle and its other two legs.

Problem 12.5 Apply to Solution

In Problems 12.1 and 12.3, we have seen two right triangles in which all three side lengths are integers. Can you find any more right triangles in which all three side lengths are integers?

Problem 12.6 Apply to Solution

In Problems 12.1 and 12.3, we have seen two right triangles in which all three side lengths are integers. Can you find any more right triangles in which all three side lengths are integers?

Problem 12.7 Apply to Solution

The length of one leg of a right triangle is 2.0 and the triangle's hypotenuse has length 10.0. What is the length of the other leg?

Problem 12.8 Apply to Solution

Two square with sides of lengths 1 and 2 are attached to the sides of a right triangle with $90°$. Therefore, a right triangle $PRST$ is now formed, as shown in the figure below.

Since triangles WXZ and PRW are identical, we have $PW = XZ = 2$. Combining this into the equation above gives

$$2^2 + 1^2 = 1^2 + 2^2$$

We are told that $WXYZ$ is a square, so $\angle XWZ = 90°$. Therefore,

$$\angle PWX = \angle PWR + \angle XWZ = 90°$$

Since triangles WXZ and PRW are identical, we have $\angle PWR = \angle XWZ$. Substituting this into the equation above gives

$$\angle PWR + \angle PWR = 90°$$

$$\angle PWR = \angle PWR + \angle PWR = \angle PWR + \angle XWZ$$

$$= \angle PWR + \angle WR' + \angle PWR$$

$$= \angle PWR + \angle WR' + \angle WR'Z$$

$$= 180°$$

Therefore, $PWR'Z$ is a straight angle. Now, we can prove that $WR' = WZ$ as follows: Since $WXYZ$ is an square, all the sides of quadrilateral $PQRST$, Each side of $PQRST$ has length $S + 1 = 1$, and each angle of $PQRST$ is the right angle of one of the triangles. So, all the sides of $PQRST$ are equal, and all the angles of $PQRST$ are right angles, which means $PQRST$ is a square.

- Since $PQRST$ is a square with side length 1, $PR = \sqrt{2}$.
- Each right triangle has area $\frac{1}{2}(1)(2) = 1$. Four triangles have the right triangles from $PQRST$ have area $WXYZ$, so we have

$$[WXYZ] = [PQRST] - 4[\triangle] = 5 - 2 = 3.$$

- The area of $WXYZ$ is the square of its side length. Because the area of $WXYZ$ is 3, its side length must be $\sqrt{3}$, which equals WZ.

Problem 12.9 Apply to Solution

Find the missing side lengths in each of the three triangles shown below.

Problem 12.10 Apply to Solution

Find the missing side lengths in each of the three triangles shown below.

Solution For Problem 12.1

(a) Each is Worth 1/2 Square, we have learned that the entire angle of a right triangle which is 90°. Therefore, in right triangle PRW, we have

$$PW^2 + R = PR^2 \Rightarrow$$

Since triangles $WR'Z$ and PRW are identical, we have $PW = WR'$. Substituting this into the equation above gives

$$PW^2 + R = PR^2$$

We are told that $WXYZ$ is a square, so $\angle XWZ = 90°$. Therefore,

$$\angle PWX = \angle PWR + \angle XWZ = 90°$$

$$= \angle PWR + WR' + \angle PWR$$

$$= \angle PWR + WR' + \angle WR'Z$$

$$= 180°$$

Therefore, $PWR'Z$ is a straight angle. Now, we can prove that $WR' = WZ$ as follows: Since $WXYZ$ is an square, all the sides of quadrilateral $PQRST$, Each side of $PQRST$ has length $S + 1 = 1$, and each angle of $PQRST$ is the right angle of one of the triangles. So, all the sides of $PQRST$ are equal, and all the angles of $PQRST$ are right angles, which means $PQRST$ is a square.

- Since $PQRST$ is a square with side length 1, $PR = \sqrt{2}$.
- Each right triangle has area $\frac{1}{2}(1)(2) = 1$. Four triangles have the right triangles from $PQRST$ have $WXYZ$ area, so we have

$$[WXYZ] = [PQRST] - 4[\triangle] = 5 - 2 = 3.$$

- The area of $WXYZ$ is the square of its side length. Because the area of $WXYZ$ is 3, its side length must be $\sqrt{3}$, which equals WZ.

Problem 12.11 Apply to Solution

In Problems 12.1 and 12.3, we have seen two right triangles in which all three side lengths are integers. Can you find any more right triangles in which all three side lengths are integers?

Problem 12.12 Apply to Solution

Find the hypotenuse of a right triangle for the longer side of the triangle's leg or any side?

Solution For Problem 12.4. Step 1: The square of the hypotenuse equals the sum of the squares of the legs. The sum of any two positive integers is always greater than each individual integer. So, the square of the hypotenuse must be greater than the square of each leg. Therefore, the hypotenuse must be longer than each leg.

Solution For Problem 12.7. There are lots and lots of right triangles in which all three side lengths are integers! To search for some, we can find the first 20 positive perfect squares:

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400.$$

Then, we look for pairs of squares that add up to another square. We immediately see $9 + 16 = 25$, which is $3^2 + 4^2 = 5^2$. We already knew about this triangle from Problem 12.1. We also see $36 + 64 = 100$, which is $6^2 + 8^2 = 10^2$. We also see $100 + 144 = 244$, which is $10^2 + 12^2 = 14^2$. We also see $144 + 169 = 313$, which is $12^2 + 13^2 = 17^2$. We also see $225 + 400 = 625$, which is $15^2 + 20^2 = 25^2$. We also see $400 + 225 = 625$, which is $20^2 + 15^2 = 25^2$. We also see $400 + 400 = 800$, which is $20^2 + 20^2 = 40^2$. Notice that we can use the 3 and 4 to get 5, we can use the 6 and 8 to get 10, we can use the 10 and 12 to get 14, we can use the 12 and 13 to get 17, we can use the 15 and 20 to get 25, we can use the 20 and 15 to get 25, and we can use the 20 and 20 to get 40. So,

Choose always the way with the lowest length (it has the cashier and soon needed it way and separate. → Pythagoras

CHAPTER 12

Right Triangles and Quadrilaterals

12.1 The Pythagorean Theorem

In a right triangle, the side of the triangle opposite the right angle is called the **hypotenuse** and the other two sides are called the **legs** of the triangle. We also refer to the terms "leg" and "hypotenuse" when we are dealing with right triangles.

In this section, we explain one of the most famous math theorems, the **Pythagorean Theorem**, which is a geometrical relationship among the three sides of a right triangle, and is called the "Pythagorean Theorem" after the name of the mathematician Pythagoras (Πυθαγόρας in pronounced "puθaɣoˈɾas").

Problems

Problem 12.1 Apply to Solution

In an isosceles right triangles with legs of lengths 3 and 4 are attached to the sides of square $WXYZ$ as shown, such that $PW = QX = RY = SZ = 1$ and $PQ = QR = RS = ST = 1$.

- Question one: $PW = 1$ and PQ and QR are equal to 1.
- (a) What is the area of $PQRST$?
- (b) What is the area of $WXYZ$?
- (c) Find W, X.

Problem 12.2 Apply to Solution

In this problem, we follow in the steps of the previous problem to prove the Pythagorean Theorem. We start again with four copies of a right triangle with legs of lengths a and b and hypotenuse of length c. We arrange the four triangles in a square with side length $a + b$, as shown, and let the hypotenuse of each right triangle have length c.

- (a) Find the area of $WXYZ$ in terms of a, b.
- (b) Find the area of $PQRST$ in terms of c and b.
- (c) Find the area of $WXYZ$ in terms of a and b.
- (d) Show that $a^2 + b^2 = c^2$.

Problem 12.3 Apply to Solution

Find the missing side lengths in each of the three triangles shown below.

Problem 12.4 Apply to Solution

Find the hypotenuse of a right triangle from the longest side of the triangle and its other two legs.

Problem 12.5 Apply to Solution

In Problems 12.1 and 12.3, we have seen two right triangles in which all three side lengths are integers. Can you find any more right triangles in which all three side lengths are integers?

Problem 12.6 Apply to Solution

In Problems 12.1 and 12.3, we have seen two right triangles in which all three side lengths are integers. Can you find any more right triangles in which all three side lengths are integers?

Problem 12.7 Apply to Solution

The length of one leg of a right triangle is 2.0 and the triangle's hypotenuse has length 10.0. What is the length of the other leg?

Problem 12.8 Apply to Solution

Two square with sides of lengths 1 and 2 are attached to the sides of a right triangle with $90°$. Therefore, a right triangle $PRST$ is now formed, as shown in the figure below.

Since triangles WXZ and PRW are identical, we have $PW = XZ = 2$. Combining this into the equation above gives

$$2^2 + 1^2 = 1^2 + 2^2$$

We are told that $WXYZ$ is a square, so $\angle XWZ = 90°$. Therefore,

$$\angle PWX = \angle PWR + \angle XWZ = 90°$$

Since triangles WXZ and PRW are identical, we have $\angle PWR = \angle XWZ$. Substituting this into the equation above gives

$$\angle PWR + \angle PWR = 90°$$

$$\angle PWR = \angle PWR + \angle PWR = \angle PWR + \angle XWZ$$

$$= \angle PWR + \angle WR' + \angle PWR$$

$$= \angle PWR + \angle WR' + \angle WR'Z$$

$$= 180°$$

Therefore, $PWR'Z$ is a straight angle. Now, we can prove that $WR' = WZ$ as follows: Since $WXYZ$ is an square, all the sides of quadrilateral $PQRST$, Each side of $PQRST$ has length $S + 1 = 1$, and each angle of $PQRST$ is the right angle of one of the triangles. So, all the sides of $PQRST$ are equal, and all the angles of $PQRST$ are right angles, which means $PQRST$ is a square.

- Since $PQRST$ is a square with side length 1, $PR = \sqrt{2}$.
- Each right triangle has area $\frac{1}{2}(1)(2) = 1$. Four triangles have the right triangles from $PQRST$ have area $WXYZ$, so we have

$$[WXYZ] = [PQRST] - 4[\triangle] = 5 - 2 = 3.$$

- The area of $WXYZ$ is the square of its side length. Because the area of $WXYZ$ is 3, its side length must be $\sqrt{3}$, which equals WZ.

Problem 12.9 Apply to Solution

Find the missing side lengths in each of the three triangles shown below.

Problem 12.10 Apply to Solution

Find the missing side lengths in each of the three triangles shown below.

Solution For Problem 12.1

(a) Each is Worth 1/2 Square, we have learned that the entire angle of a right triangle which is 90°. Therefore, in right triangle PRW, we have

$$PW^2 + R = PR^2 \Rightarrow$$

Since triangles $WR'Z$ and PRW are identical, we have $PW = WR'$. Substituting this into the equation above gives

$$PW^2 + R = PR^2$$

We are told that $WXYZ$ is a square, so $\angle XWZ = 90°$. Therefore,

$$\angle PWX = \angle PWR + \angle XWZ = 90°$$

$$= \angle PWR + WR' + \angle PWR$$

$$= \angle PWR + WR' + \angle WR'Z$$

$$= 180°$$

Therefore, $PWR'Z$ is a straight angle. Now, we can prove that $WR' = WZ$ as follows: Since $WXYZ$ is an square, all the sides of quadrilateral $PQRST$, Each side of $PQRST$ has length $S + 1 = 1$, and each angle of $PQRST$ is the right angle of one of the triangles. So, all the sides of $PQRST$ are equal, and all the angles of $PQRST$ are right angles, which means $PQRST$ is a square.

- Since $PQRST$ is a square with side length 1, $PR = \sqrt{2}$.
- Each right triangle has area $\frac{1}{2}(1)(2) = 1$. Four triangles have the right triangles from $PQRST$ have $WXYZ$ area, so we have

$$[WXYZ] = [PQRST] - 4[\triangle] = 5 - 2 = 3.$$

- The area of $WXYZ$ is the square of its side length. Because the area of $WXYZ$ is 3, its side length must be $\sqrt{3}$, which equals WZ.

Problem 12.11 Apply to Solution

In Problems 12.1 and 12.3, we have seen two right triangles in which all three side lengths are integers. Can you find any more right triangles in which all three side lengths are integers?

Problem 12.12 Apply to Solution

Find the hypotenuse of a right triangle for the longer side of the triangle's leg or any side?

Solution For Problem 12.4. Step 1: The square of the hypotenuse equals the sum of the squares of the legs. The sum of any two positive integers is always greater than each individual integer. So, the square of the hypotenuse must be greater than the square of each leg. Therefore, the hypotenuse must be longer than each leg.

Solution For Problem 12.7. There are lots and lots of right triangles in which all three side lengths are integers! To search for some, we can find the first 20 positive perfect squares:

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400.$$

Then, we look for pairs of squares that add up to another square. We immediately see $9 + 16 = 25$, which is $3^2 + 4^2 = 5^2$. We already knew about this triangle from Problem 12.1. We also see $36 + 64 = 100$, which is $6^2 + 8^2 = 10^2$. We also see $100 + 144 = 244$, which is $10^2 + 12^2 = 14^2$. We also see $144 + 169 = 313$, which is $12^2 + 13^2 = 17^2$. We also see $225 + 400 = 625$, which is $15^2 + 20^2 = 25^2$. We also see $400 + 225 = 625$, which is $20^2 + 15^2 = 25^2$. We also see $400 + 400 = 800$, which is $20^2 + 20^2 = 40^2$. Notice that we can use the 3 and 4 to get 5, we can use the 6 and 8 to get 10, we can use the 10 and 12 to get 14, we can use the 12 and 13 to get 17, we can use the 15 and 20 to get 25, we can use the 20 and 15 to get 25, and we can use the 20 and 20 to get 40. So,

12.2 Some Special Triangles

In this section, we use the Pythagorean Theorem to investigate a few common special types of triangles.

Problem 12.0 ⇧ Jump to Solution

DEF is an isosceles triangle with $DE = EF = 26$ and $DF = 20$.

(a) Suppose M is on line DF such that $\overline{EM}$ is an altitude of the isosceles triangle. Why must M be the midpoint of $\overline{DF}$?

(b) Find the area of $\triangle DEF$.

Problem 12.9 ⇧ Jump to Solution

(a) Suppose $\triangle ABC$ is equilateral with side length 6. Find the length of an altitude of $\triangle ABC$.

(b) Find the area of $\triangle ABC$.

Problem 12.10 ⇧ Jump to Solution

(a) If one leg of an isosceles right triangle has length 6, then what are the lengths of the other leg and the hypotenuse of the triangle?

(b) If one leg of an isosceles right triangle has length a, then what are the lengths of the other leg and the hypotenuse of the triangle?

(c) If we know the length of one leg of an isosceles right triangle, then what's a fast way to find the length of the hypotenuse?

Problem 12.11 ⇧ Jump to Solution

A 30-60-90 triangle is a triangle whose angle measures 30°, 60°, and 90°.

(a) Describe how to divide an equilateral triangle into two congruent 30-60-90 triangles.

(b) Suppose the leg opposite the 30° angle in a 30-60-90 triangle has length 1. Use your answer to part (a) to find the hypotenuse of the triangle. What is the length of the other leg?

(c) Suppose the leg opposite the 30° angle in a 30-60-90 triangle has length s. In terms of s, what are the lengths of the hypotenuse and the other leg of the triangle?

Problem 12.0.A ⇧ ⇄ ✓

DEF is an isosceles triangle with $DE = EF = 26$ and $DF = 20$.

Question: M is on line DF such that $\overline{EM}$ is an altitude of the isosceles triangle. Why must M be the midpoint of $\overline{DF}$?

Question: Find the area of $\triangle DEF$.

Solution for Problem 12.0

Triangles DEM and FEM are right triangles. Applying the Pythagorean Theorem to both gives

$$DM^2 + EM^2 = DE^2 = 26^2 = 676,$$
$$FM^2 + EM^2 = EF^2 = 26^2 = 676.$$

Since the right-hand sides of these equations are equal, the left-hand sides must also be equal, so $DM^2 = FM^2$. Since DM and FM are both positive, this means that $DM = FM$. Therefore, we have $DM = FM$, which means that M is the midpoint of $\overline{DF}$.

Important: The altitude to the base of an isosceles triangle divides the base into two congruent segments.

(b)

Since M is the midpoint of $\overline{DF}$, we have $DM = DF/2 = 10$. Applying the Pythagorean Theorem to $\triangle DEM$ gives $DM^2 + EM^2 = DE^2$, so

$$10^2 + EM^2 = 26^2.$$

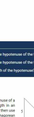
This gives us

$$EM^2 = 26^2 - 10^2 = 676 - 100 = 576.$$

We take the square root to find $EM = 24$. (Notice that we also could have found EM by noticing that $\overline{EM}$ has length 1 and proportional lengths 10 and 26.) So, using the $\{5, 12, 13\}$ Pythagorean triple, we know that the other leg $\overline{EM}$ has length 24, so the area of $\triangle DEF$ is $(20)(24)/2 = 240$ square units.

Concept: Building right triangles and applying the Pythagorean Theorem is one of the most important techniques in geometry problems.

In Problem 12.0, we discovered that we can think of an isosceles triangle as a pair of identical right triangles glued together along a common leg. This way of viewing an isosceles triangle reveals another important fact about isosceles triangles:



In an isosceles triangle, the angles opposite the equal sides have the same measure. If two sides of a triangle are equal, then the angles opposite those sides are equal. Conversely, if two angles of a triangle are equal, then the sides opposite those angles are equal.

The equal angles of an isosceles triangle are called the **base angles** of the triangle, and the other angle is called the **vertex angle** of the triangle.

Problem 12.9 ⇧ ⇄ ✓

(a) Suppose $\triangle ABC$ is equilateral with side length 1. Find the length of an altitude of $\triangle ABC$.

(b) Find the area of $\triangle ABC$.

Solution for Problem 12.9

(a)

We start by drawing altitude $\overline{CD}$ of the triangle. Because the Pythagorean Theorem is both given and useful, we'll also draw $\overline{AD}$ into our picture. So, we have $AD = DB = 1$. Applying the Pythagorean Theorem to $\triangle ADC$ gives

$$AD^2 + DC^2 = AC^2.$$

Since $AD = 1$ and $AC = 6$, we have $1^2 + DC^2 = 6^2$, which means $DC^2 = 6^2 - 1^2 = 36 - 9 = 27$.

Taking the square root gives $DC = \sqrt{27} = 3\sqrt{3} = 3\sqrt{3}$.

(b) Now that we have the length of an altitude, we can find the area:

$$\{ABC\} = \frac{(AB)(CD)}{2} = \frac{(6)(3\sqrt{3})}{2}$$
$$= \frac{6}{2} \cdot 3\sqrt{3}$$
$$= 3 \cdot 3\sqrt{3}$$
$$= 9\sqrt{3} \text{ square units.}$$

Isosceles Right Triangles

Problem 12.10 ⇧ ⇄ ✓

(a) If one leg of an isosceles right triangle has length 6, then what are the lengths of the other leg and the hypotenuse of the triangle?

(b) If one leg of an isosceles right triangle has length a, then what are the lengths of the other leg and the hypotenuse of the triangle?

(c) If we know the length of one leg of an isosceles right triangle, then what's a fast way to find the length of the hypotenuse?

Solution for Problem 12.10

(a)

Since the triangle is isosceles, two of the sides must have the same length. We know that the hypotenuse of a right triangle is the longest side of the triangle. So, it must be the leg that has the same length as an isosceles right triangle. This means the other leg of the triangle is also the problem has length 6. So, then we use the Pythagorean Theorem to find the hypotenuse. If we let c be the hypotenuse, then the Pythagorean Theorem tells us that

$$c^2 = 6^2 + 6^2 = 36 + 36 = 72.$$

Taking the square root gives us

$$c = \sqrt{36} = \sqrt{36 \cdot 2} = 6\sqrt{2}.$$

We conclude that each leg of the right triangle has length 3 and the hypotenuse has length $3\sqrt{2}$.

(b)

We follow essentially the same steps as in the previous part. Since the triangle is isosceles, the legs have the same length, so the second leg has length a. We let c be the hypotenuse, then the Pythagorean Theorem tells us that

$$a^2 + a^2 = 6^2 = 6 \times 6 = 12a.$$

Taking the square root gives us

$$c = \sqrt{12a} = \sqrt{6 \cdot 2 \cdot a} = 6\sqrt{2}.$$

We conclude that each leg of the right triangle has length a and the hypotenuse has length $a\sqrt{2}$.

(c)

So far, we have seen an isosceles right triangle with side lengths $6, 6, 6\sqrt{2}$ and one with side lengths $a, a, a\sqrt{2}$. It looks like there's a pattern. We know that the leg of an isosceles right triangle must always have the same length, but it always true that the hypotenuse is $\sqrt{2}$ times each leg?

We investigate by assigning a variable to the leg length. Suppose each leg has length x. Then, the sum of the squares of the legs is $x^2 + x^2$, which equals $2x^2$. So, the Pythagorean Theorem tells us that the square of the hypotenuse equals $2x^2$, which means the length of the hypotenuse is $\sqrt{2x^2}$. We can simplify $\sqrt{2x^2}$ as follows:

$$\sqrt{2x^2} = \sqrt{x^2} \cdot \sqrt{2} = x\sqrt{2}.$$

We therefore see that the hypotenuse is $\sqrt{2}$ times as long as each leg. (As a side note, we usually write $x\sqrt{2}$ instead of $\sqrt{2}x$ to make it clear that the x is not inside the radical.)

Important: In an isosceles right triangle, the legs are congruent and the hypotenuse is $\sqrt{2}$ times as long as each leg.

Note that the two acute angles of an isosceles right triangle must be congruent, since they are opposite the congruent sides of the triangle. Because the acute angles of a right triangle must sum to 90°, each of these two angles has measure $90^\circ/2 = 45^\circ$. For this reason, isosceles right triangles are often referred to as **45-45-90 triangles**.

Isosceles Right Triangles

Problem 12.11 ⇧ ⇄ ✓

A 30-60-90 triangle is a triangle whose angle measures 30°, 60°, and 90°.

(a) Describe how to divide an equilateral triangle into two congruent 30-60-90 triangles.

(b) Suppose the leg opposite the 30° angle in a 30-60-90 triangle has length 1. Use your answer to part (a) to find the hypotenuse of the triangle. What is the length of the other leg?

(c) Suppose the leg opposite the 30° angle in a 30-60-90 triangle has length s. In terms of s, what are the lengths of the hypotenuse and the other leg of the triangle?

Solution for Problem 12.11

(a) First, we note that all three angles of an equilateral triangle must be equal, because any two of the angles are opposite equal sides of the triangle. So, each angle of an equilateral triangle must measure $180^\circ/3 = 60^\circ$. Now, we can divide an equilateral triangle into two congruent 30-60-90 triangles by drawing an altitude to one side of the triangle. Applying the Pythagorean Theorem to $\triangle ABD$ and $\triangle CBD$ shows that $\triangle ABD$ and $\triangle CBD$ are congruent.

(b)

Inspired by our observation in part (a), we see that we can make an equilateral triangle by attaching two identical 30-60-90 triangles along the longer leg of each triangle. This is shown at the right where we have combined right triangles ABD and ACD to form equilateral triangle ABC.

Since we started with a 30-60-90 triangle with shortest side length 1, we have $AD = 1$. Since $\triangle ABC$ is equilateral, we have $AB = AC = BC = 2$. So, we have $BD = 2 - 1 = 1$. Therefore, we have lengths of the sides of $\triangle ABD$ of $1, 2, 1$. Since $\triangle ABD$ is a 30-60-90 triangle, the lengths of the sides of $\triangle ABD$ are $1, 2, 1$. Finally, we apply the Pythagorean Theorem to $\triangle ABD$ to find that

$$BD^2 + AD^2 = AB^2.$$

Since $BD = 1$ and $AB = 2$, we have $1^2 + AD^2 = 2^2$, so $AD^2 = 4^2 - 1^2 = 16 - 1 = 15$. Taking the square root gives $AD = \sqrt{15} = 1\sqrt{15} = 1\sqrt{15}$. So, we see that the shorter side of a 30-60-90 triangle has length 1, and the other leg has length $1\sqrt{15}$ and the hypotenuse has length 2.

(c)

In the previous part we found a 30-60-90 triangle with side lengths $2, 1\sqrt{3}, 1$. Look closely at Problem 12.9, and you'll see a 30-60-90 triangle with side lengths $3, 3\sqrt{3}, 6$. It looks like there's a pattern! Let's see if this pattern always holds, so we don't have to go through all this work for every triangle.

We start with an arbitrary 30-60-90 triangle, shown at the right. We can make an equilateral triangle by attaching two of these triangles along their longer legs, just as we did in the previous part. We know that the length of the shortest side of each triangle is s, so we have a triangle with side lengths s, s, s, so the equilateral triangle has length $2s$. Then, we apply the Pythagorean Theorem to $\triangle ABD$ to find that

$$BD^2 + AD^2 = AB^2.$$

Since $BD = s$ and $AB = 2s$, we have $s^2 + AD^2 = (2s)^2$, so $AD^2 = (2s)^2 - s^2 = 4s^2 - s^2 = 3s^2$. Taking the square root gives $AD = \sqrt{3s^2} = \sqrt{3} \cdot \sqrt{s^2} = s\sqrt{3}$. Finally, taking the square root gives

$$AD = \sqrt{3s^2} = \sqrt{3} \cdot \sqrt{s^2} = s\sqrt{3}.$$

This confirms the pattern we saw! For any 30-60-90 triangle with shortest leg of length s the longer leg has length $s\sqrt{3}$ and the hypotenuse has length $2s$. In other words, the side lengths of a 30-60-90 triangle always come in the ratio

$$1 : \sqrt{3} : 2.$$

Important: In a right triangle with acute angles of 30° and 60°, the side lengths are in the ratio $1 : \sqrt{3} : 2$. Equivalently, the hypotenuse is twice the shorter side and the longer side is $\sqrt{3}$ times the shorter side.

Exercises

12.2.1: ⇧ ⇄ ✓

Find the missing side lengths below:

You may type any additional notes here.

Hide Solution

Send

Your Submission Solution

Solution: Triangle XYZ is an isosceles right triangle, so its hypotenuse is $\sqrt{2}$ times the length of each leg. Therefore, $XY = \frac{6}{\sqrt{2}}$ and $x = \frac{6}{\sqrt{2}}$.

For $\triangle ZTU$, we first note that because $\angle U = \angle Y$, we have $TU = \frac{6}{\sqrt{2}}$. Next we must find TU. Let y be the unknown length of the shorter leg opposite the 30° angle in $\triangle ZTU$. Applying the Pythagorean Theorem to $\triangle ZTU$ gives $TU^2 + TU^2 = TU^2$ so

$$TU^2 = 2TU^2 = 2 \cdot y^2 = 4y^2 = 36, \text{ so } y^2 = 18.$$

Therefore, we have $TU = \sqrt{18} = 12$. We could also have used the Pythagorean triple $\{12, 16, 20\}$. Because $\triangle ZTU$ is isosceles with $\angle U = \angle Y = 45^\circ$, the sides of $\triangle ZTU$ further have TU as two congruent segments. Therefore, we have $z = 2(TU) = \frac{12}{1}$.

We immediately have $x = \frac{6}{\sqrt{2}} = \frac{6}{1}$. We just TU and two triangles by drawing $\overline{TX}$, as shown at the right. Since $QP^2 = 8^2$, we know that $\triangle QPR$ is isosceles with $\angle QPR = \angle QRP$. Since $\angle Q = 60^\circ$, we have two angles equal to $\frac{180^\circ - 60^\circ}{2} = 60^\circ$. Therefore, each measure 60°, which means $\triangle QPR$ is equilateral. Therefore, $PR = 8$. $\angle RPS = 120^\circ - \angle QPR = 60^\circ$, and $\angle PRS = 90^\circ - 60^\circ = 30^\circ$.

Turning to $\triangle PRS$, we have

$$\angle S = 180^\circ - \angle RPS - \angle PRS = 90^\circ.$$

So $\triangle PRS$ is a 30-60-90 triangle with hypotenuse $PR = 14$. The leg opposite the 30° equals half the hypotenuse, so $PS = \frac{14}{2}$. The longer leg is $\sqrt{3}$ times the shorter leg, so $RS = \frac{7\sqrt{3}}{1}$.

12.2.2: ⇧ ⇄ ✓

In $\triangle ABC$, we have $AB = BC$ and $\angle B = 60^\circ$. Find $\angle A$.

You may type any additional notes here.

Hide Solution

Send

Your Submission Solution

Solution: Since $AB = BC$, triangle ABC is isosceles. The angles opposite the congruent sides are congruent, so $\angle A = \angle C$. Since $\angle B = 60^\circ$, we know that the other two angles of the triangle sum to $180^\circ - 60^\circ = 112^\circ$. These two angles have the same measure, so each equals $112^\circ/2 = \frac{56}{1}$.

12.2.3: ⇧ ⇄ ✓

In $\triangle ABC$, we have $\angle A = 3\angle B = 90^\circ$ and $\angle C = 4\angle B$. Find $\angle C$.

You may type any additional notes here.

Hide Solution

Send

Your Submission Solution

Solution: Since $3\angle A = 90^\circ$ we have $\angle A = 30^\circ$. We also have $\angle A = 90^\circ$, so $\angle A\angle C_1$ is a 30-60-90 triangle with hypotenuse $\overline{AC_1}$, short leg $\overline{C_1C}$ (opposite the 30° angle), and long leg $\overline{AC}$. The hypotenuse of a 30-60-90 triangle is twice the short leg, so the short leg of $\triangle ABC$ is $AC = 3/2 = \frac{3}{2}$. The long leg of a 30-60-90 triangle is $\sqrt{3}$ times the short leg, so $AB = \frac{3\sqrt{3}}{2}$.

12.2.4: Source: AMC 8 ⇧ ⇄ ✓

The sum of an isosceles triangle is 24 and its area is 96. What is the length of one of the equal sides?

You may type any additional notes here.

Hide Solution

Send

Your Submission Solution

Solution: Let A be the length of the altitude to the base with length 24. Then, since the triangle's area is 96, we must have $\frac{24 \cdot A}{2} = 96$. Therefore, $12A = 96$ which means $A = 8$. As discussed in the text, the altitude to the base of an isosceles triangle divides the base into two equal segments. So, the segments on the base are $12 = 24/2$. Therefore, the hypotenuse of the right triangle ABD has lengths 8 and 12. Applying the Pythagorean Theorem, or recalling the Pythagorean triple $\{5, 12, 13\}$, tells us that the hypotenuse has length $\frac{13}{1}$.

12.2.5: ⇧ ⇄ ✓

This week in the diagram at the right two radius $\overline{CD}$ and center CD. Points A and B are on the circumference of the circle such that $\angle ACD = 60^\circ$.

(a) We call $\overline{AB}$ a **chord** of the circle because its endpoints are on the circle. Find AB.

You may type any additional notes here.

Hide Solution

Send

Your Submission Solution

Solution: Since $\overline{CD}$ and $\overline{CB}$ are radii of the circle, they both have length 10. Therefore, $\triangle ACD$ is isosceles, and $\angle A = \angle D$. Since $\angle D = 60^\circ$, we have $\angle A = \angle B = 180^\circ - \angle D = 120^\circ$. Therefore, $\angle A = \angle B$, so $\triangle ACB$ is isosceles with $AC = CB = 10^\circ$, so all three angles of $\triangle ACB$ are congruent. This means that $\triangle ACB$ is equilateral, which means $AB = OA = OB = \frac{10}{1}$.

(b) An **arc** of a circle is a portion of the circle's circumference. There are two arcs from A to B in the diagram. Find the length of the shorter of these two arcs.

You may type any additional notes here.

Hide Solution

Send

Your Submission Solution

Solution: Since $\angle ACD = \frac{60^\circ}{360^\circ}$ of a circle, the shorter arc from A to B is $\frac{1}{6}$ of the circumference of the circle. The diameter of the circle is $2 \cdot 10 = 20$, so the circumference of the circle is 20π. Therefore, the length of the shorter arc from A to B is $\frac{1}{6}(20\pi) = \frac{10\pi}{3}$.

(c) A **sector** of a circle is a portion of the circle's interior that is bounded by two radii and an arc of the circle. Just as there are two arcs from A to B, there are two sectors. Find the area of the sector in the shorter arc from A to B. (Note: Here's the "pie" shape with angle $\angle ACD$, and the other is the portion of the circle outside this angle. Find the area of the smaller of these sectors.

You may type any additional notes here.

Hide Solution

Send

Your Submission Solution

Solution: Just as the shorter arc cut off by $\angle ACD$ is $\frac{1}{6}$ of the circumference of the circle, the sector cut off by $\angle ACD$ has area equal to $\frac{1}{6}$ the area of the circle. The area of the circle is $\pi \cdot 10^2 = 100\pi$, so the area of the sector is $\frac{1}{6}(100\pi) = \frac{50\pi}{3}$.

12.2.6A: ⇧ ⇄ ✓

(a) Let $\triangle XYZ$ be an equilateral triangle with side length s. Find a formula for the area of $\triangle XYZ$ in terms of s.

Hint: When trying to find a formula, it's often best to first solve the problem for specific values. Find the area when $s = 8$.

Hint: Can you use the same steps you used to tackle the first hint to find the formula?

Preview Solution

You may type any additional notes here.

Hide Solution

Send

Your Submission Solution

Solution: Let $\triangle XYZ$ be an equilateral triangle with side length s, as shown in the diagram at the right. We draw altitude $\overline{ZD}$ and form right triangle XZD. The altitude splits base $\overline{XY}$ in half, so $XD = \frac{s}{2}$. So, $\triangle XYZ$ is equilateral, so we have $\angle X = 60^\circ$ which means $\triangle XZD$ is a 30-60-90 triangle. Therefore, $ZD = 3/2\sqrt{3} = \frac{s}{2}\sqrt{3}$, so we have

$$[XYZ] = \frac{(XY)(ZD)}{2} = \frac{(s)\left(\frac{s}{2}\sqrt{3}\right)}{2} = \left(\frac{s}{4}\sqrt{3}\right) = \frac{s^2\sqrt{3}}{4}.$$

From part (c) of the previous problem, the area of the sector is 2π, so the desired area is $\left(\frac{25\pi - 36\sqrt{3}}{3}\right)$.

(b) The circle in the diagram at the right has radius 12 and center O. Points A and B are on the circumference of the circle such that $\angle AOB = 60^\circ$.

Hint: Is the desired region part of another figure whose area you can find?

Hint: The desired region is part of sector AOB. What is the other part of the sector?

You may type any additional notes here.

Hide Solution

Send

Your Submission Solution

Solution: The region between $\overline{AB}$ and the shorter arc from A to B is what's left after removing $\triangle AOB$ from the sector in part (c) of the previous problem. So, the area of the desired region is $2\pi - \frac{36\sqrt{3}}{3}$. We can also subtract the area of $\triangle AOB$. Since $\triangle AOB$ is equilateral, we can use the formula we found in part (a) to obtain similar answer.

$$[OAB] = \frac{12^2\sqrt{3}}{4} = \frac{144\sqrt{3}}{4} = 36\sqrt{3}.$$

13.2.3 Types of Quadrilaterals

Rectangle, rhombus, square, parallelogram, trapezoid, kite, isosceles trapezoid, and isosceles kite are all types of quadrilaterals. The diagram below shows how these quadrilaterals are related to each other. For example, a square is a rectangle and a rhombus.

Figure 13.2: A Venn diagram of quadrilaterals

Figure 13.3: A chart of quadrilaterals

Use the Venn diagram and the chart to answer the following questions.

Problems

Problem 13.1 ▶ Answer in Section 13.1

Is a square a rhombus?

Problem 13.2 ▶ Answer in Section 13.1

Is a rhombus a square?

Problem 13.3 ▶ Answer in Section 13.1

Is a rectangle a parallelogram?

Problem 13.4 ▶ Answer in Section 13.1

Is a parallelogram a rectangle?

Problem 13.5 ▶ Answer in Section 13.1

Is a trapezoid a parallelogram?

Problem 13.6 ▶ Answer in Section 13.1

Is a parallelogram a trapezoid?

Problem 13.7 ▶ Answer in Section 13.1

Is a kite a trapezoid?

Problem 13.8 ▶ Answer in Section 13.1

Is a trapezoid a kite?

Problem 13.9 ▶ Answer in Section 13.1

Is a rectangle a rhombus?

Problem 13.10 ▶ Answer in Section 13.1

Is a rhombus a rectangle?

Problem 13.11 ▶ Answer in Section 13.1

Is a square a rectangle?

Problem 13.12 ▶ Answer in Section 13.1

Is a rectangle a square?

Problem 13.13 ▶ Answer in Section 13.1

Is a rectangle a parallelogram?

Problem 13.14 ▶ Answer in Section 13.1

Is a parallelogram a rectangle?

Problem 13.15 ▶ Answer in Section 13.1

Is a rhombus a parallelogram?

Problem 13.16 ▶ Answer in Section 13.1

Is a parallelogram a rhombus?

Problem 13.17 ▶ Answer in Section 13.1

Is a square a rhombus?

Problem 13.18 ▶ Answer in Section 13.1

Is a rhombus a square?

Problem 13.19 ▶ Answer in Section 13.1

Is a square a parallelogram?

Problem 13.20 ▶ Answer in Section 13.1

Is a parallelogram a square?

Problem 13.21 ▶ Answer in Section 13.1

A **trapezoid** is a quadrilateral in which both sides are parallel. Is a rectangle a trapezoid?

Problem 13.22 ▶ Answer in Section 13.1

A **trapezoid** is a quadrilateral in which both sides are parallel. Is a rhombus a trapezoid?

Problem 13.23 ▶ Answer in Section 13.1

A **trapezoid** is a quadrilateral in which both sides are parallel. Is a square a trapezoid?

Problem 13.24 ▶ Answer in Section 13.1

A **trapezoid** is a quadrilateral in which both sides are parallel. Is a parallelogram a trapezoid?

Problem 13.25 ▶ Answer in Section 13.1

A **trapezoid** is a quadrilateral in which both sides are parallel. Is a kite a trapezoid?

Problem 13.26 ▶ Answer in Section 13.1

A **trapezoid** is a quadrilateral in which both sides are parallel. Is a trapezoid a kite?

Problem 13.27 ▶ Answer in Section 13.1

A **kite** is a quadrilateral in which two pairs of adjacent sides are equal. Is a rectangle a kite?

Problem 13.28 ▶ Answer in Section 13.1

A **kite** is a quadrilateral in which two pairs of adjacent sides are equal. Is a rhombus a kite?

Problem 13.29 ▶ Answer in Section 13.1

A **kite** is a quadrilateral in which two pairs of adjacent sides are equal. Is a square a kite?

Problem 13.30 ▶ Answer in Section 13.1

A **kite** is a quadrilateral in which two pairs of adjacent sides are equal. Is a parallelogram a kite?

Problem 13.31 ▶ Answer in Section 13.1

A **kite** is a quadrilateral in which two pairs of adjacent sides are equal. Is a trapezoid a kite?

Problem 13.32 ▶ Answer in Section 13.1

A **kite** is a quadrilateral in which two pairs of adjacent sides are equal. Is a kite a trapezoid?

Problem 13.33 ▶ Answer in Section 13.1

Find the area of the shaded quadrilateral.

Problem 13.34 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.35 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.36 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.37 ▶ Answer in Section 13.2

If $\overline{AC}$ is the diagonal of the kite-shaped figure, $AC = 7$ and $BD = 12$ in the picture, what is the area of $\triangle ABC$?

Problem 13.38 ▶ Answer in Section 13.2

If $\overline{AC}$ is the diagonal of the kite-shaped figure, $AC = 7$ and $BD = 12$ in the picture, what is the area of $\triangle ADC$?

Problem 13.39 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.40 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.41 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.42 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.43 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.44 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.45 ▶ Answer in Section 13.2

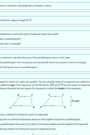
Find the area of the shaded quadrilateral.

Problem 13.46 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.47 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.



Problem 13.48 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.49 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.50 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.51 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.52 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.53 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.54 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.55 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.56 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.57 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.58 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.59 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.60 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.61 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.62 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.63 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.64 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.65 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.66 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.67 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.68 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.69 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.70 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.71 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.72 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.73 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.74 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.75 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.76 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.77 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.78 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.79 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.80 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.81 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.82 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.83 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.84 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.85 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.86 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.87 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.88 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.89 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.90 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.91 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.92 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.93 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.94 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.95 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.96 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.97 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.98 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.99 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.100 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.101 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.102 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.103 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.104 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.105 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.106 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.107 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.108 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.109 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.110 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.111 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.112 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.113 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.114 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.115 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.116 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.117 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.118 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.119 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.120 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.121 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.122 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.123 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.124 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.125 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.126 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.127 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.128 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.129 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.130 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.131 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.132 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.133 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.134 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.135 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.136 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.137 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.138 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.139 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.140 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.141 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.142 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.143 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.144 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.145 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.146 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.147 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.148 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.149 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.150 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.151 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.152 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.153 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.154 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.155 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.156 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.157 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.158 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.159 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.160 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.161 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.162 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.163 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.164 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.165 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.166 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.167 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.168 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.169 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.170 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.171 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.172 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.173 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.174 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.175 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.176 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.177 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.178 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.179 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.180 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.181 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.182 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.183 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.184 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.185 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.186 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.187 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.188 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.189 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.190 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.191 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.192 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.193 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.194 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

Problem 13.195 ▶ Answer in Section 13.2

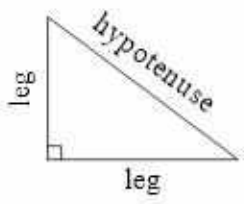
Find the area of the shaded quadrilateral.

Problem 13.196 ▶ Answer in Section 13.2

Find the area of the shaded quadrilateral.

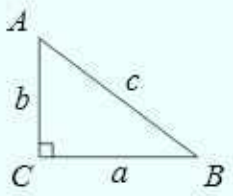
12.4 Summary

In a right triangle, the side of the triangle opposite the right angle is called the **hypotenuse** and the other two sides are called the **legs** of the triangle. We also often use the terms "legs" and "hypotenuse" to refer the lengths of the legs and hypotenuse of a right triangle.



Important: The **Pythagorean Theorem** tells us that in any right triangle, the sum of the squares of the legs equals the square of the hypotenuse. So, in the diagram to the right, we have

$$a^2 + b^2 = c^2.$$

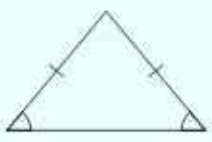


The Pythagorean Theorem also works in reverse: if the sum of the squares of two sides of a triangle equals the square of the third side, then the triangle is a right triangle.

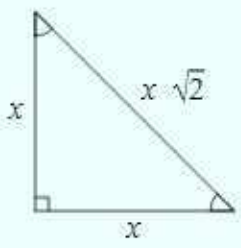
A **Pythagorean triple** is a group of three positive integers that satisfy the Pythagorean Theorem equation. So, for example, $\{3, 4, 5\}$ is a Pythagorean triple, as are $\{5, 12, 13\}$ and $\{8, 15, 17\}$. There are infinitely many Pythagorean triples.

Important: If we multiply all three side lengths of a right triangle by the same positive number, then the three new side lengths also satisfy the Pythagorean Theorem. In other words, if side lengths a, b, and c satisfy $a^2 + b^2 = c^2$, then $(na)^2 + (nb)^2 = (nc)^2$ for any number n.

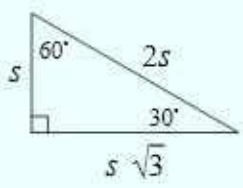
Important: In an isosceles triangle, the angles opposite the equal sides have the same measure. If two sides of a triangle are equal, then the angles opposite those sides are equal. Similarly, if two angles of a triangle are equal, then the sides opposite those angles are equal.



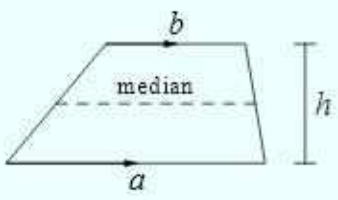
Important: In an isosceles right triangle, the legs are congruent and the hypotenuse is $\sqrt{2}$ times as long as each leg. An isosceles triangle is often called a **45-45-90 triangle**.



Important: In a right triangle with acute angles of $30°$ and $60°$, the side lengths are in the ratio $1 : \sqrt{3} : 2$ as shown to the right. Such a triangle is often called a **30-60-90 triangle**.



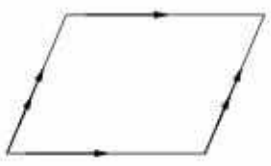
Definitions: A **trapezoid** is a quadrilateral with two parallel sides. The segment connecting the midpoints of the other two sides is the **median** of the trapezoid, and the distance between the two parallel sides is the **height** of the trapezoid.



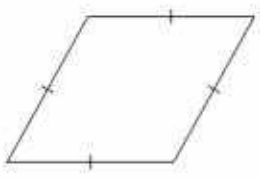
Important:
- The median of a trapezoid is parallel to the bases of the trapezoid, and equal in length to the average of the lengths of the bases.
- The area of a trapezoid equals the height of the trapezoid times the length of the median of the trapezoid. So, the area of the trapezoid above is $h(a + b)/2$.

Definition: A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel.

Important:
- The area of a parallelogram is the product of a side length (the base) and the distance between that side and the opposite side of the parallelogram. This distance between opposite sides is called a **height** of the parallelogram.
- In any parallelogram, the opposite sides are equal, and the opposite angles are equal.



A Parallelogram



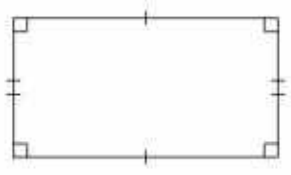
A Rhombus

Definition: A quadrilateral is a **rhombus** if all of its sides are equal.

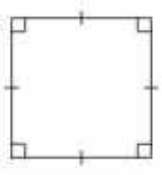
Important:
- Every rhombus is a parallelogram. Therefore, everything that is true about parallelograms is true about every rhombus.
- The diagonals of a rhombus are perpendicular. The area of a rhombus is half the product of its diagonals (and also equals its base times its height).

Definition: A quadrilateral in which all angles are equal is a **rectangle**.

Important:
- All rectangles are parallelograms, so all that is true of parallelograms is true of rectangles.
- Let two consecutive sides of a rectangle have lengths ℓ and w. The area of the rectangle is ℓw, and the diagonals of the rectangle both have length $\sqrt{\ell^2 + w^2}$.



A Rectangle



A Square

Definition: A quadrilateral in which all sides are equal and all angles are equal is a **square**.

Important:
- Each square is a parallelogram, a rectangle, and a rhombus, so all that is true of parallelograms, rectangles, or rhombuses is true of squares.
- If the side length of a square is s and its diagonal is d, then $d = s\sqrt{2}$ and the area of the square is s^2, or $d^2/2$.

WARNING!! Although every rhombus is a parallelogram, not every parallelogram is a rhombus. Therefore, if we prove a property that is true for every rhombus, this property is not necessarily true for every parallelogram. (The same warning holds for rectangles–rectangles are parallelograms, but not all parallelograms are rectangles, etc.)

12.37:

In the diagram at the left, O is the center of the circle, $SOXOP$ is a rectangle, and the area of the circle is 196π. What is the length of diagonal $\overline{PS}$ of the rectangle?

You may type any additional notes here.

Hide Solution | Reset

Your Submission Solution:

Preview Solution

The diagonals of a rectangle are congruent, so $XP = OS$. OXS is a radius of the circle. Since the area of the circle is 196π, the radius radius is $\sqrt{196} = 14$. Therefore, $XP = \boxed{14}$.

12.38:

In the diagram at the right, $\overline{ST}$ $\overline{DC}$ and the area of $\triangle ABD$ is 15. Given that the area of $\triangle BDC$ is $\frac{4}{3}$ the area of $\triangle ABC$, $AB = CD = 7$, find $\sin \angle ABD$.

Hint: we are given the equation involving AB and CD. How are $\overline{AB}$ and $\overline{CD}$ related to the triangles mentioned in the problem?

Hint: $\overline{ST}$ is a chord of $\triangle ABD$ and $\overline{CD}$ is a base of $\triangle BDC$. Notice anything interesting about these triangles?

Preview Solution

You may type any additional notes here.

Hide Solution | Reset

Your Submission Solution:

Solution: Let h be the height of the trapezoid. The height from D of $\triangle ABD$ is also h, as is the height from B of $\triangle BDC$. Since $[ABD] = \frac{3}{4}[ABC]$, we have $\frac{1}{2}AB(h) = \frac{3}{4}(\frac{1}{2}CD(h))$. Dividing both sides by h and multiplying both sides by 2 gives $AB = \frac{3}{4}CD$. Substituting this into $\frac{1}{2}AB(h) = 15$, we get $\frac{1}{2}(\frac{3}{4}CD)(h) = 15$. Since $CD = 7$, we have $\frac{1}{2}(\frac{3}{4}(7))(h) = 15$. Solving for h, we have $7(3)(h) = 30$, and dividing by 21 gives $h = \frac{30}{21} = \frac{10}{7}$. Therefore, $AB = 7(\frac{3}{4}) = \frac{21}{4}$.

12.39:

Source: MATHCOUNTS

$ABCD$ is a square, M and N are midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. What is the ratio of the area of $\triangle ADM$ to the area of $\triangle ADN$? Express your answer as a fraction in simplest form.

Hint: What do the two triangles have in common?

Preview Solution

You may type any additional notes here.

Hide Solution | Reset

Your Submission Solution:

Solution: Let $\overline{DM} \cap \overline{AN} = x$, so the side length of the square is $2x$. Therefore, we have $[MAN] = x^2/2$. We also have $AM = CN = x$, so

$$[AMD] = [CMD] = (x)(2x)/2 = x^2.$$

Finally we have

$$\begin{aligned}[MDN] &= [ABCD] - [MBN] - [AMD] - [CND] \\ &= 4x^2 - \frac{x^2}{2} - x^2 - x^2 \\ &= \frac{x^2}{2}.\end{aligned}$$

Therefore, we have

$$[MAD]/[MDN] = (x^2)/(x^2/2) = \boxed{2/1}.$$

Perhaps a quicker way to see this is to note that $[MAD]$ and $[MDN]$ share a base, $\overline{MD}$. So, the ratio of the areas of the triangles equals the ratio of their heights to that common base. The altitudes to $\overline{MD}$ of these two triangles together form diagonal $\overline{AC}$. Note that DN is half as long as DM, so the ratio of their heights is also $2/1$. Therefore, the ratio of areas is $2/1$.

12.40:

A 30-60-90 triangle is drawn on the exterior of equilateral triangle ABC as shown so that the hypotenuse is $\overline{AB}$. Given that the area of the equilateral triangle is 1, what is the area of the right triangle ABD?

Hint: What length-finding strategies do you know?

Hint: Is there a right triangle that has the desired segment as a side?

Preview Solution

You may type any additional notes here.

Hide Solution | Reset

Your Submission Solution:

Solution: Since $AC = BC$ and $\angle ABC = 60°$, we have $\angle ABD = 30° + 60° = 90°$. Therefore, we see that AD has finite length and BD, and then applying the Pythagorean Theorem to right $\triangle ABD$ gives $AD^2 + BD^2 = AB^2$. Since $AB = AC$, we have $AD^2 + BD^2 = AC^2$. Since $BC = BCD = 1$ these $\triangle ABC$ is equilateral, we have $AB = BC = 1$. Applying the Pythagorean Theorem to $\triangle ABD$ gives

$$\begin{aligned}AD^2 &= AB^2 - BD^2 \\ &= 1^2 - \left(\frac{1}{2}\right)^2 \\ &= 1 - \left(\frac{1}{4}\right) \\ &= 1 - \left(\frac{1}{4}\right) \\ &= 1 - \left(\frac{1}{4}\right)

12.21: Source: MATHCOUNTS

Two bicyclists start at the intersection of two perpendicular roads. One rides east at a rate of 9 miles per hour, while the other rides south at a rate of 12 miles per hour. How many miles are in the shortest distance between them at the end of 3 hours?

Preview Solution

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: The shortest rider travels $9 \cdot 3 = 27$ miles in three hours and the southeast rider travels $12 \cdot 3 = 36$ miles. The shortest distance between them is the hypotenuse of a right triangle with legs of lengths 27 and 36 miles. We can use the Pythagorean Theorem to find the hypotenuse. Or, we could notice that $\frac{27}{9} = 3$ and $36 = 4 \cdot 9$, so, applying the $(3, 4, 5)$ Pythagorean triple, we see that the hypotenuse is $5 \cdot 9 = \boxed{45}$ miles.

12.22: Source: MATHCOUNTS

A right triangle has one leg of length 48 and hypotenuse with length 52. What is the length of the other leg? (Challenge: Try to do this problem in your head.)

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: The hypotenuse is $4 \cdot 13$. We have $48 = 3 \cdot 16$, but the given leg is not either 3 or 4 times 16, so we cannot use the $(3, 4, 5)$ Pythagorean triple to solve the problem. Instead, we use the Pythagorean Theorem. Letting x be the length of the other leg, we have $x^2 + 48^2 = 52^2$, so

$$\begin{aligned} x^2 &= 4^2 \cdot 52^2 - 48^2 \\ &= 4^2 \cdot 100^2 - 3^2 \cdot 16^2 \\ &= (8 \cdot 100)^2 - (3 \cdot 16)^2 \\ &= (1\,000)^2, \end{aligned}$$

Taking the square root gives $x = \sqrt{(1\,000)^2} = \boxed{1\,000\sqrt{2}}$ cm.

12.23: Source: MATHCOUNTS

Tim knows that one leg of a certain right triangle is 300 cm and the hypotenuse is 400 cm. Pat notes that $300 = 3 \cdot 100$ and $400 = 4 \cdot 100$ and then uses the $(3, 4, 5)$ Pythagorean triple to determine that the other leg must be $5 \cdot 100 = 500$ cm. Explain why Pat's method doesn't work, and determine the correct length of the other leg of the triangle.

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: In the $(3, 4, 5)$ Pythagorean triple, the 3 and the 4 are both leg lengths, but in the given problem, the side with length $4 \cdot 100$ is the hypotenuse, not a leg. We do have $400 = 5 \cdot 80$, but the given leg is not either 3 or 4 times 80, so we cannot use the $(3, 4, 5)$ Pythagorean triple to solve the problem. Instead, we use the Pythagorean Theorem. Letting x be the length of the other leg, we have $x^2 + 300^2 = 400^2$, so

$$\begin{aligned} x^2 &= 400^2 - 300^2 \\ &= 4^2 \cdot 100^2 - 3^2 \cdot 100^2 \\ &= (8 \cdot 100)^2 - (3 \cdot 100)^2 \\ &= (5\,100)^2, \end{aligned}$$

Taking the square root gives $x = \sqrt{(5\,100)^2} = \boxed{100\sqrt{2}}$ cm.

12.24:

(a) What is the area of a right triangle that has legs with lengths 7 inches and 24 inches?

Preview Solution

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: The area of a right triangle is half the product of the lengths of its legs, so the area of the given triangle is $(7)(24)/2 = \boxed{84}$ in^2.

(b) What is the length of the hypotenuse of the triangle in part (a)?

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: First, we find the hypotenuse length. From the Pythagorean Theorem, or from remembering the $(7, 24, 25)$ Pythagorean triple, we find that the hypotenuse of the triangle has length 25 inches. The area of the triangle equals half the product of the hypotenuse and the desired altitude length. Let the desired altitude have length h. Using the area we found in part (a), we have $25h/2 = 84$. Multiplying both sides by 2 gives $25h = 168$, and dividing by 25 gives $h = \boxed{168/25}$ inches.

12.25:

Find the area of the quadrilateral on the right.

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: We find the areas of the two right triangles separately. We start with the triangle on the left side. Let the missing leg have length x. The Pythagorean Theorem gives $28^2 + x^2 = 26^2$, so $x^2 = 26^2 - 28^2$. We can simplify the expression on the right by noticing that $26 = 27 - 1$. So taking the square root gives

$$\begin{aligned} x &= \sqrt{27^2 - 1^2} = \sqrt{(3 \cdot 9)^2 - 1^2} = \sqrt{27 \cdot 1^2 - 1^2} \\ &= \sqrt{6(6^2) - 1^2} = \sqrt{35 \cdot 1^2} = 1\sqrt{5}. \end{aligned}$$

(We might also have noticed that the given leg is half the hypotenuse, so the triangle is a 30-60-90 triangle.) Therefore, the right triangle on the lower left has area $(14)(1\sqrt{5})/2 = 49\sqrt{5}$ in^2.

Turning to the other right triangle, we let h be the length of the missing leg and again apply the Pythagorean Theorem. We find $28^2 + h^2 = 26^2$, so

$$h^2 = 26^2 - 28^2 = 784 - 676 = 108.$$

Taking the square root gives $h = \sqrt{108} = \sqrt{36 \cdot 3} = 6\sqrt{3}$. So, the area of the upper right triangle is $(28)(6\sqrt{3})/2 = 78\sqrt{3}$ in^2.

Combining the two triangles gives a total area of $49\sqrt{5} + 78\sqrt{3} = \boxed{109\sqrt{5} + 78\sqrt{3}}$ in^2.

12.26: Source: MATHCOUNTS

A garden is laid out in the fashion shown in the diagram at the right. If only the shaded isosceles right triangles are used for planting, what is the total area, in square feet, that is to be used for planting?

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: We present two methods for finding the total area.

Method 1: The Hard Way. Because the hypotenuse of an isosceles right triangle is $\sqrt{2}$ times a leg of the triangle, the leg length of each of the triangle sections is $\frac{20}{\sqrt{2}}$. Therefore, the area of each of the triangle sections is

$$\frac{1}{2}\left(\frac{20}{\sqrt{2}}\right)\left(\frac{20}{\sqrt{2}}\right) = \frac{1}{2} \cdot \frac{10^2}{1} = \frac{1}{2} \cdot \frac{200}{1} = 100 \text{ ft}^2.$$

Combining all four sections gives a total area of $4(100) = \boxed{400}$ ft^2.

Method 2: The Easy Way. Pushing together all four right triangles forms a square with side length 16 ft. This square has area $(16)^2 = \boxed{256}$ ft^2.

12.27:

In $\triangle ABC$, we have $AA = 30^\circ$, $AB = 60^\circ$, and $AB = 8$. Find BC, AC, and the area of $\triangle ABC$.

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: We have $AC = 180^\circ - 30^\circ - 60^\circ = 90^\circ$, so $\triangle ABC$ is a right triangle. Moreover, because the acute angles of the triangle are 30° and 60°, the triangle is a 30-60-90 triangle. Therefore, the side length of the triangle is opposite the 30° angle and is half the hypotenuse, so $BC = 8/2 = \boxed{4}$. The longer leg is $\sqrt{3}$ times the shorter leg, so $AC = \boxed{4\sqrt{3}}$. The area is half the product of the legs, so $[ABC] = \boxed{8\sqrt{3}}$.

12.28:

(a) What is the greatest possible angle measure in an isosceles triangle that has an angle measuring 54 degrees?

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: Either the 54° angle is one of the two congruent angles, or it isn't.

Case 1: The 54° angle is one of the two congruent angles. Then, another angle is 54° and the third angle is $180^\circ - 54^\circ - 54^\circ = 72^\circ$.

Case 2: The 54° angle is not one of the two congruent angles. Then, the other two angles are congruent, and their measures must sum to $180^\circ - 54^\circ$, which equals 126°. Since the two congruent angles sum to 126°, they each measure $126^\circ/2 = 63^\circ$.

Considering the two cases above, the maximum possible measure of an angle of the triangle is $\boxed{54^\circ}$.

(b) What is the least possible angle measure in an isosceles triangle that has an angle measuring 54 degrees?

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: Considering the two cases in part (a), the smallest possible measure of an angle of the triangle is $\boxed{54^\circ}$.

12.29:

Find the perimeter and the area of the quadrilateral shown at the right.

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: Because $\angle A = \angle B = 180^\circ$, we know that $\overline{BC} \parallel \overline{AD}$, so $ABCD$ is a trapezoid. The bases of the trapezoid have lengths 3 and 15, and the height of the trapezoid is 8, so the area of the trapezoid is $\frac{3 + 15}{2} \cdot 8 = \boxed{72}$.

To find the perimeter of the trapezoid, we must find CD. We draw the altitude from C to $\overline{AD}$, forming the right triangle CED shown in the diagram. Since $BC = 3$ is a rectangle, we have $ED = 3$. We have $CE = 8$, so we have the Pythagorean Triple $(3, 4, 5)$. Scaling the Pythagorean Triple to $(CE, CD, ED) = (8, 10, 6)$ or recognizing the $(3, 12, 15)$ Pythagorean triple (which is 3 times the $(3, 4, 5)$ Pythagorean triple), we find that $CD = 15$, so the perimeter of $ABCD$ is $8 + 3 + 15 + 15 = \boxed{41}$.

12.30:

The lengths of the diagonals of a rhombus are 10 inches and 24 inches. What are the perimeter and the area of the rhombus?

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: The area of a rhombus is half the product of its diagonals, so the area of the rhombus is $(10)(24)/2 = \boxed{120}$ in^2. The diagonals of a rhombus bisect each other, so drawing the diagonals of the rhombus splits the rhombus into four right triangles with legs of lengths 5 and 12 inches. Recalling the $(5, 12, 13)$ Pythagorean triple, or applying the Pythagorean Theorem to these triangles, we find that each side of the rhombus has length 13 inches. So, the perimeter of the rhombus is $4(13) = \boxed{52}$ inches.

12.31: Source: MATHCOUNTS

An eleven in the diagram on the right, a 25-foot ladder reaches 24 feet up the side of a building. Then the top of the ladder slips down 4 feet, and the bottom of the ladder slides 7 feet away from the wall. How many additional feet does the bottom of the ladder slide out from the base of the building?

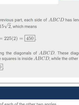
You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: Connecting the foot of the ladder to the base of the building completes a right triangle in which part of the building wall is a leg and the ladder is the hypotenuse. From the Pythagorean Theorem, or from the $(7, 24, 25)$ Pythagorean triple, we see that the base of the ladder is initially 7 feet from the building. After the top slides down 4 feet, the top is 20 feet from the ground. The ladder is still 25 feet long. Again using the Pythagorean Theorem or recognizing the $(15, 20, 25)$ Pythagorean triple (which is 5 times the $(3, 4, 5)$ Pythagorean triple), we see that the base of the ladder is 15 feet from the wall after sliding. Therefore, the base of the ladder slid $15 - 7 = \boxed{8}$ feet.

12.32: Source: MATHCOUNTS

A tessellation is composed of four regular hexagons and a rhombus as shown in the diagram on the right. How many degrees are in an acute angle of the rhombus?



You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: The sum of the angles of a hexagon is $(6 - 2)(180^\circ) = 720^\circ$, so each angle of a regular hexagon is $720^\circ/6 = 120^\circ$. We'll next compute the measure of an obtuse angle of the rhombus. Each obtuse angle of the rhombus shares its vertex with two angles of hexagons, so the sum of the obtuse angles must sum to 360°, so the measure of each obtuse angle of the rhombus is $360^\circ - 120^\circ - 120^\circ = 120^\circ$. The rhombus is also a parallelogram, which means its opposite sides are parallel. Therefore, consecutive angles in the rhombus must be supplementary, which means each acute angle has measure $180^\circ - 120^\circ = \boxed{60^\circ}$.

12.33:

Label each statement as true or false, and explain why your answer is correct.

(a) A quadrilateral can have exactly three right angles among its interior angles.

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: $\boxed{\text{True}}$ As explained in the text, every square is a rectangle and every rectangle is a parallelogram, so every square is a parallelogram.

(b) If both pairs of opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: $\boxed{\text{True}}$ Let the quadrilateral be $ABCD$, so $\angle A = \angle C$ and $\angle B = \angle D$. Therefore, we have $\angle A + \angle B = \angle C + \angle D$. Since the sum of all four angles must be 360°, we must have $\angle A + \angle B = \angle C + \angle D = 180^\circ$. Since $\angle A + \angle B = 180^\circ$, we have $\overline{BC} \parallel \overline{AD}$. Similarly we can start with $\angle A = \angle C$ and $\angle D = \angle B$ to find $\angle A + \angle D = \angle B + \angle C$, which gives $\angle A + \angle D = 180^\circ$ so $\overline{AB} \parallel \overline{CD}$. Therefore, both pairs of opposite sides of $ABCD$ are parallel, so $ABCD$ is a parallelogram.

(c) The diagonals of a rhombus have the same length.

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: $\boxed{\text{False}}$ For example, the rhombus at the right has the same four tetrahedron diagonals of different length.

12.34:

The measures of one interior angle of a rhombus is 79°. What are the measures of the other three interior angles?

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: Let the rhombus be $EFGH$, with $\angle E$ being the given 79° angle. Every rhombus is a parallelogram. Since opposite sides of the rhombus are parallel, each pair of consecutive angles is supplementary. Therefore, we have $\angle E + \angle F = \angle E + \angle H = 180^\circ$, which gives us

$$\angle F = \angle H = 180^\circ - \angle E = \boxed{101^\circ}.$$

Finally, the opposite angles of a parallelogram are congruent, so $\angle G = \angle E = \boxed{79^\circ}$.

12.35:

The midpoints of the sides of square $WXYZ$ are connected to form another quadrilateral.

(a) Explain why the new quadrilateral must also be a square.

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: Let the quadrilateral formed by connecting the midpoints of the sides of $WXYZ$ be $ABCD$, as shown in the diagram at the right. First, we show that all four sides of $ABCD$ are congruent. Each side of $ABCD$ is the hypotenuse in an isosceles right triangle whose legs are half the side length of the square. Therefore, all four sides of $ABCD$ have lengths $\frac{1}{2}$ times half the side length of the square. Next, we show that all of the angles of $ABCD$ are right angles. We note that

$$\angle DAB = 180^\circ - \angle WAD - \angle XAB.$$

Since $\angle WAD$ and $\angle XAB$ are isosceles right triangles, we have $\angle WAD = \angle XAB = 45^\circ$, so

$$\angle DAB = 180^\circ - 45^\circ - 45^\circ = 90^\circ.$$

Similarly, all four angles of $ABCD$ are right angles. Since all four angles of $ABCD$ are congruent and all four sides of $ABCD$ are congruent, $ABCD$ is a square.

(b) If the area of $WXYZ$ is 900, then what is the area of the new quadrilateral?

Preview Solution

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: The side length of $WXYZ$ is $\sqrt{900} = 30$. As described in the previous part, each side of $ABCD$ has length equal to $\sqrt{2}$ times half the side length of $WXYZ$, so the side length of $ABCD$ is $15\sqrt{2}$, which means

$$[ABCD] = \left(15\sqrt{2}\right)^2 = 15^2 \left(\sqrt{2}\right)^2 = 225(2) = \boxed{450}.$$

We also could have figured out that $[ABCD] = (WXYZ)/2$ by drawing the diagonals of $ABCD$. These diagonals divide $WXYZ$ into four smaller squares with area 225 each. Half of each of these squares is inside $ABCD$, while the other half of each is outside $ABCD$. So, the area of $ABCD$ is $\frac{1}{2} \cdot 225 \cdot 4 = \boxed{450}$.

12.36:

The measures of two angles in a parallelogram add to 204°. Find the measure of each of the other two angles.

You may type any additional notes you have here. Hide Solution Reset

Your Submission Solution

Solution: Because opposite sides of a parallelogram are parallel, each pair of consecutive angles adds to 180°. So, the two angles that add to 204° must be opposite each other. The opposite angles of a parallelogram are congruent, so each of these angles must be $204^\circ/2 = 102^\circ$. The other two angles are congruent, and each is supplementary to each of the 102° angles. Therefore, these other two angles have measure $180^\circ - 102^\circ = \boxed{78^\circ}$.

Section 13 Data and Statistics

A sample of your phones, so people may not have all of their phones accounted for, and the sample of your phones may not be representative of the population of all phones. Or, you may be interested in the population of people who have phones, so you may want to know how many people have how many phones, but the sample of the population of people who have phones may not be representative. Your teacher probably will not let you do this, but you could make a survey to find out how many people have how many phones.

Only the sample can be observed, so statistics, which are numerical values of such sample, are used to make inferences about the population, which is the population information about that population.

Concept You are "lost" in such an hotel or a hospital and you need a doctor. When referring to a doctor, you don't care which one, but you want to be sure that, for example, you find a doctor. This also is confirmed in the experiment we attached to an appendix! When referring to a doctor, you don't care which one, but you want to be sure that, for example, you find a doctor. This also is confirmed in the experiment we attached to an appendix!

13.1 Basic Statistics

Very important concept of statistics is the concept of a random variable. Random variable is a variable whose value is determined by chance. Random variable is a variable whose value is determined by chance.

- **Mean:** The mean of a group of numbers is the sum of the numbers divided by the number of numbers in the group. For example, the average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is 11 because $\frac{2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20}{19} = 11$.
  - We sometimes use the word "average" as a verb. For example, we may say that a basketball player averages 20 points per game. We also sometimes use the word "average" as a noun. For example, we may say that the average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is 11.
- **Median:** Find the number of numbers from a group, so that half of the numbers in the group are greater than the number in the middle. As the median of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is 11.
  - If there are two numbers in the middle, then the median is the average of the two numbers in the middle. For example, the median of the sequence

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21$$

is the average of 11 and 12, which is $\frac{11+12}{2} = 11.5$.

- **Mode:** The number of a group of numbers is the number that occurs most frequently in the group. For example, the mode of the sequence

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 21, 21$$

is 21 because 21 occurs more than any other number in the sequence.

Average, median, and mode are called measures of central tendency because they are in the middle of a group of numbers. Median is also called a measure of position, because it shows where a number is in a group of numbers. Mean is also called a measure of dispersion, because it shows how much the numbers in the group are spread out.

Concept Average, median, and mode are called measures of central tendency because they are in the middle of a group of numbers. Median is also called a measure of position, because it shows where a number is in a group of numbers. Mean is also called a measure of dispersion, because it shows how much the numbers in the group are spread out.

Problems

Problem 13.1 Suppose the following are your grades on tests this year:

$$75, 76, 80, 90, 92, 96, 98.$$

(a) What is the average of your test scores?

(b) What is the median of your test scores?

(c) What is the mode of your test scores?

Problem 13.2 Suppose the following are your grades on tests this year:

$$75, 76, 80, 90, 92, 96, 98, 100.$$

(a) What is the average of your test scores?

(b) What is the median of your test scores?

(c) What is the mode of your test scores?

Problem 13.3 Suppose the following are your grades on tests this year:

$$75, 76, 80, 90, 92, 96, 98, 100, 100.$$

(a) What is the average of your test scores?

(b) What is the median of your test scores?

(c) What is the mode of your test scores?

Problem 13.4 Suppose you average 90 on your first 7 tests in a class. What should be your average on the 8th test to raise your average to 92?

Problem 13.5 Suppose you average 90 on your first 7 tests in a class. What should be your average on the 8th test to raise your average to 95?

Problem 13.6 Suppose you average 90 on your first 7 tests in a class. What should be your average on the 8th test to raise your average to 98?

Concept The average of a list of numbers really depends on the number of times the list of numbers is repeated. For example, the average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is 11. The average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is still 11. The average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is still 11. The average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is still 11.

Suppose you average 90 on your first 7 tests in a class. What should be your average on the 8th test to raise your average to 92?

Concept The average of a list of numbers really depends on the number of times the list of numbers is repeated. For example, the average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is 11. The average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is still 11. The average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is still 11.

Problem 13.7 Suppose you average 90 on your first 7 tests in a class. What should be your average on the 8th test to raise your average to 95?

Problem 13.8 Suppose you average 90 on your first 7 tests in a class. What should be your average on the 8th test to raise your average to 98?

Problem 13.9 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20.

Problem 13.10 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21.

Problem 13.11 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22.

Problem 13.12 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23.

Problem 13.13 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24.

Problem 13.14 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25.

Problem 13.15 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26.

Problem 13.16 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27.

Problem 13.17 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28.

Problem 13.18 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29.

Problem 13.19 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30.

Problem 13.20 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31.

Problem 13.21 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32.

Problem 13.22 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33.

Problem 13.23 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34.

Problem 13.24 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35.

Problem 13.25 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36.

Problem 13.26 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37.

Problem 13.27 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38.

Problem 13.28 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39.

Problem 13.29 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40.

Problem 13.30 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41.

Problem 13.31 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42.

Problem 13.32 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43.

Problem 13.33 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44.

Problem 13.34 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45.

Problem 13.35 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46.

Problem 13.36 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47.

Problem 13.37 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48.

Problem 13.38 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49.

Problem 13.39 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50.

Problem 13.40 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51.

Problem 13.41 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52.

Problem 13.42 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53.

Problem 13.43 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54.

Problem 13.44 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55.

Problem 13.45 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56.

Problem 13.46 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57.

Problem 13.47 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58.

Problem 13.48 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59.

Problem 13.49 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60.

Problem 13.50 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61.

Problem 13.51 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62.

Problem 13.52 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63.

Problem 13.53 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64.

Problem 13.54 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65.

Problem 13.55 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66.

Problem 13.56 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67.

Problem 13.57 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68.

Problem 13.58 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69.

Problem 13.59 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70.

Problem 13.60 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71.

Problem 13.61 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72.

Problem 13.62 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73.

Problem 13.63 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74.

Problem 13.64 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31,

Section 13 Data and Statistics

A sample of your phones, so people may not have all of their phones accounted for, and the sample of your phones may not be representative of the population of all phones. Or, you may be interested in the population of people who have phones, so you may want to know how many people have how many phones, but the sample of the population of people who have phones may not be representative. Your teacher probably will not let you do this, but you could make a survey to find out how many people have how many phones.

Only the sample can be observed, so statistics, which are numerical values of such sample, are used to make inferences about the population, which is the population information about that population.

Concept You are "lost" in such an hotel or a hospital and you need a doctor. When referring to a doctor, you don't care which one, but you want to be sure that, for example, you find a doctor. This also is confirmed in the experiment we attached to an appendix! When referring to a doctor, you don't care which one, but you want to be sure that, for example, you find a doctor. This also is confirmed in the experiment we attached to an appendix!

13.1 Basic Statistics

Very important concept of statistics is the concept of a random variable. Random variable is a variable whose value is determined by chance. Random variable is a variable whose value is determined by chance.

- **Mean:** The mean of a group of numbers is the sum of the numbers divided by the number of numbers in the group. For example, the average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is 11 because $\frac{2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20}{19} = 11$.
  - We sometimes use the word "average" as a verb. For example, we may say that a basketball player averages 20 points per game. We also sometimes use the word "average" as a noun. For example, we may say that the average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is 11.
- **Median:** Find the number of numbers from a group, so that half of the numbers in the group are greater than the number in the middle. As the median of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is 11.
  - If there are two numbers in the middle, then the median is the average of the two numbers in the middle. For example, the median of the sequence

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21$$

is the average of 11 and 12, which is $\frac{11+12}{2} = 11.5$.

- **Mode:** The number of a group of numbers is the number that occurs most frequently in the group. For example, the mode of the sequence

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 21, 21$$

is 21 because 21 occurs more than any other number in the sequence.

Average, median, and mode are called measures of central tendency because they are in the middle of a group of numbers. Median is also called a measure of position, because it shows where a number is in a group of numbers. Mean is also called a measure of dispersion, because it shows how much the numbers in the group are spread out.

Concept Average, median, and mode are called measures of central tendency because they are in the middle of a group of numbers. Median is also called a measure of position, because it shows where a number is in a group of numbers. Mean is also called a measure of dispersion, because it shows how much the numbers in the group are spread out.

Problems

Problem 13.1 Suppose the following are your grades on tests this year:

$$75, 76, 80, 90, 92, 96, 98.$$

(a) What is the average of your test scores?

(b) What is the median of your test scores?

(c) What is the mode of your test scores?

Problem 13.2 Suppose the following are your grades on tests this year:

$$75, 76, 80, 90, 92, 96, 98, 100.$$

(a) What is the average of your test scores?

(b) What is the median of your test scores?

(c) What is the mode of your test scores?

Problem 13.3 Suppose the following are your grades on tests this year:

$$75, 76, 80, 90, 92, 96, 98, 100, 100.$$

(a) What is the average of your test scores?

(b) What is the median of your test scores?

(c) What is the mode of your test scores?

Problem 13.4 Suppose you average 90 on your first 7 tests in a class. What should be your average on the 8th test to raise your average to 92?

Problem 13.5 Suppose you average 90 on your first 7 tests in a class. What should be your average on the 8th test to raise your average to 95?

Problem 13.6 Suppose you average 90 on your first 7 tests in a class. What should be your average on the 8th test to raise your average to 98?

Concept The average of a list of numbers really depends on the number of times the list of numbers is repeated. For example, the average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is 11. The average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is still 11. The average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is still 11. The average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is still 11.

Suppose you average 90 on your first 7 tests in a class. What should be your average on the 8th test to raise your average to 92?

Concept The average of a list of numbers really depends on the number of times the list of numbers is repeated. For example, the average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is 11. The average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is still 11. The average of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 is still 11.

Problem 13.7 Suppose you average 90 on your first 7 tests in a class. What should be your average on the 8th test to raise your average to 95?

Problem 13.8 Suppose you average 90 on your first 7 tests in a class. What should be your average on the 8th test to raise your average to 98?

Problem 13.9 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20.

Problem 13.10 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21.

Problem 13.11 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22.

Problem 13.12 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23.

Problem 13.13 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24.

Problem 13.14 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25.

Problem 13.15 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26.

Problem 13.16 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27.

Problem 13.17 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28.

Problem 13.18 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29.

Problem 13.19 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30.

Problem 13.20 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31.

Problem 13.21 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32.

Problem 13.22 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33.

Problem 13.23 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34.

Problem 13.24 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35.

Problem 13.25 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36.

Problem 13.26 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37.

Problem 13.27 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38.

Problem 13.28 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39.

Problem 13.29 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40.

Problem 13.30 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41.

Problem 13.31 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42.

Problem 13.32 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43.

Problem 13.33 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44.

Problem 13.34 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45.

Problem 13.35 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46.

Problem 13.36 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47.

Problem 13.37 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48.

Problem 13.38 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49.

Problem 13.39 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50.

Problem 13.40 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51.

Problem 13.41 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52.

Problem 13.42 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53.

Problem 13.43 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54.

Problem 13.44 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55.

Problem 13.45 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56.

Problem 13.46 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57.

Problem 13.47 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58.

Problem 13.48 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59.

Problem 13.49 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60.

Problem 13.50 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61.

Problem 13.51 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62.

Problem 13.52 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63.

Problem 13.53 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64.

Problem 13.54 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65.

Problem 13.55 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66.

Problem 13.56 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67.

Problem 13.57 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68.

Problem 13.58 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69.

Problem 13.59 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70.

Problem 13.60 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71.

Problem 13.61 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72.

Problem 13.62 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73.

Problem 13.63 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74.

Problem 13.64 Find the median of the following numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31,

13.2 Limits of Basic Statistics

Now that we know how to compute average, median, and mode, we will think about what our information they tell us. And, perhaps more importantly, we will think about what sort of information they don't tell us.

Problems

Problem 13.8 ⊙ Jump to Solution

You have to choose one of four people in your math club to be your partner on a math assignment. You'd like to choose someone who is very good at math! Here are the four students' scores on their previous exam assignments:

Anna:94, 93, 90, 93, 92, 91, 0
Bob:98, 94, 33, 33, 96, 97, 23
Carol:89, 89, 86, 88, 97, 84, 85
Doug:100, 14, 3, 109, 11, 2, 21

(a) Which student has the highest average score?
(b) Which student has the highest median score?
(c) Which student has the highest mode score?
(d) Which student would you choose as your partner?

Problem 13.9 ⊙ Jump to Solution

The average wealth of a person in Richville is $150,000 and the average wealth of a person in Poorville is $20,000.

(a) Suppose a person is called "filthy stinking rich" if the person has over one million dollars. Can we tell which town has more filthy stinking rich people?
(b) Can we tell which town has the higher median wealth?
(c) If both towns have the same number of people, can we tell which town has the higher total wealth?
(d) Suppose a person with $1,000,000 and 4 people who have $0 more into Richville. Will the average wealth of Richville go up or down?

Problem 13.10 ⊙ Jump to Solution

The median wealth of a person in Goldtown is $150,000 and the median wealth of a person in Tintown is $20,000. Each town has 1000 people.

(a) Suppose a town qualifies for special government subsidies if each of at least half of its residents has less than $60,000. Does either town necessarily qualify for the subsidies?
(b) Can we tell which town has the higher average wealth?
(c) Suppose a person with $1,000,000,000 and 4 people who have $0 move into Goldtown. Will the median wealth of Goldtown go up or down?

Problem 13.8

You have to choose one of four people in your math club to be your partner on a math assignment. You'd like to choose someone who is very good at math! Here are the four students' scores on their previous exam assignments:

Anna:94, 93, 90, 93, 92, 91, 0
Bob:98, 94, 33, 33, 96, 97, 23
Carol:89, 89, 86, 88, 97, 84, 85
Doug:100, 14, 3, 109, 11, 2, 21

(a) Which student has the highest average score?
(b) Which student has the highest median score?
(c) Which student has the highest mode score?
(d) Which student would you choose as your partner?

Solution for Problem 13.8

(a) Here are the averages:

$$\text{Anna's average} = \frac{94+93+90+93+92+91+0}{7} = \frac{553}{7} = 79,$$

$$\text{Bob's average} = \frac{98+94+33+33+96+97+23}{7} = \frac{474}{7} = 67\frac{5}{7},$$

$$\text{Carol's average} = \frac{89+89+86+88+97+84+85}{7} = \frac{607}{7} = 86\frac{5}{7},$$

$$\text{Doug's average} = \frac{100+14+3+109+11+2+21}{7} = \frac{251}{7} = 35\frac{6}{7}.$$

Carol has the highest average.

(b) We find the students' median scores by putting each student's scores in order from least to greatest:

Anna: 0, 90, 91, 92, 93, 93, 94
Bob: 23, 33, 33, 94, 96, 97, 98
Carol: 84, 85, 86, 88, 89, 89, 97
Doug: 2, 3, 11, 14, 21, 100, 109

For each student, the corresponding median is the middle number in the list. Each student's median is underlined. Bob has the highest median.

(c) The mode is the most frequent score. Each student has one repeated score, so we can quickly find each student's mode score:

Anna: 93 Bob: 33 Carol: 88 Doug: 100

Doug has the highest mode.

(d) There's no clearly correct answer to this question. Carol has the highest average, Bob has the highest median, and Doug has the highest mode. But you might be best off choosing Carol.

Looking at all of the scores, it's pretty clear that you probably don't want to choose Doug the student with the highest model. Most of the time he does very poorly. Mode is usually not a very helpful statistic.

Even though Bob has the highest median, Bob appears to have considerably more likely than Anna or Carol to do badly. So choosing Bob seems to bring a higher risk of a low score.

Carol, with the highest average, seems to be more reliable than Anna, but Anna's single bad score of 0 suggests that there may be some other explanation for her poor performance on that assignment. Maybe she simply didn't turn it in.

If you can be sure that Anna will do her best, then it looks like she has a good chance to do better than Carol, because all of Anna's nonzero scores are higher than all of Carol's scores.

The top point of Problem 13.8 is that all of the basic statistics we have studied so far are usually poor substitutes for considering all of the data. Of course, sometimes we don't have all of the data, or there may be too much data to consider all of it. As you study more statistics in the future, you'll learn methods for analyzing large batches of data.

For now, we'll take a closer look at the limitations of the basic tools we have studied so far. It's particularly important to understand the strengths and weaknesses of average and median, since these are the two statistics that people use (and misuse!) the most.

Problem 13.9 ⊙ Jump to Solution

The average wealth of a person in Richville is $150,000 and the average wealth of a person in Poorville is $20,000.

(a) Suppose a person is called "filthy stinking rich" if the person has over one million dollars. Can we tell which town has more filthy stinking rich people?
(b) Can we tell which town has the higher median wealth?
(c) If both towns have the same number of people, can we tell which town has the higher total wealth?
(d) Suppose a person with $1,000,000 and 4 people who have $0 more into Richville. Will the average wealth of Richville go up or down?

Solution for Problem 13.9

(a) First notice that we aren't given any information about how many people are in each town.

WARNING!	Neither the average nor the median of a group of numbers tells us anything about how numbers are in the group.

On the one hand, it's possible that there are Richville people who have exactly $150,000, so there is no one in Richville who is filthy stinking rich. At the same time, Poorville might consist of 100 people: one person with $2,000,000, who is definitely filthy stinking rich, and 99 people with no money at all. Note that this shows that the average wealth of Poorville is $20,000 as required. In this case, no Poorville has more filthy stinking rich people.

On the other hand, it's possible that everyone in Poorville has exactly $20,000 (so that no one is filthy stinking rich), while Richville consists of one filthy stinking rich person with $15,000,000 and 99 people who each have nothing. In this case, Richville has more filthy stinking rich people.

So without any additional information, we can't tell which town has more filthy stinking rich people.

(b) On the one hand, suppose the towns' populations are:

Richville	Poorville
50 people with $150,000 each	1 person with $1,000,000 49 people with 0 each

The average and median wealth in Richville are each $150,000, whereas the average wealth in Poorville is $1,000,000/50 = $20,000 and the median wealth in Poorville is $0. In this case, Richville has a higher median wealth than Poorville.

On the other hand, suppose the towns' populations are:

Richville	Poorville
1 person worth $1,500,000 1 people with $0	10 people with $20,000 each

The average and median wealth in Poorville are each $20,000, whereas the average wealth in Richville is $1,500,000/2 = $750,000 and the median wealth in Richville is $0. In this case, Poorville has a higher median wealth than Richville.

So without any additional information, we can't tell which town has the higher median wealth.

Important:	We cannot determine anything about the median of a set of data just by knowing the average of the data.

(c) For each town, the average wealth equals the total wealth divided by the number of people. So, the total wealth equals the product of the average wealth and the number of people. If the two towns have the same number of people, then Richville must have the higher total wealth. Specifically, if both towns have n people, then Richville has a total wealth of $150,000n, and Poorville has a total wealth of $20,000n.

Note that this analysis only worked because we were told that the number of people in each town are the same. If we weren't given this additional bit of information, we couldn't tell which town has the higher total wealth. For example, if the population of Richville was just 1 and the population of Poorville was 10, then the total wealth of Richville would be $150,000 · 1 = $150,000 whereas the total wealth of Poorville would be $20,000 · 10 = $200,000. In this example, the total wealth of Poorville would be greater.

Important:	Without knowing the size of the data sets, we cannot compare the sums of the data just by knowing the averages.

(d) The average wealth equals the total wealth divided by the number of people. In other words, the average equals the amount each person would have if we divided the total wealth equally among the people. The five people who move into Richville have a total of $1,000,000. To keep the town average $150,000, the total wealth of Richville must be $150,000 · 5 = $750,000. Since there are five new people come to town, there's an extra $250,000 above the total amount needed for $150,000 per person. This means that the average wealth in the town must go up when these five people arrive.

Problem 13.9 shows us some features and limitations of average. In particular, in our examples in parts (a), (b), and (d), we see that it is possible for a single number to have a huge effect on the average. For example, if an extremely rich person moves into your neighborhood, the average wealth of your neighborhood will rise a lot! But the other person in your neighborhood hasn't gotten any richer; even though, judging by the average wealth of the neighborhood, it looks like the neighborhood has become much richer.

Problem 13.10 ⊙ Jump to Solution

The median wealth of a person in Goldtown is $150,000 and the median wealth of a person in Tintown is $20,000. Each town has 1000 people.

(a) Suppose a town qualifies for special government subsidies if each of at least half of its residents has less than $60,000. Does either town necessarily qualify for the subsidies?
(b) Can we tell which town has the higher average wealth?
(c) Suppose a person with $1,000,000,000 and 4 people who have $0 move into Goldtown. Will the median wealth of Goldtown go up or down?

Solution for Problem 13.10

(a) Suppose we live a pauper in Tintown based on how much wealth they have. The population of Tintown is even, so the median wealth, $20,000, is the average wealth of the two people in the middle. So, either the middle two people have $20,000 each, or the poorer of the two has less than $20,000 and the richer person has more than $20,000. If the middle two people have the same wealth, $20,000, Each of these people therefore has less than $60,000, so Tintown qualifies for the subsidies.

Over in Goldtown, the median wealth is $150,000 and we live like a pauper in Goldtown from poorest to richest, then each person in the richer half of the line has at least $150,000. But all we know about the people in the poorer half of the line is that each person has no more than $150,000. It is possible for each person in the poorer half to have less than $60,000. For example, suppose each person in the poorer half of Goldtown has $50,000 and each person in the richer half of Goldtown has $250,000. Then, the median wealth of Goldtown is $150,000, but Goldtown still qualifies for the subsidies.

The median value only tells us how much the "middle" person has in each town, and that half the people in town have at least this much. But the richer person in town could have any amount of money and the median wealth. For example, if's possible that everyone in Goldtown lives exactly $150,000 while everyone in Tintown has $20,000 except for one person who has tens of billions of dollars. In this case, Tintown would have much higher total wealth and therefore higher average wealth, than Goldtown.

(b) Suppose that, before the five new people arrive, we line up all the people in Goldtown based on their wealth, with the richest first and the poorest last. Since there are 1000 people, the median is the average wealth of the middle two people, who are the 499th and 500th people in the line.

When the five new people arrive and find their places in the line, the four with $0 will join the poorer end, and the person with $1,000,000,000 will be somewhere in the rich half of the line. So, in this new line there will be 8005 people. The person who joined the rich end causes the people who used to be the 499th and 500th people to move up to the 501st and 502nd places.

In this new line of 8005 people, the middle person is 4003rd in line. Most notably, this person is behind, and therefore not richer than, the two people whose average gave us the original median wealth. So, the new median cannot possibly be greater than the old median.

We can't actually say that the new median must be lower than the old median, since we might have many people in the middle with the same wealth. So, it's possible that the new median is equal to the old median, but the new median cannot be higher than the old median.

The interesting observation here is that the new people brought a total of one billion dollars to the town, but the median did not rise. That's because we added more poor people than rich people. Once again, we see that the actual values above and below the median value do not affect the median. The median only tells us what the middle is.

c

Concept:	Average and median only give us very limited information about a set of data. In particular, neither tells us anything about the size of the set, and median does not tell us anything about the total of the data in the set. Neither average nor median tells us anything about the other (that is, if we know median and average, we don't know anything about the median, and if we know the median, we don't know anything about the average).

We say that a number is a group of numbers is an outlier if it is very far from the other numbers in the group. Outliers can have a significant effect on average, but have little effect on the median.

Sometimes, when analyzing data, people will remove outliers from a group of numbers before computing the average of the group. For example, a teacher might do this by throwing out a student's lowest exam grade before computing his or her average grade.

Limits of Basic Statistics Part 1

Limits of Basic Statistics Part 2

Limits of Basic Statistics Part 3

Exercises

13.2.1:

The median height of the players on my basketball team is 6 feet, 4 inches. What is the shortest that the tallest player on the team could possibly be?

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: The median height tells us the height of the middle player on the team when they are lined up from shortest to tallest. The tallest player must be at least as tall as the middle player, so the tallest player must be at least 6 feet, 4 inches tall. It is indeed possible for the tallest player to have the same height as the median height player—suppose all the players have the same height.

13.2.2:

The average wealth of a person in Richville is $150,000 and the average wealth of a person in Poorville is $20,000. Suppose Richville and Poorville combine to form Medium.

(a) Is it possible that the average wealth in Medium is exactly $20,000?

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: No. Suppose we form Medium by starting with the people in Poorville and then adding in the people from Richville. The people from Poorville have an average wealth of $20,000, so the average before including the people from Richville is $20,000. We expect that if we add people whose average wealth is greater than $20,000, then the average must increase. So, it seems that it is impossible for the average of Medium to be $20,000. We'll now make sure this is the case by comparing the total wealth of Poorville and Richville to the total wealth that would be in Medium if the average were only $20,000.

Let p be the number of people in Poorville and r be the number of people in Richville. So, the total wealth in Poorville is $20,000p and the total wealth of Richville is $150,000r. The total wealth in Medium if the two cities combined is $20,000(p + r) = $150,000r.

There are p + r people total in Medium. So, if the average wealth were $20,000, then the total wealth would be $20,000(p + r) which equals $20,000p + $20,000r. But we know that there are $20,000p + $150,000r total in Medium. Medium's $20,000r is always less than $150,000r (since r is positive). So, there has to be more wealth in Medium than there would be if the average were just $20,000.

(b) Suppose one person from Richville refuses to join Medium. Then, it is possible that the average wealth of Medium is less than $20,000?

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: Yes. It's possible that one person in Richville had all of the wealth in Richville, and everyone else had $0. Then, if all the people with $0 merge with Poorville to form Medium, the average wealth, the average wealth of the new city will be lower than that of Poorville. This is because Medium will have the same total wealth as the original Poorville, but Medium will have more people than Poorville did.

(c) If there were the same number of people in Poorville and Richville, then what is the average wealth of Medium?

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: Suppose there are r people in each city. So, the total wealth of Poorville is $20,000r and the total wealth of Richville is $150,000r. Medium then has 2r people and a total wealth of

$$\$20{,}000r + \$150{,}000r = \$170{,}000r.$$

Therefore, the average wealth of Medium is

$$\frac{\$170{,}000r}{2r} = \frac{\$170{,}000}{2} = \$85{,}000.$$

Note that $85,000 is the average of Poorville's average wealth ($20,000) and Richville's average wealth ($150,000).

(d) If the average wealth of Medium is $120,000, then which city was larger, Richville or Poorville?

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: Richville was larger. In the previous part, we saw that if the two cities had been the same size, then the average wealth of Medium would be $85,000. We expect that to get an average wealth that is greater than this, we need to have more rich people.

To make sure that this intuition is correct, we need to see how many more people from Poorville and r people from Richville, so that the combined wealth of the two towns is $20,000p + $150,000r. Since the population of Medium is p + r and the average wealth of Medium is $120,000, the total wealth in Medium is

$$\$120{,}000(p+r) = \$120{,}000p + \$120{,}000r.$$

The total wealth in Medium must equal the combined wealth of Poorville and Richville (since that's where Medium came from), so

$$\$20{,}000p + \$150{,}000r = \$120{,}000p + \$120{,}000r.$$

Subtracting $120,000r from both sides gives

$$\$20{,}000p + \$30{,}000r = \$120{,}000p.$$

Subtracting $20,000p from both sides gives $30,000r = $100,000p. Dividing both sides by $30,000 gives

$$r = p \cdot \frac{\$100{,}000}{\$20{,}000} = p \cdot \frac{10}{3} = \left(\frac{3}{1}\right)p.$$

Therefore, r is more than 3 times p, which means that there definitely were more people in Richville!

13.2.3:

Below are the average and median of test scores for Nick and Omar.

Nick: average 70, median 50
Omar: average 50, median 70

One of the students usually does OK, but when he does badly, he does very badly. The other one usually doesn't do very well, but when he does well, he does extremely well. Which is which?

Preview: Solution

You may type any additional notes you have here.

Hide Solution
Reset

Your Submission: Solution

Solution: The median scores in the "middle score," which means that Omar scores at least 70 at least half the time. Similarly Nick gets 50 or below at least half the time. Because Omar usually does well, he probably doesn't do very badly. It appears that, although half his scores are at least 70, his bad tests are much more below 70 than his good tests are above 70. So, Omar is the student who usually does OK, but when he does badly, he does really badly. Similarly, because Nick's median is 50, he does well more than half the time, though at least half Nick's scores are 50 or below, so know that his high scores must be quite high, to pull up his average despite all those scores at 50 or below. So, Nick is the student who usually doesn't do very well, but when he does well, he does extremely well.

13.3 Tables, Graphs, and Charts

Tables, graphs, and charts are used to summarize or organize information into a form easily understood by people represented in a collection of data. In this section, we explore a variety of ways to display data taken from past choosing A-F questions.

Problems

Problem 13.1 ⏱ Guide by Solution

Use the Sports Table and the number of students to create, annotate, and graphs of two suitable scatterplots. Use the Sports Table and the number of students to create a pie chart for the sports of students in each sport (by sport, and by the sport column corresponds to the total number of students in each sport).

Students Involved in Sports Outside School by Grade

	9th Grade	10th Grade	11th Grade	12th Grade
Sport Middle School	200	210	215	220
Sport High School	300	310	315	320
Sport Middle School Total	500	520	530	540
Sport High School Total	600	620	630	640

(a) Complete the table by filling in the missing entries.

(b) How many more eighth graders are in Sports Middle School than in Sport Middle School?

(c) Which grade has the most students in Sport Middle School?

(d) Which school has the highest average number of students per grade?

(e) Which school has the highest percentage of students in the sport of soccer?

Problem 13.2 ⏱ Guide by Solution

Suppose a group of 10 Freshman Biology students and they ask about each doing Kolling class. The results are shown in the **bar chart** at the right. Describe each of the following statements as true, false, or impossible. Explain your answer.

(a) More than three times as many students like Kolling as dislike Kolling.

(b) Exactly two students like Kolling.

(c) Students both most about Kolling and hate the self-doubt Kolling can together.

(d) Kolling is the most popular class.

(e) Most students spend more time thinking about the class about Kolling.

Problem 13.3 ⏱ Guide by Solution

There are two north-bound of Street Academy, The Little Westerners and The Bird. At the end of each day, each team begins the total number of points they earned. The **pie chart** at the right shows the **ratio** of points for each of the two north-bound parts. The Little Westerners earn 2 points.

Decide the truth of the following statements. Give reasons for your answer.

(a) The Bird had fewer points than The Little Westerners at the start of Monday, but more points than The Little Westerners at the end of Monday.

(b) The Little Westerners earned more points than The Bird earned in the morning and afternoon of Monday.

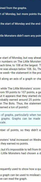
(c) There was a tie between The Little Westerners and The Bird at the end of Monday and the end of Friday.

(d) There was a tie between The Little Westerners and The Bird at the end of Monday.

(e) The Little Westerners earned more than 20 points on Friday.

Problem 13.4 ⏱ Guide by Solution

The exercise of a Kolling test are shown at the right in a **pie chart and half chart**. Each circle represents the total number of students in the class. The pie chart shows the ratio of the part of the total of the class and the half chart shows the total number of students.



(a) Kolling is the most popular class.

(b) What is the inside score?

(c) Represent the pie chart and half chart in a single chart. Note that, in one bar, a number of students from 0 to 50 another bar is a number of students from 50 to 100.

(d) Which class is the most popular?

(e) What is the inside score?

(f) Represent the pie chart and half chart in a single chart. Note that, in one bar, a number of students from 0 to 50 another bar is a number of students from 50 to 100.

(g) Which class is the most popular?

(h) What is the inside score?

Problem 13.5 ⏱ Guide by Solution

Students in Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.

Exercises for Problem 13.5

(a) The missing entry first Middle School Middle School is 200.

(b) The missing entry for Middle School Middle School is the number of 9th grade students, Sports Middle School.

(c) The missing entry for Middle School Middle School is the number of 9th grade students, Sports Middle School.

(d) This pie 9th grade students and ten pies 11th grade students. For Middle School Middle School, the missing entry is 200 + 200 = 400.

(e) The completed table is shown below with the new numbers in bold.

Students Involved in Sports Outside School by Grade

	9th Grade	10th Grade	11th Grade	12th Grade
Sport Middle School	200	210	215	220
Sport High School	300	310	315	320
Sport Middle School Total	500	520	530	540
Sport High School Total	600	620	630	640

(f) Sport Middle School has 220 students, and Sport High School has 320 students. Sport Middle School has 220 students, and Sport High School has 320 students. Sport Middle School has 220 students, and Sport High School has 320 students. Sport Middle School has 220 students, and Sport High School has 320 students.

(g) Sport Middle School has 220 students, and Sport High School has 320 students. Sport Middle School has 220 students, and Sport High School has 320 students. Sport Middle School has 220 students, and Sport High School has 320 students. Sport Middle School has 220 students, and Sport High School has 320 students.

(h) Sport Middle School has 220 students, and Sport High School has 320 students. Sport Middle School has 220 students, and Sport High School has 320 students. Sport Middle School has 220 students, and Sport High School has 320 students. Sport Middle School has 220 students, and Sport High School has 320 students.

(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.6 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.

Exercises for Problem 13.6

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

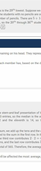
(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.7 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.



Exercises for Problem 13.7

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

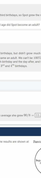
(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.8 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.



Exercises for Problem 13.8

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.9 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.

Exercises for Problem 13.9

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.10 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.

Exercises for Problem 13.10

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.11 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.

Exercises for Problem 13.11

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.12 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.

Exercises for Problem 13.12

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.13 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.

Exercises for Problem 13.13

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.14 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.

Exercises for Problem 13.14

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.15 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.

Exercises for Problem 13.15

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.16 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.

Exercises for Problem 13.16

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.17 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.

Exercises for Problem 13.17

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Problem 13.18 ⏱ Guide by Solution

Students in the Class Pie Representation were asked to draw Number pies of the. The results of the pie and are shown in the **pie chart** at the right. Pie charts are typically circular and sliced into pieces representing different parts of the whole. The pie chart shows the percentage of students who drawing each number pie. The pie chart shows that 25% of the students drawing a number pie, 15% of the students drawing a number pie, and 10% of the students drawing a number pie.

Exercises for Problem 13.18

(a) Sport Middle School Middle School has 200 students.

(b) Sport Middle School Middle School has 200 students.

(c) Sport Middle School Middle School has 200 students.

(d) Sport Middle School Middle School has 200 students.

(e) Sport Middle School Middle School has 200 students.

(f) Sport Middle School Middle School has 200 students.

(g) Sport Middle School Middle School has 200 students.

(h) Sport Middle School Middle School has 200 students.

(i) What students both most about each hate Yet and Kolling Class are together.

(j) What students both most about each hate Yet and Kolling Class are together.

(k) What students both most about each hate Yet and Kolling Class are together.

Exercises

13.1.1 ⏱ Guide by Solution

The Histogram at the right shows the results when the students in my class were called from twenty-one students.

(a) What is the modal number of pencils?

Please Solution

This Help Sign Not will without Hello Long Sales Pencils

Your Submission Solution

(b) Compute the average number of pencils for the learner's journalists.

Please Solution

This Help Sign Not will without Hello Long Sales Pencils

Your Submission Solution

(c) Compute the average number of pencils for the learner's journalists.

Please Solution

This Help Sign Not will without Hello Long Sales Pencils

Your Submission Solution

Solution: Reading the chart, the total number of students is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 100.$$

The total number of pencils is

$$1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 4 + 5 \cdot 5 + 6 \cdot 6 + 7 \cdot 7 + 8 \cdot 8 + 9 \cdot 9 + 8 \cdot 8 + 7 \cdot 7 + 6 \cdot 6 + 5 \cdot 5 + 4 \cdot 4 + 3 \cdot 3 + 2 \cdot 2 + 1 \cdot 1 = 190.$$

Therefore, the average number of pencils for the nearest hundredth is $190/100 = 1.90$.

(d) What is the modal number of pencils?

Please Solution

This Help Sign Not will without Hello Long Sales Pencils

Your Submission Solution

(e) Suppose the number of students, the median number of pencils is the 25th percentile. Suppose we let the students go to recess and we let the students go back to class. The median number of pencils is the 25th percentile. Suppose we let the students go to recess and we let the students go back to class. The median number of pencils is the 25th percentile. Suppose we let the students go to recess and we let the students go back to class. The median number of pencils is the 25th percentile.

Please Solution

This Help Sign Not will without Hello Long Sales Pencils

Your Submission Solution

(f) Suppose the number of students, the median number of pencils is the 25th percentile. Suppose we let the students go to recess and we let the students go back to class. The median number of pencils is the 25th percentile. Suppose we let the students go to recess and we let the students go back to class. The median number of pencils is the 25th percentile. Suppose we let the students go to recess and we let the students go back to class. The median number of pencils is the 25th percentile.

Please Solution

This Help Sign Not will without Hello Long Sales Pencils

Your Submission Solution

13.1.2 ⏱ Guide by Solution

Let n be the number of pencils and m the number of students. Let n be the number of pencils and m the number of students. Let n be the number of pencils and m the number of students. Let n be the number of pencils and m the number of students.

(a) Find the average, median, and mode of the number of tests each member has taken, based on the data in the table above.

Please Solution

This Help Sign Not will without Hello Long Sales Pencils

Your Submission Solution

Solution: The mean number of tests is $(1 \cdot 10 + 2 \cdot 5 + 3 \cdot 2 + 4 \cdot 1)/(10 + 5 + 2 + 1) = 1.75$. The mean and median of the data consistently already has one entry is numerical order from least to greatest. There are 20 entries so the median is the average of the two middle entries, which is $(1 + 1)/2 = 1$. The mode is the most common entry, which is 1.

(b) Find the average, and find the total number of tests. To find the sum, we add up the sum and the sum consistently. The sum of all of the entries is $1 \cdot 10 + 2 \cdot 5 + 3 \cdot 2 + 4 \cdot 1 = 30$. The sum of all of the entries is $1 \cdot 10 + 2 \cdot 5 + 3 \cdot 2 + 4 \cdot 1 = 30$. The sum of all of the entries is $1 \cdot 10 + 2 \cdot 5 + 3 \cdot 2 + 4 \cdot 1 = 30$. The sum of all of the entries is $1 \cdot 10 + 2 \cdot 5 + 3 \cdot 2 + 4 \cdot 1 = 30$.

(c) Find the average, median, and mode of the number of tests taken by students who have taken at least one test. The average is $(1 \cdot 10 + 2 \cdot 5 + 3 \cdot 2 + 4 \cdot 1)/(10 + 5 + 2 + 1) = 1.75$. The median is the average of the two middle entries, which is $(1 + 1)/2 = 1$. The mode is the most common entry, which is 1.

(d) Find the average, median, and mode of the number of tests taken by students who have taken at least two tests. The average is $(1 \cdot 10 + 2 \cdot 5 + 3 \cdot 2 + 4 \cdot 1)/(10 + 5 + 2 + 1) = 1.75$. The median is the average of the two middle entries, which is $(1 + 1)/2 = 1$. The mode is the most common entry, which is 1.

(e) Find the average, median, and mode of the number of tests taken by students who have taken at least three tests. The average is $(1 \cdot 10 + 2 \cdot 5 + 3 \cdot 2 + 4 \cdot 1)/(10 + 5 + 2 + 1) = 1.75$. The median is the average of the two middle entries, which is $(1 + 1)/2 = 1$. The mode is the most common entry, which is 1.

(f) It's with every test half of them get their tests, which students will be affected the most, average, median, and mode?

Please Solution

This Help Sign Not will without Hello Long Sales Pencils

Your Submission Solution

Solution: The mode and median go up, and the average goes down. Since the average is the average of the tests and dividing by 20, the average goes down. Since the average is the average of the tests and dividing by 20, the average goes down. Since the average is the average of the tests and dividing by 20, the average goes down.

(g) Lisa's shooting the ball is random and tests, the Lisa Rose that she slept in the week and half tests is incorrect. When this number is added to the table, which of the average, median, and mode will change? Which one will change? Which one will change? Which one will change?

Please Solution

This Help Sign Not will without Hello Long Sales Pencils

Your Submission Solution

Solution: Adding the number of tests, the total number of tests is $30 \cdot 10 + 5 \cdot 10$. So, we want to increase the total number of tests by 50 by adding changes our data in the table. Since we change from the average from 1 to more than 1, our average increases. Since we change from the median from 1 to more than 1, our median increases. Since we change from the mode from 1 to more than 1, our mode increases. Since we change from the median from 1 to more than 1, our median increases. Since we change from the median from 1 to more than 1, our median increases.

(h) Lisa's shooting the ball is random and tests, the Lisa Rose that she slept in the week and half tests is incorrect. When this number is added to the table, which of the average, median, and mode will change? Which one will change? Which one will change? Which one will change?

Please Solution

This Help Sign Not will without Hello Long Sales Pencils

Your Submission Solution

(i) Average, median, mode, average, median, average, median, average, median

13.4 Summary

We studied three **statistics**, which are numbers used to give information about groups of numbers:

- **Average**: The average of a group of numbers is the sum of the numbers divided by the number of numbers. So, the average of 3, 5, 6, and 10 is $\frac{3+5+6+10}{4}$, which equals 6. The average is also called the **mean** or the **arithmetic mean**.
- **Median**: If we list a group of numbers from least to greatest, the median of the group is the number in the middle. So, the median of the numbers in the list

  $$4, 5, 7, 8, 11$$

  is 7. If there is an even number of numbers, then the median is the average of the middle two numbers.
- **Mode**: The mode of a group of numbers is the number that appears most frequently in the group. So, the mode of

  $$3, 3, 3, 3, 4, 5, 6, 7$$

  is 3. A group can have multiple modes if there are multiple numbers that appear the same number of times.

Average and median are used much more often than mode. There are significant limitations on what these statistics can tell us about a group of data. The median only tells us what the middle number is (or the average of the middle two numbers). It doesn't tell us anything about how far the numbers in the group are from the middle. Numbers that are much greater or much less than most of the other numbers in a group are called **outliers**. Outliers tend to have a very large effect on the average of the group, but very little effect on the median.

There are many ways to display data, such as tables (Section 13.3 **here**), bar charts (Section 13.3 **here**), line graphs (Section 13.3 **here**), stem-and-leaf plots (Section 13.3 **here**), and pie charts (Section 13.3 **here**).

Challenge Problems

13.27:

Source: AMC 8

There is a list of seven numbers. The average of the first four numbers is 5, and the average of the last four numbers is 8. If the average of all seven numbers is $6\frac{4}{7}$, then what number is common to both sets of four numbers?

Hint: Problems involving averages can often be solved by thinking about sums.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The sum of the first four numbers is $4 \cdot 5 = 20$. The sum of the last four numbers is $4 \cdot 8 = 32$. If we add these two sums, we find that $20 + 32 = 52$ is the sum of all seven numbers plus an extra copy of the middle number. The sum of all seven numbers is $7 \cdot \left(6\frac{4}{7}\right) = 46$, so the middle number must be $52 - 46 = \boxed{6}$.

13.28:

The mean, median, and mode of the five numbers 5, 7, 8, A, B are equal. (The list has a single mode.) Find all possible values of $A + B$.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: In order for there to be a unique mode, one of the numbers must be repeated, but we can't have two numbers be repeated. It doesn't matter which is the repeated value, A or B, so we'll let A be the repeated value. There are four possibilities to consider:

Case 1: 5 is repeated. Our list then is 5, 5, 7, 8, B. Since the mode is 5, the median and mean must also be 5. Since 7 and 8 together are a total of $2 + 3 = 5$ greater than the mean, we know that B is 5 less than the mean. This tells us that $B = 5 - 5 = 0$, making our list 0, 5, 5, 7, 8. This list does indeed have mean, median, and mode equal to 5. In this case, $A + B = 5$.

Case 2: 7 is repeated. Our list then is 5, 7, 7, 8, B. The median and mean must also be 7. Since 8 is 1 greater than the mean and 5 is 2 less than the mean, the remaining number must also be 1 greater than the mean. That makes the list 5, 7, 7, 8, 8. But the mode of this list is not unique! So, it is impossible for 7 to be the repeated number.

Case 3: 8 is repeated. Our list then is 5, 7, 8, 8, B. The median and mean must also be 8. Since 5 is 3 less than the mean and 7 is 1 less than the mean, the remaining number must be $3 + 1 = 4$ greater than the mean. This makes our list 5, 7, 8, 8, 12. This list has mean, median, and mode equal to 8, and we have $A + B = 20$.

Case 4: A and B are equal. Our list then is 5, 7, 8, A, A. The mean and median must also be A. The mean of this list is $\frac{5 + 7 + 8 + A + A}{5}$. Simplifying this and setting it equal to A gives

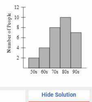
$$\frac{20 + 2A}{5} = A.$$

Multiplying both sides by 5 gives $20 + 2A = 5A$. Subtracting $2A$ from both sides gives $20 = 3A$, so $A = \frac{20}{3} = 6\frac{2}{3}$. This makes our list 5, $6\frac{2}{3}$, $6\frac{2}{3}$, 7, 8, which has mean, median, and mode equal to $6\frac{2}{3}$. In this case, $A + B = 13\frac{1}{3}$.

Combining all of our cases, the possible values of $A + B$ are $\boxed{5, 13\frac{1}{3}, \text{ and } 20}$.

13.29:

The Back In My Day club made the histogram at right to represent the ages of its members.



(a) What is the least possible median age of the members?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We start by noting that there are $2 + 4 + 8 + 10 + 7 = 31$ people total.

The number of people who are less than 80 years old is $2 + 4 + 8 = 14$. There are 7 people in their 90s and 10 in their 80s. So, less than half the people are below 80 years old and more than half are below 90 years old. This means that the median age is in the 80s. Therefore, the smallest it can be is $\boxed{80}$, which would indeed occur if all the people in their 80s are exactly 80.

(b) What is the greatest possible median age?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: As explained in the first part, the median must be in the 80s. Therefore, the greatest the median age can be is $\boxed{89}$. For example, if all the people in their 80s were 89 years old, the median would be 89.

(c) What is the least possible average age?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There are 31 people. To make the average of the ages as small as possible, we must make the sum of the ages as small as possible. The least possible sum occurs when each of the ages is as low as possible, which is the corresponding multiple of 10 for each age. So, the lowest possible sum is

$$\begin{aligned} &2 \cdot 50 + 4 \cdot 60 + 8 \cdot 70 + 10 \cdot 80 + 7 \cdot 90 \\ &= 100 \cdot 5 + 4 \cdot 6 + 8 \cdot 7 + 10 \cdot 8 + 7 \cdot 9) \\ &= 2330. \end{aligned}$$

Therefore, the least possible average is $\frac{2330}{31} = \boxed{75\frac{5}{31}} \approx 75.16$.

(d) What is the greatest possible average age?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The average is greatest if all the people are as old as possible. This will occur if all the people in their 50s are 59, all the people in their 60s are 69, and so on. To sum these ages, we add the tens and ones separately. We found the sum of these tens in part (c) as 2330. The sum of the units is $31 \cdot 9 = 279$, so the sum of all 31 ages is $2330 + 279 = 2609$, and the desired greatest possible average is $\frac{2609}{31} = \boxed{84\frac{25}{31}} \approx 84.16$. Seeing that this result is exactly 9 greater than the result for part (c), we note that adding 9 to each of the ages in part (c) results in adding 9 to the average, as expected.

13.30:

(a) Two lists of numbers that have the same average are combined to form a longer list. Must the average of the new list be the same as the average of each of the original shorter lists?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\boxed{\text{Yes}}$. The total amount by which the above-average numbers in the first list exceeds the average must equal the total amount by which the average exceeds the below-average numbers in the first list. Similarly, the total amount by which the above-average numbers in the second list exceeds the average must equal the total amount by which the average exceeds the below-average numbers in the second list. So, when we combine the two lists, the total amount by which numbers above the common average exceed the common average equals the total amount by which the common average exceeds the below-average numbers. This means that the combined list must have the same average as the common average of the two lists.

(b) Two lists of numbers that have the same median are combined to form a longer list. Must the median of the new list be the same as the median of each of the original shorter lists?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\boxed{\text{Yes}}$. Suppose we start with the first list, in order. The median must be the middle number of this list, or the average of the two middle numbers. When we combine the second list with this first list, there must be just as many new numbers above this median as below, since this median is the middle number (or between the two middle numbers) of the second list, too. So, the common median will be the median of the new list, too.

(c) Two lists of numbers that have the same mode are combined to form a longer list. Is the mode of the new list necessarily the same as the mode of each of the original shorter lists?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\boxed{\text{Yes}}$. The mode is the number that appears most often. If a number appears more than any other in the first list, and it appears more than any other in the second list, then it definitely will have more total appearances across both lists than any other number. So, the mode of the combined list is the same as the mode of each individual list.

(d) Two lists of numbers are combined to form a longer list. The mode of the new list is 3. Is it possible that neither of the original lists had 3 as its mode?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\boxed{\text{Yes}}$. Suppose the lists are 5,5,5,3,3 and 2,2,2,3,3. The mode of the first list is 5 and the mode of the second is 2. The combined list is 2,2,2,3,3,3,3,5,5,5. This list has mode 3.

13.31:

My four closest friends have weekly allowances of \$13, \$17, \$24, and \$30. What are the possible values of my weekly allowance if the median of our allowances equals the average of our allowances?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We start by noting that the total allowance of my friends is $\$13 + \$17 + \$24 + \$30 = \$84$. We then consider three cases:

Case 1: My weekly allowance is at most \$17. The median allowance then is \$17. In order for the average also to be \$17, the total of all 5 allowances must be $5 \cdot (\$17) = \$85$. So, my allowance must be $\$85 - \$84 = \$1$, which is indeed no more than \$17, and I need to have a conversation with my parents.

Case 2: My weekly allowance is at least \$24. The median allowance then is \$24. In order for the average also to be \$24, the total of all 5 allowances must be $5 \cdot (\$24) = \$120$. So, my allowance must be $\$120 - \$84 = \$36$, which is greater than \$24, as expected.

Case 3: My weekly allowance is between \$17 and \$24. The median allowance then is my allowance. Let my allowance be x dollars. The average allowance then is $\frac{84 + x}{5}$ dollars. Since this must equal the median, which is my allowance, we must have

$$\frac{84 + x}{5} = x.$$

Multiplying both sides by 5 gives $84 + x = 5x$, so $84 = 4x$ and $x = 21$, which is between 17 and 24 as desired. (My allowance is the average of my four friends' allowances in this case. Is this a coincidence?)

Combining the three cases, the possible values of my weekly allowance are $\boxed{\$1, \$21, \text{ and } \$36}$.

13.32:

We asked 124 people to rank four card games from favorite (1st) to least favorite (4th). The results are shown in the table at the right. Two of the entries in the table are wrong. Fix the errors by changing two of the numbers in the table.

	1st	2nd	3rd	4th
Magic	34	56	23	11
Pokemon	12	14	38	64
SET	57	41	23	3
Yu-Gi-Oh!	21	19	44	46

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since each card receives one vote per person, the four rows must each sum to 124. Similarly, each column must sum to 124. We include a row and a column to our table for these totals, and we have the table shown on the right. Looking at this table, we expect that the erroneous rows are those for Pokemon and Yu-Gi-Oh!, and the erroneous columns are those for 2nd and 3rd. The Pokemon row total is 4 too high, as is the total for the 3rd place column. So, we can fix both by subtracting 4 from the number of 3rd place votes received by Pokemon. Similarly, we can fix the Yu-Gi-Oh! row and the 2nd place column by subtracting 6 from the number of 2nd place votes Yu-Gi-Oh! received. The resulting table is shown below.

	1st	2nd	3rd	4th	Total
Magic	34	56	23	11	124
Pokemon	12	14	38	64	128
SET	57	41	23	3	124
Yu-Gi-Oh!	21	19	44	46	130
Total	124	130	128	124	

	1st	2nd	3rd	4th	Total
Magic	34	56	23	11	124
Pokemon	12	14	**34**	64	124
SET	57	41	23	3	124
Yu-Gi-Oh!	21	**13**	44	46	124
Total	124	124	124	124	

13.33★:

For a set of ten numbers, removing the largest number decreases the average by 1. Removing the smallest number increases the average by 2. What is the positive difference between the largest and the smallest of these ten numbers?

Hint: Sometimes thinking of the average of a list of numbers as "balancing" the list helps solve problems.

Hint: How far must the largest number be from the average of the original list?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The average of the smallest 9 numbers is 1 less than the average of all 10 numbers. So, on average, each of the 9 smallest numbers is 1 less than the average of all 10 numbers. This means that the 9 smallest numbers together are a total of $9 \cdot 1 = 9$ less than the average of all 10 numbers. So, the largest number must be 9 greater than the average of all 10 numbers.

Similarly, the average of the largest 9 numbers is 2 greater than the average of all 10 numbers. So, on average, each of the 9 largest numbers is 2 greater than the average of all 10 numbers. This means that the 9 largest numbers together are a total of $9 \cdot 2 = 18$ greater than the average of all 10 numbers. So, the smallest number must be 18 less than the average of all 10 numbers.

Since the largest number is 9 greater than the average of all 10 numbers, and the smallest number is 18 less than this average number, we know that the smallest and largest are $9 + 18 = \boxed{27}$ apart.

Review Problems

13.16:

I have four dogs whose average weight is 43 pounds. I also have three cats. The average weight of all seven of my animals is 41 pounds. What is the average weight of my cats?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The total weight of the dogs is $4 \cdot 43 = 252$ pounds. The total weight of all 7 animals is $7 \cdot 41 = 287$ pounds. So, the total weight of all three cats is $287 - 252 = 35$ pounds, which means the average weight of each cat is $\frac{35}{3} = \boxed{11\frac{2}{3}}$ pounds.

13.17:

The average age of the ten people on my basketball team was 13.5, but then a 15-year-old joined our team and an 11-year-old quit the team. What is the average age of my team now?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: When the 15-year-old joins and the 11-year-old quits, the sum of the ages of the people on the team increases by 4 years. Since there are 10 people on the team, this increases the average age by $\frac{4}{10} = 0.4$ years, to $13.5 + 0.4 = \boxed{13.9}$ years.

13.18:

Source: MOEMS

The average of five numbers is 18. Let the first number be equal to 1, the second number by 2, the third number by 3, the fourth number by 4, and the fifth number by 5. What is the average of the list of increased numbers?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since the average of the five numbers is 18, their sum is $5 \cdot 18 = 90$. The total increase of the numbers is $1 + 2 + 3 + 4 + 5 = 15$, so the sum of the increased numbers is $90 + 15 = 105$. Therefore, the average of the increased numbers is $\frac{105}{5} = \boxed{21}$.

We also could have noticed that the average increase is the average of 1, 2, 3, 4, and 5, which is 3. Since the average increase is 3, the average of the increased numbers is $18 + 3 = 21$.

13.19:

Suppose you took eight mark tests this semester. If your average score on your first six tests was 84 and your average score on all eight tests was 86, then what was the average of your last two test scores?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since the average of the five numbers is 18, their sum is $5 \cdot 18 = 90$. The total increase of the numbers is $1 + 2 + 3 + 4 + 5 = 15$, so the sum of the increased numbers is $90 + 15 = 105$. Therefore, the average of the increased numbers is $\frac{105}{5} = \boxed{21}$.

We also could have noticed that the average increase is the average of 1, 2, 3, 4, and 5, which is 3. Since the average increase is 3, the average of the increased numbers is $18 + 3 = 21$.

13.20:

Must the median of a group of consecutive integers equal the average of the group?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\boxed{\text{Yes}}$. As we saw in **13.5.9**, the average of a group of consecutive numbers equals the middle number if there are an odd number of numbers, and the average of the group equals the average of the two middle terms if there are an even number of numbers. In both cases, the average equals the median.

13.21:

Larry writes a list of numbers that has average 14, median 21 and mode 11.

(a) Mike creates a list by adding 12 to each number in Larry's list. What are the average, median, and mode of Mike's list?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Average: Suppose there are n numbers in the list. Adding 12 to each number adds $12n$ to the sum of the numbers. Since the sum is increased by $12n$ while the number of numbers stays the same, the average increases by $\frac{12n}{n} = 12$. So, the new average is $14 + 12 = \boxed{26}$.

Median: Suppose we place the numbers in Larry's list in increasing order. Adding 12 to each number in this list won't change the order; Mike's new list is in increasing order, too. So, the middle number of Mike's list is 12 more than the middle number of Larry's list. This means that Mike's list's median is $21 + 12 = \boxed{33}$. If there are an even number of numbers, then the median is the average of the middle two numbers. We can use the same argument as in our first paragraph to see that Mike's median is still 12 more than Larry's median.)

Mode: If two numbers are different in Larry's list, then the corresponding numbers are different in Mike's list. If two numbers are the same in Larry's list, then the corresponding numbers are the same in Mike's list. So, for any number x in Larry's list, the number of times x appears in Larry's list equals the number of times that $12 + x$ appears in Mike's list. Therefore, the number that appears most frequently in Mike's list is 12 more than the number that appears most frequently in Larry's list. This means that the mode of Mike's list is $11 + 12 = \boxed{23}$.

(b) Curly creates a list by multiplying each number in Larry's list by 2. What are the average, median, and mode of Curly's list?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Average: We follow essentially the same process as in the previous part. Doubling each number in a sum doubles the sum. So, the sum of the numbers in Curly's list is double the sum of the numbers in Larry's list. The lists have the same number of numbers, so Curly's average is double Larry's average. This means that Curly's list has average $2 \cdot 14 = \boxed{28}$.

Median: Suppose we place the numbers in Larry's list in increasing order. Multiplying each number in this list by 2 won't change the order; Curly's new list is in increasing order, too. So, the middle number of Curly's list is 2 times the middle number of Larry's list. This means that Curly's list's median is $2 \cdot 21 = \boxed{42}$. (If there are an even number of numbers, then the median is the average of the middle two numbers. We can use the same argument as in our first paragraph to see that Curly's median is still 2 times Larry's median.)

Mode: The number of times a certain number appears in Larry's list equals the number of times double that number appears in Curly's list. Therefore, the number that appears most frequently in Curly's list is 2 times the number that appears most frequently in Larry's list. This means that the mode of Curly's list is $2 \cdot 11 = \boxed{22}$.

13.22:

Find the average of the following numbers:

$$9420385988, 9420385994, 9420386003, 9420385981, 9420385991.$$

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: All 5 of the numbers are very close to 9402385980. If we subtract 9402385980 from each number, our list becomes

$$8, 14, 23, 1, 11.$$

The average of the numbers in this list is

$$\frac{8 + 14 + 23 + 1 + 11}{5} = \frac{57}{5} = 11.4.$$

Each number in the original list is 9402385980 greater than the corresponding number in our list with average 11.4. Therefore, the average of the numbers in the original list is

$$11.4 + 9402385980 = \boxed{9402385991.4}.$$

13.23:

Each of 9 friends chooses her favorite positive integer.

(a) The median of the chosen numbers is 91. What is the smallest the average of the 9 chosen numbers could be?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution In order to make the average as small as possible, the sum of the chosen numbers must be as small as possible. Because the median of the numbers is 91, we know that five of the nine numbers are at least 91. To make the sum as small as possible, we let all five of these numbers equal 91. The smallest the other four numbers can be is 1, so the smallest the sum of all nine numbers can be is $4 \cdot 1 + 5 \cdot 91 = 459$. This means that the smallest the average can be is $\frac{459}{9} = \boxed{51}$.

(b) The median of the chosen numbers is 91. What is the least it how large the average of the chosen numbers can be? If so, what is the largest the average can be?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There is $\boxed{\text{no maximum}}$. The median only tells us that at least one of the numbers is 91, and four other numbers are at least 91. It doesn't place any limit on how large these four other numbers are, so they can be as large as we like. Therefore, there's no limit on how large the sum of all 9 numbers is, which means there is no limit to how large the average of the numbers can be.

(c) The average of the chosen numbers is 91. What is the smallest the median of the 9 chosen numbers could be?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since the average is 91, the sum of the 9 chosen numbers is $9 \cdot 91 = 819$. We might have one friend choose 811 and each of the other eight friends choose 1. The median of these nine choices is 1 and the sum is $811 + 8 \cdot 1 = 819$. Therefore, it is possible for 1 to be the median. Since the friends must choose positive integers, no one can choose a number smaller than 1. This means that $\boxed{1}$ is the smallest possible median.

(d)★ The average of the chosen numbers is 91. What is the largest the median of the chosen numbers could be?

Hint: Problems involving averages can often be solved by thinking about sums.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: As in the previous part, the sum of the chosen numbers is $9 \cdot 91 = 819$. Suppose the median is x. Since the sum of all nine numbers is 819, we must have

$$x = 819 - (\text{sum of the other eight numbers}).$$

Therefore, to make x as large as possible, we must make the sum of the other eight numbers as small as possible.

Since x is the median of the nine numbers, there are four other numbers that are no greater than x and four others that are no less than x. The smallest the numbers in the first group can be is 1, and the smallest the numbers in the second group can be is x. So, the median is as large as possible when four of the numbers are 1 and the other five are the same. If four of the numbers are 1, then the sum of the other five is $819 - 4 = 815$. Since these other five numbers must be the same to make the median as large as possible, the greatest the median can be is $\frac{815}{5} = \boxed{163}$.

As an extra challenge, try to figure out the greatest that the median can be if we have more friends choosing favorite positive integers, but the average of the integers chosen is still 91. What if there are 100 friends? 1000 friends? 4 million friends?

13.24:

Consider the two lists of numbers below:

List A: 34, 54, 161, 443, 87, 43, 76, 339, 38, 654, 75, 164, 876
List B: 5674784, 54, 65, 12, 654, 765, 12, 34, 98, 56, 72, 34, 86

(a) Is it easy to tell quickly which list has the higher average?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\boxed{\text{Yes, List B}}$. The average of each list is the sum of the list divided by the number of numbers in the list. None of the numbers in List A is greater than 1000, but List B has a number that is greater than fifty million! All the numbers in both lists are positive, so the sum of List B is far greater than the sum of List A. Therefore, the average of List B is far greater than that of List A.

(b) Is it easy to tell quickly which list has the higher median?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\boxed{\text{No}}$. We have to put the numbers in order to figure out what the median is in each list. We can't tell at a glance which list is more likely to have a larger middle number. (For those of you keeping score at home, the median of List A is 87 and the median of list B is 65.)

(c) What do the first two parts tell us about outliers?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: One extreme value, such as the large number in List B, can have a huge effect on the average, but not a very large impact on the median. That is, outliers affect the average more than they affect the median.

13.25:

Source: MATHCOUNTS

The chart at the right shows where Meadow Lark Lane Middle School students do their homework. The options are their bedroom (BR), the kitchen (K), the dining room (D), and the family living room (FLR). Using these data, how many of the 880 students at the school do their homework in the dining room?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: $\boxed{\text{Yes}}$. As we can see in the chart, 30% of the students do their homework in the dining room. Therefore, there are $0.2 \cdot 880 = \boxed{176}$ students who do their homework in the dining room.

13.26:

Five people live along a long, straight road. The table below gives the distance in miles between pairs of houses. For example, the distance between Adrian's house and Walter's house is 11 miles.

	Adrian	Dan	Laurie	Jon	Walter
Adrian	0		15		11
Dan	3	0	12	7	8
Laurie	15		0		4
Jon	10	7	5	0	
Walter	11	8	4	1	0

(a) Fill in the missing entries.

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The number in the upper left corner is 0 because the distance between Adrian's house and Adrian's house is obviously 0. Similarly, each number on the diagonal from upper left to lower right must be 0. This allows us to fill in two of the missing entries with 0. (corresponding to Laurie's house and Walter's house).

Next, we note that the entry in Adrian's row and Dan's column is the same as the entry in Dan's row and Adrian's column, since both of these entries represent the distance between Adrian's house and Dan's house. Similarly, the distance between each pair of houses appears twice in the table. This allows us to fill in all the rest of the missing entries except the ones corresponding to the distance between Walter's house and Jon's house.

Looking at the bottom two rows, we see that the distances from each person to Walter and to Jon differ by 1 mile, so we conclude that Walter's house and Jon's house are 1 mile apart. The completed table is shown below:

	Adrian	Dan	Laurie	Jon	Walter
Adrian	0	3	15	10	11
Dan	3	0	12	7	8
Laurie	15	12	0	5	4
Jon	10	7	5	0	1
Walter	11	8	4	1	0

(b) Which two people live the farthest apart? Which two people live the closest?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The largest number in the table is 15, which tells us that $\boxed{\text{Laurie and Adrian}}$ are the farthest apart. The smallest number corresponding to a distance between two different people is 1, so $\boxed{\text{Jon and Walter}}$ are closest.

(c) Draw an accurate map showing where the people live along the road.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Since Laurie and Adrian are farthest apart, they are at opposite ends of the road. We can then use the distance between Adrian (or Laurie) and each of the others to place all 5 people along the road. Dan is closest to Adrian, so we place him first, 3 miles away. Next closest is Jon, who is 10 miles from Adrian. Jon is $10 - 3 = 7$ miles farther from Adrian than Dan, so Jon is 7 miles from Dan, as shown. Similarly, Walter is $11 - 10 = 1$ mile from Jon and Laurie is $15 - 11 = 4$ miles from Walter:

(We could also reverse the entire map, placing Adrian at the far right and Laurie at the far left.)

14 10 10 10

14 10 10 10

14.2 The Multiplication Principle

Problems

Problem 14.8 Jump to Solution

You have three shirts and four pairs of pants. How many outfits consisting of one shirt and one pair of pants can you make?

Problem 14.9 Jump to Solution

In how many ways can we form an international commission if we must choose one European country from among 6 European countries, one Asian country from among 4, one North American country from among 3, and one African country from among 7?

Problem 14.10 Jump to Solution

In how many ways can we form a license plate using only digits (0–9) and capital letters (other than O and I), given that each plate has 6 characters, the first of which is a digit, and the second of which is a letter?

Problem 14.11 Jump to Solution

In how many ways can I arrange four different books from left to right on a shelf?

Problem 14.12 Jump to Solution

Your math club has 16 members. In how many ways can it select a president, a vice-president, and a treasurer if no member can hold more than one office?

Often we'll be counting the number of outcomes of a series of events. Here's an example of this type of problem.

Problem 14.8

You have three shirts and four pairs of pants. How many outfits consisting of one shirt and one pair of pants can you make?

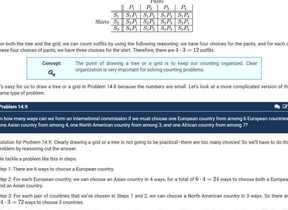
Solution for Problem 14.8: In this problem, the number of possibilities is so small that we can just list them.

If our shirts are labeled S_1, S_2, S_3 and our pants are labeled P_1, P_2, P_3, P_4, then we can list all of the possible outfits:

$$S_1P_1 \quad S_1P_2 \quad S_1P_3 \quad S_1P_4$$
$$S_2P_1 \quad S_2P_2 \quad S_2P_3 \quad S_2P_4$$
$$S_3P_1 \quad S_3P_2 \quad S_3P_3 \quad S_3P_4$$

So there are 12 outfits. ☺

Listing the outfits like this is somewhat annoying. We can better visualize the outfits by using a diagram. One sort of diagram is a **tree**, as shown below. We start at the dot at the top, and each arrow is a choice of one item; the first arrow (from the dot at the top) is the choice of pants, and the second arrow (from the choice of pants) is the choice of shirt. Each complete path of two arrows leads to a complete outfit.



Another type of diagram we can use for Problem 14.8 is a **grid**. A grid looks a bit like a multiplication table: we put the choices of pants along the top and the choices of shirts along the side. Each box in the grid is a complete outfit, as shown below:

		Pants			
		P_1	P_2	P_3	P_4
Shirts	S_1	S_1P_1	S_1P_2	S_1P_3	S_1P_4
	S_2	S_2P_1	S_2P_2	S_2P_3	S_2P_4
	S_3	S_3P_1	S_3P_2	S_3P_3	S_3P_4

For both the tree and the grid, we can count outfits by using the following reasoning: we have four choices for the pants, and for each of these four choices of pants, we have three choices for the shirt. Therefore, there are $4 \cdot 3 = 12$ outfits.

> **Concept:** The point of drawing a tree or a grid is to keep our counting organized. Clear organization is very important for solving counting problems.

It's easy for us to draw a tree or a grid in Problem 14.8 because the numbers are small. Let's look at a more complicated version of the same type of problem.

Problem 14.9

In how many ways can we form an international commission if we must choose one European country from among 6 European countries, one Asian country from among 4, one North American country from among 3, and one African country from among 7?

Solution for Problem 14.9: Clearly drawing a grid or a tree is not going to be practical—there are too many choices! So we'll have to do this problem by reasoning out the answer.

We tackle a problem like this in steps.

Step 1: There are 6 ways to choose a European country.

Step 2: For each European country, we can choose an Asian country in 4 ways, for a total of $6 \cdot 4 = 24$ ways to choose both a European and an Asian country.

Step 3: For each pair of countries that we've chosen in Steps 1 and 2, we can choose a North American country in 3 ways. So there are $24 \cdot 3 = 72$ ways to choose 3 countries.

Step 4: For each triple of countries that we've chosen in Steps 1–3, we can choose an African country in 7 ways. So there are $72 \cdot 7 = 504$ ways to choose 4 countries.

So we see that there are $6 \cdot 4 \cdot 3 \cdot 7 = 504$ possibilities. ☺

We say that the choices in Problem 14.9 are **independent**, meaning that each decision does not depend on the others and does not affect the others. Specifically, we choose a European country, then an Asian country, then a North American country, then an African country. Each choice doesn't depend on or affect the other choices.

> **Concept:** We use multiplication to count the number of outcomes for a sequence of independent events.

Problem 14.10

In how many ways can we form a license plate using only digits (0–9) and capital letters (other than O and I), given that each plate has 6 characters, the first of which is a digit, and the second of which is a letter?

Solution for Problem 14.10: Since each character does not depend on any of the other characters, our choices are independent. There are 10 choices for the first character (any digit from 0 through 9), there are 24 choices for the second character (any letter A–Z except for O or I), and there are 34 choices for each of the other four characters (any digit 0–9 or any letter A–Z except O or I).

Therefore, since the choices are independent, we multiply the number of choices at each step, and we have

$$10 \cdot 24 \cdot 34 \cdot 34 \cdot 34 \cdot 34 = 10 \cdot 24 \cdot 34^4 = 320{,}720{,}640$$

ways to form our license plate. ☺

In some counting problems, we make a series of choices, but later choices will depend on some of the earlier choices. This is unlike the problems that we've done before. For example, in Problem 14.8, our choices of shirt didn't depend at all on our choice of pants. Things are a little bit different in the next problem.

Problem 14.11

In how many ways can I arrange four different books from left to right on a shelf?

Solution for Problem 14.11: We cannot just count $4 \cdot 4 \cdot 4 \cdot 4$, because once we place the first book on the left, we no longer have 4 choices for the second book. We can't reuse the first book, so we only have 3 choices remaining for the second book. Let's carefully count our choices step by step:

Step 1: We have 4 choices for which book to place on the left.

Step 2: Regardless of which book we chose in step 1, we have 3 books remaining, so we have 3 choices for the second book. Thus we have a total of $4 \cdot 3 = 12$ choices for the first two books on the left.

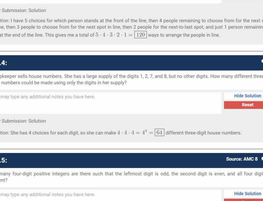
Step 3: We have 2 books remaining, so we have 2 choices for the third book. Thus we have a total of $12 \cdot 2 = 24$ choices for the first three books.

Step 4: We only have 1 book remaining, so we have 1 choice for the fourth book. Thus we have a total of $24 \cdot 1 = 24$ choices for all four books.

So the answer is 24. ☺

One important thing to note about Problem 14.11 is that although the choices themselves are not independent at each step, the number of choices at each step is independent of our previous choices. For example, no matter which book we choose in Step 1, we always have 3 remaining books to choose from in Step 2. Therefore, we can get our answer by multiplying the number of choices at each step, so that $4 \cdot 3 \cdot 2 \cdot 1 = 24$ is the answer.

If you're not yet convinced, we could represent our choices in this problem as a tree, where the books are labeled A, B, C, and D, as shown in the picture on the next page. We can see that there are 4 arrows from the starting dot for the first choice, then 3 arrows for the next choice, and so on, and indeed there are 24 possible arrangements.



Here's another example of a counting problem in which the choices are not independent.

Problem 14.12

Your math club has 16 members. In how many ways can it select a president, a vice-president, and a treasurer if no member can hold more than one office?

Solution for Problem 14.12: Once again, our choices are not independent. Once a student has been chosen president, she is not available to be chosen vice-president or treasurer. However, the number of choices we have for each position is the same no matter who is chosen.

We have 16 choices for president. Once we've chosen a president, then we have 15 people remaining to choose from for vice-president. Then, we have 14 people remaining to choose from for treasurer.

Therefore, there are

$$16 \cdot 15 \cdot 14 = 3360$$

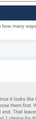
ways to fill the three offices. ☺

The last two problems contain examples of **permutations**. A permutation occurs whenever we have to choose several different items, one at a time, from a large group of items.

Counting with Multiplication Part 1

Counting with Multiplication Part 2

Factorial Introduction



Counting Permutations

With or Without Replacement

Exercises

14.2.1:

Suppose I have 7 shirts and 4 ties. How many shirt-and-tie outfits can I make?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: For each of the 7 choices I have for a shirt, I have 4 choices for a tie, giving me $7 \cdot 4 = \boxed{28}$ shirt-and-tie outfits.

14.2.2:

For each of 9 colors, I have one shirt and one tie of that color. How many shirt-and-tie outfits can I make if I refuse to wear a shirt and a tie of the same color?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: Similar to the previous exercise, I have $9 \cdot 9 = 81$ shirt-and-tie outfits, but I refuse to wear the 9 outfits that consist of a shirt and a tie of the same color. This leaves me $81 - 9 = \boxed{72}$ outfits. Another way to think of this is to suppose I choose my shirt first, and then my tie. I have 9 choices for the shirt, but then I only have 8 choices for the tie because I can't pick the one that is the same color as my shirt. So, I have $9 \cdot 8 = 72$ possible outfits, as before.

14.2.3:

In how many ways can 5 people stand in a line?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: Similar to the previous exercise, I have 5 choices for which person stands at the front of the line, then 4 people remaining to choose from for the next spot in line, then 3 people to choose from for the next spot in line, then 2 people for the next-to-last spot, and just 1 person remaining to put at the end of the line. This gives me a total of $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{120}$ ways to arrange the people in line.

14.2.4:

A shopkeeper sells house numbers. She has a large supply of the digits 1, 2, 3, and 8, but no other digits. How many different three-digit house numbers could be made using only the digits in her supply?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: She has 4 choices for each digit, so she can make $4 \cdot 4 \cdot 4 = 4^3 = \boxed{64}$ different three-digit house numbers.

14.2.5: Source: AMC 8

How many four-digit positive integers are there such that the leftmost digit is odd, the second digit is even, and all four digits are different?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: We have 5 choices for the first digit, and 5 choices for the second digit. For the third digit, we can't use either of the two digits we have already used, so there are 8 choices for the third digit. For the fourth digit, we can't use any of the three digits we have already used, so there are 7 choices for the fourth digit. This gives us a total of $5 \cdot 5 \cdot 8 \cdot 7 = \boxed{1400}$ ways to choose the digits.

14.2.6: Source: MOEMS

When the order of the digits of 2022 is reversed, the number remains the same. Such a number is called a **palindrome**. How many integers between 100 and 1000 are palindromes?

Preview: Solution

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: A three-digit number that reads the same forwards as backwards must have the same hundreds and units digit. We have 9 choices for that digit, since the hundreds digit cannot be 0. We can use any digit as the middle digit, so we have 10 choices for the middle digit. This gives us $9 \cdot 10 = \boxed{90}$ three-digit palindromes.

14.2.7★:

Suppose that I have 5 different books, 2 of which are math books. In how many ways can I place my 5 books left-to-right on a shelf if I want a math book on both ends?

Preview: Solution

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: At first, it might seem that the answer is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, since it looks like I have 5 choices for the first book, 4 for the next, and so on. But we have to have math books on the ends! So, we choose them first. We have 2 choices for which math book to put on the left end, and we have to save the other math book for the right end. That leaves 3 choices for which book to put to the right of the first math book, then 2 choices for the next non-math book, and 1 choice for the last non-math book. Then, we place our other math book on the right end of the shelf. This gives us a total of $2 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = \boxed{12}$ ways to shelve the books.

14.2.8★: Source: AMC 8

How many integers between 99 and 999 contain exactly one 0?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: There are a number of ways to tackle this problem, but perhaps the most clever is to think about making a three-digit number with one 0 as follows:

Step 1: Place the 0. There are 2 places where we can put the 0: in the tens place or the units place.

Step 2: Place digits from 1-9 in the other two places. No matter where we place the 0, there are 9 choices for each of the other two digits in our three-digit number. Therefore, for each possible placement of the 0, there are $9 \cdot 9 = 81$ ways to choose the other two digits to make a three-digit number with one zero.

So, there are 2 choices for where we can put the 0, and each gives us 81 three-digit numbers with one 0. This gives us a total of $2 \cdot 81 = \boxed{162}$ three-digit numbers with one 0.

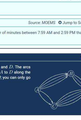
14.3 Casework

Many counting problems can be solved by considering different cases that, by dividing what we're trying to count into two or more disjoint subsets, make the problem easier to solve. Two key to solving problems using casework is to be organized, and to be careful that you don't miss any cases.

Problems

Problem 14.13 Jump to Solution

The figure to the right represents a road map between 4 villages labeled A, B, C and D. The only roads connecting the villages are the roads shown on the map. How many routes are there from A to D, if along the path, if you can only move left to right? (For example, you can't go $A B E D$, but you can go $A C D$ or $A B E C D$)



Problem 14.14 Jump to Solution

On the island of Abcdef, the alphabet has only 6 letters, and every word in their language has no more than 3 letters in it. How many words are possible? (A word can use a letter more than once, but 0 letters does not count as a word.)

Problem 14.15 Jump to Solution

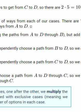
How many positive integer triples (a, b) satisfy $a^2 + b < 20$?

Problem 14.16 Source: MOEMS Jump to Solution

A digital clock shows time in the form HH:MM. On a certain day, what is the number of minutes between 7:00 AM and 2:00 PM that HH is greater than MM?

Problem 14.13

The figure to the right represents a road map between 4 villages labeled A, B, C and D. The only roads connecting the villages are the roads shown on the map. How many routes are there from A to D, if along the path, if you can only move left to right? (For example, you can't go $A B E D$, but you can go $A C D$ or $A B E C D$)



Solution for Problem 14.13. Broadly speaking, we have two ways to get from A to D: we can go through B or through C. We have to go through E no matter what, so we can't go through both. We say that these two choices are **exclusive**, because we must choose one or the other, but not both.

Our two exclusive choices will form our two cases.

Case 1: Go from A to D through B. There are 4 ways to get from A to B, and 2 ways to get from B to D, so there are $4 \cdot 2 = 12$ ways to get from A to D through B.

Case 2: Go from A to D through C. There are 2 ways to get from A to C and 3 ways to get from C to D, so there are $2 \cdot 3 = 6$ ways to get from A to D through C.

To count the total number of ways to get from A to D, we now add the number of ways from each of our cases. There are 12 ways through B and another 6 ways through C, so there are a total of $12 + 6 = 22$ ways from A to D.

We did not multiply in some of the steps of Solution 14.13 (for example, counting the paths from A to D through B), but add in other steps (when we added the results from each case)?

Yes, we multiply when we are in the middle of a case, but we add when we are moving from one case to another.

Make sure you see the difference: Don't memorize it—understand it.

Important: Don't memorize! Understand, and know.

Problem 14.14

On the island of Abcdef, the alphabet has only 6 letters, and every word in their language has no more than 3 letters in it. How many words are possible? (A word can use a letter more than once, but 0 letters does not count as a word.)

Solution for Problem 14.14. Often, the tricky part of casework problems is deciding what the cases should be. For this problem, it makes sense to use as our cases the number of letters in each word.

Case 1: one-letter words. There are 6 words because there are 6 letters to itself a 1-letter word.

Case 2: two-letter words. To form a two-letter word, we have 6 choices for our first letter, and 6 choices for our second letter. Then there are $6 \cdot 6 = 36$ two-letter words.

Case 3: three-letter words. To form a three-letter word, we have 6 choices for our first letter, 6 choices for our second letter, and 6 choices for our third letter. There are $6 \cdot 6 \cdot 6 = 216$ three-letter words possible.

So to get the total number of words in the language, we add the number of words from each of our cases. (We need to make sure that the cases are exclusive, meaning that don't overlap at all. But that's clear in this solution, since, for example, a word can't be both a 2-letter word and a 3-letter word at the same time.)

Therefore there are $6 + 36 + 216 = 258$ words possible on Abcdef. (I guess the Abcdefians don't have a lot to say.)

Again, in this problem, you have to know when to multiply and when to add. When computing the number of 3-letter words, for example, we have to use the first letter **AND** the second letter **AND** the third letter, so we **multiply** the number of choices for each letter to get the number of 3-letter words. However, when choosing a word from the language as a whole, we have to choose a 1-letter word **OR** a 2-letter word **OR** a 3-letter word, so we **add** the number of choices for each length of word to get the number of total words.

Problem 14.15

How many pairs of positive integers (x, y) satisfy $x^2 + x \leq 20$?

Solution for Problem 14.15. It may not be obvious how to proceed with this problem, but a little experimentation might lead you to determine the possible values of x.

Since $0 < x^2 \leq 20$, we can see that x must be one of $1, 2, 3, 4$ or 5 (let's use these as our cases.

Case 1: $x = 1$ Since $x^2 + x \leq 20$, we have $1 + 1 = 2 \leq 20$. Thus there are 22 possible choices for b where $x = 1$, since b can be any integer from 1 to 22.

Case 2: $x = 2$ Where $x = 2$, we must have $4 + 2 = 6 \leq 20$. Thus there are 19 possible choices for b where $x = 2$

Case 3: $x = 3$ We must have $9 + 3 = 12 \leq 20$. Thus there are 14 possible choices for b where $x = 3$.

Case 4: $x = 4$ Where $x = 4$, we must have $16 + 4 = 20 \leq 20$. Thus there are 5 possible choices for b where $x = 4$.

So to get the total number of pairs of positive integers satisfying the inequality, we add up all of our possible cases, and see that there are $22 + 19 + 14 + 5 = 60$ possible pairs.

We can see from this problem that sometimes it's not immediately clear how we want to set up our different cases. Breaking up a problem into cases, and deciding what the different cases should be is an important part of problem solving. It isn't just further art with practice. It is important to make sure that the cases are exclusive—that is, that they don't overlap—and also that every possible solution to the original problem is accounted for in one of the cases.

Problem 14.16 Source: MOEMS

A digital clock shows time in the form HH:MM. On a certain day, what is the number of minutes between 7:00 AM and 2:00 PM that HH is greater than MM?

Solution for Problem 14.16. We first notice that the hours in the time period described in the problem are 8, 9, 10, 11, 12, 1, and 2. We can then do casework in two ways.

Method 1: Use the hours as the cases. In the 8:o'clock hour, the minutes from 00 to 07 (inclusive) satisfy the condition that the hour is greater than the minutes. Thus, there are 8 minutes in the 8:o'clock hour that we count.

The same logic shows that in the 9:o'clock hour, there are 9 minutes that we need to count. So we just need to add up the hours:

$$8 + 9 + 10 + 11 + 12 + 1 + 2 = 53,$$

and our answer is that there are 53 minutes satisfying the condition of the problem.

Method 2: Use the minutes as the cases. We first notice that the hours in each case which should be counted in. For example, the minute :05 should be counted in each hour, since it is less than 8 of the hours. We can then add up the other hours: the minute :16 should only be counted in the 11:o'clock and 12:o'clock hours. Here is the list:

Minute	Hour(s)	Number of hours
00	8,9,10,11,12,1,2	7
01	8,9,10,11,12,1,2	7
02	8,9,10,11,12	5
03	8,9,10,11,12	5
04	8,9,10,11,12	5
05	8,9,10,11,12	5
06	8,9,10,11,12	5
07	8,9,10,11,12	5
08	9,10,11,12	4
09	9,10,11,12	4
10	11,12	2
11	11,12	2

Minutes that are 12 or greater will never be counted. The total number of minutes is

$$7 + 6 + 5 + 5 + 5 + 5 + 5 + 4 + 4 + 2 + 1 = 53.$$

Casework Counting Part 1

Casework Counting Part 2

Exercises

14.3.1:

Write the integers from 1 to 150 inclusive. What is the total number of digits that must be written?

Hide Solution

Reset

You may type any additional notes you have here.

Your Submission Solution

Solution: The 9 single numbers contribute 1 digit each, for a total of 9 digits. The 90 two-digit numbers from 10 to 99 contribute 2 digits each, for a total of $90 \cdot 2 = 180$ digits. There are 51 three-digit numbers from 100 to 150. These contribute a total of $51 \cdot 3 = 153$ digits. Therefore, there are $9 + 180 + 153 = \boxed{342}$ digits.

14.3.2:

I have two hats. In one hat are balls numbered 1 through 15. In the other hat are balls numbered 16 through 25. I first choose a hat, then from that hat, I select a ball, without replacing the balls between selections. How many different ordered selections of 2 balls are possible? (By "ordered selections," we mean that "ball 1 then ball 2" is considered different from "ball 2 then ball 1.")

Hide Solution

Reset

You may type any additional notes you have here.

Your Submission Solution

Solution: There are two possibilities for which hat I choose. We tackle these separately:

Case 1: I choose the hat with balls 1 through 15. In this case I pick the first ball from that hat I choose, and for each of these choices I have 14 possibilities for the second ball. Therefore, there are a total of $15 \cdot 14 = 210$ possible selections in this case.

Case 2: I choose the hat with balls 16 through 25. There are $25 - 16 + 1 = 10$ balls in this hat. So, I have 10 choices for the first ball and 9 for the second, for a total of $10 \cdot 9 = 90$ selections.

Combining the two cases, we have $210 + 90 = \boxed{300}$ different ordered selections.

14.3.3:

(a) How many paths are there from A to D in the diagram shown below, if we can only travel in the direction of the arrows?

Hide Solution

Reset

You may type any additional notes you have here.

Your Submission Solution

Solution: We go through either B or C to get to D. The number of paths going from A to D through B is $2 \cdot 3$, and the number of paths going from A to D through C is $2 \cdot 3$. Thus the total number of paths from A to D is $(2 \cdot 3) + (2 \cdot 3) = \boxed{12}$.

(b) How many paths are there from D to E in the diagram shown below, if we can only travel in the direction of the arrows?

Hide Solution

Reset

You may type any additional notes you have here.

Your Submission Solution

Solution: To get from D to E, we go through one of B, E, or G. The three cases give a total of $(1 \cdot 1) + (2 \cdot 3) + (2 \cdot 1) = \boxed{11}$ paths from D to E.

(c) How many paths are there from A to E in the diagram shown below, if we can only travel in the direction of the arrows?

Hide Solution

Reset

You may type any additional notes you have here.

Your Submission Solution

Solution: All paths must go through D. We saw in part (a) that there are 6 paths from A to D. We saw in part (b) that there are 5 paths from D to E. The choice of path from A to D is independent of the choice of path from D to E. Therefore we multiply the number of paths from A to D by the number of paths from D to E to get the number of paths from A to E. The answer is $6 \cdot 5 = \boxed{30}$.

14.3.4:

How many positive 3-digit numbers have the property that the first digit is at least three times the second digit?

Hide Solution

Reset

You may type any additional notes you have here.

Your Submission Solution

Solution: We always have 10 options for the last digit. For each choice of the second digit, we have a different number of options for the first digit. So, we consider each possible second digit as a separate case.

Case 1: The second digit is 0. Then, three times the second digit is 0, so the first digit can be any number from 1 to 9. This gives 9 choices for the first digit, and there are 10 choices for the last digit, so there are $9 \cdot 10 = 90$ three-digit numbers that satisfy this case.

Case 2: The second digit is 1. Then, three times the second digit is 3, so the first digit can be any number from 3 to 9. This gives 7 choices for the first digit, and there are 10 choices for the last digit, so there are $7 \cdot 10 = 70$ three-digit numbers that satisfy this case.

Case 3: The second digit is 2. Then, three times the second digit is 6, so the first digit can be any number from 6 to 9. This gives 4 choices for the first digit, and there are 10 choices for the last digit, so there are $4 \cdot 10 = 40$ three-digit numbers that satisfy this case.

Case 4: The second digit is 3. Then, three times the second digit is 9, so the first digit can only be 9. This gives 1 choice for the first digit, and there are 10 choices for the last digit, so there are $1 \cdot 10 = 10$ three-digit numbers that satisfy this case.

Case 5: The second digit is 4 or greater. Then, three times the second digit is greater than 10, so there are no possibilities for the first digit. Therefore, no three-digit numbers satisfy this case.

Combining our cases, we have a total of $90 + 70 + 40 + 10 = \boxed{210}$ numbers that satisfy the problem.

14.3.5:

How many positive 4-digit numbers have the last digit equal to the sum of the first two digits?

Hide Solution

Reset

You may type any additional notes you have here.

Your Submission Solution

Solution: We let the possible last digits, and the possibilities for the first two digits for each choice of last digit:

Last Digit	First two digits
0	00
1	10, 01
2	20, 11, 02
3	30, 21, 12, 03
4	40, 31, 22, 13, 04
5	50, 41, 32, 23, 14, 05
6	60, 51, 42, 33, 24, 15, 06
7	70, 61, 52, 43, 34, 25, 16, 07
8	80, 71, 62, 53, 44, 35, 26, 17, 08
9	90, 81, 72, 63, 54, 45, 36, 27, 18, 09

The third digit can be any of the 10 digits. The answer is:

$$(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) \cdot 10 = \boxed{450}$$

There is a quicker solution. Once we choose the last digit, we need the first digit to be between 1 and the chosen last digit, inclusive. So, if we choose 5 as the last digit, we can pick the first digit from 1, 2, 3, 4, 5, or 6. The last digit is 2 choices if a single digit from 0 to 9, then there are 4 choices for the first digit. The second digit is the last digit minus the first digit, so there is just one choice for the second digit. Thus, there are 4 ways for this case. The third digit can be any of the 10 digits. Thus the answer is $(1 + 2 + \cdots + 9) \cdot 10 = \boxed{450}$.

14.3.6: Source: AMC 8

How many positive two-digit numbers have digits whose sum is a perfect square?

Hide Solution

Reset

You may type any additional notes you have here.

Your Submission Solution

Solution: The greatest the sum of the digits of a two-digit integer can be is $9 + 9 = 18$. Therefore, we have four possibilities to consider if the sum of the digits is a perfect square: 1, 4, 9, and 16.

Case 1: The sum of the digits is 1. The only two-digit number that satisfies this is 10, so there is 1 possibility in this case.

Case 2: The sum of the digits is 4. For each tens digit from 1 to 4, there is one such number: 13, 22, 31, 40. So, there are 4 possibilities in this case.

Case 3: The sum of the digits is 9. For each tens digit from 1 to 9, there is one such number: 18, 27, 36, 45, 54, 63, 72, 81, 90. So, there are 9 possibilities in this case.

Case 4: The sum of the digits is 16. For each tens digit from 7 to 9, there is one such number: 79, 88, 97. So, there are 3 possibilities in this case.

Combining our cases, we have $1 + 4 + 9 + 3 = \boxed{17}$ numbers that satisfy the problem.

14.3.7a:

How many squares of any size can be formed by connecting dots in the grid shown to the right?

Hint: Don't forget squares whose sides aren't horizontal or vertical lines!

Hide Solution

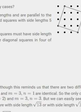
Reset

You may type any additional notes you have here.

Your Submission Solution

Solution: We can make each case be counting squares of a particular size.

Case 1: 1 × 1 "horizontal" squares There are sixteen 1×1 squares whose sides are parallel to the sides of the grid, as shown to the right.



Case 2: 2 × 2 "horizontal" squares There are nine 2×2 squares whose sides are parallel to the sides of the grid, as shown to the right.

Case 3: 3 × 3 "horizontal" squares There are four 3×3 squares whose sides are parallel to the sides of the grid, as shown to the right.

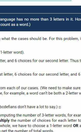
Case 4: 4 × 4 "horizontal" squares There is only one 4×4 square whose sides are parallel to the sides of the grid, namely the border of the entire grid.

This gives $16 + 9 + 4 + 1 = 30$ squares whose sides are parallel to the sides of the grid. But we also have squares which are "diagonal," meaning that their sides are not parallel to the sides of the grid.

Case 5: $\sqrt{2} \times \sqrt{2}$ "diagonal" squares There are 9 $\sqrt{2} \times \sqrt{2}$ squares, as shown to the right.

Case 6: $\sqrt{5} \times \sqrt{5}$ "diagonal" squares There are 8 $\sqrt{5} \times \sqrt{5}$ squares, as shown to the right. Note that these squares come in two different orientations.

Case 7: $\sqrt{8} \times \sqrt{8}$ "diagonal" squares There is only 1 $\sqrt{8} \times \sqrt{8}$ square—it is shown at right.



Case 8: $\sqrt{10} \times \sqrt{10}$ "diagonal" squares There are 2 $\sqrt{10} \times \sqrt{10}$ squares, shown to the right.

This gives a total of

$$16 + 9 + 4 + 1 + 9 + 8 + 1 + 2 = 50$$

squares in the grid.

This seems like all of the squares, but how can we be sure that we didn't miss any cases?

We can be certain that we have all of the squares by considering the possible integer side lengths and are parallel to the sides of the grid. We counted all the squares with side lengths 1, 2, 3, and 4, and squares with side lengths 5 and $\sqrt{2}$ are the only possibilities.

We can also use, because of the Pythagorean Theorem, that all of the diagonal squares must have side length $\sqrt{n}$ for some n and is are positive integers. We can also determine the diagonal squares in four of these cases:

m	n	Side Length
1	1	$\sqrt{2}$
1	2	$\sqrt{5}$
1	3	$\sqrt{10}$
2	2	$\sqrt{8}$

Note that the case $m = 1, n = 2$ and the case $m = 2, n = 1$ are identical (although this reminds us that there are two different orientations of squares with side length $\sqrt{5}$). Similarly the cases $m = 1, n = 3$ and $m = 3, n = 1$ are identical. So the only cases that we might have missed are $m = 2, n = 3$ (and its companion $m = 3, n = 2$) and $m = 3, n = 3$. But we can easily see that such squares will not fit into the grid: there is no way to insert into the grid a square with side length $\sqrt{13}$ or with side length $\sqrt{18}$.

Thus we're accounted for all the cases, and counted all the squares: the answer is $\boxed{50}$.

14.4 Counting Pairs

Problems

Problem 14.17 Jump to Solution

A round-robin tennis tournament consists of each player playing every other player exactly once. How many matches will be held during an 8-person round-robin tennis tournament?

Problem 14.18 Jump to Solution

A round-robin tennis tournament consists of each player playing every other player exactly once. Find a formula for the number of matches that will be held during a n-person round-robin tennis tournament, where $n \geq 2$.

Problem 14.19 Jump to Solution

Five women and four men are at a party.

(a) If each person shakes hands with each other person, then how many total handshakes are there?

(b) If no two men shake hands, but each woman shakes hands with each other person, then how many total handshakes are there?

Problem 14.20 Jump to Solution

(a) How many diagonals does a triangle have?

(b) How many diagonals does a quadrilateral have?

(c) How many diagonals does a pentagon have?

(d) How many diagonals does a hexagon have?

(e) Find a formula for the number of diagonals in a polygon with n sides, where n is an integer greater than 2.

Problem 14.17

A round-robin tennis tournament consists of each player playing every other player exactly once. How many matches will be held during an 8-person round-robin tennis tournament?

Solution for Problem 14.17: Say you're one of the players. How many matches will you play?

Each player plays 7 matches, one against each of the other 7 players.

So what's wrong with the following reasoning?

> **Bogus Solution:** Each of the eight players plays 7 games, so there are $8 \cdot 7 = 56$ total games played.

Suppose two of the players are Alice and Bob. Among Alice's 7 matches is a match against Bob. Among Bob's 7 matches is a match against Alice. When we count the total number of matches as $8 \cdot 7$, the match between Alice and Bob is counted twice, once for Alice and once for Bob.

Therefore, since $8 \cdot 7 = 56$ counts each match twice, we must divide this total by 2 to get the total number of matches. Hence the number of matches in an 8-player round-robin tournament is $\frac{8 \cdot 7}{2} = 28$.

Here's another way to solve Problem 14.17. We'll go one player at a time, and keep a running count of the number of matches involving those players.

Let's start with Alice. We already know that Alice plays 7 matches.

Next, we move on to Bob. We also know that Bob plays 7 matches, but we've already counted the match that he plays with Alice. So Bob plays in 6 more matches that we haven't already counted. Therefore, our running total—the number of matches involving Alice or Bob—is $7 + 6 = 13$.

Then, we go to the next player, Carol. We know that Carol plays 7 matches, but we've already counted two of them: the one between Carol and Alice, and the one between Carol and Bob. So Carol plays in 5 more matches that we haven't already counted. Therefore, our running total—the number of matches involving Alice, Bob, or Carol—is $7 + 6 + 5 = 18$.

If we keep going in this manner, we'll eventually arrive at a total of

$$7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$$

matches.

Problem 14.17 is an example of counting pairs of objects—in this case, pairs of players in a tennis tournament. Now that we know what happens with 8 players, let's take a look at what happens when we play a n-player tournament. This is an example of **generalizing** a problem: we have solved the problem for a specific number, 8, and now we're going to find a formula that will work for any positive integer number of players.

Problem 14.18

A round-robin tennis tournament consists of each player playing every other player exactly once. Find a formula for the number of matches that will be held during a n-person round-robin tennis tournament, where $n \geq 2$.

Solution for Problem 14.18: If we proceed as in the solution to Problem 14.17, we see that each of the n players must play every other player, so each player must play $n - 1$ matches. This leads to a preliminary count of $n(n-1)$ matches.

But this method counts each match twice, once for each player. Thus we must divide by 2, to get the total number of matches $\frac{n(n-1)}{2}$.

Another way to approach the general tournament problem in Problem 14.18 is as follows:

Number the players from 1 through n.

Player 1 plays players $2, 3, \ldots, n$ for a total of $n - 1$ matches.

Player 2 plays players $1, 3, 4, \ldots, n$. But we've already counted the match between players 1 and 2. So we only need to add on the matches between player 2 and players $3, 4, \ldots, n$, for another $n - 2$ matches.

Similarly, for player 3, we've already counted the matches against players 1 and 2, so we need to add on the matches between player 3 and players $4, 5, \ldots, n$, for another $n - 3$ matches.

And so on.

So the total number of matches is the sum of our counts above, which is

$$(n-1) + (n-2) + (n-3) + \cdots + 2 + 1.$$

Since this must match the count that we had before, we've just shown that

$$(n-1) + (n-2) + (n-3) + \cdots + 2 + 1 = \frac{n(n-1)}{2}.$$

Our work above gives us a formula for the sum of the first k positive integers! If we let $n = k + 1$ in the equation we just found (and reverse the order of the sum so our sum goes from 1 to k), then we get

$$1 + 2 + \cdots + (k-2) + (k-1) + k = \frac{(k+1)(k)}{2}.$$

So, our counting explanations have given us an algebra formula!

> **Concept:** We can sometimes use counting explanations to find algebraic relationships.

We also could have used algebra to find a formula for the sum of the first k positive integers. We write

$$S = 1 + 2 + \cdots + (k-1) + k.$$

We can, of course, just as easily write S in reverse order:

$$S = k + (k-1) + \cdots + 2 + 1.$$

This may seems like a useless thing to do, but watch what happens when we add them together:

$$\begin{array}{rcccccccccc} S & = & 1 & + & 2 & + & \cdots & + & k-1 & + & k \\ + \ S & = & k & + & k-1 & + & \cdots & + & 2 & + & 1 \\ \hline 2S & = & k+1 & + & k+1 & + & \cdots & + & k+1 & + & k+1 \end{array}$$

The last line has k copies of $k + 1$, and hence

$$2S = k(k+1).$$

So, we divide by 2 to get $S = \frac{k(k+1)}{2}$.

> **Important:** For any positive integer k,
>
> $$1 + 2 + \cdots + (k-1) + k = \frac{k(k+1)}{2}.$$

> **Sidenote:** Legend has it that the famous German mathematician Carl Friedrich Gauss (1777–1855) once used the above formula to very quickly compute the sum of the integers from 1 to 100. What is remarkable about this is that Gauss was only 7 years old at the time! His schoolmaster gave his class, as an exercise, the assignment to calculate
>
> $$1 + 2 + 3 + \cdots + 99 + 100,$$
>
> figuring that it would take the students many minutes. The teacher was quite surprised when young Gauss came up with the answer almost immediately! Gauss, even at age 7, recognized that there was a quick way to add consecutive integers, and used the same process that we used to prove our formula.

In some problems, we don't necessarily want to count all pairs, but just some of them.

Problem 14.19

Five women and four men are at a party.

(a) If each person shakes hands with each other person, then how many total handshakes are there?

(b) If no two men shake hands, but each woman shakes hands with each other person, then how many total handshakes are there?

Solution for Problem 14.19:

(a) There are 9 people total, and each must shake hands with each other person. So, each person shakes hands 8 times. This gives us a preliminary count of $9 \cdot 8$ handshakes. But this counts each handshake twice, once for each person in the handshake. So, we must divide by 2 to get a total of $9 \cdot 8/2 = 36$ handshakes. (Notice that this problem is essentially the same as the round-robin tournament problem, where we've replaced "matches" with "handshakes.")

(b) What's wrong with this solution:

> **Bogus Solution:** Each woman shakes hands with 8 other people, and there are 5 women, for a preliminary total of $9 \cdot 5 = 45$ handshakes. But this counts each handshake twice, once for each person in the handshake. So, we must divide by 2 to get a total of $45/2 = 22\frac{1}{2}$ handshakes.

The problem here is that the count $9 \cdot 5$ does not count each handshake twice! A handshake between two women is counted twice, but a handshake between a man and a woman is counted only once, just for the woman.

> **WARNING!** When dividing by 2 in a counting problem to correct for counting each item twice, make sure you really were counting each item twice!

We have to handle the handshakes between two women separately from handshakes between a man and a woman. Each of the 5 women shakes hands with 4 other women, for a preliminary count of $5 \cdot 4$ handshakes. Suppose Alice and Barb are two of the women. Our count $5 \cdot 4$ counts the handshake between Alice and Barb twice—once among Alice's 4 handshakes with other women, and once among Barb's 4 handshakes with other women. So, here we have to divide by 2, since $5 \cdot 4$ counts each handshake twice, once for each woman. This gives us $5 \cdot 4/2 = 10$ handshakes between two women.

Turning to handshakes between a man and a woman, each of the women shakes hands with 4 men, for a preliminary count of $5 \cdot 4 = 20$ handshakes. Suppose Alice is one of the women and Carl is one of the men. The handshake between Alice and Carl is only counted once, just among Alice's 4 handshakes with men. So, we don't divide the preliminary count by 2, because each handshake between a woman and a man is only counted once. There are 20 handshakes between a woman and a man.

Combining the handshakes between two women with those between a woman and a man, we have $10 + 20 = 30$ handshakes.

Problem 14.20

(a) How many diagonals does a triangle have?

(b) How many diagonals does a quadrilateral have?

(c) How many diagonals does a pentagon have?

(d) How many diagonals does a hexagon have?

(e) Find a formula for the number of diagonals in a polygon with n sides, where n is an integer greater than 2.

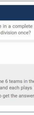
Solution for Problem 14.20:

(a) A triangle has 0 diagonals.

(b) A quadrilateral has 2 diagonals.

(c) As shown at the left below, a pentagon has 5 diagonals.

A Pentagon and Its Diagonals



A Hexagon and Its Diagonals

(d) A hexagon and its diagonals are shown at the right above. Rather than counting the diagonals directly, we count the diagonals by thinking about each vertex. Each vertex can connect to the other 5 vertices, but 2 of these connections are edges, not diagonals. So each vertex has $5 - 2 = 3$ diagonals. (This is a bit like the handshaking problem! Thinking of a diagonal as a handshake between two vertices, each vertex "shakes hands" with 3 other vertices.)

There are 6 vertices of a hexagon and each connects to 3 vertices by a diagonal, so it appears that there are $6 \cdot 3 = 18$ diagonals. However, this counts each diagonal twice, once for each vertex at its endpoints. To correct for this, we must divide by 2, giving $18/2 = 9$ diagonals.

(e) We can use our process from part (d) for any polygon. A polygon with n sides has n vertices. As in part (c), each vertex can connect to the other $n - 1$ vertices, but 2 of these connections are edges, not diagonals. So each vertex has $(n-1) - 2 = n - 3$ diagonals.

Since each of the n vertices has $n - 3$ diagonals, it appears that there are $n(n-3)$ diagonals. But again, this counts each diagonal twice, once for each vertex at its endpoints. So, we correct our count by dividing by 2, giving $n(n-3)/2$ diagonals for a polygon with n sides. Checking, we see that this formula does give us the values we found in the first four parts.

Counting Pairs

Exercises

14.4.1: Source: MOEMS

There are five girls in a tennis class. How many different doubles teams of two girls each can be formed from the students in the class?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There are 5 ways to choose the first girl for a team and 4 ways to choose the second girl. This gives us $5 \cdot 4$ ways to choose the two girls, but this counts each possible team twice, once for each order in which we choose the girls. Therefore, there are $(5 \cdot 4)/2 = \boxed{10}$ different doubles teams possible.

14.4.2:

A club has 12 members and needs to choose 2 members to be co-presidents. How many different pairs of co-presidents are possible?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There are 12 ways to choose the first person to be a co-president, and then 11 ways to choose the second person. This gives us a total of $12 \cdot 11$ ways to choose the two people, but this counts each pair of co-presidents twice, once for each order of choosing them. Therefore, the possible number of pairs of co-presidents is $(12 \cdot 11)/2 = \boxed{66}$.

14.4.3: Source: MOEMS

Two students are needed to work in the school store during the lunch hour every day and four students volunteer for this work. What is the greatest number of days that can be arranged in which no pair of the four students work together more than once?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: The greatest number of days for which no pair works together more than once occurs when each student works with each other student exactly once. So, we need to count the number of different pairs of students we can form from a group of 4 students. There are 4 students to choose as the first student and 3 left to choose as the second. This gives us a total of $(4 \cdot 3)/2 = \boxed{6}$ pairs. We divide by 2 because $4 \cdot 3$ counts each pair twice, once for each order in which we can pick the students in a given pair.

14.4.4: Source: MOEMS

Six people participated in a checkers tournament. Each participant played exactly three games with each of the other participants. How many games were played in all?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Each person plays 3 other people 3 times, so each of the 6 people plays $5 \cdot 3 = 15$ games. In multiplying $6 \cdot 15$, each game is counted twice, once for each player in the game, so we divide by two to get the answer of $(6 \cdot 15)/2 = \boxed{45}$ games.

14.4.5:

A sports conference has 12 teams in two divisions of 6. How many games are in a complete season for the conference if each team must play every other team in its own division and every team in the other division once?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Each team plays the other 5 teams in its division twice and each of the 6 teams in the other division once. This gives a total of $5 \cdot 2 + 6 \cdot 1 = 16$ games played by each team. There are 12 teams total, and each plays 16 games. In multiplying $12 \cdot 16$, each game is counted twice, once for each team in the game, so we divide by two to get the answer of $(12 \cdot 16)/2 = \boxed{96}$ games.

14.4.6+:

Find a formula for the sum of the first n even integers: $2 + 4 + 6 + \cdots + 2n$.

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We'll by the same strategy we used in the book to compute the sum of the first n positive integers. We let S equal the sum of the first n even integers, and then write the sum both forwards and backwards. We then add the two sums:

$$\begin{array}{rcccccccccc} S & = & 2 & + & 4 & + & \cdots & + & 2(n-1) & + & 2n \\ + \ S & = & 2n & + & 2(n-1) & + & \cdots & + & 4 & + & 2 \\ \hline 2S & = & 2(n+1) & + & 2(n+1) & + & \cdots & + & 2(n+1) & + & 2(n+1) \end{array}$$

The last line has n copies of $2(n+1)$, and hence $2S = n(2(n+1))$, so $2S = 2n(n+1)$. We divide by 2 to get $S = \boxed{n(n+1)}$.

Notice that our final result is two times our formula for the sum $1 + 2 + 3 + \cdots + n$. Watch what happens when we multiply this sum by 2:

$$2(1 + 2 + 3 + \cdots + n) = 2 + 4 + 6 + \cdots + 2n.$$

So, the sum of the first n positive even integers is indeed twice the sum of the first n positive integers.

14.4.7+:

How many interior diagonals does an icosahedron have? (An **icosahedron** is a 3-dimensional figure with 20 triangular faces and 12 vertices, with 5 faces meeting at each vertex. An **interior diagonal** is a segment connecting two vertices that do not lie on a common face.)

Hint: Consider a simpler problem; how did we solve the problem about polygon diagonals in the text?

Hint: Each vertex is connected to how many other vertices by an edge of the icosahedron?

Hint: Each vertex is connected to how many other vertices by an interior diagonal of the icosahedron?

Preview: Solution

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: There are 12 vertices in the icosahedron, so from each vertex there are potentially 11 other vertices to which we could extend a diagonal. However, 5 of these 11 points are connected to the original point by an edge of the icosahedron, so they are not connected by interior diagonals. So each vertex is connected to 6 other points by interior diagonals. This gives a preliminary count of $12 \cdot 6 = 72$ interior diagonals. However, we have counted each diagonal twice (once for each of its endpoints), so we must divide by 2 to correct for this overcounting, and the answer is $\frac{12 \cdot 6}{2} = \boxed{36}$ diagonals.

14.5 Probability

Now we turn from counting to **probability**. Unfortunately, while even most three-year-olds know what "counting" means, the concept of probability is a bit difficult to describe precisely.

Let's start with an example: Suppose we have a coin, where one side is heads, and the other side is tails. If we flip the coin over and over and over again, we expect that about half the time we'll flip heads and about half the time we'll flip tails. We express this mathematically by saying that the **probability** that a coin flip will turn up heads is $\frac{1}{2}$.

This is what we mean by probability: of all the outcomes, or events, the fraction of times that we expect an outcome to occur if we perform the experiment over and over. Notice that we can't say the exact fraction of times we have success. This is the difficulty in defining probability. Probability only makes sense if we are trying to measure an event that is not definite.

In basic examples, we'll compute probability by counting the number of equally likely outcomes, then counting how many of these outcomes are "successes". Our probability of success is the ratio of the number of successful outcomes to the total number of possible outcomes. For instance, in our coin-flipping example above, there are 2 possible outcomes (heads and tails), and 1 successful outcome (heads), so the probability of flipping heads is $\frac{1}{2}$.

But we have a very important caveat:

> **Important:** We must be sure that each possible outcome is equally likely

Problems

Problem 14.21 Jump to Solution

The faces of a fair 6-sided die are numbered $\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6}$. ("Fair" means that each side is equally likely to be rolled.) What is the probability that when the die is rolled, a $\boxed{5}$ is facing up?

Problem 14.22 Jump to Solution

What is the probability that when a fair 6-sided die is rolled, an odd number faces up?

Problem 14.23 Jump to Solution

What is the greatest possible probability for any event? What is the least possible probability for any event?

Problem 14.24 Jump to Solution

A standard deck of cards has 52 cards divided into 4 suits, each of which has 13 cards. Two of the suits (♡ and ♢, called "hearts" and "diamonds") are red, and the other two (♤ and ♧, called "spades" and "clubs") are black. The cards in the deck are placed in random order (usually by a process called "shuffling"). What is the probability that the top two cards are both red?

Problem 14.25 Jump to Solution

Suppose we have 2 fair 6-sided dice numbered from $\boxed{1}$ to $\boxed{6}$. If we roll them both, then what is the probability that the two numbers shown sum to 7?

Let's start with a basic example.

Problem 14.21

The faces of a fair 6-sided die are numbered $\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6}$. ("Fair" means that each side is equally likely to be rolled.) What is the probability that when the die is rolled, a $\boxed{5}$ is facing up?

Solution for Problem 14.21: When we roll the die, there are 6 equally likely outcomes: $\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{5}, \boxed{6}$, or $\boxed{6}$. In this problem, there is only one "successful" outcome, namely when a $\boxed{5}$ comes up. Therefore, the probability that a $\boxed{5}$ is rolled is the 1 successful outcome divided by the total 6 equally likely outcomes, $\frac{1}{6}$.

As we discussed in the introduction to this section, the way to interpret this probability is that if we were to roll our die many, many times, we would expect that about $\frac{1}{6}$ of our rolls would come up $\boxed{5}$.

In Problem 14.21 we ran the basic way to calculate probabilities:

$$\text{Concept:} \quad \text{If all outcomes are } \textbf{equally likely}, \text{ then the probability of success is } \frac{\text{Number of successful outcomes}}{\text{Number of possible outcomes}}.$$

Here's another example with our fair deck:

Problem 14.22

What is the probability that when a fair 6-sided die is rolled, an odd number faces up?

Solution for Problem 14.22: In this problem, as in Problem 14.21, there are 6 equally likely outcomes. Three of those outcomes are successful: $\boxed{1}, \boxed{3}, \boxed{5}$. Therefore, the probability is $\frac{3}{6} = \frac{1}{2}$.

Now that we've done a couple of basic probability problems, let's ask some important general questions about probability.

Problem 14.23

What is the greatest possible probability for any event? What is the least possible probability for any event?

Solution for Problem 14.23: Since the probability of an event occurring is essentially the proportion of times that we expect the event to occur, the largest the probability could be is 1. This occurs when the event happens every time.

Another way to look at this is to recall that for any event, we have

$$\text{Probability of success} = \frac{\text{Number of successful outcomes}}{\text{Number of possible outcomes}}$$

Clearly the number of successful outcomes cannot be greater than the number of possible outcomes, so the probability must be no greater than 1. The smallest it can be is 0 (no successful outcomes) when the number of successful outcomes equals the number of possible outcomes, and the probability is 1. For example, if the event is "a single die roll is less than 10," then the probability is 1, since there are 6 successful outcomes out of 6 possible outcomes.

Similarly, the smallest the probability could ever be is 0. This occurs when the event cannot happen, so the number of successful outcomes is 0. For example, if the event is "a single die roll is 7," then the probability is 0, because there are 0 successful outcomes out of 6 possible outcomes.

> **Important:** When working on a probability problem, if you ever get a probability that's less than 0 or greater than 1, then you've made a mistake! All probabilities are greater than or equal to 0, and less than or equal to 1.

Let's try a slightly more complicated problem.

Problem 14.24

A standard deck of cards has 52 cards divided into 4 suits, each of which has 13 cards. Two of the suits (♡ and ♢, called "hearts" and "diamonds") are red, and the other two (♤ and ♧, called "spades" and "clubs") are black. The cards in the deck are placed in random order (usually by a process called "shuffling"). What is the probability that the top two cards are both red?

Solution for Problem 14.24: Since all choices of the first two cards are equally likely, we need to count the total number of possibilities for the first two cards, and the number of those possibilities in which both cards are both red.

First we count the total number of ways to draw two cards. There are 52 ways to pick the first card, then 51 ways to pick the second card, for a total of 52 · 51 possibilities.

Next we count the number of successful possibilities, meaning the number of ways to draw two red cards. There are 26 ways to pick a red card first (since there are 26 total red cards), then there are 25 ways to pick a second card (since there are 25 red cards remaining after we've chosen the first card). Thus, there are a total of 26 · 25 successful possibilities.

Therefore, the probability is

$$\frac{\text{Number of successful outcomes}}{\text{Number of possible outcomes}} = \frac{26 \cdot 25}{52 \cdot 51} = \frac{25}{102}.$$

Problem 14.25

Suppose we have 2 fair 6-sided dice numbered from $\boxed{1}$ to $\boxed{6}$. If we roll them both, then what is the probability the two numbers shown sum to 7?

Solution for Problem 14.25: We might incorrectly think like this:

> **Bogus Solution:** Any number from 2 to 12 could occur, so there are 11 possibilities. 7 is one of those, so the probability is $\frac{1}{11}$.

The error here is that the 11 possibilities are **not equally likely**. For example, rolling a sum of 7 is more likely than rolling a sum of 2 in order to correctly compute probability, we need to count equally likely outcomes, and then determine how many of those possibilities result in a sum of 7. But how can we count equally likely outcomes? Why is the sum of two dice?

There are 6 equally likely outcomes for each die, so there are 6 · 6 = 36 total equally likely outcomes for both dice together. It may be easier to see this if you think of one of the dice as being red and the other as being white. There are 6 outcomes for the black die and 6 outcomes for the white die, for a total of 6 · 6 = 36 outcomes for the pair of dice. The outcomes that sum to 7 are $\boxed{1}\boxed{6}, \boxed{2}\boxed{5}, \boxed{3}\boxed{4}, \boxed{4}\boxed{3}, \boxed{5}\boxed{2}$, and $\boxed{6}\boxed{1}$, for a total of 6 equally likely successful outcomes. Therefore, the probability of rolling a sum of 7 is $\frac{6}{36} = \frac{1}{6}$.

We can see all of the possible equally likely outcomes for the two dice in the following table:

	Die #1					
	■	■	■	■	■	■
Die #2 ■	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$
■	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$
■	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$
■	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$	$\boxed{9}$
■	$\boxed{5}$	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$	$\boxed{9}$	$\boxed{10}$
■	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$	$\boxed{9}$	$\boxed{10}$	$\boxed{11}$

Notice that 6 of the entries in the above table are 7, and there are 36 total entries in the table, so the probability of rolling a 7 is $\frac{6}{36} = \frac{1}{6}$.

Probability and Equally Likely Events

Exercises

14.5.1:

Suppose we roll a fair 6-sided die. What is the probability that

(a) $\boxed{5}$ is rolled?

Hide Solution

Reset

Solution: Rolling a $\boxed{5}$ is 1 out of 6 possible equally-likely outcomes, so its probability is $\frac{1}{6}$.

(b) an even number is rolled?

Hide Solution

Reset

Solution: 3 of the 6 equally-likely possible outcomes are even (namely, $\boxed{2}$, $\boxed{4}$, and $\boxed{6}$), so the probability is $\frac{3}{6} = \frac{1}{2}$.

(c) a perfect square is rolled?

Hide Solution

Reset

Solution: A $\boxed{1}$ or $\boxed{4}$ can be rolled for success, which is 2 out of 6 possible outcomes, so the probability is $\frac{2}{6} = \frac{1}{3}$.

14.5.2:

Suppose that we have an 8-sided die with 2 red sides, 3 yellow sides, and a blue side. What is the probability of rolling a yellow side?

Hide Solution

Reset

Solution: There are 3 yellow sides and 8 equally-likely sides in total, so the probability of rolling a yellow side is $\frac{3}{8}$.

14.5.3:

Consider a standard deck of 52 cards (as described in Problem 14.24). The deck is randomly arranged. What is the probability that

(a) the top card is a ♣?

Hide Solution

Reset

Solution: There are 13 ♣'s and 52 cards total, so the probability that the top card is a ♣ is $\frac{13}{52} = \frac{1}{4}$.

(b) the top card is a ♥?

Preview: Solution

Hide Solution

Reset

Solution: There are 3 · 8 = 12 face cards and 52 cards total, so the probability that the top card is a face card is $\frac{12}{52} = \frac{3}{13}$.

(c) the top card is black and the second card is red?

Hide Solution

Reset

Solution: There are 26 ways to choose the first card to be black, then 26 ways to choose the second card to be red. There are 52 · 51 ways to choose any two cards. So the probability is $\frac{26 \cdot 26}{52 \cdot 51} = \frac{26}{51}$.

(d) the top card is a 2 and the second card is an 8?

Hide Solution

Reset

Solution: There are 4 ways to choose the first card to be 2, then 4 ways to choose the second card to be an 8. There are 52 · 51 ways to choose any two cards. So the probability is $\frac{4 \cdot 4}{52 \cdot 51} = \frac{4}{663}$.

(f) the top two cards are both Aces?

Hide Solution

Reset

Solution: There are 4 ways to choose the first card to be an Ace, then 3 ways to choose the second card to be another Ace. There are 52 · 51 ways to choose any two cards. So the probability is $\frac{4 \cdot 3}{52 \cdot 51} = \frac{1}{221}$.

14.5.4:

Suppose we flip four coins simultaneously: a penny, a nickel, a dime, and a quarter. What is the probability that

(a) they all come up heads?

Hide Solution

Reset

Solution: There are $2^4 = 16$ possible outcomes, since each of the 4 coins can land 2 different ways (heads or tails).

There is only 1 way that they can all come up heads, so the probability of this is $\frac{1}{16}$.

(b) the penny and nickel both come up heads?

Hide Solution

Reset

Solution: There are $2^4 = 16$ possible outcomes, since each of the 4 coins can land 2 different ways (heads or tails).

There are 2 possibilities for the dime and 2 for the quarter, so there are $2 \cdot 2 = 4$ successful outcomes, so the probability is $\frac{4}{16} = \frac{1}{4}$.

(c) at least 1 tic worth of coins come up heads?

Hide Solution

Reset

Solution: There are $2^4 = 16$ possible outcomes, since each of the 4 coins can land 2 different ways (heads or tails).

If the quarter is heads, then the nickel and dime must be heads, so there are 2 successful outcomes, since the penny can be heads or tails.

So there are $8 + 2 = 10$ total successful outcomes, and the probability of success is $\frac{10}{16} = \frac{5}{8}$.

14.5.5:

Suppose that we roll two fair 6-sided dice. What is the probability that the two numbers rolled sum to 8?

Hide Solution

Reset

Solution: There are 6 ways to roll a sum of 8 ($\boxed{2}$ on the first die and $\boxed{6}$ on the second die, $\boxed{3}$ on the first die and $\boxed{5}$ on the second die, $\boxed{4}$ on the first die and $\boxed{4}$ on the second die, $\boxed{5}$ on the first die and $\boxed{3}$ on the second die, $\boxed{6}$ on the first die and $\boxed{2}$ on the second die. There are 36 total possibilities, so the probability is $\frac{6}{36} = \frac{1}{6}$.

14.5.6: Source: AMC 8

Two dice are thrown. What is the probability that the product of the two numbers is a multiple of 5?

Hide Solution

Reset

Solution: There are 6 · 6 = 36 possible rolls. If the product of the rolls is a multiple of 5, then at least one of the two dice shows a 5. We handle separately the case of both dice being 5 and the case of exactly one die being 5. There is one way for both dice to be 5. For each die, there are 5 ways for that die to be a 5 and the other to be not a 5. So, there are 5 + 5 = 11 ways for the roll to have at least one 5. This means the desired probability is $\frac{11}{36}$.

Alternatively, we can compute that there are 5 · 5 = 25 rolls in which neither die is a 5 (because each die has 5 possibilities to be any number other than 5). Therefore, this leaves 36 − 25 = 11 rolls that have at least one 5, and again the desired probability is $\frac{11}{36}$.

14.5.7a: Source: AMC 8

A pair of 6-sided dice has the sides of each die numbered 1 through 6. Each side has the same probability of landing face up. What is the probability that the product of the two numbers on the sides that land face up is greater than 36?

Hide Solution

Reset

Solution: There are 6 · 6 = 64 possible rolls. Next, we count the number of rolls in which the product of the numbers showing exceeds 36.

Suppose the first die shows a 4 or smaller. Then, the largest possible product is 4 · 6 = 22. So, neither die can be 4 or smaller if the product of the dice is greater than 36. Consider each possible result for the first die:

Case 1: The first die shows 5. Only 6 in the second die results in a product greater than 36. There is 1 possibility in this case.

Case 2: The first die shows 6. Only 7 and 8 on the second die result in a product greater than 36. There are 2 possibilities in this case.

Case 3: The first die shows 7. Only 6, 7, and 8 on the second die result in a product greater than 36. There are 3 possibilities in this case.

Case 4: The first die shows 8. Only 5, 6, 7, and 8 on the second die result in a product greater than 36. There are 4 possibilities in this case.

Combining these cases, there are 1 + 2 + 3 + 4 = 10 rolls that result in a product greater than 36, so the desired probability is $\frac{10}{64} = \frac{5}{32}$.

14.5.8a: Source: AMC 8

At a party, there are only single women and married men with their wives. The probability that a randomly selected woman is single is $\frac{2}{3}$. What Fraction of the people in the room are married men?

Preview: Solution

Hide Solution

Reset

Solution: Because the probability that a randomly-chosen woman is single is $\frac{2}{3}$, we know that $\frac{2}{3}$ of the women are single. Thus, $\frac{2}{3}$ of the women are married. So, for every 2 single women, there are 3 married women. There are just as many men as there are married women, so for every 2 single women, there are 3 married women and 3 married men. As a ratio, we can write this as

$$\text{single women : married women : married men} = 2 : 3 : 3.$$

So, there are 3 married men out of every 2 + 3 + 3 = 8 people. Therefore, $\frac{3}{8}$ of the people are married men.

14.6 Summary

In this chapter, we learned how to use arithmetic to solve counting problems.

> **Important:** Don't just blindly add, subtract, multiply, and divide–think about what you're doing!

Counting a list of numbers like $1, 2, 3, \ldots, 38$ is easy! Counting many other lists of numbers is easy too: use arithmetic to convert it into a list that looks like 1,2,3,. . .. The number of items in the list $a, a+1, \ldots, b-1, b$, where $a \leq b$ are integers, is $b - a + 1$.

> **Concept:** When presented with a complicated problem, try to reduce or simplify it to a simpler problem that you know how to solve.

Counting items which lie in one or more overlapping sets requires a thoughtful use of addition and subtraction. We can use a Venn diagram to help keep track of the thought process. When using a Venn diagram, we should initially try to fill in numbers that correspond exactly to a single region of the diagram. We may have to place variables in our diagram: if so, it's usually wise to choose our variable to represent the answer to the problem.

> **Concept:** When assigning a variable, it's usually best to let your variable represent the answer to the problem.

When counting multiple independent events, we multiply: the number of ways that A **and** B can occur is

$$\begin{aligned}&(\text{the number of ways } A \text{ can occur})\\ &\quad \textbf{times } (\text{the number of ways } B \text{ can occur}).\end{aligned}$$

Sometimes our later choices depend on our earlier choices, so our counting must take that into consideration.

Casework can often help us break a counting problem into more manageable pieces. We split the possibilities into exclusive cases, meaning that every outcome shows up in exactly one case. Then, to count the total, you simply add the counts for each case. Casework requires you to be organized and careful, and to make sure that you don't omit any outcomes in your cases.

> **Concept:** When faced with a series of independent choices, one after the other, we **multiply** the number of options at each step. When faced with exclusive options (meaning we can't choose more than one), we **add** the number of options.

To count pairs of items, we first count the number of ways to select 2 items from our group, and then divide by 2 because we've counted each pair twice (once in each order). In general, if we have n items, then there are $\dfrac{n(n-1)}{2}$ pairs of items.

> **Important:** For any positive integer n,
>
> $$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

If all outcomes are *equally likely*, then the probability of success is

$$\frac{\text{Number of successful outcomes}}{\text{Number of possible outcomes}}.$$

It is very important that all of the outcomes are equally likely!

When working on a probability problem, if you ever get a probability that's less than 0 or greater than 1, then you've made a mistake! All probabilities are greater than or equal to 0, and less than or equal to 1.

Most probability problems are just two counting problems: counting the number of total outcomes and counting the number of successful outcomes. So we can take advantage of all of the counting techniques that we've learned to solve probability problems as well.

14.6 Summary

In this chapter, we learned how to use arithmetic to solve counting problems.

> **Important:** Don't just blindly add, subtract, multiply, and divide–think about what you're doing!

Counting a list of numbers like $1, 2, 3, \ldots, 38$ is easy! Counting many other lists of numbers is easy too: use arithmetic to convert it into a list that looks like 1,2,3,. . .. The number of items in the list $a, a+1, \ldots, b-1, b$, where $a \leq b$ are integers, is $b - a + 1$.

> **Concept:** When presented with a complicated problem, try to reduce or simplify it to a simpler problem that you know how to solve.

Counting items which lie in one or more overlapping sets requires a thoughtful use of addition and subtraction. We can use a Venn diagram to help keep track of the thought process. When using a Venn diagram, we should initially try to fill in numbers that correspond exactly to a single region of the diagram. We may have to place variables in our diagram: if so, it's usually wise to choose our variable to represent the answer to the problem.

> **Concept:** When assigning a variable, it's usually best to let your variable represent the answer to the problem.

When counting multiple independent events, we multiply: the number of ways that A **and** B can occur is

$$\begin{aligned}&(\text{the number of ways } A \text{ can occur})\\ &\quad \textbf{times } (\text{the number of ways } B \text{ can occur}).\end{aligned}$$

Sometimes our later choices depend on our earlier choices, so our counting must take that into consideration.

Casework can often help us break a counting problem into more manageable pieces. We split the possibilities into exclusive cases, meaning that every outcome shows up in exactly one case. Then, to count the total, you simply add the counts for each case. Casework requires you to be organized and careful, and to make sure that you don't omit any outcomes in your cases.

> **Concept:** When faced with a series of independent choices, one after the other, we **multiply** the number of options at each step. When faced with exclusive options (meaning we can't choose more than one), we **add** the number of options.

To count pairs of items, we first count the number of ways to select 2 items from our group, and then divide by 2 because we've counted each pair twice (once in each order). In general, if we have n items, then there are $\dfrac{n(n-1)}{2}$ pairs of items.

> **Important:** For any positive integer n,
>
> $$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

If all outcomes are *equally likely*, then the probability of success is

$$\frac{\text{Number of successful outcomes}}{\text{Number of possible outcomes}}.$$

It is very important that all of the outcomes are equally likely!

When working on a probability problem, if you ever get a probability that's less than 0 or greater than 1, then you've made a mistake! All probabilities are greater than or equal to 0, and less than or equal to 1.

Most probability problems are just two counting problems: counting the number of total outcomes and counting the number of successful outcomes. So we can take advantage of all of the counting techniques that we've learned to solve probability problems as well.

14.6 Summary

In this chapter, we learned how to use arithmetic to solve counting problems.

> **Important:** Don't just blindly add, subtract, multiply, and divide–think about what you're doing!

Counting a list of numbers like $1, 2, 3, \ldots, 38$ is easy! Counting many other lists of numbers is easy too: use arithmetic to convert it into a list that looks like 1,2,3,. . .. The number of items in the list $a, a+1, \ldots, b-1, b$, where $a \leq b$ are integers, is $b - a + 1$.

> **Concept:** When presented with a complicated problem, try to reduce or simplify it to a simpler problem that you know how to solve.

Counting items which lie in one or more overlapping sets requires a thoughtful use of addition and subtraction. We can use a Venn diagram to help keep track of the thought process. When using a Venn diagram, we should initially try to fill in numbers that correspond exactly to a single region of the diagram. We may have to place variables in our diagram: if so, it's usually wise to choose our variable to represent the answer to the problem.

> **Concept:** When assigning a variable, it's usually best to let your variable represent the answer to the problem.

When counting multiple independent events, we multiply: the number of ways that A **and** B can occur is

$$\begin{aligned}&(\text{the number of ways } A \text{ can occur})\\ &\quad \textbf{times } (\text{the number of ways } B \text{ can occur}).\end{aligned}$$

Sometimes our later choices depend on our earlier choices, so our counting must take that into consideration.

Casework can often help us break a counting problem into more manageable pieces. We split the possibilities into exclusive cases, meaning that every outcome shows up in exactly one case. Then, to count the total, you simply add the counts for each case. Casework requires you to be organized and careful, and to make sure that you don't omit any outcomes in your cases.

> **Concept:** When faced with a series of independent choices, one after the other, we **multiply** the number of options at each step. When faced with exclusive options (meaning we can't choose more than one), we **add** the number of options.

To count pairs of items, we first count the number of ways to select 2 items from our group, and then divide by 2 because we've counted each pair twice (once in each order). In general, if we have n items, then there are $\dfrac{n(n-1)}{2}$ pairs of items.

> **Important:** For any positive integer n,
>
> $$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

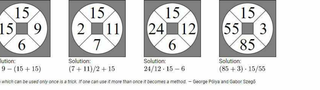
If all outcomes are *equally likely*, then the probability of success is

$$\frac{\text{Number of successful outcomes}}{\text{Number of possible outcomes}}.$$

It is very important that all of the outcomes are equally likely!

When working on a probability problem, if you ever get a probability that's less than 0 or greater than 1, then you've made a mistake! All probabilities are greater than or equal to 0, and less than or equal to 1.

Most probability problems are just two counting problems: counting the number of total outcomes and counting the number of successful outcomes. So we can take advantage of all of the counting techniques that we've learned to solve probability problems as well.



Solution: $6 \cdot 9 - (15 + 15)$ | Solution: $(7 + 11)/2 + 15$ | Solution: $24/12 \cdot 15 - 6$ | Solution: $(85 + 3) \cdot 15/55$

An idea which can be used only once is a trick. If one can use it more than once it becomes a method. — George Pólya and Gábor Szegö

CHAPTER 15

Problem-Solving Strategies

We finish the book with a chapter covering common general strategies for tackling problems. These strategies have a common goal:

> **Concept:** Simplify the problem.

15.1 Find a Pattern

Much of learning consists simply of observing patterns. Solving math problems is no different. An excellent problem solver is playful and experiments with problems. Often these experiments reveal patterns that lead to a solution.

Problems

Problem 15.1 Jump to Solution

What is the 2010th letter in the sequence below?

$$ABCDEDCBAABCDEDCBAABCDEDCBAABCDE\ldots$$

Problem 15.2 Jump to Solution

In this problem, we find the units digit (ones digit) of 2^{2011}.

(a) Find the units digit of each of the first 8 powers of 2, starting with 2^1.

(b) Find the units digit of 2^{2011}.

Problem 15.3 *Source: MOEMS* Jump to Solution

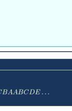
Michelle's Number Recycling Machine obeys two rules:

1. If an inserted number has exactly one digit, double the number.
2. If an inserted number has exactly two digits, compute the sum of the digits.

The first number Michelle inserts is 1. Then every answer she gets is inserted back into the machine until fifty numbers are inserted. What is the fifteenth number to be inserted?

Problem 15.4 Jump to Solution

In the figure below, each row of x's has two more x's than the row above it. Altogether, how many x's are contained in the first 30 rows?



Problem 15.1

What is the 2010th letter in the sequence below?

$$ABCDEDCBAABCDEDCBAABCDEDCBAABCDE\ldots$$

Solution for Problem 15.1: We certainly don't want to write out 2010 letters! Fortunately, the letters follow a clear pattern. The same 9-letter block repeats over and over:

$$ABCDEDCBA.$$

We'll have to write this block over and over many times before reaching the 2010th letter. To figure out how many times, we divide 2010 by 9, which has quotient 223 and remainder 3. So, in writing the first 2010 letters, we write this repeating block 223 times, and then write 3 more letters. Therefore, the 2010th letter is C. ☐

In Problem 15.1, the pattern is part of the problem statement. In many problems, the pattern isn't given in the problem, and we must discover it by experimenting.

Problem 15.2

What is the units digit (ones digit) of 2^{2011}?

Solution for Problem 15.2: Computing 2^{2011} by hand will take a while. Many calculators aren't able to output all the digits of such a large number, so even using a calculator may not help solve the problem. Instead, we'll have to play with the problem a bit by computing several powers of 2 and looking for a pattern. Starting from 2^1, the first eight powers of 2 are

$$2, 4, 8, 16, 32, 64, 128, 256.$$

At first, it may not be obvious that there's a pattern. But remember, the problem asks for the units digit of 2^{2011}, so let's look at just the units digits of the numbers in this list:

$$2, 4, 8, 6, 2, 4, 8, 6.$$

The same block of 4 digits repeats twice. Will this pattern continue forever?

> **Important:** Noticing a pattern isn't enough. We have to be sure that the pattern continues in order to use it to solve a problem. This usually means that we have to figure out why the pattern occurs.

The units digit of the product of two integers is the same as the units digit of the product of the units digits of the integers. For example, the units digit of $342 \cdot 486$ is the same as the units digit of $2 \cdot 6$. So, all we have to consider are the units digits of the powers of 2.

We start with 2, and multiplying any number that ends in 2 by 2 gives a number that ends in 4. This means that any 2 in the list of units digits of powers of 2 must be followed by a 4. Similarly, 4 must be followed by 8, and 8 must be followed by 6. Since $6 \cdot 2$ ends in 2, the next units digit is 2 again. We already know that after 2 comes 4, after 4 comes 8, and after 8 comes 6. Then, we're back to 2 yet again. So, we see that the units digits repeat the cycle 2, 4, 8, 6 over and over.

Now our goal is to figure out which digit is 2011th if we repeat the cycle 2, 4, 8, 6 until we have 2011 terms. 2011 divided by 4 has quotient 502 and remainder 3. So, we will write the cycle 502 times, and then write 3 more terms to get to the 2011th term. Therefore, the 2011th term is 8, so the units digit of 2^{2011} is 8.

Notice that we don't even have to find the quotient when 2011 is divided by 4. All we really care about is the remainder. Since the remainder is 3, we know that to get to the 2011th term, we have to write three more terms after writing the last complete cycle 2, 4, 8, 6. ☐

It's very easy to be off by one term when using patterns to solve a problem as we did in Problem 15.2. For example, suppose we wanted to find the units digit of 2^{2000}. Following our work in the previous problem, we find the remainder when we divide 2000 by 4. The remainder is 0, so do we take the first term in the cycle 2, 4, 8, 6, or do we take the last?

To get to the 2000th term, we write the cycle $2000/4 = 500$ times and then write no *more extra* terms. We write the 2000th term when we write the last term of the cycle for the 500th time. So, the units digit of 2^{2000} is 6.

> **WARNING!** Be careful not to be off by one term when using patterns to solve a problem.

One way to check that you're not off by one is to test small cases. For example, to test our reasoning above for the units digit of 2^{2000}, we can consider what happens when 2 is raised to smaller exponents that are multiples of 4. Both 2^4 and 2^8 end in 6, which gives us more confidence that our answer for 2^{2000} is correct.

Problem 15.3 *Source: MOEMS*

Michelle's Number Recycling Machine obeys two rules:

1. If an inserted number has exactly one digit, double the number.
2. If an inserted number has exactly two digits, compute the sum of the digits.

The first number Michelle inserts is 1. Then every answer she gets is inserted back into the machine until fifty numbers are inserted. What is the fifteenth number to be inserted?

Solution for Problem 15.3: As with the previous problem, we experiment a bit and hope we find a pattern. Following the two given rules, the first several numbers Michelle's machine produces are

$$1, 2, 4, 8, 16, 7, 14, 5, 10, 1, 2, 4, 8.$$

Once the machine produces 1 for a second time, we know that the machine will repeat the first 9 numbers over and over. Since 50 divided by 9 leaves a remainder of 5, the 50th number in the list is the same as the 5th in this repeating block of numbers. This means that the 50th number in the list is 16. ☐

Problem 15.4

In the figure below, each row of x's has two more x's than the row above it. Altogether, how many x's are contained in the first 30 rows?

Solution for Problem 15.4: First, we simplify the problem by getting rid of the symbols, and instead write the number of x's in each row. The numbers of x's in the rows are

$$1, 3, 5, 7, 9, 11, \ldots.$$

So, to count all the x's in the first 30 rows, we could add the first 30 positive odd numbers. But instead, let's add up shorter lists and look for a pattern.

> **Concept:** Considering a simpler version of a problem can help us solve the original problem.

The first 5 such sums are

$$\begin{aligned} 1 &= 1, \\ 1 + 3 &= 4, \\ 1 + 3 + 5 &= 9, \\ 1 + 3 + 5 + 7 &= 16, \\ 1 + 3 + 5 + 7 + 9 &= 25. \end{aligned}$$

Each sum is a perfect square! The sum of the first 2 odd numbers is 2^2, the sum of the first 3 odd numbers is 3^2, the sum of the first 4 odd numbers is 4^2, and so on.

We now expect that the sum of the first 30 positive odd numbers is $30^2 = 900$. But we still have to figure out why summing the first n positive odd numbers results in n^2. Now that we know what to look for, there are a number of ways to see why it is true.

One way is to use what we learned back in Section 2.1 here about consecutive squares. There, we learned that

$$a^2 + 2a + 1 = (a + 1)^2.$$

The first square is just the first positive odd number: $1^2 = 1$. To get from 1^2 to 2^2, we add the odd number $2 \cdot 1 + 1$:

$$1^2 + 2 \cdot 1 + 1 = 2^2.$$

Then, to get from 2^2 to 3^2, we add the next odd number, $2 \cdot 2 + 1$:

$$2^2 + 2 \cdot 2 + 1 = 3^2.$$

And so on:

$$\begin{aligned} 3^2 &+ 2 \cdot 3 + 1 = 4^2, \\ 4^2 &+ 2 \cdot 4 + 1 = 5^2, \\ 5^2 &+ 2 \cdot 5 + 1 = 6^2. \end{aligned}$$

We can also use some clever diagrams to explain why summing the first n positive odd numbers equals n^2:

Now, we can confidently state that there are 900 x's total in the first 30 rows. ☐

Find a Pattern

Exercises

15.1.1: Source: MATHCOUNTS

It takes exactly 74 colored beads on a string to make a necklace. The beads are strung in the following order: one red, one orange, two yellow, one green, and one blue. Then the pattern repeats, starting again with one red bead. If the first bead of the necklace is red, what is the color of the last bead used to make the necklace?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solve

Solution: The beads on the necklace repeat the same pattern of 6 bead colors over and over. Since 74 divided by 6 has quotient 12 and remainder 2, the 74th bead is 2 beads into this 6-bead pattern. The second bead of the pattern is orange.

15.1.2:

The **Fibonacci sequence** begins

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55.$$

Each term of the sequence after the second term is the sum of the two terms before it. The sequence continues forever. For example, the eleventh term is $34 + 55$, or 89. How many of the first 60 terms of this sequence are odd numbers?

Preview: Solve

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solve

Solution: In the first ten terms shown, it looks like there's a pattern of two odd numbers followed by an even number. To see why this pattern will continue, note that the sum of two odd numbers is even. So, two odd numbers must be followed by an even number. Similarly, the sum of an odd and even is odd. So, both "odd then even" and "even then odd" will be followed by an odd number. Putting these observations together, we see that the Fibonacci sequence consists of blocks of two odd numbers and an even number. In the first 60 terms, there are $60/3 = 20$ such blocks, for a total of $20 \cdot 2 = 40$ odd numbers.

15.1.3:

In the table of numbers below, what number is directly above 119?

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 2 & 3 & 4 & & & \\ & & 5 & 6 & 7 & 8 & 9 & & \\ & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \\ \cdot^{\cdot^{\cdot}} & & & & \vdots & & & & \ddots \end{array}$$

Preview: Solve

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solve

Solution: The last number in each row appears to be a perfect square. To see why this is the case, note that there is 1 number in the first row, 3 numbers in the second row, 5 numbers in the third row, and so on. In each row, we add the next odd number amount of numbers. So, the total number of numbers on or above each row is the sum of consecutive odd numbers starting from 1. As explained in the text, such a sum is always a perfect square. This means the last number in each row is a perfect square.

Since 119 is 2 less than the perfect square 121, we know that 119 is two numbers before the last number of the row that ends with 121. The row above 119's row ends in the previous square, 100, and this final number is directly above the next-to-last number in 119's row, namely 120. So, the number just to 100's left is directly above 119, which means that the number directly above 119 is 99.

15.1.4:

All of the even numbers from 2 through 288, except those ending in 0, are multiplied together. What is the units digit (ones digit) of the product?

Hint: We aren't multiplying the same units digit over and over again. But we are multiplying the same group of units digits over and over again.

Preview: Solve

You may type any additional notes you have here.

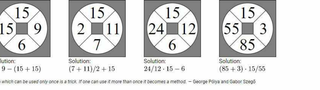
Hide Solution | Reset

Your Submission: Solve

Solution: We only care about the units digits of the numbers in the product. The first four numbers are $2 \cdot 4 \cdot 6 \cdot 8$. The product of the units digits of the next four numbers is also $2 \cdot 4 \cdot 6 \cdot 8$. Similarly, the product of the units digits in the rest of the product just repeats this block over and over.

We have $2 \cdot 4 \cdot 6 \cdot 8 = 8 \cdot 6 \cdot 8 = 48 \cdot 8$, which has units digit 4 (since $8 \cdot 8 = 64$). The rest of the product in the problem consists of 28 more blocks of $2 \cdot 4 \cdot 6 \cdot 8$. Each block has a product with units digit 4. The entire product consists of 29 such blocks, each with a product with units digit 4. Therefore, the product has the same units digit as 4^{29}.

4^1 ends in 4, 4^2 ends in 6, and the units digit of 4^3 is back to 4. So, the units digits of the powers of 4 cycle back and forth between 4 and 6, starting with 4^1. Since 29 is odd, we see that 4^{29} has 4 as its units digit. Therefore, the original product has units digit 4.



Solution: $6 \cdot 9 - (15 + 15)$ | Solution: $(7 + 11)/2 + 15$ | Solution: $24/12 \cdot 15 - 6$ | Solution: $(85 + 3) \cdot 15/55$

An idea which can be used only once is a trick. If one can use it more than once it becomes a method. — George Pólya and Gábor Szegö

CHAPTER 15

Problem-Solving Strategies

We finish the book with a chapter covering common general strategies for tackling problems. These strategies have a common goal:

> **Concept:** Simplify the problem.

15.1 Find a Pattern

Much of learning consists simply of observing patterns. Solving math problems is no different. An excellent problem solver is playful and experiments with problems. Often these experiments reveal patterns that lead to a solution.

Problems

Problem 15.1 Jump to Solution

What is the 2010th letter in the sequence below?

$$ABCDEDCBAABCDEDCBAABCDEDCBAABCDE\ldots$$

Problem 15.2 Jump to Solution

In this problem, we find the units digit (ones digit) of 2^{2011}.

(a) Find the units digit of each of the first 8 powers of 2, starting with 2^1.

(b) Find the units digit of 2^{2011}.

Problem 15.3 *Source: MOEMS* Jump to Solution

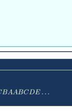
Michelle's Number Recycling Machine obeys two rules:

1. If an inserted number has exactly one digit, double the number.
2. If an inserted number has exactly two digits, compute the sum of the digits.

The first number Michelle inserts is 1. Then every answer she gets is inserted back into the machine until fifty numbers are inserted. What is the fifteenth number to be inserted?

Problem 15.4 Jump to Solution

In the figure below, each row of x's has two more x's than the row above it. Altogether, how many x's are contained in the first 30 rows?



Problem 15.1

What is the 2010th letter in the sequence below?

$$ABCDEDCBAABCDEDCBAABCDEDCBAABCDE\ldots$$

Solution for Problem 15.1: We certainly don't want to write out 2010 letters! Fortunately, the letters follow a clear pattern. The same 9-letter block repeats over and over:

$$ABCDEDCBA.$$

We'll have to write this block over and over many times before reaching the 2010th letter. To figure out how many times, we divide 2010 by 9, which has quotient 223 and remainder 3. So, in writing the first 2010 letters, we write this repeating block 223 times, and then write 3 more letters. Therefore, the 2010th letter is C. ☐

In Problem 15.1, the pattern is part of the problem statement. In many problems, the pattern isn't given in the problem, and we must discover it by experimenting.

Problem 15.2

What is the units digit (ones digit) of 2^{2011}?

Solution for Problem 15.2: Computing 2^{2011} by hand will take a while. Many calculators aren't able to output all the digits of such a large number, so even using a calculator may not help solve the problem. Instead, we'll have to play with the problem a bit by computing several powers of 2 and looking for a pattern. Starting from 2^1, the first eight powers of 2 are

$$2, 4, 8, 16, 32, 64, 128, 256.$$

At first, it may not be obvious that there's a pattern. But remember, the problem asks for the units digit of 2^{2011}, so let's look at just the units digits of the numbers in this list:

$$2, 4, 8, 6, 2, 4, 8, 6.$$

The same block of 4 digits repeats twice. Will this pattern continue forever?

> **Important:** Noticing a pattern isn't enough. We have to be sure that the pattern continues in order to use it to solve a problem. This usually means that we have to figure out why the pattern occurs.

The units digit of the product of two integers is the same as the units digit of the product of the units digits of the integers. For example, the units digit of $342 \cdot 486$ is the same as the units digit of $2 \cdot 6$. So, all we have to consider are the units digits of the powers of 2.

We start with 2, and multiplying any number that ends in 2 by 2 gives a number that ends in 4. This means that any 2 in the list of units digits of powers of 2 must be followed by a 4. Similarly, 4 must be followed by 8, and 8 must be followed by 6. Since $6 \cdot 2$ ends in 2, the next units digit is 2 again. We already know that after 2 comes 4, after 4 comes 8, and after 8 comes 6. Then, we're back to 2 yet again. So, we see that the units digits repeat the cycle 2, 4, 8, 6 over and over.

Now our goal is to figure out which digit is 2011th if we repeat the cycle 2, 4, 8, 6 until we have 2011 terms. 2011 divided by 4 has quotient 502 and remainder 3. So, we will write the cycle 502 times, and then write 3 more terms to get to the 2011th term. Therefore, the 2011th term is 8, so the units digit of 2^{2011} is 8.

Notice that we don't even have to find the quotient when 2011 is divided by 4. All we really care about is the remainder. Since the remainder is 3, we know that to get to the 2011th term, we have to write three more terms after writing the last complete cycle 2, 4, 8, 6. ☐

It's very easy to be off by one term when using patterns to solve a problem as we did in Problem 15.2. For example, suppose we wanted to find the units digit of 2^{2000}. Following our work in the previous problem, we find the remainder when we divide 2000 by 4. The remainder is 0, so do we take the first term in the cycle 2, 4, 8, 6, or do we take the last?

To get to the 2000th term, we write the cycle $2000/4 = 500$ times and then write no *more extra* terms. We write the 2000th term when we write the last term of the cycle for the 500th time. So, the units digit of 2^{2000} is 6.

> **WARNING!** Be careful not to be off by one term when using patterns to solve a problem.

One way to check that you're not off by one is to test small cases. For example, to test our reasoning above for the units digit of 2^{2000}, we can consider what happens when 2 is raised to smaller exponents that are multiples of 4. Both 2^4 and 2^8 end in 6, which gives us more confidence that our answer for 2^{2000} is correct.

Problem 15.3 *Source: MOEMS*

Michelle's Number Recycling Machine obeys two rules:

1. If an inserted number has exactly one digit, double the number.
2. If an inserted number has exactly two digits, compute the sum of the digits.

The first number Michelle inserts is 1. Then every answer she gets is inserted back into the machine until fifty numbers are inserted. What is the fifteenth number to be inserted?

Solution for Problem 15.3: As with the previous problem, we experiment a bit and hope we find a pattern. Following the two given rules, the first several numbers Michelle's machine produces are

$$1, 2, 4, 8, 16, 7, 14, 5, 10, 1, 2, 4, 8.$$

Once the machine produces 1 for a second time, we know that the machine will repeat the first 9 numbers over and over. Since 50 divided by 9 leaves a remainder of 5, the 50th number in the list is the same as the 5th in this repeating block of numbers. This means that the 50th number in the list is 16. ☐

Problem 15.4

In the figure below, each row of x's has two more x's than the row above it. Altogether, how many x's are contained in the first 30 rows?

Solution for Problem 15.4: First, we simplify the problem by getting rid of the symbols, and instead write the number of x's in each row. The numbers of x's in the rows are

$$1, 3, 5, 7, 9, 11, \ldots.$$

So, to count all the x's in the first 30 rows, we could add the first 30 positive odd numbers. But instead, let's add up shorter lists and look for a pattern.

> **Concept:** Considering a simpler version of a problem can help us solve the original problem.

The first 5 such sums are

$$\begin{aligned} 1 &= 1, \\ 1 + 3 &= 4, \\ 1 + 3 + 5 &= 9, \\ 1 + 3 + 5 + 7 &= 16, \\ 1 + 3 + 5 + 7 + 9 &= 25. \end{aligned}$$

Each sum is a perfect square! The sum of the first 2 odd numbers is 2^2, the sum of the first 3 odd numbers is 3^2, the sum of the first 4 odd numbers is 4^2, and so on.

We now expect that the sum of the first 30 positive odd numbers is $30^2 = 900$. But we still have to figure out why summing the first n positive odd numbers results in n^2. Now that we know what to look for, there are a number of ways to see why it is true.

One way is to use what we learned back in Section 2.1 here about consecutive squares. There, we learned that

$$a^2 + 2a + 1 = (a + 1)^2.$$

The first square is just the first positive odd number: $1^2 = 1$. To get from 1^2 to 2^2, we add the odd number $2 \cdot 1 + 1$:

$$1^2 + 2 \cdot 1 + 1 = 2^2.$$

Then, to get from 2^2 to 3^2, we add the next odd number, $2 \cdot 2 + 1$:

$$2^2 + 2 \cdot 2 + 1 = 3^2.$$

And so on:

$$\begin{aligned} 3^2 &+ 2 \cdot 3 + 1 = 4^2, \\ 4^2 &+ 2 \cdot 4 + 1 = 5^2, \\ 5^2 &+ 2 \cdot 5 + 1 = 6^2. \end{aligned}$$

We can also use some clever diagrams to explain why summing the first n positive odd numbers equals n^2:

Now, we can confidently state that there are 900 x's total in the first 30 rows. ☐

Find a Pattern

Exercises

15.1.1: Source: MATHCOUNTS

It takes exactly 74 colored beads on a string to make a necklace. The beads are strung in the following order: one red, one orange, two yellow, one green, and one blue. Then the pattern repeats, starting again with one red bead. If the first bead of the necklace is red, what is the color of the last bead used to make the necklace?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solve

Solution: The beads on the necklace repeat the same pattern of 6 bead colors over and over. Since 74 divided by 6 has quotient 12 and remainder 2, the 74th bead is 2 beads into this 6-bead pattern. The second bead of the pattern is orange.

15.1.2:

The **Fibonacci sequence** begins

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55.$$

Each term of the sequence after the second term is the sum of the two terms before it. The sequence continues forever. For example, the eleventh term is $34 + 55$, or 89. How many of the first 60 terms of this sequence are odd numbers?

Preview: Solve

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solve

Solution: In the first ten terms shown, it looks like there's a pattern of two odd numbers followed by an even number. To see why this pattern will continue, note that the sum of two odd numbers is even. So, two odd numbers must be followed by an even number. Similarly, the sum of an odd and even is odd. So, both "odd then even" and "even then odd" will be followed by an odd number. Putting these observations together, we see that the Fibonacci sequence consists of blocks of two odd numbers and an even number. In the first 60 terms, there are $60/3 = 20$ such blocks, for a total of $20 \cdot 2 = 40$ odd numbers.

15.1.3:

In the table of numbers below, what number is directly above 119?

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 2 & 3 & 4 & & & \\ & & 5 & 6 & 7 & 8 & 9 & & \\ & 10 & 11 & 12 & 13 & 14 & 15 & 16 & \\ \cdot^{\cdot^{\cdot}} & & & & \vdots & & & & \ddots \end{array}$$

Preview: Solve

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solve

Solution: The last number in each row appears to be a perfect square. To see why this is the case, note that there is 1 number in the first row, 3 numbers in the second row, 5 numbers in the third row, and so on. In each row, we add the next odd number amount of numbers. So, the total number of numbers on or above each row is the sum of consecutive odd numbers starting from 1. As explained in the text, such a sum is always a perfect square. This means the last number in each row is a perfect square.

Since 119 is 2 less than the perfect square 121, we know that 119 is two numbers before the last number of the row that ends with 121. The row above 119's row ends in the previous square, 100, and this final number is directly above the next-to-last number in 119's row, namely 120. So, the number just to 100's left is directly above 119, which means that the number directly above 119 is 99.

15.1.4:

All of the even numbers from 2 through 288, except those ending in 0, are multiplied together. What is the units digit (ones digit) of the product?

Hint: We aren't multiplying the same units digit over and over again. But we are multiplying the same group of units digits over and over again.

Preview: Solve

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission: Solve

Solution: We only care about the units digits of the numbers in the product. The first four numbers are $2 \cdot 4 \cdot 6 \cdot 8$. The product of the units digits of the next four numbers is also $2 \cdot 4 \cdot 6 \cdot 8$. Similarly, the product of the units digits in the rest of the product just repeats this block over and over.

We have $2 \cdot 4 \cdot 6 \cdot 8 = 8 \cdot 6 \cdot 8 = 48 \cdot 8$, which has units digit 4 (since $8 \cdot 8 = 64$). The rest of the product in the problem consists of 28 more blocks of $2 \cdot 4 \cdot 6 \cdot 8$. Each block has a product with units digit 4. The entire product consists of 29 such blocks, each with a product with units digit 4. Therefore, the product has the same units digit as 4^{29}.

4^1 ends in 4, 4^2 ends in 6, and the units digit of 4^3 is back to 4. So, the units digits of the powers of 4 cycle back and forth between 4 and 6, starting with 4^1. Since 29 is odd, we see that 4^{29} has 4 as its units digit. Therefore, the original product has units digit 4.

15.2 Make a List

If a strategy is good enough to allow Santa Claus to distribute presents to all the children in the world, it ought to be good enough to help us solve some math problems.

Problems

Problem 15.5 *Source: MOEMS* Jump to Solution

Find a two-digit positive integer with all these properties:

- the tens digit is larger than the ones digit (units digit),
- the difference between the digits is greater than 3,
- the sum of the digits is greater than 10, and
- the integer is a multiple of 12.

Problem 15.6 Jump to Solution

Elizabeth spent exactly $7 for some 15-cent stamps and some 31-cent stamps. How many 15-cent stamps did she buy?

Problem 15.7 Jump to Solution

A **partition** of a positive integer n is a way of writing n as the sum of positive integers in which the order of the numbers in the sum doesn't matter. For example, the partitions of 3 are 3, $2+1$, and $1+1+1$. We consider $2+1$ and $1+2$ to be the same, so there are 3 partitions of 3. How many partitions of 6 are there?

Problem 15.5 *Source: MOEMS*

Find a two-digit positive integer with all these properties:

- the tens digit is larger than the ones digit (units digit),
- the difference between the digits is greater than 3,
- the sum of the digits is greater than 10, and
- the integer is a multiple of 12.

Solution for Problem 15.5: It would take way too long to test every single two-digit number. We narrow our search by first focusing on just one of the properties. But which one?

> **Concept:** When faced with several restrictions in a problem, it's often best to consider the most restrictive first.

Each of the first three properties is satisfied by quite a few two-digit numbers, but there aren't many two-digit multiples of 12. So, we start by listing the multiples of 12:

$$12, 24, 36, 48, 60, 72, 84, 96.$$

Now that we have a short list of possible solutions, we can quickly go through the other properties and eliminate possibilities from this list. Since the tens digit must be larger than the ones digit, we can eliminate the first four numbers in the list, leaving

$$\cancel{12}, \cancel{24}, \cancel{36}, \cancel{48}, 60, 72, 84, 96.$$

The next property, that the difference between the digits is greater than 3, eliminates 96:

$$\cancel{12}, \cancel{24}, \cancel{36}, \cancel{48}, 60, 72, 84, \cancel{96}.$$

Finally, since the sum of the digits must be greater than 10, we can eliminate 60 and 72, and we are left with 84 as the answer to the problem. □

Problem 15.6

Elizabeth spent exactly $7 for some 15-cent stamps and some 31-cent stamps. How many 15-cent stamps did she buy?

Solution for Problem 15.6: We often start word problems by assigning variables and writing equations. Let's try that here. First, we note that $7 is 700 cents. If we let a be the number of 15-cent stamps and b be the number of 31-cent stamps, then we must have

$$15a + 31b = 700.$$

That's all we know about a and b, and this equation isn't terribly helpful. We could just guess values of a or b and hope we get lucky, but there are a lot of possibilities.

Instead, we take a more organized approach. After paying for all the 31-cent stamps, the amount remaining must be divisible by 15 cents. So, we imagine that Elizabeth starts with 700 cents and pays for the 31-cent stamps one at a time. We then list the number of cents she has remaining at each step, looking for a multiple of 15:

$$700, 669, 638, 607, 576, 545, 514, 483, 452, 421, 390.$$

Since 390 is divisible by both 3 and 5, it is divisible by 15. Elizabeth spent $700 - 390 = 310$ cents on 31-cent stamps, so she bought 10 31-cent stamps. With the other 390 cents, she bought $390/15 = 26$ 15-cent stamps.

You might have noticed that we could have saved a lot of work with a little bit of thinking. Instead of buying the 31-cent stamps one at a time, we might have noticed that the number of 31-cent stamps she bought must have been a multiple of 5. This is the only way the number of cents left for 15-cent stamps could be a multiple of 5. So, we can shorten our list by buying five 31-cent stamps at a time. Each block of five 31-cent stamps costs $5 \cdot 31 = 155$ cents, so our list is

$$700, 545, 390.$$

We reach the desired multiple of 15 much faster this way! Furthermore, it's easy with this method to see why there are no other combinations she could have bought. If we continue the list, we have

$$700, 545, 390, 235, 80.$$

The only multiple of 15 in this list is 390. So, the only way she can buy 31-cent stamps and 15-cent stamps is if she spends 390 cents on 15-cent stamps and 310 cents on 31-cent stamps. □

In the previous problem, we saw that doing a little bit of thinking before listing can sometimes shorten the amount of listing you have to do. In the next problem, we see that sometimes we need to do some careful thinking just to make sure we have a complete list.

Problem 15.7

A **partition** of a positive integer n is a way of writing n as the sum of positive integers in which the order of the numbers in the sum doesn't matter. For example, the partitions of 3 are 3, $2+1$, and $1+1+1$. We consider $2+1$ and $1+2$ to be the same, so there are 3 partitions of 3. How many partitions of 6 are there?

Solution for Problem 15.7: If we just start listing partitions of 6 as we think of them, it will be hard to know when we've found them all. So, we should take an organized approach.

> **Concept:** Organized lists are much more useful than disorganized lists.

One way to organize the partitions is to group them based on the largest number in the partition. Obviously, there's only one partition that has 6, namely

$$6.$$

Similarly, there's only one partition in which 5 is the largest number:

$$5+1.$$

The partitions with 4 as the greatest number are

$$4+2,$$
$$4+1+1.$$

The partitions with 3 as the greatest number are

$$3+3,$$
$$3+2+1,$$
$$3+1+1+1.$$

The partitions with 2 as the greatest number are

$$2+2+2,$$
$$2+2+1+1,$$
$$2+1+1+1+1.$$

The only partition with 1 as the greatest number is

$$1+1+1+1+1+1.$$

Altogether, we count 11 partitions of 6. □

Make a List

Exercises

15.2.1:

In Problem 15.6, why did we choose to imagine buying the 31-cent stamps one at a time, rather than starting with buying the 15-cent stamps?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: We tackled the problem by counting backwards from 700 by 31's while looking for a multiple of 15. This is much easier than counting backwards from 700 by 15's while looking for a multiple of 31 for two reasons. First, it is far easier to spot multiples of 15 than it is to spot multiples of 31. Only numbers that end in 0 or 5 can possibly be a multiple of 15. So, we don't have to do anything besides look at the last digit to discard most of the numbers we hit as we subtract 31's. Moreover, we cover a lot more ground as we subtract 31's than when we subtract 15's, so we have fewer numbers to check.

15.2.2:

I am less than 6 feet tall but more than 2 feet tall. My height in inches is a multiple of 7 and is also 1 more than a multiple of 6. What is my height?

Preview: Solution

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: My height is a number of inches that is greater than 24 (2 feet), less than 72 (6 feet), a multiple of 7, and 1 more than a multiple of 6. To find my height, I list the multiples of 7 greater than 24 and less than 72:

$$28, 35, 42, 49, 56, 63, 70.$$

The only number in this list that is 1 more than multiple of 6 is 49. So, I'm $\boxed{4 \text{ feet, } 1 \text{ inch}}$ tall.

15.2.3: *Source: MOEMS*

List all two-digit positive integers that satisfy both of the following:

1. The tens and ones digits are consecutive numbers, and
2. The integer is the product of two consecutive numbers.

Preview: Solution

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: It's easiest to start with the second condition, since there aren't many two-digit numbers that are the product of two consecutive numbers. (It's also much easier to multiply numbers than to find factorizations, which is what we'd have to do if we started by listing numbers whose digits are consecutive numbers.) We build a list of two-digit numbers that satisfy the second condition by multiplying pairs of consecutive numbers. The full list is

$$3 \cdot 4 = 12,$$
$$4 \cdot 5 = 20,$$
$$5 \cdot 6 = 30,$$
$$6 \cdot 7 = 42,$$
$$7 \cdot 8 = 56,$$
$$8 \cdot 9 = 72,$$
$$9 \cdot 10 = 90.$$

The only numbers in this list with consecutive numbers as digits are $\boxed{12 \text{ and } 56}$.

15.2.4:

If a number is divided by 3, the remainder is 0. If the number is divided by 5, the remainder is 4. If the number is divided by 11, the remainder is 7. The number has two digits. What is the number?

Preview: Solution

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: We could start by listing all the two-digit multiples of 3, but that would be a pretty long list. Instead, we start by listing all the two-digit numbers that are 7 more than a multiple of 11. We can do this by starting with 7 and counting by 11's. The two-digit numbers we thus create are

$$18, 29, 40, 51, 62, 73, 84, 95.$$

Next, we note that because the number leaves a remainder of 4 when divided by 5, the number must end in 4 or 9. That leaves only 29 and 84 as possibilities. 29 is not divisible by 3, but $\boxed{84}$ is.

15.2.5★:

The three-digit number 114 has a **digit sum** of $1+1+4$, or 6. How many three-digit positive integers have a digit sum of 6?

Hint: What is a good way to organize the three-digit numbers whose digit sum is 6?

Hint: How many start with 6? How many start with 5?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: We make an organized list, where we organize the numbers by the hundreds digit. Clearly, there are no numbers starting with 7 or greater that have a digit sum of 6. There is one such number starting with 6, namely 600. Continuing in this manner, we produce the following organized list.

Hundreds digit	Numbers with digit sum 6
6	600
5	510, 501
4	420, 411, 402
3	330, 321, 312, 303
2	240, 231, 222, 213, 204
1	150, 141, 132, 123, 114, 105

Notice how each row is nicely organized. This makes us confident that we didn't miss any, and didn't count any twice. It also reveals a nice pattern. We see that there are $1+2+3+4+5+6 = \boxed{21}$ three-digit numbers with digit sum 6.

15.3 Draw a Picture

The old adages "Seeing is believing" and "A picture is worth a thousand words" are applicable to many math problems. Naturally, this is often true of geometry problems, but it's also true of some problems that don't appear to be about pictures at all.

Problems

Problem 15.8 Jump to Solution

Wally the wandering walrus swims 6 miles north, then 3 miles east, then 3 miles north, then 15 miles west. How far is he from where he started? (You can assume Earth is flat for this problem.)

Problem 15.9 Jump to Solution

A frog is at the bottom of a 12-meter well. Each morning he climbs up 5 meters. Each night he slides down 3 meters. If he starts climbing on a Sunday, on which day will he reach the top of the well and escape?

Problem 15.10 Jump to Solution

Abel, Bernoulli, Cantor, and Descartes have a race. Bernoulli finished between Abel and Descartes. Cantor is happy that he didn't finish last, and Descartes bragged all day long about beating Abel. If no one finished between Cantor and Abel, then in what order did the participants finish?

Problem 15.11 Jump to Solution

Below are the pairs of cities connected by direct flights by GetUThere Airlines. For each pair of cities, there are flights in both directions. Is it possible to get from Boston to Philadelphia on a series of GetUThere flights?

Boston–New York	Baltimore–Washington	Buffalo–Newark
Buffalo–Albany	Philadelphia–Erie	New York–Washington
New York–Baltimore	Erie–Newark	Albany–Philadelphia
Newark–Albany	Baltimore–Boston	Newark–Philadelphia

We start with a problem in which drawing a picture is a pretty obvious first step.

Problem 15.8

Wally the wandering walrus swims 6 miles north, then 3 miles east, then 3 miles north, then 15 miles west. How far is he from where he started? (You can assume Earth is flat for this problem.)

Solution for Problem 15.8: At the right, the solid arrows show Wally's path. As shown in the dashed right triangle, Wally wanders 9 miles north and 12 miles west in total. Applying the Pythagorean Theorem, or recalling the $\{9, 12, 15\}$ Pythagorean triple, we find that Wally ends 15 miles from where he started. □

Problem 15.9

A frog is at the bottom of a 12-meter well. Each morning he climbs up 5 meters. Each night he slides down 3 meters. If he starts climbing on a Sunday, on which day will he reach the top of the well and escape?

Solution for Problem 15.9: What's wrong with this solution:

> **Bogus Solution:** Since he climbs up 5 meters each morning and slides back 3 meters each night, he moves upward 2 meters total each day of the week. Therefore, he will climb 12 meters in $12/2 = 6$ days. He starts climbing on a Sunday, so his sixth day of climbing is the following Friday.

We see the flaw in this reasoning when we draw a picture that shows both his climbs and his slides. Our picture is shown at the right. We draw up arrows for the climbs and down arrows for the slides, and note the total height climbed at each step. We see that his height first goes over 12 meters after the fifth climb, so he escapes the well on the fifth day of climbing. He starts climbing on a Sunday, so his fifth and last day of climbing is a Thursday. □

> **Concept:** A quick sketch can help prevent errors.

Problem 15.10

Abel, Bernoulli, Cantor, and Descartes have a race. Bernoulli finished between Abel and Descartes. Cantor is happy that he didn't finish last, and Descartes bragged all day long about beating Abel. If no one finished between Cantor and Abel, in what order did the participants finish?

Solution for Problem 15.10: We need a good way to organize the information in the problem. Each piece of information eliminates possibilities: For example, because Bernoulli finished between Abel and Descartes, we know that Bernoulli did not finish first or last. We visualize this information by making a table with a row for each racer and a column for each position. We place X's in the 1st and 4th columns of Bernoulli's row to indicate that Bernoulli didn't finish in either of these positions.

	1st	2nd	3rd	4th
Abel				
Bernoulli	X			X
Cantor				
Descartes				

Next, we place an X in Cantor's 4th column since Cantor wasn't last. Because Descartes beat Abel, we know that Abel wasn't first and Descartes wasn't last. Placing X's for these facts as well, we now have the table at the right:

	1st	2nd	3rd	4th
Abel	X			
Bernoulli	X			X
Cantor				X
Descartes				X

Now, it's clear that Abel must have been 4th, since none of the other runners could have been 4th. We then immediately know that Cantor was 3rd, since no one was between Cantor and Abel. We place O's in Abel's 4th column and Cantor's 3rd column to indicate that we know these placements. We then can place X's in the remainder of Abel's and Cantor's rows, and in the remainder of the 3rd column.

	1st	2nd	3rd	4th
Abel	X			O
Bernoulli	X			X
Cantor	X		O	X
Descartes			X	X

	1st	2nd	3rd	4th
Abel	X	X	X	O
Bernoulli	X		X	X
Cantor	X	X	O	X
Descartes			X	X

Now we have the full story. There's only one possibility left for Bernoulli, 2nd place, and only Descartes could have been in 1st place. The completed chart is shown at the left, and the order of the racers was

Descartes, Bernoulli, Cantor, Abel.

We can check our answer by making sure that this order agrees with all the information in the problem. In the finishing order we found above, Bernoulli is between Abel and Descartes. Cantor is not last, Descartes beats Abel, and no one is between Cantor and Abel. □

Problem 15.11

Below are the pairs of cities connected by direct flights by GetUThere Airlines. For each pair of cities, there are flights in both directions. Is it possible to get from Boston to Philadelphia on a series of GetUThere flights?

Boston–New York	Baltimore–Washington	Buffalo–Newark
Buffalo–Albany	Philadelphia–Erie	New York–Washington
New York–Baltimore	Erie–Newark	Albany–Philadelphia
Newark–Albany	Baltimore–Boston	Newark–Philadelphia

Solution for Problem 15.11: We know that Boston and Philadelphia are not connected by a direct flight, but we can't tell whether or not there's a sequence of flights that goes from Boston to Philadelphia. To help hunt for such a sequence, we draw a picture that includes all the flights. We make each city a point, and connect each pair of points that is connected by a direct flight. The result is shown below:

Now it's obvious that there is no way to get from Boston to Philadelphia. □

> **Sidenote:** **Graph theory** is a powerful field of mathematics that we can use to study connections between pairs of items. In Problem 15.11, we constructed a **graph** to represent connections between cities. The points in the graph for the cities are called **vertices** and the connections are **edges** of the graph.

Draw a Picture

Exercises

15.3.1:

A lumberjack can cut a log into five pieces in 20 minutes. How long would it take to cut a log of the same size and shape into seven pieces?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We might think that 5 pieces requires 5 cuts, but a quick sketch makes us realize that we only need 4 cuts. In the diagram at the right, the log is represented by a line segment, and the dots on the log are the cuts.

The lumberjack makes 4 cuts in 20 minutes, so it takes $20/4 = 5$ minutes to make each cut. Just as he needs 4 cuts to make 5 pieces, he needs 6 cuts to make 7 pieces. Each cut takes 5 minutes, so he needs $5 \cdot 6 =$ 30 minutes to cut the log into 7 pieces.

15.3.2:

The lengths of three rods are 6 cm, 10 cm, and 11 cm. How can you use these rods to measure a length of 15 cm?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We start by drawing $\overline{AB}$ with length 11 to represent the 11 cm rod. We can add the 10 cm rod to the end of this either by continuing past B, or by starting at B and going back towards A. Let's try extending past B, as shown at the right. We now have $AC = AB + BC = 21$. So, if we add the 6 cm rod starting at C and going back towards B, the other end of the rod will be $21 - 6 = 15$ cm from A, as desired.

15.3.3:

Albert's house is 5 miles east of Bella's house and 3 miles west of Carnot's house. Dolly's house is 6 miles east of Carnot's house, and 4 miles east of Ella's house. Frank's house is 5 miles north of Ella's house and 8 miles north of Greta's house. To the nearest tenth of a mile, how far apart are Belle's house and Greta's house?

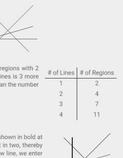
You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We draw a diagram to figure out how the houses are situated. The resulting diagram is on the right. Each person's house is represented by a point labeled with the first letter of the person's name. Therefore, B is 5 miles to the left of A, and C is 3 miles to the right of A. Since Dolly is 6 miles east of Carnot and 4 miles east of Ell, we know that Ell is $6 - 4 = 2$ miles east of Carnot. Finally, since Frank is 5 miles north of Ell and Greta is 8 miles south of Frank, we know that Greta is 3 miles south of Ell.



As shown at the right, we find that Greta is 10 miles east and 3 miles south of Belle. Applying the Pythagorean Theorem to $\triangle BEG$, we have $BG^2 = 10^2 + 3^2$, so $BG = \sqrt{109} \approx 10.4$. Therefore, to the nearest tenth of a mile, Belle's house and Greta's house are 10.4 miles apart.

15.3.4:

A man wearing red pants has red shoes. A man with blue pants has blue shoes. A man with green pants has green shoes. They exchange shoes so that each man is wearing one shoe from each of the other two men. After they leave, you only remember that the man in green pants had a red shoe on his right foot. What color shoe is each man wearing on each foot?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: We start with a quick sketch of the shoes, and we name each person based on the color of their pants. (Yes, those are shoes in the diagram.) We label Green's right shoe "Red," since we know Green's right shoe is red.

We know that Green's shoes have different colors, and that neither is green. So, his left shoe must be blue. Also, we know that Blue's right shoe must be red or green, but the red right shoe is already taken by Green. So, Blue's right shoe is green.

Only the blue shoe remains for Red's right foot. Blue's left shoe can't be green or blue, so it must be Red. Red's left shoe can't be blue (Green has that shoe) and it can't be red, so it must be green. We now have the full arrangement of shoes:

15.3.5★:

One line divides a plane into two regions. Two lines can divide a plane into at most four regions. What is the maximum number of regions possible using eight lines?

Hint: Draw a picture. Keep adding lines. Count the regions you have each time. Do you see a pattern in your results?

You may type any additional notes you have here.

Hide Solution

Reset

Your Submission: Solution

Solution: Below are diagrams for three lines and four lines.

We organize the information we have so far in the table at the right. The number of regions with 2 lines is 2 more than the number of regions with 1 line. The number of regions with 3 lines is 3 more than the number of regions with 2 lines. The number of regions with 4 lines is 4 more than the number of regions with 3 lines. Does this pattern continue?

# of Lines	# of Regions
1	2
2	4
3	7
4	11

To see why this pattern continues, consider what happens when we add a fifth line, as shown in bold at the right. We start the line from the top of the page, dividing the region where we start in two, thereby adding one more new region. Each time we cross one of the four old lines with our new line, we enter another region of the 4-line configuration. Then we cut that region into two pieces, again adding a new region. We therefore create 5 more new regions: one for the region we start in, plus one more for each of the 4 lines we cross. Two lines can only intersect in at most one point, so we can't make any additional new regions.

Similarly, we add 6 new regions when we draw the sixth line, 7 new regions when we draw the seventh line, and 8 new regions when we draw the eighth line. This gives a total of

$$2 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \boxed{37}$$

regions.

15.4 Work Backwards

In some math problems, we are given the situation at the end of a process and must figure out what happened to bring about that end. We can often solve these problems by working backwards from the end, reversing each step in the process described in the problem.

Problems

Problem 15.12 *Source: MOEMS* Jump to Solution

David buys a Beanie Baby. He later sells it to Jessica and loses $3 on the deal. Jessica makes a profit of $6 by selling it to Bryan for $25. How much did David pay for the Beanie Baby?

Problem 15.13 Jump to Solution

Danielle chooses a number. She multiplies it by 3, then adds 6, then divides by 3, and finally subtracts 6. Her end result is 4. What number did she choose?

Problem 15.14 *Source: MATHCOUNTS* Jump to Solution

The surface of a pond is being covered by oil leaking from a pipeline. The amount of surface area covered by the oil is doubling each day. At this rate, the surface of the pond will be completely covered with oil at the end of the twenty-fifth day. What fraction of the surface area of the pond will be covered by oil at the end of 20 days?

Problem 15.15 Jump to Solution

Alice gave Bob as many dollars as Bob had. Bob then gave Alice as many dollars as Alice then had. At this point, each had 24 dollars. How much did Alice have at the beginning?

Problem 15.16 *Source: MATHCOUNTS* Jump to Solution

In a sequence of five integers, the third integer is the sum of the previous two, the fourth integer is the sum of the previous three, and the fifth integer is the sum of the previous four. If the sum of the five integers is 248, what is the third integer in the sequence?

Problem 15.12

David buys a Beanie Baby. He later sells it to Jessica and loses $3 on the deal. Jessica makes a profit of $6 by selling it to Bryan for $25. How much did David pay for the Beanie Baby?

Solution for Problem 15.12: Rather than starting at the beginning by assigning a variable to the price David paid for the Beanie Baby, we start at the end. Jessica made a profit of $6 by selling the Beanie Baby for $25, so she must have bought it for $25 − $6 = $19. David lost $3 when he sold the Beanie Baby for $19, so he must have bought it for $19 + $3 = $22. □

Problem 15.13

Danielle chooses a number. She multiplies it by 3, then adds 6, then divides by 3, and finally subtracts 6. Her end result is 4. What number did she choose?

Solution for Problem 15.13: Just for fun, let's try working this problem "forwards." Let x be Danielle's number. We build an expression step by step for her final number:

Danielle's action	Resulting expression
Multiply by 3	$3x$
Add 6	$3x+6$
Divide by 3	$(3x+6)/3$
Subtract 6	$(3x+6)/3-6$

We know that this final number must equal 4, so we have an equation:

$$\frac{3x+6}{3}-6=4.$$

It would be awfully easy to make a mistake while setting up or solving this equation. Let's try going backwards and see if that's easier.

We'll start with the end result, 4, and we'll go backwards through the steps, undoing each. For example, Danielle's last step was "subtract 6." Going backwards, we add 6 to see that Danielle's number was 10 before she subtracted 6. Here's what we find when we start with 4 and go backwards through the steps, undoing each:

Danielle's action	Undo action	Resulting number
Subtract 6	Add 6	10
Divide by 3	Multiply by 3	30
Add 6	Subtract 6	24
Multiply by 3	Divide by 3	8

Therefore, Danielle started with 8. (Note that $x=8$ satisfies our earlier equation.) □

Our "work backwards" solution to Problem 15.13 is closely related to solving the equation we wrote for the problem. Here's the equation we wrote for the problem:

$$\frac{3x+6}{3}-6=4.$$

Now, let's go through the "Undo actions" from our table, in order:

Undo action	Resulting equation
Add 6	$\frac{3x+6}{3}=10$
Multiply by 3	$3x+6=30$
Subtract 6	$3x=24$
Divide by 3	$x=8$

So, our "working backwards" is essentially just solving the equation without writing the equation in the first place!

Problem 15.14 *Source: MATHCOUNTS*

The surface of a pond is being covered by oil leaking from a pipeline. The amount of surface area covered by the oil is doubling each day. At this rate, the surface of the pond will be completely covered with oil at the end of the twenty-fifth day. What fraction of the surface area of the pond will be covered by oil at the end of 20 days?

Solution for Problem 15.14: What's wrong with this solution:

> **Bogus Solution:** Since 20 days is $\frac{20}{25}=\frac{4}{5}$ of the number of days it takes to cover the whole pond, $\frac{4}{5}$ of the pond must be covered at the end of 20 days.

This solution assumes that the amount the oil spreads each day is the same. But that's not the case! For example, on the fourth day the amount covered by oil doubles, which means that it increases by the same amount as was covered on all three previous days combined.

The portion of the pond covered doubles each day when going forwards in time. So, if we go backwards in time, the amount of the pond covered is halved each day. Since the pond is completely covered at the end of the twenty-fifth day, $\frac{1}{2}$ is covered at the end of the twenty-fourth day. Going back another day, the portion covered at the end of the twenty-third day is $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$. Similarly, we have:

End of Day	Portion Covered
25	1
24	$\frac{1}{2}$
23	$\frac{1}{4}$
22	$\frac{1}{8}$
21	$\frac{1}{16}$
20	$\frac{1}{32}$

The pond is $\frac{1}{32}$ covered with oil at the end of 20 days. □

Problem 15.15

Alice gave Bob as many dollars as Bob had. Bob then gave Alice as many dollars as Alice then had. At this point, each had 24 dollars. How much did Alice have at the beginning?

Solution for Problem 15.15: Solving this with algebra would require two variables and great care in setting up equations. Let's try going backwards. They both finish with $24. In the final step, Alice receives as much money as she has. So, she doubles her money to end up with $24. Therefore, she must have previously had only $12, and Bob gave her $12 more. Since Bob has $24 dollars after giving Alice $12, he must have had $36 before giving Alice any money.

Going back one more step, we know that Alice has $12 and Bob has $36 after Alice gives Bob money. Since Alice gave Bob as much money as Bob already had, Bob doubled his money to get to $36. Therefore, he originally had $18 before Alice gave him $18 more. Alice had $12 after giving the $18 to Bob, so she started with $12 + $18 = $30 dollars.

> **Important:** It's particularly important to check our answer when working backwards to solve a complicated problem. We can usually do so by using our answer and working forwards through the problem.

We check that Alice starting with $30 and Bob with $18 does indeed end with both of them having $24:

Action	Alice's amount	Bob's amount
Start	$30	$18
Alice gives Bob the amount Bob had	$12	$36
Bob gives Alice the amount Alice had	$24	$24

Our answer checks out, so Alice started with $30. □

All four of these problems have one key feature in common. Each tells a story in which we know the end, and are trying to figure out what the situation was sometime earlier. This is a huge clue to try working backwards from the known end situation.

> **Concept:** When we know a lot about the end state of a problem, and want to know where we started, we should try working backwards.

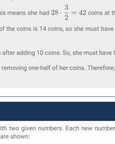
Problem 15.16 *Source: MATHCOUNTS*

In a sequence of five integers, the third integer is the sum of the previous two, the fourth integer is the sum of the previous three, and the fifth integer is the sum of the previous four. If the sum of the five integers is 248, what is the third integer in the sequence?

Solution for Problem 15.16: We know something about the end situation, so let's work backwards. The sum of all five integers is 248. But the sum of the first four integers equals the fifth integer. So, the sum of the first five integers is double the fifth integer. Therefore, the fifth integer is $248/2=124$ and the sum of the first four integers is 124.

Similarly, the fourth integer is the sum of the first three integers, so the sum of the first four integers is double the fourth integer. This makes the fourth integer $124/2=62$, and the sum of the first three integers is also 62.

We take one more step backwards. The third integer is the sum of the first two integers, so the sum of the first three integers is double the third integer. Therefore, the third integer is $62/2=31$. □



Exercises

15.4.1: *Source: AMC 8*

Granny Smith has $63. Elberta has $2 more than Anjou and Anjou has one-third as much as Granny Smith. How many dollars does Elberta have?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: Anjou has one-third as much as Granny Smith, so Anjou has $\frac{1}{3}(\$63)=\$21$. Elberta has $2 more than Anjou, so Elberta has $\$21+\$2=\boxed{\$23}$.

15.4.2:

Suppose you enter an elevator at a certain floor. Then the elevator moves up 6 floors, down 4 floors, and up 2 floors. You are then at floor 7. At which floor did you initially enter the elevator?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: You ended on floor 7 after going up 2 floors, so you were on floor 5 before going up 2 floors. You were on floor 5 after going down 4 floors, so you were on floor 9 before going down 4 floors. You were on floor 9 after going up 6 floors, so you were on floor $\boxed{3}$ before going up 6 floors.

15.4.3:

Early one morning, Joy took out one-half of the coins from her coin bank, and in the evening she put in 10 coins. The next morning, she took out one-third of the coins in the bank, and that evening she put in 4 coins. The next morning, she took out one-half the coins in the bank, leaving 16 coins. How many coins were in the bank to begin with?

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: Working backwards, Joy had 16 coins after taking one-half of the coins out, so she had $2\cdot 16=32$ coins at the start of the last day.

The previous evening, she put 4 coins in the bank, so she had $32-4=28$ coins in the coin bank before that evening.

That morning, she had these 28 coins after taking one-third of the coins out of the coin bank. Therefore, these 28 coins are $\frac{2}{3}$ of the coins she had at the start of the morning. This means she had $28\cdot\frac{3}{2}=42$ coins at the start of the morning. (We also could have reasoned that if 28 is $\frac{2}{3}$ of the coins, then $\frac{1}{3}$ of the coins is 14 coins, so she must have had $28+14=42$ coins before removing a third of them.)

The previous evening, she had these 42 coins after adding 10 coins. So, she must have had 32 coins before that evening.

Finally, these 32 coins are what she had after removing one-half of her coins. Therefore, she started out with twice this amount, $\boxed{64}$ coins.

15.4.4: *Source: AMC 8*

A list of 8 numbers is formed by beginning with two given numbers. Each new number in the list is the product of the two previous numbers. Find the first number if the last three are shown:

$$\underline{\ ?\ },\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{16},\ \underline{64},\ \underline{1024}.$$

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: Working backwards, the fifth number times 16 is 64, so the fifth number is 4:

$$\underline{\ ?\ },\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{4},\ \underline{16},\ \underline{64},\ \underline{1024}.$$

The fourth number times 4 is 16, so the fourth number is also 4:

$$\underline{\ ?\ },\ \underline{\quad},\ \underline{\quad},\ \underline{4},\ \underline{4},\ \underline{16},\ \underline{64},\ \underline{1024}.$$

Continuing backwards, we find that the third number is 1, the second number is 4, and the first number is $\boxed{\frac{1}{4}}$:

$$\underline{\tfrac{1}{4}},\ \underline{4},\ \underline{1},\ \underline{4},\ \underline{4},\ \underline{16},\ \underline{64},\ \underline{1024}.$$

15.4.5:

Serena and Visala had a combined total of $180. Serena then gave Visala $20, and then Visala gave Serena a quarter of the money Visala had. After this, they each had the same amount. How much money did Serena start with?

Preview: Solution

You may type any additional notes you have here. Hide Solution Reset

Your Submission: Solution

Solution: Since Serena and Visala together have $180 at the start of the problem, they will always have a total of $180 throughout the problem. At the end, they each have $90. Visala has this $90 after giving $\frac{1}{4}$ of her money to Serena. Therefore, $90 is $\frac{3}{4}$ of the money she had before giving any money to Serena. So, Visala had $\frac{4}{3}(\$90)=\$120$ before giving any money to Serena. Therefore, Visala had $120 after receiving $20 from Serena, which means Visala had $100 before receiving money from Serena. This means that Serena started with the other $180 − $100 = $\boxed{\$80}$.

15.5 Summary

In this chapter, we discussed four powerful problem-solving strategies. As you continue your math studies, you will use these strategies on all sorts of problems:

- Find a pattern
- Make a list
- Draw a picture
- Work backwards

All four of these strategies have a common goal:

Concept: Simplify the problem.

Challenge Problems

15.36:

Source: AMC 8 ✎ ✓

There are 24 four-digit positive integers that use each of the four digits 2, 4, 5, and 7 exactly once. Listed in numerical order from smallest to largest, what is the sum of the 8th and 16th?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: For each choice of thousands digit, there are 3 choices remaining for the hundreds digit, then 2 choices remaining for the tens digit, and then finally 1 choice for the ones digit. Therefore, there are $3 \cdot 2 \cdot 1 = 6$ numbers with each thousands digit. So, there are 6 four-digit numbers that start with 2. This means that the 7th in the list is the fifth number that starts with 4. We list the six numbers that start with 4 in increasing order:

$$5247,\ 5274,\ 5427,\ 5472,\ \boxed{5724},\ 5742.$$

15.37:

The half of the water is poured out of a full container. Then one third of the remainder is poured out. Continue the process: one fourth of the remainder, then one fifth of the remainder, then one sixth of the remainder, and so on. After how many pourings does exactly one ninth of the original water remain?

Hint: What fraction of the water remains after 1 pour? After 2 pours? After 3 pours?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: After the first pour, $\frac{1}{2}$ of the water remaining. On the next pour, $\frac{1}{3}$ is removed, leaving $\frac{2}{3}$ of the container remaining. Therefore, $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$ of the container remains. On the next pour, $\frac{1}{4}$ of the amount remaining is poured out, leaving $\frac{3}{4}$ of the container remaining. Therefore, the container is $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$ full after three pours.

We see a pattern: After the fourth pour, $\frac{1}{5}$ of the remaining water, there is $\frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}$ of the water remaining. Similarly, after the fifth pour, $\frac{1}{6}$ of the water remains. After the sixth pour, $\frac{1}{7}$ of the water remains. After the seventh pour, $\frac{1}{8}$ of the water remains. And after pour $\boxed{9}$, exactly $\frac{1}{10}$ of the water remains.

15.38:

Consider all pairs of positive integers in which both numbers are less than 10. The two integers in each pair can be the same or be different. How many different products are possible if the two integers are multiplied?

Hint: We have $2 \cdot 3 = 6 = 3 \cdot 2$ and $4 \cdot 6 = 3 \cdot 8$. So, we have to be careful, since some products will appear multiple times. How can we be organized in writing all the possible products?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Clearly all the numbers from 1 to 9 can be achieved, by pairing 1 with the appropriate number. To count others in an organized manner, we make a table. We list all the single-digit numbers as well as any instance of a number after its first appearance:

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2		4	6	8	10	12	14	16	18
3			9	12	15	18	21	24	27
4				16	20	24	28	32	36
5					25	30	35	40	45
6						36	42	48	54
7							49	56	63
8								64	72
9									81

We count 27 numbers in the table, and combining these with our 9 one-digit numbers, we have $\boxed{36}$ possible products.

15.39:

Source: AMC 8 ✎ ✓

A 3 by 3 square is divided into four 1 by 1 squares. Each of the 1 by 1 squares is to be painted either white or gray. In how many different ways can the 1 by 1 squares be painted so that no two gray square shares its top or right side with any other square? There may be as few as zero or as many as four gray squares.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: We make an organized list of pictures. First, we consider the configurations in which the bottom left square is gray, since both of the restrictions on gray squares apply in this square. Neither the square above it nor the square to its right can be white, so these two squares must also be gray. The upper right corner is both above and to the right of a gray square, so it must be gray too. This gives us the first configuration shown on the right.

Now, we let the bottom left be white and we start with the upper right corner. It must be gray, but there is no restriction on the bottom right square. This gives us the two configurations shown on the right.

Having covered the cases in which at least one of the left squares is gray, we now consider those in which the left squares are both white. The upper right corner must be gray because of the top and the upper right square white, but the other three configurations with both squares on the left white are OK.

This gives us a total of $\boxed{7}$ possible configurations.

15.40:

Source: AMC 8 ✎ ✓

A certain calculator has only two keys [×2] and [+2]. When you press one of the keys, the calculator automatically displays the result of applying the operation to the number on the screen. For example, if the number 5 is on the screen, then pressing [×2] would display "10", and pressing [+2] would display "7". What is the smallest number that can be displayed if you press the keys exactly five times?

Hint: Its not immediately obvious what four first few pressures should be if we want to get from 1 to 200 as fast as possible. If we just double 7 times, we get to 128. Then what? If it's not obvious how to proceed going forwards, ...

Hint: What is probably the last button we'll press if we get from 1 to 200 as fast as possible?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Working forwards from 1, it's not at all clear what a good strategy is to get to 200. But if we work backwards, it becomes more clear. So, instead going from 1 to 200 by adding 1 or doubling at each step, we start from 200 and work our way down to 1 by subtracting 1 or halving at each step.

If we simply halve whenever possible, we get

$$200, 100, 50, 25, 24, 12, 6, 3, 2, 1.$$

This gives us $\boxed{10}$ steps.

To see why we cannot get to 200 in 9 steps, we first note that pressing either [×2] or [+2] on the first step results in the new number being even. So, we must have pressed the operation that we pressed on the last step on the second step. Now, if we pressed 3 times, we have pressed the operation that we pressed on the last step, and we pressed 2 times. Pressing [×2] on the first step, we pressed 3 times, we get 24. What if we instead start with 2 displayed, and press [+2] three times and [×2] once? If we first press [×2] and then [+2] twice, we get $3 \cdot 2^2 = 192$. Pressing [+2] on any other step results in $\frac{1}{2}$ of the other results in the previous sentence, so we can't get to 200 in 9 total steps. The highest we can reach in 7 steps is $2^7 = 128$, so we can't reach 200 in 7 or fewer steps, either.

15.41:

The product of three positive integers (not necessarily different) is 40. How many sets of 3 integers have this property if the order of the 3 integers is a set does not matter?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: We make an organized list. We list the numbers in each set from least to greatest. We'll start with the sets that begin with 1.

$$\{1, 1, 40\}\quad \{1, 2, 20\}\quad \{1, 4, 10\}\quad \{1, 5, 8\}.$$

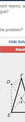
Next, we move on to those that begin with 2 (so the other two numbers multiply to 20, and 1 is not among the numbers):

$$\{2, 2, 10\}\quad \{2, 4, 5\}.$$

There are no sets in which 4 is the smallest number, since such a set would have a product of at least $4 \cdot 4 \cdot 4 = 64$. Therefore, there are $\boxed{7}$ sets of numbers that satisfy the property in the problem.

15.42:

How many paths with length 9 units are there from *A* to *B* along the grid at the right? Notice that one segment in the path along the upper left corner must be along the horizontal.



Hint: For the dots that are 1 step away from *B*, it's obvious how many paths there are to *B* that only go upward or rightward at each step. What about the dots that are 2 steps away from *B*? Or the dots that are 3 steps away from *B*?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Our path can only be 3 units, 5 units, and 7 units of length. We'll start with 3 units, 5 units, and 7 units along the grid. Therefore, each step must go up or to the right. We start backwards to count the number of such paths from A to B. We'll label points in the grid with letters so we can refer to them. To the upper right of A point, we'll place the number of ways to get from that point to point B along only upward or rightward steps. We start by putting 1's along the top and the right, since there's only one path to B from each of these points.

Then, we turn to point C. We can either go up one step from C or right one step from C. We know the number of ways to get to the point above C and the point to the right of C. So, we can add these two numbers to get the number of ways to finish from C. We get 2.

We place a 2 at C in the grid, and we continue in this manner backwards. From point D, we can either go up to C or right one step. We each we know there are 2 paths, or right, from which point we know there is 1 path. So, there are $2 + 1 = 3$ paths from D. From E, we can either go up to D from which there are 3 paths, or go right 1, from which there's one way to finish. So, there are $3 + 1 = 4$ paths to E.

Going down the next column, we see that from F, we can go right to C, from which there are 2 ways to finish, or we can go up 1, from which there is 1 way to finish. This gives $2 + 1 = 3$ paths to F. From G, we can go up to F, from which there are 3 ways to finish, or we can go right 1, from which there are 4 ways to finish. So, we have $3 + 4 = 7$ paths from G. From H, we can go up to G, or to E, so there are $4 + 4 = 10$ paths from H.

Continuing in this manner, we can work our way backwards throughout the entire grid. Notice that we are careful to remember that we cannot go up from point J in the grid. We can only go rightward from J, so the number of paths from J to B is the same as the number of paths from D to B. We find that there are $\boxed{53}$ paths from A to B.

15.43:

(a) Maria starts hiking on a path at noon, and stops at 8 p.m. He rests until noon the following day, and then starts hiking back towards his original starting point along the same path. He gets back to the starting point 8 hours after he starts walking. He walks at a constant speed on the way out and the same speed on the way back. Show that there is a point on the path that Marcia visits at the exact same time on both days.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

*Solution: We make an organized list of time (vertical) versus distance from the starting point (horizontal). The bottom of the graph is noon and the top is 8 p.m. On the outbound trip, the graph must go from the bottom left corner (noon at the starting point) to the upper right corner (8 p.m. at the end of his first hike). Her possible such graph is shown in the picture on the right. On the way back, the graph must go from the bottom right corner (noon at the second day) to point *B* in the upper left (8 p.m. back at the original starting point). Clearly the graph for the second day must intersect the graph for the first day at some point. At that time, Marcia is at the same place on both days. Points *A* and *B* are the same on both days, so Marcia is at the same place on the trail at the same time.*

Another way to think about this problem is to imagine that Marcia took both trips on the same day. Then, Marcia and his alter ego must meet on the path somewhere!

(b) Suppose Marcia only takes 4 hours on his return trip. Must there still be a point on the path that Marcia visits at exactly the same time on both days?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

*Solution: The situation is essentially the same as in part (a). The only difference here is that Marcia's return path graph ends at point *C* instead of point *B*, which is at 4 p.m., back at the original starting point. Points *A* and *C* will always be on opposite sides of Marcia's first day graph (or *C* will be on the first day graph if Marcia just stays at the start point for four hours on the second day), so the second day graph and the first must intersect.*

15.44:

In the game of Tribal, each team has three players. In my Tribal league, each player is on exactly two different teams, and no two players are represented on two different teams. What is the maximum possible number of teams in my Tribal league?

Hint: Draw a picture. Let each player be a point. What will teams be?

Hint: Each team is a triangle connecting three points. How many points do you need to satisfy the conditions in the problem?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

*Solution: We start by drawing a picture. Each player is represented by a point, and each team is represented by a triangle connecting the points of the players on that team. Suppose the players on the first team are *A*, *B*, and *C*. We draw triangle *ABC* to represent this team. Player *A* must be on another team, but *B* and *C* cannot be on this team. So, we need two more players, *D* and *E*, on this team. Draw triangle *ADE* dashed to distinguish it from team *ABC*.*

*Next, *D* must be on another team. *D* cannot be on a team with *B* or with *C*. So, we need two more players, *F* and *G*, to complete the second team. The second team is shown in *DFG*. We then form team *DGFC*, which we represent with a dotted triangle. Finally, we form team *CEF* in red.*

We therefore see that it is impossible to have more than 4 players, and the diagram at the right shows that it is possible to have a league with exactly $\boxed{4}$ players.

15.45:

Find the last two digits (the tens digit and the units digit) of 2^{2011}.

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Just as the units digit of a product depends only on the units digits of the numbers being multiplied, the last two digits of a product depend only on the last two digits of the numbers being multiplied. So, rather than computing powers of 2, we just start with 2 and only keep track of the last two digits as we multiply by 2 over and over. These are the first few results:

$$07,\ 49,\ 43,\ 03,\ 07,\ 49.$$

We see that the first two digits of powers of 2 repeat in a cycle of 4 terms. Since 2011 divided by 4 has a remainder of 3, the last two digits of 2^{2011} are the third term, which is $\boxed{43}$.

15.46:

Suppose is a country with twelve cities. Some of the cities are connected by a direct road, and there is no duplicate route. Explain why there must be three towns such that all three are directly connected to each other by the same mode of transportation.

Hint: Let each town be a point.

Hint: Is it possible for one town to be connected to exactly two towns by train and exactly two towns by airplane?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

*Solution: Let *A* be one of the cities. Only *A* must be connected to at least three of the other cities by the same mode of transportation. To see why, consider what happens if *A* is not connected to at least three of the other cities by the same mode of transportation. Then, *A* can be connected to at most 2 by train and at most 2 by airplane. That's a total of 4 connections, but *A* must be connected directly to 11 other towns. So, there must be at least 3 towns that are all directly connected to each other by the same mode of transportation.*

*Now, we need to show that *B*, *C*, and *D*, like illustrate that *A*, is connected to them by the same mode of transportation with solid line segments in the diagram. The dashed lines represent the other mode of transportation. Actually, it could also be connected to *D*, and/or *F* by solid lines, but all that matters is that it is connected to at least three of the towns in *B*, *C*, and *D*.*

*Now, we need to show that *B*, *C*, and *D* are connected by the same mode. Then, these two cities and *A* are directly connected to each other by the same mode of transportation. However, if it has not *B*, *C*, and *D* are connected by a solid line segment, then they must all be connected by dashed segment. Then, cities *B*, *C*, and *D* are directly connected to each other by the same mode of transportation.*

Therefore, there must be three towns that are directly connected to each other by the same mode of transportation.

15.47:

A *cryptarithm* is a math puzzle in which the digits in a simple equation are replaced with letters. Each digit is represented by only one letter, and each letter represents a different digit. So, for example, we might represent $53 + 50 = 101$ as $AB + BC = BCD$ in the cryptarithm *TEN TY + TONEY = SEVEN TY*, what digit does the letter *Y* represent?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

*Solution: The sum of two four-digit numbers cannot be 20000 or greater. So, if the sum of two four-digit numbers is a five-digit number, then the first digit of the five-digit number is 1. This tells us that *M* is 1. We place that in the appropriate row in the table.*

*Next, we note that the sum $S + 1$ in the thousands place, plus a carry from the hundreds place if there is one, must be 10 or greater. So, the thousands place in the sum must be 1. Therefore, the thousands digit of the sum is 1, so *O* is 0. Furthermore, we know that *S* is 9.*

*We turn to the hundreds place. There is a carry from the hundreds place to the thousands in the sum, then *E* + 9 + 9 and we have a carry from the tens to the hundreds. Then, *T*0 would have to be 1, which is already taken. Therefore, there is no carry from the hundreds place to the thousands place. This tells us that *E* is 9.*

*Still looking at the hundreds place, we see that *N* is one more than *E*, and that there must be a carry from the tens to the hundreds digit. Therefore, we have $N + N + 1$ equal to 10 plus *E*. The letter in the tens place tell us that either *E* is 5, or *E* is 6 and there is a carry in the units digit. Since *N* is already taken, we know *E* is 6.*

*Now, we know that *N* is one greater than *E*, and that there must be a carry in the units place. The greatest *D* can be is 7, since 8 and 9 are taken. So, *E* cannot be 3, since this would make *y* carry in the units place impossible. We continue with guess-and-check, trying values of *E* to see which works. We only have to check 3, 4, 5, and 6, since *E* = 7 leads to *D* being 8, which is taken.*

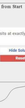
*Case 1: *E* is 3. Then *D* must be 7, which forces *Y* to be 0. But 0 is already taken, so *E* cannot be 3.*

*Case 2: *E* is 4. Then *D* must be 6 or 7, which forces *Y* to be 1 or 2. Again, this is impossible, so *E* cannot be 4.*

*Case 3: *E* is 5. Then *D* could be 6 (since this forces *Y* to be 1), but *D* can't be 7. We can complete the cryptarithm as follows:*

*Case 4: *E* is 6. This is just to make sure there aren't two solutions! Then, *N* must be 7, so *D* can only be 4 or 5. But this forces *Y* to be 0 or 1. Once again, this is impossible.*

*We conclude that the cryptarithm stands for 9567 + 1085 = 10652, so *Y* is $\boxed{2}$.*



15.48:

Five married couples get together at a party. At the start of the party each person shakes hands with everyone they didn't know before the party. After all the handshakes, Kyle, one of the husbands, asks everyone else how many hands they shook. He received each number from 0 to 8 as an answer once. How many hands did Kyle shake?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

*Solution: We draw a picture. We represent each person with a point, and we draw a segment between two points if the two people shook hands. Since Kyle shook hands with everyone except his or his spouse, the person who shook hands with 8 people must have shaken hands with everyone except his or her spouse. Let's call this person *X*, and let *Y* represent his or her spouse. *X* and *Y* shook hands with everyone else. We now have 8 points and 27 segments.*

*We note the number of hands shaken by each of *A* and *D* with numbers in parentheses.*

*Next, we note that the person who shook 7 hands must have shaken hands with everyone except *D* and his or her own spouse. We call this person *C*, and connect this person to everyone except his spouse and *D*. Now, everyone but *D* and *C*'s spouse has shaken hands at least twice, which means *C*'s spouse must be the person who shook hands with just 1 person. We'll call this spouse *D*.*

*Continuing in the same manner, the person who shook hands with 6 others must have shaken hands with everyone not labeled so far (besides his or her spouse), and the spouse of this person must be the person who shook hands with 2 others. We label these people *E* (6) and *F* (2).*

*Next, the person who shook hands with 5 others must have shaken hands with the two unlabeled people, and the spouse of this person must be the person who shook hands with 3 people. We label these people *G* (5) and *H* (3). We now have all 10 people in the remaining couple shook 4 hands. The numbers 0 through 8 were each spoken once, where Kyle asked everyone except himself. So, we know that Kyle must have shook 0s. Kyle must have the repeated number of handshakes, which is $\boxed{4}$.*

Challenge Problems

15.36:

Source: AMC 8 ✎ ✓

There are 24 four-digit positive integers that use each of the four digits 2, 4, 5, and 7 exactly once. Listed in numerical order from smallest to largest, what is the sum of the 8th and 16th?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: For each choice of thousands digit, there are 3 choices remaining for the hundreds digit, then 2 choices remaining for the tens digit, and then finally 1 choice for the ones digit. Therefore, there are $3 \cdot 2 \cdot 1 = 6$ numbers with each thousands digit. So, there are 6 four-digit numbers that start with 2. This means that the 7th in the list is the fifth number that starts with 4. We list the six numbers that start with 4 in increasing order:

$$5247,\ 5274,\ 5427,\ 5472,\ \boxed{5724},\ 5742.$$

15.37:

The half of the water is poured out of a full container. Then one third of the remainder is poured out. Continue the process: one fourth of the remainder, then one fifth of the remainder, then one sixth of the remainder, and so on. After how many pourings does exactly one ninth of the original water remain?

Hint: What fraction of the water remains after 1 pour? After 2 pours? After 3 pours?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: After the first pour, $\frac{1}{2}$ of the water remaining. On the next pour, $\frac{1}{3}$ is removed, leaving $\frac{2}{3}$ of the container remaining. Therefore, $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$ of the container remains. On the next pour, $\frac{1}{4}$ of the amount remaining is poured out, leaving $\frac{3}{4}$ of the container remaining. Therefore, the container is $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$ full after three pours.

We see a pattern: After the fourth pour, $\frac{1}{5}$ of the remaining water, there is $\frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}$ of the water remaining. Similarly, after the fifth pour, $\frac{1}{6}$ of the water remains. After the sixth pour, $\frac{1}{7}$ of the water remains. After the seventh pour, $\frac{1}{8}$ of the water remains. And after pour $\boxed{9}$, exactly $\frac{1}{10}$ of the water remains.

15.38:

Consider all pairs of positive integers in which both numbers are less than 10. The two integers in each pair can be the same or be different. How many different products are possible if the two integers are multiplied?

Hint: We have $2 \cdot 3 = 6 = 3 \cdot 2$ and $4 \cdot 6 = 3 \cdot 8$. So, we have to be careful, since some products will appear multiple times. How can we be organized in writing all the possible products?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Clearly all the numbers from 1 to 9 can be achieved, by pairing 1 with the appropriate number. To count others in an organized manner, we make a table. We list all the single-digit numbers as well as any instance of a number after its first appearance:

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2		4	6	8	10	12	14	16	18
3			9	12	15	18	21	24	27
4				16	20	24	28	32	36
5					25	30	35	40	45
6						36	42	48	54
7							49	56	63
8								64	72
9									81

We count 27 numbers in the table, and combining these with our 9 one-digit numbers, we have $\boxed{36}$ possible products.

15.39:

Source: AMC 8 ✎ ✓

A 3 by 3 square is divided into four 1 by 1 squares. Each of the 1 by 1 squares is to be painted either white or gray. In how many different ways can the 1 by 1 squares be painted so that no two gray square shares its top or right side with any other square? There may be as few as zero or as many as four gray squares.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: We make an organized list of pictures. First, we consider the configurations in which the bottom left square is gray, since both of the restrictions on gray squares apply in this square. Neither the square above it nor the square to its right can be white, so these two squares must also be gray. The upper right corner is both above and to the right of a gray square, so it must be gray too. This gives us the first configuration shown on the right.

Now, we let the bottom left be white and we start with the upper right corner. It must be gray, but there is no restriction on the bottom right square. This gives us the two configurations shown on the right.

Having covered the cases in which at least one of the left squares is gray, we now consider those in which the left squares are both white. The upper right corner must be gray because of the top and the upper right square white, but the other three configurations with both squares on the left white are OK.

This gives us a total of $\boxed{7}$ possible configurations.

15.40:

Source: AMC 8 ✎ ✓

A certain calculator has only two keys [×2] and [+2]. When you press one of the keys, the calculator automatically displays the result of applying the operation to the number on the screen. For example, if the number 5 is on the screen, then pressing [×2] would display "10", and pressing [+2] would display "7". What is the smallest number that can be displayed if you press the keys exactly five times?

Hint: Its not immediately obvious what four first few pressures should be if we want to get from 1 to 200 as fast as possible. If we just double 7 times, we get to 128. Then what? If it's not obvious how to proceed going forwards, ...

Hint: What is probably the last button we'll press if we get from 1 to 200 as fast as possible?

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Working forwards from 1, it's not at all clear what a good strategy is to get to 200. But if we work backwards, it becomes more clear. So, instead going from 1 to 200 by adding 1 or doubling at each step, we start from 200 and work our way down to 1 by subtracting 1 or halving at each step.

If we simply halve whenever possible, we get

$$200, 100, 50, 25, 24, 12, 6, 3, 2, 1.$$

This gives us $\boxed{10}$ steps.

To see why we cannot get to 200 in 9 steps, we first note that pressing either [×2] or [+2] on the first step results in the new number being even. So, we must have pressed the operation that we pressed on the last step on the second step. Now, if we pressed 3 times, we have pressed the operation that we pressed on the last step, and we pressed 2 times. Pressing [×2] on the first step, we pressed 3 times, we get 24. What if we instead start with 2 displayed, and press [+2] three times and [×2] once? If we first press [×2] and then [+2] twice, we get $3 \cdot 2^2 = 192$. Pressing [+2] on any other step results in $\frac{1}{2}$ of the other results in the previous sentence, so we can't get to 200 in 9 total steps. The highest we can reach in 7 steps is $2^7 = 128$, so we can't reach 200 in 7 or fewer steps, either.

15.41:

The product of three positive integers (not necessarily different) is 40. How many sets of 3 integers have this property if the order of the 3 integers is a set does not matter?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: We make an organized list. We list the numbers in each set from least to greatest. We'll start with the sets that begin with 1.

$$\{1, 1, 40\}\quad \{1, 2, 20\}\quad \{1, 4, 10\}\quad \{1, 5, 8\}.$$

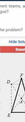
Next, we move on to those that begin with 2 (so the other two numbers multiply to 20, and 1 is not among the numbers):

$$\{2, 2, 10\}\quad \{2, 4, 5\}.$$

There are no sets in which 4 is the smallest number, since such a set would have a product of at least $4 \cdot 4 \cdot 4 = 64$. Therefore, there are $\boxed{7}$ sets of numbers that satisfy the property in the problem.

15.42:

How many paths with length 9 units are there from *A* to *B* along the grid at the right? Notice that one segment in the path along the upper left corner must be along the horizontal.



Hint: For the dots that are 1 step away from *B*, it's obvious how many paths there are to *B* that only go upward or rightward at each step. What about the dots that are 2 steps away from *B*? Or the dots that are 3 steps away from *B*?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Our path can only be 3 units, 5 units, and 7 units of length. We'll start with 3 units, 5 units, and 7 units along the grid. Therefore, each step must go up or to the right. We start backwards to count the number of such paths from A to B. We'll label points in the grid with letters so we can refer to them. To the upper right of A point, we'll place the number of ways to get from that point to point B along only upward or rightward steps. We start by putting 1's along the top and the right, since there's only one path to B from each of these points.

Then, we turn to point C. We can either go up one step from C or right one step from C. We know the number of ways to get to the point above C and the point to the right of C. So, we can add these two numbers to get the number of ways to finish from C. We get 2.

We place a 2 at C in the grid, and we continue in this manner backwards. From point D, we can either go up to C or right one step. We each we know there are 2 paths, or right, from which point we know there is 1 path. So, there are $2 + 1 = 3$ paths from D. From E, we can either go up to D from which there are 3 paths, or go right 1, from which there's one way to finish. So, there are $3 + 1 = 4$ paths to E.

Going down the next column, we see that from F, we can go right to C, from which there are 2 ways to finish, or we can go up 1, from which there is 1 way to finish. This gives $2 + 1 = 3$ paths to F. From G, we can go up to F, from which there are 3 ways to finish, or we can go right 1, from which there are 4 ways to finish. So, we have $3 + 4 = 7$ paths from G. From H, we can go up to G, or to E, so there are $4 + 4 = 10$ paths from H.

Continuing in this manner, we can work our way backwards throughout the entire grid. Notice that we are careful to remember that we cannot go up from point J in the grid. We can only go rightward from J, so the number of paths from J to B is the same as the number of paths from D to B. We find that there are $\boxed{53}$ paths from A to B.

15.43:

(a) Maria starts hiking on a path at noon, and stops at 8 p.m. He rests until noon the following day, and then starts hiking back towards his original starting point along the same path. He gets back to the starting point 8 hours after he starts walking. He walks at a constant speed on the way out and the same speed on the way back. Show that there is a point on the path that Marcia visits at the exact same time on both days.

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

*Solution: We make an organized list of time (vertical) versus distance from the starting point (horizontal). The bottom of the graph is noon and the top is 8 p.m. On the outbound trip, the graph must go from the bottom left corner (noon at the starting point) to the upper right corner (8 p.m. at the end of his first hike). Her possible such graph is shown in the picture on the right. On the way back, the graph must go from the bottom right corner (noon at the second day) to point *B* in the upper left (8 p.m. back at the original starting point). Clearly the graph for the second day must intersect the graph for the first day at some point. At that time, Marcia is at the same place on both days. Points *A* and *B* are the same on both days, so Marcia is at the same place on the trail at the same time.*

Another way to think about this problem is to imagine that Marcia took both trips on the same day. Then, Marcia and his alter ego must meet on the path somewhere!

(b) Suppose Marcia only takes 4 hours on his return trip. Must there still be a point on the path that Marcia visits at exactly the same time on both days?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

*Solution: The situation is essentially the same as in part (a). The only difference here is that Marcia's return path graph ends at point *C* instead of point *B*, which is at 4 p.m., back at the original starting point. Points *A* and *C* will always be on opposite sides of Marcia's first day graph (or *C* will be on the first day graph if Marcia just stays at the start point for four hours on the second day), so the second day graph and the first must intersect.*

15.44:

In the game of Tribal, each team has three players. In my Tribal league, each player is on exactly two different teams, and no two players are represented on two different teams. What is the maximum possible number of teams in my Tribal league?

Hint: Draw a picture. Let each player be a point. What will teams be?

Hint: Each team is a triangle connecting three points. How many points do you need to satisfy the conditions in the problem?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

*Solution: We start by drawing a picture. Each player is represented by a point, and each team is represented by a triangle connecting the points of the players on that team. Suppose the players on the first team are *A*, *B*, and *C*. We draw triangle *ABC* to represent this team. Player *A* must be on another team, but *B* and *C* cannot be on this team. So, we need two more players, *D* and *E*, on this team. Draw triangle *ADE* dashed to distinguish it from team *ABC*.*

*Next, *D* must be on another team. *D* cannot be on a team with *B* or with *C*. So, we need two more players, *F* and *G*, to complete the second team. The second team is shown in *DFG*. We then form team *DGFC*, which we represent with a dotted triangle. Finally, we form team *CEF* in red.*

We therefore see that it is impossible to have more than 4 players, and the diagram at the right shows that it is possible to have a league with exactly $\boxed{4}$ players.

15.45:

Find the last two digits (the tens digit and the units digit) of 2^{2011}.

Preview Solution

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

Solution: Just as the units digit of a product depends only on the units digits of the numbers being multiplied, the last two digits of a product depend only on the last two digits of the numbers being multiplied. So, rather than computing powers of 2, we just start with 2 and only keep track of the last two digits as we multiply by 2 over and over. These are the first few results:

$$07,\ 49,\ 43,\ 03,\ 07,\ 49.$$

We see that the first two digits of powers of 2 repeat in a cycle of 4 terms. Since 2011 divided by 4 has a remainder of 3, the last two digits of 2^{2011} are the third term, which is $\boxed{43}$.

15.46:

Suppose is a country with twelve cities. Some of the cities are connected by a direct road, and there is no duplicate route. Explain why there must be three towns such that all three are directly connected to each other by the same mode of transportation.

Hint: Let each town be a point.

Hint: Is it possible for one town to be connected to exactly two towns by train and exactly two towns by airplane?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

*Solution: Let *A* be one of the cities. Only *A* must be connected to at least three of the other cities by the same mode of transportation. To see why, consider what happens if *A* is not connected to at least three of the other cities by the same mode of transportation. Then, *A* can be connected to at most 2 by train and at most 2 by airplane. That's a total of 4 connections, but *A* must be connected directly to 11 other towns. So, there must be at least 3 towns that are all directly connected to each other by the same mode of transportation.*

*Now, we need to show that *B*, *C*, and *D*, like illustrate that *A*, is connected to them by the same mode of transportation with solid line segments in the diagram. The dashed lines represent the other mode of transportation. Actually, it could also be connected to *D*, and/or *F* by solid lines, but all that matters is that it is connected to at least three of the towns in *B*, *C*, and *D*.*

*Now, we need to show that *B*, *C*, and *D* are connected by the same mode. Then, these two cities and *A* are directly connected to each other by the same mode of transportation. However, if it has not *B*, *C*, and *D* are connected by a solid line segment, then they must all be connected by dashed segment. Then, cities *B*, *C*, and *D* are directly connected to each other by the same mode of transportation.*

Therefore, there must be three towns that are directly connected to each other by the same mode of transportation.

15.47:

A *cryptarithm* is a math puzzle in which the digits in a simple equation are replaced with letters. Each digit is represented by only one letter, and each letter represents a different digit. So, for example, we might represent $53 + 50 = 101$ as $AB + BC = BCD$ in the cryptarithm *TEN TY + TONEY = SEVEN TY*, what digit does the letter *Y* represent?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

*Solution: The sum of two four-digit numbers cannot be 20000 or greater. So, if the sum of two four-digit numbers is a five-digit number, then the first digit of the five-digit number is 1. This tells us that *M* is 1. We place that in the appropriate row in the table.*

*Next, we note that the sum $S + 1$ in the thousands place, plus a carry from the hundreds place if there is one, must be 10 or greater. So, the thousands place in the sum must be 1. Therefore, the thousands digit of the sum is 1, so *O* is 0. Furthermore, we know that *S* is 9.*

*We turn to the hundreds place. There is a carry from the hundreds place to the thousands in the sum, then *E* + 9 + 9 and we have a carry from the tens to the hundreds. Then, *T*0 would have to be 1, which is already taken. Therefore, there is no carry from the hundreds place to the thousands place. This tells us that *E* is 9.*

*Still looking at the hundreds place, we see that *N* is one more than *E*, and that there must be a carry from the tens to the hundreds digit. Therefore, we have $N + N + 1$ equal to 10 plus *E*. The letter in the tens place tell us that either *E* is 5, or *E* is 6 and there is a carry in the units digit. Since *N* is already taken, we know *E* is 6.*

*Now, we know that *N* is one greater than *E*, and that there must be a carry in the units place. The greatest *D* can be is 7, since 8 and 9 are taken. So, *E* cannot be 3, since this would make *y* carry in the units place impossible. We continue with guess-and-check, trying values of *E* to see which works. We only have to check 3, 4, 5, and 6, since *E* = 7 leads to *D* being 8, which is taken.*

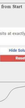
*Case 1: *E* is 3. Then *D* must be 7, which forces *Y* to be 0. But 0 is already taken, so *E* cannot be 3.*

*Case 2: *E* is 4. Then *D* must be 6 or 7, which forces *Y* to be 1 or 2. Again, this is impossible, so *E* cannot be 4.*

*Case 3: *E* is 5. Then *D* could be 6 (since this forces *Y* to be 1), but *D* can't be 7. We can complete the cryptarithm as follows:*

*Case 4: *E* is 6. This is just to make sure there aren't two solutions! Then, *N* must be 7, so *D* can only be 4 or 5. But this forces *Y* to be 0 or 1. Once again, this is impossible.*

*We conclude that the cryptarithm stands for 9567 + 1085 = 10652, so *Y* is $\boxed{2}$.*



15.48:

Five married couples get together at a party. At the start of the party each person shakes hands with everyone they didn't know before the party. After all the handshakes, Kyle, one of the husbands, asks everyone else how many hands they shook. He received each number from 0 to 8 as an answer once. How many hands did Kyle shake?

You may type any additional notes you have here.

Hide Solution | Reset

Your Submission Solution

*Solution: We draw a picture. We represent each person with a point, and we draw a segment between two points if the two people shook hands. Since Kyle shook hands with everyone except his or his spouse, the person who shook hands with 8 people must have shaken hands with everyone except his or her spouse. Let's call this person *X*, and let *Y* represent his or her spouse. *X* and *Y* shook hands with everyone else. We now have 8 points and 27 segments.*

*We note the number of hands shaken by each of *A* and *D* with numbers in parentheses.*

*Next, we note that the person who shook 7 hands must have shaken hands with everyone except *D* and his or her own spouse. We call this person *C*, and connect this person to everyone except his spouse and *D*. Now, everyone but *D* and *C*'s spouse has shaken hands at least twice, which means *C*'s spouse must be the person who shook hands with just 1 person. We'll call this spouse *D*.*

*Continuing in the same manner, the person who shook hands with 6 others must have shaken hands with everyone not labeled so far (besides his or her spouse), and the spouse of this person must be the person who shook hands with 2 others. We label these people *E* (6) and *F* (2).*

*Next, the person who shook hands with 5 others must have shaken hands with the two unlabeled people, and the spouse of this person must be the person who shook hands with 3 people. We label these people *G* (5) and *H* (3). We now have all 10 people in the remaining couple shook 4 hands. The numbers 0 through 8 were each spoken once, where Kyle asked everyone except himself. So, we know that Kyle must have shook 0s. Kyle must have the repeated number of handshakes, which is $\boxed{4}$.*