Mathematical Modelling and Controller Design of Inverted Pendulum

Přemysl Strakoš
Department of Control Systems and Instrumentation
VŠB-TU Ostrava
Ostrava, Czech Republic
premysl.strakos@vsb.cz

Jiří Tůma
Department of Control Systems and Instrumentation
VŠB-TU Ostrava
Ostrava, Czech Republic
jiri.tuma@vsb.cz

***Abstract* - The balancing of inverted pendulum along a vertical position by applying force to the cart is a typical problem in the area of automatic control. Its popularity derives from the fact that it is unstable without control. This article describes creation of the mathematical model, analyzing its behavior and state space controller design for subsequent use in programmable logic controller. Mathematical model was created using the Lagrange equations of second kind. After linearization in working point state-space model and transfer function were derived. Finally controller was designed and simulated in MATLAB Simulink.**

Keywords - Identification, modeling, simulation of processes and systems, Theory and application of control systems

I. INTRODUCTION

In this post will be described one of the common constructions – inverted pendulum on a cart. This construction arise as models of many real-world systems e.g. segway vehicles (rocking forward and backward), bicycles (rocking side to side), unicycles (both directions), rockets (during liftoff) and even skyscrapers.

The possible example of real model comprises of a cart driven by a DC motor, via a rack and pinion mechanism to ensure consistent and continuous traction. The cart is equipped with a rotary shaft to which a free turning pendulum is attached. The linear base unit system has two encoders – one encoder is used to measure the cart´s position and the other is used to sense the position of the pendulum shaft [7].

The aim is to stabilize this highly non-linear and unstable system upright always during disturbances such that the position of the cart on the track is controlled quickly and accurately. This system is an ideal platform for verifying the effectiveness of many control algorithms. The system is shown in fig. 1.

II. MODELLING OF PHYSICAL SYSTEM

For a more detailed study of the dynamic behavior of the system and subsequent controller parameters design it is necessary to identify the system, respectively create its mathematical model.

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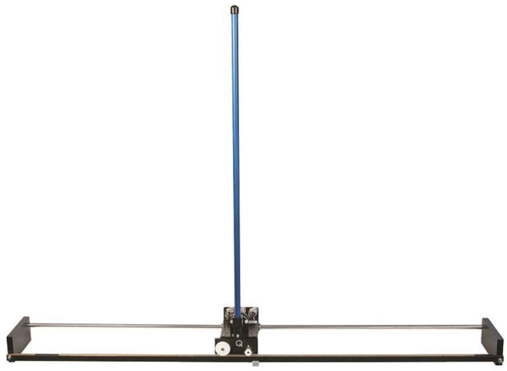

Fig. 1 Example of real model of inverted pendulum on a cart

In this paper we find the mathematical model of the system using the Lagrange equations of second kind. They represent the most widely used analytical mechanics method for creation kinetic equations of the more complex mechanical systems, which can be considered as a system of common connected mass points acting a general spatial motion.

Let us consider these system parameters:

TABLE 1 PARAMETERS OF THE SYSTEM

Parameter	Value	Unit
Mass of the cart (M)	0.5	kg
Mass of the pendulum (m)	0.2	kg
Length of the pendulum (l)	0.3	m
Coefficient of friction for the cart (b1)	0.2	$Nm^{-1}s^{-1}$
Coefficient of friction for the pendulum (b2)	0.002	$Nrad^{-1}s^{-1}$
Mass moment of inertia of the pendulum (I)	0.006	kg/m^2
Gravitation force (g)	9.81	m/s^2
Force applied to the cart (F)	-	N
Position of the cart (x)	-	m
Pendulum angle (φ)	-	rad

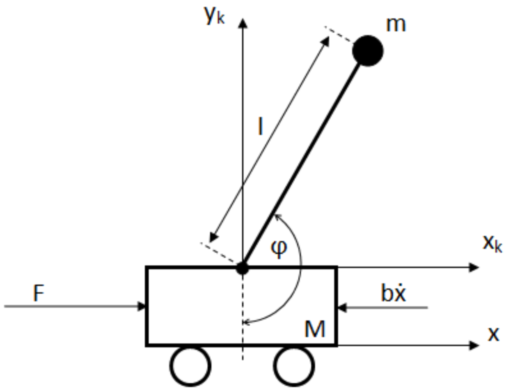

Fig. 2 The inverted pendulum system

Kinetic energy of the cart is

$$E_{KV} = \frac{1}{2}M\dot{x}^2 .$$

The coordinates x_k, y_k indicates position of the pendulum,

$$x_k = x + l\sin\varphi$$

$$y_k = l\cos\varphi .$$

Derivations of positions by the time are velocities

$$v_{kx} = \dot{x} + l\dot{\varphi}\cos\varphi$$

$$v_{ky} = -l\dot{\varphi}\sin\varphi .$$

The square of the velocity of the pendulum is

$$\left|v_k{}^2\right| = v_{kx}{}^2 + v_{ky}{}^2 = \dot{x}^2 + 2l\dot{x}\dot{\varphi}\cos\varphi + l^2\dot{\varphi}^2\cos^2\varphi + l^2\dot{\varphi}^2\sin^2\varphi ,$$

after simplifying

$$\left|v_k{}^2\right| = \dot{x}^2 + 2l\dot{x}\dot{\varphi}\cos\varphi + l^2\dot{\varphi}^2 .$$

Finally **kinetic energy** of the pendulum is

$$E_{KK} = \frac{1}{2}m\dot{x}^2 + ml\dot{x}\dot{\varphi}\cos\varphi + \frac{1}{2}ml^2\dot{\varphi}^2$$

and **total kinetic energy** of the system is

$$E_K = L = E_{KV} + E_{KK} = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}\left(I + ml^2\right)\dot{\varphi}^2 + ml\dot{x}\dot{\varphi}\cos\varphi .$$

Individual derivations of Lagrange function for position x of the cart are

$$\frac{\partial E_K}{\partial \dot{x}} = (M+m)\dot{x} + ml\dot{\varphi}\cos\varphi$$

$$\frac{d}{dt}\left(\frac{\partial E_K}{\partial \dot{x}}\right) = (M+m)\ddot{x} + ml\ddot{\varphi}\cos\varphi - ml\dot{\varphi}^2\sin\varphi$$

$$\frac{\partial E_K}{\partial x} = 0$$

$$Q_x = F - b_1\dot{x}$$

and also individual derivations of Lagrange function for the angle φ are

$$\frac{\partial E_K}{\partial \dot{\varphi}} = \left(I + ml^2\right)\dot{\varphi} + ml\dot{x}\cos\varphi$$

$$\frac{d}{dt}\left(\frac{\partial E_K}{\partial \dot{\varphi}}\right) = \left(I + ml^2\right)\ddot{\varphi} + ml\ddot{x}\cos\varphi - ml\dot{x}\dot{\varphi}\sin\varphi$$

$$\frac{\partial E_K}{\partial \varphi} = -ml\dot{x}\dot{\varphi}\sin\varphi$$

$$Q_\varphi = -mgl\sin\varphi - b_2\dot{\varphi}$$

The general Lagrange´s equation for the cart velocity x and pendulum angle φ are

$$\frac{d}{dt}\left(\frac{\partial E_K}{\partial \dot{x}}\right) - \frac{\partial E_K}{\partial x} = Q_x$$

$$\frac{d}{dt}\left(\frac{\partial E_K}{\partial \dot{\varphi}}\right) - \frac{\partial E_K}{\partial \varphi} = Q_\varphi$$

Using these two above equations and putting the individual derivations, we get after simplifying the **resultant equations** of motion.

$$\ddot{x} = -\frac{ml}{(M+m)}\ddot{\varphi}\cos\varphi - \frac{ml}{(M+m)}\dot{\varphi}^2\sin\varphi - \frac{b_1}{(M+m)}\dot{x} + \frac{F}{(M+m)}$$

$$\ddot{\varphi} = -\frac{ml}{\left(I + ml^2\right)}\ddot{x}\cos\varphi - \frac{b_2}{\left(I + ml^2\right)}\dot{\varphi} - \frac{mgl}{\left(I + ml^2\right)}\sin\varphi$$

The above equations show the dynamics of the system.

III. Linear Differential Equations

For further work, we need to linearize the system differential equations. It is a linearization around the equilibrium point, in our case about vertical position $\theta = \pi$.

The prerequisite is that the system remains in the vicinity of this position. Let ϕ represents a deviation from the equilibrium point, we get $\theta = \pi + \phi$. When pendulum is in upright position

$$\cos\theta = \cos(\pi + \phi) \approx -1$$

$$\sin\theta = \sin(\pi + \phi) \approx -\phi$$

$$\dot{\theta}^2 = \dot{\phi}^2 \approx 0 \cdot$$

Using above relation we obtained linearized equations of the system

$$\ddot{x} = \frac{ml}{M+m}\ddot{\phi} - \frac{b_1}{M+m}\dot{x} + \frac{F}{M+m}$$

$$\ddot{\phi} = \frac{ml}{I+ml^2}\ddot{x} - \frac{b_2}{I+ml^2}\dot{\phi} + \frac{mgl}{I+ml^2}\phi .$$

IV. STATE SPACE

By substituting linearized equation $\ddot{\phi}$ into $\ddot{x}$, after rearranging we get

$$\ddot{x} = \frac{F\left(I+ml^2\right) - b_1\left(I+ml^2\right)\dot{x} - mlb_2\dot{\phi} + m^2l^2g\phi}{q}$$

and for $\ddot{x}$ into $\ddot{\phi}$

$$\ddot{\phi} = \frac{mlF - mlb_1\dot{x} - b_2\left(M+m\right)\dot{\phi} + m\lg\left(M+m\right)\phi}{q},$$

where

$$q = \left[\left(I+ml^2\right)\left(M+m\right) - \left(ml^2\right)\right].$$

Now we can define state variables, x_1 through x_4

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x}_1 = \dot{x} \qquad & y_1 &= x_1 \\ x_3 &= \phi & y_2 &= x_3 \\ x_4 &= \dot{\phi}, & & . \end{aligned}$$

Finally state space model, which is general represented as

$$x(t) = Ax(t) + Bu(t)$$

$$y(t) = C^T x(t) + Du(t)$$

will appear, system matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{b_1\left(I+ml^2\right)}{q} & \frac{m^2l^2g}{q} & -\frac{mlb_2}{q} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mlb_1}{q} & \frac{m\lg\left(M+m\right)}{q} & -\frac{b_2\left(M+m\right)}{q} \end{bmatrix},$$

input, output and feedforward matrices are

$$B = \begin{bmatrix} 0 \\ \frac{I+ml^2}{q} \\ 0 \\ \frac{ml}{q} \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note u has been substituted for the input force F.

V. TRANSFER FUNCTION

At the beginning, we need the rearranged linearized equations of the system

$$\left(I+ml^2\right)\ddot{\phi} + b_2\dot{\phi} - mgl\phi = ml\ddot{x}$$

$$(M+m)\ddot{x} + b_1\dot{x} - ml\ddot{\phi} = u$$

To obtain the transfer functions of the linearized system equations, we must first take the Laplace transform of the system equations assuming zero initial conditions. The resulting Laplace transforms are shown below. [4]

$$\left(I+ml^2\right)\Phi(s)s^2 + b_2\Phi(s)s - mgl\Phi(s) = mlX(s)s^2$$

$$(M+m)X(s)s^2 + b_1X(s)s - ml\Phi(s)s^2 = U(s)$$

Transfer function represents the relationship between a single input and a single output at a time. To find first transfer function for the output $\Phi(s)$ and an input of $U(s)$ we need to eliminate $X(s)$ from the above equations. Solve the first equation for $X(s)$

$$X(s) = \left[\frac{I+ml^2}{ml} + \frac{b_2}{mls} - \frac{g}{s^2}\right]\Phi(s).$$

By substituting the above into the second equation and rearranging we get following transfer function

$$\frac{\Phi(s)}{U(s)} = \frac{\frac{ml}{q}s}{s^3 + \frac{b_1\left(I+ml^2\right)+b_2(M+m)}{q}s^2 - \frac{b_1b_2 + m\lg(M+m)}{q}s - \frac{b_1m\lg}{q}}$$

where the angle of pendulum $\Phi(s)$ is the output and

$$q = \left[\left(I+ml^2\right)\left(M+m\right) - \left(ml^2\right)\right].$$

Second, the transfer function with the cart position $X(s)$ as the output can be derived in a similar manner to arrive at the following

$$\frac{X(s)}{U(s)} = \frac{\frac{\left(I+ml^2\right)}{q}s^2 + \frac{b_2}{q}s - \frac{m\lg}{q}}{s^4 + \frac{b_1\left(I+ml^2\right)+b_2(M+m)}{q}s^3 - \frac{b_1b_2 + m\lg(M+m)}{q}s^2 - \frac{b_1m\lg}{q}s}$$

For the transfer functions above are valid these units

$$P_K(s) = \frac{\Phi(s)}{U(s)}\left[\frac{rad}{N}\right], P_V(s) = \frac{X(s)}{U(s)}\left[\frac{m}{N}\right].$$

VI. BEHAVIOR ANALYSIS

The chapter deals with simulation of dynamic behavior of the pendulum using pulse and step response, analysis of stability and conditions of system controllability and observability.

Recall that the above two transfer functions are valid only for small values of the angle *phi* where ϕ is the deviation of the pendulum from the vertically upward position. Also, the absolute pendulum angle θ is equal to $\pi + \phi$.

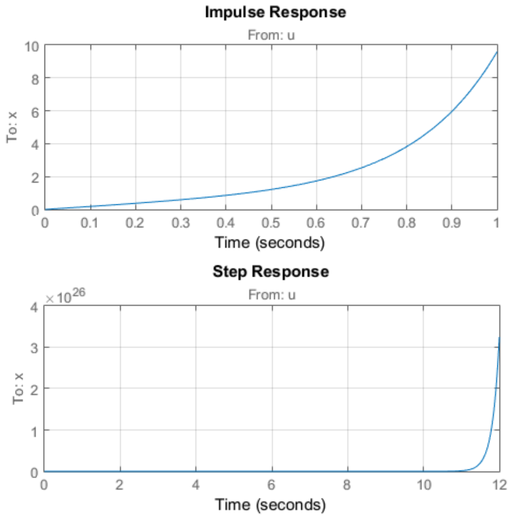

Fig. 3 Open-loop system responses

The characteristics were obtained in MATLAB by *step* and *impulse* commands. Both outputs are unstable in open-loop, which is also confirmed by poles location from function *pzmap,* one pole is situated on the right half plane. [3][5]

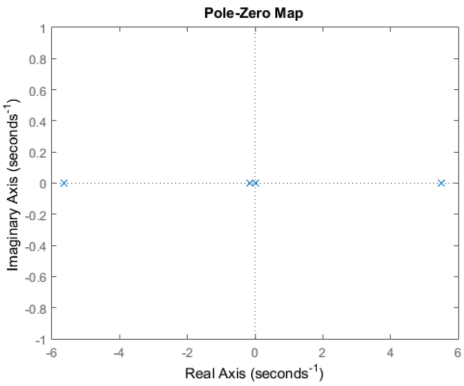

Fig. 4 Poles situation

The above results confirm our expectation. For further design of controller, we have to verify that the system is controllable. Controllability matrix must have rank *n,* which corresponds to the number of state variables of the system [6].

$$rank(Q_{CO}) = rank\begin{bmatrix} b & Ab & A^2 b & A^3 b \end{bmatrix} = n$$

Using the function *ctrb* the rank Q_{CO} is equal to 4, which means for us that the system is **controllable**.

Similarly, for the system to be completely observable, the observability matrix must have rank *n.* Matrix is defined as follows.

$$rank(Q_{OB}) = rank\begin{bmatrix} c^T \\ c^T A \\ c^T A^2 \\ c^T A^3 \end{bmatrix} = n$$

Using the function *obsv* the rank Q_{OB} is equal to 4, which means for us that our system is **observable**.

VII. CONTROLLER DESIGN

The pendulum system can be controlled using full state feedback. The schematic diagram of this type of control is shown in figure 5.

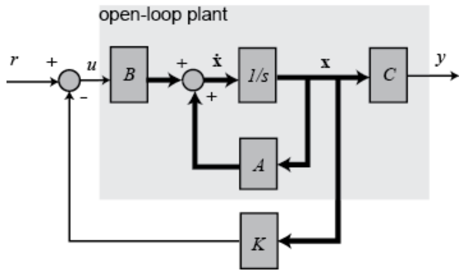

Fig. 5 Full-state feedback

The feedback control law is determined by finding the feedback gain vector K according to the desired closed loop poles location. This can be accomplished by using **pole placement** method [1][2].

The state of the system is to be feedback as an input, the controller dynamics will be

$$u = r - Kx$$
$$x = Ax + B(r - Kx) = (A - BK)x + Br$$
$$y = Cx$$

This method depends on the performance criteria, such as settling time and overshoot used in the design. The design criteria for this system are - settling time for booth outputs $T_S < 5s$ and angle overshoot $\%OS < 20$.

Using comlex domain specifications to locate dominant poles-roots of $s^2 + 2\zeta\omega s + \omega^2 = 0$. This is done by using the following formulas and finding the dominant poles at $-\sigma \pm j\omega_d$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}},\ T_S < \frac{4}{\zeta\omega_n},\ \omega_d = \omega_n\sqrt{1-\zeta^2},$$

$$\sigma = \zeta\omega_n,\ \beta = \cos^{-1}\zeta.$$

The resulting values

$$\zeta = 0{,}45595, \omega_n = 1{,}75rad, \beta = 62{,}87, \sigma = 0.798, \omega_d = 1.558.$$

The complex conjugates dominant poles are

$$S_{1,2} = -0{,}798 \pm 1{,}558i.$$

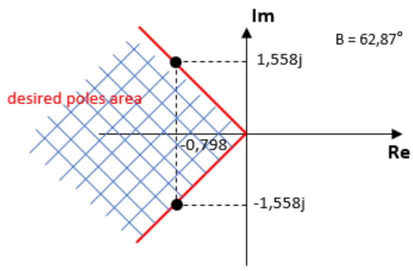

Fig. 6 Dominant poles allocation

Using *acker* function, the following gainvectors K was calculated for different sets of desired poles P:

>> *P1 =[-2.4 -1.6 -0.8 + 1.55i -0.8 - 1.55i];*
>> *K1 =[-0.2620 -0.6116 9.9077 1.3733];*

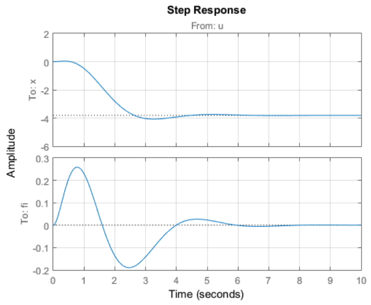

Fig. 7 Remaining poles are 2 and 3 times faster then a real part of dominant poles

>> *P2 = [-8 -4 -0.8 + 1.55i -0.8 - 1.55i];*
>> *K2 =[-2.1834 -2.1744 19.7395 3.7584];*

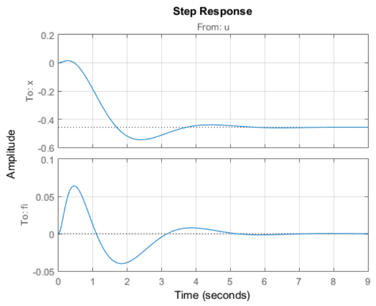

Fig. 8 Remaining poles are 5 and 10 times faster then a real part of dominant poles

>> *P3 = [-9.6 -8 -0.8 + 1.55i -0.8 - 1.55i];*
>> *K3 =[-5.2402 -4.1744 32.8561 5.7904];*

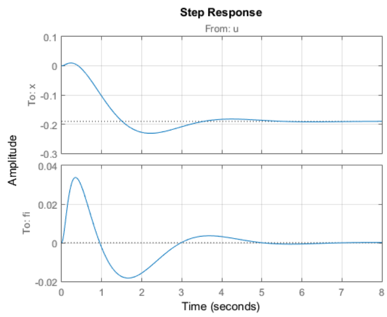

Fig. 9 Remaining poles are 10 and 12 times faster then a real part of dominant poles

Position value for the first case reaches 4 meters, which is not applicable. For other variants the selected requirements are satisfied and the system response tends to be faster when the real poles go farther to the left from the real part of the dominant poles. This requires a larger actuating signal.

VIII. Full Order State Observer Design

For dynamic systems state variables often cannot be measured. In these cases it is necessary to use an observer, which gives estimate of system states. Here is the full state observer designed, which implies that it is not possible to measure any state variable.

For our system the observer poles can be chosen *P1= -8, P2= -9, P3= -10, P4-11*. The observer matrix was couted by function *place*.

The characteristic polynomial of observer is

$$\det(sI - A_L) = s^4 + (l_1 + l_3 + 0.2879)s^3 + (0.2879 l_1 + l_2 + 0.2879 l_3 + l_4 - 31.17)s^2 + (-31.17 l_1 - 0.3485 l_2 + 2.6879 l_3 + 0.1727 l_4 - 4.4545)s - 4.4545 l_1 - 31.18 l_2 + 2.6727 l_4$$

where $A_L = A - lc^T$

Comparing the coefficients in the same squared complex variable S of observer characteristic polynomial we gets a system of linear algebraic equations respected the unknown parts l_1, l_2, l_3 and l_4 of correction vector i.e.

$$\left.\begin{aligned} &-4.4545 l_1 - 31.18 l_2 + 2.6727 l_4 = 4096 \\ &-31.17 l_1 - 0.3485 l_2 + 2.6879 l_3 + 0.1727 l_4 = 2052{,}4545 \\ &0.2879 l_1 + l_2 + 0.2879 l_3 + l_4 = 415.17 \\ &l_1 + l_3 = 31.7121 \end{aligned}\right\} \Rightarrow \begin{aligned} l_1 &= -54.7729 \\ l_2 &= -81.7303 \\ l_3 &= 86.4850 \\ l_4 &= 487.7704 \end{aligned}$$

The simulation results are shown below in fig. 10 and 11.

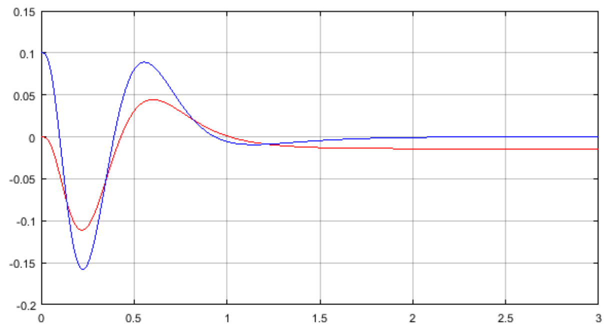

Fig. 10 System responses with full-state observer angle [rad], cart position [m]

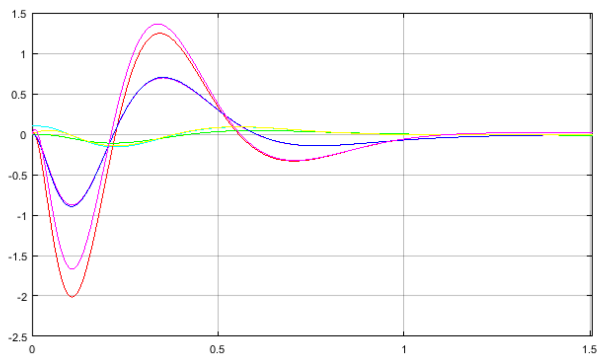

Fig. 11 Comparison of real states and their estimations angle [rad]/estimate[rad]; cart position [m]/estimate[m]; angular velocity[rad/s]/estimate[rad/s]; velocity[m/s]/estimate[m/s]

IX. Conclusion

The post dealt with mathematical modeling of linear inverted pendulum and its controller design. Firstly, the details of model and parameters were chosen and then using Lagrange equations of second kind was derived equations of motion of the system. These equations were verified by simulation. Because it is a non-linear system, the linearization close to the vertical position was done.

By adjusting the linearized system equations were derived state-space matrices and transfer functions. Then the stability was checked and ranks of controllability and observability matrices were calculated. The system was unstable, but controllable and observable.

By pole-placement method were calculated three variants of status feedback, the first was not applicable due to excessive position of the cart. In case it is not possible to measure the conditions full state observer was designed.

During the simulation was observer initial conditions set at zero and the pendulum was set at non-zero initial angle. The regulation process showed that the control was set up correctly.

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References

[1] MODRLÁK, O. *Non-linear Systems.* Studying Materials. Liberec. Technical University of Liberec, Faculty of Mechartonics, 2004. 38 s.

[2] KUREČKOVÁ, E. *Selected Methods of Non-linear Control.* Zlín. Tomáš Baťa University in Zlín, Faculty of Applied Informatics, 2013. 71 s. Diploma thesis, Thesis head: Dostál, P.

[3] KUPKA, L.*Matlab & Simulink, Studijní Study materials for Fundamentals of Cybernetics*. Liberec: Technical University of Liberec. 2008. 224 s.

[4] VÍTEČKOVÁ, M., VÍTEČEK, A. *Fundamentals of Automatic Control*. Ostrava : VŠB – Technical University of Ostrava. 2008. 244 s. ISBN 978-80-248-1924-2, 2. edition.

[5] NOSKIEVIČ, P . *Modeling and Systems Identification*. 1999. Ostrava : MONTANEX, 1999. 276 s. ISBN 80-7225-030-2, 1.reprint.

[6] SKALICKÝ, J.*Theory of Control*, UVEE FEKT VUT in Brno, 2002, 98 s., ISBN: 80-214-2112-6

[7] MAKOVÝ, V. *Design and Realization of Inverted Pendulum: master thesis.* Ostrava: VŠB – Technical University of Ostrava, Faculty of Mechanical Engineering, The Department of Control Systems and Instrumentation, 2014, 48 p. Thesis head: Wagnerová, R.