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APPLIED MATHEMATICS - III

Semester III – Computer Engineering

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INDEX

Table of Contents

□□□

Applied Mathematics-III

Statistical Analysis

Chapter 1	25 Marks	19 Marks	12 Marks	18 Marks	23 Marks	17 Marks
Chapter 2	23 Marks	31 Marks	31 Marks	27 Marks	31 Marks	27 Marks
Chapter 3	25 Marks	31 Marks	29 Marks	33 Marks	21 Marks	33 Marks
Chapter 4	35 Marks	25 Marks	31 Marks	26 Marks	25 Marks	28 Marks
Chapter 5	14 Marks	14 Marks	11 Marks	11 Marks	12 Marks	18 Marks
Repeated Questions	–	–	–	06 Marks	19 Marks	12 Marks

Dec. 2013

Chapter 1 : Complex Variable and Mapping [Total Marks - 25]

Q. 1(b) Find the constant a, b, c, d and e, if $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic. **(5 Marks)**

Ans. : We have $f(z) = u + iv$

$$\therefore \quad u = ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2 \qquad v = 4x^3y - exy^3 + 4xy$$

$$\therefore \quad u_x = 4ax^3 + 2bxy^2 + 2dx \qquad \text{and} \qquad u_y = 2bx^2y + 4cy^3 - 4y$$

$$\text{Also} \quad v_x = 12x^2y - ey^3 + 4y \qquad v_y = 4x^3 - 3exy^2 + 4x$$

Since f(z) is analytic, it satisfies C-R equations.

$$\therefore \quad u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

Thus using $u_x = v_y$

$$4ax^3 + 2bxy^2 + 2dx = 4x^3 - 3exy^2 + 4x$$

Equating coefficients of x^3, xy^2 and x, we get

$$4a = 4; \qquad 2b = -3e, \qquad 2d = 4$$

$$a = 1 \qquad 2b = -3e \text{ and } d = 2 \qquad \text{...(1)}$$

Also, $u_y = -v_x$

$$2bx^2y + 4cy^3 - 4y = -[12x^2y - ey^3 + 4y]$$

$$\therefore \quad 2bx^2y + 4cy^3 = -12x^2y + ey^3$$

Comparing coefficient of x^2y and y^3, we get

$$2b = -12 \quad \text{and} \quad 4c = e \qquad b = -6$$

$$\text{if} \quad b = -6 \Rightarrow c = \frac{-2}{3}b = \frac{-2}{3}(-6) = 4 \qquad \text{and} \quad 4c = e$$

$$\therefore \quad c = 1$$

Thus we get $\mathbf{a = 1, \quad b = -6, \quad c = 1, \quad d = 2, \quad e = 4}$

Q. 2(a) Find the analytic function $f(z) = u + iv$ if $3u + 2v = y^2 - x^2 + 16xy$. **(6 Marks)**

Ans. : $u + iv = 3u + 2v = y^2 - x^2 + 16xy$

$$u_x = v_y \text{ and } u_y = -v_x \qquad \ldots(1)$$

Differentiating partially w.r.t 'x'

$$\therefore 3u_x + 2v_x = -2x + 16y \qquad \ldots(2)$$

Similarly, differentiating '3u + 2v' partially w.r.t. 'y'.

$$\therefore 3u_y + 2u_y = 2y + 16x$$

$$\therefore -3v_x + 2u_x = 16x + 2y \qquad \ldots(3)$$

Multiply (2) by '2' and (3) by 3 and then subtract

$$0 + 13v_x = -52x + 26y$$

$$\therefore v_x = -4x + 2y \qquad \ldots(4)$$

Substitute Equations (4) in (2)

$$\therefore 3u_x + 2(-4x + 2y) = -2x + 16y; \qquad \therefore 3u_x - 8x + 4y = -2x + 16y$$

$$\therefore 3u_x = 6x + 12y; \qquad \therefore u_x = 2x + 4y$$

$$\therefore f'(z) = u_x + i v_x = (2x + 4y) + i(-4x + 2y)$$

By Milne Thompson's method, put x = z and y = 0

$$\therefore f'(z) = (2z + 0) + i(-4z + 0) = 2z - 4iz = 2z(1 - 2i)$$

$$\therefore f(z) = \int f'(z)\,dz = \int 2z(1 - 2i)\,dz = 2(1 - 2i)\cdot\frac{z^2}{2} + k = (1 - 2i)z^2 + k$$

Q. 3(b) Find the orthogonal trajectories of the family of curve $e^{-x}\cos y + xy = \alpha$ where α is the real constant in xy – plane. **(6 Marks)**

Ans. :

$$\text{Let } u = e^{-x}\cos y + xy - \alpha$$

$$\therefore u_x = -e^{-x}\cos y + y \text{ and } u_y = -e^{-x}\sin y + x$$

Let f(z) = u + iv be analytic function.

$$\therefore f'(z) = u_x - i u_y = (-e^{-x}\cos y + y) - i(-e^{-x}\sin y + x)$$

By Miline Thomposn's method, put x = z and y = 0

$$\therefore f'(z) = -e^{-z}\cdot 1 + 0 - i(-0 + z) = -e^{-z} - iz$$

$$\therefore f(z) = f'(z)\,dz = \int f'(z)\,dz = \int(-e^{-z} - iz)\,dz = e - z - \frac{iz^2}{2} + c$$

$$\therefore u + iv = e^{-(x+iy)} - \frac{i}{2}(x + iy)^2 + c = e^{-x}e^{-iy} - \frac{i}{2}(x^2 + 2xi\,y + i^2y^2) + c$$

$$= e^{-x}(\cos y - i\sin y) - \frac{i}{2}(x^2 - y^2) - \frac{i}{2}\times i\,2xy + c$$

$$= (e^{-x}\cos y + xy) + i\left[-e^{-x}\sin y - \frac{1}{2}(x^2 - y^2)\right] + c = u + iv$$

$\therefore$ Imaginary parts are

$$u = e^{-x}\cos y + xy - \alpha \text{ and } v = -e^{-x}\sin y - \frac{1}{2}(x^2 - y^2)$$

Thus the required orthogonal trajectories are

$$-e^{-x}\sin y - \frac{1}{2}(x^2 - y^2) = \beta$$

$$\text{i.e. } -\frac{1}{2}[2e^{-x}\sin y + x^2 - y^2] = \beta$$

$$\text{i.e. } 2e^{-x}\sin y + x^2 - y^2 = \gamma$$

Q. 6(c) Find the image of the circle $x^2 + y^2 = 1$, under the transformation $w = \dfrac{5-4z}{4z-2}$. **(8 Marks)**

Ans. :

Let $w = u + iv$; $\quad z = x + iy,\ x^2 + y^2 = |z|$

Given $w = \dfrac{5-4z}{4z-2}$; $\quad \therefore w(4z-2) = (5-4z)$

$$4wz - 2w = 5 - 4z; \qquad 4wz + 4z = 5 + 2w$$

$$4z(w+1) = 5 + 2w; \qquad z = \frac{5+2w}{4(w+1)}$$

$$|z| = \frac{|5+2w|}{4|w+1|}$$

Given, $|z| = 1$; $\quad \therefore |5+2w| = 4|w+1|$

$$|5 + 2(u+iv) = 4|(u+iv)+1|$$

$$\therefore |(5+2u) + 2iv| = 4|(u+1)+iv|$$

$$\sqrt{(5+2u)^2 + (2v)^2} = 4\left(\sqrt{(u+1)^2 + v^2}\right)$$

Squaring both sides

$$(5+2u)^2 + (2v)^2 = 16((u+1)^2 + v^2)$$

$$25 + 4u^2 + 20u + 4v^2 = 16[u^2 + 2u + 1 + v^2]$$

$$25 + 4u^2 + 20u + 4v^2 = 16u^2 + 32u + 16 + 16v^2$$

$$12u^2 + 12v^2 + 12u - 9 = 0$$

$$u^2 + u + v^2 - \frac{3}{4} = 0$$

$$u^2 + u + \frac{1}{4} + v^2 - 1 = 0$$

$$\left(u + \frac{1}{2}\right)^2 + v^2 = 1$$

It is a circle with radius 1 and centre $\left(\dfrac{-1}{2}, 0\right)$

Chapter 2 : Laplace Transform [Total Marks - 23]

Q. 1(a) Find $L^{-1}\left[\dfrac{e^{4-3s}}{(s+4)^{5/2}}\right]$. **(5 Marks)**

Ans. :

$$F(s) = \frac{e^{4-3s}}{(s+4)^{5/2}}$$

$$\text{Let } f(t) = L^{-1}[F(s)] = L^{-1}\left\{\frac{e^{4-3s}}{s^{5/2}}\right\} = e^{-4t} L^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{5/2}}\right\} = e^{-4t}\frac{t^{3/2}}{\overline{|5/2}}$$

$$= e^{-4t}\frac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\overline{|1/2}} = \frac{4e^{-4t}\, t^{3/2}}{3\sqrt{\pi}}$$

$$f(t) = \frac{4}{3\sqrt{\pi}} e^{-4(t-4)} (t-3)^{3/2} H(t-3)$$

Q. 3(a) Find : (i) $L^{-1}\left\{\frac{s}{(2s+1)^2}\right\}$; (ii) $L^{-1}\left\{\log\frac{s^2+a^2}{\sqrt{s+b}}\right\}$. **(6 Marks)**

Ans. :

(i) Let $L^{-1}\left[\frac{s}{(2s+1)^2}\right] = L^{-1}\left[\frac{s}{2^2(s+1/2)^2}\right] = \frac{1}{4}L^{-1}\left[\frac{(s+1/2)-1/2}{(s+1/2)^2}\right]$

$$= \frac{1}{4}e^{-t/2}L^{-1}\left[\frac{s-1/2}{s^2}\right] \quad \text{...(First shifting property)}$$

$$= \frac{1}{4}e^{-t/2}L^{-1}\left[\frac{s}{s^2}-\frac{1/2}{s^2}\right] = \frac{e^{-t/2}}{4}\left\{L^{-1}\left[\frac{1}{s}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{s^2}\right]\right\}$$

$$= \frac{1}{4}e^{-t/2}\left\{1-\frac{1}{2}t\right\} = \frac{2-t}{8}e^{-t/2}$$

(ii) $$F(s) = \log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right) = \log(s^2+a^2) - \frac{1}{2}\log(s+b)$$

$$\therefore -t f(t) = L^{-1}\left\{\frac{d}{ds}(s^2+a^2) - \frac{1}{2}\frac{d}{ds}(s+b)\right\} = L^{-1}\left[\frac{2s}{s^2+a^2}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{s+b}\right]$$

$$= 2\cos at - \frac{1}{2}e^{-bt}$$

$$\therefore \quad f(t) = \frac{\frac{1}{2}e^{-bt} - 2\cos at}{t}$$

Q. 5(a) Solve using Laplace transform $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-x}$, $y(0) = 1$. **(6 Marks)**

Ans. :

$$L\left[\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y\right] = L[e^{-t}]$$

$$\therefore \quad L\left[\frac{d^2y}{dx^2}\right] + 4L\left[\frac{dy}{dx}\right] + 3L[y] = \frac{1}{s+1}$$

$$\therefore [s^2\bar{y} - s\,y(0) - y'(0)] + 4[s\bar{y} - y(0)] + 3\bar{y} = \frac{1}{s+1}$$

$$\therefore \quad s^2\bar{y} - s(1) - 1 + 4s\bar{y} - 4(1) + 3\bar{y} = \frac{1}{s+1}$$

$$\therefore \quad s^2\bar{y} - s - 1 + 4s\bar{y} - 4 + 3\bar{y} = \frac{1}{s+1}$$

$$\therefore \quad \bar{y}(s^2+4s+3) = \frac{1}{s+1} + s + 5$$

$$\therefore \quad \bar{y}(s+1)(s+3) = \frac{1 + s(s+1) + 5(s+1)}{(s+1)}$$

$$\therefore \quad \bar{y} = \frac{1+s^2+s+5s+5}{(s+1)(s+1)(s+3)}$$

$$\therefore Y(s) = \left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right]$$

By partial fractions, $\dfrac{s^2+6s+6}{(s+1)^2(s+3)} = \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+3}$

$$\therefore \quad B = \left.\frac{s^2+6s+6}{s+3}\right|_{s=-1} = \frac{(-1)^2+6(-1)+6}{(-1+3)} = \frac{1}{2}$$

$$\therefore C = \left.\frac{s^2+6s+6}{(s+1)^2}\right|_{s=-3} = \frac{(-3)^2+6(-3)+6}{(-3+1)^2} = \frac{-3}{4}$$

$$\frac{0+0+6}{(0+1)^2(0+3)} = \frac{A}{0+1} + \frac{1/2}{(0+1)^2} + \frac{3/4}{0+3}$$

$$2 = A + \frac{1}{2} - \frac{1}{4}\,; \qquad \therefore A = 2 - \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

$$y = L^{-1}\left[\frac{7/4}{s+1} + \frac{1/2}{(s+1)^2} - \frac{3/4}{s+3}\right]$$

$$= \frac{7}{4} L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{(s+1)^2}\right] - \frac{3}{4} L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{7}{4} e^{-t} + \frac{1}{2} e^{-t} L^{-1}\left[\frac{1}{s^2}\right] - \frac{3}{4} e^{-3t} \qquad \text{...(First shifting property)}$$

$$= \frac{7}{4} e^{-t} + \frac{1}{2} e^{-t}\, t - \frac{3}{4} e^{-3t}$$

$$= \frac{1}{4}(7e^{-t} + 2t\,e^{-t} - 3e^{-3t})$$

Q. 6(a) Find the inverse Laplace transform by using convolution theorem

$$L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right\}.$$ **(6 Marks)**

Ans. :

$$F(s) = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \frac{s^2+2s+1+2}{(s^2+2s+1+4)(s^2+2s+1+1)}$$

$$= \frac{(s+1)^2+2}{((s+1)^2+4)((s+1)^2+1)}$$

$$\therefore f(t) = e^{-t} L^{-1}\left\{\frac{s^2+2}{(s^2+4)(s^2+1)}\right\} = e^{-t} L^{-1}\left\{\frac{s}{s^2+4}\cdot\frac{s}{s^2+1} + \frac{2}{s^2+4}\cdot\frac{1}{s^2+1}\right\}$$

$$= e^{-t} L^{-1}\{F_1(s)\,G_1(s) + F_2(s)\,G_2(s)\} = e^{-t}\{\cos 2t \ \cos t + \sin 2t \sin t\}$$

$$= e^{-t}\left\{\int_0^t \cos 2u \cos(t-u)\,dy + \int_0^t \sin 2u \sin(t-u)\right\}$$

Solving the integrals, we get,

$$f(t) = \frac{e^{-t}}{3}\{2\sin 2t - \sin t + 2\sin t - \sin 2t\} = \frac{e^{-t}}{3}(\sin 2t + \sin t)$$

Chapter 3 : Fourier Series [Total Marks - 25]

Q. 1(c) Obtain half range Fourier cosine series for $f(x) = \sin x, x, x \in (0, \pi)$. **(5 Marks)**

Ans. : For half range cosine series $b_n = 0$

Here, $l = \pi$;

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{\pi}\int_0^\pi \sin x\,dx = \frac{2}{\pi}[-\cos x]_0^\pi = \frac{2}{\pi}[-\cos\pi + \cos 0] = \frac{2}{\pi}[1+1] = \frac{4}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{\pi}\int_0^\pi \sin x \cos\frac{n\pi x}{\pi}dx \qquad ...(1)$$

$$= \frac{1}{\pi}\int_0^\pi [\sin(nx+x) - \sin(nx-x)]dx = \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^\pi$$

$$= \frac{1}{\pi}\left\{\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right] - \left[\frac{-\cos 0}{n+1} + \frac{\cos 0}{n-1}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{+(-1)^\pi}{n+1} + \frac{-(-1)^n}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right\} \qquad [\because \cos(n\pm1)\pi = -(-1)^n]$$

$$= \frac{1}{\pi}\left\{(-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right] + \left[\frac{1}{n+1} - \frac{1}{n-1}\right]\right\}$$

$$= \frac{1}{\pi}[(-1)^n + 1]\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = \frac{(-1)^n + 1}{\pi} \times \left[\frac{n-1-n-1}{(n-1)(n+1)}\right]$$

$$= \frac{(-1)^n + 1}{\pi} \times \frac{-2}{n^2-1} = \frac{-2[(-1)^n + 1]}{\pi(n^2-1)} \quad (n \neq 1)$$

Put $n = 1$ in Equation (1)

$$a_1 = \frac{1}{\pi}\int_0^1 2\cos\cdot x \sin x \cdot dx = \frac{1}{\pi}\left[\frac{\cos 2x}{2}\right]_0^\pi = \frac{1}{\pi}\left[\frac{\cos 2\pi}{2} - \frac{\cos 0}{2}\right] = \frac{1}{\pi}\left[\frac{1}{2} - \frac{1}{2}\right] = 0$$

$\therefore$ Half range Fourier cosine series is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$\therefore f(x) = \frac{a_0}{2} + a_1\cos\frac{1\pi x}{\pi} + \sum_{n=2}^{\infty} a_n\cos\frac{n\pi x}{\pi}$$

$$\therefore \sin x = \frac{2}{\pi} + 0 + \sum_{n=2}^{\infty}\frac{-2[(-1)^\pi + 1]}{\pi(n^2-1)}\cos nx = \frac{2}{\pi} - \frac{2}{\pi}\left[\frac{2\cos 2x}{2^2-1} + 0 + \frac{2\cos 4x}{4^2-1} + \dots\right]$$

$$\therefore \sin x = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \dots\right]$$

Q. 2(c) Obtain Fourier series expansion for $f(x) = \sqrt{1 - \cos x}$, $x \in (0, 2\pi)$ and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$. **(8 Marks)**

Ans.: Let $f(x) = \sqrt{1 - \cos x} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Here, $f(x) = \sqrt{2} \cdot \sin \frac{x}{2}$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{2} \cdot \sin \frac{x}{2}\, dx$$

$$= \frac{1}{\sqrt{2} \cdot \pi} \left[-2 \cos \frac{x}{2} \right]_0^{2\pi} = \frac{1}{\sqrt{2} \cdot \pi} [-2(-1-1)] = \frac{4}{\sqrt{2} \cdot \pi} = \frac{2\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{2\pi} \sqrt{2} \cdot \sin \frac{x}{2} \cdot \cos nx\, dx$$

$$= \frac{\sqrt{2}}{2\pi} \int_0^{2\pi} \left[\sin\left(\frac{1}{2} + n\right) x + \sin\left(\frac{1}{2} + n\right) x \right] dx$$

$$= \frac{\sqrt{2}}{2\pi} \left[-\frac{2}{1+2n} \cos\left(\frac{1+2n}{2}\right) x - \frac{2}{1-2n} \cos\left(\frac{1-2n}{2}\right) x \right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi} \left[-\frac{1}{2n+1} \cos\left(\frac{2n+1}{2}\right) x - \frac{1}{2n-1} \cos\left(\frac{2n-1}{2}\right) x \right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi} \left[\frac{2}{2n+1} - \frac{2}{2n-1} \right] = -\frac{4\sqrt{2}}{\pi(4n^2 - 1)}$$

$$b_n = \frac{1}{x} \int_0^{2\pi} f(x) \sin nx\, dx = \frac{1}{x} \int_0^{2\pi} \sqrt{2} \sin \frac{x}{2} \sin nx\, dx$$

$$= \frac{-\sqrt{2}}{2\pi} \int_0^{2\pi} \left[\cos\left(\frac{1}{2} + n\right) x - \cos\left(\frac{1}{2} - n\right) x \right] dx$$

$$= \frac{-\sqrt{2}}{2\pi} \int_0^{2\pi} \left[\frac{2}{1+2n} \sin\left(\frac{1+2n}{2}\right) x - \frac{2}{1-2n} \sin\left(\frac{1-2n}{2}\right) x \right]_0^{2\pi}$$

$$= \frac{-\sqrt{2}}{2\pi} \left[\frac{1}{2n+1} \sin\left(\frac{2n+1}{2}\right) x - \frac{1}{2n-1} \sin\left(\frac{2n-1}{2}\right) x \right]_0^{2\pi} = 0$$

$\therefore$ Putting these valued in (1)

$$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cdot\cos nx$$

Putting $x = 0$, $\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$

Q. 4(b) Find the half range sine series for the function $f(x) = \begin{cases} 2kx/l & 0 \le x \le l/2 \\ 2k/l^{(l-x)} & l/2 \le x \le l \end{cases}$ **(6 Marks)**

Ans. :

$$f(x) = \frac{2kx}{l}\ 0 \le x \le \frac{l}{2} = \frac{2k}{l}(l-x)\cdot\frac{l}{2} \le x \le l$$

let $f(x) = \sum b_n \sin\frac{n\pi x}{l}$; $\qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

$$b_n = \frac{2}{l}\left[\int_0^{\frac{l}{2}}\frac{2kx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{\frac{l}{2}}^{l}\frac{2k}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\frac{2k}{l}\int_0^{\frac{l}{2}}x\sin\frac{n\pi x}{l}\,dx + \frac{2k}{l}\int_{\frac{l}{2}}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4k}{l^2}\left\{\left[x\cdot\frac{l}{n\pi}\cdot\left(-\cos\frac{n\pi x}{l}\right) - \int\frac{l}{n\pi}\cdot\left(-\cos\frac{n\pi x}{l}\right)dx\right]_0^{l/2}\right.$$
$$\left. + \left[(l-x)\frac{l}{n\pi}\cdot\left(-\cos\frac{n\pi x}{l}\right) - \int\frac{l}{n\pi}\cdot(-1)\left(-\cos\frac{n\pi x}{l}\right)dx\right]_{l/2}^{l}\right\}$$

$$= \frac{4k}{l^2}\left\{\left[x\cdot\frac{l}{n\pi}\cdot\left(-\cos\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\cdot\sin\frac{n\pi x}{l}\right]_0^{l/2} + \left[(l-x)\cdot\frac{l}{n\pi}\cdot\left(-\cos\frac{n\pi x}{l}\right) - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_{l/2}^{l}\right\}$$

$$= \frac{4k}{l^2}\left[\frac{l^2}{2n\pi}\left(-\cos\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - 0\right] + \left[0 - \frac{l}{2}\cdot\frac{l}{n\pi}\left(-\cos\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4k}{l^2}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l} = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$$

$$f(x) = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l} - \frac{1}{7^2}\sin\frac{7\pi x}{l} + \ldots\right]$$

Q. 5(b) Express $f(x) = \frac{\pi}{2} e^{-x} \cos x$ for $x > 0$ as Fourier sine integral and show that

$$\int_0^\infty \frac{\omega^3 \sin \omega x}{\omega^4 + 4} dw = \frac{\pi}{2} e^{-x} \cos x.$$ **(6 Marks)**

Ans. :

Since, the given function f(x) is an odd function by using fourier sine integral.

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin \omega x \int_0^\infty f(s) \sin \omega s \, ds \, d\omega = \frac{2}{\pi}\int_0^\infty \sin \omega x \int_0^\infty \frac{\pi}{2} e^{-s} \cos s \cdot \sin \omega s \, ds \, d\omega \times \frac{2}{2}$$

$$= \frac{1}{2}\int_0^\infty \sin \omega x \int_0^\infty e^{-s} [\sin (s + \omega s) - \sin (s - \omega s)] \, ds \, d\omega$$

$$= \frac{1}{2}\int_0^\infty \sin \omega x \left\{ \int_0^\infty e^{-s} \sin (1 + \omega) s \, ds - \int_0^\infty e^{-s} \sin (1 - \omega) s \, ds \right\} d\omega$$

$$= \frac{1}{2}\int_0^\infty \sin \omega x \left\{ \frac{e^{-s}[-\sin (1 + \omega)s - (1 + \omega) \cos (1 + \omega)s]}{1 + (\omega + 1)^2} - \frac{e^{-s}[-1\sin (1 - \omega) s - (1 - \omega) \cos (1 - \omega)s]}{(-1)^2 + (1 - \omega)^2} \right\}_0^\infty d\omega$$

$$= \frac{1}{2}\int_0^\infty \sin \omega x \left\{ (0 - 0) - \left(\frac{e^0 [-0 - (1 + \omega) \times 1]}{1 + \omega^2 + 2\omega + 1} - \frac{e^{-0}[-0 - (1 - \omega) \times 1]}{1 + \omega^2 - 2\omega + 1} \right) \right\} d\omega$$

$$= \frac{1}{2}\int_0^\infty \sin \omega x \left\{ \frac{1 + \omega}{\omega^2 + 2\omega + 2} + \frac{-(1 - \omega)}{\omega^2 - 2\omega + 2} \right\} d\omega$$

$$= \frac{1}{2}\int_0^\infty \sin \omega x \left\{ \frac{(1 + \omega)(\omega^2 - 2\omega + 2) - (1 - \omega)(\omega^2 + 2 + 2)}{(\omega^2 + 2 + 2\omega)(\omega^2 + 2 - 2\omega)} \right\} d\omega$$

$$= \frac{1}{2}\int_0^\infty \sin \omega x \left\{ \frac{(\omega^2 - 2\omega + 2 + \omega^3 - 2\omega^2 + 2\omega) - (\omega^2 + 2\omega + 2 - \omega^3 - 2\omega^2 - 2\omega)}{(\omega^2 + 2)^2 - (2\omega)^2} \right\} d\omega$$

$$= \frac{1}{2}\int_0^\infty \sin \omega x \left\{ \frac{\omega^2 + 2 + \omega^3 - 2\omega^2 - \omega^2 - 2 + \omega^3 + 2\omega^2}{\omega^4 + 4\omega^2 + 4 - 4\omega^2} \right\} d\omega = \frac{1}{2}\int_0^\infty \sin \omega x \times \frac{2\omega^3}{\omega^4 + 4} d\omega$$

$$\therefore \frac{\pi}{2} e^{-x} \cos x = \int_0^\infty \frac{\omega^3}{\omega^4 + 4} \sin \omega x \, d\omega$$

Chapter 4 : Vector Algebra & Calculus [Total Marks - 33]

Q. 1(d) If r and $\bar{r}$ have their usual meaning and $\bar{a}$ is constant vector, prove that

$$\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^n} \right) = \frac{(2 - n)\,\bar{a}}{r^n} + \frac{m\,(\bar{a} \cdot \bar{r})\,\bar{r}}{r^{n+2}}.$$ **(5 Marks)**

Ans. :

$$\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = \nabla \times \left[\frac{1}{r^n}(\bar{a} \times \bar{r})\right] = \left(\nabla \frac{1}{r^n}\right) \times (\bar{a} \times \bar{r}) + \frac{1}{r^n} \nabla \times (\bar{a} \times \bar{r})$$

Since, $\nabla \times (\phi \bar{u}) = (\nabla \phi) \times \bar{u} + \phi (\nabla \times \bar{u}) = (\nabla r^{-n}) \times (\bar{a} \times \bar{r}) + \frac{1}{r^n} \nabla \times (\bar{a} \times \bar{r})$

$$= \left[-n \frac{r^{-n-1}}{r} \bar{r}\right] \times (\bar{a} \times \bar{r}) + \frac{1}{r^n} \left[(\nabla \cdot \bar{r}) \bar{a} - (\bar{a} \cdot \nabla) \bar{r}\right]$$

$$= (-n r^{-n-2}) \bar{r} \times (\bar{a} \times \bar{r}) + \frac{1}{r^n} (3\bar{a} - \bar{a}) = (-n r^{-n-2}) \left[(\bar{r} \cdot \bar{r}) \bar{a} - (\bar{a} \cdot \bar{r}) \bar{r}\right] + \frac{2\bar{a}}{r^n}$$

$$= (-n r^{-n-2}) \left[r^2 \bar{a} - (\bar{a} \cdot \bar{r}) \bar{r}\right] + \frac{2\bar{a}}{r^n}$$

$$= -n r^{-n} \bar{a} + \frac{n (\bar{a} \cdot \bar{r}) \bar{r}}{r^{n+2}} + \frac{2\bar{a}}{r^n} = \left(\frac{2-n}{r^n}\right) \bar{a} + \frac{n (\bar{a} \cdot \bar{r})}{r^{n+2}}$$

$$= \frac{(2-n) \bar{a}}{r^n} + \frac{m (\bar{a} \cdot \bar{r}) \bar{r}}{r^{n+2}} = \text{R.H.S}$$

Q. 3(c) Show that $\vec{F} = (ye^{xy} \cos z)\, \vec{i} + (xe^{xy} \cos z)\, \vec{j} - (e^{xy} \sin z)\, \vec{k}$ is irrotational and find the scalar potential for $\vec{F}$ and evaluate $\int_C \vec{F} \cdot \vec{dr}$ along the curve joining the points (0, 0, 0) and $(-1, 2, \pi)$.

(8 Marks)

Ans. : A vector field $\bar{F}$ is irrotational if $\nabla \times \bar{F} = \bar{0}$

Consider, $$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{xy} \cos z & xe^{xy} \cos z & -e^{xy} \sin z \end{vmatrix}$$

$$= \bar{i} \left[\frac{\partial}{\partial y}(-e^{xy} \sin z) - \frac{\partial}{\partial z}(xe^{xy} \cos z)\right] - \bar{j} \left[\frac{\partial}{\partial x}(-e^{xy} \sin z) - \frac{\partial}{\partial z}(ye^{xy} \cos z)\right]$$
$$+ \bar{k} \left[\frac{\partial}{\partial x}(xe^{xy} \cos z) - \frac{\partial}{\partial y}(ye^{xy} \cos z)\right]$$

$$= \bar{i}\ [-\sin z\, (+e^{xy} (x)) - xe^{xy} (-\sin z)] - \bar{j}\ [-\sin z\, (e^{xy} (y)) - ye^{xy} (-\sin z)]$$
$$+ \bar{k}\ [\cos z\, (xe^{xy} (y) + e^{xy} (1)) - \cos z\, (ye^{xy} (x) + e^{xy} (1))]$$

$$= \bar{i}\ [-xe^{xy} \sin z + xe^{xy} \sin z] - \bar{j}\ [-ye^{xy} \sin z + ye^{xy} \sin z]$$
$$+ \bar{k}\ [xy\, e^{xy} \cos z + e^{xy} \cos z - xy\, e^{xy} \cos z - e^{xy} \cos z]$$

$\therefore \quad \nabla \times \bar{F} = \bar{0}$

$\therefore$ A vector field $\bar{F}$ is irrotational.

$\therefore$ There exists a scalar potential (ϕ) of $\vec{F}$ such that $\vec{F} = \nabla \phi$

$$\therefore \vec{F} = (ye^{xy} \cos z)\,\bar{i} + (x\,e^{xy} \cos z)\,\bar{j} - (e^{xy} \sin z)\,\bar{k} = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

Comparing both sides, we have

$$\frac{\partial \phi}{\partial x} = ye^{xy} \cos z\,; \qquad \frac{\partial \phi}{\partial y} = x\,e^{xy} \cos z\,;$$

$$\frac{\partial \phi}{\partial z} = -e^{xy} \sin z$$

Now, $$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$\therefore d\phi = (ye^{xy} \cos z)\,dx + (xe^{xy} \cos z)\,dy - (e^{xy} \sin z)\,dz$$

On integration, $$\therefore \phi = y \cdot \frac{e^{xy}}{y} \cdot \cos z + 0 - 0 + c$$

$\therefore$ Scalar potential of $\vec{F} = \phi = e^{xy} \cos z + c$

Now, $$\int_c \vec{F} \cdot \vec{dr} = \int_{(0,0,0)}^{(-1,2,\pi)} [(ye^{xy} \cos z)\,dx + (xe^{xy} \cos z)dy - (e^{xy} \sin z)dz]$$

Integrating as above,

$$\int_c \vec{F} \cdot \vec{dr} = [\,e^{xy} \cos z]_{0,0,0}^{(-1,2,\pi)} = e^{-2} \cos \pi - e^{0} \cos 0 = -e^{-2} - 1 = -(e^{-2} + 1)$$

Q. 4(a) Evaluate by Green's theorem : $\int_c (e^{-x} \sin y\, dx + e^{-x} \cos y)\, dy$ where c is the rectangle whose vertices are $(0,0) : (\pi,0); \left(\pi\frac{\pi}{2}\right); \left(0\cdot\frac{\pi}{2}\right)$. **(6 Marks)**

Ans. : Let $P = e^{-x} \sin y$ and $Q = e^{-x} \cos y$

$$\therefore \frac{\partial P}{\partial y} = e^{-x} \cos y \text{ and } \frac{\partial Q}{\partial x} = \cos y \cdot e^{-x} \cdot -1$$

By Green's Theorem, $$\int_c P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$\therefore \int_c (e^{-x} \sin y dx + e^{-x} \cos y)\, dy = \int_0^{\pi/2} \int_0^{\pi} [-e^{-x} \cos y - (e^{-x} \cos y)]\, dxdy$$

$$= \int_0^{\pi/2} \int_0^{\pi} -2\,e^{-x} \cos y\, dx\, dy$$

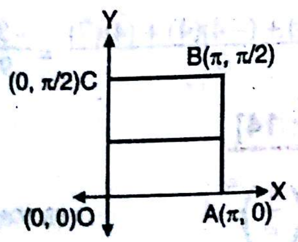

Fig. 1

$$= -2 \int_0^{\pi} e^{-x}\, dx \int_0^{\pi/2} \cos y\, dy$$

$$= -2\left[\frac{e^{-x}}{-1}\right] [\sin y]_0^{\pi/2} = 2\,[e^{-x} - 1] \times [1 - 0] = 2\,[e^{-x} - 1]$$

Q. 5(c) Evaluate $\iint_S \bar{F} \cdot \hat{n} ds$, where $\vec{F} = x\vec{i} - y\vec{j} + (z^2 - 1)\vec{k}$ and s is the cylinder formed by the surface $z = 0$, $z = 1$, $x^2 + y^2 = 4$, using the Gauss – Divergence theorem. **(6 Marks)**

Ans. : By Gauss divergence Theorem,

$$\iint_S \bar{F} \cdot \hat{n} ds = \iiint_V \nabla \cdot \vec{F} \, dV$$

$$\therefore \iint_S \bar{F} \cdot \hat{n} ds = \iiint_V 2z \, dx \, dy \, dz$$

$$\therefore \iint_S \bar{F} \cdot \hat{n} ds = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{1} = 2 \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{2} r dr \int_{z=0}^{1} z dz$$

$$= 2 [\theta]_0^{2\pi} \times \left[\frac{r^2}{2}\right]_0^2 \times \left[\frac{z^2}{2}\right]_0^1$$

$$= 2[2\pi - 0] \times \frac{1}{2}[2^2 - 0] \times \frac{1}{2}[1^2 - 0] = 4\pi \times 2 \times \frac{1}{2} = 4\pi$$

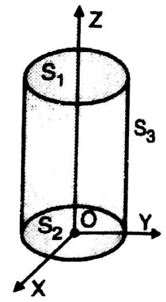

Fig. 2

Q. 6(b) Find the directional derivative of $\phi = 4 e^{2x - y + x}$ at the point (1, 1, – 1) in the direction towards the point (– 3, 5, 6). **(6 Marks)**

Ans. :

$$\phi = 4 e^{2x-y+z}; \qquad \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}(e^{2x-y+z} \cdot 2) + \vec{j}(4 e^{2x-y+z} - 1) + \vec{k}(4 e^{2x-y+z} \cdot 1)$$

At point A(1, 1, – 1)

$$\nabla f = \vec{i}(8 e^{2-1-1}) - \vec{j}(4 e^{2-1-1}) + \vec{k}(4 e^{2-1-1}) = 8\vec{i} - 4\vec{j} + 4\vec{k}$$

Let B = (– 3, 5, 6)

$$\therefore \vec{AB} = \vec{b} - \vec{a} = (-3-1)\vec{i} + (5-1)\vec{j} + (6+1)\vec{k} = -4\vec{i} + 4\vec{j} + 7\vec{k}$$

$$\therefore |\vec{AB}| = \sqrt{(-4)^2 + 4^2 + 7^2} = \sqrt{81} = 9$$

Directional derivative ϕ in the direction of $\vec{AB} = \nabla\phi \cdot \frac{\vec{AB}}{|\vec{AB}|}$

$$= (8\vec{i} - 4\vec{j} + 4\vec{k}) \cdot \frac{(-4\vec{i} + 4\vec{j} + 7\vec{k})}{9} = \frac{(8)(-4) + (-4)(4) + (4)(7)}{9} = \frac{-20}{9}$$

Chapter 5 : Z - Transform [Total Marks : 14]

Q. 2(b) Find the z-transform of $\{a^{|k|}\}$ and hence find the z-transform of $\left\{\left(\frac{1}{2}\right)^{|k|}\right\}$. **(6 Marks)**

Ans. :

Here, $f(k) = a^{|k|}$; $\quad \therefore\ f(k) = \begin{cases} a^{k} & k \geq 0 \\ a^{-k} & k < 0 \end{cases}$

By definition, $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) \cdot z^{-k}$

$$\therefore\ Z\{a^{|k|}\} = \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^{k} \cdot \frac{1}{z^{k}} = (\ldots + a^3 z^3 + a^2 z^2 + a^1 z^1) + \left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ldots\right)$$

$$= \frac{az}{1-az} + \frac{1}{1-a/z} = \frac{az}{1-az} + \frac{z}{z-a} = \frac{az^2 - a^2 z + z - az^2}{(1-az)(z-a)} = \frac{z(1-a^2)}{(1-az)(z-a)}$$

$$Z\{a^{|k|}\} = \frac{z(1-a^2)}{(1-az)(z-a)}$$

Deduction : Put $a = \frac{1}{2}$

$$\therefore\ Z\left\{\left(\frac{1}{2}\right)^{|k|}\right\} = z\left(1 - \frac{1}{2^2}\right) \Big/ \left(1 - \frac{1}{2}z\right)\left(z - \frac{1}{2}\right)$$

$$= \frac{3z}{4} \Big/ \left(\frac{2-z}{2}\right)\left(\frac{2z-1}{2}\right) = \frac{3z}{4} \times \frac{4}{(2-z)(2z-1)}$$

$$\therefore\ Z\left\{\left(\frac{1}{2}\right)^{|k|}\right\} = \frac{3z}{(2-z)(2z-1)}$$

Q. 4(c) Find the invers z-trans form of $\frac{1}{(z-2)(z-3)}$ (i) $|z| < 2$; (ii) $2 < |z| < 3$ (iii) $|z| > 3$. **(8 Marks)**

Ans. :

By Partial fractions, $F(z) = \frac{1}{(z-2)(z-3)} = \frac{-1}{z-2} + \frac{1}{z-3}$

Case 1 : For $|z| < 2$,

$$F(z) = \frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2} \left\{\because\ \frac{1}{(z-a)(z-b)} = \frac{1}{b}\left[\frac{1}{z-a} - \frac{1}{z-b}\right]\right\}$$

$\because\ |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$; Also, $|z| < 3 \Rightarrow |z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$

$$\therefore\ F(z) = \frac{\frac{1}{3}}{\frac{z}{3} - 1} - \frac{\frac{1}{2}}{\frac{z}{2} - 1} = \frac{\frac{1}{2}}{1 - \frac{z}{2}} - \frac{\frac{1}{3}}{1 - \frac{z}{3}} = \frac{1}{2}\left[\frac{1}{1 - \frac{z}{2}}\right] - \frac{1}{3}\left[\frac{1}{1 - \frac{z}{3}}\right] \left|\frac{z}{3}\right| < 1 ; \left|\frac{z}{2}\right| < 1$$

$$F(z) = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^{k} - \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{z}{3}\right)^{k}$$

$$\text{Input } z = -m = \frac{1}{2} \sum_{m=-\infty}^{0} \left(\frac{z}{2}\right)^{-m} - \frac{1}{3} \sum_{m=-\infty}^{0} \left(\frac{z}{3}\right)^{-m}$$

$$F(z) = \frac{1}{2} \sum_{m=-\infty}^{0} 2^m z^{-m} - \frac{1}{3} \sum_{m=-\infty}^{0} 3^m z^{-m}$$

Operating inverse Z - transform

$$f(k) = \frac{1}{2} 2^k_{\ k \le 0} - \frac{1}{3} 3^k \quad k \le 0$$

$$f(k) = 2^{k-1} - 3^{k-1} \qquad k \le 0$$

Case 2 : For $2 < |z| < 3$

$$F(z) = \frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2} \left\{ \because \frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left[\frac{1}{z-a} - \frac{1}{z-b}\right]\right\}$$

$$\because \quad 2 < |z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1 \text{ and } \left|\frac{2}{z}\right| < 1$$

$$\therefore \quad F(z) = \frac{\frac{1}{3}}{\frac{z}{3} - 1} - \frac{\frac{1}{z}}{1 - \frac{2}{z}} = -\frac{1}{3}\left[\frac{1}{1-\frac{z}{3}}\right] - \frac{1}{2}\left[\frac{\frac{2}{z}}{1-\frac{2}{z}}\right] = -\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^k - \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{2}{z}\right)^k$$

In the 1st term put $k = -m$

$$= -\frac{1}{3}\sum_{m=-\infty}^{0}\left(\frac{z}{3}\right)^{-m} - \frac{1}{2}\sum_{k=1}^{\infty} 2^k z^{-k} = -\frac{1}{3}\sum_{m=-\infty}^{0} 3^m z^{-m} - \frac{1}{2}\sum_{k=1}^{\infty} 2^k z^{-k}$$

Operating inverse Z - transform

$$\therefore \quad f(k) = -\frac{1}{3} \ 3^k_{\ (k \le 0)} - \frac{1}{2} 2^k_{\ (k > 0)}$$

$$\therefore \quad f(k) = \begin{cases} -3^{k-1} & k \le 0 \\ -2^{k-1} & k > 0 \end{cases}$$

Case 3 : For $|z| > 3$

$$F(z) = \frac{1}{(z-2)(z-3)} = \frac{1}{(z-3)} - \frac{1}{z-2} \left\{ \because \frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\left[\frac{1}{z-a} - \frac{1}{z-b}\right]\right\}$$

Since $|z| > 3 \Rightarrow |z| > 2$

$$\therefore \quad \left|\frac{3}{z}\right| < 1 \quad ; \quad \left|\frac{2}{z}\right| < 1$$

$$\therefore \quad F(z) = \frac{1}{z-3} - \frac{1}{z-2} = \frac{\frac{1}{z}}{1-\frac{3}{z}} - \frac{\frac{1}{z}}{1-\frac{2}{z}} = \frac{1}{3}\left[\frac{\frac{3}{z}}{1-\frac{3}{z}}\right] - \frac{1}{2}\left[\frac{\frac{2}{z}}{1-\frac{2}{z}}\right]$$

$$F(z) = \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{3}{z}\right)^k - \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{2}{z}\right)^k = \frac{1}{3}\sum_{k=1}^{\infty} 3^k z^{-k} - \frac{1}{2}\sum_{k=1}^{\infty} 2^k z^{-k}$$

Take inverse Z - transform,

$$f(k) = \frac{1}{3} 3^k_{\ (k \ge 1)} - \frac{1}{2} 2^k \ (k \ge 1)$$

$$\therefore f(k) = 3^{k-1} - 2^{k-1} \qquad (k \ge 1)$$

May 2014

Chapter 1 : Complex Variable and Mapping [Total Marks - 19]

Q. 1(b) State true or false with proper justification. There does not exist an analytic function whose real part is $x^3 - 3x^2 - y - y^3$". **(5 Marks)**

Ans. : Let $f(z) = u + iv$ be analytic.

$$\text{Let real part } u = x^3 - 3x^2 y - y^3$$

Differentiating partially w.r.t. 'x'

$$\therefore \quad u_x = 3x^2 - 6xy - 0$$

Again differentiating partially w.r.t. 'x'

$$\therefore \quad u_{xx} = 6x - 6y$$

Similarly differentiating 'u' partially w.r.t. 'y'

$$\therefore \quad u_y = 0 - 3x^2 - 3y^2$$

Again differentiating partially w.r.t. 'y'

$\therefore \quad u_{yy} = -6y \qquad \therefore \quad u_{xx} + u_{yy} \neq 0$

$\therefore$ u does not satisfies Laplace's equation and hence it is not harmonic function

So u is not a part of analytic function $f(z) = u + iv$'

Hence the statement, 'There does not exist an analytic function whose real part is $x^3 - 3x^2 y - y^3$" is true.

Q. 5(a) If the imaginary part of the analytic function $W = f(z)$ is $V = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$.

Find the real part U. **(6 Marks)**

Ans. :

$$V = x^2 - y^2 + \frac{x}{x^2 + y^2}$$

$$\frac{\partial v}{\partial x} = 2x + \frac{(x^2 + y^2)(1) - x(2x)}{(x^2 + y^2)} = 2x - \frac{x^2 - y^2}{(x^2 + y^2)^2} = \phi_2(x, y)$$

$$\frac{\partial v}{\partial y} = -2y - \frac{2xy}{(x^2 + y^2)^2} = \phi_1(x, y)$$

By Milen-Thompson method,

$$\phi_1(z, 0) = 0 \quad \text{and} \quad \phi_2(z, 0) = 2z - \frac{z^2}{z^4} = 2z - \frac{1}{z^2}$$

$$f'(z) = \phi_1(z, 0) + i\,\phi_2(z, 0) = i\left(2z - \frac{1}{z^2}\right)$$

$$f(z) = \int f'(z)\,dz = i\int\left(2z - \frac{1}{z^2}\right)dz = i\left(2\frac{z^2}{2} + \frac{1}{z}\right) + C = i\left(z^2 - \frac{1}{z}\right)$$

$$z = x + iy$$

$$\therefore f(z) = i\left((x + iy)^2 + \frac{1}{x + iy}\right) + C = \left(-2xy + \frac{y}{x^2 + y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2 + y^2}\right) + C = u + iv$$

$$\therefore u = -2xy + \frac{y}{x^2 + y^2} + C$$

Q. 6(c) Find the bilinear transformation under which 1, i, –1 from the z-plane are mapped onto 0, 1, ∞ of w plane. Also show that under this transformation the unit circle in w-plane is mapped onto a straight line in the z-plane. Write the name of this line. **(8 Marks)**

Ans. :

First part :

$$(z_1, z_2, z_3) = (1, i, -1)$$
$$(w_1, w_2, w_3) = (0, 1, \infty)$$

By linear transformation, $w = \dfrac{az+b}{cz+d}$ ...(1)

Put z = – 1 and w = ∞ in (1)

$$\frac{1}{w} = \frac{cz+d}{az+b} \;; \qquad \therefore 0 = \frac{-c+d}{-a+b}$$

$\therefore -c + d = 0$; $\qquad \therefore c = d$ ...(2)

Put z = 1 and w = 0 $\qquad 0 = \dfrac{a+b}{c+d}$; $\qquad \therefore a + b = 0$

$\therefore a = -b$ ...(3)

Put z = i and w = 1 in Equation (1)

$$\therefore 1 = \frac{a(i)+b}{c(i)+d}$$

$\therefore ci + d = ai + b$; $\qquad \therefore di + d = -bi + b$ [from 2 and 3]

$\therefore d(i+1) = b(-i+1)$

$$\therefore d = \left(\frac{1-i}{1+i}\right) b = -ib$$ ...(4)

From Equations (2) and (4)

$c = -ib$ ...(5)

Substitute Equations (2), (3) and (5) in (1)

$$\therefore w = \frac{-bz+b}{-ibz-ib} = \frac{-b(z-1)}{-ib(z+1)} = \frac{-b(z-1)}{-ib(z+1)} = \frac{z-1}{i(z+1)}$$

is the required bilinear transformation.

Second part: $|w| = 1$ is the unit circle in w-plane.

$$\therefore \left|\frac{z-1}{i(z+1)}\right| = 1$$

$$\therefore |z-1| = |i| \times |z+1|$$

$$\therefore |x+iy-1| = 1 \times |x+iy+1|$$

$$\therefore |(x-1)+iy| = |(x+1+iy)|$$

$$\therefore \sqrt{(x-1)^2+y^2} = \sqrt{(x+1)^2+y^2}$$

On squaring, $x^2 - 2x + 1 + y^2 = x^2 + 2x + 1 + y^2$

$\therefore 0 = 4x$

$\therefore x = 0$, which is a straight is a straight line in z-plane

Line x = 0 is the Y-axis

Chapter 2 : Laplace Transform [Total Marks - 31]

Q. 1(a) Find $L^{-1} \dfrac{se^{-\pi s}}{s^2+3s+2}$ (chp Laplace II) **(5 Marks)**

Ans. : First we find inverse L.T. of $\frac{s}{s^2+3s+2}$ and then use the theorem

$$L^{-1} e^{-as} \bar{f}(s) = f(t-a)\, H(t-a)$$

Now, $$L^{-1}\frac{s}{s^2+3s+2} = L^{-1}\frac{s}{(s+2)(s+1)} = L^{-1}\left[\frac{2}{s+2}-\frac{1}{s+1}\right] \text{ by partial fractions}$$

$$= 2L^{-1}\frac{1}{s+2} - L^{-1}\frac{1}{s+1} = 2e^{-2t} - e^{-t}$$

Q. 2(b) Verify Laplace's equation for $U\left(r+\frac{a^2}{r}\right)\cos\theta$. Also find V and f(z). **(6 Marks)**

Ans. :

$$u = \left(r+\frac{a^2}{r}\right)\cos\theta$$

$$\therefore \frac{\partial u}{\partial r} = \left(1-\frac{a^2}{r^2}\right)\cos\theta, \quad \frac{\partial^2 u}{\partial r^2} = \frac{2a^2}{r^3}\cos\theta$$

$$\frac{\partial u}{\partial \theta} = -\left(r+\frac{a^2}{r}\right)\sin\theta, \quad \frac{\partial^2 u}{\partial \theta^2} = -\left(r+\frac{a^2}{r}\right)\cos\theta$$

$$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{2a^2}{r^3}\cos\theta + \frac{1}{r}\cdot\left(1-\frac{a^2}{r^2}\right)\cos\theta - \frac{1}{r^2}\left(r+\frac{a^2}{r}\right)\cos\theta$$

$\therefore$ Laplace's equation is satisfied.

By Cauchy-Riemann equations in polar form

$$u_r = \frac{1}{r} v_\theta \qquad \therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\therefore \left(1-\frac{a^2}{r^2}\right)\cos\theta = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta} \qquad \therefore \frac{\partial v}{\partial \theta} = \left(r-\frac{a^2}{r}\right)\cos\theta$$

Integrating w.r.t. θ,

$$v = \left(r-\frac{a^2}{r}\right)\sin\theta + C$$

Hence, $$f(z) = u+iv = \left(r+\frac{a^2}{r}\right)\cos\theta + i\left(r-\frac{a^2}{r}\right)\sin\theta + C$$

$$= r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta) - C = z + \frac{a^2}{r} - C$$

$$v = \left(r-\frac{a^2}{r}\right)\sin\theta \; ; \; z + \frac{a^2}{z}$$

Q. 2(c) Solve the following equation by using Laplace transform $\frac{dy}{dt} + 2y + \int_0 y\,dt = \sin t$ given that $y(0) = 1$. **(8 Marks)**

Ans. :

$$\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t$$

Taking laplace transform of both sides, we get

$$L(y')+2L(y)+L\left[\int_0^t y\cdot dt\right] = L(\sin t)$$

But,

$$L(y') = sL(y)-y(0)=s\bar{y}-1$$

$$L\left[\int_0^t y\cdot dt\right] = \frac{1}{s}L(y)=\frac{1}{s}\bar{y}$$

$$L(\sin t) = \frac{1}{s^2+1}$$

$\therefore$ The equation becomes,

$$s\bar{y}-1+2\bar{y}+\frac{1}{s}\bar{y} = \frac{1}{s^2+1}$$

$$\left(s+2+\frac{1}{s}\right)\bar{y} = \frac{1}{s^2+1}+1 \qquad \therefore \quad \bar{y}=\frac{s(s^2+2)}{(s+1)^2(s^2+1)}$$

Let $\dfrac{s(s^2+2)}{(s+1)^2(s^2+1)} = \dfrac{a}{s+1}+\dfrac{b}{(s+1)^2}+\dfrac{cs+d}{s^2+1}$

$$s(s^2+2) = a(s+1)(s^2+1)+b(s^2+1)+(cs+d)(s+1)^2$$

Putting $s=-1$; $\quad -3=2b \Rightarrow b=\frac{-3}{2}$

$s=0$; $\quad a+b+d=0$

Equating the coefficients of s^2 and s^3.

$a+b+2c+d = 0$ and $a+c=1$

$b = -\frac{3}{2}$; $\qquad a+d = 3/2$

and $\quad a+2c+d = 3/2 \Rightarrow c=0$

$a+c = 1 \Rightarrow a=1$

and $\quad d = 1/2$

$$\bar{y} = \frac{1}{s+1}-\frac{3}{2(s+1)^2}+\frac{1}{2(s^2+1)}$$

$$y = L^{-1}\left(\frac{1}{s+1}\right)-\frac{3}{2}e^{-t}L^{-1}\left(\frac{1}{s^2}\right)+\frac{1}{2}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= e^{-t}-\frac{3}{2}e^{-t}\cdot t+\frac{1}{2}\sin t$$

Q. 4(b) Given : $L(\text{erf}\sqrt{t})=\dfrac{1}{s\sqrt{s+1}}$, evaluate $\int_0^\infty te^{-t}\,\text{erf}(\sqrt{t})\,dt$. **(6 Marks)**

Ans. : $L(\text{erf}\sqrt{t}) = \dfrac{1}{s\sqrt{s+1}}=\dfrac{1}{\sqrt{s^2(s+1)}}=(s^3+s^2)^{-1/2}$

$$\therefore L[t \cdot \text{erf}\sqrt{t}] = (1)^t \frac{d}{ds}(s^3+s^2)^{-1/2}$$

$$\therefore L[t \cdot \text{erf}\sqrt{t}] = 1 \times \frac{-1}{2}(s^3+s^2)^{-3/2}(3s^2+2s)$$

$$\therefore \text{By definition,} \quad \int_0^\infty te^{-t}\, t\, \text{erf}(\sqrt{t})\, dt = \frac{3s^2+2s}{2(s^3+s^2)^{3/2}}$$

$$\text{Put } s = 1, \int_0^\infty e^{-t}\, t\text{erf}(\sqrt{t})\, dt = \frac{3(1)^2+2(1)}{2[1^3+2^2]^{3/2}} = \frac{5}{2\times 2^{3/2}}$$

$$\therefore \int_0^\infty e^{-t}\, t\text{erf}(\sqrt{t})\, dt = \frac{5}{4\sqrt{2}}$$

Q. 6(b) Find the inverse Laplace transform of $\frac{(s-1)^2}{(s^2-2s+5)^2}$. **(6 Marks)**

Ans. :

$$F(s)\,G(s) = \left\{\frac{(s-1)^2}{(s^2-2s+5)^2}\right\} = \left\{\frac{s}{s^2+4} \times \frac{s}{s^2+4}\right\}$$

$$\therefore F(s) = G(s) = L^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t$$

By convolution theorem,

$$L^{-1}[\phi_1(s)\,\phi_2(s)] = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$\therefore L^{-1}\left\{\frac{s}{s^2+2^2} \times \frac{s}{s^2+2^2}\right\} = \int_0^t \cos 2u \cdot \cos 2(t-u)\, du \times \frac{2}{2}$$

$$= \frac{1}{2}\int_0^t \cos(2u+2t-2u) + \cos(2u-2t+2u)\, du$$

$$= \frac{1}{2}\left[\cos 2t \int_0^t du + \int_0^t \cos(4u-2t)\, du\right] = \frac{1}{2}\cos 2t\, [u]_0^t + \frac{1}{2}\left[\frac{\sin(4u-2t)}{4}\right]_0^t$$

$$= \frac{1}{2}\cos 2t\,[t-0] + \frac{1}{8}[\sin 2t - \sin(-2t)] = \frac{1}{2}t\cos 2t + \frac{1}{8}\times 2\sin 2t$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] = \frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t$$

$\therefore$ From Equations (1) and (2),

$$\therefore L^{-1}\left\{\frac{(s-1)^2}{(s^2-2s+5)^2}\right\} = e^t \times \frac{2t\cos + \sin 2t}{4}$$

$$\therefore L^{-1}\left\{\frac{(s-1)^2}{(s^2-2s+5)^2}\right\} = \frac{e^t}{4}[2t\cos 2t + \sin 2t]$$

Chapter 3 : Fourier Series [Total Marks - 31]

Q. 1(c) Prove that $f_1(x) = 1$; $f_2(x) = x$; $f_3(x) = \frac{3x^2-1}{2}$ are orthogonal over $(-1, 1)$. **(5 Marks)**

Ans. :

Case I : $m \neq n$

$$\int_{-1}^{1} f_1(x)\, f_2(x)\, dx = \int_{-1}^{1} 1 \cdot x\, dx = \int_{-1}^{1} x\, dx = 0$$

$$\int_{-1}^{1} f_1(x)\, f_3(x)\, dx = \int_{-1}^{1} 1 \cdot \frac{3x^2-1}{2}\, dx = 2\int_{0}^{1} \frac{3x^2-1}{2}\, dx = \left[3\frac{x^3}{3} - x\right]_0^1 = (1^3-1)-(0-0) = 0$$

$$\int_{-1}^{1} f_2(x)\, f_3(x)\, dx = \int_{-1}^{1} x \cdot \frac{3x^2-1}{2}\, dx = \frac{1}{2}\int_{-1}^{1} (3x^3 - x)\, dx = 0$$

Case II : $m = n$

$$\int_a^b [f_1(x)]^2\, dx = \int_{-1}^{1} 1^2\, dx = 2\int_0^1 dx = 2\,[x]_0^1 = 2\,(1-0) = 2 \neq 0$$

$$\int_a^b [f_2(x)]^2\, dx = \int_{-1}^{1} x^2\, dx = 2\int_0^1 x^2\, dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}\,(1^3-0) = \frac{2}{3} \neq 0$$

$$\int_a^b [f_3(x)]^2\, dx = \int_{-1}^{1} \left(\frac{3x^2-1}{2}\right)^2 dx = \frac{1}{4} \times 2\int_0^1 (9x^4 - 6x^2 + 1)\, dx$$

$$= \frac{1}{2}\left[9\frac{x^5}{5} - 6\frac{x^3}{3} + x\right]_0^1 = \frac{1}{2}\left[\frac{9}{5} - 2 + 1\right] - (0+0+0) = \frac{1}{2} \times \frac{4}{5} = \frac{2}{5} \neq 0$$

From case I and II. The set of functions $f_1(x)$; $f_2(x)$; $f_3(x)$ is orthogonal in $(-1, 1)$.

Q. 2(a) Find the Fourier Cosine Integral representation of the function $f(x) = e^{-ax}$, $x > 0$ and hence show that $\int_0^\infty \frac{\cos wx}{1+w^2}\, dw = \frac{\pi}{2} e^{-x}$, $x \geq 0$. **(6 Marks)**

Ans. : $f(x) = e^{-ax}$

By definition of Fourier Cosine Integral, $f(x) = \frac{2}{\pi}\int_0^\infty \cos wx = \int_0^\infty f(s) \cos ws\, ds\, dw$

$$\therefore\ e^{-ax} = \frac{2}{\pi}\int_0^\infty \cos w x \int_0^\infty e^{-as} \cos ws\, ds\, dw = \frac{2}{\pi}\int_0^\infty \cos wx$$

$$\left[\frac{e^{-as}}{(-a)^2 + w^2}\,[-a\cos ws + w \sin ws]\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{\cos wx}{a^2+w^2}\,\{0 - 1\,[-a \times 1 + 0]\}\, dw$$

$$\therefore\ e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{a \cos wx}{a^2+\lambda^2}\, dw$$

Deduction :

From above, $\displaystyle\int_0^\infty \frac{a\cos wx}{a^2+\lambda^2}\,dw = \frac{\pi}{2}e^{-ax}$

Put $a = 1$ $\quad\therefore \displaystyle\int_0^\infty \frac{\cos wx}{1+w^2}\,dw = \frac{\pi}{2}e^{-x}$

Q. 3(a) Expand $f(x) = \begin{cases} \pi x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$ with period 2 into a Fourier series. (Fourier series) **(6 Marks)**

Ans. : Let $c = 0$ and $c + 2l = 2$

$\therefore\ 0 + 2l = 2\ ;\ \therefore\ l = 1$

Now,

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{l}\left[\int_0^1 \pi x\,dx + \int_1^2 0\,dx\right] = \pi\left[\frac{x^2}{2}\right]_0^1 = \pi\left[\frac{1}{2}-0\right] = \frac{\pi}{2}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$= \frac{1}{1}\left[\int_0^1 \pi x\cdot\cos\frac{n\pi x}{1}\,dx + \int_1^2 0\,dx\right] = \pi\left[x\cdot\frac{\sin n\pi x}{n\pi} - 1\cdot\frac{-\cos n\pi x}{n^2\pi^2}\right]_0^1$$

$$= \pi\left[\left(-1\cdot\frac{\cos n\pi}{n\pi} + \frac{\sin n\pi}{n^2\pi^2}\right) - \left(0 + \frac{\sin 0}{n^2\pi^2}\right)\right] = \pi\left[-\frac{(-1)^n}{n\pi} + 0 - 0 - 0\right]$$

$$= -\frac{(-1)^n}{n}$$

In Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n\cos \sum_{n=1}^\infty b_n \sin\frac{n\pi x}{l}$$

$$= \frac{\pi}{4} + \sum_{n=1}^\infty \frac{[(-1)^n - 1]}{\pi n^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^\infty \frac{-(-1)^n}{n}\sin\frac{n\pi x}{l}$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\left[\frac{-2\cos 1\pi x}{1^2} + 0 - \frac{2\cos 3\pi x}{3^2} + 0 - \ldots\right] + \left[\frac{\sin 1\pi x}{1} - \frac{\sin 2\pi x}{2} + \ldots\right]$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 1\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2}\ \ldots\right]$$
$$+\left[\frac{\sin 1\pi x}{1} + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3}\ \ldots\right]$$

Q. 4(c) Obtain the expansion of as $f(x) = x(\pi - x)$, $0 < x < \pi$ as a half range cosine series. Hence show that (i) $\displaystyle\sum_1^\infty \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ (ii) $\displaystyle\sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$ **(8 Marks)**

Ans. : $f(x) = x(\pi - x) = \pi x - x^2$

For half range cosine series, $b_n = 0$; Here, $l = x$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\,dx$$

$$= \frac{2}{\pi}\left[\pi \cdot \frac{\pi^2}{2} \cdot \frac{\pi^3}{3}\right]_0^\pi = \frac{2}{\pi}\left[\left(\pi \cdot \frac{\pi^2}{2} \cdot \frac{\pi^3}{3}\right) - (0-0)\right] = \frac{2}{\pi} \cdot \frac{\pi^3}{6} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^\pi (\pi x - x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\cdot\frac{\sin nx}{n} - (\pi - 2x)\frac{-\cos nx}{n^2\pi^2} + (0-2)\frac{-\sin nx}{n^3}\right]_0^\pi$$

$$= \frac{2}{\pi}\left\{\left[(\pi^2 - \pi^2)\frac{\sin n\pi}{n} + (\pi - 2x)\frac{-\cos n\pi}{n^2} + \frac{\sin n\pi}{n^3}\right]\right.$$

$$\left. - \left[(0-0)\frac{\sin 0}{n} + (\pi - 0)\frac{\cos 0}{n^2} + 2\frac{\sin 0}{n^3}\right]\right\}$$

$$= \frac{2}{\pi}\left[\left(0 - \pi\frac{(-1)^n}{n^2} + 0\right) - \left(0 + \pi\frac{1}{n^2}\right) + 0\right] = \frac{2}{\pi} \times \frac{-\pi}{n^2}\left[(-1)^n + 1\right]$$

$$= \frac{-2\left[(-1)^n + 1\right]}{n^2}$$

$\therefore$ Half range Fourier cosine series is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$\therefore x(\pi - x) = \frac{\pi^2}{x} + \sum_{n=1}^{\infty}\frac{-2\left[(-1)^n + 1\right]}{n^2}\cos\frac{n\pi x}{\pi}$$

$$\therefore x(\pi - x) = \frac{\pi^2}{6} - 2\left[\frac{2\cos 2x}{2^2} + 0 + \frac{2\cos 4x}{4^2} + 0 \ldots\ldots\right]$$

$$\therefore x(\pi - x) = \frac{\pi^2}{6} - 2 \times \frac{2}{2^2}\left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \ldots\ldots\right]$$

$$\therefore x(\pi - x) = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \ldots\ldots\right] \qquad \ldots(1)$$

Deduction I : Put $x = \frac{\pi}{2}$ in Equation (1),

$$\therefore \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{6} - \left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{2^2} \ldots\right]$$

$$\therefore \frac{\pi}{2}\cdot\frac{\pi}{2} - \frac{\pi^2}{6} = -\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{2^2} + \ldots\right]$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\ldots$$

$$\therefore \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

Deduction 2 : Using Parseval's Identity

$$\frac{1}{l}\int_0^l [f(x)]^2\,dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\therefore \frac{1}{\pi}\int_0^{\pi}(\pi x - x^2)^2\,dx = \left(\frac{\pi^2}{6}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{-2[(-1)^n + 1]}{n^2}\right]^2 + 0$$

$$\therefore \frac{1}{\pi}\int_0^{\pi}(\pi^2 x^2 - 2\pi x^3 + x^4)\,dx = \frac{\pi^4}{36} + \frac{1}{2}\left[0 + \frac{16}{2^4} + 0 + \frac{16}{4^4} + 0 + \frac{16}{6^4}\right]$$

$$\therefore \frac{1}{\pi}\left[\frac{\pi^2 x^3}{3} - \frac{2\pi x^4}{4} + \frac{x^5}{5}\right]_0^{\pi} = \frac{\pi^4}{36} + \frac{1}{2}\times\frac{16}{2^4}\left[\frac{1}{1^4} + \frac{1}{2^4} \frac{1}{3^4} + \ldots\ldots\right]$$

$$\therefore \frac{1}{\pi}\left[\left(\frac{x^5}{3} - \frac{\pi^5}{2} + \frac{\pi^5}{5}\right) - (0 - 0 + 0)\right] = \frac{\pi^4}{36} + \frac{1}{2}\sum_{1}^{\infty}\frac{1}{n^4}$$

$$\therefore \frac{1}{\pi}\times\frac{\pi^5}{30} = \frac{\pi^4}{36} + \frac{1}{2}\sum_{1}^{\infty}\frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{30} - \frac{\pi^4}{36} = \frac{1}{2}\sum_{1}^{\infty}\frac{1}{n^4} \quad \therefore \frac{\pi^4}{180} = \frac{1}{2}\sum_{1}^{\infty}\frac{1}{n^4} \quad \therefore \sum_{1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

Q. 6(a) Obtain complex form of Fourier series for $f(x) = \cosh 3x + \sinh 3x$ in $(-3, 3)$. **(6 Marks)**

Ans. : Let, $c = -3$ and $c + 2l = 3$; $\therefore -3 + 2l = 3$

$\therefore 2l = 6$; $\therefore l = 3$

Let $f(x) = \cosh 3x + \sinh 3x$

$$= \frac{e^{3x} + e^{-3x}}{2} + \frac{e^{3x} + e^{-3x}}{2} = \frac{2e^{3x}}{2} = e^{3x}$$

$$C_n = \frac{1}{2l}\int_c^{c+2l} f(x)\,e^{-n\pi x/l}\,dx = \frac{1}{6}\int_{-3}^{3} e^{3x}\,e^{-n\pi x/3}\,dx$$

$$= \frac{1}{6}\int_{-3}^{3} e^{(3 - in\pi/3)x}\,dx = \frac{1}{6}\int_{-3}^{3} e^{(9 - in\pi)/3}\,dx = \frac{1}{6}\left[\frac{e^{(9 - in\pi)x/3}}{(9 - in\pi)/3}\right]_{-3}^{3}$$

$$= \frac{1}{6}\times\frac{3}{9 - in\pi}\left[e^{(9 - in\pi)} - e^{-(9 - in\pi)}\right]$$

$$= \frac{1}{2(9 - in\pi)}\times\left[e^9 e^{-in\pi} - e^{-9}e^{-in\pi}\right]\times\frac{(9 + in\pi)}{(9 + in\pi)}$$

Consider, $e^{\pm in\pi} = \cos in\pi \pm i \sin n\pi = (-1)^n \pm i0 = (-1)^n$

$$\therefore C_n = \frac{(9+in\pi)}{2(9^2 - i^2 n^2 \pi^2)} \times [e^9 (-1)^n - e^{-9} (-1)^n]$$

$$= \frac{(9+in\pi)(-1)^n}{81+n^2\pi^2} \times \frac{e^9 - e^{-9}}{2} = \frac{(9+in\pi)(-1)^n}{81+n^2\pi^2} \sinh 9$$

In Complex Fourier series,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}$$

$$\therefore \cosh 3x + \sinh 3x = \sum_{n=-\infty}^{\infty} \frac{(9+in\pi)(-1)^n}{81+n^2\pi^2} \sinh 9 \cdot e^{in\pi x/3}$$

$$\therefore \cosh 3x + \sinh 3x = \sinh 9 \sum_{n=-\infty}^{\infty} \frac{(9+in\pi)(-1)^n}{81+n^2\pi^2} e^{in\pi x/3}$$

Chapter 4 : Vector Algebra & Calculus [Total Marks - 23]

Q. 1(d) Using Green's theorem in the plane evaluate $\int_C (x^2 - y)\,dx + (2y^2 + x)\,dy$, around the boundary of the region defined by $y = x^2$ and $y = 4$. (Chp : Vector Integration) **(5 Marks)**

Ans.: Given, $y = 4$ and $y = x^2$

Solving simultaneously we get,

$x^2 = 4$; $x = \pm 2$

Two curves intersect at (2, 4) and (2, – 4)

Let $P = x^2 - y$ and $Q = \frac{\partial \phi}{\partial x} = 1$

By Green's Theorem,

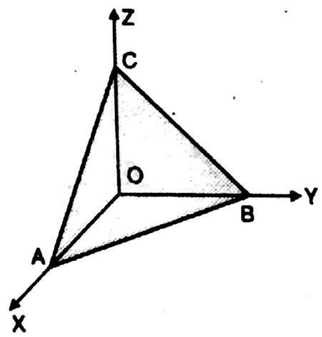

Fig. 1

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$\therefore \int (x^2 - y)\,dx + (2y^2 + x)\,dy = \iint_R [1+1]\,dx\,dy = 2 \int_0^4 \int_{\sqrt{y}}^{\sqrt{y}} dx\,dy = 2 \int_0^4 [\,y\,]_{\sqrt{y}}^{\sqrt{y}}\,dy$$

$$= 2 \int_0^4 [\sqrt{y} + \sqrt{y}\,]\,dy = 2 \int_0^4 2y^{1/2}\,dy = \left[\frac{y^{3/2}}{3/2} \right]_0^4$$

$$= 4 \times \frac{2}{3} [4^{3/2} - 0] = \frac{64}{3}$$

$$\int_c (x^2 - y)\,dx + (2y^2 + x)\,dy = \frac{64}{3}$$

Q. 3(b) A vector field is given by $\vec{F} = (x^2 + xy^2)\vec{i} + (y^2 + x^2 y)\vec{j}$. Show that $\vec{F}$ is irrotational and find its scalar potential. **(6 Marks)**

Ans. :

$$\vec{F} = (x^2 + xy^2)\vec{i} + (y^2 + x^2 y)\vec{j}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2 y & 0 \end{vmatrix} = \vec{i}\,[0-0] - \vec{j}\,[0-0] + \vec{k}\,[2xy - 2xy] = 0$$

$\therefore$ $\vec{F}$ is irrotational.

$\therefore$ There exists a scalar potential (ϕ) of $\vec{F}$ such that $\vec{F} = \nabla\phi$

$$\therefore \quad (x^2 - xy^2)\vec{i} + (y^2 + x^2 y)\vec{j} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Comparing both sides, we have

$$\frac{\partial\phi}{\partial x} = x^2 + xy^2 \cdot \frac{\partial\phi}{\partial y} = y^2 + x^2 y\,;\ \frac{\partial\phi}{\partial z} = 0$$

Now, $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$\therefore$ $d\phi = (x^2 - xy^2)\,dx + (y^2 + x^2 y)\,dy + 0\,dz$

On integration,

$$\therefore \quad \phi = \left(\frac{x^3}{3} + y^2 \cdot \frac{x^3}{3}\right) + \left(\frac{y^3}{3}\right) |0 + c$$

$$\therefore \quad \text{Scalar potential of } \vec{F} = \phi = \frac{x^3}{3} + \frac{x^2 y^2}{2} + \frac{y^3}{3} + c$$

Q. 4(a) Find the constants 'a' and 'b' so that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the surface $4x^2 y + z^3 = 4$ at $(1, -1, 2)$. **(6 Marks)**

Ans. :

Let $\phi = ax^2 - byz - (a+2)\,x$

$$\nabla\phi = [2ax - (a+2)]\,\hat{i} - bz\,\hat{j} - by\,\hat{k}$$

$$\nabla\phi_{(1,-1,2)} = [(a-2)\,\hat{i} - 2b\,\hat{j} + b\hat{k}\,]$$

Let $\phi_1 = 4x^2 y + z^3 - 4$

$$\nabla\phi_1 = 8xy\,\hat{i} + 4x^2\,\hat{j} + 3z^2\,\hat{k}$$

$$\nabla\phi_{1(1,-1,2)} = 8(-1)\,\hat{i} + 4\,(1)^2\hat{j} + 3\,(2)^2\,\hat{k}$$

$$\nabla\phi_1 = -8\hat{i} + 4\hat{j} + 12\hat{k}$$

Since they are orthogonal,

$$\nabla\phi \cdot \nabla\phi_1 = 0$$

$$[(a-2)\hat{i} - 2b\hat{j} + b\hat{k}] \cdot [-8\hat{i} + 4\hat{j} + 12\hat{k}] = 0$$

$$-8(a-2) - 8b + 12b = 0 \qquad -8(a-2) + 4b = 0$$

$$2(a-2) = b \qquad 2a - 4 = b$$

$$\mathbf{2a - b = 4} \qquad \text{...(1)}$$

Similarly point $(1, -1, 2)$ is on ϕ, thus

$$ax^2 - byz = (a+2)x$$

$$a(1)^2 - b(-1)(2) = (a+2) \qquad a + 2b = a + 2$$

$$2b = 2 \Rightarrow \mathbf{b = 1}$$

Putting $b = 1$ in Equation (1), we get

$$2a - 1 = 4 \Rightarrow 2a = 5$$

$$\mathbf{a = \frac{5}{2}}$$

Q. 5(c) Use Gauss Divergence Theorem to evaluation $\iint_S \bar{N} \cdot \bar{F}\, ds$ where $\vec{F} = 4x\,\vec{i} + 3y\,\vec{j} - 4z^2\,\vec{k}$ and S is the surface boundary by $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$. **(6 Marks)**

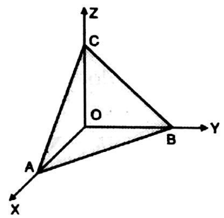

Fig. 2

Ans. :

$$\vec{F} = 4x\,\vec{i} + 3y\,\vec{j} - 4z^2\,\vec{k}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(3y) - \frac{\partial}{\partial z}(4z^2)$$

$$= 4 + 3 - 8z = 7 - 8z$$

By Gauss Divergence theorem

$$\iint_S \bar{F} \cdot n\, ds = \iiint_V \nabla \cdot \vec{F}\, dV$$

$$= \int_{z=0}^{4} \int_{y=0}^{(4-z)/2} \int_{x=0}^{(4-2y-z)/2} (7 - 8z)\, dx\, dy\, dz$$

$$= \int_{z=0}^{4} \int_{y=0}^{(4-z)/2} (7 - 8z)[x]_0^{(4-2y-z)/2}\, dy\, dz$$

$$= \int_{z=0}^{4} \int_{y=0}^{(4-z)/2} (7-8z)\left[\frac{4-2y-z}{2} - 0\right] dydz$$

$$= \int_{z=0}^{4} \int_{y=0}^{(4-z)/2} (7 - 8z) \times \frac{1}{2}\,[(4-z) - 2y]\, dy\, dz$$

$$= \frac{1}{2} \int_{z=0}^{4} (7 - 8z)\left[(4-z)\,y - \frac{2y^2}{2}\right]_0^{(4-z)/2} dz$$

$$= \frac{1}{2} \int_{z=0}^{4} (7-8z) \left\{ \left[\frac{(4-z)^2}{2} - \frac{(4-z)^2}{4} \right] - 0 \right\} dz$$

$$= \frac{1}{2} \int_{z=0}^{4} (7-8z) \times \frac{(4-z)^2}{4} dz = \frac{1}{8} \int_{z=0}^{4} (7-8z)(16-8z+z^2) dz$$

$$= \frac{1}{8} \int_{z=0}^{4} (112 - 56z + 7z^2 - 128z + 64z^2 - 8z^3) dz$$

$$= \frac{1}{8} \int_{z=0}^{4} (-8z^3 + 71z^2 - 184z + 112) dz = \frac{1}{8} \left[\frac{-8z^4}{4} + \frac{71z^3}{3} - \frac{184z^2}{2} + 112z \right]_0^4$$

$$= \frac{1}{8} \left\{ \left[-2(4)^4 + \frac{71(4)^2}{3} - 92(4)^2 + 112(4) \right] - 0 \right\} = \frac{-8}{3}$$

$$\therefore \iint_S \bar{N} \cdot \bar{F} \, ds = \frac{-8}{3}$$

Chapter 5 : Z – Transform [Total Marks - 14]

Q. 3(c) Find the inverse z-transform of $f(z) = \dfrac{z+2}{z^2-2z+1}$, $|z| > 1$. **(8 Marks)**

Ans. :

$$F(x) = \frac{z+2}{z^2-2z+1} = \frac{z+2}{(z-1)^2} = \frac{(Z-1)+3}{(z-1)^2}$$

$$= \frac{1}{(z-1)} + \frac{3}{(z-1)^2} \qquad \because |Z| > 1 \Rightarrow \frac{1}{|Z|} < 1$$

$$= \frac{1}{Z\left(1-\frac{1}{Z}\right)} + \frac{3}{z^2\left(1-\frac{1}{z}\right)^2} = \frac{1}{Z}\left(1-\frac{1}{Z}\right)^{-1} + \frac{3}{z^2}\left(1-\frac{1}{z}\right)^{-2}$$

$$= \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}\cdots\cdots\right] + \frac{3}{z^2}\left[1+\frac{2}{z}+\frac{6}{2!}\frac{1}{z}+\cdots\cdots\right]$$

$$= \left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}\cdots\cdots\right) + \frac{3}{z^2}\left(1+\frac{2}{z}+\frac{3}{z^2}+\frac{4}{z^3}\cdots\cdots\right)$$

$$= \left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\cdots+\frac{1}{z^{k-1}}\cdots\cdots\right) + 3\left(\frac{1}{z^2}+\frac{2}{z^3}+\frac{3}{z^4}+\frac{4}{z^5}\cdots\cdots\right)$$

$$= \left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\cdots+\frac{1}{z^k}\cdots\cdots\right) + 3\left(\frac{1}{z^2}+\frac{2}{z^3}+\frac{3}{z^4}\cdots\cdots+\frac{k-1}{z^k}\cdots\cdots\right)$$

$$= \frac{1}{z} + \frac{3+1}{z^2} + \cdots + \frac{3k-3+1}{z^k}\cdots\cdots \qquad k \geq 1$$

$$\text{Coefficient of } z^{-k} = (3k-2) \qquad k \geq 1$$

$$z^{-1}[F(z)] = (3k-2),$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-\cos(2n\pi+\pi)+1}{\left(n+\frac{1}{2}\right)} + \frac{\cos(2n\pi-\pi)-1}{\left(n-\frac{1}{2}\right)}\right]$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{\cos(2n\pi)+1}{\frac{(2n+1)}{2}} + \frac{-\cos 2n\pi-1}{\frac{(2n-1)}{2}}\right] = \frac{\sqrt{2}}{\pi}\left[\frac{(-1)^{2n}+1}{2n+1} - \frac{(-1)^{2n}+1}{2n-1}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1+1}{2n+1} - \frac{1+1}{2n-1}\right] = \frac{2\sqrt{2}}{\pi}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{\cancel{2n}-1-\cancel{2n}-1}{4n^2-1}\right] = \frac{2\sqrt{2}}{\pi}\left[\frac{(-2)}{4n^2-1}\right] = \frac{(-4\sqrt{2})}{\pi(4n^2-1)} = \frac{4\sqrt{2}}{\pi(1-4n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\ \sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} \sqrt{1-\cos x}\ \ \sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\ \sin\frac{x}{2}\ \sin nx\, dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin nx \sin\frac{x}{2}\, dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(nx-\frac{x}{2}\right) - \cos\left(nx+\frac{x}{2}\right)\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{\sin\left(n-\frac{1}{2}\right)x}{\left(n-\frac{1}{2}\right)} - \frac{\sin\left(n+\frac{1}{2}\right)x}{\left(n+\frac{1}{2}\right)}\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{\sin\left(n-\frac{1}{2}\right)2\pi-0}{\left(n-\frac{1}{2}\right)} - \frac{\sin\left(n+\frac{1}{2}\right)2\pi-0}{\left(n+\frac{1}{2}\right)}\right]$$

$$= \frac{\sqrt{2}}{2\pi}\ [(0-0)-(0-0)] = 0$$

$$\therefore\ \sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{\pi(1-4n^2)}\cos nx$$

Put, $x = 0$

$$\frac{\lim\limits_{x\to 0^+}\sqrt{1-\cos x} + \lim\limits_{x\to 0^-}\sqrt{1-\cos x}}{2} = \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(1-4n^2)}$$

$$\frac{\lim\limits_{x\to 0^+}\sqrt{1-\cos x} + \lim\limits_{x\to 2\pi^-}\sqrt{1-\cos x}}{2} = \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(1-4n^2)}$$

$$0+0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)} = \frac{2\sqrt{2}}{\pi}$$

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

Q. 5(b) If $f(k) = 4^k U(k)$ and $g(k) + 5^k U(k)$, then find the z-transform of $f(k) \cdot g(k)$. **(6 Marks)**

Ans. : $U(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$

$$\therefore Z\{U(k)\} = \sum_{n=-\infty}^{\infty} U(k) \cdot z^{-k}$$

$$\therefore Z\{U(k) = \sum_{k=-\infty}^{-1} 0 + \sum_{n=-\infty}^{\infty} 1 \cdot \frac{1}{z^k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots\dots = \frac{1}{1 - 1/z}$$

$$= \frac{z}{z-1} \quad (\text{ROC} : 1 < |z| \text{ or } |z| > 1)$$

By Scaling property of Z-Transform,

If $Z\{f(k)\} = F(z)$ then $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

$$\therefore Z\{4^k U(k)\} = \frac{z/4}{z/4 - 1}$$

$$\therefore F(z) = Z\{4^k U(k)\} = \frac{z}{z-4}$$

Similarly, for $g(k) = 5^k U(k)$.

$$\therefore G(z) = \frac{z}{z-5} \quad (\text{ROC} : |z| > 5)$$

$\therefore$ By convolution theorem, $H(z) = F(z)\, G(z)$

$$\therefore H(z) = \frac{z}{z-4} \times \frac{z}{z-5}$$

$$\therefore Z\{f(k) * g(k)\} = \frac{z^2}{(z+4)(z-5)}$$

❑❑❑

Dec. 2014

Chapter 1 : Complex Variable and Mapping [Total Marks - 12]

Q. 2(a) Find an analytic function f(z) whose real part is $u = e^x (x \cos y - y \sin y)$. **(6 Marks)**

Ans. :

$$\text{Let } f(z) = u + iv \text{ be analytic function.}$$

$$\text{Since, } u = e^x (x \cos y - y \sin y)$$

$$\therefore \frac{\partial u}{\partial x} = e^x (\cos y \cdot 1 - 0) + (x\cos y - y\sin y) \cdot e^x = e^x (\cos y + x\cos y - y\sin y)$$

$$\therefore \frac{\partial u}{\partial y} = e^x [x \cdot -\sin y - (y \cdot \cos y + \sin y \cdot 1)] = -e^x (x\sin y + y\cos y + \sin y)$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(by C-R equations)}$$

$$= e^x (\cos y + x\cos y - y\sin y) + ie^x (x\sin y + y\cos y + \sin y)$$

By Milne Thompson's method, put $x = z$ and $y = 0$

$$f'(z) = e^z (1 + z \cdot 1 - 0) + ie^z (0 + 0 + 0) = e^z (1 + z)$$

$$\therefore f(z) = \int f'(z)\, dz = \int (1 + z) \cdot e^z\, dz = (1 + z) \cdot e^z - (0 + 1) \cdot e^z + C$$

$$= e^z (1 + z - 1) + C$$

$$\therefore \text{Analytic function } f(z) = z e^z + C$$

Q. 3(b) Find the image of the real axis under the transformation $w = \frac{2}{z + i}$. **(6 Marks)**

Ans. :

Suppose, $w = \frac{2}{z+i}$ $\qquad \therefore z + i = \frac{2}{w}$

$$\therefore x + iy + i = \frac{2}{(u + iv)} \times \frac{(u - iv)}{(u - iv)} \qquad \therefore x + i(y + 1) = \frac{2u - 2iv}{u^2 - v^2}$$

Comparing imaginary part on both sides,

$$y + 1 = \frac{-2v}{u^2 + v^2} \quad \text{...(1)}$$

Now, equation of real axis in z-plane is $y = 0$ ...(2)

$\therefore$ From (1) and (2),

$$0 + 1 = \frac{-2v}{u^2 + v^2}$$

$$\therefore u^2 + v^2 = -2v$$

$$\therefore u^2 + v^2 + 2v = 0, \text{ which is a circle in the W-plane.}$$

Comparing above equation with $u^2 + v^2 + 2gu + 2fv + c = 0$,

we get, $2g = 0$, $2f = 2$, $c = 0$

$\therefore$ Centre $= (-g, -f) = (0, -1)$ and Radius $= \sqrt{g^2 + f^2 - c} = \sqrt{0^2 + (-1)^2 - 0} = 1$

Q. 1(a) Find the Laplace Transform of sint cos2t cosht. **(5 Marks)**

Ans. :

$$L[\cosh 2t \cdot (\sin t \cos 2t)] = L\left[\left(\frac{e^{t}+e^{-t}}{2}\right)(\sin t \cos 2t)\right]$$

$$= \frac{1}{2}\{L[e^{t}(\sin t \cos 2t)] + L[e^{-t}(\sin t \cos 2t)]\}$$

By First shifting property

$$= \frac{1}{2}\left\{\frac{1}{2}\left[\frac{3}{(s-1)^2+9} - \frac{1}{(s-1)^2+1}\right] + \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]\right\}$$

$$= \frac{1}{2}\times\frac{1}{2}\left\{\frac{3}{s^2-2s+10} - \frac{1}{s^2-2s+2} + \frac{3}{s^2-2s+10} - \frac{1}{s^2-2s+2}\right\}$$

$$= \frac{1}{4}\left\{3\left[\frac{1}{s^2-2s+10} + \frac{1}{s^2+2s+10}\right] - \left[\frac{1}{s^2-2s+2} + \frac{1}{s^2+2s+2}\right]\right\}$$

$$= \frac{3}{4}\left[\frac{(s^2+10+2s)+(s^2+10+2s)}{(s^2+10-2s)+(s^2+10+2s)}\right] - \frac{1}{4}\left[\frac{(s^2+2+2s)+(s^2+2+2s)}{(s^2+2-2s)+(s^2+2+2s)}\right]$$

$$= \frac{3}{4}\times 2\left[\frac{s^2+10}{(s^2+10)^2-(2s)^2}\right] - \frac{1}{4}\times 2\left[\frac{s^2+2}{(s^2+2)^2-(2s)^2}\right]$$

$$= \frac{3}{2}\left[\frac{s^2+10}{s^4+20s^2+100-4s^2}\right] - \frac{1}{2}\left[\frac{s^2+2}{s^4+4s^2+4-4s^2}\right]$$

$$\therefore L[\cosh 2t \cdot (\sin t \cos 2t)] = \frac{1}{2}\left[\frac{3(s^2+10)}{s^4+16s^2+100} - \frac{s^2+2}{s^4+4}\right]$$

Q. 2(b) Find inverse Laplace Transform by using convolution theorem $\frac{1}{(s-3)(s+4)^2}$. **(6 Marks)**

Ans. :

$$F(s)\,G(s) = \frac{1}{(s-3)(s+4)^2} = \frac{1}{(s-3)} \times \frac{1}{(s+4)^2}$$

$$\therefore f(s) = L^{-1}\left[\frac{1}{(s+4)^2}\right] = e^{-4t}\,t$$

$$G(s) = L^{-1}\left[\frac{1}{s-3}\right] = e^{3t}$$

By convolution theorem,

$$\therefore L^{-1}\left[\frac{1}{(s+4)^2}\times\frac{1}{s-3}\right] = \int_0^t e^{-4u}\,u \times e^{3(t-u)}\,du = \int_0^t e^{-4u}\,u \times e^{3t}e^{-3u}\,du$$

$$= e^{3t}\int_0^t ue^{-7u}\,du = e^{3t}\left[u\cdot\frac{-e^{-7u}}{-7} - 1\cdot\frac{e^{-7u}}{(-7)^2}\right]_0^t$$

$$= e^{3t}\left[\frac{-e^{-7u}}{49}(7u+1)\right]_0^t = \frac{-1}{49}e^{3t}\,[e^{-7t}(7t+1) - e^{0}(0+1)]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1+1}{2n+1} - \frac{1+1}{2n-1}\right] = \frac{2\sqrt{2}}{\pi}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{2\cancel{n}1 - 2\cancel{n}1}{4n^2-1}\right] = \frac{2\sqrt{2}}{\pi}\left[\frac{(-2)}{4n^2-1}\right] = \frac{(-4\sqrt{2})}{\pi(4n^2-1)} = \frac{4\sqrt{2}}{\pi(1-4n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\ \sin nx\ dx = \frac{1}{\pi}\int_0^{2\pi}\sqrt{1-\cos x}\ \ \sin nx\ dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\ \sin\frac{x}{2}\ \sin nx\ dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin nx\sin\frac{x}{2}\,dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(nx-\frac{x}{2}\right) - \cos\left(nx+\frac{x}{2}\right)\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{\sin\left(n-\frac{1}{2}\right)x}{\left(n-\frac{1}{2}\right)} - \frac{\sin\left(n+\frac{1}{2}\right)x}{\left(n+\frac{1}{2}\right)}\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{\sin\left(n-\frac{1}{2}\right)2\pi - 0}{\left(n-\frac{1}{2}\right)} - \frac{\sin\left(n+\frac{1}{2}\right)2\pi - 0}{\left(n+\frac{1}{2}\right)}\right]$$

$$= \frac{\sqrt{2}}{2\pi}\ [(0-0)-(0-0)] = 0$$

$$\therefore\ \sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{\pi(1-4n^2)}\cos nx$$

Put, $x = 0$

$$\frac{\lim\limits_{x\to 0^+}\sqrt{1-\cos x} + \lim\limits_{x\to 0^-}\sqrt{1-\cos x}}{2} = \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(1-4n^2)}$$

$$\frac{\lim\limits_{x\to 0^+}\sqrt{1-\cos x} + \lim\limits_{x\to 2\pi^-}\sqrt{1-\cos x}}{2} = \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(1-4n^2)}$$

$$0+0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)} = \frac{2\sqrt{2}}{\pi}$$

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

Q. 5(b) If $f(k) = 4^k U(k)$ and $g(k) + 5^k U(k)$, then find the z-transform of $f(k) \cdot g(k)$. **(6 Marks)**

Ans. : $U(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$

$$\therefore Z\{U(k)\} = \sum_{n=-\infty}^{\infty} U(k) \cdot z^{-k}$$

$$\therefore Z\{U(k) = \sum_{k=-\infty}^{-1} 0 + \sum_{n=-\infty}^{\infty} 1 \cdot \frac{1}{z^k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots\dots = \frac{1}{1 - 1/z}$$

$$= \frac{z}{z-1} \quad (\text{ROC} : 1 < |z| \text{ or } |z| > 1)$$

By Scaling property of Z-Transform,

If $Z\{f(k)\} = F(z)$ then $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

$$\therefore Z\{4^k U(k)\} = \frac{z/4}{z/4 - 1}$$

$$\therefore F(z) = Z\{4^k U(k)\} = \frac{z}{z-4}$$

Similarly, for $g(k) = 5^k U(k)$.

$$\therefore G(z) = \frac{z}{z-5} \quad (\text{ROC} : |z| > 5)$$

$\therefore$ By convolution theorem, $H(z) = F(z)\, G(z)$

$$\therefore H(z) = \frac{z}{z-4} \times \frac{z}{z-5}$$

$$\therefore Z\{f(k) * g(k)\} = \frac{z^2}{(z+4)(z-5)}$$

□□□

Q. 2(a) Find an analytic function f(z) whose real part is $u = e^x (x \cos y - y \sin y)$. (6 Marks)

Ans. :

$$\text{Let } f(z) = u + iv \text{ be analytic function.}$$

$$\text{Since, } u = e^x (x \cos y - y \sin y)$$

$$\therefore \frac{\partial u}{\partial x} = e^x (\cos y \cdot 1 - 0) + (x\cos y - y\sin y) \cdot e^x = e^x (\cos y + x\cos y - y\sin y)$$

$$\therefore \frac{\partial u}{\partial y} = e^x [x \cdot - \sin y - (y \cdot \cos y + \sin y \cdot 1)] = -e^x (x\sin y + y\cos y + \sin y)$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{(by C-R equation)}$$

$$= e^x (\cos y + x\cos y - y\sin y) + ie^x (x\sin y + y\cos y + \sin y)$$

By Milne Thompson's method, put $x = z$ and $y = 0$

$$f'(z) = e^z (1 + z \cdot 1 - 0) + ie^z (0 + 0 + 0) = e^z (1 + z)$$

$$\therefore f(z) = \int f'(z)\, dz = \int (1 + z) \cdot e^z\, dz = (1 + z) \cdot e^z - (0 + 1) \cdot e^z + C$$

$$= e^z (1 + z - 1) + C$$

$$\therefore \text{Analytic function } f(z) = z e^z + C$$

Q. 3(b) Find the image of the real axis under the transformation $w = \dfrac{2}{z + i}$ (6 Marks)

Ans. :

$$\text{Suppose,} \quad w = \frac{2}{z + i} \qquad \therefore z + i = \frac{2}{w}$$

$$\therefore x + iy + i = \frac{2}{(u + iv)} \times \frac{(u - iv)}{(u - iv)} \qquad \therefore x + i(y + 1) = \frac{2u - 2iv}{u^2 - v^2}$$

Comparing imaginary part on both sides,

$$y + 1 = \frac{-2v}{u^2 + v^2}$$

Now, equation of real axis in z-plane is $y = 0$

$\therefore$ From (1) and (2),

$$0 + 1 = \frac{-2v}{u^2 + v^2}$$

$$\therefore u^2 + v^2 = -2v$$

$$\therefore u^2 + v^2 + 2v = 0, \text{ which is a circle in the W-plane.}$$

Comparing above equation with $u^2 + v^2 + 2gu + 2fv + c = 0$,

we get, $2g = 0$, $2f = 2$, $c = 0$

$\therefore$ Centre $= (-g, -f) = (0, -1)$ and Radius $= \sqrt{g^2 + f^2 - c} = \sqrt{0^2 + (-1)^2 - 0} = 1$

(5 Marks)

Q. 1(a) Find the Laplace Transform of sint cos2t cosht.

Ans. :

$$L[\cosh 2t \cdot (\sin t \cos 2t)] = L\left[\left(\frac{e^t+e^{-t}}{2}\right)(\sin t \cos 2t)\right]$$

$$= \frac{1}{2}\{L[e^t(\sin t \cos 2t)] + L[e^{-t}(\sin t \cos 2t)]\}$$

By First shifting property

$$= \frac{1}{2}\left\{\frac{1}{2}\left[\frac{3}{(s-1)^2+9} - \frac{1}{(s-1)^2+1}\right] + \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]\right\}$$

$$= \frac{1}{2}\times\frac{1}{2}\left\{\frac{3}{s^2-2s+10} - \frac{1}{s^2-2s+2} + \frac{3}{s^2-2s+10} - \frac{1}{s^2-2s+2}\right\}$$

$$= \frac{1}{4}\left\{3\left[\frac{1}{s^2-2s+10} + \frac{1}{s^2+2s+10}\right] - \left[\frac{1}{s^2-2s+2} + \frac{1}{s^2+2s+2}\right]\right\}$$

$$= \frac{3}{4}\left[\frac{(s^2+10+2s)+(s^2+10+2s)}{(s^2+10-2s)+(s^2+10+2s)}\right] - \frac{1}{4}\left[\frac{(s^2+2+2s)+(s^2+2+2s)}{(s^2+2-2s)+(s^2+2+2s)}\right]$$

$$= \frac{3}{4}\times 2\left[\frac{s^2+10}{(s^2+10)^2-(2s)^2}\right] - \frac{1}{4}\times 2\left[\frac{s^2+2}{(s^2+2)^2-(2s)^2}\right]$$

$$= \frac{3}{2}\left[\frac{s^2+10}{s^4+20s^2+100-4s^2}\right] - \frac{1}{2}\left[\frac{s^2+2}{s^4+4s^2+4-4s^2}\right]$$

$$\therefore L[\cosh 2t \cdot (\sin t \cos 2t)] = \frac{1}{2}\left[\frac{3(s^2+10)}{s^4+16s^2+100} - \frac{s^2+2}{s^4+4}\right]$$

Q. 2(b) Find inverse Laplace Transform by using convolution theorem $\frac{1}{(s-3)(s+4)^2}$. **(6 Marks)**

Ans. :

$$F(s)\,G(s) = \frac{1}{(s-3)(s+4)^2} = \frac{1}{(s-3)} \times \frac{1}{(s+4)^2}$$

$$\therefore f(s) = L^{-1}\left[\frac{1}{(s+4)^2}\right] = e^{-4t}\,t$$

$$G(s) = L^{-1}\left[\frac{1}{s-3}\right] = e^{3t}$$

By convolution theorem,

$$\therefore L^{-1}\left[\frac{1}{(s+4)^2} \times \frac{1}{s-3}\right] = \int_0^t e^{-4u}\,u \times e^{3(t-u)}\,du = \int_0^t e^{-4u}\,u \times e^{3t}e^{-3u}\,du$$

$$= e^{3t}\int_0^t ue^{-7u}\,du = e^{3t}\left[u\cdot\frac{-e^{-7u}}{-7} - 1\cdot\frac{e^{-7u}}{(-7)^2}\right]_0^t$$

$$= e^{3t}\left[\frac{-e^{-7u}}{49}(7u+1)\right]_0^t = \frac{-1}{49}e^{3t}\left[e^{-7t}(7t+1) - e^0(0+1)\right]$$

$$= \frac{-1}{49}[-e^{-4t}(7t+1)-e^{3t}] = \frac{1}{49}[e^{3t}-e^{-4t}(7t+1)]$$

$$\therefore L^{-1}\left[\frac{1}{(s+4)^2}\times\frac{1}{s-3}\right] = \frac{1}{49}[e^{3t}-e^{-4t}(7t+1)]$$

Q. 4(a) Find the Laplace transform of $f(t) = \begin{cases} E & 0 \le t \le p/2 \\ -E & p/2 \le t \le p \end{cases} \cdot f(t+p) = f(t).$ **(6 Marks)**

Ans. : Since $f(t+p) = f(t)$, $f(t)$ is periodic function with period $(a) = p$.

$\therefore$ By definition of Laplace transform for periodic function,

$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt = \frac{1}{1-e^{-ps}}\left[\int_0^{p/2} e^{-st}\cdot E\,dt + \int_{p/2}^{p} e^{-st}\cdot -E\,dt\right]$$

$$= \frac{1}{1-e^{-ps}}\times E\left\{\left[\frac{e^{-st}}{-s}\right]_0^{p/2} - \left[\frac{e^{-st}}{-s}\right]_{p/2}^{p}\right\} = \frac{E}{1-e^{-ps}}\left\{-\frac{1}{s}[e^{-sp/2}-1]+\frac{1}{s}[e^{-sp}-e^{-sp/2}]\right\}$$

$$= \frac{E}{1-e^{-ps}}\times\frac{1}{s}\{-e^{-sp/2}+1+e^{-sp}-e^{-sp/2}\}$$

$$= \frac{E}{1-(e^{-ps/2})^2}\times\frac{1}{s}\{1-2e^{-sp/2}+e^{-sp/2}+(e^{-sp/2})^2\}$$

$$= \frac{E}{(1+e^{-sp/2})(1-\cancel{e}^{-sp/2})}\times\frac{1}{s}\times(1-e^{-sp/2})^2 = \frac{E}{s}\times\frac{(1-e^{-sp/2})}{(1+e^{-sp/2})}$$

$$= \frac{E}{s}\times\frac{\cancel{e^{-sp/4}}\,(e^{sp/4}-e^{-sp/4})}{\cancel{e^{-sp/4}}\,(e^{sp/4}+e^{-sp/4})} = \frac{E}{s}\cdot\frac{2\sinh(ps/4)}{2\cosh(ps/4)} = \frac{E}{s}\tanh\left(\frac{ps}{4}\right)$$

Q. 5(c) Solve the differential equation using Laplace transform. **(6 Marks)**

$(D^2+2D+5)y = e^{-t}\sin t \cdot y(0) = 0,\ y'(0) = 1.$

Ans. :

$$\text{Consider, } L[\sin t] = \frac{1}{s^2+1^2}$$

$$\therefore L[e^{-t}\sin t] = \frac{1}{(s+1)^2+1} \quad \text{(First shifting)} \qquad \text{...(1)}$$

$$\text{Given, } (D^2+2D+5)\,y = e^{-t}\sin t$$

$$\therefore [D^2y] = 2L[Dy]+5L[y] = L[e^{-t}\sin t]$$

$$\therefore \left[s^2\bar{y}-sy(0)-y'(0)\right]+2\left[s\bar{y}-y(0)\right]+5\bar{y} = \frac{1}{(s+1)^2+1} \quad \text{(From Equation (1))}$$

$$\therefore s^2\bar{y}-s(0)-1+2s\bar{y}-(0)+5\bar{y} = \frac{1}{s^2+2s+2}$$

$$\therefore \bar{y}(s^2+2s+5) = \frac{1}{s^2+2s+2}+1$$

$$\therefore \bar{y}(s^2+2s+5) = \frac{1+s^2+2s+2}{s^2+2s+2}$$

$$\therefore \bar{y} = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$\therefore y = L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right] = L^{-1}\left[\frac{s^2+2s+1+2}{(s^2+2s+1+4)(s^2+2s+1+1)}\right]$$

$$= L^{-1}\left\{\frac{(s+1)^2+2}{[(s+1)^2+4]\cdot[(s+1)^2+1]}\right\} = e^{-t}L^{-1}\left\{\frac{s^2+2}{(s^2+4)\cdot(s^2+1)}\right\} \quad \text{(First shifting)} \quad \ldots(2)$$

$$\text{Let } \frac{s^2+2}{(s^2+4)\cdot(s^2+1)} = \frac{A}{s^2+4}+\frac{B}{s^2+1} \quad \ldots(3)$$

$$\therefore A = \left.\frac{s^2+2}{s^2+1}\right|_{s^2=-4} = \frac{-4+2}{-4+1} = \frac{2}{3} \quad \ldots(4)$$

$$\therefore B = \left.\frac{s^2+2}{s^2+4}\right|_{s^2=-1} = \frac{-1+2}{-1+4} = \frac{1}{3} \quad \ldots(5)$$

From Equations (2), (3), (4) and (5)

$$\therefore \quad y = e^{-t}L^{-1}\left\{\frac{2/3}{s^2+4}+\frac{1/3}{s^2+1}\right\} = e^{-t}\left\{\frac{2}{3}\times\frac{1}{2}\sin 2t+\frac{1}{3}\sin t\right\} = \frac{e^{-t}}{3}(\sin 2t+\sin t)$$

Q. 6(a) If $\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\,dt = \frac{3}{8}$ find the value of α. **(6 Marks)**

Ans. :

We have $\int_0^\infty e^{-st}\sin(t+\alpha)\cos(t-\alpha)\,dt$

$$= \frac{1}{2}\int_0^\infty e^{-st}[\sin 2t+\sin 2\alpha]\,dt = \frac{1}{2}[L(\sin 2t)+\sin 2\alpha\cdot L(1)]$$

$$= \frac{1}{2}\left[\frac{2}{s^2+4}+\sin 2\alpha\left(\frac{1}{s}\right)\right]$$

Putting $s = 2$ we get

$$\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)\,dt = \frac{1}{2}\left[\frac{2}{8}+\frac{\sin 2\alpha}{2}\right] = \frac{1}{8}+\frac{\sin 2\alpha}{4}$$

But this equals to $\frac{3}{8}$ $\quad \therefore \frac{3}{8} = \frac{1}{8}+\frac{\sin 2\alpha}{4} \Rightarrow \sin 2\alpha = 1$

$$2\alpha = \frac{\pi}{2} \qquad \alpha = \frac{\pi}{4}$$

Chapter 3 : Fourier Series [Total Marks - 29]

Q. 1(b) Find the Fourier series expansion of $f(x) = x^2$ in $(-\pi, \pi)$. **(5 Marks)**

Ans. : Let $x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ in $(-\pi, \pi)$

$f(x) = x^2 \qquad \therefore f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore$ f(x) is even function $\therefore b_n = 0$ Here, $l = \pi$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \frac{2}{\pi}\left[\frac{x^3}{3} - 0\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{\pi}\int_0^\pi x^2 \cos\frac{n\cancel{\pi}x}{\cancel{\pi}}\,dx$$

By integration,

$$= \frac{2}{\pi}\left[x^2 \cdot \frac{\sin nx}{n} - 2x \cdot \frac{-\cos nx}{n^2} + 2\cdot\frac{-\sin nx}{n^3}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\left(x^2 \cdot \frac{\sin n\pi}{n} + 2\pi \cdot \frac{\cos n\pi}{n^2} - 2\cdot\frac{\sin n\pi}{n^3}\right) - \left(0 - 0 + 2\cdot\frac{\sin 0}{n^3}\right)\right]$$

$$= \frac{2}{\pi}\left[0 + 2\pi \cdot \frac{(-1)^2}{n^2} + 0 - 0\right] = \frac{4(-1)^n}{n^2}$$

In Fourier series,
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\therefore x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos\frac{n\pi x}{\pi} + 0$$

$$\therefore x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$\therefore x^2 = \frac{\pi^2}{3} + 4\left(\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots\right)$$

$$\therefore x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\right)$$

Q. 3(c) Obtain the Fourier series expansion of $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}$

Here, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi}{8}$ **(8 Marks)**

Ans. :

$$f(x) = \pi x \qquad 0 < x \le \pi$$
$$= 2\pi - x \qquad \pi \le x < 2\pi$$

Let $c = 0$ and $c + 2l = 2$ $\qquad \therefore 0 + 2l = 2$

$l = 2$

$$\text{Now, } a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx = \frac{1}{1}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx\right]$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2 = \pi\left[\frac{1}{2} - 0\right] + \pi\left[\left(4 - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \pi\left[\frac{1}{2} + 4 - 2 - 2 + \frac{1}{2}\right] = \pi$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{1}\left[\int_0^1 \pi x \cdot \cos\frac{n\pi x}{1}\,dx + \int_1^2 \pi(2-x)\cos\frac{n\pi x}{1}\,dx\right]$$

$$= \pi\left[x \cdot \frac{\sin(n\pi x)}{n\pi} - 1 \cdot \frac{-\cos(n\pi x)}{n^2\pi^2}\right]_0^1 + \pi\left[(2-x)\cdot\frac{\sin(n\pi x)}{n\pi} - (0-1)\frac{-\cos(n\pi x)}{n^2\pi^2}\right]_1^2$$

$$= \pi\left[\left(1\cdot\frac{\sin n\pi}{n\pi} + \frac{\cos n\pi}{n^2\pi^2}\right) - \left(0 + \frac{\cos 0}{n^2\pi^2}\right)\right]$$

$$+ \pi\left[\left(0 - \frac{\cos 2n\pi}{n^2\pi^2}\right) - \left(1\cdot\frac{\sin n\pi}{n\pi} - \frac{\cos n\pi}{n^2\pi^2}\right)\right]$$

$$= \pi\left[0 + \frac{(-1)^n}{n^2\pi^2} - 0 - \frac{1}{n^2\pi^2}\right] + \pi\left[0 - \frac{1}{n^2\pi^2}0 + \frac{(-1)^n}{n^2\pi^2}\right] = 2\pi \times \frac{1}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{1}\left[\int_0^1 \pi x \cdot \sin\frac{n\pi x}{1}\,dx + \int_1^2 \pi(2-x)\sin\frac{n\pi x}{1}\,dx\right]$$

$$= \pi\left[x \cdot \frac{-\cos(n\pi x)}{n\pi} - 1 \cdot \frac{-\sin(n\pi x)}{n^2\pi^2}\right]_0^1 + \pi\left[(2-x)\frac{-\cos(n\pi x)}{n\pi} - (0-1)\frac{-\sin(n\pi x)}{n^2\pi^2}\right]_1^2$$

$$= \pi\left[\left(1\cdot\frac{-\cos n\pi}{n\pi} + \frac{\sin n\pi}{n^2\pi^2}\right) - \left(0 + \frac{\sin 0}{n^2\pi^2}\right)\right]$$

$$+ \pi\left[\left(0 - \frac{\sin 0}{n^2\pi^2}\right) - \left(-1\cdot\frac{-\cos n\pi}{n\pi} - \frac{\sin n\pi}{n^2\pi^2}\right)\right]$$

$$= \pi\left[\frac{(-1)^n}{n\pi} + 0 - 0 - 0\right] + \pi\left[0 - 0 + \frac{(-1)^n}{n\pi} - 0\right] = 0$$

In Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}\sum_{n=1}^{\infty} a_n\sin\frac{n\pi x}{l} = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2\pi}{n^2\pi^2}[(-1)^n - 1]\cos\frac{n\pi x}{l} + 0$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2\cos 1\pi x}{1^2} + 0 - \frac{2\cos 3\pi x}{3^2} + 0 - \ldots\right]$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos 1\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2}\ldots\right] \qquad \ldots(1)$$

Deduction : From question, $f(0) = \pi x = \pi(0) = 0$

$$\therefore f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2}\ldots\right]$$

$$\therefore 0 - \frac{\pi}{2} = -\frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\ldots\right]$$

$$\therefore -\frac{\pi}{2}\times-\frac{\pi}{4} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\ldots\ldots$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\ldots\ldots$$

Q. 4(c) Find the Fourier integral for $f(x) = \begin{cases} 1-x^2 & 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases}$. Hence evaluate $\int_0^\infty \left[\frac{\lambda\cos\lambda - \sin\lambda}{\lambda^3}\right] \cos\left(\frac{\lambda}{2}\right) d\lambda$. **(8 Marks)**

Ans. :

$$f(x) = \begin{cases} 1-x^2 & \text{when } |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

$f(x)$ is even since $f(-x) = f(x)$

hence, Fourier cosine integral for $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \left\{\int_0^\infty f(t)\cos\omega t\, dt\right\} d\omega = \frac{2}{\pi}\int_0^\infty \cos\omega x \left\{\int_0^1 f(t)\cos\omega t\, dt\right\} d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left\{\int_0^1 f(1-t^2)\cos\omega t\, dt\right\} d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left\{(1-t)^2\frac{\sin\omega t}{\omega} - (-2t)\left(\frac{-\cos\omega t}{\omega^2}\right) + (-2)\left[\frac{-\sin\omega t}{\omega^3}\right]_0^1\right\} d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left\{\frac{-2\cos\omega}{\omega^2} + \frac{2\sin\omega}{\omega^3}\right\} d\omega$$

$$f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cos\omega x \cdot d\omega$$

$$\text{thus, } 1-x^2 = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cos\omega x \cdot d\omega$$

Put $x = \frac{1}{2}$ $\quad 1-\frac{1}{4} = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cos\frac{\omega}{2}\, d\omega$

Put $\omega = x$ $\quad \therefore\ d\omega = dx$

$$\frac{3}{4} = \frac{4}{\pi}\int_0^\infty \frac{\sin x - x\cos x}{x^3} \cos\frac{x}{2}\, d\omega = \int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx = \frac{-3\pi}{16}$$

Q. 6(c) Find half range sine series for $f(x) = lx - x^2$, (0, 1). Hence prove that $\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots = \frac{\pi^6}{960}$.

(8 Marks)

Ans. :

For half range sine series, $a_0 = a_n = 0$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} dx = \frac{2}{l}\int_0^l (lx - x^2) \sin\frac{n\pi x}{l} dx$$

$$= \frac{2}{l}\left[(lx - x^2) - \cos\frac{n\pi x}{l} \cdot \frac{l}{n\pi} - (l - 2x) \cdot - \sin\frac{n\pi x}{l} \cdot \frac{l^2}{n^2\pi^2} + (-2) \cos\frac{n\pi x}{l} \cdot \frac{l}{n^3\pi^3}\right]_0^1$$

$$= \frac{2}{l}\left\{\left[- (l^2 - l^2) \cos n\pi \cdot \frac{l}{n\pi} + (1 - 2l) \sin n\pi \cdot \frac{l^2}{n^2\pi^2} - 2\cos n\pi \cdot \frac{l^3}{n^3\pi^3}\right] - \left[0 + 0 - 2\cdot 1 \cdot \frac{1}{n^3\pi^3}\right]\right\}$$

$$= \frac{2}{l}\left[0 + 0 - 2(-1)^n \frac{l^3}{n^3\pi^3} - 0 + 0 + 2 \cdot \frac{l^3}{n^3\pi^3}\right]$$

$$= \frac{2}{l} \times \frac{2l^3}{n^3\pi^3} [1 - (-1)^n] = \frac{4l^2}{n^3\pi^3} [1 - (-1)^n]$$

$\therefore$ Half range Fourier sine series is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\therefore lx - x^2 = \sum_{n=1}^{\infty} \frac{4l^2}{n^3\pi^3} [1 - (-1)^n] \sin\frac{n\pi x}{l} = \frac{4l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n^3} \sin\frac{n\pi x}{l}$$

$$= \frac{4l^2}{\pi^3}\left[\frac{2}{l^3} \sin\frac{\pi x}{l} + 0 + \frac{2}{3^3} \sin\frac{3\pi x}{l} + 0 \ldots\right] = \frac{8l^2}{\pi^3}\left[\frac{1}{l^3} \sin\frac{\pi x}{l} + \frac{1}{3^3} \sin\frac{3\pi x}{l} + \ldots\right]$$

Deduction :

Using Parseval's identity, $\frac{1}{l}\int_0^1 [f(x)]^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + a_n^2)$

$$\therefore \frac{1}{l}\int_0^1 (lx - x^2)^2 dx = 0 + \frac{1}{2}\sum_{n=1}^{\infty} 0 + \left\{\frac{4l^2 [1 - (-1)^n]}{n^3\pi^3}\right\}^2$$

$$\frac{1}{l}\int_0^1 (l^2x^2 - 2lx^3 + x^4) dx = \frac{1}{2}\sum_{n=1}^{\infty} \frac{16 l^4 [1 - (-1)^n]^2}{n^6\pi^6}$$

$$\therefore \frac{1}{l}\left[l^2 \cdot \frac{x^3}{3} - 2l \cdot \frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{8l^4}{\pi^6}\left[\frac{2^2}{1^6} + 0 + \frac{2^2}{3^6} + 0 \ldots\right]$$

$$\therefore \frac{1}{l}\left[\left(\frac{l^5}{3} - \frac{l^5}{4} + \frac{l^5}{5}\right) - (0 - 0 + 0)\right] = \frac{8l^4}{\pi^6}\left[\frac{2^2}{1^6} + 0 + \frac{2^2}{3^6} + \frac{2^2}{5^6} \ldots\right]$$

$$\therefore \frac{1}{l} \times \frac{l^5}{30} = \frac{8l^4}{l^6} \times 2^2 \left[\frac{1}{1^6} + 0 + \frac{1}{3^6} + \frac{1}{5^6} \ldots\right]$$

$$\therefore \frac{\ell^4}{30} \times \frac{\pi^6}{8\ell^4 \times 2^2} = \frac{8\ell^4}{\ell^6} \times 2^2 \frac{1}{1^6} + \frac{1}{3^6} + \ldots$$

$$\therefore \frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots$$

Chapter 4 : Vector Algebra & Calculus [Total Marks - 31]

Q. 1(d) Find the directional derivative of $4x^2 + x^2yz$ at $(1, -2, -1)$ in the direction $2\vec{i} - \vec{j} - 2\vec{k}$.

(5 Marks)

Ans. :

$$\text{Let } \phi = 4xz^2 + x^2yz$$

$$\therefore \nabla\phi = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z} = \vec{i}\,(4z^2 + 2xyz) + \vec{j}\,(0 + x^2z) + \vec{k}\,(8xz + x^2y)$$

At point $(1, -2, -1)$

$$\nabla f = \vec{i}\,(4 \times 1^2 + 2 \times 1 \times -2 \times -1) + \vec{j}\,(1^2 \times -1) + \vec{k}\,(8 \times 1 \times -1 + 1^2 \times -2)$$

$$= 8\vec{i} - 1\vec{j} - 10\vec{k}$$

$$\text{Let } \vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$$

$$\therefore |\vec{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = 3$$

Directional derivative of f in the direction of $\vec{a} = \nabla f \cdot \dfrac{\vec{a}}{|\vec{a}|}$

$$= (8\vec{i} - 1\vec{j} - 10\vec{k})\frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{1}{3}[8 \times 2 + (-1) \times (-1) + (-10) \times (-2)] = \frac{37}{3}$$

Q. 2(c) Prove that $\vec{F} = (6xy^2 - 2z^3)\vec{i} + (6x^2y + 2yz)\vec{j} + (y^2 - 6z^2x)\vec{k}$ is a conservative field. Find the scalar potential ϕ such that $\vec{F} = \nabla\phi$. Hence find the work done by $\vec{F}$ in displacing a particle from A (1, 0, 2) to B(0, 1, 1) along AB. (Vector Integration)

Ans. : **(8 Marks)**

$$\vec{F} = (6xy^2 - 2z^3)\vec{i} + (6x^2y + 2yz)\vec{j} + (y^2 - 6z^2x)\vec{k}$$

$$\therefore \text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6xy^2 - 2z^3 & 6x^2y + 2yz & y^2 - 6z^2x \end{vmatrix}$$

$$= \vec{i}\,[2y - 2y] - \vec{j}\,[-6z^2 + 6z^2] + \vec{k}\,[12xy - 12xy] = 0 - 0 + 0 = 0$$

$\therefore$ $\vec{F}$ is irrotational.

$\therefore$ There exists a scalar potential (ϕ) of $\vec{F}$ such that $\vec{F} = \nabla\phi$

$$\therefore (6xy^2 - 2z^3)\vec{i} + (6x^2y + 2yz)\vec{j} + (y^2 - 6z^2x)\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}\vec{k}$$

Comparing both sides, we have

$$\frac{\partial\phi}{\partial x} = 6xy^2 - 2z^2;\quad \frac{\partial\phi}{\partial y} = 6x^2y + 2yz \text{ and } \frac{\partial\phi}{\partial z} = y^2 - 6z^2x$$

$$\text{Now, } d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\therefore d\phi = (6xy^2 - 2z^3)\,dx + (6x^2y + 2yz)\,dx + (y^2 - 6z^2x)\,dz$$

On integration,

$$\therefore \phi = \left(6y^2\cdot\frac{x^2}{2} - 2z^3\cdot x\right) + \left(0 + 2z\cdot\frac{y^2}{2}\right) + (0 - 0) + c$$

$$\therefore \text{Scalar potential of } \vec{F} = \phi\ 3x^2y^2 - 2xz^3 + y^2z + c;$$

$$\text{Work done} = \int_C \vec{F}\cdot d\vec{r} = \int_{(1,0,2)}^{(0,1,1)} (6xy^2 - 2z^3)\,dx + (6x^2y + 2yz)\,dy + (y^2 - 6z^2x)\,dz$$

$$= \left[3x^2y^2 - 2xz^3 + y^2z\right]_{(1,0,2)}^{(0,1,1)}$$

$$= [0 - 0 + (1)^2(1)] - [0 - 2(1)(2)^3 + 0] = 17 \text{ units}$$

$$\therefore \text{Scalar potential of } \vec{F} = 3x^2y^2 - 2xz^3 + y^2z + c;$$

$$\text{Work done} = 17 \text{ units}$$

Q. 4(b) Using Green's theorem evaluate $\int_c \frac{1}{y}dx\,\frac{1}{x}dx$ where c is the boundary of the region bounded by $x = 1$, $x = 4$, $y = 1$, $y = \sqrt{x}$. **(6 Marks)**

Ans. :

Using Green's theorem, $\oint_C u\,dx + V\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

$y = \sqrt{x}$. $\quad\therefore y^2 = x$ ...(1)

$y = 1$ ...(2)

$x = 4$ ...(3)

From Equations (1) and (2), $1^2 = x$ $\quad\therefore x = 1$

$\therefore A = (1, 1)$

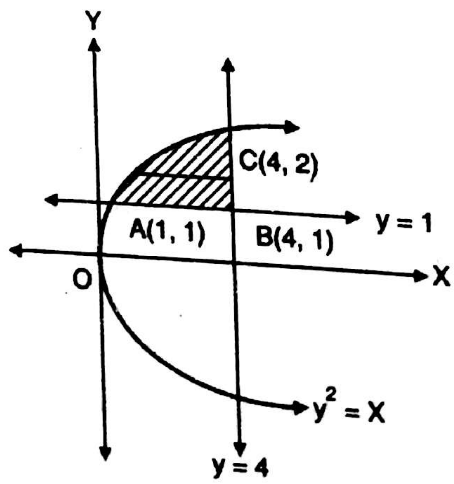

Fig. 1

From Equations (1) and (3), $y^2 = 4$ $\therefore y = \pm 2$

$\therefore C = (4, 2)$

From Equations (2) and (3), $y^2 = 4$ Let $p = \frac{1}{y}$ and $\frac{1}{x}$

$$\therefore \frac{\partial p}{\partial y} = \frac{-1}{y^2} \text{ and } \frac{\partial Q}{\partial x} = \frac{-1}{x^2}$$

By Green's Theorem,

$$\therefore \oint_C u\,dx + V\,dy = \oint_C \int_c \frac{1}{y}dx \frac{1}{x}dy \int_c \frac{1}{y}dx \frac{1}{x}dx$$

$$\int_c P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\therefore \int_c \frac{1}{y}dx + \frac{1}{x}dy = \int_1^2 \int_{x=y^2}^4 \left[-\frac{-1}{x^2} + \frac{-1}{y^2}\right] dx\,dy = \int_1^2 \int_{x=y^2}^4 [y^{-2} - x^{-2}]\,dx\,dy$$

$$= \int_1^2 \left[y^{-2} \cdot x - \frac{x^{-1}}{-1}\right]_{y^2}^{0} dy = \int_1^2 [\,(4y^{-2} + 4^{-1}) - (1 + y^{-2})\,]\,dy$$

$$= \int_1^2 \left[3y^{-2} - \frac{3}{4}\right] dy = \left[3\frac{y^{-1}}{-1} - \frac{3}{4}\right]_1^2 = \left(-3 \times 2^{-1} - \frac{3}{4} \times 2\right) - \left(-3 - \frac{3}{4}\right)$$

$$= \frac{3}{4}$$

$$\therefore \int_c \frac{1}{y}dx + \frac{1}{x} = \frac{3}{4}$$

Q. 5(a) If $\vec{F} = x^2\vec{i} + (x-y)\vec{j} + (y+z)\vec{k}$ move a particle from A (1, 0, 1) to B (2, 1, 2) along line AB. Find the work done. **(6 Marks)**

Ans. :

Given, $\vec{F} = x^2\vec{i} + (x-y)\vec{j} + (y+z)\vec{k}$

And $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

Using two point form, Equation of line is $\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}$

$$\therefore \frac{x-1}{1-2} = \frac{y-0}{0-1} = \frac{z-1}{1-2}$$

$$\therefore \frac{x-1}{-1} = \frac{y-0}{-1} = \frac{z-1}{-1}$$

$$\therefore x - 1 = y = z - 1$$

$$\therefore x = y + 1 \text{ and } z = y + 1$$

$\therefore \quad dx = dy$ and $dz = dy$

$$\therefore \text{ work done } = \int_c \vec{F}\cdot d\vec{r} = \int_{(1,0,1)}^{(2,1,2)} (x^2)\,dx + (x-y)\,dy + (y+z)\,dz$$

$$= \int_0^1 (y+1^2)\,dy + (y+1-y)\,dy + (y+y+1)\,dy$$

$$= \int_0^1 [\,y^2+2y+1+1+2y+1]\,dy = \int_0^1 [\,y^2+4y+3]\,dy$$

$$= \left[\frac{y^3}{3}+\frac{4y^2}{2}+3y\right]_0^1 = \left[\frac{1}{3}+\frac{4}{2}+3\right]-[0+0+0] = \frac{16}{3}$$

$$\therefore \text{Work done } = \frac{16}{3}$$

Q. 6(b) Evaluate $\iint_s \left(y^2z^2\,\bar{i}+z^2x^2\,\bar{j}+z^2y^2\,\bar{k}\right)\cdot\bar{n}\,ds$ where s is the hemisphere $x^2+y^2+z^2=1$ above xy – plane and bounded by this plane. **(6 Marks)**

Ans. :

$$\iint_s \vec{F}\cdot\bar{N}\,ds = \iint_s \left(y^2z^2\,\bar{i}+z^2x^2\,\bar{j}+z^2y^2\,\bar{k}\right)\cdot\bar{n}\,ds$$

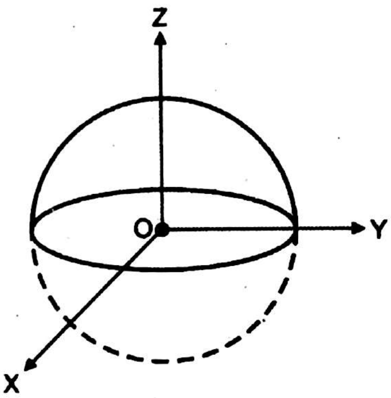

Fig. 2

Thus, $\quad \vec{F} = y^2z^2\,\vec{i}+z^2x^2\,\vec{j}+z^2y^2\,\vec{k}$

$$\therefore \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(y^2z^2)+\frac{\partial}{\partial x}(z^2x^2)+\frac{\partial}{\partial z}(z^2y^2)$$

$$= 0+0+2y^2z = 2y^2z$$

By Gauss Divergence Theorem,

$$\iint_s \vec{F}\cdot\bar{N}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV$$

$$\iint_s \left(y^2z^2\,\bar{i}+z^2x^2\,\bar{j}+z^2y^2\,\bar{k}\right)\cdot\bar{n}\,ds = \iiint_V 2y^2 z\,dx\,dy\,dz$$

Using spherical polar co-ordinates for hemisphere $x^2+y^2+z^2=1$ [centre = (0, 0, 0) and radius = 1]

Put $x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

limits of r are 0 to 1 ; limits of θ are 0 to $\pi/2$

limits of ϕ are 0 to 2π ;

$$\therefore \iint_s \left(y^2z^2\,\bar{i}+z^2x^2\,\bar{j}+z^2y^2\,\bar{k}\right)\cdot\bar{n}\,ds = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} 2(r\sin\theta\sin\phi)^2\,(r\cos\theta)\cdot r^2\sin\theta\,d\phi\,d\theta\,dr$$

$$= 2 \int_{r=0}^{1} r^2 dr \int_{\theta=0}^{\pi/2} \sin^3\theta \cos\theta \, d\theta \int_{\phi=0}^{2\pi} \sin^2\phi \, d\phi$$

In second integral, put u = sinθ,

∴ du = cosθ dθ

$$\therefore \iint_s \left(y^2 z^2 \bar{i} + z^2 x^2 \bar{j} + z^2 y^2 \bar{k} \right) \cdot \bar{n} \, ds = 2\left[\frac{r^6}{6}\right]_0^1 \cdot \int_{\theta=0}^{1} u^3 du \int_{\phi=0}^{2\pi} \frac{1-\cos 2\phi}{2} d\phi$$

$$= \frac{2}{6}[1-0] \times \left[\frac{u^4}{4}\right]_0^1 \times \frac{1}{2}\left[\phi - \frac{\sin 2\phi}{2}\right]_0^{2\pi}$$

$$= \frac{1}{3} \times \frac{1}{4}[1-0] \times \frac{1}{2}\left[\left(2\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{12} \times \frac{1}{2} \times (2\pi - 0 - 0 - 0)$$

$$\therefore \iint_s \left(y^2 z^2 \bar{i} + z^2 x^2 \bar{j} + z^2 y^2 \bar{k} \right) \cdot \bar{n} \, ds = \frac{\pi}{12}$$

Chapter 5 : Z - Transform [Total Marks - 11]

Q. 1(c) Find the z-transform of $\left(\frac{1}{3}\right)^{|k|}$. **(5 Marks)**

Ans. :

We have, the Z - transforms,

$$Z\{ f(k) \} = \sum_{k=-\infty}^{\infty} f(k) \cdot z^{-k}$$

$$\therefore Z\left\{\left(\frac{1}{3}\right)^{|k|}\right\} = \sum_{k=-\infty}^{-1} \left(\frac{1}{3}\right)^{k} z^{-k} + \sum_{k=\infty}^{\infty} \left(\frac{1}{3}\right)^{k} \cdot \frac{1}{z^k} = \sum_{k=-\infty}^{-1} \left(\frac{z}{3}\right)^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{3z}\right)^{k}$$

$$= \left(\dots + \frac{z^3}{3^3} + \frac{z^3}{3^3} + \frac{z}{3}\right) + \left(1 + \frac{1}{3z} + \frac{1}{3^2 z^2} + \frac{1}{3^3 z^3} + \dots\right)$$

$$= \frac{\frac{z}{3}}{\frac{1-z}{3}} + \frac{1}{\frac{1-1}{3z}}$$

$$= \frac{z}{3-z} + \frac{3z}{3z-1}$$

$$= \frac{3z^2 - z + 9z - 3z^2}{(3-z)(3z-1)} = \frac{8z}{(3-z)(3z-1)}$$

$$\therefore Z\left\{\left(\frac{1}{3}\right)^{|k|}\right\} = \frac{8z}{(3-z)(3z-1)}$$

Q. 3(a) Find the inverse z-transform of $F(z) = \dfrac{z^3}{(z-3)(z-2)^3}$

(i) $2 < |z| < 3$; (ii) $|z| < 3$.

(6 Marks)

Ans. :

$$F(z) = \frac{z^3}{(z-3)(z-2)^3}; \quad \therefore \frac{F(z)}{z} = \frac{z^2}{(z-3)(z-2)^2} \quad \text{...(1)}$$

By partial fraction

$$\frac{z^2}{(z-3)(z-2)^2} = \frac{A}{z-3} + \frac{B}{z-2} + \frac{C}{(z-2)^2} \quad \text{...(2)}$$

$$\therefore A = \cancel{(z-3)} \frac{z^2}{\cancel{(z-3)}(z-2)^2}\Bigg|_{z=3} = \frac{3^2}{(3-2)^2} = 9$$

$$\therefore C = \cancel{(z-3)} \frac{z^2}{\cancel{(z-3)}(z-2)^2}\Bigg|_{z=2} = \frac{3^2}{(3-2)^2} = -4$$

Put $z = 0$, $A = 9$ and $B = -4$ in Equation

$$\therefore 0 = \frac{9}{0-3} + \frac{B}{0-2} - \frac{4}{(0-2)^2}$$

$$\therefore \frac{B}{2} = 3 - 1$$

$$\therefore B = -8$$

$\therefore$ From Equations (1) and (2), $\dfrac{F(z)}{z} = \dfrac{9}{z-2} - \dfrac{4}{(z-2)^2}$

$$F(z) = \frac{9z}{z-3} - \frac{8z}{z-3} - \frac{4}{(z-2)^2}$$

Case 1 : For $2 < |z| < 3$ $\quad \therefore 2 < |z|$ and $|z| < 3$

$$\therefore \left|\frac{2}{z}\right| < 1 \text{ and } \left|\frac{2}{3}\right| < 1$$

$$\therefore F(z) = \frac{9z}{3\left(\frac{z}{3}-1\right)} - \frac{8z}{3\left(\frac{1-2}{z}\right)} - \frac{4z}{z^2\left(\frac{1-2}{z}\right)^2}$$

$$= -3z\left(1-\frac{z}{3}\right)^{-1} - 8\left(1-\frac{2}{z}\right)^{-1} - \frac{4}{z}\left(1-\frac{2}{z}\right)^{-2}$$

$$= -3z\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\ldots\right) - 2^3\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\ldots\right) - \frac{2^2}{z}\left(1+2\cdot\frac{2}{z}+3\frac{2^2}{z^2}+\ldots\right)$$

$$= \left(-3z - z^2 - \frac{z^3}{3} - \ldots - \frac{z^k}{3^{k-2}} - \ldots\right) + \left(-2^3 - \frac{2^4}{z} - \frac{2^5}{z^2} - \ldots - \frac{2^{k+3}}{z^k} - \ldots\right)$$

$$-\left(-1\frac{2^2}{z} - 2\cdot\frac{2^3}{z^2} - 3\cdot\frac{2^4}{z^3} - \ldots - k\cdot\frac{2^{k+1}}{z^k} - \ldots\right)$$

In first series, Coefficient of $z^k = \dfrac{-1}{3^{k-2}}$ (for $k > 0$).

Replace 'k' by '– k'

$$\therefore \text{Coefficient of } z^{-k} = \frac{-1}{3^{k-2}} \text{ (for } k > 0) = -3^{k+2} \text{ (for } k > 0)$$

In second and third series,

$$\text{Ccoefficient of } z^k \left(\text{or } \frac{1}{z^k}\right) = -2^{k+3} - k \cdot 2^{k+1} = -2^{k+1}(2^2 + k) \text{ (for } k > 0)$$

$$\therefore z^{-1}\left[\frac{z^3}{(z-3)(z-2)^2}\right] = \begin{cases} -3^{k+2} & k < 0 \\ -2^{k+1}(k+4) & k \geq 0 \end{cases}$$

Case 2 : For $|z| > 3$

Obviously, $|z| > 2$

$\therefore 2 < |z|$ and $3 < |z|$

$$\therefore \left|\frac{2}{z}\right| < 1 \text{ and } \left|\frac{3}{z}\right| < 1$$

$$\therefore F(z) = \frac{9z}{z\left(\frac{1-3}{z}\right)} - \frac{8z}{z\left(\frac{1-2}{z}\right)} - \frac{4z}{z^2\left(\frac{1-2}{z}\right)^2} = 9\left(1-\frac{3}{z}\right)^{-1} - 8\left(1-\frac{2}{z}\right)^{-1} - \frac{4}{z}\left(1-\frac{2}{z}\right)^{-2}$$

$$= 3^2\left(1+\frac{3}{z}+\frac{3^2}{z^2}+\ldots\right) - 2^3\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\ldots\right) - \frac{2^2}{z}\left(1+2\cdot\frac{2}{z}+3\frac{2^2}{z^2}+\ldots\right)$$

$$= \left(3^2+\frac{3^3}{z}+\frac{3^4}{z^2}+\ldots+\frac{3^{k+2}}{z}+\ldots\right) + \left(-2^3-\frac{2^4}{z}-\frac{2^5}{z^2}+\ldots+\frac{2^{k+2}}{z^k}-\ldots\right)$$

$$+\left(-1\cdot\frac{2^2}{z}-2\cdot\frac{2^3}{z^2}-3\cdot\frac{2^4}{z^3}\ldots-k\frac{2^{k+1}}{z^k}-\ldots\right)$$

In all three series, Coefficient of $z-k\left(\text{or } \frac{1}{z^k}\right) = 3^{k+2} - 2^{k+3} - k \cdot 2^{k+1}$ (for $k \geq 0$)

$$= 3^{k+2} - 2^{k+1}(2^2 + k)$$

$$\therefore z^{-1}\left[\frac{z^3}{(z-3)(z-2)^2}\right] = 3^{k+2} - 2^{k+1}(k+4) \quad k \geq 0$$

□□□

May 2015

Chapter 1 : Complex Variable and Mapping [Total Marks - 18]

Q. 3(a) Find the Analytic function $f(z) = u + iv$ if $u + v = \frac{x}{x^2 + y^2}$. **(6 Marks)**

Ans. :

$$\because \quad f(z) = u + iv$$

$$if(z) = iu - v$$

$$(1 + i)\, f(z) = (u - v) + i\,(u + v)$$

$$F(z) = U + iV \qquad \text{where } (1 + i)\, f(z) = F(z)$$

$$u - v = U$$

$$u + v = V$$

$\because$ F(z) is analytic

$$\therefore \quad F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} \qquad \text{...(i)}$$

$$\because \quad u + v = \frac{x}{x^2 + y^2}$$

$$V = \frac{x}{x^2 + y^2} \Rightarrow \frac{\partial V}{\partial x} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2}$$

$$\frac{\partial V}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \qquad \text{...(ii)}$$

$$\frac{\partial V}{\partial y} = -\frac{x}{(x^2 + y^2)^2}\frac{\partial}{\partial y}(x^2 + y^2)$$

$$\frac{\partial V}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2} \qquad \text{...(iii)}$$

$$F'(z) = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$$

$$F'(z) = \frac{-2xy}{(x^2 + y^2)^2} + i\left(\frac{y^2 - x^2}{(x^2 + y^2)^2}\right)$$

Put x = z and y = 0

$$F'(z) = 0 + i\left(\frac{-z^2}{z^4}\right)$$

$$F'(z) = \frac{i}{z^2}$$

$$F(z) = -i\int z^{-2}\,dz + C = -i\frac{z^{-1}}{(-1)} + C = \frac{i}{z} + C$$

$$(1 + i)\, f(z) = \frac{i}{z} + C$$

$$f(z) = \frac{i}{(1 + i)}\frac{1}{z} + \frac{C}{(1 + i)}$$

Q. 4(a) Find the Orthogonal Trajectory of $3x^2 y - y^3 = k$. **(6 Marks)**

Ans. :

Let
$$u = 3x^2y - y^3$$
$$\frac{\partial u}{\partial x} = 6xy$$
$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2$$
$$\because \quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$
$$dv = \left(-\frac{\partial u}{\partial y}\right)dx + \left(-\frac{\partial u}{\partial x}\right)dy$$
$$dv = (-3x^2 + 3y^2)\,dx + 6xy\,dy$$
$$v = -3\int x^2\,dx + 3y^2\int dx + C$$
$$v = -x^3 + 3xy^2 + C$$
is the family of orthogonal trajectory.

Q. 5(b) Find the Bilinear Transformation that maps the points $z = 1, i, -1$ into $w = i, 0, -i$. **(6 Marks)**

Ans. :

$z = 1, i, -1$, and $w = i, 0, -i$

Let bilinear transformation
$$w = \frac{az+b}{cz+d}$$
$$i = \frac{a+b}{c+d} \Rightarrow a + b = ic + id$$
$$\Rightarrow a + b - ic - id = 0 \quad \ldots(i)$$
$$0 = \frac{ai+b}{ci+d} \Rightarrow ai + b = 0 \Rightarrow b = -ia \quad \ldots(ii)$$
$$-i = \frac{-a+b}{-c+d} \Rightarrow -a + b = ic - id$$
$$\Rightarrow -a + b - ic + id = 0 \quad \ldots(iii)$$
(i) + (iii)
$$2b - 2ic = 0$$
$$b = ic \Rightarrow c = \frac{b}{i} = -bi$$
$$c = -i(-ia)$$
$$c = -a$$
(i) – (iii)
$$2a - 2id = 0$$
$$\Rightarrow a = id \Rightarrow d = \frac{a}{i} = -ia$$
$$w = \frac{az - ia}{-az - ia} = \frac{z-i}{-z-i} = \frac{i-z}{i+z}$$

Chapter 2 : Laplace Transform [Total Marks - 27]

Q. 1(a) Find Laplace Transform of $\frac{\sin t}{t}$. **(5 Marks)**

Ans. :

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty L(\sin t)\, ds = \int_s^\infty \frac{1}{s^2+1} ds = [\tan^{-1} s]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

Q. 2(c) Find inverse Laplace Transform of (i) $\frac{s+29}{(s+4)(s^2+9)}$ (ii) $\frac{e^{-2x}}{s^2+8s+25}$ **(8 Marks)**

Ans. :

(i) $L^{-1}\left[\frac{(s+29)}{(s+4)(s^2+9)}\right]$

$$\frac{(s+29)}{(s+4)(s^2+9)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+9}$$

$$s + 29 = A(s^2+9) + (Bs+C)(s+4)$$

$$s + 29 = A(s^2+9) + (Bs+C)(s+4)$$

$$A + B = 0 \quad 4B + C = 1, 9A + 4C = 29$$

$$B = -A$$

$$-4A + C = 1$$

$$-16A + 4C = 4$$

$$\underline{9A \pm 4C = -29}$$

$$25A = -25$$

$$A = -1$$

$$B = -A = 1$$

$$C = -3$$

$$= L^{-1}\left[\frac{-1}{(s+4)} + \frac{s-3}{s^2+9}\right] = -L^{-1}\left[\frac{1}{s+4}\right] + L^{-1}\left[\frac{s}{s^2+9}\right] - L^{-1}\left[\frac{3}{s^2+9}\right]$$

$$= -e^{-4t} + \cos 3t - \sin 3t$$

(ii) $L^{-1}\left[\frac{e^{-2s}}{s^2+8s+25}\right]$

$$L^{-1}\left[\frac{e^{-2s}}{s^2+8s+25}\right] = L^{-1}\left[\frac{1}{(s^2+8s+16)+9}\right] = L^{-1}\left[\frac{1}{(s+4)^2+9}\right]$$

$$= e^{-4t} L^{-1}\left[\frac{1}{s^2+9}\right] =$$

$$\therefore L^{-1}\left[\frac{e^{-2s}}{s^2+8s+25}\right] = e^{-4t} \cdot \frac{1}{3} \sin 3t$$

Q. 3(c) Solve the differential Equation $\frac{d^2y}{dt^2} + y = t$, $y(0) = 1$, $y'(0) = 0$, using Laplace Transform. **(8 Marks)**

Ans. :

$$\frac{d^2y}{dt^2} + y = t, \qquad y(0) = 1, y'(0) = 0$$

Taking Laplace of it

$$L\left[\frac{d^2y}{dt^2}\right] + L[y] = L[t]$$

$$[s^2L(y) - s\,y(0) - y'(0)] + L(y) = \frac{1}{s^2}$$

$$(s^2 + 1)\,L(y) - s = \frac{1}{s^2}$$

$$L(y) = \frac{1}{s^2(s^2+1)} + \frac{s}{(s^2+1)}$$

$$y(t) = L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] + L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{(s^2+1)-s^2}{s^2(s^2+1)}\right] + \cos t = L^{-1}\left[\frac{1}{s^2} - \frac{1}{(s^2+1)}\right] + \cos t$$

$$= L^{-1}\left[\frac{1}{s^2}\right] - L^{-1}\left[\frac{1}{s^2+1}\right] + \cos t = t - \sin t + \cos t$$

Q. 5(a) Find Inverse Laplace Transform using Convolution theorem $\frac{s}{(s^4 + 8s^2 + 16)}$. **(6 Marks)**

Ans. :

$$L^{-1}\left[\frac{s}{s^4 + 8s^2 + 16}\right]$$

$$= L^{-1}\left[\frac{s}{s^4 + 16 + 8s^2}\right] = L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+4)} \cdot \frac{s}{s^2+4}\right]$$

$$= \int_0^t \frac{\sin 2u}{2} \cos 2(t-u)\, du$$

$$= \frac{1}{4}\int_0^t 2 \sin 2u \cos (2t - 2u)\, du$$

$$= \frac{1}{4}\int_0^t \sin [(2u + 2t - 2u) + \sin (2u - 2t + 2u)]\, du$$

$$= \frac{1}{4}\int_0^t [\sin 2t + \sin (4u - 2t)\, du$$

$$= \frac{1}{4}\left[\sin 2t\,(u)_0^t + \left(-\frac{\cos(4u-2t)}{4}\right)_0^t\right]$$

$$= \frac{1}{4}\left[t \sin 2t + \frac{1}{4}(-\cos 2t + \cos(-2t))\right]$$

$$= \frac{1}{4}\left[t \sin 2t + \frac{1}{4}(-\cos 2t + \cos 2t)\right]$$

$$= \frac{1}{4}\sin 2t$$

Chapter 3 : Fourier Series [Total Marks - 33]

Q. 1(c) Find fourier series for $f(x) = 9 - x^2$ over $(-3, 3)$. **(5 Marks)**

Ans. :

$$f(x) = 9 - x^2 \quad (-3, 3)$$

$$2l = 3-(-3) \Rightarrow 2l = 6 \Rightarrow l = 3$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right)$$

$$f(x) = 9 - x^2 \Rightarrow f(-x) = 9-(-x)^2 = 9 - x^2 = f(x) \Rightarrow f(x) \text{ is even}$$

$$\Rightarrow b_n = 0$$

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{2}{3}\int_0^3 (9-x^2)\,dx$$

$$= \frac{2}{3}\left[9\,(x)_0^3 - \left(\frac{x^3}{3}\right)_0^3\right] = \frac{2}{3}\left[27 - \frac{27}{3}\right] = \frac{2}{3}[27-9] = \frac{2}{3}\times 18 = 12$$

$$a_n = \frac{1}{3}\int_{-3}^{3}(9-x^2)\cos\left(\frac{n\pi x}{3}\right)dx = \frac{2}{3}\int_0^3 (9-x^2)\cos\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(9-x^2)\cancel{\left(\frac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}}\right)} - (-2x)\left(\frac{-\cos\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2}\right) + (-2)\cancel{\left(\frac{-\sin\frac{n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^3}\right)}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-(-2\times 3)(-\cos n\pi)}{\left(\frac{n\pi}{3}\right)^2}\right] = \frac{2}{\cancel{3}} \times \frac{6(-\cos n\pi)\times \cancel{3}}{n^2\pi^2} = \frac{-36(-1)^n}{n^2\pi^2} = \frac{36(-1)^{n+1}}{n^2\pi^2}$$

$$9 - x^2 = 6 + \sum_{n=1}^{\infty}\frac{36(-1)^{n+1}}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right)$$

Q. 2(b) Find the Fourier series for $f(x) = \frac{\pi - x}{2}$; $0 \le x \le 2\pi$. **(6 Marks)**

Ans. :

$$2l = 2\pi - 0$$

$$2l = 2\pi \Rightarrow l = \pi$$

The Fourier series will be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)dx = \frac{1}{\pi}\left[\frac{\pi}{2}(x)_0^{2\pi} - \frac{1}{2}\left(\frac{x^2}{2}\right)_0^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}(2\pi) - \frac{1}{4}\times 4\pi^2\right] = \frac{1}{\pi}[\pi^2 - \pi^2] = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\cos nx\,dx = \frac{1}{2\pi}\left[(\pi - x)\left(\cancel{\frac{\sin nx}{n}}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-\cos 2n\pi + 1}{n^2}\right] = \frac{1}{2\pi}\left[\frac{-1+1}{n^2}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\sin nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\cancel{\frac{-\sin nx}{n^2}}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{(-\pi)(-\cos 2n\pi)}{n} - \frac{\pi(-1)}{n}\right] = \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right] = \frac{1}{n}$$

$$\therefore \frac{\pi - x}{2} = \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

Q. 4(c) Find Fourier Integral of $f(x) = \begin{cases} \sin x & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$ Hence show that $\int_0^{\infty}\frac{\cos(\lambda\pi/2)}{1-\lambda^2}d\lambda = \frac{\pi}{2}$

(8 Marks)

Ans. :

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\lambda(t - x)\,dt\right]d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\int_0^{\pi} f(t)\cos\lambda(t - x)\,dt\right]d\lambda$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\int_0^{\pi} \sin t\cos\lambda(t - x)\,dt\right]d\lambda$$

$$= \frac{1}{2\pi}\int_0^{\infty}\left[\int_0^{\pi} 2\,\text{sint}\cos(\lambda t - \lambda x)\,dt\right]d\lambda$$

$$= \frac{1}{2\pi}\int_0^{\infty}\left[\int_0^{\pi}\{\sin(t+\lambda t-\lambda x)+\sin(t-\lambda t+\lambda x)\}\,dt\right]d\lambda$$

$$= \frac{1}{2\pi}\int_0^{\infty}\left[-\frac{\cos(t+\lambda t+\lambda x)}{1+\lambda}-\frac{\cos(t-\lambda t+\lambda x)}{1-\lambda}\right]_0^{\pi}d\lambda$$

$$= \frac{1}{2\pi}\int_0^{\infty}\left[\left\{-\frac{\cos\{(1+\lambda)\pi-\lambda x\}}{1+\lambda}+\frac{\cos\lambda x}{1+\lambda}\right\}-\left\{-\frac{\cos\{(1-\lambda)\pi+\lambda x\}}{1-\lambda}-\frac{\cos\lambda x}{1-\lambda}\right\}\right]d\lambda$$

$$= \frac{1}{2\pi}\int_0^{\infty}\left[\left(\frac{\cos(\lambda\pi-\lambda x)}{1+\lambda}+\frac{\cos\lambda x}{1+\lambda}\right)-\left(-\frac{\cos(\lambda\pi-\lambda x)}{(1-\lambda)}-\frac{\cos\lambda x}{1-\lambda}\right)\right]d\lambda$$

$$= \frac{1}{2\pi}\int_0^{\infty}(\cos(\lambda\pi-\lambda x)+\cos\lambda x)\left(\frac{1}{1+\lambda}+\frac{1}{1-\lambda}\right)d\lambda$$

$$= \frac{1}{2\pi}\int_0^{\infty}\{\cos(\lambda\pi-\lambda x)+\cos\lambda x\}\left(\frac{1-\cancel{\lambda}+1+\cancel{\lambda}}{1+\lambda^2}\right)d\lambda$$

$$f(x) = \frac{1}{2\pi}\int_0^{\infty}\frac{\cos(\lambda\pi-\lambda x)+\cos\lambda x}{(1-\lambda^2)}d\lambda \Rightarrow \int_0^{\infty}\frac{\cos(\lambda\pi-\lambda x)+\cos\lambda x}{1-\lambda^2}d\lambda = \frac{\pi}{1}f(x)$$

put $x = \frac{\pi}{2}$

$$\int_0^{\infty}\frac{\cos\frac{\lambda\pi}{2}+\cos\frac{\lambda\pi}{2}}{1-\lambda^2}d\lambda = \pi\ (\sin x)\Big|_{\frac{\pi}{2}}$$

$$2\int_0^{\infty}\frac{\cos\frac{\lambda\pi}{2}}{1-\lambda^2}d\lambda = \pi \Rightarrow \int_0^{\infty}\frac{\cos\frac{\lambda\pi}{2}}{1-\lambda^2}d\lambda = \frac{\pi}{2}$$

Q. 6(b) Find complex form of Fourier Series for e^{2x}; $0 < x < 2$ **(6 Marks)**

Ans. :

$$f(x) = e^2x : 0 < x < 2$$
$$2l = 2-0$$
$$2l = 2 \Rightarrow l = 1$$
$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{-in\pi x}$$

$$C_n = \frac{1}{2}\int_0^2 f(x)\, e^{-in\pi x}\, dx = \frac{1}{2}\int_0^2 e^{2x}\, e^{-in\pi x}\, dx = \frac{1}{2}\int_0^2 e^{(2-in\pi)x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{(2-in\pi)x}}{(2-in\pi)}\right]_0^2 = \frac{1}{2}\left[\frac{e^{(2-in\pi)2}-1}{(2-in\pi)}\right] = \frac{1}{2}\left[\frac{e^4 e^{-i2n\pi}-1}{(2-in\pi)}\right]$$

$$= \frac{1}{2}\left[\frac{e^4-1}{(2-in\pi)}\right]$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[\frac{e^4-1}{(2-in\pi)}\right] e^{in\pi x}$$

Q. 6(c) Find Half Range Cosine Series for $f(x) = \begin{cases} kx & 0 \le x \le l/2 \\ k(l-x) & l/2 \le x \le l \end{cases}$, hence find $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots..$ **(8 Marks)**

Ans. :

$$f(x) = \begin{cases} kx & ;\ 0 \le x \le l/2 \\ k(l-x) & ;\ l/2 \le x \le l \end{cases}$$

let us define f(x) as even function in $[-l, l]$

$$2l = 2l \Rightarrow l = l$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$\therefore$ f(x) in even $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx = \frac{2}{l}\int_0^l f(x)\, dx = \frac{2}{l}\left[\int_0^{l/2} kx\, dx + \int_{l/2}^{l} k(l-x)\, dx\right]$$

$$= \frac{2}{l}\left[k\left(\frac{x^2}{2}\right)_0^{l/2} + k\left(\frac{(l-x)^2}{-2}\right)_{l/2}^{l}\right] = \frac{2}{l}\left[\frac{K}{2}\left(\frac{l}{2}\right)^2 + k\left(0+\frac{1}{2}\left(\frac{l}{2}\right)^2\right)\right]$$

$$= \frac{2}{l}\left[\frac{kl^2}{8}+\frac{kl^2}{8}\right] = \frac{\cancel{2}}{\cancel{l}}\left[\frac{k\cancel{l^2}}{\cancel{4}}\right] = \frac{kl}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{x\pi x}{l}\, dx = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} f(x)\cos\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^{l} \cos\left(\frac{n\pi x}{l}\right) dx\right]$$

$$= \frac{2}{l}\left[\int_0^{l/2} kx\cos\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^{l} k(l-x)\cos\left(\frac{n\pi x}{l}\right) dx\right]$$

$$= \frac{2}{l}\left[k\left\{ x\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - \frac{\left(-\cos\left(\frac{n\pi x}{l}\right)\right)}{\left(\frac{n\pi}{l}\right)^2} \right\}_0^{l/2} + k\left\{ (l-x)\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - (-1)\frac{\left(-\cos\left(\frac{n\pi x}{l}\right)\right)}{\left(\frac{n\pi}{l}\right)^2} \right\}_{l/2}^{l} \right]$$

$$= \frac{2k}{l}\left[\left(l/2\,\cancel{\frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{2}\right)}} + \frac{\cos\frac{n\pi}{2}}{\left(\frac{n\pi}{l}\right)^2} - \frac{1}{\left(\frac{n\pi}{l}\right)^2}\right) + \left(\frac{-\cos n\pi}{\left(\frac{n\pi}{l}\right)^2} - l/2\,\cancel{\frac{\sin\frac{n\pi}{2}}{\left(\frac{n\pi}{l}\right)}} + \frac{\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2}\right)\right]$$

$$= \frac{2k}{l}\left[\frac{2\cos\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{l}\right)^2} - \frac{1}{\left(\frac{n\pi}{l}\right)^2} - \frac{\cos n\pi}{\left(\frac{n\pi}{l}\right)^2}\right]$$

$$= \frac{2k}{l} \times \frac{l^2}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - \cos n\pi - 1\right]$$

$$= \frac{2kl}{n^2.\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - \cos n\pi - 1\right]$$

$$\therefore f(x) = \frac{kl}{4} + \sum_{n=1}^{\infty}\frac{2kl}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}\right) - \cos n\pi - 1\right]\cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{kl}{4} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(2\cos-\left(\frac{n\pi}{2}\right) - \cos n\pi - 1\right)}{n^2}\cos\left(\frac{n\pi x}{l}\right)$$

put $x = \frac{l}{2}$

$$\frac{kl}{2} = \frac{kl}{4} + \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(2\cos\frac{n\pi}{2} - (-1)^n - 1\right)}{n^2}\cos\left(\frac{n\pi x}{l}\right)\Bigg|_{x=\frac{l}{2}}$$

$$kl\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(2\cos\frac{n\pi}{2} - (-1)^n - 1\right)}{n^2}\cos\left(\frac{n\pi}{2}\right)$$

$$\frac{kl}{4} = \frac{2kl}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(2\cos\frac{n\pi}{2} - (-1)^n - 1\right)}{n^2}\cos\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi^2}{8} = 0 + \frac{(-2-1-1)}{2^2}(-1) + 0 + 0 + 0 + \frac{(-4)}{6^2}(-1) + 0 + 0 + 0 + 0 + \frac{(-4)}{10^2}(-1)$$

$$\frac{\pi^2}{8} = \frac{4}{2^2} + \frac{4}{6^2} + \frac{4}{10^2}\cdots$$

$$\frac{\pi^2}{8} = \frac{4}{2^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\cdots\cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\cdots\cdots$$

Chapter 4 : Vector Algebra & Calculus [Total Marks - 26]

Q. 2(a) A prove that $\bar{F} = ye^{xy}\cos z\,i + xe^{xy}\cos z\,j - e^{xy}\sin z\,k$ is irrotational. Find scalar potential for $\bar{F}$ hence evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve C joining the points (0, 0, 0) and $(-1, 2, \pi)$. **(6 Marks)**

Ans. : **Please refer Q. 3(c) of Dec. 2013.**

Q. 4(b) Using Greens theorem evaluate $\int_C (xy + y^2)\,dx + x^2\,dy$, C is closed path formed by $y = x$, $y = x^2$.

(6 Marks)

Ans. :

By Greens theorem

$$\int_C P(x, y)\,dx + Q(x, y)\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\Rightarrow \quad \int_C (xy + y^2)\,dx + x^2\,dy = \iint_S (2x - x - 2y)\,dx\,dy \qquad \text{...(i)}$$

$$\begin{aligned} C : y &= x \quad \text{and} \quad y = x^2 \\ x^2 &= x \\ x^2 - x &= 0 \\ x(x-1) &= 0 \\ x &= 0, \; x = 1 \\ y &= 0, \; y = 1 \end{aligned}$$

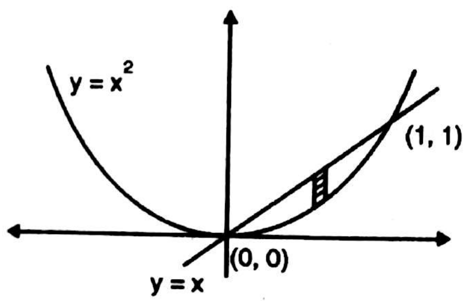

Fig. 1-Q. 4(b)

R.H.S. of (i)

$$= \int_{x=0}^{1}\int_{y=x^2}^{y=x} (x - 2y)\,dx\,dy = \int_0^1 \left[x(y)_{x^2}^{x} - \cancel{2}\left(\frac{y^2}{\cancel{2}}\right)_{x^2}^{x} \right] dx$$

$$= \int_0^1 \{x[x - x^2] - (x^2 - x^4)\}\,dx = \int_0^1 \{x^2 - x^3 - x^2 + x^4\}\,dx$$

$$= \int_0^1 (x^4 - x^3)\,dx = \left(\frac{x^5}{5}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = -\frac{1}{20}$$

Q. 5(c) Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where C is the boundary of the plane 2x + y + z = 2 cut off by co-ordinate planes and $\bar{F} = (x + y)i + (y + z)j - xk$. **(8 Mark)**

Ans. :

By stocks theorem

$$\int_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot d\bar{s} \qquad \ldots(i)$$

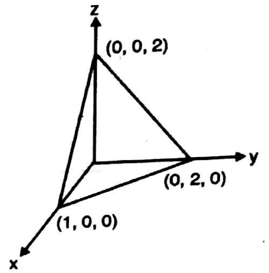

Fig. 1-Q. 5(c)

$$\bar{F} = (x+y)\hat{i} + (y+z)\hat{j} - x\hat{k}$$

$$\nabla \times \bar{F} = \begin{vmatrix} i & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & y+z & -x \end{vmatrix}$$

$$= i(-1) - j(-1) + k(0-1)$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

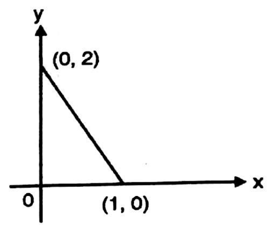

Fig. 2-Q. 5(c)

$$2x + y + z = 2$$

$$\phi : 2x + y + z - 2$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = \hat{i}(2) + j(1) + k(1)$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{4+1+1}} = \frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$$

R.H.S. of (i)

$$= \iint_S (-\hat{i}+\hat{j}-\hat{k}) \cdot \left(\frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}\right) ds$$

$$= \iint_R \left(\frac{-2+1-1}{\sqrt{6}}\right) \frac{dx\,dy}{\left|\frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}} \cdot \hat{k}\right|} = \iint_R \left(-\frac{2}{\sqrt{6}}\right) \frac{dx\,dy}{\left(-\frac{1}{\sqrt{6}}\right)} = (-2)\frac{1}{2}(1)(2) = -2$$

Q. 6(a) Find the Directional derivative of $\phi = x^2 + y^2 + z^2$ in the direction of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at (1, 2, 3) **(6 Marks)**

Ans. :

$$\phi = x^2 + y^2 + z^2$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = i(2x) + j(2y) + k(\partial 2z)$$

$$\nabla\phi\Big|_{1,2,3} = i(2) + j(4) + k(6) \qquad \ldots(i)$$

$\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$

Vector in the duration of the line

$$= 3\hat{i} + 4\hat{j} - 5\hat{k}$$

$\therefore$ Directional derivative

$$= (2\hat{i} + 4\hat{j} - 6\hat{k}) \cdot \frac{(3\hat{i} + 4\hat{j} + 5\hat{k})}{\sqrt{9 + 16 + 25}}$$

$$= \frac{(6 + 16 + 30)}{\sqrt{50}} = \frac{52}{5\sqrt{2}}$$

Chapter 5 : Z – Transform [Total Marks - 11]

Q. 1(d) Find Z{ f(k) * g(k) } if $f(k) = \frac{1}{3^k}$, $g(k) = \frac{1}{5^k}$. **(5 Marks)**

Ans. :

$$z\left[\frac{1}{3^k}\right] = \sum_{k=0}^{\infty} \frac{1}{3^k} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{3^k z^k} = \sum_{k=0}^{\infty} \frac{1}{(3z)^k}$$

$$= 1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + \ldots..$$

$$= \frac{1}{1 - \frac{1}{3z}} = \frac{3z}{(3z-1)}, \frac{1}{3} < |z|$$

$$z\left[\frac{1}{5^k}\right] = \sum_{k=0}^{\infty} \frac{1}{5^k} z^{-k} = \sum_{k=0}^{\infty} \frac{1}{(5z)^k}$$

$$= 1 + \frac{1}{5z} + \frac{1}{(5z)^2} + \frac{1}{(5z)^3} + \ldots..$$

$$= \frac{1}{1 - \frac{1}{5z}} = \frac{5z}{(5z-1)}, \quad \frac{1}{5} < |z|$$

$$z\left[\frac{1}{3^k} \times \frac{1}{5^k}\right] = z\left[\frac{1}{3^k}\right] z\left[\frac{1}{5^k}\right] = \frac{3z}{(3z-1)} \cdot \frac{5z}{5z-1}, |z| > \frac{1}{3}$$

$$= \frac{15z^2}{(3z-1)(5z-1)}, |z| > \frac{1}{3}$$

Q. 3(b) Find inverse Z transform of $\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \frac{1}{3} < |z| < \frac{1}{2}$. **(6 Marks)**

Ans. :

$$F(z) = \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \frac{1}{3} < |z| < \frac{1}{2}$$

$$= \frac{\frac{1}{6}}{\frac{1}{6}\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} = \frac{\left(z-\frac{1}{3}\right)-\left(z-\frac{1}{2}\right)}{\frac{1}{6}\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= 6\left[\frac{1}{\left(z-\frac{1}{2}\right)} - \frac{1}{\left(z-\frac{1}{3}\right)}\right]$$

$\frac{1}{3} < |z| \Rightarrow \frac{1}{3|z|} < 1$ and $|z| < \frac{1}{2} \Rightarrow 2|z| < 1$

$$F(z) = 6\left[\frac{2}{-(1-2z)} - \frac{1}{z\left(1-\frac{1}{3z}\right)}\right]$$

$$F(z) = 6\left[\frac{-2}{(1-2z)} - \frac{1}{z\left(1-\frac{1}{3z}\right)}\right] = -12\,(1-2z)^{-1} - \frac{6}{z}\left(1-\frac{1}{3z}\right)^{-1}$$

$$= -12\,(1+2z+(2z)^2+(2z)^3 \ldots\ldots) - \frac{6}{z}\left(1+\frac{1}{3z}+\left(\frac{1}{3z}\right)^2+\left(\frac{1}{3z}\right)^3 \ldots\ldots\right)$$

Coefficient of $z^k = -12\,(2)^k$ where $k \geq 0$ ...(i)

Coefficient of $z^{-k} = \frac{-6}{3^{k-1}}$ where $k \geq 1$ ...(ii)

In (i) coefficient of $z^{-k} = (-12)\,2^{-k}$ where $k \leq 0$

$\therefore$ Coefficient of $z^{-k} = (-12)\,2^{-k}$ when $k \leq 0 - 6 \times 3^{-k+1}$ when $k \geq 1$

$$z^{-1}[F(z)] = \begin{array}{l} (-12)2^{-k} \text{ when } k \leq 0 \\ (-6)\,3^{-k+1} \text{ when } k \geq 1 \end{array}$$

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Dec. 2015

Chapter 1 : Complex Variable and Mapping [Total Marks - 23]

Q. 1(c) Find a, b, c, d, e if,

$f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic **(5 Marks)**

Ans. : **Please refer Q. 1(b) of Dec. 2013.**

Q. 2(a) If $f(z) = u + iv$ is analytic and $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$, find f(z) **(6 Marks)**

Ans. :

$$u + v = \frac{3\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x} = \frac{3\sin 2x}{2\left(\frac{e^{2y} + e^{-2y}}{2}\right) - 2\cos 2x}$$

$$u + v = \frac{3\sin 2x}{2\cosh 2y - 2\cos 2x}$$

$$f(z) = u + iv \qquad \text{if}(z) = iu - v$$

$$(1+i) f(x) = (u - v) + i(u + v) \qquad f(x) = u + iv \quad (\text{where } U = u - v, V = u + v)$$

Given that

$$u + v = V = \frac{3\sin 2x}{2\cosh 2y - 2\cos 2x} \qquad \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left[\frac{3\sin 2x}{2\cosh 2y - 2\cos 2x}\right]$$

$$= \frac{(2\cosh 2y - 2\cos 2x)\frac{\partial}{\partial x}(3\sin 2x) - 3\sin 2x\frac{\partial}{\partial x}(2\cosh 2y = \cos 2x)}{(2\cosh 2y - 2\cos 2x)^2}$$

$$= \frac{(2\cosh 2y - 2\cos 2x)(6\cos 2x) - 3\sin 2x(4\sin 2x)}{4(\cosh 2y - \cos 2x)^2}$$

$$= \frac{12\cosh 2y\cos 2x - 2\cos^2 2x - 12\sin^2 2x}{4(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial v}{\partial x} = \frac{12\cosh 2y\cos 2x - 12}{4(\cosh 2y - \cos 2x)^2} = \frac{3(\cosh 2y\cos 2x - 1)}{(\cosh 2y - \cos 2x)^2} \qquad \ldots(i)$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[\frac{3\sin 2x}{2(\cosh 2y - \cos 2x)}\right] = \frac{-3}{\cancel{2}}\frac{\sin 2x}{(\cosh 2y - \cos 2x)^2}(\cancel{2}\sinh 2y)$$

$$\frac{\partial v}{\partial y} = \frac{-3\sin 2x\sinh 2y}{(\cosh 2y - \cos 2x)^2} \qquad \ldots(ii)$$

$$F'(Z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \left(\frac{-3\sin 2x\sinh 2y}{(\cosh 2y - \cos 2x)^2}\right) + i\left(\frac{3(\cosh 2y\cos 2x - 1)}{(\cosh 2y - \cos 2x)^2}\right)$$

Put $x = z, y = 0$

$$F'(Z) = i\left(\frac{3(\cos 2z - 1)}{(1 - \cos 2z)^2}\right) = i\frac{(-1)(1 - \cancel{\cos} 2z)}{(1 - \cos 2z)^{\cancel{2}}}$$

$$= \frac{-i}{(1-\cos 2z)} = \frac{-i}{(2\sin^2 z)} = \frac{-i}{2}\operatorname{cosec}^2 z$$

$$F'(Z) = \frac{i}{2}\cot Z + C$$

$$(1+i)\,f(z) = \frac{i}{2}\cot Z + C$$

$$f(x) = \frac{i}{2(1+i)}\cot Z + \frac{C}{1+i}$$

Q. 3(b) Prove that $f_1(X) = 1$, $f_2(x) = x$, $f_3(X) = (3x^2 - 1)/2$ are orthogonal over $(-1, 1)$ **(6 Marks)**

Ans. :

$$f_1(x) = 1, \qquad f_2(x) = x, \qquad f_3(x) = \frac{3x^2-1}{2}$$

$$\int_{-1}^{1} f_1(x)\,f_2(x)\,dx = \int_{-1}^{1} 1\cdot x\,dx = [x]_{-1}^{1} = 1-(-1) = 1+1 = 2$$

$$\int_{-1}^{1} f_2(x)\,f_2(x)\,dx = \int_{-1}^{1} x\cdot x\,dx = 2\int_{0}^{1} x^2\,dx = 2\left(\frac{x^3}{3}\right)_0^1 = \frac{2}{3}$$

$$\int_{-1}^{1} f_3(x)\,f_3(x)\,dx = \int_{-1}^{1}\left(\frac{3x^2-1}{2}\right)\left(\frac{3x^2-1}{2}\right)dx = \frac{1}{4}\int_{-1}^{1}(3x^2-1)^2\,dx$$

$$= \frac{1}{4}\int_{-1}^{1}(9x^4-6x^2+1)\,dx$$

$$= \frac{2}{4}\times 9\int_0^1 x^4\,dx - \frac{2}{4}\times 6\int_0^1 x^2\,dx + \frac{2}{4}\int_0^1 dx$$

$$= \frac{9}{2}\left(\frac{x^5}{5}\right)_0^1 - 3\left(\frac{x^3}{3}\right)_0^1 + \frac{1}{2}[x]_0^1 = \frac{9}{2}\,\frac{1}{5} - \frac{3}{3} + \frac{1}{2}$$

$$= \frac{9}{10} - 1 + \frac{1}{2} = \frac{9}{10} - \frac{1}{2} = \frac{9-5}{10} = \frac{4}{10} = \frac{2}{5}$$

$$\int_{-1}^{1} f_1(x)\,f_2(x)\,dx = \int_{-1}^{1} 1\cdot x\,dx = \int_{4}^{1} x\,dx = 0$$

$$\int_{-1}^{1} f_2(x)\,f_3(x)\,dx = \int_{-1}^{1} x\left(\frac{3x^2-1}{2}\right)dx = \frac{3}{2}\int_{-1}^{1} x^3\,dx - \frac{1}{2}\int_{-1}^{1} x\,dx$$

$$= 0 - 0 = 0$$

$$\int_{-1}^{1} f_1(x)\,f_3(x)\,dx = \int_{-1}^{1} 1\left(\frac{3x^2-1}{2}\right)dx = \frac{3}{2}\int_{-1}^{1} x^2\,dx - \frac{1}{2}\int_{-1}^{1} dx$$

$$= \frac{3}{\cancel{2}}\cdot\cancel{2}\int_0^1 x^2\,dx - \frac{1}{\cancel{2}}\cdot\cancel{2}\int_0^1 dx$$

$$= 3\left(\frac{x^3}{3}\right)_0^1 - (x)_0^1 = 1 - 1 = 0$$

$\Rightarrow$ $\{f_1(x), f_2(x), f_3(x)\}$ are orthogonal on $[-1, 1]$

Q. 6(b) Find the bilinear transformation which maps the ponts 2, j, –2 onto point 1, j, –1 using cross-ratio property. **(6 Marks)**

Ans. :

$$z = \underset{z_1}{2},\ \underset{z_2}{i},\ \underset{z_3}{-2},\ \underset{z_4}{z}, \qquad w = \underset{w_1}{1},\ \underset{w_2}{i},\ \underset{w_3}{-1},\ \underset{w_4}{w}$$

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)}$$

$$\frac{(2 - i)(-2 - z)}{(i + 2)(z - 2)} = \frac{(1 - i)(-1 - w)}{(i + 1)(w - 1)}$$

$$\frac{(2 - i)(i + 1)(\cancel{-1})(z + 2)}{(i + 2)(1 - i)(z - 2)} = \frac{(\cancel{-1})(1 + w)}{(w - 1)}$$

$$\frac{(2i + 1 + 2 - i)(z + 2)}{(i + 2 + 1 - 2i)(z - 2)} = \frac{w + 1}{w - 1}$$

$$\frac{(i + 3)(z + 2)}{(3 - i)(z - 2)} = \frac{w + 1}{w - 1} \Rightarrow \frac{(i + 3)(z + 2) + (3 - i)(z - 2)}{(i + 3)(z + 2) - (3 - i)(z - 2)}$$

$$= \frac{w + \cancel{1} + w - \cancel{1}}{\cancel{w} + 1 - \cancel{w} + 1}$$

$$\frac{(\cancel{i} + 3 + 3 - \cancel{i})z + 2i + \cancel{6} - \cancel{6} + 2i}{(i + \cancel{3} - \cancel{3} - i)z + \cancel{2i} + 6 + 6 - \cancel{2i}} = \frac{2w}{2}$$

$$\frac{6z + 4i}{2iz + 12} = w \qquad \Rightarrow \qquad w = \frac{3z + 2i}{iz + 6}$$

Chapter 2 : Laplace Transform [Total Marks - 31]

Q. 1(a) Find Laplace of $\{t^5 \cosh t\}$ **(5 Marks)**

Ans. :

$L[t^5 \cos ht]$

$$= L\left[t^5\left(\frac{e^t + e^{-t}}{2}\right)\right] = \frac{1}{2}L[t^5 e^t] + \frac{1}{2}L[t^5 e^{-t}] = \frac{1}{2}\frac{s!}{(s-1)^6} + \frac{1}{2}\frac{5!}{(s+1)^6}$$

$$= \frac{5!}{2}\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right] = \frac{120}{2}\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right] = 60\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

Q. 3(a) Find $L^{-1}\left\{\frac{1}{(s-2)^4(s+3)}\right\}$ using Convolution theorem. **(6 Marks)**

Ans. :

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right] \qquad \text{...(i)}$$

$$\therefore \quad L^{-1}\left[\frac{1}{(s-2)^4}\right] = e^{2t} L^{-1}\left[\frac{1}{s^4}\right] = e^{2t}\frac{t^3}{3!}$$

$$L^{-1}\left[\frac{1}{(s+3)}\right] = e^{-3t}$$

$\therefore$ From Equation (i)

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right] = \int_0^t e^{2u}\frac{u^3}{3!}\, e^{-3(t-u)}\, du$$

$$= \frac{1}{6}\int_0^t e^{2u} e^{-3t} e^{3u} u^3\, du$$

$$= \frac{e^{-3t}}{6}\int_0^t e^{5u} u^3\, du$$

$$= \frac{e^{-3t}}{6}\left[u^3\frac{e^{5u}}{5} - 3u^2\frac{e^{5u}}{5^2} + 6u\frac{e^{5u}}{5^3} - 6\frac{e^{5u}}{5^4}\right]_0^t$$

$$= \frac{e^{-3t}}{6}\left[\frac{t^3 e^{5t}}{5} - \frac{3t^2 e^{5t}}{5^2} + \frac{6t\, e^{5t}}{5^3} - \frac{6e^{5t}}{5^4} + \frac{6}{5^4}\right]$$

$$= \frac{1}{6}\left[\frac{t^3 e^{2t}}{5} - \frac{3t^2}{25}e^{2t} + \frac{6t}{5^3}e^{2t} - \frac{6}{5^4}e^{2t} + \frac{6}{5^4}e^{-3t}\right]$$

Q. 4(a) Find Laplace transform f(t) = |sin pt|, t ≥ 0. **(6 Marks)**

Ans. :

$$f(t) = |\sin pt|, \qquad t \geq 0$$

Period of $|\sin pt|$ is $\frac{\pi}{p}$

$$L[f(t)] = \frac{1}{1-e^{-s\pi/p}}\int_0^{\pi/p} e^{-st} f(t)\, dt = \frac{1}{1-e^{-s\pi/p}}\int_0^{\pi/p} e^{-st}|\sin pt|\, dt$$

Put $pt = u$

$p\,dt = du \Rightarrow dt = \frac{du}{p}$

t	0	$\frac{\pi}{p}$
u	0	π

$$= \frac{1}{1-e^{-s\pi/p}}\int_0^{\pi} e^{-s(u/p)}|\sin u|\frac{du}{p} = \frac{1}{p(1-e^{-s\pi/p})}\int_0^{\pi} e^{-(s/p)u}\sin u\, du$$

$$= \frac{1}{p(1-e^{-s\pi/p})}\left[\frac{e^{-(s/p)u}}{\frac{s^2}{p^2}+1}\left(\frac{-s}{p}\sin u - \cos u\right)\right]_0^{\pi}$$

$$= \frac{1}{p(1-e^{-s\pi/p})}\left[\frac{e^{-su/p}}{\frac{(s^2+p^2)}{p^2}}\left(\frac{-s\sin u - p\cos u}{p}\right)\right]_0^{\pi}$$

$$= \frac{1}{p(1-e^{-s\pi/p})}\left[\frac{p^2 e^{-s\pi/p}}{(s^2+p^2)}\frac{(o+p)}{p} - \frac{p^2}{s^2+p^2}(-1)\right]$$

$$= \frac{1}{\cancel{p}\,(1-e^{-s\pi/p})}\left[\frac{e^{-s\pi/p}+1}{s^2+p^2}\right]p^{\cancel{2}} = \frac{p\,(e^{-s\pi/p}+1)}{(s^2+p^2)\,(1-e^{-s\pi/p})}$$

$$= \frac{p}{(s^2+p^2)}\ \frac{\cancel{e^{-s\pi/2p}}\,(e^{s\pi/2p}+e^{-s\pi/2p})}{\cancel{e^{-s\pi/2p}}\,(e^{s\pi/2p}-e^{-s\pi/2p})}$$

$$= \frac{p}{(s^2+p^2)}\ \coth h\left(\frac{s\pi}{2p}\right)$$

Q. 5(c) Solve $(D^2+2D+5)y = e^{-t}\sin t$ With $y(0) = 0$ and $y'(0) = 1$ **(8 Marks)**

Ans. : **Please refer Q. 5(c) of Dec. 2014.**

Q. 6(a) Find $L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$ **(6 Marks)**

Ans. :

$$L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right] = \left(\frac{-1}{t}\right)L^{-1}\left[\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right]$$

$$= \left(\frac{-1}{t}\right)L^{-1}\left[\frac{1}{1+\frac{4}{s^4}}\frac{d}{ds}\left(\frac{2}{s^2}\right)\right] = \left(\frac{-1}{t}\right)L^{-1}\left[\frac{s^4}{s^4+4}\left(\frac{-4}{s^3}\right)\right]$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{s^4+4}\right] = \frac{4}{t}L^{-1}\left[\frac{s}{(s^4+4s^2+4)-4s^2}\right]$$

$$= \frac{4}{t}L^{-1}\left[\frac{s}{(s^2+2)^2-(2s)^2}\right] = \frac{4}{t}L^{-1}\left[\frac{s}{(s^2+2s+2)\,(s^2-2s+2)}\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{(s^2+2s+2)-(s^2-2s+2)}{(s^2+2s+2)\,(s^2-2s+2)}\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{1}{(s^2-2s+2)}-\frac{1}{(s^2+2s+2)}\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{1}{(s^2-2s+2)}\right]-\frac{1}{t}L^{-1}\left[\frac{1}{(s^2+2s+2)}\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{1}{(s+1)^2+1}\right]-\frac{1}{t}L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{t}e^{t}L^{-1}\left[\frac{1}{s^2+1}\right]-\frac{1}{t}e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{e^t}{t}\sin t-\frac{e^x}{t}\sin t = \frac{2}{t}\frac{(e^t-e^{-t})}{2}\sin t = \frac{2}{t}\sin ht\ \sin t$$

Chapter 3 : Fourier Series [Total Marks - 21]

Q. 1(b) Find Fourier series for $f(x) = 1-x^2$ in $(-1, 1)$ **(5 Marks)**

Ans. :

$f(x) = 1-x^2$ in $[-1, 1]$

$2l = 1-(-1)$

$2l = 2 \Rightarrow l = 1$

Let Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$\because f(x) = 1 - x^2 \qquad f(-x) = 1-(-x^2) = 1 - x^2 = f(x)$$

$$\Rightarrow \quad b_n = 0 \qquad a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = 2\int_0^1 (1-x^2)\,dx$$

$$= 2\left[x - \frac{x^3}{3}\right]_0^1 = 2\left(1-\frac{1}{3}\right) = 2 \times \frac{2}{3} = \frac{4}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx = 2\int_0^1 f(x)\cos(n\pi x)\,dx = 2\int_0^1 (1-x^2)\cos(n\pi x)\,dx$$

$$= 2\left[\cancel{(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right)} - (-2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\cancel{\frac{-\sin n\pi x}{n^3\pi^3}}\right)\right]_0^1$$

$$= 2\left[-(-2)\frac{(-\cos n\pi)}{n^2\pi^2}\right] = 2\left[\frac{-2(-1)^n}{n^2\pi^2}\right] = \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$(1-x^2) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2\pi^2}\cos(n\pi x)$$

Q. 4(c) Obtain Fourier expansion of $f(x) = x + \frac{\pi}{2}$ where $-\pi < x < 0 = \frac{\pi}{2} - x$ where $0 < x < \pi$

Hence deduce that (i) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

(ii) $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$ **(8 Marks)**

Ans. :

$$f(x) = x + \frac{\pi}{2} : \qquad -\pi < x < 0$$

$$\frac{\pi}{2} - x : \qquad 0 < x < \pi$$

$$2l = 2\pi \qquad \Rightarrow \qquad l = \pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(-x) = -x + \frac{\pi}{2} \quad : \quad -\pi < -x < 0$$

$$\frac{\pi}{2} + x \quad : \quad 0 < -x < \pi$$

$$= \frac{\pi}{2} - x \quad : \quad \pi > x > 0 = f(x) \Rightarrow f(x)\ 0 \text{ even}$$

$$\frac{\pi}{2}+x \quad : \quad 0 > x > -\pi$$

$$\Rightarrow \quad b_n = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}-x\right)dx$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\times x - \frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2}{2}-\frac{\pi^2}{2}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}-x\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\cancel{\left(\frac{\pi}{2}-x\right)\left(\frac{\sin nx}{n}\right)} - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\cos n\pi + 1}{n^2}\right] = \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]$$

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx$$

(i) Put $x = 0$

$$\frac{\lim\limits_{x\to 0^+} f(x) + \lim\limits_{x\to 0^-} f(x)}{2} = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]$$

$$\frac{\lim\limits_{x\to 0^+}\left(\frac{\pi}{2}-x\right) + \lim\limits_{x\to 0^-}\left(\frac{\pi}{2}+x\right)}{2} = \frac{2}{\pi}\left[\frac{2}{1^2}+\frac{2}{3^2}+\frac{2}{5^2}\cdots\right]$$

$$\frac{\frac{\pi}{2}+\frac{\pi}{2}}{2} = \frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots\right]$$

$$\frac{\pi}{2}\times\frac{\pi}{4} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

(ii)

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$$

$$\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}-x\right)^2dx = \sum_{n=1}^{\infty}\frac{4}{\pi^2}\left[\frac{1-(-1)^n}{n^2}\right]^2$$

$$\frac{2}{\pi}\left[\frac{\left(\frac{\pi}{2}-x\right)^3}{-3}\right]_0^{\pi} = \frac{4}{\pi^2}\left[\frac{4}{1^4}+\frac{4}{3^4}+\frac{4}{5^4}\cdots\cdots\right]$$

$$\frac{2}{\pi}\left[\frac{-\pi\frac{3}{8}}{-3}-\frac{\pi\frac{3}{8}}{-3}\right] = \frac{16}{\pi^2}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}\cdots\cdots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^3}{24}+\frac{\pi^3}{24}\right]\times\frac{\pi^2}{16} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}\cdots\cdots$$

$$\frac{\pi^2}{6}\times\frac{\pi^2}{16} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}\cdots\cdots$$

$$\frac{\pi^4}{96} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\cdots$$

Q. 6(c) Find Fourier sin integral representation for $f(x) = \frac{e^{-ax}}{x}$. **(8 Marks)**

Ans. :

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\left[\int_0^{\infty} f(t)\ \sin\lambda t\ dt\right]\sin\lambda x\ d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty}\left[\int_0^{\infty}\frac{e^{-at}}{t}\sin\lambda t\ dt\right]\sin\lambda x\ d\lambda \qquad \ldots.(1)$$

Let, $$I = \int_0^{\infty}\frac{e^{-at}}{t}\sin\lambda t\ dt$$

$$\frac{dI}{d\lambda} = \int_0^{\infty}\frac{e^{-at}}{t}\frac{\partial}{\partial\lambda}\sin\lambda t\ dt = \int_0^{\infty}\frac{e^{-at}}{\cancel{t}}\cos\lambda t\ (\cancel{t})\ dt = \int_0^{\infty}e^{-at}\cos\lambda t\ dt$$

$$= \left[\frac{e^{-at}}{a^2+\lambda^2}(-a\cos\lambda t+\lambda\sin\lambda t)\right]_0^{\infty} = \left[0-\frac{1}{a^2+\lambda^2}(-a)\right]$$

$$= \frac{a}{a^2+\lambda^2}$$

$$\frac{dI}{d\lambda} = \frac{a}{\lambda^2+a^2} \qquad \Rightarrow \qquad I = a\int\frac{d\lambda}{a^2+\lambda^2}$$

$$I = a\,\frac{1}{a}\tan^{-1}\left(\frac{\lambda}{a}\right)+C$$

$$I = \tan^{-1}\left(\frac{\lambda}{a}\right)+C$$

$$\int_0^{\infty}\frac{e^{-at}}{t}\sin\lambda t\ dt = \tan^{-1}\left(\frac{\lambda}{a}\right)+C$$

Put $\lambda = 0$

$$0 = 0+C \qquad \Rightarrow \qquad C = 0$$

$$I = \tan^{-1}\left(\frac{\lambda}{a}\right) \text{ Put in Equation (1)}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \tan^{-1}\left(\frac{\lambda}{a}\right) \sin \lambda x \, d\lambda$$

Chapter 4 : Vector Algebra & Calculus [Total Marks - 25]

Q. 1(d) Prove that $\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^3}$ **(5 Marks)**

Ans. :

$$\nabla\left(\frac{1}{r}\right) = \left(i\frac{\partial}{\partial x}+J\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) = i\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+J\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+k\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= i\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}+J\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y}+k\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= \left(\frac{-1}{r^2}\right)\left(i\frac{\partial r}{\partial x}+J\frac{\partial r}{\partial y}+k\frac{\partial r}{\partial z}\right) \quad \text{...(i)}$$

$$\because \bar{r} = x\,\hat{i}+y\,\hat{y}+z\,\hat{k} \qquad r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow r^2 = x^2+y^2+z^2 \qquad \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r}$

$\therefore$ From (i)

$$\nabla\left(\frac{1}{r}\right) = \left(\frac{-1}{r^2}\right)\left(i\frac{x}{r}+J\frac{y}{r}+k\frac{z}{r}\right) = \left(\frac{-1}{r^2}\right)\frac{(x\,\hat{i}+y\,\hat{y}+z\,\hat{k})}{r} = -\frac{\hat{r}}{r^3}$$

Q. 3(c) Verify Green's theorem for $\int_C \bar{F}\cdot\bar{dr}$ where $\bar{F} = (x^2-y^2)\,I + (x+y)j$ and C is the triangle with vertices (0, 0), (1, 1), (2, 1) **(8 Marks)**

Ans. :

$$\int_C \bar{F}\cdot\bar{dr} = \int_C (x^2-y^2)\,dx + (x+y)\,dy$$

By green theorem,

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \left(\frac{\partial \theta}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\int_C (x^2-y^2)\,dx + (x+y)\,dy = \iint_R (1+2y)\,dx\,dy \quad \text{...(i)}$$

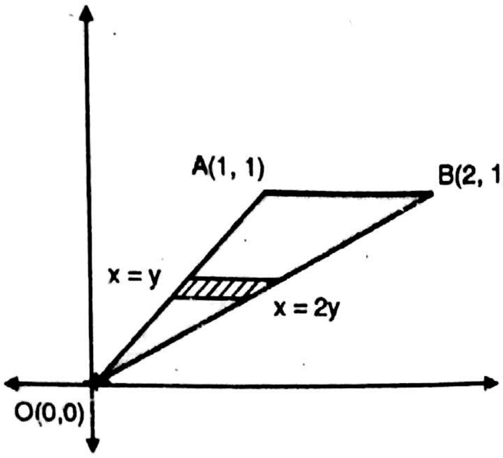

Fig. 1-Q. 3(c)

L.H.S. of (i) A long OB : $x = 2y$

$dx = 2dy$

$$= \int_0^1 (4y^2 - y^2)(2\,dy) + (2y + y)\,dy = \int_0^1 (3y^2)\,2dy + 3y\,dy$$

$$= 6\left(\frac{y^3}{3}\right)_0^1 + 3\left(\frac{y^2}{2}\right)_0^1 = 2 + \frac{3}{2} = \frac{7}{2}$$

A long BA : $y = 1 \Rightarrow dy = 0$

$$= \int_2^1 (x^2 - 1)\,dx + (x + 1) \times 0 = \left(\frac{x^3}{3}\right)_2^1 - (x)_2^1 = \left(\frac{1}{3} - \frac{8}{3}\right) - (1 - 2)$$

$$= \frac{-7}{3} + 1 = \frac{-4}{3}$$

Along AO : $Y = x \Rightarrow dy = dx$

$$= \int_1^0 (x^2 - x^2)\,dx + 2x\,dx = 2\left(\frac{x^2}{\cancel{2}}\right)_1^0 = 0 - 1 = -1$$

$$\therefore \text{ L.H.S.} = \frac{7}{2} - \frac{4}{3} - 1 = \frac{21 - 8 - 6}{6} = \frac{21 - 14}{6} = \frac{7}{6}$$

R.H.S. of (1)

$$= \int_0^1 \int_{x=y}^{x=2y} (1 + 2y)\,dx\,dy = \int_0^1 \left[(x)_y^{2y} + 2y\,(x)_y^{2y}\right] dy$$

$$= \int_0^1 [y + 2y\,(y)]\,dy = \int_0^1 [y + 2y^2]\,dy = \left(\frac{y^2}{2}\right)_0^1 + 2\left(\frac{y^3}{3}\right)_0^1$$

$$= \frac{1}{2} + \frac{2}{3} = \frac{3 + 4}{6} = \frac{7}{6}$$

$\therefore$ L.H.S. = R.H.S. in equation (i)

$\Rightarrow$ Greens theorem is verified.

Q. 4(b) Show that $\bar{F} = (y \sin z - \sin x)\, i + (x\sin z + 2yz)\, j + (xy \cos z + y^2)\, k$ is irrotational. Hence, find its scalar potential. **(6 Marks)**

Ans. :

$$\bar{F} = (y \sin z - \sin x)\, \hat{i} + (x \sin z + 2yz)\, \hat{j} + (xy \cos z + y^2)\, \hat{k}$$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y \sin z - \sin x & x \sin z + 2yz & xy \cos z + y^2 \end{vmatrix}$$

$$= 1\,(x \cos z + 2y - x \cos z - 2y) + J\,(y \cos z - y \cos z) + k\,(\sin z - \sin z)$$

$$= i\,(0) + j\,(0) + k\,(0) = 0 \;\Rightarrow\; \bar{F} \text{ is vocotational}$$

$$\bar{F} = \nabla \phi$$

$$(y \sin z - \sin x)\, \hat{i} + (x \sin z + 2yz)\, \hat{j} + (xy \cos z + y^2)\, \hat{k} = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = y \sin z - \sin x$$

$$\frac{\partial \phi}{\partial y} = x \sin z + 2y\, z$$

$$\frac{\partial \phi}{\partial z} = xy \cos z + y^2$$

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy + \frac{\partial \phi}{\partial z}\, dz$$

$$d\phi = (y \sin z - \sin x)\, dx + (x \sin z + 2yz)\, dy + (xy \cos z + y^2)\, dz$$

$$\int d\phi = \int d\,(xy \sin z + \cos x + y^2 z)$$

$$\phi = xy \sin z + \cos x + y^2 z \text{ is the scaler potential.}$$

Q. 5(a) Using Gauss Divergence theorem to evaluate $\iint_S \bar{N}\, \bar{F}\, dz$ where $\bar{F} = 4xi - 2y^2 j + z^2 k$ and S is the region bounded by $x^2 + y^2 = 4$, $z = 0$, $z = 3$. **(6 Marks)**

Ans. :

$$\bar{F} = 4x\, \hat{i} - 2y^2\, \hat{j} + z^2\, \hat{k}$$

$$\nabla \cdot \bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(4x\, \hat{i} - 2y^2\, \hat{j} + z^2\, \hat{k}\right)$$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= (4 - 4y + 2z)$$

By divergence theorem

$$\iint_S \bar{F} \cdot \overline{dS} = \iiint_V \left(\nabla \cdot \bar{F}\right) dv$$

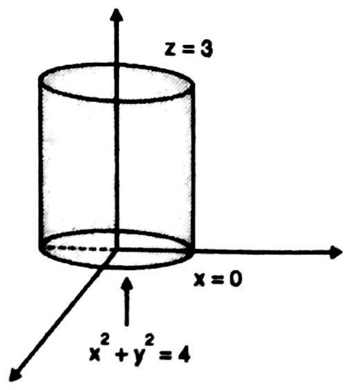

Fig. 1-Q. 5(a)

$$= \iint \int_{z=0}^{z-3} (4-4y+2z)\,dx\,dy\,dz = \iint\left[(4-4y)(z)_0^3 + 2\left(\frac{x^2}{2}\right)_0^3\right] dx\,dy$$

$$= \iint [3(4-4y)+9]\,dx\,dy = \iint [12-12y+9]\,dx\,dy = \iint [21-12y]\,dx\,dy$$

$$= \int_0^{2\pi}\int_{r=0}^{2} (21-12r\sin\theta)\,r\,dr\,d\theta \qquad \text{(using cylindrical co-ordinate system)}$$

$$= \int_0^{2\pi}\left[21\left(\frac{r^2}{2}\right)_0^2 - 12\sin\theta\left(\frac{r^3}{3}\right)_0^2\right] d\theta = \int_0^{2\pi}\left[21\times\frac{4}{2} - \frac{12\times 8}{3}\sin\theta\right] d\theta$$

$$= \int_0^{2\pi} [42-32\sin\theta]\,d\theta = 42\int_0^{2\pi} d\theta - 32\int_0^{2\pi}\sin\theta\,d\theta$$

$$= 42\,(\theta)_0^{2\pi} - 32\times 0 = 42\,(2\pi) - 0 = 84\pi$$

Chapter 5 : Z – Transform [Total Marks - 12]

Q. 2(b) Find inverse Z-transform of $f(z) = \frac{z+2}{z^2-2z+1}$ for $|z| > 1$. **(6 Marks)**

Ans. : **Please refer Q. 3(c) of May 2014.**

Q. 5(b) Find Z $\{2^K \cos(3k+2)\}$, $k \geq 0$. **(6 Marks)**

Ans. :

$$z\,[2k\cos(3k+2)] \qquad \ldots(i)$$

$$z\,[\cos(3k+2)] = z\,[\cos 3k\cos 2 - \sin 3k \sin 2] = \cos 2z\,[\cos 3k] - \sin 2z\,[\sin 3k]$$

$$= \frac{\cos 2z\,(z-\cos 3)}{z^2-2z\cos 3+1} - \frac{\sin 2\,z\sin 3}{z^2-2z\cos 3+1} = \frac{z^2\cos\theta - z\cos 2\cos 3 - z\sin 2\sin 3}{z^2-2z\cos 3+1}$$

$$= \frac{z^2\cos 2 - z\cos 1}{z^2-2z\cos 3+1} \Rightarrow z\,[2^k\cos(3k+2)] = \frac{\left(\frac{z}{2}\right)^2\cos 2 - \left(\frac{z}{2}\cos 1\right)}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3 + 1}$$

□□□

May 2016

Chapter 1 : Complex Variable and Mapping [Total Marks - 17]

Q. 1(C) If u(x, y) is a harmonic function then prove that $f(z) = u_x - iu_y$ is an analytic function.

(5 Marks)

Ans. :

Given that u (x, y) is an harmonic function.

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

$$f(z) = u_x - i\,u_y$$

It's real part $= u_x$ and imaginary part $b - u_y$

$$\frac{\partial}{\partial x} u_x = \frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad \frac{\partial}{\partial y} u_x = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial}{\partial y}(-u_y) = -\frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial}{\partial x}(-u_y) = -\frac{\partial^2 u}{\partial x \partial y}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \left(\text{as } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ from Equation (1)}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial y}(-u_y)$$

Hence first C-R equation is satisfied

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = -\left(-\frac{\partial^2 u}{\partial x \partial y}\right)$$

$$\frac{\partial}{\partial y}(u_x) = \frac{\partial}{\partial x}(-u_y)$$

Hence second C-R equation is satisfied and $\frac{\partial}{\partial x}(u_x), \frac{\partial}{\partial y}(u_x), \frac{\partial}{\partial y}(-u_y), \frac{\partial}{\partial x}(-u_y)$ are continues function therefore function,

$$f(z) = u_x - i\,u_y$$

is an analytic function.

Q. 2(A) If $v = e^x$ siny, prove that v is a harmonic function. Also find the corresponding analytic function.

(6 Marks)

Ans. :

$$\text{If } v = e^x \sin y \qquad ...(1)$$

Differentiate it with respect to x

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}[e^x \sin y] = \sin y \frac{\partial}{\partial x}(e^x) = \sin y\, e^x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \sin y \frac{\partial}{\partial x}(e^x)$$

$$\frac{\partial^2 v}{\partial x^2} = e^x \sin y \qquad \ldots(i)$$

Differentiate Equation (1) with respect to y

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}[e^x \sin y] = e^x \frac{\partial}{\partial y}(\sin y) = e^x (\cos y)$$

$$\frac{\partial^2 v}{\partial y^2} = e^x \frac{\partial}{\partial y}(\cos y)$$

$$\frac{\partial^2 v}{\partial y^2} = e^x (-\sin y) \qquad \ldots(ii)$$

From Equations (i) and (ii)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^x \sin y - e^x \sin y = 0$$

$\Rightarrow$ v is harmonic.

To find the analytic function

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = e^x \cos y + i\, e^x \sin y$$

but $x = z,\ y = 0$

$$f'(z) = e^z \Rightarrow f(z) = e^z + C$$

Q. 4(B) Show that the map of the real axis of the z- plane is a circle under the transformation $w = \frac{2}{z+i}$. Find its centre and the radius. **(6 Marks)**

Ans. : **Please refer Q. 3(b) of Dec. 2014.**

Chapter 2 : Laplace Transform [Total Marks - 27]

Q. 1(A) If $\int_0^\infty e^{-2t} \sin(t+\alpha)\cos(t-\alpha)dt = \frac{1}{4}$. Find 'α'. **(5 Marks)]**

Ans. :

$$L[\sin(t+\alpha)\cos(t-\alpha)] = L\left[\frac{1}{2} 2\sin(t+\alpha)\cos(t-\alpha)\right]$$

$$= \frac{1}{2} L[\sin(t+\alpha+t-\alpha) + \sin(t+\alpha-t+\alpha)]$$

$$= \frac{1}{2} L\ [\sin 2t + \sin 2\alpha] = \frac{1}{2}[L(\sin 2t) + L(\sin 2\alpha)]$$

$$= \frac{1}{2}\left[\frac{2}{s^2+2^2} + (\sin 2\alpha) L(1)\right] = \frac{1}{2}\left[\frac{2}{s^2+4} + \frac{\sin 2\alpha}{s}\right]$$

$$\therefore \int_0^\infty e^{-s} \sin(t+\alpha)\cos(t-\alpha)dt = \frac{1}{2}\left[\frac{2}{s^2+4} + \frac{\sin 2\alpha}{s}\right]$$

Put $s = 2$

$$\therefore \int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)dt = \frac{1}{2}\left[\frac{2}{4+4}+\frac{\sin 2\alpha}{2}\right]=\frac{1}{4}$$

$$\Rightarrow \frac{1}{4}+\frac{\sin 2\alpha}{2}=\frac{1}{2}\Rightarrow \frac{\sin 2\alpha}{2}=\frac{1}{2}-\frac{1}{4}$$

$$\Rightarrow \frac{\sin 2\alpha}{2}=\frac{1}{4}\Rightarrow \sin 2\alpha=\frac{1}{2}\Rightarrow 2\alpha=\frac{\pi}{6}\Rightarrow \alpha=\frac{\pi}{12}$$

Q. 3(A) Find inverse Laplace of $\frac{(s+3)^2}{(s^2+6s+5)^2}$ using Convolution theorem. **(6 Marks)**

Ans. :

$$L^{-1}\left[\frac{(s+3)^2}{(s^2+6s+5)^2}\right] = L^{-1}\left[\frac{(s+3)}{(s^2+6s+5)}\cdot\frac{(s+3)}{(s^2+6s+5)}\right] \qquad ...(1)$$

$$L^{-1}\left[\frac{(s+3)}{(s^2+6s+5)}\right] = L^{-1}\left[\frac{(s+3)}{(s^2+6s+9)-4}\right] = L^{-1}\left[\frac{(s+3)}{(s+3)^2-4}\right]$$

$$= e^{-3t}L^{-}\left[\frac{s}{s^2-4}\right] = e^{-3t}\cosh 2t$$

$\therefore$ From Equation (1)

$$L^{-1}\left[\frac{(s+3)}{(s^2+6s+5)}\cdot\frac{(s+3)}{(s^2+6s+5)}\right] = \int_0^t e^{-3u}\cosh 2u\, e^{-3(t-u)}\cosh 2(t-u)\, du$$

$$= \int_0^t e^{-3u}\; e^{-3t+3u}\cosh 2u \cosh(2t-2u)\, du$$

$$= \int_0^t e^{-3u-3t+3u}\cosh 2u \cosh(2t-2u)\, du$$

$$= \frac{e^{-3t}}{2}\int_0^t 2\cosh 2u\cosh(2t-2u)\, du$$

$$= \frac{e^{-3t}}{2}\int_0^t [\cosh(2u+2t-2u)+\cosh(2u-2t+2u)]\, du$$

$$= \frac{e^{-3t}}{2}\int_0^t [\cosh 2t + \cosh(4u-2t)]\, du$$

$$= \frac{e^{-3t}}{2}\left[\cosh 2t\int_0^t du + \int_0^t \cosh(4u-2t)\, du\right]$$

$$= \frac{e^{-3t}}{2}\left[\cosh 2t\,(u)_0^t + \left\{\frac{\sin h\,(4u-2t)}{4}\right\}_0^t\right]$$

$$= \frac{e^{-3t}}{2}\left[t\cosh 2t + \frac{1}{4}\{\sin h\,(2t) - \sin h\,(-2t)\}\right]$$

$$= \frac{e^{-3t}}{2}\left[t\cosh 2t + \frac{1}{4}\{\sin h\,2t + \sin h\,2t\}\right]$$

$$= \frac{e^{-3t}}{2}\left[t\cosh 2t + \frac{1}{4}2\sinh 2t\right]$$

$$= \frac{e^{-3t}}{2}\left[t\cosh 2t + \frac{1}{2}\sinh 2t\right]$$

$$= \frac{e^{-3t}}{4}\,[\,2t\cosh 2t + \sinh 2t\,]$$

Q. 5(C) Solve $(D^2 + 3D + 2)\,y = e^{-2t}\sin t$, with $y(0) = 0$ and $y'(0) = 0$ **(8 Marks)**

Ans. :

$(D^2 + 3D + 2)\,y = e^{-2t}\sin t$, $y(0) = 0$, and $y'(0) = 0$

Given differential equation

$$(D^2 + 3D + 2)\,y = e^{-2t}\sin t$$

Taking Laplace tranform of it

$$L[D^2 y] + 3L[Dy] + 2L[y] = L[e^{-2t}\sin t]$$

$$[s^2 L(y) - sy(0) - y'(0)] + 3[sL(y) - y(0)] + 2L(y) = \frac{1}{(s+2)^2 + 1}$$

$$(s^2 + 3s + 2)\,L(y) = \frac{1}{s^2 + 4s + 4 + 1}$$

$$L(y) = \frac{1}{(s^2 + 4s + 5)(s^2 + 3s + 2)}$$

$$y(t) = L^{-1}\left[\frac{1}{(s^2 + 4s + 5)(s^2 + 3s + 2)}\right] \qquad \ldots(1)$$

$$\frac{1}{(s^2 + 4s + 5)(s^2 + 3s + 2)} = \frac{As + B}{(s^2 + 4s + 5)} + \frac{Cs + D}{(s^2 + 3s + 2)}$$

$$1 = (As + B)(s^2 + 3s + 2) + (Cs + D)(s^2 + 4s + 5)$$

$$A + C = 0 \;;\; 2B + 5D = 1$$

$$3A + B + 4C + D = 0 \;;\; 2A + 3B + 5C + 4D = 0$$

$$\begin{aligned}
A + C &= 0 \Rightarrow A = -C &;\quad -2C + 3B + 5C + 4D &= 0\\
3A + B + 4C + D &= 0 &;\quad 3B + 3C + 4D &= 0\\
-3C + B + 4C + D &= 0 &;\quad 3(B + C) + 4D &= 0\\
B + D + C &= 0 &;\quad 3(-D) + 4D &= 0\\
B + C &= -D & D &= 0\\
& & \text{and}\quad 2B + 5D &= 1\\
\therefore\; B + C &= 0 & &\\
\Rightarrow\; 2B &= 1 & B &= 1/2
\end{aligned}$$

$$C = -B = -1/2 \quad ; \quad A = 1/2$$

$$y(t) = L^{-1}\left[\frac{1/2\,s + 1/2}{(s^2+4s+5)} + \frac{-1/2\,s}{(s^2+3s+2)}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{(s+1)}{(s+2)^2+1}\right] - 1/2\,L^{-1}\left[\frac{5}{s^2+3s+2}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{(s+2)}{(s+2)^2+1} + \frac{1}{(s+2)^2+1}\right] - \frac{1}{2}L^{-1}\left[\frac{s}{(s+2)(s+1)}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{(s+2)}{(s+2)^2+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s+2)^2+1}\right] - \frac{1}{2}L^{-1}\left[\frac{2(s+1)-(s+2)}{(s+1)(s+2)}\right]$$

$$= \frac{1}{2}e^{-2t}L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{1}{2}e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right] - \frac{1}{2}L^{-1}\left[\frac{2}{s+2} - \frac{1}{(s+1)}\right]$$

$$= \frac{1}{2}e^{-2t}\cos t + \frac{1}{2}e^{-2t}\sin t - \frac{1}{2}\,2\,L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right]$$

$$= \frac{1}{2}e^{-2t}\cos t + \frac{1}{2}e^{-2t}\sin t - e^{-2t} + \frac{1}{2}e^{-t}$$

Q. 6(C) Find (i) $L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\}$

(ii) $L^{-1}\left(\frac{e^{-\pi s}}{s^2-2s+2}\right)$ **(8 Marks)**

Ans. :

(i)

$$L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}\tan^{-1}\left(\frac{a}{s}\right)\right\}$$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{1+a^2/s^2}\frac{d}{ds}\left(\frac{a}{s}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{s^2}{s^2+a^2}\left(-\frac{a}{s^2}\right)\right] = \frac{1}{t}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{1}{t}\sin at$$

(ii) $L^{-1}\left(\frac{e^{-\pi s}}{s^2-2s+2}\right)$ ...(1)

$$L^{-1}\left[\frac{1}{s^2-2s+2}\right] = L^{-1}\left[\frac{1}{(s^2-2s+1)+1}\right]$$

$$= L^{-1}\left[\frac{1}{(s-1)^2+1}\right] e^t L^{-1}\left[\frac{1}{s^2+1}\right] = e^t \sin t$$

$$\therefore L^{-1}\left[\frac{e^{-\pi s}}{s^2-2s+2}\right] = e^{(t-\pi)}\sin(t-\pi) + 1\,(t-\pi)$$

Chapter 3 : Fourier Series [Total Marks - 33]

Q. 1(B) Find half range Fourier cosine series for $f(x) = x, 0 < x < 2$ **(5 Marks)**

Ans. :

Let us assume f (x) as even function on the interval $-2 < x < 2$.

$$2l = 2-(-2) = 4 \Rightarrow l = 2$$

Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$\therefore$ f (x) b even $\Rightarrow b_n = 0$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = \frac{2^2}{2} = 2$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^2 x\cos\frac{n\pi x}{2}\,dx = \left[\cancel{x\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}} - 1\frac{\left(-\cos\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$$

$$= \frac{4}{n^2\pi^2}[\cos n\pi - 1] = \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

$$f(x) = \frac{2}{2} + \sum_{n=1}^{\infty} \frac{4[(-1)^n - 1]}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$$

$$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2}\cos\left(\frac{n\pi x}{2}\right)$$

Q. 2(C) Obtain Fourier series for $f(x) = \frac{3x^2 - 6x\pi + 2\pi^2}{12}$ in $(0, 2\pi)$, where $f(x + 2\pi) = f(x)$. Hence deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ **(8 Marks)**

Ans. :

$$f(x) = \frac{3x^2 - 6x\pi + 2\pi^2}{12} \text{ in } (0, 2\pi)$$

$$2l = 2\pi - 0 = 2\pi \Rightarrow l = \pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2 - 6\pi x + 2\pi^2}{12}\right)dx$$

$$= \frac{1}{12\pi}\int_0^{2\pi}(3x^2 - 6\pi x + 2\pi^2)\,dx = \frac{1}{12\pi}\left[3\left(\frac{x^3}{3}\right)_0^{2\pi} - 6\pi\left(\frac{x^2}{2}\right)_0^{2\pi} + 2\pi^2(x)_0^{2\pi}\right]$$

$$=\frac{1}{12\pi}\left[\cancel{3}\times\frac{8\pi^3}{\cancel{3}}-6\pi\times\frac{4\pi^2}{2}+2\pi^2(2\pi)\right]=\frac{1}{12\pi}[8\pi^3-12\pi^3+4\pi^3]=0$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2-6\pi x+2\pi^2}{12}\right)\cos nx\,dx$$

$$=\frac{1}{12\pi}\int_0^{2\pi}(3x^2-6\pi x+2\pi^2)\cos nx\,dx$$

$$=\frac{1}{12\pi}\left[(3x^2-6\pi x+2\pi^2)\cancel{\left(\frac{\sin nx}{n}\right)}-(6x-6\pi)\left(\frac{-\cos nx}{n^2}\right)+(6)\cancel{\left(\frac{-\sin nx}{n^3}\right)}\right]_0^{2\pi}$$

$$=\frac{1}{12\pi}\left[-\frac{(6\pi)(-\cos 2n\pi)}{n^2}+\frac{(-6\pi)(-1)}{n^2}\right]=\frac{1}{12\pi}\left[\frac{6\pi(-1)^{2n}}{n^2}+\frac{6\pi}{n^2}\right]=\frac{1}{12\pi}\times\frac{12\pi}{n^2}=\frac{1}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2-6\pi x+2\pi^2}{12}\right)\sin nx\,dx$$

$$=\frac{1}{12\pi}\int_0^{2\pi}(3x^2-6\pi x+2\pi^2)\sin nx\,dx$$

$$=\frac{1}{12\pi}\left[(3x^2-6\pi x+2\pi^2)\left(\frac{-\cos nx}{n}\right)-(6x-6\pi)\cancel{\left(\frac{-\sin nx}{n^2}\right)}+(6)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{12\pi}\left[\left\{(12\pi^2-12\pi^2+2\pi^2)\left(\frac{-\cos 2n\pi}{n}\right)-\frac{(2\pi^2)(-1)}{n}\right\}+6\left(\frac{1}{n^3}-\frac{1}{n^3}\right)\right]$$

$$=\frac{1}{12\pi}\left[\left(\frac{-2\pi^2}{n}+\frac{2\pi^2}{n}\right)+6\left(\frac{1}{n^3}-\frac{1}{n^3}\right)\right]=0$$

$$\therefore f(x)=\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$\frac{3x^2-6\pi x+2\pi^2}{12}=\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

Put $x = 0$

$$\frac{\lim\limits_{x\to 0}\dfrac{3x^2-6\pi x+2\pi^2}{12}+\lim\limits_{x\to 2\pi}\left(\dfrac{3x^2-6\pi x+2\pi^2}{12}\right)}{2}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\dfrac{2\pi^2}{12}+\dfrac{(12\pi^2-12\pi^2+2\pi^2)}{12}}{2}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\frac{\pi^2}{6}+\frac{\pi^2}{6}}{2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\ldots$$

Q. 3(B) Show that the set of functions {sinx, sin3x, sin5x,.....} is orthogonal over $[0,\pi/2]$. Hence construct orthonormal set of functions. **(6 Marks)**

Ans. :

{sinx, sin3x, sin5x,.....} over $[0, \pi/2]$

$$\int_0^{\pi/2} \sin(2n+1)x \sin(2m+1)x\, dx = \frac{1}{2}\int_0^{\pi/2} 2\sin(2n+1)x\sin(2m+1)x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} [\cos(2nx+x-2mx-x)+\cos(2nx+x+2mx+x)]\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} [\cos(2n-2m)x+\cos(2n+2m+2)x]\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} [\cos 2(n-m)x+\cos 2(n+m+1)x]\, dx \qquad \ldots(1)$$

If m $= n$

$$= \frac{1}{2}\left[\frac{\sin 2(n-m)x}{2(n-m)}+\frac{\sin 2(n+m+1)x}{2(n+m+1)}\right]_0^{\pi/2}$$

$$= \frac{1}{2}[(0-0)+(0-0)] = 0$$

$$= \frac{1}{2}\int_0^{\pi/2} dx + \frac{1}{2}\int_0^{\pi/2} \cos(4n+2)x\, dx$$

If n $= m$ then from Equation (1)

$$= \frac{1}{2}\int_0^{\pi/2} [1+\cos 2(2n+1)x]\, dx$$

$$= \frac{1}{2}(x)_0^{\pi/2} + \frac{1}{2}\left[\frac{\sin(4n+2)x}{4n+2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}(\pi/2)+\frac{1}{2(4n+2)}[\sin(2n+1)\pi - \sin 0] = \frac{\pi}{4}+0 = \frac{\pi}{4}$$

$\Rightarrow$ [sinx, sin3x, sin5x,.....} is orthogonal function.

To construct orthonormal function in above integral in the case of m = n we should have integral value 1. Hence to cosntruct the orthonormal set will be $\left\{\sqrt{\frac{4}{\pi}}\sin x, \sqrt{\frac{4}{\pi}}\sin 3x, \sqrt{\frac{4}{\pi}}\sin 5x....\right\}$

Q. 4(C) Express the function $f(x) = \begin{cases} \sin x & ; \quad |x| < \pi \\ 0 & ; \quad |x| > \pi \end{cases}$ as Fourier sine integral. **(8 Marks)**

Ans. :

$$f(x) = \begin{cases} \sin x & ; \quad |x| < \pi \\ 0 & ; \quad |x| > \pi \end{cases}$$

Fourier sin integral representation of the function.

$$f(x) = \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(t)\sin\lambda + dt\right]\sin\lambda x\, d\lambda$$

Given function

$$f(x) = \begin{cases} \sin x & x < (\pi, \pi) \\ 0 & x > (\infty, \pi) \cup (\pi, \infty) \end{cases}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \left[\int_{-\infty}^{\pi} f(t)\sin\lambda t\, dt + \int_{-\pi}^{\pi} f(t)\sin\lambda t\, dt + \int_{\pi}^{\infty} f(t)\sin\lambda t\, dt\right]\sin\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \left[\int_{-\pi}^{\pi}\sin t\sin\lambda t\, dt\right]\sin\lambda x\, d\lambda = \frac{1}{\pi}\int_0^\infty\left[\int_{-\pi}^{\pi} 2\sin t\sin\lambda t\, dt\right]\sin\lambda x\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty\left[\int_{-\pi}^{\pi}\{\cos(\lambda-1)t - \cos(\lambda+1)t\}\, dt\right]\sin\lambda x\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty\left[\frac{\sin(\lambda-1)t}{\lambda-1} - \frac{\sin(\lambda+1)t}{\lambda+1}\right]_{\pi}^{\pi}\sin\lambda x\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty\left[\left(\frac{\sin(\lambda-1)\pi}{\lambda-1} - \frac{\sin(\lambda+1)\pi}{\lambda+1}\right) - \left(\frac{\sin(\lambda-1)(-\pi)}{\lambda-1} - \frac{\sin(\lambda+1)(-\pi)}{\lambda+1}\right)\right]$$

$\sin\lambda x\, d\lambda$

$$= \frac{1}{\pi}\int_0^\infty\left[\frac{\sin(\lambda-1)\pi}{\lambda-1} - \frac{\sin(\lambda+1)\pi}{\lambda+1} + \frac{\sin(\lambda-1)\pi}{\lambda-1} - \frac{\sin(\lambda+1)\pi}{\lambda+1}\right]\sin\lambda x\, d\lambda$$

$$= \frac{1}{\pi}2\int_0^\infty\left[\frac{\sin(\lambda-1)\pi}{\lambda-1} - \frac{\sin(\lambda+1)\pi}{\lambda+1}\right]\sin\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \left[\frac{\sin(\lambda-1)\pi}{\lambda-1} - \frac{\sin(\lambda+1)\pi}{\lambda+1}\right]\sin\lambda x \, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \left[\frac{\sin(\lambda\pi-\pi)}{\lambda-1} - \frac{\sin(\lambda\pi+\pi)}{\lambda+1}\right]\sin\lambda x \, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \left[\frac{-\sin\lambda\pi}{\lambda-1} + \frac{\sin\lambda\pi}{\lambda+1}\right]\sin\lambda x \, d\lambda = \frac{2}{\pi}\int_0^\infty \left[\frac{\lambda-1-\lambda-1}{(\lambda-1)(\lambda+1)}\right]\sin\lambda\pi \sin\lambda x \, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\pi \sin\lambda x}{1-\lambda^2} d\lambda$$

Q. 6(A) Find Fourier expansion of $f(x) = 4 - x^2$ in the interval $(0, 2)$ **(6 Marks)**

Ans. : Total length of the interval

$2l = 2 - 0, \quad 2l = 2, \quad l = 1$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^2 (4-x^2)\,dx = 4(x)_0^2 - \left(\frac{x^3}{3}\right)_0^2 = 4(2) - 8/2 = 16/2$$

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos(n\pi x)\,dx = \int_0^2 (4-x^2)\cos(n\pi x)\,dx = \int_0^2 (4-x^2)\cos(n\pi x)\,dx$$

$$= \left[(4-x^2)\cancel{\left(\frac{\sin n\pi x}{n\pi}\right)} - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\cancel{\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)}\right]_0^2$$

$$= \left[-(-2\times 2)\left(\frac{-\cos 2n\pi}{n^2\pi^2}\right)\right] = \frac{-4(-1)^{2n}}{n^2\pi^2} = \frac{-4}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_0^2 (4-x^2)\sin(n\pi x)\,dx$$

$$= \left[(4-x^2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-2x)\cancel{\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)} + (-2)\left(-\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[\left(0 - 4\frac{(-1)}{n\pi}\right) - 2\left(\frac{\cos 2n\pi - 1}{n^3\pi^3}\right)\right] = \frac{4}{n\pi} - 2\left[\frac{(-1)^{2n}-1}{n^3\pi^3}\right]$$

$$= \frac{4}{n\pi} - 2\left[\frac{1-1}{n^3\pi^3}\right] = \frac{4}{n\pi}$$

$$\therefore\ 4 - x = \frac{8}{3} + \sum_{n=1}^{\infty} \frac{(-4)}{n^2\pi^2}\cos(n\pi x) + \sum_{n=1}^{\infty}\left(\frac{4}{n\pi}\right)\sin(n\pi x)$$

Chapter 4 : Vector Algebra & Calculus [Total Marks - 28]

Q. 1(D) Prove that $\nabla f(r) = f'(r)\frac{\hat{r}}{r}$. **(5 Marks)**

Ans. :

$$\nabla f(r) = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right) f(r) = i\frac{\partial}{\partial x}f(r)+j\frac{\partial}{\partial y}f(r)+k\frac{\partial}{\partial z}f(r)$$

$$= i f'(r)\frac{\partial r}{\partial x}+j f'(r)\frac{\partial r}{\partial y}+k f'(r)\frac{\partial r}{\partial y} = f'(r)\left[i\frac{\partial r}{\partial x}+j\frac{\partial r}{\partial y}+k\frac{\partial r}{\partial y}\right] \quad ...(1)$$

$$\because \quad r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

Differentiate it with respect to x

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \text{similarly } \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$\therefore$ From Equation (1)

$$\nabla f(r) = f'(r)\left[i\frac{x}{r}+j\frac{y}{r}+k\frac{z}{r}\right]$$

$$= f'(r)\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{r} = f'(r)\frac{\bar{r}}{r}$$

Q. 3(C) Verify Green's theorem for $\int_C \frac{1}{y}dx + \frac{1}{x}dy$ where C is the boundary of region defined by x = 1, x = 4, y = 1 and $y = \sqrt{x}$ **(8 Marks)**

Ans. : By Greens theorem,

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \left(\frac{\partial \theta}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy$$

$$\therefore \int_C \frac{1}{y}dx + \frac{1}{x}dy = \iint_R \left(-\frac{1}{x^2} + \frac{1}{y^2}\right) dx\,dy \quad ...(1)$$

$x = 1,\ x = 4,\ y = 1,\ y = \sqrt{x} \Rightarrow y^2 = x$

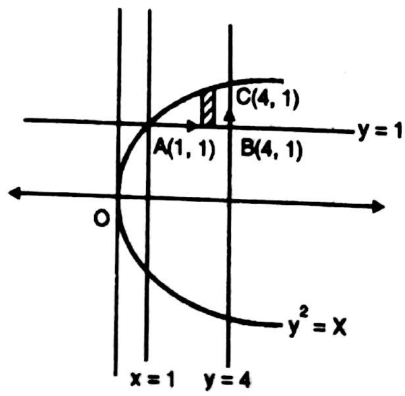

Fig. 1-Q. 3(C)

L.H.S. of Equation (1) on the region ABC

(i) Along AB : $y = 1 \Rightarrow dy = 0$

$\therefore$ From LHS of Equation (1)

$$\int_{x=1}^{4} \left(\frac{1}{1} dx\right) = [x]_1^4 = 4 - 1 = 3$$

(ii) Along BC : $x = 4 \Rightarrow dx = 0$

$$\int_{y=1}^{2} \left(\frac{1}{4} dy\right) = \frac{1}{4}(y)_1^2 = \frac{1}{4}(2-1) = 1/4$$

(iii) Along CA : $x = y^2 \Rightarrow dx = 2ydy$

$$\int_{y=2}^{y=1} \left(\frac{1}{y}\cdot 2ydy + \frac{1}{y^2}dy\right) = 2\ (y)_2^1 + \left[\frac{y^{-1}}{-1}\right]_2^1 = 2\,(1-2) + (-1+1/2)$$

$$= 2\,(-1) + (-1/2) = -2 - 1/2 = -5/2$$

$$\therefore \text{ LHS of Equation (1)} = 3 + \frac{1}{4} - 5/2 = \frac{12+1-10}{4} = \frac{3}{4}$$

$$\text{RHS of Equation (1)} = \int_{x=1}^{4}\int_{y=1}^{y=\sqrt{x}} \left(-\frac{1}{x^2} + \frac{1}{y^2}\right) dx\,dy$$

$$= \int_1^4 \left[-\frac{1}{x^2}(y)_1^{\sqrt{x}} + (-y^{-1})_1^{\sqrt{x}}\right] dx$$

$$= \int_1^4 \left[-\frac{1}{x^2}(\sqrt{x}-1) + \left(\frac{1}{\sqrt{x}}+1\right)\right] dx$$

$$= \int_1^4 \left[-\frac{1}{x^2}(x^{1/2}-1) + \left(-\frac{1}{x^{1/2}}+1\right)\right] dx$$

$$= \int_1^4 \left[-\frac{1}{x^{2-1/2}} + \frac{1}{x^2} - \frac{1}{x^{1/2}} + 1\right] dx$$

$$= \int_1^4 \left[-\frac{1}{x^{3/2}} + \frac{1}{x^2} - \frac{1}{x^{1/2}} + 1\right] dx$$

$$= -\left(\frac{x^{-1/2}}{-1/2}\right)_1^4 + \left(\frac{x^{-1}}{-1}\right)_1^4 - \left(\frac{x^{1/2}}{1/2}\right)_1^4 + (x)_1^4$$

$$= 2\,(1/2-1) - (1/4-1) - 2\,(2-1) + (4-1)$$

$$= 2\,(-1/2) - (-3/4) - 2 + 3 = -1 + 3/4 - 2 + 3 = 3/4$$

$\therefore$ LHS = RHS Hence Greens theorem is verified

Q. 5(A) Using Gauss divergence theorem evaluate $\iint_S \bar{N} \cdot \bar{F}\, ds$ where $\bar{F} = x^2 i + zj + yzk$ and S is the cube bounded by x = 0, x = 1, y = 0, y = 1, z = 0, z = 1. **(6 Marks)**

Ans. :

$$\bar{F} = x^2\,\hat{i} + z\hat{j} + yz\hat{k}$$

s is the cube bounded by x = 0, x = 1, y = 0

y = 1, z = 0, z = 1

By Gauss divergence theorem

$$\iint_s \bar{F} \cdot \bar{ds} = \iiint_v (\nabla \cdot \bar{F})\, dv$$

$$\nabla \cdot \bar{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(x^2\,\hat{i} + z\,\hat{j} + yz\,\hat{k}\right) = 2x + 0 + y = (2x + y)$$

$$\iint_s \bar{F} \cdot \bar{ds} = \int_{z=0}^{z=1}\int_{y=0}^{y=1}\int_{x=0}^{x=1} (2x + y)\, dx\, dy\, dz = \int_0^1\int_0^1 \left[\,2\,(x^2/2)_0^1 \cdot y\ (x)_0^1\right] dy\, dz$$

$$= \int_0^1\int_0^1 (1 - y)\, dy\, dz = \int_0^1 \left[\,(y - y^2/2\,)\right]_0^1 dz = (1\ 1/2)\,(x)_0^1 = 1/2$$

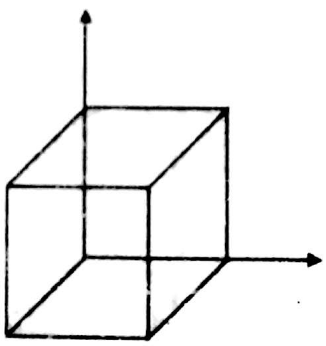

Fig. 1-Q. 5(a)

Q. 6(B) A vector field is given by $\bar{F} = (x^2 + xy^2)\, i + (y^2 + x^2y)\, j$. Show that $\bar{F}$ is irrotational and find its scalar potential. **(6 Marks)**

Ans. : **Please refer Q. 3(b) of May 2014.**

Chapter 5 : Z – Transform [Total Marks - 18]

Q. 2(B) Find Z-transform of $f(k) = b^k, k \geq 0$ **(6 Marks)**

Ans. :

$$f(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=0}^{\infty} f(k)\, z^{-k} \text{ (as } f(k) \text{ is defined for } k \geq 0)$$

$$= \sum_{k=0}^{\infty} b^k z^{-k} = \sum_{k=0}^{\infty} \frac{b^k}{z^k} = 1 + \frac{b}{z} + \frac{b^2}{z^2} + \frac{b^3}{z^3} + \ldots.$$

$$= \frac{1}{1-\frac{b}{z}} = \frac{z}{z-b} \quad , \left|\frac{b}{z}\right| < 1$$

$$= \frac{z}{z-b}, \quad b < |z|$$

Q. 4(a) Find $Z\{k^2 a^{k-1} U(k-1)\}$ **(6 Marks)**

Ans. : $Z[k^2 a^{k-1} U(k-1)]$

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right) z[a^{k-1} U(k-1)] = \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right) z^{-1} z[a^k U(k)]$$

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\frac{1}{z}\frac{\frac{z}{a}}{\frac{z}{a}-1} = \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\frac{1}{(z-a)} = \left(-z\frac{d}{dz}\right)(z)\frac{(-1)}{(z-a)^2}$$

$$= z\frac{d}{dz}\frac{z}{(z-a)^2} = z\left[\frac{(z-a)^2 - z\,2(z-a)}{(z-a)^4}\right] = z\left[\frac{(z-a)-2z}{(z-a)^3}\right] = \frac{z(-z-a)}{(z-a)^3} = \frac{z(z+a)}{(a-z)^3}$$

Q. 5(B) Find inverse Z-transform of $F(z) = \frac{z}{(z-1)(z-2)}$, $|z| > 2$ **(6 Marks)**

Ans. :

$$F(z) = \frac{z}{(z-1)(z-2)}, \quad |z| > 2$$

$$F(z) = \frac{2(z-1)-(z-2)}{(z-1)(z-2)} = \frac{2}{(z-2)} - \frac{1}{(z-1)}$$

$$\therefore \; |z| > 2 \Rightarrow \frac{2}{|z|} < 1 \text{ and therefore } |z| > 1 \Rightarrow \frac{1}{|z|} < 1$$

$$F(z) = \frac{2}{z(1-2/z)} - \frac{1}{z(1-1/z)} = \frac{2}{z}[1-2/z]^{-1} - 1/z\,[1-1/z]^{-1}$$

$$= \frac{2}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}\ldots\ldots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}\ldots\ldots\right]$$

$$= \frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\frac{2^4}{z^4}\ldots - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4}\ldots$$

$$= \frac{(2-1)}{z} + \frac{(2^2-1)}{z^2} + \frac{(2^3-1)}{z^3} + \frac{(2^4-1)}{z^4} \ldots$$

$$\therefore \; f(k) = 2^k - 1 \text{ for } k \geq 1$$

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Dec. 2013

Q. 1 (a) Find $L^{-1}\left\{\frac{e^{4-3}}{(s+4)^{5/2}}\right\}$ **(5 Marks)**

(b) Find the constant a, b, c, d and e if

$f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i\,(4x^3y - exy^3 + 4xy)$ is analytic. **(5 Marks)**

(c) Obtain half range Fourier cosine series for $f(x) = \sin x,\ x \in (0, \pi)$. **(5 Marks)**

(d) If r and $\bar{r}$ have their usual meaning and a is constant vector, prove that **(5 Marks)**

$$\nabla \times \left[\frac{a \times \bar{r}}{r^n}\right] = \frac{(2-n)}{r^n}\,a + \frac{n(a \cdot r)\,\bar{r}}{r^{n+2}}$$

Q. 2 (a) Find the analytic function $f(c) = u + iv$ If $3u + 2v = y^2 - x^2 + 16xy$. **(6 Marks)**

(b) Find the z-transform $\{a^{|k|}\}$ and hence find the z-transform of $\left\{\left(\frac{1}{2}\right)^{|k|}\right\}$. **(6 Marks)**

(c) Obtain Fourier series expansion for $f(x) = \sqrt{1-\cos x}$, $x \in (0, 2\pi)$ and hence deduce that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$$ **(8 Marks)**

Q. 3 (a) Find : (i) $L^{-1}\left\{\frac{s}{(2s+1)^2}\right\}$ **(3 Marks)**

(ii) $L^{-1}\left\{\log \frac{s^2+a^2}{\sqrt{s+b}}\right\}$ **(3 Marks)**

(b) Find the orthogonal trajectories of the family of curves $e^{-x}\cos y + xy = \infty$ where ∞ is the real constant in xy – plane. **(6 Marks)**

(c) Show that $\bar{F} = (ye^{xy}\cos z)i + (xe^{xy}\cos z)\,j - (e^{xy}\sin z)\,k$ is irrotational and find the scalar potential for $\bar{F}$ and evaluate $\int_c \bar{F} \cdot dr$ along the curve joining the point (0, 0, 0) and (– 1, 2, π). **(8 Marks)**

Q. 4 (a) Evaluate by Green theorem $\int e^{-x}\sin y\,dx + e^{-x}\cos y\,dy$ where c is the rectangle whose vertices are $(0, 0)(\pi, 0)\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$. **(6 Marks)**

(b) Find the half range sine series for the function **(6 Marks)**

$$f(x) = \frac{2kx}{l}, \quad 0 \le x \le l/2$$
$$= \frac{2k}{l}(l-x), \quad \frac{l}{2} \le x \le l$$

(c) Find the inverse z-transform of $\frac{1}{(z-3)(z-2)}$. **(8 Marks)**

(i) $|z| < 2$ (ii) $2 < |z| < 3$ (iii) $|z| > 3$

Q. 5 (a) Solve using Laplace transform $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-x}$, $y(0) = 1$, $y'(0) = 1$. **(6 Marks)**

(b) Express $f(x) = \frac{\pi}{2}e^{-x} \cos x$ for $x > 0$ as Fourier sine integral and show that

$$\int_0^{\infty} \frac{w^3 \sin wx}{w^4 + 4} dw = \frac{\pi}{2} e^{-x} \cos x.$$ **(6 Marks)**

(c) Evaluate $\iint_s \bar{F} \cdot n ds$, where $\bar{F} = xi - yj + (z^2 - 1) k$ and s is the cylinder formed by the surface $z = 0, z = 1, x^2 + y^2 = 4$, using the Gauss – Divergence theorem. **(8 Marks)**

Q. 6 (a) Find the inverse Laplace transform by using convolution theorem **(6 Marks)**

$$L^{-1}\left\{\frac{s^2 + 2s + 3}{(s^2 + 2s + 5)(s^2 + 2s + 2)}\right\}$$

(b) Find the directional derivative of $\phi = 4e^{2x - y + z}$ at the point (1, 1, – 1) in the direction towards the point (–3, 5, 6). **(6 Marks)**

(c) Find the image of the circle $x^2 + y^2 = 1$, under the transformation $w = \frac{5 - 4z}{4z - 2}$. **(8 Marks)**

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May 2014

Q.1 (a) Find $L^{-1}\left[\frac{Se^{-\pi s}}{S^2 + 2S + 2}\right]$ **(5 Marks)**

(b) State true or false with proper justification "There does not exist an analytic function whose real part is $x^3 - 3x^2y - y^3$" **(5 Marks)**

(c) Prove that $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = \frac{(3x^2 - 1)}{2}$ are orthogonal over (–1, 1) **(5 Marks)**

(d) Using Green's theorem in the plane, evaluate $\int_C (x^2 - y) dx + (2y^2 + x) dy$ around the boundary of the region defined by $y = x^2$ and $y = 4$. **(5 Marks)**

Q. 2 (a) Find the fourier cosine integral representation of the function $f(x) = e^{-ax}$, $x > 0$ and hence show that $\int_0^{\infty} \frac{\cos ws}{1 + w^2} dw = \frac{\pi}{2} e^{-x}, x \geq 0$. **(6 Marks)**

(b) Verify laplaces equation for $U = \left[r + \frac{a^2}{r}\right] \cos \theta$. Also find V and f(z) **(6 Marks)**

(c) Solve the following equation by using laplace transform $\frac{dy}{dt} + 2y + \int_0^t y dt = \sin t$ given that $y(0) = 1$. **(8 Marks)**

Q. 3 (a) Expand $f(x) = \begin{cases} \pi x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$ with period 2 into a fourier series. **(6 Marks)**

(b) A vector field is given by $\bar{F} = (x^2 + xy^2)\, i + (y^2 + x^2y) j$ show that $\bar{F}$ is irrotational and find its scalar potential. **(6 Marks)**

(c) Find the inverse z-transform of $f(z) = \dfrac{z+2}{z^2 - 2z + 1}$, $|z| > 1$ **(8 Marks)**

Q. 4 (a) Find the constants 'a' and 'b' so that the surface $ax^2 - byz = (a + 2)\,x$ will orthogonal to the surface $4x^2y + z^3 + 4$ at $(1, -1, 2)$ **(6 Marks)**

(b) Given $L(\text{erf}\sqrt{t}) = \dfrac{1}{S\sqrt{S+1}}$ evaluate $\int_0^\infty te^{-t}\,\text{erf}(\sqrt{t})dt$ **(6 Marks)**

(c) Obtain the expansion of $f(x) = x(\pi - x)$, $0 < x < \pi$ as a half range cosine series.

Hence show that

(i) $\sum_1^\infty \dfrac{(-1)^{n+1}}{n^2} = \dfrac{\pi^2}{12}$ (ii) $\sum_1^\infty \dfrac{1}{n^4} = \dfrac{\pi^4}{90}$ **(8 Marks)**

Q. 5 (a) If the imaginary part of the analytic function $W = f(z)$ is $V = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$. Find the real part U. **(6 Marks)**

(b) If $f(k) = 4^k\, U(K)$ and $g(K) = 5^k\, U(K)$, then find the z-transform of $f(k)\,.\,g(k)$ **(6 Marks)**

(c) Use Gauss's Divergence theorem to evaluate $\iint_S \bar{N}\cdot\bar{F}\,ds$ where $\bar{F} = 4xi + 3yj - 2z\hat{k}$ and S is the surface bounded by $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$. **(8 Marks)**

Q. 6 (a) Obtain complex form of fourier series for $f(x) = \cosh 3x + \sinh 3x$ in $(-3,3)$ **(6 Marks)**

(b) Find the inverse laplace transform of $\dfrac{(S-1)^2}{(S^2 - 2S + 5)^2}$ **(6 Marks)**

(c) Find the bilinear transformation under which $1, i, -1$ from the z-plane are mapped onto $0, 1, \infty$ of w-plane. Also show that under this transformation the unit circle in the w-plane is mapped onto a straight line in the z-plane. Write the name of this line. **(8 Marks)**

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Dec. 2014

Q. 1 (a) Find the Laplace Ttransform of sint cos2t cosht. **(5 Marks)**

(b) Find the Fourier series expansion of $f(x) = x^2\ (-\pi, \pi)$ **(5 Marks)**

(c) Find the z-transform of $\left(\dfrac{1}{3}\right)^{|k|}$. **(5 Marks)**

(d) Find the directional derivative of $4xz^2 + x^2yz$ at $(1, -2, -1)$ in the direction $2\vec{i} - \vec{j} - 2\vec{k}$. **(5 Marks)**

Q. 2 (a) Find an analytic function f(z) whose real part is $u = e^x (x \cos y - y \sin y)$. **(6 Marks)**

(b) Find inverse Laplace Transform by using convolution theorem $\frac{1}{(s-3)(s+4)^2}$. **(6 Marks)**

(c) Prove that $\vec{F} = (6xy^2 - 2z^3)\vec{i} + (6x^2y + 2yz)\vec{j} + (y^2 - 6z^2x)\vec{k}$ is a conservative field. Find the scalar potential ϕ such that $\vec{F} = \nabla\phi$. Hence find the work done by $\vec{F}$ in displacing a particle from A (1, 0, 2) to B(0, 1, 1) along AB. **(8 Marks)**

Q. 3 (a) Find the inverse z-transform of $F(z) = \frac{z^3}{(z-3)(z-2)^3}$.

(i) $2 < |z| < 3$; (ii) $|z| > 3$ **(6 Marks)**

(b) Find the image of the real axis under the transformation $w = \frac{2}{z+i}$. **(6 Marks)**

(c) Obtain the Fourier series expansion of $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}$

Here, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \ldots = \frac{\pi}{8}$ **(8 Marks)**

Q. 4 (a) Find the Laplace transform of $f(t) = \begin{cases} E & 0 \le t \le p/2 \\ -E & p/2 \le t \le p \end{cases}$ · $f(t+p) = f(t)$. **(6 Marks)**

(b) Using Green's theorem evaluate $\int_c \frac{1}{y} dx \frac{1}{x} dx$ where c is the boundary of the region bounded by $x = 1$, $x = 4$, $y = 1$, $y = \sqrt{x}$. **(6 Marks)**

(c) Find the Fourier integral for $f(x) = \begin{cases} 1 - x^2 & 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases}$. Hence evaluate $\int_0^\infty \left[\frac{\lambda \cos \lambda - \sin \lambda}{\lambda^3}\right] \cos\left(\frac{\lambda}{2}\right) d\lambda$. **(8 Marks)**

Q. 5 (a) If $\vec{F} = x^2 \vec{i} + (x-y)\vec{j} + (y+z)\vec{k}$ move a particle from A (1, 0, 1) to B (2, 1, 2) along line AB. Find the work done. **(6 Marks)**

(b) Find the complex form of fourier series $f(x) = \sinh ax \ (-l, l)$. **(6 Marks)**

(c) Solve the differential equation using Laplace transform.

$(D^2 + 2D + 5)y = e^{-t} \sin t \cdot y(0) = 0, y'(0) = 1$. **(8 Marks)**

Q. 6 (a) If $\int_0^{\infty} e^{-2t} \sin(t+\alpha)\cos(t-\alpha)\,dt = \frac{3}{8}$ find the value of α. **(6 Marks)**

(b) Evaluate $\iint_s \left(y^2z^2\bar{i} + z^2x^2\bar{j} + z^2y^2\bar{k}\right)\cdot\bar{n}\,ds$ where s is the hemisphere $x^2 + y^2 + z^2 = 1$ above xy – plane and bounded by this plane. **(6 Marks)**

(c) Find half range sine series for $f(x) = lx - x^2$, $(0, l)$. Hence prove that $\frac{1}{1^6} + \frac{1}{3^6} + \ldots = \frac{\pi^6}{960}$. **(8 Marks)**

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May 2015

Q. 1 (a) Find Laplace Transform of $\frac{\sin t}{t}$. **(20 Marks)**

(b) Prove that $f(z) = \sinh z$ is analytic and find its derivative.

(c) Find fourier series for $f(x) = 9 - x^2$ over $(-3, 3)$

(d) Find $Z\{ f(k) * g(k) \}$ if $f(k) = \frac{1}{3^k}$, $g(k) = \frac{1}{5^k}$

Q. 2 (a) A prove that $\bar{F} = ye^{xy}\cos zi + xe^{xy}\cos zj - e^{xy}\sin zk$ is irrotational. Find scalar potential for $\bar{F}$ hence evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve C joining the points (0, 0, 0) and $(-1, 2, \pi)$. **(6 Marks)**

(b) Find the Fourier series for $f(x) = \frac{\pi - x}{2}$; $0 \le x \le 2\pi$. **(6 Marks)**

(c) Find inverse Laplace Transform of (i) $\frac{s+29}{(s+4)(s^2+9)}$ (ii) $\frac{e^{-2x}}{s^2+8s+25}$ **(8 Marks)**

Q. 3 (a) Find the Analytic function $f(z) = u + iv$ if $u + v = \frac{x}{x^2+y^2}$ **(6 Marks)**

(b) Find inverse Z transform of $\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$, $\frac{1}{3} < |z| < \frac{1}{2}$ **(6 Marks)**

(c) Solve the differential Equation $\frac{d^2y}{dt^2} + y = t$, $y(0) = 1$, $y'(0) = 0$, using Laplace Transform. **(8 Marks)**

Q. 4 (a) Find the Orthogonal Trajectory of $3x^2 y - y^3 = k$ **(6 Marks)**

(b) Using Greens theorem evaluate $\int_C (xy + y^2)\, dx + x^2\, dy$, C is closed path formed by $y = x$, $y = x^2$.

(6 Marks)

(c) Find Fourier Integral of $f(x) = \begin{cases} \sin x & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$ Hence show that $\int_0^\infty \frac{\cos(\lambda\pi/2)}{1-\lambda^2}\, d\lambda = \frac{\pi}{2}$

(8 Marks)

Q. 5 (a) Find Inverse Laplace Transform using Convolution theorem $\frac{s}{(s^4 + 8s^2 + 16)}$. **(6 Marks)**

(b) Find the Bilinear Transformation that maps the points $z = 1, i, -1$ into $w = i, 0, -i$. **(6 Marks)**

(c) Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where C is the boundary of the plane $2x + y + z = 2$ cut off by co-ordinate planes and $\bar{F} = (x + y)i + (y + z)j - xk$. **(8 Mark)**

Q. 6 (a) Find the Directional derivative of $\phi = x^2 + y^2 + z^2$ in the direction of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at (1, 2, 3)

(6 Marks)

(b) Find complex form of Fourier Series for e^{2x} ; $0 < x < 2$ **(6 Marks)**

(c) Find Half Range Cosine Series for $f(x) = \begin{cases} kx & 0 \le x \le l/2 \\ k(l - x) & l/2 \le x \le l \end{cases}$, hence find $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

(8 Marks)

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Dec. 2015

Q. 1 (a) Find Laplace of $\{t^5 \cosh t\}$ **(5 Marks)**

(b) Find Fourier series for $f(x) = 1 - x^2$ in $(-1, 1)$ **(5 Marks)**

(c) Find a, b, c, d, e if,

$f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i\,(4x^3y - exy^3 + 4xy)$ is analytic **(5 Marks)**

(d) Prove that $\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^3}$ **(5 Marks)**

Q. 2 (a) If $f(z) = u + iv$ is analytic and $u + v = \frac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x}$, find f(z) **(6 Marks)**

(b) Find inverse Z-transform of $f(z) = \frac{z+2}{z^2 - 2z + 1}$ for $|z| > 1$. **(6 Marks)**

(c) Find Fourier series for $f(x) = \sqrt{1 - \cos x}$ in $(0, 2\pi)$, hence deduce that $\frac{1}{2} = \sum_{1}^{\infty} \frac{1}{4\pi^2 - 1}$

(8 Marks)

Q. 3 (a) Find $L^{-1}\left\{\frac{1}{(s-2)^4 (s+3)}\right\}$ using Convolution theorem. **(6 Marks)**

(b) Prove that $f_1(X) = 1$, $f_2(x) = x$, $f_3(X) = (3x^2 - 1)/2$ are orthogonal over $(-1, 1)$

(6 Marks)

(c) Verify Green's theorem for $\int_C \overline{F} \cdot \overline{dr}$ where $\overline{F} = (x^2 - y^2)\, I + (x + y) j$ and C is the triangle with vertices (0, 0), (1, 1), (2, 1) **(8 Marks)**

Q. 4 (a) Find Laplace transform $f(t) = |\sin pt|$, $t \geq 0$. **(6 Marks)**

(b) Show that $\overline{F} = (y \sin z - \sin x)\, i + (x \sin z + 2yz)\, j + (xy \cos z + y^2)\, k$ is irrotational. Hence, find its scalar potential. **(6 Marks)**

(c) Obtain Fourier expansion of $f(x) = x + \frac{\pi}{2}$ where $-\pi < x < 0 = \frac{\pi}{2} - x$ where $0 < x < \pi$

Hence deduce that (i) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ (ii) $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$ **(8 Marks)**

Q. 5 (a) Using Gauss Divergence theorem to evaluate $\iint_S \overline{N}\, \overline{F}\, dz$ where $\overline{F} = 4xi - 2y^2 j + z^2 k$ and S is the region bounded by $x^2 + y^2 = 4$, $z = 0$, $z = 3$. **(6 Marks)**

(b) Find $Z\{2^K \cos(3k + 2)\}$, $k \geq 0$. **(6 Marks)**

(c) Solve $(D^2+2D+5)y = e^{-t} \sin t$ With $y(0) = 0$ and $y'(0) = 1$ **(8 Marks)**

Q. 6 (a) Find $L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$ **(6 Marks)**

(b) Find the bilinear transformation which maps the ponts 2, j, –2 onto point 1, j, –1 using cross-ratio property. **(6 Marks)**

(c) Find Fourier singe integral representation for $f(x) = \frac{e^{-ax}}{x}$. **(8 Marks)**

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May 2016

Q. 1 (A) If $\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)dt = \frac{1}{4}$. Find '$\alpha$'. **(5 Marks)**

(B) Find half range Fourier cosine series for $f(x) = x,\ 0 < x < 2$ **(5 Marks)**

(C) If $u(x, y)$ is a harmonic function then prove that $f(z) = u_x - iu_y$ is an analytic function.

(5 Marks)

(D) Prove that $\nabla f(r) = f'(r)\frac{\hat{r}}{r}$. **(5 Marks)**

Q. 2 (A) If $v = e^x \sin y$, prove that v is a harmonic function. Also find the corresponding analytic function.

(6 Marks)

(B) Find Z-transform of $f(k) = b^k,\ k \geq 0$ **(6 Marks)**

(C) Obtain Fourier series for $f(x) = \frac{3x^2 - 6x\pi + 2\pi^2}{12}$ in $(0, 2\pi)$, where $f(x + 2\pi) = f(x)$. Hence deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ **(8 Marks)**

Q.3 (A) Find inverse Laplace of $\frac{(s+3)^2}{(s^2+6s+5)^2}$ using Convolution theorem. **(6 Marks)**

(B) Show that the set of functions {sinx, sin3x, sin5x,.....} is orthogonal over $[0,\pi/2]$. Hence construct orthonormal set of functions. **(6 Marks)**

(C) Verify Green's theorem for $\int_c \frac{1}{y}dx + \frac{1}{x}dy$ where C is the boundary of region defined by $x= 1$, $x = 4$, $y = 1$ and $y = \sqrt{x}$ **(8 Marks)**

Q. 4 (A) Find $Z\{k^2 a^{k-1} U(k-1)\}$ **(6 Marks)**

(B) Show that the map of the real axis of the z- plane is a circle under the transformation $w = \frac{z}{z+i}$. Find its centre and the radius. **(6 Marks)**

(C) Express the function $f(x) = \begin{cases} \sin x & |x| < \pi \\ 0 & |x| > \pi \end{cases}$ as Fourier sine integral. **(8 Marks)**

Q. 5 (A) Using Gauss divergence theorem evaluate $\iint_S \bar{N} \cdot \bar{F}\, ds$ where $\bar{F} = x^2 i + zj + yzk$ and S is the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$. **(6 Marks)**

(B) Find inverse Z-transform of $F(z) = \frac{z}{(z-1)(z-2)}$, $|z| > 2$ **(6 Marks)**

(C) Solve $(D^2 + 3D + 2)y = e^{-2t} \sin t$, with $y(0) = 0$ and $y'(0) = 0$ **(8 Marks)**

Q. 6 (A) Find Fourier expansion of $f(x) = 4 - x^2$ in the interval $(0, 2)$ **(6 Marks)**

(B) A vector field is given by $\bar{F} = (x^2 + xy^2)\, i + (y^2 + x^2 y)\, j$. Show that $\bar{F}$ is irrotational and find its scalar potential. **(6 Marks)**

(C) Find (i) $L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\}$ **(8 Marks)**

(ii) $L^{-1}\left(\frac{e^{-\pi s}}{s^2 - 2s + 2}\right)$

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