Complex Variables

1) $z = x + iy$
$\bar{z} = x - iy$

$$|z| = \sqrt{x^2 + y^2} = \sqrt{(RP)^2 + (IP)^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{IP}{RP}\right)$$

2) $z = re^{i\theta}$
$\bar{z} = re^{-i\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$

3) $f(z) = u + iv$

4) Analytic function
Cartesian form

$$\begin{array}{cc} u & v \\ \downarrow & \downarrow \\ \partial x & \partial y \end{array} \qquad \begin{array}{ccc} u & & v \\ & \times & \\ \partial x & & \partial y \end{array}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Polar form

$$\begin{array}{cc} u & v \\ \downarrow & \downarrow \\ \partial r & r\partial\theta \end{array} \qquad \begin{array}{ccc} u & & v \\ & \times & \\ \partial r & & r\partial\theta \end{array}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

5) Laplace equation (Harmonic function)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

6) $i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

- Step to find Analytic fn.

Step 1: Differentiate partially w.r.t x and y.

Step 2: In above partiall diff. put $x = z$ and $y = 0$

Step 3: Write down analytic condition

Step 4: Differentiate $f(z)$ partially w r t x.

Step 5: Apply M.T method i.e put $x = z$ and $y = 0$ and substitute the value from Step 2

Step 6: Integrate w r t z

Type of Q.

1) Only u or v is given
2) Find orthogonal trajectory of family of curves
3) u−v or u+v is given

Q1) Find the analytic function and it's harmonic conjugate.

If $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$ $\qquad$ $\frac{\partial u}{\partial y} = -6xy - 6y$

$x \xrightarrow{\text{put}} z$ $\qquad$ $\left.\frac{\partial u}{\partial y}\right|_{(z,0)} = 0$

$\left.\frac{\partial u}{\partial x}\right|_{(z,0)} = 3z^2 - 3y^2 + 6z$

$= 3z^2 - 0 + 6z$

$u \quad v$

$\downarrow \quad \downarrow$

$\partial x \quad \partial y$ $\qquad$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

Step 4) $f(z) = u + iv$

diff partially wrt x

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

✓ we have this $\qquad$ find this

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

apply MT method

Put $x = z,\ y = 0$

$$f'(z) = \frac{\partial u}{\partial x}_{(z,0)} - i\frac{\partial u}{\partial y}_{(z,0)}$$

$$= (3z^2 + 6z) - i(0)$$

$$f'(z) = 3z^2 + 6z$$

Step ⑥ $\int f'(z)\,dz = \int (3z^2 + 6z)\,dz$

$$f(z) = z^3 + 3z^2 + c$$

put $z = x+iy$

$$f(z) = (x+iy)^3 + 3(x+iy)^2$$

$$= x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3 + 3(x^2 + 2xiy + i^2y^2)$$

$$= x^3 + 3x^2iy - 3xy^2 - iy^3 + 3x^2 + 6xiy - 3y^2 + c$$

$$= (x^3 - 3xy^2 + 3x^2 - 3y^2 + c) + i(3x^2y - y^3 + 6xy)$$

$\therefore\ u = x^3 - 3xy^2 + 3x^2 + c - 3y^2 \quad v = 3x^2y - y^3 + 6xy$

same as ⑧

it is harmonic conj.

) If $v = e^{x}\cdot \sin y$ ST v is a harmonic fn. also find analytic fn. and harmonic conj.

$V = e^{x}\cdot \sin y$

$x \to 2$

$\frac{\partial v}{\partial x} = e^{x}\cdot \sin y$ $\qquad \frac{\partial v}{\partial x}_{(2,0)} = e^{2}\cdot \sin 0 = 0$

$\frac{\partial v}{\partial y} = e^{x} \cos y$ $\qquad \frac{\partial v}{\partial y}_{(2,0)} = e^{2}\cdot \cos 0 = e^{2}$

$z = u + iv$ $\qquad$ u $\quad$ v $\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ $\qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = e^{2} + i\,(e^{x}\cdot \sin y)$

$f'(z)$ apply MT rule $x = 2,\ y = 0$

$f'(z) = \frac{\partial v}{\partial y}_{(2,0)} + i\,\frac{\partial v}{\partial x}_{(2,0)} = e^{2} + i(0)$

$f'(z) = e^{2}$

$\int f'(z)\,dz = \int e^{2}\,dz \Rightarrow f(z) = e^{z} + c$

put $z = x + iy$

$f(z) = e^{x+iy} + c = e^{x}\cdot e^{iy} + c$

$= e^{x}[\cos y + i \sin y] + c$

$= (e^{x}\cos y + c) + i(\sin y\, e^{x})$

$f(z) = u + iv$

$u = e^{x}\cos y + c \qquad v = e^{x} \sin y$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = e^x \cdot \sin y \rightarrow (A)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = e^x(-\sin y) \rightarrow (B)$$

A + B

$$e^x \cdot \sin y - e^x \sin y = 0$$

$\therefore$ v is a harmonic fn.

3) $v = e^x(x\sin y + y\cos y)$ analytic fn., harmonic conj.

$$v = e^x \cdot x\sin y + e^x y\cos y$$

$$\frac{\partial v}{\partial x} = e^x \sin y + e^x \cdot x\sin y + e^{x} \cdot y\cos y$$

$$\frac{\partial v}{\partial x}_{(2,0)} = e^2 \cdot 0 + e^2 \cdot 2 \cdot 0 + 0 = 0$$

$$\frac{\partial v}{\partial y} = e^x \cdot x \cdot \cos y + e^x \cos y + e^x \cdot y(-\sin y)$$

$$\frac{\partial v}{\partial y}_{(2,0)} = e^2 \cdot 2 \cdot \cos 0 + e^2 \cos 0 + e^2 \cdot (0)$$

$$= 2 \cdot e^2 + e^2$$

$$\begin{array}{ccc} u & & v \\ \downarrow & \times & \downarrow \\ \partial x & & \partial y \end{array} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$f(z) = u + iv$ $\qquad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$f'(z) = \frac{\partial v}{\partial y}_{(2,0)} + i\frac{\partial v}{\partial x}_{(2,0)} = 2 \cdot e^2 + e^2 + i(0)$$

$$= 2 \cdot e^2 + e^2 + 0$$

$$\int f'(z)\,dz = \int z\cdot e^{z} + e^{z}\,dz$$

$$= z\cdot e^{z} - e^{z} + e^{z} + c$$

$$f(z) = z\cdot e^{z} + c$$

$$z = x+iy$$

$$= (x+iy)\,e^{x}\cdot e^{iy} + c$$

$$= (x+iy)\,e^{x}(\cos y + i\sin y) + c$$

$$= (xe^{x} + iye^{x})(\cos y + i\sin y) + c$$

$$= xe^{x}\cos y + ixe^{x}\sin y + iye^{x}\cos y - ye^{x}\sin y + c$$

$$= (xe^{x}\cos y - ye^{x}\sin y + c) + i(xe^{x}\sin y + ye^{x}\cos y)$$

$$z = u+iv$$

$$u = xe^{x}\cos y - ye^{x}\sin y + c$$

$$v = e^{x}(x\sin y + y\cos y)$$

4) $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$ analytic fn.?

$\cosh 0 = 1 \quad \sinh 0 = 0$

$\cancel{\dfrac{\partial u}{\partial x}} = \quad \int \sec^2 x\,dx = \tan x \quad \int \operatorname{cosec}^2 x\,dx = -\cot x$

$$\frac{\partial u}{\partial x} = \frac{2(\cosh 2y + \cos 2x)\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$\frac{\partial u}{\partial x}_{(z,0)} = \frac{(2\cosh 2y + 2\cos 2x)\cos 2x + \sin 2x(2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{(2\cosh 0 + 2\cos 2z)\cos 2z + 2\sin^2 2z}{(1+\cos 2z)^2}$$

$$= \frac{2\cos 2z + 2\cos^2 2z + 2\sin^2 2z}{(1+\cos 2z)^2}$$

$$= \frac{2(\cos 2z + \cos^2 2z + \sin^2 2z)}{(1+\cos 2z)^2}$$

$$= \frac{2(\cancel{\cos 2z + 1})}{(1+\cos 2z)^{\cancel{2}}} = \frac{2}{(1+\cos 2z)}$$

$$= \frac{\cancel{2}}{(\cancel{2}\cos^2 z)} = \sec^2 z$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{(\cosh 2y + \cos 2x)}\right)(\sin 2x)$$

$$= \frac{-1}{(\cosh 2y + \cos 2x)^2} \cdot (2\sinh 2y + 0)\sin 2x$$

$$\frac{\partial u}{\partial y}_{(z,0)} = \frac{-2\sinh 2y \cdot \sin 2x}{(\cosh 2y + \cos 2x)^2} = 0$$

$u \qquad v$

$\downarrow \qquad \downarrow$

$\partial x \qquad \partial y$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$f(z) = u + iv \qquad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \cancel{\sec^2 z + i(0)} \cdot \sec^2 z + i\left(-\frac{\partial u}{\partial y}\right) = \sec^2 z$$

$$\int f'(z)\,dz = \int \sec^2 z\,dz = \tan z + c$$

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5) Find orthogonal trajectory family of curve
$x^3y - xy^3 = c$

Sol. We have to find harmonic conj.
Let $u = x^3y - xy^3$

$\frac{\partial u}{\partial x} = 3x^2y - y^3$ $\quad \frac{\partial u}{\partial x}_{(z,0)} = 0$

$\frac{\partial u}{\partial y} = x^3 - 3y^2$ $\quad \frac{\partial u}{\partial y}_{(z,0)} = z^3$

$u \quad v$

$\frac{\partial}{\partial x} \times \frac{\partial}{\partial y}$ $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$

$f(z) = u+iv$ $\quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 0 - i(z^3)$

$\int f'(z)\,dz = -i\int z^3\,dz = \cancel{-iz^3} = f(z)$

$-\frac{iz^4}{4} + c$

put $z = (x+iy)$

$f(z) = \frac{(-i)}{4}(x+iy)^4 + c$

$= \frac{-i}{4}(x^4 + 4x^3iy + 6x^2i^2y^2 + 4xi^3y^3 + i^4y^4) + c$

$= -i(x^4 + 4x^3iy - 6x^2y^2 - i4xy^3 + y^4) + c$

$$= \frac{-ix^4}{4} + \frac{4x^3y}{4} + \frac{6x^2y^2i}{4} - \frac{4xy^3}{4} - \frac{iy^4}{4} + c$$

$$= -\frac{ix^4}{4} + x^3y + i6x^2y^2 - xy^3 - \frac{iy^4}{4} + c$$

$$= (x^3y - xy^3 + c) + i\left(-\frac{x^4}{4} + 6x^2y^2 - \frac{y^4}{4}\right)$$

$$u = x^3y - xy^3 + c \qquad v = -\frac{x^4}{4} - \frac{y^4}{4} + 6x^2y^2$$

- If $u-v$ and $u+v$ is given.

$$f(z) = u + iv \longrightarrow ①$$

$$if(z) = iu + i^2v$$

$$if(z) = iu - v \longrightarrow ②$$

① + ②

$$f(z) + if(z) = u + iv + iu - v$$

$$f(z)(1+i) = (u-v) + i(u+v)$$

$$\boxed{f(z) = \frac{1}{(1+i)}[(u-v) + i(u+v)] = \frac{1}{(1+i)}[U + iV] \qquad U = u - v,\ V = u + v}$$

1) If $u+v = \dfrac{2\sin 2x}{(e^{2y} + e^{-2y} - 2\cos 2x)}$

let $u + v = V$

$$V = \frac{2\sin 2x}{(e^{2y} + e^{-2y} - 2\cos 2x)} = \frac{\sin 2x}{\left(\frac{e^{2y} + e^{-2y}}{2} - \cos 2x\right)}$$

$$= \frac{\sin 2x}{(\cosh 2y - \cos 2x)}$$

$$\frac{\partial v}{\partial x} = \frac{2(\cosh 2y - \cos 2x)\cancel{\sin 2x}\cos 2x - \sin 2x(0+\sin 2x)2}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2(\cosh 2y - \cos 2x)\cos 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial v}{\partial x}_{(z,0)} = \frac{2(1-\cos 2z)\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2(\cos 2z - (\cos^2 2z + \sin^2 2z))}{(1-\cos 2z)^2}$$

$$= \frac{2(\cos 2z - 1)}{(1-\cos 2z)^2} = \frac{-2\cancel{(1-\cos 2z)}}{(1-\cos 2z)^{\cancel{2}}}$$

$$= \frac{-2}{(1-\cos 2z)} = \frac{-\cancel{2}}{\cancel{2}\sin^2 z} = -\operatorname{cosec}^2 z$$

$$\frac{\partial v}{\partial y} = \sin 2x \cdot \frac{(-1)}{(\cosh 2y - \cos 2x)^2} \cdot (2\sinh 2y)$$

$$\frac{\partial v}{\partial y}_{(z,0)} = \frac{-2\sin 2x(\sinh 2y)}{(\cosh 2y - \cos 2x)^2} = \frac{-2\sin 2z(0)}{(1)} = 0$$

$$\begin{array}{cc} U & V \\ \downarrow & \downarrow \\ \partial x & \partial y \end{array} \qquad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

$$f(z) = U + iV$$

$$f'(z) = \left(\frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}\right)\frac{1}{1+i}$$

$$= \left(\frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}\right)\frac{1}{1+i}$$

$= 0 + i(-\operatorname{cosec}^2 z)$

$f'(z) = (-i\operatorname{cosec}^2 z)\frac{1}{1+i}$

$\int f'(z)\,dz = \frac{-i}{1+i}\int \operatorname{cosec}^2 z\,dz$

$= \frac{-i}{1+i}(-\cot z)\cancel{x} = \frac{i\cot z + c}{1+i}$

$\cancel{f(z) =}$ $z = x+iy$

$\cot(A+B) = \frac{\cot A\cot B - 1}{\cot A + \cot B}$

$\frac{i}{1+i}\cot(x+iy)$

2) $u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ analytic fn.

$U = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$

$\frac{\partial U}{\partial x} = \frac{(2\cos x - e^{y} - e^{-y})(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})(-2\sin x)}{(2\cos x - e^{y} - e^{-y})^2}$

$\frac{\partial u}{\partial x}_{(z,0)} = \frac{(2\cos z - 1 - 1)(-\sin z + \cos z) - (\cos z + \sin z - 1)(-2\sin z)}{(2\cos z - 1 - 1)^2}$

$= \frac{(2\cos z - 2)(\cos z - \sin z) - (\cos z + \sin z - 1)(-2\sin z)}{(2\cos z - 2)^2}$

$= \cancel{2(\cos z - (2\cos^2 z - 2\cos z - 2\cos}$ $+ 2\sin^2 z - 2\sin$

$= 2(\cos z - 1)(\cos z - \sin z) + (2\cos z \sin z + 2\sin^2 z - 2\sin$

$= 2\{\cos^2 z - \cancel{2\cos z\sin z} - 2\cos z + 2\cancel{\sin z} + 2\cos z$

$+ 2\sin^2 z - 2\cancel{\sin z}$

$$= \frac{2\left((\cos^2 z + \sin^2 z) - \cancel{2}\cos z\right)}{\left(4(\cos z - 1)^2\right)}$$

$$-\frac{\cancel{2}\,\cancel{(\cos z - 1)}}{2\cancel{4}(\cos z - 1)^{\cancel{2}}} = \frac{\pm 1}{2(\cancel{1})(2\sin^2 z/2)}$$

$$= \frac{\pm 1}{4}\operatorname{cosec}^2(z/2)$$

$$\frac{\partial u}{\partial y} = \frac{(2\cos x - e^y - e^{-y})(+e^{-y}) - (\cos x + \sin x - e^{-y})(-e^{y} + e^{-y})}{(2\cos x - e^{y} - e^{-y})^2}$$

$$\frac{\partial u}{\partial y}_{(z,0)} = \frac{(2\cos z - 2)(1) - (\cos z + \sin z)(0)}{(2\cos z - 2)^2}$$

$$= \frac{2\cancel{(\cos z - 1)}}{4(\cos z - 1)^{\cancel{2}}} \quad -\frac{1}{4}\operatorname{cosec}^2(z/2)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad = -\frac{1}{4}\operatorname{cosec}^2(z/2)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$\int f'(z) = \frac{1}{4}\int \operatorname{cosec}^2(z/2)\,dz + i\,\frac{1}{4}\int \operatorname{cosec}^2(z/2)\,dz$$

$$= \frac{-1}{4}\cot(z/2)\times 2 - \frac{i}{4}\cot(z/2)\times 2$$

$$= \frac{1}{1+i}\left(\frac{-1}{2}\cot(z/2) - \frac{i}{2}\cot(z/2)\right)$$

$$= \frac{1}{\cancel{(1+i)}}\left[\frac{\cancel{(1+i)}}{2}\left(-\cot(z/2)\right)\right]$$

$$= \frac{-\cot(z/2)}{2} + C$$

3) Find the value of a, b, c, d, e

If $f(z) = (ax^3 + bxy^2 + 3x^2 + c(y^2+x) + i(dx^2y - 2y^3 + exy + y)$ is analytic fn.

$f(z) = u + iv$

$(u+iv) = (ax^3 + bxy^2 + 3x^2 + cy^2 + x) + i(dx^2y - 2y^3 + exy + y)$

$u = ax^3 + bxy^2 + 3x^2 + cy^2 + x$

$v = dx^3y - 2y^3 + exy + y$

$\frac{\partial u}{\partial x} = 3ax^2 + by^2 + 6x + 0 + 1$

$\frac{\partial u}{\partial y} = 0 + 2bxy + 0 + 2cy + 0$

$\frac{\partial v}{\partial x} = 2dx^2y - 0 + ey + 0$

$\frac{\partial v}{\partial y} = dx^2 - 6y^2 + ex + 1$

Since f(z) is analytic fn.

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$3ax^2 + by^2 + 6x + \cancel{1} = dx^2 - 6y^2 + ex + \cancel{1}$

$3a = d$

$b = -6$

$e = 6$

$\therefore \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow 2bxy + 2cy = -2dx^2y - ey$

$2c = -e$

$c = -3$

~~$-12xy + 6y = -3dx^2y - 6y$~~

~~$+3dx^2 - 12x + 12 = 0$~~

~~$3d = 0 \quad d = 0$~~

$d = 6$

$a = 2$

$b = -6$

$c = -3$

$d = 6$

$e = 6$

4) $\because f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic fn. then find a, b, c, d.

$u + iv = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$

$u = (x^2 + axy + by^2)$

$v = (cx^2 + dxy + y^2)$

$\frac{\partial u}{\partial x} = 2x + ay + 0$ $\qquad$ $\frac{\partial u}{\partial y} = ax + 2by$

$\frac{\partial v}{\partial x} = 2cx + dy$ $\qquad$ $\frac{\partial v}{\partial y} = dx + 2y$

$\because f(z)$ is analytic

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\qquad$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore 2x + ay = dx + 2y$ $\qquad$ $ax + 2by = -2cx - dy$

$d = 2 \quad a = 2$ $\qquad$ $a = -d$

$c = -1$ $\qquad$ $2b = -2 \quad b = -1$

$a = d$

$2b = -d = -1$

$c = -1, \; b = -1, \; a = 2 \; d = 2$

5) Find value of k such that $w = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\frac{kx}{y}\right)$ is analytic fn.

$w = u + iv$

$u = \frac{1}{2}\log(x^2 + y^2)$ $\qquad$ $v = \tan^{-1}\left(\frac{kx}{y}\right)$

$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$

$\frac{\partial v}{\partial y} = \frac{1}{1 + \frac{k^2x^2}{y^2}} \cdot -\frac{kx}{y^2} = \frac{-\cancel{y}\cdot kx}{y^2 + k^2x}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{x}{x^2+y^2} = \frac{-kx}{y^2+kx^2} \qquad -k=1 \qquad k=-1$$

Find value of p if $f(z) = r^5 \cos p\theta + i r^p \sin(5\theta)$

$f(z) = r^5 \cos p\theta + i r^p \sin(5\theta)$

$f(z) = u+iv$

$u = r^5 \cos p\theta \qquad v = r^p \sin(5\theta)$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial r} = 5r^4 \cos p\theta \qquad \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left[r^p \cos 5\theta \cdot 5\right]$$

$$\cancel{5}r^4 \cos p\theta = \cancel{5}r^{p-1}\cos 5\theta$$

$$r^4 = r^{p-1}$$

$$4 = p-1 \qquad p=5$$

Derive C.R (cauchyreman) eq. for analytic fn. in polar form.

$\because f(z) = u+iv$

z in polar $\quad z = re^{i\theta}$

$\therefore f(re^{i\theta}) = u+iv \longrightarrow (A)$

differentiate A partially wrt r

$$f'(re^{i\theta}) \cdot e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \longrightarrow (1)$$

differentiate A partially wrt θ

$$f'(re^{i\theta}) \cdot r \cdot e^{i\theta} \cdot i = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$f'(re^{i\theta})\, e^{i\theta} = \frac{1}{ri}\left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right]$$

$$= \frac{1}{ri}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$= \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{i}{ri^2}\frac{\partial u}{\partial \theta} = \frac{1}{r}\frac{\partial v}{r\partial \theta} - \frac{i\partial u}{r\partial \theta} \rightarrow ②$$

Comparing ① and ②

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Hence proved.

1) Prove that RP and IP of analytic fn. is harmonic.

$f(z) = u+iv$ is analytic fn.

Conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \quad \text{i.e} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \longrightarrow ①$$

$$\cancel{\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)} \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) \longrightarrow ②$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \longrightarrow ②$$

① + ②

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

The real part is harmonic.

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \quad \text{i.e} \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \rightarrow ①$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \quad \text{i.e} \quad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \rightarrow ②$$

① + ②

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y} = 0$$

The imaginary part is harmonic.

- <u>Formulae</u>

1) $$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{4\,\partial^2}{\partial z\, \partial \bar{z}}$$

2) $z\bar{z} = |z|^2 = |\bar{z}|^2$

3) $f(z)\, f(\bar{z}) = |f(z)|^2 = |f(\bar{z})|^2$

4) $f'(z)\, f'(\bar{z}) = |f'(z)|^2 = |f'(\bar{z})|^2$

Q1) T.P $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

formula 1 $\quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{4\partial^2}{\partial z \partial \bar{z}}$

i.e $\left(\frac{4\,\partial^2}{\partial z \partial \bar{z}}\right)|f(z)|^2 = \frac{4\partial^2}{\partial z \partial \bar{z}} f(z) f(\bar{z})$

$$= 4\left[\frac{\partial}{\partial z} f(z) \quad \frac{\partial}{\partial \bar{z}} f(\bar{z})\right] = 4(f'(z) f'(\bar{z}))$$

$$= 4|f'(z)|^2$$

2) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^n = n^2 |f(z)|^{n-2} |f'(z)|^2$

$\therefore \frac{4\,\partial^2}{\partial z \partial \bar{z}} [|f(z)|^2]^{n/2} = \frac{4\,\partial^2}{\partial z \partial \bar{z}} \left[[f(z) f(\bar{z})]\right]^{n/2}$

$= \frac{4\,\partial}{\partial z} f(z)^{n/2} \cdot \frac{\partial}{\partial \bar{z}} f(\bar{z})^{n/2}$

$= \cancel{4} \cdot \frac{n}{\cancel{2}} [f(z)]^{n/2-1} \cdot f'(z) \cdot \frac{n}{\cancel{2}} [f(\bar{z})]^{n/2-1} \cdot f'(\bar{z})$

$= n^2 [f(z)]^{\frac{n-2}{2}} \cdot [f(\bar{z})]^{\frac{n-2}{2}} \cdot [f'(z) \cdot f'(\bar{z})]$

$= n^2 [f(z) \cdot f(\bar{z})]^{\frac{n-2}{2}} \cdot |f'(z)|^2$

$= n^2 [|f(z)|^2]^{\frac{n-2}{2}} \cdot |f'(z)|^2$

$= n^2 |f(z)|^{n-2} \cdot |f'(z)|^2$

3) Verify u is a harmonic fn. and $u = \left(r + \frac{a}{r}\right)\cos\theta$ also find v and analytic fn.

$\frac{\partial u}{\partial r} = \left(1 - \frac{a}{r^2}\right)\cos\theta \quad \therefore \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\cos\theta}{r} - \frac{a\cos\theta}{r^3}$

$\frac{\partial^2 u}{\partial r^2} = \left(0 - \frac{(-2)\,a}{r^{4}}\,(2r)\right)\cos\theta$

$= \frac{2a}{r^3}\cos\theta$

$\frac{\partial u}{\partial \theta} = \left(r + \frac{a}{r}\right)(-\sin\theta) \quad \frac{\partial^2 u}{\partial \theta^2} = -\cos\theta\left(r + \frac{a}{r}\right)$

LHS $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

$= \frac{2a\cos\theta}{r^3} + \frac{\cos\theta}{r} - \frac{a\cos\theta}{r^3} - \frac{\cos\theta}{r} - \frac{a\cos\theta}{r^3}$

$= 0$

Hence u is a harmonic fn.

$\therefore u$ is also satisfying the analytic function condition.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial r} \cdot r = \frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial \theta} = r\left(1 - \frac{a}{r^2}\right)\cos\theta = \left(r - \frac{a}{r}\right)\cos\theta$$

$$\partial v = \left(r - \frac{a}{r}\right)\cos\theta \, \partial\theta$$

$$\int \partial v = \int \left(r - \frac{a}{r}\right)\cos\theta \, \partial\theta$$

$$\therefore v = \left(r - \frac{a}{r}\right)\sin\theta + c$$

$$f(z) = u + iv$$

$$f(z) = \left(r + \frac{a}{r}\right)\cos\theta + i\left[\left(r - \frac{a}{r}\right)\sin\theta + c\right]$$

$$= r\cos\theta + \frac{a}{r}\cos\theta + ir\sin\theta - \frac{ia}{r}\sin\theta + ic$$

$$= r(\cos\theta + i\sin\theta) + \frac{a}{r}(\cos\theta - i\sin\theta) + ic$$

$$= re^{i\theta} + \frac{a}{r}e^{-i\theta} + ci = re^{i\theta} + \frac{a}{re^{i\theta}} + ci$$

$$f(z) = z + \frac{a}{z} + ci$$

- Conformal Mapping

Step 1: Find z in terms of ω
Step 2: Put $\omega = u+iv$ and remove the imaginary part from denominator using rationalisation.
Step 3: Put $z = x+iy$ and compare real and img. part on both side to find value of x and y
Step 4: Substitute the value of x and y in given condition.

$$\frac{a}{b} = \frac{c}{d}$$

only Compo. $\frac{a+b}{b} = \frac{c+d}{d}$

only Dividendo $\frac{a-b}{b} = \frac{c-d}{d}$

Compo and dividen $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

$\frac{z+a}{z-a}$ same terms with both signs use componendo and dividendo

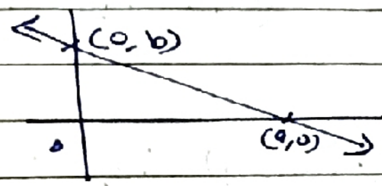
$\frac{z+2}{1-z}$ different term ~~&~~ different sign of z, apply componendo

$\frac{z}{z+1}$ different term with same sign apply dividendo only.

Equation of line $y = mx$ line passing through origin

$y = mx + c$ not passing through origin. convert it into

$\frac{x}{a} + \frac{y}{b} = 1$

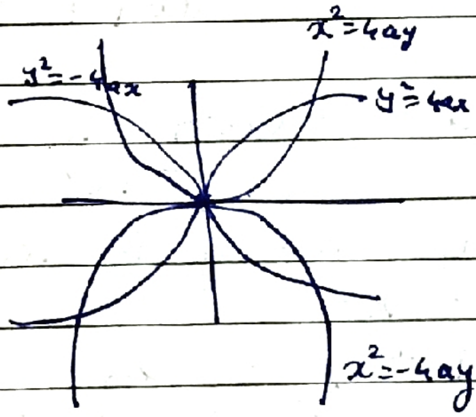


Equation of circle 1) $x^2+y^2=a^2$ center $(0,0)$
radius $= a$

2) $(x-h)^2 + (y-k)^2 = a^2$
center $= (h, k)$
radius $= a$

General form of circle
$x^2+y^2+2gx+2fy+c=0$ center $= (-g, -f)$
radius $= \sqrt{(g^2+f^2-c)}$

3) Eq. of parabola
$y^2 = 4ax$
$y^2 = -4ax$
$x^2 = 4ay$
$x^2 = -4ay$



Q1) Under the transformation of $w = \frac{z-1}{z+1}$
S.T the map of straight line $x=y$
is a circle. Find its center and radius.

$\frac{w}{1} = \frac{z-1}{z+1}$ compo. divider.

$\frac{w+1}{w-1} = \frac{z-\cancel{1}+z+\cancel{1}}{\cancel{z}-1-\cancel{z}-1} = \frac{2z}{-2} = -z$

$$\frac{\omega+1}{\omega-1} = -z$$

$$\omega = u+iv$$

$$\frac{((u+iv)+1)}{((u+iv)-1)} = \frac{((u+1)+iv)}{((u-1)+iv)}$$

$$= \frac{((u+1)+iv)((u-1)-iv)}{((u-1)+iv)((u-1)-iv)} = \frac{(u+1)(u-1)+iv(u-1) - iv(u+1) - i^2v^2}{(u-1)^2 - i^2v^2}$$

$$= \frac{u^2-1 + iv(u-1-u-1)+v^2}{(u-1)^2+v^2}$$

$$= \frac{u^2-1 - 2iv+v^2}{(u-1)^2+v^2} = \frac{(u^2+v^2-1) - 2iv}{(u-1)^2+v^2}$$

$$-z = -(x+iy)$$

$$-x-iy = \frac{(u^2+v^2-1)}{(u-1)^2+v^2} - \frac{i2v}{(u-1)^2+v^2}$$

$$x = \frac{-(u^2+v^2-1)}{(u-1)^2+v^2} \qquad y = \frac{2v}{(u-1)^2+v^2}$$

using eq. of line in Q

$$x = y$$

$$\frac{-(u^2+v^2-1)}{\cancel{(u-1)^2+v^2}} = \frac{2v}{\cancel{(u-1)^2+v^2}}$$

$$0 = 2v + u^2+v^2-1$$

$u^2+v^2+2v-1 = 0$ is the general eq. of center.

$$u^2+v^2+2gu+2fv-1=0$$

$$g=0 \quad f=1 \quad c=-1$$

center $= (0,-1)$

radius $= \sqrt{0^2+1^2+1} = \underline{\sqrt{2}}$ units

2) ~~Set~~ Find the image of real axis under the tranformation of $w = \dfrac{2}{z+i}$. Also find its center and radius.

$$w = \frac{2}{z+i}$$

$$z+i = \frac{2}{w} \qquad z = \frac{2}{w} - i$$

$$z = \frac{2-iw}{w}$$

put $w = u+iv$

$$\therefore z = \frac{2-(u+iv)i}{(u+iv)} = \frac{2-ui+v}{u+iv}$$

$$= \frac{(2-ui+v)(u-iv)}{u^2+v^2} = \frac{2u-u^2 i+\cancel{uv}+2iv-\cancel{vu}-iv^2}{u^2+v^2}$$

$$= \frac{2u-u^2 i-iv^2}{u^2+v^2} = \frac{2u-i(u^2+v^2-2v)}{u^2+v^2}$$

$\cancel{z = x+iy}$

$\cancel{x+iy = \frac{2u}{u^2+v^2} - i}$ $\qquad x+iy = \dfrac{2u-i(u^2+v^2+2v)}{u^2+v^2}$

$\cancel{x+i}$ $x = \dfrac{2u}{u^2+v^2} \qquad y = -\dfrac{(u^2+v^2+2v)}{u^2+v^2}$

Image on real axis in z-plane.
i.e x axis i.e $y=0$

$$\therefore -\frac{(u^2+v^2+2v)}{u^2+v^2} = 0$$

$$u^2+v^2+2v = 0$$

center $(0,-1)$ radius $= 1$

is line $6v+1=0$

$w = \frac{1}{z}$ $\qquad z = \frac{1}{w}$

$w = u+iv$

$z = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$

$z = x+iy$

$x+iy = \frac{u}{u^2+v^2} - \frac{iv}{u^2+v^2}$

$x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}$

$|z-3i| = 3$

$z = x+iy$

$|x+iy-3i| = |x+i(y-3)|$

$= \sqrt{x^2+(y-3)^2} = 3$

$x^2+(y-3)^2 = 9$

$x^2+y^2-6y+\cancel{9} = \cancel{9}$

$x^2+y^2-6y=0$

$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 + \frac{6v}{u^2+v^2} = 0$

$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{6v}{u^2+v^2} = 0$

$\frac{u^2}{u^2+v^2} + \frac{v^2}{u^2+v^2} + 6v = 0$

$\frac{u^2+v^2}{u^2+v^2} + 6v = 0$

$1+6v=0$

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4) $w = i\left(\frac{1-z}{1+z}\right)$ transform a circle $|z|=1$ on to the real axis of w plane.

Note: If $|z|$ = constant given then no need to find x and y. Take modulus on both sides.

$$\frac{w}{i} = \left(\frac{1-z}{1+z}\right)$$

compo. dividen.

$$\frac{w+i}{w-i} = \frac{1-\cancel{z}+1+\cancel{z}}{\cancel{1}-z-\cancel{1}-z} = \frac{\cancel{2}}{-\cancel{2}z}$$

$$-z = \frac{w-i}{w+i}$$

put $w = u+iv$

$$-z = \frac{u+iv-i}{u+iv+i} = \frac{u+i(v-1)}{u+i(v+1)}$$

taking modulus on both sides

$$|z| = \frac{|u+i(v-1)|}{|u+i(v+1)|}$$

$$|z| = \frac{\sqrt{u^2+(v-1)^2}}{\sqrt{u^2+(v+1)^2}}$$

$$1 = \frac{\sqrt{u^2+(v-1)^2}}{\sqrt{u^2+(v+1)^2}} \qquad \sqrt{u^2+(v+1)^2} = \sqrt{u^2+(v-1)^2}$$

squaring both sides

$$\cancel{u^2}+(v+1)^2 = \cancel{u^2}+(v-1)^2$$

$$\cancel{v^2}+2v+\cancel{1} = \cancel{v^2}-2v+\cancel{1}$$

$$0 = -4v$$

$v=0$ line of real axis

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5) Find img of circle $|z|=2$ under the transform of $w = z+3+2i$

$z = w-3-2i$

$w = u+iv$

$z = u+iv-3-2i$

$z = (u-3)+i(v-2)$

$|z| = |(u-3)+i(v-2)|$

$2 = \sqrt{(u-3)^2+(v-2)^2}$

$4 = (u-3)^2+(v-2)^2$

center $= 3,2$ radius $= 2$

6) Find the l.S.T the relation $w = \frac{5-4z}{4z-2}$

Transform a circle $|z|=1$ into the circle of radius unity in w plane. Find centre of circle.

$\frac{w}{1} = \frac{5-4z}{4z-2}$

$\frac{w+1}{1} = \frac{5-\cancel{4z}+\cancel{4z}-2}{4z-2} = \frac{3}{4z-2}$

$4z-2 = \frac{3}{w+1}$

$4z = \frac{3+2w+2}{w+1}$ $z = \frac{5+2w}{4(w+1)}$

$w = u+iv$

$z = \frac{5+2u+2iv}{4u+4iv+4} = \frac{(5+2u)+2iv}{(4u+4)+4iv}$

$$|z| = \frac{\sqrt{(5+2u)^2 + (2v)^2}}{\sqrt{(4u+4)^2 + (4v)^2}}$$

$$1 = \frac{\sqrt{(5+2u)^2 + (2v)^2}}{\sqrt{(4u+4)^2 + (4v)^2}}$$

$$(4u+4)^2 + (4v)^2 = (5+2u)^2 + (2v)^2$$

$$16u^2 + 32u + 16 + 16v^2 = 25 + 20u + 4u^2 + 4v^2$$

$$16u^2 - 4u^2 + 16v^2 - 4v^2 + 32u - 20u + 16 - 25 = 0$$

$$12u^2 + 12v^2 + 12u - 9 = 0$$

$$4u^2 + 4v^2 + 4u - 3 = 0$$

$$u^2 + v^2 + u - \frac{3}{4} = 0$$

center $= \left(-\frac{1}{2}, 0\right)$ radius $= 1$ units //

7 Show that under the tranformation of $w = \frac{z-i}{z+i}$ real axis of z-plane is mapped onto the circle $w = 1$.

$$\frac{w}{1} = \frac{z-i}{z+i}$$

comp. dividen. $\frac{w+1}{w-1} = \frac{z-\cancel{i}+z+\cancel{i}}{\cancel{z}-i-\cancel{z}-i} = \frac{\cancel{2}z}{-\cancel{2}i}$

$$\frac{-i(w+1)}{(w-1)} = z$$

put $w = u+iv$

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$$\frac{-i(u+iv+1)}{(u+iv-1)} = z$$

$$\frac{i\cancel{u} - v + 1i}{u+iv-1} = z$$

$$\frac{-v+i(u+1)}{(u-1)+iv} = z$$

$$= \frac{[-v+i(u+1)][(u-1)-iv]}{((u-1)+iv)((u-1)-iv)} = \frac{-v(u-1)+iv^2+i(u^2-1)+v(u+1)}{(u-1)^2+iv\cancel{(u-1)}-iv\cancel{(u-1)}+v^2}$$

$$= \frac{-\cancel{vu}+v+iv^2+iu^2-i+\cancel{vu}+v}{(u-1)^2+v^2}$$

$$z = \frac{2v+i(u^2+v^2-1)}{(u-1)^2+v^2}$$

$z = x+iy$

IP $y = \dfrac{u^2+v^2-1}{(u-1)^2+v^2}$

$y=0$ (real axis)

$$\therefore \frac{u^2+v^2-1}{(u-1)^2+v^2} = 0$$

$$\therefore u^2+v^2=1$$

as $w=1$ $w=u+iv$

$|w| = 1 = \sqrt{u^2+v^2}$

$$\therefore 1 = u^2+v^2$$

8) Find the image of square bounded by line $x=0$, $x=2$, $y=0$, $y=2$ under the tranformation of $w=(1+i)z+2-i$

$$w=(1+i)z+2-i$$

$$\frac{w-2+i}{1+i}=z$$

$$w=u+iv$$

$$\frac{u+iv-2+i}{1+i}=\frac{(u-2)+i(v+1)}{1+i}=z$$

$$\frac{[(u-2)+i(v+1)](1-i)}{(1+i)(1-i)}=\frac{u-ui-2+2i+iv+v+i+1}{1+1}$$

$$=\frac{(u-2+v+1)+i(-u+2+v+1)}{2}$$

$$=\frac{(u+v-1)}{2}+i\frac{(-u+3+v)}{2}$$

$z=x+iy$

compairing RP and IP

$$x=\frac{u+v-1}{2} \qquad y=-\frac{u+3+v}{2}$$

$x=0$

$\frac{u+v-1}{2}=0 \qquad u+v=1 \rightarrow (1)$

$x=2 \qquad \frac{u+v-1}{2}=2 \qquad u+v-1=4$

$u+v=5 \rightarrow (2)$

$y=0 \therefore \frac{-u+v+3}{2}=0 \therefore v-u=-3 \rightarrow (3)$

$y=2 \qquad \frac{-u+v+3}{2}=2 \quad -u+v=4 \quad v-u=1 \rightarrow (4)$

Solving ① and ③

$u+v=1 \qquad u-v=3$

$2u=4$

$u=2 \qquad v=-1$

Solving ② ④

$u+v=5 \qquad v-u=1$

$2v=6 \qquad v=3$

$u=2$

9) Find the image of triangular region whose vertices are ~~(i, 1+i)~~ i, 1+i, 1-i under the transformation of $\omega = z+4-2i$

$z = \omega - 4 + 2i$

$\omega = u+iv$

$z = u+iv-4+2i$

$= (u-4)+i(v+2)$

$z = x+iy$

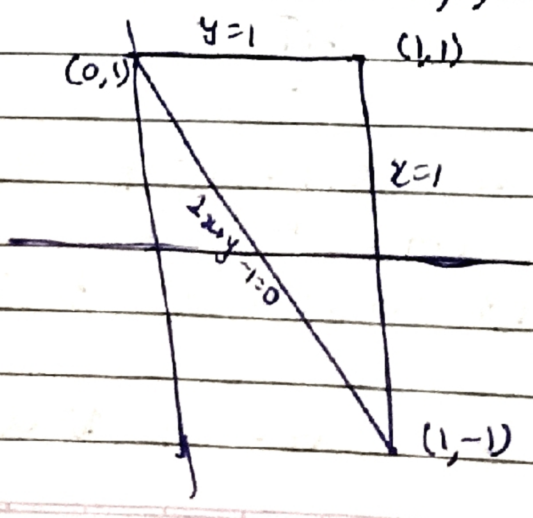
$x+iy = (u-4)+i(v+2)$

compairing RP, IP

$x = u-4 \qquad y = v+2$

Vertices of Δ are $(0+i)$, $1+i$, $1-i$

coordinates $(0,1)$ $(1,1)$ $(1,-1)$



eq. of line passing through (x_1, y_1) and (x_2, y_2)

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$(1,1)$ $(1,-1)$

$$\frac{y-1}{1-1} = \frac{x-1}{+1-1}$$

$(0,1)$ $(1,-1)$

$$\frac{y-1}{-1-1} = \frac{x-0}{\cancel{0-1}\ 1-0} \Rightarrow \frac{y-1}{-2} = \frac{x}{+1}$$

$y-1=-2x$

$y=-2x+1$

$2x+y-1=0$

$y=1$ i.e) $v+2=1$

$v=-1$

$x=1$ i.e) $u-4=1$

$u=5$

$2x+y-1=0$ i.e $2(u-4)+v+2=1$

$=2u-8+v+2=1$

$2u+v-6=1$

$2u+v=7$

10) Find the image of circle $|z|=k$ under the transformation of $w = 3z+4+2i$

$w = 3z+4+2i$

$$z = \frac{w-4-2i}{3}$$

$w = u+iv$

$z = \frac{u+iv-4-2i}{3}$

$= \frac{(u-4)}{3} + i\frac{(v-2)}{3}$

~~$z = x+iy$~~

~~$x+iy = \frac{(u-4)}{3} + i\frac{(v-2)}{3}$~~

taking mod on both sides

$|z| = \left| \frac{(u-4)}{3} + i\frac{(v-2)}{3} \right|$

$k = \sqrt{\frac{(u-4)^2}{9} + \frac{(v-2)^2}{9}}$

$k^2 = \frac{(u-4)^2}{9} + \frac{(v-2)^2}{9}$

$9k^2 = (u-4)^2 + (v-2)^2$

center $(4,2)$ radius $= 3k$

Under the transformation of $w = \frac{3-z}{z-2}$

transform the circle with centre $(\frac{5}{2}, 0)$ and radius $\frac{1}{2}$ in the z plane into the imaginary axis of w plane

$w = \frac{3-z}{z-2}$

~~$\frac{w}{1}$~~ applying compo.

$\frac{w+1}{1} = \frac{3-\cancel{z}+\cancel{z}-2}{z-2} = \frac{1}{z-2}$

$$z = \frac{2\omega+3}{\omega+1}$$

Note: equation of circle in complex form is given by $|z-(h+ik)| = a$ having radius 'a'. And centre (h, k)

$$|z-(h+ik)| = a$$

$$\left|z-\left(\frac{5}{2}+0i\right)\right| = \frac{1}{2}$$

$$\left|z-\frac{5}{2}\right| = \frac{1}{2} \qquad \text{as } z = \frac{2\omega+3}{\omega+1}$$

$$\left|\frac{2\omega+3}{\omega+1}-\frac{5}{2}\right| = \frac{1}{2}$$

$$\left|\frac{4\omega+6-5\omega-5}{2(\omega+1)}\right| = \frac{1}{2} \quad \text{i.e} \left|\frac{-\omega+1}{2(\omega+1)}\right| = \frac{1}{2}$$

$\omega = u+iv$

$$\left|\frac{-u-iv+1}{2(u+iv+1)}\right| = \frac{1}{2} \quad \text{i.e} \left|\frac{(-u+1)-iv}{2((u+1)+iv)}\right| = \frac{1}{2}$$

$$\frac{\sqrt{(1-u)^2+(-v)^2}}{\sqrt{(2u+2)^2+(2v)^2}} = \frac{1}{2}$$

$$2\sqrt{(1-u)^2+v^2} = \sqrt{(2u+2)^2+(2v)^2}$$

$$4(1-u)^2+4v^2 = (2u+2)^2+4v^2$$

$$\cancel{4}-8u+\cancel{4u^2}+\cancel{4v^2} = \cancel{4u^2}+\cancel{4}+8u+\cancel{4v^2}$$

$$u = 0$$

12) Under the transformation of $w+2i = z+\frac{1}{z}$

S.T the map of circle $|z|=2$ is an ellipse in w plane

$|z|=2$ is the eq. of circle in complex form of radius $=2$.

$w+2i = z+\frac{1}{z}$

use polar form $z = re^{i\theta}$ where $r=2$

$w+2i = re^{i\theta} + \frac{1}{re^{i\theta}}$

$= 2e^{i\theta} + \frac{1}{2e^{i\theta}} = 2e^{i\theta} + \frac{1}{2}e^{-i\theta}$

$w = u+iv$

$u+iv+2i = 2e^{i\theta} + \frac{1}{2}e^{-i\theta}$

$u+i(v+2)$

$u+i(v+2) = 2(\cos\theta + i\sin\theta) + \frac{(\cos\theta - i\sin\theta)}{2}$

$= 2\cos\theta + 2i\sin\theta + \frac{\cos\theta}{2} - \frac{i\sin\theta}{2}$

$= \left(2+\frac{1}{2}\right)\cos\theta + i\sin\theta\left(2-\frac{1}{2}\right)$

$u+i(v+2) = \frac{5}{2}\cos\theta + \frac{i3}{2}\sin\theta$

$u = \frac{5}{2}\cos\theta$ $\qquad v+2 = \frac{3}{2}\cancel{\cos}\sin\theta$

$\cos\theta = \frac{2u}{5}$ $\qquad \sin\theta = \frac{3}{2}(v+2)$

~~eq. of ellipse~~ we know

$\cos^2\theta + \sin^2\theta = 1$

$\therefore \left(\frac{u^2}{2.5}\right) + \left(\frac{v+2}{1.5}\right)^2 = 1$ eq. of ellipse

- Bilinear Transformation

Formula

$$\frac{(w-w_1)}{(w_1-w_2)}\frac{(w_2-w_3)}{(w_3-w)} = \frac{(z-z_1)}{(z_1-z_2)}\frac{(z_2-z_3)}{(z_3-z)}$$

$\frac{1+i}{1-i} = i$ $\qquad \frac{1-i}{1+i} = -i$ $\qquad \frac{1}{i} = -i$

1) Find the bilinear transformation which maps the pt. $z = 1, i, -1$ onto the point $w = i, 0, -i$. Also find fixed pt.

$z = \underset{z_1}{1}, \underset{z_2}{i}, \underset{z_3}{-1}$ $\qquad w = \underset{w_1}{i}, \underset{w_2}{0}, \underset{w_3}{-i}$

$$\frac{(w-w_1)}{(w_1-w_2)}\frac{(w_2-w_3)}{(w_3-w)} = \frac{(z-z_1)}{(z_1-z_2)}\frac{(z_2-z_3)}{(z_3-z)}$$

$$\left(\frac{w-i}{i-0}\right)\left(\frac{0+i}{-i-w}\right) = \frac{z-1}{1-i}\cdot\frac{i+1}{-1-z}$$

$$\left(\frac{w-i}{\cancel{i}}\right)\left(\frac{\cancel{i}}{-(i+w)}\right) = \frac{z-1}{(-)(1+z)}\cdot\frac{i+1}{1-i}$$

$$\frac{w-i}{\cancel{-}(i+w)} = \frac{(z-1)}{1+z}\,i = \frac{zi-i}{\cancel{(-1)}(+z)}$$

$$\frac{w-i}{w+i} = \frac{zi-i}{1+z}$$

$$\frac{w-i+w+i}{w-i-w-i} = \frac{zi-i+1+z}{zi-i-1-z}$$

$$\frac{\cancel{7}\omega}{-\cancel{7}i} = \frac{z(i+1)+(1-i)}{-z(1-i)-(1+i)}$$

$$\frac{\omega}{-i} = \frac{z\frac{(\cancel{i}+1)}{(i+\cancel{1})} + \frac{(1-i)}{(1+i)}}{-z\frac{(1-i)}{(1+i)} - \frac{(\cancel{1}+i)}{(\cancel{1}+i)}} = \frac{z - i}{-z(-i)-1}$$

$$= \frac{z-i}{zi-1}$$

$$\omega = (-i)\frac{(z-i)}{(zi-1)} = \frac{-zi+i^2}{zi-1} = \frac{i(-z+i)}{i(z\cancel{i}-\frac{1}{i})}$$

$$\omega = \frac{-(z-i)}{(z+i)} = \frac{i-z}{i+z}$$

for fixed point put $\omega = z$

$$\omega = \frac{i-z}{i+z}$$

$$z = \cancel{t+i}\ \frac{i-z}{i+z}$$

$zi + z^2 = i - z$

$zi + z^2 - i + z = 0$

$z^2 + \cancel{z(i+)} \; z(i+1) - i = 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad a = 1 \quad b = \cancel{z}\ i+1 \quad c = -i$$

$$\frac{-(i+1) \pm \sqrt{(i+1)^2 - 4(-i)}}{2} = \frac{-i-1 \pm \sqrt{i^2+2i+1+4i}}{2}$$

$$= \frac{-i-1 \pm \sqrt{\cancel{2i+}6i}}{2}$$

2) $z = 0, i, -2i \qquad w = -4i, \infty, 0$

Note: The term which is ∞ take common from formula

$$\frac{(w-w_1)}{(w_1-w_2)} \frac{(w_2-w_3)}{(w_3-w)} = \frac{z-z_1}{z_1-z_2} \cdot \frac{z_2-z_3}{z_3-z}$$

$$\frac{w-w_1}{\cancel{w_2}\left(\frac{w_1}{w_2}-1\right)} \cdot \frac{\cancel{w_2}\left(1-\frac{w_3}{w_2}\right)}{w_3-w} = \left(\frac{z-0}{0-i}\right)\left(\frac{i+2i}{-2i+z}\right)$$

$$\left(\frac{w+4i}{-1}\right)\left(\frac{1}{0-w}\right) = \frac{z}{\cancel{i}}\left(\frac{3\cancel{i}}{+2i+z}\right)(-1)$$

$$\frac{w+4i}{w} = \frac{-z \cdot 3}{(-2i+z)}$$

$$\frac{w+4i-w}{w} = \frac{4i}{w} = \frac{+3z-(z+2i)}{z+2i}$$

$$\frac{w}{4i} = \frac{z+2i}{3z-z-2i} = \frac{z+2i}{2z-2i}$$

$$w = \frac{2i\,(z+2i)}{(z-i)}$$

fixed pt. $w = z$

$$z = \frac{2i\,(z+2i)z}{z-1}$$

$$z^2 - z = 2iz^2 - 4z$$

$$z^2 - 2iz^2 - z + 4z = 0$$

$$z^2(1-2i) - 3z = 0$$