COMPLEX VARIABLES

⇒ Analytic Function:-

* A function $f(z) = u + iv$ is said to be analytic at z_0 if it is differentiable at z_0 and some neighbourhood of z_0

Condition for function $f(z) = u + iv$ to be analytic is.

i) u_x, u_y, v_x, v_y are continuous fns.

ii) $u_x = v_y$, $u_y = -v_x$ } Cauchy-Riemann condition (CR equations)

Note

1) If $f(z)$ is analytic, then $f'(z) = u_x + i v_x$.

2) If $f(z)$ is analytic fn in z, then it can be differentiated & integrated in usual manner.

* CR equation in polar form:-

$f(z) = u(r, \theta) + i v(r, \theta)$, then CR eqn,

$$u_r = \frac{1}{r} v_\theta$$

$$u_\theta = -r v_r$$

* A function $\phi(x,y)$ is said to be Harmonic if it satisfies Laplace's equation.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

* If $f(z)$ be an ~~harmo~~ analytic fn, then u & v are harmonic functions called Harmonic conjugates.

Laplace eqn in polar form:-

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

Eg. 8) If $f(z)$ & $\overline{f(z)}$ are both analytic, prove that $f(z)$ is constant.

Let $f(z) = u + iv$ & $\overline{f(z)} = u - iv$.

$\because f(z)$ is analytic,

$u_x = v_y$ & $u_y = -v_x$ (CR eqn)

$\because \overline{f(z)}$ is also analytic

$u_x = -v_y$ & $u_y = -(-v_x) = v_x$ (CR eqn)

Adding $u_x = v_y$ & $u_x = -v_y$

$\therefore 2u_x = 0$

$\therefore u_x = 0$

$v_y = 0$

Adding $u_y = -v_x$ & $u_y = v_x$

$\therefore 2u_y = 0$

$\therefore u_y = 0$

$\therefore v_x = 0$

$\therefore u_x = 0$ and $u_y = 0$, u = a constant

Also $v_x = 0$ and $v_y = 0$, $\therefore v$ = a constant.

⇒ Method of Finding Analytic function (Either u or v will be given)

Steps:-

1) Find u_x, u_y or v_x, v_y
2) Use $f'(z) = u_x + iv_x$
3) Apply Milne - Thompson's Method, ie put $x = z$, $y = 0$.
4) Integrate.

⇒ To verify whether fn is harmonic.

Condition is

$u_{xx} + u_{yy} = 0$ or $v_{xx} + v_{yy} = 0$

→ To find orthogonal trajectory of family of curves, ① find analytic fn. ② find real & imaginary part ③ Orthogonal trajectory to u = real part, ~~orth~~ is v = imaginary part.

—*—*—*—*—

⇒ **MAPPING**

Imp $w = u + iv$ $z = x + iy$

Eg:- Determine the 'D' region in w-plane corresponding to the region D in the z-plane given by $x = 0$, $y = 0$, $x = 1$, $y = 1$ under transformation $w = z + (2 - i)$

→ We have $w = z + (2 - i)$

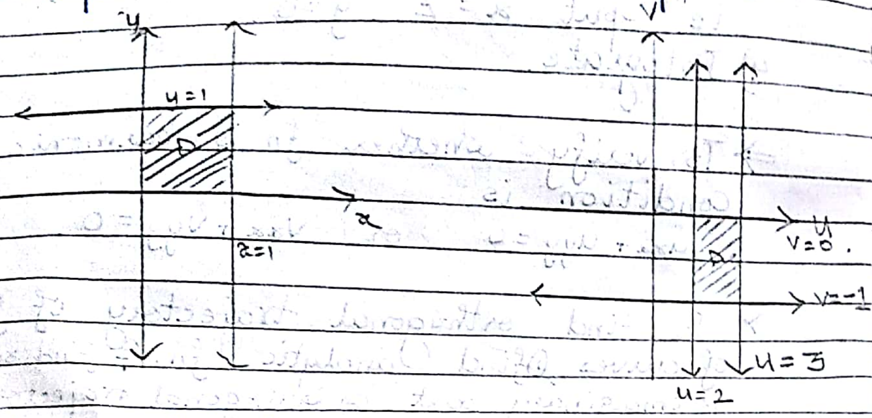
$= x + iy + 2 - i = (x+2) + i(y-1)$

$w = u + iv$

$\therefore u = x + 2$, $v = y - 1$

$\therefore$ When $x=0$, then $u=2$
When $y=0$, then $v=-1$
When $x=1$, then $u=3$
When $y=1$, then $v=0$.

$\therefore$ Z-plane   W-plane.



$\Rightarrow$ Bilinear Transformation:

A transformation of the type $w = \frac{az+b}{cz+d}$

where a, b, c, d are complex constants & $ad-cb \neq 0$, is called a Bilinear Transformation.

$\rightarrow$ Cross Ratio:-

$\frac{z_1-z_2}{z_2-z_3} \; \frac{z_3-z_4}{z_4-z_1}$ is called cross ratio.

$\rightarrow$ Preservation of Cross Ratio property:

$$\left(\frac{z_1-z_2}{z_2-z_3}\right)\left(\frac{z_3-z_4}{z_4-z_1}\right) = \left(\frac{w_1-w_2}{w_2-w_3}\right)\cdot\left(\frac{w_3-w_4}{w_4-w_1}\right)$$

$\rightarrow$ Find the fixed points of bilinear transformation.

1) $w = \frac{z-1}{z+1}$

Put $w = z$

$$z = \frac{z-1}{z+1}$$
$$z(z+1) = z-1$$
$$z^2+z = z-1$$
$$z^2 = -1$$
$$z = i, i$$

$\rightarrow$ Find the bilinear transformation under which the points $z = 1, i, -1$ are mapped onto points $w = 0, 1, \infty$. Further show that under this transformation the unit circle in w-plane ($|w| = 1$) is mapped onto straight line in z-plane. Write name of this line.

Soln:- Let $w = \frac{az+b}{cz+d}$ ... ①

Put $z = 1$ & $w = 0$

$\therefore 0 = \frac{a+b}{c+d}$

$\therefore a+b = 0$ ... ①

Put $z = i$ & $w = 1$

$1 = \frac{ai+b}{ci+d}$

$ai+b = ci+d$

$\therefore$ ~~a=c & b=d~~ ... ②

$\therefore \cancel{a = a = -b = d}$

Put $z = -1$, $\omega = \infty$

$\therefore \infty = \dfrac{a(-1)+b}{c(-1)+d}$

$\therefore -c+d = 0$

$\therefore c = d$ ④

Put $a = -b$, $c = d$ in ③

$\therefore id + d = -ib + b$

$d(1+i) = b(1-i)$

$d = \left(\dfrac{1-i}{1+i}\right) b$

$d = \dfrac{(1-i)^2}{1^2+1} \cdot b = \dfrac{(1-i)^2 b}{2}$

$d = \left(\dfrac{1-2i+i^2}{2}\right) b$

$d = \left(\dfrac{1-2i-1}{2}\right) b = -ib$

$\therefore c = d = -ib$.

$\therefore \omega = \dfrac{az+b}{cz+d} = \dfrac{-bz+b}{-ibz-ib}$

$= \dfrac{-b(z-1)}{-ib(z+1)}$

$= \dfrac{+(z-1)}{i(z+1)}$

$\omega = \dfrac{z-1}{iz+i}$

Now, $|\omega| = 1$

$\left|\dfrac{z-1}{iz+i}\right| = 1$

$\dfrac{|z-1|}{|i(z+1)|} = 1$

$\therefore \dfrac{|z-1|}{|z+1|} = 1$

$|z-1| = |z+1|$

$|x+iy-1| = |x+iy+1|$

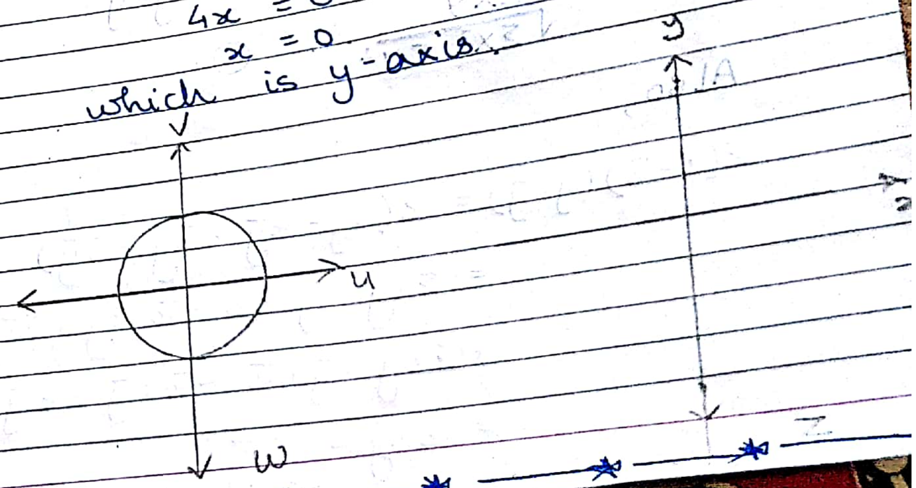
$\sqrt{(x-1)^2+y^2} = \sqrt{(x+1)^2+y^2}$

$x^2-2x+1+y^2 = x^2+2x+1+y^2$

$4x = 0$

$x = 0$

which is y-axis.



CORRELATION, REGRESSION & CURVE FITTING

1) Karl-Pearson's Co-efficient of Correlation (Denoted by 'r'):-

$$r = \frac{cov(x,y)}{\sigma_x \sigma_y} \quad \dots (1)$$

where $\sigma_x^2 = \frac{\Sigma(x-\bar{x})^2}{n}$

$\sigma_y^2 = \frac{\Sigma(y-\bar{y})^2}{n}$ are std deviations in x & y

$$cov(x,y) = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n}$$

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2 \ \Sigma(y-\bar{y})^2}} \quad (2)$$

Take $X = x-\bar{x}$, $Y = y-\bar{y}$

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2 . \Sigma Y^2}} \quad (3)$$

Also,

$$\Sigma(x-\bar{x})(y-\bar{y}) = \Sigma(xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y})$$
$$= \Sigma xy - \bar{y}\Sigma x - \bar{x}\Sigma y + \bar{x}\bar{y}\Sigma 1$$
$$= \Sigma xy - \bar{y}n\bar{x} - \cancel{\bar{x}n\bar{y}} + \cancel{\bar{x}\bar{y}n}$$
$$= \Sigma xy - n\bar{x}\bar{y}$$

$$\therefore r = \frac{\Sigma xy - n\bar{x}\bar{y}}{\sqrt{(\Sigma x^2 - n\bar{x}^2).(\Sigma y^2 - n\bar{y}^2)}} \quad (4)$$

$$r = \frac{\Sigma xy - n\bar{x}\bar{y}}{\sqrt{\left(\Sigma x^2 - \frac{1}{n}(\Sigma x)^2\right)\left(\Sigma y^2 - \frac{1}{n}(\Sigma y)^2\right)}} \quad (5)$$

Results:-
1) $-1 \le r \le 1$
2) $r = \pm 1$ → Perfect Correlation.
3) $r = 0$ → No relation.

Q) Calculate Karl-Pearson's Co-eff.
X : 28, 45, 40, 38, 35, 33, 40, 32, 36, 33.
Y : 23, 34, 33, 34, 30, 26, 28, 31, 36, 35.

→ Here, $n = 10$

$\bar{X} = \frac{\Sigma x}{n} = \frac{360}{10} = 36.$

$\bar{Y} = \frac{\Sigma Y}{n} = \frac{310}{10} = 31$

X	Y	$x = \bar{X} - X$	$y = \bar{Y} - Y$	x^2	y^2	xy

$\Sigma x^2 =$ $\Sigma y^2 =$ $\Sigma xy =$

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2 . \Sigma Y^2}} =$$

For assumed mean

$$r = \frac{\Sigma dx\,dy - \frac{1}{n}(\Sigma dx)(\Sigma dy)}{\sqrt{\left[\Sigma dx^2 - \frac{1}{n}(\Sigma dx)^2\right]\left[\Sigma dy^2 - \frac{1}{n}(\Sigma dy)^2\right]}}$$

2) Spearman's Rank Co-efficient (Denoted by R).

$$R = 1 - \frac{6\Sigma D^2}{n^3 - n}$$

where $D = R_1 - R_2$

If ranks are repeating

$$R = 1 - \frac{6}{n^3 - n}\left\{\Sigma D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) \dots\right\}$$

—*—*—*—*—*—*—

⇒ REGRESSION :-

→ Line of Regression of Y on X

$y = a + bx$ OR $y - \bar{y} = b_{yx}(x - \bar{x})$

→ Line of Regression of X on Y

$x = a + by$ OR $x - \bar{x} = b_{xy}(y - \bar{y})$

Note:- 1) Line of Regression always passes through the mean point.

2) The slopes b_{yx} & b_{xy} are called regression coefficients.

$b_{yx} = r\frac{\sigma y}{\sigma x}$ $\quad b_{xy} = r\frac{\sigma x}{\sigma y}$

$b_{yx} = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma x^2}$ $\quad b_{xy} = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma y^2}$

$b_{yx} = \frac{\Sigma XY}{\Sigma X^2}$ $\quad b_{xy} = \frac{\Sigma XY}{\Sigma Y^2}$

$\tan\theta = \frac{1 - r^2}{r}\left(\frac{\sigma x\,\sigma y}{\sigma x^2 + \sigma y^2}\right)$

Note:-

1) $r^2 = b_{xy} \cdot b_{yx}$

- Regression co-efficient must be of same sign.

⇒ CURVE FITTING

1) Fitting a straight line:-

Let $y = a + bx$

∴ Normal equations,

$\Sigma y = na + b\Sigma x$

$\Sigma xy = a\Sigma x + b\Sigma x^2$

Similarly for $x = a + by$

$\Sigma x = na + b\Sigma y$

$\Sigma xy = a\Sigma y + b\Sigma y^2$

2) Fitting a parabolic curve

$\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$

$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$

~~For assumed mean.~~

$$\cancel{r = \frac{\sum dx\,dy - \frac{1}{n}\sum dx \sum dy}{\sqrt{\left(\sum dx^2 - \frac{1}{n}(\sum dx)^2\right)\left(\sum dy^2 - \frac{1}{n}(\sum dy)^2\right)}}}$$

Z - TRANSFORM.

$$F(Z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) \cdot z^{-k}$$

$$\therefore f(k) = Z^{-1}\{F(Z)\}$$