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SEM -3-All Branch

Maths-III

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(For Private Circulation Olny)

APPLIED MATHEMATICS - III

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INDEX

Important Mathematical Formulae

Trigonometry

$\sin^2\theta + \cos^2\theta = 1$ $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$ $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\sec^2\theta = 1 + \tan^2\theta$ $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$ $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$\text{cosec}^2\theta = 1 + \cot^2\theta$ $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$ $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\sin(-\theta) = -\sin\theta$ $\sin 2A = 2\sin A\cos A$ $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$\cos(-\theta) = \cos\theta$ $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$ $\cos 2A = \cos^2 A - \sin^2 A$

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$ $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$ $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$ $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$ $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$ $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

$\sin 3A = 3\sin A - 4\sin^3 A$ $\cos 3A = 4\cos^3 A - 3\cos A$

$\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}$ $\tan\frac{A}{2} = \frac{1-\cos A}{\sin A}$ $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$ $\sin^{-1}(-x) = -\sin^{-1}x$

$\cos^{-1}(-x) = \pi - \cos^{-1}x$ $\tan^{-1}(-x) = -\tan^{-1}x$ $\tan^2 A = \pm\left(\frac{1-\cos 2A}{1+\cos 2A}\right)$

$\sin(90-\theta) = \cos\theta$ $\cos(90-\theta) = \sin\theta$ $\cos(180+\theta) = -\cos\theta$
$\sin(90+\theta) = \cos\theta$ $\cos(90+\theta) = \sin\theta$ $\cos(180-\theta) = -\cos\theta$
$\sin(180-\theta) = \sin\theta$ $\sin(180+\theta) = -\sin\theta$ $\tan(90-\theta) = \cot\theta$

$\sin 0 = 0$ $\sin\frac{\pi}{6} = \frac{1}{2}$ $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\sin\frac{\pi}{2} = 1$ (where $\pi = 180^0$)

$\cos 0 = 1$ $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\cos\frac{\pi}{3} = \frac{1}{2}$ $\cos\frac{\pi}{2} = 0$

$\tan 0 = 0$ $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ $\tan\frac{\pi}{4} = 1$ $\tan\frac{\pi}{3} = \sqrt{3}$ $\tan\frac{\pi}{2} = \infty$

Logarithms

$\log_a mn = \log_a m + \log_a n$ $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$ $\log_a m^n = n\log_a m$

$\log_n m = \frac{\log_a m}{\log_a n}$ for any base a {change of base theorem}

Binomial Theorem and Related

$(a+b)^n = a^n + {}^nC_1 a^{n-1}b^1 + {}^nC_2 a^{n-2}b^2 + \dots\dots {}^nC_{r-1}a^{n-(r-1)}b^{(r-1)} + \dots\dots + b^n$ where ${}^nC_{r-1}a^{n-(r-1)}b^{(r-1)}$ is the r^{th} term of the series and where n is an integer.

$(a+b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{1.2}a^{n-2}b^2 + \dots\dots \frac{n(n-1)\dots(n-(r-2))}{1.2\dots\dots(r-1)}a^{n-(r-1)}b^{(r-1)} + \dots\dots + b^n$ using ${}^nC_r = \frac{n!}{r!(n-r)!}$

$(a\pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$ $\quad (a\pm b)^2 = a^2 \pm 2ab + b^2$ $\quad a^2 - b^2 = (a+b)(a-b)$

$a^2 + b^2 = (a+ib)(a-ib)$ $\quad {}^nP_r = \frac{n!}{(n-r)!}$ $\quad a^3 \pm b^3 = (a\pm b)(a^2 \mp ab + b^2)$

Derivatives

$\frac{d}{dx}(\sin x) = \cos x$ $\quad \frac{d}{dx}(\cos x) = -\sin x$ $\quad \frac{d}{dx}(\tan x) = \sec^2 x$ $\quad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ $\quad \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\quad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{\pm 1}{x\sqrt{x^2-1}}$ $\quad \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{\mp 1}{x\sqrt{x^2-1}}$ $\quad \frac{d}{dx}(\log x) = \frac{1}{x}$

$\frac{d}{dx}(\log_a x) = \frac{1}{x\log a}$ $\quad \frac{d}{dx}(a^x) = a^x \log a$ $\quad \frac{d}{dx}(e^x) = e^x$

u, v are functions of x then $\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$ and $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

For composite functions, we have $\frac{d}{dx}[f(g(x))] = \frac{d}{dx}f(x)\Big|_{x=g(x)} \frac{d}{dx}g(x) = f'(g(x))g'(x)$

$\frac{du}{dx} = \frac{1}{dx/du}$ $\quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ $\quad \frac{dy}{dx} = \frac{dy/du}{dx/du}$

Generalised uv for derivatives is $D^n(u.v) = u(D^n v) + {}^nC_1(Du)(D^{n-1}v) + {}^nC_2(D^2u)(D^{n-2}v) + \dots + (D^n u)v$; $D^n = \frac{d^n}{dx^n}$

Integration (for sem 2)

$\int(a)dx = ax + c$ $\quad \int(x^n)dx = \frac{x^{n+1}}{n+1} + c$ $\quad \int(\frac{1}{x})dx = \log x + c$

$\int(e^x)dx = e^x + c$ $\quad \int(a^x)dx = \frac{a^x}{\log a} + c$ $\quad \int(\sin x)dx = -\cos x + c$

$\int(\cos x)dx = \sin x + c$ $\quad \int(\tan x)dx = \log(\sec x) + c$ $\quad \int(\cot x)dx = \log(\sin x) + c$

$\int(\sec^2 x)dx = \tan x + c$ $\quad \int(\operatorname{cosec}^2 x)dx = -\cot x + c$ $\quad \int f(ax)dx = \frac{1}{a}\int f(u)du + c$

$\int \sec x dx = \log(\sec x + \tan x) + c$ $\quad \int \operatorname{cosec} x dx = \log(\operatorname{cosec} x - \cot x) + c$

$\int \sec x \tan x dx = \sec x + c$ $\int_a^b f(x)dx = -\int_b^a f(x)dx$ $\int e^{ax}\cos(bx)dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx)+b\sin(bx)]+c$

$\int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx)-b\cos(bx)]+c$ $\int(\operatorname{cosec} x \cot x)dx = -\operatorname{cosec} x + c$

$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$ $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\frac{x-a}{x+a}+c$ $\int e^{ax}[af(x)+f'(x)]dx = e^{ax}f(x)$

$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\frac{a+x}{a-x}+c$ $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}+c$ $\int \frac{1}{\sqrt{x^2+a^2}}dx = \log(x+\sqrt{x^2+a^2})+c$

$\int \frac{1}{\sqrt{x^2-a^2}}dx = \log(x+\sqrt{x^2-a^2})+c$ $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log(x+\sqrt{x^2+a^2})+c$

$\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log(x+\sqrt{x^2-a^2})+c$ $\int(uv)dx = u\int vdx - \int\left(\frac{du}{dx}\int vdx\right)dx + c$

$\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$ $\int_a^a f(x)dx = 0$ $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}+c$

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ $\int \frac{f'(x)}{f(x)}dx = \log f(x)+c$ $\int e^{f(x)}f'(x)dx = e^{f(x)}+c$

$\int uv\, dx = u(v_1)-(u')(v_2)+(u'')(v_3)..+(-1)^{n-1}\left(u^{(n-1)}\right)(v_n)$ till nth term where $u^{(n-1)}=0$

where $v_n = \int v_{n-1}dx$ and $u^{(n-1)} = \frac{d}{dx}u^{(n-2)}$ (generalized uv rule for integrals)

Complex Numbers

$e^{i\theta} = \cos\theta + i\sin\theta$ $e^{-i\theta} = \cos\theta - i\sin\theta$ $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ $\tan\theta = \frac{-i(e^{i\theta}-e^{-i\theta})}{e^{i\theta}+e^{-i\theta}}$

$e^{i\frac{\pi}{2}} = i$ $e^{i\pi} = -1$ $e^{-i\frac{\pi}{2}} = -i$ $e^{i2\pi} = 1$

$e^{i\theta} = e^{i\theta+i2k\pi}$ where k is an integer and $2k\pi i$ is the period.

$z = x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$ where $r =$ modulus and $\theta =$ argument.

$|z| = r = \sqrt{x^2+y^2}$; $Arg(z) = \theta = \pm\tan^{-1}\frac{y}{x}$

$\bar{z} = x-iy$ $z\bar{z} = x^2+y^2$ $z+\bar{z} = 2x$ $z-\bar{z} = 2iy$

$\log z = \log(re^{i\theta}) = \log(r)+i\theta = \frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{y}{x}$ is called the principal value of $\log z$.

$Log\, z = \frac{1}{2}\log(x^2+y^2)+i\left(2k\pi+\tan^{-1}\frac{y}{x}\right)$ is called the general value of $\log z$ with $k =$ integer.

$i = \frac{1}{2}(1+i)^2$ $\sqrt{i} = \frac{\pm1}{\sqrt{2}}(1+i)$ $\sqrt{-i} = \frac{\pm1}{\sqrt{2}}(1-i)$

$(\cos\theta+i\sin\theta)^n = \cos n\theta + i\sin n\theta$ {DeMoivre's Theorem}.

nth roots of a complex no u is z then it is given by $z^n = u$. Taking $u = re^{i\theta}$, then we get $z = r^{1/n} e^{i\left(\theta + 2k\pi/n\right)}$ for $k = 0$ to $n-1$.

Hyperbolic Functions

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ $\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ $\cosh^2 x - \sinh^2 x = 1$

$\text{sech}^2 x = 1 - \tanh^2 x$ $\text{cosech}^2 x = \coth^2 x - 1$ $\sinh(-x) = -\sinh x$ $\cosh(-x) = \cosh x$

$\tanh(-x) = -\tanh x$ $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

$-\infty < \sinh x < +\infty$ $1 \le \cosh x < +\infty$ $-1 \le \tanh x \le +1$

$\sinh 2x = 2 \sinh x \cosh x$ $\cosh 2x = 2\cosh^2 x - 1$ $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

$\sinh 3x = 3\sinh x + 4\sinh^3 x$ $\cosh 3x = 4\cosh^3 x - 3\cosh x$ $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

$\sinh^{-1} x = \log(x + \sqrt{x^2 + 1})$ $-\infty < x < +\infty$ $\cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$ $x \ge 1$

$\tanh^{-1} x = \dfrac{1}{2}\log\dfrac{1+x}{1-x}$ $-1 < x < 1$ $\sin(ix) = i\sinh x$ $\cos(ix) = \cosh x$

$\tan(ix) = i\tanh x$ $\sinh(ix) = i\sin x$ $\cosh(ix) = \cos x$ $\tanh(ix) = i\tan x$

$2k\pi i$ is the period for $\cosh x$ and $\sinh x$ and $k\pi i$ is the period for $\tanh x$

$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ $\dfrac{d}{dx}\sinh x = \cosh x$ $\dfrac{d}{dx}\cosh x = \sinh x$

$\dfrac{d}{dx}\tanh x = \text{sech}^2 x$ $\dfrac{d}{dx}\coth x = -\text{cosech}^2 x$ $\int \sinh(x)dx = \cosh x$

$\int \cosh(x)dx = \sinh x$ $\int \tanh(x)dx = \log(\cosh x)$ $\int \coth(x)dx = \log(\sinh x)$

Differentiating an Integral

$$\frac{d}{d\alpha}\int_{\phi_1(\alpha)}^{\phi_2(\alpha)} F(x,\alpha)dx = \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} \frac{\partial F}{\partial \alpha}dx + F(\phi_2(\alpha),\alpha)\frac{d\phi_2}{d\alpha} - F(\phi_1(\alpha),\alpha)\frac{d\phi_1}{d\alpha}$$ {Leibnitz Formula} (for second year)

Series

$e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots..$ $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots...$ $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots..$

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots$ $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \ldots$ $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots$

$\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots$ $\tanh x = x - \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \ldots$ $\log(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \ldots$

$\tanh^{-1} x = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots$ $(1+x)^m = 1 + mx + \dfrac{m(m-1)}{2!}x^2 + \ldots$ $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

$\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ $f(x+h) = f(x) + hf'(x) + \dfrac{h^2}{2!}f''(x) + \ldots$ {Taylor S}

$f(x+h)=f(h)+xf'(h)+\frac{x^2}{2!}f''(h)+\ldots$ {Taylor Series} $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots$ {Maclaurin S}

Beta and Gamma Functions

$\Gamma(n+1)=n!=\int_0^\infty e^{-x}x^n dx$ {Gamma Function} $\Gamma(n+1)=n\Gamma(n)$ for $n>0$ $\frac{\Gamma(n+1)}{n}=\Gamma(n)$ for $n<0$

$\Gamma(\frac{1}{2})=\sqrt{\pi}$ $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$ for $(m>0,n>0)$ {Beta Function} $\beta(m,n)=\beta(n,m)$

$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ $\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta=\int_0^\infty\frac{t^{m-1}}{(1+t)^{m+n}}dt$ $\Gamma(p)\Gamma(1-p)=\frac{\pi}{\sin(p\pi)}$

LAPLACE TRANSFORMS

Forward Laplace Transforms

Chapter Summary: 1. LAPLACE TRANSFORMS

Definition of a Laplace Transform: $L[f(t)] = \int_0^{\infty} e^{-st} f(t)dt = F(s)$

Linearity of Laplace Transforms: $L[af(t)+b(g(t)] = aL[f(t)]+bL[g(t)]$

Elementary Laplace Transforms

$L[e^{at}] = \frac{1}{s-a}$ $\quad L[e^{-at}] = \frac{1}{s+a}$ $\quad L[1] = \frac{1}{s}$ $\quad L[\sin(at)] = \frac{a}{s^2+a^2}$ $\quad L[\cos(at)] = \frac{s}{s^2+a^2}$

$L[\sinh(at)] = \frac{a}{s^2-a^2}$ $\quad L[\cosh(at)] = \frac{s}{s^2-a^2}$ $\quad L[t^n] = \frac{\overline{|n+1}}{s^{n+1}}$ $\quad L[erf(\sqrt{t})] = \frac{1}{s\sqrt{s+1}}$

Properties of Laplace Transforms

Given that $L[f(t)] = F(s)$ then

1. $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$ (Change of Scale Property)
2. $L[e^{-at}f(t)] = F(s+a)$ (First Shifting Theorem)
3. Let $g(t) = \{f(t-a)$ if $t \geq a$ and 0 if $t < a\}$ then $L[g(t)] = e^{-as}F(s)$ (Second Shifting Theorem)
4. $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$ (Effect of Multiplication by t)
5. $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(s)ds$ (Effect of Division by t)

Laplace Transforms for Derivatives and Integrals of Functions

Given that $L[f(t)] = F(s)$ then

1. $L[f'(t)] = sL[f(t)] - f(0)$ $\quad L[f^{(n)}(t)] = s^n L[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots\ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

..(Generalised Form)

$L[f^{(n)}(t)] = s^n L[f(t)]$ if $f(0) = f'(0) = .. = f^{(n-1)}(0) = 0$

2. $L\left[\int_0^t f(u)du\right] = \frac{F(s)}{s}$

Chapter Summary: 2. INVERSE LAPLACE TRANSFORMS

Some Elementary Inverse Laplace Transforms . Given that $L^{-1}[F(s)] = f(t)$.

$L^{-1}[\frac{1}{s}] = 1$ $\quad L^{-1}[\frac{1}{s+a}] = e^{-at}$ $\quad L^{-1}[\frac{1}{s-a}] = e^{+at}$ $\quad L^{-1}[\frac{1}{s^n}] = \frac{t^{n-1}}{\Gamma(n)}$

$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin(at)$

$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at)$ $\quad L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh(at)$ $\quad L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh(at)$

$L^{-1}\left[\frac{1}{(s-b)^n}\right] = e^{bt}L^{-1}\left[\frac{1}{s^n}\right] = e^{bt}\frac{t^{n-1}}{(n-1)!} = e^{bt}\frac{t^{n-1}}{\overline{|n}}$ $\quad L^{-1}\left[\frac{1}{(s-b)^2+a^2}\right] = e^{bt}L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{e^{bt}}{a}\sin(at)$

$L^{-1}\left[\frac{s-b}{(s-b)^2+a^2}\right]=e^{bt}L^{-1}\left[\frac{s}{s^2+a^2}\right]=e^{bt}\cos(at)$ $\quad L^{-1}\left[\frac{s-b}{(s-b)^2-a^2}\right]=e^{bt}L^{-1}\left[\frac{s}{s^2-a^2}\right]=e^{bt}\cosh(at)$

$L^{-1}\left[\frac{1}{(s-b)^2-a^2}\right]=e^{bt}L^{-1}\left[\frac{1}{s^2-a^2}\right]=\frac{e^{bt}}{a}\sinh(at)$ $\quad L^{-1}\left[F\left(\frac{s}{a}\right)\right]=af(at)$

$L^{-1}[F_1(s)]=f_1(t)$ and $L^{-1}[F_2(s)]=f_2(t)$ then $L^{-1}[F_1(s)\times F_2(s)]=\int_0^t f_1(u)f_2(t-u)du=f_1(t)\otimes f_2(t)$

(Convolution Th.)

$L^{-1}[F(s)]=-\frac{1}{t}L^{-1}[F'(s)]$ $\quad L[F(s)]=tL^{-1}\left[\int_s^\infty F(s)ds\right]$ (Diff. & Int. Inverse Transforms)

$L[s^nF(s)]=\frac{d^n}{dt^n}L^{-1}[F(s)]$ iff $f(0)=f'(0)=..=f^{(n-1)}(0)=0$ $\quad L^{-1}\left[\frac{1}{s}F(s)\right]=\int_0^t L^{-1}[F(s)]dt$

Chapter Summary: 3. LAPLACE TRANSFORMS OF SPECIAL FUNCTIONS

$L[f(t)]=\frac{1}{1-e^{as}}\int_0^a e^{-st}f(t)dt$ if $f(t)=f(t+a)\ \forall\ t$ {Laplace Transform of a periodic functions}

$H(t-a)=0$ for $t<a$ and $H(t-a)=1$ for $t\geq a$ {Heaviside Unit Step Function at a}

$L[H(t-a)]=\frac{1}{s}e^{-as}$ $\quad L[(H(t)]=\frac{1}{s}$ where $H(t)$ is Heaviside Unit Step Function at 0

$L[f(t-a)H(t-a)]=e^{-as}L[f(t)]$ $\quad L^{-1}[\frac{1}{s}e^{-as}]=H(t-a)$ $\quad L^{-1}[\frac{1}{s}]=H(t)$

$L^{-1}[e^{-as}F(s)]=f(t-a)H(t-a)$

$\delta(t-a)\to\infty$ when $|t-a|\leq\varepsilon$ and $\delta(t-a)=0$ when $|t-a|>\varepsilon$; $\varepsilon\to 0$ {Dirac Delta Function or Impulse Function}

$L[\delta(t-a)]=e^{-as}$ $\quad L[\delta(t)]=1$ $\quad L[f(t)\delta(t-a)]=e^{-as}f(a)$ $\quad L^{-1}[e^{-as}]=\delta(t-a)$ $\quad L^{-1}[1]=\delta(t)$

Chapter Summary: 4. APPLICATIONS OF LAPLACE TRANSFORMS

$$L[y']=sL(y)-y(0) \qquad L[y'']=s^2L(y)-sy(0)-y'(0)$$
$$L[y''']=s^3L(y)-s^2y(0)-sy'(0)-y''(0)$$

Forward Laplace Transforms

Type 1A Problems on elementary laplace transforms

1a. Define Laplace transform of $f(t)$, $t>0$. Find laplace transform of $f(t)=t^{3/2}+5^{6t}+\sinh 6t+\cos 3t+e^{-11t}$.

1b. State and prove the linearity property of laplace transforms.

2. Find Laplace transforms of the following i) $\sin^5 t$ *Ans.* $\frac{5!}{(s^2+1)(s^2+9)(s^2+25)}$

ii) $\cos t\cos 2t\cos 3t$ **(1995)** *Ans.* $\frac{1}{4}\left[\frac{1}{s}+\frac{s}{s^2+2^2}+\frac{s}{s^2+4^2}+\frac{s}{s^2+6^2}\right]$

iii) $\cos(wt+\beta)$ **(1999)** *Ans.* $\cos\beta.\frac{5}{s^2+w^2}-\sin\beta.\frac{w}{s^2+w^2}$

iv) $\sin^4 t$ **(02, 03)** ***Ans.*** $\frac{1}{4}\left[\frac{3s}{2}-\frac{2s}{s^2+4}+\frac{1}{s^2+16}\right]$ v) $\cosh 5t$ ***Ans.*** $\frac{1}{8}\left[\frac{3}{s}+\frac{4s}{s^2-4}+\frac{s}{s^2+16}\right]$

vi) $\sqrt{1+\sin t}$ **(2004)** ***Ans.*** $\frac{s}{s^2+(1/2)^2}+\frac{1/2}{s^2+(1/2)^2}$ vii) $\left(\sqrt{t}-1\right)^2$ ***Ans.*** $\frac{1}{s^2}-2\frac{(1/2)\sqrt{\pi}}{s^{3/2}}+\frac{1}{s}$

Type 1B Problems to use Definition of laplace transforms

1. Find the Laplace transform of

i) $f(t)=\cos t$, for $0<t<\pi$ and $f(t)=\sin t$, for $t>\pi$. **(93, 02, 06)** ***Ans.*** $\frac{1}{s^2+1}.\left[s+(s-1)e^{-s\pi}\right]$

ii) $f(t)=t, 0<t<1/2$; $f(t)=t-1,\ 1/2<t<1$; $f(t)=0,\ t>1$. **(2003)** ***Ans.*** $\frac{2}{s^2}-\frac{2e^{-s}}{s^2}-\frac{e^{-s/2}}{s}$

iii) $f(t)=\cos t, 0<t<2\pi$; $f(t)=0,\ t>2\pi$. **(2002)** ***Ans.*** $\left(1-e^{-2\pi s}\right)\frac{s}{s^2+1}$

iv) $f(t)=t^2, 0<t<1$; $f(t)=1, t>1$. **(2003)** ***Ans.*** $\frac{2}{s^3}\left(1-e^{-s}\right)-\frac{2}{s^2}e^{-s}$

v) $f(t)=0,\ 0<t<\pi$; $f(t)=\sin^2(t-\pi),\ t>\pi$. **(2003)** ***Ans.*** $\frac{e^{-\pi s}}{2s}-\frac{s.e^{-\pi s}}{s^2+4}$

Type 1C Use series method to find Laplace Transforms

1. Find $L\left(\sin\sqrt{t}\right)$ **(1996)** ***Ans.*** $\frac{\sqrt{\pi}}{2s^{3/2}}.e^{-1/(4s)}$ 2. Find $L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right)$ **(2004)** ***Ans.*** $\frac{\sqrt{\pi}}{\sqrt{s}}.e^{-1/4s}$

3. Find $L\left(erf\sqrt{t}\right)$. **(96, 97, 03)** ***Ans.*** $\frac{1}{s\sqrt{s+1}}$

Type 2A Theory problems on properties of laplace transforms

1. Define Laplace transform of $f(t)$, $t>0$. **(95, 02)**
2. Define Laplace transform of a function of t and state the rule of change of scale with one example.
3. State and prove first shifting theorem. Hence, find $L\left[e^{2t}\cos t\cos 2t\right]$. **(2003)**

Ans. $\frac{(s-2)\left(s^2-4s+9\right)}{\left(s^2-4s+13\right)\left(s^2-4s+5\right)}$

4. If $L[f(t)]=\phi(s)$, prove that $L\left[e^{-at}f(t)\right]=\phi(s+a)$. **(1995)**
5. If $g(t)=\begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$ and $L[f(t)]=\phi(s)$, prove taht $L[g(t)]=e^{-as}\phi(s)$. **(1995)**
6. If $L[f(t)]=\phi(s)$, prove that $L\left[t^n f(t)\right]=(-1)^n\frac{d^n}{ds^n}\phi(s)$. **(1997)** Hence find $L(t\cos at)$. **(CIT 04)**
7. State and prove First Shifting Theorem. **(97, 99, 03)**
9. If $L[f(t)]=\phi(s)$, prove that $L[t\,f(t)]=-\phi'(s)$. **(1993)**
10. If $L[f(t)]=\phi(s)$, prove that $L\left[\frac{1}{t}f(t)\right]=\int_s^\infty\phi(s)\,ds$. **(94, 98, 00, 06)**

11. State and prove second shifting theorem. (2002)
12. Define an error function and a complimentary error function.
13. State and prove the first shifting property of laplace transf. Hence find $L(e^{-t}\sin 3t\cos t)$. (2006)

Type 2B Problems on properties of laplace transforms

1. i) If $L\left(erf_c\left(\sqrt{t}\right)\right)=\frac{1}{\sqrt{s+1}+1}$, find $L\left(erf_c\left(2\sqrt{t}\right)\right)$. *Ans.* $\frac{1}{2\sqrt{s+4}+4}$

   ii) If $L(\sin\sqrt{t})=\frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-1/(4s)}$, find $L(\sin 2\sqrt{t})$ (2004)

2. i) Find Laplace transform of $\sin h\, at \sin at$. (2003) *Ans.* $\frac{2a^2 s}{s^4+4a^4}$

   ii) Show that $L\left[\sinh(t/2)\sin\left(\sqrt{3}\,t/2\right)\right]=\frac{2\sqrt{3}s}{\left(4s^4+16s^2+25\right)}$ (93,02,03)

   iii) $L\left(\frac{\cos 2t\sin t}{e^t}\right)$ (93, 03, 04) *Ans.* $\frac{3}{2(s+1)^2+18}-\frac{1}{2(s+1)^2+2}$

   iv) $L\left(e^{-4t}\sinh t\sin t\right)$ (1995) *Ans.* $\frac{1}{2(s+3)^2+2}-\frac{1}{2(s+5)^2+2}$

   v) $L\left(e^t\sin 2t\sin 3t\right)$ (1997) *Ans.* $\frac{s-1}{2(s-1)^2+2}-\frac{s-1}{2(s-1)^2+50}$

   vi) $e^{-3t}\cosh 5t\sin 4t$. (1998) *Ans.* $\frac{2}{(s-2)^2+16}+\frac{2}{(s+8)^2+16}$

   vii) Find Laplace transform of $\sin 2t\cos t\cosh 2t$. (94)
   Ans. $\frac{1}{\frac{4}{3}(s-2)^2+12}+\frac{1}{\frac{4}{3}(s+2)^2+12}+\frac{1}{4(s-2)^2+4}+\frac{1}{4(s+2)^2+4}$

   viii) $L\left(e^{-4t}\cosh t\sin t\right)$ (1995) *Ans.* $\frac{1}{2(s+3)^2+2}+\frac{1}{2(s+5)^2+2}$

   ix) $L\left(e^{-3t}\cosh 4t\sin 3t\right)$ (1999) *Ans.* $\frac{3}{2(s-1)^2+18}+\frac{3}{2(s+7)^2+18}$

   x) $L\left(e^{2t}\cos 2t\cos t\right)$ (02, 03) *Ans.* $\frac{s-2}{2(s-2)^2+2}+\frac{s-2}{2(s-2)^2+18}$

   xi) $L\left[e^{2t}(1+t)^2\right]$ (02, 03) *Ans.* $\frac{1}{s-2}+\frac{2}{(s-2)^2}+\frac{2}{(s-2)^3}$

   xii) If $L[f(t)]=\frac{s}{s^2+s+4}$, find $L\left[e^{-3t}f(2t)\right]$ (2003) *Ans.* $\frac{s+3}{(s+3)^2+2s+22}$

3. Find $L(f(t))$ i) $f(t)=(t-1)^3, t>1$ and $f(t)=0, t<1$. *Ans.* $e^{-s}.\frac{3!}{s^4}$

   ii) $f(t)=(t-2)^2, t>2$ and $f(t)=0, t<2$. *Ans.* $e^{-2s}.\frac{2}{s^3}$

   iii) $f(t)=0,\ 0<t<\pi$; $f(t)=\sin^2(t-\pi),\ t>\pi$. (2003) *Ans.* $\frac{2e^{-\pi s}}{s\left(s^2+4\right)}$

4. i) $L\left(t\,e^{-4t}.\sin 3t\right)$ *Ans.* $\dfrac{3(2s+8)}{\left((s+4)^2+9\right)^2}$ ii) $L\left(t^2\,e^{-t}\sin 4t\right)$. *Ans.* $\dfrac{32(s+1)^2}{\left((s+1)^2+16\right)^3}-\dfrac{8}{\left((s+1)^2+16\right)^2}$

iii) $L\left(t\sin^3 t\right)$ **(1994)** *Ans.* $\dfrac{12s}{(s^2+1)(s^2+9)}\left(\dfrac{1}{s^2+1}+\dfrac{1}{s^2+9}\right)$

iv) $L\left(t\sin 2t\cosh t\right)$ **(1995)** *Ans.* $\dfrac{2s-2}{\left((s-1)^2+4\right)^2}+\dfrac{2s+2}{\left((s+1)^2+4\right)^2}$

v) $L\left(t\cos^2 t\right)$ **(1993)** *Ans.* $\dfrac{1}{2s^2}+\dfrac{1}{2}\cdot\dfrac{s^2-2^2}{\left(s^2+2^2\right)^2}$ vi) $L\left(t^5\cosh t\right)$ *Ans.* $60\left[\dfrac{1}{(s-1)^6}+\dfrac{1}{(s+1)^6}\right]$

vii) $L\left(t\,e^{3t}\sin 4t\right)$ **(97)** *Ans.* $\dfrac{8(s-3)}{\left((s-3)^2+16\right)^2}$ viii) $L\left(t\,e^{3t}\sin t\right)$ **(98, 02)** *Ans.* $\dfrac{2(s-3)}{\left((s-3)^2+1\right)^2}$

ix) $L\left(t\sqrt{1+\sin t}\right)$ *Ans.* $4\dfrac{\left(4s^2+4s-1\right)}{\left(4s^2+1\right)^2}$ x) $L\left(t\,e^{3t}erf\sqrt{t}\right)$ **(2000)** *Ans.* $\dfrac{3s-7}{2(s-3)^2(s-2)^{3/2}}$

xi) $L\left(t\left(\dfrac{\sin t}{e^t}\right)^2\right)$ **(2004)** *Ans.* $\dfrac{2}{(s+2)^2\left((s+2)^2+4\right)}+\dfrac{4}{\left((s+2)^2+4\right)^2}$

xii) If $L\left[f(t)\right]=\dfrac{s+3}{s^2+s+1}$, find : $\left[t\,f(2t)\right]$. **(2004)** *Ans.* $\dfrac{s^2+12s+8}{\left(s^2+2s+4\right)^2}$

xiii) If $L\left[erf\sqrt{t}\right]=\dfrac{1}{s\sqrt{s+1}}$, find $L\left[t\,erf\,3\sqrt{t}\right]$. **(2002)** *Ans.* $\dfrac{9}{2}\cdot\dfrac{s+6}{s^2(s+9)^{3/2}}$

xiv) $L\left(t\,e^{3t}\sin 3t\right)$ **(2004)** *Ans.* $6\cdot\dfrac{(s-3)}{\left((s-3)^2+9\right)^2}$

xv) $L\left(t\,e^{t}\sin 2t\cos t\right)$ **(2003)** *Ans.* $\dfrac{3(s-1)}{\left((s-1)^2+9\right)^2}+\dfrac{s-1}{\left((s-1)^2+1\right)}$

xvi) $L\left(t\sqrt{1-\sin t}\right)$ *Ans.* $4\cdot\dfrac{\left(4s^2-4s-1\right)}{\left(4s^2+1\right)^2}$ xvii) $L\left(t\,e^{-2t}\sinh 4t\right)$ *Ans.* $\dfrac{8(s+2)}{\left((s+2)^2-16\right)^2}$

xviii) $L\left(t\,e^{3t}\sinh 2t\right)$ **(03, 03)** *Ans* $\dfrac{4(s-3)}{\left((s-3)^2-4\right)^2}$ xix) $L\left(t\sqrt{1+\sin 2t}\right)$ *Ans.* $\dfrac{s^2+2s-1}{\left(s^2+1\right)^2}$

xx) $L\left(t\cos(wt-\alpha)\right)$ **(2003)** *Ans.* $\dfrac{\left(s^2-w^2\right)\cos\alpha+2w\,s\sin\alpha}{\left(s^2+w^2\right)^2}$

xxi) $L\left(\left(t+\sin 2t\right)^2\right)$ **(2004)** Also find $L\left(\left(t+\sin t\right)^2\right)$.

Ans. $\dfrac{2}{s^3}+\dfrac{8s}{\left(s^2+4\right)^2}+\dfrac{8}{s\left(s^2+16\right)}$ *Ans.* $\dfrac{2}{s^3}+\dfrac{4s}{\left(s^2+1\right)^2}+\dfrac{2}{s\left(s^2+4\right)}$

xxiii) $L\left(\left(t\sinh 2t\right)^2\right)$ (2003) $Ans. \frac{1}{2}\left[\frac{1}{(s-4)^3}+\frac{1}{(s+4)^3}\right]$

xxiv) Find $L(t^4 \sinh 2t \cosh 2t)$ and hence find $L(t^4 \sinh t \cosh t)$. *Ans*

$192\left(\frac{1}{(2s-4)^5}-\frac{1}{(2s+4)^5}\right)$

xxv) If $L\left[erf\sqrt{t}\right]=\frac{1}{s\sqrt{s+1}}$, find $L\left[t\ erf 2\sqrt{t}\right]$.(2003) $Ans. \frac{3s+8}{s^2(s+4)^{3/2}}$

5. i) $L\left(\frac{1}{t}(1-\cos t)\right)$ (95, 04) $Ans. \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$ ii) $L\left(\frac{1}{t}\left(e^{-at}-e^{-bt}\right)\right)$ (1997) $Ans. \log\left(\frac{s+b}{s+a}\right)$

iii) $L\left(\frac{\sin^2 2t}{t}\right)$ (1993) $Ans. \frac{1}{4}\log\left(\frac{s^2+4^2}{s^2}\right)$

iv) $L\left(\frac{e^{-2t}\sin 2t\cosh t}{t}\right)$ (1996) $Ans. \frac{\pi}{2}-\frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right)-\frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right)$

v) $L\left(\frac{\sin t}{t}\right)$ (1994) $Ans. \cot^{-1}s$ vi) $L\left(\frac{e^{2t}\sin t}{t}\right)$ (1999) $Ans. \cot^{-1}(s-2)$

vii) $L\left(\frac{\cosh 2t\sin 2t}{t}\right)$ (1999) $Ans. \pi-\tan^{-1}\left(\frac{s-2}{2}\right)-\tan^{-1}\left(\frac{s+2}{2}\right)$

viii) $L\left(\frac{2\sin t\sin 2t}{t}\right)$ $Ans. \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)$ ix) $L\left(\frac{\sin t\sin 5t}{t}\right)$ $Ans. \frac{1}{4}\log\left(\frac{s^2+36}{s^2+16}\right)$

x) $L\left(\frac{\sin^2 t}{t}\right)$ (2003) $Ans. \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$ xi) $L\left[\frac{(1-\cos 2t)}{t}\right]$ (2003) $Ans. \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$

xii) Find $L(\sinh 2t/t)$. $Ans. \frac{1}{2}\log\frac{s-2}{s+2}$ xiii) $L\left(\frac{\sin^2 t}{t^2}\right)$ (2005) $Ans. \int_0^\infty \frac{1}{2}\log\frac{\sqrt{s^2+4}}{s}ds$

Type 3A Theory: Laplace Transforms of derivatives and integrals

1. Prove that if $L(f(t))=F(s)$ then $L(f'(t))=sF(s)-f(0)$. **(2004)**.Also show that generally, $L(f^n(t))=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-.....-sf^{n-1}(0)-f^{n-1}(0)$.

2. Prove that if $L(f(t))=F(s)$ then $L\left(\int_0^t f(u)du\right)=\frac{F(s)}{s}$. **(1996)**

3. What is an error function ? Find $L\left(erf\sqrt{t}\right)$? **(96, 97, 03)**

Type 3B Problems on Laplace Transforms of derivatives and integrals

I) 1. Given $f(t)=t+1,\ 0\le t\le 2$ and $f(t)=3,\ t>2$, find $L[f(t)]$, $L[f'(t)]$ and $L[f''(t)]$ **(2003)**

$Ans. \frac{1}{s}+\frac{1}{s^2}\left(1-e^{-2s}\right)$ $Ans. \frac{1}{s}\left(1-e^{-2s}\right)$ $Ans. -e^{-2s}$

2. Given $f(t)=\frac{\sin t}{t}$, find $L[f'(t)]$. **(2002)** $Ans. s\cot^{-1}s-1$.

3. Find Laplace transform of $\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)$. **(2003)** $Ans. s\log\left(\frac{\sqrt{s^2+2^2}}{s}\right)$

4. Find the Laplace transform of $\frac{d}{dt}\left(\frac{\sin t}{t}\right)$. *Ans.* $s\cot^{-1}s-1$

5.Find $L(f(t)$ and hence find $L(f'(t))$ where $f(t)=\begin{Bmatrix} t; & 0\le t\le 3 \\ 6; & t>3 \end{Bmatrix}$. **(2005)**

II) 1. $L\left(\int_0^t u\cos^2 u\,du\right)$ **(1995)** *Ans.* $\frac{1}{2s^3}+\frac{1}{2}\cdot\frac{1}{s\left(s^2+2^2\right)}$

2. $L\left(\int_0^t ue^{-3u}\cos^2 2u\,du\right)$ **(1993)** *Ans.* $\frac{1}{2s(s+3)^2}+\frac{1}{2}\frac{s^2+6s-7}{s\left(s^2+6s+25\right)^2}$

3a. $L\left(\int_0^t u^{-1}e^{-u}\sin u\,du\right)$ **(1996)** *Ans.* $\frac{1}{s}\cot^{-1}(s+1)$

3b. $L\left(\int_0^t u\cosh au\,du\right)$ **(98, 99)** *Ans.* $\frac{s^2+a^2}{s\left(s^2-a^2\right)^2}$

4) $L\left(\int_0^t\frac{1-e^{-au}}{u}du\right)$ **(97, 03)** *Ans.* $\frac{1}{s}\log\left(\frac{s+a}{s}\right)$ 5) $L\left(\int_0^t\frac{\sin u}{u}du\right)$ **(97, 02)** *Ans.* $\frac{1}{s}\cot^{-1}s$

6) $L\left(\int_t^\infty\frac{\cos u}{u}du\right)$ **(1997)** *Ans.* $\frac{1}{2s}\log\left(s^2+1\right)$ 7) $L\left(\int_0^t u^2\sin u\,du\right)$ **(03)** *Ans.* $\frac{6s^2-2}{s\left(s^2+1\right)^3}$

8) $L\left(e^{-4t}\int_0^t u\,\sin 3u\,du\right)$ **(03, 04)** *Ans.* $\frac{6}{\left(s^2+8s+25\right)^2}$

9) $L\left(\cosh t\int_0^t e^u\cosh u\,du\right)$ **(1996)** *Ans.* $\frac{1}{2}\left[\frac{s-2}{\left((s-2)^2-1\right)(s-1)}+\frac{s}{\left(s^2-1\right)(s+1)}\right]$

10) $L\left(\int_0^t ue^{-3u}\sin 4u\,du\right)$ **(2002)** *Ans.* $\frac{8(s+3)}{s\left(s^2+6s+25\right)^2}$

11) $L\left(\int_0^t\frac{1-e^{-au}}{u}du\right)$ **(2003)** *Ans.* $\frac{1}{s}\log\left(\frac{s+a}{s}\right)$ 12) $L\left(\int_0^t\frac{e^u\sin u}{u}du\right)$ **(1997)** *Ans.* $\frac{1}{s}\cot^{-1}(s-1)$

13) $L\left(\int_0^t e^u\frac{\sin 4u}{u}du\right)$ **(2000)** *Ans.* $\frac{1}{s}.\cot^{-1}\left(\frac{s-1}{4}\right)$

$L\left(\int_0^t ue^{-2u}\sin 3u\,du\right)$ **(02)** *Ans.* $\frac{1}{s}\frac{3(2s+4)}{\left(s^2+4s+13\right)^2}$

14a) $L\left(\int_0^t ue^u\sin u\,du\right)$ *Ans.* $\frac{2(s-1)}{s\left((s-1)^2+1\right)^2}$ 15) Find $L\int_0^t\int_0^t\int_0^t t\sin t\,(dt)(dt)(dt)$ *Ans.* $\frac{2}{s^2\left(s^2+1\right)^2}$

Type 4 Particular value of 0 to ∞ integrals using laplace transforms

1. If $J_0(t)=\frac{1}{\pi}\int_0^\pi\cos(t\cos\theta)\,d\theta$, prove that $L\left[J_0(t)\right]=\frac{1}{\sqrt{s^2+1}}$. **(2003)**. Also find value of

$\frac{1}{\pi}\int_0^\infty e^{-t}\left(\int_0^\pi\cos(t\cos\theta)d\theta\right)dt$ *Ans.* $\frac{1}{\sqrt{s^2+1}}$ *Ans.* $\frac{1}{\sqrt{2}}$

2. Evaluate $\int_0^\infty e^{-2t}.\sin^3 t\,dt$. (85, 02) $Ans.\ \frac{6}{65}$

3. Find $L\left(erf\sqrt{t}\right)$ and hence, obtain $\int_0^\infty erf\sqrt{t}\,e^{-t}dt$. (00, 06) $Ans.\ \frac{1}{s\sqrt{s+1}}$ $Ans.\ \frac{1}{\sqrt{2}}$

4. $\int_0^\infty e^{-3t}.\cos^3 t\,dt$. (2003) $Ans.\ \frac{4}{15}$

5. Find $L\left[erf\sqrt{t}\right]$ and then evaluate $\int_0^\infty e^{-2t}erf\left(2\sqrt{t}\right)d\,t$. (03, 03, 05) $Ans.\ \frac{1}{\sqrt{6}}$

6. Find $\int_0^\infty e^{-3t}\sin t.dt$. (2002) $Ans.\ \frac{1}{50}$. 7. Find $\int_0^\infty e^{-t}\sin t\,dt$. (1996) $Ans.\ \frac{1}{2}$

8. If $L\left[J_0(t)\right]=\frac{1}{\sqrt{1+s^2}}$, prove that i) $\int_0^\infty J_0(t)\,dt=1$. ii) $L\left[t\,J_0(at)\right]=\frac{s}{\left(s^2+a^2\right)^{3/2}}$.

iii) $L\left[e^{-bt}J_0(at)\right]=\frac{1}{\sqrt{(s+b)^2+a^2}}$. (1998) iv) $\int_0^\infty t\,e^{-3t}J_0(4t)\,dt=\frac{3}{125}$. (96, 03)

9. Evaluate $\int_0^\infty \frac{t^2\sin 3t}{e^{2t}}d\,t$. (2002) $Ans.\ \frac{18}{2197}$ 10. Evaluate $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt$. $Ans.\ \log\frac{b}{a}$

11. Evaluate $\int_0^\infty \frac{\cos at-\cos bt}{t}dt$. (2004) $Ans.\ \log\frac{b}{a}$ 12. Prove that $\int_0^\infty e^{-t}.\frac{\sin^2 t}{t}dt=\frac{1}{4}\log 5$.

13. Prove that $\int_0^\infty e^{-\sqrt{2}\,t}\sin t\sinh t=\frac{\pi}{8}$. (2002)

14. Find the L.T. of $\frac{e^{-at}-\cos at}{t}$. Hence, $\int_0^\infty \frac{e^{-t}-\cos t}{t\,e^{4t}}dt$. (04) $Ans.\ \log\left(\frac{\sqrt{s^2+a^2}}{s+a}\right)$; $\log\frac{\sqrt{17}}{5}$

15 a). Prove that $\int_0^\infty \left(\frac{\sin 2t+\sin 3t}{t\,e^t}\right)dt=\frac{3\pi}{4}$. (1995) 15 b) Show that $\int_0^\infty \frac{\sin at}{t}dt=\frac{\pi}{2}$. (1997)

16 a). $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt$. (97, 02) $Ans.\ \log 3$ 16 b). $\int_0^\infty \frac{\cos 6t-\cos 4t}{t}dt$. (1997) $Ans.\ \log(2/3)$

17. $\int_0^\infty \frac{e^{-t}\sin t}{t}dt$. (2004) $Ans.\ \pi/4$ 18. Find $\int_0^\infty e^{-t}erf\sqrt{t}dt$. (1997) $Ans.\ \frac{1}{\sqrt{2}}$.

20. Find $\int_0^\infty e^{-t}erf\sqrt{t}dt$. $Ans.\ \frac{1}{\sqrt{6}}$. 21. $\int_0^\infty e^{-t}\left(\int_0^t u^2\sinh u\cosh u\,du\right)dt$ $Ans.\ -\frac{4}{27}$

22. If $\int_0^\infty e^{-2t}\sin(t+a)\cos(t-a)dt=3/8$ then find 'a' using Laplace Transforms. (2004)

23. If $J_0(t)=\frac{1}{\pi}\int_0^\pi \cos(t\sin\theta)d\theta$, prove that $L\left[J_0(t)\right]=\frac{1}{\sqrt{s^2+1}}$. (2006).

24. Using L.T. show that $\int_0^\infty e^{-\sqrt{2}t}\frac{\sinh t\sin t}{t}dt=\frac{\pi}{8}$. (06, 06) 25. Find using L.T. $\int_0^\infty\int_0^t \frac{e^{-t}\sin u}{u}du\,dt$

25a. Evaluate $\int_0^\infty \frac{\cos 4t-\cos 6t}{t}dt$. (2006) $Ans.\ \log\frac{3}{2}$

Inverse Laplace Transforms

Type 1 Elementary inverse laplace transforms using standard formulae

1. If $L\left[f(t)\right]=\dfrac{s+2}{s^2+2}$, find $L\left[f'(t)\right]$. **(2003)** $\quad Ans.\ \dfrac{2(s-2)}{s^2+2}$

2. $L\left[f(t)\right]=\dfrac{s+3}{s^2+4}$, find $L\left[f'(t)\right]$ $\quad Ans.\ \dfrac{3s-4}{s^2+4}$

3. Find L^{-1} of $\dfrac{1}{s^{3/2}}+\dfrac{1}{s^2+9}+\dfrac{1}{s^2-1}+\dfrac{s}{s^2-36}+\dfrac{1}{s+4}$.

4. Find the inverse Laplace transform of the following

i) $\dfrac{3s+4}{s^2+16}$. $Ans.\ 3.\cos 4t+\sin 4t$ $\quad$ ii) $\dfrac{\left(s^2-1\right)^2}{s^5}$ $\quad Ans.\ 1-t^2+\dfrac{t^4}{24}$ $\quad$ iii) $\dfrac{(s-3)}{(s-3)^2-2^2}$ $Ans.$ $e^{3t}\cosh 2t$

iv) $\dfrac{s+2}{s^2-4s+13}$ $Ans.\ e^{2t}.\cos 3t+\dfrac{4}{3}e^{2t}\sin 3t$ $\quad$ v) $\dfrac{4s+12}{s^2+8s+12}$ **(2003)** $Ans.\ 3e^{-6t}+e^{-2t}$

vi) $L^{-1}\left[\dfrac{s+2}{s^2+4s+7}\right]$ **(1993)** $\quad Ans.\ e^{-2t}.\cos\sqrt{3}\,t.$ $\quad$ vii) $L^{-1}\left[\dfrac{2s+3}{s^2+2s+2}\right]$ $Ans.\ e^{-t}\left[2\cos t+\sin t\right]$

viii) $L^{-1}\left[\dfrac{3s+7}{s^2-2s-3}\right]$ **(1999)** $\quad Ans\ 4e^{3t}-e^{-t}$. $\quad$ ix) $\dfrac{s}{s^2+2s+2}$ **(2004)** $Ans.\ e^{-t}\left[\cos t-\sin t\right]$

x) $\dfrac{s}{(2s+1)^2}$ **(2003)** $\quad Ans.\ \dfrac{1}{4}\left(1-\dfrac{t}{2}\right)e^{\frac{-t}{2}}$

Type 2 Inverse Laplace transforms using Partial Fractions

1. Find inverse laplace transforms of the following

i) $\dfrac{s+29}{(s+4)\left(s^2+9\right)}$ **(1999)** $Ans.\ e^{-4t}-\cos 3t+\dfrac{5}{3}\sin 3t$ $\quad$ ii) $\dfrac{s+2}{(s+3)(s+1)^3}$ $Ans.\ \dfrac{1}{8}e^{-3t}+\dfrac{1}{4}\left(t^2+t-\dfrac{1}{2}\right)e^{-t}$

iii) $\dfrac{s^2+2s-4}{\left(s^2+2s+5\right)\left(s^2+2s+2\right)}$ $Ans.\ \dfrac{3}{2}e^{-t}\sin 2t-2e^{-t}\sin t$

iv) $\dfrac{3s+1}{(s+1)\left(s^2+2\right)}$ **(1995)** $\quad Ans.\ \dfrac{-2}{3}e^{-t}+\dfrac{2}{3}\cos\sqrt{2}\,t+\dfrac{7\sqrt{2}}{6}\sin\sqrt{2}\,t$

v) $\dfrac{5s^2-15s-11}{(s+1)(s-2)^2}$ **(1995)** $Ans.\ e^{-t}+4e^{2t}-7te^{2t}$ $\quad$ vi) $\dfrac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}$ $Ans.\ \dfrac{a\sin at-b\sin bt}{a^2-b^2}$

vii) $\dfrac{s}{\left(s^2+a^2\right)\left(s^2+b^2\right)}$ $Ans.\ \dfrac{(\cos at-\cos bt)}{b^2-a^2}$ $\quad$ viii) $\dfrac{5s^2+8s-1}{(s+3)\left(s^2+1\right)}$ **(96, 02)** $Ans.\ 2e^{-3t}+3\cos t-\sin t$

ix) $\dfrac{2s}{s^4+4}$ **(97, 02)** $Ans.\ \sin t\sinh t$. $\quad$ x) $\dfrac{s+2}{s^2(s+3)}$ **(97, 03)** $Ans.\ \dfrac{2}{3}\left(t+\dfrac{1}{3}e^{\frac{-3}{2}t}\sinh\left(\dfrac{3}{2}t\right)\right)$

xi) $\dfrac{s^2+2s+3}{\left(s^2+2s+5\right)\left(s^2+2s+2\right)}$ $Ans.\ \dfrac{e^{-t}}{3}(\sin 2t+\sin t)$ $\quad$ xii) $\dfrac{s+4}{\left(s^2-1\right)(s+1)}$ $Ans.\ \dfrac{5}{4}e^{t}-\dfrac{5}{4}e^{-t}+\dfrac{3}{2}te^{-t}$

xiii) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$ $Ans.\ \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$ xiv) $\frac{1}{s^2(s+2)}$ $Ans.\ \frac{t}{2} - \frac{e^{-t}}{2}\sinh t$

xv) $\frac{1}{s^2(s+3)^2}$ $Ans\ 2e^{\frac{-3}{2}t}\left(\frac{1}{9}t\cosh\left(\frac{3}{2}t\right) - \frac{2}{27}\sinh\left(\frac{3}{2}t\right)\right)$ xvi) $\frac{2s^2-1}{\left(s^2+1\right)\left(s^2+4\right)}$ $Ans.\ \frac{3}{2}\sin 2t - \sin t$

xvii) $\frac{s}{(s-3)\left(s^2+4\right)}$ $Ans.\ \frac{6}{13}e^{\frac{3}{2}t}\cosh\left(\frac{3}{2}t\right) - \frac{6}{13}\cos^2 t + \frac{2}{13}\sin 2t$

xviii) $\frac{s}{(s+1)^2\left(s^2+1\right)}$ **(04)** $Ans.\ \frac{1}{2}\sin t - \frac{1}{2}te^{-t}$ xix) $\frac{s\left(s^2+2\right)}{(s+1)^2\left(s^2+1\right)}$ **(05)** $Ans.\ \frac{1}{2}\sin t - \left(1 - \frac{3}{2}\right)e^{-t}$

xx) $\frac{s^2+16s-24}{\left(s^4+20s^2+64\right)}$ **(06)** $Ans.\ \frac{-4}{3}\cos 4t + \frac{5}{6}\sin 4t + \frac{4}{3}\cos 2t - \frac{7}{6}\sin 2t$

xxi) $\frac{s^2+2s-1}{(s-3)(s^2+2s+5)}$

Type 3 Inverse Laplace transforms using convolution theorem

Theory: State and prove convolution theorem. **(2003)**

1. Find the inverse of the following by using convolution theorem.

i) $\frac{s^2}{\left(s^2+a^2\right)^2}$ $Ans.\ \frac{1}{2a}(\sin at + at\cos at)$ ii) $\frac{s^2}{\left(s^2-a^2\right)^2}$ **(02)** $Ans.\ \frac{1}{2a}(\sinh at + at\cosh at)$

iii) $\frac{1}{(s-3)(s+3)^2}$ $Ans.\ \frac{1}{36}\left(e^{3t} - e^{-3t} - 6te^{-3t}\right)$ iv) $\frac{1}{s^2(s+1)^2}$ **(97,99)** $Ans.\ 2e^{\frac{-t}{2}}\left(t\cosh\left(\frac{t}{2}\right) - 2\sinh\left(\frac{t}{2}\right)\right)$

v) $\frac{s^2}{\left(s^2+1\right)\left(s^2+4\right)}$ $Ans.\ \frac{1}{3}(2\sin 2t - \sin t)$ vi) $\frac{(s+2)^2}{\left(s^2+4s+8\right)^2}$ Ans. $\frac{e^{-2t}}{4}(2t\cos 2t + \sin 2t)$

vii) $\frac{1}{(s+3)\left(s^2+2s+2\right)}$ **(2004)** $Ans.\ \frac{1}{5}\left[e^{-t}(2\sin t - \cos t) + e^{-3t}\right]$

viii) $\frac{1}{(s-2)^4(s+3)}$ **(93, 98, 99)** $Ans.\ \frac{e^{-3t}}{625} - e^{2t}\left[\frac{1}{625} - \frac{t}{125} + \frac{t^2}{50} - \frac{t^3}{30}\right]$

ix) $\frac{1}{\left(s^2+4s+13\right)^2}$ $Ans.\ \frac{e^{-2t}}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$ x) $\frac{1}{(s-2)(s+2)^2}$ $Ans.\ \frac{1}{16}\left[e^{2t} - e^{-2t} - 4te^{-2t}\right]$

xi) $\frac{s}{\left(s^2+4\right)\left(s^2+1\right)}$ $Ans.\ \frac{1}{3}[\cos t - \cos 2t]$ xii) $\frac{s}{\left(s^2+a^2\right)^2}$ **(84, 87, 03)** $Ans.\ \frac{1}{2a}t\sin at$

2. Find the inverse Laplace transform of the following by convolution theorem.

i) $\frac{1}{s\sqrt{s+4}}$ **(2002)** $Ans.\ \frac{1}{2}erf\left(2\sqrt{t}\right)$. ii) $\frac{1}{s}\log\left(\frac{s+3}{s+2}\right)$ $Ans.\ \int_0^t \frac{e^{-2u}-e^{-3u}}{u}du$

iii) $\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)$ **(96)** $Ans.\ \int_0^t \frac{2}{u}(1-\cos u)\,du$. iv) $\frac{s^2}{\left(s^2+2^2\right)^2}$ $Ans.\ \frac{1}{4}(\sin 2t + 2t\cos 2t)$

v) $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$ (2002) $Ans.\ 2e^{\frac{-t}{2}}\left(\frac{1}{5}\sin t\cosh\left(\frac{t}{2}\right)+\frac{3}{5}\cos t\sinh\left(\frac{t}{2}\right)\right)$

vii) $\dfrac{s+3}{(s^2+6s+13)^2}$ (2004) $Ans.\ \frac{1}{4}e^{-3t}\,t\sin 2t$ viii) $\dfrac{s}{s^4+8s^2+16}$ (2003) $Ans.\ \frac{1}{4}t\sin 2t$

ix) $\dfrac{s}{(s^2-a^2)^2}$ (2003) $Ans.\ \dfrac{t\sinh at}{2a}$ x) $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ (2004) $Ans.\ \dfrac{\cos at-\cos bt}{b^2-a^2}$

xi) $L^{-1}\dfrac{2s}{(s^2+1)^2}$ (2004) $Ans.\ t\sin t$

xii) Find $L^{-1}\left(\dfrac{s}{s^4+13s^2+36}\right)$ and hence find $L^{-1}\left(\dfrac{1}{s^4+13s^2+36}\right)$ and $L^{-1}\left(\dfrac{s^2}{s^4+13s^2+36}\right)$. (2005)

$Ans.\ \frac{1}{5}(\cos 2t-\cos 3t)$ $Ans.\ \frac{1}{5}\left(\frac{\sin 2t}{2}-\frac{\sin 3t}{3}\right)$ $Ans.\ \frac{3}{5}\sin 3t-\frac{2}{5}\sin 2t$

3. Using convolution theorem, prove the following

i) $L^{-1}\left[\frac{1}{s}\log\left(a^2+\frac{b^2}{s^2}\right)\right]=\int_0^t\frac{2}{u}\left[1-\cos\left(\frac{b}{a}\right)u\right]du$. (94) ii) $L^{-1}\left[\frac{1}{s}\log\left(\frac{s+a}{s+b}\right)\right]=\int_0^t\frac{e^{-bu}-e^{-au}}{u}du$.

(04)

iii) $L^{-1}\left[\frac{1}{s}\tan^{-1}\frac{2}{s}\right]=\int_0^t\frac{1}{u}\sin 2u\,du$ iv) $L^{-1}\left[\frac{1}{s}\cot^{-1}(s+1)\right]=\int_0^t\frac{1}{u}e^{-u}\sin u\,du$

v) $L^{-1}\left[\frac{1}{s}\log\left(\frac{s+1}{s+2}\right)\right]=\int_0^t\left(\frac{e^{-2u}-e^{-u}}{u}\right)du$ (04) vi) $L^{-1}\left[\frac{1}{s}\log\left(\frac{s+2}{s+3}\right)\right]=\int_0^t\left(\frac{e^{-3u}-e^{-2u}}{u}\right)du$

Type 4A Inverse Laplace transforms using derivatives and integrals of F(s)

1. Find inverse Laplace transform of

i) $\log\left(\frac{s+a}{s+b}\right)$ (93, 96) $Ans.\ -\frac{1}{t}\left[e^{-at}-e^{-bt}\right]$ ii) $\log\left(1+\frac{a^2}{s^2}\right)$ (99, 02) $Ans.\ \frac{2}{t}[1-\cos at]$

iii) $\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)$ (03, 04) $Ans.\ \frac{1}{t}\left[\frac{1}{2}e^{-bt}-2\cos at\right]$ iv) $2\tanh^{-1}s$ (03, 06) $Ans.\ \frac{2}{t}\sinh t$

v) $\tan^{-1}\left(\frac{2}{s^2}\right)$ (95, 06) $Ans.\ \frac{2}{t}\sin t\sinh t$ vi) $\tan^{-1}\left(\frac{2}{s}\right)$ (03, 04) $Ans.\ \frac{1}{t}\sin 2t$

vii) $\cot^{-1}(s+1)$ $Ans.\ \frac{1}{t}e^{-t}\sin t$ viii) $\log\left[\frac{s^2-4}{(s-3)^2}\right]$ (2003) $Ans.\ \frac{2}{t}\left[e^{3t}-\cosh 2t\right]$

ix) $\frac{1}{2}\log\left[1-\frac{a^2}{s^2}\right]$ (1999) $Ans.\ \frac{1}{t}(1-\cosh at)$ x) $\log\sqrt{\frac{s^2+a^2}{s^2}}$ (95, 98, 04) $Ans.\ \frac{1}{t}(1-\cos at)$

xi) $\log\left[\frac{s^2+a^2}{(s+b)^2}\right]$ (03, 04) $Ans.\ \frac{2}{t}\left(e^{-bt}-\cos at\right)$ xii) $\log\left[1+\frac{a}{s}\right]$ (2003) $Ans.\ \frac{1}{t}\left(1-e^{-at}\right)$

xiii) $\tan^{-1}\frac{1}{s}$ (2003) $Ans.\ \frac{1}{2t}\sin 2t$ xiv) $\cot^{-1}s$ (96, 98) $Ans.\ \frac{1}{t}\sin t$

xvi) $\log\left[1-\frac{1}{s^2}\right]$ (1997) $Ans.\ \frac{2}{t}(1-\cosh t)$ xvii) $\tan^{-1}\frac{s+a}{b}$ (04, 05) $Ans.\ \frac{e^{-at}}{-t}\sin bt$

xviii) $\log\left[\frac{(s-2)^2}{s^2+1}\right]$ (2005) $Ans.\ \frac{2\left(\cos t - e^{2t}\right)}{t}$ xix) $\cot^{-1}\frac{2}{s^2}$ (2006) $Ans.\ \frac{2}{t}\sin t \sinh t$

2. Find inverse laplace transform of i) $\frac{s}{\left(s^2+a^2\right)^2}$ (2000) $Ans.\ \frac{t}{2a}\sin at$

ii) $\frac{s+3}{\left(s^2+6s+13\right)^2}$ (2004) $Ans.\ \frac{t}{4}e^{-3t}\sin 2t$

Type 4B Inverse Laplace transforms using effect of multiplication of s

i) $\frac{s^2}{\left(s^2+a^2\right)^2}$ $Ans.\ \frac{1}{2a}[\sin at + at\cos at]$ ii) $\frac{s^2}{(s+a)^3}$ $Ans.\ \frac{1}{2}\left[a^2t^2-4at+2\right]e^{-at}$

iii) $\frac{(s+2)^2}{\left(s^2+4s+8\right)^2}$ $Ans.\ \frac{1}{4}e^{-2t}(2t\cos 2t+\sin 2t)$

Type 4C Inverse Laplace transforms using effect of division by s

1. Find i) $L^{-1}\frac{1}{s^3\left(s^2+a^2\right)}$ (1993) $Ans.\ \frac{1}{a^2}\left[\frac{t^2}{2}\right]+\frac{1}{a^4}[\cos at - 1]$

ii) $L^{-1}\left[\frac{1}{s^3\left(s^2+1\right)}\right]$ $Ans.\ \frac{t^2}{2}+\cos t-1$ iii) $\frac{54}{s^3(s-3)}$ (2003) $Ans.\ \frac{1}{8}\left[1-\left(2t^2+2t+1\right)e^{-2t}\right]$

Type Misc

1. Find $\int_0^\infty \cos\left(tx^2\right)dx$ and hence, find $\int_0^\infty \cos x^2 .\, dx.$ $Ans.\ \frac{\sqrt{\pi}}{2\sqrt{2}}.\frac{1}{\sqrt{t}}\ ;\ \frac{1}{2}\sqrt{\frac{\pi}{2}}.$

2. Find $\int_0^\infty \sin\left(tx^2\right)dx$ and hence, find $\int_0^\infty \sin x^2 dx.$ (2003) $Ans.\ \frac{\sqrt{\pi}}{2\sqrt{2}}.\frac{1}{\sqrt{t}};\ \frac{1}{2}\sqrt{\frac{\pi}{2}}.$

3. Find $\int_0^\infty e^{-tx^2}dx$ and hence, find $\int_0^\infty e^{-x^2}dx$. (2004) $Ans.\ \frac{1}{2}\sqrt{\frac{\pi}{t}}.\ ;\ \frac{\sqrt{\pi}}{2}.$

4. Find i) $\int_0^\infty \frac{\sin x}{x^n}dx \quad (n<1)$ ii) $\int_0^\infty x\sin x^3 dx$ iii) $\int_0^\infty x\cos x^3 dx$ iv) $\int_0^\infty \frac{\sin^2 x}{x^2}dx$.

Laplace Transforms of Special Functions

Type 1 Laplace transforms for periodic functions

Theory: If $f(t)$ is a periodic function of period a, prove that $L[f(t)]=\frac{1}{1-e^{-as}}\int_0^a e^{-st}f(t)\,dt.$

1. $f(t)=1$, for $0\le t<a$ and $f(t)=-1, a<t<2a$ and $f(t)$ is periodic with period 2a. *Ans.* $\frac{1}{s}\tanh\left\{\frac{as}{2}\right\}$

2. Find Laplace transform of $f(t)=a\sin pt,\ 0<t<\pi/p, f(t)=0,\ \pi/p<t<2\pi/p$ and $f(t)=f(t+2\pi/p)$. **(1995)** *Ans.* $\frac{ap}{1-e^{-(s\pi/p)}}\cdot\frac{1}{s^2+p^2}$

3. Find the Laplace transform of $f(t)=|\sin pt|,\ t\ge 0$. **(2003)** *Ans.* $\frac{p}{s^2+p^2}.\coth\left(\frac{\pi s}{2p}\right)$

4. Find Laplace transform of $f(t)=\sin 2t, 0<t<\pi/2, f(t)=0, \pi/2<t<\pi$ and $f(t)=f(t+\pi)$. **(95)**

   Ans. $\frac{2}{1-e^{-\pi s/2}}\cdot\frac{1}{s^2+2^2}$

5. Find $Lf(t)$ where, $f(t)=t, 0<t<1; f(t)=0, 1<t<2\ \&\ f(t+2)=f(t)$ for $t>0$.

   Ans. $\frac{1-e^{-s}-se^{-s}}{s^2\left(1-e^{-2s}\right)}$

6. Find the Laplace transform of

   i) $f(t)=\frac{t}{a}, 0<t\le a; f(t)=\frac{1}{a}(2a-t),\ a<t<2a$ where $f(t)=f(t+2a)$ **(2004)** *Ans.* $\frac{1}{as^2}\tanh\left(\frac{as}{2}\right)$

   ii) $f(t)=\begin{cases} E, & 0\le t\le (p/2) \\ -E, & (p/2)\le t\le p\end{cases}, f(t+p)=f(t)$. **(02, 03)** *Ans.* $\frac{E}{s}\tanh\left(\frac{sp}{4}\right)$

Type 2 Laplace transforms for unit step functions

Theory: If $f(t-a)\cdot H(t-a)=\begin{cases}0 & t<a\\ f(t-a) & t>a\end{cases}$ then prove that

$L[f(t-a)H(t-a)]=e^{-as}L[f(t)]=e^{-as}\phi(s)$ where $\phi(s)=L[f(t)]$. **(96, 04)**

Theory: Define heaviside unit step function. Obtain its laplace transform. Hence find $L(e^{t}\sin t\,H(t-\pi))$. **(2006)**

1. Find the Laplace transform of $\sin t.H\left(t-\frac{\pi}{2}\right)-H\left(t-\frac{3\pi}{2}\right)$. **(1996)** *Ans.* $\frac{e^{-\pi s/2}s}{s^2+1}-\frac{e^{-3\pi s/2}}{s}$

2. Find Laplace transform of $\left(1+2t-3t^2+4t^3\right)H(t-2)$. **(1998)** *Ans.* $e^{-2s}\left[\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right]$

3. Using Laplace transform evaluate $\int_0^\infty e^{-t}\left(1+2t-t^2+t^3\right)H(t-1)\,dt$ **(2003)** *Ans.* $\frac{16}{e}$.

4a. Write $f(t)$ in terms of heaviside function and find its laplace transform;

$$f(t)=\begin{cases}\sin t; & 0<t<\pi \\ \sin 2t; & \pi<t<2\pi \\ \sin 3t; & t>2\pi\end{cases} \text{ (2005)}$$

4b. Find $L\left[t^2 H(t-2)\right]$ (1997) *Ans.* $e^{-2s}\left[\frac{4}{s}+\frac{4}{s^2}+\frac{2}{s^3}\right]$

4c. Express $f(t)=\sin 2t;\ 2\pi<t<4\pi$ and 0 , otherwise , in terms of unit step function and hence find its L.T. (06)

5. Find the inverse Laplace transform of i) $\frac{e^{4-3s}}{(s+4)^{5/2}}$ *Ans.* $\frac{4}{3\sqrt{\pi}}e^{-4(t-4)}(t-3)^{3/2}H(t-3)$

ii) $\frac{(s+1)e^{-s}}{s^2+s+1}$ (1997) *Ans.* $e^{-(t-1)/2}\left[\cos\left(\sqrt{3}(t-1)/2\right)+\frac{1}{\sqrt{3}}\sin\left(\sqrt{3}(t-1)/2\right)\right]H(t-1)$

iii) $\frac{8e^{-3s}}{s^2+4}$ (1998) *Ans.* $4\sin 2(t-3)H(t-3)$

iv) $\frac{e^{-5s}}{(s-2)^4}$ (1999) *Ans.* $\frac{1}{3!}e^{2(t-5)}(t-5)^3 H(t-5)$

v) $\frac{e^{-as}}{(s+b)^{5/2}}$ (02) *Ans.* $\frac{4e^{-b(t-a)}(t-a)^{3/2}H(t-a)}{3\sqrt{\pi}}$

6. Find the inverse Laplace transform of i) $\frac{e^{-\pi s}}{s^2-2s+2}$ *Ans.* $e^{(t-\pi)}\sin(t-\pi)H(t-\pi)$

ii) $\frac{e^{-3s}}{(s+4)^3}$ (96) *Ans.* $e^{-4(t-3)}\frac{(t-3)^2}{2}H(t-3)$

iii) $\frac{e^{-2s}}{s^2+8s+25}$ (99) *Ans.* $\frac{1}{3}e^{-4(t-2)}\sin 3(t-2)H(t-2)$

iv) $\frac{e^{-4s}}{\sqrt{2s+7}}$ (03) *Ans.* $\frac{e^{-7(t-4)/2}}{\sqrt{2\pi(t-4)}}H(t-4)$

v) $\frac{s.e^{-2s}}{s^2+2s+2}$ (03) *Ans.* $e^{-(t-2)}\left[\cos(t-2)-\sin(t-2)\right]H(t-2)$

vi) $L^{-1}\left(\frac{(s+1)e^{-2s}}{s^2+2s+2}\right)$ (2006) vii) $L^{-1}\left(\frac{(s+2)e^{-2s}}{2s^2+2s+1}\right)$ (2005)

Type 3- Laplace Transform of Dirac-Delta Functions

Theory: i) Find $L\left[\delta(t-a)\right]$ where $\delta(t-a)\begin{cases}=0 & \text{for } t<a \\ =1/\varepsilon & \text{for } a<t<a+\varepsilon. \\ =0 & \text{for } t>a+\varepsilon\end{cases}$ (1998)

ii) Define Dirac-Delta Function and obtain its laplace transform. (99, 04)

1. Find laplace transforms for i) $\sin t.\delta(t-\pi/2)-t^2.\delta(t-2)$ ii) $t^4H(t-2)+t^2\delta(t-2)$
iii) $\sin(2t).\delta(t-2)$ iv) $tH(t-2)+t^2\delta(t-4)$

Type 4 Laplace Transform of Differential equations

1. Using Laplace transform solve the following differential equation $\frac{dx}{dt}+x=\sin\omega t, x(0)=2$. **(1993)**

   Ans. $\frac{1}{1+\omega^2}\left[-e^{-t}-\omega\sin\omega t+\cos\omega t\right]+2e^{-t}$

2. Solve $\left(D^2-3D+2\right)y=4e^{2t}$, with $y(0)=-3$ and $y'(0)=5$. *Ans.* $y=-7e^t+4e^{2t}+4t\,e^{2t}$

3. Use Laplace transform to solve, $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+8y=1$ where $y(0)=0, y'(0)=1$. **(1994)**

   Ans. $y=\frac{1}{8}-\frac{1}{8}e^{-2t}\cos 2t-\frac{1}{8}e^{-2t}\sin 2t$.

4. Using Laplace transform solve $\frac{d^2y}{dt^2}+y=t,\ y(0)=1,\ y'(0)=0$. **(1995)** *Ans.* $t-\cos t-\sin t$.

5. Solve by using Laplace transform $\left(D^2+2D+5\right)y=e^{-t}\sin t$, where $y(0)=0, y'(0)=0$. **(95, 03, 06)**

   Ans. $\frac{e^{-t}}{3}(\sin 2t+\sin t)$.

6. Solve using Laplacetransform $\frac{d^2y}{dt^2}+9y=18t$, given that $y(0)$ and $y(\pi/2)=0$. **(96, 98)**

   Ans. $y=2t+\pi\sin 3t$.

7. Solve the following equation by using Laplace transform $\frac{dy}{dt}+2y+\int_0^t y\,dt=\sin t$, given that $y(0)=1$. **(98, 99)** *Ans.* $y=e^{-t}-\frac{3}{2}e^{-t}t+\frac{1}{2}\sin t$.

8. Solve using laplace transforms $\frac{d^2y}{dt^2}+4y=f(t)$ where $y(0)=0$ and $y'(0)=1$ where $f(t)=H(t-2)$. **(2002)** *Ans.* $\frac{1}{2}\sin 2t+\frac{1}{4}H(t-2)-\frac{1}{4}\cos 2(t-2)H(t-2)$

9. Solve using L.T. i) $\left(D^2+D\right)y=t^2+2t$ at $t=0=y=4$ and $Dy=2$. *Ans.* $y=\frac{t^3}{3}+2e^{-t}+2$

   ii) $\left(D^2-2D+1\right)x=e^t$ with the conditions $x=2, Dx=-1$ at $t=0$. *Ans.* $x=2e^t-3e^t\,.\,t+\frac{t^2}{2}e^t$

   iii) $(D+1)^2 y=6t\,e^{-t}$ with $y(0)=2, y'(0)=5$. **(1994)** *Ans.* $y=e^{-t}\left(t^3+7t+2\right)$

10. Solve $\frac{dx}{dt}+y=\sin t, \frac{dy}{dt}+x=\cos t$ where $x=0, y=2$ at $t=0$. **(1993)** *Ans.* $\sin t+2\cosh t$

11. Solve using laplace transforms $\frac{d^2y}{dt^2}-4y=3e^t$ where $y(0)=0$ and $y'(0)=3$. **(04, 06)**

12. Solve $y'''-3y''+3y'-y=e^t t^2$, where $y(0)=y'(0)=0$; $y''(0)=-2$ by using L.T. **(05, 06)**

13. Solve $y'''-2y''+5y'=0$ where $y(0)=y'(0)=0$; $y''(0)=1$ by using L.T. **(2006)**

LAPLACE TRANSFORMS

Some Examination Questions

1. Find the Laplace transform of $f(t)=\cos t$, for $0<t<\pi$ and $f(t)=\sin t$, for $t>\pi$.

Sol.: $L[f(t)]=\int_0^\infty e^{-st}f(t)dt=\int_0^\pi e^{-st}\cos t\,dt+\int_\pi^\infty e^{-st}\sin t\,dt$. But $\int e^{ax}\cos bx\,dx=\frac{1}{(a^2+b^2)}.e^{ax}(a\cos bx+b\sin bx)$ and

$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{(a^2+b^2)}(a\sin bx-b\cos bx)$ ...(A)

$\therefore Lf(t)=\frac{1}{s^2+1}.\left[e^{-st}(-s\cos t+\sin t)\right]_0^\pi+\frac{1}{s^2+1}.\left[e^{-st}(-s\sin t-\cos t)\right]_\pi^\infty=\frac{1}{s^2+1}.\left[e^{-s\pi}(s)-(-s)\right]+\frac{1}{s^2+1}\left[-e^{-s\pi}\right]$

$Lf(t)=\frac{1}{s^2+1}.\left[s+(s-1)e^{-s\pi}\right]$.

2. Find the Laplace transform of $\cos t\cos 2t\cos 3t$.

Sol.: $L(\cos t\cos 2t\cos 3t)=L\left[\frac{1}{2}(\cos 3t+\cos t)\cos 3t\right]=\frac{1}{2}L\left[\cos^2 3t+\cos 3t\cos t\right]=\frac{1}{2}L\left[\frac{1}{2}(1+\cos 6t)+\frac{1}{2}\{\cos 4t\cos 2t\}\right]$

$=\frac{1}{4}L[1+\cos 2t+\cos 4t+\cos 6t]=\frac{1}{4}L[L(1)+L(\cos 2t)+L(\cos 4t)+L(\cos 6t)]=\frac{1}{4}\left[\frac{1}{s}+\frac{s}{s^2+2^2}+\frac{s}{s^2+4^2}+\frac{s}{s^2+6^2}\right]$

3. Find the Laplace transform of $\sin\sqrt{t}$

Sol.: Since, $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-.....$ then $\sin\sqrt{t}=t^{1/2}-\frac{t^{3/2}}{3!}+\frac{t^{5/2}}{5!}-.....$ $\therefore L[\sin\sqrt{t}]=L(t^{1/2})-\frac{1}{3!}L(t^{3/2})+\frac{1}{5!}L(t^{5/2})+\_\_$

But $L(t^n)=\frac{\overline{|n+1}}{s^{n+1}}$ and $\overline{|n}=n\overline{|n-1}$, $\overline{|1/2}=\sqrt{\pi}$. $\therefore L[\sin\sqrt{t}]=\frac{\overline{|3/2}}{s^{3/2}}-\frac{1}{3!}.\frac{\overline{|5/2}}{s^{5/2}}+\frac{1}{5!}.\frac{\overline{|7/2}}{s^{7/2}}-.....$

$L[\sin\sqrt{t}]=\frac{(1/2)\overline{|1/2}}{s^{3/2}}-\frac{1}{3!}.\frac{(3/2)(1/2)\overline{|1/2}}{s^{5/2}}+\frac{1}{5!}.\frac{(5/2)(3/2)(1/2)\overline{|1/2}}{s^{7/2}}=\frac{\overline{|1/2}}{2s^{3/2}}\left[1-\left(\frac{1}{2^2s}\right)+\frac{1}{2!}\left(\frac{1}{2^2s}\right)^2-.....\right]$ $\left[\because\overline{|n}=n\overline{|n-1}\right]$

$L[\sin\sqrt{t}]=\frac{\sqrt{\pi}}{2s^{3/2}}.e^{-1/(4s)}$ $\left[\because\overline{|1/2}=\sqrt{\pi}\right]$

4. If $J_0(t)=\frac{1}{\pi}\int_0^\pi\cos(t\cos\theta)d\theta$, prove that $L[J_0(t)]=\frac{1}{\sqrt{s^2+1}}$. Hence, evaluate $\frac{1}{\pi}\int_0^\infty e^{-t}\left[\int_0^\pi\cos(t\cos\theta)\,d\theta\right]dt$.

Sol.: We have $J_0(t)=\frac{1}{\pi}\int_0^\pi\cos(t\cos\theta)d\theta=\frac{2}{\pi}\int_0^{\pi/2}\cos(t\cos\theta)d\theta$

Taking Laplace transforms of both sides.

$L[J_0(t)]=\frac{2}{\pi}\int_0^\infty e^{-st}\left[\int_0^{\pi/2}\cos(t\cos\theta)d\theta\right]dt=\frac{2}{\pi}\int_0^{\pi/2}\left[\int_0^\infty e^{-st}\cos(t\cos\theta)d\theta\right]dt=\frac{2}{\pi}\int_0^{\pi/2}[L\cos(t\cos\theta)]\,dt$

$L[J_0(t)]=\frac{2}{\pi}\int_0^{\pi/2}\frac{s}{s^2+\cos^2\theta}\,d\theta=\frac{2}{\pi}\int_0^{\pi/2}\frac{\sec^2\theta}{s^2\sec^2\theta+1}.d\theta=\frac{2}{\pi}\int_0^{\pi/2}\frac{\sec^2\theta}{(s^2+1)+s^2\tan^2\theta}d\theta$

Put $s\tan\theta=t\Rightarrow s\sec^2\theta d\theta=dt$. When $\theta=0$, $t=0$; when $\theta=\pi/2$, $t=\infty$.

$\therefore L[J_0(t)]=\frac{2}{\pi}\int_0^\infty\frac{dt}{t^2+(s^2+1)}=\frac{2}{\pi}.\frac{1}{\sqrt{s^2+1}}\left[\tan^{-1}\left(\frac{t}{\sqrt{s^2+1}}\right)\right]_0^\infty=\frac{2}{\pi}.\frac{1}{\sqrt{s^2+1}}.\left[\frac{\pi}{2}-0\right]=\frac{1}{\sqrt{s^2+1}}$

By definition of Laplace transform this means $\int_0^\infty e^{-st}J_0(t)dt=\frac{1}{\sqrt{s^2+1}}$ i.e. $\int_0^\infty e^{-st}\left[\frac{1}{\pi}\int_0^\pi\cos(t\cos\theta)d\theta\right]d\,t=\frac{1}{\sqrt{s^2+1}}$

Putting $s=1$, we get $\frac{1}{\pi}\int_0^\infty e^{-t}\left[\int_0^\infty\cos(t\cos\theta)d\theta\right]dt=\frac{1}{\sqrt{1+1}}=\frac{1}{\sqrt{2}}$.

5. Find $L\{erf\sqrt{t}\}$ and then evaluate $\int_0^\infty e^{-2t}erf(2\sqrt{t})dt$.

Sol.: We know that $L\{erf\sqrt{t}\}=\frac{1}{s\sqrt{s+1}}$. But $L\{f(at)\}=\frac{1}{a}\phi\left[\frac{s}{a}\right]$. $\therefore L\{erf(2\sqrt{t})\}=L\{erf.(\sqrt{4t})\}$

$=\frac{1}{4}.\frac{1}{(s/4)\sqrt{(s/4)+1}}=\frac{2}{s\sqrt{s+4}}$

$\therefore \int_0^\infty e^{-st} erf\left(2\sqrt{t}\right) dt = \frac{2}{s\sqrt{s+4}}$. Putting $s=2$: $\int_0^\infty e^{-2t} erf\left(2\sqrt{t}\right) dt = \frac{2}{2\sqrt{2+4}} = \frac{1}{\sqrt{6}}$.

6. Show that $L\left[\sinh(t/2)\sin\left(\sqrt{3}\,t/2\right)\right] = \frac{\sqrt{3}}{2}.\frac{s}{s^4+s^2+1}$.

Sol.: We have, $\sinh(t/2).\sin\left(\sqrt{3}t/2\right) = \left(\frac{e^{t/2}-e^{-t/2}}{2}\right).\sin\frac{\sqrt{3}}{2}t$. Now $L\sin\left(\frac{\sqrt{3}}{2}t\right) = \frac{\sqrt{3}/2}{s^2+(3/4)}$. By first shifting theorem,

$$\therefore L\left[e^{t/2}.\sin\left(\frac{\sqrt{3}}{2}t\right)\right] = \frac{\sqrt{3}/2}{\left[s-(1/2)\right]^2+3/4} = \frac{\sqrt{3}/2}{s^2+1-s}\text{; also } L\left[e^{-t/2}.\sin\left(\frac{\sqrt{3}}{2}t\right)\right] = \frac{\sqrt{3}/2}{\left[s+(1/2)\right]^2+3/4} = \frac{\sqrt{3}/2}{s^2+1+s}.$$

$$\therefore L\left[\sinh(t/2)\sin\left(\sqrt{3}t/2\right)\right] = \frac{1}{2}\left[\frac{\sqrt{3}/2}{(s^2+1)-s} - \frac{\sqrt{3}/2}{s^2+1+s}\right] = \frac{1}{2}.\frac{\sqrt{3}}{2}\left[\frac{(s^2+1+s)-(s^2+1-s)}{(s^2+1)^2-s^2}\right] = \frac{\sqrt{3}}{2}.\frac{s}{s^4+s^2+1}.$$

7. Find the Laplace transform of the following (i) $\frac{\cos 2t\ \sin t}{e^t}$ (ii) $e^t\sin 2t\sin 3t$

Sol.: (i) $\cos 2t\sin t = \frac{1}{2}.2\sin t\cos 2t = \frac{1}{2}\left[\sin(1+2)t+\sin(1-2)t\right] = \frac{1}{2}\left[\sin 3t - \sin t\right]$

$\therefore L(\cos 2t\sin t) = \frac{1}{2}\left[L(\sin 3t)-L(\sin t)\right] = \frac{1}{2}.\left[\frac{3}{s^2+9}-\frac{1}{s^2+1}\right]$. Now by shifting theorem,

$$L\left[e^{-t}(\cos 2t\sin t)\right] = \frac{1}{2}\left[\frac{3}{(s+1)^2+9}-\frac{1}{(s+1)^2+1}\right] = \frac{1}{2}\left[\frac{3}{s^2+2s+10}-\frac{1}{s^2+2s+2}\right] = \frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}.$$

(iii) $\sin 2t\sin 3t = \frac{1}{2}.2\sin 3t\sin 2t = -\frac{1}{2}\left[\cos 5t - \cos t\right]$.

$$\therefore L(\sin 2t\sin 3t) = -\frac{1}{2}\left[L(\cos 5t)-L(\cos t)\right] = -\frac{1}{2}\left[\frac{s}{s^2+5^2}-\frac{s}{s^2+1^2}\right]$$

Now by shifting theorem, $L\left[e^t\sin 2t\sin 3t\right] = -\frac{1}{2}\left[\frac{s-1}{(s-1)^2+5^2}-\frac{s-1}{(s-1)^2+1^2}\right] = \frac{1}{2}\left[\frac{s-1}{s^2-2s+2}-\frac{s-1}{s^2-2s+26}\right]$

$$L\left[e^t\sin 2t\sin 3t\right] = \frac{12(s-1)}{(s^2-2s+2)(s^2-2s+26)}$$

8. Find Laplace transform of $\sin 2t\cos t\cosh 2t$.

Sol.: $\sin 2t\cos t = \frac{1}{2}.2\sin 2t\cos t = \frac{1}{2}\left[\sin 3t+\sin t\right]$. $\cosh 2t = \frac{e^{2t}+e^{-2t}}{2}$. $\therefore \sin 2t\cos t\cos h2t = \frac{1}{2}\left(e^{2t}+e^{-2t}\right)(\sin 3t+\sin t)$ ....(1)

$\therefore L(\sin 3t) = \frac{3}{s^2+9}$. $\therefore L\left[e^{2t}\sin 3t\right] = \frac{3}{(s-2)^2+9}$, $L\left[e^{-2t}\sin 3t\right] = \frac{3}{(s+2)^2+9}$

$$\therefore L\left(e^{2t}\sin 3t\right)+L\left(e^{-2t}\sin 3t\right) = 3\left[\frac{1}{(s-2)^2+9}+\frac{1}{(s+2)^2+9}\right] = \frac{3.2(s^2+13)}{s^4+10s^2+13^2}\text{ ....(2)}$$

Now $L(\sin t) = \frac{1}{s^2+1}$. $\therefore L\left(e^{2t}\sin t\right) = \frac{1}{(s-2)^2+1}$, $L\left(e^{-2t}\sin t\right) = \frac{1}{(s+2)^2+1}$.

$\therefore L\left(e^{2t}\sin t\right)+L\left(e^{-2t}\sin t\right) = \frac{2(s^2+5)}{s^4-6s^2+5^2}$ ...(3). From (1), (2), (3), we get

$$L\left[\sin 2t\cos t\cosh 2t\right] = \frac{3(s^2+13)}{s^4+10s^2 13^2}+\frac{s^2+5}{s^4-6s^2+5^2}$$

9. If $L\left[erf\sqrt{t}\right] = \frac{1}{s\sqrt{s+1}}$, find $L\left[t\ erf 2\sqrt{t}\right]$.

Sol.: By change of scale property if $L\left[f(t)\right] = \phi(s)$, then $L\left[f(at)\right] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$ .Since, $L\left[erf\sqrt{t}\right] = \frac{1}{s\sqrt{s+1}}$,

$L\left[erf\,2\sqrt{t}\right] = L\left[erf\sqrt{4t}\right] = \frac{1}{4}\frac{1}{(s/4)\sqrt{(s/4)+1}} = \frac{2}{s\sqrt{s+4}} = \phi(s)$. By the effect of multiplication of t

$$L\left[t\,f(t)\right] = -\phi'(s) = -\frac{d}{ds}\left[2\left(s^3+4s^2\right)^{-1/2}\right] = -2.\left(-\frac{1}{2}\right)\left(s^3+4s^2\right)^{-3/2}\left(3s^2+8s\right).$$

$$\therefore L\left[t\,erf\,2\sqrt{t}\right] = \frac{s(3s+8)}{s^3(s+4)^{3/2}} = \frac{3s+8}{s^2(s+4)^{3/2}}$$

10. Find the Laplace transforms of (i) $t\,e^{-4t}.\sin 3t$ (ii) $t^2e^{-t}\sin 4t$.

Sol.: (i) $L\left[t.e^{-4t}\sin 3t\right] = (-1)\frac{d}{ds}L\left[e^{-4t}\sin 3t\right] = -\frac{d}{ds}\left[\frac{3}{(s+4)^2+9}\right]$. By shifting theorem, $= -\frac{d}{ds}\left[\frac{3}{s^2+8s+25}\right] = \frac{6(s+4)}{(s^2+8s+25)}$

(ii) $L\left[t^2.e^{-t}\sin 4t\right] = (-1)^2.\frac{d^2}{ds^2}L\left[e^{-t}\sin 4t\right] = \frac{d^2}{ds^2}\left[\frac{4}{(s+1)^2+16}\right] = \frac{d^2}{ds^2}\left[\frac{4}{s^2+2s+17}\right] = \frac{8\left(3s^2+6s-13\right)}{\left(s^2+2s+17\right)^3}$

11. Find the Laplace transforms of the following (i) $t\sin^3 t$ (ii) $t^5\cosh t$

Sol.: (i) $\sin 3t = 3\sin t - 4\sin^3 t$. $L\left[\sin^3 t\right] = \frac{1}{4}\left[L(3\sin t) - L(\sin 3t)\right] = \frac{1}{4}.\left[\frac{3}{s^2+1} - \frac{3}{s^2+9}\right]$. $\therefore L\left[t\sin^3 t\right] = -\frac{3}{4}\frac{d}{ds}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$

$$= -\frac{3}{4}\left[-\frac{2s}{\left(s^2+1\right)^2} + \frac{2s}{\left(s^2+9\right)^2}\right] = \frac{3s}{2}\left[\frac{1}{\left(s^2+1\right)^2} - \frac{1}{\left(s^2+9\right)^2}\right] = \frac{3s}{2}\left[\frac{s^4+18s^2+81-s^4-2s^2-1}{\left(s^2+1\right)^2\left(s^2+9\right)^2}\right] = \frac{3s}{2}.\frac{16(s+5)}{\left(s^2+1\right)^2\left(s^2+9\right)^2}$$

$$L\left[t\sin^3 t\right] = 24.\frac{s(s+5)}{\left(s^2+1\right)^2\left(s^2+9\right)^2}.$$

(ii) $L\left(t^5\cosh t\right) = L\left[t^5\left(\frac{e^t+e^{-t}}{2}\right)\right] = \frac{1}{2}L\,t^5\left(e^t\right) + \frac{1}{2}L\,t^5\left(e^{-t}\right) = \frac{1}{2}.(-1)^5.\frac{d^5}{ds^5}\left(\frac{1}{s-1}\right) + \frac{1}{2}.(-1)^5.\frac{d^5}{ds^5}\left(\frac{1}{s+1}\right)$

But $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$, $\frac{d^2}{dx^2}\left(\frac{1}{x}\right) = \frac{2}{x^3}$; $\frac{d^3}{dx^3}\left(\frac{1}{x}\right) = -\frac{2.3}{x^4}$, $\frac{d^4}{dx^4}\left(\frac{1}{x}\right) = \frac{2.3.4}{x^5} = \frac{4!}{x^5}$; $\frac{d^5}{dx^5}\left(\frac{1}{x}\right) = -\frac{5!}{x^6}$.

$$\therefore L\left(t^5\cosh t\right) = -\frac{1}{2}.\frac{(-5!)}{(s-1)^6} - \frac{1}{2}.\frac{(-5!)}{(s+1)^6} = 60\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right].$$

12. Find the Laplace transforms of the following (i) $t\sqrt{1+\sin t}$ (ii) $t\left(\frac{\sin t}{e^t}\right)^2$

(i) we have $\sqrt{1+\sin t} = \sqrt{\left[\sin^2(t/2)+\cos^2(t/2)+2\sin(t/2)\cos(t/2)\right]} = \sqrt{\left[\sin(t/2)+\cos(t/2)\right]^2} = \sin(t/2)+\cos(t/2)$

$$\therefore L\sqrt{1+\sin t} = L\left[\sin(t/2)+\cos(t/2)\right] = \frac{1/2}{s^2+(1/2)^2} + \frac{s}{s^2+(1/2)^2} = \frac{1}{2}.\frac{4}{\left(4s^2+1\right)} + \frac{4s}{\left(4s^2+1\right)} = \frac{4s+2}{\left(4s^2+1\right)} = \frac{2(2s+1)}{\left(4s^2+1\right)}$$

$$\therefore L\left[t\sqrt{1+\sin t}\right] = -\frac{d}{ds}\left[\frac{2(2s+1)}{\left(4s^2+1\right)}\right] = -2\left[\frac{\left(4s^2+1\right)2-(2s+1)8s}{\left(4s^2+1\right)^2}\right] = -2\frac{\left[-8s^2-8s+2\right]}{\left(4s^2+1\right)} = 4\frac{\left(4s^2+4s-1\right)}{\left(4s^2+1\right)^2}$$

(ii) We have $t\left(\frac{\sin t}{e^t}\right)^2 = t.e^{-2t}\sin^2 t = te^{-2t}\left[\frac{1-\cos 2t}{2}\right] = \frac{1}{2}te^{-2t}\left[1-\cos 2t\right]$. Now $L(1-\cos 2t) = L(1) - L(\cos 2t)$

$= \frac{1}{s} - \frac{s}{s^2+2^2}$. By first shifting theorem, $L\left\{e^{-2t}(1-\cos 2t)\right\} = \frac{1}{s+2} - \frac{s+2}{(s+2)^2+2^2} = \frac{1}{s+2} - \frac{s+2}{s^2+4s+8}$.

$$\therefore L\left\{\frac{1}{2}t.e^{-2t}(1-\cos 2t)\right\} = \frac{1}{2}\frac{d}{ds}\left\{\frac{1}{s+2} - \frac{s+2}{s^2+4s+8}\right\} = \frac{1}{2}\left[-\frac{1}{(s+2)^2} - \frac{\left(s^2+4s+8\right).1-(s+2)(2s+4)}{\left(s^2+4s+8\right)^2}\right]$$

$$L\left\{\frac{1}{2}t.e^{-2t}(1-\cos 2t)\right\} = \frac{1}{2}\left[-\frac{1}{(s+2)^2} + \frac{s^2+4s}{\left(s^2+4s+8\right)}\right].$$

13. Find $\int_0^\infty e^{-t}t^3 \sin t\, dt$.

Sol.: By definition $\int_0^\infty e^{-st}t^3 \sin t\, dt = L\left(t^3 \sin t\right) = (-1)^3 \frac{d^3}{ds^3} L(\sin t) = (-1)\frac{d^3}{ds^3}\left[\frac{1}{s^2+1}\right]$. Now $\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = -\frac{2s}{\left(s^2+1\right)^2}$

$$\frac{d}{ds}\left[-\frac{2s}{\left(s^2+1\right)^2}\right] = -2\left[\frac{\left(s^2+1\right)^2.1 - s.2\left(s^2+1\right).2s}{\left(s^2+1\right)^4}\right] = -2\left[\frac{\left(s^2+1\right)-4s^2}{\left(s^2+1\right)^3}\right] = -2.\frac{\left(-3s^2+1\right)}{\left(s^2+1\right)^3} = 2\frac{\left(3s^2-1\right)}{\left(s^2+1\right)^3}$$

$$\frac{d}{ds}\left[2\frac{\left(3s^2-1\right)}{\left(s^2+1\right)^3}\right] = 2\left[\frac{\left(s^2+1\right)^3.6s-\left(3s^2-1\right).3\left(s^2+1\right)^2.2s}{\left(s^2+1\right)^6}\right] = 2\left[\frac{6s\left(s^2+1\right)-6s\left(3s^2-1\right)}{\left(s^2+1\right)^4}\right] = -24s.\frac{\left(s^2+1\right)}{\left(s^2+1\right)^4}$$

Thus, we have $\int_0^\infty e^{-st}t^3 \sin t\, dt = 24s.\frac{\left(s^2-1\right)}{\left(s^2+1\right)^4}$. To find the value of the given integral we put $s = 1$. $\therefore \int_0^\infty e^{-t}t^3 \sin t\, dt = 0$.

14. Find the Laplace transform of $\frac{1}{t}e^{-t}\sin t$.

Sol.: $L\left(e^{-t}\sin t\right) = \frac{1}{(s+1)^2+1}$ $\therefore L\left[\frac{1}{t}\left(e^{-t}\sin t\right)\right] = \int_s^\infty \frac{1}{(s+1)^2+1}ds = \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$.

15. Find the Laplace transform of $\frac{1-\cos t}{t^2}$.

Sol.: $L[1-\cos t] = L(1) - L(\cos t) = \frac{1}{s} - \frac{s}{s^2+1}$ $\therefore L\left(\frac{1-\cos t}{t}\right) = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right]ds = \left[\log s - \frac{1}{2}\log\left(s^2+1\right)\right]_s^\infty$

$L\left(\frac{1-\cos t}{t}\right) = \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+1}\right)\right]_s^\infty = -\frac{1}{2}\log\frac{s^2}{s^2+1} = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$ $\therefore L\left(\frac{1-\cos t}{t^2}\right) = \int_s^\infty \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)ds$

Integrating by parts $L\left(\frac{1-\cos t}{t^2}\right) = \frac{1}{2}\left[\log\left(\frac{s^2+1}{s^2}\right)s - \int s.\left(\frac{s^2}{s^2+1}\right)\left(\frac{s^2.2s-\left(s^2+1\right)2s}{s^4}\right)ds\right]_s^\infty$

$$= \frac{1}{2}\left[s\log\left(\frac{s^2+1}{s^2}\right) + 2\int\frac{ds}{s^2+1}\right]_s^\infty = \frac{1}{2}\left[s\log\left(\frac{s^2+1}{s^2}\right) + 2\tan^{-1}s\right]_s^\infty = \frac{1}{2}\left[0 + 2.\frac{\pi}{2} - s\log\left(\frac{s^2+1}{s^2}\right) - 2\tan^{-1}s\right].$$

$$L\left(\frac{1-\cos t}{t^2}\right) = \frac{\pi}{2} - \frac{s}{2}\log\left(\frac{s^2+1}{s^2}\right) - \tan^{-1}s.$$

16. Evaluate $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt$.

Sol.: Consider $f(t) = e^{-at} - e^{-bt}$ $\therefore L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+a}\right)ds = \left[\log(s+a) - \log(s+b)\right]_s^\infty$

$\therefore L\left[\frac{1}{t}f(t)\right] = -\left[\log\left(\frac{s+b}{s+a}\right)\right]_s^\infty = \log\left(\frac{s+b}{s+a}\right)$...(1). The equation (1) means $\int_0^\infty e^{-st}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = \log\left(\frac{s+b}{s+a}\right)$.

Putting $s = 0$, $\int_0^\infty \left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = \log\frac{b}{a}$.

17. Prove that $\int_0^\infty e^{-st}.\frac{\sin t \sinh at}{t} = \frac{1}{2}\tan^{-1}\left[\frac{2a}{1+\left(s^2-a^2\right)}\right]$.

Sol.: We have, $\sin t \sinh at = \left(\frac{e^{at}-e^{-at}}{2}\right)\sin t$. Now $L\sin t = \frac{1}{s^2+1}$. $\therefore L\left(e^{at}\sin t\right) = \frac{1}{(s-a)^2+1}$, $L\left(e^{-at}\sin t\right) = \frac{1}{(s+a)^2+1}$.

$\therefore L\sin t \sinh at = \frac{1}{2}\left[\frac{1}{(s-a)^2+1} - \frac{1}{(s+a)^2+1}\right]$; $\therefore L\left(\frac{\sin t \sinh at}{t}\right) = \frac{1}{2}\int_s^\infty \left[\frac{1}{(s-a)^2+1} - \frac{1}{(s+a)^2+1}\right]ds$.

$$L\sin t\sinh at = \frac{1}{2}\left[\tan^{-1}(s-a)-\tan^{-1}(s+a)\right]_s^{\infty} = \frac{1}{2}\left[\left\{\frac{\pi}{2}-\tan^{-1}(s-a)\right\}-\left\{\frac{\pi}{2}-\tan^{-1}(s+a)\right\}\right]$$

$= \frac{1}{2}\left[\tan^{-1}(s+a)-\tan^{-1}(s-a)\right]$. Now let $\tan^{-1}(s+a)=\alpha,\ \tan^{-1}(s-a)=\beta$. $\therefore \tan\alpha = s+a,\ \tan\beta = s-a$

$$\therefore \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} = \frac{(s+a)-(s-a)}{1+(s^2-a^2)} = \frac{2a}{1+(s^2-a^2)}.\ \therefore\ \alpha-\beta = \tan^{-1}\frac{2a}{1+(s^2+a^1)}$$

$$\therefore\ \tan^{-1}(s+a)-\tan^{-1}(s-a) = \tan^{-1}\frac{2a}{1+(s^2-a^2)}.\ \therefore L\left(\frac{\sin t\sin h\ at}{t}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2a}{1+(s^2-a^2)}\right).$$

This means $\int_0^\infty e^{-st}.\frac{\sin t\sin h\ at}{t}dt = \frac{1}{2}\tan^{-1}\left(\frac{2a}{1+(s^2-a^2)}\right).$

18. Find Laplace transform of $\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)$.

Sol.: We have $L(1-\cos 2t) = L(1)-L(\cos 2t) = \frac{1}{s}-\frac{s}{s^2+2^2}$. $\therefore L\left(\frac{1-\cos 2t}{t}\right) = \int_0^\infty \phi(s)\,ds = \int_0^\infty\left[\frac{1}{s}-\frac{s}{s^2+2^2}\right]ds$

$$L\left(\frac{1-\cos 2t}{t}\right) = \left[\log s-\frac{1}{2}\log(s^2+2^2)\right]_s^\infty = \left[\log\frac{s}{\sqrt{s^2+2^2}}\right] = \left[0-\log\frac{s}{\sqrt{s^2+2^2}}\right] = \log\left(\frac{\sqrt{s^2+2^2}}{s}\right)$$

But $L[f'(t)] = s\,L[f(t)]-f(0)$ where $f(t) = \frac{1-\cos 2t}{t} = \frac{2\sin^2 t}{t}$. $\therefore f(0) = \lim \sin t\left(\frac{\sin t}{t}\right) = 0$.

$$\therefore L\left[\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)\right] = s\log\left(\frac{\sqrt{s^2+2^2}}{s}\right).$$

19. Find the Laplace transform of $\int_0^t \sin 2u\ du$.

Sol.: Since $L\sin 2t = \frac{2}{s^2+4} = \phi(s)$, say $\therefore L\left[\int_0^t \sin 2u du\right] = \frac{1}{s}\phi(s) = \frac{2}{s(s^2+4)}$.

20. Find the Laplace transform of $erf\sqrt{t}$.

Sol.: We have $erf\ x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$. Hence, $erf\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}du$. Now put $u^2 = v$ $\therefore u=\sqrt{v}$ $\therefore du = \frac{1}{2\sqrt{v}}dv$.

When $u=0,\ v=0$; When $u=\sqrt{t},\ v=t$; $\therefore erf\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^t e^{-v}.\frac{1}{2\sqrt{v}}dv = \frac{1}{\sqrt{\pi}}\int_0^t e^{-v}v^{-1/2}dv$

$L(v^{-1/2}) = \frac{\lceil 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$ $\therefore L(e^{-v}v^{-1/2}) = \frac{\sqrt{\pi}}{\sqrt{s+1}}$ (By first shifting theorem)

$\therefore L\left[\int_0^t e^{-v}v^{-1/2}dv\right] = \frac{\sqrt{\pi}}{s\sqrt{s+1}}$; $\therefore L\left[erf\sqrt{t}\right] = \frac{1}{\sqrt{\pi}}.\frac{\sqrt{\pi}}{s\sqrt{s+1}} = \frac{1}{s\sqrt{s+1}}$.

21. Find the Laplace transform of the following. (iv) $\int_0^t \frac{1-e^{-au}}{u}du$.

Sol.: $L(1-e^{au}) = L(1)-L(e^{au}) = \frac{1}{s}-\frac{1}{s-a}$. $\therefore L\left(\frac{1-e^{au}}{u}\right) = \int_s^\infty\left[\frac{1}{s}-\frac{1}{s-a}\right]ds = \left[\log\frac{s}{s-a}\right]_s^\infty = \left[0-\log\frac{s}{s-a}\right] = \log\left(\frac{s-a}{s}\right)$

$$\therefore L\int_0^t\left(\frac{1-e^{-u}}{u}\right)du = \frac{1}{s}\phi(s) = \frac{1}{s}\log\left(\frac{s-a}{s}\right).$$

22. Find the Laplace transforms of $\cosh t\int_0^t e^u\cosh u\ du$.

Sol.: We have $L(\cos hu) = \frac{s}{s^2-1}$. $\therefore L(e^u\cos hu) = \frac{s-1}{(s-1)^2-1} = \frac{s-1}{s^2-2s+1-1} = \frac{s-1}{s(s-2)}$. $\therefore L\left[\int_0^t e^u\cos hu\ du\right] = \frac{1}{s}\cdot\frac{s-1}{s(s-2)}$

$$\therefore L\left[\int_0^t e^u\cos hu\ du\right] = \frac{s-1}{s^2(s-2)};\ \therefore L\left[\cos hu\int_0^t e^u\cos hu\ du\right] = L\left[\frac{e^u+e^{-u}}{2}\int_0^t e^u\cos hu\ du\right]$$

$$\therefore L\left[\cos hu\,.\int_0^t e^u \cos hu\,du\right]=\frac{1}{2}\left[L\left\{e^u\int_0^t e^u\cos hu\,du\right\}+L\left\{e^{-u}\int_0^t e^u\cos hu\,du\right\}\right]$$

$$=\frac{1}{2}\left[\frac{(s-1)-1}{(s-1)^2(s-1-2)}+\frac{(s+1)-1}{(s+1)^2(s+1-2)}\right]$$

$$\therefore L\left[\cos hu\,.\int_0^t e^u\cos hu\,du\right]=\frac{1}{2}\left[\frac{s-2}{(s-1)^2(s-3)}+\frac{s}{(s+1)^2(s-1)}\right].$$

23. Evaluate the following integrals by using Laplace transforms.

(i) $\int_0^\infty e^{-2t}\left(\int_0^t\frac{e^{-u}\sin u}{u}du\right)dt$ (ii) $\int_0^\infty e^{-t}\left(\int_0^t u^2\sin hu\cos hu\,du\right)dt$

Sol.: (i) $L\left[\int_0^t\frac{e^{-u}\sin u}{u}du\right]=\frac{1}{s}\cot^{-1}(s+1)$; By definition of Laplace transform this means

$$\int_0^\infty e^{-st}\left[\int_0^t\frac{e^{-u}\sin u}{u}du\right]dt=\frac{1}{s}.\cot^{-1}(s+1)$$

Putting $s=2$, we get $\int_0^\infty e^{-2t}\left[\int_0^t\frac{e^{-u}\sin u}{u}du\right]dt=\frac{1}{2}\cot^{-1}(3)$.

(ii) We have $L(\sin hu\cos hu)=L\left(\frac{1}{2}\sin h2u\right)=\frac{1}{2}.\frac{2}{s^2+2^2}=\frac{1}{s^2+4}$. $\therefore L(u^2\sin hu\cos hu)=(-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s^2+4}\right)$

$$\therefore L(u^2\sin hu\cos hu)=\frac{d}{ds}\left(-\frac{2s}{(s^2+4)^2}\right)=-2\left[\frac{(s^2+4)^2-s.2(s^2+4).2s}{(s^2+4)^4}\right]=-2\left[\frac{s^2+4-4s^2}{(s^2+4)^3}\right]=\frac{3s^2-4}{(s^2+4)^3}=\phi(s)\text{ say}$$

$\therefore L\left[\int_0^t u^2\sin hu\cos hu\,du\right]=\frac{1}{s}\phi(s)=\frac{2}{s}.\frac{(3s^2-4)}{(s^2+4)^3}$. By definition of Laplace transform, this means,

$\int_0^\infty e^{-st}\left[\int_0^t u^2\sin hu\cos hu\,du\right]dt=\frac{2}{s}\frac{(3s^2-4)}{(s^2+4)^3}$. Putting $s=1$, we get

$$\int_0^\infty e^{-t}\left[\int_0^t u^2\sin hu\cos hu\,du\right]dt=\frac{2}{1}\frac{(3.1-4)}{(1+4)^3}=-\frac{2}{125}$$

24. Find $\int_0^\infty\cos(tx^2)\,dx$ and hence, find $\int_0^\infty\cos x^2.dx$

Sol.: Let $f(t)=\int_0^\infty\cos(tx^2)\,dx$. $\therefore Lf(t)=\int_0^\infty e^{-st}f(t)\,dt=\int_0^\infty e^{-st}\int_0^\infty\left[\cos(tx^2)\,dx\right]dt=\int_0^\infty\left[\int_0^\infty e^{-st}\cos(tx^2)\,dt\right]dx$

$\therefore Lf(t)=\int_0^\infty\left[L\cos(tx^2)\right]dx=\int_0^\infty\frac{s}{s^2+x^4}dx$. Now put $x=\sqrt{s}\tan\theta$; $\therefore dx=\frac{s.\sec^2\theta\,d\theta}{2\sqrt{s\tan\theta}}$

$$\therefore L[f(t)]=\int_0^{\pi/2}\frac{s}{s^2+s^2\tan^2\theta}.\frac{s.\sec^2\theta\,d\theta}{2\sqrt{s\tan\theta}}=\int_0^{\pi/2}\frac{1}{2\sqrt{s\tan\theta}}d\theta=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}(\sin\theta)^{-1/2}(\cos\theta)^{-1/2}d\theta$$

$$\therefore L[f(t)]=\frac{1}{2\sqrt{s}}\frac{\Gamma(1/4)\,\Gamma(3/4)}{2.1!}\left[\int_0^{\pi/2}\sin^p\theta\cos^p\theta\,d\theta=\frac{\Gamma((p+1)/2)\,\Gamma((q+1)/2)}{2\Gamma((p+q+2)/2)}\right]=\frac{1}{2\sqrt{s}}.\frac{\sqrt{2}\pi}{2}\quad\left[\because\Gamma(1/4)\,\Gamma(3/4)=\sqrt{2}\pi\right]$$

$\therefore L(f(t))=\frac{\pi}{2\sqrt{2}\sqrt{s}}$. Now taking a laplace inverse, we get $f(t)=\frac{\pi}{2\sqrt{2}}L^{-1}\left(\frac{1}{\sqrt{s}}\right)=\frac{\pi}{2\sqrt{2}}.\frac{t^{-1/2}}{\Gamma(1/2)}=\frac{\sqrt{\pi}}{2\sqrt{2}}.\frac{1}{\sqrt{t}}$.

Now put $t=1$, $\therefore\int_0^\infty\cos x^2dx=\frac{\sqrt{\pi}}{2\sqrt{2}}=\frac{1}{2}\sqrt{\frac{\pi}{2}}$.

25. Find $\int_0^\infty\sin(tx^2)\,dx$ and hence, find $\int_0^\infty\sin x^2\,dx$.

Sol.: Let $f(t)=\int_0^\infty\sin(tx^2)\,dx$. $\therefore Lf(t)=\int_0^\infty e^{-st}f(t)\,dt=\int_0^\infty e^{-st}\left[\int_0^\infty\sin(tx^2)\,dx\right]dt=\int_0^\infty\left[\int_0^\infty e^{-st}\sin(tx^2)\,dt\right]dx$

$\therefore L[f(t)]=\int_0^\infty\left[L\sin(tx^2)\right]dx=\int_0^\infty\frac{x^2}{s^2+x^4}dx$. Now put $x=\sqrt{s}\tan\theta$, $dx=\frac{s.\sec^2\theta\,d\theta}{2\sqrt{s\tan\theta}}$.

$$\therefore Lf(t)=\int_0^{\pi/2}\frac{s\tan\theta}{s^2+s^2\tan^2\theta}\cdot\frac{s.\sec^2\theta}{2\sqrt{s\tan\theta}}.d\theta=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta$$

$$=\frac{1}{2\sqrt{s}}\frac{\overline{|3/4}\,\overline{|1/4}}{2.1}=\frac{1}{2\sqrt{s}}\frac{\sqrt{2}.\pi}{2}$$

$\therefore L[f(t)]=\frac{\pi}{2\sqrt{2}}\cdot\frac{1}{\sqrt{s}}$. Taking laplace inverse, we get $\therefore f(t)=\frac{\pi}{2\sqrt{2}}L^{-1}\left(\frac{1}{\sqrt{s}}\right)=\frac{\pi}{2\sqrt{2}}\cdot\frac{t^{-1/2}}{\overline{|1/2}}=\frac{\sqrt{\pi}}{2\sqrt{2}}\cdot\frac{1}{\sqrt{t}}$.

Now put $t=1$, $\therefore\int_0^\infty\sin x^2\,dx=\frac{\sqrt{\pi}}{2\sqrt{2}}=\frac{1}{2}\sqrt{\frac{\pi}{2}}$.

26. Find $\int_0^\infty e^{-tx^2}\,dx$ and hence, find $\int_0^\infty e^{-x^2}\,dx$.

Sol.: Let $f(t)=\int_0^\infty e^{-tx^2}dx$. $\therefore Lf(t)=\int_0^\infty e^{-st}\left[\int_0^\infty e^{-tx^2}dx\right]dt=\int_0^\infty\left[\int_0^\infty e^{-st}.e^{-tx^2}dt\right]dx$.

$L(f(t))==\int_0^\infty\left[L\left(e^{-tx^2}\right)\right]dx=\int_0^\infty\frac{dx}{s+x^2}\quad\left[\because Le^{-at}=\frac{1}{s+a}\right]=\left[\frac{1}{\sqrt{s}}\tan^{-1}\frac{x}{\sqrt{s}}\right]_0^\infty=\frac{\pi}{2\sqrt{s}}$. Taking laplace inverse, we get

$\therefore f(t)=\frac{\pi}{2}L^{-1}\left(\frac{1}{\sqrt{s}}\right)=\frac{\pi}{2}\frac{t^{-1/2}}{\overline{|1/2}}=\frac{\pi}{2}\frac{1}{\sqrt{\pi}\sqrt{t}}=\frac{1}{2}\sqrt{\frac{\pi}{t}}$. Now, put $t=1$, $\therefore\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

27. Find the inverse Laplace transform of $\frac{s+2}{s^2-4s+13}$

Sol.: $L^{-1}\left[\frac{s+2}{s^2-4s+13}\right]=L^{-1}\left[\frac{s+2}{(s-2)^2+3^2}\right]=L^{-1}\left[\frac{(s-2)+4}{(s-2)^2+3^2}\right]=e^{2t}L^{-1}\left[\frac{s+4}{s^2+3^2}\right]$ (Inverse of First Shifting Th.)

$=e^{2t}L^{-1}\frac{s}{s^2+3^2}+4.e^{2t}L^{-1}\left(\frac{1}{s^2+3^2}\right)=e^{2t}.\cos 3t+\frac{4}{3}e^{2t}\sin 3t$.

28. Find inverse Laplace transform of(i) $\frac{s^2+16s-24}{s^4+20s^2+64}$ (ii) $\frac{s}{(s^2+1)(s^2+4)(s^2+9)}$

Sol.: (i) $\phi(s)=\frac{s^2+16s-24}{s^4+20s^2+64}=\frac{s^2+16s-24}{(s^2+4)(s^2+16)}$. Let $\phi(s)=\frac{as+b}{s^2+4}+\frac{cs+d}{s^2+16}$.

$\therefore s^2+16s-24=(as+b)(s^2+16)+(cs+d)(s^2+4)$

$=(a+c)s^3+(b+d)s^2+(16a+4c)s+(16b+4d)$. Equating the coefficients of like powers of a s we get,
$a+c=0,\ b+d=1,$

$16a+4c=16,\ 16b+4d=-24$ $\therefore a=4/3,\ c=-4/3,\ b=-7/3,\ d=10/3$. $\therefore\phi(s)=\frac{1}{3}\cdot\frac{4s-7}{s^2+4}-\frac{1}{3}\cdot\frac{4s-10}{s^2+16}$

$\therefore L^{-1}[\phi(s)]=\frac{4}{3}L^{-1}\left(\frac{s}{s^2+4}\right)-\frac{7}{3}L^{-1}\left(\frac{1}{s^2+4}\right)-\frac{4}{3}L^{-1}\left(\frac{s}{s^2+16}\right)+\frac{10}{3}L^{-1}\left(\frac{1}{s^2+16}\right)\quad\therefore L^{-1}[\phi(s)]$
$=\frac{4}{3}\cos 2t-\frac{7}{3}\cdot\frac{1}{2}\sin 2t-\frac{4}{3}.\cos 4t+\frac{10}{3}\cdot\frac{1}{4}\sin 4t$

(ii) Let us first consider $\frac{1}{(s^2+1)(s^2+4)(s^2+9)}=\frac{1/24}{s^2+1}-\frac{1/15}{s^2+4}+\frac{1/40}{s^2+9}$ (short cut method)

Now $\frac{1}{(s^2+1)(s^2+4)(s^2+9)}=\frac{1}{24}\cdot\frac{s}{s^2+1}-\frac{1}{15}\cdot\frac{s}{s^2+4}+\frac{1}{40}\cdot\frac{s}{s^2+9}$

$\therefore L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right]=\frac{1}{24}L^{-1}\left(\frac{s}{s^2+1}\right)-\frac{1}{15}L^{-1}\left(\frac{s}{s^2+4}\right)+\frac{1}{40}L^{-1}\left(\frac{s}{s^2+9}\right)=\frac{1}{24}\cos t-\frac{1}{15}\cos 2t+\frac{1}{40}\cos 3t$.

29. Find the inverse Laplace transform of the following (i) $\frac{3s+1}{(s+1)(s^2+2)}$ (ii) $\frac{5s^2-15s-11}{(s+1)(s+2)^2}$

Sol.: (i) $\frac{3s+1}{(s+1)(s^2+2)}=\frac{a}{s+1}+\frac{bs+c}{s^2+2}$. $\therefore 3s+1=a(s^2+2)+(s+1)(bs+c)=(a+b)s^2+(b+c)s+2a+c$.

Equating the coefficients of like powers of s; $\therefore a+b=0,\ b+c=3,\ 2a+c=1$ $\therefore a=-\frac{1}{2},\ b=\frac{1}{2},\ c=\frac{5}{2}$

$$\therefore L^{-1}\left[\frac{3s+1}{(s+1)(s^2+2)}\right]=-\frac{1}{2}L^{-1}\frac{1}{s+1}+\frac{1}{2}L^{-1}\frac{s}{s^2+2}+\frac{5}{2}L^{-1}\frac{1}{s^2+2}=-e^{-t}L^{-1}\frac{1}{s}+\frac{1}{2}L^{-1}\frac{s}{s^2+(\sqrt{2})^2}+\frac{5}{2}L^{-1}\frac{1}{s^2+(\sqrt{2})^2}$$

$$\therefore L^{-1}\left[\frac{3s+1}{(s+1)(s^2+2)}\right]=-e^{-t}.1+\frac{1}{2}\cos\sqrt{2}\,t+\frac{5}{2\sqrt{2}}\sin\sqrt{2}\,t.$$

(ii) Let $\frac{5s^2-15s-11}{(s+1)(s-2)^2}=\frac{a}{s+1}+\frac{b}{s-2}+\frac{c}{(s-2)^2}$. $\therefore 5s^2-15s-11=a(s-2)^2+b(s+1)(s-2)+c(s+1)$.

Equating the coefficient of like power of s; $a+b=5,\ -4a-b-c=-15,\ 4a-2b+c=-11$. $\therefore a=1,\ b=4,\ c=-7$.

$$\therefore L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^2}\right]=L^{-1}\frac{1}{s+1}+4L^{-1}\frac{1}{s-2}-7L^{-1}\frac{1}{(s-2)^2}=e^{-t}L^{-1}\frac{1}{s}+4e^{2t}L^{-1}\frac{1}{s}-7e^{2t}L^{-1}\frac{1}{s^2}=e^{-t}+4e^{2t}-7e^{2t}t.$$

30. If $L^{-1}\left\{\frac{s^2+4}{(s^2-4)^2}\right\}=t\cos h2t$, hence find $L^{-1}\left\{\frac{s^2+9}{(s^2-9)^2}\right\}$.

Sol.: By the inverse change of scale rule, since $L^{-1}\left\{s^2+4\big/(s^2-4)^2\right\}=t\cosh 2t$, replacing s by $\frac{s}{a}$ and t by at (after dividing the given function by a). $L^{-1}\left\{\frac{1}{a}\cdot\frac{(s/a)^2+4}{\left[(s/a)^2-4\right]^2}\right\}=at\cosh 2at$. $\therefore L^{-1}\left\{\frac{1}{a^3}\cdot\frac{s^2+4a^2}{(s^2-4a^2)^2}.a^4\right\}=at\cosh 2at$.

$\therefore L^{-1}\left\{\frac{s^2+4a^2}{(s^2-4a^2)^2}\right\}=t\cosh 2at$. Comparing the l.h.s. with the required result, we put $a=3/2$. $L^{-1}\left\{\frac{s^2+9}{(s^2-9)^2}\right\}=t\cos h3t$.

31. Find the inverse of the following by using convolution theorem $\frac{1}{(s-3)(s+3)^2}$.

Sol.: Let $\phi_1(s)=\frac{1}{(s-3)},\ \phi_2(s)=\frac{1}{(s+3)^2}$. $\therefore L^{-1}\phi_1(s)=L^{-1}\frac{1}{(s-3)}=e^{3t},\ L^{-1}\phi_2(s)=L^{-1}\frac{1}{(s+3)^2}=e^{-3t}L^{-1}\frac{1}{s^2}=e^{-3t}t$

$$\therefore L^{-1}\phi(s)=\int_0^t e^{3u}.e^{-3(t-u)}.(t-u)\,du=\int_0^t e^{6u-3t}(t-u)\,du=\left[t-u\frac{e^{6u-3t}}{6}-\left(\frac{e^{6u-3t}}{36}\right)(-1)\right]_0^t=\left[\frac{e^{3t}}{36}-t\frac{e^{-3t}}{6}-\frac{e^{-3t}}{36}\right]$$

$$\therefore L^{-1}\phi(s)=\frac{1}{36}\left[e^{3t}-e^{-3t}-6t\,e^{-3t}\right].$$

32. Find the inverse of the following by using convolution theorem (i) $\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ (iii) $\frac{(s+3)^2}{(s^2+6s+5)^2}$

Sol.: (i) $L^{-1}\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}=L^{-1}\frac{(s+1)^2+2}{\left[(s+1)^2+1\right]\left[(s+1)^2+4\right]}=e^{-t}L^{-1}\frac{s^2+2}{(s^2+1)(s^2+4)}$

$$L^{-1}\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}=e^{-t}L^{-1}\frac{s^2}{(s^2+1)(s^2+4)}+e^{-t}L^{-1}\left[\frac{2}{(s^2+1)(s^2+4)}\right]\quad L^{-1}\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$=e^{-t}L^{-1}\frac{s^2}{(s^2+1)(s^2+4)}+\frac{2e^{-t}}{3}.L^{-1}\left[\frac{1}{(s^2+1)}-\frac{1}{(s^2+4)}\right]$$

$$L^{-1}\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{e^{-t}}{3}[2\sin 2t-\sin t]+\frac{2}{3}e^{-t}\left[\sin t-\frac{1}{2}\sin 2t\right] = \frac{e^{-t}}{3}[\sin 2t+\sin t]$$

(ii) $L^{-1}\frac{(s+3)^2}{(s^2+6s+5)^2} = L^{-1}\frac{(s+3)^2}{\left[(s+3)^2-2^2\right]^2} = e^{-3t}.L^{-1}\frac{s^2}{(s^2-2^2)^2}$ (By first shifting theorem). We find $L^{-1}\frac{s^2}{(s^2-2^2)^2}$ by convolution theorem. Let $\phi_1(s)=\frac{s}{s^2-2^2}$, $\phi_2(s)=\frac{s}{s^2-2^2}$. $\therefore L^{-1}\phi_1(s)=\cosh 2u$, $L^{-1}\phi_2(s)=\cosh 2u$

$$\therefore L^{-1}\phi(s)=\int_0^t \cosh 2u.\cosh 2(t-u)\,du = \frac{1}{2}\int_0^t\left[\cosh 2t+\cosh 2(2u-t)\right]du = \frac{1}{2}\left[u\cosh 2t+\frac{\sinh 2(2u-t)}{4}\right]_0^t$$

$$\therefore L^{-1}\phi(s)=\frac{1}{2}\left[t\cosh 2t+\frac{1}{4}\sinh 2t+\frac{1}{4}\sinh 2t\right]=\frac{1}{4}[2t\cosh 2t+\sinh 2t].\ \therefore L^{-1}\frac{(s+3)^2}{(s^2+6s+5)^2}=\frac{1}{4}e^{-3t}[2t\cosh 2t+\sinh 2t]$$

33. Find the inverse Laplace transform of the following by convolution theorem (i) $\frac{1}{s(s^2-a^2)}$ (ii) $\frac{1}{5}\log\left(\frac{s+3}{s+2}\right)$

Sol.: (i) $L^{-1}\left(\frac{1}{s^2-a^2}\right)=\frac{1}{a}\sin h\,au$, $L^{-1}.\frac{1}{s}=1$. $\therefore L^{-1}\left[\frac{1}{(s^2-a^2)}.\frac{1}{s}\right]=\int_0^t\frac{1}{a}\sin h\,au.1.du=\frac{1}{a^2}[\cos h\,au]_0^t=\frac{1}{a^2}[\cos h\,at-1]$

(ii) $L^{-1}\log\left(\frac{s+3}{s+2}\right)=-\frac{1}{u}L^{-1}\left\{\frac{d}{ds}\log\left(\frac{s+3}{s+2}\right)\right\}=-\frac{1}{u}L^{-1}\left[\frac{d}{ds}\{\log(s+3)-\log(s+2)\}\right]$

$L^{-1}\log\left(\frac{s+3}{s+2}\right)=-\frac{1}{u}L^{-1}\left[\frac{1}{u+3}-\frac{1}{u+2}\right]=-\frac{1}{u}\left(e^{-3u}-e^{-2u}\right)$ And $L^{-1}\frac{1}{s}=1$. $\therefore L^{-1}\frac{1}{s}\log\left(\frac{s+3}{s+2}\right)=\int_0^t\frac{e^{-2u}-e^{-3u}}{u}.du.$

34. Find inverse Laplace transform of, (i) $\log\left[\frac{s^2+a^2}{\sqrt{s+b}}\right]$ (ii) $s\log\left(\frac{s+1}{s-1}\right)$

Sol.: (i) $L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left\{\log(s^2+a^2)-\frac{1}{2}\log(s+b)\right\}\right]$

$L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right]=-\frac{1}{t}L^{-1}\left[\frac{2s}{s^2+a^2}-\frac{1}{2}\cdot\frac{1}{s+b}\right]=-\frac{1}{t}\left[2\cos at-\frac{1}{2}e^{-bt}\right]=\frac{1}{t}\left[\frac{1}{2}e^{bt}-2\cos at\right]$

(i) $L^{-1}\left[s\log\left(\frac{s+1}{s-1}\right)\right]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}s\log\left(\frac{s+1}{s-1}\right)\right]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\{s\log(s+1)-s\log(s-1)\}\right]$

$=-\frac{1}{t}L^{-1}\left[\frac{s}{s+1}+\log(s+1)-\frac{s}{s-1}-\log(s-1)\right]=-\frac{1}{t}L^{-1}\left[\frac{s}{s+1}-\frac{s}{s-1}\right]-\frac{1}{t}L^{-1}\left[\log(s+1)-\log(s-1)\right]$

$=-\frac{1}{t}L^{-1}\left(-\frac{2s}{s^2-1}\right)-\frac{1}{t}\left(-\frac{1}{t}\right)L^{-1}\left\{\frac{d}{ds}\left[\log(s+1)-\log(s-1)\right]\right\}=\frac{2}{t}L^{-1}\left(\frac{s}{s^2-1}\right)+\frac{1}{t^2}L^{-1}\left[\frac{1}{s+1}-\frac{1}{s-1}\right]$

$L^{-1}\left[s\log\left(\frac{s+1}{s-1}\right)\right]=\frac{2}{t}\cos ht+\frac{1}{t^2}\left(e^{-t}-e^{t}\right)=\frac{2}{t}\cos ht-\frac{2}{t^2}\left(\frac{e^t-e^{-t}}{2}\right)=\frac{2}{t}\cos ht-\frac{2}{t^2}\sin ht$

35. Find inverse Laplace transform of $\cot^{-1}as$

Sol. $L^{-1}\left[\cot^{-1}(as)\right]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\cot^{-1}(as)\right]=-\frac{1}{t}L^{-1}\left[\frac{-a}{1+a^2s^2}\right]=\frac{a}{t}L^{-1}\left(\frac{1}{1+a^2s^2}\right)=\frac{a}{t}L^{-1}\left(\frac{1}{a^2}\cdot\frac{1}{s^2+(1/a)^2}\right)$

$L^{-1}\left[\cot^{-1}(as)\right]=\frac{1}{t}L^{-1}\left(\frac{1/a}{s^2+(1/a)^2}\right)=\frac{1}{t}\sin\frac{t}{a}.$

36. Find inverse Laplace transform of $\frac{s+1}{(s^2+2s-15)^2}$

Sol: $L^{-1}\left[\frac{s+1}{(s^2+2s-15)^2}\right]=L^{-1}\left\{\frac{s+1}{\left[(s+1)^2-4^2\right]^2}\right\}=e^{-t}.L^{-1}\left[\frac{s}{(s^2-4^2)^2}\right]=e^{-t}.t.L^{-1}\int_s^\infty\frac{s}{(s^2-4^2)^2}ds$

To find the integral put $s^2-4^2=x$. ∴ $\int \frac{s}{\left(s^2-4^2\right)^2}ds = \frac12\int\frac{dx}{x^2} = -\frac12\cdot\frac1x = -\frac12\cdot\frac{1}{s^2-4^2}$

∴ $L^{-1}\left\{\frac{s+1}{\left(s^2+2s-15\right)^2}\right\} = e^{-t}t.\frac12.L^{-1}\left[\frac{1}{s^2-4^2}\right] = \frac12 e^{-t}t.\frac14\sinh 4t = \frac18 t e^{-t}\sinh 4t.$

37. Using convolution theorem, prove that $L^{-1}\left[\frac1s\tan^{-1}\frac2s\right] = \int_0^t\frac1u.\sin 2u\,du$

Sol.: Let $\phi_1(s)=\tan^{-1}\frac2s$ and $\phi_2(s)=\frac1s$. ∴ $\phi_1'(s) = \frac{1}{1+\left(4/s^2\right)}\left(-\frac{2}{s^2}\right) = -\frac{2}{s^2+4}$. ∴ $L^{-1}\phi_1'(s) = -\sin 2t$

∴ By the theorem proved in (f) on page 9-30. $L^{-1}\left[\phi_1(s)\right] = -\frac1t L^{-1}\left[\phi_1'(s)\right]$. ∴ $L^{-1}\phi_1(s) = \frac1t\sin 2t$.

Now $\phi_2(s)=\frac1s$ ∴ $L^{-1}\phi_2(s)=1$. ∴ By the corollary of convolution theorem $L^{-1}\phi(s) = \int_0^t\frac1u\sin 2u.1.du$

38. Find inverse Laplace transform of $\frac{s^2}{\left(s^2+a^2\right)^2}$

Sol.: Let $\phi(s)\frac{1}{s^2+a^2}$ ∴ $\phi'(s) = -\frac{2s}{\left(s^2+a^2\right)^2}$. But $L^{-1}\phi(s) = \frac1a\sin at$ and $L^{-1}\left[\phi'(s)\right] = -tL^{-1}\phi(s)$

∴ $L^{-1}\left[-\frac{2s}{\left(s^2+a^2\right)^2}\right] = -t.\frac1a\sin at$. ∴ $L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right] = \frac{1}{2a}t\sin at$

But $L^{-1}\left[s.\phi(s)\right] = \frac{d}{dt}\left[L^{-1}\phi(s)\right]$. ∴ $L^{-1}\left[s.\frac{s}{\left(s^2+a^2\right)^2}\right] = \frac{d}{dt}\left[\frac{1}{2a}t\sin at\right] = \frac{1}{2a}\left[\sin at + at\cos at\right]$.

39. Find $L^{-1}\left[\frac{1}{s\left(s^2+4\right)}\right]$

Sol.: $L^{-1}\left[\frac{1}{s\left(s^2+4\right)}\right] = \int_0^t L^{-1}\left[\frac{1}{\left(s^2+4\right)}\right]du = \int_0^t\frac12\sin 2u\,du = \frac12\left[\frac{-\cos 2u}{2}\right]_0^t = \frac14\left[1-\cos 2t\right]$

40. Find Laplace transform of $f(t) = a\sin pt,\ 0<t<\pi/p,\ f(t)=0,\ \pi/p<t<2\pi/p$ and $f(s) = f(t+2\pi/p)$.

Sol.: $Lf(t) = \frac{1}{1-e^{-(2\pi/p)s}}\int_0^{\pi/p}e^{-st}a\sin pt\,dt = \frac{a}{1-e^{-(2\pi/p)s}}\left[\frac{1}{s^2+p^2}.e^{-st}\left(-s\sin pt - p\cos pt\right)\right]_0^{\pi/p}$

$Lf(t) = \frac{a}{1-e^{-(2\pi/p)s}}.\frac{1}{s^2+p^2}.\left[e^{-s\pi/p}\left(-s\sin\pi - p\cos\pi\right) - e^0\left(-s\sin 0 - p\cos 0\right)\right] = \frac{1}{1-e^{-(2\pi/p)s}}.\frac{1}{s^2+p^2}\left[e^{-s\pi/p}+p\right]$

$Lf(t) = \frac{ap}{1-e^{-(s\pi/p)}}.\frac{1}{s^2+p^2}$

41. Find $Lf(t)$ where, $f(t)=t,\ 0<t<1;\ f(t)=0,\ 1<t<2$ and $f(t+2)=f(t)$ for $t>0$.

Sol.: Since $f(t)$ is periodic with period $a=2$, we have ∴ $Lf(t) = \frac{1}{1-e^{-as}}\int_0^a e^{-st}f(t)dt = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)dt$

∴ $Lf(t) = \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}t\,dt + \int_1^2 e^{-st}.0.dt\right] = \frac{1}{1-e^{-2s}}\left[t\left(-\frac{e^{-st}}{s}\right)-\left(-\frac{e^{-st}}{s^2}\right)(1)\right]_0^1 = \frac{1}{1-e^{-2s}}\left[-\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}+\frac{1}{s^2}\right]$

∴ $Lf(t) = \frac{1}{s^2\left(1-e^{-2s}\right)}\left(1-e^{-s}-s\,e^{-s}\right)$.

42. Find inverse Laplace transform of $e^{-s}\left\{\frac{1-\sqrt{s}}{s^2}\right\}^2$

Sol.: $f(t)=L^{-1}\phi(s)=L^{-1}\left(\frac{1-\sqrt{s}}{s^2}\right)^2 \therefore f(t)=L^{-1}\left[\frac{1-2\sqrt{s}+s}{s^4}\right]=L^{-1}\left[\frac{1}{s^4}-\frac{2}{s^{7/2}}+\frac{1}{s^3}\right]$

$=\frac{t^3}{3!}-2\frac{t^{5/2}}{\overline{|7/2}}+\frac{t^2}{2!}=\frac{t^3}{6}-\frac{16}{15\sqrt{\pi}}t^{5/2}+\frac{t^2}{2}\quad (\because a=1).\ \therefore L^{-1}e^{-s}\left\{\frac{1-\sqrt{s}}{s}\right\}^2=\left[\frac{(t-1)^3}{6}-\frac{16}{15\sqrt{\pi}}(t-1)^{5/2}+\frac{(t-1)^2}{2}\right]H(t-1)$

43. Find the inverse Laplace transform of the following (i) $\frac{(s+1)e^{-s}}{s^2+s+1}$ (ii) $\frac{8e^{-3s}}{s^2+4}$

Sol.: (i) $f(t)=L^{-1}\phi(s)=L^{-1}\left(\frac{s+1}{s^2+s+1}\right)=L^{-1}\frac{[s+(1/2)]+(1/2)}{[s+(1/2)]^2+(3/4)}=L^{-1}\frac{[s+(1/2)]}{[s+(1/2)]^2+(\sqrt{3}/2)^2}+\frac{1}{2}L^{-1}\frac{1}{[s+(1/2)]^2+(\sqrt{3}/2)^2}$

$=e^{-t/2}L^{-1}\frac{s}{s^2+(\sqrt{3}/2)^2}+\frac{1}{2}e^{-t/2}L^{-1}\frac{1}{s^2+(\sqrt{3}/2)^2}=e^{-t/2}\cos(\sqrt{3}t/2)+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}t/2)$

$\therefore L^{-1}\frac{(s+1)e^{-s}}{s^2+s+1}=e^{-(t-1)/2}\left[\cos(\sqrt{3}(t-1)/2)+\frac{1}{\sqrt{3}}\sin(\sqrt{3}(t-1)/2)\right]H(t-1).$

(ii) $f(t)=L^{-1}\phi(s)=L^{-1}\frac{8}{s^2+4}=8.\frac{1}{2}\sin 2t=4\sin 2t.\ \therefore L\frac{8e^{-3s}}{s^2+4}=4\sin 2(t-3)H(t-3)$

44. Find Laplace transform of $t[H(t-4)]+t^2\delta(t-4)$.

Sol.: For, $L[tH(t-4)]$ we write $t=4+(t-4)$. $\therefore L[tH(t-4)]=L[\{4+(t-4)\}H(t-4)]=L[f\{t-4\}H(t-4)]$

$\therefore L[tH(t-4)]=e^{-4s}Lf(t)$ where, $f(t)=4+t$. $\therefore L[tH(t-4)]=e^{-4s}\left[\frac{4}{s}+\frac{1}{s^2}\right]$

For, $L[t^2\delta(t-4)]$, we take $f(t)=t^2$ and $a=4$. $\therefore L[t^2\delta(t-4)]=Lf(t)\delta(t-4)=e^{-as}f(a)=e^{-4s}.16$

$\therefore L[t.H(t-4)+t^2\delta(t-4)]=e^{-4s}\left[\frac{4}{s}+\frac{1}{s^2}\right]+16e^{-4s}=\frac{e^{-4s}}{s^2}\left[1+4s+16s^2\right].$

45. Solve $\frac{dy}{dx}+3y=2+e^{-t}$, if $y=1$ at $t=0$.

Sol.: Taking Laplace transform of both sides, $L(y')+3L(y)=L(2)+L(e^{-t})$; $s\bar{y}-y(0)+3\bar{y}=2\frac{1}{s}+\frac{1}{s+1}$. But $y(0)=1$

$\therefore (s+3)\bar{y}=\frac{2}{s}+\frac{1}{s+1}+1=\frac{s^2+4s+2}{s(s+1)}$. $\therefore \bar{y}=\frac{s^2+4s+2}{s(s+1)(s+3)}$. $\therefore$ By partial fractions $\bar{y}=\frac{2}{3}\cdot\frac{1}{s}+\frac{1}{2}\cdot\frac{1}{s+1}-\frac{1}{6}\cdot\frac{1}{s+3}$

Taking inverse Laplace transforms, $y=\frac{2}{3}+\frac{1}{2}e^{-t}-\frac{1}{6}e^{-3t}$.

46. Solve $Dx-2y-x=-2te^{-t}+e^{-t}-6t$, $D^2x-Dy=te^{-t}-2e^{-t}-3$ given that $x=0$, $Dx=1$ when $t=0$.

Sol.: Taking Laplace transforms of both the equations we get, $L(x')-2L(y)-L(x)=-2L(te^{-t})+L(e^{-t})-6L(t)$.

$L(x'')-L(y')=L(te^{-t})-2L(e^{-t})-3L(t)$. If $L(x)=\bar{x}$ and $L(y)=\bar{y}$, we have $L(x')=s\bar{x}-x(0)=s\bar{x}$

$L(x'')=s^2\bar{x}-sx(0)-x'(0)=s^2\bar{x}-1$. $L(y')=s\bar{y}-y(0)=s\bar{y}$. $L(te^{-t})=(-1)\frac{d}{ds}\left[L(e^{-t})\right]=-\frac{d}{ds}\left(\frac{1}{s+1}\right)=\frac{1}{(s+1)^2}$

$L(e^{-t})=\frac{1}{s+1}$, $L(t)=\frac{1}{s^2}$, $L(1)=\frac{1}{s}$. $\therefore$ The equations become $s\bar{x}-2\bar{y}-\bar{x}=-\frac{2}{(s+1)^2}+\frac{1}{s+1}-\frac{6}{s^2}$

i.e. $(s-1)\bar{x}-2\bar{y}=\frac{s-1}{(s+1)^2}-\frac{6}{s^2}$ .....(1)And $s^2\bar{x}-1-s\bar{y}=\frac{1}{(s+1)^2}-\frac{2}{s+1}-\frac{3}{s}$; $s^2\bar{x}-s\bar{y}=1+\frac{1}{(s+1)^2}-\frac{2}{s+1}-\frac{3}{s}$

$=\frac{s^2+2s+1+1-2s-2}{(s+1)^2}-\frac{3}{s}$. i.e. $s^2\bar{x}-s\bar{y}=\frac{s^2}{(s+1)^2}-\frac{3}{s}$ .....(2). Now, multiply (1) by s and (2) by 2 and subtract,

$\therefore (s^2+s)\bar{x}=\frac{2s^2}{(s+1)^2}-\frac{6}{s}-\frac{s(s-1)}{(s+1)^2}+\frac{6}{s}$. $\therefore s(s+1)\bar{x}=\frac{s(s+1)}{(s+1)^2}$ $\therefore \bar{x}=\frac{1}{(s+1)^2}$ $\therefore x=L^{-1}\frac{1}{(s+1)^2}=e^{-t}L\frac{1}{s^2}=te^{-t}$

Now, putting the value of $\bar{x}$ in (2) we get, $s^2.\frac{1}{(s+1)^2}-s\bar{y}=\frac{s^2}{(s+1)^2}-\frac{3}{s}$. $\therefore s\bar{y}=\frac{3}{s}$ $\therefore \bar{y}=\frac{3}{s^2}$ $\therefore y=L^{-1}\frac{3}{s^2}=3t$.

47. Solve $\left(D^2-D-2\right)y=20\sin 2t$, with $y(0)=1$ and $y'(0)=2$.

Sol: Let $L(y)=\bar{y}$. Then, taking Laplace transform, $L(y'')L(y')-2L(y)=20L(\sin 2t)$. But $L(y')=s\bar{y}-y(0)=s\bar{y}-1$

And $L(y'')=s^2\bar{y}-sy(0)-y'(0)=s^2\bar{y}-s-2$. $\therefore$ The equation becomes, $\left(s^2\bar{y}-s-2\right)-\left(s\bar{y}-1\right)-2\bar{y}=20\dfrac{2}{s^2+4}$

$\left(s^2\bar{y}-s-2\right)\bar{y}=\dfrac{40}{s^2+4}+s+1=\dfrac{s^3+s^2+4s+44}{s^2+4}$. $\therefore \bar{y}=\dfrac{s^3+s^2+4s+44}{\left(s^2+4\right)\left(s^2-s-2\right)}=-\dfrac{8}{3}\cdot\dfrac{1}{s+1}+\dfrac{8}{3}\cdot\dfrac{1}{s-2}+\dfrac{s-6}{s^2+4}$

Taking inverse Laplacetransform, $y=\dfrac{-8}{3}L^{-1}\left(\dfrac{1}{s+1}\right)+\dfrac{8}{3}L^{-1}\left(\dfrac{1}{s-2}\right)+L^{-1}\dfrac{s}{s^2+4}-6L^{-1}\dfrac{s}{s^2+4}$

$\therefore y=-\dfrac{8}{3}e^{-t}+\dfrac{8}{3}e^{-2t}+\cos 2t-3\sin 2t.$

48. Solve the equation $y+\int_0^t y\,dt=1-e^{-t}$.

Sol: Let $L(y)=\bar{y}$. Taking the Laplace transform of both sides, we get, $L(y)+L\left[\int_0^t y\,dt\right]=L(1)+L\left(e^{-t}\right)$

Since, $L\left[\int_0^t y\,dt\right]=\int_0^\infty e^{-st}\int_0^\infty y\,dt=L\left[\int_0^t y\,dt.\dfrac{e^{-st}}{s}\right]_0^\infty-\int_0^\infty-\dfrac{e^{-st}}{s}.y\,dt=0+\dfrac{1}{s}\int_0^\infty e^{-st}y\,dt=\dfrac{1}{s}L(y)=\dfrac{1}{s}\bar{y}$

and $Le^{-t}=\dfrac{1}{s+1}$, the equation becomes $\bar{y}+\dfrac{\bar{y}}{s}=\dfrac{1}{s}-\dfrac{1}{s+1}=\dfrac{1}{s(s+1)}$. $\therefore \bar{y}\dfrac{(s+1)}{s}=\dfrac{1}{s(s+1)}$ $\therefore \bar{y}=\dfrac{1}{(s+1)^2}$

$\therefore y=L^{-1}\dfrac{1}{(s+1)^2}=e^{-t}L^{-1}\dfrac{1}{s^2}=e^{-t}t$. $\therefore y=t\,e^{-t}$.

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Chap: Fourier Series and Fourier Integral

Theory Define the Fourier Series and derive a formulae for its coefficients or Derive Eulers Formulae for Fourier Coefficients. **(99, 02, 04)**

Theory State the Dirichlet's Conditions for the Fourier Series to exist. **(03, 05)**

Theory State and prove Parseval's Identity for a General Fourier Series. **(93, 99, 00, 03, 05)**

Theory Derive the formulae for fourier coefficients in complex form.

Theory Define Fourier Sine and Cosine Integrals. **(98, 05)**

Type 1A Fourier Series for interval $[0,2\pi]$

1) Find the Fourier expansion for $f(x)=\sqrt{1-\cos x}$ in $(0,2\pi)$. Hence deduce that $\frac{1}{2}=\sum_{n=1}^{n=\infty}\frac{1}{4n^2-1}$

(94, 99, 05, 06)

$$Ans.\left[a_0=\frac{4\sqrt{2}}{\pi}, a_n=\frac{-4\sqrt{2}}{\pi\left(4n^2-1\right)}, b_n=0, \text{ put } x=0\right]$$

2) Obtain the Fourier expansion of $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in the interval $0\le x\le 2\pi$ and $f(x+2\pi)=f(x)$.

Deduce that i) $\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$ ii) $\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ldots$ iii) $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

iv) $\frac{\pi^4}{90}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots$. **(99, 00, 02, 03, 03)** $Ans.\left[a_0=\frac{\pi^2}{6}, a_n=\frac{1}{n^2}, b_n=0, \text{ put } x=0 \text{ and } x=\pi\right]$

3) Find the Fourier series expansion for $f(x)=x\sin x$ in $(0,2\pi)$. **(98, 01, 02, 02, 03)**. Deduce that $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\ldots=\frac{\pi-2}{4}$. **(03)**

$$Ans.\left[a_0=-2, a_n=\frac{2}{n^2-1}, a_1=-\frac{1}{2}, b_1=\pi, b_n=0\right]$$

4) If $f(x)=2x,\ 0<x<2\pi$ and $f(x)=a_0+\sum_1^{\infty}a_n\cos nx+\sum_1^{\infty}b_n\sin nx$. Find b_{10} and a_4. **(2002)**

$$Ans.\left[a_0=4\pi, a_n=0, b_n=-\frac{4}{n}, b_{10}=-\frac{2}{5}, a_4=0\right]$$

5) Obtain the Fourier expansion of $f(x)=x(2\pi-x)$ for $0<x<2\pi$ and hence deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. **(2003)**

$$Ans.\left[a_0=4\pi^2/3, a_n=\frac{-4}{n^2}, b_n=0\right]$$

6) Find Fourier series to represent $f(x)=x^2$ in $(0,2\pi)$ and hence deduce that

$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$ **(2003)**

$$Ans.\left[a_0=\frac{8\pi^2}{3}, a_n=\frac{4}{n^2}, b_n=\frac{-4\pi}{n}\right]$$

7) Find a Fourier series for $f(x)$ which is defined as, $f(x)=\begin{cases} x & 0<x<\pi \\ 2\pi-x & \pi<x<2\pi \end{cases}$. Hence deduce that $\sum_0^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$ **(04, 06)**. Also deduce that $\frac{\pi^4}{96}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots$ **(05, 06)**

$$Ans.\left[a_0=\pi/4, a_n=\frac{2\left[1-(-1)^n\right]}{\pi n^2}, b_n=0\right]$$

8) Find the Fourier series of the function $f(x)=e^{-x},\ 0<x<2\pi$ and $f(x+2\pi)=f(x)$. **(2005)**

9) In interval $0 < x < 2\pi$, prove that $\frac{3x^2 - 6\pi x + 3\pi^2}{12} = \sum_1^{\infty} \frac{\cos nx}{n^2}$. Hence find the sum of the Series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$ **(2005)**

Type 1B Fourier Series for interval $[-\pi, \pi]$

1) Find fourier series of $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$ hence, deduce that i) $\frac{1}{2} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$

ii) $\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots$ iii) Show that $f(x) = \frac{1}{\pi} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{(4n^2 - 1)} + \frac{1}{2} \sin x$ **(00, 03, 04, 05, 06)**

Ans.

$$\left[a_0 = \frac{2}{\pi}, b_n = 0, b_1 = \frac{1}{2}, a_n = \frac{1 + (-1)^n}{\pi\left(1 - n^2\right)}, \text{put } x = 0 \text{ and put } x = \frac{\pi}{2} \right]$$

2) Find the Fourier Series for periodic function

$f(x) = \begin{cases} -\pi; & -\pi < x < 0 \\ x; & 0 < x < \pi \end{cases}$. State the value of $f(x)$ at $x = 0$ and hence, deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

$$\textit{Ans.} \left[a_0 = -\frac{\pi}{2}, a_n = \frac{(-1)^n - 1}{\pi n^2}, b_n = \frac{1 - 2(-1)^n}{n}, \text{put } x = 0, \text{ discontinuity at } x = 0 \right]$$

3) If $f(x) = \begin{cases} x - \pi & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$ Then find Fourier series of $f(x)$. Hence deduce that

$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$. **(03, 03)** $\textit{Ans.} \left[a_0 = -\pi, a_n = \frac{2\left(1 - (-1)^n\right)}{\pi n^2}, b_n = \frac{2\left(1 - (-1)^n\right)}{\pi n} \right]$

4) Find the Fourier series for, $f(x) = \begin{cases} \cos x, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$ **(2004)**

$$\textit{Ans.} \left[a_0 = 2/\pi, a_n = \frac{1 + (-1)^n}{\pi\left(1 - n^2\right)}, a_1 = 1/2, b_n = \frac{n\left[1 + (-1)^n\right]}{\pi\left(1 - n^2\right)}, b_1 = 1/2 \right]$$

Type 1C Fourier Series for general interval $[0, 2l]$

1) Find Fourier Series for $f(x) = \begin{cases} kx, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ $\textit{Ans.} \left[a_0 = k/2, a_n = \frac{k\left((-1)^n - 1\right)}{n^2\pi^2}, b_n = \frac{-(-1)^n k}{\pi n} \right]$

2) Expand $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ of period 2 into a Fourier Series. *Ans*

[subs $k = \pi$ in Q1 Type 1C]

3) Find fourier series for $f(x) = \begin{cases} \pi x, & 0 \le x < 1 \\ 0, & x = 1 \\ \pi(x - 2), & 1 < x \le 2 \end{cases}$ Hence, deduce that $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

(97)

$$Ans.\left[a_0 = 0, a_n = 0, b_n = \frac{-2(-1)^n}{n}\right]$$

4) Find Fourier series to represent $f(x) = 4 - x^2$ in the interval $(0, 2)$. Also state the values of series for $x = 1, 2, 11$. **(02, 03)**

$$Ans.\left[a_0 = \frac{16}{3}, a_n = -\frac{4}{n^2\pi^2}, b_n = \frac{4}{n\pi}, \text{discontinuity at } x = 0\right]$$

5) If $x^2 = \frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_1^\infty \frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right) - \frac{4l^2}{\pi}\sum_1^\infty \frac{1}{n}\sin\left(\frac{n\pi x}{l}\right)$ for $0 < x < 2l$, find the sum of the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ **(2003)**

$$Ans.\left[\frac{\pi^2}{3}\right]$$

6) If $f(x) = \begin{cases} \pi x & \text{for } 0 \le x \le 1 \\ \pi(2-x) & \text{for } 1 \le x \le 2 \end{cases}$ and hence prove that $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x}{(2n+1)^2}$ **(03, 03)**

$$Ans.\left[a_0 = \pi, a_n = \frac{2\left((-1)^n - 1\right)}{\pi n^2}, b_n = 0\right]$$

8) Express $f(x) = 2x - x^2$ in $(0,3)$ as a fourier series and deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ **(05)**

9) Find half range cosine series of $f(x) = x, 0 < x < 2$ and deduce that $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$ **(06)**

Type 1D Fourier Series for interval $[-l, l]$

1) Find the fourier series for $f(x) = e^{-x}$ in $(-a, a)$. **(2005)**

$$Ans.\left[a_0 = (2/a)\sinh a, a_n = \frac{2a(-1)^n \sinh a}{a^2 + n^2\pi^2}, b_n = \frac{2n\pi(-1)^n \sinh a}{a^2 + n^2\pi^2}\right]$$

Type 2 Odd and Even functions in Type 1

1) Find the Fourier expansion of $f(x) = \begin{Bmatrix} 0, & -2 < x < -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{Bmatrix}$ **(93, 03)**

Ans.

$$\left[\text{even}, a_0 = 1/2, a_n = \frac{4}{\pi^2 n^2}\{1 - \cos(n\pi/2)\}\right]$$

2) Obtain Fourier series for

$f(x) = x + \frac{\pi}{2}, \quad -\pi < x < 0$

$= \frac{\pi}{2} - x, \quad 0 < x < \pi$. Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ Also deduce that $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$ **(1999)**

Ans.
$$\left[\text{even}, b_n = 0, a_0 = 0, a_n = \frac{2\left[1-(-1)^n\right]}{\pi n^2}, \text{put } x = 0\right]$$

3) Obtain series for $f(x) = \cos px(-\pi, \pi)$, where p is not an integer. Hence, prove that
$$\cot p\pi = \frac{2p}{\pi}\left[\frac{1}{2p^2} + \frac{1}{p^2-1^2} + \frac{1}{p^2-2^2} + \frac{1}{p^2-3^2} + \ldots\right].$$
Also deduce that $\frac{1}{2} - \frac{\pi\sqrt{3}}{18} = \frac{1}{9.1^2-1} + \frac{1}{9.2^2-1} + \frac{1}{9.3^2-1} + \ldots$ (93, 96)

$$\textit{Ans.}\left[b_n = 0, a_0 = \frac{2\sin p\pi}{p\pi}, a_n = \frac{2p(-1)^n \sin p\pi}{\pi\left(p^2-n^2\right)}, x = \pi \;\&\; p = 1/3\right]$$

4) Find fourier series for $f(x) = x + x^2$ in $(-1,1)$ *Ans.* $\left[a_0 = 2/3, a_n = \frac{4(-1)^n}{\pi^2 n^2}, bn = \frac{2(-1)^{n+1}}{n\pi}\right]$

5) Find the Fourier expansion of $f(x) = x + x^2$ when $-\pi \le x \le \pi$ and $f(x+2\pi) = f(x)$.

Hence, deduce that i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ (ii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ (1996)

Ans.
$$\left[a_0 = \frac{2\pi^2}{3}, a_n = \frac{4(-1)^n}{n^2}, b_n = \frac{-2(-1)^n}{n}, \right.$$ i) put $x = 0$ for ii) put $x = \pi$ and add the results i) and ii) to get series. $]$

6) $f(x) = x - x^2$, $-\pi < x < \pi$. Hence, deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ (98,03,03,03)

$$\textit{Ans.}\left[a_0 = -\frac{2\pi^2}{3}, a_n = \frac{-4(-1)^n}{n^2}, b_n = \frac{-2(-1)^n}{n}, \text{put } x = 0\right]$$

7) Obtain Fourier Series for the function $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$. Deduce that

$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ (95, 97, 01, 04, 05) *Ans.* $\left[\text{even}, b_n = 0, a_0 = 0, a_n = \frac{4\left[1-(-1)^n\right]}{\pi^2 n^2}, \text{put } x = 0\right]$

8) Find fourier series for $f(x) = \sin mx$, $-\pi < x < \pi$ (97, 03) *Ans.* $\left[b_n = \frac{2n(-1)^{n+1}\sin m\pi}{\pi\left(n^2-m^2\right)}\right]$

9) Find F.Series for $f(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ (96) *Ans.* $\left[a_0 = k, a_n = \frac{2k}{n\pi}\sin\frac{n\pi}{2}, b_n = 0\right]$

10) $f(x) = x\sin x$ in $(-\pi, \pi)$. Hence, deduce that $\frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots$ (2000)

$$\textit{Ans.}\left[\text{even}, b_n = 0, a_0 = 2, a_n = \frac{-2(-1)^n}{\left(n^2-1\right)}, a_1 = -\frac{1}{2}; x = \frac{\pi}{2}\right]$$

11) Prove that $\sinh ax = \frac{2}{\pi}\sinh a\pi\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2+a^2}\sin nx\right]$ in $[-\pi,\pi]$. **(1998)**

12) Find fourier series for $f(x)=\begin{cases}-\sin\frac{\pi x}{c}, & -c<x<0\\ \sin\frac{\pi x}{c}, & 0<x<c\end{cases}$ **(1999)**

Ans. $\left[\text{even, } b_n=0, a_0=\frac{4}{\pi}, a_n=\frac{2\left(1+(-1)^n\right)}{\pi\left(1-n^2\right)}, a_1=0\right]$

13) Obtain the Fourier expansion of x^2 from $x=-l$ to $x=l$ and hence, deduce that:

$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+....$ and $\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....$

Ans. $\left[\text{even, } a_0=\frac{2l^2}{3}, a_n=\frac{4l^2(-1)^n}{n^2\pi^2}\right]$

14) Find F.Series for $f(x)=x^2-2,(-2\le x\le 2)$ ***Ans.*** $\left[\text{even, } a_0=-\frac{4}{3}, a_n=\frac{16(-1)^n}{n^2\pi^2}\right]$

15) Find fourier series for $f(x)=\begin{cases}\pi-x & \text{for} & 0<x<\pi\\ -\pi-x & \text{for} & -\pi<x<0\end{cases}$ **(03)** ***Ans.***

$\left[\text{odd }, b_n=\frac{2}{n}\right]$

16) P.T. for $-\pi<x<\pi$; $\frac{x\left(\pi^2-x^2\right)}{12}=\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-..$ **(03,03,04,05)** ***Ans***

$\left[b_n=\frac{(-1)^{n+1}}{n^3}\right]$

17) Obtain the Fourier expansion of $f(x)=\begin{cases}\cos x\ ; & -\pi<x<0\\ -\cos x\ ; & 0<x<\pi\end{cases}$ and $f(x+2\pi)=f(x)$. **(2003)**

Ans. $\left[\text{odd, } b_n=\frac{2n\left(1+(-1)^n\right)}{\pi\left(1-n^2\right)}, b_1=0\right]$

18) Find Fourier series for $f(x)=x\cos x$ for $-\pi\le x\le\pi$. **(03)** ***Ans.*** $\left[\text{odd, } b_n=\frac{2n(-1)^n}{(n^2-1)}, b_1=\frac{-1}{2}\right]$

19) Assuming that for $-\pi<x<\pi$, $x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$, prove that $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$. **(03, 04)**

20) Find Fourier series for $f(x)=|\sin x|,-\pi<x<\pi$. **(2003)**

Ans.

$\left[\text{even, } a_0=4/\pi, a_n=\frac{2\left[1+(-1)^n\right]}{\pi\left(1^2-n^2\right)}, a_1=0\right]$

21) Find F.Series for $f(x)=|x|,-\pi<x<\pi$. (2003) *Ans.*

$$\left[\text{even}, a_0=\pi/2, a_n=\frac{2\left[(-1)^n-1\right]}{\pi n^2}\right]$$

22) Find Fourier series of $f(x)=|x|$ in $(-3,3)$ and hence deduce that $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$. (1997)

$$\text{Ans}\left[a_0=3, a_n=\frac{6\left((-1)^n-1\right)}{n^2\pi^2}, x=0\right]$$

23) Find Fourier expansion for $f(x)=x-x^2,\ -1<x<1$. (2005)

24) Find the Fourier expansion of x^2 in $(0,a)$. (2005)

25) Find the Fourier series expansion of $f(x)=4-x^2;\ 0<x<2$ and hence deduce that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.....=\frac{\pi^2}{6}$. (2004)

26) Find the Fourier series of $f(x)=|x|;\ -1\le x\le 1$. (04,05) *Ans.*

$$\left[a_0=2\ ;\ a_n=\frac{2\left[(-1)^n-1\right]}{n^2\pi^2}\right]$$

27) Find fourier series for $f(t)=1-t^2, -1<t<1$. (2006)

28) Expand $f(x)=|\cos x|$ in a Fourier series. (2005)

29) Expand $f(x)=|\sin x|$ in a Fourier series. (2004)

Type 3 HalfRange Cosine and Sine Series

1) Expand $f(x)=lx-x^2.\ 0<x<l$ in a half-range (i) cosine series, (ii) sine series. Hence, from sine series deduce that $\frac{\pi^3}{32}=1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+....$ (94, 05, 06)

Ans.

$$\left[\text{i) } a_0=\frac{2l^2}{6}, a_n=\frac{-l^2\left[1+(-1)^n\right]}{n^2\pi^2}\quad \text{ii) } b_n=\frac{4l^2\left[1-(-1)^n\right]}{n^3\pi^3}; x=l/2\right]$$

2) Find half range sine series of period $2l$ for $f(x)=\begin{cases}\frac{2x}{l}, & 0\le x\le\frac{1}{2}\\ \frac{2}{l}(l-x), & \frac{l}{2}\le x\le l\end{cases}$ (98).

$$\text{Ans}\left[b_n=\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

3) Find a cosine series of period 2π to represent $\sin x$ in $0 \le x \le \pi$ (95, 99). Hence, deduce that $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\dots=\frac{1}{2}$ (1996). From the above expansion deduce that $\frac{1}{1^2 3^2}+\frac{1}{3^2 5^2}+\frac{1}{5^2 7^2}+\dots=\frac{\pi^2-8}{16}$ (97, 04, 05). Also deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$(2005)

$$Ans.\left[a_0=\frac{4}{\pi}, a_n=\frac{-2\left[1+(-1)^n\right]}{\pi\left(n^2-1\right)}, a_1=0, \text{ put } x=0, \text{ use Parseval's identity}\right]$$

4) Obtain the expansion of $f(x)=x(\pi-x),\ 0<x<\pi$ as a half-range cosine series. Hence, show that

(i) $\sum_{1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, (ii) $\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$, (iii) $\sum_{1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$, (93, 96, 99, 03)

$$Ans.\left[a_0=\frac{\pi^2}{3}, a_n=\frac{-2\left(1+(-1)^n\right)}{n^2} \quad \text{Use Parseval's identity}\right]$$

4a) Obtain the expansion of $f(x)=x(\pi-x),\ 0<x<\pi$ as a half-range sine series. Hence, show that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6}=\frac{\pi^6}{960}$. (2005)

5) Obtain half range sine series for $f(x)=\begin{cases}(1/4)-x, & 0<x<(1/2)\\ x-(3/4), & (1/2)<x<1\end{cases}$ (01)

$$Ans\left[b_n=-\frac{4\sin\frac{n\pi}{2}}{n^2\pi^2}+\frac{1-(-1)^n}{2n\pi}\right]$$

6) Find half range sine series for $f(x)$ where $f(x)=\begin{cases}x, & 0<x\le(\pi/2)\\ \pi-x, & (\pi/2)<x<\pi\end{cases}$. Hence, deduce that

$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots$(1993)Hence, find the sum of $\sum\frac{1}{(2n-1)^4}$. Hence, prove that

$\frac{\pi^4}{96}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots\dots\dots$(2004) Ans.

$$\left[b_n=\frac{4\sin(n\pi/2)}{\pi n^2}\right]$$

7) Show that in the interval $0<x<\pi$, $x\sin x=1-\frac{1}{2}\cos x-2\left[\frac{\cos 2x}{1.3}-\frac{\cos 3x}{2.4}+\frac{\cos 4x}{3.5}-\dots\right]$ (95,97)

$$Ans.\left[a_0=2, a_n=\frac{2(-1)^n}{1-n^2}, a_1=-\frac{1}{2}\right]$$

7a) Expand in half range sine series for $x\sin x$ in $(0,\pi)$ and hence deduce that $\frac{\pi^2}{8\sqrt{2}}=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots$

8) Find half range cosine series for $f(x)=x,\ 0<x<2$. Using Parseval's Identity, deduce that $\frac{\pi^4}{96}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots$ (94, 96)

Ans. $\left[a_0=2, a_n=\frac{4((-1)^n-1)}{n^2\pi^2}\right]$

9) Obtain half-range sine series in $(0,\pi)$ for $x(\pi-x)$ and hence, find the value of $\sum\frac{(-1)^n}{(2n+1)^3}$. (96, 04)

Ans. $\left[a_n=\frac{4(1-(-1)^n)}{n^3\pi}, \text{ put } x=\frac{\pi}{2}\right]$

11) Prove that in the interval $0<x<\pi$, $\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}}=\frac{2}{\pi}\left[\frac{\sin x}{a^2+1}-\frac{2\sin 2x}{a^2+4}+\frac{3\sin 3x}{a^2+9}-\ldots\right]$ (1996)

12) Obtain half range cosine series for $f(x)=\sin\left(\frac{\pi x}{l}\right)$ in $0<x<l$. (2000)

Ans. $\left[a_0=\frac{4}{\pi}, a_n=-\frac{2(1+(-1)^n)}{\pi(n^2-1)}, a_1=0\right]$

13) Obtain half range cosine series for $f(x)=x$ in $0<x<2$. **Ans.** $\left[a_0=2, a_n=\frac{4((-1)^n-1)}{\pi^2n^2}\right]$

14) Expand $f(x)=\begin{cases}kx & 0<x<l/2\\ 0 & l/2<x<l\end{cases}$ into half range cosine series. Deduce the sum of the series $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$ (1998)

Ans. $\left[a_0=lk/4,\ a_n=\frac{lk\sin\frac{n\pi}{2}}{n\pi}+\frac{2lk\left(\cos\frac{n\pi}{2}-1\right)}{n^2\pi^2}\right]$

15 Obtain half range sine series form $f(x)=x-x^2$ for $0\le x\le 1$ (1994) **Ans.** $\left[b_n=\frac{4(1-(-1)^n)}{\pi^3n^3}\right]$

15a) Obtain half range cosine series for $f(x)=x-x^2$ for $0\le x\le 1$. (2003)

16) Expand $f(x)=\begin{cases}kx & 0<x<\frac{L}{2}\\ k(L-x) & \frac{L}{2}<x<L\end{cases}$ into half range cosine series. Deduce the sum of the series $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$ (01, 02, 03, 03) **Ans.** $\left[a_0=\frac{kL}{2}, a_n=\frac{-2Lk}{n^2\pi^2}\left(1+(-1)^n-2\cos\frac{n\pi}{2}\right), \text{ put } x=0 \ \&\text{ sum}=\frac{\pi^2}{8}\right]$

16a) Expand $f(x)=Kx,\quad 0<x<\frac{1}{2}$

$=K(1-x),\ \frac{1}{2}<x<1$. into half range cosine series. Hence find sum of series $:\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$ (2003

Ans. Take L = 1 in 16)

17) Show that, if $0<x<\pi$ $\cos x=\frac{8}{\pi}\sum_{m=1}^{\infty}\left(\frac{m}{4m^2-1}\right)\sin 2mx.$ *Ans.* $\left[b_n=\frac{2n\left(1+(-1)^n\right)}{\pi\left(n^2-1\right)},\ b_1=0\right]$

17a) Obtain the Half range sine series for $f(x)=\cos x$ in $0<x<\pi$. **(2004)**

18) In $(0,\pi)$ show that $x^2=\frac{2}{\pi}\left[\left(\frac{\pi^2}{1}-\frac{4}{1^3}\right)\sin x-\frac{\pi^2}{2}\sin 2x+\left(\frac{\pi^2}{3}-\frac{4}{3^3}\right)\sin 3x+\ldots\right]$ **(03, 03)**

19) Find half range sine (and cosine) series of $f(x)=x(2-x)$ in $0<x<2$ and hence deduce that $\sum_{1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{960}$ **(03, 05)** *Ans.*

$$\left[b_n=\frac{16\left(1-(-1)^n\right)}{\pi^3 n^3}\text{, use Parseval's identity}\right]$$

20) Show that the Fourier Series expansion of the function

$$f(x)=\begin{cases}\frac{3kx}{L} & ;\ 0\le x\le\frac{L}{3}\\ \frac{3k(L-2x)}{L} & ;\ \frac{L}{3}\le x\le\frac{2L}{3}\\ \frac{3k(x-L)}{L} & ;\ \frac{2L}{3}\le x\le L\end{cases}\quad\text{is } f(x)=\frac{9k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{2n\pi}{3}\sin\frac{2n\pi x}{L}.$$ **(2003)**

21) Find half range cosine series for $f(x)=e^x$, $0<x<1$. **(2005)**

22) Expand $f(x)=\frac{1}{4}-x$, $0<x<\frac{1}{2}$

$=x-\frac{3}{4}$, $\frac{1}{2}\le x\le 1$ in the Fourier series of (i) Sine (ii) Cosine terms. **(2004)**

Type 4 Complex form of Fourier Series

1) Find the complex form of Fourier Series of $\cosh 3x+\sinh 3x$ in $(-\pi,\pi)$. **(1999)**

Ans.

$$\left[c_n=\frac{(-1)^n(3+in)\sinh 3\pi}{\pi\left(n^2+9\right)}\right]$$

2) Find complex form of Fourier Series for $f(x)=\sinh x$ in $(-l,l)$. *Ans.* $\left[c_n=\frac{in\pi(-1)^n\sinh l}{l^2+n^2\pi^2}\right]$

3) Obtain the complex form of Fourier Series for $f(x)=e^{ax}$ in $(-l,l)$. **(1993)**

Ans.

$$\left[c_n=\frac{(-1)^n\sinh al\left(al+in\pi\right)}{a^2l^2+n^2\pi^2}\right]$$

4) Find complex form of $f(x)=e^x$ in $(-\pi,\pi)$. **(99,00,04)** *Ans.*

$$\left[c_n=(-1)^n\frac{\sinh\pi}{\pi}\frac{(1+in)}{\left(1+n^2\right)}\right]$$

5) Obtain complex form of Fourier Series for $f(x)=e^{ax}$ in $(-\pi,\pi)$ where a is not an integer.

Ans. $\left[c_n = \frac{(-1)^n \sinh a\pi\,(a+in)}{\pi\left(a^2+n^2\right)}\right]$

6) Obtain complex form of Fourier series for $f(x)=e^{ax},\quad -1<x<1$. **(96, 03, 03, 04)**

Ans. $\left[c_n = \frac{(-1)^n \sinh a\,(a+in\pi)}{\left(a^2+n^2\pi^2\right)}\right]$

7) Find complex form of Fourier Series for $f(x)=\cosh(2x)+\sinh(2x)$ in $(-5,5)$. **(00, 03, 03)**

Ans. $\left[c_n = \frac{(-1)^n(10+in\pi)\sinh 10}{\left(100+n^2\pi^2\right)}\right]$

8) Find the complex form of fourier series for $f(x)=\begin{cases}1; & 0<x<1\\ 0; & 1<x<2\end{cases}$ **Ans.** $\left[c_n = \frac{i\left[(-1)^n-1\right]}{2n\pi}\right]$

9) Find complex form $f(x)=\sinh ax$ in $[-l,+l]$ **Ans.** $\left[c_n = \frac{(-1)^n in\pi \sinh al}{a^2l^2+n^2\pi^2}\right]$

10) Find the complex form of i) $\sin ax$ ii) $\cos ax$ in the interval $[-\pi,+\pi]$ **(03, 04)**

Ans. $\left[\text{i) } c_n = \frac{(-1)^{n+1} in \sin a\pi}{\pi\left(a^2-n^2\right)},\ \text{ii) } c_n = \frac{a(-1)^n \sin a\pi}{\pi\left(a^2-n^2\right)}\right]$

11) Derive the complex form of the Fourier series for $f(x)=e^{ax},\quad -\pi<x<\pi$ given that 'a' is real constant. Hence deduce that when α is a constant other than an integer

i) $\cos\alpha x = \frac{\sin\pi\alpha}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{\alpha}{\alpha^2-n^2}e^{inx}$ ii) $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+a^2}=\frac{\pi}{a\sinh a\pi}$ **(00, 01, 02, 02, 03)**

Ans. $\left[c_n = \frac{(-1)^n}{\pi}\frac{\sinh a\pi}{a^2+n^2}(a+in)\right]$ **Ans.** $\left[\text{for i) put } a=i\alpha \text{ to find R.P } e^{i\alpha x}\right]$

12) Find complex Form of fourier series for $f(x)=e^{-x}$ in $(-1,1)$. **(2005)**

13) Obtain complex form of Fourier series for $f(x)=e^{ax}, 0<x<2$. **(2005)**

14) Find the complex form of Fourier Series of $\cosh ax+\sinh ax$ in $(-l,l)$. **(2006)**

15) Find the complex form of Fourier Series of $\cosh ax$ in $(-l,l)$. **(2005)**

Type 5 Orthogonal and Orthonormal Set of Functions

Theory If $f(x)=c_1\,\phi_1(x)+c_2\,\phi_2(x)+c_3\,\phi_3(x)$, where c_1,c_2,c_3 constants and ϕ_1,ϕ_2,ϕ_3 are orthogonal sets on (a,b) show that $\int_a^b\left[f(x)\right]^2 dx=c_1^2+c_2^2+c_3^2$. **(04, 05)**

Theory If $f_i(x)$, $i=1,2,3,\ldots\ldots$ is a set of orthogonal functions on $[a,b]$ and $g(x)=\sum_{i=1}^{\infty}a_i f_i(x)$, then find a_i. **(2003)**

1) Show that the set of functions $\sin(2n+1)x$, $n=0,1,2,\ldots$ is orthogonal over $[0,\pi/2]$. Hence, construct orthonormal set of functions. **(1993)**

2) Show that the set of functions $\sin x,\sin 2x,\sin 3x,\ldots$ is orthogonal on the interval $[0,\pi]$.

3) Show that the set of functions $1,\ \sin\frac{\pi x}{L},\ \cos\frac{\pi x}{L},\ \sin\frac{2\pi x}{L},\ \cos\frac{2\pi x}{L},\ \ldots$ Form an orthogonal set in $(-L,L)$ and construct an orthonormal set. **(1998)**

4) Show that the set of functions $\cos x,\cos 2x,\cos 3x,\ldots$ is a set of orthogonal functions over $[-\pi,\pi]$. Construct a set of orthonormal functions. **(95, 98)** ***Ans.*** $\left[f_n(x)=\frac{1}{\sqrt{\pi}}\cos nx\right]$

5) Show that the set of functions $\sin\left(\frac{\pi x}{2L}\right),\ \sin\left(\frac{3\pi x}{2L}\right),\ \sin\left(\frac{5\pi x}{2L}\right),\ldots$ is orthogonal over $(0,L)$. **(96, 05)**

6) Prove that $\sin x,\sin 2x,\sin 3x,\ldots$ is orthogonal on $[0,2\pi]$ and construct orthonormal set of functions. **(94, 97, 99, 00, 02)** ***Ans.*** $\left[\frac{2}{\sqrt{\pi}}\sin(2n+1)x\ i.e.\ \frac{2}{\sqrt{\pi}}\sin x\ \frac{2}{\sqrt{\pi}}\sin 3x\right]$

7) Prove that $f_1(x)=1,\ f_2(x)=x,\ f_3(x)=(3x^2-1)/2$ are orthogonal over $(-1,1)$. **(97, 05, 05, 06)**

8) Show the set of functions $e^{\frac{-x}{2}},\ e^{\frac{-x}{2}}(1-x),\ e^{\frac{-x}{2}}(2-4x+x^2)$ are orthogonal over $(0,\infty)$. **(03,04)**

9) Show that the functions $f_1(x)=1, f_2(x)=x$ are orthogonal on $(-1,1)$. Determine the constants 'a' and 'b' such that the function $f_3(x)=-1+ax+bx^2$ is orthogonal to both f_1 and f_2 on that interval. **(03, 05)**

10) Show that the set of functions $\frac{\text{Cos}\,x}{\sqrt{\pi}},\ \frac{\text{Cos}\,2x}{\sqrt{\pi}},\ \frac{\text{Cos}\,3x}{\sqrt{\pi}},\ \ldots$ from orthonormal set in the interval $[-\pi,\pi]$. **(2003)**

11) Determine if $\{\sin\pi x,\sin 3\pi x,\sin 5\pi x,\ldots\}$ is orthogonal over [0,2] **(2005)**

12) Show that set of $\{1,\sin x,\cos,\sin 2x,\cos 2x,\ldots\}$ is orthogonal over $(0,2\pi)$ and construct the corresponding orthonormal set. **(2006)**

Type 6 Fourier Cosine and Sine Integral Representations

1) Express the function $f(x)=\begin{cases}\sin x, & 0\le x\le\pi\\ 0, & x>\pi\end{cases}$ as Fourier Sine Integral and evaluate

$\int_0^\infty \frac{\sin\omega x\ \sin\pi\omega}{1-\omega^2}d\omega$ (00,06). Also evaluate it as a fourier integral and prove that

$$f(x)=\frac{1}{\pi}\int_0^\infty \frac{\cos\omega x+\cos\omega(\pi-x)}{1-\omega^2}d\omega.\ \textbf{(2001)}$$

2) Find Fourier Integral representation for $f(x)=\begin{cases}1-x^2 & \text{for } |x|\le 1\\ 0 & \text{for } |x|>1\end{cases}$. Evaluate

$\int_0^\infty \frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}\,dx$ (98, 03, 04, 04, 05) Ans.

$$\left[f(x)=\frac{4}{\pi}\int_0^\infty \frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\cos\lambda x\,d\lambda\right]$$

3) Express the function $f(x)=\begin{cases}\pi/2 & \text{for } 0<x<\pi\\ 0 & \text{for } x>\pi\end{cases}$ as Fourier Sine Integral. Hence show that

$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x(d\omega)=\frac{\pi}{2}$ when $0<x<\pi$. (1998)

4) Find Fourier integral representation of $f(x)=\begin{cases}e^{ax} & x\le 0,\ a>0\\ e^{-ax} & x\ge 0,\ a>0\end{cases}$. Hence, show that

$\int_0^\infty \frac{\cos\omega x}{\omega^2+a^2}d\omega=\frac{\pi}{2a}e^{-ax},\ x>0,\ a>0.$ (96, 97, 02)

5) Express the function $f(x)=\begin{cases}1 & \text{for } |x|<1\\ 0 & \text{for } |x|>1\end{cases}$ as Fourier Integral. Hence evaluate $\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}d\omega$.

(97, 99).Show that $\int_0^\infty \frac{\sin\omega\cos\omega d\omega}{\omega}=\frac{\pi}{4}$. (2003)

6) Find Fourier Integral representation of $f(x)=\begin{cases}x, & 0<x<a\\ 0, & x>a\end{cases}$ with $f(-x)=f(x)$ (95, 05)

$$\textit{Ans.}\left[f(x)=\frac{2}{\pi}\int_0^\infty (a\lambda\sin\lambda a+\cos\lambda a-1)\frac{\cos\lambda x}{\lambda^2}d\lambda\right]$$

7) Find Fourier Sine integral of $f(x)=\begin{cases}x, & 0<x<1\\ 2-x, & 1<x<2\\ 0, & x>2\end{cases}$ (05)

$$\textit{Ans.}\left[f(x)=\frac{2}{\pi}\int_0^\infty \frac{(2\sin\lambda-\sin 2\lambda)}{\lambda^2}\sin\lambda x\,d\lambda\right]$$

8) Express the function $f(x)=\begin{cases}-e^{kx}; & x<0\\ e^{-kx} & ; x>0\end{cases}$ as a fourier integral and hence prove that

$$\int_0^\infty \frac{\omega \sin \omega x}{\omega^2+k^2}\,d\omega=\frac{\pi}{2}e^{-kx} \text{ if } x>0; k>0.$$ **(2002)**

9) Find the fourier cosine integral of $f(x)=1-x^2;\ 0\le x\le 1$ and 0 otherwise. Prove that

$$\int_0^\infty \frac{\omega\cos\omega-\sin\omega}{\omega^3}\cos\frac{\omega}{2}\,d\omega=\frac{3\pi}{16}.$$ **(2003)**

10) Find the fourier cosine and sine integral of $f(x)=xe^{-ax}$.

11) Find the fourier sine integral of $f(x)=\frac{\pi}{2}e^{-x}\cos x;\ x>0$ and show that

$$\int_0^\infty \frac{\omega^3 \sin \omega x}{\omega^4+4}\,d\omega=\frac{\pi}{2}e^{-x}\cos x$$

12) Using Fourier Cosine integral prove that $e^{-x}\cos x=\frac{2}{\pi}\int_0^\infty \frac{(w^2+2)}{(w^4+4)}\cos wx\,dw$ **(2002)**

13) Find the Fourier transform of $f(x)$, defined as $\begin{array}{ll} f(x)=1 & \text{for } |x|<a \\ 0= & \text{for } |x|>a\end{array}$ **(2003)**

14) Find the Fourier transforms of $f(x)=\begin{cases}1-x^2 & \text{for } |x|\le 1\\ 0 & \text{for } |x|>1\end{cases}$ **(03, 03)**

15) Using Fourier sine integral for the function $f(x)=\frac{e^{-kx}}{x}$. Show that

$$\frac{e^{-kx}}{x}=\frac{2}{\pi}\int_0^\infty \tan^{-1}\left(\frac{\lambda}{k}\right)\sin\lambda x\,d\lambda.$$ **(2004)**

Some Fourier Integrals Solved Examples
(Not fourier series)

Ex. 1: Expose the function $f(x)=\begin{cases}1 & \text{for } |x|<1\\ 0 & \text{for } |x|>1\end{cases}$ as Fourier Integral.

Hence, evaluate $\int_0^{\infty}\frac{\sin\omega\sin\omega x}{\cos\omega}d\omega$.

Sol.: The Fourier Integral for $f(x)$ is

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(s)\cos\omega(s-x)\,ds\,d\omega$$

[By data $f(s)=0$ from $-\infty$ to -1, $f(s)=1$ from -1 to 1 and $f(s)=0$ from 1 to ∞].

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(s)\cos\omega(s-x)\,ds\,d\omega$$

Substituting for $f(s)$ in above formula, we get

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-1}^{1}1\cdot\cos\omega(s-x)\,ds\,d\omega$$

$$=\frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin\omega(s-x)}{\omega}\right]_{-1}^{1}d\omega$$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{\sin\omega(1-x)-\sin\omega(-1-x)}{\omega}d\omega$$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{\sin\omega(1-x)+\sin\omega(1+x)}{\omega}d\omega$$

$$=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega\cos\omega x}{\omega}d\omega$$

$$\therefore\int_0^{\infty}\frac{\sin\omega\cdot\cos\omega x}{\omega}d\omega=\frac{\pi}{2}\cdot f(x)$$

$$=\begin{cases}\frac{\pi}{2} & \text{for } f(x)=1 \quad |x|<1\\ 0 & \text{for } f(x)=0 \quad |x|>1\end{cases}$$

At $|x|=1$ i.e. $x=\pm1$, $f(x)$ is discontinuous and the integral

$$=\frac{\pi}{2}\cdot\frac{1}{2}\left\{\lim_{x\to1^{-1}}f(x)+\lim_{x\to1^{+1}}f(x)\right\}=\frac{\pi}{4}[1+0]=\frac{\pi}{4}$$

Ex. 2: Expressing the above function as Fourier Integral, evaluate

$$\int_0^{\infty}\frac{\sin\omega}{\omega}d\omega$$

Sol.: In the final result obtained above put $x=0$.

$$\therefore\int_0^{\infty}\frac{\sin\omega}{\omega}d\omega=\frac{\pi}{2}f(0)=\frac{\pi}{2}\qquad[\because f(0)=1]$$

Note: Unfortunately there is no uniformity in the notation of Fourier Integral and Fourier transforms. Some authors use λ or α in place of ω and t in place of s.

Ex. 3: Express the function

$$f(x)=\begin{cases}-e^{kx} & \text{for } x<0\\ e^{-kx} & \text{for } x>0\end{cases}$$

asFourier Integral and hence, prove that

$$\int_0^\infty \frac{\omega\sin\omega x}{\omega^2+k^2}\,d\omega=\frac{\pi}{2}e^{-kx}\quad \text{if } x>0,\ k>0$$

Sol.: The Fourier Integral for $f(x)$ is

$$f(x)\ =\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(s)\cos\omega(s-x)\,d\omega\,ds$$

But since the given function $f(x)$ is an odd function we use (3), of \$ 3(b).

$$f(x)\ =\frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\infty e^{-ks}\sin\omega s\,d\omega\,ds$$

$$=\frac{2}{\pi}\int_0^\infty \sin\omega x\left[\frac{1}{k^2+\omega^2}e^{-ks}\left(-k\sin\omega s-\omega\cos\omega s\right)\right]_0^\infty d\omega$$

$$f(x)\ =\frac{2}{\pi}\int_0^\infty \sin\omega x\cdot\frac{\omega}{k^2+\omega}\,d\omega$$

$$\therefore\quad \int_0^\infty \frac{\omega\sin\omega x}{\omega^2+k^2}\,d\omega=\frac{\pi}{2}f(x)=\frac{\pi}{2}e^{-kx}\quad \text{if } x>0$$

Ex. 4: Express the function

$$f(x)=\begin{cases}\sin x & 0\le x\le\pi\\ 0 & 0\end{cases}$$

as Fourier sine integral and evaluate, $\int_0^\infty \frac{\sin\omega x\cdot\sin\pi\omega}{1-\omega^2}\,d\omega$

Sol.: The Fourier Sine Integral for $f(x)$ is

$$f(x)\ =\frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\infty f(s)\sin\omega s\,d\omega\,ds$$

$$=\frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\pi \sin s\sin\omega s\,d\omega\,ds$$

$$=\frac{2}{\pi}\int_0^\infty \sin\omega x\cdot\left(-\frac{1}{2}\right)\int_0^\pi\left[\cos s(1+\omega)-\cos s(1-\omega)\right]d\omega\,ds$$

$$=\frac{2}{\pi}\int_0^\infty \sin\omega x\left(-\frac{1}{2}\right)\left[\frac{\sin s(1+\omega)}{1+\omega}-\frac{\sin s(1-\omega)}{1-\omega}\right]_0^\pi d\omega$$

$$=\frac{2}{\pi}\int_0^\infty \sin\omega x\cdot\left(-\frac{1}{2}\right)\left[-\frac{2\sin\pi\omega}{1-\omega^2}\right]d\omega$$

$$\left[\because\ \sin(\pi+\theta)=-\sin\theta\ \text{ and }\ \sin(\pi-\theta)=\sin\theta\right]$$

$$\therefore\quad f(x)=\frac{2}{\pi}\int_0^\infty \frac{\sin\omega x\sin\pi\omega}{1-\omega^2}\,dx$$

$$\therefore\quad \int_0^\infty \frac{\sin\omega x\cdot\sin\pi\omega}{1-\omega^2}\,dx\ =\frac{\pi}{2}f(x)$$

$$=\frac{\pi}{2}\begin{cases}\sin x & 0\le x\le\pi\\ 0, & 0\end{cases}$$

Ex. 5: Find the Fourier cosine integral representation of the function $f(x)=e^{-ax}, x>0$ and hence, show that

$$\int_0^\infty \frac{\cos\omega s}{1+\omega^2}d\omega=\frac{\pi}{2}e^{-x}, x\geq 0$$

Sol.: Fourier cosine Integral representation of $f(x)$ is

$$f(x) =\frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty f(s)\cos\omega s\, d\omega\, ds$$

$$=\frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty e^{-as}\cos\omega x\, d\omega\, ds$$

$$=\frac{2}{\pi}\int_0^\infty \cos\omega x\left[\frac{e^{-as}}{a^2+\omega^2}(-a\cos\omega x+s\sin\omega s)\right]_0^\infty d\omega$$

$$=\frac{2}{\pi}\int_0^\infty \cos\omega x\left[\frac{a}{a^2+\omega^2}\right]d\omega$$

$$=\frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{a^2+\omega^2}d\omega$$

For deduction put $a=1$,

$$\therefore\ e^{-x}=\frac{2}{\pi}\int_0^\infty \frac{\cos\omega x}{1+\omega^2}d\omega$$

$$\therefore\ \int_0^\infty \frac{\cos\omega x}{1+\omega^2}d\omega=\frac{\pi}{2}e^{-x}$$

Ex. 6: Find the Fourier cosine integral representation of the function

$$f(x)=\begin{cases}\cos x & |x|<(\pi/2)\\ 0, & |x|>(\pi/2)\end{cases}$$

Sol.: Fourier cosine Integral representation of $f(x)$ is

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty f(s)\cos\omega s\, d\omega\, ds$$

$$=\frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^{\pi/2} \cos s\cos\omega s\, d\omega\, ds$$

$$=\frac{2}{\pi}\int_0^\infty \cos\omega x\left(\frac{1}{2}\right)\int_0^{\pi/2}\left[\cos(1+\omega)s+\cos(1-\omega)s\right]d\omega\, ds$$

$$=\frac{2}{\pi}\int_0^\infty \cos\omega x\left(\frac{1}{2}\right)\left[\frac{\sin(1+\omega)s}{1+\omega}+\frac{\sin(1-\omega)s}{1-\omega}\right]_0^{\pi/2} ds$$

$$=\frac{2}{\pi}\int_0^\infty \cos\omega x\left(\frac{1}{2}\right)\left[\frac{\sin\pi(1+\omega)/2}{1+\omega}+\frac{\sin\pi(1-\omega)/2}{1-\omega}\right]d\omega$$

But $\sin\left(\frac{\pi}{2}+\frac{\pi\omega}{2}\right)=\cos\frac{\pi\omega}{2}$ and $\sin\left(\frac{\pi}{2}-\frac{\pi\omega}{2}\right)=\cos\frac{\pi\omega}{2}$

$$\therefore f(x)=\frac{2}{\pi}\int_0^\infty \cos\omega x\cdot\left(\frac{1}{2}\right)\cdot\left[\frac{\cos(\pi\omega/2)}{1+\omega}+\frac{\cos(\pi\omega/2)}{1-\omega}\right]d\omega$$

$$=\frac{2}{\pi}\int_0^\infty \cos\omega x\left(\frac{1}{2}\right)\cdot\frac{2\cdot\cos(\pi\omega/2)}{1-\omega^2}d\omega$$

$$=\frac{2}{\pi}\int_0^\infty \frac{\cos\omega x\cdot\cos(\pi\omega/2)}{1-\omega^2}d\omega$$

COMPLEX VARIABLES Part I

C-R Equations, Analytic Functions, Harmonic Functions

Type 1A Problems on limits, continuity and differentiability of a complex function $f(z)=u+iv$

1. Show that the function $f(z)=x+i4y$ is nowhere differentiable.
2. Express the given function in the form $u+iv$; i) $f(z)=7z-9i\bar{z}-3+2i$ ii) $f(z)=z^2-3z+4i$
3. Does limit of the following function exist ; $\lim\limits_{z\to 0}\frac{z}{\bar{z}}$?
4. Discuss continuity of $f(z)=|z|^2$. At which points is $f(z)$ differentiable ?
5. Show that $\lim\limits_{z\to 0}\frac{xy}{x^2+y^2}$ does not exist ?

Type 1B Theory questions on analytic functions

1. Show that Cauchy-Riemann conditions are necessary conditions for analyticity of a complex function but not a sufficient condition.
2. Define analytic function, differentiable function, analyticity at a point, analyticity in a domain D.
3. State and prove C-R equations in polar form. (99, 02, 04, 05, 06, 06)
4. State and prove the conditions for a function $w=f(z)$ to be analytic. (97, 03, 04)
5. State and prove Cauchy – Riemann equations in cartesian form. (05)
6. If $f(z)=u+iv$ is analytic and $z=re^{i\theta}$, show that $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$. (96, 02, 03)
7. If $f(z)$ and $\overline{f(z)}$ are both analytic, prove that $f(z)$ is constant. (93, 03)
8. If $f(z)$ is an analytic function, show that $\frac{\partial f}{\partial \bar{z}}=0$. (1996)
9. If $f(z)$ is an analytic and $|f(z)|$ is constant, prove that $f(z)$ is constant. (97, 99, 02, 03, 05, 05)
   OR If $f(z)$ is analytic and if the amplitude of $f(z)$ is constant, prove that $f(z)$ is constant. (2003)
10. State true or false with proper justification; If $f(z)$ is an analytic function such that $f'(z)=0$, then $f(z)$ is a constant function. (2004)
11. If $f(z)=u+iv$ is analytic in R show that (i) $\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=|f'(z)|^2$ (2006)
   (ii) $\left[\frac{\partial|f(z)|}{\partial x}\right]^2+\left[\frac{\partial|f(z)|}{\partial y}\right]^2=|f'(z)|^2$ (95, 97, 04)
12. If $f(z)$ be a regular function in any domain then prove that $\nabla^2|f(z)|^2=4|f'(z)|^2$. (93, 96, 98, 03, 06)
13. If $f(z)$ is analytic then show that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^n=n^2|f(z)|^{n-2}|f'(z)|^2$. (2004)

14. If $f(z)=u+iv$ is an analytic function in any domain, prove

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u)^n=n(n-1)(u)^{n-2}\left|f'(z)\right|^2.$$

15. If $f(z)=u+iv$ is analytic then prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\left|f'(z)\right|=0.$

16a. If $f(z)$ is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\text{Re}[f(z)]^2=2\left|f'(z)\right|^2.$

16b. Prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=4\frac{\partial^2}{\partial z\partial \bar{z}}.$

17. If $f(z)=\bar{z}$ then show that $f'(z)$ does not exist. **(2002)**

18. State true or false with proper justification " If $f(z)$ and $\overline{f(z)}$ are both analytic then $f(z)$ is a constant function." **(1995)**

19. If $w=f(z)$ is an analytic function and $z=x+iy$ where $x=r\cos\theta, y=r\sin\theta$ then show that $\frac{dw}{dz}=(\cos\theta-i\sin\theta)\frac{\partial w}{\partial r}$. **(2002)**

Type 1C Problems on proving analyticity or non-analyticity of complex functions

1. Are the following functions analytic ; if yes ; then find their derivatives. (i) $e^{\bar{z}}$ (ii) $\sinh \bar{z}$.(iii) $f(z)=z^3$ (iv) $f(z)=ze^{\bar{z}}$ (v) $f(z)=\sin z$. **(03)** (vi) x^2-y^2+2ixy **(04)** (xi) $\left(x^3-3xy^2+3x\right)+i\left(3x^2y-y^3+3y\right)$ **(04)**

1a. Show that the function $f(z)=2x^2+y+i\left(y^2-x\right)$ is not analytic at any point.

2. Discuss the analyticity of $f(z)$ for the following : (i) $\bar{z}$ (ii) $2x+ixy^2$ (iii) $f(z)=|z|^2$ (iv) $f(z)=\frac{x}{x^2+y^2}-\frac{iy}{x^2+y^2}$ (v) $f(z)=\frac{xy^2(x+iy)}{x^2+y^4}$.

3. Find k such that the following function is analytic $\frac{1}{2}\log\left(x^2+y^2\right)+i\tan^{-1}\frac{kx}{y}$.**(2006)**

4. Determine the constants a, b, c, d if $f(z)=x^2+2axy+by^2+i\left(cx^2+2dxy+y^2\right)$ is analytic. **(1998)**

Ans [$a=1,\ b=-1,\ c=-1,\ d=1$.]

5. Show that $f(z)=z\bar{z}=|z|^2$ satisfies C-R equations at $z=0$ and yet is not analytic anywhere

6. Prove that $f(z)=\frac{x^2(1+i)-y^2(1-i)}{x+y}$, $z\neq 0$

$f(0)=0$; Is not analytic although Cauchy-Riemann equations are satisfied at $z=0$.

7. Find the constants a, b, c, d, e if (i) $f(z)=\left(ax^3+by^2+3x^2+cy^2+x\right)+i\left(dx^2y-2y^3+exy+y\right)$

(ii) $f(z)=\left(ax^4+bx^2y^2+cy^4+dx^2-2y^2\right)+i\left(4x^3y-exy^3+4xy\right)$ are analytic. **(02, 03)**

Ans. (i) $[a=2,\ b=-6,\ c=-3,\ d=6,\ e=6]$ *Ans.* (ii) $[a=1,\ b=-6,\ c=1,\ d=2,\ e=4]$

8. Show that the following function is not analytic although Cauchy-Riemann equations are satisfied $f(z)=\dfrac{xy(y-ix)}{x^2+y^2},\ z\neq 0;\ f(0)=0$ **(2004)**

9. Verify that the real and imaginary parts of $f(z)=ze^{2z}$ satisfy the Cauchy-Riemann equations. **(2003)**

10. Find p if $f(z)=r^2\cos 2\theta+ir^2\sin p\theta$ is analytic. **(98, 03)** *Ans.* $[p=2.]$

11. State true or false with proper justification $f(z)=(x^3-3xy^2+3x)+i(3x^2y-y^3+3y)$ is an analytic function. **(2004)**

12. Find the constants a, b, c, d, e if (i) $f(z)=(ax^3+bxy^2+3x^2+cy^2+x)+i(dx^2y-2y^3+exy+y)$ is analytic. **(2006)**

Type 2 Finding Real and Imaginary parts of an analytic function

1. Construct an analytic function whose real part is $e^x\cos y$. *Ans.* $[e^z+c]$

2. Construct the analytic function whose real part is $e^{2x}(x\cos 2y-y\sin 2y)$ *Ans.* $[e^{2z}z+c.]$

3. Find the analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y+\cos 2x}$. **(02, 06)** *Ans.* $[\tan z+c.]$

4. Find an analytic function whose imaginary part is $e^{-x}(y\cos y-x\sin y)$. **(2006)** *Ans.* $[ze^{-z}+c.]$

5. Find an analytic function $f(z)$ whose imaginary part is $e^{-x}(y\sin y+x\cos y)$. *Ans.* $[ie^{-z}+c.]$

6. If $f(z)=u+iv$ is analytic and $u-v=e^x(\cos y-\sin y)$, find $f(z)$ in terms of z. *Ans.* $[e^z+c.]$

7. If $f(z)=u+iv$ is analytic and $u+v=\dfrac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$, find $f(z)$. **(98, 03, 04)**

8. Find the analytic function $f(z)=u+iv$ such that $u-v=\dfrac{\cos x+\sin x-e^{-y}}{2\cos x-e^y-e^{-y}}$ when $f(\pi/2)=0$. **(93, 03)** *Ans.* $\left[\dfrac{1}{2}\left(1-\cot\dfrac{z}{2}\right).\right]$

9. Find the analytic function $f(z)=u+iv$ if $3u+2v=y^2-x^2+16xy$. **(2002)** *Ans.* $[(1-2i)z^2+c]$

11a. If $f(z)=u+iv$ is analytic and $u-v=\dfrac{e^y-\cos x+\sin x}{\cosh y-\cos x}$ when $f(\pi/2)=\dfrac{3-i}{2}$; find $f(z)$ in terms of z. **(06)**

11b. If $f(z)=u+iv$ is analytic and $u-v=\dfrac{-e^{-y}+\cos x+\sin x}{2\cos x-e^y-e^{-y}}$ when $f(\pi/2)=0$; find $f(z)$ in terms of z. **(06)**

12. Find the analytic function $f(z)=u+iv$ if $u+v=e^x(\cos y+\sin y)+\dfrac{x-y}{x^2-y^2}$ **(05, 05)**

14. Find the analytic functions whose real parts are (i) $u = e^{-x}(x\sin y - y\cos y)$ *Ans.* $\left[ie^{-z} + c\right]$

(ii) $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ **(1995)** *Ans.* $\left[\dfrac{i}{1+i}\cot z + c\right]$

(iii) $u = \dfrac{x}{2}\log\left(x^2+y^2\right) - y\tan^{-1}\left(\dfrac{y}{x}\right) + \sin x\cosh y$ **(2003)** *Ans.* $[z\log z + \sin z + c.]$

15. Find the analytic functions whose imaginary parts are

(i) $v = \log\left(x^2+y^2\right) + x - 2y$ *Ans.* $\left[(i-2)\log z + i\log z + c\right]$ (ii) $v = \dfrac{x-y}{x^2+y^2}$ *Ans.* $\left[\dfrac{(1-i)}{z} + c\right]$

(iii) $v = \dfrac{\sinh 2y}{\cos 2x + \cosh 2y}$ *Ans.* $[\tan z + c]$

(iv) $v = e^{-x}\left[2xy\cos y + \left(y^2 - x^2\right)\sin y\right]$ **(2003)** *Ans.* $\left[e^{-z}.z^2 + c\right]$

(v) $\dfrac{x}{x^2+y^2} + \cosh x\cos y$ **(2002)** *Ans.* $\left[i\left(\dfrac{1}{z} + \cos hz\right) + c\right]$

16. Find the analytic function $f(z) = u + iv$ in terms of z if

(i) $u - v = (x-y)\left(x^2+4xy+y^2\right)$ **(99, 03, 04, 06)** *Ans.* $\left[f(z) = -iz^3 + c\right]$

(ii) $u + v = \dfrac{x}{x^2+y^2}$ **(2003)** *Ans.* $\left[\dfrac{1}{1+i}\left(\dfrac{i}{z} + 1\right)\right]$

(iii) $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ and $f(\pi/2) = 0$ **(1996)** *Ans.* $\left[\dfrac{1-i}{2} + \cot\left(\dfrac{z}{2}\right).\right]$

17. Let $f(z) = u(r,\theta) + iv(r,\theta)$ be an analytic function. If $u = -r^3\sin 3\theta$ then find the analytic function in terms of z. **(2004)** *Ans.* $\left[iz^3 + c.\right]$

Type 3A Theory for harmonic functions and laplace's equation for analytic functions

1. Show that a harmonic function satisfies the differential equation $\dfrac{\partial^2 u}{\partial z\partial \bar{z}} = 0.$
2. If $u(x,y)$ is a harmonic function then prove that $f(z) = u_x - iu_y$ is an analytic function. **(2004)**
3. If u, v are harmonic conjugate functions, show that uv is a harmonic function. **(2003)**
4. If ϕ and ψ are functions of x and y satisfying Laplace equation and if $u = \phi_y - \psi_y$ and $v = \phi_x + \psi_y$ prove that $u + iv$ is analytic (holomorphic). **(03, 05)**
5. Prove that real and imaginary parts of an analytic function satisfy Laplace equation. **(1993)**
6. Show that real and imaginary parts of an analytic function are harmonic. **(1996)**
7. If $f(z) = u + iv$ is analytic in a region R then show that (i) u and v are harmonic functions and (ii) u = constant and v = constant intersect orthogonally. **(94, 97, 98, 03, 05, 06)**
8. If $f(z) = u + iv$ is analytic in a region R show that

   (i) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ (ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y} = 0,\ \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0.$ **(1995)**
9. Prove that the real part of an analytic function $f(z) = u(r,\theta) + iv(r,\theta)$ is an harmonic function. **(05)**
10. True or false : If u and v are harmonic functions then $f(z) = u + iv$ is analytic function. **(1995)**

Type 3B Problems on Harmonic functions, Laplace's Equation and Orthogonal Trajectories

I) 1. Show that the following functions are harmonic and find its harmonic conjugate

(i) $e^{x}\cos y+x^{3}-3xy^{2}$ **(03)**(ii) $e^{2x}\left(x\cos 2y-y\sin 2y\right)$ **(03)** (iii) $\log\sqrt{x^{2}+y^{2}}$ **(03, 06)**

2. Prove that $u(x,y)=x^{2}-y^{2}$ and $v(x,y)=-y/\left(x^{2}+y^{2}\right)$ are both harmonic functions but $u+iv$ is not analytic. **(2003)**

3. Show that the function $u=\sin x\cosh y+2\cos x\sinh y+x^{2}-y^{2}+4xy$ satisfies Laplace's equation and find its corresponding analytic function $f(z)=u+iv$. ***Ans.*** $\left[\sin z+z^{2}-i\left(2\sin z+2z^{2}\right)+c.\right]$

4. Prove that $u=x^{2}-y^{2}$, $v=-\dfrac{y}{x^{2}+y^{2}}$ both u and v satisfy Laplace's equation, but that $u+iv$ is not an analytic function of z. **(96, 03)**

5. Show that the following function satisfies Laplace's equation and find its corresponding analytic function and the harmonic conjugate, $u=\dfrac{1}{2}\log\left(x^{2}+y^{2}\right)$. ***Ans.*** $\left[\tan^{-1}\dfrac{y}{x}+c\right]$.

6. Prove that $u=e^{x}\cos y+x^{3}-3xy^{2}$ is harmonic. **(2003)**

7. Check whether $u=x+e^{xy}+y+e^{-xy}$ is harmonic. **(2004)**

8. Show that $u=\dfrac{1}{2}\log\left(x^{2}+y^{2}\right)$ is harmonic and find its harmonic conjugate. ***Ans.*** $\left[e^{z}+c.\right]$

9. Show that $u=\dfrac{\sin 2x}{\cosh 2y+\cos 2x}$ is harmonic. Hence, find its harmonic conjugate v and corresponding analytic function $f(z)=u+iv$. **(1997)** ***Ans.*** $\left[f(z)=\tan z+c.\right]$

10. Show that $u=3x^{2}y+2x^{2}-y^{3}-2y^{2}$ is harmonic. Find its harmonic conjugate and corresponding analytic function. **(06)**

11. Show that $v=e^{-x}\left(x\cos y+y\sin y\right)$ is harmonic and find its corresponding $f(z)=u+iv$. **(2006)**

12. Verify Laplace's equation for $u=\left(r+\dfrac{a^{2}}{r}\right)\cos\theta$. Also find v and $f(z)$. **(2004)**

13. State true or false with proper justification "There does not exist analytic function whose real part is $x^{3}-3x^{2}y-y^{3}$". **(95, 04)** ***Ans.*** [~~False~~] True

14. Show that there does not exists an analytic function whose real part is

(i) $3x^{2}+\sin x+y^{2}+5y+4$ **(02)** (ii) $3x^{2}-2x^{2}y+y^{2}$ **(03)** iii) $x^{2}+3x+y^{2}-4y+6$.

15. Show that $u=r-\dfrac{a^{2}}{r}\sin\theta$ cannot be the real part of an analytic function $f\left(re^{i\theta}\right)=u(r,\theta)+iv(r,\theta)$. **(03, 04)**

16. Verify Laplace's equation for $u=r^{2}\cos 2\theta$. Also find v and $f(z)$.

II) 1.Find the orthogonal trajectory of the family of curves $x^3y - xy^3 = c$. **(03, 05, 06)**

Ans. $\left[x^4 - 6x^2y^2 + y^4 = c.\right]$

2. Find the orthogonal trajectories of the family of curves $e^{-x}\cos y + xy = \alpha$, where α is the real constant in the xy-plane. **(05, 06)** *Ans.* $\left[e^{-x}\sin y + \frac{1}{2}\left(x^2 - y^2\right) = c_2.\right]$

3. Find the orthogonal trajectory of the family of curves given by $2x - x^3 + 3xy^2 = a$. **(02, 04)**

Ans. $\left[2y - 3x^2y + y^3 = c.\right]$

4. Find the orthogonal trajectories of the family of curvesi) $3x^2y - y^3 = c$ *Ans.* $\left[3xy^2 - x^3 = c\right]$

ii) $x^2 - y^2 - 2xy + 2x - 3y = c$. **(2004)** *Ans.* $\left[e^x \sin y + \left(x^2 - y^2\right)/2 = c\right]$

iii) $x^2 - y^2 - 2xy + 2x - 3y = c$. **(2004)** *Ans.* $\left[x^2 - y^2 + 2xy + 3x + 2y = c\right]$

iv) $3x^2y + 2x^2 - y^3 - 2y^2 = c$. **(2003)** *Ans.* $\left[4y - 3x^2 + 3y^2 = c.\right]$

5. If $w = \phi + i\psi$ represents complex potential for an electric field and $\psi = x^2 - y^2 + \frac{x}{x^2 + y^2}$, determine ϕ. **(03)**

6. Show that the family of curves are orthogonal trajectories ; $r^n = \alpha \sec n\theta$, $r^n = \beta \operatorname{cosec} n\theta$ **(05)**

Conformal Mapping

Type 1 Theory for Conformal Mapping

1. Define conformal mapping and show that the mapping defined by an analytic function is conformal at all points where $f'(z) \neq 0$. **(1993)**
2. If $f(z)$ is analytic and $f'(z) \neq 0$ in R, prove that $w = f(z)$ is conformal in R. **(1997)**

Type 1A Problems on transformation $w = az + b$

1. Consider the transformation $w = (1+i)z + (2-i)$ and determine the region in the w-plane into which the rectangular region bounded by $x = 0,\ y = 0,\ x = 1,\ y = 2$ in the z-plane is mapped under this transformation. Sketch the regions. **(2000)*Ans.***
$[u+v=1, u-v=3, u+v=3,\ u-v=-1]$
2. Find the image of the circle $|z| = 2$ under the transformation $w = z+3+2i$ **(2002)**
Ans. $\left[(u-3)^2 + (v-2)^2 = 4.\right]$
3. Find the image of the rectangle bounded by $x = 0,\ y = 0,\ x = 2,\ y = 3$ under the transformation $w = \sqrt{2}\ .\ e^{i\pi/4}.\ z$.**(1999)** ***Ans.*** [The rectangle $u = -v,\ u = v,\ u+v = 4,\ u-v = -6$]
4. Find the image of the circle $|z| = k$ under the transformation $w = 3z+4+2i$.
Ans. $(u-4)^2 + (v-2)^2 = 9k^2$
5. Determine the region in the w-plane corresponding to the triangular region bounded by the lines $x = 0, y = 0, x+y = 1$ in the z-plane under the transformation $w = e^{\frac{i\pi}{4}} z$. **(99, 05)**
Ans. $[u = -v]$; $[u = v]$; $\left[v = \frac{1}{\sqrt{2}}\right]$
6. Find the image of the circle $|z| = 2$ under the transformation $w = (1+2i)z + (3+4i)$. **(2004)**
7. Find the region in the w-plane into which the region bounded by the lines $x = 0,\ y = 0,\ x + y = 1$ in the z-plane is mapped under the transformation $w = 4z$. Is this map conformal ?
Ans. $[u = 0,\ v = 0,\ u + v = 4]$

Type 1B Problems on transformation $w = \frac{a}{z+b}$

1. Find the images of the following under the transformation $w = \frac{1}{z}$. (i) $z = \frac{\sqrt{5}}{2} + i$ (ii) $|z| < 1$
2. Find the image of $|z - ai| = a$ under the transformation $w = 1/z$.**(99, 03, 04)*Ans.*** $\left[v = -\frac{1}{2a}\right]$
3. Find the image of the circle $(x-3)^2 + y^2 = 2$ under the transformation $w = 1/z$.**(96)**
Ans. $\left[\left(u - \frac{3}{7}\right)^2 + v^2\right]$ ***Ans.*** $\left[\frac{2}{49}\right]$

4. Show that the function $w=4/z$ transforms the straight lines $x=c$ in the z-plane into circles in the w-plane. **(1993)** ***Ans.***

$$\left[\left(u-\frac{2}{c}\right)^2+v^2=\frac{4}{c^2}=\left(\frac{2}{c}\right)^2\right]$$

5. Show that the image of the rectangular hyperbola $x^2-y^2=1$ under the transformation $w=1/z$ is the lemniscate $\rho^2=\cos 2\phi$. **(1997)** ***Ans.*** $\left[\rho^2=\cos(-2\phi)=\cos 2\phi\right]$

6. Prove that the circle $|z-3|=5$ is mapped onto the circle. $\left|w+\frac{3}{16}\right|=\frac{5}{16}$ under the transformation $w=1/z$. **(96)**

7. Find the image of the line $y-x+1=0$ under the transformation $w=1/z$. Also find the image of $y-x=0$. Draw rough sketches. ***Ans.*** [The circle with centre $(1/2,\ 1/2)$ and radius $1/\sqrt{2}$.]

8. Show that the transformation $w=1/z$ maps the circle $|z-a|=a,\ a>0$ into a straight line. **(1995)**

10. Find the image of (i) $x^2+y^2=2x$ and (ii) $x=2y$ under the transformation $w=1/z$. **(2004)** ***Ans.*** [(i) $v=-1/2$, (ii) $u-2v$]

11. Under the transformation $w=\frac{1}{z}$ prove that circles in the z-plane are mapped onto circles in w-plane. **(2003)**

12. Under the mapping $w=\frac{1}{z}$ show that the image of i) the circle $|z-3i|=3$ is the line $6v+1=0$ and ii) the hyperbola $x^2-y^2=1$ is the lemniscate $r^2=\cos 2\theta$. **(2005)**

13. Show that the map of the real axis of the z-plane on the w-plane by the transformation $w=\frac{1}{z+i}$ is a circle. **(1998)** ***Ans.*** $\left[u^2+(v+1/2)^2=(1/2)^2\right]$

14. Show that the transformation $w=\frac{1}{z}$ transforms the circle passing through the origin in the z-plane into a straight line not passing through the origin in w-plane. **(03, 04)**

15. Find the image of $|z-1|=1$ under the mapping $w=\frac{1}{z}$. **(2003)**

Type 1C Problems on transformation i) $w=az+\frac{b}{z}$; ii) $w=az^2$ and iii) $w=e^z, \sin z, \cos z, \cosh z, z^2+2z$

I) 1. Show that the transformation $w=z+\frac{(a^2-b^2)}{z}$ transforms the circle $|z|=\frac{1}{2}(a+b)$ in the z-plane into theellipse with semi-axes a and$b$ in the w-plane. **(2003)**

2. Find the image of the circle $|z|=a$ under the transformation, $w=z+1/z$. What is the image when $a=1$? **(95,97)**

3. Determine the image of the circle $x^2+y^2=a^2$ under the transformation $w=z+1/z$ where $a\neq 1$. Also determine the image when $a=1$. ***Ans.*** $\frac{u^2}{\left(a+\frac{1}{a}\right)^2}+\frac{v^2}{\left(a-\frac{1}{a}\right)^2}=1$; for $a=1$ we have $v=0$.

4. If $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$, then prove that in general circles $|z|=$ constant and lines arg. $z=$ constant correspond to conics with foci at $w=\pm1$. Is the mapping conformal? **(2003)**

5. Find the image of the line $z=\pi/3$, under the transformation $w=z+1/z$. *Ans.* $\left[\frac{u^2}{1}-\frac{v^2}{3}=1.\right]$

6. Under the transformation $w=z+1/z$, find the image of line $\theta=\pi/3$. **(2003)**

7. Show that the transformation $w=z+1/z$ converts the st. line $\arg z=\alpha$ $\left(|\alpha|<\pi/2\right)$ into hyperbola of eccentricity $\sec\alpha$. **(04, 05)**

II) 1. Find the image of $|z-1|=1$ under the transformation $w=z^2$. **(98, 04)**

2. Find the image of the triangular region bounded by $x=1,\ y=1,\ x+y=1$ under the transformation $w=z^2$. *Ans.* $\left[u^2=1-2v\right]$

3. Find the image of the region bounded by $x=4,\ y=1,\ x-y=1$ under the transformation $w=z^2$. **(1996)** *Ans.* $\left[v^2=4(u+1),\ u^2=2v+1.\right]$

4. Find the map in the w-plane of the square with vertices $(0,0),(2,0),(2,2),(0,2)$ in the z-plane under the transformation $w=z^2$. **(1996)** *Ans.* $\left[v=0,\ v^2=16(4-u),\ v^2=16(4+u)\right]$

5. Find the images of the lines $x=1,\ y=1,$ under the transformation $w=iz^2$. **(2004)**

6. Draw the region in w-plane into the which the region bounded by the lines $x=0, y=0, x=1, y=1$ in the z-plane is mapped under the transformation $w=z^2$. Are the angles preserved ? **(2004)**

III) 1a. Prove that the map of the straight line $y=mx$ due to the transformation $w=e^z$ is an equiangular spiral. **(1993)**

1b. Find the images of $x=2\ \&\ 3y=\pi$ under the transformation $w=e^z$. **(2004)**

2. Find the image of the region bounded by $a\le x\le b,\ c\le y\le d$ $\left(a,\ b>0,\ a<b<\pi/2;\ c,\ d>0,\ c<d\right)$ in the z-plane by the transformation $w=\cos z$ in the w-plane. **(02, 03)**

Ans. $\left[\begin{array}{l}\frac{u^2}{\cos^2 a}-\frac{v^2}{\sin^2 a}=1,\quad \frac{u^2}{\cos^2 b}-\frac{v^2}{\sin^2 b}=1.\\ \frac{u^2}{\cosh^2 c}+\frac{v^2}{\sinh^2 c}=1,\quad \frac{u^2}{\cosh^2 d}+\frac{v^2}{\sinh^2 d}=1.\end{array}\right]$

3. Prove that the circle $|z|=1$ in the z-plane is mapped onto the cardioide in the w-plane under the transformation $w=z^2+2z$. **(2002)** *Ans.* $\left[r=2(1+\cos\theta)\right]$

4. Find the image of the line $x=\pi/4$ in the z-plane under the map $w=\sin z$. *Ans.* $\left[u^2-v^2=1/2\right]$

5. Show that $w=\cosh z$ is conformal for all z except at $z=0$ and under this transformation the straight lines $x=$ constant, $y=$ constant are respectively mapped onto confocal ellipses and confocal hyperbolas in w-plane.

(97, 04) *Ans.* $\left[\frac{u^2}{\cos^2 k}+\frac{v^2}{\sin^2 k}=1\right]$ *Ans.* $\left[\frac{u^2}{\cos^2 k'}-\frac{v^2}{\sin^2 k'}=1.\right]$

Type 2 Bilinear or Mobius Transformations: Theory Questions

1. Every bilinear transformation with two finite fixed points α, β can be put in the form $\dfrac{w-\alpha}{w-\beta}=\lambda\cdot\dfrac{z-\alpha}{z-\beta}$ **(02, 05, 06)**
2. Every bilinear transformation which has only one fixed point α can be put in the form, called normal form. $\dfrac{1}{w-\alpha}=\dfrac{1}{z-\alpha}+k$ **(96, 05)**
3. State and prove that cross-ratio preservation property of a bilinear transformation. Hence, find the bilinear transformation which maps the points $z=2,i,-2$ onto the points $w=1,i,-1$. **(1993)**
4. Define the cross-ratio of the numbers z_1,z_2,z_3 and z_4. Prove that the cross ratio remains invariant under a bilinear transformation. **(94, 97, 02, 03, 03)**
5. If w_1,w_2,w_3,w_4 are distinct images of z_1,z_2,z_3,z_4 (all different) under the transformation $w=\dfrac{az+b}{cz+d}.(ad-bc\neq 0)$, then show that $\dfrac{(z_1-z_2)}{(z_2-z_3)}\cdot\dfrac{(z_3-z_4)}{(z_4-z_1)}=\dfrac{(w_1-w_2)}{(w_2-w_3)}\cdot\dfrac{(w_3-w_4)}{(w_4-w_1)}$. **(1995)**
6. Prove that, in general, a bilinear transformation maps a circle into a circle. **(96, 03, 04)**
7. Define bilinear transformation. Show that every bilinear transformation is a combination of (i) translation, (ii) Rotation and Magnification, (iii) Inversion. **(1999)**
8. Show that every bilinear transformation is a one to one conformal mapping.

Type 2A Finding Bilinear/Mobius Transformations from given set of points in z and w.

1. Find the bilinear transformation which maps the points 2, i, -2 onto the points 1, $i,-1$. **(93, 00, 02. 04)** *Ans.* $\left[w=\dfrac{3z+2i}{zi+6}\right]$
2. Find the bilinear transformation which maps the points $z=\infty,i,0$ onto the points 0, i,∞. **(04)** *Ans.* $\left[w=\dfrac{-i^2}{-z}=-\dfrac{1}{z}\right]$
3. Find the bilinear transformation which maps $z=2,1,0$ onto $w=1,0,i$. *Ans.* $\left[w=\dfrac{2(z-1)}{(1-z)z+2i}\right]$
4. Find the bilinear transformation which maps the points $z=1,i,-1$ onto the points $w=i,0,-i$. **(02, 04, 04, 06)** Hence, find the fixed points of the transformation and the image of $|z|=1$. **(2004)** *Ans.* $\left[w=\dfrac{i-z}{i+z}\right]$
5. Find the bilinear transformation which maps the points $z=1,i,-1$ onto the points $w=i,0,-i$. Hence, find the image of $|z|<1$ onto the w-plane. **(1999)** *Ans.* $\left[z=i\cdot\dfrac{1-w}{1+w}\right]$
6. Find the bilinear transformation which maps the points $1,-i,2$ on z-plane onto $0,2,-i$ respectively of w-plane. **(1993)** *Ans.* $\left[w=\dfrac{2(z-1)}{(1+i)z-2}\right]$
7. Find the bilinear transformation under which $1,i,-1$ from the z-plane are mapped onto $0,1,\infty$ of z-plane.Further show that under this transformation the unit circle in w-plane is mapped onto a straight line in the z-plane. Write the name of this line. **(1997)** *Ans.* $\left[-i\dfrac{(z-1)}{(z+1)}=w\right]$

8. Find the bilinear transformation which maps the points

(i) $z=-1,1,\infty$ onto the points $w=-i,-1,i$. (2003) *Ans.* $\left[w=\dfrac{iz-2+i}{z+1-2i}\right]$

(ii) $z=2,1,0$ onto the points $w=1,0,i$. (2002) *Ans.* $\left[w=\dfrac{2i(z-1)}{z(1+i)-2}\right]$

(iii) $z=-1,0,1$ onto the points $w=-1,-i,1$ (2003) *Ans.* $\left[w=\dfrac{(1+i)z+1-i}{(1-i)z+1+i}\right]$

(iv) $z=2,i,-2$ onto the points $w=1,i,-1$. (2002) *Ans.* $\left[w=\dfrac{3z+2i}{zi+6}\right]$

9. Find the bilinear transformation which maps $z=\infty,i,0$ onto the points $w=0,i,\infty$. *Ans.* $[w=-1/z]$

10. Find the bilinear transformation which maps the points $0,i,-2i$ of z-plane onto the points $-4i,\infty,0$ respectively of w-plane. Also obtain fixed points of the transformation. (95)

Ans. $\left[w=-\dfrac{2z+4i}{iz+1},\ z=-\dfrac{4}{i},\ -i\right]$

11. Find the bilinear transformation which maps the points $z=0,i,-1$ onto $w=i,1,0$. (99)

Ans. $w=\dfrac{z+1}{(2-i)z-i}$

12. Find the bilinear transformation which maps the points $z=1,-i,-1$ into points $w=i,-\infty,-i$ and hence find the image of $|z|>1$. (2005)

13. Find the bilinear transformation which maps the points $z=0,-1,i$ as $w=i,0,\infty$. (2003)

Type 2B Finding fixed points (invariant points) of Bilinear/Mobius Transformations

1. Find the fixed points of $w=\dfrac{3z-4}{z-1}$. Also express it in the normal form $\dfrac{1}{w-\alpha}=\dfrac{1}{z-\alpha}+\lambda$, where λ is a constant and λ is the fixed point. (03, 05) *Ans.* $\left[\dfrac{1}{z-2}+1\right]$

2. Find the fixed points of the bilinear transformation

(i) $w=\dfrac{1+3iz}{i+z}$ (93, 02) *Ans.* $[z=i]$ (ii) $w=\dfrac{z-4}{2z-5}$ (95, 02) *Ans.* $[z=1,2]$

3. Find the fixed points of bilinear transformation

(i) $\dfrac{2z-2+iz}{i+z}$ (1998) *Ans.* $[z=1\pm i]$ (ii) $\dfrac{2z-5}{z+4}$ (1997) *Ans.* $[z=-1\pm 2i]$

(iii) $\dfrac{2z+4i}{iz+1}$. Also show these two points with one more point and image have const. cross-ratio

(02, 03, 04, 05) *Ans.* $[z=-i,-4/i]$

(iv) $w=\dfrac{2z+6}{z+7}$. Is this transformation a conformal mapping. (2003) *Ans.* $[-6,\ 1]$

4. Find the two fixed points of the transformation $w=-\dfrac{2z+4i}{iz+1}$. Further, prove that these points with any point z and its image form a set of four points having a constant cross-ratio. (03, 05)

5. Find fixed points of bilinear transformation $w=\dfrac{3z-5i}{iz-1}$. (2003)

Type 2C Finding image of Bilinear/Mobius Transformations

1. Find the image of the circle $|z| = k$ where k is real under the bilinear transformation $w = \dfrac{5-4z}{4z-3}$. (2002) *Ans.* $\left[\left(16k^2-9\right)\left(u^2+v^2\right)+\left(32k^2-30\right)u+\left(16k^2-25\right)=0\right]$

2. Under the transformation $w = \dfrac{z-1}{z+1}$, show that the map of the straight line $y = x$ is a circle and find its centre and radius. (94, 02) *Ans.* $\left[u^2+(v+1)^2=2\right]$

3. Show that $w = i\left(\dfrac{1-z}{1+z}\right)$ transforms the circle $|z|<1$ onto the upper half of the w-plane. (2003) *Ans.* $[v>0]$

4. Show that under the transformation $w = \dfrac{2z+3}{z-4}$, the circle $x^2+y^2=4x$ in the z-plane is transformed into the straight line $4u+3=0$ in the w-plane. Explain why straight line is obtained.

5. Show that under the transformation $w = \dfrac{5-4z}{4z-2}$ the circle $|z|=1$ in the z-plane is transformed into a circle of unity in the w-plane. Also find the centre of the circle. (1998)
Ans. $\left[\left|\dfrac{2w+5}{w+1}\right|=4\right]$ *Ans.* $\left[u^2+v^2+u-3/4=0.\right]$

6. Show that $w = i\left(\dfrac{1-z}{1+z}\right)$ transforms the circle $|z|<1$ onto the upper half of the w-plane and transforms circle $z=1$ on real axis. (2005)

7. Show that the transformation $w = \dfrac{iz+2}{4z+1}$ maps the real axis in the z-plane into a circle in the w-plane. Find the centre and radius of the circle. Also find the points on the z-plane which is mapped on to the centre of the circle. (00, 02) *Ans.*
$\left[\text{Centre } (0,-7/8),\ \text{radius } = 9/8\,(0,1/4)\right]$

8. Find the image of the circle $x^2+y^2=1$, under the transformation $w = \dfrac{5-4z}{4z-2}$. (02, 03)
Ans. $\left[\text{Circle with centre}\left(-1/2,0\right) \text{ and radius } 1\right]$

COMPLEX VARIABLES PART I

Some Examination Questions and Theory.

Ex. 1: If $f(z)$ and $\overline{f(z)}$ and both analytic, prove that $f(z)$ is constant.

Sol.: Let $f(z)=u+iv$ then $\overline{f(z)}=u-iv=u+i(-v)$

Since, $f(z)$ is analytic ; $u_x=v_y$ and $u_y=-v_x$ C-R equations.

Since, $\overline{f(z)}$ is analytic ; $u_x=(-v_y)$ and $u_y=-(-v_x)$ C-R equation.

Adding $u_x=v_y$ and $u_x=-v_y$, we get $u_x=0$. Adding $u_y=-v_x$ and $u_y=v_x$, we get $u_y=0$.

Since, $u_x=0$ and $u_y=0$, $u=a$ constant. Similarly by subtraction we can prove that $v_x=0$ and $v_y=0$ $\therefore v=a$ constant. Hence, $f(z)=u+iv=a$ constant.

Ex. 2 Show that the following function is analytic and find its derivative $f(z)=\sin z$.

Sol.: $f(z)=\sin z=\sin(x+iy)=\sin x\cos iy+\cos x\sin iy=\sin x\cosh y+i\cos x\sinh y$

$\therefore u=\sin x\cosh y,\ v=\cos x\sinh y$; $\dfrac{\partial u}{\partial x}=\cos x\cosh y,\ \dfrac{\partial v}{\partial x}=-\sin x\sinh y$

$\dfrac{\partial u}{\partial y}=\sin x\sinh y,\ \dfrac{\partial v}{\partial y}=\cos x\cosh y$; $\therefore u_x=v_y$ and $u_y=-v_x$

$\therefore f(z)=\sin z$ is analytic and can be differentiated as usual $\therefore f'(z)=\cos z$.

Ex. 3: Determine the constants *a, b, c, d* if $f(z)=x^2+2axy+by^2+i(cx^2+2dxy+y^2)$ is analytic.

Sol.: We have $f(z)=u+iv$ and $u=x^2+2axy+by^2$; $V=cx^2+2dxy+y^2$

$\therefore u_x=2x+2ay,\ u_y=2ax+2by$; $v_x=2cx+2dy,\ v_y=2dx+2y$

Since, $f(z)$ is analytic, Cauchy-Riemann Equations are satisfied.

$\therefore u_x=v_y$, and $u_y=-v_x$; $\therefore 2x+2ay=2dx+2y$ and $2ax+2by=-2cx-2dy$

Equating the coefficients of *x* and *y*, we get $a=1,\ d=1$ and $a=-c,\ b=-d$

$\therefore a=1,\ b=-1,\ c=-1,\ d=1$

Ex. 4: If $f(z)=u+iv$ is analytic in R show that (i) $\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}=|f'(z)|^2$ (ii) $\left[\dfrac{\partial|f(z)|}{\partial x}\right]^2+\left[\dfrac{\partial|f(z)|}{\partial y}\right]^2=|f'(z)|^2$

Sol.: First we note that

If $f(z)u+iv,\ f'(z)=\dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x}$ ....(i)

(i) l.h.s. $=\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y}\cdot\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial v}{\partial x}$. But by C-R equations $u_x=v_y,\ u_y=-v_x$

$\therefore$ l.h.s. $=\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial v}{\partial x}\right)^2$. From (i) r.h.s $=|f'(z)|^2=\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial v}{\partial x}\right)^2$. Hence, the result.

(ii) Now, $|f(z)|=\sqrt{u^2+v^2}$, $\therefore \dfrac{\partial}{\partial x}|f(z)|=\dfrac{1}{2\sqrt{u^2+v^2}}\left(2u\dfrac{\partial u}{\partial x}+2v\dfrac{\partial v}{\partial x}\right)$

$$\frac{\partial}{\partial y}|f(z)| = \frac{1}{2\sqrt{u^2+v^2}}\left(2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}\right).$$

$$\therefore \text{l.h.s.} = \frac{1}{u^2+v^2}\left[u^2\left(\frac{\partial u}{\partial x}\right)^2+v^2\left(\frac{\partial v}{\partial x}\right)^2+2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+u^2\left(\frac{\partial u}{\partial y}\right)^2+v^2\left(\frac{\partial v}{\partial y}\right)^2+2uv\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right]$$

Using C-R equations

$$\therefore \text{l.h.s.} = \frac{1}{u^2+v^2}\left[u^2\left(\frac{\partial u}{\partial x}\right)^2+v^2\left(\frac{\partial v}{\partial x}\right)^2+2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+u^2\left(\frac{\partial v}{\partial x}\right)^2+v^2\left(\frac{\partial u}{\partial x}\right)^2-2uv\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x}\right]$$

$$= \frac{1}{u^2+v^2}\left(u^2+v^2\right)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2 = \text{r.h.s.}$$

Ex.5: Using Cauchy-Riemann equations in polar form prove that $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$

Sol.: We know that Cauchy-Riemann equations in polar form are

$u_r = \frac{1}{r}v_\theta$ ......(i) and $u_\theta = -rv_r$ ....(ii). Differentiating (i) w.r.t. r, we get,

$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v}{\partial \theta\,\partial r}$ ....(iii) . Differentiating (ii) w.r.t. θ, we get,

$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial \theta\,\partial r}$ ....(iv)

Now, using (iii) and (iv), we get, $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = \left(-\frac{1}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v}{\partial \theta\,\partial r}\right)+\frac{1}{r}\cdot\frac{1}{r}\frac{\partial v}{\partial \theta}-\frac{1}{r^2}r\frac{\partial^2 v}{\partial \theta\,\partial r}$

$$= -\frac{1}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v}{\partial \theta\,\partial r}+\frac{1}{r^2}\frac{\partial v}{\partial \theta}-\frac{1}{r}\frac{\partial^2 v}{\partial \theta\,\partial r}=0$$

Ex.6: Construct an analytic function whose real part is $x^4-6x^2y^2+y^4$.

Sol: Milne-Thompson Method : we have as above

$u_x = 4x^3-12xy^2$; $u_y = -12xy^2+4y^3$, $\therefore f'(z) = u_x(z,0)-iu_y(z,0)$

$= 4z^3 - i(0)$ [putting $x=z, y=0$ above] . $\therefore f(z) = \int 4z^3dz = z^4+c$ as before.

Ex.7: Construct an analytic function whose real part is $e^x\cos y$.

Sol.: Let $u = e^x\cos y$. $\therefore u_x = e^x\cos y$ and $u_y = -e^x\sin y$

$u_x = e^x\cos y$, $u_y = -e^x\sin y$, By Milne –Thompson method

$\therefore f'(z) = u_x(z,0)-iu_y(z,0) = e^z - i(0)$

$\therefore f(z) = \int e^z dz = e^z + c$ which is the required analytic function.

Ex.8: Find an analytic function whose imaginary part is $e^{-x}(y\cos y - x\sin y)$.

Sol.: We have $v = e^{-x}(y\cos y - x\sin y)$. $\therefore v_y = \psi_1(x,y) = e^{-x}(\cos y - y\sin y - x\cos y)$

$v_x = -e^{-x}(y\cos y - x\sin y)+e^{-x}(-\sin y) = e^{-x}(-\sin y - y\cos y + x\sin y)$

We use Milne-Thompson Method

$\therefore v_y(z,0) = e^{-z}(1-z)$. $\therefore v_x(z,0) = 0$. $\therefore f(z) = \int(1-z)e^{-z}dz$

$$=(1-z)\left(-e^{-z}\right)-\int\left(-e^{-z}\right)(-1)\,dz=-e^{-z}+ze^{-z}+e^{-z}=ze^{-z}+c.$$

Ex.9: Find an analytic function $f(z)=u+iv$ where $u+v=e^{x}(\cos y+\sin y)$

Sol.: We have $f(z)=u+iv \quad \therefore i f(z)=iu-v$

Adding the above two we get $\therefore (1+i)f(z)=(u-v)+i(u+v)=U+iV$, say

$$\therefore \frac{\partial V}{\partial x}=\frac{\partial}{\partial x}(u+v)=e^{x}(\cos y+\sin y)=\psi_1(x,y)$$

$$\therefore \frac{\partial V}{\partial y}=\frac{\partial}{\partial y}(u+v)=e^{x}(-\sin y+\cos y)=\psi_2(x,y)$$

By Milne-Thompson method

$$(1+i)f'(z)=\frac{\partial V}{\partial y}+i\frac{\partial V}{\partial x}=v_y(z,0)+iv_x(z,0)$$

$$e^{z}(0+1)+ie^{z}(1+0)=(1+i)e^{z}\ .\ \therefore f'(z)=e^{z} \qquad f(z)=\int e^{z}dz=e^{z}+c.$$

Ex. 10: Verify Laplace's equation for $u=\left(r+\frac{a^2}{r}\right)\cos\theta$. also find v and $f(z)$.

Sol.: $\because u=\left(r+\frac{a^2}{r}\right)\cos\theta \ \therefore \frac{\partial u}{\partial r}=\left(1-\frac{a^2}{r^2}\right)\cos\theta,\quad \frac{\partial^2 u}{\partial r^2}=\frac{2a^2}{r^3}\cos\theta$

$$\frac{\partial u}{\partial \theta}=-\left(r+\frac{a^2}{r}\right)\sin\theta,\quad \frac{\partial^2 u}{\partial \theta^2}=-\left(r+\frac{a^2}{r}\right)\cos\theta \ \therefore \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$=\frac{2a^2}{r^3}\cos\theta+\frac{1}{r}\cdot\left(1+\frac{a^2}{r^2}\right)\cos\theta-\frac{1}{r^2}\left(r+\frac{a^2}{r}\right)\cos\theta$$

$\therefore$ Laplace's equation is satisfied.

By Cauchy – Riemann equations in polar form

$$=u_r=\frac{1}{r}v_\theta \qquad \therefore \frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \ \therefore \left(1-\frac{a^2}{r^2}\right)\cos\theta=\frac{1}{r}\cdot\frac{\partial v}{\partial \theta} \quad \therefore \frac{\partial v}{\partial \theta}=\left(r-\frac{a^2}{r}\right)\cos\theta$$

Integrating w.r.t. θ, $v=\left(r-\frac{a^2}{r}\right)\sin\theta+c$

Hence, $f(z)=u+iv=\left(r+\frac{a^2}{r}\right)\cos\theta+i\left(r-\frac{a^2}{r}\right)\sin\theta+c \quad =r(\cos\theta+i\sin\theta)+\frac{a^2}{r}(\cos\theta-i\sin\theta)+c$

$$=z+\frac{a^2}{z}+c.$$

Ex. 11: Show that the function $u=\sin x\cosh y+2\cos x\sinh y+x^2-y^2+4xy$ satisfiesLaplace's equation and find it corresponding analytic function $f(z)=u+iv$.

Sol.: We have $\frac{\partial u}{\partial x}=\cos x\cos hy-2\sin x\sin hy+2x+4y$

$$\frac{\partial^2 u}{\partial x^2}=-\sin x\cos hy-2\cos x\sin hy+2\ ;\ \frac{\partial u}{\partial y}=\sin x\sin hy+2\cos x\cos hy-2x+4y$$

$$\frac{\partial^2 u}{\partial y^2}=\sin x\cos hy+2\cos x\sin hy-2\ ,\ \therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Hence, u satisfies Laplace's equation.

Now $u_x = \cos x \cos hy - 2\sin x \sin hy + 2x + 4y$

$u_x(z,0) = \cos z + 2z$; $u_y = \phi_2(x,y) = \sin x \sin hy + 2\cos x \cos hy - 2y + 4x$; $u_y(z,0) = 2\cos z + 4z$

Now we use Milne-Thompson Method , $\therefore f'(z) = u_x(z,0) - iu_y(z,0) = (\cos z + 2z) - i(2\cos z + 4z)$

$\therefore f(z) = \int[(\cos z + 2z) - i(2\cos z + 4z)]\,dz = \sin z + z^2 - i(2\sin z + 2z^2) + c$

Ex. 12 :Prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$.

Sol.: We have $z = x + iy$, $\bar{z} = x - iy$; $\therefore x = \frac{1}{2}(z + \bar{z})$, $y = \frac{1}{2i}(z - \bar{z})$

Treating z and $\bar{z}$ as independent variables $\therefore \dfrac{\partial x}{\partial z} = \dfrac{1}{2}, \dfrac{\partial x}{\partial \bar{z}} = \dfrac{1}{2}, \dfrac{\partial y}{\partial z} = \dfrac{1}{2i}, \dfrac{\partial y}{\partial \bar{z}} = -\dfrac{1}{2i}$.

Now, $\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial x}\cdot\dfrac{\partial x}{\partial z} + \dfrac{\partial}{\partial y}\cdot\dfrac{\partial y}{\partial z} = \dfrac{\partial}{\partial x}\cdot\dfrac{1}{2} + \dfrac{\partial}{\partial y}\cdot\dfrac{1}{2i} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$

And $\dfrac{\partial}{\partial \bar{z}} = \dfrac{\partial}{\partial x}\cdot\dfrac{\partial x}{\partial \bar{z}} + \dfrac{\partial}{\partial y}\cdot\dfrac{\partial y}{\partial \bar{z}} = \dfrac{\partial}{\partial x}\cdot\dfrac{1}{2} + \dfrac{\partial}{\partial y}\cdot\left(-\dfrac{1}{2i}\right) = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$

$\therefore \dfrac{\partial^2}{\partial z \partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)\cdot\dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right) = \dfrac{1}{4}\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)$

Ex. 13 If $f(z) = u + iv$ is an analytic function, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2|f'(z)|^2$.

Sol.: We first note that if $f(z) = u + iv$, $f(\bar{z}) = u - iv$ $\therefore |Rf(z)|^2 = \left|\dfrac{1}{2}\left[f(z) + f(\bar{z})\right]\right|^2$ .....(1)

$= \dfrac{1}{4}\left|f(z) + f(\bar{z})\right|^2$. And $|f(z)|^2 = u^2 + v^2 = (u + iv)(u - iv) = f(z).f(\bar{z})$.....(2)

$\therefore |Rf(z)|^2 = \dfrac{1}{4}\left[f(z) + f(\bar{z})\right]\left[f(\bar{z}) + f(\bar{z})\right] = \dfrac{1}{4}\left[f(z) + f(\bar{z})\right].\left[f(\bar{z}) + f(z)\right] = \dfrac{1}{4}\left[f(z) + f(\bar{z})\right]^2$

$\therefore \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}|Rf(z)|^2$

$= 4\dfrac{\partial^2}{\partial z \partial \bar{z}}\left\{\dfrac{1}{4}\left[f(z) + f(\bar{z})\right]^2\right\} = \dfrac{\partial^2}{\partial z \partial \bar{z}}\left[f(z) + f(\bar{z})\right]^2 = \dfrac{\partial^2}{\partial z}2\left[f(z) + f(\bar{z})\right].f'(\bar{z})$

$= 2f'(\bar{z}).\dfrac{\partial}{\partial z}\left[f(z) + f(\bar{z})\right] = 2f'(\bar{z}).f(z) = 2|f'(z)|^2$ [By (2)]

Ex. 14 Find the orthogonal trajectories of the family of curves $e^{-x}\cos y + xy = \alpha$, where α is the real constant in the *xy*-plane.

Sol.: We have seen above that the orthogonal trajectories of $u = c_1$ are given by $v = c_2$ where v is the harmonic conjugate of u. $\therefore u = e^{-x}\cos y + xy$

$u_x = -e^{-x}\cos y + y$ and $u_y = -e^{x}\sin y + x$

Also $f'(z) = u_x + iv_x = u_x - iu_y$ (By C-R equations) $= \left(-e^{-x}\cos y + y\right) - i\left(-e^{-x}\sin y + x\right)$

By Milne-Thompson's method, we replace x by z and y by zero.

$f'(z) = -e^{-z} - iz$. By integration $f(z) = e^{-z} - i\dfrac{z^2}{2} + c$. Take $z = x + iy$,

$\therefore f(z)=e^{-(x+iy)}-i\frac{(x+iy)^2}{2}+c=e^{-x}(\cos y-i\sin y)-\frac{i}{2}(x^2+2ixy-y^2)+c$

$\therefore$ Imaginary part, $v=-e^{-x}\sin y-\frac{1}{2}(x^2-y^2)$

Hence, the required orthogonal trajectories are $e^{-x}\sin y+\frac{1}{2}(x^2-y^2)=c_2$

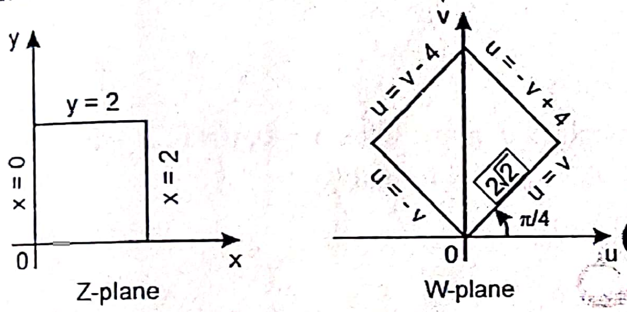
Conformal Mapping in a Complex Plane

Ex. 1 Find the image of the region bounded by $x=0, x=2, y=0, y=2$ in the z-plane under transformation $w=(1+i)z$.

Sol.: Since $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}.e^{i\pi/4}$

$\therefore w=cz$ becomes $\text{Re}i\phi=\sqrt{2}.e^{i\pi/4}.re^{i\theta}=\sqrt{2}.re^{i(\theta+\pi/4)}$

The square in z-plane is transformed into a square in the w-plane but its sides are magnified by $\sqrt{2}$ and it is rotated through an angle of $\pi/4$.



Alternatively We may write $z=x+iy$.

Then $u+iv=(1+i)(x+iy)=(x-y)+i(x+y)$

$\therefore u=x-y, v=x+y$

When $x=0, u=-y, v=y$ $\qquad \therefore u=-v$

When $x=2, u=2-y, v=2+y$ $\qquad \therefore u+v=4$

When $y=0, u=x, v=x$ $\qquad \therefore u=v$

When $y=2, u=x-2, v=x+2$ $\qquad \therefore u-v=-4$

Thus, we get the same square as above.

Ex. 2 Find the bilinear transformation which maps the points $2, i, -2$ onto the points $1, i, -1$.

Sol.: Method mentioned here is an alternative method, but method of cross-ratio discussed in the classroom is preferred over this method.

Let the required transformation be $w=\frac{az+b}{cz+d}$

By data, when $z=2, i, -2,\ w=1, i, -1$

Putting these values, we get

$1=\frac{2a+b}{2c+d}$ $\quad \therefore 2a+b=2c+d$ $\quad \therefore 2a+b-2c-d=0$ .....(1)

$i=\frac{ai+b}{ci+d}$ $\quad \therefore ai+b=-c+di$ $\quad \therefore ai+b+c-di=0$ ....(2)

$-1=\frac{-2a+b}{-2c+d}$ $\quad \therefore -2a+b=2c-d$ $\quad \therefore -2a+b-2c+d=0$ .....(3)

From (1) and (3), we get

$2b-4c=0$ $\quad \therefore b=2c$ .....(4)

Multiply (1) by i $\quad \therefore 2ai+bi-2ci-di=0$

Subtract (2) from (4)

$ai+b(i-1)-c(2i+1)=0$

But $b=2c$, $\quad \therefore ai+2c(i-1)-c(2i+1)=0$

$\therefore ai-3c=0 \quad \therefore a=3c/i=-3ci$

Putting the values of a and b in (1), we get

$-6ci+2c-2c-d=0 \qquad \therefore d=-6ci$

Putting the values of a, b, d

$$w=\frac{-3ciz+2c}{cz-6ci}=\frac{-3zi+2}{z-6i}$$

Multiply by i in the numerator and denominator of r.h.s.

$$w=\frac{-3z(i)^2+2i}{zi-6(i)^2}=\frac{3z+2i}{zi+6}.$$

Ex.3 Find the fixed points of the bilinear transformation $w=\dfrac{z-1}{z+1}$

Sol.: The fixed points are given by

$z=\dfrac{z-1}{z+1} \qquad \therefore z^2+z=z-1$

$\therefore z^2+1=0 \quad \therefore z^2-i^2=0$

$\therefore (z+i)(z-i)=0 \qquad \therefore z=i,-i$ are the finite fixed points.

Ex. 4 Find the region in the w-plane into which the region bounded by the lines $x=0, y=0, x+y=1$ in the z-plane is mapped under the transformation $w=4z$. Is this map conformal ?

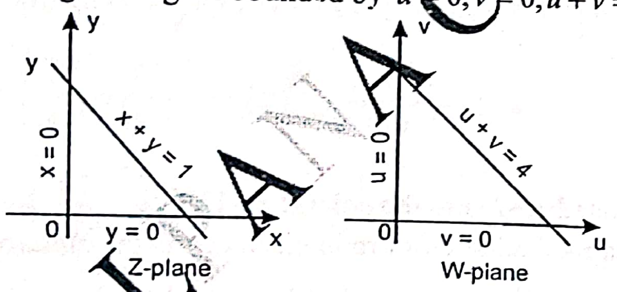
Sol.: We have $w=u+iv=4z=4(x+iy) \therefore u=4x, v=4y$

When $x=0, u=0$, which is v-axis.

When $y=0, v=0$, which is u-axis.

When $x+y=1$, we get $u+v=4(x+y)=4$

Hence, the triangular region bounded by $x=0, y=0, x+y=1$ in the z-plane is mapped onto the triangular region bounded by $u=0, v=0, u+v=4$ in the w-plane.



Further since $w=4z, \dfrac{dw}{dz}=4\neq 0$

$\therefore u=4x, v=4y$

$u_x=4, u_y=0, v_x=0, v_y=4$

$\therefore u_x=v_y$ and $u_y=-v_x$

$\therefore$ Cauchy-Riemann equations are satisfied and $f'(z)$ exists.

Hence, $w=f(z)$ is conformal over z-plane.

Ex. 5 Find the image of the circle $(x-3)^2+y^2=2$ under the transformation $w=1/z$.

Sol.: The given circle has center at $(3,0)$ and radius $\sqrt{2}$.

Now $w=\dfrac{1}{z} \qquad \therefore z=\dfrac{1}{w} \qquad \therefore x+iy=\dfrac{1}{u+iv}=\dfrac{u-iv}{u^2+v^2}$

$\therefore x=\dfrac{u}{u^2+v^2}$ and $y=-\dfrac{v}{u^2+v^2}$ .....(i)

The equation of the circle is

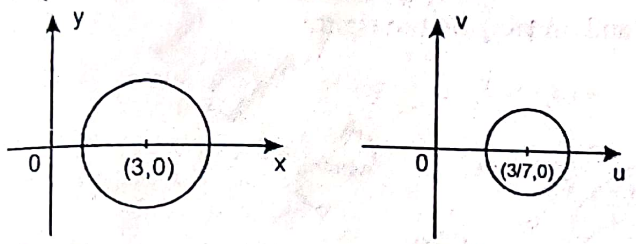
$x^2+y^2-6x+7=0$ .....(ii)

Eliminating x and y from (i) and (ii) we get

$$\frac{u^2}{\left(u^2+v^2\right)^2}+\frac{v^2}{\left(u^2+v^2\right)^2}-\frac{6u}{\left(u^2+v^2\right)}+7=0\ .\ \therefore \frac{u^2+v^2}{\left(u^2+v^2\right)^2}-\frac{6u}{\left(u^2+v^2\right)}+7=0$$

$$\therefore \frac{1}{u^2+v^2}-\frac{6u}{u^2+v^2}+7=0\ ,\ 1-6u+7\left(u^2+v^2\right)=0\ .\ \therefore u^2-\frac{6}{7}u+\frac{1}{7}+v^2=0$$

$\therefore \left(u-\dfrac{3}{7}\right)^2+v^2=\dfrac{9}{49}-\dfrac{1}{7}=\dfrac{2}{49}$ which is a circle with center $(3/7,0)$ and radius $\sqrt{2}/7$



Z-plane W-plane

Ex. 6. Find the image of the circle $x^2+y^2=c^2$, under the transformation $w=z+\left(k^2/z\right)$.

Sol.: Since $\dfrac{dw}{dz}=1-\dfrac{k^2}{z^2}, \dfrac{dw}{dz}=0$ when $z=\pm k$.

Hence, the mapping is conformal everywhere except at $z=\pm k$.

Now, putting $z=re^{i\theta}$, we get $w=re^{i\theta}+\dfrac{k^2}{re^{i\theta}}=re^{i\theta}+\dfrac{k^2}{e}e^{-i\theta}$

$$\therefore u+iv=\left(r+\frac{k^2}{r}\right)\cos\theta+i\left(r-\frac{k^2}{r}\right)\sin\theta$$

$$\therefore u=\left(r+\frac{k^2}{r}\right)\cos\theta,\ v=\left(r-\frac{k^2}{r}\right)\sin\theta \quad .....(1)$$

Now, $x^2+y^2=c^2$ i.e. $r^2=c^2$ i.e. $r=c$.

Hence, from (1) we get,

$$u=\left(c+\frac{k^2}{c}\right)\cos\theta,\ v=\left(c-\frac{k^2}{c}\right)\sin\theta \quad ....(2)$$

Eliminating θ from (2), we get,

$$\frac{u^2}{\left[c+\left(k^2/c\right)\right]^2}+\frac{v^2}{\left[c+\left(k^2/c\right)\right]^2}=1 \quad .....(3)$$

This represents the family of ellipses with center at the origin.

Their foci are given by

$$\left(\pm\sqrt{\left[c+\left(k^2/c\right)\right]^2-\left[c-\left(k^2/c\right)\right]^2},0\right) \text{ i.e. } (\pm 2k,0)$$

Since, the foci are independent of c, the circles with center at the origin in the z-plane are mapped onto the confocal ellipses in the w-plane except when $r=\pm k$.

When $r=k$, we get from (1), $u=2k\cos\theta, v=0$ i.e. $-2k<u<2k, v=0$. This is a segment of the u-axis.

When $r=-k$, we get from (1), $u=0, v=2k\sin\theta$ i.e. $u=0$, $-2k<v<2k$, this is a segment of the v-axis.

Ex. 7 Find the image of the triangular region bounded by $x=1, y=1, x+y=1$ under the transformation $w=z^2$.

Sol.: Let $z=x+iy, w=u+iv$. $\therefore w=z^2$ gives $u+iv=(x+iy)^2$

i.e. $u=x^2-y^2$ and $v=2xy$

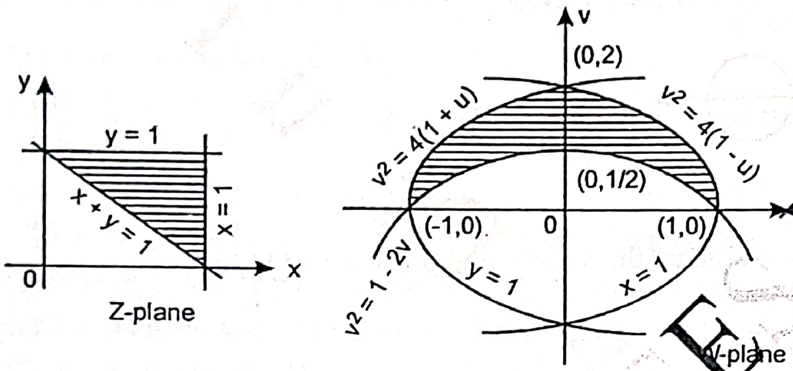
(i) When $x=1, u=1-y^2, v=2y$.

Eliminating $y, u=1-\left(v^2/4\right)$ $\therefore 4u=4-v^2$ $\therefore v^2=4(1-u)=-4(u-1)$

which is a parabola with vertex at $(1,0)$ and opening to the left.

(ii) When $y=1, u=x^2-1, v=2x$. Eliminating $x, u=\left(v^2/4\right)-1$ $\therefore v^2=4(1+u)=4(u+1)$

which again is a parabola with vertex at $(-1,0)$ and opening on the right.



(iii) When $x+y=1, u=x^2-(1-x)^2$ $\therefore u=2x-1$ and $v=2x(1-x)$

$\therefore v=2x-2x^2$. Put $x=\dfrac{u+1}{2}$ $\therefore$ Eliminating x, $v=2\dfrac{(u+1)}{2}-2\left(\dfrac{u+1}{2}\right)^2$

$\therefore 2v=2u+2-u^2-2u-1=1-u^2$, $\therefore u^2=1-2v=-2(v-1/2)$

which is a parabola with vertex at $(0,1/2)$ and opening downwards.

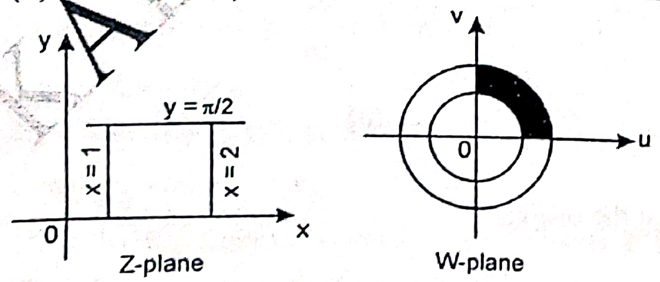
Ex. 8 Find the region in the w-plane into which the region bounded by $x=1, y=0, x=2, y=\pi/2$ in the z-plane is mapped under the transformation $w=e^z$.

Sol.: Using the cartesian form for z and polar form for w, we get,

$\rho e^{i\phi}=e^{x+iy}=e^x.e^{iy}$ $\therefore \rho=e^x$ and $\phi=y$

(i) When $x=1, \rho=e$. Hence, the line $x=1$ is mapped onto the circle $\rho=e$.

(ii) When $x=2, \rho=e^2$. Hence, the line $x=2$ is mapped onto the circle $\rho=e^2$.



(iii) When $y=0, \phi=0$. Hence, the line $y=0$ is mapped onto the u-axis.

(iv) When $y=\pi/2, \phi=\pi/2$. Hence, the line $y=\pi/2$ is mapped onto the v-axis.

Hence, the rectangular region $x=1, y=0, x=2, y=\pi/2$ in the z-plane is mapped onto the region bounded by the circles and the axes.

Ex. 9 Find the image of the circle $|z|=a$ under the transformation, $w=z+1/z$. What is the image when $a=1$?

Sol.: The circle $|z|=a$ can be written as $z=ae^{i\theta}$

Now $w=z+\frac{1}{z}$ gives $u+iv=ae^{i\theta}+\frac{1}{a}e^{-i\theta}$. $\therefore u+iv=a(\cos\theta+i\sin\theta)+\frac{1}{a}(\cos\theta-i\sin\theta)$

$=\left(a+\frac{1}{a}\right)\cos\theta+i\left(a-\frac{1}{a}\right)\sin\theta$. $\therefore u=\left(a+\frac{1}{a}\right)\cos\theta,\ v=\left(a-\frac{1}{a}\right)\sin\theta.$

i.e. $u=l\cos\theta, v=m\sin\theta$, say which are parametric equations of an ellipse.

Thus, the circle is transformed into an ellipse.

If $a=1$, we get $u=2\cos\theta$ and $v=0$. Since $-1\le\cos\theta\le 1,\ -2\le u\le 2.$

Hence, if $a=1$, the circle $|z|=1$ is transformed into a segment of the u-axis of length 4.

Ex. 10 Show that the map of the real axis of the z-plane is a circle under the transformation $w=\frac{2}{z+i}$. Find its center and the radius.

Sol.: We have $w=\frac{2}{z+i}$ $\therefore z+i=\frac{2}{w}$. $\therefore(x+iy)+i=\frac{2}{u+iv}=\frac{2}{u+iv}\cdot\frac{(u-iv)}{(u-iv)}$

$\therefore x+i(y+1)=\frac{2(u-iv)}{u^2+v^2}$. $\therefore x=\frac{2u}{u^2+v^2}$ and $y+1=-\frac{2v}{u^2+v^2}$

For the real axis i.e. x-axis, $y=0$. $\therefore 1=-\frac{2v}{u^2+v^2}$

$\therefore u^2+v^2+2v=0$ $\quad\therefore u^2+(v+1)^2=1$

$\therefore$ The map is a circle with center at $(0,-1)$ and radius 1.

Ex. 11 Prove that $w=i\left(\frac{z-i}{z+i}\right)$ maps upper half of the z-plane into the interior of the unit circle in the w-plane.

Sol.: Since $w=i\left(\frac{z-i}{z+i}\right)$, we get, $zw+iw=iz+1$ $\therefore z(w-i)=1-iw$

$\therefore z=\frac{1-iw}{w-i}$ $\quad\therefore x+iy=\frac{1-i(u+iv)}{u+iv-i}$;

$\therefore x+iy=\frac{(1+v)-iu}{u+i(v-1)}\cdot\frac{u-i(v-1)}{u-i(v-1)}=\frac{(1+v)u-u(v-1)}{u^2+(v-1)^2}-i\cdot\frac{u^2+v^2-1}{u^2+(v-1)^2}$

$\therefore y=-\frac{u^2+v^2-1}{u^2+(v-1)^2}$. For the upper half of the z-plane $y>0$,

$\therefore -\frac{(u^2+v^2-1)}{u^2+(v-1)^2}>0$ $\therefore -(u^2+v^2-1)>0$

$\therefore u^2+v^2-1<0$; $\therefore u^2+v^2<1$; $\therefore |w|<1$

$\therefore$ The upper of the z-plane is mapped onto the interior of the unit circle with center at the origin.

Ex. 12 Find the bilinear transformation under which $1,i,-1$ from the z-plane are mapped onto $0,1,\infty$ of w-plane. Further show that under this transformation the unit circle in w-plane is mapped onto straight line in the z-plane. Write the name of this line.

Sol.: Let the transformation be $w=\frac{az+b}{cz+d}$

Putting the given values of z and w, we get,

$0=\frac{a+b}{c+d}$, $1=\frac{ai+b}{ci+d}$, $\infty=\frac{-a+b}{-c+d}$

From the first we get, $a+b=0$ $\quad\therefore b=-a$

From the last we get, $-c+d=0$ $\quad\therefore d=c$

From the second we get,

$ai+b=ci+d \quad ai-a=ci+c$

$$a(i-1)=c(i+1) \qquad \therefore c=a\frac{(i-1)}{(i+1)}$$

$$\therefore c=a\frac{(i-1)}{(i+1)}\cdot\frac{(i-1)}{(i-1)}=a\frac{(i^2-2i+1)}{i^2-1}=ai$$

$\therefore d=c=ai$

$$\therefore w=\frac{az-a}{aiz+ai}=\frac{z-1}{i(z+1)}=-i\frac{(z-1)}{(z+1)}$$

Image of w – plane:

Now, when $|w|=1, \left|-i.\frac{(z-1)}{(z+1)}\right|=1$

$\therefore |z-1|=|z+1|$ $\qquad [\because |i|=1]$

$\therefore |(x-1)-iy|=|(x+1)+iy|$

$\therefore (x-1)^2+y^2=(x+1)^2+y^2$

$-2x=2x \quad \therefore 4x=0 \quad \therefore x=0$

Hence, the map is the y-axis.