3) $f(x) = \begin{cases} x & , \quad 0 < x \leq \pi \\ 2\pi - x & , \quad \pi \leq x < 2\pi \end{cases}$

$\rightarrow$ $f(x) = a_0 + \sum a_n \cos nx + \sum b_n \sin nx \ldots$ ①

$a_0 = \frac{\pi}{2}$

$a_n = \begin{cases} 0 & , \text{ if } n \text{ is even} \\ \frac{-4}{\pi n^2} & \text{ if } n \text{ is odd.} \end{cases}$

$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \sin nx \, dx$

$= \frac{1}{\pi} \left\{ \int_0^{\pi} x \sin nx \, dx + \int_{\pi}^{2\pi} (2\pi - x) \sin nx \, dx \right\}$

$= \frac{1}{\pi} \left[\left\{ x \left(\frac{-\cos nx}{n} \right) - 1 \left(\frac{-\sin nx}{n^2} \right) \right\}_0^{\pi} + \left\{ (2\pi - x) \left(\frac{-\cos nx}{n} \right) - (-1) \left(\frac{-\sin nx}{n^2} \right) \right\}_{\pi}^{2\pi} \right]$

$= \frac{1}{\pi} \left[\left\{ \left(\frac{-\pi (-1)^n}{n} + 0 \right) - (0+0) \right\} + \left\{ (0+0) - \left(\frac{-\pi (-1)^n}{n} - 0 \right) \right\} \right]$

$= \frac{1}{\pi} (0)$

$\boxed{b_n = 0}$

Putting above values in ①

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right\}$$

By Parseval's Identity,

$$\frac{1}{2\pi}\int_0^{2\pi} \{f(x)\}^2\,dx = a_0^2 + \frac{1}{2}\sum(a_n^2 + b_n^2)$$

Consider,

$$\frac{1}{2\pi}\int_0^{2\pi}\{f(x)\}^2 dx = \frac{1}{2\pi}\left[\int_0^{\pi} x^2\,dx + \int_{\pi}^{2\pi}(2\pi - x)^2\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{x^3}{3}\right)_0^{\pi} + \left[\frac{(2\pi - x)^3}{-3}\right]_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left\{\frac{\pi^3}{3} - \frac{1}{3}\left[0 - \pi^3\right]\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{\pi^3}{3} + \frac{\pi^3}{3}\right\}$$

$$= \frac{2\pi^2}{3} \times \frac{1}{2} = \frac{\pi^3}{3}$$

Thus,

$$\frac{2\pi^2}{3} = \left(\frac{\pi}{2}\right)^2 + \frac{1}{2}\cdot\frac{16}{\pi^2}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right\}$$

$$\therefore \frac{2\pi^2}{3} - \frac{\pi^2}{4} = \frac{8}{\pi^2}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right\}$$

$$\frac{5\pi^2}{12} = \frac{8}{\pi^2}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right\}$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$

4) ⇒ Find the Fourier Expansion of $f(x) = \sqrt{1 - \cos x}$ in $(0, 2\pi)$. Hence deduce that $\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$

→ Let,

$$\sqrt{1 - \cos x} = \sqrt{2}\sin\frac{x}{2} = a_0 + \sum a_n \cos nx + \sum b_n \sin nx \quad \dots ①$$

Now,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\,dx$$

$$= \frac{\sqrt{2}}{2\pi}\left\{-2\cos\frac{x}{2}\right\}_0^{2\pi}$$

$$= -\frac{\sqrt{2}}{\pi}\left\{-1 - 1\right\} = \frac{2\sqrt{2}}{\pi}$$

$$\therefore \boxed{a_0 = \frac{2\sqrt{2}}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\cos nx\,dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left\{\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right\}dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\left\{\sin\left(\frac{2n+1}{2}\right)x - \sin\left(\frac{2n-1}{2}\right)x\right\}dx$$

$$= \frac{1}{\sqrt{2}\pi}\left\{\frac{-2}{2n+1}\cos\left(\frac{2n+1}{2}\right)x + \frac{2}{2n-1}\cos\left(\frac{2n-1}{2}\right)\right\}_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left\{\frac{1}{2n+1} - \frac{1}{2n-1} + \frac{1}{2n+1} - \frac{1}{2n-1}\right\}$$

$$= \frac{2\sqrt{2}}{\pi}\left\{\frac{1}{2n+1} - \frac{1}{2n-1}\right\}$$

$$= \frac{2\sqrt{2}}{\pi}\left\{\frac{2n-1-2n-1}{(2n+1)(2n-1)}\right\}$$

$$= \frac{2\sqrt{2}}{\pi}\left\{\frac{-2}{4n^2-1}\right\}$$

$$= \frac{-4\sqrt{2}}{\pi(4n^2-1)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\,\sin nx\, dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left(2\sin nx.\sin\frac{x}{2}\right)dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\left\{\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\right\}dx$$

$$= \frac{1}{\sqrt{2}\pi}\left\{\frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x\right\}_0^{2\pi}$$

$$= \frac{1}{\sqrt{2}\pi}\{0-0\} = \frac{1}{\sqrt{2}\pi}(0) = 0.$$

$$\boxed{b_n = 0}$$

Putting above values in ①

$$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty}\frac{-4\sqrt{2}}{\pi(4n^2-1)}\cos nx$$

Put $x=0$

$$\therefore\quad 0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\therefore\quad \frac{4\cancel{\sqrt{2}}}{\cancel{\pi}}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{2\cancel{\sqrt{2}}}{\cancel{\pi}}$$

$$\therefore\quad 2\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = 1$$

$$\boxed{\therefore\quad \frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}}$$

⑤ $f(x) = \begin{cases} -\pi & , -\pi < x < 0 \\ x & . \; 0 < x < \pi \end{cases}$

State the value of $f(x)$ at $x = 0$ & hence deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{3\pi^2}{8}$

→ Let,

$f(x) = a_0 + \sum a_n \cos nx + \sum b_n \sin nx$ ... ①

Now,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi} \left\{ \int_{\pi}^{0} -\pi\,dx + \int_0^{\pi} x\,dx \right\}$$

$$= \frac{1}{2\pi} \left\{ -\pi (x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi} \right\}$$

$$= \frac{1}{2\pi} \left\{ (-\pi)(0+\pi) + \left(\frac{\pi^2}{2} - 0\right) \right\}$$

$$= \frac{1}{2\pi} \left\{ -\pi^2 + \frac{\pi^2}{2} \right\}$$

$$a = \frac{-\pi}{4}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x).\cos nx\,dx$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} -\pi \cos nx\,dx + \int_0^{\pi} x \cos nx\,dx \right\}$$

$$= \frac{1}{\pi} \left[-\pi \left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left\{ x \left(\frac{\sin nx}{n}\right) - 1 \left(\frac{\cos nx}{n^2}\right) \right\}_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left\{ -\pi(0-0) + \left(0 + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2}\right) \right\}$$

$$= \frac{1}{\pi n^2} \left\{ (-1)^n - 1 \right\}$$

$$a_n = \begin{cases} 0 & , \text{ if } n \text{ is even} \\ \frac{-2}{\pi n^2} & , \text{ if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} -\pi \sin nx\,dx + \int_0^{\pi} x \sin nx\,dx \right\}$$

$$= \frac{1}{\pi} \left\{ -\pi \left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left\{ x \left(\frac{-\cos nx}{n}\right) - (1).\left(\frac{-\sin nx}{n^2}\right) \right\}_0^{\pi} \right.$$

$$= \frac{1}{\pi} \left\{ -\pi \left(-\frac{1}{n} + \frac{(-1)^n}{n}\right) + \left(\frac{-\pi(-1)^n}{n} + 0 + 0 + 0\right) \right\}$$

$$= \frac{1}{n} - \frac{2(-1)^n}{n}$$

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$$b_n = \frac{1}{n}\left\{1-2(-1)^n\right\}$$

Putting above values in ①,

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right\} + \sum_{n=1}^{\infty}\frac{1}{n}\left\{1-2(-1)^n\right\}\sin nx$$

Now,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -\pi = -\pi$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$$

$$f(0) = \frac{1}{2}\left\{\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right\}$$

$$= \frac{1}{2}(-\pi+0) = -\frac{\pi}{2}$$

Put $x = 0$

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right\}$$

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

f(x)	E	O	E	O
g(x)	E	O	O	E
f(x)/g(x)	E	E	O	O

Function $f(x)$ is said to be even if

$$f(-x) = f(x)$$

eg:- $f(x) = \cos x$

Function $f(x)$ is said to be odd if

$$f(-x) = -f(x)$$

eg $f(x) = \sin x$

⇒ In case of even function:-

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx,$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = 0$$

⇒ In case of odd function:-

$a_0 = 0$,

$a_n = 0$,

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Note:-

f(x)	E	O	O	E
g(x)	E	O	E	O
f(x).g(x)	E	E	O	O

⇒ Obtain Fourier series for

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$$

Deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

→ Here,

$$f(-x) = \begin{cases} 1 - \frac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1 + \frac{2x}{\pi}, & 0 \le -x \le \pi \end{cases}$$

$$= \begin{cases} 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \\ 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \end{cases}$$

$$= f(x)$$

$\therefore$ $f(x)$ is even function

$\therefore$ $b_n = 0$

Let,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad \dots \text{(1)}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$$

$$= \frac{1}{\pi}\left\{x - \frac{x^2}{\pi}\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\{\pi - \pi - 0\}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\left(1 - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{-2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left(0 - \frac{2(-1)^n}{\pi n^2}\right) - \left(0 - \frac{2}{\pi n^2}\right)\right\}$$

$$= \frac{4}{\pi^2 n^2}\left\{1 - (-1)^n\right\}$$

$$\therefore a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{8}{\pi^2 n^2}, & \text{if } n \text{ is odd.} \end{cases}$$

putting above values in (1)

$$f(x) = \frac{8}{\pi^2}\left\{\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right\}$$

Put $x = 0$

$$F(0) = \frac{8}{\pi^2}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\}$$

$$1 = \frac{8}{\pi^2}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

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