LAPLACE TRANSFORM

Definition

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$$

Linearity Property

$$L[c_1 F_1(t) + c_2 F_2(t)] = c_1 L[F_1(t)] + c_2 L[F_2(t)]$$

Problems

01)Using the definition , find the Laplace Transform of the following functions

(a) $F(t) = \begin{cases} t & 0 < t < 4 \\ 5 & t > 4 \end{cases}$ (b) $F(t) = \begin{cases} \sin t & 0 < t < \pi \\ \cos t & t > \pi \end{cases}$ **(D-08)**

Homework

02) Using the definition , find the Laplace Transform of the following functions

(a) $F(t) = \begin{cases} (t-a)^3 & 0 < t < a \\ 0 & t > a \end{cases}$ **(M-11)** (b) $F(t) = \begin{cases} \cos t & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases}$

Standard Formulae

Function F(t)	**Laplace Transform f(s)**
1	$\frac{1}{s}$
T	$\frac{1}{s^2}$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$ if n is a positive integer
sin at	$\frac{a}{s^2 + a^2}$
cos at	$\frac{s}{s^2 + a^2}$
e^{at}	$\frac{1}{s-a}$
sinh at	$\frac{a}{s^2 - a^2}$
cosh at	$\frac{s}{s^2 - a^2}$

Problems

03) Find the Laplace Transform of following functions

(a) $(t+1)^3$ (b) $\cos^4 t$ (c) $\sin 2t \sin 4t \sin 6t$ (d) $\dfrac{\sin\sqrt{t}}{\sqrt{t}}$ **(D-09)**

04) Show that

$(a)\ L[\sin\sqrt{t}] = \dfrac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$ **(D-13,D-11)** $(b)\ L\{\sin^3 t\} = \dfrac{3!}{(s^2+1)(s^2+9)}$

$(c)\ \alpha = \dfrac{\pi}{4}$ Using Laplace Transform if $\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)dt = \dfrac{3}{8}$ **(D-14,D-10)**

Homework

05) Find the Laplace Transform of following functions

(a) $(\sqrt{t}-1)^4$ (b) cos2tcos4tcos6t **(M-09)** (c) $\cosh^4 t$ (d) $\cos^5 t$

06) Show that

$(a)\ L[\dfrac{\cos\sqrt{t}}{\sqrt{t}}] = \dfrac{\sqrt{\pi}}{\sqrt{s}} e^{-1/4s}$ **(M-12,M-11)**

$(b)\ L\{\sin^5 t\} = \dfrac{5!}{(s^2+1)(s^2+9)(s^2+25)}$ *and hence find* $L\{\sin^5 2t\}$

$(c)\ L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$ *if* $J_0(t) = \sum_0^\infty \dfrac{(-1)^r}{(r!)^2}\left(\dfrac{t}{2}\right)^{2r}$ **(M – 10)**

First Shift Theorem

If L{F(t)}= f(s) then $L\{e^{at}F(t)\} = f(s-a)$ **(M-11)**

Problems

07) Find the Laplace transform of the following functions

(a) $(1+te^{-t})^3$ (b) $(t^2\sinh 2t)^2$ **(M-08)** (c) $e^{-2t}\sin^2 4t$

Homework

08) Find the Laplace transform of the following functions

(a) $\sin t\cos 2t\cosh t$ **(D-14)** (b) $\sinh\dfrac{1}{2}t\sin\dfrac{\sqrt{3}}{2}t$ **(M-10)** (c) $\left(\dfrac{\cos t+\sin t}{e^t}\right)^2$ (d) $te^{-t}\cosh 2t$ **(M-15)**

09) Evaluate $\int_0^\infty e^{-\sqrt{2}\,t}\sin t\sinh t\,dt$

Second Shift Theorem

If L{F(t)}= f(s) then
$G(t) = 0 \text{ for } 0 < t < a \text{ and } F(t-a) \text{ for } t > a \text{ then } L\{G(t)\} = e^{-as} f(s)$

Problem

10) Find $L\{G(t)\}$ where $G(t) = 0$ for $0 < t < \frac{2\pi}{3}$ and $\cos(t - \frac{2\pi}{3})$ for $t > \frac{2\pi}{3}$

Homework

11) Find $L\{G(t)\}$ where $G(t) = 0$ for $0 < t < \frac{2\pi}{3}$ and $\sin^2(t - \frac{2\pi}{3})$ for $t > \frac{2\pi}{3}$

Change of Scale Theorem

If L{F(t)}= f(s) then $L\{F(at)\} = \frac{1}{a} f(\frac{s}{a})$

Problem

12)Find $L\{F(3t)\}$ *and* $L\{F(\frac{t}{2})\}$ if given $L\{F(t)\} = \frac{1-3s}{s^2 - 4s + 2}$

Homework

13) Find $L\{e^{-t} F(2t)\}$ if given $L[tF(t)] = \frac{1}{s(s^2+1)}$

14) Find $L\{e^{-2t} f(2t)\}$ if given $L\{F(t)\} = \frac{s}{s^2 + s + 4}$

Multiplication By t Theorem

If L{F(t)}= f(s) then $L\{t^n F(t)\} = (-1)^n f^{(n)}(s)$

Problems

15) Find the Laplace transform of the following functions & hence evaluate the integral given

(a) $t\sin^2 t$; $\int_0^\infty e^{-2t} t \sin^2 t \, dt = \frac{1}{8}$ (b) $t\sqrt{1+\sin t}$; $\int_0^\infty e^{-t} t\sqrt{1+\sin t}\, dt = \frac{28}{25}$

(c) $t^2 \sin 3t$; $\int_0^\infty e^{-2t} t^2 \sin 3t \, dt$ **(M-13,M-12)** (d) $t^3 \sin t$; $\int_0^\infty e^{-t} t^3 \sin t \, dt = 0$

Homework

16) Find the Laplace transform of the following functions

(a) $te^{-2t}\sin(at-b)$ (b) $t\sin^3 t$ (c) $te^{3t}\sinh 4t$ **(d)** $t\sqrt{1-\sin t}$ **(D-13)**

(e) $te^{3t}\cos 2t$;and hence show that $\int_0^\infty e^{3t}t\cos 2t\,dt=\frac{5}{169}$ (f) $te^{-t}\cosh 2t$ **(M-15)**

$(g)\, tJ_0(4t)$ & *hence show that* $\int_0^\infty te^{-3t}J_0(4t)dt=\frac{3}{125}$ *where* $L[J_0(t)]=\frac{1}{\sqrt{s^2+1}}$ **(D – 09)**

Division By t Theorem

If L{F(t)}= f(s) then $L\{\frac{F(t)}{t}\}=\int_s^\infty f(u)du$ provided $\lim_{t\to 0^+}\frac{F(t)}{t}$ exists

Problems

17) Find the Laplace transform of the following functions & hence evaluate the integral

$(a)\frac{\sin^2 t}{t}$; $\int_0^\infty e^{-t}\frac{\sin^2 t}{t}dt=\frac{1}{4}\log 5$ **(D-14,D-13,D-11)**

(b) $\frac{\sin 2t+\sin 3t}{t}$; $\int_0^\infty e^{-t}(\frac{\sin 2t+\sin 3t}{te^t})dt=\frac{3\pi}{4}$ **(M-08)**

(c) $\frac{e^{-at}-e^{-bt}}{t}$; $\int_0^\infty\left(\frac{e^{-3t}-e^{-6t}}{t}\right)dt=\log 2$

(d) $\frac{\cos at-\cos bt}{t}$ **(D-09)** ; $\int_0^\infty\left(\frac{\cos 6t-\cos 4t}{t}\right)dt=\log\frac{2}{3}$ **(M-14,M-09)**

(e) $\{e^{-t}F(2t)\}$ if given $L[tF(t)]=\frac{1}{s(s^2+1)}$

Homework

18) Find the Laplace transform of the following functions & hence evaluate the integral

(a) $\frac{\sin t}{t}$ *and* $\int_{t=0}^\infty\int_{u=0}^t\frac{e^{-t}\sin u}{u}dudt$ **(D-12)** $(b)\frac{e^{2t}\sin t}{t}$ **(M-11)** (c) $\frac{\sin t\sin 5t}{t}$ **(M-09)**

(d) $\frac{\sin t\sinh t}{t}$; $\int_0^\infty e^{-\sqrt{2}t}\frac{\sin t\sinh t}{t}dt=\frac{\pi}{8}$ **(D-08)** **(e)** $t\left(\frac{\sin t}{e^t}\right)^2$ **(f)** $\frac{e^{-t}\sin t}{t}$

(g) $\frac{e^{-at}-\cos at}{t}$; $\int_0^\infty\left(\frac{e^{-t}-\cos t}{te^{4t}}\right)dt=\log\frac{\sqrt{17}}{5}$ **(D-11)**

Laplace Transform Of Integral

If L{F(t)}= f(s) then $L\{\int_0^t F(u)du\} = \frac{f(s)}{s}$

Problem

19) Find Laplace Transform of the following functions

$(a)\int_0^t e^{-u}\frac{\sin 4u}{u}du$ **(M – 09)** $(b)\int_0^t \frac{1-e^{-u}}{u}du$ **(M – 08)**

$(c)\int_0^t u\cos^2 u\, du$ (d) $\cosh t\int_0^t e^{u}\cosh u\, du$

Homework

20) Find Laplace Transform of the following functions

$(a)\int_0^t \frac{1-\cos u}{u}du$ $(b) e^{-3t}\int_0^t u\sin 3u\, du$ **(M-15,D-09)** (c) $\int_0^t ue^{-2u}\sin 3u\, du$ **(D-08)**

$(d)\, t\int_0^t e^{-4u}\cos u du$ *and evaluate* $\int_0^\infty e^{-t}(t\int_0^t e^{-4u}\cos u du)dt$ **(D – 10)** (e)$\int_t^\infty \frac{\cos u}{u}du$ $(f)\int_0^t u^2\sin u\, du$

Laplace Transform of Derivative

$L\{F'(t)\} = sL\{F(t)\} - F(0)$

$L\{F''(t)\} = s^2L\{F(t)\} - sF(0) - F'(0)$

Problems

21) (a)Find $L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right)$ given $L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s}$

(b) Show that f $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$ and deduce that $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$ **(D-12)**

Homework

22) (a) If $L\{t\sin\omega t\} = \frac{2\omega}{(s^2+\omega^2)^2}$ *find* $L\{\sin\omega t + \omega t\cos\omega t\}$

(b) Find $L\left\{\frac{d}{dt}\left(\frac{1-\cos t}{t}\right)\right\}$ **(M-12)**

Convolution Theorem

$$\text{If } L\{F(t)\}=f(s) \text{ and } L\{G(t)\}=g(s) \text{ then } L\left\{\int_0^t F(u)G(t-u)\,du\right\}=f(s)g(s)$$

Problem

23) Verify Convolution theorem for the function $F(t)=t^2, G(t)=e^{2t}$

Homework

24) Verify Convolution theorem for the function $F(t)=\sin at, G(t)=\sin bt$

Periodic Function

$$\text{If } F(t+T)=F(t) \text{ then } L\{F(t)\}=\frac{\int_0^T e^{-st}F(t)\,dt}{1-e^{-sT}}$$

Problems

25) Find the Laplace transform of the following functions with period equal to length of the given interval

(a) $F(t)=k\dfrac{t}{T} \qquad 0 \le t \le T$

(b) $F(t)=|\sin \omega t|$

(c) $F(t)=\begin{cases} E & 0<t<a/2 \\ -E & a/2<t<a \end{cases}$ (D-14)

(d) $F(t)=\begin{cases} \dfrac{t}{a} & 0<t<a \\ \dfrac{2a-t}{a} & a<t<2a \end{cases}$ **(M-13,M-11)**

Homework

26) Find the Laplace transform of the following functions with period equal to length of the given interval

(a) $F(t)=\begin{cases} a\sin pt; & 0<t<\pi/p \\ 0 \quad ; & \pi/p<t<2\pi/p \end{cases}$ **and** $\mathbf{f(t)}=\mathbf{f}\left(\mathbf{t}+\dfrac{\mathbf{2}\pi}{p}\right)$ **(D-13)**

(b) F(t)=3t ; 0<t<2 and 6; 2<t<4 and F(t+4)=F(t) for t>0 **(D-12)**

(c) $F(t)=\begin{cases} t & 0<t<\pi \\ \pi - t & \pi<t<2\pi \end{cases}$

Heavyside’s Unit Step Function

$$H(t-a)=\begin{cases}0 \text{ for } t<a\\ 1 \text{ for } t>a\end{cases}$$

Problems

27) Prove the following results

$$(a)\quad L[F(t).H(t-a)]=e^{-as}L[F(t+a)]$$

$$(b)\quad L[H(t-a)]=\frac{e^{-as}}{s}$$

28)Find the Laplace transform of the following functions

(a) $L[t^4H(t-1)]$

(b) $L[(1+2t-3t^2+4t^3)H(t-2)]$

29) Express the following function using Unit step functions and hence find its Laplace transform

$$F(t)=\begin{cases}t^2 & 0<t<2\\ 4t & t>2\end{cases}$$

Homework

30) Prove the following results

(a) $L[F(t).H(t)]=L[F(t)]=f(s)$

(b) $L[F(t-a).H(t-a)]=e^{-as}L[F(t)]$

31) Find the Laplace transform of the following functions

(a) $L[t^2H(t-3)]$

(b) $L[(1+3t-t^2+t^3)H(t-4)]$

32) Express the following function using Unit step functions and evaluate the Laplace transform

$$F(t)=\begin{cases}\cos t & 0<t<\pi\\ \cos 2t & \pi<t<2\pi\\ \cos 3t & t>2\pi\end{cases}$$ **(D-10)**

Unit impulse (or Dirac delta) function

Dirac's delta function is denoted by $\delta(t-a)$ and is defined as

$$\delta(t-a) = \lim_{\varepsilon \to 0} F_\varepsilon(t-a) \text{ where } F_\varepsilon(t-a) = \begin{cases} 0; \ t < a \\ \dfrac{1}{\varepsilon}; \ a < t < a+\varepsilon \\ 0; \ t > a+\varepsilon \end{cases}$$

Problems

33)Prove the following result

$(a)\int_0^\infty F(t)\delta(t-a)dt = F(a)$ (b) $L[F(t)\delta[F-a)] = e^{-as}F(a)$

34)Find the following

(a) $L[\sin 2t\delta(t-\pi/4) - t^2\delta(t-4)]$ $(b)\,L[\cos t \log t\delta(t-\pi)]$

Homework

35)Prove the following result

$(a)\ L[\delta(t-a)] = e^{-as}$ $(b) L[\delta(t)] = 1$

34)Find $L[t\,U(t-4) - t^3\delta(t-2)]$

INVERSE LAPLACE TRANSFORM

If $L\{F(t)\}=f(s)$ then $L^{-1}\{f(s)\}=F(t)$

Linearity Property

$L^{-1}\{af(s)+bg(s)\} = aL^{-1}\{f(s)\}+bL^{-1}\{g(s)\}$

Standard Inverse Laplace Transforms

f(s)	$L^{-1}\{f(s)\}$=F(t)
$\frac{1}{s}$	1
$\frac{1}{s^2}$	t
$\frac{1}{s^{n+1}}$	$\frac{t^n}{\Gamma(n+1)}$ $=\frac{t^n}{n!}$ if n is a positive integer
$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
$\frac{s}{s^2+a^2}$	cos at
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
$\frac{s}{s^2-a^2}$	cosh at

Problems

36) Find (**a**)$\boldsymbol{L^{-1}}\left\{\frac{6}{3-2s}-\frac{3+4s}{9s^2+16}+\frac{8-6s}{16s^2-9}\right\}$ (**b**)$\boldsymbol{L^{-1}}\left\{\frac{3s-2}{s^{\frac{5}{2}}}-\frac{3+4s}{9s^2+16}+\frac{8-6s}{16s^2-9}\right\}$ (**c**) $\frac{s^2+5}{(s^2+4s+13)^2}$ **(M-14)**

Homework

37)Find(a) $L^{-1}\left(\frac{3s-8}{s^2+4}+\frac{4s-24}{s^2-16}\right)$ **(D-12)** (b) $L^{-1}\left(\frac{3s-2}{s^{5/2}}-\frac{7}{3s+2}\right)$**(D-12)** (d) $\frac{s}{(s-2)^6}$ **(D-14)**

Standard Theorems on Inverse Laplace Transform

f(s)	$L^{-1}\{f(s)\}$=F(t)
$f(s-a)$	$e^{at}F(t)$

$e^{-as}f(s)$	$F(t-a)$ **H(t-a)**
$f^{(n)}(s)$	$(-1)^n t^n F(t)$
$\frac{f(s)}{s}$	$\int_0^t F(u)du$
$sf(s)$	$F'(t)$ **if F(0)=0**
$f(s)g(s)$	$\int_0^t F(u)G(t-u)du$

Problems

38)Find the Inverse Laplace Transform of the following functions

(a) $\frac{1}{s\sqrt{s+4}}$ (b) $\frac{2s^2-3s+4}{(s+3)^4}$ (c) $\frac{1}{s}\cos\frac{1}{s}$ **(D-12)**

39)Find the Inverse Laplace transform of the following functions

(a) $\frac{e^{4-3s}}{(s+4)^{5/2}}$ **(M-09,M-08)** (b) $\frac{e^{-3s}}{s^2-2s+5}$ **(M-10,M-07)** (c) $\frac{se^{-2s}}{s^2-6s+25}$ **(M-13)**

Homework

40)Find the Inverse Laplace transform of the following functions

(a) $\frac{e^{4-3s}}{(s+4)^{5/2}}$ (b) $\frac{se^{-4\pi s}}{s^2+4}$ **(M-10)** (c) $\frac{(s+1)e^{-\pi s}}{s^2+s+1}$ **(D-13,D-08)** (d) $\frac{(s+1)e^{-2s}}{s^2+2s+2}$

Problems

41)Find the Inverse Laplace Transform of the following functions using partial fraction method

(a) $\frac{s+2}{s^2+4s+7}$ **(M-12)** (b) $\left\{\frac{-3s^2+20s-24}{(s-1)(s-2)^2}\right\}$ (c) $\frac{1}{s^3+1}$

(d) $\frac{s^3+2s}{(s+1)^2(s^2+1)}$ (e) $\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ **(D-09,D-08)** (f) $\frac{2s}{s^4+4}$ **(M-15)**

Homework

42)Find the Inverse Laplace Transform of the following functions using partial fraction method

(a) $\frac{11s^2-2s+5}{2s^3-3s^2-3s+2}$ **(D-13)**

(b) $\frac{s+2}{s^2(s+3)}$

(c) $\frac{3s+1}{(s+1)(s^2+1)}$ **(M-11)**

(d) $\frac{1}{(s+1)^2(s^2+4)}$

(e) $\frac{5s^2-15s-1}{(s+1)(s-2)^2}$ **(D-11, M-10)**

(f) $\frac{2s^2-1}{(s^2+1)(s^2+4)}$

(g) $\frac{s}{(s^2+1)(s^2+4)(s^2+9)}$

$(h)\frac{s}{s^4+s^2+1}$

(i) $\frac{6s+3}{s^4+5s^2+4}$

(J) $\frac{1}{s^2(s+a)^2}$ **(D-13)**

Problems

43)Find the Inverse Laplace Transform of the following functions using convolution theorem

(a) $\frac{1}{(s^2+4)(s^2+9)}$ (b) $\left\{\frac{s}{s^4+13s^2+36}\right\}$ (c) $\frac{1}{(s^2+4s+13)^2}$ **(D-13)** **(d)** $\frac{1}{s\sqrt{s+4}}$ **(D-10)**

(d) $\frac{1}{(s^2+a^2)^2}$ (*e*) $\frac{s}{(s^2+a^2)^2}$ (f) $\frac{s^2}{(s^2+a^2)^2}$ $(D-09)$ **(g)** $\frac{1}{s^2(s+1)^2}$ **(D-14)**

Homework

44)Find the Inverse Laplace Transform of the following using convolution theorem

(a) $\frac{1}{(s^2+1)(s^2+4)}$ **(M-08)**

(b) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ **(M-12, M-10, D-09)**

(c) $\frac{s^2}{(s^2-a^2)^2}$ **(M-14)**

(d) $\frac{1}{(s+4)^2(s-3)}$ **(D-14)**

(e) $\frac{1}{(s+3)(s^2+2s+2)}$ **(D-11)**

(f) $\frac{s^2+s}{(s^2+1)(s^2+2s+2)}$ **(M-13)**

(g) $\frac{(s+3)^2}{(s^2+6s+5)^2}$

(h) $\frac{1}{s(s+a)^2}$

(i) $\frac{s}{(s^2-a^2)^2}$ **(M-15)**

Problems

45) Find the Inverse Laplace Transform of

(a) $\tan^{-1}(s+1)$ (b) $\cot^{-1}\frac{2}{s^2}$ **(D-10)**

(c) $\frac{1}{s}\log\left(\frac{s+3}{s+2}\right)$ **(D-09)** $(d)\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)$ **(M - 12, M – 09, D - 08, M - 08)**

Homework

46)Find the Inverse Laplace Transform of

(a) $\cot^{-1} as$ **(D-09)** (b) $2\tanh^{-1} s$ **(D-14)** (c) $\log(1+\frac{a^2}{s^2})$

(d) $\frac{1}{s}\log\sqrt{\left(\frac{s^2+a^2}{s^2+b^2}\right)}$ **(M-11)** (e) $\frac{1}{s}\log\left(\frac{s^2+a^2}{(s+b)^2}\right)$

Application of Laplace transform

$$L\{F'(t)\} = sL\{F(t)\} - F(0) \text{ and}$$
$$L\{F''(t)\} = s^2L\{F(t)\} - sF(0) - F'(0)$$

Problems

47)Solve the following intial value differential equations:

(a) $y''+2y'+5y = e^{-t}\sin t$;y(0)=0,y'(0)=1 (D-14)

(b) $y'+2y+\int_0^t ydt = \sin t$; y(0)=1 **(D-11,D-10)**

(c) $y''+9y = \cos 2t$; y(0)=1, $y(\pi/2) = -1$

(d) $(D^2-3D+2)y = 4e^{2t}$; y(0)= -3,y'(0)=5 **(M-14,M-09)**

(e) $y''-y'-2y = 20\sin 2t$ y(0)= 1,y'(0)=2 **(M-12)**

(f) $y''+2y'-3y = 0$ y(0)= 0,y'(0) =4 **(M-11)**

(g) $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+8y = t, y(0)=0, y'(0)=1$ **(D-13)**

Homework

48)Solve the following equations

(a) $y''-2y'+y = e^t$;y(0)=2,y'(0)=0-1 **(M-08)**

(b) $y + \int_0^t y\,dt = 1 - e^{-t}$ **(M-15)**

(c) $y'' + 9y = 18t$; y(0)=1, $y(\pi/2) = 0$ **(M-13)**

(d) $y'' + 3y' + 2y = t\,\delta(t-1)$; y(0)=

(e) $y'' + y' = t^2 + 2t$; y(0)= 4,y'(0)=-2 **(D-08)**

(f) 3 $y' + 2y = e^{3t}$; y(0)=1 **(M-10, D-09)**

(g) $\frac{d^2y}{dt^2} + y = t, y(0) = 1, y'(0) = 0$ **(D-13)**

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