$$\boxed{\mu_4 = 6\mu^2\sigma^2 + 3\sigma^4 + \mu^4} \quad , \quad \mu_4 = 3\sigma^4$$

$$\alpha_4 = \frac{3\sigma^4}{(\sigma^2)^2} = 3 \, //$$

Two Dimensional R.V's

let x_1, x_2 be two R.V's (Ω, S, P) then

$$X = (X_1, X_2)$$

Joint distribution function

$$F(x,y) = \text{Prob}\{X \le x, Y \le y\}$$

∴ probability is a dependent variable.

Conditions :- (1) F is non decreasing & right continuous w.r.t x & y

(2) $F(-\infty, y) = F(x, -\infty) = 0$

(3) $F(\infty, \infty) = 1$

(4) $\forall [(x_1, y_1) \,\&\, (x_2, y_2)]$

$$x_1 < x_2 \,,\, y_1 < y_2$$

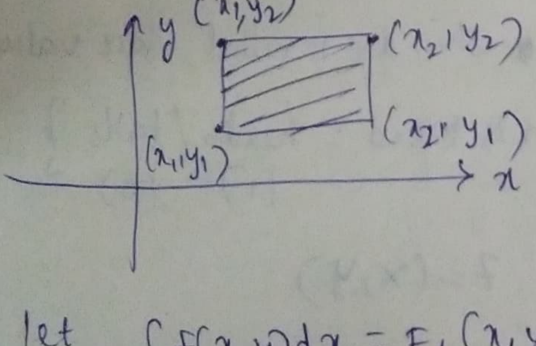
(1), (2), (3) do not imply (4)

$$\text{Prob}\{x_1 < X < x_2, \; y_1 < Y < y_2\} \ge 0$$

$$F(x_2, y_2) + F(x_1, y_1) - F(x_1, y_2) - F(x_2, y_1) \ge 0$$

Verify :- Area probability

$$\text{Prob}\{x_1 < X < x_2, y_1 < Y < y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy$$



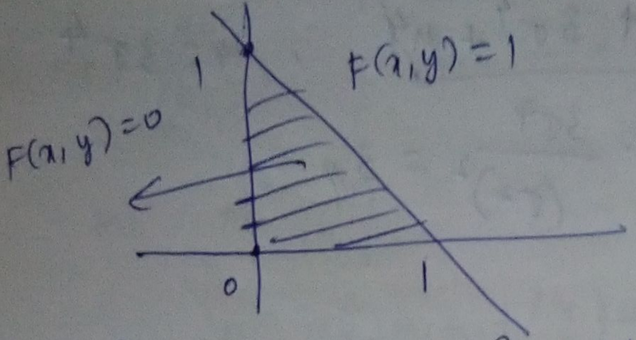
$$= \int_{y_1}^{y_2} F_1(x,y)\Big|_{x_1}^{x_2} dy$$

let $\int f(x,y)\,dx = F_1(x,y)$

$$= \int_{y_1}^{y_2} [F_1(x_2, y) - F_1(x_1, y)]\,dy$$

$\int F_1(x,y)\,dy = F(x,y)$

$$= F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$$



$$F(x,y) = \begin{cases} 0, & x<0,\ x+y<1,\ y<0 \\ 1, & \text{elsewhere} \end{cases}$$

$$\text{Prob}\left\{\frac{1}{3} < X < 1, \frac{1}{3} < Y < 1\right\} = F(1,1) + F\left(\frac{1}{3},\frac{1}{3}\right) - F\left(1,\frac{1}{3}\right) - F\left(\frac{1}{3},1\right)$$

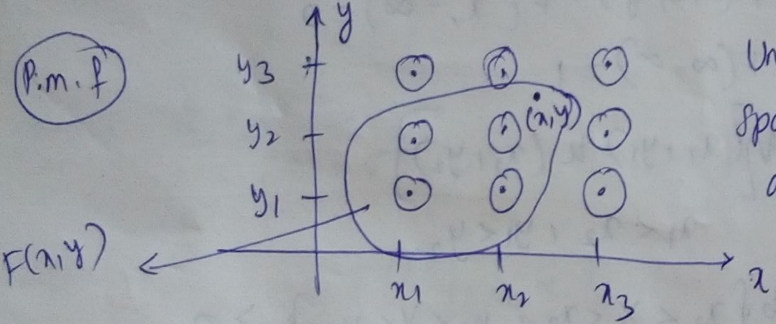
$$= 1 + 0 - 1 - 1$$

$$= -1$$ Not a proper joint distribution function.

[Not valid]

Discrete 2 Dimensional R.v's

P.m.f



Uniformly spaced grid points as gap is same

$$\sum_j \sum_i (P_{ij}) = 1 \qquad F(x,y) = \sum_{X \in B} (P_{ij})$$

$$B = \{(i,j),\ x_i < x,\ y_i < y\}$$

Experiment

X → {event when die is rolled, face value}

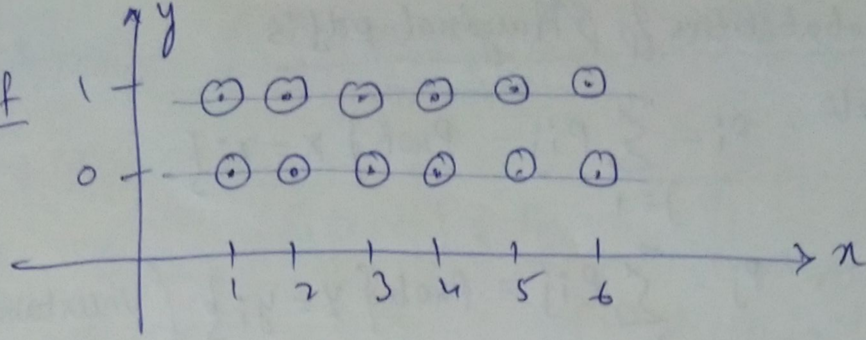
Y → {A coin is tossed : heads / tails}
(1) (0)

uniformly distributed → $P_{ij} = \frac{1}{12}$ $\qquad Z = (X,Y)$

probability of one joint R.v' outcome

$$\Omega = \{(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)\}$$

pmf



Continuous 2 Dimensional R.V's

R.V = (X, Y)

(cdf) $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,dv\,du$

Distribution function

$F(+\infty, \infty) = 1$

Joint pdf :- $f(x,y) = \dfrac{\partial^2}{\partial x \partial y}\left(F(x,y)\right)$

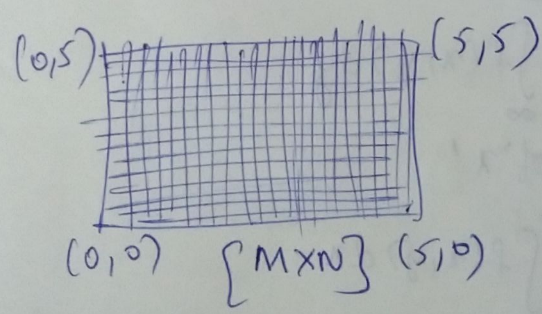
Ex:- $f(x,y) = \begin{cases} e^{-x-y}, & 0<x<\infty,\ 0<y<\infty \\ 0, & \text{elsewhere.} \end{cases}$

$x: 0 \to 5 \quad 0, 0.01, 0.02 \cdots 5 \to M$

$y: 0 \to 5 \quad 0, 0.01, \cdots \quad N$

$\boxed{(X_{mat}, Y_{mat}) = \text{Meshgrid}(x, y)}$

creates a 2D grid



surf (xmat, ymat, c)

$F(x,y) = ? \quad = \int_{-\infty}^{x}\int_{-\infty}^{y} e^{-u-v}\,dv\,du$

$= \int_{0}^{y} e^{-v}\,dv \int_{0}^{x} e^{-u}\,du = \left[\dfrac{e^{-v}}{-1}\right]_0^y \left[\dfrac{e^{-u}}{-1}\right]_0^x$

$= \left[\dfrac{e^{-y}}{-1} - \left(\dfrac{1}{-1}\right)\right]\left[\dfrac{e^{-x}}{-1} - \left(\dfrac{1}{-1}\right)\right]$

$\boxed{F(x,y) = (1-e^{-y})(1-e^{-x})}$

Marginal probabilities & Marginal pdf's

For discrete, $p_i = \sum_{j=1}^{\infty} p_{ij} = \text{Prob}\{X = x_i\}$

$p_j = \sum_{i=1}^{\infty} p_{ij} = \text{Prob}\{Y = y_i\}$ (Uncertainity w.r.t 'y' is eliminated)

Experiment: Fair coin tossed thrice

X: No. of heads in 3 tosses

Y: Absolute difference b/w no. of heads & tails.

$X = \{0, 1, 2, 3\}$, $Y = \{1, 3\}$

Y\X	0	1	2	3	P(Y=y)
1	0	3/8	3/8	0	6/8
3	1/8	0	0	1/8	2/8
P(X=x)	1/8	3/8	3/8	1/8	

Marginal

Marginal

Stem 2 for 2D discrete

Stem 3 for 3D discrete

$P(X, Y)$ = value of the point $= x_1, y_1$

For continuous,

$f_1(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$

Marginal pdf of 'x'

$f_2(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$

marginal pdf of 'y'

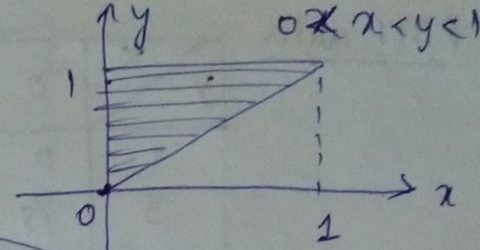
Univariate:- $x \sim f(x)$

Bivariate:- $(x, y) \sim f(x, y)$

Multivariate:- $X = [x_1, x_2, x_3, \cdots x_n]$

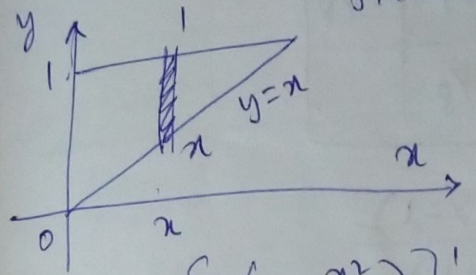
$X \sim f(\bar{x}) \rightarrow$ nD pdf

a) $f(x,y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{elsewhere} \end{cases}$



$f_1(x) = ?$

$$f_1(x) = \int_x^1 2\,dy \longrightarrow , 0 < x < 1$$



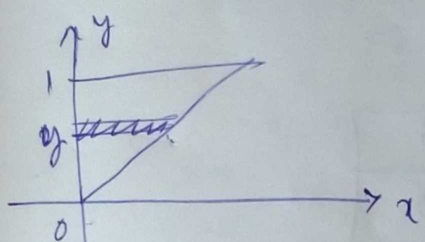
$$f_1(x) = 2(1-x) \rightarrow 0 < x < 1$$
$$0, \rightarrow \text{elsewhere.}$$

proper pdf $\left[2\left(x - \frac{x^2}{2}\right)\right]_0^1 = 2\left[1 - \frac{1}{2}\right] - 2(0)$

$$2\left[\frac{1}{2}\right] - 2(0) = \underline{\underline{1}}$$

$f_2(y) = ?$

$$f_2(y) = \int_0^y 2\,dx \;, 0 < y < 1 = 2y$$
$$0, \text{ elsewhere}$$



$$\text{proper pdf} = 2\left[\frac{y^2}{2}\right]_0^1$$
$$= 2\left(\frac{1}{2}\right) = \underline{\underline{1}}$$

Marginal distribution function

$$F_1(x) = F(x,\infty) = \lim_{y \to \infty} F(x,y)$$

p_i is the marginal probability mass function of X

$$\begin{cases} \sum_{x_i \le x} p_i & (x,y) \text{ discrete} \\ \int_{-\infty}^{x} f_1(t)\,dt & (x,y) \text{ continuous} \end{cases}$$

Conditional probability

For discrete, $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = P\left\{X = x_i / Y = y_j\right\}$

$P(A/B) = \dfrac{p_{ij}}{p_j}$ $\rightarrow$ joint probability mass function; $p_j \rightarrow$ marginal probability mass function

$A : \{X = x_i\}$

$B = \{Y = y_j\}$

$A \cap B = \{X = x_i, Y = y_j\}$

PMF data

y \ x	0	1	2	3	P(Y=y)
1	0	3/8	3/8	0	6/8
3	1/8	0	0	1/8	2/8
P(X=x)	1/8	3/8	3/8	1/8	

(i) $P\{X=i/Y=1\} = \dfrac{P(X=i, Y=1)}{P(Y=1)}$

$P(X=i, Y=1) =$

i = 0	1	2	3
0	3/8	3/8	0

$P(Y=1) = \dfrac{6}{8}$

$P\{X=i/Y=1\}$

i=0	1	2	3
0	$\frac{3/8}{6/8}$	$\frac{3/8}{6/8}$	0
0	1/2	1/2	0

(ii) $P(X=i/Y=3) = \dfrac{P(X=i, Y=3)}{P(Y=3)}$

$P(X=i, Y=3)$

0	1	2	3
1/8	0	0	1/8

$P(Y=3) = 2/8$

$P\{X=i/Y=3\}$

0	1	2	3
1/2	0	0	1/2

(iii) $P\left(Y=j / X=0\right) = $

0,	1/8
1/8	

$= [0, 1]$

(iv)

$$P\left(Y=j / X=0\right) = \begin{cases} 0, & j=1 \\ 1, & ~~elsewhere~~\ j=3 \end{cases}$$

Conditional D.F for RV X

Given, $Y=y$

$$F_{X/Y}(x/y) = \lim_{\epsilon \to 0} P\{X \le x \mid Y \in (y-\epsilon, y+\epsilon)\}$$

$$= F_{X/Y}(x/y) = F_{X/Y}(x/Y=y)$$

$$= \lim_{\epsilon \to 0} \frac{P\{X \le x, Y \in (y-\epsilon, y+\epsilon)\}}{P\{Y \in (y-\epsilon, y+\epsilon)\}}$$

$$= \lim_{\epsilon \to 0} \frac{\int_{-\infty}^{x} \int_{y-\epsilon}^{y+\epsilon} f(u,v)\,dv\,du}{\int_{y-\epsilon}^{y+\epsilon} f_2(v)\,dv}$$

$$F_1(u,y) = \int_{-\infty}^{y} f(u,v)\,dv$$

$$= \lim_{\epsilon \to 0} \frac{\int_{-\infty}^{x} F_1(u,v) \Big|_{y-\epsilon}^{y+\epsilon} du}{g_2(v) \Big|_{y-\epsilon}^{y+\epsilon}}$$

$$g_2(v) = \int_{-\infty}^{v} f_2(w)\,dw$$

$$f(u,y) = \frac{d}{dy} F_1(x,y)$$

$$= \frac{\lim_{\epsilon \to 0} \int_{-\infty}^{x} \frac{F_1(u, y+\epsilon) - F_1(u, y-\epsilon)}{2\epsilon}\,du}{\lim_{\epsilon \to 0} \frac{g_2(y+\epsilon) - g_2(y-\epsilon)}{2\epsilon}}$$

$$= \frac{\int_{-\infty}^{x} f(u,y)\,du}{f_2(y)}$$

$$F_{X/Y}(x/y) = \frac{\int_{-\infty}^{x} f(u,y)\,du}{f_2(y)}$$

$$\frac{d}{dx}\left(F_{X/Y}(x/y)\right) = f(x/y)$$

cdf of continuous R.V X

$$f(x/y) = \frac{f(x,y)}{f_2(y)}$$ $\to f(y)$

p.d.f of continuous R.V X (conditional)

$$\boxed{f(x,y) = f(y)\cdot f(x/y)}$$ → Baye's rule valid for density functions also

$$\boxed{f(x,y) = f(x)\cdot f(y/x)}$$

Integrate w.r.t 'x' on B.S.

$$\int_{-\infty}^{x} f(u,y)\,du = f(y)\cdot \int f(x/y)\,dx$$

$$\int_{-\infty}^{x} f(u,y)\,du = f(y)\, F_{X/Y}(x/y)$$

marginal c.d.f $\quad F_1(x) = ? = \int_{-\infty}^{\infty}\int_{-\infty}^{x} f(u,y)\,du\,dy = \int_{-\infty}^{\infty} f(y)\, F_{X/Y}\left(\frac{x}{y}\right) dy$

Differentiat w.r.t 'x'

Marginal density $\quad f_1(x) = \int_{-\infty}^{\infty} f(y)\, f(x/y)\,dy$

$$\frac{d}{dx}\left(F_1(x)\right) = f_1(x)$$

Q) $f(x,y) = \begin{cases} 2, & 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$

$f_1(x) = 2(1-x), \quad 0<x<1$
$\qquad\quad 0, \quad \text{else}$

$f_2(y) = 2y, \quad 0<y<1$
$\qquad\quad 0, \quad \text{else}$

$$f_{Y/X}(y/x) = \frac{f(x,y)}{f(x)} = \frac{2}{2(1-x)}, \quad x<y<1$$
$\downarrow$ R.V

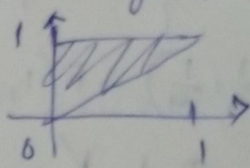
$$f_{X/Y}(x/y) = \frac{f(x,y)}{f(y)} = \frac{1}{y}, \quad 0<x<y$$
$\downarrow$ R.V

Q) i) $P\{Y \geq 1/2 \mid X = \frac{1}{2}\}$. 1

ii) $P\{Y \geq 1/2 \mid X = \frac{1}{4}\}$ 2/3

iii) $P\{X \geq 1/3 \mid Y = \frac{2}{3}\}$ 1/2

$0<x<y<1$



(i) $1 - P\{Y < 1/2 \mid X = 1/2\}$

$$= 1 - \int_{1/2}^{1/2} f(y/x = 1/2)\, dy = 1 - 0 = \underline{\underline{1}}$$

(ii) $P\{Y \geq 1/2 \mid X = \frac{1}{4}\}$

$1 - P\{Y < 1/2 \mid X = \frac{1}{4}\}$

$$= 1 - \int_{1/4}^{1/2} f(y/x)\, dy$$

$$= 1 - \frac{1}{1-1/4}\, y \Big|_{1/4}^{1/2}$$

$$= 1 - \frac{4}{3}\left[\frac{1}{2} - \frac{1}{4}\right]$$

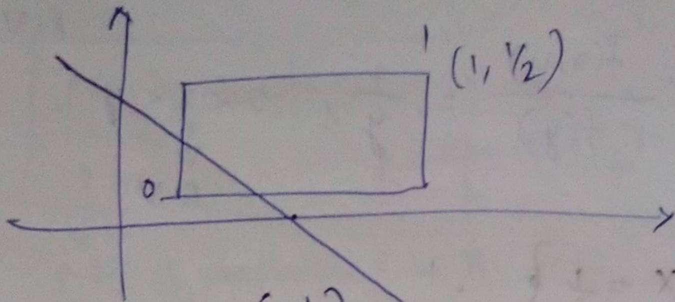
$$= 1 - \frac{4}{3}\left[\frac{1}{4}\right] = 2/3$$

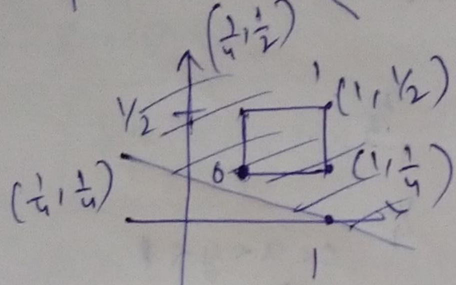
iii) $P\{X \geq 1/3 \mid Y = 2/3\} = 1 - P\{X < 1/3 \mid Y = 2/3\}$

$$= 1 - \int_{0}^{1/3} f(x/y)\, dx$$

$$1 - \frac{1}{y}\left(\frac{1}{3}\right) = \frac{1}{2}$$

Q) $F(x,y) = \begin{cases} 1, & x+2y \geq 1 \\ 0, & \text{elsewhere} \end{cases}$ verify if it is a valid D.F



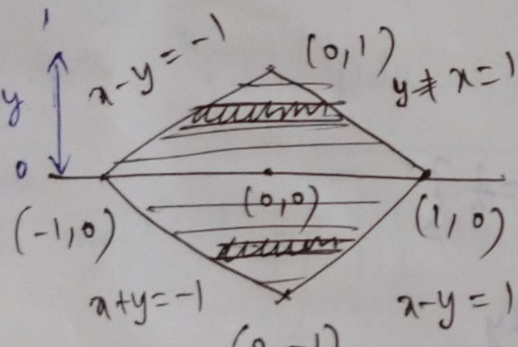


$$F\left(\tfrac{1}{4},\tfrac{1}{4}\right) + F\left(1,\tfrac{1}{2}\right) - F\left(\tfrac{1}{4},\tfrac{1}{2}\right) - F\left(1,\tfrac{1}{4}\right)$$

$$0 + \cancel{1} - \cancel{1} - 1$$

$$= -1 \quad [\text{Area is -ve}]$$

Q)



$f(x,y) \begin{cases} \frac{1}{2} \\ = 0 \text{ else} \end{cases}$

square diamond with corners at (−1,0), (0,−1), (1,0), (0,1)

For upper Δle

$0<y<1$

$y-1<x<1-y$ } $f(x,y) = \frac{1}{2}$

$= 0$ elsewhere

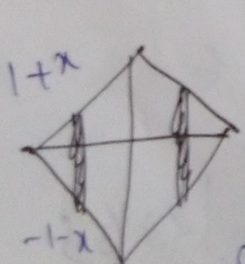
For lower Δle

$-1<y<0$

$-(1+y)<x<1+y$

$f(x,y) = \frac{1}{2}$

$= 0$ elsewhere



$$f(x) = \int f(x,y)\, dy$$

$$f(x,y) = \begin{cases} 1/2, & x-1<y<1-x,\ 0<x<1 \\ 1/2, & -(1+x)<y<(1+x),\ -1<x<0 \\ 0 & \text{else} \end{cases}$$

$$f(x) = \int_{-(1+x)}^{1+x} \frac{1}{2}\,dy \;,\quad -1 < x < 0$$

$$\int_{x-1}^{1-x} \frac{1}{2}\,dy \;,\quad 0 < x < 1$$

$$f(x) = \begin{cases} \frac{1}{2}\left[(1+x) + (1+x)\right], & -1<x<0 \\ \frac{1}{2}\left[(1-x) - (x-1)\right], & 0<x<1 \\ 0, & \text{else} \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2}\left[2+2x\right], & -1<x<0 \\ \frac{1}{2}\left[1-x-x+1\right], & 0<x<1 \\ 0, & \text{else} \end{cases}$$

$$\boxed{f(x) = \begin{cases} 1+x, & -1<x<0 \\ 1-x, & 0<x<1 \\ 0, & \text{else} \end{cases}}$$

$$f(y) = \int f(x,y)\,dx$$

$$f(x,y) = \begin{cases} 1/2, & 0<y<1,\; y-1<x<1-y \\ 1/2, & -1<y<0,\; -(1+y)<x<1+y \\ 0, & \text{else} \end{cases}$$

$$\boxed{f(y) = \begin{cases} 1+y, & -1<y<0 \\ 1-y, & 0<y<1 \\ 0, & \text{else} \end{cases}}$$

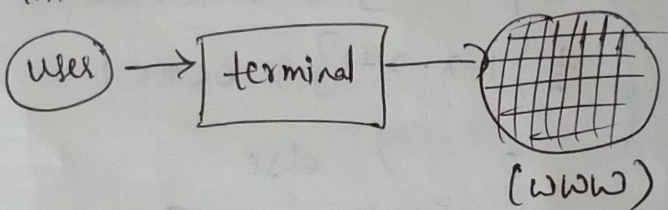
$$f_{X/Y}(x/y) = \frac{f(x,y)}{f(y)} = \frac{1/2}{1+y} = \frac{1}{2(1+y)}, \quad -(1+y)<x<(1+y), \; -1<y<0$$

$$= \frac{1}{2(1-y)}, \quad y-1<x<1-y, \; 0<y<1$$

$$= 0, \text{ elsewhere}$$

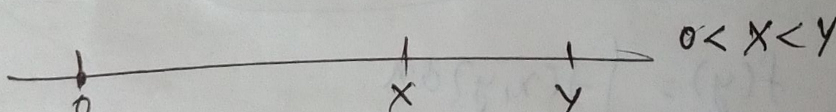
$$f_{Y/X}(y/x) = \begin{cases} \frac{1}{2(1+x)}, & -(1+x)<y<(1+x), \; -1<x<0 \\ \frac{1}{2(1-x)}, & x-1<y<1-x, \; 0<x<1 \\ 0, & \text{else.} \end{cases}$$

Q) $X \to$ Time until a server connects to your terminal



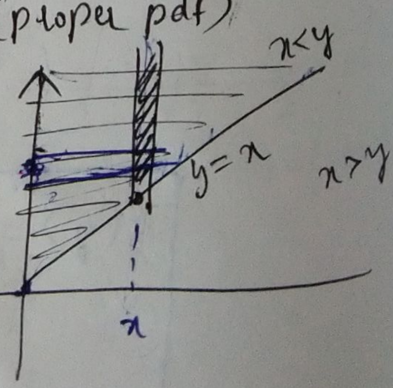
$Y \to$ time until the server authorises you.

X and Y are measured from a common point



$$f(x,y) = 6\times 10^{-6} \exp(-10^{-3}x - 2\times 10^{-3}y), \quad 0<x<y$$

(i) Show that $\iint f(x,y)\,dx\,dy = 1$ (proper pdf)



$$= \int_0^\infty \int_x^\infty 6\times 10^{-6} e^{(-10^{-3}x - 2\times 10^{-3}y)} \, dy\, dx$$

$$= \int_0^\infty 6\times 10^{-6} e^{-10^{-3}x} \cdot \left. \frac{e^{-2\times 10^{-3}y}}{-2\times 10^{-3}} \right|_x^\infty$$

$$= 3\times10^{-3}\int_0^\infty e^{-10^{-3}x}\, e^{-2\times10^{-3}x}\,dx$$

$$= 1.$$

(ii) $P\{X \le 1000,\ Y \le 2000\}$

$$= \int_0^{1000}\int_x^{2000} f(x,y)\,dy\,dx$$

$$= \int_0^{1000}\int_x^{2000}\left(6\times10^{-6}\, e^{-10^{-3}x}\, e^{-6\times10^{-3}y}\,dy\,dx\right)$$

$$= 6\times10^{-6}\int_0^{1000} e^{-10^{-3}x}\, \frac{e^{-2\times10^{-3}y}}{-2\times10^{-3}}\Bigg|_x^{2000} dx$$

$$= 6\times10^{-6}\int_0^{1000} e^{-10^{-3}x}\left[\frac{e^{-4}}{-2\times10^{-3}} - \frac{e^{-2\times10^{-3}x}}{-2\times10^{-3}}\right]dx$$

$$= \frac{\cancel{6}^{3}\times10^{-6}\times\cancel{10^{3}}}{-\cancel{2}\times10^{-3}}\int_0^{1000} e^{-10^{-3}x}\left(e^{-4} - e^{-2\times10^{-3}x}\right)dx$$

$$= -3\times10^{-3}\int_0^{1000}\left(e^{-4}e^{-10^{-3}x} - e^{-2\times10^{-3}x}\, e^{-10^{-3}x}\right)dx$$

$$= -3\times10^{-3}\left[\frac{-e^{-4}\,e^{-10^{-3}x}}{10^{-3}} - \frac{e^{-3\times10^{-3}x}}{-3\times10^{-3}}\right]_0^{1000}$$

$$= \frac{-3\times\cancel{10^{-3}}}{\cancel{10^{-3}}}\left[e^{-4}e^{-10^{-3}x} + \frac{e^{-3\times10^{-3}x}}{3}\right]_0^{1000}$$

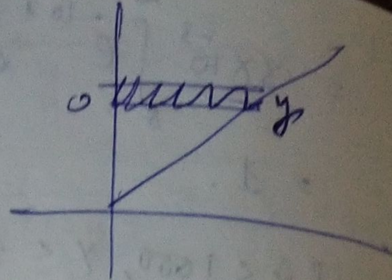
$$= -3\left[e^{-4}e^{-1} + \frac{e^{-3}}{3} - \left(e^{-4} + \frac{1}{3}\right)\right]$$

$$= -3\left[e^{-5} + \frac{e^{-3}}{3} + e^{-4} - \frac{1}{3}\right] = +3e^{-5} - e^{-3} - 3e^{-4} + 1$$

$$= 0.915$$

iii) $P(Y > 2000) = ?$

Sol:- $f(y) = \int_0^y f(x,y)\,dx$



$= \int_0^y f(x,y)\,dx$

$P(Y > 2000) = 1 - P(Y \le 2000)$

$= 1 - F(a)$

$f(y) = \int_0^y \left(6\times10^{-6} e^{-10^{-3}x} e^{-2\times10^{-3}y}\,dx \right)$

$= 6\times10^{-6} e^{-2\times10^{-3}y} \int_0^y e^{-10^{-3}x}\,dx$

$= 6\times10^{-6} e^{-2\times10^{-3}y} \cdot \left. \frac{e^{-10^{-3}x}}{-10^{-3}} \right|_0^y$

$f(y)\, F(a) = \frac{6\times10^{-6}}{10^{-3}} e^{-2\times10^{-3}y} \left[1 - e^{-10^{-3}y} \right], \quad y > 0$

$= \frac{6\times10^{-6} e^{-2\times10^{-3}y}}{10^{-3}} - \frac{6\times10^{-6} e^{-3\times10^{-3}y}}{10^{-3}}$

$F(y) = \int_0^{2000} f(y)\,dy$

$\frac{6\times10^{-6}}{10^{-3}} \int_0^{2000} \left(e^{-2\times10^{-3}y} - e^{-3\times10^{-3}y} \right)$

$= \frac{6\times10^{-6}}{10^{-3}} \left[\frac{e^{-2\times10^{-3}y}}{-2\times10^{-3}} - \frac{e^{-3\times10^{-3}y}}{-3\times10^{-3}} \right]_0^{2000}$

$= \frac{6\times10^{-6}}{10^{-3}} \left[\frac{e^{-4}}{-2\times10^{-3}} - \frac{e^{-6}}{-3\times10^{-3}} - \left(\frac{1}{-2\times10^{-3}} - \frac{1}{-3\times10^{-3}} \right) \right.$

$= \frac{6\times10^{-6}}{10^{-3}} \left[\frac{e^{-4}}{-2\times10^{-3}} + \frac{e^{-6}}{3\times10^{-3}} + \frac{1}{2\times10^{-3}} - \frac{1}{3\times10^{-3}} \right]$

$$= \frac{6\times10^{-6}}{10^{-3}}\left[\frac{e^{-4}}{-2\times10^{-3}} + \frac{1}{2\times10^{-3}} + \frac{e^{-6}-1}{3\times10^{-3}}\right]$$

$$= \frac{6\times10^{-6}}{10^{-3}}\left[\frac{e^{-4}}{-2\times10^{-3}} + \frac{e^{-6}}{3\times10^{-3}} + \frac{1}{6\times10^{-3}}\right]$$

$$= \frac{-3e^{-4}\times10^{-3} + 2e^{-6}\times10^{-3} + 10^{-3}}{10^{-3}}$$

$$= -3e^{-4} + 2e^{-6} + 1 = 0.95$$

(ii) $$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$f(x) = \int f(x,y)\,dy$$

$$= \int_x^\infty 6\times10^{-6} e^{-10^{-3}x} e^{-2\times10^{-3}y}\,dy$$

$$= 6\times10^{-6} e^{-10^{-3}x} \int_x^\infty e^{-2\times10^{-3}y}\,dy$$

$$= 6\times10^{-6} e^{-10^{-3}x}\left[\frac{e^{-2\times10^{-3}y}}{-2\times10^{-3}}\right]_x^\infty$$

$$= 6\times10^{-6}\times e^{-10^{-3}x}\left[0 - \frac{e^{-2\times10^{-3}x}}{-2\times10^{-3}}\right]$$

$$= \frac{3\times10^{-6}}{10^{-3}}\times e^{-10^{-3}x} e^{-2\times10^{-3}x}$$

$$= 3\times10^{-3} e^{-3\times10^{-3}x}, \quad x>0$$

$$f(y/x) = \frac{\cancel{6}^{2}\times10^{-6} e^{-10^{-3}x} e^{-2\times10^{-3}y}}{\cancel{3}\times10^{-3} e^{-3\times10^{-3}x}}$$

$$= \frac{2\times10^{-3} e^{-10^{-3}x} e^{-2\times10^{-3}y}}{e^{-3\times10^{-3}x}} = \begin{cases} 2\times10^{-3} e^{-2\times10^{-3}(y-x)}, & 0<x<y \\ 0, & \text{else} \end{cases}$$

$$P(Y>2000/X=1500) = 1 - P(Y<2000/X=1500)$$

$$= 1 - \int_0^{2000} \left(2\times10^{-3} e^{-2\times10^{-3}(y-x)} dy\right)\Bigg|_{x=1500}$$

$$= 1 - \frac{e^{3}(e^{-4} - e^{-3})}{-1}$$

$$= 0.367$$

Independent R.V's

X, Y are two R.Vs

$$f(x,y) = f(x) \cdot f(y)$$

$$\boxed{f(x,y) = 2\times10^{-6} e^{-10^{-3}x} e^{-2\times10^{-3}y}} \quad x\geq 0,\ y\geq 0$$

$$f(x) = 2\times10^{-6} \times e^{-10^{-3}x} \int_0^{\infty} e^{-2\times10^{-3}y} dy$$

$$f(x) = 2\times10^{-6} \times e^{-10^{-3}x} \left[\frac{e^{-2\times10^{-3}y}}{-2\times10^{-3}}\right]_0^{\infty}$$

$$f(x) = 2\times10^{-6} e^{-10^{-3}x} \left[\frac{1}{2\times10^{-3}}\right]$$

$$\boxed{f(x) = 10^{-3} e^{-10^{-3}x}, \quad x>0}$$

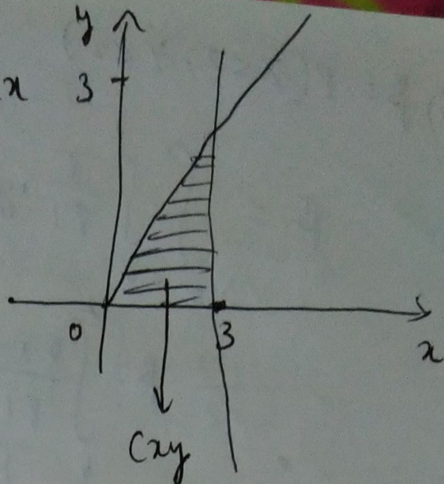
$$f(y) = \int_0^{\infty} 2\times10^{-6} \times e^{-10^{-3}x} e^{-2\times10^{-3}y} dx$$

$$f(y) = 2\times10^{-6} \times e^{-2\times10^{-3}y} \left[\frac{e^{-10^{-3}x}}{-10^{-3}}\right]_0^{\infty}$$

$$f(y) = 2\times10^{-6} \times e^{-2\times10^{-3}y} \left[\frac{1}{10^{-3}}\right]$$

$$\boxed{f(y) = 2\times10^{-3} \times e^{-2\times10^{-3}y}, \quad y>0}$$

Q) Given $f(x,y) = c \cdot xy$, $0 < x < 3$
$0 < y < x$



(i) Find c?

$$\int_0^3 \int_0^x f(x,y)\,dx\,dy = 1$$

$$\int_0^3 \int_0^x (cxy)\,dy\,dx = 1$$

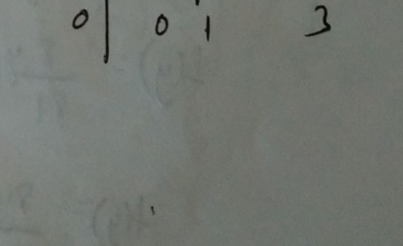
$$\int_0^3 cx\left[\frac{y^2}{2}\right]_0^x dx = 1$$

$$\int_0^3 cx\left[\frac{x^2}{2}\right] dx = 1$$

$$\Rightarrow c\int_0^3 \frac{x^3}{2}\,dx = 1 \Rightarrow \frac{c}{2}\left[\frac{x^4}{4}\right]_0^3 = 1$$

$$\Rightarrow \frac{c}{2}\left[\frac{81}{4}\right] = 1 \Rightarrow \boxed{c = \frac{8}{81}}$$

(ii) $P(X<1, Y<2) = \alpha$



$$\alpha = \int_0^1 \int_0^x \left(\frac{8}{81}xy\right) dy\,dx$$

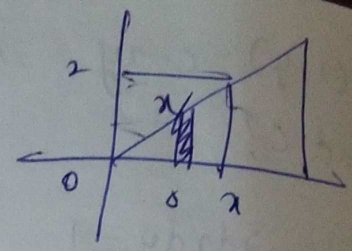
$$\alpha = \int_0^1 \frac{8}{81}x\left[\frac{y^2}{2}\right]_0^x dx$$

$$\alpha = \frac{8}{81}\int_0^1 x\left[\frac{x^2}{2}\right] dx$$

$$\alpha = \frac{4}{81}\left[\frac{x^4}{4}\right]_0^1$$

$$\alpha = \frac{4}{81}\left[\frac{1}{4}\right] \Rightarrow \boxed{\alpha = \frac{1}{81}}$$

(iii) $\beta = P(X<2, Y<2)$



$$\beta = \int_0^2 \int_0^x \frac{8}{81} xy \, dy \, dx$$

$$\beta = \int_0^2 \frac{8x}{81} \left(\frac{x^2}{2}\right) dx = \frac{4}{81} \times \frac{x^4}{4} \Big|_0^2$$

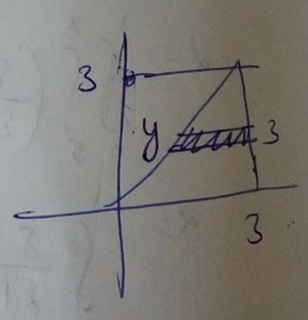
$$\boxed{\beta = \frac{16}{81}}$$

<u>Marginals</u>

$$f(x) = \int_0^x f(x,y) \, dy$$

$$f(x) = \frac{8}{81} \int_0^x xy \, dy = \frac{8x}{81} \left[\frac{y^2}{2}\right]_0^x$$

$$f(x) = \frac{8x}{81} \left[\frac{x^2}{2}\right] = \frac{4x^3}{81} \,//$$



$$f(y) = \int_y^3 f(x,y) \, dx$$

$$f(y) = \frac{8y}{81} \int_y^3 x \, dx$$

$$f(y) = \frac{8y}{81} \left[\frac{x^2}{2}\right]_y^3$$

$$f(y) = \frac{8y}{81} \left[\frac{9}{2} - \frac{y^2}{2}\right]$$

$$f(y) = \frac{8y}{81} \left[\frac{9-y^2}{2}\right] = \frac{4y(9-y^2)}{81} \,//$$

$$E(X) = \int_0^3 x f(x)\,dx = \int_0^3 \frac{4x^3}{81} x\,dx$$

$$= \frac{4}{81}\left[\frac{x^5}{5}\right]_0^3$$

$$= \frac{4}{\cancel{81}}\left[\frac{\cancel{81}\times 3}{5}\right] = \frac{12}{5}$$

$$E(Y) = \int_0^3 y f(y)\,dx$$

$$= \int_0^3 \frac{4y^2(9-y^2)}{81}\,dy$$

$$= \frac{4}{81}\int_0^3 (9y^2 - y^4)\,dy$$

$$= \frac{4}{81}\left[\frac{9y^3}{3} - \frac{y^5}{5}\right]_0^3$$

$$= \frac{4}{81}\left[\frac{9\times 27}{3} - \frac{81\times 3}{5}\right]$$

$$= \frac{4}{81}\left[81 - \frac{81\times 3}{5}\right]$$

$$= \frac{4}{\cancel{81}}\times\cancel{81}\left[1-\frac{3}{5}\right] = 4\left[\frac{2}{5}\right] = \frac{8}{5}$$

Given a conditional pdf

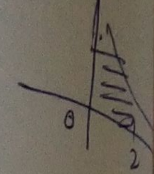
$$f_{Y/X}(y/x) = x e^{-xy}, \quad y>0$$

$$f(x) = \begin{cases} \frac{1}{10}, & 0<x<10 \\ 0, & \text{elsewhere} \end{cases}$$

(i) plot $f(y/x)$

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

(ii) $P\left(Y<2/X=2\right) = \int_0^2 \left(x e^{-xy}\right) dy \Big|_{X=2}$



$P\left(Y<2/X=2\right) = x \int_0^2 e^{-xy} dy \Big|_{X=2}$

$P\left(Y<2/X=2\right) = 2 \int_0^2 e^{-2y} dy = 2 \left[\frac{e^{-2y}}{-2}\right]_0^2$

$P\left(Y<2/X=2\right) = 2\left[\frac{e^{-4}}{-2} + \frac{1}{2}\right]$

$\boxed{P\left(Y<2/X=2\right) = 1-e^{-4}}$

(iii) $E\left[Y/X=2\right] = \int_0^\infty y\, f\left(y/x=2\right) dy$

$= \int_0^\infty y\left(x e^{-xy}\right) dy$

$= x\int_0^\infty y\, e^{-xy}\, dy$

$= x\left\{\left[y\frac{e^{-xy}}{-x}\right]_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-xy}}{-x} dy\right\}$

$= x\left\{\frac{1}{x}\int_0^\infty e^{-xy} dy\right\} = \left[\frac{e^{-xy}}{-x}\right]_0^\infty$

$\boxed{E\left[Y/X=2\right] = \frac{1}{x}}$, $0<x<10$

(iv) $f(x,y) = ?$ $\quad f(x,y) = f(x) \cdot f\left(y/X=x\right)$

$\boxed{f(x,y) = x e^{-xy} \cdot \frac{1}{10}}$, $y>0$, $0<x<10$

(v) $f(y) = \int_{0}^{10} f(x,y)\,dx$

$$= \int_0^{10} x e^{-xy} \cdot \frac{1}{10}\,dx$$

$$= \frac{1}{10} \int_0^{10} x e^{-xy}\,dx$$

$$= \frac{1}{10}\left[\left[x \frac{e^{-xy}}{-y}\right]_0^{10} - \int_0^{y} (1)\,\frac{e^{-xy}}{-y}\,dx\right]$$

$$= \frac{1}{10}\left[\left(y\,\frac{e^{-y(10)}}{-y}\right) + \frac{1}{y}\int_0^{y} e^{-xy}\,dx\right]$$

$$= \frac{1}{10}\left[e^{-y(10)} + \frac{1}{y}\left[\frac{e^{-xy}}{-y}\right]_0^{10}\right]$$

$$= \frac{1}{10}\left[e^{-y(10)} + \frac{1}{y}\left[\frac{e^{-y(10)}}{-y} + \frac{1}{y}\right]\right]$$

$$= \frac{1}{10y^2}\left[1 - e^{-10y}(1+10y)\right]$$

Q) A clothing manufacturing company

user 1 — $\boxed{\text{S.S.1}}$ → $\boxed{\text{order}}$ → perce. center ?

user 2 — $\boxed{\text{S.S.2}}$ → $\boxed{\text{order}}$ → $\boxed{\text{pa.2}}$

$X \to$ time b/w order for site 1

$Y \to$ time b/w for site 2

$X \sim \exp(\mu_1 = 3.2\,\text{min})$

$Y \sim \exp(\mu_2 = 3.2\,\text{min})$

a) Prob { No order will be processed in a 5 min period }

$P\{X>5, Y=5\}$

$= \text{Prob}\{X>5\} \cdot \text{Prob}\{Y>5\}$

$P\{X>5, Y=5\} = 1 - \text{Prob}\{X<5\}$

$f(x) = \frac{1}{\mu_1} e^{-\frac{1}{\mu_1}x}, \quad x>0$

$\text{Prob}\{X<5\} = \int_0^5 \frac{1}{3.2} e^{-\frac{x}{3.2}} dx$

$= \frac{1}{3.2}\left[\frac{e^{-\frac{x}{3.2}}}{-\frac{1}{3.2}}\right]_0^5$

$= -\frac{3.2}{3.2}\left[e^{-5/3.2} - 1\right]$

$\text{Prob}\{X<5\} = 1 - e^{-5/3.2}$

$P\{X>5, Y=5\} = \left(1-\left(1-e^{-5/3.2}\right)\right) \times \left(1-\left(1-e^{-5/3.2}\right)\right)$

$= e^{-5/3.2} = 0.2 \times 0.2 = \underline{\underline{0.04}}$

for $P\{Y>5\}$

b) Prob { No order received for 10min }

$\text{Prob}\{X>10\} \cdot \text{Prob}\{Y>10\}$

$= \left(e^{-10/3.2}\right)^2 = 0.0019$

c) Prob { both systems receive $\underline{1}$ order each b/w 10 & 15 min after business is open }

$\text{Prob}\{10<X<15, 10<Y<15\} = P\{10<X<15\} \cdot P\{10<Y<15\}$

$$\text{Prob}\{10 < X < 15\} = F(15) - F(10)$$

$$= \frac{1}{3.2}\int_{10}^{15} e^{-x/3.2}\,dx$$

$$= e^{-10/3.2} - e^{-15/3.2} = 0.0347$$

$$\text{Prob}\{10 < X < 15,\ 10 < Y < 15\} = 0.0347 \times 0.0347$$

$$= 0.001204$$

→ let $\bar{X} = (X_1, X_2, X_3, \ldots X_n)$

n – Discrete R.V $\quad \mathbb{R}_n \to \mathbb{R}_n$

↓ for x ↓ for y

Continuous R.V $f(x_1, x_2, x_3 \ldots x_n)$

a)
$$y_1 = g_1(x_1, x_2, \ldots x_n)$$
$$y_2 = g_2(x_1, x_2, \ldots x_n)$$
$$\vdots$$
$$y_n = g_n(x_1, x_2, \ldots x_n)$$

$$x_1 = h_1(y_1, y_2, \ldots y_n)$$
$$x_2 = h_2(y_1, y_2, \ldots y_n)$$
$$\vdots$$
$$x_n = h_n(y_1, y_2, \ldots y_n)$$

b) g_i and h_i are all continuous functions

c) Partial derivatives $\frac{\partial x_i}{\partial y_j}$, $1 \le i, j \le n$

d) Jacobian

$$J = \frac{\partial(x_1, x_2, \ldots x_n)}{\partial(y_1, y_2, \ldots y_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

$J \neq 0$ $\quad$ |det|

absolute

$$\boxed{w(y_1, y_2, \ldots y_n) = |J|\, f_{\bar{X}}(h_1, h_2, \ldots h_n)}$$

$$w(y_1, y_2, \dots y_n) = |J| \, f(h_1, h_2, \dots h_n)$$

$$n = 1, \; y = g(x) \Rightarrow x = g^{-1}(y) = h$$

$$J = \frac{\partial x}{\partial y} = \frac{\partial g^{-1}(y)}{\partial y}$$

$$w(y) = \left| \frac{\partial g^{-1}(y)}{\partial y} \right| f(g^{-1}(y))$$

Q) let $X_1, X_2, X_3 \dots$ are Distribu R.V (iiD)

identitical independent distributed R.V

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$Y_1 = X_1 + X_2 + X_3$$

$$Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}$$

$$Y_3 = \frac{X_1}{X_1 + X_2}$$

Find p.d.f of $w(y_1, y_2, y_3) = ?$

(i) Find h_1, h_2, h_3 ?

$$x_1 + x_2 + x_3 = y_1 \quad \text{— (1)}$$

$$\frac{x_1 + x_2}{x_1 + x_2 + x_3} = y_2 \quad \text{— (2)} \qquad \frac{x_1 + x_2}{y_1} = y_2 \quad \boxed{x_1 + x_2 = y_1 y_2}$$

$$\frac{x_1}{x_1 + x_2} = y_3 \quad \text{— (3)} \qquad x_1 = (x_1 + x_2) y_3$$

$$\boxed{x_1 = y_1 y_2 y_3} = h_1$$

$$x_1 + x_2 = y_1 y_2$$

$$x_2 = y_1 y_2 - y_1 y_2 y_3$$

$$\boxed{x_2 = y_1 y_2 (1 - y_3)} = h_2$$

$$x_3 = y_1 - y_1 \cancel{y_2 y_3} - y_1 y_2 + y_1 \cancel{y_2 y_3}$$

(ii) Find Jacobion

$$\boxed{x_3 = y_1 (1 - y_2)} = h_3$$

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix}$$

$$J = \begin{vmatrix} y_2 y_3 & y_1 y_2 & y_1 y_2 \\ y_2(1-y_3) & y_1(1-y_3) & -y_1 y_2 \\ 1-y_2 & -y_1 & 0 \end{vmatrix}$$

$$J = y_2 y_3 \{-y_1^2 y_2] - y_1 y_2 [y_1 y_2 (1-y_2)] + y_1 y_2 \{-y_1 y_2 (1-y_3) - y_1(1-y_2)(1-y_3)]$$

$$J = -y_1^2 y_2^2 y_3 - (y_1 y_2)^2 (1-y_2) - (y_1 y_2)^2 (1-y_3) - y_1^2 y_2 (1-y_2) + y_1 y_2 (1-y_3)$$

$$J = \cancel{-y_1^2 y_2^2 y_3} - \cancel{y_1^2 y_2^2} + y_1^2 y_2^3 - \cancel{y_1^2 y_2^2} + \cancel{y_1^2 y_2^2 y_3} - y_1^2 y_2 + \cancel{y_1^2 y_2^2} + y_1 \cancel{y_2 y_3} y_1 \cancel{y_2} y_3$$

$$\boxed{J = -y_1^2 y_2}$$

(iii) Find ranges:-

$y_1 = x_1 + x_2 + x_3$, $y_1 > 0$

$y_2 = \dfrac{x_1 + x_2}{x_1 + x_2 + x_3}$, $0 < y_2 < 1$

$y_3 = \dfrac{x_1}{x_1 + x_2}$, $0 < y_3 < 1$

Independent R.V

$f(x_1, x_2, x_3) =$ product

$e^{-x_1 - x_2 - x_3}$, $x_i > 0$
$i = 1, 2, 3$

$$w(y_1 y_2 y_3) = |J| f(h_1, h_2, h_3)$$

$$\boxed{w(y_1, y_2 y_3) = y_1^2 y_2 \exp\{-y_1 y_2 y_3 - y_1 y_2 (1-y_3) - y_1(1-y_2)\}}$$

$$w(y_1, y_2, y_3) = y_1^2 y_2 \exp\{-y_1\}$$

$$w(y_1, y_2, y_3) = y_1^2 y_2 e^{-y_1}$$

$, y_1 > 0$
$, 0 < y_2 < 1$
$, 0 < y_3 < 1$

Q) x_1, x_2 are two R.Vs

1) $Y_1 = X_1 + X_2$

2) $Y_2 = X_1 - X_2$

$$x_1 + x_2 = y_1 \quad \Rightarrow x_1 = \frac{y_1 + y_2}{2} = h_1$$
$$x_1 - x_2 = y_2$$
$$\Rightarrow x_2 = \frac{y_1 - y_2}{2} = h_2$$

$$J = \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix} = \frac{-1}{4} - \frac{1}{4} = \frac{-1}{2}$$

$$w(y_1, y_2) = |J| \, f(h_1 h_2)$$

$$\boxed{w(y_1, y_2) = \frac{1}{2} f\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right)}$$

$$w(y_1) = \int_{-\infty}^{\infty} w(y_1, y_2)\, dy_2$$

$$w(y_1) = \int_{-\infty}^{\infty} \frac{1}{2} f\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right) dy_2$$

$$\frac{y_1 + y_2}{2} = u \Rightarrow y_2 = 2u - y_1$$

$$dy_2 = 2du \qquad \frac{y_1 - y_2}{2} = \frac{y_1 - 2u + y_1}{2}$$

$$\frac{y_1 - y_2}{2} = y_1 - u$$

$$\boxed{w(y_1) = \int_{-\infty}^{\infty} f(u, y_1 - u)\, du}$$

$y - u = t$

$u = y - t$

let x_1, x_2 are independent

$$\boxed{w(y) = \int_{-\infty}^{\infty} \underbrace{f(u) f(y-u)\, du}_{\text{convolution integral}}}$$

$$w(y_2) = \int_{-\infty}^{\infty} \omega(y_1, y_2)\, dy_1$$

$$\frac{y_1 - y_2}{2} = v, \quad dy_1 = 2dv$$

$$\frac{y_1 + y_2}{2} \Rightarrow \frac{2v + y_2 + y_2}{2} = v + y_2$$

$$\boxed{w(y_2) = \int_{-\infty}^{\infty} f(v + y_2, v)\, dv}$$

let x_1, x_2 are independent then

$$\boxed{\underbrace{w(y) = \int_{-\infty}^{\infty} f(v + y_2) \cdot f(v)\, dv}_{\text{convolution integral.}}}$$

Q) $Y_1 = X_1 X_2$

$Y_2 = \frac{X_1}{X_2}$

$X_1 X_2 = Y_1 \Rightarrow \cancel{X_1 = \frac{Y_1}{}} \quad X_1 \left[\frac{X_1}{Y_2}\right] = Y_1 \Rightarrow \boxed{X_1 = \sqrt{Y_1 Y_2}} = h_1$

$\frac{X_1}{X_2} = Y_2 \quad X_2 = \frac{\sqrt{Y_1 Y_2}}{Y_2} \Rightarrow \boxed{X_2 = \sqrt{\frac{Y_1}{Y_2}}} = h_2$

$$w(y_1) = \int_{-\infty}^{\infty} \frac{1}{|\alpha|} f\left(\alpha, \frac{y_1}{\alpha}\right) d\alpha$$

$$w(y_2) = \int_{-\infty}^{\infty} f(u\beta, \beta)\, |\beta|\, d\beta.$$

Q) If $g_j(x_1, x_2, \cdots x_n) \to h_i(y_1, y_2, \cdots y_n)$ has multiple solutions then

$x_1 = h_{1i}(\cdots)$

$x_2 = h_{2i}(\cdots)$

$x_3 = h_{3i}(\cdots)$

$x_n = h_{ni}(\cdots)$

$i = 1, 2, 3, \cdots k$

$$J_i = \begin{vmatrix} \frac{\partial h_{1i}}{\partial y_1} & \frac{\partial h_{2i}}{\partial y_1} & \cdots \\ \vdots & & \\ \frac{\partial h_{ni}}{\partial y_n} & \cdots & \frac{\partial h_{ni}}{\partial y_n} \end{vmatrix}$$

$i = 1, 2, \cdots k$

$$w(y_1, y_2 \cdots y_n) = \sum_{i=1}^{k} |J_i| f(h_{1i}, h_{2i}, \cdots h_{ni})$$

Bivariate Normal Distribution

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} *$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

x_1, x_2 are jointly gaussian

$\rho > 0$, $|\rho| < 1$ $\rho \rightarrow$ correlation coefficient

μ_1, μ_2 are means

σ_1, σ_2 are standard deviations.

Q) $U_1 = \sqrt{X_1^2 + X_2^2}$, $U_2 = \frac{X_1}{X_2}$ $Z = X + iY$

$U_1 \rightarrow$ Magnitude of transmitted signal.

$\tan^{-1}\left(\frac{X_2}{X_1}\right) \rightarrow$ phase

$u_1 > 0$, $\sqrt{x_1^2 + x_2^2} = u_1$

$\frac{x_1}{x_2} = u_2 \Rightarrow x_1^2 = x_2^2 u_2^2$

$\boxed{i=1}$

$\sqrt{x_2^2(1+u_2^2)} = u_1$

$$\boxed{x_2 = \pm \frac{u_1}{\sqrt{1+u_2^2}}}$$

$x_{11} = \frac{u_1 u_2}{\sqrt{1+u_2^2}}$

$x_{21} = \frac{u_2}{\sqrt{1+u_2^2}}$

$$\boxed{x_1 = \pm \frac{u_2 u_1}{\sqrt{1+u_2^2}}}$$

$\boxed{i=2}$

$x_{12} = -x_{11}$

$x_{22} = -x_{21}$

$$J_1 = \begin{vmatrix} \dfrac{u_2}{\sqrt{1+u_2^2}} & \dfrac{u_1}{\sqrt{1+u_2^2}} - \dfrac{1}{2}\left(\dfrac{-2u_2}{(1+u_2^2)^{3/2}}(u_1u_2)\right) \\ \dfrac{1}{\sqrt{1+u_2^2}} & u_1\left(\dfrac{-1}{2}\right)\dfrac{2u_2}{(1+u_2^2)^{3/2}} \end{vmatrix}$$

$$J_1 = \begin{vmatrix} \dfrac{u_2}{\sqrt{1+u_2^2}} & \dfrac{u_1}{\sqrt{1+u_2^2}}\left[1 - \dfrac{u_2^2}{1+u_2^2}\right] \\ \dfrac{1}{\sqrt{1+u_2^2}} & \dfrac{-u_1u_2}{(1+u_2^2)^{3/2}} \end{vmatrix}$$

$$J_1 = \frac{-u_1u_2^2}{\sqrt{1+u_2^2}} \cdot \frac{1}{(1+u_2^2)^{3/2}} - \left(\frac{u_1}{1+u_2^2}\right)\left[1 - \frac{u_2^2}{1+u_2^2}\right]$$

$$J_1 = \cancel{\frac{-u_1u_2^2}{(1+u_2^2)^2}} - \frac{u_1}{1+u_2^2} + \cancel{\frac{u_2^2u_1}{(1+u_2^2)^2}}$$

$$\boxed{J_1 = \frac{-u_1}{1+u_2^2}}$$

$$J_2 = J_1 = \frac{-u_1}{1+u_2^2}$$

$$w(u_1,u_2) = \begin{cases} \dfrac{u_1}{1+u_2^2}\left[f\left(\dfrac{u_1u_2}{\sqrt{1+u_2^2}}, \dfrac{u_1}{\sqrt{1+u_2^2}}\right) + f\left(\dfrac{-u_1u_2}{\sqrt{1+u_2^2}}, \dfrac{-u_1}{\sqrt{1+u_2^2}}\right)\right] & u_1 > 0,\ u_2 \in R \\ 0, & \text{otherwise} \end{cases}$$

Q) Joint pdf of U_1 & U_2

$$U_1 = \sqrt{x^2+y^2}, \quad U_2 = \frac{x}{y}$$

$$w(u_1,u_2) = \begin{cases} \dfrac{u_1}{1+u_2^2}\left[f\left(\dfrac{u_1u_2}{\sqrt{1+u_2^2}}, \dfrac{u_1}{\sqrt{1+u_2^2}}\right) + f\left(\dfrac{-u_1u_2}{\sqrt{1+u_2^2}}, \dfrac{-u_1}{\sqrt{1+u_2^2}}\right)\right], & u_1>0,\ u_2 \in R \\ 0, \text{ elsewhere} \end{cases}$$

Special case:- $\mu_1 = \mu_2 = 0$, $\rho = 0$, $\sigma_1 = \sigma_2 = \sigma$

$$f(x,y) = \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{1}{2} \frac{x^2+y^2}{\sigma^2} \right\}$$

$$f(x,y) = f(x) \cdot f(y)$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ \frac{-x^2}{2\sigma^2} \right\} = N(0,\sigma^2)$$

$$w(u_1,u_2) = \frac{u_1 \times 2}{1+u_2^2} \cdot \exp\left\{ \frac{-1}{2\sigma^2}\left(\frac{u_1^2u_2^2}{1+u_2^2} + \frac{u_1^2}{1+u_2^2} \right) \right\} \times \frac{1}{2\pi\sigma^2}$$

$$w(u_1,u_2) = \frac{u_1}{\sigma^2} \exp\left\{ \frac{-u_1^2}{2\sigma^2} \right\} \cdot \frac{1}{\pi(1+u_2^2)}, \quad u_1>0,\ u_2 \in R$$

$$w(u_1,u_2) = w(u_1) \cdot w(u_2)$$

$$w(u_1) = \frac{u_1}{\sigma^2} \exp\left\{ \frac{-u_1^2}{2\sigma^2} \right\}$$ Rayleigh distribution

$$w(u_2) = \frac{1}{\pi(1+u_2^2)}, \quad u_2 \in R$$

$\hookrightarrow$ Cauchy pdf

Multivariate Gaussian Distribution (vector form)

$\bar{x} = (x_1, x_2, \ldots, x_n)^T \quad (N \times 1)$

$\bar{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)^T \quad (N \times 1)$

$\Sigma = \mathrm{cov}(\bar{x}) \quad (N \times N)$

Covariance matrix

$\mathrm{cov}(\bar{x}) = E[(\bar{x} - \bar{\mu})(\bar{x} - \bar{\mu})^T]$

$$f_{\bar{x}}(\bar{x}) = \frac{1}{(2\pi)^{n/2}} \cdot \frac{1}{|\Sigma|^{1/2}} \cdot \exp\left\{ -\frac{1}{2} (\bar{x} - \bar{\mu})^T \Sigma^{-1} (\bar{x} - \bar{\mu}) \right\}$$

let $n = 2$

$\bar{\mu} = [\mu_1, \mu_2]^T$

$\bar{x} = [x_1, x_2]^T$

$$f_{\bar{x}}(\bar{x}) = \frac{1}{2\pi} \cdot \frac{1}{|\Sigma|^{1/2}} \cdot \exp\left\{ -\frac{1}{2} (x_1 - \mu_1, x_2 - \mu_2) \Sigma^{-1} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} \right\}$$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

Covariances

pdf is never a vector
It is a +ve scalar.

$|\Sigma| = \sigma_1^2 \sigma_2^2 (1 - \rho^2)$

$$\Sigma^{-1} = \frac{1}{\sigma_1^2 \sigma_2^2 (1 - \rho^2)} \begin{bmatrix} \sigma^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

$f_{\bar{x}}(\bar{x}) =$

$$\frac{1}{2\pi} \cdot \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{ [(x_1 - \mu_1), (x_2 - \mu_2)] \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \right\}$$

Exponent of

$$f_x(\bar{x}) = [\sigma_2^2 (x_1 - \mu_1) - \rho_1 \sigma_1 \sigma_2 (x_2 - \mu_2) \quad -\rho\sigma_1\sigma_2(x_1 - \mu_1) + \sigma_1^2 (x_2 - \mu_2)] \begin{bmatrix} x_1 - \mu_1 \\ x_1 - \mu_2 \end{bmatrix}$$

$$= (x_1 - \mu_1)^2 \sigma_2^2 - 2\rho\sigma_1\sigma_2(x_1 - \mu_1)(x_2 - \mu_2) + \sigma_1^2 (x_2 - \mu_2)^2$$

$$E \quad \frac{(x_1 - \mu_1)^2 \sigma_2^2 - 2\rho\sigma_1\sigma_2(x_1 - \mu_1)(x_2 - \mu_2) + \sigma_1^2 (x_2 - \mu_2)^2}{\sigma_1^2 \sigma_2^2 (1 - \rho^2)}$$

$$N.C = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$
$\downarrow$
normalising constant

$$f(x_1, x_2) = \frac{N.C}{\cancel{\quad}} \exp\left\{\frac{-1}{2}\cdot A\right\}$$

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

$$\boxed{\rho = \frac{cov(x,y)}{\sigma_x \sigma_y}}$$

$$cov(x,y) = E[(x-E(x))(y-E(y))]$$

$E(XY) \rightarrow$ correlation

$\underline{-1<\rho<1}$

Find $f_1(x) = ?$

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

$$f_1(x) = \int f(x,y)\,dy$$

$$f_1(x) = \int f(x)\, f(y/x)\,dy \quad \text{[using Baeyer's theorem]}$$

$$Q(x,y) = \frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]$$

$$(1-\rho^2)\,Q(x,y) = (1-\rho^2+\rho^2)\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}$$

$$(1-\rho^2)\,Q(x,y) = (1-\rho^2)\frac{(x-\mu_1)^2}{\sigma_1^2} + \left(\frac{\rho(x-\mu_1)}{\sigma_1}\right)^2 - \frac{2\rho(x_1-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2$$

$$(1-\rho^2)\,Q(x,y) = (1-\rho^2)\frac{(x-\mu_1)^2}{\sigma_1^2} + \left(\frac{y-\mu_2}{\sigma_2} - \frac{\rho(x-\mu_1)}{\sigma_1}\right)^2$$

$$(1-\rho^2)\,Q(x,y) = \frac{(1-\rho^2)(x-\mu_1)^2}{\sigma_1^2} + \left(\frac{y-\beta_x}{\sigma_2}\right)^2$$

$$\beta_x = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$$

$$f_1(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1}\exp\left\{\frac{-(x-\mu_1)^2}{2\sigma_1^2}\right\} \times \underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\,\sigma_2(1-\rho^2)^{1/2}}\exp\left\{\frac{-(y-\beta_x)^2}{2\sigma_2^2(1-\rho^2)}\right\}dy}_{1}$$

$$\boxed{f_1(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1}\exp\left\{\frac{-(x-\mu_1)^2}{2\sigma_1^2}\right\}}$$

$$f_1(x) = \int f(x,y)\,dy$$

$$= f(x)\int f(y/x)\,dy$$

$$f_1(x) = f(x)\,1.$$

$$y/x \sim N\left(\beta_x,\ \sigma_2^2(1-\rho^2)\right)$$

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$f(y/x) = \frac{1}{\sqrt{2\pi}\,\sigma_1}\exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right)$$

$$f(y/x) = \frac{1}{\sqrt{2\pi}\,\sigma_2(1-\rho^2)^{1/2}}\exp\left(-\frac{(y-\beta_x)^2}{2\sigma_2^2(1-\rho^2)}\right)$$

$$N\left(\beta_x,\ \sigma_2^2(1-\rho^2)\right)$$

$$E\left[Y/X=x\right] = \beta_x, \quad \mathrm{Var}\left[Y/X=x\right] = (1-\rho^2)\,\sigma_2^2$$

$$(X_1, X_2) \sim N^{(2)}(\mu_1 \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

$\rho \downarrow$ correlation coefficient

$\rho = 0 \rightarrow X_1$ & X_2 are independent

$$Z = X_1 + X_2$$

$$f(z) = \int f(x, z-x)\,dx$$

$$f(z) = \int f_1(x) \cdot f_2(z-x)\,dx$$

$$f(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left\{\frac{-1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}\right\} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left\{\frac{-1}{2}\frac{(z-x-\mu_2)^2}{\sigma_2^2}\right\} dx$$

$$Z \sim N(\underbrace{\mu_1 + \mu_2}_{\text{for all}}, \underbrace{\sigma_1^2 + \sigma_2^2}_{\text{only for independent variables}})$$

$$f(z) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left[\frac{-\mu_2^2}{2\sigma_2^2}\right] \exp\left[-\frac{z^2}{2\sigma_2^2}\right] \exp\left[\frac{z\mu_2}{\sigma_2^2}\right]$$

$$\int \exp\left[-\frac{x^2 + \mu_1^2 - 2x\mu_1}{2\sigma_1^2} - \frac{x^2 - 2xz + 2x\mu_2}{2\sigma_2^2}\right] dx$$

$$\exp\left[-\frac{(x-\mu_{x/z})^2}{2\sigma_{x/z}^2}\right]$$

$$\frac{1}{\sigma_{x/z}^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \Rightarrow \sigma_{x/z}^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad \text{— (1)}$$

$$\frac{\mu_{x/z}}{\sigma_{x/z}^2} = \frac{\mu_1}{\sigma_1^2} + \frac{z-\mu_2}{\sigma_2^2}$$

$$\mu_{x/z} = \sigma_{x/z}^2\left[\frac{\mu_1}{\sigma_1^2} + \frac{z-\mu_2}{\sigma_2^2}\right] \quad \text{— (2)}$$

$$\mu_{x/z} = \frac{\sigma_2^2\mu_1}{\sigma_1^2 + \sigma_2^2} + \frac{\sigma_2^2(z-\mu_2)}{\sigma_1^2 + \sigma_2^2}$$

$$\mu_{x/z} = A(z)\int \frac{1}{\sqrt{2\pi\sigma^2_{x/z}}} \exp\left\{\frac{-(x-\mu_{x/z})^2}{2\sigma^2_{x/z}}\right\}$$

$$A_1(z) = A(z)\exp\left\{\frac{-\mu_1^2}{2\sigma_1^2}\right\} \quad \text{--- (3)}$$

$$f_2(z) = \frac{1}{2\pi\sigma_1\sigma_2}\exp\left\{\frac{-(z-\mu_1)^2}{2\sigma_2^2}\right\}\exp\left\{\frac{-\mu_1^2}{2\sigma_1^2}\right\}\cdot$$

$$\sqrt{2\pi\sigma^2_{x/z}}\ \exp\left\{\frac{\mu^2_{x/z}}{2\sigma^2_{x/z}}\right\}$$

$$f_2(z) = \frac{1}{\sqrt{2\pi\sigma_1^2\sigma_2^2}}\cdot\sqrt{\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}\exp\left\{\frac{-(z-\mu_2)^2}{2\sigma_2^2}\right\}e^{-\frac{\mu_1^2}{2\sigma_1^2}}\cdot$$

$$\exp\left\{\sigma^2_{x/z}\left(\frac{\mu_1}{\sigma_1^2}+\frac{z-\mu_2}{\sigma_2^2}\right)^2\cdot\frac{1}{2\sigma^2_{x/z}}\right\}$$

$$f_2(z) = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\exp\left\{\frac{-(z-\mu_2)^2}{2\sigma_2^2}\right\}\exp\left\{\frac{-\mu_1^2}{2\sigma_1^2}\right\}\cdot$$

$$\exp\left\{\frac{\sigma_1^2\sigma_2^2}{2(\sigma_1^2+\sigma_2^2)}\left[\frac{\mu_1^2}{\sigma_1^4}+\frac{(z-\mu_2)^2}{\sigma_2^4}+\frac{2(z-\mu_2)\mu_1}{\sigma_1^2\sigma_2^2}\right]\right\}$$

compare with $\frac{1}{\sqrt{2\pi\sigma_2^2}}\exp\left\{\frac{-(z-\mu_2)^2}{2\sigma_2^2}\right\}$

$$\frac{1}{\sigma_z^2} = \frac{1}{\sigma_2^2} - \frac{1}{\sigma_2^2}\left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right) = \frac{1}{\sigma_1^2+\sigma_2^2}$$

$$\boxed{\sigma_z^2 = \sigma_1^2+\sigma_2^2}$$

$$\frac{\mu_z}{\sigma_z^2} = \frac{\mu_2}{\sigma_2^2} - \frac{\mu_2}{\sigma_1^2+\sigma_2^2} + \frac{\mu_1}{\sigma_1^2+\sigma_2^2}$$

$$\mu_z = \mu_1+\mu_2$$

$$X, Y \sim N^{(2)}(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, 0)$$

$$z = X+Y$$

$$f(z) = N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

$$= \frac{\mu_2^2}{\sigma_2^2} - \frac{\mu_1^2 \sigma_2^2}{\sigma_1^2(\sigma_1^2+\sigma_2^2)} - \frac{\mu_2^2 \sigma_1^2}{\sigma_2^2(\sigma_1^2+\sigma_2^2)} + \frac{2\mu_1\mu_2}{\sigma_1^2\sigma_2^2} \cdot \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} + \frac{\mu_1^2}{\sigma_1^2}$$

$$= \frac{\mu_2^2}{\sigma_2^2}\left(1 - \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right) + \frac{\mu_1^2}{\sigma_1^2}\left(1 - \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right) + \frac{2\mu_1\mu_2}{\sigma_1^2+\sigma_2^2}$$

$$= \frac{1}{\sigma_1^2+\sigma_2^2}(\mu_1+\mu_2)^2$$

$$\boxed{= \frac{\mu_Z^2}{\sigma_Z^2}}$$

Covariance & Correlation

let X, Y be identically distributed

$$P_k = P\{X = k\} = \frac{1}{N}, \quad k = 1, 2, \cdots N$$

$E[X] = ?$

$$= \sum_k k\, P_k = \sum_{k=1}^{N} k \cdot \frac{1}{N}$$

$$= \frac{1}{N} \times \frac{N(N+1)}{2} = \frac{N+1}{2}$$

$$E[X^2] = \sum_k k^2 \cdot P_k = \sum_{k=1}^{N} k^2 \cdot \frac{1}{N} = \frac{1}{N} \frac{N(N+1)(2N+1)}{6}$$

$$= \frac{(N+1)(2N+1)}{6}$$

$E[XY] = ?$

$$\sum_k \sum_m k \cdot m \cdot P_{k,m} \rightarrow \text{Joint PMF Not available}$$

(i) If X and Y are independent

$$E[XY] = E[X] \cdot E[Y] = \left(\frac{N+1}{2}\right)^2$$

(ii) $Y = aX + b$

$$E[X(aX+b)] = aE[X^2] + b[E[X]]$$

$$= \frac{a(N+1)(2N+1)}{6} + b\left(\frac{N+1}{2}\right)$$

$$= \frac{N+1}{2}\left[\frac{a(2N+1)}{3} + b\right]$$

$\mathrm{Cov}[X, Y] = ?$

(i) Independent case: $\mathrm{Cov}[X, Y] = 0$

(ii) $Y = aX + b$

$$\mathrm{Cov}[X, Y] = E[XY] - E[X]E[Y]$$

$$= \frac{N+1}{2}\left[\frac{a(2N+1)}{3} + b\right] - \left(\frac{N+1}{2}\right)\left(a\frac{N+1}{2} + b\right)$$

$$= a\left(\frac{N^2-1}{12}\right)$$

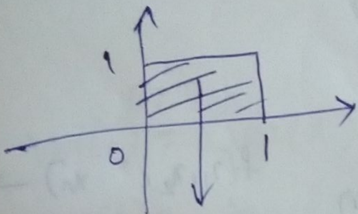
(i) $\rho = 0$ let X, Y be identically distributed.

(ii) $\rho = 1, \; a > 0$

$\quad\;\; -1, \; a < 0$

Q) let X, Y be jointly distributed

$$f(x,y) = \begin{cases} x+y, & 0<x<1, \; 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$



Find $M_{01}, M_{10}, M_{20}, M_{02}, \mu_{20}, \mu_{02}, M_{11}, \mu_{11}, \rho$?

$$E[X^l Y^m] = \int_0^1 \int_0^1 x^l y^m (x+y)\, dx\, dy$$

Joint moment of order $l+m$

$$= \int_0^1 \frac{x^{l+2}}{l+2}\Big|_0^1 y^m \, dy$$

$$= \int_0^1 x^{l+1} dx \int_0^1 y^m dy + \int_0^1 x^l dx \int_0^1 y^{m+1} dy$$

$$= \left[\frac{x^{l+2}}{l+2}\right]_0^1 \left[\frac{y^{m+1}}{m+1}\right]_0^1 + \left[\frac{x^{l+1}}{l+1}\right]_0^1 \left[\frac{y^{m+2}}{m+2}\right]_0^1$$

$$E[X^l Y^m] = \frac{1}{l+2} \cdot \frac{1}{m+1} + \frac{1}{l+1} \cdot \frac{1}{m+2}$$

$E[Y] = M_{01}, \; l=0, m=1$

$$\frac{1}{2} \times \frac{1}{2} + \frac{1}{1} \cdot \frac{1}{3} = \frac{7}{12}$$

$E[X] = M_{10} = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$

$M_{20} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$

$M_{02} = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$

$\mu_{02} = M_{02} - (M_{01})^2$

$\mu_{02} = \frac{5}{12} - \left(\frac{7}{12}\right)^2$

$\mu_{02} = \frac{5}{12} - \frac{49}{144} = \frac{60-49}{144}$

$\mu_{02} = \frac{11}{144} = 0.076$

$\mu_{20} = \mu_{02}$

$M_{11} = E[XY] = \frac{1}{3} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{3}$ //

$\mu_{11} = M_{11} - (M_{01} M_{10}) = \frac{1}{3} - \left(\frac{7}{12}\right)^2$

$\mu_{11} = \frac{1}{3} - \frac{49}{144} = \frac{-1}{144}$ //

$\rho = \frac{\mu_{11}}{\sqrt{\mu_{02}}\sqrt{\mu_{20}}} = \frac{-1}{11}$ /

Theorem:-

let $X_1, X_2, X_3, \dots X_n$ be n RV's

$E[X_i] < \infty$, $i = 1, 2, \dots n$

$f(x_1, x_2, \dots x_n) \rightarrow$ Joint pdf.

let $a_1, a_2, \dots a_n \in R$

(i) $S = \sum_{i=1}^{n} a_i X_i$ then $E[S] = \sum_{i=1}^{n} a_i E[X_i]$

Proof:- $E[S] = \int \dots \int \sum_{i=1}^{n} a_i x_i f(x_1, x_2, \dots x_n)\, dx_1 dx_2 \dots dx_n$

$$E[S] = \int a_1 x_1 \Big[\underbrace{\int\int \dots \int f(x_1, x_2, \dots x_n)\, dx_2 dx_3 \dots dx_n}_{\text{marginal}} \Big] dx_1 +$$

$$\int a_2 x_2 \Big[\underbrace{\int\int \dots \int f(x_1, x_2, \dots x_n)\, dx_1 dx_3 \dots dx_n}_{\text{marginal}} \Big] dx_2 +$$

$$\dots + $$

$$\int a_n x_n \Big[\underbrace{\int\int \dots \int f(x_1, x_2, \dots x_n)\, dx_1 dx_2 \dots dx_{n-1}}_{\text{marginal}} \Big] dx_n$$

$$E[S] = \int a_1 x_1 f(x_1)\, dx_1 + \int a_2 x_2 f(x_2)\, dx_2 + \dots + \int a_n x_n f(x_n)\, dx_n$$

$$E[S] = a_1 E[x_1] + a_2 E[x_2] + \dots + a_n E[x_n]$$

$$\boxed{E[S] = \sum_{i=1}^{n} a_i E[X_i]}$$

(ii) $var[S] = ?$ $\quad S = ax + by$

$$\underline{var[S^{(2)}] = a^2 var(x) + b^2 var(y) + 2ab\, Cov(x, y)}$$

$$var(S) \overset{(2)}{=} var(ax + by)$$

$$= E[(ax+by)^2] - (E[ax+by])^2$$

$$var(S) \overset{(2)}{=} a^2 E[x^2] + b^2 E[y^2] + 2ab E[xy]$$
$$- a^2(E(x))^2 - b^2(E(y))^2 - 2ab E(x)E(y)$$

$$var(S) \overset{(2)}{=} a^2 var(x) + b^2 var(y) + 2ab[E[xy] - E[x]E[y]]$$

$$var(S) \overset{(2)}{=} a^2 var(x) + b^2 var(y) + 2ab[Cov(x, y)]$$

$$\boxed{var[S] = \sum_{i=1}^{n} a_i^2 var(x_i) + \sum_i \sum_j a_i a_j\, cov(x_i, x_j) \quad i \neq j}$$

$$var[S_n] = var\left[\sum_{i=1}^{n} a_i x_i\right]$$

$$var[S_n] = E\left[\left(\sum_{i=1}^{n} a_i x_i\right)^2\right] - \left[E\left(\sum_{i=1}^{n} a_i x_i\right)\right]^2$$

$$var[S_n] = E\left\{\left[\sum_{i=1}^{n} a_i (x_i - E(x_i))\right]^2\right\}$$

let $y_i = [x_i - E(x_i)]\, a_i$

$$\left(\sum_{i=1}^{n} y_i\right)^2 = \sum_i y_i \sum_j y_j$$

$$\left(\sum_{i=1}^{n} y_i\right)^2 = \sum_{i,\ i=j} y_i^2 + \sum_i \sum_{j,\ i \neq j} y_i y_j$$

$$\begin{bmatrix} y_1^2 & y_2 y_1 & y_3 y_1 & \cdots & y_n y_1 \\ y_1 y_2 & y_2^2 & & & \vdots \\ y_1 y_3 & & y_3^2 & & \vdots \\ \vdots & & & \ddots & \\ y_1 y_n & \cdots & \cdots & \cdots & y_n^2 \end{bmatrix}$$

$$\text{var}[S_n] = E\left\{ \sum_{i=1}^{n} a_i^2 (X_i - E(X_i))^2 + \sum_i \sum_j a_i a_j (X_i - E(X_i))(X_j - E(X_j)) \right\}$$

$$\text{var}[S_n] = \sum_{i=1}^{n} E[X_i - E(X_i)]^2 + \sum_i \sum_{\substack{j \\ i \neq j}} a_i a_j \underbrace{E[(X_i - E(X_i))(X_j - E(X_j))]}_{\Downarrow \; \text{cov}(X_i, X_j)}$$

In case of uncorrelated X_i and X_j

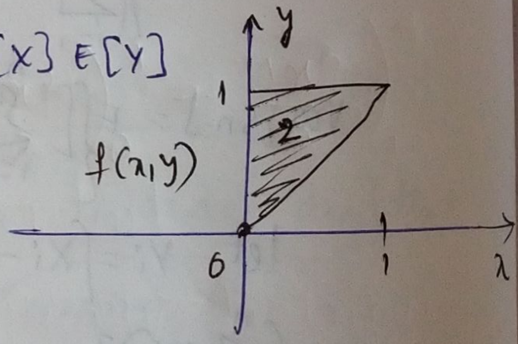
$\text{cov}(X_i, X_j) = 0$

$$\Rightarrow \text{var}[S_n] = \sum_{i=1}^{n} a_i^2 E[X_i - E[X_i]]^2$$

$$\boxed{\text{var}[S_n] = \sum_{i=1}^{n} a_i^2 \text{var}[X_i]}$$

Q) $f(x,y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$ Find $\text{cov}[X, Y] = ?$

$$\text{Cov}[X, Y] = E[XY] - E[X]E[Y]$$



$$E[X^j Y^k] = \iint x^j y^k f(x,y)\, dy\, dx$$

$$E[x^j y^k] = \int_0^1 \int_x^1 x^j y^k (2\, dy)\, dx$$

$$E[x^j y^k] = \int_0^1 2x^j \left. \frac{y^{k+1}}{k+1} \right|_x^1 dx$$

$$E[x^j y^k] = \int_0^1 2x^j \left(\frac{1}{k+1} - \frac{x^{k+1}}{k+1} \right) dx$$

$$E[x^j y^k] = \frac{2}{k+1} \int_0^1 x^j (1 - x^{k+1})\, dx$$

$$E[x^j y^k] = \frac{2}{k+1} \left[\frac{x^{j+1}}{j+1} - \frac{x^{j+k+2}}{j+k+2} \right]_0^1$$

$$E[x^j y^k] = \frac{2}{k+1}\left[\frac{1}{j+1} - \frac{1}{j+k+2}\right]$$

$$E[XY] = \frac{2}{2}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{1}{4} \Rightarrow M_{11} = \frac{1}{4}$$

$$E[X] = \frac{2}{1}\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3} \Rightarrow M_{10} = \frac{1}{3}$$

$j=1, k=0$

$$E[Y] = \frac{2}{2}\left[\frac{1}{1} - \frac{1}{3}\right] = \frac{2}{3} \Rightarrow M_{01} = \frac{2}{3}$$

$j=0, k=1$

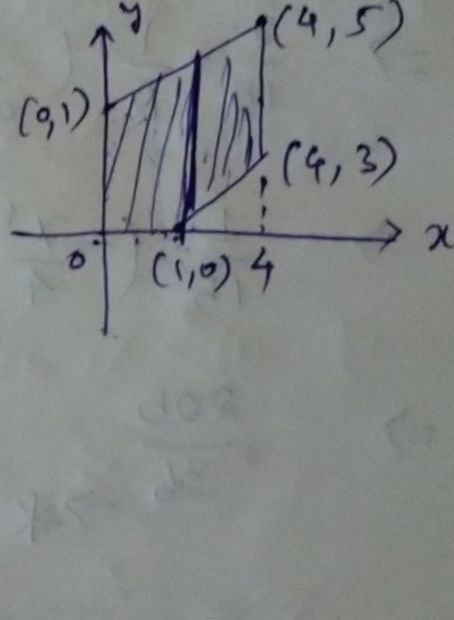
$$\mu_{11} = M_{11} - M_{01}M_{10} = \frac{1}{4} - \frac{1}{3} \times \frac{2}{3}$$

$$\mu_{11} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36} \quad //$$

$$\therefore \operatorname{Cov}(X,Y) = \frac{1}{36} \quad //$$

Q) $f(x,y) = \begin{cases} c, & 0<x<4, (y>0 \;\&\; x-1<y<x+1) \\ 0, & \text{otherwise} \end{cases}$

Find c, $\operatorname{Cov}(X,Y)$



$$\int_0^{1}\int_0^{x+1} f(x,y)\,dy\,dx + \int_1^4\int_{x-1}^{x+1} f(x,y)\,dy\,dx = 1$$

$$\int_0^1\int_0^{x+1} c\,dy\,dx + \int_1^4\int_{x-1}^{x+1} c\,dy\,dx = 1$$

$$\int_0^1 c\{x+1\}\,dx + \int_1^4 c[\cancel{x}+1-\cancel{x}+1]\,dx = 1$$

$$\int_0^1 (cx+c)\,dx + \int_1^4 2c\,dx = 1$$

$$\left[c\frac{x^2}{2} + cx\right]_0^1 + 2c[4-1] = 1 \qquad \boxed{c = \frac{2}{15}}$$

$$\frac{c}{2} + c + 6c = 1 \qquad 7c + \frac{c}{2} = 1 \Rightarrow \frac{15c}{2} = 1$$

$E[X,Y] = ?$ $Cov[X,Y] = ?$

$$M_{jk} = \frac{2}{15} \int\int x^j y^k \, dy\, dx$$

$$= \frac{2}{15}\left(\int_0^1 x^j \int_0^{x+1} y^k dy\, dx + \int_1^4 x^j \int_{x-1}^{x+1} y^k dy\, dx \right)$$

$$= \frac{2}{15}\left(\int_0^1 x^j \times \frac{y^{k+1}}{k+1}\Big)_0^{x+1} dx \right) + \int_1^4 x^j \times \frac{y^{k+1}}{k+1}\Big)_{x-1}^{x+1} dx \Big)$$

$$\Rightarrow \frac{2}{15}\left(\int_0^1 x^j \frac{(x+1)^{k+1}}{k+1} dx \right) + \int_1^4 \frac{x^j (x+1)^{k+1} - (x-1)^{k+1}}{k+1} dx$$

$$\Rightarrow \frac{2}{15} \times \frac{1}{2} \left(\int_0^1 x(x+1)^2 dx + \int_1^4 x(x+1)^2 - (x-1)^2 dx \right)$$

$$\Rightarrow \frac{1}{15}\left(\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} \Big)_0^1 + \frac{4x^3}{3}\Big)_1^4 \right)$$

$$\Rightarrow \frac{205}{36} \rightarrow *$$

$* \; Cov[x,y] = E[xy] - E[x]E[y]$

$E[x] = M_{10}$

$$= \frac{2}{15} \times 1 \left(\int_0^1 x(x+1) dx + \int_1^4 x((x+1)-(x-1)) dx \right)$$

$$= \frac{2}{15}\left(\frac{x^3}{3} + \frac{x^2}{2} \Big)_0^1 + x^2 \Big)_1^4 \right)$$

$\Rightarrow M_{10} = \frac{19}{9}$

$E[y] = \cancel{M_{14}}\ M_{01}$

$\Rightarrow \frac{2}{15}\cancel{\frac{1}{2}}\left(\int_0^1 (x^{\cancel{01}}+1)^2 dx + \int_1^4 4x\, dx\right)$

$\Rightarrow \frac{1}{15}\left(\left(\frac{x^3}{3} + x^2 + x\right)_0^1 + \left.\frac{4x^2}{2}\right)_1^4\right)$

$\Rightarrow \frac{1}{15}\left(\frac{7}{3} + \frac{\cancel{2}}{\cancel{2}}(15)\times 2\right)$

$\Rightarrow \frac{1}{15}\left(\frac{\cancel{97}\ 52}{3}\right) = \frac{\cancel{97}(52+45)}{15\times 3} = \frac{52+45}{45} = \frac{97}{45}$

$Cov[X,Y] = \frac{205}{36} - \frac{97}{45} \times \frac{19}{9}$

$= 1.1438$

9) X, Y Joint PDF

$f(x,y) = x^2 + \frac{xy}{2}, \quad 0<x<1,\ 0<y<2$

$E[x^j y^k] \quad 0, \quad$ otherwise $\bar{x}$

$\Rightarrow E[x]\ E[y]$

$M_{jk} = \frac{2^{k+1}}{(k+1)(3+j)} + \frac{2^{k+2}}{3(k+2)(j+2)}$

$M_{11} = \frac{43}{54} \quad M_{20} = \frac{17}{30} \quad M_{02} = \frac{4}{3}$

Sampling distribution

↳ central limit theorem

Population → sampling → Random Sample

Truth

$P(x;\theta)$

Random Sample ↓ statistical inference ↓ Decision

{P.S of soft drink containes} $\xrightarrow{\text{sample}}$ 25 Containers

Random sample

R.V's $\Rightarrow$ $X_1, X_2, \ldots X_n$

are a random sample of size 'n'

if a) X_i are independent

b) X_i are identically distributed

$$\bar{x} = \sum_{1}^{n} \frac{x_i}{n}$$

* sample proportion:-

$\Rightarrow$ $\hat{p} = \dfrac{\text{no. of individuals satisfy cond}^n}{\text{SAMPLE SIZE "N"}}$

Importance of sampling

Purpose :- obtain info about the properties of a population.

In practice 'p' is not easy to determine

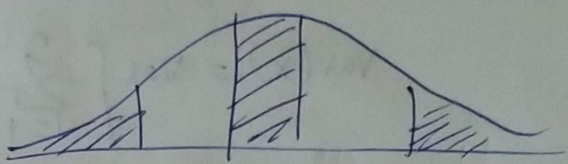
Reasonable solution :-

- Random sample of population
- compute sample proportions

$\hat{p} \rightarrow$ changes based on nature of Rample sample.
(random variables)

Statistic :- Any function of the observation in a random sample.

Eg :- $X = x_1, x_2, x_3, \ldots x_n$
a set of observations
sample size 'n'



$-\infty$ ∞

$x_r \sim N(\mu, \sigma^2)$

i) Sample mean $\bar{x} = \frac{1}{N} \sum_{i=1}^{n} x_i$ population mean $= \mu$

Sample variance $= s^2 = \frac{1}{N} \sum_{i=1}^{n} (x_i - \bar{x})^2$ σ^2

sample standard deviation $= s = \sqrt{s^2}$ σ

$\bar{x} \rightarrow f_{\bar{x}}(\bar{x})$

Sampling Distribution $\rightarrow$ The probability distribution of statistics.

$f(\bar{x}) \rightarrow$ sampling distribution of mean, properties of population

$\downarrow$

Sample site $\rightarrow$ sampling distribution

sample selection method $\nearrow$

Central limit theorem

Consider a sampling distribution of mean $X = x_1, x_2, \ldots x_n \rightarrow$ R.S

$x_i \sim N(\mu, \sigma^2)$, $i = 1$ to n

x's are all IID (identical & independently distributed)

$\bar{x} = \frac{1}{N} \sum_i x_i$

$\bar{x} \sim N(\mu, \sigma^2/n)$

Proof: Recursive application of $z_2 = x_1 + x_2$

$$f(z) = \int f_1(x) f_2(z-x)\,dx$$

$$z_3 = z_1 + x_3, \quad z_n = \sum_i x_i = z_{n-1} + x_n$$

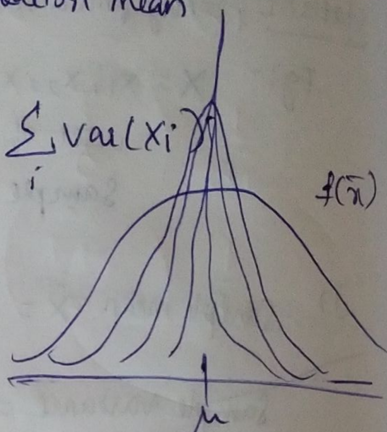
$z_n \sim N(\mu_n, \sigma_n^2)$

$\bar{x} \sim N(\mu_{\bar{x}}, \sigma_{\bar{x}}^2)$

$$E[\bar{x}] = E\left[\frac{1}{N}\sum_i x_i\right] = \frac{1}{N}\sum_i E[x_i] = \frac{1}{N}\sum_i \mu$$

$$\boxed{E[\bar{x}] = \mu}$$

Samplemean — population mean



$$Var(\bar{x}) = Var\left[\sum_{i=1}^{n} \frac{1}{n} x_i\right] = \frac{1}{n^2}\sum_i Var(x_i)$$

$$= \frac{1}{n^2}\sum_i \sigma^2 = \frac{\sigma^2}{n}$$

$$\boxed{\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}}$$

$\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ for $\bar{x} \sim N(\mu, \sigma)$

__Central limit theorem__

If $x_1, x_2, x_3, \ldots x_n$ is a random sample of size 'n' from a population with finite mean μ and variance σ^2 and if $\bar{x}$ is a sample mean. The limiting form of distribution of $\bar{x}$

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}, \quad E(z) = 0, \; Var(z) = 1$$

μ also follows normal distribution

Standardisation.

$\lim_{n\to\infty} z \sim N(0,1)$ standard normal distribution

$\bar{x}$ also follows normal distribution with $N\left(\mu, \frac{\sigma^2}{n}\right)$

__Exercises:__ An electronics company make resistors $\mu_R = 100\,\Omega$

$\sigma_R = 10\,\Omega$

$R \sim N(\mu_R, \sigma_R^2)$

8) Find probability $\left\{ \begin{array}{l} \text{that a random sample } n = 25 \\ \text{will have an avg. resistance less than } 95\Omega \end{array} \right\}$

sol:- $\text{Prob}\{\bar{x} < 95\} = ?$

$$\bar{x} \sim N\left(\mu_R, \frac{\sigma_R^2}{25}\right)$$

$$E(\bar{x}) = \mu_R = 100\Omega$$

$$\sigma_{\bar{x}}^2 = \frac{\sigma_R^2}{25} = \frac{(10)^2}{25} = 4$$

$$\sigma_{\bar{x}} = 2, \quad \boxed{\bar{x} \sim N(100, 4)}$$

standard deviation.

$$\text{Prob}\{\bar{x} < 95\} = F(95)$$

$$F(95) = \int_{-\infty}^{95} \frac{1}{\sqrt{2\pi(4)}} \exp\left\{-\frac{(\cancel{95}-100)^2}{2 \cdot 4}\right\} d\alpha$$

$$= \frac{1}{2\sqrt{2\pi}} \left[\frac{1}{2}\left(1 + \text{erf}\left(\frac{x-\mu}{\sqrt{2} \times \sigma}\right)\right)\right]_{-\infty}^{25}$$

$$=$$

$$\boxed{\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt}$$

$$= \frac{1}{4\sqrt{2\pi}} \left\{1 + \text{erf}\left(\frac{95-100}{\sqrt{2} \times 2}\right)\right\}$$

$$= \frac{1}{4\sqrt{2\pi}} \left\{1 + \text{erf}\left(\frac{-\cancel{05}}{2\sqrt{2}}\right)\right\}$$

$$= \frac{1}{4\sqrt{2\pi}} \text{erf}\left(\frac{-5}{2\sqrt{2}}\right) = 0.0062$$

$$\text{Prob}_{\bar{x}}\{\bar{x} < 95\} = \text{Prob}_z\{z < -2.5\}$$

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{95-100}{2/\sqrt{1\cancel{00}}} = \frac{-5}{2/\cancel{10}} = -2.5$$

General case

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right\} dx$$

$$F(x) = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-z^2}{2}\right\} dz$$

$$\frac{x-\mu}{\sigma} = z$$

$$\frac{dx}{\sigma} = dz$$

$$F(x) = F_{stan}\left(\frac{x-\mu}{\sigma}\right)$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\phi(x) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]$$

$$\frac{x-\mu}{\sqrt{2}\sigma} = z$$

CDF:-

$$\phi(x) = \int_{-\infty}^{\frac{x-\mu}{\sqrt{2}\sigma}} \frac{1}{\sqrt{\pi}} \exp(-z^2)\, dz$$

$$\delta = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$\phi(x) = \int_{-\infty}^{\delta} \frac{1}{\sqrt{\pi}} \exp\{z^2\}\, dz$$

If $\delta > 0$

$$\phi(x) = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi}} e^{-z^2} dz + \int_0^{\delta} \frac{1}{\sqrt{\pi}} e^{-z^2} dz$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz = 1$$

$$\phi(x) = \frac{1}{2} + \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_0^{\delta} e^{-z^2} dz\right]$$

So, as normal distribution with mean $= 1/2$

$$\boxed{\phi(x) = \frac{1}{2}(1 + \text{erf}(\delta))}$$

$$\int_{-\infty}^{0} \frac{1}{\sqrt{\pi}} e^{-z^2} dz = 1/2$$

as it is a even function

If $\delta < 0$

$$\phi(x) = \int_{-\infty}^{\delta} \frac{1}{\sqrt{\pi}} e^{-z^2} dz - \int_{\delta}^{0} \frac{1}{\sqrt{\pi}} e^{-z^2} dz$$

$$\phi(x) = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_{-\rho}^{0} e^{-z^2}(-du)$$

let $\rho = -\beta$

let $z = -u$

$$\phi(x) = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_{0}^{\rho} e^{-z^2} du$$

$$\phi(x) = \frac{1}{2}\left(1 - \operatorname{erf}(-\rho)\right) = \frac{1}{2}\left(1 + \operatorname{erf}(\rho)\right)$$

Q) Consider a random variable X

$$f(x) = \begin{cases} 1/2, & 4 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}$$

R.S with $n = 40$. Find the distribution of mean

$f(\bar{x}) \Longrightarrow$ sampling distribution

$$E(\bar{x}) = \frac{6+4}{2} = 5$$

$$\operatorname{var}(\bar{x}) = \frac{\sigma_x^2}{n} = \frac{(6-4)^2}{12} \times \frac{1}{40} = \frac{1}{120}$$

$$\bar{x} \sim N\left(5, \frac{1}{120}\right)$$

Q) $\mu = 8.2\text{min}$, $\sigma = 6\text{ min}$

$n = 36$ Probf

Find the following probabilities $\bar{x}$

(i) $P\{\bar{x} < 10\} = F_{\bar{x}}(10)$

$E(x) = 8.2\text{min}$, $\sigma_x^2 = \frac{(6)^2}{36} = 1$

$$\text{Prob}\{\bar{x} < 10\} = F_z\left(\frac{10-8.2}{\sqrt{2}\cdot 1}\right) = F_z\left(\frac{1.8}{\sqrt{2}}\right)$$

$$\text{prob}\{\bar{x} < 10\} = \phi\left(\frac{1.8}{\sqrt{2}}\right)$$

$$\text{Prob}\{\bar{x} < 10\} = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{1.8}{\sqrt{2}}\right)\right) = 0.964$$

$$\text{Prob}\{5 < X < 10\} = F_X(10) - F_X(5)$$

$$= F_z\left(\frac{1.8}{\sqrt{2}}\right) - F_z\left(\frac{5-8.2}{\sqrt{2}}\right) = F_z\left(\frac{1.8}{\sqrt{2}}\right) - F_z\left(\frac{-3.2}{\sqrt{2}}\right)$$

$= 0.964 - \phi\left(\frac{-3.2}{\sqrt{2}}\right)$

$= 0.964 - \left[\frac{1}{2}\left[1 + \text{erf}\left(\frac{-3.2}{\sqrt{2}}\right)\right]\right]$

$= 0.964 - 6.8713 \times 10^{-4}$

$= 0.96338$

$\text{Prob}\{x < 6\} = F_{\bar{x}}(6) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{-2.2}{\sqrt{2}}\right)\right)$

$= 0.0139$

Q) Psychology course $X \to$ no. of students enrolled

$X \sim \text{poisson}(\lambda)$

$E(X) = \text{var}(X) = \lambda \quad | \quad \lambda = 100$

Condition:-

(i) If $X > 120$ - split the class into 2 sections

$X < 120$ - only 1 section.

Prob{professor will teach 2 sections}

$P_x = \text{PMF} \to \frac{e^{-\lambda}\lambda^x}{x!}$

$P\{\bar{x} < 120\} = \sum_{k=0}^{120} P_x$

$P\{\bar{x} > 120\} = \sum_{k=120}^{\infty} P_x = e^{-100} \sum_{k=120}^{\infty} \frac{(100)^x}{x!}$

$Y \sim \text{Poisson}(1)$

$i = 1, 2, \ldots 100, n = 100$

$\text{mean}(X_i) = \sum_{i=1}^{100} Y_i = 100(\lambda) = 100(1) = 100$

$\text{var}(X_i) = \sum_{i=1}^{100} \text{var}(Y_i) = 100$

Sum $x_i \rightarrow$ apply CLT

$x_i \sim N(100, 100)$

prob $\{x_i > 120\} = 1 - P\{x < 120\} = 1 - \phi(120)$

$= 1 - \left(\frac{1}{2}\left(1 + \text{erf}\left(\frac{120-100}{10\sqrt{2}}\right)\right)\right)$

$= 1 - \frac{1}{2}\left(1 + \text{erf}(\sqrt{2})\right) = 0.022$

Consider a binomial distribution example:-

Consider a MCQ test $n = 100$ questions each question is awarded randomly $p = 1/4$, $X \rightarrow$ no. of correct questions

Q) $P\{X > 20\} = ? = \sum_{k=21}^{100} P_k = \sum_{k=21}^{100} \binom{100}{k} p^k (1-p)^{n-k}$

$= 1 - \sum_{k=0}^{20} \binom{100}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{100-k} = 0.85116$

$X \sim N(np, npq)$

$P\{X > 20\} = 1 - \phi(20) = 1 - \frac{1}{2}\left(1 + \text{erf}\left(\frac{20-25}{\sqrt{2}\sqrt{\frac{75}{4}}}\right)\right)$

$= 0.8758$

Law of large numbers

Weak law

Strong law

conditions

Average & statistics of random sample $\rightarrow$ Converge

Expected values properties of people time

$\bar{X} = \sum_i x_i$, $X = \{x_1, x_2, \ldots x_n\}$

$E(\bar{X})$ var(S)

Eg:- $N(\mu, \sigma^2)$

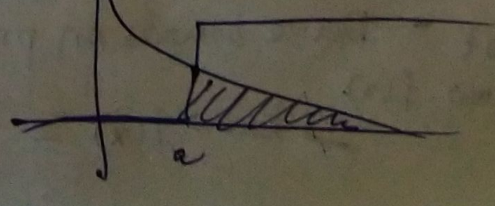
Markov's Inequality

$X \rightarrow$ R.V

$X \geq 0$

$a > 0$

$P\{X \geq a\} \leq \frac{E(X)}{a}$



Proof: For $a > 0$, $I = \begin{cases} 1 & , \text{if } X \geq a \\ 0 & , X \leq a \end{cases}$

Since $X \geq 0$, $I \leq \frac{X}{a}$

i) $X < a$, $I = 0$, $\frac{X}{a} \geq 0$

ii) $X \geq a$ $I = 1$, $\frac{X}{a} \geq 1$

$I \leq \frac{X}{a} \Rightarrow E(I) \leq \frac{E(x)}{a}$

$\int_0^\infty I f(x)\,dx \leq \frac{E(x)}{a}$

$\int_0^a 0\, f(x)\,dx + \int_a^\infty 1\, f(x)\,dx \leq \frac{E(x)}{a}$

$P\{X \geq a\} \leq \frac{E(x)}{a}$

Chebyshev's Inequality

If X is R.V, $E(X) = \mu$, $Var(X) = \sigma^2$

For $k > 0$, $P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$

Proof: From Markov's Inequality

$P\{Y \geq a\} < \frac{E(Y)}{a}$

$Y = (X-\mu)^2$ $\quad a = k^2$

$P\{(X-\mu)^2 \geq k^2\} \leq \frac{E((X-\mu)^2)}{k^2}$

$$\boxed{P\{|X-\mu| \geq k\} < \frac{\sigma^2}{k^2}}$$

Role of Markov's & Chebyshev's inequalities

Donot know $f(x)$ – Derive bounds on probabilities if first 2 ~~[illegible]~~ moments are known

– know $f(x)$ – If true pdf's are known bounds are not so tight

Q) $X \to$ no. of items produced in a week

$E(X) = \mu = 50$

a) prob{week's production > 75}

$P\{X > 75\} \le \frac{E(X)}{a} = \frac{50}{75} = \frac{2}{3}$

$\therefore \sigma^2 = 25$

P{weeks production $\in (40, 60)$}

a) $P\{40 < X < 60\} = P\{-10 < X - 50 < 10\}$

$P\{|X - 50| < 10\}$?

$P\{|X - \mu| \ge k\} \le \frac{\sigma^2}{k^2}$

$P\{|X - 50| \ge 10\} \le \frac{25}{100} = \frac{1}{4}$

$1 - P\{|X - \mu| \le 10\} \le \frac{1}{4}$

$P\{|X - 50| \le 10\} \ge 3/4$

Q) $X \sim U(0, 10)$

$E(X) = \frac{0+10}{2} = 5$, $Var(X) = \frac{(10-0)^2}{12} = \frac{25}{3}$.

$P\{|X-5| > 4\} \le \frac{\sigma^2}{k^2} = \frac{25}{3} / 16 = \frac{25}{48} = 0.52$

$P\{|X-5| > 4\} = 1 - P\{|X-5| < 4\}$

$= 1 - P\{-4 < X - 5 < 4\}$

$= 1 - P\{1 < X < 9\}$

$= 1 - 0.8 = 0.2$

weak law of large numbers

Random sample $X_1, X_2, \ldots X_n$

$E[X_i] = \mu < \infty$,

For $\epsilon > 0$, $\lim_{n\to\infty} P\{|\bar{X} - \mu| \geq \epsilon\} \to 0$

$$\boxed{\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i}$$

Random sample $X_1, X_2, \ldots X_n$ be IID

$E[X] = \mu < \infty$,

For any $\epsilon > 0$

$$\lim_{n\to\infty} P\{|\bar{X} - \mu| \geq \epsilon\} \to 0$$

Proof :- $E[(X-\mu)^2] = \sigma^2 < \infty$

From chebyshev's inequality

$$P\{|\bar{X} - \mu| \geq \epsilon\} \leq \frac{\sigma_{\bar{X}}^2}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$

$$\sigma_{\bar{X}}^2 = \frac{\sigma^2}{n}$$

$$\lim_{n\to\infty} P\{|\bar{X} - \mu| \geq \epsilon\} \leq \lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2}$$

$$\lim_{n\to\infty} P\{|\bar{X} - \mu| \geq \epsilon\} \cancel{\leq} \ 0$$

(approaches)

Strong law of large numbers

The sample mean of a R.S with probability '1'. Converge to the mean of distribution ($n \to \infty$)

Theorem :- R.S $X_1, X_2, \ldots X_n$ (IID X_i)

$E[X_i] = \mu$

$$\lim_{n \to \infty} P\{\bar{x} \to \mu\} \to 1, \quad \boxed{\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

Proof:- Initially $E[x_i] = \mu = 0$

Assume: The fourth moment

$$E[x^4] = \hat{K} < \infty$$

$$S_n = \sum_{i=1}^{n} x_i$$

$$E[S_n^4] = E\left[\left(\sum_{i=1}^{n} x_i\right)^4\right]$$

$$E[S_n^4] = E\left[\sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j \sum_{l=1}^{n} x_l \sum_{m=1}^{m} x_m\right]$$

$$x_i^4,\ x_i^3 x_j,\ x_i^2 x_l^2,\ x_i^2 x_j x_l,\ x_i x_j x_l x_m$$

$$E\{x_i^3 x_j\} = E[x_i^3]\, E[x_j] = 0$$

$$E[x_i^2 x_l x_m] = E[x_i^2]\, E[x_l]\, E[x_m] = 0$$

$$E[x_i x_j x_l x_m] = E[x_i]\, E[x_j]\, E[x_l]\, E[x_m] = 0$$

1) How many $x_i^4 = n \qquad n \cdot E[x_i^4] = nK$

$x_i^2 x_j^2$ for any pair i & j

$$\binom{4}{2} = 6$$

$$(x_1+x_2)(x_1+x_2)(x_1+x_2)(x_1+x_2)$$

$$(x_1^2+x_2^2+2x_1x_2)(x_1^2+x_2^2+2x_1x_2)$$

Total no. of $E[x_i^2 x_j^2]$ are found in $E(S_n^4)$

$\equiv$ No. of pairs i & j from 'n'

$$= \binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2}$$

$$E[x_i^2 x_j^2] \rightarrow k \frac{6n(n-1)}{2} = 3n(n-1)\, E[x_i^2 x_j^2]$$

$$E[x_i] = 0$$

$$\boxed{E[x_i^2] = \sigma^2}$$

$$E[x_i^2]\, E[x_j^2] = \sigma^2 \times \sigma^2 = \sigma^4$$

$$\boxed{E\left((x_i^2)^2\right)^2 = \sigma^4 \leq E[x_i^4] = k \qquad \sigma^4 \leq k}$$

$$E[S_n^4] \leq nk + 3n(n-1)k$$
$$\leq k(n + 3n(n-1))$$

$$E\left[\frac{S_n^4}{n^4}\right] \leq \frac{k}{n^3} + \frac{3k}{n^2}, \quad k < \infty$$

$$\sum_{n=1}^{\infty} E\left(\frac{S_n^4}{n^4}\right) \leq k\left[\sum_{n=1}^{\infty} \frac{1}{n^2} + 3\sum_{n=1}^{\infty} \frac{1}{n^2}\right] < \infty$$

$$\sum_{n=1}^{\infty} \beta_n < \infty$$ β_n is very small

$$\lim_{n\to\infty} \frac{S_n^4}{n^4} \rightarrow 0$$

$$\lim_{n\to\infty} \frac{S_n}{n} \rightarrow 0$$

$$w_i = u_i - \mu \qquad S_n = \sum_{i=1}^{n} x_i$$

$$S_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{(\mu_i - \mu)}{n} \rightarrow 0$$

with probability 1

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mu_i = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mu$$

$$\boxed{\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = \mu}$$

$$\lim_{n\to\infty} \bar{x} = \mu$$

Q) $X \to R.V$, $E(X) = Var(X) = 20$

$P\{0 < X < 40)$

$P\{|X-20|<20\} = P(-20 < X-20 < 20)$

$P\{|X-20| < 20\}$

$P\{|X-20|<20\} = 1 - P\{|X-20| \geq 20\}$

$\alpha = P\{|X-20| \geq 20\} \leq \frac{20}{400} = \frac{1}{20}$

$\cancel{\alpha = \frac{1}{20}}$. $1-\alpha \leq \frac{1}{20}$

$\alpha \geq \frac{19}{20}$ $\Rightarrow$ $\boxed{\frac{19}{20} < \alpha < 1}$

Q) $X = [X_1, X_2, \cdots X_{20}]$

Poisson R.V $E[X_i] = 1$

$P\{\sum_{i=1}^{20} X > 15\}$

$Y = \sum_{i=1}^{20} X_i$ $\boxed{E[Y] = 20}$

$P\{\sum_{i=1}^{20} X > 15\} \leq \frac{E(X)}{a} \leq \frac{20}{15} \leq 1$

As probability > 1

Using CLT

$P\{Y > 15\}$

$Y \sim N(20, 20)$

$P\{Y>15\} = 1 - \phi(15)$

$= 1 - \frac{1}{2}\left(1 + erf\left(\frac{15-20}{\sqrt{2}\sqrt{20}}\right)\right)$

$= 1 - 0.1317 = 0.86822$