Moments of a distribution fn.

(i) Let X - discrete RV.

$$P_k = P\{X = x_k\} \quad k = 1, 2, 3, \ldots$$

and $\sum_{k=1}^{\infty} |x_k| P_k < \infty$

then the expected value of X.

$$\mu = E[X] = \sum_{k=1}^{\infty} x_k P_k.$$

eg1: Let X have PMF.

$$\cancel{P_j = P\left\{X = (-1)^{j+1} \cdot \frac{3^j}{j}\right\}}$$

$$\cancel{= \frac{2}{3^j} \quad j = 1, 2, \ldots}$$

$$\cancel{\sum_{j=1}^{\infty} |x_j| p_j}$$

$$\cancel{= \sum_{j=1}^{\infty} \left|(-1)^{j+1} \frac{3^j}{j}\right| \cdot \frac{2}{3^j}}$$

$$\cancel{= 2 \sum_{j=1}^{\infty} \frac{1}{j} \to \infty.}$$

$$p_j = P\left\{X = (-1)^{j+1} \frac{3^j}{2^j}\right\} = \frac{2}{3^j}$$

$$j = 1, 2, \ldots$$

$$\sum_{j=1}^{\infty} |x_j| \cdot p_j = \sum_{j=1}^{\infty} \left|(-1)^{j+1} \frac{3^j}{2^j}\right| \frac{2}{3^j}$$

$$= 2 \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = 2\left[\sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j - 1\right]$$

$$= 2 \cdot \left[\frac{1}{1-\frac{1}{2}} - 1\right] = 2.$$

$$E(X) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{2}{2^j} = 2 \cdot (-1) \cdot \sum_{j=1}^{\infty} \left(\frac{-1}{2}\right)^j$$

$$= -2 \cdot \frac{1}{1+\frac{1}{2}} = -\frac{4}{3}.$$

For continuous RV. $f(x)$ pdf

If $\int |x| f(x)\, dx < \infty$.

then.

$$E(X) = \int x \cdot f(x)\, dx.$$

Mean.

In general. $h(x)$ ie $Y = h(X)$
a fn. of X.

$$E[h(X)] = \int h(x) f(x)\, dx \to \text{moment.}$$

if $\int |h(x)| f(x)\, dx < \infty$

---

e.g. $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$ $\quad -\infty < x < \infty$

Cauchy Pdf.

Check.

$$\int_{-\infty}^{\infty} |x| \cdot \frac{1}{1+x^2} \, dx$$

$$= \int_{-\infty}^{0} (-x) \frac{1}{1+x^2} \, dx + \int_{0}^{\infty} \frac{x}{1+x^2} \, dx.$$

$\rightarrow$ diverges.

$\therefore$ $E(X)$ does not exist.

For discrete RV.

$X.$; $Y = g(X)$.

required

$\Rightarrow E|g(x)| < \infty$

moments. $\sum_{j=1}^{\infty} g(x_j) P\{X = x_j\} \rightarrow$

$= \sum_{k=1}^{\infty} y_k P\{Y = y_k\}$.

For continuous RV.

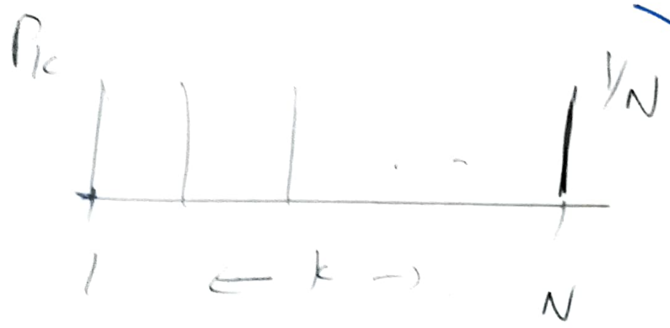
X, $Y = g(X)$.

$\int g(x) f(x) dx = \int y h(y) dy$.

Summary: Moments are same irrespective of X or Y space.

Provided $E\left[|g(x)|\right] < \infty$.

e.g. Given $P\{X=k\}=\frac{1}{N}$. $k=1,\dots,N$ → Discrete Uniform PMF.



$E(X) = ?$

$$E[X] = \sum_{k=1}^{N} k \cdot \frac{1}{N} = \frac{1}{N} \cdot \frac{N(N+1)}{2}$$

$$= \frac{N+1}{2}.$$

$$E[X^2] = \sum_{k=1}^{N} k^2 \frac{1}{N} = \frac{1}{N} \frac{N(N+1)(2N+1)}{6}$$

$$= \frac{(N+1)(2N+1)}{6}.$$

Let X be an RV with pdf.

$$f(x) = \begin{cases} \frac{2}{x^3} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

$$E(X) = \int_1^\infty x \cdot \frac{2}{x^3}\, dx$$

$$= 2 \cdot \int_1^\infty x^{-2}\, dx$$

$$= 2 \left. \frac{x^{-2+1}}{-2+1} \right|_1^\infty = 2.$$

$$E(X^2) = \int_1^\infty x^2 \frac{2}{x^3}\, dx$$

$$= 2 \int_1^\infty \frac{1}{x}\, dx.$$

$$= 2 \cdot \ln x \Big|_1^\infty = 2 \ln \infty. \rightarrow \infty.$$

eg.4)

A & B play coin-toss game.

o/p: Heads $A \xrightarrow{\text{Pays}} B$ 1\$

Tails $B - A$ 1\$

$P\{\text{Heads}\} = p.$

Expected gain of A = ?

Q: Find $E[G(A)]$?

Let X → RV denoting gain / loss of 1 \$. by A.

$P(X=1)$

X = 1 gain by player A

$= P\{\text{Tails}\}.$

$= 1-p.$

$P(X=-1)$

X = -1. loss 1\$ by player A/B.

$= \text{Prob}\{\text{Heads}\}.$

$\therefore E(X) = (-1)p + (1)(1-p) = 1-2p.$

$E(X) = 0$ for $p = \frac{1}{2}$ Fair coin.

Interpretation of mean & Variance.

(i) $\mu = E(x) = \sum x f(x)$

$\therefore \mu = \sum_k x_k f(x_k). = \sum_k x_k \cdot \omega_k$

[$\omega_k = f(x_k) \rightarrow$ weights.

$\mu \rightarrow$ weighted average. of x.]

Variance

(ii) $\sigma^2 = V(x) = E[(x-\mu)^2]$. (center of dist.) *

standard deviation.

$\sigma = \sqrt{\sigma^2} = \sqrt{V(x)}$.

[Measure of dispersion / scatter in possible value of x. (from μ).]

$V(x) = E[(x-\mu)^2] = E[x^2 + \mu^2 - 2x\mu]$

$= E[x^2] + \mu^2 - 2\mu E(x)$

$\therefore V(x) = E[x^2] - \mu^2$.

$\rightarrow$ second-moment.

$\therefore V(x) \to E[(x-\mu)^2] \to 2^{nd}$ central moment.

Marketing - Problem.

Design.	A	B.
Mean. Revenue.	\$3 mil	?
Variance	0. (certain)	?

Prob { revenue from design. } = 0.3.

revenue (design) = 7.0. mil.

Prob { revenue } = 0.7

revenue (no design) = 2 mil.

Is design B better. or A?

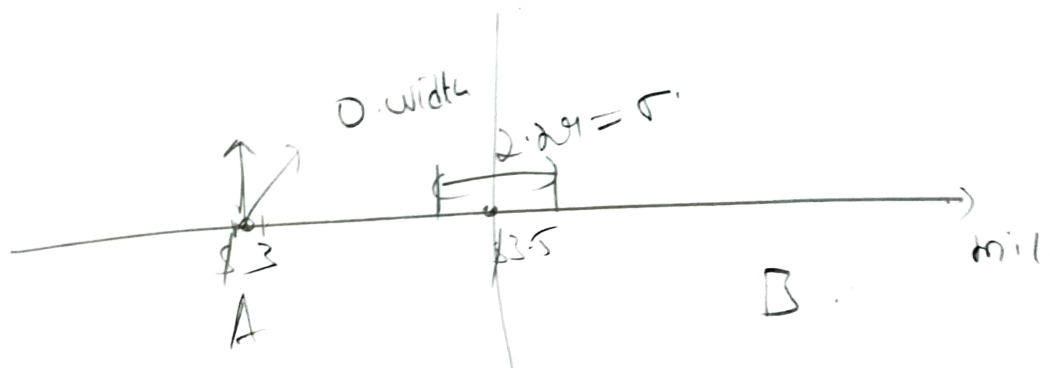
~~E[A]~~ $E[X_A] = 3$. $V[X_A] = 0$.

$E[X_B] = 0.3(7) + 0.7(2)$

$= 3.5$ mil.

$$V[X_B] = (7-3.5)^2\, 0.3 + (2-3.5)^2 \cdot 0.7 = 5.25.$$

$$\therefore \sigma(X_B) = 2.29 \text{ mil.}$$



eg: A → E_A → \$3 mil.

B → $3.5 - 2.29 \to \$2.21$ mil

to

$3.5 + 2.29 \to \$5.79$ mil

B → better mean but greater v variance / uncertainty.

A → No uncertainty * (better design)

$\boxed{E(X_A) = 3 \text{ mil}}$

B → ok If: Average is higher

- higher uncertainty okay

es 3.11

No. of email messages / hour.

x	10	11	12	13	14	15
$f(x)$	0.08	0.15	0.3	0.2	0.2	0.07

Find / in $E(x) = \sum x f(x)$

$= 10(0.08) + 11(0.15) + \cdots + 15(0.07)$

$\boxed{E(x) = 12.5}$

$V(x) = \sum x^2 f(x) - (E(x))^2$

$= 10^2(0.08) + 11^2(0.15) + \cdots + 15^2(0.07) - (12.5)^2$

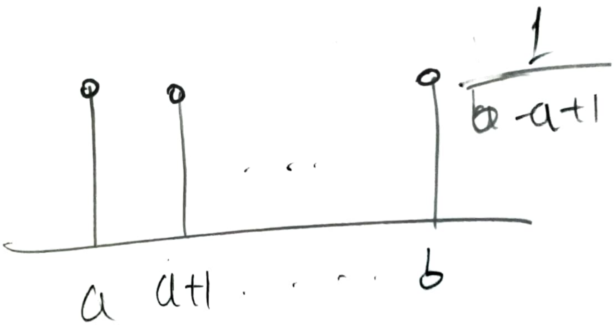
$\boxed{V(x) = 1.85}$

$\boxed{\sigma = \sqrt{V(x)} = \sqrt{1.85} = 1.36}$

Discrete Uniform distribution:

General case:

$$f(x) = \begin{cases} \dfrac{1}{b-a+1} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$



How: no. of value $= b-a+1$.

$$E(X) = \sum_{a \le x \le b} x f(x)$$

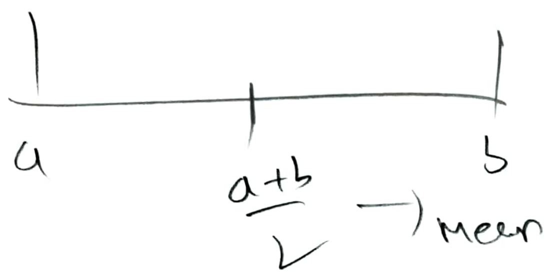
$$= \sum_{x=a}^{b} x \cdot \frac{1}{b-a+1}$$

$x-a=y$
$x=a$
$y=0$.
$x=b;\ y=b-a$

$$= \frac{1}{b-a+1} \sum_{y=0}^{b-a} (y+a)$$

$$= \frac{1}{b-a+1}\left[\frac{(b-a)(b-a+1)}{2} + a(b-a+1)\right]$$

$$\therefore \boxed{E(X) = \frac{a+b}{2}}$$



Variance $E[X^2] - (E(X))^2$

$$E[X^2] = \sum_{x} x^2 f(x)$$

$$= \sum_{x=a}^{b} x^2 \cdot \frac{1}{b-a+1}.$$

$x-a=y$
$x=y+a$

$$= \sum_{y=0}^{b-a} (y+a)^2 \frac{1}{b-a+1}$$

$$= \left[\sum_{y=0}^{b-a} y^2 + a^2 \sum_{y=0}^{b-a} 1 + 2a \sum_{y=0}^{b-a} y\right] \frac{1}{b-a+1}$$

$$= \left[\frac{(b-a)(b-a+1)(2(b-a)+1)}{6} + a^2(b-a+1) + 2a\frac{(b-a)(b-a+1)}{2}\right] \frac{1}{b-a+1}$$

$$= \frac{2(b-a)^2 + b-a}{6} + a^2 + a(b-a)$$

$$V(x) = \frac{1}{3}(b^2+a^2-2ab) + \frac{b-a}{6} + \cancel{2a^2} + ab - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{12}\left[4b^2 + 4a^2 - 8ab + 2b - 2a + \cancel{12a^2} - 12ab - 3a^2 - 3b^2 - 6ab\right].$$

$$= \frac{b^2 + a^2 - 2ab + 2b - 2a + 1 - 1.}{12}$$

$$\therefore \boxed{V(X) = \frac{(b-a+1)^2 - 1.}{12}}$$

Binomial distribution:

Def: A random exp. consists of n Bernoulli trials such that

(1) The trials are independent

(2) Each trials results in only two possible outcomes. success or failure

(3) Prob{ success in each trial } = p. a constant.

The r.v. X that equals the no. of trials with result = success within n no. of trials. is a Binomial random variable

The PMF

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}. \qquad x = 0, 1, \ldots, n \quad p > 0.$$

Example.

(1) Flip a coin 10 times.

Let X = no. of heads. out of 10 trials.

(2) Let $p = 0.1$ for Prob. of a rare molecule in a sample of air.

X = no. of air samples that have rare molecule out of 18 trials.

(3) A MCQ test → 10 questions

each question - response is guessed

Let X → no. of questions answered correctly.

P → prob { a question is answered correctly }.

Organic pollution problem:

Let Prob{ a sample of water has a particular organic pollutant } = p.

Assume samples are independent.

$X \rightarrow$ no. of samples that has pollutant.

Q1: Given $n = 18$, $p = 0.1$.

find the prob. $\{X = 2\}$. ie; exactly two samples are polluted.

Sol: Find prob $\{X = 2\}$.

$$P(X=2) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{18}{2} (0.1)^2 (0.9)^{16}.$$

$$\boxed{P(X=2) = 0.2835}$$

Q2: Determine prob{ At least 4 samples are polluted }

Sol: To find $P\{X \geq 4\}$.

note: $P\{X \geq 4\} = 1 - P\{X < 4\}$.

$P\{X<4\} = P(X=0) + P(X=1) + \dots + P(X=3)$

$= \sum_{x=0}^{3} \binom{n}{x} p^x (1-p)^{n-x}$

$= \binom{18}{0} 0.1^0 \, 0.9^{18} + \binom{18}{1} (0.1)^1 \, 0.9^{17}$

$+ \binom{18}{2} 0.1^2 \, 0.9^{16} + \binom{18}{3} 0.1^3 \cdot 0.9^{15}$

$= 0.9018.$

$\therefore \boxed{P\{X \geq 4\} = 0.0981}$

Mean of binomial rv:

(i) $\mu = E(X) = ?$

$$E(X) = \sum_k k \binom{n}{k} p^k (1-p)^{n-k}.$$

Sol:

Recall $\sum_k \binom{n}{k} p^k (1-p)^{n-k} = 1.$ —①

Proof: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$a = p,\ b = 1-p$

$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (1)^n = 1.$

diff. ① w.r.t. p.

$$\sum_k \binom{n}{k} k \cdot p^{k-1} (1-p)^{n-k} - \sum_k \binom{n}{k} p^k (n-k) p^{n-k-1} = 0.$$

$$\Rightarrow \left(\frac{1}{p} + \frac{1}{1-p}\right) E(X) = \frac{n}{1-p}.$$

$\therefore \boxed{E(X) = np}$

For Variance

$E(X^2) \rightarrow ?$

start with $\sum_k k\binom{n}{k}p^k(1-p)^{n-k} = np$

diff. w.r.t p.

$$\sum_k k\binom{n}{k}k p^{k-1}(1-p)^{n-k} - \sum_k k(n-k)\binom{n}{k}p^k(1-p)^{n-k-1}$$

$$= n.$$

$$\Rightarrow \quad E[X^2]\left(\frac{1}{p}+\frac{1}{1-p}\right) = \frac{n.np}{1-p} + n.$$

$$\Rightarrow \quad E(X^2) = n^2p^2 + np(1-p).$$

$$\therefore Var(X) = E(X^2) - (np)^2$$

$$\therefore \boxed{Var(X) = np(1-p)}.$$

ex: 3.23 pg 93

In a semiconductor manufacturing process.

3 wafers form a lot. ← (testing).

Each wafer → pass / fail in test.

$P\{\text{wafer passes a test}\} = 0.7 \quad \therefore p = 0.7$

Wafers are independent.

Find the PMF of no. of wafers from a lot that pass the test.

Sol: $n = 3, \quad p = 0.7$

$k = 0, \ldots, 3$ 0 pass to 3 pass.

$$\boxed{f(X=k) = P_k = \binom{n}{k} 0.7^k (0.3)^{n-k} \quad k = 0, \ldots, 3.}$$

$\therefore$ "X → binomial RV".

3.22

An optical inspection system.

↳ distinguish → Part types.

Prob $\{$correct classification$\}$ = $p = 0.95$.

Let 3 parts are inspected.

and tests are independent.

$X \rightarrow$ no. of parts that are correctly classified

$X \rightarrow$ Binomial RV

$$\boxed{p_k = P\{X = k\} = \binom{3}{k} (0.95)^k (0.05)^{3-k}.}$$

Exercises on Binomial. dist.

Yes | No

(1) From a batch of 70 temperature transducers a sample of size 40 is selected without replacement. X denotes the no. of non confirming transducers.

Ans: No. Sample size is not uniform - trials are not identical.

(2) X -> denotes a no. of correct answers by a student in MCQ.

(i) In some questions, student can eliminate a few incorrect answers

(ii) In a few questions, Student can eliminate all incorrect answers

X -> Not a binomial RV.

reason:- trials are not Bernoulli "variable p".

$X \to$ Binomial RV if

all questions are answered in a similar manner. [strategy*].

(b) A filling operation - fills detergent Packages to standard weight of 1kg.

X - denotes the no. of Packages that are underfilled.

$X \to$ Binomial. (Yes)

if $P\{X \Rightarrow \text{underfilled}\} = p$. const.

- Filling process is independent.

- Some process/tech used for filling.

(d) $X \to$ denotes the no. of accidents occur along federal highways. in Arizona. during a one month period.

Yes:- - If $\text{prob}\left\{\frac{\text{accident}}{\text{day}}\right\} = p$.

- Independence of accidents across days.

No:

If: multiple accidents / day. is possible.

3-80 Given $X \rightarrow$ Binomial RV.

$n = 10, \; p = 0.5$

Find $P(3 \le x < 5)$

$$\therefore P(3 \le x < 5) = \sum_{k=3}^{4} \binom{n}{k} 0.5^k (0.5)^{n-k}.$$

$$= \binom{10}{3}(0.5)^{10} + \binom{10}{4}(0.5)^{10}.$$

$$= 0.322$$

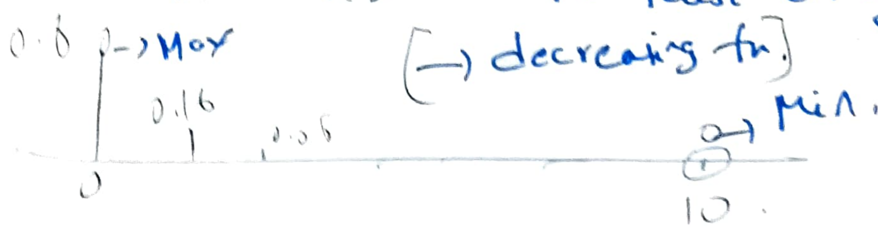
3-83 Sketch the PMF of a Binomial dist.

$n = 10, \quad p = 0.02.$ Comment on shape.

(i) what value of x is most likely

(ii) " " " " is least likely.

[→ decreasing fn]



3.33 MCQ-test. 30 questions.

$\downarrow$ 4 Answer (possible)

Assume student guesses the answers.

(a) what is prob { student answer. > 20 correctly }.

Sol: Prob{ $X > 20$ }. | $P = \frac{1}{4}$.

$$P(x>20) = 1 - P(x \leq 20).$$

$$= \sum_{k=21}^{30} \binom{30}{k} p^k (1-p)^{30-k}.$$

$$= \sum_{k=21}^{30} \binom{30}{k} (0.25)^k (0.75)^{30-k}.$$

$$= 2.8 \times 10^{-7}$$

(b) What is prob { student answer less than 5 questions correctly }

$$P(x<5) = \sum_{k=0}^{4} \binom{30}{k} 0.25^k (0.75)^{30-k}$$

$$= 0.0978.$$

Geometric Distribution.

- A Series of Bernoulli trials with constant prob. of success.

- Trials are conducted until a success is obtained.

$X \to$ no. of trials until a success.

Fail Fail Fail . . . . Success

1 2 3 . . . . STOP

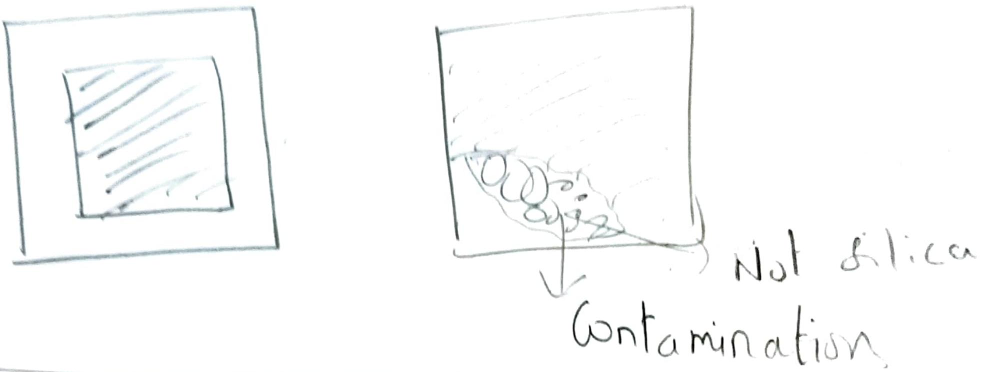
Geometric RV. $- X$.

$$f(x) = (1-p)^{x-1} \cdot p. \qquad x = 1, 2, \ldots$$

Note: (1) No combination

(2) No fixed 'n' no. of trials.

eg: Prob{a wafer contains a large particle of contamination} = 0.01



What is the prob. that exactly 125 wafers are analyzed before a large particle is detected.

$$f(x) = (1-p)^{x-1}p. \qquad p = 0.01.$$

$$P\{X = 125\} = (0.99)^{124}(0.01)$$

$$= 0.0029.$$

Mean of geometric RV.

$$\mu = E(X) = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p.$$

$$= p \cdot \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1}. \qquad \text{Let } 1-p = q.$$

$$= p \sum_{k=1}^{\infty} k \cdot q^{k-1}.$$

Consider $\displaystyle\sum_{k=1}^{\infty} q^k = \frac{1}{1-q} - 1.$ $\quad |q| < 1.$

$$\sum_{k=1}^{\infty} q^k = \frac{q}{1-q}.$$

$$\frac{\partial}{\partial q} \sum_k q^k = \frac{\partial}{\partial q} \frac{q}{1-q}.$$

$$\Rightarrow \sum_k k q^{k-1} = \frac{1}{1-q} + \frac{q(+1)}{(1-q)^2}.$$

$$= \frac{1}{1-q} \cdot \left[1 + \frac{q}{1-q}\right] = \frac{1}{(1-q)^2}.$$

$$\therefore p \sum_k k q^{k-1} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

$$\therefore \boxed{E(X) = 1/p.}$$

$$E(x^2) = \sum_{k} k^2 \cdot (1-p)^{k-1} \cdot p \ . \quad ?$$

$$\sum k (1-p)^{k-1} p = \frac{1}{p} .$$

$$\therefore \sum k (1-p)^{k-1} = \frac{1}{p^2} .$$

$$\frac{\partial}{\partial p} \sum k (1-p)^{k-1} = \frac{\partial}{\partial p} \frac{1}{p^2} .$$

$$\therefore \sum k(k-1)(1-p)^{k-2}\cancel{(-1)} = \cancel{+}\frac{2}{p^3} .$$

$$\sum k^2 (1-p)^{k-2} = \sum k (1-p)^{k-1} + \frac{2}{p^3} .$$

$$\frac{1}{p} \cdot E(x^2) = \left(\frac{1}{p^2} + \frac{2}{p^3}\right)(1-p)$$

$$E(x^2) = \left(\frac{1}{p} + \frac{2}{p^2}\right)(1-p)$$

$$V(x) = \frac{(1-p)\cancel{p}}{p^2} + \frac{2(1-p)}{p^2} - \frac{1}{p^2} \ . =$$

$$\frac{1}{p^2} - \frac{1}{p} .$$

To find $E(X^2) = \sum_k k^2 (1-p)^{k-1} p.$

start with mean:

$$\sum k(1-p)^{k-1} p = \frac{1}{p}.$$

$$\frac{\partial}{\partial p} \sum_k k(1-p)^{k-1} p = \frac{\partial}{\partial p} \frac{1}{p}.$$

$$\Rightarrow \sum_k k(k-1)(1-p)^{k-2} \cdot p(-1).$$

$$+ \sum_k k(1-p)^{k-1} = \frac{-1}{p^2}.$$

$$\sum_k k^2 (1-p)^{k-1} \cdot p \left(\frac{-1}{1-p}\right) + \sum_k k(1-p)^{k-1} \cdot p \left(\frac{1}{1-p}\right)$$

$$+ \frac{1}{p} \cdot \sum_k k \cdot (1-p)^{k-1} p = \frac{-1}{p^2}$$

$$\Rightarrow \cancel{E(X^2)} \ \frac{-1}{1-p} \cdot E(X^2) + \frac{1}{1-p} \cdot \frac{1}{p} = \frac{-2}{p^2}.$$

$$\therefore E(X^2) = (1-p)\left[\frac{2}{p^2} + \frac{1}{p(1-p)}\right]$$

$$E(X^2) = \frac{2(1-p)}{p^2} + \frac{1}{p}.$$

$$\therefore V(X) = \frac{2}{p^2} - \frac{2p}{p^2} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{p}{p^2} = \frac{1}{p^2}(1-p)$$

$$\boxed{\therefore V(X) = \frac{1-p}{p^2}.}$$

18

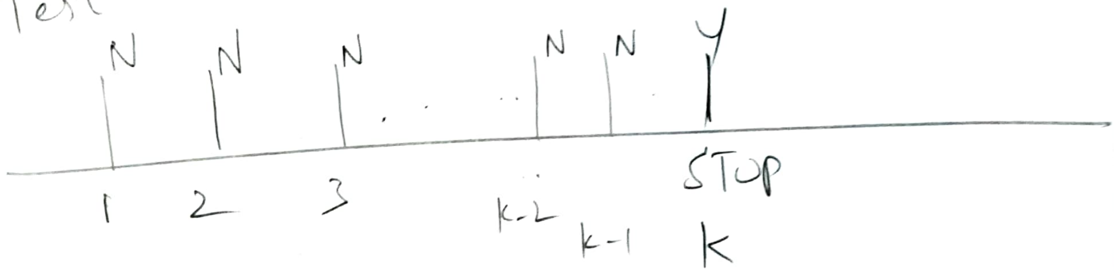
Lack of memory Property.

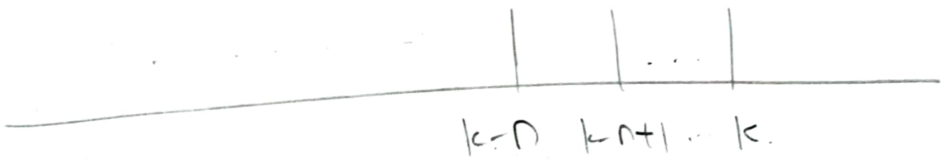
(1) Def: No. of trials till the 1^{st} success.

Note: Trials are independent.

Hence: The count of no. of trials until next success can start any-where

Test result:

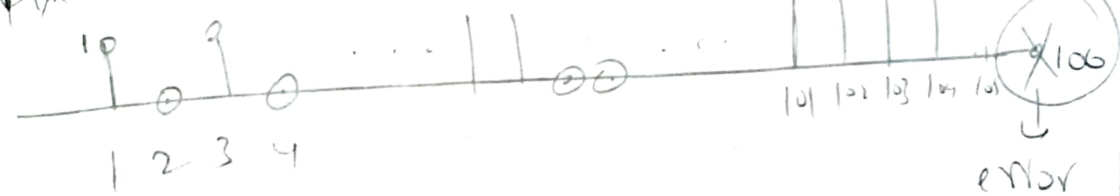




Start counting at k-n. -> still Geometric

eg: Consider transmission of bits. (1, 0)

~~Rx~~ data



Proposition:

$$P(X=6) = P\left(X=106 \,\middle|\, \begin{matrix} N=100 \\ \text{trials tx} \\ \text{(no. error)} \end{matrix}\right).$$

Validation:

$$P(X=6) = (1-p)^5 \cdot p. \quad - ①$$

$$P(X=106 \mid N=100 \text{ (trial) bits transmitted}).$$

$$= \frac{P(X=106, \underset{\text{no error}}{\text{100 bit Rx}})}{P(\text{100 bits Rx. no error})} \rightarrow \text{Joint Prob.}$$

Since individual outcomes are independent

$$= \frac{\prod_{i=1}^{105} P[\text{bit tx. with no err}] \cdot P[\text{bit with err}].}{\prod_{i=1}^{100} P[\text{bit tx. with no err}]}$$

$$= \frac{(1-p)^{105} \cdot p}{(1-p)^{100}} = (1-p)^5 \cdot p. \quad - ②$$

$$① \equiv ②$$

2.1. Given that a random variable

$X \sim \text{Bern}(x \mid \mu)$. $P(X=1) = \mu$.

i.e., $f_X(x) = \mu^x (1-\mu)^{1-x}$. $x \in \{0,1\}$

(i). To show that

$$\sum_x f_X(x) = 1.$$

Proof: $$\sum_{x=0}^{1} \mu^x (1-\mu)^{1-x}$$

$$= \mu^0 (1-\mu) + \mu (1-\mu)^0 = 1.$$

(ii) S.T. $E[X] = \mu$.

Proof. $$E[X] = \sum_x x f_X(x)$$

$$= 0 \cdot \mu^0 (1-\mu) + 1 \cdot \mu (1-\mu)^0$$

$$= \mu.$$

(iii) S.T $\text{Var}[X] = \mu(1-\mu)$

Proof: $\text{Var}[X] = E[X^2] - (E[X])^2$

$$\therefore E[x^2] = \sum x^2 f_X(x)$$

$$= 0\cdot \mu^0(1-\mu) + 1\cdot \mu(1-\mu)^0$$

$$\therefore E[x^2] = \mu .$$

$$\therefore Var[x] = \mu - \mu^2 = \mu(1-\mu). \quad \blacksquare$$

(iv) To S.T. Entropy of X.

$$H[X] = -\mu\ln\mu - (1-\mu)\ln(1-\mu).$$

Proof. Def: $H[X] = -\sum_{x} f_X(x)\ln f_X(x)$

$$\therefore H[X] = -\sum_{x=0}^{1} \mu^x(1-\mu)^{1-x}\cdot \ln\left(\mu^x(1-\mu)^{1-x}\right).$$

$$= -\left[1\cdot(1-\mu)\cdot\ln(1\cdot(1-\mu)) + \mu\cdot 1\cdot \ln(\mu\cdot 1)\right]$$

$$\therefore H[X] = -\mu\ln\mu - (1-\mu)\ln(1-\mu) \quad \blacksquare$$

Negative binomial distribution.

- generalization of geometric dist.
- RV. X – no. of trials required to obtain r – successes.

PMF: $P(X=x) = \binom{x-1}{r-1} p^{r-1} \cdot (1-p)^{x-r} \cdot p$. — 1

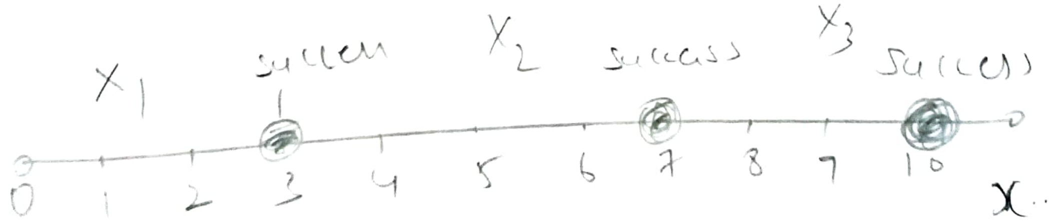
$\boxed{(;\ 0<p<1) \quad (x \geq r)}$

$P(X=x)$ = Prob{ $r-1$ successes from $x-1$ trials }

· Prob{ (r^{th}) success in x^{th} trial }.

binomial rv.

$= \left[\binom{x-1}{r-1} \cdot p^{r-1} \cdot (1-p)^{x-r} \right] \cdot p \cdot = \rightarrow$ 1



$X = X_1 + X_2 + X_3 \rightarrow$ NBD. RV

Relation between geometric RVs

and negative binomial · RV.

Let $X \to$ no. of trial required for

r- successes.

X_1	$\to$	Success 1	—
X_2	$\to$	2	—
$\vdots$		$\vdots$	
X_r	$\to$	r.	—

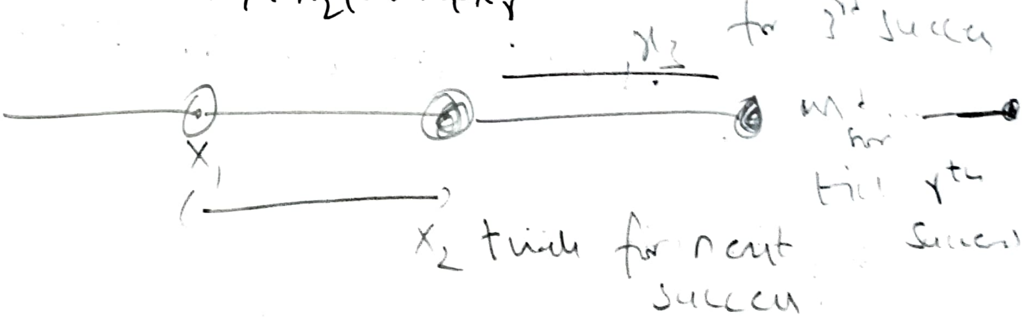
random. geometric variables

due to.

Memory less property.

The total no. of trials

$$X = X_1 + X_2 + \cdots + X_r$$



$$\therefore \boxed{E(X) = r/p}$$

$$V[X] = E(X^2) - (E(X))^2$$

$$E(X^2) = \sum x^2 \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

start with

$$\sum x \binom{x-1}{r-1} p^r (1-p)^{x-r} = r/p.$$

$$\frac{\partial}{\partial p} \sum x \binom{x-1}{r-1} p^r (1-p)^{x-r} = r \cdot \frac{\partial}{\partial p}\left(\frac{1}{p}\right)$$

$$\sum x \binom{x-1}{r-1} \left(r p^{r-1} (1-p)^{x-r} - p^r (x-r)(1-p)^{x-r-1} \right)$$

$$= \frac{-r}{p^2}$$

$$\Rightarrow \frac{r}{p} \sum_x x \cdot f_{NB}(x) - \frac{1}{1-p} \sum_x x^2 f_{NB}(x)$$

$$+ \frac{r}{1-p} \sum_x x f_{NB}(x) = \frac{-r}{p^2}$$

$$\therefore \frac{1}{1-p} E(X^2) = \frac{r^2}{p^2} + \frac{r}{p^2} + \frac{r^2}{p(1-p)}.$$

$$E(X^2) = \frac{r}{p^2} \cdot \left[r + 1 + \frac{r \cdot p}{1-p} \right] (1-p)$$

$$= \frac{r}{p^2} \left[r - \cancel{rp} + 1 - p + \cancel{rp} \right]$$

Mean of NBin. - r.v.

$$E(X) = \sum x f(x)$$

$$= \sum_{x=r}^{\infty} x \cdot \binom{x-1}{r-1} (1-p)^{x-r} p^r.$$

Recall $\displaystyle\sum_{x=r}^{\infty} \binom{x-1}{r-1} (1-p)^{x-r} p^r = 1.$

$$\frac{\partial}{\partial p} \sum_{x=r}^{\infty} \binom{x-1}{r-1} (1-p)^{x-r} p^r = 0.$$

$$\sum_{x=r}^{\infty} \binom{x-1}{r-1} (x-r) (1-p)^{x-r-1} (-1)\, p^r$$

$$+ \sum_{x} \binom{x-1}{r-1} r p^{r-1} \cdot (1-p)^{x-r} = 0$$

$$\Rightarrow -\frac{1}{1-p} \sum_{x} x \cdot \binom{x-1}{r-1} (1-p)^{x-r} \cdot p^r + \frac{r}{1-p} \sum_{x} \binom{x-1}{r-1} (1-p)^{x-r} p^r$$

$$+ \frac{r}{p} \sum_{x} \binom{x-1}{r-1} p^r (1-p)^{x-r} = 0.$$

$$\Rightarrow \frac{1}{1-p} \sum_{x} x f_{NB}(x) = \frac{r}{1-p} \sum_{x} f_{NB}(x) + \frac{r}{p} \sum_{x} f(x).$$

$$\Rightarrow E(X) = (1-p)\left[\frac{r}{1-p} + \frac{r}{p}\right] = r\cdot\left(1 + \frac{1-p}{p}\right) = \frac{r}{p}.$$

ex 25

Prob{ A camera passes a test } = 0.8.

Assume the camera works independently.

Prob { the third failure is obtained in five or fewer tests } ?

Sol: X → no. of cameras tested until 3 failures are obtained.

To find $P\{X \leq 5\}$

X → NBn. rv

$$f(x) = \binom{x-1}{r-1}(1-p)^{x-r} p^r \quad \Big| \; p = 0.2.$$

$$\therefore P\{X \leq 5\} = \sum_{x=3}^{5} \binom{x-1}{2} (0.8)^{x-3} \cdot (0.2)^3.$$

$$= \binom{2}{2}(0.8)^0 (0.2)^3 + \binom{3}{2}(0.8)(0.2)^3 + \binom{4}{2}(0.8)^2 (0.2)^3 = 0.056$$

$$\therefore V(X) = \frac{r^2}{p^2} + \frac{r}{p^2} - \frac{r}{p} - \frac{r^2}{p^2}$$

$$= \frac{r}{p}\left(\frac{1}{p} - 1\right) = r\left(\frac{1-p}{p^2}\right)$$

$$\therefore \boxed{V(X) = r\,\frac{1-p}{p^2}}$$

Using sum-relation between Geometric & NB r.v.

$$X_{NB} = \sum_{i=1}^{r} X_G^{(i)}.$$

$$E(X_{NB}) = \sum_{i=1}^{r} E(X_G^{(i)}) = \sum_{i=1}^{r} \frac{1}{p}.$$

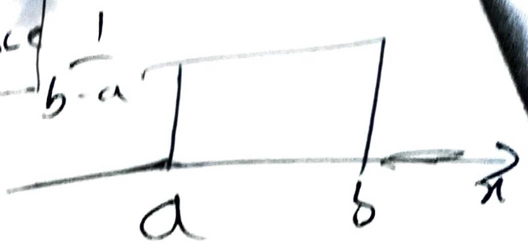
$$\boxed{\therefore E(X_{NB}) = \frac{r}{p}.}$$

$$V(X_{NB}) = \sum_{i=1}^{r} V(X_G^{(i)}) = \sum_{i=1}^{r} \left(\frac{1-p}{p^2}\right)$$

$$\boxed{\therefore V(X_{NB}) = \frac{r(1-p)}{p^2}.}$$

(1)

① $\boxed{\text{Uniform r.v: mean \& variance}}$



$$f_X(x) = \frac{1}{b-a}$$

$$E(X) = \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b = \frac{1}{2}\frac{b^2-a^2}{b-a}$$

$$\therefore E(X) = \frac{a+b}{2}$$

$$V[X] = \int (x - E(X))^2 f_X(x)\,dx$$

$$= E[X^2] - \mu_X^2$$

$$\int_a^b x^2 \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a} \cdot \frac{1}{3} \cdot \left. x^3 \right|_a^b$$

$$= \frac{(b-a)(a^2+b^2+ab)}{(b-a)3}$$

$$\therefore V[X] = \frac{a^2+b^2+ab}{3} - \frac{(a+b)^2}{4} = \frac{(a-b)^2}{12}$$

Given the Beta distribution.

$$f_{\mu}(\mu) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a-1} (1-\mu)^{b-1} \quad ; \mu \in [0,1].$$

Motivation: Beta dist. is a prior for the Bernoulli dist.

ex: (1). S.T. $\int f_{\mu}(\mu)\, d\mu = 1$.

proof: To S.T. $\displaystyle\int_0^1 \mu^{a-1}(1-\mu)^{b-1}\, d\mu = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ — (1)

∴ start with,

$$\Gamma(a)\Gamma(b) = \int_0^\infty e^{-x} x^{a-1}\, dx \int_0^\infty e^{-y} y^{b-1}\, dy.$$

$$= \int_0^\infty x^{a-1} \int_0^\infty e^{-y-x} \cdot y^{b-1}\, dy\, dx.$$

now let $y = t - x$.

$dy = dt$

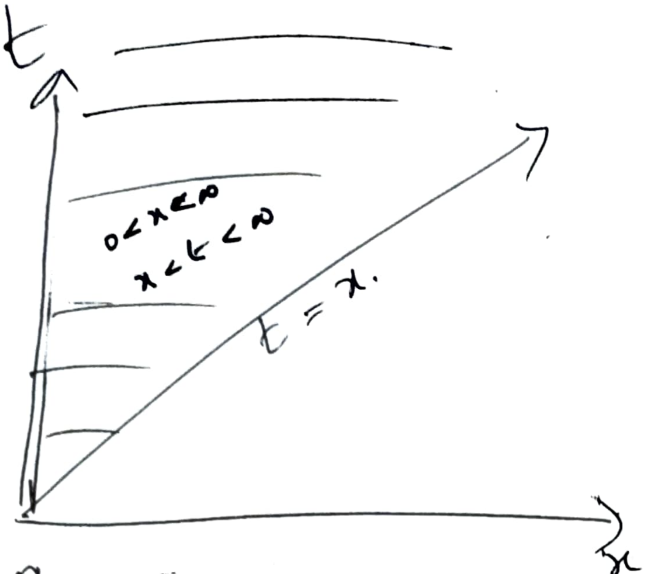
$y=0, \ t=x$

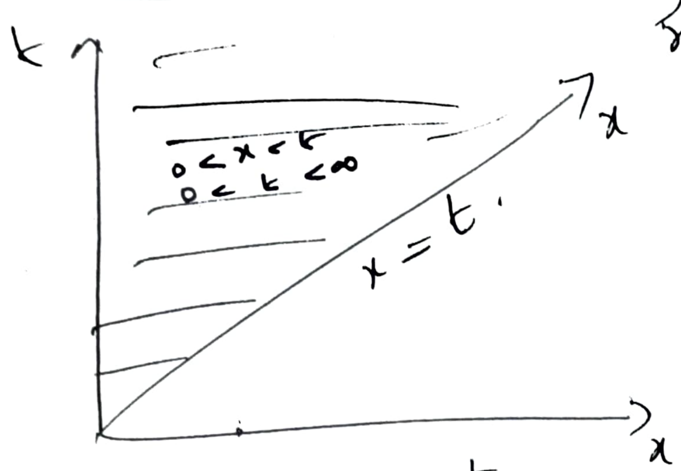
$y=\infty \ \ t \to \infty$

$$= \int_0^\infty x^{a-1} \int_x^\infty e^{-t} \cdot (t-x)^{b-1}\, dt\, dx.$$

$x: 0 \to \infty$

$t: x \to \infty$





$$\Gamma(a)\Gamma(b) = \int_0^\infty e^{-t}\int_0^t x^{a-1}(t-x)^{b-1}\,dx\,dt$$

let $x = t\mu$.
$dx = t\,d\mu$.
$x=0 \to \mu=0$
$x\to t \to \mu=1$.

$$= \int_0^\infty e^{-t}\int_0^1 (t\mu)^{a-1}(t-t\mu)^{b-1}\,t\,d\mu\,dt$$

$$= \int_0^\infty e^{-t}\, t^{a-1}\, t^{b-1}\cdot t\,dt \int_0^1 \mu^{a-1}(1-\mu)^{b-1}\,d\mu.$$

$$= \int_0^\infty e^{-t}\, t^{a+b-1}\,dt \int_0^1 \mu^{a-1}(1-\mu)^{b-1}\,d\mu.$$

$$\therefore \Gamma(a)\Gamma(b) = \Gamma(a+b)\cdot\int_0^1 \mu^{a-1}(1-\mu)^{b-1}\,d\mu.$$

$\therefore$ Proves eq. (1)

Given $\int_0^1 \mu^{a-1}(1-\mu)^{b-1}\,d\mu = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ — ①

or $\int_0^1 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a-1}(1-\mu)^{b-1}\,d\mu = 1$

$f_x(x) = \mathrm{Beta}(\mu|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a-1}(1-\mu)^{b-1}$.

(i) S.T $E[\mu] = \frac{a}{a+b}$

Proof: $E[\mu] = \int \mu \; \mathrm{Beta}(\mu/a,b)\,d\mu$.

$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \mu \cdot \mu^{a-1}(1-\mu)^{b-1}\,d\mu$.

$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \mu^{a+1-1}(1-\mu)^{b-1}\,d\mu$

from ①

$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+1)\cdot\Gamma(b)}{\Gamma(a+b+1)} = \frac{a\cancel{\Gamma(a)}\cdot\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)}(a+b)\cancel{\Gamma(a+b)}}$

$\boxed{E[x] = a/(a+b)}$

(ii) S.T $Var[\mu] = \frac{ab}{(a+b)^2(a+b+1)}$.

$$E[\mu^2] = \int \mu^2 \,\mathrm{Beta}(\mu/a,b)\, d\mu$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \mu^2 \mu^{a-1} (1-\mu)^{b-1} \, d\mu$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+2)} = \frac{a(a+1)}{(a+b+1)(a+b)}.$$

$$\therefore \mathrm{Var}[\mu] = \frac{a(a+1)}{(a+b)(a+b+1)} - \frac{a^2}{(a+b)^2}$$

$$= \frac{a}{a+b}\left[\frac{a+1}{a+b+1} - \frac{a}{a+b}\right]$$

$$\therefore \mathrm{var}[\mu] = \frac{a \cdot (\cancel{a^2} + \cancel{a} + \cancel{ab} + b - \cancel{a^2} - \cancel{ab} - \cancel{a})}{(a+b)^2 (a+b+1)}$$

$$\therefore \mathrm{var}[\mu] = \frac{ab}{(a+b)^2(a+b+1)}$$

To prove that normalizing constant of the Gaussian pdf. is $\sqrt{2\pi\sigma^2}$.

Sol: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$\sqrt{2\pi\sigma^2} \rightarrow$ derive this const.

let $I = \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx.$

$= \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2\sigma^2}\right) dt$

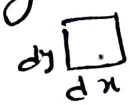
then $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right) \exp\left(-\frac{y^2}{2\sigma^2}\right) dx\,dy$

let $x^2+y^2 = r^2$. or $x = r\cos\theta$, $y = r\sin\theta$

$\theta \rightarrow 0$ to 2π

$r \Rightarrow 0$ to ∞

$dx\,dy = r\,dr\,d\theta$.



$\therefore I^2 = \int_0^{2\pi}\int_0^{\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) . r\,dr\,d\theta$

let $u = r^2$

$du = 2r\,dr$.

$\therefore I^2 = 2\pi . \int_0^{\infty} \exp\left(-\frac{u}{2\sigma^2}\right) \frac{du}{2}$.

$$I^2 = \pi \int_0^\infty \exp\left(-\frac{u}{2\sigma^2}\right) du$$

$$= \pi \cdot \frac{1}{\left(\frac{-1}{2\sigma^2}\right)} \cdot \exp\left(-\frac{u}{2\sigma^2}\right)\Big|_0^\infty$$

$$I^2 = 2\pi\sigma^2$$

$$\Rightarrow \boxed{I = \sqrt{2\pi\sigma^2}}$$

② Prove $E(x) = \mu$.

diff. ① w.r.t μ.

$$\int_{-\infty}^{\infty} \frac{d}{d\mu} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = \frac{d}{d\mu} \sqrt{2\pi\sigma^2}.$$

$$\Rightarrow \int_{-\infty}^{\infty} -\frac{2(x-\mu)(-1)}{2\sigma^2} \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx}{\sqrt{2\pi\sigma^2}} = \mu \cdot \frac{\int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx}{\sqrt{2\pi\sigma^2}}.$$

$$\Rightarrow \boxed{E[x] = \mu}$$

Given

$$\int_{-\infty}^{\infty} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx = \sqrt{2\pi\sigma^2} \quad \text{—} ①$$

①. For $E[x-\mu^2] = \sigma^2$.

diff. ① w.r.t. σ^2

$$\int_{-\infty}^{\infty} \frac{d}{d\sigma^2}\left[\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx\right] = \frac{d}{d\sigma^2}\sqrt{2\pi\sigma^2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{2} \cdot \frac{1}{(\sigma^2)^2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx$$

$$= \sqrt{2\pi} \cdot \frac{1}{2}(\sigma^2)^{-1/2}.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} (x-\mu)^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx$$

$$= (\sigma^2)^{3/2 - 1/2}$$

$$\Rightarrow \boxed{E[(x-\mu)^2] = \sigma^2}$$

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Mode of the Gaussian distribution.

- $$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Mode $\rightarrow$ $\hat{x} = \underset{x}{\arg\max}\, f_X(x)$.

from $\frac{d}{dx} f_X(x) = 0$

$$\frac{d}{dx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = 0$$

$$\Rightarrow \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}}_{>0} \underbrace{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}_{>0} \cdot \frac{-2(x-\mu)}{2\sigma^2} = 0$$

$$\Rightarrow x - \mu = 0$$

$\therefore$ mode $\boxed{\hat{x} = \mu.}$