Signal and systems

Signals & Systems		LTPC (2-1-3)
		134
COURSE OBJECTIVES	The major aim of this course is to give an overall understanding of the fundamentals of signals and systems including various signal and system classifications, basic signal transformations, properties of linear systems, sampling and introduce various continuous and discrete Fourier representations of signals. These fundamentals form the foundation for understanding topics such as digital signal processing, control systems, digital image processing, computer vision and some key aspects of machine learning. Hence this course is required for students pursuing careers in the topics mentioned above.	
EXPECTED OUTCOMES	On completion of the course, students have the ability to represent signals and analyse them efficiently, Understand the application of Fourier transform, Understand the concept of spectrum	
Prerequisite:	Calculus	

Introduction to Signals

Introduction, Classification of Signals: Continuous time and discrete time, Even and odd, Periodic and non-periodic, Deterministic and random, Energy and Power Basic operations on signals: scaling, shifting, reflection, precedence rule for time shifting and time scaling

Elementary signals: Exponential, Sinusoidal, step, pulse, impulse, ramp, relationship between sinusoidal and complex exponential signals, Exponentially damped sinusoid signals

Linear Systems and Convolution

Convolution sum, Convolution integral, Interconnection of LTI systems, impulse response, step response, Relationship between impulse response and system properties, Properties of systems: Stability, Memory, Causality, Invertibility, Time invariance

Laplace transforms

Eigen Function property, Laplace transform representation, Convergence, S-place, Unilateral Laplace transform, ROC, properties

Fourier Series and Transforms (continuous)

Periodic signal-Fourier Series, Non Periodic signal-Fourier Transform, Properties of Fourier Representations, Parseval's Relationships, Duality property and its applications, Hilbert transform, Pre-envelope, Phase and Group delay

Fourier Representation of aperiodic discrete time signals

Periodic discrete signals, properties, Discrete time fourier transform, relation with the fourier series, properties

Sampling Theory

Sampling continuous time signals, aliasing, Reconstruction-Ideal, practical

Signal and Systems

Lab experiments:

1. Generation of basic signals and their classification, energy computation , Even and Odd signals, Unit impulse, step
2. Signal transformations (scaling, shifting)
3. Linear Convolution
4. Linear convolution integral
5. Fourier analysis and synthesis
6. Hilbert transform
7. Fourier Transform
8. Sampling Theory -1
9. Sampling Theory -2

Matlab - https://www.mathworks.com/

Text-books

1. Alan V.Oppenheim, Alan S.Willsky, et al. Signal and Systems. Pearson Education India; 2nd edition (1 January 2015)
2. John G. Proakis, Dimitris G. Manolakis. Digital Signal Processing. Pearson Education India; 4th edition (1 January 2007)

References

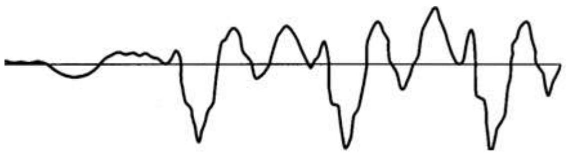
1. Luis Chaparro. Signals and Systems using MATLAB. Academic Press; 2nd edition (2 April 2014)
2. Vinay K. Ingle , John G. Proakis. Digital Signal Processing Using MATLAB: A Problem Solving Companion. Cl-Engineering;
3. Hahn. Essential MATLAB for Engineers and Scientists. Elsevier; Fifth edition (10 January 2013)
4. Barry Van Veen Simon Haykin. Signals and Systems. Wiley; Second edition (1 January 2007)
5. H Hsu, R Ranjan. Signals & Systems. McGraw Hill Education; 2nd edition (1 July 2017)

Assesement components

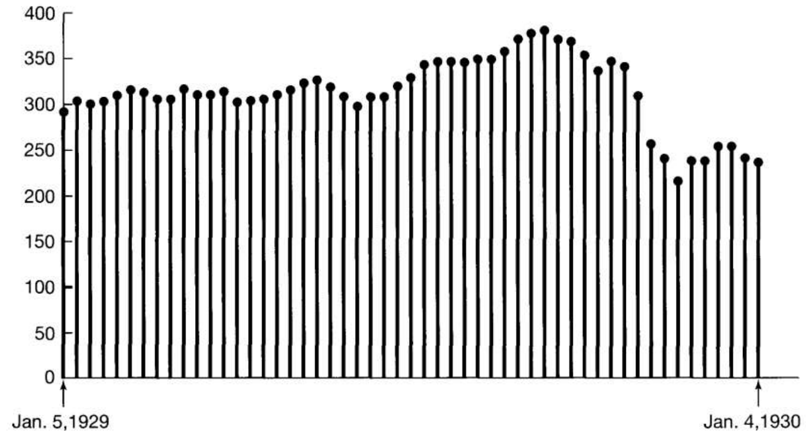
Mid-sem	20
End-sem	30
Surpize quiz	10
Scheduled Quiz	15
Lab	15
Assignments	10

What is signal?

- Any physical quantity that varies
  - with time, space, or any other independent variable or variables (e.g. voice signal, video)



Sound as a function of time



An example of a discrete-time signal: The weekly Dow-Jones stock market index from January 5, 1929, to January 4, 1930.

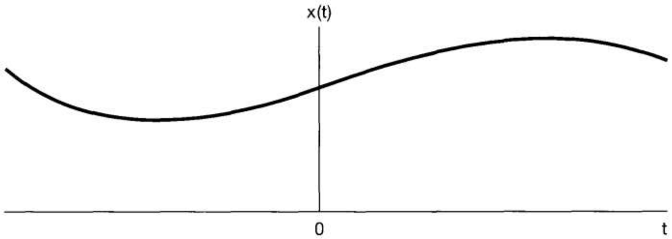
Examples: A telephone or a television signal, monthly sales of a corporation, pressure as a function of altitude, Radio signal

Representation of signal

- Mathematically, a signal is represented as a function of an independent variable t.
  - Thus, a signal is denoted by $S_1(t), S_2(t)$

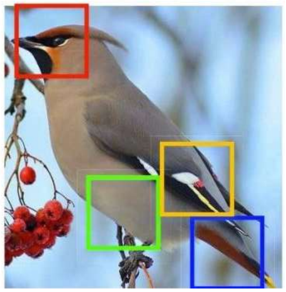
$$s_1(t) = 5t$$

$$s_2(t) = 20t^2$$



- **A signal of two independent variables x and y** that could represent the **two spatial** coordinates in a plane

$$s(x, y) = 3x + 2xy + 10y^2$$



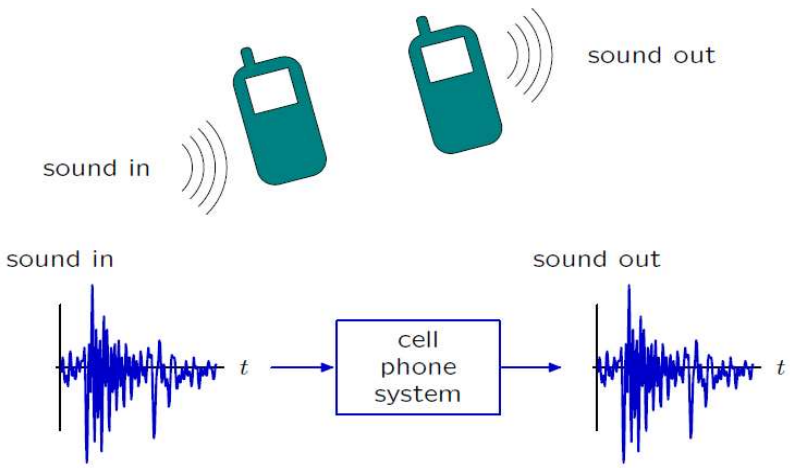
- **Other examples:**

$$x(t) = \cos(2\pi t),\ x(t) = 4\sqrt{t} + t^3,\ x(m, n) = (m + n)^2$$

What is system?

- System is a device or algorithm which process or transforms an input signal into an desired output signal

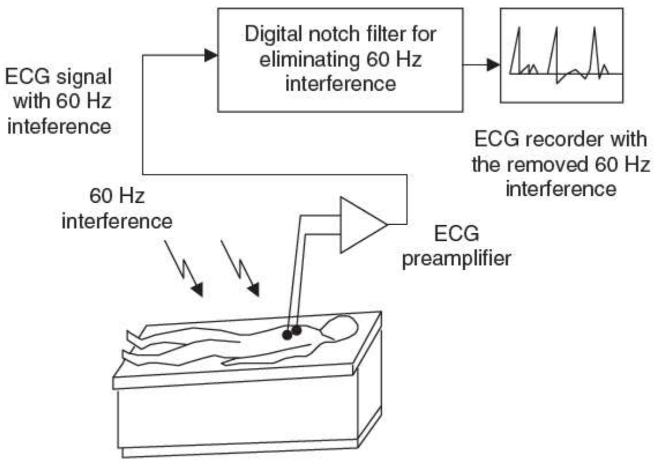
$x(t)$ $h(.)$ $y(t) = h(x(t))$



Examples: $y(t) = -4x(t)$, $\frac{dy(t)}{dt} + 3y(t) = -\frac{dx(t)}{dt} + 6x(t)$, $y(n) - \frac{1}{2}y(n-2) = 3x(n) + x(n-2)$

- radio receiver
- audio amplifier
- modem
- microphone
- cell telephone
- cellular metabolism
- national and global economies

ECG recording system



Seizure detection system with AI

Brain-computer interface (BCI)

Signal and Systems

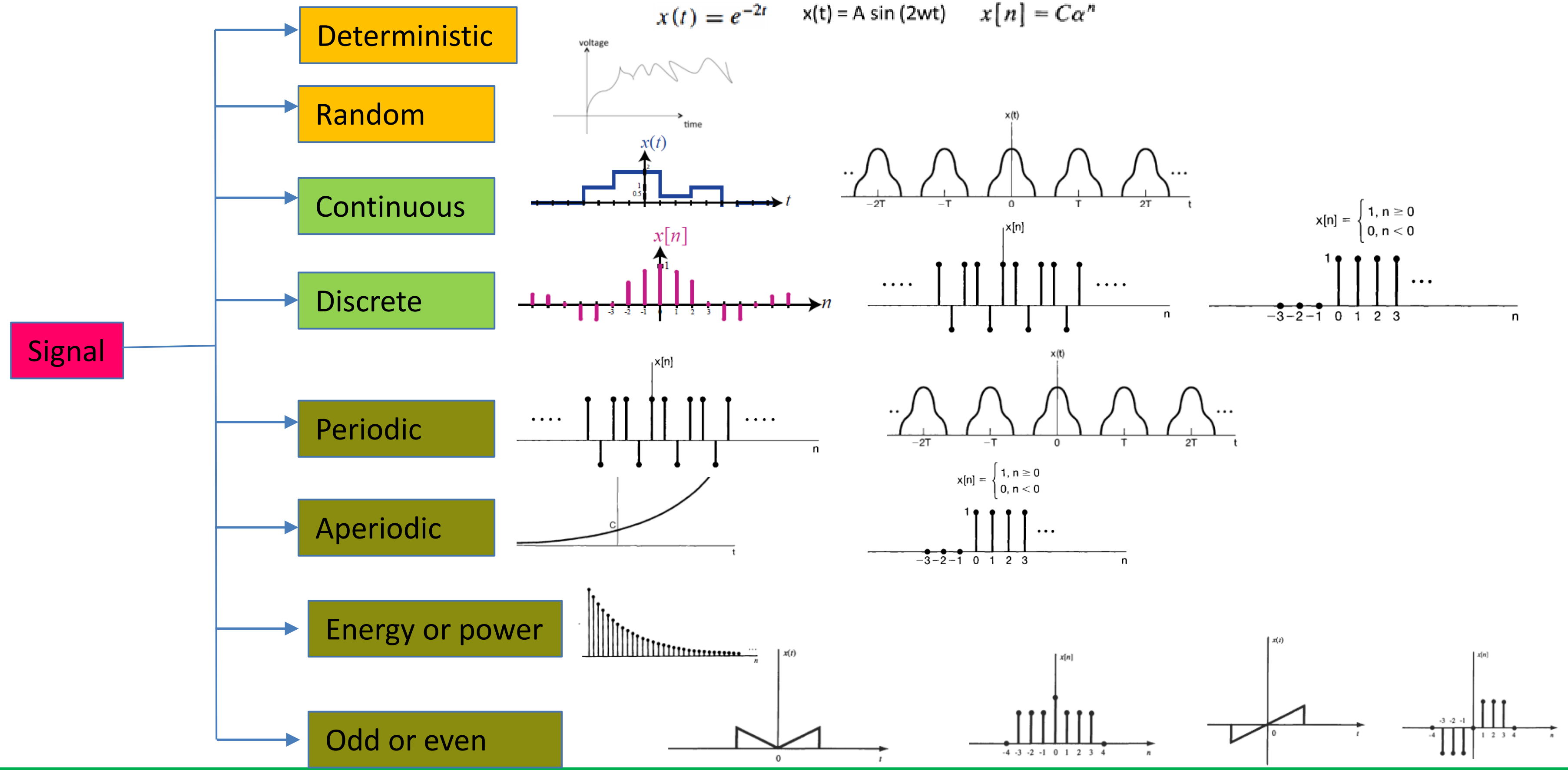
Applications of signal processing

- Speech/audio (speech recognition/synthesis etc.),
- Image/video (enhancement, coding for storage and transmission, robotic vision, animation, etc.),
- Military/space (radar processing, secure communication, missile guidance, sonar processing, etc.),
- Biomedical/health care (scanners, ECG analysis, X-ray analysis, EEG brain mappers, etc.)
- Consumer electronics (cellular/mobile phones, digital television, digital camera, Internet voice/music/video, interactive entertainment systems, etc) and many more

Write brief answers to the following questions

- What has been a valuable or helpful content in a session
- What is the “most unclear point” in a session

Different types of signals



Deterministic Vs Random signals

Deterministic: A signal whose physical description is known completely either in

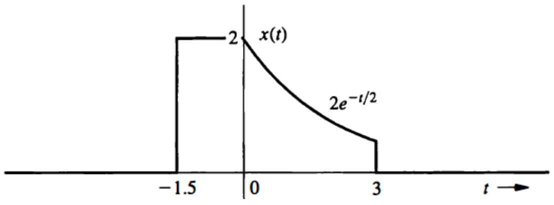
- A mathematical form or a graphical form

e.g.

$$x(t) = e^{-2t}$$

x(t) = A sin (2wt)

$$x[n] = C\alpha^n$$



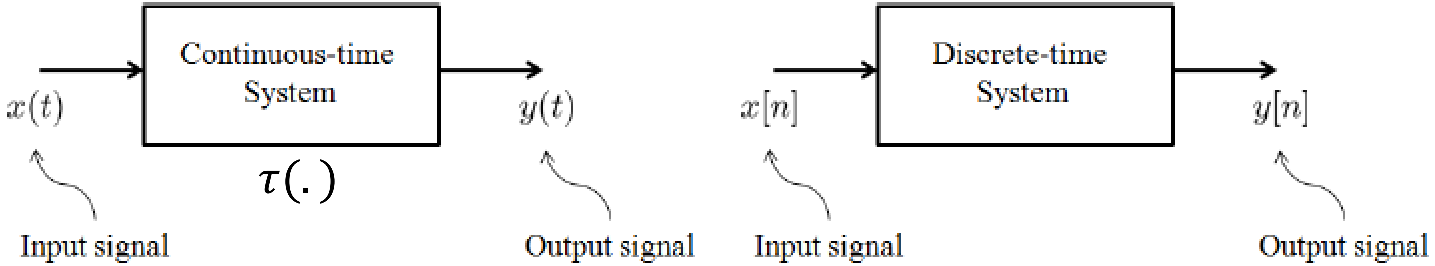
Random:

- If signal has uncertainty with respect to its value at some point of time
- Existence of signals are in random
- Signals are not able explained by a explicit mathematical equation as they are modeled in probabilistic terms

Example: **S**peech, thermal noise (random of movement of electron)

Types of systems

- System is a device or algorithm which process or transforms an input signal into an desired output signal

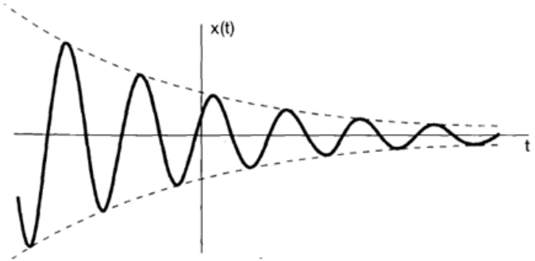


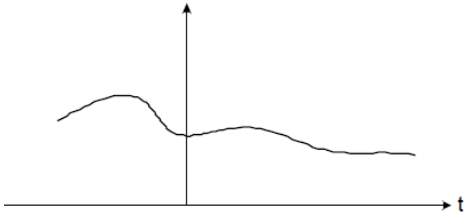
A system is an operator $\mathcal{T}$ which maps input into output:

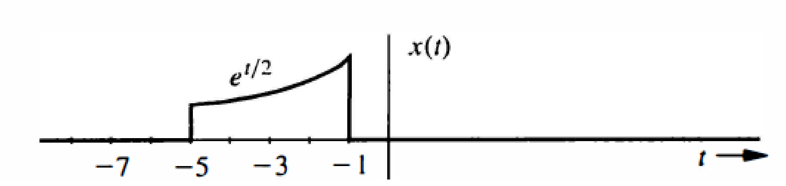
$$y(t) = \mathcal{T}\{x(t)\} \quad \text{or} \quad y[n] = \mathcal{T}\{x[n]\}$$

Continuous-time signals

- Are defined for every value of time and
- They take on values in the continuous interval $(a\,,b\,)$, *where* a *can be* $-\infty$ *and* b can be $+\infty$







$$x(t) = \begin{cases} e^{t/2} & -1 \geq t > -5 \\ 0 & \text{otherwise} \end{cases}$$

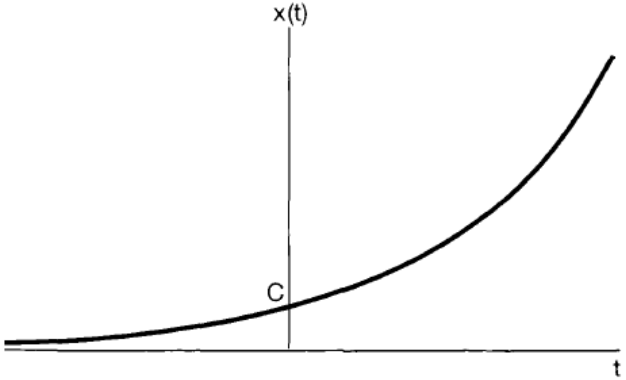
Continuous-time exponential signal

- **Continuous exponential**: $x(t) = Ce^{at}$

Condition-1: if C and a **are real** → real exponential

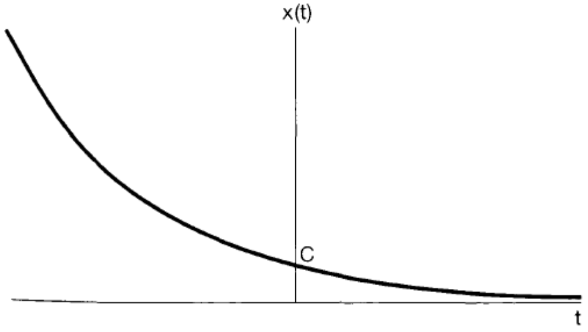
→ Two types of behavior based on value of "a"

→ (a) growing exponential or (b) decaying exponential



$x(t) = Ce^{at}: \quad a > 0$

Growing exponential

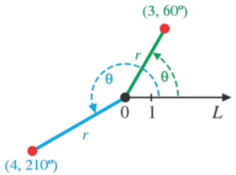


$a < 0.$

Decaying exponential

Cont..

- **Continuous exponential**: $x(t) = Ce^{at}$



Condition-2: if C and a is inform $C = |C|e^{j\theta}$ (in polar form)

$$a = r + j\omega_0.$$

$$x(t) = Ce^{at} \Rightarrow Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

Using Euler's relation, $Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta).$

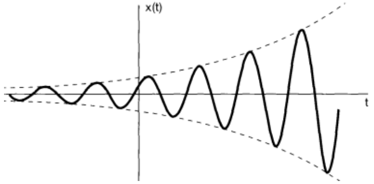
$$e^{i\phi} = \cos\phi + i\sin\phi$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Case 1: For r > 0 $Ce^{at} = \boxed{|C|e^{rt}}\cos(\omega_0 t + \theta) + j\boxed{|C|e^{rt}}\sin(\omega_0 t + \theta).$

Growing sinusoid = "sinusoidal signals" x "Growing exponential"

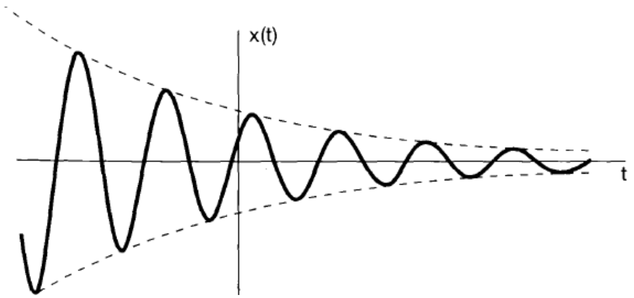


$x(t) = Ce^{rt}\cos(\omega_0 t + \theta),$

Case 2: For r < 0

$$Ce^{at} = \boxed{|C|e^{rt}}\cos(\omega_0 t + \theta) + j\boxed{|C|e^{rt}}\sin(\omega_0 t + \theta).$$

Decaying/damped sinusoid = "sinusoidal signals" x "decaying exponential"



$$x(t) = Ce^{rt}\cos(\omega_0 t + \theta),$$

Example:

viscosity in mechanical systems (brake), automotive suspension (spring) systems,
Exponential growth is unstable since nothing can *grow exponentially* forever

Write brief answers to the following questions

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- What is the "most unclear point" in a session

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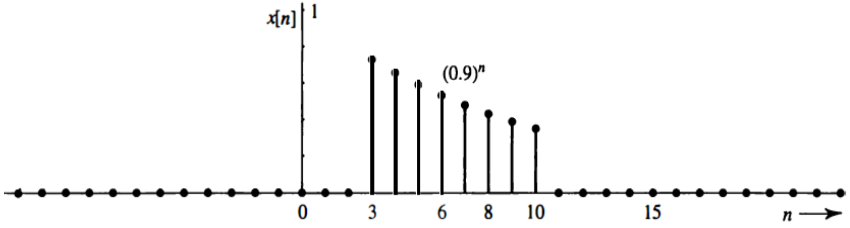
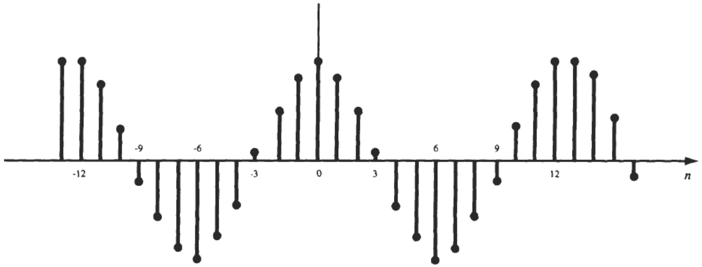
- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

Discrete-time signals

- Are defined **only** at **certain specific values of time**
- These time instants need not be equidistant n **(integer value)**
- In practice, they are usually taken at equally spaced intervals for computational convenience and mathematical tractability



$x[n] = (0.9)^n$ for $3 \leq n \leq 10$



$x[n] = \cos\left(\frac{n}{2}\right)$

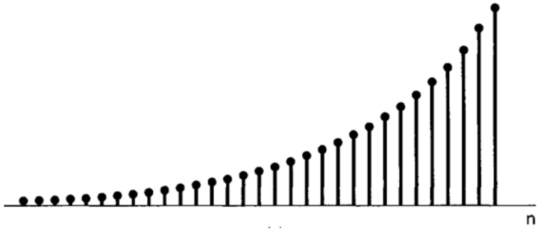
Notation: $x[n]$ is analogous to $x(n)$

Discrete-time exponential signal

- **Discrete exponential signal:** $x[n] = Ce^{\beta n}$

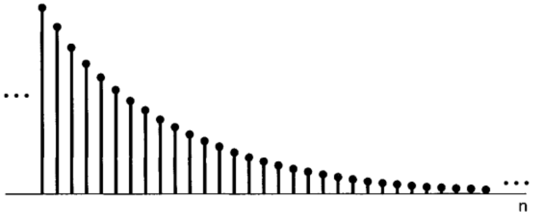
Condition-1: if C and β **are real and $e^{\beta} > 1$**

→ the magnitude of the signal grows exponentially with n



Condition-2: if C and β **are real and $e^{\beta} < 1$**

→ the magnitude of the signal decaying exponentially with n



Condition-3: if β is purely **imaginary and** $C = |C|e^{j\theta}$ →

$$x[n] = Ce^{j\omega_0 n} = |C|e^{j\theta}e^{j\omega_0 n}$$

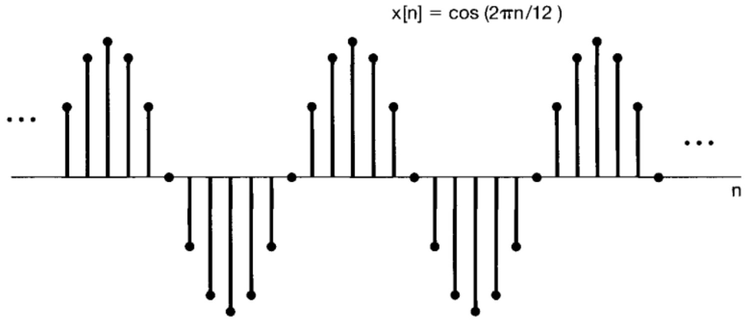
$$= |C|e^{j\theta}[\cos(\omega_0 n) + j\,sin(\omega_0 n)]$$

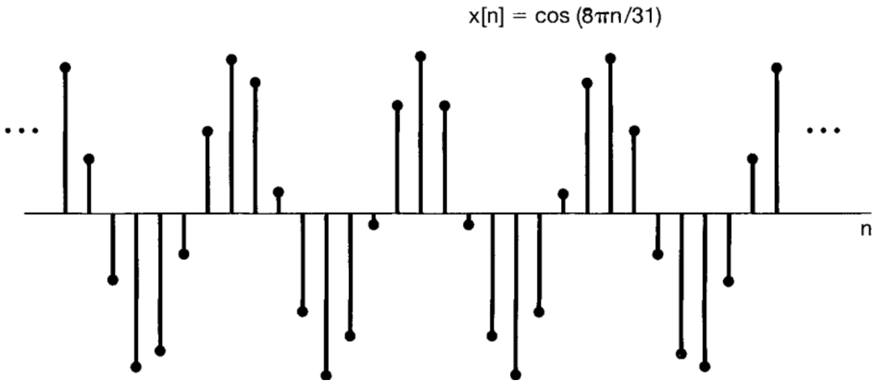
Cont..

Condition-3: if β is purely **imaginary and** $C = |C|e^{j\theta}$ → $x[n] = Ce^{j\omega_0 n} = |C|e^{j\theta}e^{j\omega_0 n}$

$$= |C|e^{j\theta}[\cos(\omega_0 n) + j\,sin(\omega_0 n)]$$

Illustration graphically



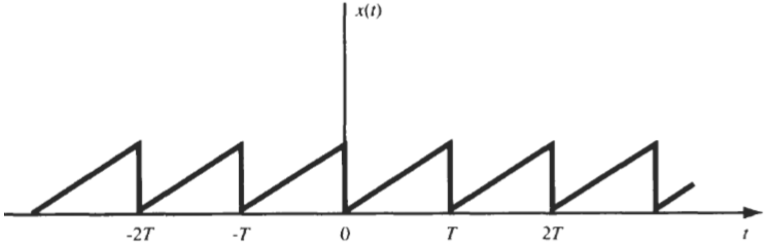


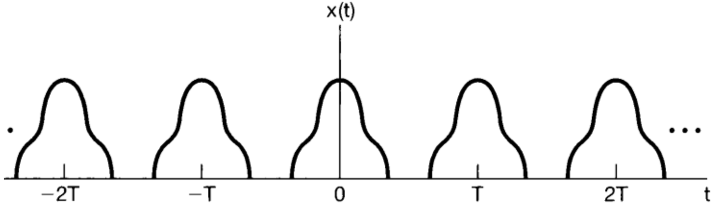
Periodic signal

- **Continuous signal**

A continuous-time signal $x(t)$ is said to be *periodic with period* T if there is a positive nonzero value of T for which

$$x(t+T) = x(t) \qquad \text{all } t$$





It can be also write, $\boldsymbol{x(t+mT) = x(t)}$ for all t and any integer **m**

The *fundamental period* (say, T_0) $of\ x(t)$ → $is\ the$ smallest positive value of $\boldsymbol{T}$

Is the signal is periodic $x(t) = e^{j\omega t}$? If so what is fundamental period of the signal?

Using the relation for period $of\ a\ signal,\ x(t) = x(t+T)$

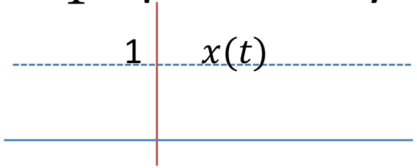
$$\rightarrow e^{j\omega t} = e^{j\omega(t+T)}$$

$$\rightarrow e^{j\omega t} = e^{j\omega t} . e^{j\omega T}$$

$$\rightarrow e^{j\omega T} = 1 \quad = e^{j.0.T}$$

Case 1: $\omega = 0,\ e^{j.0.t} = \cos(0.t) + j\sin(0.t) = 1$

$\rightarrow$ $x(t) = e^{j\omega=0t} = 1$ **periodic** for **any value** of **T**



Case 2: $e^{j\omega T} = 1$

$$\rightarrow e^{j\omega T} = 1 = e^{j2\pi m} \quad (m \text{ integer})$$

Short-cut

$$\rightarrow j\omega T = j2\pi m$$

$$\rightarrow T = \frac{2\pi m}{\omega}$$

$$\omega = 2\pi$$

$$e^{j.2\pi.m} = \cos(2\pi.m) + j\sin(2\pi.m)$$

$$e^{j2\pi m} = \cos(4.\frac{\pi}{2}m + 0) + j\sin(4.\frac{\pi}{2}m + 0)$$

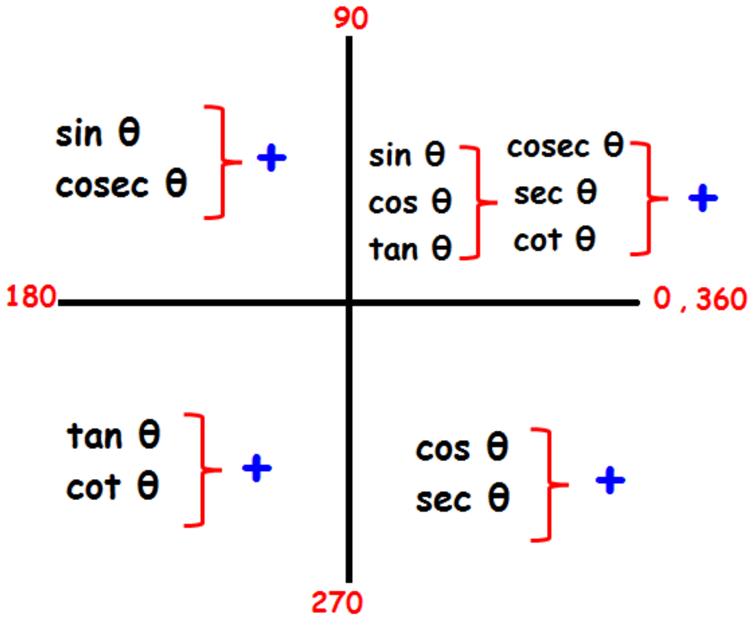
$$e^{j2\pi m} = \cos(0) + j\sin(0) = 1$$

The *fundamental period* (say, T_0) $of\ x(t)$

$\rightarrow$ *is the* smallest positive value of ***T***

$$\boldsymbol{T_0 = \frac{2\pi(m=1)}{|\omega|} = \frac{2\pi}{|\omega|}}$$

Should be rational



$$\boxed{\sin 2A} = \sin(A + A) = \sin A\cos A + \cos A\sin A = 2\sin A\cos A = 2\sin A\sqrt{1-\sin^2 A}$$

$$\boxed{\cos 2A} = \cos(A + A) = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$$

$$\boxed{\tan 2A} = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A\tan A} = \frac{2\tan A}{1 - \tan^2 A}$$

or

$$\boxed{\tan 2A} = \frac{\sin 2A}{\cos 2A} = \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A} = \frac{2\,\frac{\sin A\cos A}{\cos^2 A}}{\frac{\cos^2 A}{\cos^2 A} - \frac{\sin^2 A}{\cos^2 A}} = \frac{2\tan A}{1 - \tan^2 A}$$

Evaluate the fundamental period of signal $x(t) = e^{-j\omega t}$

Hints.

$$x(t) = x(t+T)$$
$$\rightarrow \; e^{-j\omega t} = e^{-j\omega(t+T)}$$
$$\rightarrow \; e^{-j\omega t} = e^{-j\omega t}.e^{-j\omega T}$$
$$\rightarrow \; e^{-j\omega T} = 1$$

1. For ω=0, x(t) = 1 → **periodic for** any value of **T**
2. For ω=2πm

$$e^{-j\omega T} = 1 = e^{j2\pi m}$$
$$\rightarrow \; -j\omega T = j2\pi m$$
$$\rightarrow T = \frac{2\pi m}{-\omega}$$

Short-cut

As per the definition, the *fundamental period* (say, T_0) $of\ x(t)$

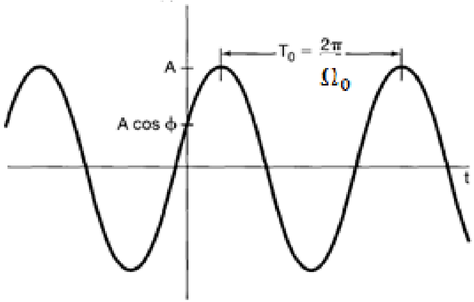
→ $is\ the$ smallest positive value of $\boldsymbol{T}$

$$T_0 = \frac{2\pi(m=1)}{|-\omega|} = \frac{2\pi}{|\omega|}$$

Periodic Continuous sinusoidal signal

- Sinusoidal signal is **closely related** toType equation here. the periodic complex exponential

$$x(t) = A\cos(\Omega_0 t + \emptyset)$$



where ***A*** is the amplitude (real)

Ω_0 is the frequency in radians per second

$\emptyset$ is the phase angle in radians.

T_0 the fundamental period of x(t)

- The reciprocal of the fundamental period T_0 is called the fundamental frequency F_0 (Hz) - **cycle/second**

$$F_0 = \frac{1}{T_0} \text{ hertz (Hz)}$$

A half circle there are π radians

- The relation between angular frequency (Ω_0) and F_0

$$\Omega_0 = 2\pi F_0$$

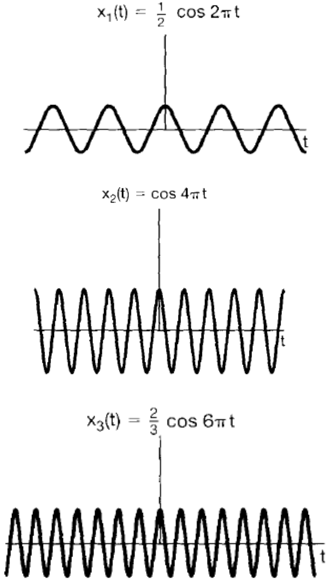
$$\boxed{\frac{2\pi\, radian}{cycle} \times \frac{cycle}{second} = \frac{radian}{second}}$$

Harmonically Related Complex Exponentials

Let's consider sinusoidal signal $x(t) = e^{j\omega_0 t}$

The fundamental period of the signal = ? $T = \frac{2\pi}{\omega_0}$

Fundamental angular frequency ω_0



Let's consider a signal,

$$\emptyset_k(t) = e^{jk\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$k = 0 \rightarrow \emptyset_0(t) = e^{j.0.\omega_0 t}$$
$$k = 1 \rightarrow \emptyset_1(t) = e^{j.1.\omega_0 t}$$
$$k = 2 \rightarrow \emptyset_2(t) = e^{j.2.\omega_0 t}$$

Each of these signals has a fundamental frequency that is a **multiple of ω_0** → **called harmonically related**

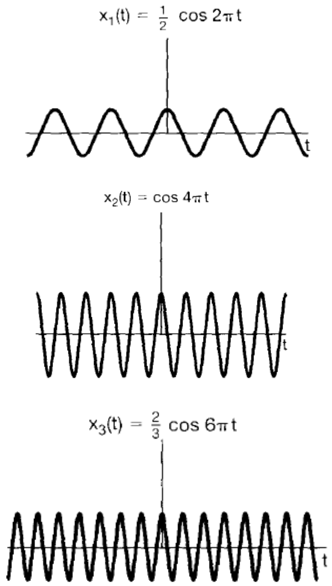
$|k| \geq 2$, the fundamental period of $\emptyset_k(t)$ is fraction of $\boldsymbol{T}$

Harmonically Related Complex Exponentials

Let's consider sinusoidal signal $x(t) = e^{j\omega_0 t}$

The fundamental period of the signal = ? $T = \dfrac{2\pi}{\omega_0}$

Fundamental angular frequency ω_0



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Each of these signals has a fundamental frequency that is a **multiple of ω_0** → **called harmonically related**

$|k| \geq 2$, the fundamental period of $\emptyset_k(t)$ is fraction of $\boldsymbol{T}$

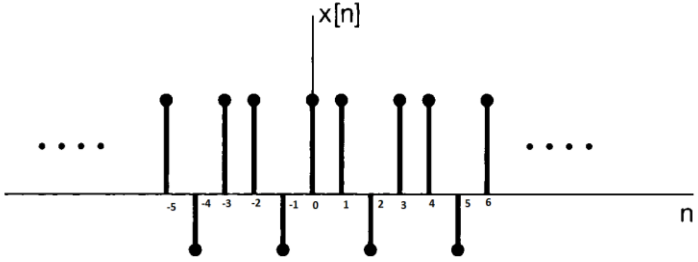
Periodic discrete signal

- **Discrete signal:** a discrete time signal x(n) *is* periodic *with period N*
  - *where N is a positive integer, if it is unchanged* by a time shift of ***N***, *i.e.*

$$x(n) = x(n+N) \quad \text{for all values of } n$$

- The **fundamental period** N_0 is **the smallest positive value of *N*** for which it holds the above relation

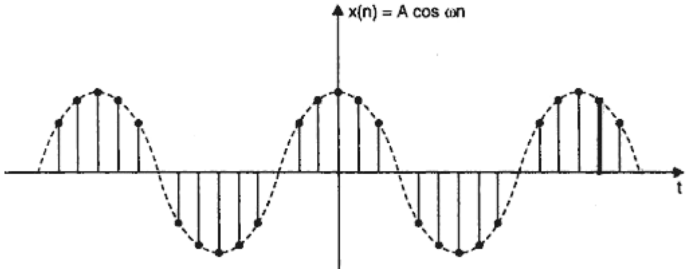
What is the fundamental period of the below given signal?



The fundamental period $\boldsymbol{N_0 = 3}$

Discrete periodic sinusoidal signal

- A discrete-time sinusoidal signal is represented as,



$$x(n) = A\cos(\omega n + \emptyset), \qquad -\infty < n < \infty$$
$$= A\cos(2\pi f n + \emptyset)$$

A is the aplitute
ω is the frequency in radians per sample
∅ is the phase in radians

For the periodicity, $x(n) = x(n + N)$

$$= \cos(2\pi f(n + N) + \emptyset)$$

$$= \cos(2\pi f n + 2\pi f N + \emptyset)$$

$$= \cos[2\pi f N + (2\pi f n + \emptyset)]$$ If **2πfN** is multiple of $2\pi k$, **k** is an integer

$$= \cos(2\pi f n + \emptyset)$$

$$2\pi f N = 2\pi k$$
$$\rightarrow f = \frac{k}{N}$$

Should be rational number

- Determine whether the following signal are periodic? If so, what is the value of period?

$$(a)\cos(\frac{\pi n}{2})$$

$$(b)\sin(\frac{\pi n}{8})$$

Short-cut:

$$\frac{\pi N}{2} = 2\pi k$$

$$\rightarrow N = 2 \times 2k$$

N_0 = 2x2 =4 (k=1)

(a)

$$x(n) = x(n+N)$$

N *is a* positive integer

$$\rightarrow \cos(\frac{\pi n}{2}) = \cos(\frac{\pi}{2}(n+N)) = \cos(\frac{\pi}{2}n)\cos(\frac{\pi}{2}N) - \sin(\frac{\pi n}{2})\sin(\frac{\pi N}{2}) = \cos(\frac{\pi}{2}n)$$

If $\cos(\pi N/2) = 1$, i.e. $(\pi N/2)$ **should have** $2\pi k$ **form for integer value of** k

$$\frac{\pi N}{2} = 2\pi k$$

$$\rightarrow N = 2 \times 2k$$

Fundamental period N_0
= 2x2(k=1)=4

- The **fundamental period** N_0 is **the smallest positive value of** N for which it holds
$$x(n) = x(n+N)$$

- Determine whether the following signal are periodic? If so, what is the value of period?

$$x(n) = e^{j(2\pi/3)n} + e^{j(3\pi/4)n}$$

We can write, $x(n) = x_1(n) + x_2(n)$

For the fundamental period of $x_1(n)$, it should satisfy

$x(n) = x(n+N)$ for all values of n *where* N *is a positive integer*

Now, we can get $x(n) = x(n+N)$

$$\rightarrow e^{j\left(\frac{2\pi}{3}\right)n} = e^{\frac{j2\pi}{3}(n+N)} = e^{j\left(\frac{2\pi}{3}\right)n} . e^{j\left(\frac{2\pi}{3}\right)N}$$

$$\rightarrow e^{j\left(\frac{2\pi}{3}\right)N} = 1$$

$\rightarrow e^{j\left(\frac{2\pi}{3}\right)N} = 1 = e^{j2\pi m}$ (m integer)

$$\rightarrow j\left(\frac{2\pi}{3}\right)N = j2\pi m$$

$$\rightarrow N = \frac{2\pi m}{\left(\frac{2\pi}{3}\right)} = 2\pi m \times \left(\frac{3}{2\pi}\right) = 3m = 3 \ (m = 1)$$

$$\omega = 2\pi m$$
$$e^{j.2\pi.m} = \cos(2\pi.m) + j\sin(2\pi.m)$$
$$e^{j2\pi m} = \cos(4.\frac{\pi}{2}m + 0) + j\sin(4.\frac{\pi}{2}m + 0)$$
$$e^{j2\pi m} = \cos(0) + j\sin(0) = 1$$

The fundamental period of $x_1(n)$ is **3**

Similarly, for the $x_2(n) = e^{j\left(\frac{3\pi}{4}\right)n}$

$$x(n) = x(n+N)$$

$$\rightarrow e^{j\left(\frac{3\pi}{4}\right)n} = e^{\frac{j3\pi}{4}(n+N)} = e^{j\left(\frac{3\pi}{4}\right)n} . e^{j\left(\frac{3\pi}{4}\right)N}$$

$$\rightarrow e^{j\left(\frac{3\pi}{4}\right)N} = 1$$

$$\rightarrow e^{j\left(\frac{3\pi}{4}\right)N} = 1 = e^{j2\pi m}$$

$\rightarrow j\left(\frac{3\pi}{4}\right)N = j2\pi m$ for m integer

$$\rightarrow N = \frac{2\pi m}{\left(\frac{3\pi}{4}\right)} = 2\pi m \times \left(\frac{4}{3\pi}\right) = 8m/3$$

$$= 8\ (m = 3)$$

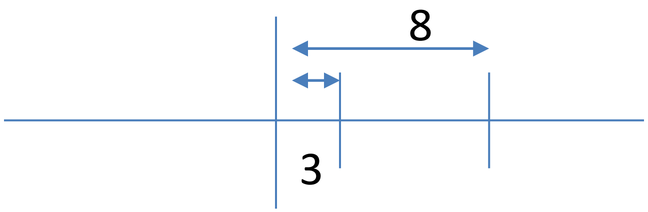
The fundamental period of $x_2(n)$ is 8

The fundamental period of $x_1(n)$ is **3**

Shot-cut:

$$\rightarrow j\left(\frac{3\pi}{4}\right)N = j2\pi m$$

$$\rightarrow N = \frac{2\pi m}{\left(\frac{3\pi}{4}\right)} = 2\pi m \times \left(\frac{4}{3\pi}\right) = 8m/3 = 8\ (m = 3)$$



What is the period of x[n]? LCM(3,8) = 24

What is the Least Common Multiple of 4 and 6?

Multiples of 4 are:

4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, ...

and the multiples of 6 are:

6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, ...

Common multiples of 4 and 6 are simply the numbers that are in both lists:

12, 24, 36, 48, 60, 72, ...

So, from this list of the first few common multiples of the numbers 4 and 6,

their *least common multiple* is 12.

Determine whether signal is periodic or not?

- $x(n) = cos(\frac{\pi}{8}n^2)$

Determine whether the following signal is periodic?

$$x(t) = 2sin\left(\frac{2}{3}\right)t + 3\ cos\left(\frac{2\pi}{5}\right)t$$

We express $x(t) = x_1(t) + x_2(t)$

For $x_1(t) = x_1(t+T)$

$\rightarrow 2\sin\left(\frac{2}{3}\right)t = 2\sin(\frac{2}{3})(t+T)$

With short-cut

$\rightarrow \left(\frac{2}{3}\right)T = 2\pi m,$ (m integer)

$\rightarrow T = 2\pi m \times \left(\frac{3}{2}\right) = 3\pi m$

$\rightarrow T_0 = 3\pi$ (for smallest positive value of T, m = 1)

The fundamental period of $x_1(t) = 2\sin\left(\frac{2}{3}\right)t = 3\pi$

Similarly, for $x_2(t) = x_2(t+T)$

$\rightarrow \left(\frac{2\pi}{5}\right)T = 2\pi m,$ (m integer)

$\rightarrow T = 2\pi m \times \left(\frac{5}{2\pi}\right) = 5m$

$\rightarrow T_0 = 5$ (for smallest positive value of T, m = 1)

The fundamental period of $x_2(t) = 2\cos\left(\frac{2\pi}{5}\right)t = 5$

The fundamental period of $x(t)$ *is*

LCM$(3\pi, 5) = ?$ *(does not exist)*

$\rightarrow$ *Signal Aperiodic*

Determine whether the following signal is periodic?

$$y(t) = 3\ sin\ (t) + 5\ cos\ \left(\frac{4}{3}\right) t$$

The period of $3\ sin\ (t)$ is

$\rightarrow T = 2\pi m,$ (m integer)

$\rightarrow\ T_0 = 2\pi$ (for smallest positive value of T, m = 1)

Method-1:

The period of $5\ cos\ (4/3)t$ is

$\rightarrow (\frac{4}{3})T = 2\pi m,$ (m integer)

$\rightarrow T = 2\pi m \times \left(\frac{3}{4}\right) = (3\pi m/2)$

$\rightarrow\ T_0 = 3\pi/2$ (for smallest positive value of T, m = 1)

The period of $y(t)$

LCM$(2\pi, 3\pi/2)$

$= 6\pi$

Method-2: The ratio of fundamental periods of both signal $= \frac{T_1}{T_2} = \frac{2\pi}{\frac{3\pi}{2}} = 2\pi \times \frac{2}{3\pi} = \frac{4}{3}$, a rational number

$\rightarrow$ y(t)is periodic

A rational number in the form $\frac{p}{q}$; where p and q are integers and q ≠0

e.g. $\frac{3}{5}, \frac{-3}{10}, \frac{11}{-15}$

e.g. not rational $\frac{0}{0}$

Difference between continuous and discrete exponential

In discrete: **"n"** is always integer value

$$e^{j(\omega_0 \pm 2\pi)n} = e^{j(\omega_0 \pm 4\pi)n} = e^{j\omega_0 n}.$$

the signal with ω_0 is identical to the signals with frequencies $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on

In continuous, the period say **"T"** can be a **rational number/float value**

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} . e^{jw_0 T} = e^{j\omega_0 t}$$
$$\rightarrow e^{j\omega_0 T} = 1 = e^{j(2\pi k)}$$
$$\rightarrow j\omega_0 T = j2\pi k$$
$$\rightarrow T = \frac{2\pi k}{\omega_0}$$

k is an **integer**

Some cases "T" **may not** rational -> signal can not be periodic

Representation of discrete & continuous signals

1. Functional representation, such as

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases}$$

2. Tabular representation, such as

n	$\cdots$	-2	-1	0	1	2	3	4	5	$\ldots$
$x(n)$	$\cdots$	0	0	0	1	4	1	0	0	$\ldots$

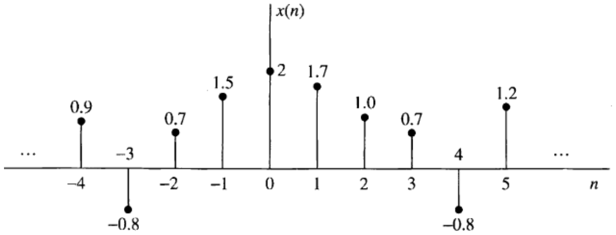
3. Sequence representation

Signal or sequence with the time origin (**n =0**) is represented by ↑ as

$$x(n) = \{\ldots 0, \underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\}$$

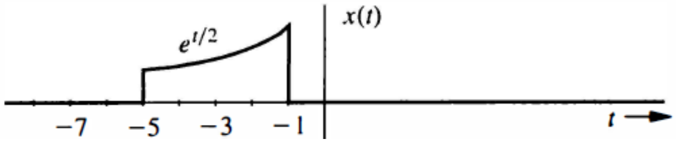
A sequence x(n), which **is zero** for **n < 0**, $x(n) = \{\underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\}$

4. Graphical



Continuous

$$x(t) = \begin{cases} e^{t/2} & -1 \geq t > -5 \\ 0 & \text{otherwise} \end{cases}$$

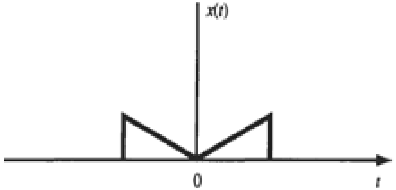


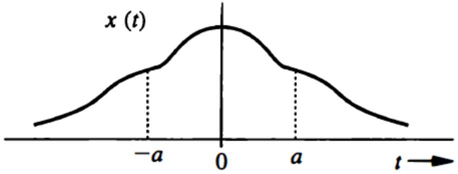
Even and odd signals

Continuous:

- A **continuous** signal **x(t)** is said to be an **<u>even</u>** signal if satisfy

$$x(-t) = x(t)$$

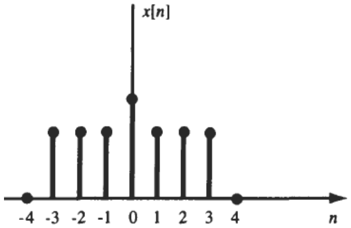




Discrete:

- A **discrete** signal **x[n]/x(n)** is said to be an **<u>even</u>** signal if satisfy

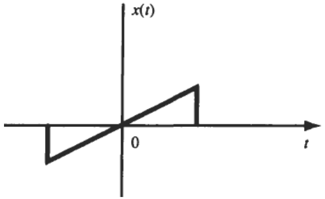
$$x[-n] = x[n]$$

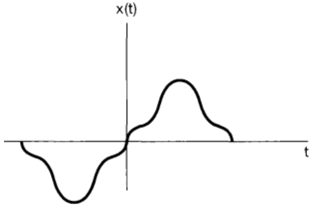


$$x[n] = \begin{cases} \frac{1}{2}, & n < 0 \\ 1, & n = 0 \\ \frac{1}{2}, & n > 0 \end{cases}$$

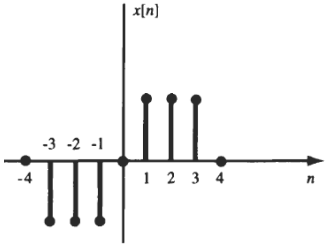
<u>Odd</u>

$$x(-t) = -x(t)$$

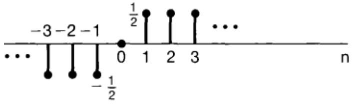




$$x[-n] = -x[n]$$

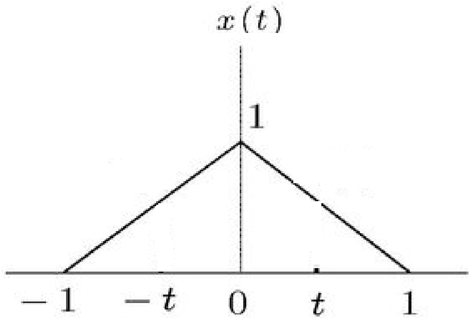


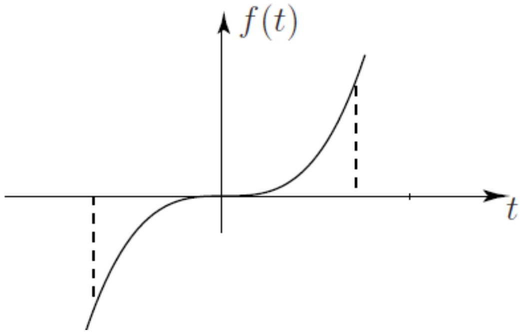
$$x[n] = \begin{cases} -\frac{1}{2}, & n < 0 \\ 0, & n = 0 \\ \frac{1}{2}, & n > 0 \end{cases}$$



Verify signals are even or odd?

- $Sin(wt)$
- $x(t) = Cos(wt)$
- $Sin(wn)$
- $Cos(wn)$
- $x(t) = |t|$
- $x(t) = t^4$





Let $x(t) = Sin(wt)$,

$$x(-t) = Sin(w.-t) = Sin(-wt) = -Sin(wt) = -x(t)$$

$x(t) \neq x(-t)$ >> **not even signal**

Representation of signal as even and odd parts

If signal is Even, $x(n) + x(-n) = ?$ $\rightarrow \boldsymbol{x(n) + x(n) = 2x(n)}$ [as signal is even, x(-n) = x(n)]

$$x(n) + x(-n) = 2\,x(n)$$

$$\rightarrow x(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$Even[x(n)] = x_e[n] = \frac{1}{2}[x(n) + x(-n)] \;\ldots\ldots \quad \ldots \; (\mathbf{1}) \quad [\text{also } x_e(n) = x_e(-n)]$$

If signal is Odd, $x(n) - x(-n) = ?$ $\rightarrow \boldsymbol{x(n) - (-x(n)) = 2\,x(n)}$ [as signal is odd, $\boldsymbol{x(-n) = -x(n)}$]

$$x(n) - x(-n) = 2\,x(n)$$

$$\boldsymbol{Odd[x(n)] = x_o(n) = \frac{1}{2}[x(n) - x(-n)] \;\ldots\; \ldots.\; \ldots.(2)}$$

Using Eq. (1) & (2)

$$\begin{aligned}
&\text{Even }[x(n)] + \text{Odd }[x(n)] \\
&= \frac{1}{2}[x(n) + x(-n)] + \frac{1}{2}[x(n) - x(-n)] \\
&= \frac{1}{2}[x(n) + \cancel{x(-n)}] + \frac{1}{2}[x(n) - \cancel{x(-n)}] \\
&= x(n)
\end{aligned}$$

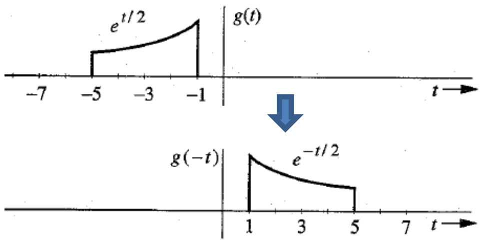
$$x(t) = \underbrace{\tfrac{1}{2}[x(t) + x(-t)]}_{\text{even}} + \underbrace{\tfrac{1}{2}[x(t) - x(-t)]}_{\text{odd}}$$

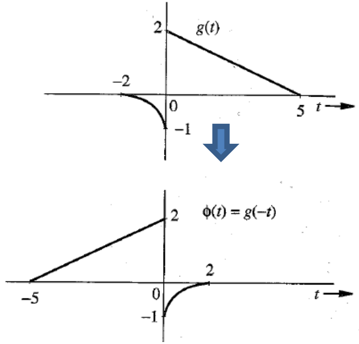
$$= x_e(t) + x_o(t)$$

Reflection (Time Reversal/Inversion)/folding

- Generate an **mirror image of signal** with respect to Y-axis (at $t = 0$)

$$\phi(t) = g(-t)$$

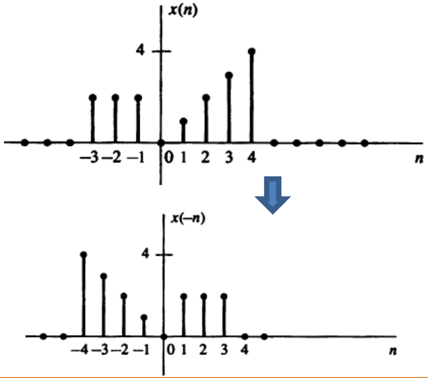




Mathematically,

$$g(-t)\ at\ t = -1, \rightarrow g(1)$$
$$g(-t)\ at\ t = 0\ \rightarrow g(0)$$
$$g(-t)\ at\ t = 1\ \rightarrow g(-1)$$

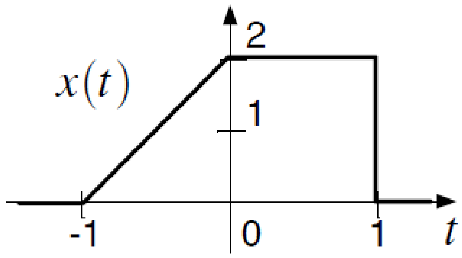
For discrete: $\emptyset(n) = x(-n)$ (at time $n = 0$ with respect to the Y- axis)

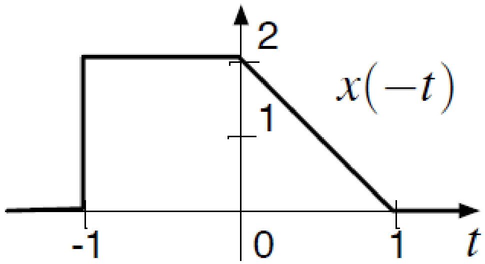


Mathematically,

$$x(-n)\ at\ n = -1\ \rightarrow x(1)$$
$$x(-n)\ at\ n = 0\ \rightarrow x(0)$$
$$x(-n)\ at\ n = 1\ \rightarrow x(-1)$$

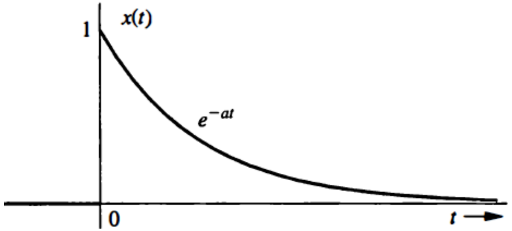
Draw the signal x(-t)





Expressing this function as a sum of the even and odd components $x_e(t)$ and $x_o(t)$

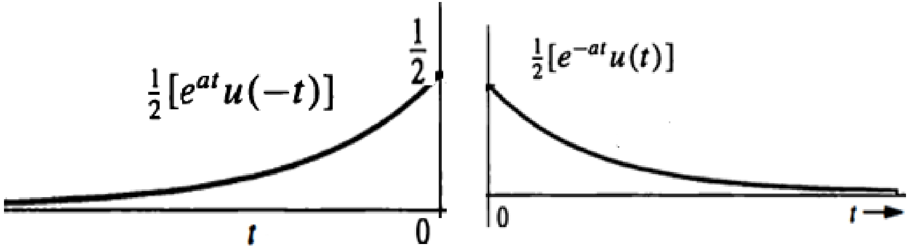
$$x(t) = e^{-at}u(t)$$

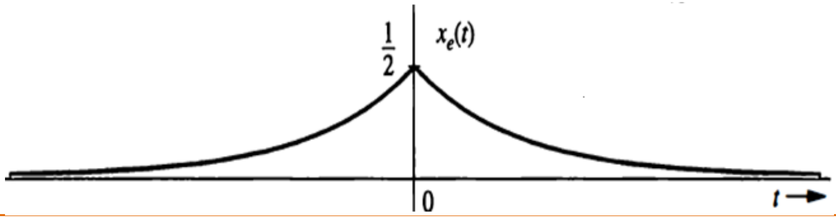


We know that x(t) $=$ $x_e(t)$ $+$ $x_o(t)$

We have to determine the $x_e(t)$ and $x_o(t)$

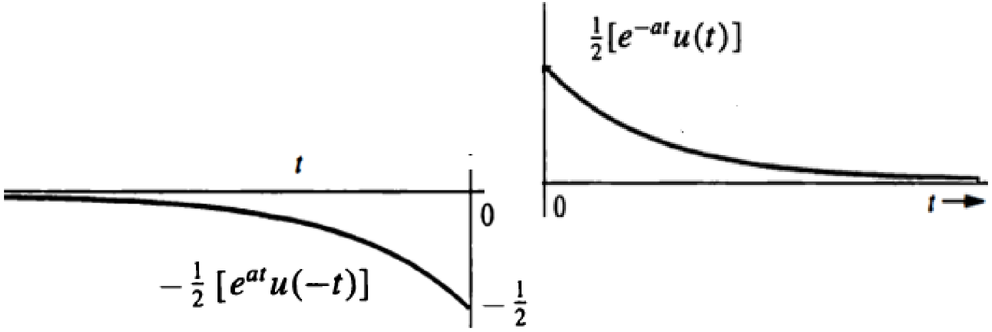
$$x_e(t) = \tfrac{1}{2}[x(t) + x(-t)] = \tfrac{1}{2}[e^{-at}u(t) + e^{at}u(-t)]$$

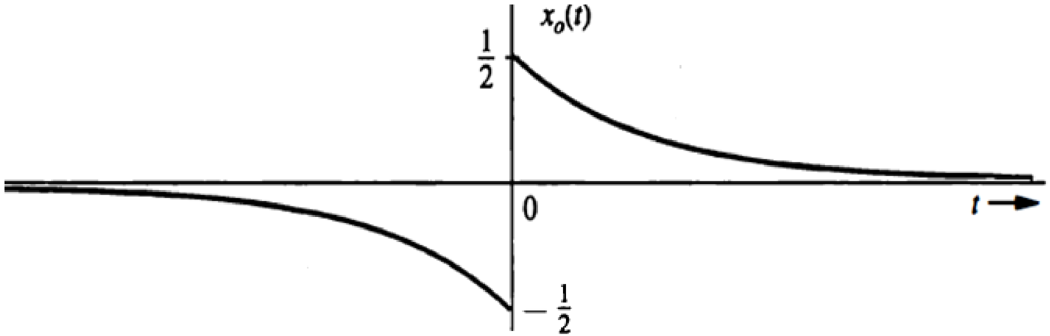




$$x_o(t) = \tfrac{1}{2}[x(t) - x(-t)] = \tfrac{1}{2}[e^{-at}u(t) - e^{at}u(-t)]$$

$$= \tfrac{1}{2}[e^{-at}u(t)] - \tfrac{1}{2}[e^{at}u(-t)]$$





Find the even and odd components of e^{jt}.

We know that x(t) = $x_e(t) + x_o(t)$

We have to determine the $x_e(t)$ *and* $x_o(t)$

Even:

$$x_e(t) = \tfrac{1}{2}[x(t) + x(-t)]$$

$$x_e(t) = \tfrac{1}{2}[e^{jt} + e^{-jt}] = \cos t$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Odd:

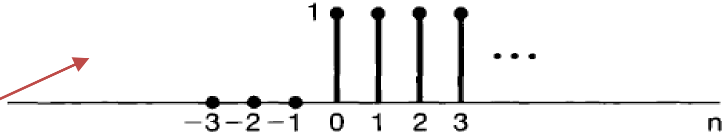
$$x_o(t) = \tfrac{1}{2}[x(t) - x(-t)]$$

$$x_o(t) = \tfrac{1}{2}[e^{jt} - e^{-jt}] = j \sin t$$

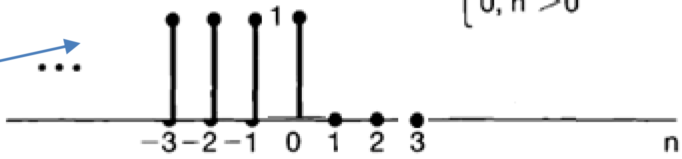
Sketch and label the even and odd components of the signals

$$x[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$x[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$x[-n] = \begin{cases} 1, & n <= 0 \\ 0, & n > 0 \end{cases}$$

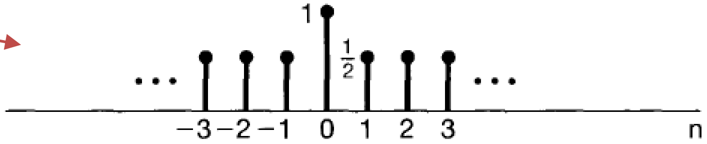


We know that x(n) = $x_e(n) + x_o(n)$

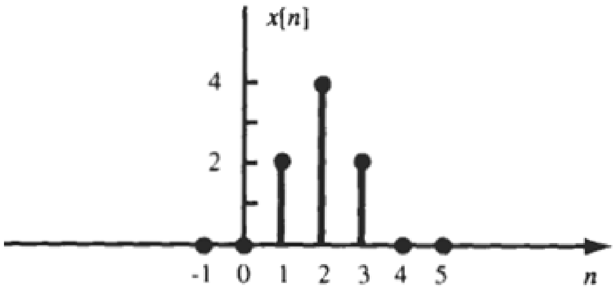
We have to determine the $x_e(n)$ and $x_o(n)$

$$x_e[n] = \frac{1}{2}[x(n) + x(-n)]$$

$$\mathcal{E}v\{x[n]\} = \begin{cases} \frac{1}{2}, & n < 0 \\ 1, & n = 0 \\ \frac{1}{2}, & n > 0 \end{cases}$$



Sketch and label the even and odd components of the signals

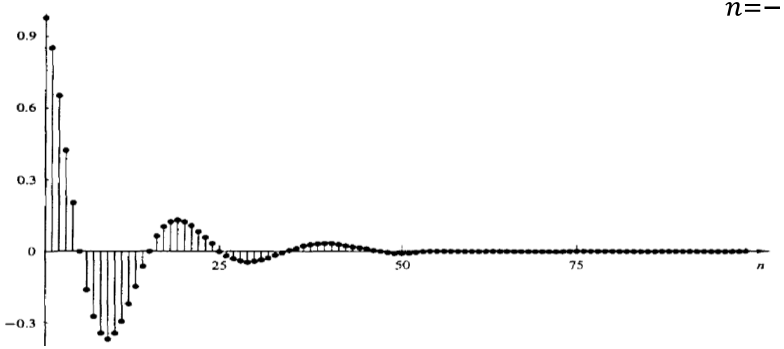


Energy signal

- The energy (E) of a signal can be **finite or infinite**.
- A **signa**l with **finite energy** is an energy signal (if E *is finite* $(0 < E < \infty)$ *, the signal is called an energy signal)*

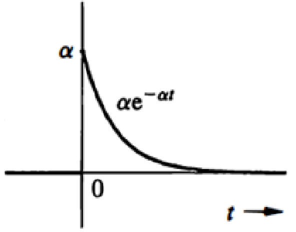
Discrete signal: The energy of a signal $x(n)$ is defined as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$



Continuous signal: The energy of continuous-time signal x(t) is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$



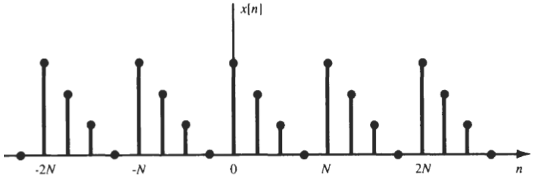
Power signal

- Signal with **finite and nonzero power** is a power signal.
- Many signals that possess infinite energy, have a finite average power
- The signal power is the time average of its energy

Discrete signal: The average power of a discrete-time signal $x(n)$ *is defined as*

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

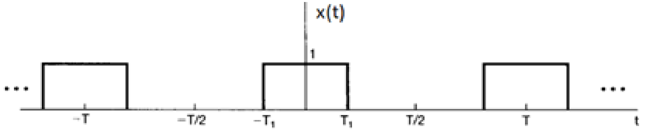
$$= \lim_{N\to\infty} \frac{1}{2N+1} E$$



N is the fundamental period of signal

Continuous signal: The average power of a continuous-time signal $x(t)$ *is* defined as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$



T is the fundamental period of the signal

Check whether the following signals are energy/power signals?

$(a)\quad x(t) = \begin{cases} A\,; 0 < t < T_0 \\ 0; \quad otherwise \end{cases}$ $\qquad (b)\quad x[n] = 2e^{j3n}$ $\qquad (c)\quad x(t) = \text{A}\cos(\omega_0 t + \theta)$

Energy of the signal:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$E = \int_{-\infty}^{0} |x(t)|^2\, dt + \int_{0}^{T_0} |x(t)|^2 dt + \int_{T_0}^{\infty} |x(t)|^2 dt$$

$$= 0 + \int_{0}^{T_0} |A|^2 dt + 0 = A^2 [t]_0^{T_0}$$

$$= A^2 [T_0 - 0]$$

$$= A^2 T_0$$

Energy has finite value

Let's calculate Power of the signal:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

$$P = \lim_{T\to\infty} \frac{1}{T} \left[\int_{-T/2}^{0} |x(t)|^2\, dt + \int_{0}^{T_0} |x(t)|^2\, dt + \int_{T_0}^{T/2} |x(t)|^2\, dt \right]$$

$$= \lim_{T\to\infty} \frac{1}{T} \left[\int_{-T/2}^{0} |0|^2\, dt + \int_{0}^{T_0} |A|^2\, dt + \int_{T_0}^{T/2} |0|^2\, dt \right]$$

$$= \lim_{T\to\infty} \frac{1}{T} \left[\int_{0}^{T_0} |A|^2\, dt = A^2 T_0 \right]$$

$$= \lim_{T\to\infty} \frac{1}{T} A^2\, T_0 = 0$$

Power =0

Cont..

$(b)\quad x[n] = 2e^{j3n}$

Power of the signal:

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|2e^{j3n}\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |2|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}\ 4 \sum_{n=-N}^{N} 1$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}\ 4\,(2N+1) = 4 < \infty$$

$$\left|e^{j3n}\right| = |\cos 3\,n + j\sin 3\,n|$$
$$= \sqrt{\cos^2 3\,n + \sin^2 3n}$$
$$= 1$$

Signal has finite power → power signal

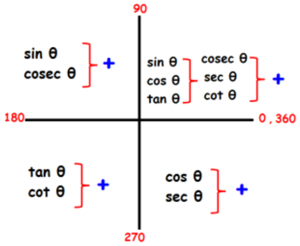
Cont..

$(c)\quad x(t) = \text{A}\cos(\omega_0 t + \theta)$

The signal is periodic. So it is a power signal

Let's calculate the fundamental period for the justification

By definition of periodic of a signal, it should satisfy: $x(t) = x(t+T)$



$$A\cos(\omega_0 t + \theta) = A\cos(\omega_0(t+T) + \theta)$$
$$\rightarrow A\cos(\omega_0 t + \theta) = A\cos((\omega_0 t + \theta) + \omega_0 T)$$
$$= A[\cos(\omega_0 t + \theta).\cos(\omega_0 T) - \sin(\omega_0 t + \theta).\sin(\omega_0 T)]$$

If $(\omega_0 T)$ is in the form of $(2\pi k)$ for a integer value **of k**
$\rightarrow \cos(2\pi k) = 1 \; and \; \sin(2\pi k) = 0$

$$= A\cos(\omega_0 t + \theta)$$

$$\omega_0 T = 2\pi k$$
$$T = \frac{2\pi k}{\omega_0}$$

Fundamental period is the smallest value of **T** =(2π.k=1)/ω_0

$$i.e.\;\; T_0 = \frac{2\pi}{|\omega_0|}$$

Cont..

(c) $x(t) = A\cos(\omega_0 t + \theta)$ $\quad\quad T = \dfrac{2\pi}{|\omega_0|}$

Using the definition power of continuous-time signal:

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt \quad = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |A\cos(\omega_0 t + \theta)|^2\, dt$$

$$[1 + \cos 2A = 2\cos^2 A]$$

$$\int \cos A = \sin A$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A^2 cos^2(\omega_0 t + \theta)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \frac{A^2}{2}[1 + cos2\,(\omega_0 t + \theta)] \quad dt$$

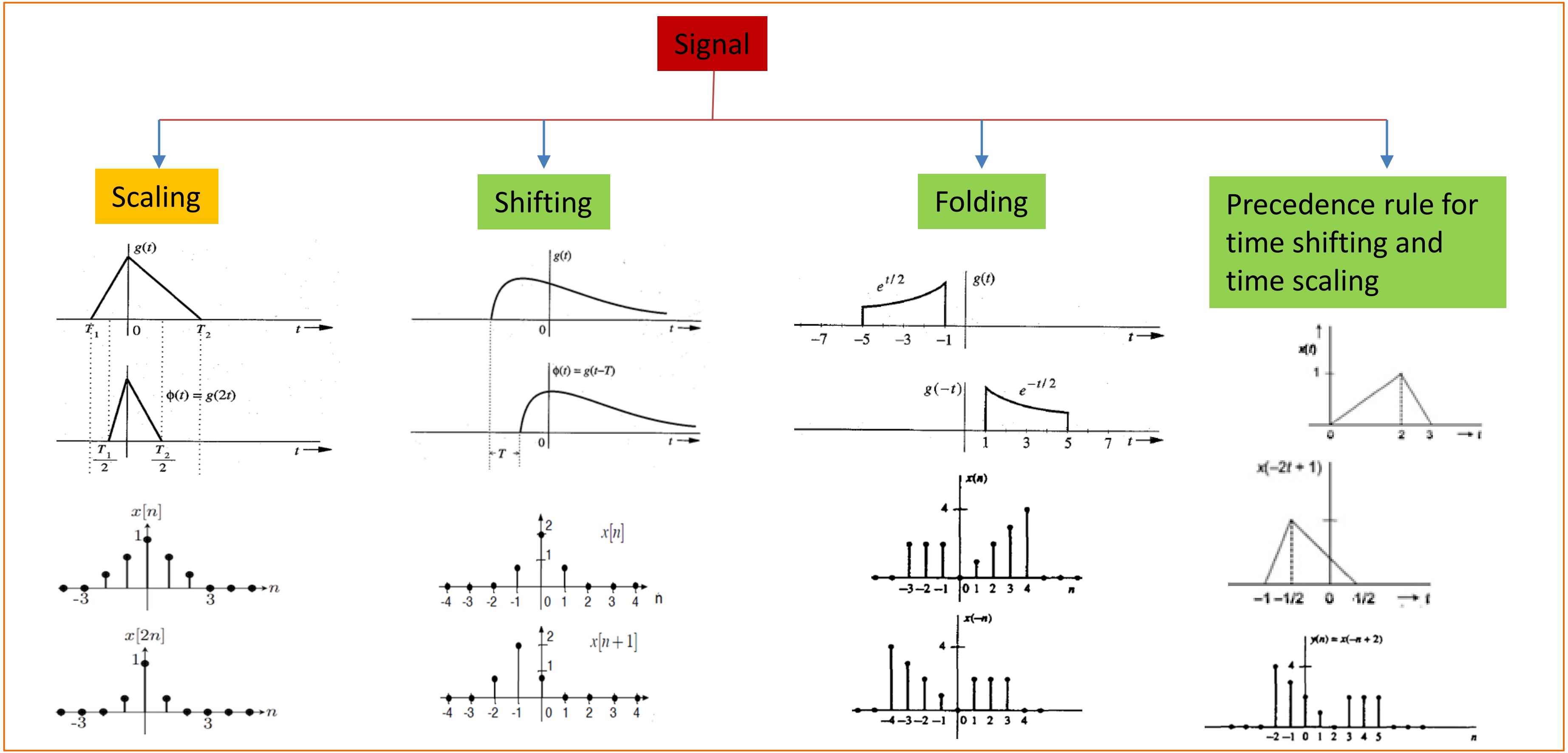
$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \frac{A^2}{2}\, dt + \lim_{T\to\infty} \frac{A^2}{2T}\int_{-T/2}^{T/2} \cos 2\,(\omega_0 t + \theta)]\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\,\frac{A^2}{2}[t]_{-T/2}^{T/2} + \lim_{T\to\infty} \frac{1}{4\omega_0 T}[\sin\ 2(\omega_0 t + \theta)]\,_{-T/2}^{T/2}$$

$$= \lim_{T\to\infty} \frac{1}{T}\,\frac{A^2}{2} T + 0 = \frac{A^2}{2} < \infty$$

Signal has finite power

Basic operations on the signals



Scaling

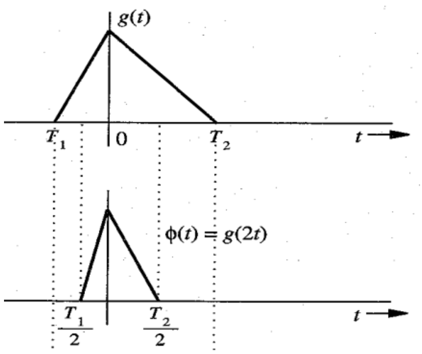
- Time scaling compresses or dilates a signal g(t) by multiplying the time variable (t) by some quantity (a)
- If quantity (a) is greater than one, the signal becomes narrower and the operation is called **compression.**

a>1

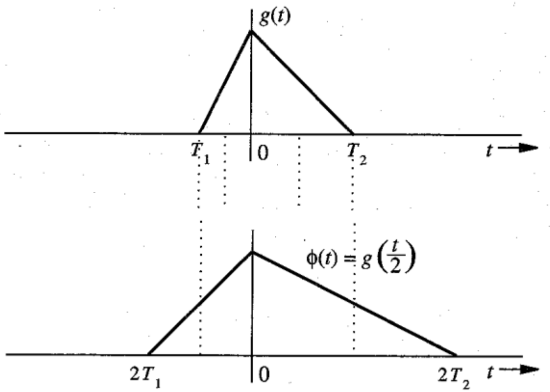
$$\varphi(t) = g(at)$$

$$\varphi(t) = g(\frac{t}{a})$$

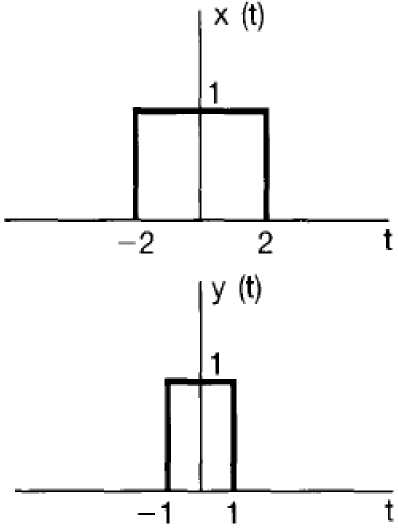
$$y(t) = x(2t)$$



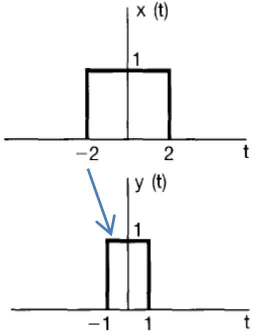
Compress



Dilates



$$y(t) = x(2t)$$



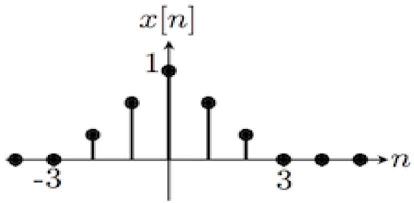
$$y(t) = x(2t)$$

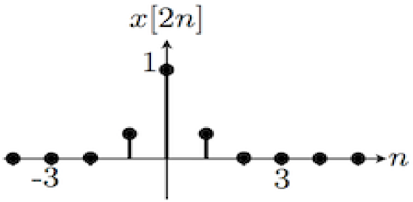
t	-3	-2	-1	0	1	2	3
$2t$	2*-1.5	2*-1	2*-0.5	2*0	2*.5	2*1	2*1.5
$x(t)$	0	1	1	1	1	1	0
$y(t) = x(2t)$	0	1	2	1	1	1	0

a>1

$y[n] = x\,[2n]$

Discrete signal **only consider** the integer value of **“index” i.e. n**

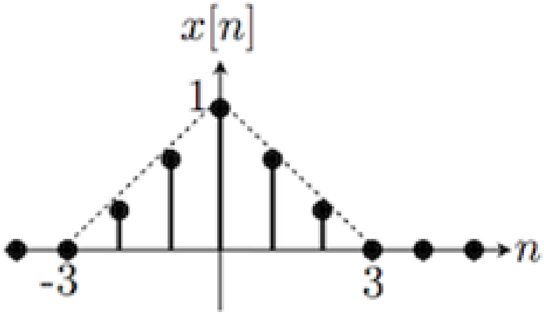




n	-3	-2	-1	0	1	2	3	4	5
2n	2* ~~-1.50~~	2*-1	2*~~-0.5~~	2*0	2*~~0.5~~	2*1	2*~~1.5~~	2*2	2*~~2.5~~
$x[n]$	0	.25	0.8	1	.8	.25	0	0	0
$y[n] = x[2n]$	-	.25	-	1	-	.25	-	0	-

a <1

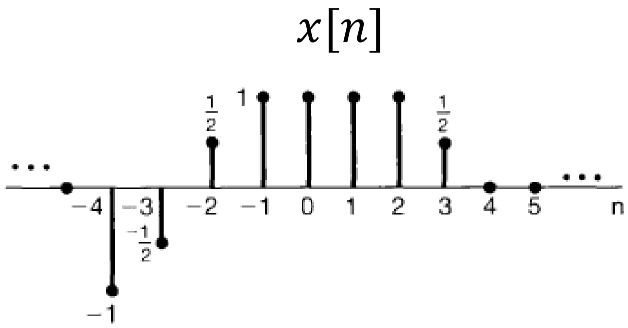
$$y[n] = x[\frac{n}{2}]$$



Dilates

If the quantity (a) is less than one, the signal becomes wider and is called dilation.

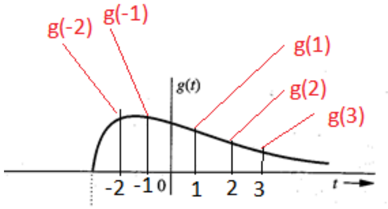
$(a)\ x[3n] = ?$



Time-Shifting

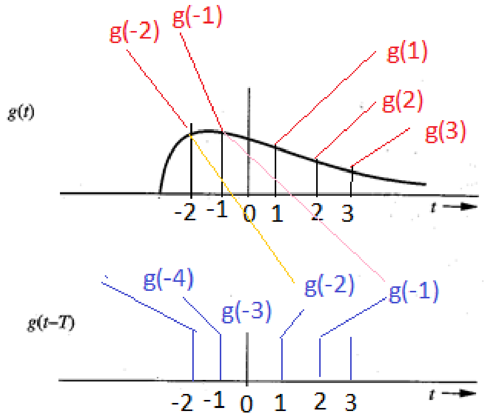
- Time shifting is the shifting of a signal in time
- This is done by **adding** or **subtracting a quantity (say. T)** of the shift to the time variable in the function

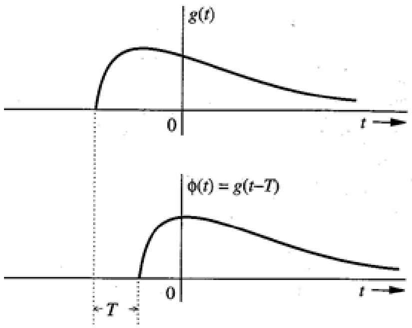
$\varphi(t) = g(t - T)$ For **positive value of T >0** ----- right-shift/delay



Let's to determine signal $g(t-3)$ for $T = 3$

0. $g(t-3)\ at\ t = -1 \rightarrow g(-1-3) = g(-4)$
1. $g(t-3)\ at\ t = 0 \rightarrow g(0-3) = g(-3)$
2. $g(t-3)\ at\ t = 1 \rightarrow g(1-3) = g(-2)$
3. $g(t-3)\ at\ t = 2 \rightarrow g(2-3) = g(-1)$





For a discrete time signal $x[n]$, and an integer $n_1 > 0$

$$y[n] = x[n - n_1] \longrightarrow$$ **Delay/ right-shift**

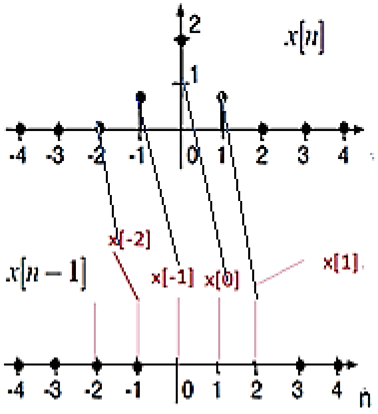
Let's given x[n], determine $x[n-1]$

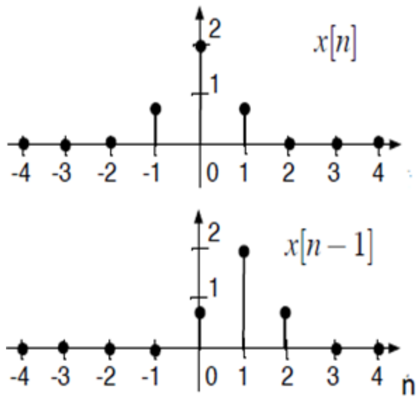
0. $x[n-1]$ at $n = -1$ $\rightarrow x[-1-1] = x[-2]$

1. $x[n-1]$ at $n = 0$ $\rightarrow x[0-1] = x[-1]$

2. $x[n-1]$ at $n = 1$ $\rightarrow x[1-1] = x[0]$

3. $x[n-1]$ at $n = 2$ $\rightarrow x[2-1] = x[1]$

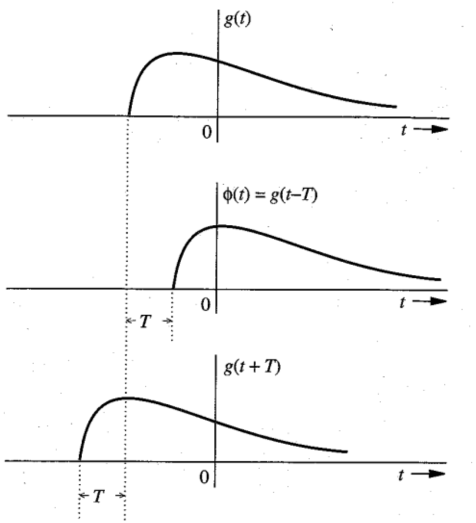


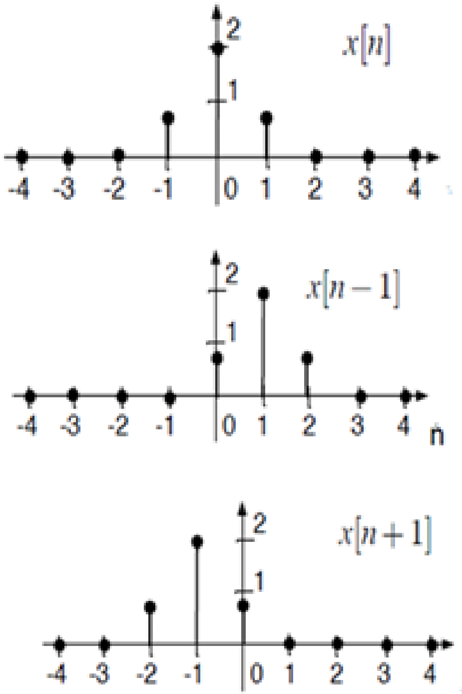


Similarly, left-shift/advance

$$\varphi(t) = g(t+T)$$

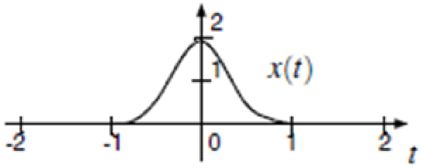
$$y[n] = x[n+n_1]$$

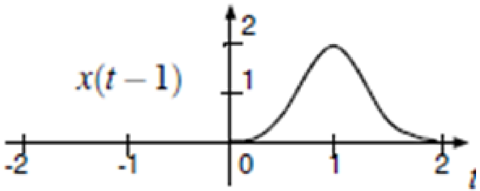




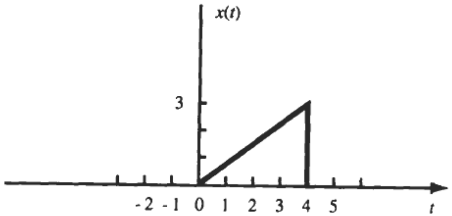
Draw the following signals

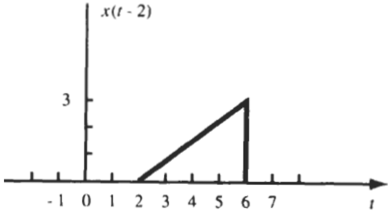
$(a)\quad y(t) = x(t-1)$

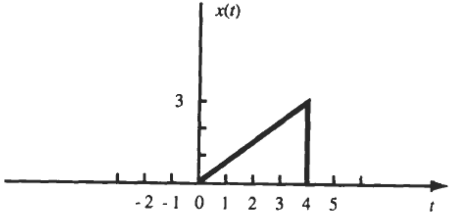


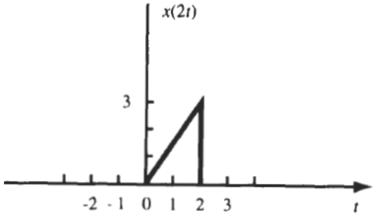


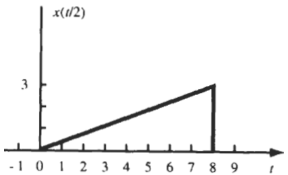
$(b)\ x(t-2) \qquad (c)\ x(2t) \qquad (d)\ x(\frac{t}{2})$



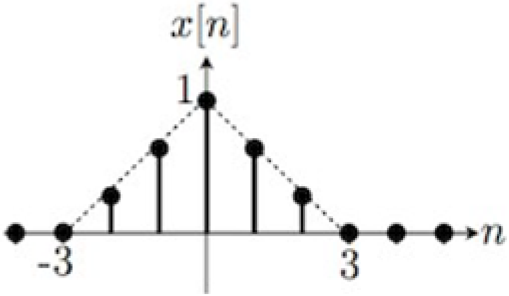




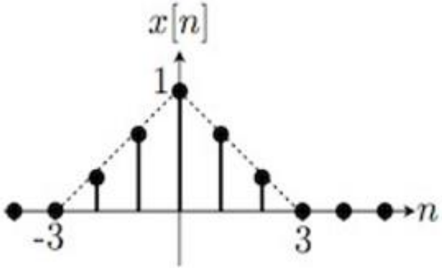


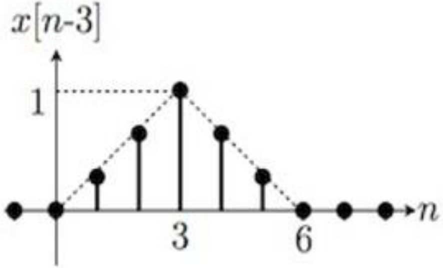


Given,



Determine the signal $y[n] = x[n-3]$?

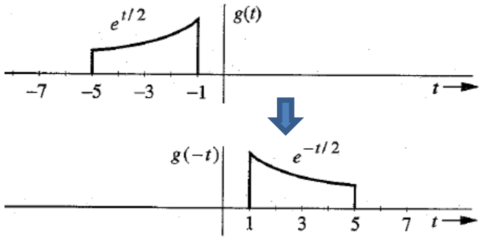


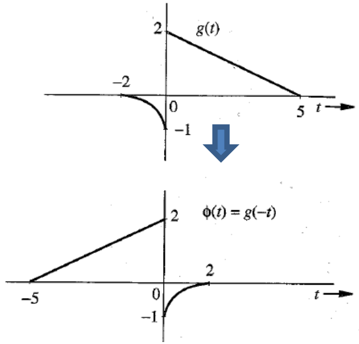


Reflection (Time Reversal/Inversion)/folding

- Generate an **mirror image of signal** with respect to Y-axis (at t =0)

$$\phi(t) = g(-t)$$

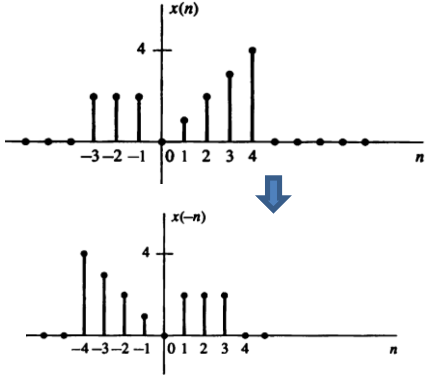




Mathematically,

$$g(-t)\ at\ t = -1\ \rightarrow g(1)$$
$$g(-t)\ at\ t = 0\ \rightarrow g(0)$$
$$g(-t)\ at\ t = 1\ \rightarrow g(1)$$

For discrete: $\emptyset(n) = x(-n)$ (at time n=0 with respect to the Y- axis)



Mathematically,

$$x(-n)\ at\ n = -1\ \rightarrow x(1)$$
$$x(-n)\ at\ n = 0\ \rightarrow x(0)$$
$$x(-n)\ at\ n = 1\ \rightarrow x(-1)$$

Shifting and scaling

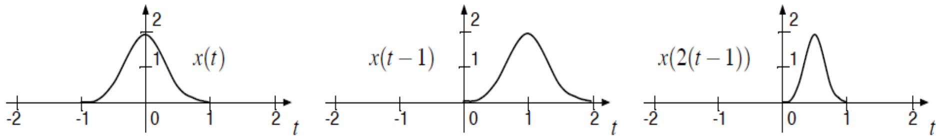
$y(t) = x(2t - 2)$

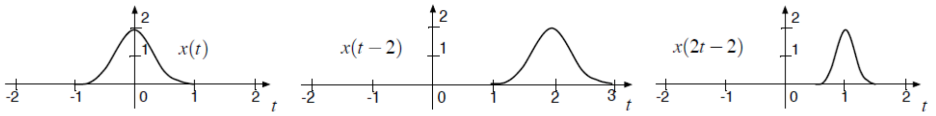
scaling shifting

Precedence rule of **time shifting** and **Scaling**

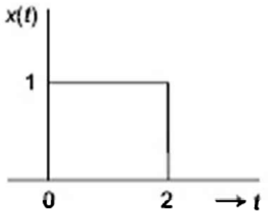
$$y(t) = x(2t-2) \rightarrow \begin{array}{ll} 1.\ x(t-2) & \text{- First shifting} \\ 2.\ x(2t-2) & \text{– Second scaling} \end{array}$$

The following which one is correct form of $x(2t-2)$?

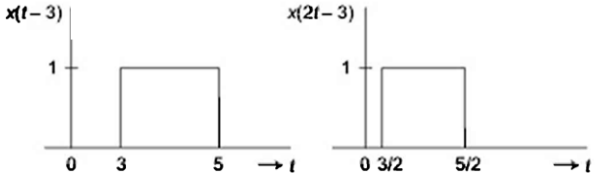




Given $x(t)$:

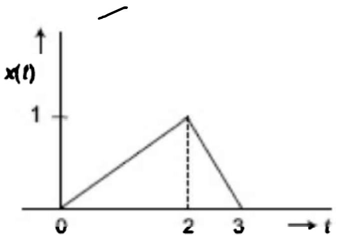


Draw signal $x(2t-3)$



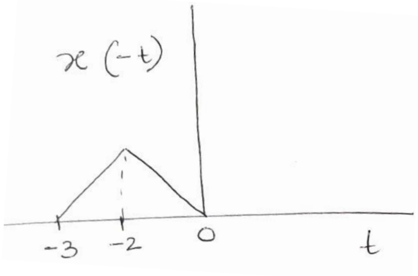
Precedence rule for time shifting and time scaling

Given, $x(t)$

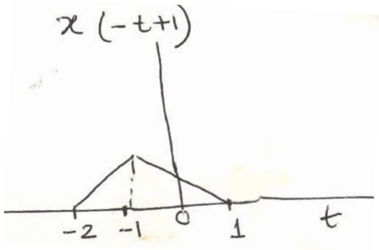


Precedence rule of time shifting and Scaling

1. Reflection/folding : $x(-t)$
2. Shifting: $x(-t+1)$ >> delay [right]
3. Scaling : $x(-2t+1)$

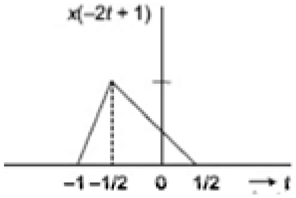


Draw the signal $\boldsymbol{x(-2t+1)}$

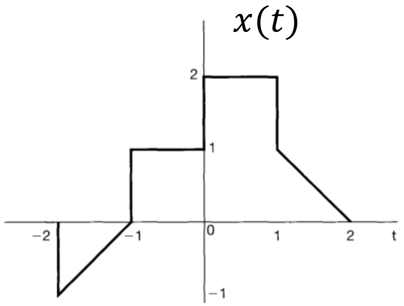


After rewriting $x(-2t+1) = x(2(-t) + 1)$

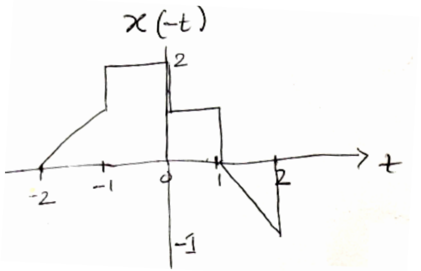
Scaling Folding Shifting

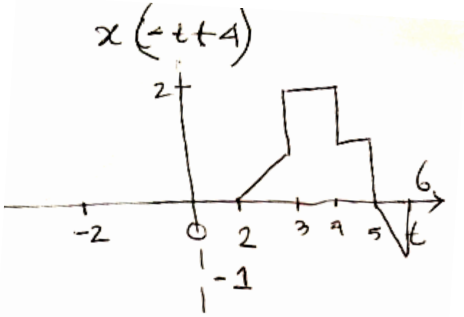


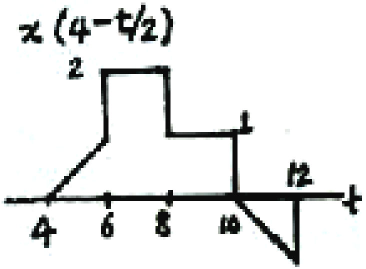
Given,



Plot the signal $x(4-t/2)$?



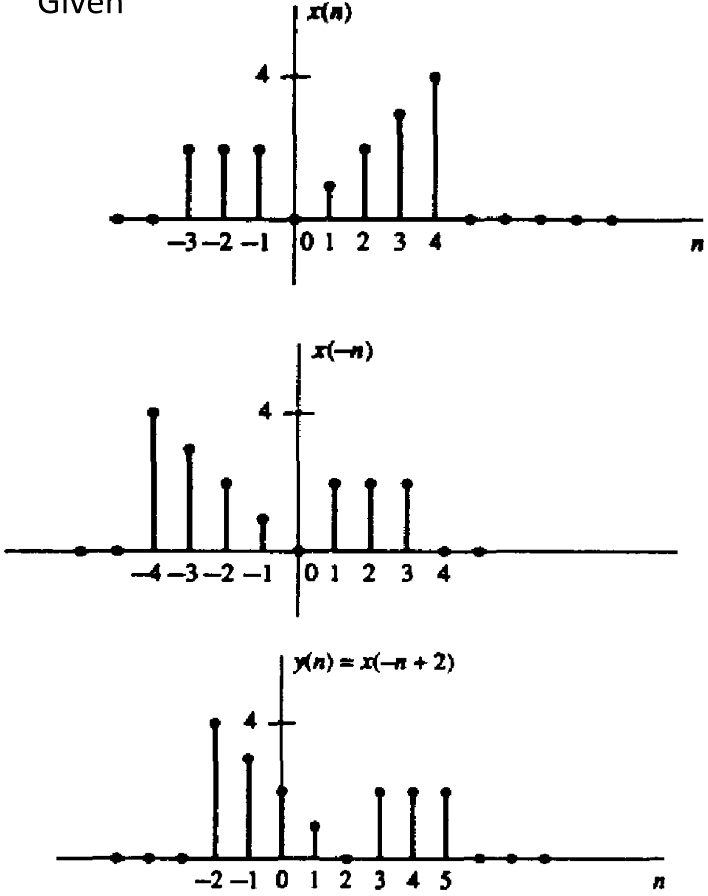




As per the Precedence rule of time shifting and Scaling

1. Reflection/folding : $x(-t)$
2. Shifting: $x(-t+4)$ >> delay [right]
3. Scaling : $x(-t/2+4)$

Given



Evaluate the signal $x[-n+2]$

Q2.

$$x(t) = \begin{cases} e^{t/2} & -1 \geq t > -5 \\ 0 & \text{otherwise} \end{cases}$$

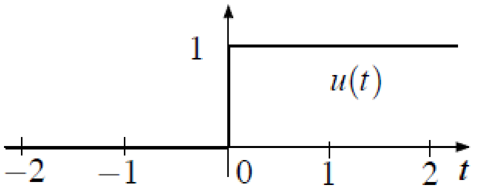
$$x(-t) =$$

Step function

Continuous:

The *unit step function* $u(t)$ is defined as

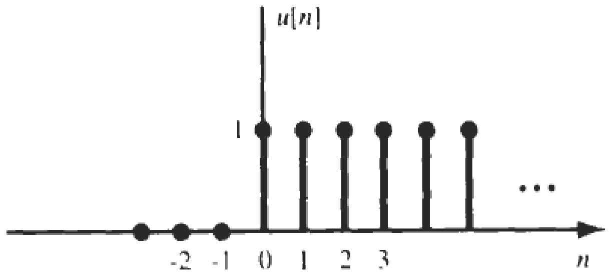
$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



Discrete:

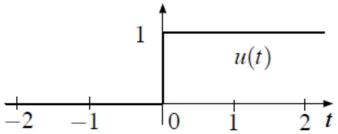
The *unit step* sequence $u[n]$ is defined as

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Q1. Express the following function using $u(t)$

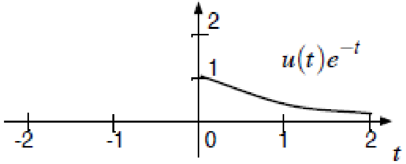
$$x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



Step function

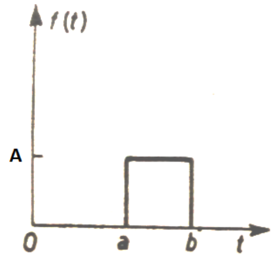
can be written as

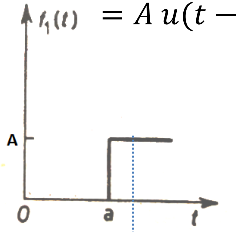
$$x(t) = u(t)e^{-t}$$



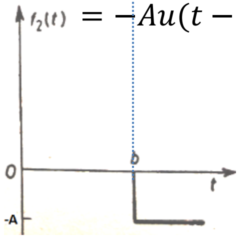
Q2. Represent the following function using $u(t)$

$$f(t) = \begin{cases} 0 & t < a \\ A & a < t < b \\ 0 & t > b \end{cases}$$



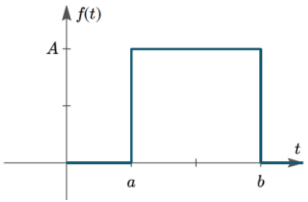


$= A\,u(t-a)$

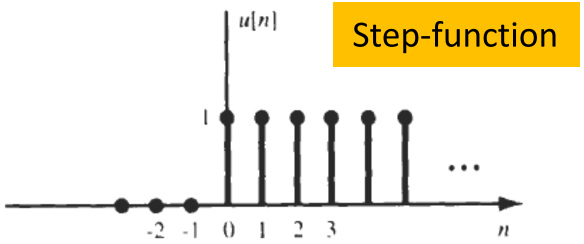


$= -Au(t-b)$

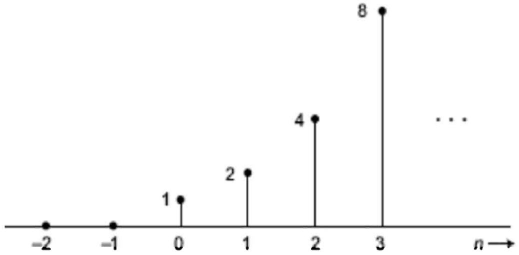
$$f(t) = A \cdot [u(t-a) - u(t-b)]$$



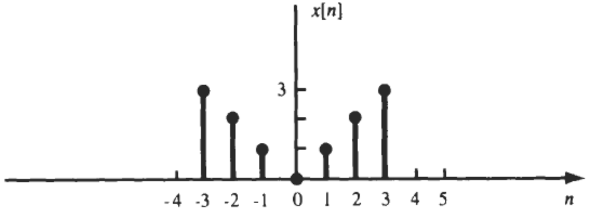
Given, $x(n) = a^n$



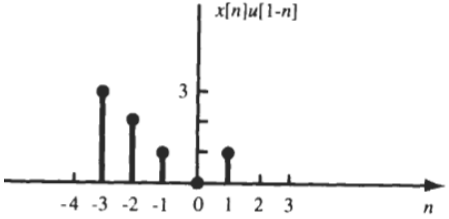
Q1. Determine the signal $x(n) = a^n u(n)\ for\ a = 2.$



Q2. Evaluate $x[n]\, u[1-n]$.



$$u[1-n] = \begin{cases} 1 & n \le 1 \\ 0 & n > 1 \end{cases}$$

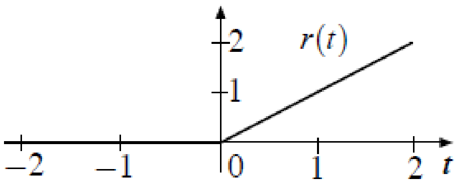


Ramp function

Continuous-time signal:

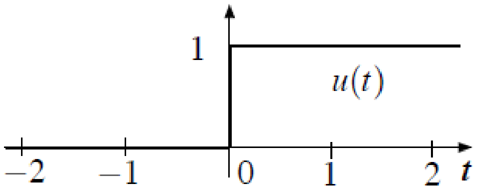
The *unit ramp* is defined as

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



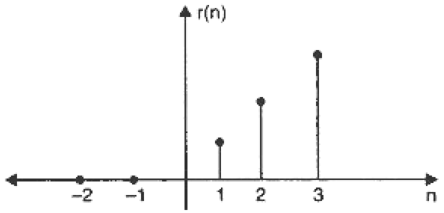
The unit ramp is the integral of the unit step,

$$r(t) = \int_{-\infty}^{t} u(\tau) d\tau$$

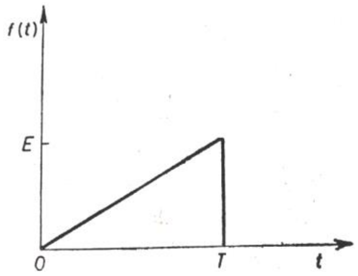


Discrete-time signal: The unit –ramp signal is defined as,

$$r(n) = \begin{cases} 0, & n < 0 \\ n, & n > 0 \end{cases}$$

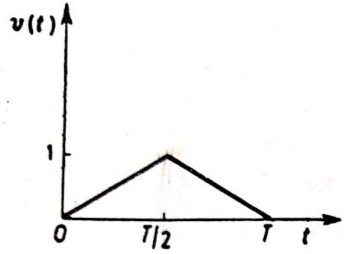


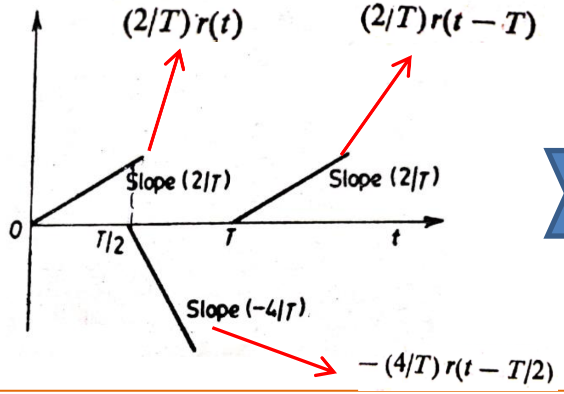
Q1. Express the following signal using $u(t)$



$$f(t) = \frac{E}{T} t[u(t) - u(t-T)]$$

Q2. Express the following signal using ramp r(t)





$$v(t) = (2/T)\,r(t) - (4/T)\,r(t - T/2) + (2/T)\,r(t - T)$$

Determine whether signals are energy/power signal

$Q1.\ \ x(t) = e^{-at}\, u(t), \qquad a > 0$

We know,

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

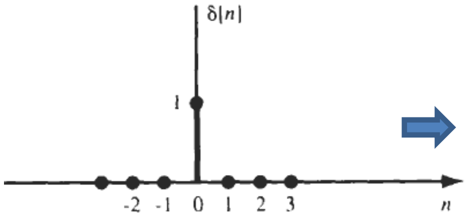
$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \quad = \int_0^{\infty} e^{-2at}\, dt \quad = \frac{1}{2a} < \infty$$

$Q2.\ \ x[n] = (-0.5)^n u[n]$

Impulse function

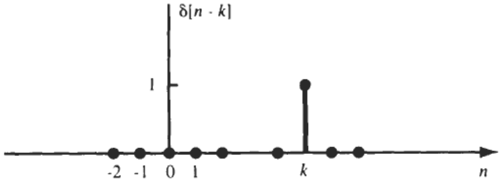
Discrete: The unit-impulse signal is called "unit-sample signal" and is defined as

$$\delta(n) = \begin{cases} 1, n = 0 \\ 0, n \neq 0 \end{cases}$$



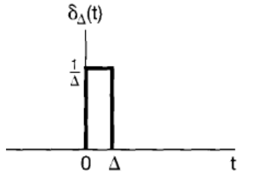
$$\delta[n] = u[n] - u[n-1]$$

The delayed of "k" unit of unit-impulse signal



Continuous: The unit-impulse function in the continues-time signal is defined as

$\delta_\Delta(t)$ is a short pulse, of duration Δ and with unit area for any value of Δ. As $\Delta \to 0$, $\delta_\Delta(t)$ becomes narrower and higher, maintaining its unit area.



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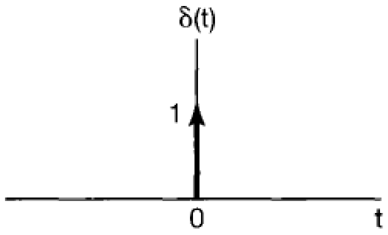
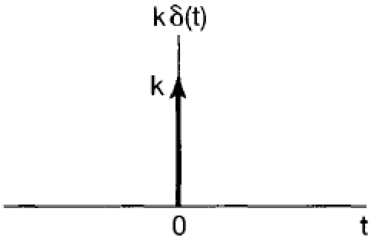
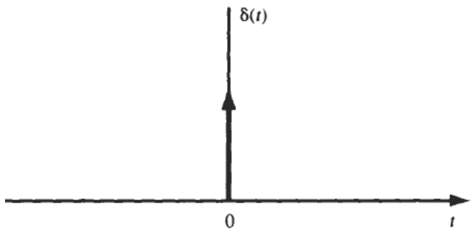
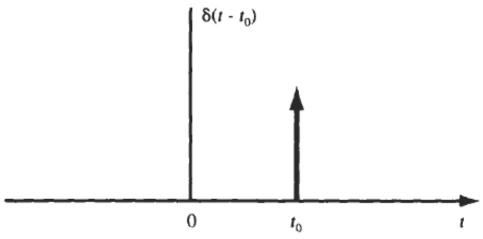




$$\int_{-\infty}^{t} k\delta(t)dt = k$$

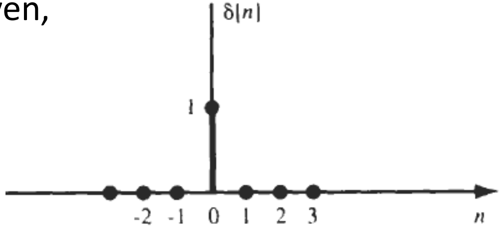
In general, we can write: $\boxed{\int_{-\infty}^{\infty} \emptyset(t)\delta(t)dt = \emptyset(0)}$

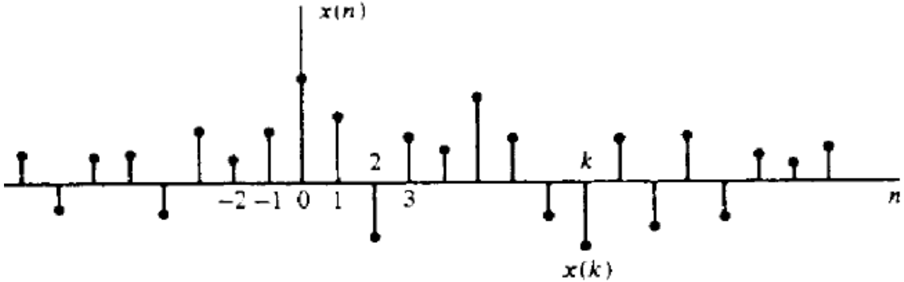




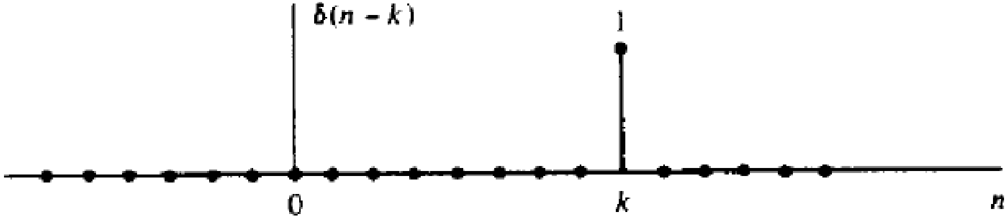
(a) $t\delta(t) = ?$
(b) $\sin t\, \delta(t) = ?$
(c) $\cos t\delta(t - \pi) = ?$

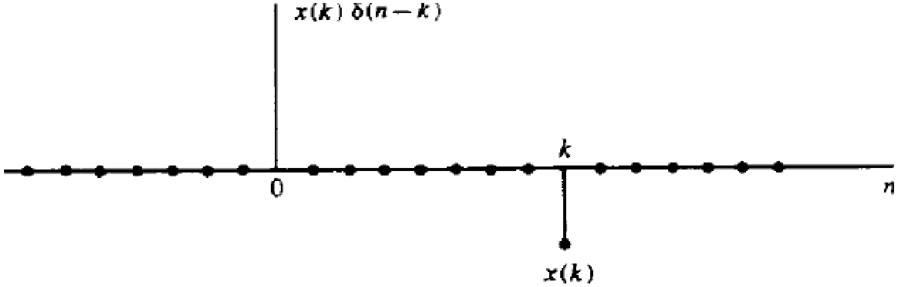
Given,





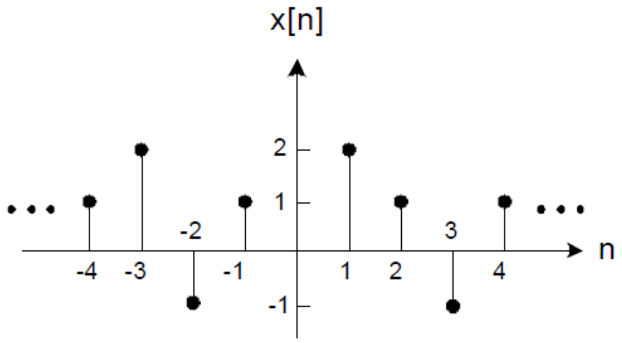
Evaluate the signal, $x(n)\delta(n-k) = ?$ $x(k)\delta(n-k)$

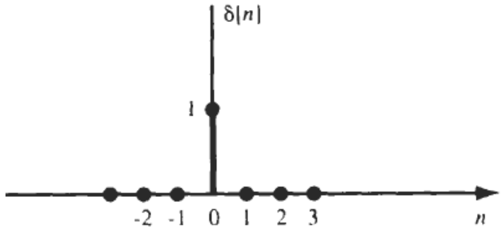




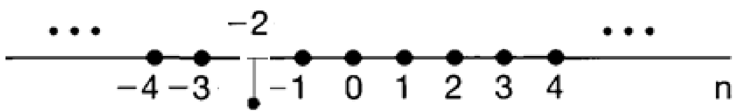
Representation of an arbitrary discrete signal

- A discrete-time signal can be decomposed into a **sequence of individual impulses**



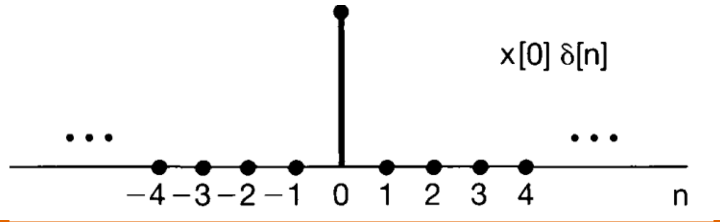


x[−2] δ[n + 2]

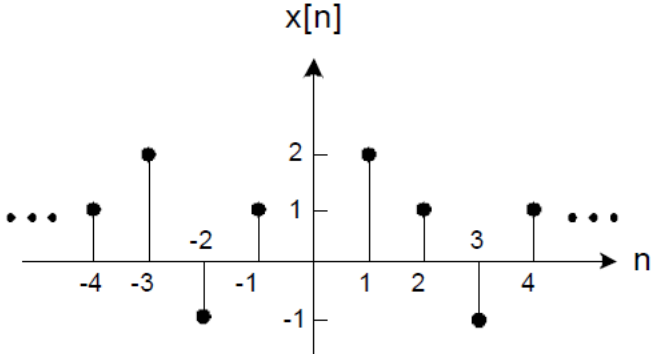


$$x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases}$$

$$x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Cont..



The signal in Fig. can **be expressed** as a **sum of the *shifted impulses***

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

or in a **more compact** form

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Right-hand side represents an arbitrary signal x(n) as weighted (scaled) sum of shifted impulse sequence

Evaluate the following integrals

$(a)\ \int_{-1}^{1}(3t^2+1)\delta(t)\,dt$

$(b)\ \int_{1}^{2}(3t^2+1)\delta(t)dt$

$(c)\ \int_{-\infty}^{\infty}(t^2+\cos(\pi t))\delta(t-1)dt$

$(d)\ \int_{-\infty}^{\infty} e^{-t}\delta(2t\ -2)\ dt$

$(e)\ \int_{-\infty}^{\infty} e^{-t}\ \dot{\delta}\ (\mathrm{t})\ \mathrm{dt}$

(a) $\int_{-1}^{1}(3t^2+1)\delta(t)dt = (3t^2+1)|_{t=0}$

$(b)\ \int_{1}^{2}(3t^2+1)\delta(t)dt = 0$

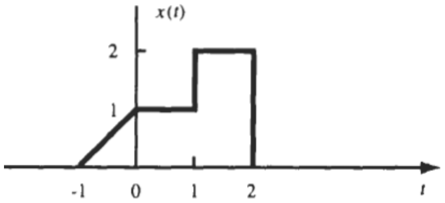
$$(c)\ \int_{-\infty}^{\infty}(t^2+\cos\quad \pi t)\ \delta(t-1)dt = (t^2+\cos\pi t)|_{t=1}$$
$$= 1+\cos\pi = 1\ -1 = 0$$

$$\int F(x)G'(x)\,dx = F(x)G(x) - \int F'(x)G(x)\,dx + C.$$

- Draw the following signal for given x(t)

$(a)\ x(t)\delta\left(t-\dfrac{3}{2}\right)$

$(b)\ x(t)u(1-t)$



Evaluate the following integrals

$(a)\ \int_{-1}^{1}(3t^2+1)\delta(t)\,dt$

$(b)\ \int_{1}^{2}(3t^2+1)\delta(t)dt$

$(c)\ \int_{-\infty}^{\infty}(t^2+\cos(\pi t))\delta(t-1)dt$

$(d)\ \int_{-\infty}^{\infty}e^{-t}\delta(2t-2)\,dt$

$(e)\ \int_{-\infty}^{\infty}e^{-t}\,\dot{\delta}\,(\mathrm{t})\,\mathrm{dt}$

(a) $\int_{-1}^{1}(3t^2+1)\delta(t)dt=(3t^2+1)|_{t=0}$

$$(b)\int_{1}^{2}(3t^2+1)\delta(t)dt=0$$

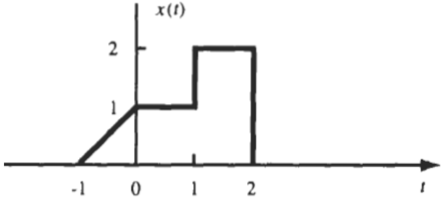
$$(c)\int_{-\infty}^{\infty}(t^2+\cos\pi t)\,\delta(t-1)dt=(t^2+\cos\pi t)|_{t=1}$$
$$=1+\cos\pi=1-1=0$$

$$\int F(x)G'(x)\,dx=F(x)G(x)-\int F'(x)G(x)\,dx+C.$$

$$\int_{-\infty}^{\infty}\emptyset(t)g^n(t)dt=(-1)^n\int_{-\infty}^{\infty}\emptyset^n(t)\,g(t)\,dt$$

- Draw the following signal for given x(t)

$$(a)\ x(t)\delta\left(t-\frac{3}{2}\right)$$
$$(b)\ x(t)u(1-t)$$



Properties (of unit-impulse function continuous)

$$\delta(at) = \frac{1}{|a|}\,\delta(t)$$

$$\delta\left(\frac{t-t_o}{a}\right) = |a|\,\delta(t-t_o)$$

$$\delta(at-t_o) = \frac{1}{|a|}\,\delta\left(t-\frac{t_o}{a}\right)$$

$$\delta(-t+t_o) = \delta(t-t_o)$$

$$\delta(-t) = \delta(t); \qquad \delta(t) = \text{even function}$$

$$\int_{-\infty}^{\infty} \delta(t)\,f(t)\,dt = f(0)$$

$$\int_{-\infty}^{\infty} \delta(t-t_o)\,f(t) = f(t_o)$$

$$f(t)\,\delta(t) = f(0)\,\delta(t)$$

$$\int_{-\infty}^{\infty} \frac{d\delta(t)}{dt}\,f(t)\,dt = -\frac{df(0)}{dt}$$

$$\int_{-\infty}^{\infty} \frac{d\delta(t-t_o)}{dt}\,f(t)\,dt = -\frac{df(t_o)}{dt}$$

$$\int_{-\infty}^{\infty} \frac{d^n\delta(t)}{dt^n}\,f(t)\,dt = (-1)^n\,\frac{d^n f(0)}{dt^n}$$

$$f(t)\frac{d\delta(t)}{dt} = -\frac{df(0)}{dt}\,\delta(t) + f(0)\,\frac{d\delta(t)}{dt}$$

$$t\frac{d\delta(t)}{dt} = -\delta(t)$$

$(a)\ \frac{d}{dt}\{[2-u(t)]cost\}$=?

$$= \frac{d}{dt}[2\cos t \ - u(t)\cos t]$$

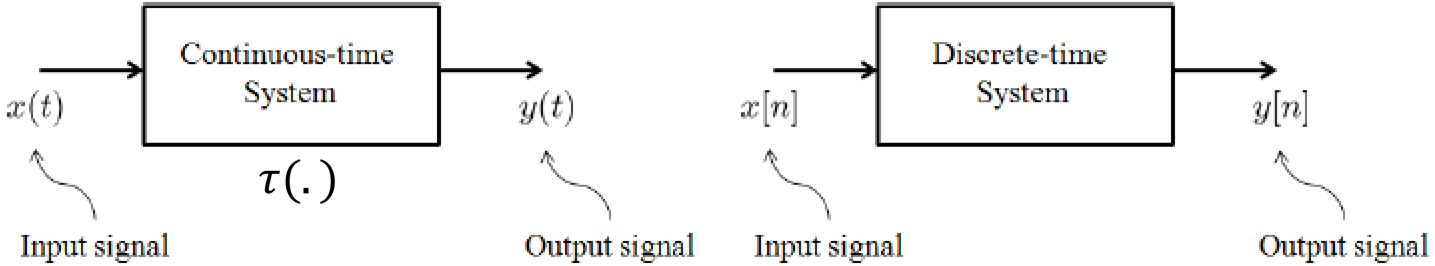
$$= \text{-2 sin t - } \delta(t)\cos t \ - [-u(t).\sin t]$$

$$= \text{-2 sin t - } \delta(t)\cos t + u(t).\sin t$$

$$= \sin t\,[u(t) - 2\,] \ - \ \delta(t)$$

Types of systems

- System is a device or algorithm which process or transforms an input signal into an desired output signal

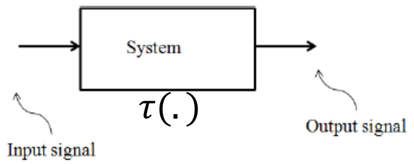


A system is an operator $\mathcal{T}$ which maps input into output:

$$y(t) = \mathcal{T}\{x(t)\} \quad \text{or} \quad y[n] = \mathcal{T}\{x[n]\}$$

Linearity- property

- A *linear system, in* ***continuous time or discrete time****, is a system that possesses the important* property of **additivity, Scaling or superposition**:



a) Additivity:

Given that $\tau\{x_1\} = y_1$ and $\tau\{x_2\} = y_2$

$$\tau\{x_1 + x_2\} = y_1 + y_2$$

b) Homogeneity (or Scaling):

$$\tau\{\alpha x\} = \alpha y$$

- **(a) & (b) Superposition:**

$$\tau\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 y_1 + \alpha_2 y_2$$

where α_1 and α_2 are arbitrary scalars

Determine whether systems are linear or non-linear

$(a)\ y(t) = t\,x(t)$

$(b)\ y[n] = 2\,x[n] + 3$

$(c)\ y[n] = Re\{x[n]\}$

$(d)\ y(n) = x(n^2)$

(a) Let's considers two arbitrary inputs $x_1(t)$ & $x_2(t)$

$$x_1(t) \rightarrow y_1(t) = t x_1(t)$$

$$x_2(t) \rightarrow y_2(t) = t x_2(t)$$

$$Let\ , x_3\,(t) = a x_1(t) + b\, x_2(t)$$

another inputs linear combination $\boldsymbol{of}$ $x_1\,(t)$ and $x_2(t)$,

$$x_3(t) \rightarrow y_3(t) = t x_3(t)$$

$$= t(a x_1(t) + b\, x_2(t))$$

$$= a t x_1(t) + b t\, x_2(t)$$

$$= a y_1(t) + b\, y_2(t)$$

As per the superposition law of linearity, it satisfies it i.e.

$$\tau\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

The system (a) is linear

Cont..

$(b)\ y[n] = 2\,x[n] + 3$

Let's considers two arbitrary inputs $x_1[n]$ & $x_2[n]$

$$x_1[n] \rightarrow y_1[n] = 2x_1[n] + 3$$
$$x_2[n] \rightarrow y_2[n] = 2x_2[\mathrm{n}] + 3$$

$$Let\ , x_3\,[n] = ax_1[n] + b\,x_2[n]$$

another inputs linear combination $\boldsymbol{of}$ $x_1[n] and\ x_2[n]$

$$x_3[n] \rightarrow y_3[n] = 2x_3[n] + 3$$
$$= 2[ax_1[n] + b\,x_2[n]] + 3$$
$$= a\,2x_1[n] + b\,2x_2[n] + 3$$
$$\neq a\,y_1[n] + b\,y_2[n]$$

It does not satisfy the superposition law for the linearity

The system (b) is not linear.

Cont..

$(c)\ y[n] = Re\{x[n]\}$

Let's input $x_1[n] = r[n] + js[n]$

$$x_1[n] \rightarrow y_1[n] = \mathrm{Re}\{x_1[n]\} = \mathrm{Re}\{r[n] + js[n]\}$$

(scaling property)

Let's another input scaled of version $x_2[n] = ax_1[n]$

$$x_2[n] \rightarrow y_2[n] = \mathrm{Re}\{x_2[n]\}$$
$$= \mathrm{Re}\{a(r[n] + js[n])\}$$
$$= \mathrm{Re}\{ar[n] + jas[n]\}$$

$$= \mathrm{Re}\{jr[n] - s[n]\}$$ If a=j
$$= -s[n]$$

$$ax_1[n] \rightarrow\neq ay_1[n]$$

System is not linear

Cont..

$(d)\ y(n) = x(n^2)$

Let's considers two arbitrary inputs $x_1[n]$ & $x_2[n]$

$$x_1[n] \to y_1[n] = x_1[n^2]$$
$$x_2[n] \to y_2[n] = x_2[n^2]$$

$$Let\ , x_3\,[n] = ax_1[n] + b\ x_2[n]$$

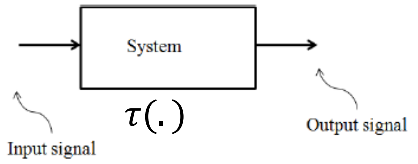
another inputs linear combination $\boldsymbol{of}$ $x_1[n] and\ x_2[n]$

$$x_3[n] \to y_3[n] = x_3[n^2]$$
$$= ax_1[n^2] + b\ x_2[n^2]$$
$$= a\ y_1[n] + b\ y_2[n]$$

As per the superposition law of linearity, it satisfies it i.e. $\tau\{\alpha_1 x_1[n] + \ \alpha_2 x_2[n]\} = \alpha_1 y_1[n] + \alpha_2 y_2[\mathrm{n}]$

The system (d) is linear

Time invariance



$\tau(.)$

Continuous system:

A system is called time-invariant

- if a time shift (delay or advance) in the input signal causes the **same time shift** in the output signal

Mathematically,

$$\tau\{x(t-\eta)\} = y(t-\eta) \qquad \text{for any real value of } \eta$$

Discrete system: The system is time-invariant (or shift-invariant) if

$$\tau\{x[n-k]\} = y[n-k] \qquad \text{for any integer } \boldsymbol{k}$$

If a system which does not satisfy the above relation is called time-varying system

Verify whether the following systems are time invariant

$(a)\ \ y(t) = x(t)\cos(\omega_c t)$

$(b)\ y(n) = n\, x(n)$

$(c)\ y(n) = x(n) - x(n-1)$

(a)

$Let's,$ $\boldsymbol{x(t)}$ be an **arbitrary input** *to* this system, and ***output*** $y(t) = \tau\{x(t)\} = x(t)\cos\omega_c t$

Let's a second input obtained by shifting the x(t) by t_0 unit in time $x_1(t) = x(t - t_0)$

$$x_1(t) \rightarrow y_1(t) = \tau\{x_1(t)\} = \tau\{x(t-t_0)\} = x(t-t_0)\cos\omega_c t \ \ldots (1)$$

If, we delay directly the output $y(t)$ *by* t_0 *unit in time*, we get $y(t-t_0) = x(t-t_0)\cos\ \omega_c(t-t_0)\ \ \ldots\ \ (2)$

As per the definition of time-invariance, it should satisfy $\tau\{x(t-\eta)\} = y(t-\eta)$

The equation $(1) \neq (2)$ System is time-variant

Verify whether the following systems are time invariant

$(a)\ \ y(t) = x(t)\cos(\omega_c t)$

$(b)\ y(n) = n\,x(n)$

$(c)\ y(n) = x(n) - x(n-1)$

(a)

$Let's,$ $\boldsymbol{x(t)}$ be an **arbitrary input** *to* this system, and ***output*** $y(t) = \tau\{x(t)\} = x(t)\cos\omega_c\,t$

Let's a second input obtained by shifting the x(t) by t_0 unit in time $x_1(t) = x(t - t_0)$

$$x_1(t) \rightarrow\ y_1(t) = \tau\{x_1(t)\} = \tau\{x(t-t_0)\}\ = x(t-t_0)\cos\omega_c\,t \ldots (1)$$

If, we delay directly the output $y(t)$ *by* t_0 *unit in time*, we get $y(t-t_0) = x(t-t_0)\cos\ \omega_c(t-t_0)\ \ldots\ \ (2)$

As per the definition of time-invariance, it should satisfy $\tau\{x(t-\eta)\} = y(t-\eta)$

The equation (1) $\neq$ (2) System is time-variant

Cont..

$(b)\ y(n) = nx(n)$ **(Eq. 0)**

Let $x(n)$ *be an* **arbitrary input** to this system, and **output** $y(n) = \tau\{x(n)\} = nx(n)$

Let, a **second input** obtained by **shifting** ***x(n) by*** $\boldsymbol{n_0}$ *in time* $x_1(n) = x(n - n_0)$

The output corresponding to the input $x_1(n)$ *is*

$$x_1(n) \rightarrow y_1(n) = \tau\{x_1(n)\} = \tau\{x(n - n_0)\}$$

$$= nx(n - n_0) \qquad \textbf{(1)}$$

If, we delay directly the output $y(t)$ **[Eq. 0]** $by\ n_0\ unit\ in\ time,$ we get $y(n - n_0) = (n - n_0)x(n - n_0)$ **(2)**

$$= nx(n - n_0) - n_0x(n - n_0)$$

As per the definition of time-invariance, it should satisfy $\tau\{x(t - \eta)\} = y(t - \eta)$

The equation (1) $\neq$ (2) System is time-variant

Cont..

$(c)\ y(n) = x(n) - x(n-1)$ ... (0)

Let $x(n)$ *be an* **arbitrary input** to this system, and **output** $y(n) = \tau\{x(n)\} = x(n) - x(n-1)$

Let a **second input** obtained by **shifting** $\boldsymbol{x_1(n)}$ ***by*** $\boldsymbol{n_0}$ *in time* $x_1(n) = x(n - n_0)$

$$x_1(n) \rightarrow y_1(n) = \tau\{x_1(n)\} = \tau\{x(n-n_0)\}$$
$$= x(n-n_0) - x(n-1-n_0)$$
$$= x(n-n_0) - x(n-n_0-1) \quad \text{... (1)}$$

If, we delay directly the output $y[n]$ **[Eq. 0]** $by\ n_0\ unit\ in\ time,$ we get

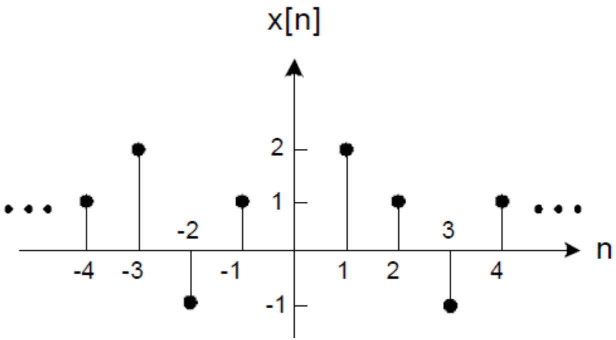
$$y(n-n_0) = x(n-n_0) - x(n-n_0-1) \quad \text{... (2)}$$

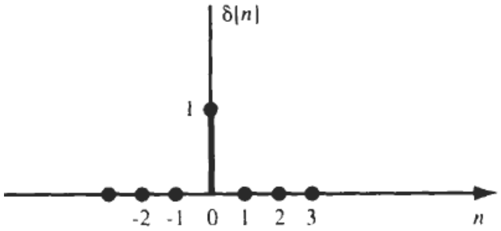
As per the definition of time-invariance, it should satisfy $\tau\{x(t-\eta)\} = y(t-\eta)$

The equation (1) = (2) System is time-invariant

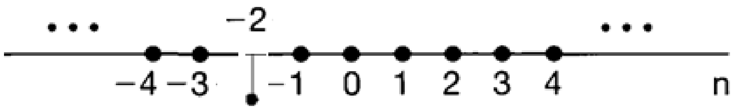
Representation of an arbitrary discrete signal

- A discrete-time signal can be decomposed into a **sequence of individual impulses**



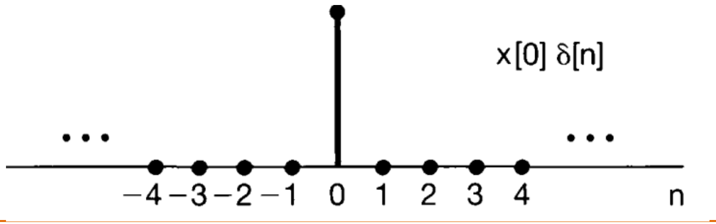


x[−2] δ[n + 2]



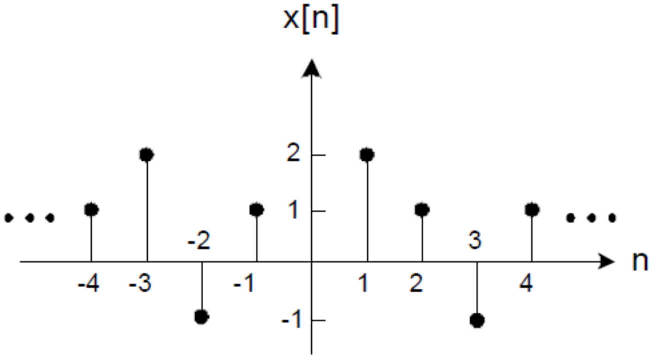
$$x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases}$$

x[0] δ[n]



$$x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Cont..



The signal in Fig. can **be expressed** as a **sum of the *shifted impulses***

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \ldots$$

or in a **more compact** form

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Right-hand side represents an arbitrary signal x(n) as weighted (scaled) sum of shifted impulse sequence

Let,

$$\delta[n] \longrightarrow \boxed{T} \longrightarrow h[n]$$

input　　　　　　　　output

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow y[n] = T\{x[n]\}$$

Amplitude of signal at "k"

Now, **system response y(n)** for the **input x(n)** can be written as,

$$y(n) = T\{x(n)\} = T\left\{\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right\}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, T\{\delta(n-k)\}$$

If **system is "Linear"** then it will hold the "**superposition rule**"

(output is weighted combination of input)

$$= \sum_{k=-\infty}^{\infty} x(k)h(n,k)$$

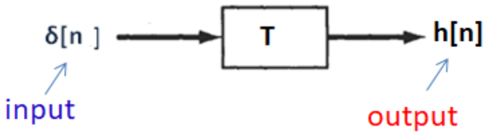
Cont..

For time-invariant system, we know that it holds

$$\tau\{x[n-k]\} = y[n-k] \quad \text{for any } \textbf{integer } \boldsymbol{k}$$

i.e. (delay or advance) in the input signal causes the **same time shift** in the output signal

If the system T is time-invariant



Then we can write,

$$if \;\; T\{\delta(n)\} = h(n) \quad \rightarrow T\{\delta(n-k)\} = h(n-k) \quad \text{for any } \textbf{integer } \boldsymbol{k}$$

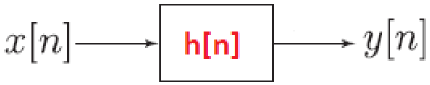
$$y(n) = T\{x(n)\} = T\left\{\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right\} = \sum_{k=-\infty}^{\infty} x(k)\, T\{\delta(n-k)\} = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

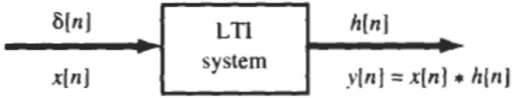
h[n] called "**impulse response ($\boldsymbol{\delta[n]}$) of LTI system**"

Input-output relation of an "LTI" system (discrete)

$$y(n) = T\{x(n)\} = \sum_{k=-\propto}^{\propto} x(k)h(n-k) = x(n) * h(n) = h(n) * x(n)$$

- This is called convolution sum/ response of system
- $h[n]$ is called Impulse response of LTI system

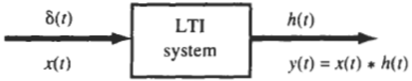




- **Suppose**, we wish to compute the output of system at **some time** $n=n_0$,

$$y(n_0) = \sum_{k=-\propto}^{\propto} x(k)h(n_0-k)$$

In continuous-time system: **(Convolution integral)**



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad = h(t) * x(t) = \int_{-\propto}^{\propto} h(\tau)x(t-\tau)d\tau$$

Steps involve in "convolution sum"

$$y(n) = T\{x(n)\} = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n) = h(n) * x(n)$$

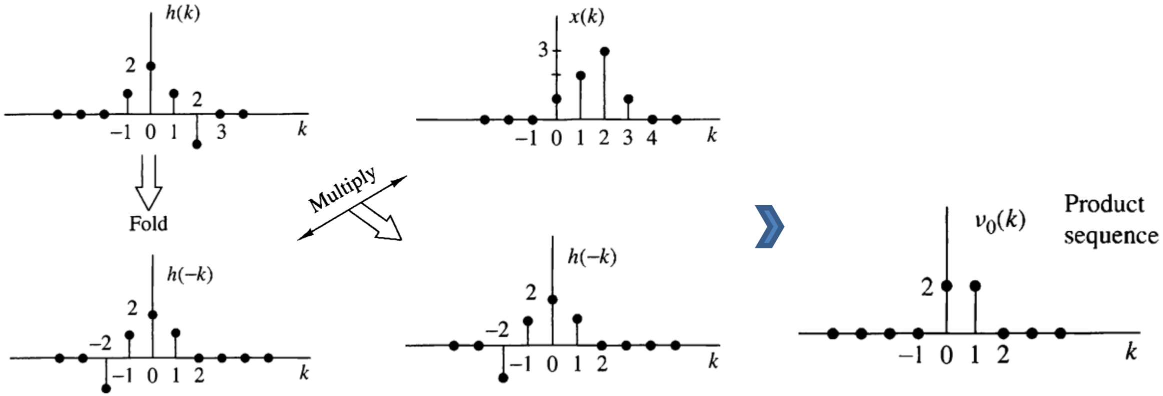
1. Folding: Fold $h(k)$ about $k = 0$ → $\boldsymbol{h(-k)}$
2. Shifting: Shift $h(-k)$ by n_0 to the right (left) if n_0 positive (negative) → $\boldsymbol{h(n_0 - k)}$
3. Multiplication: Multiply $x(k)$ by $h(n_0 - k)$ → the product sequence $\boldsymbol{x(k)\, h(n_0 - k)}$
4. Summation: **Sum all the values** of the product sequence $\boldsymbol{x(k)h(n_0 - k)}$ for input at **time** $\boldsymbol{n = n_0}$

Determine the response of the Linear-time invariant system to the input signal $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$

The impulse response of a linear time-invariant system is $h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$

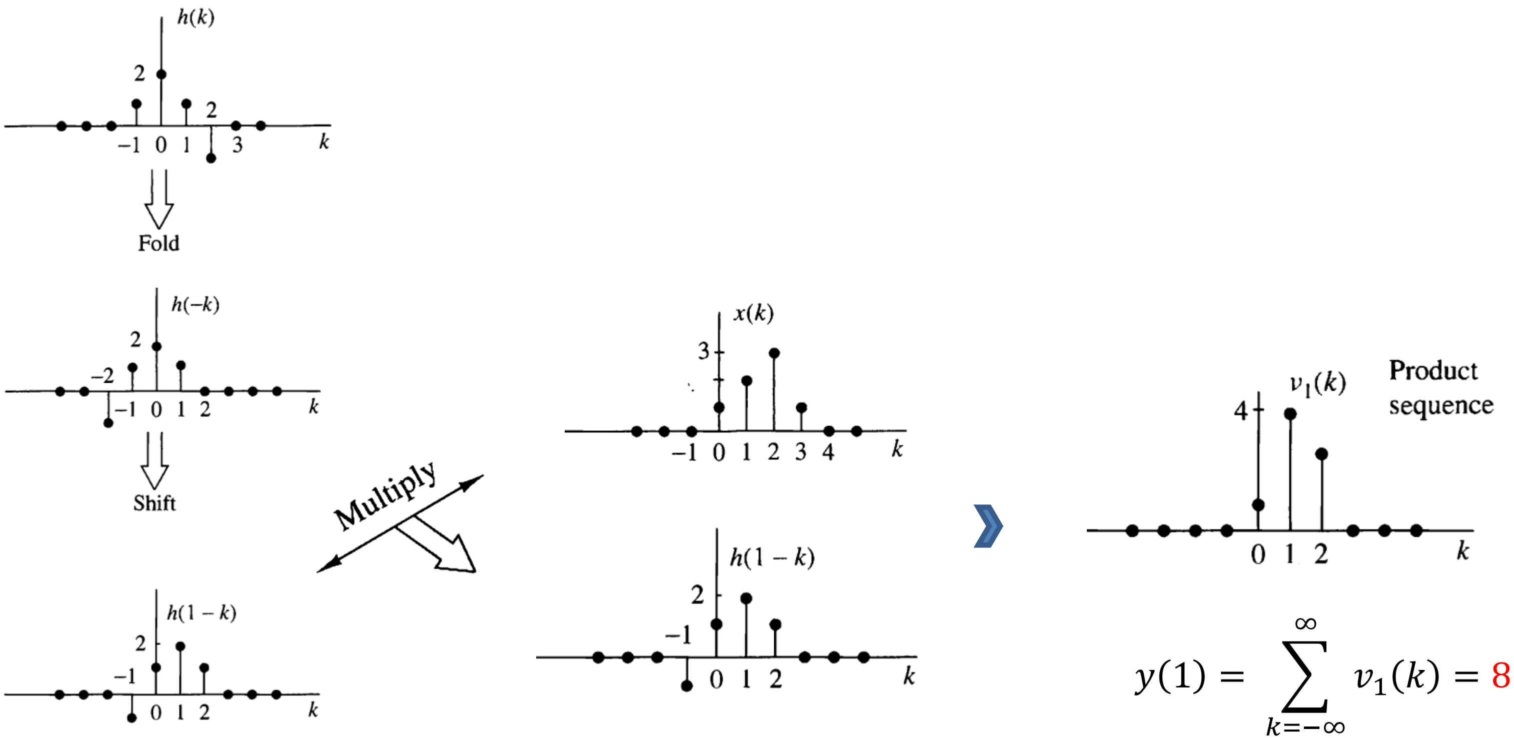
We know for the LTI system, $$y(n) = \sum_{k=-\propto}^{\propto} x(k)h(n-k)$$ for all value of n

$n = 0 \rightarrow y(0) = \sum_{k=-\propto}^{\propto} x(k)\, h(0-k) = \sum_{k=-\propto}^{\propto} x(k)\, h(-k)$ = 4



$$y(0) = \sum_{k=-\infty}^{\infty} v_0(k) = 4$$

For $\boldsymbol{n = 1}$, $y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 8$



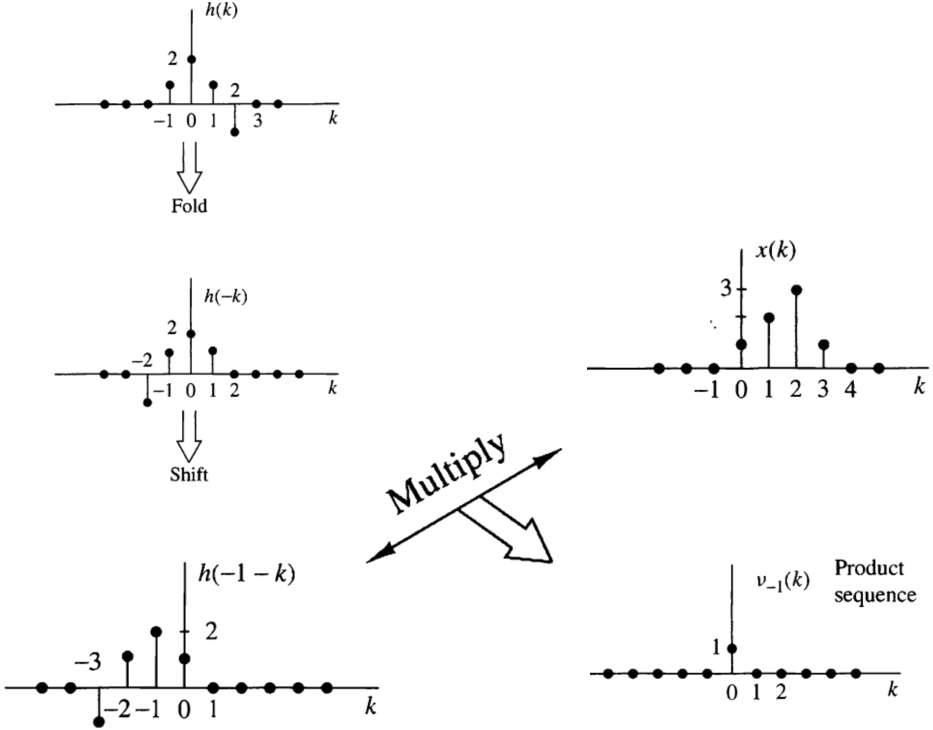
$$y(1) = \sum_{k=-\infty}^{\infty} v_1(k) = 8$$

For $n < 0$, $n = -1$

$$y(-1) = \sum_{k=-\propto}^{\propto} x(k)h(-1-k) = 1$$

Similarly for $-\propto < n < \propto$

$$y(n) = \{\ldots, 0, 0, 1, \underset{\uparrow}{4}, 8, 8, \ldots\}$$

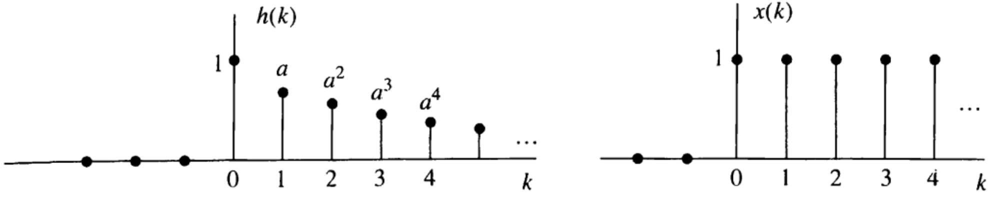


$$y(-1) = \sum_{k=-\infty}^{\infty} v_{-1}(k) = 1$$

Determine the output of a system for the given

$$h(n) = a^n u(n), \qquad |a| < 1$$

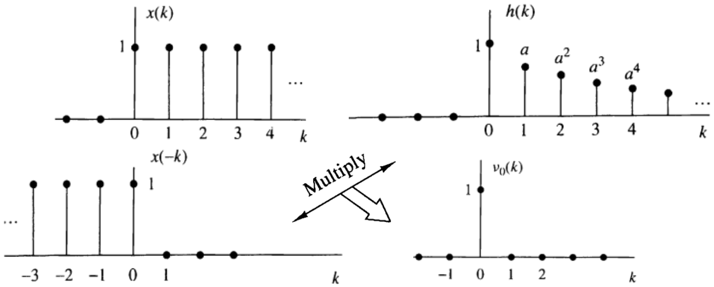
$$x(n) = u(n)$$



We know for the LTI system, $y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$ for all value of n

For $n = 0$ $\quad y(0) = \sum_{k=-\infty}^{\infty} h(k)\, x(0-k) = \sum_{k=-\infty}^{\infty} h(k) x(-k)$

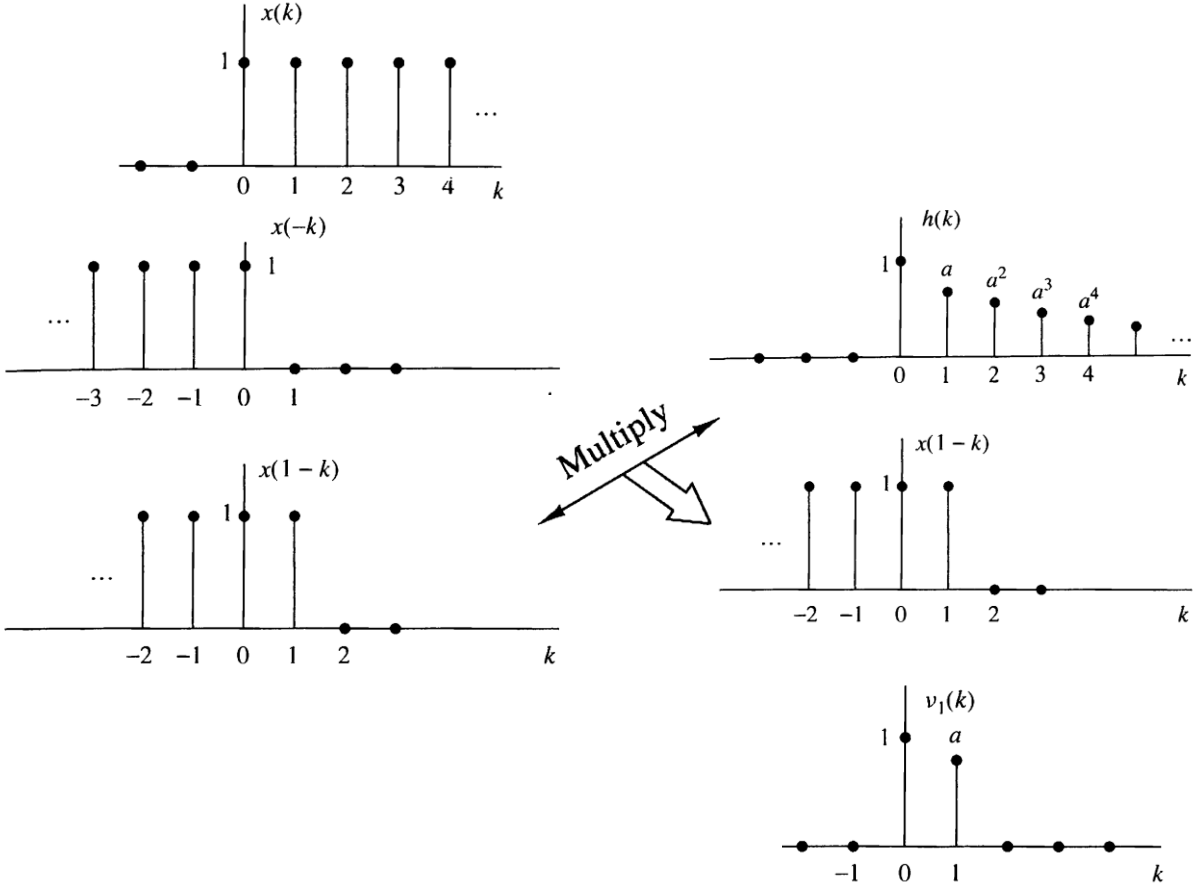
$$= 1$$



For $\boldsymbol{n} = \mathbf{1}$,

$$y(1) = \sum_{k=-\infty}^{\infty} h(k)\, x(1-k)$$

$$= 1 + a$$

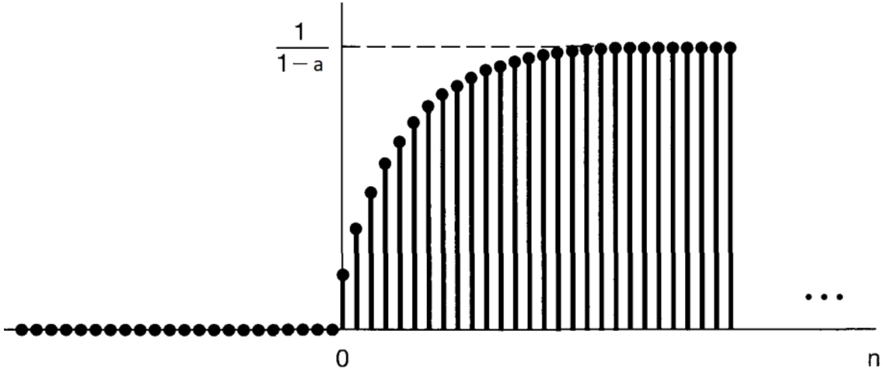


Similarly,

$$y(0) = 1$$
$$y(1) = 1 + a$$
$$y(2) = 1 + a + a^2$$

$$y(n) = 1 + a + a^2 + \ldots\ldots + a^n$$
$$= \frac{1 - a^{n+1}}{1-a}$$

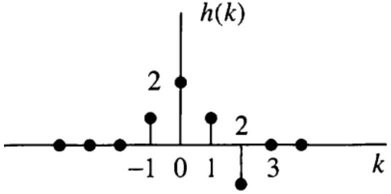
$$y[n] = \left(\frac{1 - a^{n+1}}{1 - a}\right) u[n]$$

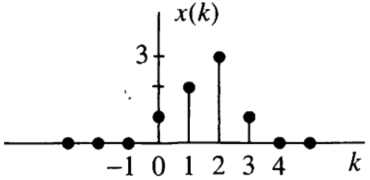


Determine the response of the Linear-time invariant system to the input signal $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$

The impulse response of a linear time-invariant system is $h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$

We know for the LTI system, $$y(n) = \sum_{k=-\propto}^{\propto} x(k)h(n-k)$$ for all value of n





Method-2:

$n = 0 \rightarrow y(0) = \sum_{k=-\propto}^{\propto} x(k)\ h(0-k) = \sum_{k=-\propto}^{\propto} x(k)\ h(-k)$

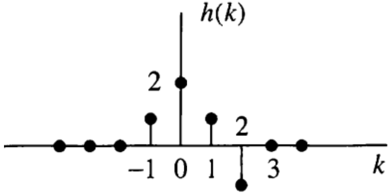
$$= \ldots + x(-1)h(-(-1) + x(0)\ h(-0) + x(1)\ h(-1) + x(2)\ h(-2) + x(3)\ h(-3) + \ldots$$

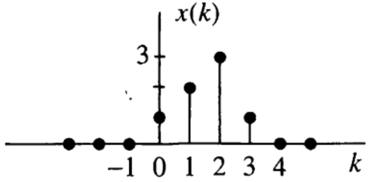
$$= \ldots + x(-1)\ h(1) + x(0)\ h(0) + x(1)\ h(-1) + x(2)\ h(-2) + x(3)\ h(-3) + \ldots$$

$$= \ldots + 0 * 1 + 1 * 2 + 2 * 1 + 3 * 0 + 1 * 0 + \ldots$$

$$= \ldots + 0 * 1 + 1 * 2 + 2 * 1 + 3 * 0 + 1 * 0 + \ldots$$

$$= 4$$





$n = -1 \rightarrow y(-1) = \sum_{k=-\propto}^{\propto} x(k)\, h(-1-k) = \sum_{k=-\propto}^{\propto} x(k)\, h(-1-k)$

$= \ldots + x(-1)h(-1+1) + x(0)\, h(-1-0) + x(1)\, h(-1-1) + x(2)\, h(-1-2) + x(3)\, h(-1-3) + \ldots$

$= \ldots + x(-1)\, h(0) + x(0)\, h(-1) + x(1)\, h(-2) + x(2)\, h(-3) + x(3)\, h(-4) + \ldots$

$= \ldots + 0 * 2 + 1 * 1 + 2 * 0 + 3 * 0 + 1 * 0 + \ldots$

$= 1$

$$y(n) = T\{x(n)\} = \sum_{k=-\propto}^{\propto} x(k)h(n-k) = x(n) * h(n) = h(n) * x(n) = \sum_{k=-\propto}^{\propto} x(n-k)h(k)$$ for all value of n

Let, $m = n - k\ ,\ k = n - m$

$$y(n) = \sum_{m=-\propto}^{\propto} x(n-m)h(m)$$

As "m" is a dummy index, we can again replace the m by k

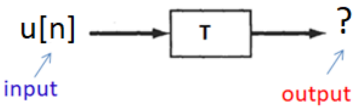
$$y(n) = \sum_{k=-\propto}^{\propto} x(n-k)h(k)$$

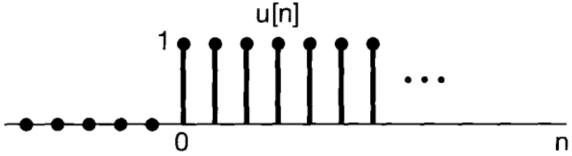
$$y(n) = T\{x(n)\} = x(n) * h(n) = h(n) * x(n)$$

Step response of LTI system

We have to calculate the response (output) of the system for input

$$x[n] = u[n]$$





$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

As per definition of convolution sum,

$$x[n] \rightarrow y[n] = s[n] = h[n] * x[n] = \sum_{k=-\propto}^{\propto} h[k]\, u[n-k]$$

$$s[n] = \sum_{k=-\propto}^{n} h[k]$$

$u[n-k] = 1 \quad if \ \ [\boldsymbol{n-k}] \geq \boldsymbol{0},$

$\rightarrow \ n - k \geq 0$

$\rightarrow k \leq n$

By induction method,

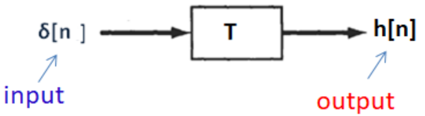
$$s[n-1] = \sum_{k=-\propto}^{n-1} h[k]$$

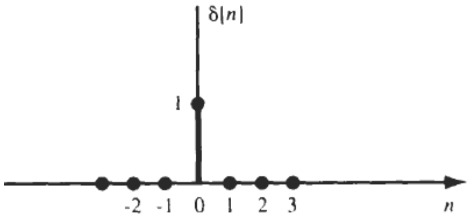
$$s[n] - s[n-1] = \sum_{k=-\propto}^{n} h[k] - \sum_{k=-\propto}^{n-1} h[k] = h[n]$$

Impulse response of the LTI

We have to calculate the response (output) of the system for input

$$x[n] = \delta[n]$$





$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

As per definition of convolution sum,

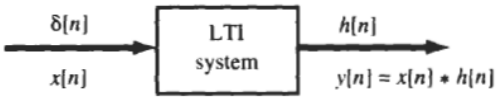
$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, \delta[n-k]$$

$$= h[n]$$

$\delta(n-k) = 1,\ if\ \boldsymbol{n-k=0}$
$\rightarrow k = n$

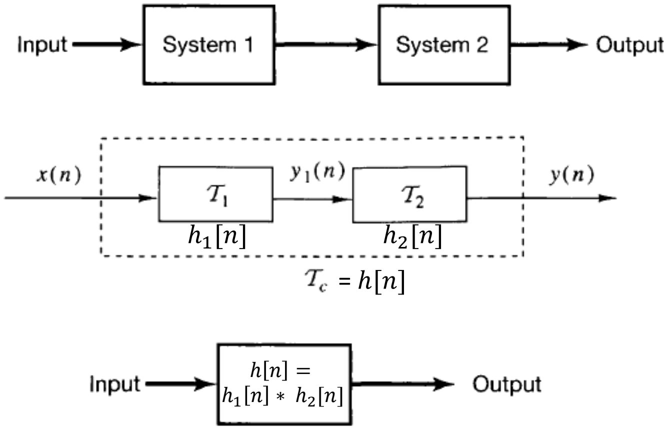
Inter connection of LTI systems

Commutative:



$$y[n] = x[n] * h[n] = h[n] * x[n]$$

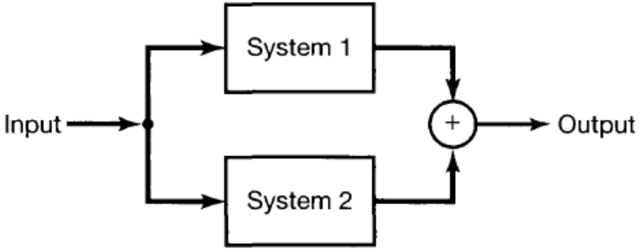
Associative:

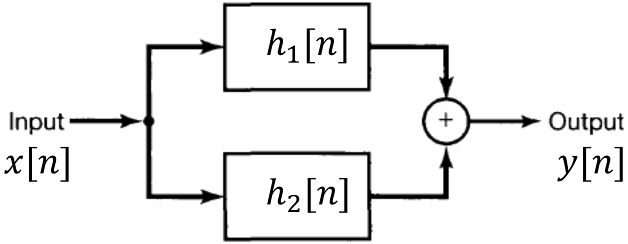


$$y[n] = \{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$$

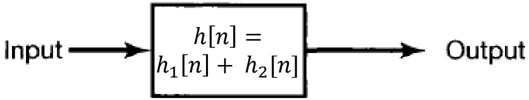
(Proof yourself)

Distributive:





$$
\begin{aligned}
y[n] &= x[n] * \{h_1[n] + h_2[n]\} \\
&= x[n] * h_1[n] + x[n] * h_2[n]
\end{aligned}
$$



Evaluate the response of the system

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$$

$$\mathrm{h}[n] = u[n]$$

As per definition of convolution sum,

$$y[n] = x[n] * h[n]$$

Let, $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$ and $x_2[n] = 2^n u[-n]$

then, $y[n] = (x_1[n] + x_2[n]) * h[n]$

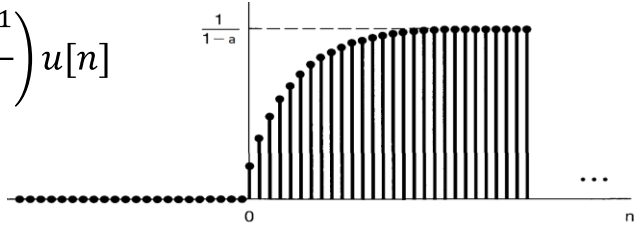
$= y_1[n] + y_2[n]$ (using distribution property)

$$\boxed{\begin{aligned} y_1[n] &= x_1[n] * h[n] \\ y_2[n] &= x_2[n] * h[n] \end{aligned}}$$

$x_1[n] = a^n u(n), a = 1/2$

$h[n] = u[n]$

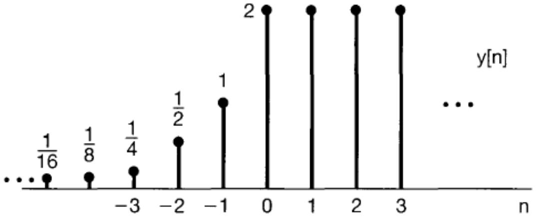
$$y_1[n] = \left(\frac{1 - a^{n+1}}{1 - a}\right) u[n]$$



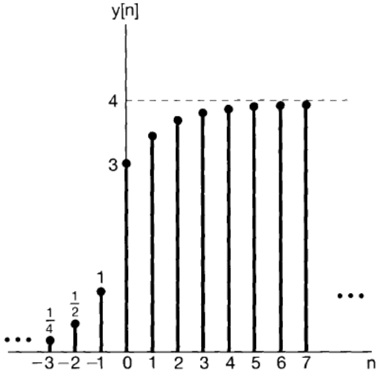
$x_2[n] = 2^n u[-n]$

$h[n] = u[n]$

$$y_2[n] = 2^{n+1}$$



$y[n] = y_1[n] + y_2[n]$



Memory/static or memoryless

A system is said to be *memory-less*

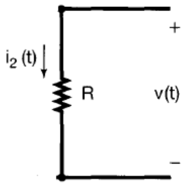
- *if the output at any time depends on* ***only the input*** at **that same time**

  $(y(n))$ at any particular time n0 depends only on the value of $x[n]$ at that time n0)

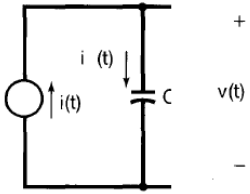
  In memoryless: no need to store any of the past inputs or outputs in order to compute the present output

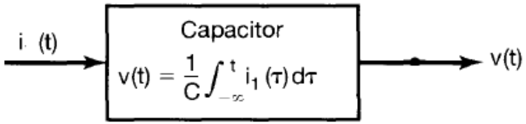
- Otherwise, the system is said **to have *memory***

e.g. a memory less system is a resistor R with the input *x(t) taken as the voltage and current at* the output *y(t)*



A **system with memory** is a capacitor ***C with the current as the input*** $x(t)$ and the voltage as the output $y(t)$





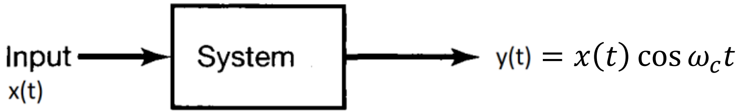
Determine whether systems are memoryless

$(a)\ \ y(t) = x(t)\cos\omega_c\, t$

$(b)\ y[n] = x[n-1]$

(a)

$y(t) = x(t)\cos\omega_c\, t \quad \Rightarrow$



Both same time index

$t = -1 \quad \rightarrow y(-1) = x(-1)\cos(\omega_c)$

$t = 0 \quad \rightarrow y(0) = x(0)\cos(\omega_c)$

$t = 1 \quad \rightarrow y(1) = x(1)\cos(\omega_c)$

....

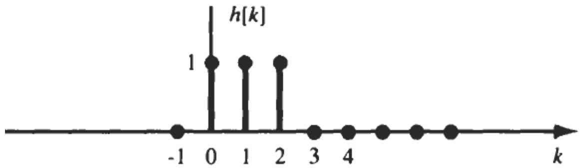
Both same time index

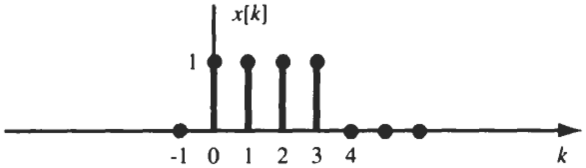
Since, the value of the output $y(t)$ *depends*

⇓

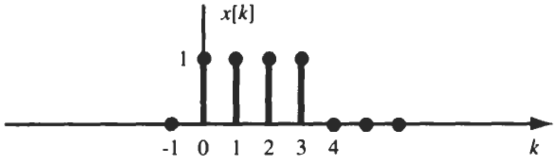
on only the current values of the input $x(t)$

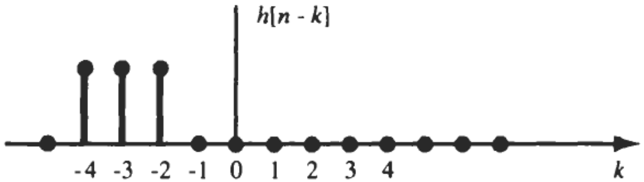
the system ***is* memoryless**



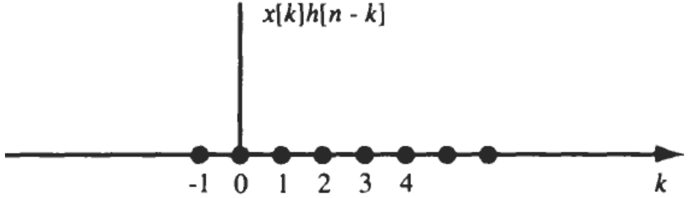


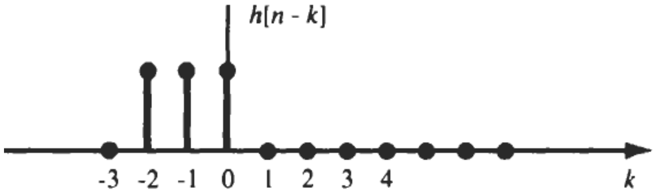
We know for the LTI system, $y(n) = \sum_{k=-\propto}^{\propto} x(k)h(n-k)$ for all value of n



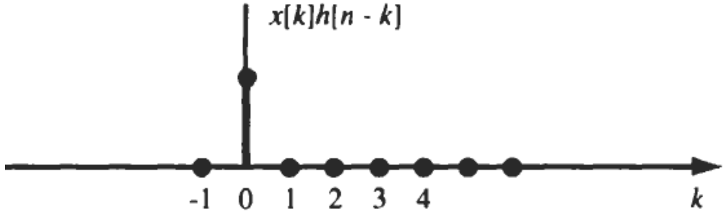


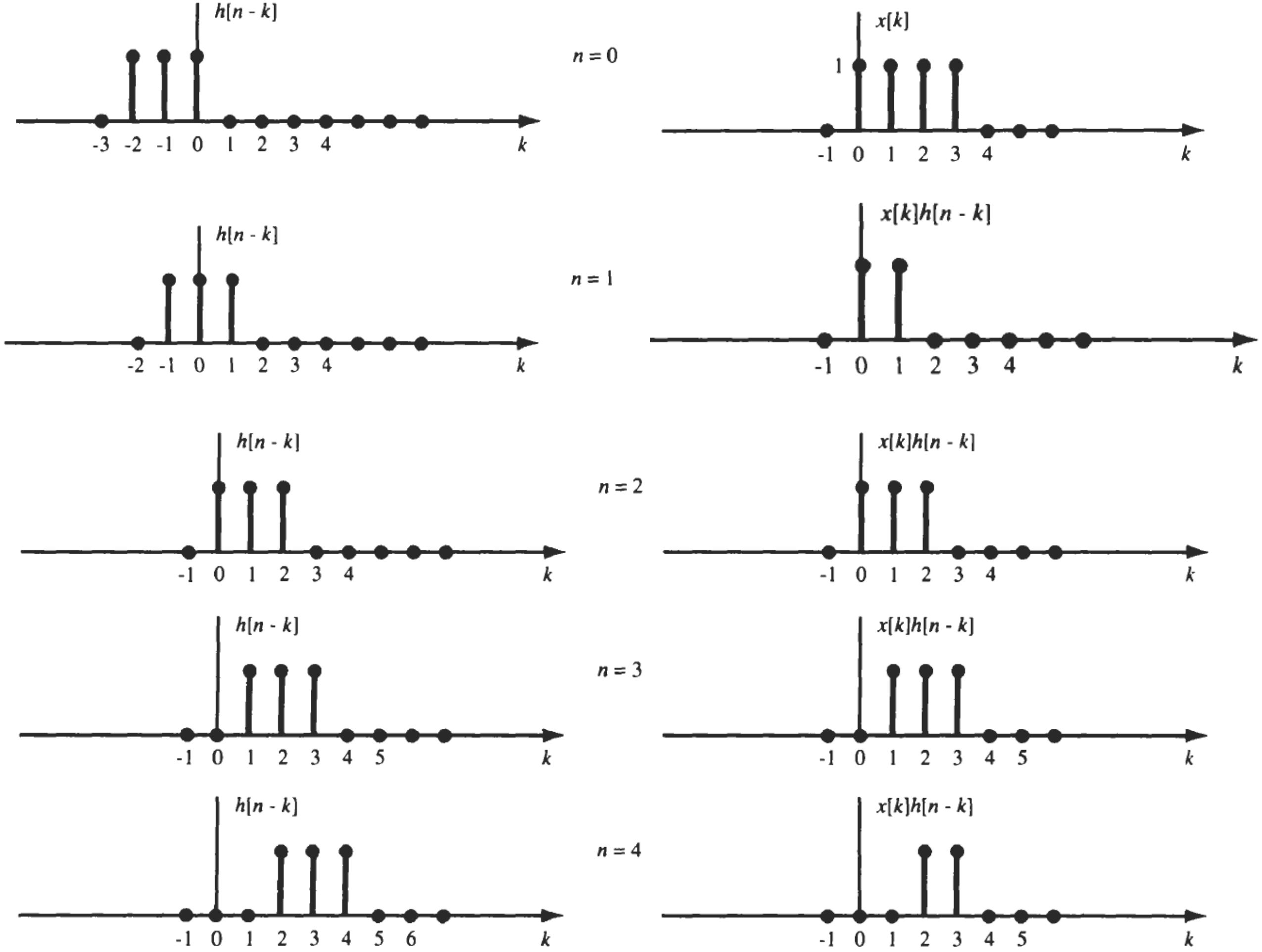
$n < 0$

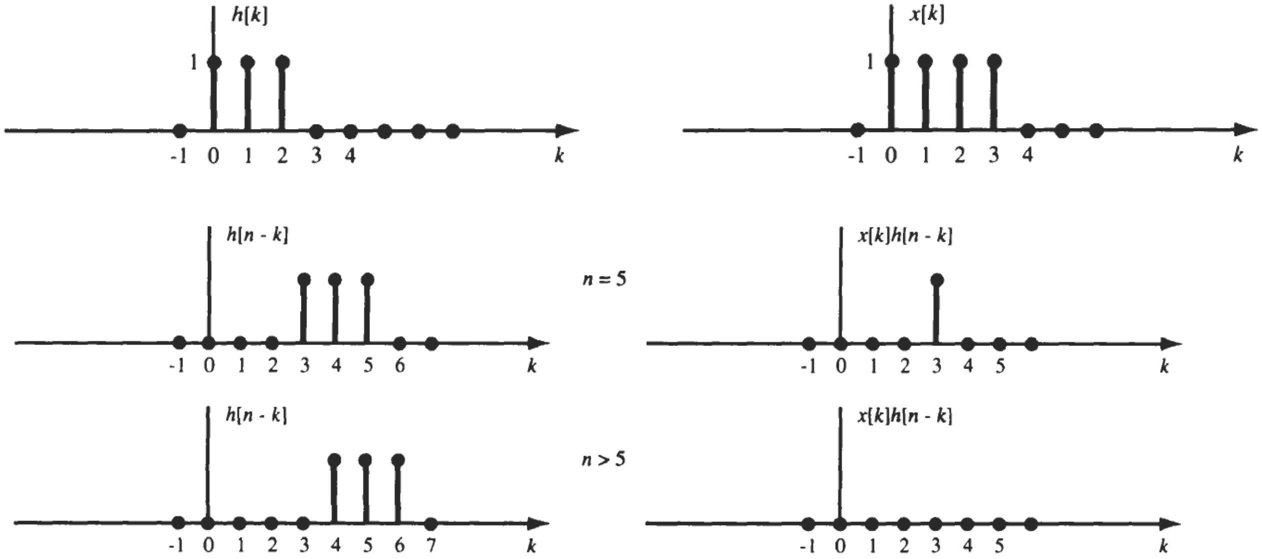




$n = 0$




h[n - k]
n = 0
x[k]
1
-3 -2 -1 0 1 2 3 4
k
-1 0 1 2 3 4
k
h[n - k]
n = 1
x[k]h[n - k]
-2 -1 0 1 2 3 4
-1 0 1 2 3 4
h[n - k]
n = 2
x[k]h[n - k]
-1 0 1 2 3 4
h[n - k]
n = 3
x[k]h[n - k]
-1 0 1 2 3 4 5
h[n - k]
n = 4
x[k]h[n - k]
-1 0 1 2 3 4 5 6
-1 0 1 2 3 4 5



$$y[n] = \{1, 2, 3, 3, 2, 1\}$$

$y[0] = 1$ $y[1] = 2$ $y[2] = 3$ $y[3] = 3$ $y[4] = 2$ $y[5] = 1$

Condition for LTI system to be memoryless

We know for the **LTI system input-output** is related

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Output time-instant

$$= \sum_{k=0}^{\infty} h[k]\, x[n-k] + \sum_{k=-\infty}^{-1} h[k]\, x[n-k]$$

Past inputs

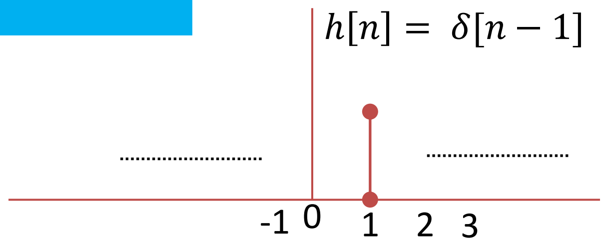
$$= (h[0]\, x[n] + h[1]\, x[n-1] + h[2]\, x[n-2] + \ldots\ldots)$$

$$+\underbrace{(h[-1]\, x[n+1] + h[-2]\, x[n+2] + \ldots\ldots)}_{\text{Future inputs}}$$

As per definition of memory less system: **output** at **any time-instant depends only on the value of the input** at **that same time-instant**

For **memoryless LTI system,** $h[n] = 0;\, n \neq 0$

Q1. The impulse response of a LTI system is expressed as $h[n] = \delta[n-1]$. Determine whether system is memoryless.



For **memoryless LTI system**, $h[n] = 0; n \neq 0$

The given $h[n]$ has 1 value at $n = 1$. Therefore, it does not satisfy the condition for memoryless. The system has memory.

Q2. The input-output of a LTI system is given by $y(n) = x(n-1)$. Verify whether system is memoryless.

For **memoryless LTI system**, $h[n] = 0; n \neq 0$

We need to calculate the $h[n]$ which is *the output of system* for $x(n) = \delta(n)$

$$\mathrm{h}(n) = \tau\{x(n) = \delta(n)\} = \delta(n-1)$$

The given $h[n]$ has 1 value at $n = 1$. Therefore, it does not satisfy the condition for memoryless. The system has memory.

Causality

- A system is *causal* if the **output at any time** depends only on values of the input at the present time and in the past (does not depend on future inputs)

Mathematically,

Output: $y(n)$ at time − instant $n = n_0$ **depends only on value** of $x(n)$ for $n \le n_0$

All **memory less** systems are causal.

Example of non-causal/anti-causal systems:

$$(a)\ y(n) = x(-n)$$
$$(b)\ y(t) = x(t+1)$$

Output depending on future value of input-time

$$\text{For} \quad n=-1, \quad y(-1) = x(-(-1)) = x(1)$$
$$n=0, \quad y(0) = x(0)$$
$$n=1, \quad y(1) = x(-1)$$

Condition for LTI system to be Causal

We know for the LTI system input-output is related

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Let's estimate output for $n = n_0$

$$y[n_0] = \sum_{k=-\infty}^{\infty} h[k]x[n_0 - k] \quad = \sum_{k=0}^{\infty} h[k]\, x[n_0 - k] + \sum_{k=-\infty}^{-1} h[k]\, x[n_0 - k]$$

$$= (h[0]\ x[n_0] + h[1]\, x[n_0 - 1] + h[2]\ x[n_0 - 2] + \ldots\ldots)$$

$$+(h[-1]\ x[n_0 + 1] + h[-2]\, x[n_0 + 2] + \ldots\ldots)$$

Input at **past** time-instant

Input at **present** time-instant

Input at **future** time-instant

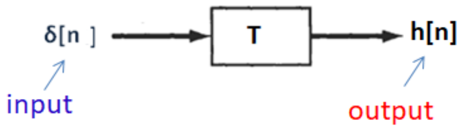
For causal system, output at time n = n_0 can depend only on the present and past inputs. Therefore, the causality condition for a discrete-time LTI system is

$$h[n] = 0, \qquad n < 0$$

Is the following system causal?

$$y[n] = \sum_{k=-\infty}^{n} 2^{k-n} x[k+1]$$

We need to calculate **the $h[n]$** which is *the* ***output of system*** for $x[n] = \delta[n]$



Impulse response of the system can be written as $h[n] = T\{\delta[n]\} = y[n] = \sum_{k=-\infty}^{n} 2^{k-n} \delta[k+1]$

For causality of LTI system, it should satisfy the condition $h[n] = 0, \quad n < 0$

$$h[n] = \sum_{k=-\infty}^{n} 2^{k-n} \delta[k+1]$$

$$= \sum_{m=-\infty}^{n+1} 2^{m-1-n} \delta[m]$$

changing the variable, $k + 1 = m$

$$= 2^{-(n+1)} \sum_{m=-\infty}^{n+1} 2^{m} \delta[m]$$

Let's consider, $n = -1, \quad h[-1] \neq 0$

System is not causal

Determine whether systems are causal or anti-causal

$$(a)\ \ h[n] = \left(\frac{1}{2}\right)^{n} u[n]$$

$$(b)\ \ h[n] = \left(\frac{1}{2}\right)^{n} u[n-1]$$

$$(c)\ \ h[n] = \left(\frac{1}{2}\right)^{|n|}$$

$$(d)\ \ h[n] = u[n+2] - u[n-2]$$

$$(e)\ \ h[n] = \left(\frac{1}{3}\right)^{n} u[n] + 3^{n}\, u[-n-1]$$

$|x| = x$ if x is positive,
$|x| = -x$ if x is negative (in which case $-x$ is positive),
$|0| = 0$

Stability

A system is said **to be stable** if bounded input (BI) (finite) produces the bounded output (BO) (finite)

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \qquad\qquad x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

For discrete case: if

$$|x[n]| \le M_x < \infty$$

$$\rightarrow |y[n]| \le M_y < \infty \quad \text{for } all\ n$$

For continuous case: if

$$|x(t)| \le M_x < \infty$$

$$\rightarrow |y(t)| \le M_y < \infty \ \text{ for } all\ t$$

Determine whether the following system is stable ?

$$(a)\ \ y(t) = x(t)\cos\omega_c t \quad \text{for all } t$$

$$(b)\ y[n] = x[n-1]$$

$$|y(t)| = |x(t)\cos\omega_c t| \ \le |x(t)|\,|\cos\omega_c t| \qquad\qquad |\cos\omega_c t| \ \le 1$$

$$\le x(t)$$

If input $x(t)$ is **bounded** → $y(t)$ will be **bounded** i.e. BIBO

$(b)\ y[n] = x[n-1]$

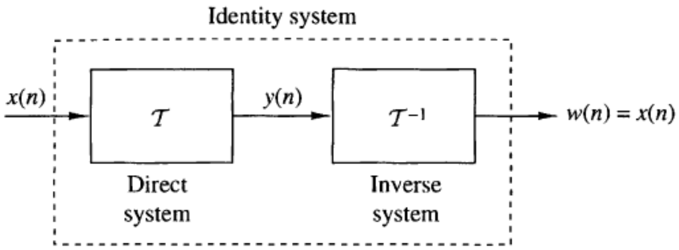
$$|y[n]| = |x[n-1]| \le k \qquad if\ |x[n]| \le k \ \ \text{for}\ all\ n$$

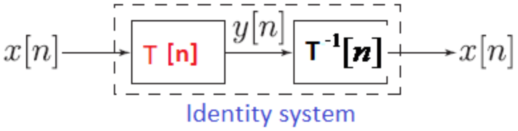
If input $x(n)$ is **bounded (finite)** → $y(n)$ will be **bounded (finite)** i.e. BIBO

the system is BIBO stable

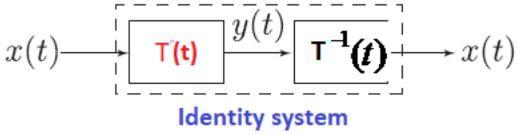
Invertibility

- A system is **invertible**, if an inverse system exists that when cascaded with the original system yields an output equal to input
- A system is invertible if **distinct inputs lead** to **distinct outputs**





Continuous case:



Determine if the following systems are invertible or not

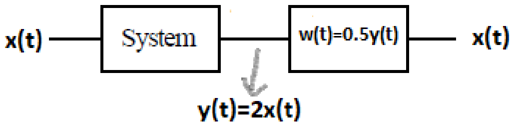
$(a)\ y(t) = 2\,x(t)$

$(b)\ y(t) = \ x^2(t)$

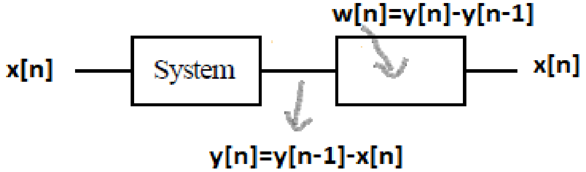
$(c)\ y[n] = \sum_{k=-\infty}^{\infty} x[k]$

$(d)\ y[n] = 0$

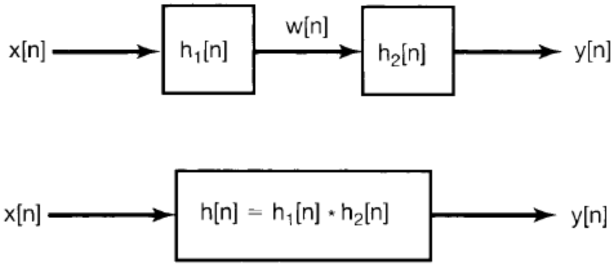
(a)



$$(c)\quad y[n] = \sum_{k=-\infty}^{n} x[k] = [\cdots x[n-2] + x[n-1]] + x[n] = y[n-1] + x[n]$$



Invertibility – LTI system



$$h[n] * h_1[n] = \delta[n]$$

(identity)

- Consider an LTI system with impulse response $h[n] = u[n]$. Determine whether inverse system of it is exist.

We know for the LTI system input-output is related

$$y[n] = \sum_{k=-\propto}^{\propto} x[k]\, h[n-k] = \sum_{k=-\propto}^{\propto} x[k]\, u[n-k]$$

$$y[n] = \sum_{k=-\propto}^{n} x[k] \quad \dots (\mathbf{1})$$

for $(n-k) > 0, u[n-k] = 1$

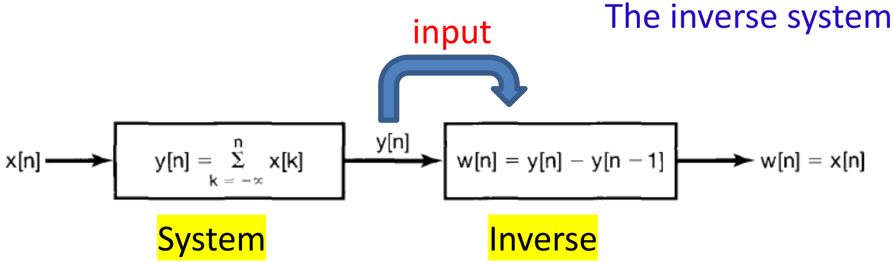
$$y[n] = \sum_{k=-\infty}^{n} x[k] \quad \ldots (1)$$

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k] \quad \ldots (2)$$

Using Eq. (1) & (2)

$$y[n] - y[n-1] = \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k] = x[n]$$

The inverse system is $y[n] - y[n-1] = x[n]$



To evaluate the impulse response of **the inverse system**, consider input to the system $y[n] = \delta[n]$

$$h_1[n] = \tau\{\delta[n]\} = \delta[n] - \delta[n-1]$$

Cross-check: $h_I[n] = h[n] * h_1[n] \quad = u[n] * (\delta[n] - \delta[n-1]) \quad = u[n] * \delta[n] - u[n] * \delta[n-1]$

$$= u[n] * \delta[n] - u[n] * \delta[n-1]$$

$1 \ \text{for} \ [n-k] = 0,$
$\rightarrow$ k=n

$1 \ \text{for} \ [n-k-1] = 0,$
$\rightarrow \ k = n-1$

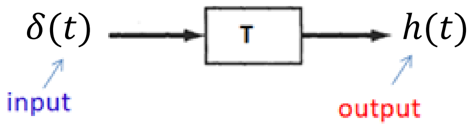
$$= \sum_{k=-\infty}^{\infty} u[k]\, \delta[n-k] \ + \sum_{k=-\infty}^{\infty} u[k]\, \delta[n-k-1]$$

$$= u[n] - u[n-1]$$

$$= \delta[n]$$

$$h_I[n] = h[n] * h_1[n] = \delta\ [n]$$

Response of LTI systems to complex exponentials (continuous-time signal)



From convolution integral of LTI continuous-time system, we can write

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau$$

Let's consider complex exponential input to the system $x(t) = e^{st}$

Then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{st} e^{-s\tau}\, d\tau \quad = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau$$

$$y(t) = e^{st} H(s)$$

(response in the form of e^{st})

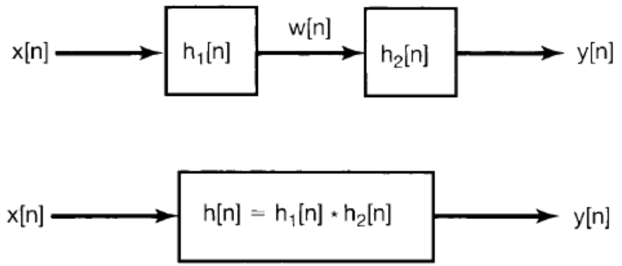
Eigen function

Eigen value

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

Shows that **complex exponentials** are **eigenfunctions** of LTI systems

Invertibility – LTI system



$$h[n] * h_1[n] = \delta[n]$$

(identity)

- Consider an LTI system with impulse response $h[n] = u[n]$. Determine whether inverse system of it is exist.

We know for the LTI system input-output is related

$$y[n] = \sum_{k=-\propto}^{\propto} x[k]\, h[n-k] = \sum_{k=-\propto}^{\propto} x[k]\, u[n-k]$$

$$y[n] = \sum_{k=-\propto}^{n} x[k] \quad \dots(\mathbf{1})$$

for $(n-k) > 0\,, u[n-k] = 1$

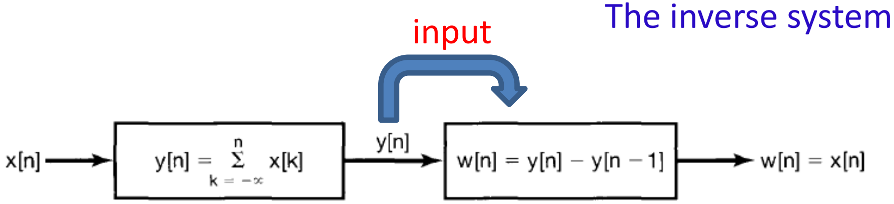
$$y[n] = \sum_{k=-\infty}^{n} x[k] \quad \ldots (1)$$

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k] \quad \ldots (2)$$

Using Eq. (1) & (2)

$$y[n] - y[n-1] = \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k] = x[n]$$

The inverse system is $y[n] - y[n-1] = x[n]$



System

Inverse

To evaluate the impulse response of the inverse system, consider input to the system $y[n] = \delta[n]$

$$h_1[n] = \tau\{\delta[n]\} = \delta[n] - \delta[n-1]$$

Cross-check: $h_I[n] = h[n] * h_1[n] \quad = u[n] * (\delta[n] - \delta[n-1]) \quad = u[n] * \delta[n] - u[n] * \delta[n-1]$

$$= u[n] * \delta[n] - u[n] * \delta[n-1]$$

$1 \text{ for } [n-k] = 0,$ $\rightarrow$ k=n

$1 \text{ for } [n-k-1] = 0,$ $\rightarrow$ $k = n-1$

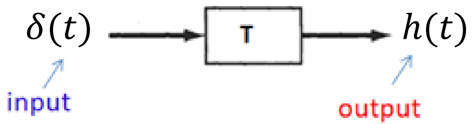
$$= \sum_{k=-\infty}^{\infty} u[k]\, \delta[n-k] - \sum_{k=-\infty}^{\infty} u[k]\, \delta[n-k-1]$$

$$= u[n] - u[n-1]$$

$$= \delta[n]$$

$$\boxed{h_I[n] = h[n] * h_1[n] = \delta\, [n]}$$

Response of LTI systems to complex exponentials (continuous-time signal)



From convolution integral of LTI continuous-time system, we can write

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

Let's consider complex exponential input to the system $x(t) = e^{st}$

Then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{st} e^{-s\tau}\,d\tau \quad = e^{st}\int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\,d\tau$$

$$y(t) = e^{st} H(s)$$

(response in the form of e^{st})

Eigen function

Eigen value

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\,d\tau$$

Shows that **complex exponentials** are **eigenfunctions** of LTI systems

The input-output relation of a continuous-time LTI system is given by

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} x(\tau) d\tau$$

(a) Evaluate the impulse response $h(t)$ of the system
(b) Show that complex exponential function e^{st} is an eigen function of the system
(c) Evaluate the eigen value of the system for e^{st} using the impulse response obtained in (a)

Impulse-response nothing but the output of the system for input $x(t) = \delta(t)$

(a)

$$x(t) = \delta(t) \rightarrow y(t) = h(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} \delta(t) \, d\tau \quad = e^{-(t-\tau)}|_{\tau=0}$$

$$= e^{-t}, \quad (say\ for\ t \geq 0)$$

$$h(t) = e^{-t} u(t)$$

(b) Let $x(t) = e^{st}$

Using convolution integral, we know $y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) \, d\tau = \int_{-\infty}^{\infty} e^{-\tau} u(\tau) \, e^{s(t-\tau)} \, d\tau$

$$y(t) = \int_0^{\infty} e^{-\tau}\, e^{st} e^{-s\tau}\, d\tau$$

$$= e^{st} \int_0^{\infty} e^{-(1+s)\tau}\, d\tau$$

$$= e^{st} . \frac{e^{-(1+s)\tau}}{-(1+s)} \Big|_0^{\tau=\infty}$$

$$= \frac{e^{st}}{-(1+s)} [e^{-(1+s).\infty} - e^{-(1+s).0}]$$

$$= \frac{e^{st}}{-(1+s)} . [0 - 1]$$

$$= e^{st} . \underbrace{\left(\frac{1}{1+s}\right)}_{\text{Eigen-value}}$$

(c) Using the relation

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$$Since, \qquad h(t) = e^{-t} u(t)$$

$$= \int_{-\infty}^{\infty} e^{-\tau} u(\tau) . e^{-s\tau} d\tau$$

$$= \int_0^{\infty} e^{-\tau} . e^{-s\tau}\, d\tau$$

$$= \int_0^{\infty} e^{-(1+s)\tau}\, d\tau$$

$$= \frac{1}{1+s}$$

Decomposing of signals in terms of eigenfunctions

Let's $x(t)$ is a linear combination of three exponential signals

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$
$$= x_1(t) + x_2(t) + x_3(t)$$
$$= \sum_{k=1}^{3} a_k \, e^{s_k t}$$

If $x(t)$ is applied to a LTI system, the response of the system ?

Using the eigen function concept of continuous LTI system, we can write

$$x_1(t) = a_1 e^{s_1 t} \rightarrow y_1(t) = e^{s_1 t} a_1 H(s_1)$$

$$x_2(t) = a_2 e^{s_2 t} \rightarrow y_2(t) = e^{s_2 t} a_2 H(s_2)$$

$$x_3(t) = a_3 e^{s_3 t} \rightarrow y_3(t) = e^{s_3 t} a_3 H(s_3)$$

Using the superposition property

$$x(t) \rightarrow y(t) = y_1(t) + y_2(t) + y_3(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t} = \sum_{k=1}^{3} a_k H(s_k) \, e^{s_k t}$$

Observation:

If the **input** to a continuous-time LTI system is a linear combination of complex exponentials i.e.

$$x(t) = \sum_k a_k\, e^{s_k t}$$

$$\Downarrow$$

Output:

$$y(t) = \sum_k a_k H(s_k)\, e^{s_k t}$$

(a linear combination of the input complex exponential signals)

Harmonically Related Complex Exponentials

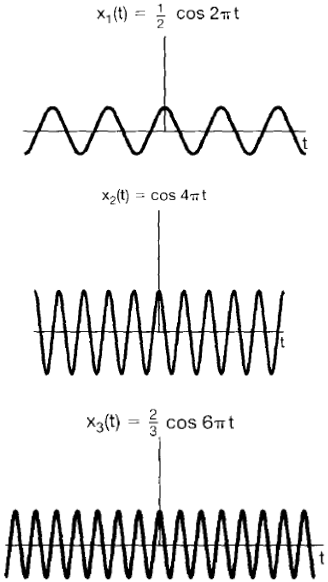
Let's consider a signal,

$$\emptyset_k(t) = e^{jk\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$k = 0 \rightarrow \emptyset_0(t) = e^{j.0.\omega_0 t}$$

$$k = 1 \rightarrow \emptyset_1(t) = e^{j.1.\omega_0 t}$$

$$k = 2 \rightarrow \emptyset_2(t) = e^{j.2.\omega_0 t}$$



Each of these signals has a fundamental frequency that is a **multiple of ω_0** → **called harmonically related**

$|k| \geq 2$, the fundamental period of $\emptyset_k(t)$ is fraction of $\boldsymbol{T}$

The fundamental period of the signal $\emptyset_k(t)$ is T

Fourier-series representation

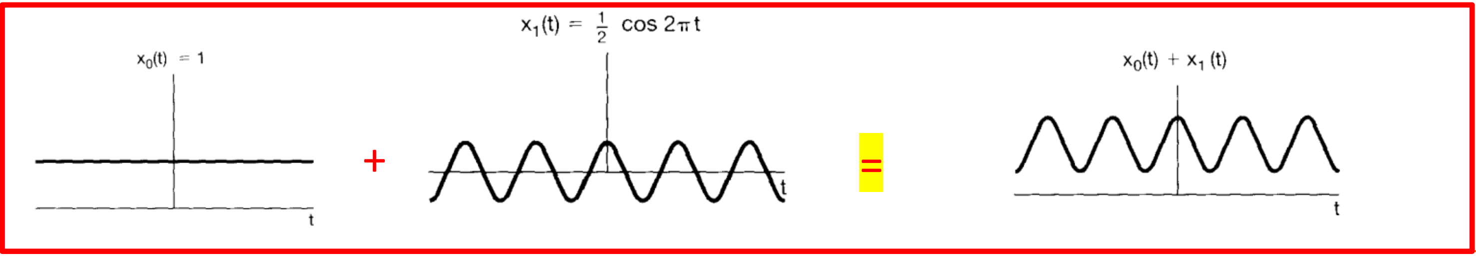
- A **linear combination of harmonically** related **complex exponentials can be written as,**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$, ω_0 is the fundamental frequency and $T = \frac{2\pi}{\omega_0}$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$ (called **synthesis equation**)

$$= \ldots + a_{-1}e^{j.-1.\left(\frac{2\pi}{T}\right)t} + a_0 e^{j.0.\left(\frac{2\pi}{T}\right)t} + a_1 e^{j.1.\left(\frac{2\pi}{T}\right)t} + a_2 e^{j.2.\left(\frac{2\pi}{T}\right)t} + \ldots\ldots$$

$\boldsymbol{a_k}$ are called the *Fourier series coefficients*



- Determine the Fourier series coefficient $x(t) = \sin \omega_0 t$

From **Fourier-series synthesis equation**, we know

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \ldots + a_{-1} e^{j.-1.\omega_0 t} + a_0 e^{j.0.\omega_0 t} + a_1 e^{j\omega_0 t} + \ldots..$$

$$= \ldots + a_{-1} e^{-j.\omega_0 t} + a_0 e^{0} + a_1 e^{j\omega_0 t} + \ldots..$$

sin=e^jwt-e^-jwt/2

Expanding the given equation,

$$x(t) = \sin \omega_0 t$$

$$= \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}]$$

cos=e^jwt+e^-jwt/2

$$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Comparing two equations

$$a_{-1} = -\frac{1}{2j}, \quad a_1 = \frac{1}{2j}, \quad a_k = 0, \qquad k \neq \pm 1$$

Determine the Fourier series coefficient

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos(2\omega_0 t + \frac{\pi}{4})$$

From **Fourier-series synthesis equation**, we know

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \ldots + a_{-1}e^{-j.\omega_0 t} + a_0 e^0 + a_1 e^{j\omega_0 t} + \ldots\ldots$$

Comparing two equations

Expanding the given equation $x(t)$

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \frac{\pi}{4})}\right]$$

$$= 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{\frac{j\pi}{4}}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{\frac{-j\pi}{4}}\right)e^{-j2\omega_0 t}$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

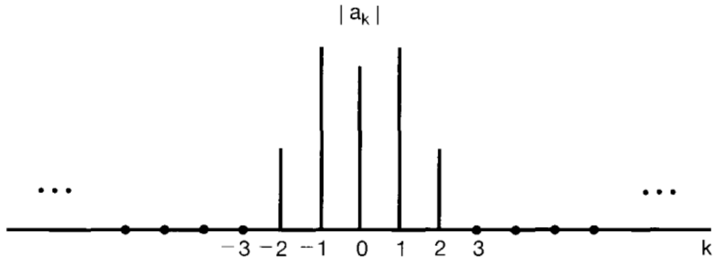
$$a_0 = 1,\ a_1 = \left(1 + \frac{1}{2j}\right), a_{-1} = \left(1 - \frac{1}{2j}\right), a_2 = \frac{1}{2}e^{j\pi/4}, a_{-2} = \frac{1}{2}e^{-j\pi/4},\ a_k = 0,\ \text{for } |k| > 2$$

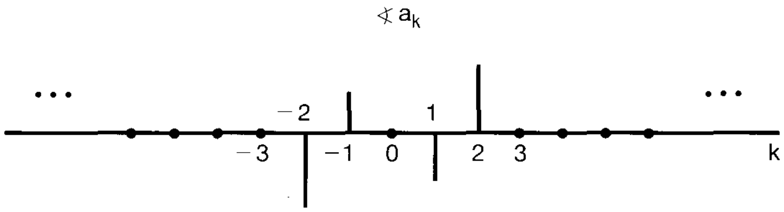
Cont..

- Magnitude and phase plot of a_k

$$a_0 = 1,$$
$$a_1 = \left(1 + \frac{1}{2j}\right),$$
$$a_{-1} = \left(1 - \frac{1}{2j}\right),$$
$$a_2 = \frac{1}{2}\, e^{j\pi/4},$$
$$a_{-2} = \frac{1}{2}\, e^{-j\pi/4},$$

$$a_k = 0, \text{ for } |k| > 2$$





$$z = x + jy$$
$$Magnitude\ |z| = \sqrt{x^2 + y^2}$$
$$Phase\ \measuredangle z = \tan^{-1}\frac{y}{x}$$

Fourier-series (analysis equation)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$ (synthesis equation)

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk\left(\frac{2\pi}{T}\right)t}dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk.2\pi f_0 t}dt$$ (Analysis equation)

Derivation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\rightarrow x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . e^{-jn\omega_0 t}$$ (multiplying both side $e^{-jn\omega_0 t}$)

$$\rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . e^{-jn\omega_0 t}\, dt$$ (Integrating both side from 0 to T)

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt$$

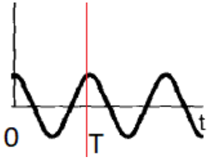
$$\rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . e^{-jn\omega_0 t}\, dt \quad = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \left(\int_0^T \cos(k-n)\omega_0 t \; dt + j \int_0^T \sin(k-n)\omega_0 t \; dt \right)$$

For $k = n$ $\left(\int_0^T \cos(k-n)\omega_0 t \; dt + j \int_0^T \sin(k-n)\omega_0 t \; dt \right) = T$

For $k \neq n$ $\left(\int_0^T \cos(k-n)\omega_0 t \; dt + j \int_0^T \sin(k-n)\omega_0 t \; dt \right) = 0$

(integration -> area under function)



In one period,
$+ve$ and $-ve$ value ,
area will zero

$$\rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt = a_n T$$

$$a_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

(**Analysis equation**)

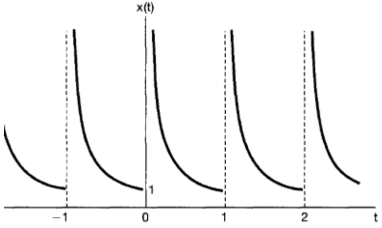
Fourier -series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

(synthesis equation)

$$a_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

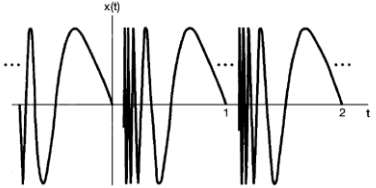
(Analysis equation)



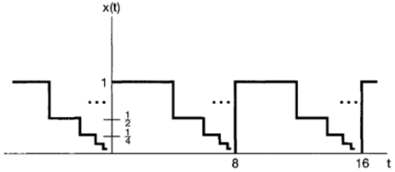
Convergence of Fourier Series (Dirichlet condition):

1. $x(t)$ is absolutely integrable over any period, that is,

$$\int_{T_0} |x(t)|\, dt < \infty$$



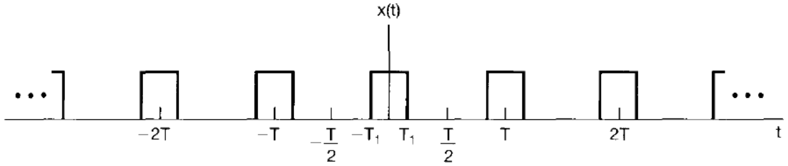
2. $x(t)$ has a finite number of maxima and minima within any finite interval of t.
3. $x(t)$ has a finite number of discontinuities within any finite interval of t, and each of these discontinuities is finite.



- The periodic square wave defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

Determine the Fourier series coefficients for $x(t)$.



Fundamental period $\boldsymbol{T}$,

Fundamental frequency $\omega_0 = 2\pi F_0 = \frac{2\pi}{T}$

We have to calculate,

$$a_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

n=0

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} 1.\, e^{-j.(n=0).\omega_0 t}\, dt = \frac{2T_1}{T}$$

n ≠ 0

$$a_n = \frac{1}{T}\int_{-T_1}^{T_1} 1.\, e^{-j.n.\omega_0 t}\, \mathrm{dt}$$

$$= \frac{1}{T}\left[\frac{e^{-jn\omega_0\, t}}{-jn\omega_0}\right]_{-T_1}^{T_1}$$

n ≠ 0

$$a_n = \frac{1}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-T_1}^{T_1}$$

$$= \frac{1}{-jn\omega_0 T}\left[e^{-jn\omega_0 T_1} - e^{jn\omega_0 T_1}\right]$$

$$= \frac{2}{n\omega_0 T}\left[\frac{e^{jn\omega_0 T_1} - e^{-jn\omega_0 T_1}}{2j}\right]$$

$$= \frac{2}{n\omega_0 T}\sin n\omega_0 T_1$$

$$= \frac{2}{n2\pi}\sin n\omega_0 T_1$$

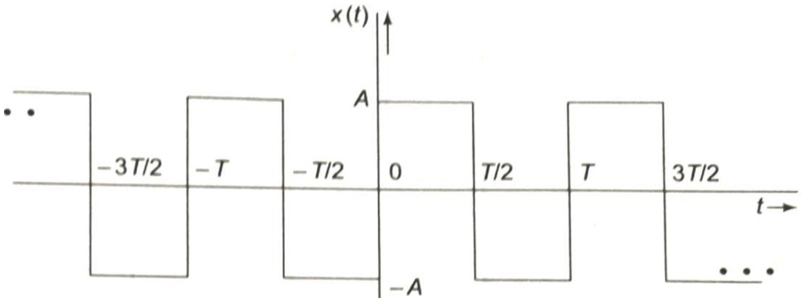
$$= \frac{\sin n\omega_0 T_1}{n\pi}$$

$$as\ \omega_0 = 2\pi F_0 = \frac{2\pi}{T}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Determine the Fourier series coefficients



Fundamental period $\boldsymbol{T}$,

Fundamental frequency $\omega_0 = 2\pi F_0 = \frac{2\pi}{T}$

We have to calculate, $$a_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

n=0 $$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t).\, e^{-j.(n=0).\omega_0 t}\, dt \quad = \frac{1}{T}\left[\int_{-\frac{T}{2}}^{0} -A.\,1dt + \int_0^{\frac{T}{2}} A.\,1dt\right] = 0$$

n ≠ 0

$$a_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\omega_0 t}\,dt = \frac{1}{T}\left[\int_{-\frac{T}{2}}^{0} -A.\,e^{-jn\omega_0 t}\,dt + \int_{0}^{\frac{T}{2}} A.\,e^{-jn\omega_0 t}\,dt\right]$$

$$= \frac{-A}{T}\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\Bigg|_{\frac{-T}{2}}^{0} + \frac{A}{T}\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\Bigg|_{0}^{T/2}$$

$$as\ \omega_0 = 2\pi F_0 = \frac{2\pi}{T}$$

$$= \frac{A}{jn2\pi F_0.T}\left[1 - e^{j\pi n}\right] + \frac{A}{-jn2\pi F_0 T}\left[1 - e^{-j\pi n}\right]$$

$$= \frac{A}{j2\pi n}\left[2 - \left(e^{j\pi n} + e^{-j\pi n}\right)\right]$$

$$= \frac{A}{j\pi n}\left[1 - \cos\pi\, n\right]$$

Properties

Linearity: Let $x(t)$ and $y(t)$ denote two periodic signals with *period* T and which have *Fourier-series coefficients*

$$a_k, b_k$$

$$x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$y(t) \xrightarrow{FS} b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$\alpha\, x(t) + \beta\, y(t) \xrightarrow{FS} \alpha\, a_k + \beta\, b_k$$

Proof:

$$\alpha\, a_k + \beta\, b_k = \frac{1}{T}\int_T \alpha\, x(t)\, e^{-jk\omega_0 t}\, dt + \frac{1}{T}\int_T \beta\, y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T [\alpha\, x(t) + \beta\, y(t)]\, e^{-jk\omega_0 t}\, dt \quad \xrightarrow{FS} \alpha\, a_k + \beta\, b_k$$

Cont..

Time-shift:

$If \quad x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$ Then, $x(t-t_0) \xrightarrow{FS} a_k e^{-jk\omega_0 t_0}$

Let, $y(t) = x(t-t_0)$

$$y(t) \xrightarrow{FS} b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

$$t - t_0 = \tau; dt = d\tau; t = \tau + t_0$$

$$= \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0(\tau+t_0)}\, d\tau \quad = e^{-jk\omega_0 t_0} \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau}\, d\tau \quad = a_k e^{-jk\omega_0 t_0}$$

Cont..

Frequency shift:

$$If \qquad x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

Then, $e^{jL\omega_0 t}\, x(t) \xrightarrow{FS} a_{k-L}$

Proof: Let, $y(t) = e^{jL\omega_0 t} x(t)$

$$y(t) \overset{FS}{\rightarrow} b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T e^{jL\omega_0 t}\, x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T x(t)\, e^{-j(k-L)\omega_0 t}\, dt \xrightarrow{FS} a_{k-L}$$

Cont..

Convolution:

If

$$x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$y(t) \xrightarrow{FS} b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$x(t)$ and $y(t)$ denote two periodic signals with *period* T

Then, $z(t) = x(t) * y(t)$

$$z(t) \xrightarrow{FS} a_k b_k T$$

Proof:

$$z(t) \xrightarrow{FS} b_k = \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt \quad = \frac{1}{T}\int_T \left[\int x(\tau) y(t-\tau)\, d\tau\right] e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int x(\tau)\left[\int_T y(t-\tau) e^{-jk\omega_0 t}\, dt\right] d\tau$$

$$\left(z(t) = x(t) * y(t) = \int x(\tau)\, y(t-\tau)\, d\tau \right)$$

$$t - \tau = L; dt = dL; t = \tau + L$$

$$= \frac{1}{T}\int x(\tau)\left[\int_T y(L) e^{-jk\omega_0(\tau+L)}\, dL\right] d\tau = \frac{1}{T}\int x(\tau)\left[e^{-jk\omega_0\tau}\int_T y(L) e^{-jk\omega_0 L}\, dL\right] d\tau = T\, a_k b_k$$

Multiplication:

If

$$x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$y(t) \xrightarrow{FS} b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$x(t)$ and $y(t)$ denote two periodic signals with *period* T

Then, $z(t) = x(t)\, y(t)$

$$z(t) \xrightarrow{FS} c_k = a_k * b_k = \sum_{l=-\infty}^{\infty} a_l\, b_{k-l}$$

(Proof yourself)

Differentiation:

If $x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$

Then

$$y(t) = \frac{d}{dt}x(t) \xrightarrow{FS} j\,k\,\omega_0 a_k = 2\pi k F_0 a_k$$

Proof:

As per synthesis equation, we can write $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right]$$

$$= \sum_{k=-\infty}^{\infty} a_k . jk\omega_0 . e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} (a_k . jk\omega_0) . e^{jk\omega_0 t}$$

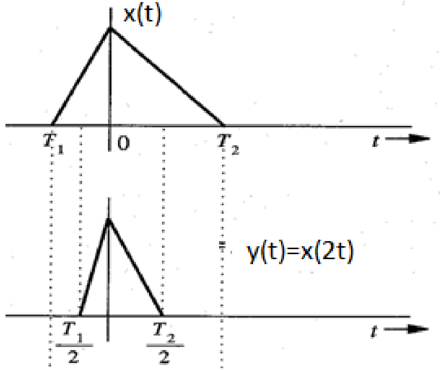
$$= \sum_{k=-\infty}^{\infty} b_k\, e^{jk\omega_0 t}$$

Comparing equation

$$y(t) = \frac{d}{dt}x(t) \xrightarrow{FS} j\,k\,\omega_0 a_k = 2\pi k F_0 a_k$$

Scaling:

If $x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$ Then $y(t) = x(at) \xrightarrow{FS} ?$



$If\ x(t)$ has period T, then the period of $x(at)$ =?

$$T_1 = \frac{T}{a}$$

$$F_1 = \frac{1}{T_1} = \frac{1}{T/a} = \frac{a}{T} = aF_0$$

$$\omega_1 = 2\pi F_1 = 2\pi\, aF_0 = a\omega_0$$

$$y(t) = x(at) \xrightarrow{FS} \frac{1}{T_1}\int_{T_1} x(at)\, e^{-jk\omega_1 t}\, dt$$

Let, $at = \tau$, $dt = \frac{1}{a}d\tau$, t= $^{\tau}/_{a}$

$$= \frac{1}{T/a}\int x(\tau)\, e^{-jka.\omega_0\frac{\tau}{a}} . \frac{1}{a}\, d\tau = \frac{1}{T}\int x(\tau)\, e^{-jk\omega_0\tau}\, d\tau = a_k$$

Fourier coefficients does not changed

Time Reversal:

If $x(t) \xrightarrow{FS} a_k$ Then $x(-t) \xrightarrow{FS} a_{-k}$

As per synthesis equation, we can write $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 -t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j(-k)\omega_0 t}$$

Replacing $-k$ by m

$$= \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_0 t}$$

$$x(-t) \xrightarrow{FS} a_{-k}$$

Conjugate:

As per synthesis equation, we can write $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$x^*(t) = \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right)^* = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$ (using time-reversal property)

If $x(t)$ is **real valued** → $\overset{*}{x}(t) = x(t)$

$$\rightarrow \overset{*}{x}(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\rightarrow a_{-k}^* = a_k$$

Try yourself for if $x(t)$ is pure imaginary

- A periodic signal $x(t)$ with fundamental period T_0 has complex-exponential Fourier- Series coefficients a_k. Express the following signal in terms of a_k.

$(a)\ x(t-t_0)$

$(b)\ \dfrac{dx(t)}{dt}$

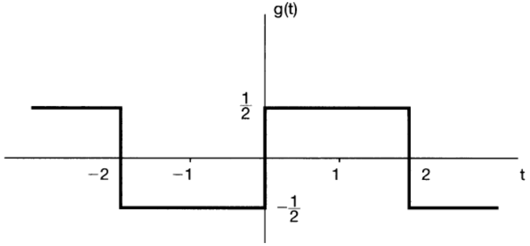
(a) As per time-shift property, we know $if\ x(t) \xrightarrow{FS} a_k$ $then,\ x(t-t_0) \xrightarrow{FS} b_k = a_k e^{-jk\omega_0 t_0}$

$$y(t) = x(t-t_0) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k . e^{-jk\omega_0 t_0} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k . e^{jk\omega_0 (t-t_0)}$$

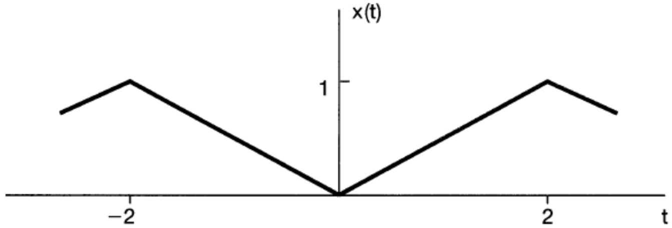
(b) Let, $y(t) = \dfrac{dx(t)}{dt}$ As per derivative property, we know $if\ x(t) \xrightarrow{FS} a_k$

$then, y(t) = \dfrac{d}{dt} x(t) \xrightarrow{FS} b_k = j\,k\,\omega_0 a_k = j2\pi k F_0 a_k$

So, $y(t) = \dfrac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = \displaystyle\sum_{k=-\infty}^{\infty} (jk\omega_0 a_k) e^{jk\omega_0 t}$



$$d_k = \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, \quad \text{for } k \neq 0$$



- The triangular wave signal $x(t) with\ period\ T = 4$, Evaluate the F.S. Coefficient of above signal?

Using derivative property, $y(t) = \frac{d}{dt}x(t) \xrightarrow{FS} b_k\,(of\ derivative) = j\,k\,\omega_0 a_k$ (without derivative)

$$d_k = jk\omega_0 a_k$$

$$\rightarrow a_k = \frac{d_k}{jk\omega_0} = \frac{d_k}{jk\frac{2\pi}{T}} = \frac{2d_k}{jk\pi} \;;\ k \neq 0$$

$$\rightarrow a_k = \frac{2\sin(\frac{\pi k}{2})}{j(k\pi)^2} e^{-j\pi/2}$$

$$a_0 = \frac{1}{T}\int_T x(t)e^{-jn=0.\omega_0}\, dt$$

$$= 1/2$$

A periodic signal $x(t)$ with fundamental period T_0 has complex-exponential Fourier- Series coefficients a_k. Express the following signal in terms of a_k.

(a) $x^*(t)$

By the definition of Fourier-series for $given\ x(t)$ and a_k, we can write

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Let $y(t) = x^*(t)$

Now

$$y(t) = x^*(t) = \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right)^*$$

$$= \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

(using conjugate property)

Parseval's theorem (continuous-time periodic signals)

- The average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$ is

$$P = \frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\propto}^{\propto} |a_k|^2$$

Proof: $|x(t)|^2 = x(t)\,x^*(t) \quad = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\left(\sum_{L=-\infty}^{\infty} a_L^* e^{-jL\omega_0 t}\right)$

$$P = \frac{1}{T}\int_T |x(t)|^2\,dt = \frac{1}{T}\int_T \left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\left(\sum_{L=-\infty}^{\infty} a_L^* e^{-jL\omega_0 t}\right)\right\} dt$$

$$\int_T e^{j(k-L)\omega_0 t}\,dt = T, \qquad if\ k = L$$
$$= 0, \qquad k \neq L$$

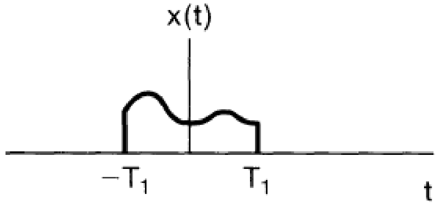
$$= \frac{1}{T}\left[\sum_{k=-\infty}^{\infty} a_k \sum_{L=-\infty}^{\infty} a_L^* \left\{\int_T e^{j(k-L)\omega_0 t}\,dt\right\}\right] = \frac{1}{T}.T \sum_{k=-\propto}^{\propto} |a_k|^2 \quad = \sum_{k=-\propto}^{\propto} |a_k|^2$$

Fourier Transform – (continuous signal)

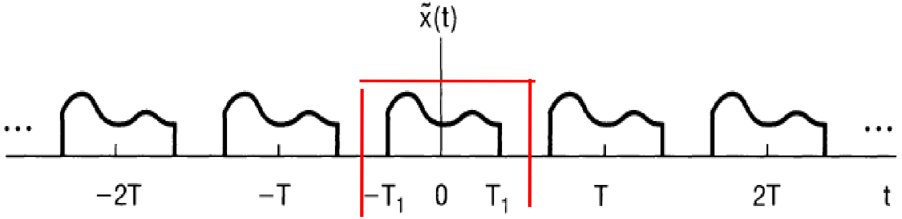
Fourier series of periodic continuous-time signal (with period T):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

Consider a aperiodic signal:



we can **construct a periodic signal** for which $x(t)$ *is one period*



$-\frac{T}{2}$ $\quad$ $\frac{T}{2}$

Observations:

1. If $T \to \infty$ ⇨ $x(t) = \tilde{x}(t)$ i.e. $x(t)$ repeat itself in infinite

2. $x(t) = \tilde{x}(t), \quad |t| < \frac{T}{2}$ *since* $x(t) = 0$ *outside this* interval

Cont..

As $\tilde{x}(t)$ is a periodic signal with $T \to \infty$, so using the concept of Fourier series, we can write

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \Rightarrow \quad a_k = \frac{1}{T}\int_T \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

Using the observation 2. $x(t) = \tilde{x}(t),\quad |t| < \frac{T}{2}$ *since* $x(t) = 0$ *outside this* interval

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} X(\omega_0 k)$$

Where, $\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt}$

Fourier-transform:
Forward Fourier Transform
Analysis Equation

$$\boxed{a_k = \frac{1}{T} X(\omega_0 k)}$$

(**Relation** between Fourier series and X($\boldsymbol{\omega}$))

Cont..

$a_k = \frac{1}{T} X(\omega_0 k)$ i.e. we can get Fourier-series (FS) from Fourier-transform (FT) $a_k = \frac{1}{T} X(\omega)|_{\omega = k\omega_0}$

$x(t) \xrightarrow{FT} \mathrm{X}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$ **(Fourier-transform)**

$FT[x(t)] = \mathrm{X}(\omega)$

- Now, we would like to derive $\boldsymbol{x(t)} \leftarrow \boldsymbol{X(\omega)}$

Now consider again

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(\omega_0 k) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi} X(\omega_0 k) e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega_0 k) e^{jk\omega_0 t} \omega_0$$

(replacing a_k by $\frac{1}{T} X(\omega_0 k)$) (replacing $\mathrm{T} = \frac{2\pi}{\omega_0}$)

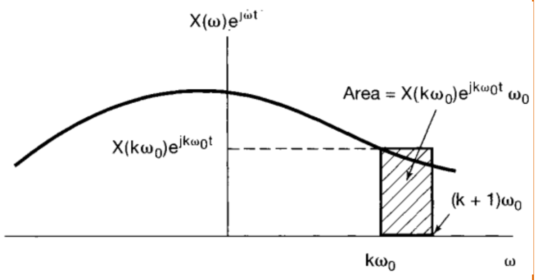
Observation 1.

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega_0 k) e^{jk\omega_0 t} \omega_0$$

$T \to \infty$ $\quad x(t) = \tilde{x}(t)$

i.e. $x(t)$ repeat itself in infinite

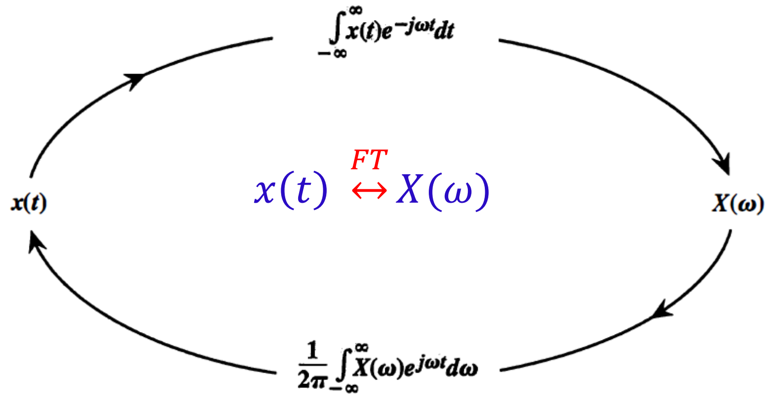
(passes to integral) as $T \to \infty$, $\omega_0 \to 0$



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

$$x(t) = \textit{Inverse FT}\,[X(\omega)] = FT^{-1}[X(\omega)]$$

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$



Properties of Fourier Transform

- **Linearity**

If $x(t) \xleftrightarrow{FT} X(\omega)$

$y(t) \xleftrightarrow{FT} Y(\omega)$

Then $ax(t) + by(t) \xleftrightarrow{FT} aX(\omega) + bY(\omega)$

- **Time shifting**

If $x(t) \xleftrightarrow{FT} X(\omega)$ **Then** $x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(\omega)$

$$FT[x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt \quad = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} . e^{-j\omega t_0} dt$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

$$= e^{-j\omega t_0} \ \mathrm{X}(\omega)$$

- **Frequency shift**

If $x(t) \xleftrightarrow{FT} X(\omega)$ **Then** $x(t)\, e^{j\omega_c t} \overset{FT}{\leftrightarrow} X(\omega - \omega_c)$

$$x(t)\left[e^{j\omega_c t} + e^{-j\omega_c t}\right] \overset{FT}{\leftrightarrow} X(\omega - \omega_c) + X(\omega + \omega_c)$$

- **Time scaling** **If** $x(t) \xleftrightarrow{FT} X(\omega)$ **Then** $x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

$$\mathfrak{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at) e^{-j\omega t} dt.$$

Using the substitution $\tau = at$, we obtain

$$\mathfrak{F}\{x(at)\} = \begin{cases} \frac{1}{a}\int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a > 0 \\ -\frac{1}{a}\int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a < 0 \end{cases},$$

Differentiation (time domain)

If $x(t) \overset{FT}{\leftrightarrow} X(\omega)$ **then**

We know, $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$ $\quad i.e.\ \ x(t) = Inverse\ FT\ [X(\omega)] = FT^{-1}[X(\omega)]$

$$\rightarrow \frac{d}{dt}x(t) = \frac{d}{dt}[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}[j\omega\ X(\omega)]\ e^{j\omega t} d\omega \ = FT^{-1}[j\omega\ X(\omega)]$$

$$\boxed{\frac{dx(t)}{dt} \overset{FT}{\longleftrightarrow} j\omega X(\omega)}$$

- ## Differentiation (frequency domain)

We know, $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$

$$\frac{d}{d\omega}X(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \ = \int_{-\infty}^{\infty} x(t).-jt.e^{-j\omega t}\ dt \ = -jt.FT\{x(t)\}$$

$$\boxed{-jt\ x(t) \overset{FT}{\longleftrightarrow} \frac{dX(\omega)}{d\omega}}$$

- **Convolution (time)**

If $x(t) \xleftrightarrow{FT} X(\omega)$

$h(t) \xleftrightarrow{FT} H(\omega)$

Then $y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(\omega) = ?$

From convolution theorem, we can write $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$$Y(\omega) = FT[y(t)] = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt \quad = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, FT[h(t-\tau)]\, d\tau$$

$$= H(\omega) \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau = H(\omega) X(\omega)$$

$$y(t) = x(t) * h(t) \xleftrightarrow{FT} X(\omega) H(\omega)$$

- Conjugate property

If $x(t) \xleftrightarrow{FT} X(\omega)$ **then** $x^*(t) \xleftrightarrow{FT} X^*(-\omega)$

If x(t) is real $X^*(\omega) = X(-\omega)$

Example: $x(t) = e^{-at}u(t)$

$$X(\omega) = \frac{1}{a+j\omega} = \frac{a-j\omega}{(a+j\omega)(a-j\omega)} = \frac{a}{a^2+\omega^2} + j\frac{-\omega}{a^2+\omega^2}$$

$$x(t) \xleftrightarrow{FT} X(\omega) = \frac{1}{a+j\omega}$$

$$X(-\omega) = \frac{1}{a+j.-\omega} = \frac{a+j\omega}{(a+j\omega)(a-j\omega)} = \frac{a}{a^2+\omega^2} + j\frac{\omega}{a^2+\omega^2}$$

$$X(-\omega) = \frac{1}{a-j\omega} = X^*(\omega)$$

$$= \mathrm{Re}\{X(\omega)\} - \mathrm{Im}\{X(\omega)\}$$

If x(t) real: $\mathrm{Re}\{X(\omega)\} = \mathrm{Re}\{X(-\omega)\}$

$$x(t) = x_e(t) + x_o(t).$$

$$\mathfrak{F}\{x(t)\} = \mathfrak{F}\{x_e(t)\} + \mathfrak{F}\{x_o(t)\},$$

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(\omega),$$

$$\mathcal{E}v\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} \mathfrak{Re}\{X(\omega)\}.$$

$$\mathcal{O}d\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} j\mathfrak{Im}\{X(\omega)\}$$

Given, $e^{-at}u(t) \overset{FT}{\longleftrightarrow} \dfrac{1}{a + j\omega}$

$$x(t) = e^{-a|t|}$$

$$= e^{-at}u(t) + e^{at}u(-t)$$

$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right]$$

$$= 2\ Even\ \{e^{-at}u(t)\}$$

Parseval's theorem (continuous-time periodic signals)

- The average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$ is

$$P = \frac{1}{T}\int_T |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Proof: $|x(t)|^2 = x(t)\, x^*(t) \quad = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\left(\sum_{L=-\infty}^{\infty} a_L^* e^{-jL\omega_0 t}\right)$

$$P = \frac{1}{T}\int_T |x(t)|^2\, dt = \frac{1}{T}\int_T \left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\left(\sum_{L=-\infty}^{\infty} a_L^* e^{-jL\omega_0 t}\right)\right\} dt$$

$$\int_T e^{j(k-L)\omega_0 t}\, dt = T, \quad if\ k = L$$
$$= 0, \quad k \neq L$$

$$= \frac{1}{T}\left[\sum_{k=-\infty}^{\infty} a_k \sum_{L=-\infty}^{\infty} a_L^* \left\{\int_T e^{j(k-L)\omega_0 t}\, dt\right\}\right] = \frac{1}{T}.T \sum_{k=-\infty}^{\infty} |a_k|^2 \quad = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- Evaluate the complex-exponential Fourier-series expansion of the signal

$$x(t) = 2 + 3\cos 2\pi t + 4\sin 3\pi t$$

And then verify the Parseval's theorem.

By the definition of synthesis equation, we can write

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

ω_0 is *unknown* i.e. we have to *first determine* the T

$2 \rightarrow$ *periodic* any value of T

$\cos 2\pi t \rightarrow T_1 = 1$

$sin\, 3\pi t \rightarrow T_2 = \frac{2}{3}$

$$T = \text{Least} - \text{common multiplier}\left(1, \frac{2}{3}\right) = 2$$

$$\omega_0 = 2\pi F_0 = 2\pi . \frac{1}{2} = \pi$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk.\pi.t} = \ldots + a_{-2}e^{-2j\pi t} + a_{-1}e^{-j\pi t} + a_0 + a_1 e^{j\pi t} + a_2 e^{2j\pi t} + \ldots.$$

Using Euler relation, we can expand the following equation $x(t) = 2 + 3\cos 2\pi t + 4\sin 3\pi t$

$$x(t) = 2 + 3.\frac{1}{2}\left[e^{j2\pi t} + e^{-j2\pi t}\right] + 4.\frac{1}{2j}\left[e^{j3\pi t} - e^{-j3\pi t}\right]$$

$$= 2 + \frac{3}{2}e^{-j2\pi t} + \frac{3}{2}e^{-j2\pi t} - \frac{4}{2j}e^{-j3\pi t} + \frac{4}{2j}e^{j3\pi t}$$

Comparing two equations

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \dots + a_{-2}e^{-2j\pi t} + a_{-1}e^{-j\pi t} + a_0 + a_1 e^{j\pi t} + a_2 e^{2j\pi t} + \dots.$$

Fourier-series expansion

$$a_{-2} = \frac{3}{2} = a_2$$
$$a_{-1} = 0 = a_1$$
$$a_0 = 2$$
$$a_3 = \frac{4}{2j}$$
$$a_{-3} = -\frac{4}{2j}$$

To verify Parseval's theorem:

$x(t)$ has *period* 2

$$a_{-2} = \frac{3}{2} = a_2$$
$$a_{-1} = 0 = a_1$$
$$a_0 = 2$$

$$a_3 = \frac{4}{2j}$$
$$a_{-3} = -\frac{4}{2j}$$

As per definition of power of a signal, we can write:

$$P = \frac{1}{T}\int_0^T |x(t)|^2 dt \qquad = \frac{1}{T}\int_0^T |2 + 3\cos 2\pi t + 4\sin 3\pi t|^2\, dt \qquad =?$$

From Parseval's theorem:

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = \sum_{k=-3}^{3} |a_k|^2 = |a_{-3}|^2 + |a_{-2}|^2 + |a_{-1}|^2 + |a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2$$
$$= 2^2 + \left(\frac{3}{2}\right)^2 + 0^2 + 2^2 + 0^2 + \left(\frac{3}{2}\right)^2 + 2^2$$
$$= \frac{33}{2}$$
$$= 16.2$$

PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting	$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration	$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Trigonometric Fourier-series

$$x(t) = a_0 + \sum_{n=1}^{\propto} a_n \cos n\,\omega_0 t + \sum_{n=1}^{\propto} b_n \sin n\,\omega_0 t;\ \omega_0 = \frac{2\pi}{T};\ t_1 < t < t_1 + T$$

$$\boxed{\begin{aligned} a_0 &= \frac{1}{T}\int_{t_1}^{t_1+T} x(t)dt \\ a_n &= \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\cos n\,\omega_0 t dt\ ;\ n = 1,2,3,\ldots \\ b_n &= \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\sin n\,\omega_0 t dt\ ;\ n = 1,2,3,\ldots \end{aligned}}$$

If x(t) is even

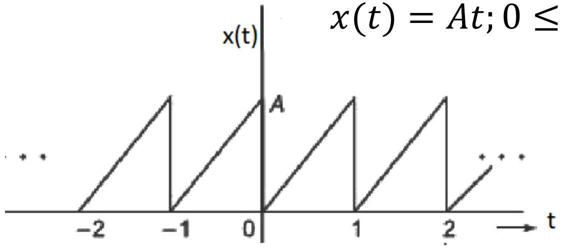
$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\cos\omega_0 t dt \qquad b_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\sin\omega_0 t dt = 0$$ (odd function)

If x(t) is odd

$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\cos\omega_0\, t dt = 0$$ (odd function)

$$b_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\sin\omega_0\, t dt$$

Find the trigonometric Fourier series coefficient:



$$x(t) = At; 0 \le t \le 1$$

$$T = 1$$

$$\omega_0 = \frac{2\pi}{T} = 2\pi$$

$$a_0 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)dt$$

$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\cos n\,\omega_0 t dt\,; n = 1,2,3,\ldots$$

$$b_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\sin n\,\omega_0 t dt\,; n = 1,2,3,\ldots$$

$$a_0 = \frac{1}{1}\int_0^1 Atdt = \frac{A}{2}$$

$$a_n = \frac{2}{1}\int_0^1 At\cos n\,\omega_0 tdt = 2A\left[t\frac{\sin n\,\omega_0 t}{n\omega_0}\bigg|_0^1 - \int_0^1 1.\frac{\sin n\,\omega_0 t}{n\omega_0}dt\right] \quad = 0$$

$$\boxed{\int F(x)G'(x)\,dx = F(x)G(x) - \int F'(x)G(x)\,dx + C.}$$

$$b_n = 2A\int_0^1 t\sin n\,\omega_0 tdt = 2A\left[\frac{t.-\cos n\,\omega_0 t}{n\omega_0}\bigg|_0^1 - \int_0^1 1.-\frac{\cos n\,\omega_0 t}{n\omega_0}dt\right]$$

$$= -2.\frac{A}{n\omega_0}\cos 2\,\pi n + \frac{2A}{n\omega_0}.0 = -\frac{2A}{n.2\pi} = -\frac{A}{n\pi}$$

Fourier Series versus Fourier Transform

	Continuous time	Discrete time
Periodic	Fourier Series	Discrete Fourier Transform
Aperiodic	Continuous Fourier Transform	Discrete Fourier Transform

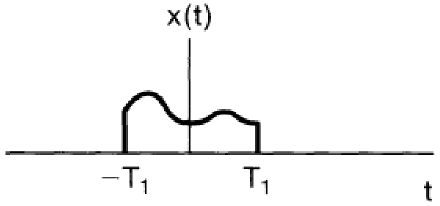
A **spectrum analyzer** measures the magnitude of an input signal versus frequency within the full frequency range of the instrument. It measures frequency, power, harmonics, distortion, noise, spurious signals and bandwidth.

Fourier Transform – (continuous signal)

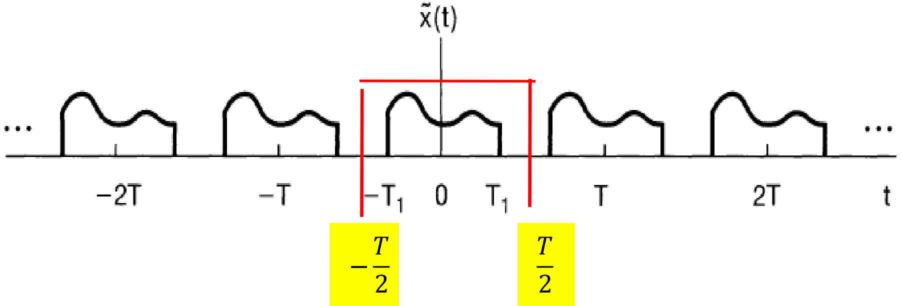
Fourier series of periodic continuous-time signal (with period T):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right) t}$$

Consider a aperiodic signal:



we can **construct a periodic signal** for which $x(t)$ *is one period*



Observations:

1. If $T \to \infty$ $\Rightarrow$ $x(t) = \tilde{x}(t)$ i.e. $x(t)$ repeat itself in infinite

2. $x(t) = \tilde{x}(t)$, $|t| < \frac{T}{2}$ *since* $x(t) = 0$ *outside this* interval

Cont..

As $\tilde{x}(t)$ is a periodic signal with $T \rightarrow \infty$, so using the concept of Fourier series, we can write

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \Rightarrow \quad a_k = \frac{1}{T}\int_T \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

Using the observation 2. $x(t) = \tilde{x}(t), \quad |t| < \dfrac{T}{2}$ *since* $x(t) = 0$ *outside this* interval

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} X(\omega_0 k)$$

Where, $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$

Fourier-transform:
Forward Fourier Transform
Analysis Equation

$$a_k = \frac{1}{T} X(\omega_0 k)$$

(**Relation** between Fourier series and $X(\boldsymbol{\omega})$)

Cont..

$a_k = \frac{1}{T} X(\omega_0 k)$ i.e. we can get Fourier-series (FS) from Fourier-transform (FT) $a_k = \frac{1}{T} X(\omega)|_{\omega = k\omega_0}$

$x(t) \xrightarrow{FT} \mathrm{X}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$ **(Fourier-transform)**

$FT[x(t)] = \mathrm{X}(\omega)$

- Now, we would like to derive $\boldsymbol{x(t)} \leftarrow \boldsymbol{X(\omega)}$

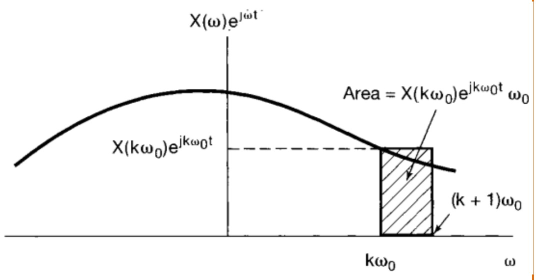
Now consider again

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(\omega_0 k) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi} X(\omega_0 k) e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega_0 k) e^{jk\omega_0 t} \omega_0$$

(replacing a_k by $\frac{1}{T} X(\omega_0 k)$) (replacing $\mathrm{T} = \frac{2\pi}{\omega_0}$)

Observation 1.

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(\omega_0 k)e^{jk\omega_0 t}\omega_0$$



$T \to \infty$ $\quad x(t) = \tilde{x}(t)$

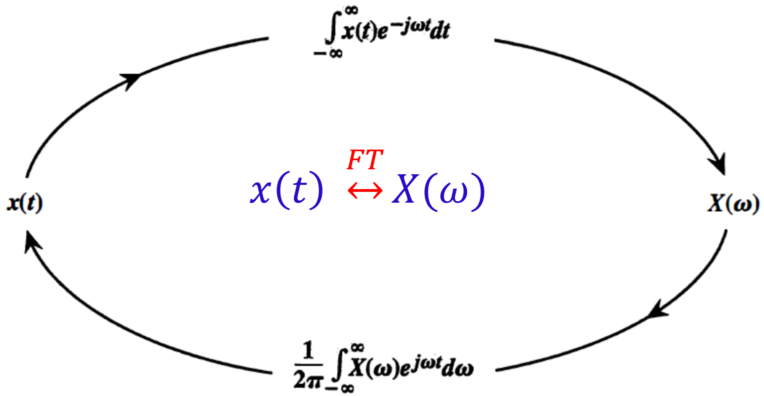
i.e. $x(t)$ repeat itself in infinite

(passes to integral)
as $T \to \infty$, $\omega_0 \to 0$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$x(t) = \textit{Inverse FT}\,[X(\omega)] = FT^{-1}[X(\omega)]$$

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$



Properties of Fourier Transform

- **Linearity**

If $x(t) \xleftrightarrow{FT} X(\omega)$

$y(t) \xleftrightarrow{FT} Y(\omega)$

Then $ax(t) + by(t) \xleftrightarrow{FT} aX(\omega) + bY(\omega)$

- **Time shifting**

If $x(t) \xleftrightarrow{FT} X(\omega)$ **Then** $x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0}\, X(\omega)$

$$FT[x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt \;=\; \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} . e^{-j\omega t_0} dt$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

$$= e^{-j\omega t_0}\; \mathrm{X}(\omega)$$

- **Frequency shift**

If $x(t) \xleftrightarrow{FT} X(\omega)$ **Then** $x(t)\, e^{j\omega_c t} \overset{FT}{\leftrightarrow} X(\omega - \omega_c)$

$$x(t)\left[e^{j\omega_c t} + e^{-j\omega_c t}\right] \overset{FT}{\leftrightarrow} X(\omega - \omega_c) + X(\omega + \omega_c)$$

- **Time scaling** **If** $x(t) \xleftrightarrow{FT} X(\omega)$ **Then** $x(at) \xleftrightarrow{FT} \dfrac{1}{|a|} X\left(\dfrac{\omega}{a}\right)$

$$\mathfrak{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at) e^{-j\omega t} dt.$$

Using the substitution $\tau = at$, we obtain

$$\mathfrak{F}\{x(at)\} = \begin{cases} \dfrac{1}{a} \displaystyle\int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a > 0 \\ -\dfrac{1}{a} \displaystyle\int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a < 0 \end{cases},$$

Differentiation (time domain)

If $x(t) \overset{FT}{\leftrightarrow} X(\omega)$ **then**

We know, $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$ $\quad i.e.\ x(t) = Inverse\ FT\ [X(\omega)] = FT^{-1}[X(\omega)]$

$$\rightarrow \frac{d}{dt}x(t) = \frac{d}{dt}[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}[j\omega\, X(\omega)]\, e^{j\omega t} d\omega \ = FT^{-1}[j\omega\, X(\omega)]$$

$$\boxed{\frac{dx(t)}{dt} \overset{FT}{\longleftrightarrow} j\omega X(\omega)}$$

- **Differentiation (frequency domain)**

We know, $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$

$$\frac{d}{d\omega}X(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \ = \int_{-\infty}^{\infty} x(t).-jt.e^{-j\omega t}\ dt \ = -jt.FT\{x(t)\}$$

$$\boxed{-jt\, x(t) \overset{FT}{\longleftrightarrow} \frac{dX(\omega)}{d\omega}}$$

- **Convolution (time)**

If $x(t) \xleftrightarrow{FT} X(\omega)$

$h(t) \xleftrightarrow{FT} H(\omega)$

Then $y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(\omega) = ?$

From convolution theorem, we can write $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$$Y(\omega) = FT[y(t)] = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt \quad = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t}\, dt \right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, FT[h(t-\tau)]\, d\tau$$

$$= H(\omega) \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau = H(\omega) X(\omega)$$

$$y(t) = x(t) * h(t) \xleftrightarrow{FT} X(\omega) H(\omega)$$

- Conjugate property

If $x(t) \xleftrightarrow{FT} X(\omega)$ **then** $x^*(t) \xleftrightarrow{FT} X^*(-\omega)$

If x(t) is real $X^*(\omega) = X(-\omega)$

Example: $x(t) = e^{-at}u(t)$

$$x(t) \xleftrightarrow{FT} X(\omega) = \frac{1}{a + j\omega}$$

$$X(-\omega) = \frac{1}{a - j\omega} = X^*(\omega)$$

$$X(\omega) = \frac{1}{a + j\omega} = \frac{a - j\omega}{(a + j\omega)(a - j\omega)} = \frac{a}{a^2 + \omega^2} + j\frac{-\omega}{a^2 + \omega^2}$$

$$X(-\omega) = \frac{1}{a + j.-\omega} = \frac{a + j\omega}{(a + j\omega)(a - j\omega)} = \frac{a}{a^2 + \omega^2} + j\frac{\omega}{a^2 + \omega^2}$$

$$= \mathrm{Re}\{X(\omega)\} - \mathrm{Im}\{X(\omega)\}$$

If x(t) real: $\mathrm{Re}\{X(\omega)\} = \mathrm{Re}\{X(-\omega)\}$

$$x(t) = x_e(t) + x_o(t).$$

$$\mathfrak{F}\{x(t)\} = \mathfrak{F}\{x_e(t)\} + \mathfrak{F}\{x_o(t)\},$$

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(\omega),$$

$$\mathcal{E}v\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} \mathcal{R}e\{X(\omega)\}.$$

$$\mathcal{O}d\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} j\mathcal{I}m\{X(\omega)\}$$

Given, $e^{-at}u(t) \overset{FT}{\longleftrightarrow} \dfrac{1}{a + j\omega}$

$$x(t) = e^{-a|t|}$$

$$= e^{-at}u(t) + e^{at}u(-t)$$

$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right]$$

$$= 2\ Even\ \{e^{-at}u(t)\}$$

Evaluate the Fourier transform of the signals

$(a)\ x(t) = e^{-at}u(t), a > 0$

$(b)\ x(t) = \begin{cases} 1, |t| < T_1 \\ 0, |t| > T_1 \end{cases}$

(a) As per the definition of Fourier-transform, we can write

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-at}u(t).\,e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-at}\,e^{-j\omega t}dt$$

since $u(t) = 1, t \geq 0$
= 0, else

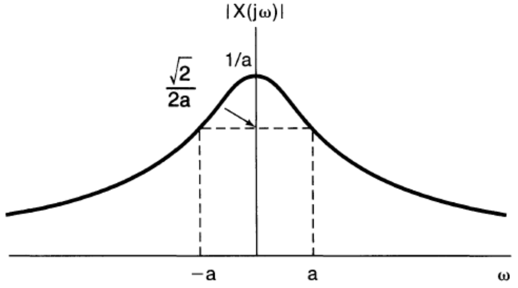
$$= -\frac{1}{(a+j\omega)}\,e^{-(a+j\omega)}\Big|_0^{\infty}$$

$$X(\omega) = \frac{1}{a+j\omega}$$

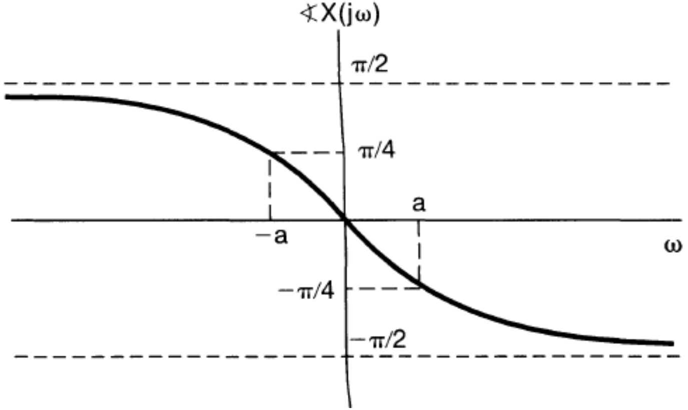
Magnitude of spectra

$$|X(\omega)| = \left|\frac{1}{a+j\omega}\right| = \left|\frac{a-j\omega}{(a+j\omega)(a-j\omega)}\right|$$

$$= \left|\frac{a}{a^2+\omega^2} + j\frac{-\omega}{a^2+\omega^2}\right| = \frac{1}{\sqrt{a^2+\omega^2}}$$

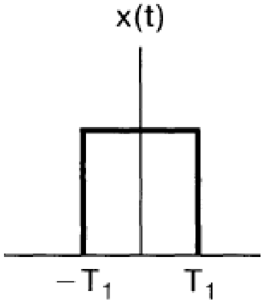


$$X(\omega) = \frac{1}{a + j\omega} \quad = \frac{\overbrace{a}^{x}}{a^2 + \omega^2} + j\frac{\overbrace{-\omega}^{y}}{a^2 + \omega^2}$$



Phase: $$\angle X(\omega) = \tan^{-1}\left(\frac{y}{x}\right) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

(b) $$x(t) = \begin{cases} 1, |t| < T_1 \\ 0, |t| > T_1 \end{cases}$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \;= \int_{-T_1}^{T_1} 1.e^{-j\omega t}\; dt$$

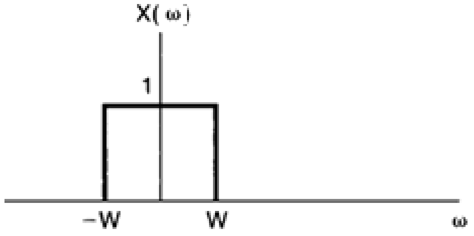
$$= -\frac{1}{j\omega}\; e^{-j\omega t}\big|_{-T_1}^{T_1}$$

$$= -\frac{1}{j\omega}\left[e^{-j\omega T_1} - e^{-j\omega.-T_1}\right]$$

$$= 2\frac{\sin\omega\, T_1}{\omega}$$

Determine the signals x(t)

$$X(\omega) = \begin{cases} 1, |\omega| < w \\ 0, |\omega| > w \end{cases}$$



Using the relation, we can write $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$

$$= \frac{1}{2\pi}\int_{-w}^{w} 1.\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}.\frac{1}{jt}.\left[e^{j\omega t}\right]\Big|_{-w}^{w}$$

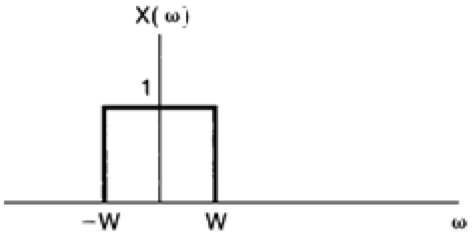
$$= \frac{1}{2\pi}.\frac{1}{jt}.\left[e^{jwt} - e^{-jwt}\right]$$

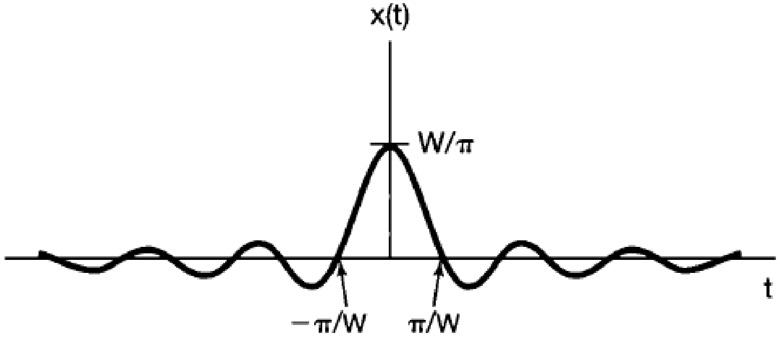
$$= \frac{\sin w\, t}{\pi t}$$

$$\frac{\sin wt}{\pi t} = \frac{w}{\pi}\frac{\sin(wt)}{wt} = \frac{w}{\pi}\text{sinc}(wt)$$

$\text{sinc}(x) = 0; \quad x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots.$

$\text{sinc}(x = 0) = 1;$ (L'Hopotal's rule)

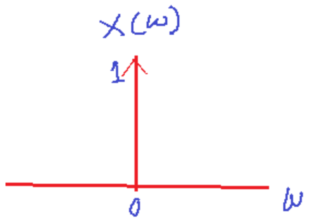




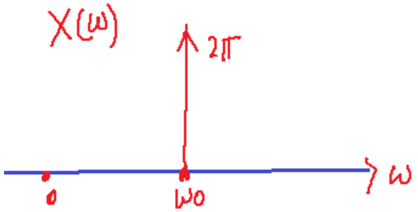
$$\frac{\sin wt}{\pi t} = \frac{w}{\pi}\frac{\sin(wt)}{wt} = \frac{w}{\pi}\text{sinc}(wt)$$

$$\text{sinc}(x) = 0; \quad x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots.$$

Fourier transform (FT) of "1"



$X(\omega) = \delta(\omega)$



$X(\omega) = 2\pi\,\delta(\omega-\omega_0)$

As per the definition of Inverse Fourier Transform, we can write

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$\rightarrow x(t) = FT^{-1}\,[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t}\,d\omega$$

$$\rightarrow FT^{-1}\{\delta(\omega)\} = \frac{1}{2\pi}$$

$$\rightarrow \delta(\omega) = FT\left\{\frac{1}{2\pi}\right\}$$

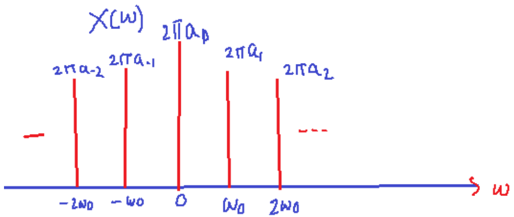
$$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

$$\rightarrow x(t) = FT^{-1}\,[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}\,d\omega$$

$$= e^{j\omega_0 t}$$

$$2\pi\delta(\omega-\omega_0) \leftrightarrow e^{j\omega_0 t}$$

Inverse FT of linear combination of impulse signal



As per the definition of Inverse Fourier Transform,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$$

$$\rightarrow x(t) = FT^{-1}\,[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} 2\pi\, a_k \delta(\omega - k\omega_0)\, e^{j\omega t}\, d\omega$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k \delta(\omega - k\omega_0) \quad \Longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Fourier transform of a periodic signal with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies

Determine the FT of $cos\omega_0 t$?

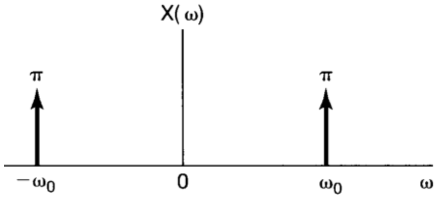
$$cos\omega_0 t = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

$$e^{j\omega_0 t} \overset{FT}{\leftrightarrow} ?$$

$$1.\, e^{j\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega - \omega_0)$$

$$cos\omega_0 t = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

$$FT\,[cos\omega_0 t] = \frac{1}{2}.\,2\pi\delta(\omega - \omega_0) + \frac{1}{2}.\,2\pi\delta(\omega + \omega_0)$$



Fourier series coefficient

$$a_1 = \frac{1}{2}$$

$$a_{-1} = \frac{1}{2}$$

$$a_k \neq 1, -1$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k \delta(\omega - k\omega_0)$$

$$\leftrightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$X(\omega) = 2\pi\, a_1 \delta(\omega - \omega_0) + 2\pi\, a_{-1} \delta(\omega + \omega_0)$$

$$= 2\pi\, \frac{1}{2}\delta(\omega - \omega_0) + 2\pi \frac{1}{2}\delta(\omega + \omega_0)$$

$$= \pi\, \delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

Properties of the Fourier Transform

Property	Signal	Fourier transform
	$x(t)$	$X(\omega)$
	$x_1(t)$	$X_1(\omega)$
	$x_2(t)$	$X_2(\omega)$
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\frac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\pi X(0)\delta(\omega)+\frac{1}{j\omega}X(\omega)$
Convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$
Real signal	$x(t)=x_e(t)+x_o(t)$	$X(\omega)=A(\omega)+jB(\omega)$ $X(-\omega)=X^*(\omega)$
Even component	$x_e(t)$	$\mathrm{Re}\{X(\omega)\}=A(\omega)$
Odd component	$x_o(t)$	$j\,\mathrm{Im}\{X(\omega)\}=jB(\omega)$
Parseval's relations		

$$\int_{-\infty}^{\infty} x_1(\lambda)X_2(\lambda)\,d\lambda=\int_{-\infty}^{\infty} X_1(\lambda)x_2(\lambda)\,d\lambda$$

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega)\,d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

Common Fourier Transforms Pairs

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\frac{1}{j\omega}$
$e^{-at}u(t),\ a>0$	$\frac{1}{j\omega+a}$
$te^{-at}u(t),\ a>0$	$\frac{1}{(j\omega+a)^2}$
$e^{-a\lvert t\rvert},\ a>0$	$\frac{2a}{a^2+\omega^2}$
$\frac{1}{a^2+t^2}$	$e^{-a\lvert\omega\rvert}$
$e^{-at^2},\ a>0$	$\sqrt{\frac{\pi}{a}}\,e^{-\omega^2/4a}$
$p_a(t)=\begin{cases}1 & \lvert t\rvert<a\\ 0 & \lvert t\rvert>a\end{cases}$	$2a\frac{\sin\omega a}{\omega a}$
$\frac{\sin at}{\pi t}$	$p_a(\omega)=\begin{cases}1 & \lvert\omega\rvert<a\\ 0 & \lvert\omega\rvert>a\end{cases}$
$\mathrm{sgn}\,t$	$\frac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0),\ \omega_0=\frac{2\pi}{T}$

Duality

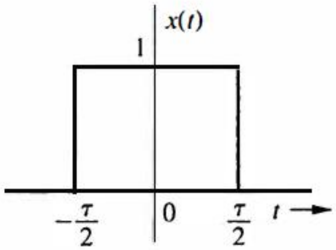
If

$$x(t) \xleftrightarrow{FT} X(\omega)$$

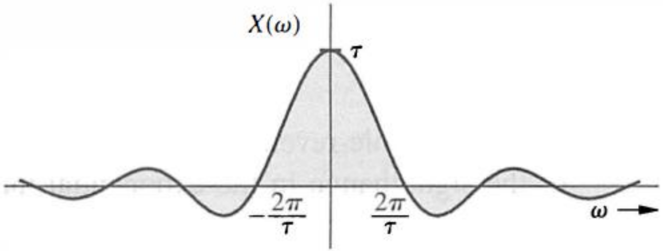
$$\boldsymbol{X(t)} \xleftrightarrow{FT} ? \qquad \int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt = ?$$

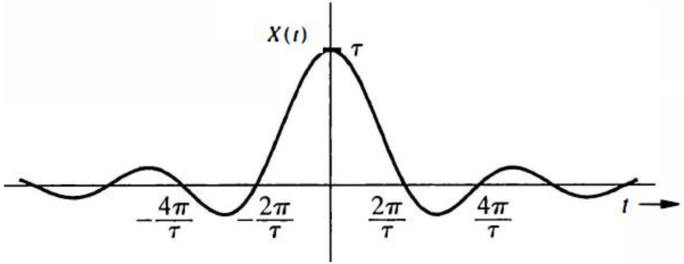
Called duality of time and frequency.

The duality principle may be compared with a photograph and its negative. A photograph can be obtained from its negative, and by using an identical procedure, a negative can be obtained from the photograph



$\Longleftrightarrow$

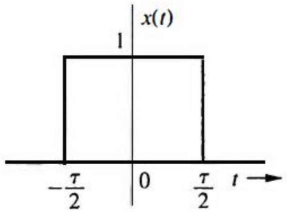


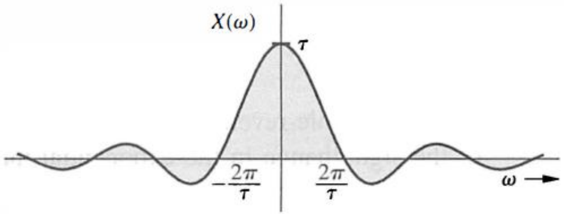


$\Longleftrightarrow$

?

Proof: We know, $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)e^{j\tau t}\,d\tau$





$$2\pi\, x(t) = \int_{-\infty}^{\infty} X(\tau)e^{j\tau t}\,d\tau$$

Putting, $t = -\omega$

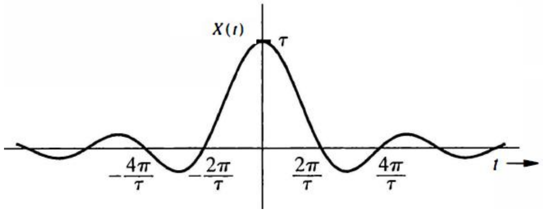
$$x(t) = \begin{cases} 1, |t| < \frac{\tau}{2} \\ 0, else \end{cases} \longleftrightarrow \boldsymbol{X(\omega)} = \tau \frac{\sin\left(\omega\frac{\tau}{2}\right)}{\omega\frac{\tau}{2}} = \tau \sin c\left(\frac{\omega\tau}{2}\right)$$

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(\tau)e^{j\tau.-\omega}\,d\tau$$

Putting, $\tau = \mathrm{t}$

$$= \int_{-\infty}^{\infty} X(t)e^{-j\omega t}\,dt$$

$$= FT\{X(t)\}$$

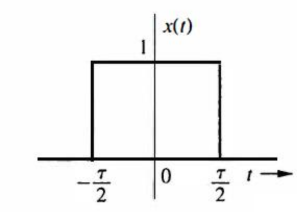


$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega) = x(-f)$$

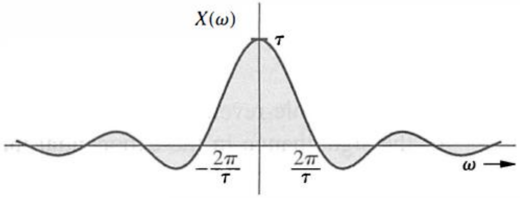
$\boldsymbol{X(t)}$ is the same as $\boldsymbol{X(\omega)}$ with ω replaced by $\boldsymbol{t}$

$$x(t) = \begin{cases} 1, |t| < \frac{\tau}{2} \\ 0, else \end{cases}$$

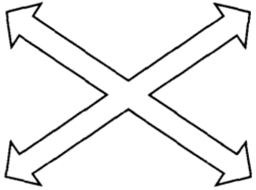
$$x(t) = rect(\frac{t}{\tau})$$

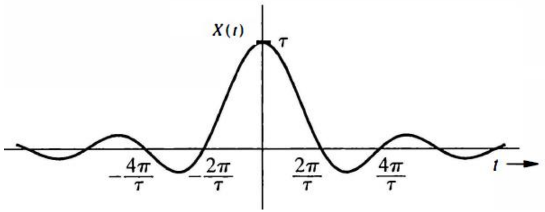


$\Longleftrightarrow$



$$X(\omega) = \tau \frac{\sin\left(\omega \frac{\tau}{2}\right)}{\omega \frac{\tau}{2}}$$
$$= \tau \sin c\left(\frac{\omega\tau}{2}\right)$$

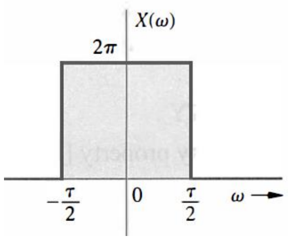




$\Longleftrightarrow$

$$2\pi x(-\omega)$$

$x(-\omega)$ is the same as $x\ (t)$ with t replaced by $-\omega$.



$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega) = x(-f)$$

$$\underbrace{\tau \,\mathrm{sinc}\left(\frac{\tau t}{2}\right)}_{X(t)} \Longleftrightarrow \underbrace{2\pi \,\mathrm{rect}\left(\frac{-\omega}{\tau}\right)}_{2\pi x(-\omega)} = 2\pi \,\mathrm{rect}\left(\frac{\omega}{\tau}\right)$$

- Evaluate Fourier transform of the signal $x(t) = \dfrac{1}{1+t^2}$

$$y(t) = e^{-a|t|}; a > 0$$

$$y(t) \xleftrightarrow{FT} Y(\omega) = \frac{2a}{a^2+\omega^2}$$

$$a = 1$$

$$y(t) = e^{-|t|} \xleftrightarrow{FT} Y(\omega) = \frac{2}{1+\omega^2}$$

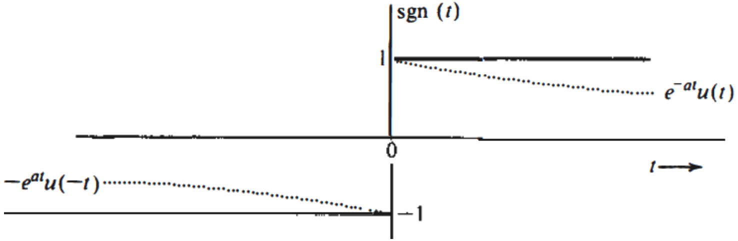
Using dual property: $z(t) \xleftrightarrow{FT} 2\pi z(-\omega)$

$$Y(t) = \frac{2}{1+t^2} \overset{FT}{\leftrightarrow} 2\,\pi\, \boldsymbol{y(-\omega)}$$

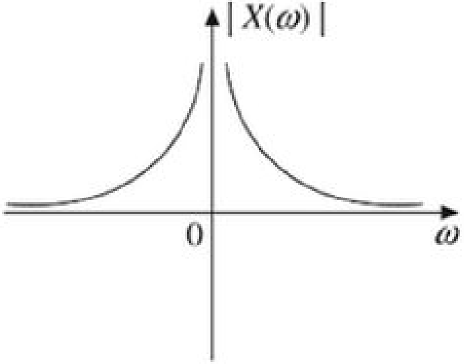
$$\Rightarrow \frac{1}{1+t^2} \overset{FT}{\leftrightarrow} \pi \;\; e^{-a|-\omega|}$$

$$\Rightarrow \frac{1}{1+t^2} \overset{FT}{\leftrightarrow} \pi \;\; e^{-a|\omega|}$$

- Evaluate the Fourier transform of the signal



$$\operatorname{sgn} t = \lim_{a \to 0} [e^{-at}u(t) - e^{at}u(-t)]$$



$$\begin{aligned}\mathcal{F}[\operatorname{sgn}(t)] &= \lim_{a\to 0}\{\mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)]\} \\ &= \lim_{a\to 0}\left(\frac{1}{a+j2\pi f} - \frac{1}{a-j2\pi f}\right) \\ &= \lim_{a\to 0}\left(\frac{-j4\pi f}{a^2+4\pi^2 f^2}\right) = \frac{1}{j\pi f} = \frac{2}{j\omega}\end{aligned}$$

Perseval's theorem/ Energy spectrum

Energy of the signal $x(t)$can be defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad = \int_{-\infty}^{\infty} x(t)x^*(t)dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t}d\omega\right] dt$$

$$= \int_{-\infty}^{\infty} X^*(\omega)d\omega \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]$$

$$= \int_{-\infty}^{\infty} X^*(\omega).\frac{1}{2\pi}X(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \quad = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Express the principle conversation of energy in time and frequency domains

Given, $X(\omega)=\dfrac{1}{(1+\omega^2)}e^{-\frac{2\omega^2}{(1+\omega^2)}}$. **Determine the Fourier transform** of the following signals:

$(a)\ x(t-2)e^{jt}$

$(b)\ x(1-t)$

$(c)\ x\left(\dfrac{t}{2}-2\right)$

(a) Time shift property: $x(t-t_0)\overset{FT}{\leftrightarrow} e^{-j\omega t_0}X(\omega)$ ⇒ $y(t)=x(t-2)\overset{FT}{\leftrightarrow} Y(\omega)=e^{-j\omega.2}X(\omega)$

Frequency shift property: $x(t)\ e^{j\omega_c t}\ \overset{FT}{\leftrightarrow}\ X(\omega-\omega_c)$ ⇒ $y(t)e^{j.\omega_c=1.t}\ \overset{FT}{\leftrightarrow}\ Y(\omega-\omega_c)$

$y(t)\ e^{j.\omega_c=1.t}=x(t-2)e^{j\omega_c=1.t}\ \overset{FT}{\leftrightarrow}\ Y(\omega-\omega_c)=e^{-j2(\omega-\omega_c)}X(\omega-\omega_c)$ where $\omega_c=1$

(b) Scaling property:

$$x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$\Rightarrow \quad x(-t) \xleftrightarrow{FT} \frac{1}{|-1|} X\left(\frac{\omega}{-1}\right) = X(-\omega)$$

(for $a=-1$) [folding]

$$x(-t+1) \overset{FT}{\leftrightarrow} \int_{-\infty}^{\infty} x(-t+1)\, e^{-j\omega t} dt$$

(replacing $-t+1 = \tau,\ dt = -d\tau$)

$$\overset{FT}{\leftrightarrow} \int_{+\infty}^{-\infty} x(\tau)\, e^{-j\omega(-\tau+1)} .-d\tau$$

$$\overset{FT}{\leftrightarrow} e^{-j\omega} \int_{+\infty}^{-\infty} x(\tau)\, e^{j\omega\tau} .-d\tau$$

$$\overset{FT}{\leftrightarrow} e^{-j\omega} \int_{-\infty}^{\infty} x(\tau)\, e^{j\omega\tau}\, d\tau = e^{-j\omega} \int_{-\infty}^{\infty} x(\tau)\, e^{-j.(-\omega).\tau}\, d\tau = e^{-j\omega} X(-\omega)$$

Using Time shift property (delay):

$$x(t-t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$$

$$\boxed{x(-t+1) \overset{FT}{\leftrightarrow} e^{-j\omega.1} X(-\omega)}$$

(c) Using Time shift property (delay): $x(t-t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$

$$y(t) = x(t-2) \overset{FT}{\leftrightarrow} Y(\omega) = e^{-j\omega.2} X(\omega)$$ [shifting]

Using scaling property $x(at) \overset{FT}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$ [Scaling]

$$y\left(\frac{t}{2}\right) = x\left(\frac{t}{2}-2\right) \overset{FT}{\leftrightarrow} \frac{1}{1/2} Y\left(\frac{\omega}{\frac{1}{2}}\right) = 2.Y(2\omega) = 2e^{-j.2\omega.2}X(2\omega)$$

Given, $X(\omega) = \frac{1}{(1+\omega^2)} e^{-\frac{2\omega^2}{(1+\omega^2)}}$. **Determine the Fourier transform** of the following signals:

$(a)\ x(t-2)e^{jt}$

$(b)\ x(1-t)$

$(c)\ x\left(\frac{t}{2}-2\right)$

(a) Time shift property: $x(t-t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$ $\Rightarrow$ $y(t) = x(t-2) \overset{FT}{\leftrightarrow} Y(\omega) = e^{-j\omega.2} X(\omega)$

Frequency shift property: $x(t)\ e^{j\omega_c t} \overset{FT}{\leftrightarrow} X(\omega-\omega_c)$ $\Rightarrow$ $y(t)e^{j.\omega_c=1.t} \overset{FT}{\leftrightarrow} Y(\omega-\omega_c)$

$y(t)\ e^{j.\omega_c=1.t} = x(t-2)e^{j\omega_c=1.t} \overset{FT}{\leftrightarrow} Y(\omega-\omega_c) = e^{-j2(\omega-\omega_c)} X(\omega-\omega_c)$ where $\omega_c = 1$

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(for $a=-1$) [folding]

$$x(-t+1) \overset{FT}{\leftrightarrow} \int_{-\infty}^{\infty} x(-t+1)\, e^{-j\omega t} dt$$

(replacing $-t+1 = \tau,\ dt = -d\tau$)

$$\overset{FT}{\leftrightarrow} \int_{+\infty}^{-\infty} x(\tau)\, e^{-j\omega(-\tau+1)} . -d\tau$$

$$\overset{FT}{\leftrightarrow} e^{-j\omega} \int_{+\infty}^{-\infty} x(\tau)\, e^{j\omega\tau} . -d\tau$$

$$\overset{FT}{\leftrightarrow} e^{-j\omega} \int_{-\infty}^{\infty} x(\tau)\, e^{j\omega\tau}\, d\tau = e^{-j\omega} \int_{-\infty}^{\infty} x(\tau)\, e^{-j.(-\omega).\tau}\, d\tau = e^{-j\omega} X(-\omega)$$

Using Time shift property (delay):

$$x(t - t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$$

$$x(-t+1) \overset{FT}{\leftrightarrow} e^{-j\omega.1} X(-\omega)$$

(c) Using Time shift property (delay): $x(t-t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$

$$y(t) = x(t-2) \overset{FT}{\leftrightarrow} Y(\omega) = e^{-j\omega.2} X(\omega)$$ [shifting]

Using scaling property $x(at) \overset{FT}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$ [Scaling]

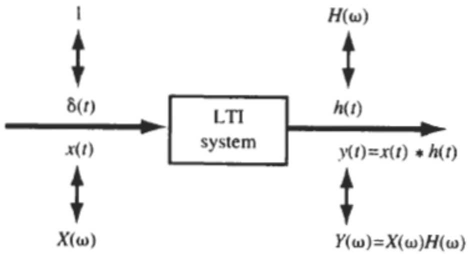
$$y\left(\frac{t}{2}\right) = x\left(\frac{t}{2} - 2\right) \overset{FT}{\leftrightarrow} \frac{1}{1/2} Y\left(\frac{\omega}{\frac{1}{2}}\right) = 2.Y(2\omega) = 2e^{-j.2\omega.2} X(2\omega)$$

3/6/21

- Evaluate the response of the LTI system with impulse response

$$h(t) = e^{-at}u(t), \qquad a > 0$$

To the input signal $\quad x(t) = e^{-bt}u(t),\ b > 0$



From convolution theorem, we can write $\quad y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

(using convolution property) $\quad y(t) = x(t) * h(t) \overset{FT}{\longleftrightarrow} \boldsymbol{X(\omega)H(\omega)}$

Fourier transform of $x(t)$ and $h(t)$ as, $\quad X(\omega) = \dfrac{1}{b+j\omega}$

$$\mathrm{H}(\omega) = \frac{1}{a+j\omega}$$

$$Y(\omega) = X(\omega)H(\omega)$$
$$= \frac{1}{(b+j\omega)(a+j\omega)}$$

Partial-fraction expansion

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

we can evaluate the coefficients A and B by

$$A = [(s+1)X(s)]|_{s=-1} = 1,$$
$$B = [(s+2)X(s)]|_{s=-2} = -1.$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

- **Multiplication (time)**

If

$$x(t) \xleftrightarrow{FT} X(\omega)$$
$$y(t) \xleftrightarrow{FT} Y(\omega)$$

Then $z(t) = x(t)y(t) \xleftrightarrow{FT} Z(\omega) = \frac{1}{2\pi} X(\omega) * Y(\omega)$

$$= \frac{1}{2\pi} \int_{-\propto}^{\propto} X(\tau)H(\omega - \tau)d\tau$$

Example:

$$x(t) = \frac{\sin(t)\sin(\frac{t}{2})}{\pi t^2}$$

$$= \pi\left(\frac{\sin t}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

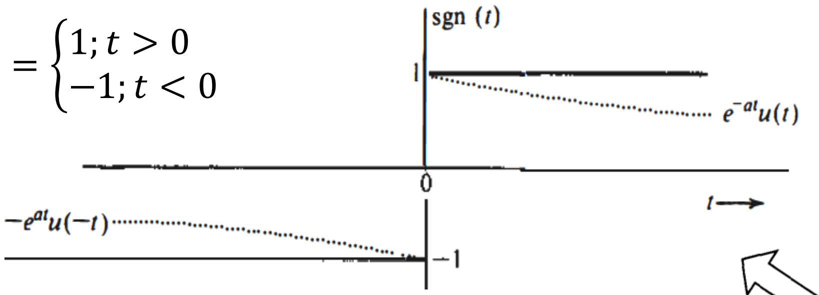
$$= \pi\underbrace{\left(\frac{\sin t}{\pi t}\right)}_{signal-1}\underbrace{\left(\frac{\sin(t/2)}{\pi t}\right)}_{signal-2}$$

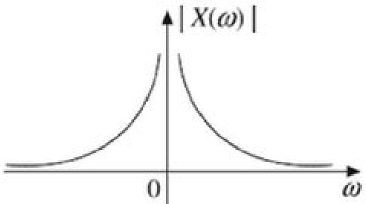
$x(t) \xleftrightarrow{FT} X(\omega) = \frac{1}{2\pi}.\pi.$(signal-1)$*$ (signal-2)

$$= \frac{1}{2} FT\left\{\frac{\sin t}{\pi t}\right\} * FT\left\{\frac{\sin(t/2)}{\pi t}\right\}$$

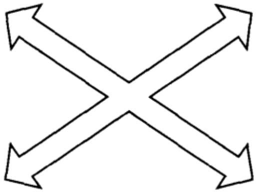
Hilbert Transform (concept)

$$\operatorname{sgn} t = \begin{cases} 1; t > 0 \\ -1; t < 0 \end{cases}$$





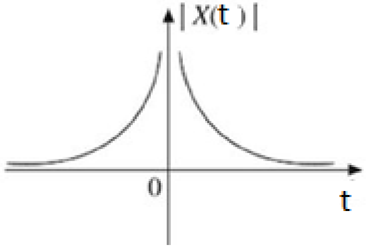
$$X(f) = \frac{1}{j.\pi f} = \frac{2}{j\omega}$$



$$2x(-f) = \operatorname{sgn}(-f) = -\operatorname{sgn}(f)$$

$$= \begin{cases} -1, f > 0 \\ 1, f < 0 \end{cases}$$

Using dual property,



$X(t) = \frac{1}{j\pi t} \overset{FT}{\leftrightarrow} \operatorname{sgn}(-f) = -\operatorname{sgn} f$

$\frac{1}{\pi t} \overset{FT}{\leftrightarrow} -j\, sgn(f)$

($\operatorname{sgn}(f)$ **odd function**)

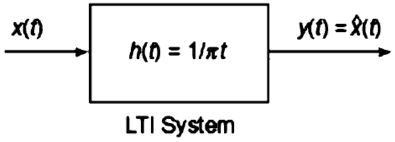
- Hilbert transform of a **signal** $\boldsymbol{x}(\boldsymbol{t})$ is defined as,

(time domain)

$$\hat{x}(t) = HT[x(t)] = x(t) * \frac{1}{\pi t}$$

$$= \int_{-\infty}^{\infty} x(\tau).\frac{1}{\pi(t-\tau)}\, d\tau$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} x(t-\tau).\frac{1}{\tau}\, d\tau \quad \text{(putting } t-\tau=\tau\text{)}$$



(Frequency domain)

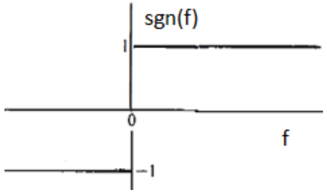
$$FT\left\{\hat{x}(t)\right\} = FT\{x(t)\}.\,FT\left\{\frac{1}{\pi t}\right\}$$

$$\hat{X}(f) = \quad X(f).\quad -j\,\mathrm{sgn}\, f$$

$$\hat{X}(f) = \begin{cases} -j\,X(f) & ; f > 0 \\ j\,X(f) & ; f < 0 \end{cases}$$

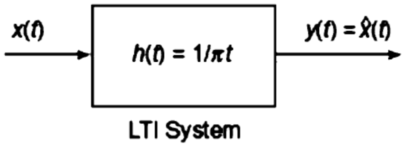
$$FT\left\{\frac{1}{\pi t}\right\} \overset{FT}{\leftrightarrow} j\,\mathrm{sgn}(-f) = -j\,\mathrm{sgn}\, f$$

$$= \begin{cases} -j, f > 0 \\ j, f < 0 \end{cases}$$



Hilbert transform

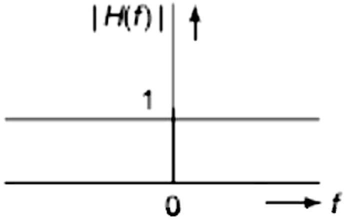
$$\hat{X}(f) = \begin{cases} -j\,X(f) & ; f > 0 \\ j\,X(f) & ; f < 0 \end{cases}$$

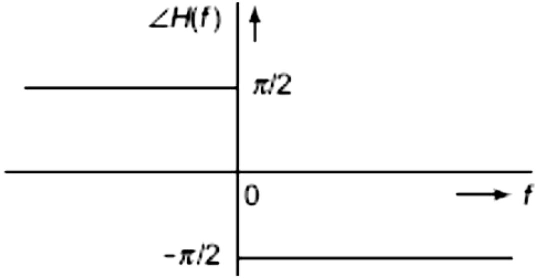


$$h(t) \;=\; FT\left\{\frac{1}{\pi t}\right\} \overset{FT}{\leftrightarrow} j\,\mathrm{sgn}(-f) = -j\,\mathrm{sgn}\,f \;=\; \begin{cases} j, f > 0 \\ -j, f < 0 \end{cases}$$

$$\left|\hat{X}(f)\right| = |X(f)|$$

(magnitude unchanged)





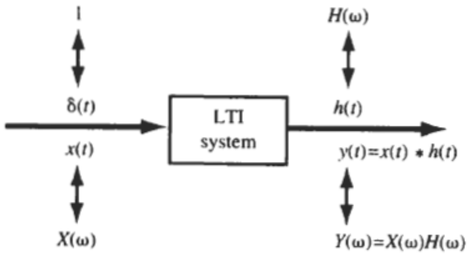
Applications:

Phase shift, representation of band pass signals, single side-band, band pass to low pass etc

- Evaluate the response of the LTI system with impulse response

$$h(t) = e^{-at}u(t), \qquad a > 0$$

To the input signal $\quad x(t) = e^{-bt}u(t),\ b > 0$



From convolution theorem, we can write $\quad y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

(using convolution property) $\quad y(t) = x(t) * h(t) \overset{FT}{\longleftrightarrow} \boldsymbol{X(\omega)H(\omega)}$

Fourier transform of $x(t)$ and $h(t)$ as, $\quad X(\omega) = \dfrac{1}{b + j\omega}$

$$\mathrm{H}(\omega) = \frac{1}{a+j\omega}$$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{(b + j\omega)(a + j\omega)}$$

Partial-fraction expansion

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

we can evaluate the coefficients A and B by

$$A = [(s+1)X(s)]\big|_{s=-1} = 1,$$
$$B = [(s+2)X(s)]\big|_{s=-2} = -1.$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

- The input output of a system is described by

$$y'(t) + 2y(t) = x(t) + x'(t)$$

Compute the impulse response h(t) of the system

Taking Fourier-transform both side of the equation

$$y'(t) + 2y(t) = x(t) + x'(t)$$

$$\rightarrow j\omega Y(\omega) + 2\,Y(\omega) = X(\omega) + j\omega X(\omega)$$

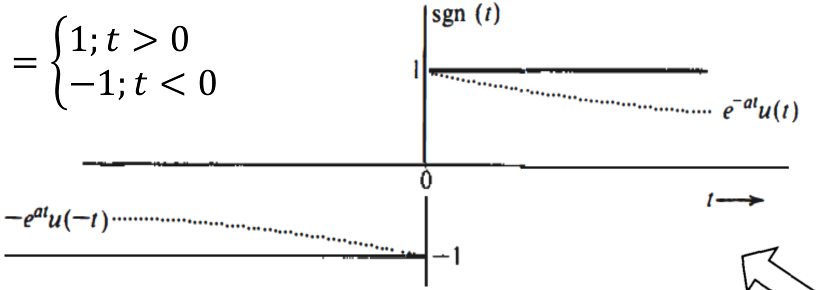
$$\rightarrow (j\omega + 2)Y(\omega) = (1 + j\omega)X(\omega)$$

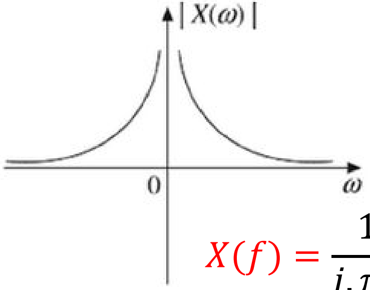
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1+j\omega}{2+j\omega} = \frac{2+j\omega-1}{2+j\omega} = 1 - \frac{1}{2+j\omega}$$

$$Taking\ Inverse:\ \ h(t) = \delta(t) - e^{-2t}u(t)$$

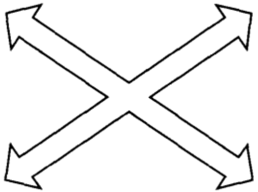
Hilbert Transform (concept)

$$\operatorname{sgn} t = \begin{cases} 1; t > 0 \\ -1; t < 0 \end{cases}$$





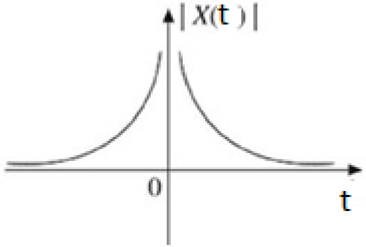
$$X(f) = \frac{1}{j.\pi f} = \frac{2}{j\omega}$$



$$2x(-f) = \operatorname{sgn}(-f) = -\operatorname{sgn}(f)$$

$$= \begin{cases} -1, f > 0 \\ 1, f < 0 \end{cases}$$

Using dual property,



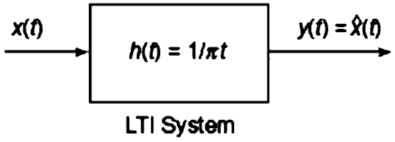
$X(t) = \frac{1}{j\pi t} \overset{FT}{\leftrightarrow} \operatorname{sgn}(-f) = -\operatorname{sgn} f$ $\quad \frac{1}{\pi t} \overset{FT}{\leftrightarrow} -j\, sgn(f)$

($\operatorname{sgn}(f)$ odd function)

- **Hilbert transform of a signal $x(t)$ is defined as,**

(time domain)

$$\hat{x}(t) = HT[x(t)] = x(t) * \frac{1}{\pi t}$$



$$= \int_{-\infty}^{\infty} x(\tau).\frac{1}{\pi(t-\tau)} d\tau$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} x(t-\tau).\frac{1}{\tau} d\tau \quad (\text{putting } t-\tau=\tau)$$

(Frequency domain)

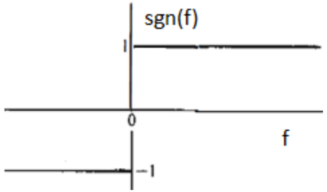
$$FT\{\hat{x}(t)\} = FT\{x(t)\}.\,FT\left\{\frac{1}{\pi t}\right\}$$

$$\hat{X}(f) = \quad X(f). \quad -j\,\text{sgn}\, f$$

$$FT\left\{\frac{1}{\pi t}\right\} \overset{FT}{\leftrightarrow} j\,\text{sgn}(-f) = -j\,\text{sgn}\, f$$

$$= \begin{cases} -j, f > 0 \\ j, f < 0 \end{cases}$$

$$\hat{X}(f) = \begin{cases} -j\,X(f) & ; f > 0 \\ j\,X(f) & ; f < 0 \end{cases}$$

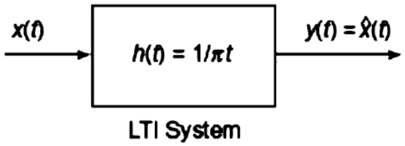


Hilbert transform

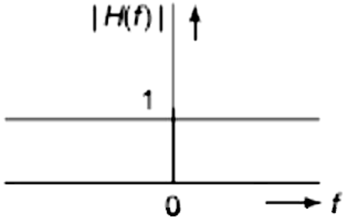
$$\hat{X}(f) = \begin{cases} -j\,X(f) & ; f > 0 \\ j\,X(f) & ; f < 0 \end{cases}$$

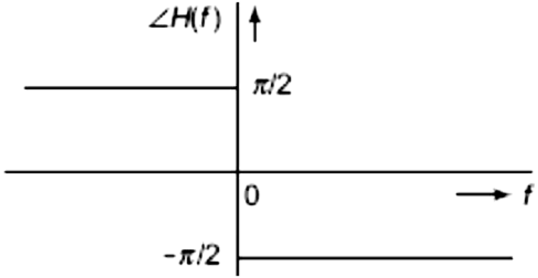
$$\left|\hat{X}(f)\right| = |X(f)|$$

(magnitude unchanged)



$$h(t) = FT\left\{\frac{1}{\pi t}\right\} \overset{FT}{\leftrightarrow} j\,\mathrm{sgn}(-f) = -j\,\mathrm{sgn}\,f = \begin{cases} j, f > 0 \\ -j, f < 0 \end{cases}$$





Applications:

Phase shift, representation of band pass signals, single side-band, band pass to low pass etc

- **Evaluate the Hilbert transform** of the signal $x(t) = \sin \omega_0 t$

$$\hat{X}(f) = \begin{cases} -j\,X(f) & ; f > 0 \\ j\,X(f) & ; f < 0 \end{cases}$$

Using Euler's relation, $\sin \omega_0 t$

$$x(t) = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$= \frac{1}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right]$$

Hilbert transform: $FT\left\{\hat{x}(t)\right\} = FT\{x(t)\}.FT\left\{\frac{1}{\pi t}\right\}$

$$\mathrm{FT}[\mathrm{x(t)}] = \mathrm{FT}[\sin \omega_0 t] = \frac{1}{2j}\,FT\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right]$$

$$= \frac{1}{2j}\,FT\left[e^{j2\pi f_0 t}\right] - FT\left[e^{-j2\pi f_0 t}\right]$$

We know,

$$1.\ e^{j\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega - \omega_0) = \delta(f - f_0)$$

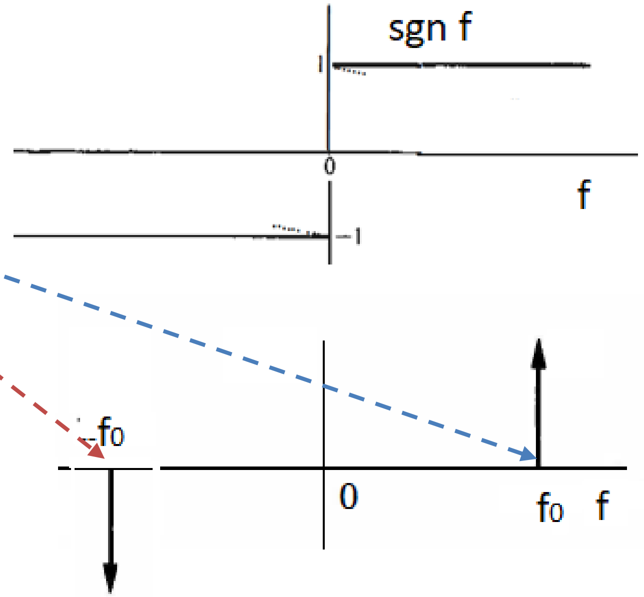
$$X(f) = \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$$

$$\rightarrow \hat{X}(f) = -j\,\mathrm{sgn}\,f\,.X(f)$$

$$\rightarrow \hat{X}(f) = -j\,\mathrm{sgn}\,f\,.\frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$$

$$= -\frac{1}{2}[\delta(f - f_0) - \delta(f + f_0)]\,\mathrm{sgn}\,f$$

$$\hat{X}(f) = -\frac{1}{2}[\delta(f - f_0) - \delta(f + f_0)]\,\mathrm{sgn}\, f$$



$$\hat{X}(f) = -\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$

We know, $1.\, e^{j\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega - \omega_0) = \delta(f - f_0)$

With inverse FT $\hat{x}(t) = -\frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] = -\cos\omega_0\, t$

$$X_+(f) = X(f) + j\hat{X}(f)$$

$$= \begin{cases} 2X(f), f \geq 0 \\ 0, f < 0 \end{cases}$$

Similarly, we can defined as,

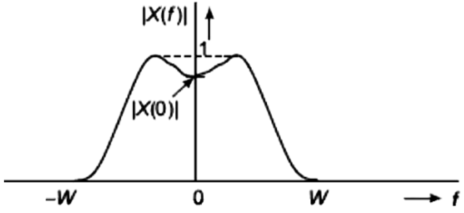
$$x_-(t) = x(t) - j\hat{x}(t)$$

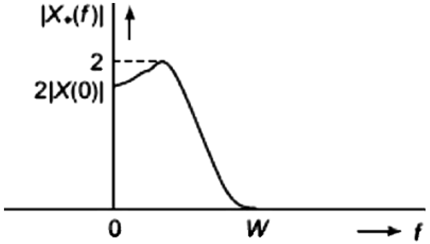
$$X_-(f) = \begin{cases} 0, f > 0 \\ 2X(f), f \leq 0 \end{cases}$$

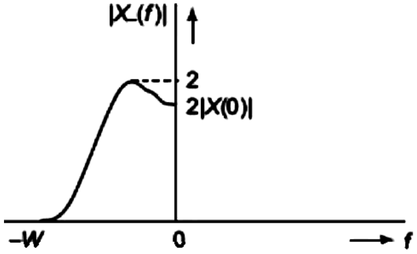
(Negative frequency pre-envelope)

Observations:

1. Spectrum of $x_+(t)$/analytical signal = 2 × positive frequency part of spectrum x(t)
2. Spectrum of $x_+(t)$/analytical signal = zero for all negative frequencies
3. Called positive frequency pre-envelope







Pre-envelope/Analytical signal

Pre-envelope / Analytical signal of x(t) **real** valued signal is defined as,

$$x_+(t) = x(t) + j\hat{x}(t)$$

Real Part itself

Imaginary part: Hilbert transform of x(t)

In frequency domain:

$$\boxed{X_+(f) = X(f) + j\hat{X}(f)}$$

$$= X(f) + j[-j\,\mathrm{sgn}\,f\; X(f)]$$

$$= \begin{cases} X(f) + j.-j.X(f), & f \geq 0 \\ X(f) + j.j.X(f), & f < 0 \end{cases}$$

$$= \begin{cases} X(f) + X(f), f \geq 0 \\ X(f) - X(f), f < 0 \end{cases} \qquad \boxed{= \begin{cases} 2X(f), f \geq 0 \\ 0, f < 0 \end{cases}}$$

$$\hat{X}(f) = -j\,\mathrm{sgn}\,f\,.\,X(f)$$

$$= \begin{cases} -j.1.X(f) = -jX(f), f \geq 0 \\ -j.-1.X(f) = jX(f), f < 0 \end{cases}$$

$$X_+(f) = X(f) + j\hat{X}(f)$$

$$= \begin{cases} 2X(f), f \geq 0 \\ 0, f < 0 \end{cases}$$

Similarly, we can defined as,

$$x_-(t) = x(t) - j\hat{x}(t)$$

$$X_-(f) = \begin{cases} 0, f > 0 \\ 2X(f), f \leq 0 \end{cases}$$

(Negative frequency pre-envelope)

Observations:

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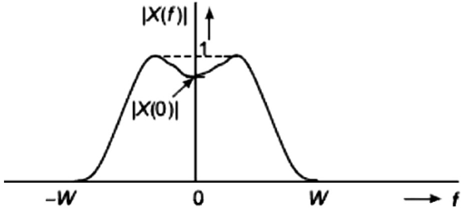
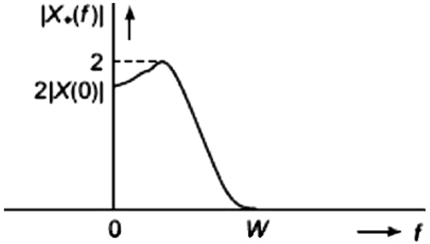
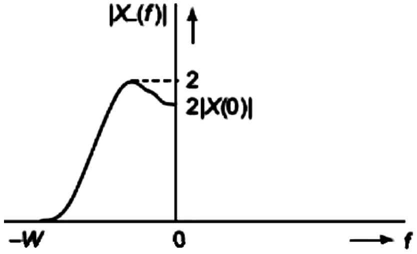
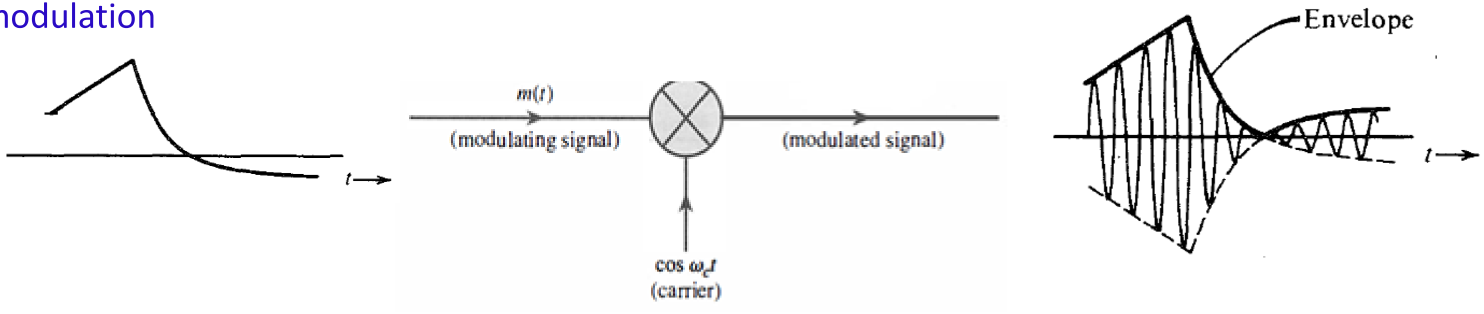




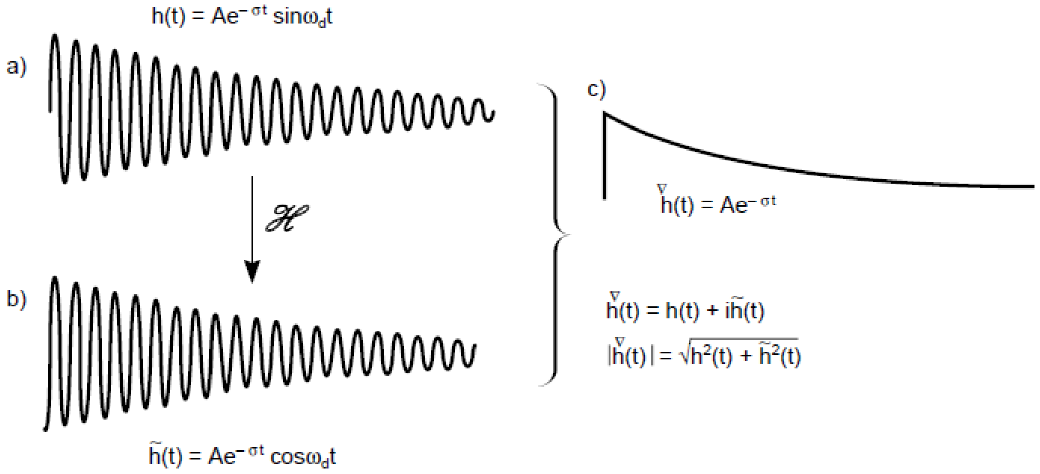


Envelope detector:

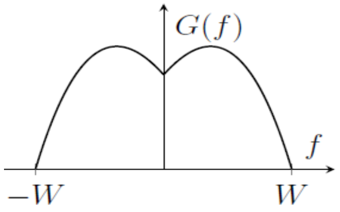
- Amplitude modulation



- Removal of the oscillations



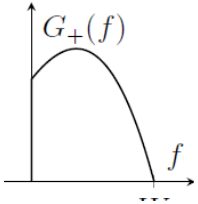
Let $g(t)$ is the message signal, with spectrum

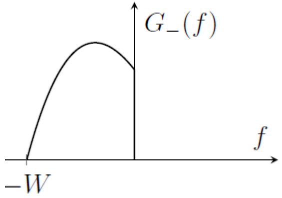


Analytical signals:

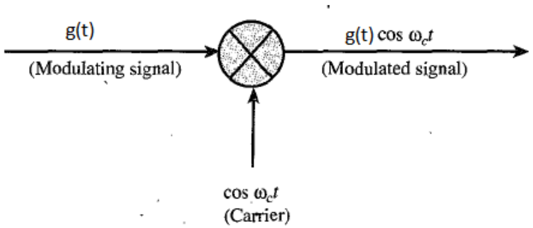
$$g_+(t) = \frac{1}{2}[g(t) + j\hat{g}(t)] \quad \overset{FT}{\longleftrightarrow} \quad G_+(f) = G(f) + j\hat{G}(f) = \begin{cases} G(f), f \geq 0 \\ 0, f < 0 \end{cases}$$

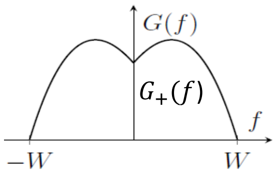
$$g_-(t) = \frac{1}{2}[g(t) - j\hat{g}(t)] \quad \overset{FT}{\longleftrightarrow} \quad G_-(f) = \begin{cases} 0, f > 0 \\ G(f), f \leq 0 \end{cases}$$



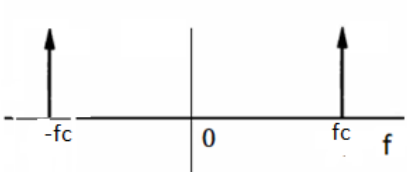


Single-side band modulation





$$\mathrm{FT}[\cos(\omega_c t)] = \frac{1}{2}\delta(f - f_c) + \frac{1}{2}\delta(f + f_c)$$

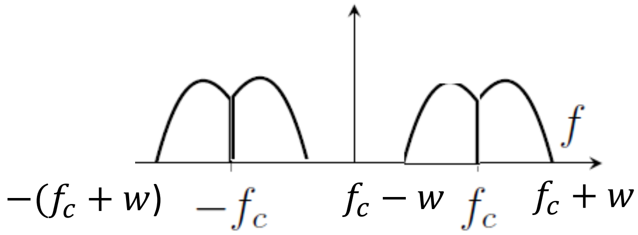


$$g(t)\cos(\omega_c t) \overset{FT}{\leftrightarrow} \frac{1}{2} FT\,[g(t)e^{-j\omega_c t} + g(t)e^{j\omega_c t}]$$

$$= \frac{1}{2}[FT\{g(t)e^{-j\omega_c t}\} + FT\{g(t)e^{j\omega_c t}\}]$$

$$= \frac{1}{2}[G(f + f_c) + G(f - f_c)]$$

As $e^{j\omega_c t} \leftrightarrow 2\pi\delta(\omega - \omega_c) = \delta(f - f_c)$



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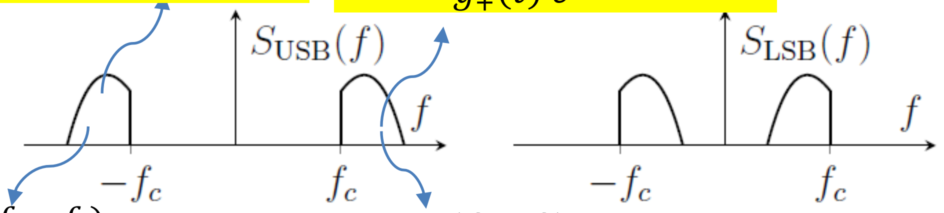
$g_-(t)\, e^{-j2\pi f_c t}$ $\quad$ $g_+(t)\, e^{j2\pi f_c t}$



$G_-(f - f_c)$ $\quad$ $G_+(f - f_c)$

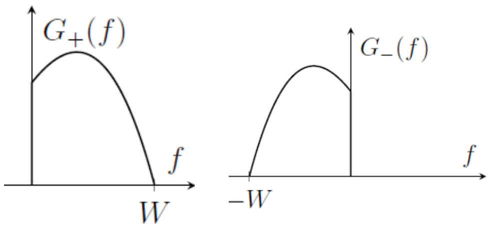
Analytical signals:

$$g_+(t) = \frac{1}{2}[g(t) + \mathrm{j}\hat{g}(t)]$$

$$g_-(t) = \frac{1}{2}[g(t) - \mathrm{j}\hat{g}(t)]$$

$\overset{FT}{\leftrightarrow}$

$$G_+(f) = G(f) + j\hat{G}(f) = \begin{cases} G(f), f \geq 0 \\ 0, f < 0 \end{cases}$$

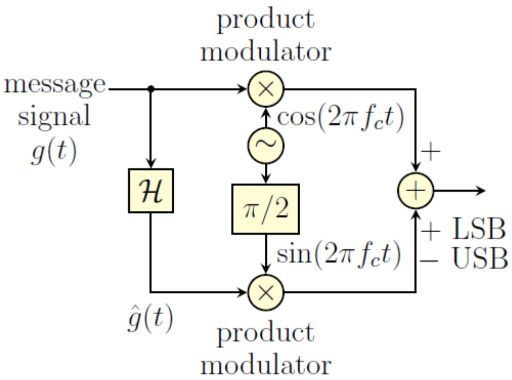


$$s_{usb}(t) = g_+(t)e^{2j\pi f_c t} + g_-(t)e^{-2j\pi f_c t}$$

$$= \frac{1}{2}[g(t) + j\hat{g}(t)]e^{2j\pi f_c t} + \frac{1}{2}[g - j\hat{g}(t)]e^{-2j\pi f_c t}$$

$$= g(t)\frac{1}{2}\left[e^{2j\pi f_c t} + e^{-2j\pi f_c t}\right] + \hat{g}(t)\frac{1}{2}\left[je^{2j\pi f_c t} - je^{-2j\pi f_c t}\right]$$

$$= g(t)\cos(2\pi f_c t) - \hat{g}(t)\sin(2\pi f_c t)$$



Hilbert transform pairs.

Time-domain signal $g(t)$	Hilbert transform $\hat{g}(t)$
$a_1 g_1(t) + a_2 g_2(t);\ a_1, a_2 \in \mathbb{C}$	$a_1 \hat{g}_1(t) + a_2 \hat{g}_2(t)$
$h(t - t_0)$	$\hat{h}(t - t_0)$
$h(at);\ a \neq 0$	$\operatorname{sgn}(a)\hat{h}(at)$
$\frac{\mathrm{d}}{\mathrm{d}t} h(t)$	$\frac{\mathrm{d}}{\mathrm{d}t} \hat{h}(t)$
$\delta(t)$	$\frac{1}{\pi t}$
$e^{\mathrm{j}t}$	$-\mathrm{j}e^{\mathrm{j}t}$
$e^{-\mathrm{j}t}$	$\mathrm{j}e^{-\mathrm{j}t}$
$\cos(t)$	$\sin(t)$
$\operatorname{rect}(t)$	$\frac{1}{\pi} \ln \lvert(2t+1)/(2t-1)\rvert$
$\operatorname{sinc}(t)$	$\frac{\pi t}{2} \operatorname{sinc}^2(t/2) = \sin(\pi t/2) \operatorname{sinc}(t/2)$
$1/(1+t^2)$	$t/(1+t^2)$

Laplace transform

- The Fourier transform is applicable to a large variety of functions and is widely used as a mathematical tool in engineering science, when it satisfy the relation

$$\int_{-\propto}^{\propto} |f(t)|\, dt < \propto$$

- Many functions which are interest in engineering work cannot be handled by this method

  - Ramp, parabolic etc (integral is not converging and functions are not Fourier transformable)

- In order handle these functions, the Laplace transform is proposed by introducing a convergence factor $e^{-\sigma}$, σ where is real number and large enough to ensure absolute convergence

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\propto}^{\propto} x(t)e^{-\sigma t}\, e^{-j\omega t}dt$$

called the **bilateral (or two-sided)**

$$s = \sigma + j\omega$$

Unilateral (or one-sided) Laplace transform

$$X(s) = \int_0^{\propto} x(t)e^{-\sigma t}\, e^{-j\omega t} dt = \int_0^{\propto} x(t)e^{-st}\, dt$$

Clearly the bilateral and unilateral transforms are equivalent only if x(t) = **0 for t < 0.**

- **The Region of Convergence (ROC):** The range of values of the complex variables s for which the Laplace transform converges is called the region of convergence (ROC).

➢ ROC of the Laplace-transform (LS) of $x(t)$ consists of those value of **s** for which x(t)$e^{-\sigma t}$ is absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)|e^{-\sigma t} dt < \infty$$

This condition depends on σ, *real part of s*, $Re\{s = \sigma + j\omega\}$

- Evaluate the Laplace transform of signal

$$x(t) = e^{-at}u(t) \quad , a \text{ real}$$

As per definition of Laplace transform,

$$X(s) = \int_{-\propto}^{\propto} x(t)\ e^{-st} dt$$

$$= \int_{-\propto}^{\propto} e^{-at}\, u(t) e^{-st} dt = \int_{0}^{\propto} e^{-at}\, e^{-st} dt$$

$$= \int_{0}^{\propto} e^{-(a+s)t}\ dt$$

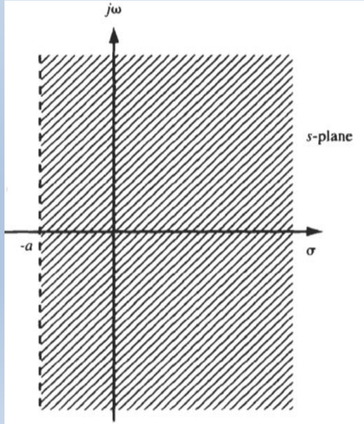
$$= -\frac{1}{s+a} e^{-(s+a)t}\Big|_0^\infty = \frac{1}{s+a}$$

For Convergence, $\int_{-\infty}^{\infty} |x(t)| e^{-\sigma t} dt < \infty$

$e^{-(s+a)t} \xrightarrow{t \to \propto} 0$ if $\mathrm{Re}\{s+a\} > 0$

$\to \mathrm{Re}\{s\} > -a$

ROC



- Evaluate the Laplace transform of signal

$$x(t) = -e^{-at}u(-t) \quad \text{, a real}$$

As per definition of Laplace transform,

$$X(s) = \int_{-\propto}^{\propto} x(t)e^{-\sigma t}\, e^{-j\omega t} dt$$

$$= \int_{-\propto}^{\propto} -e^{-at}\, u(-t)e^{-st} dt = \int_{-\propto}^{0} -e^{-at}\, e^{-st} dt$$

$$= \int_{-\propto}^{0} -e^{-(a+s)t}\, dt$$

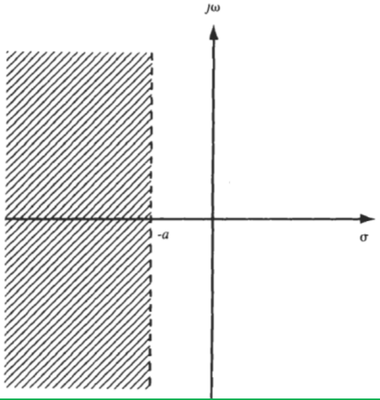
$$= \frac{1}{s+a}, \qquad Re\{s\} < -a$$

For Convergence, $\int_{-\infty}^{\infty} |x(t)|e^{-\sigma t} dt < \infty$

$$e^{-(s+a)t} \xrightarrow{t \to -\propto} 0 \quad \textbf{if} \quad \mathrm{Re}\{s+a\} < 0$$

$$\to \mathrm{Re}\{s\} < -a$$

ROC



- Evaluate the Laplace-transform (LS) of the signal $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

As per definition of Laplace transform, $X(s) = \int_{-\propto}^{\propto} x(t)e^{-st}\,dt = \int_{-\propto}^{\propto} [3e^{-2t}u(t) - 2e^{-t}u(t)]\,e^{-st}dt$

$$= 3\int_0^{\propto} e^{-2t}\,e^{-st}dt - 2\int_0^{\infty} e^{-t}\,e^{-st}dt$$

$$= 3\int_0^{\propto} e^{-(s+2)t}\,dt - 2\int_0^{\infty} e^{-(s+1)t}\,dt$$

For Convergence, $\int_{-\infty}^{\infty} |x(t)|e^{-\sigma t}dt < \infty$

$e^{-(s+2)t} \xrightarrow{t\to\propto} 0$ if $\mathrm{Re}\{s+2\} > 0$

$\to \mathrm{Re}\{s\} > -2$

$e^{-(s+1)t} \xrightarrow{t\to\propto} 0$ if $\mathrm{Re}\{s+1\} > 0$

$\to \mathrm{Re}\{s\} > -1$

$Re\{s\} \succ -1$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}$$

Pole and Zeros

- A **rational** X(s) can be written as,

$$X(s) = \frac{a_0 s^m + a_1 s^{m-1} + \cdots + a_m}{b_0 s^n + b_1 s^{n-1} + \cdots + b_n} = \frac{a_0}{b_0} \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

- The coefficients a_k, and b_k, are real constants, and m and n are positive integers.
- The X(s) is called a proper rational function if $\boldsymbol{n} > \boldsymbol{m}$, and an improper rational function if $\boldsymbol{n} \leq$ m.
- The roots of the numerator polynomial are called, z the zeros of X(s) because X(s) = **0** for those values of s.
- The roots of the denominator polynomial are called poles because X(s) is infinite for those values of s
- The **poles of X(s) lie outside** the ROC since X(s) does not converge at the poles
- The **zeros** may lie inside or outside the ROC

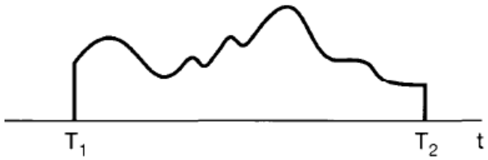
$$X(s) = \frac{2s + 4}{s^2 + 4s + 3} = 2\frac{s + 2}{(s + 1)(s + 3)} \qquad \text{Re}(s) > -1$$

X(s) has one zero at s = - 2 and two poles at s = - 1 and s = - 3

Properties

1. The ROC does not contain any poles
2. If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s-plane

$x(t)$ is absolutely integrable for any value of σ



$$x(t) = \begin{cases} e^{-at}, 0 < t < T \\ 0, else \end{cases}$$

$$X(s) = \int_0^T e^{-at} e^{-st} dt = \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]$$

It looks $X(s)$ *has a pole at* $s = -a$

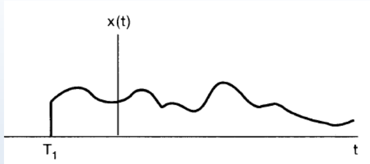
To find the value of $X(s)$ at $s = -a$, consider L'Hopital rule

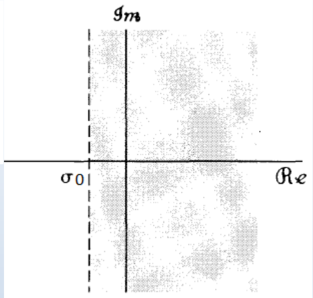
$$\lim_{s \to -a} X(s) = \lim_{s \to -a} \left[\frac{\frac{d}{ds}\left(1 - e^{-(s+a)T}\right)}{\frac{d}{ds}(s+a)} \right] = \lim_{s \to -a} Te^{-aT} e^{-sT} = T \text{ (finite)}$$

the ROC is the entire s-plane

Properties

3. If $x(t)$ is **right sided**, and if the line $Re\{s\} = \sigma_0$ is in the **ROC**,



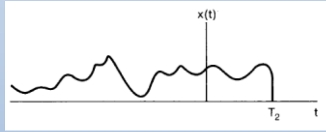


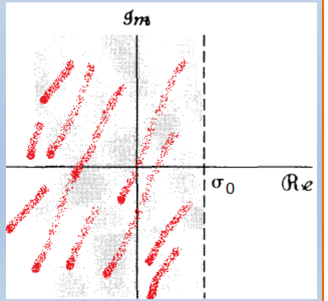
→ then all values of s for which $Re\{s\} > \sigma_0$ will also be in the ROC

$$if\ \sigma_1 > \sigma_0 \qquad \int_{-\infty}^{\infty} |x(t)| e^{-\sigma_1 t} dt < \int_{-\infty}^{\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$$

4. If $x(t)$ is **left sided**, and if the line $Re\{s\} = \sigma_0$ is in the **ROC**

→ then all values of s for which $Re\{s\} < \sigma_0$ will also be in the ROC

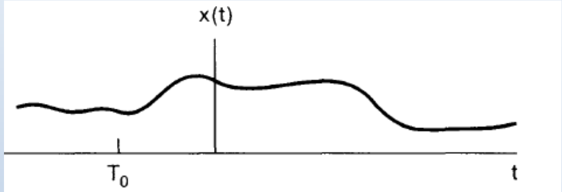


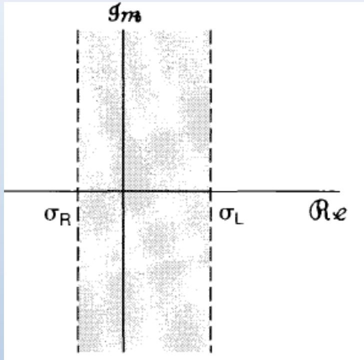


Properties

5. If $x(t)$ is **both sided**, and if the line $Re\{s\} = \sigma_0$ is in the **ROC**,

→ then the ROC consist of **a strip** in the s-plane that includes the line $Re\{s\} = \sigma_0$





5. If x(t) **is right sided** and its Laplace transform $X(s)$ is rational, then the ROC in s-plane is right of the rightmost pole

$$X(s) = \frac{2s+4}{s^2+4s+3} = 2\frac{s+2}{(s+1)(s+3)} \qquad \mathrm{Re}(s) > -1$$

X(s) has one zero at s = - 2 and two poles at s = - 1 and s = - 3

Properties

6. If x(t) **is left sided** and its Laplace transform $X(s)$ is rational, then the ROC in s-plane is right of the leftmost pole

- Find the Laplace and ROC of the signal $x(t) = e^{-b|t|}$

From definition of Laplace transform

$$X(s) = \int_{-\propto}^{\propto} x(t)e^{-st}\,dt = \int_{-\propto}^{\propto} e^{-b|t|}e^{-st}\,dt$$

$$= \int_{-\propto}^{0} e^{-b(-t)}e^{-st}\,dt + \int_{0}^{\propto} e^{-b(t)}e^{-st}\,dt$$

$$= \int_{-\propto}^{0} e^{-(s-b)t}\,dt + \int_{0}^{\propto} e^{-(s+b)t}\,dt$$

$$= -\frac{1}{s-b} + \frac{1}{s+b}$$

$\int_{-\propto}^{0} e^{-(s-b)t} \xrightarrow{t\to-\propto} 0$ If Re{s-b}<0 → Re{s} < b

$\int_{0}^{\propto} e^{-(s+b)t}\,dt \xrightarrow{t\to\propto} 0$ If Re{s+b} > 0 → Re{s} > - b

-b < Re{s}< b

PROPERTIES OF THE LAPLACE TRANSFORM

Property	Signal	Laplace Transform	ROC
	$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Initial- and Final-Value Theorems

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t - T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2 + \omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2 + \omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

Inverse Laplace transform

$$X(s) = \int_{-\propto}^{\propto} x(t)e^{-st}\,dt$$

Putting $s = \sigma + j\omega$ $\quad X(\sigma + j\omega) = \int_{-\propto}^{\propto} x(t)e^{-(\sigma+j\omega)t}\,dt = = \int_{-\propto}^{\propto} x(t)e^{-\sigma t}\,e^{-j\omega t}dt = FT\{x(t)e^{-\sigma t}\}$

We can invert this relationship using the inverse Fourier transform as given $\quad x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\propto}^{\propto} X(\omega)e^{j\omega t}\,d\omega$

$$x(t)e^{-\sigma t} = FT^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\propto}^{\propto} X(\sigma + j\omega)e^{j\omega t}\,d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\propto}^{\propto} X(\sigma + j\omega)e^{(\sigma+j\omega)t}\,d\omega$$

$s = \sigma + j\omega, ds = jd\omega$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-\propto}^{\sigma+j\propto} X(s)e^{st}\,ds$$

Evaluate inverse Laplace transform

$$X(s) = \frac{1}{(s+1)(s+2)}, \qquad \Re e\{s\} > -1$$

we first perform a partial-fraction expansion

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

we can evaluate the coefficients A and B by

$$A = [(s+1)X(s)]|_{s=-1} = 1$$

$$\mathrm{B} = [(s+2)X(s)]|_{s=-2} = -1$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Since the ROC for $X(s)$ is $\Re e\{s\} > -1$ ROC is to the right of the pole.

$$e^{-at}u(t) \overset{L}{\leftrightarrow} \frac{1}{s+a}, Re\{S\} \succ -a$$

$x(t)$ = Inverse Laplace (X(s))= $e^{-t}u(t)$ - $e^{-2t}u(t)$

Evaluate inverse Laplace transform

$$X(s) = \frac{1}{(s+1)(s+2)}; \quad Re\{s\} < -2$$

we first perform a partial-fraction expansion

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = [(s+1)X(s)]|_{s=-1} = -1$$

$$\mathrm{B} = [(s+2)X(s)]|_{s=-2} = -1$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

As $x(t) = -e^{-at}u(-t)$ $\rightarrow \mathrm{Re}\{s\} < -a$

$$-e^{-t}u(-t) \overset{L}{\leftrightarrow} \frac{1}{s+1}, Re\{s\} < -1$$

$$-e^{-2t}u(-t) \overset{L}{\leftrightarrow} \frac{1}{s+2}, Re\{s\} < -2$$

$$x(t) = [-e^{-t} + e^{-2t}]u(-t)$$

- Evaluate the Inverse Laplace transform

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad -2 < \Re e\{s\} < -1.$$

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

$$-e^{-t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \Re e\{s\} < -1$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \Re e\{s\} > -2.$$

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \quad -2 < \Re e\{s\} < -1.$$

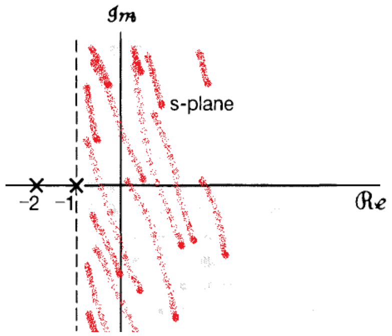
Linearity:

If

$$x_1(t) \longleftrightarrow X_1(s) \qquad \text{ROC} = R_1$$

$$x_2(t) \longleftrightarrow X_2(s) \qquad \text{ROC} = R_2$$

Then $$a_1x_1(t) + a_2x_2(t) \longleftrightarrow a_1X_1(s) + a_2X_2(s) \qquad R' \supset R_1 \cap R_2$$



Example:

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt = \int_{-\infty}^{\infty} [3e^{-2t}u(t) - 2e^{-t}u(t)]\,e^{-st}dt$$

$$= 3\int_0^{\infty} e^{-2t}\,e^{-st}dt - 2\int_0^{\infty} e^{-t}\,e^{-st}dt$$

$$= 3\int_0^{\infty} e^{-(s+2)t}\,dt - 2\int_0^{\infty} e^{-(s+1)t}\,dt$$

$e^{-(s+2)t} \xrightarrow{t\to\infty} 0$ if $\text{Re}\{s+2\} > 0$ $\rightarrow \text{Re}\{s\} > -2$

$e^{-(s+1)t} \xrightarrow{t\to\infty} 0$ if $\text{Re}\{s+1\} > 0$ $\rightarrow \text{Re}\{s\} > -1$

$\text{Re}\{s\} > -1$

Properties

Time-shift:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

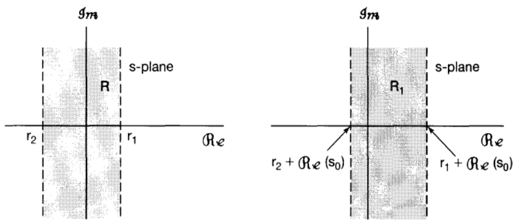
then $\quad x(t - t_0) \leftrightarrow e^{-st_0} X(s) \qquad R' = R$

Shift in S-domain:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $\quad e^{s_0 t} x(t) \leftrightarrow X(s - s_0) \qquad R' = R + \mathrm{Re}(s_0)$



ROC is shifted by

Scaling:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $\quad x(at) \leftrightarrow \dfrac{1}{|a|} X\left(\dfrac{s}{a}\right) \qquad R' = aR$

Time reversal:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $x(-t) \leftrightarrow X(-s) \qquad R' = -R$

Differentiation in S-domain:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $-tx(t) \leftrightarrow \dfrac{dX(s)}{ds} \qquad R' = R$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty}(-t)x(t)e^{-st}\,dt = \int_{-\infty}^{\infty}[-tx(t)]e^{-st}\,dt$$

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad R' = R$$

Differentiation in time domain:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $\dfrac{dx(t)}{dt} \leftrightarrow sX(s) \qquad R' \supset R$

Conjugate:

$$x(t) \longleftrightarrow X(s), \quad \text{with ROC} = R$$

$$x^*(t) \longleftrightarrow X^*(s^*), \quad \text{with ROC} = R.$$

$$X(s) = X^*(s^*) \quad \text{when } x(t) \text{ is real.}$$

Integration:

If

$$x(t) \longleftrightarrow X(s) \qquad \text{ROC} = R$$

then

$$\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{1}{s} X(s) \qquad R' = R \cap \{\mathrm{Re}(s) > 0\}$$

$$\mathfrak{L}\left\{\int_{0^-}^{t} f(\tau)\tau\right\} = \int_{0^-}^{\infty} \underbrace{\left(\int_{0^-}^{t} f(\tau)\mathrm{d}\tau\right)}_{u(t)} \underbrace{e^{-st}}_{v'(t)}\mathrm{d}t$$

$$= \left[\left(\int_{0^-}^{t} f(\tau)\mathrm{d}\tau\right)\left(-\frac{1}{s}e^{-st}\right)\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)\left(-\frac{1}{s}e^{-st}\right)\mathrm{d}t$$

Convolution:

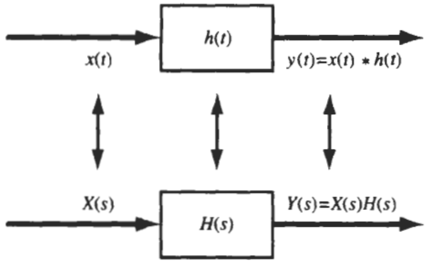
If

$$x_1(t) \longleftrightarrow X_1(s) \qquad \text{ROC} = R_1$$

$$x_2(t) \longleftrightarrow X_2(s) \qquad \text{ROC} = R_2$$

then

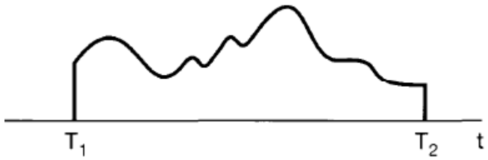
$$x_1(t) * x_2(t) \longleftrightarrow X_1(s)X_2(s) \qquad R' \supset R_1 \cap R_2$$



Properties

1. The ROC does not contain any poles
2. If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s-plane

$x(t)$ is absolutely integrable for any value of σ



$$x(t) = \begin{cases} e^{-at}, 0 < t < T \\ 0, else \end{cases}$$

$$X(s) = \int_0^T e^{-at} e^{-st} dt = \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]$$

It looks $X(s)$ *has a pole at* $s = -a$

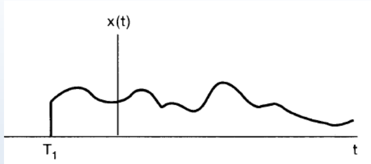
To find the value of $X(s)$ *at* $s = -a$, consider L'Hopital rule

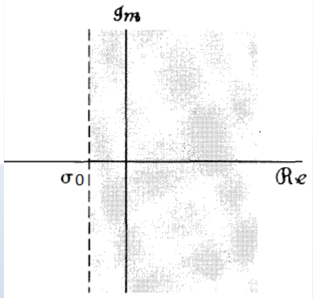
$$\lim_{s \to -a} X(s) = \lim_{s \to -a} \left[\frac{\frac{d}{ds}\left(1 - e^{-(s+a)T}\right)}{\frac{d}{ds}(s+a)} \right] = \lim_{s \to -a} T e^{-aT} e^{-sT} = T \text{ (finite)}$$

the ROC is the entire s-plane

Properties

3. If $x(t)$ is **right sided**, and if the line $Re\{s\} = \sigma_0$ is in the **ROC**,



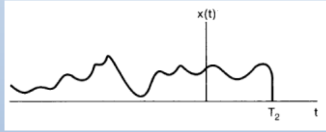


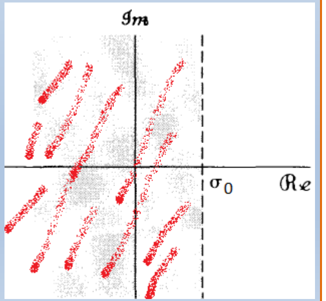
→ then all values of s for which $Re\{s\} > \sigma_0$ will also be in the ROC

$$if\ \sigma_1 > \sigma_0 \qquad \int_{-\infty}^{\infty} |x(t)| e^{-\sigma_1 t} dt < \int_{-\infty}^{\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$$

4. If $x(t)$ is **left sided**, and if the line $Re\{s\} = \sigma_0$ is in the **ROC**

→ then all values of s for which $Re\{s\} < \sigma_0$ will also be in the ROC

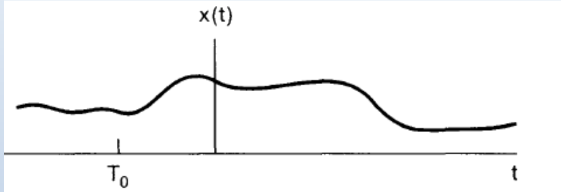


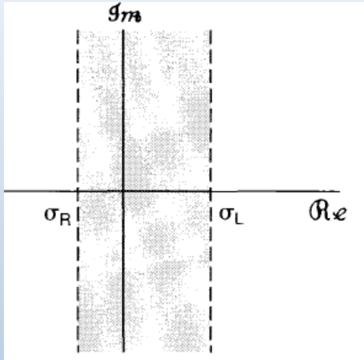


Properties

5. If $x(t)$ is **both sided**, and if the line $Re\{s\} = \sigma_0$ is in the **ROC**,

→ then the ROC consist of **a strip** in the s-plane that includes the line $Re\{s\} = \sigma_0$





5. If x(t) **is right sided** and its Laplace transform $X(s)$ is rational, then the ROC in s-plane is right of the **rightmost** pole

$$X(s) = \frac{2s+4}{s^2+4s+3} = 2\frac{s+2}{(s+1)(s+3)} \qquad \mathrm{Re}(s) > -1$$

X(s) has one zero at s = - 2 and two poles at s = - 1 and s = - 3

Properties

6. If x(t) **is left sided** and its Laplace transform $X(s)$ is rational, then the ROC in s-plane is right of the **leftmost pole**

Using the various Laplace transform properties, derive the Laplace transforms of the following signals from the **Laplace transform of u(t)**

$(a)\ \delta(t)$
(b) $\delta'(t)$
(c) $e^{-at}u(t)$
(d) t$e^{-at}u(t)$
(e) cos$\omega_0 t\ u(t)$

Laplace transform of u(t):

$$U(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt \quad = \int_{0}^{\infty} e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_0^{\infty} = \frac{1}{s};\ \ \mathrm{Re}\{s\} > 0$$

(a)

$$\delta(t) = \frac{du(t)}{dt}$$

Using differentiation property

$$\frac{dx(t)}{dt} \overset{L}{\leftrightarrow} sX(s)$$

$$\delta(t) \leftrightarrow sU(s) = s.\frac{1}{s} = 1$$

/ROC all **S**

(b)

$$\delta'(t) = \frac{d}{dt}\left(\frac{du(t)}{dt}\right)$$

Using differentiation property

$$\delta'(t) \leftrightarrow s(sU(s)) = s.s.\frac{1}{s} = s$$

ROC all **S**

(c) Using shifting property $e^{s_0 t}x(t) \overset{L}{\leftrightarrow} X(s - s_0)$ $e^{-at}u(t) \overset{L}{\leftrightarrow} U(s + a) = \frac{1}{s+a}$, $Re\{S\} \succ -a$

(d) Using differentiation property $-tx(t) \overset{L}{\leftrightarrow} \dfrac{dX(s)}{ds}$

We know, $e^{-at}u(t) \overset{L}{\leftrightarrow} \dfrac{1}{s+a},$ $Re\{S\} \succ -a$

$$t[e^{-at}u(t)] \leftrightarrow -\frac{d}{ds}\left(\frac{1}{s+a}\right); \quad \mathrm{Re}\{s\} > -a$$

(e) $\cos(\omega_0 t)\, u(t) = \frac{1}{2}\left(e^{j\omega_0 t +} e^{-j\omega_0 t}\right) u(t) = \frac{1}{2} e^{j\omega_0 t}u(t) + \frac{1}{2} e^{-j\omega_0 t}$u(t)

Using shifting property $e^{s_0 t}x(t) \overset{L}{\leftrightarrow} X(s - s_0)$ ➡ $e^{j\omega_0 t}u(t) \overset{L}{\leftrightarrow}$ U(s-jω_0) $= \frac{1}{s-j\omega_0}$

$X(s) = \frac{1}{2}\frac{1}{s-j\omega_0} + \frac{1}{2}\frac{1}{s+j\omega_0} = \frac{s}{s^2+{\omega_0}^2},$ **Re{s}>0**

- The input-output of a LTI system is represented by $y'(t) + 3\,y(t) = x(t)$.

  Determine the impulse response of the system

Taking Laplace transform both side of the equation

$$sY(s) - \boldsymbol{y}(\mathbf{0}+) + 3Y(s) = X(s)$$

$$Y(s)[s+3] - y(0+) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)},$$ considering initial condition zero

$$= \frac{1}{s+3}$$

$$\boxed{h(t) = e^{-3t}u(t)}$$

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

$$u = x(t), \qquad du = \left[\frac{dx(t)}{dt}\right]dt$$

$$dv = e^{-st}dt, \qquad v = \frac{1}{-s}e^{-st}$$

$$\int_0^\infty u\,dv = \mathrm{uv}|_0^\infty - \int_0^\infty v\,du$$

$$X(s) = -\frac{1}{s}x(t)e^{-st}|_0^\infty - \frac{1}{-s}\int_0^\infty e^{-st}\left[\frac{dx(t)}{dt}\right]dt$$

$$\left[\frac{dx(t)}{dt}\right] \overset{L}{\leftrightarrow} -\mathrm{x}(0+) + sX(s)$$

$Task$: $y''(t) + 3y'(t) + 2\,y(t) = x'(t) + 3\,x(t)$

- Express $X = \frac{3x+1}{(x-1)^2(s+2)}$ as sum of partial fractions

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$= \frac{5}{9(x-1)} + \frac{4}{3(x-1)^2} - \frac{5}{9(x+2)}$$

$$3x + 1 = A(x-1)(x+2) + B(x+2) + c(x-1)^2$$

If $x = 1$, $3 \times 1 + 1 = A \times 0 \times 3 + B \times 3 + C \times 0 = 3B,$ $B = \frac{4}{3}$

If $x = -2$, $3 \times -2 + 1 = A \times -3 \times 0 + B \times 0 + C \times 9 = 9C$ $C = \frac{-5}{9}$

$$3x + 1 = A(x-1)(x+2) + B(x+2) + c(x-1)^2$$
$$= A(x^2 + x - 2) + B(x+2) + c(x^2 - 2x + 1)$$
$$= (A + C)x^2 + (A + B - 2C)x + (-2A + 2B + C)$$ $A = \frac{5}{9}$

Find the Laplace transform and the associated ROC for each of the following signals

(a) $x(t) = \delta(t - t_0)$

(b) $x(t) = u(t - t_0)$

(c) $x(t) = e^{-2t}[u(t) - u(t-5)]$

(d) $x(t) = \sum_{k=0}^{\infty} \delta(t - kT)$

(e) $x(t) = \delta(at + b)$, a, b real constants

(e)

$$x(t) = \delta(t) \rightarrow X(s) = 1, \forall s$$

$$x(t+b) = \delta(t - (-b)) \rightarrow e^{-(-b)s} X(s)$$

$$x(at+b) = \delta(at+b) \rightarrow \frac{1}{|a|} X(\frac{s}{a}) = \frac{1}{|a|} e^{b\frac{s}{a}}$$

ROC S all

(d)

$$X(s) = \int_{-\propto}^{\propto} \sum_{k=0}^{\propto} \delta(t-kT)\, e^{-st} dt \qquad = \sum_{k=0}^{\propto} \int_{-\propto}^{\propto} \delta(t-kT)\, e^{-st} dt \qquad = \sum_{k=0}^{\propto} e^{-skT} = \frac{1}{1-e^{-sT}}$$

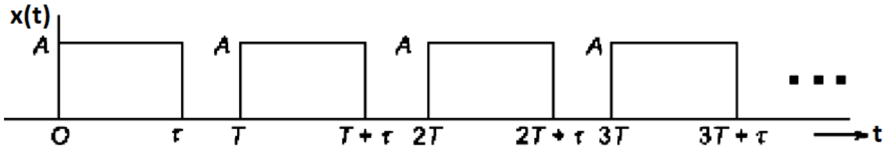
$\mathrm{Re}\{sT\} > 0$

$\mathrm{Re}\{s\} > 0$

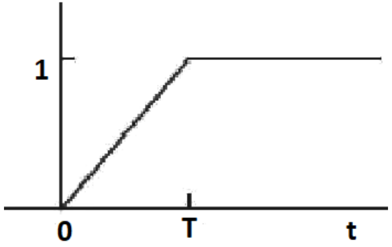
Evaluate the Laplace transform and ROC of following signals

$(a)\ x(t) = 5e^{-3t}$
$(b)\ x(t) = (2e^{-2t} + 3e^{-3t})u(t)$
$(c)\ x(t) = tu(t)$

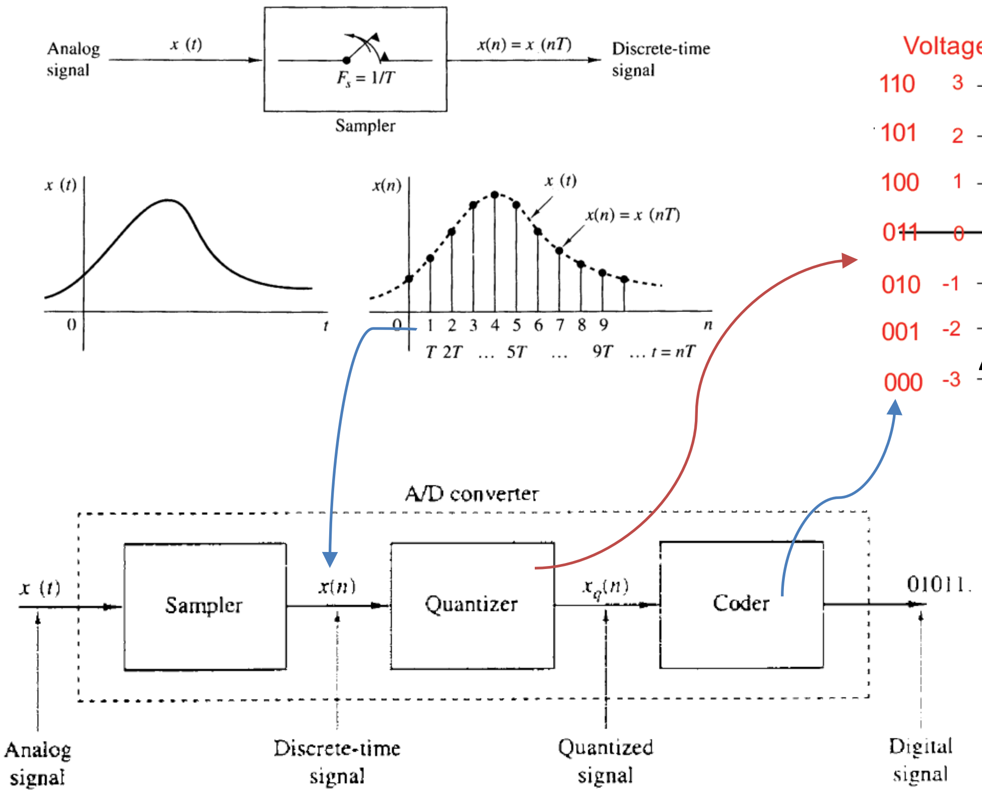
(d)

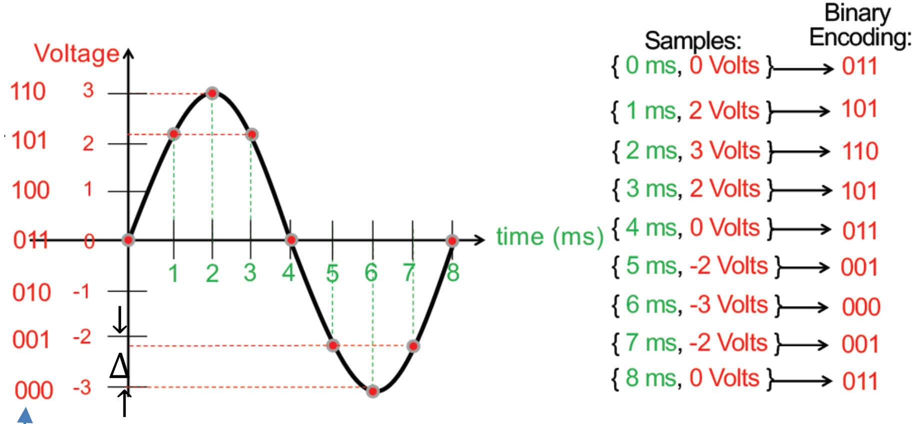


(c)



Periodic sampling of an analog signal



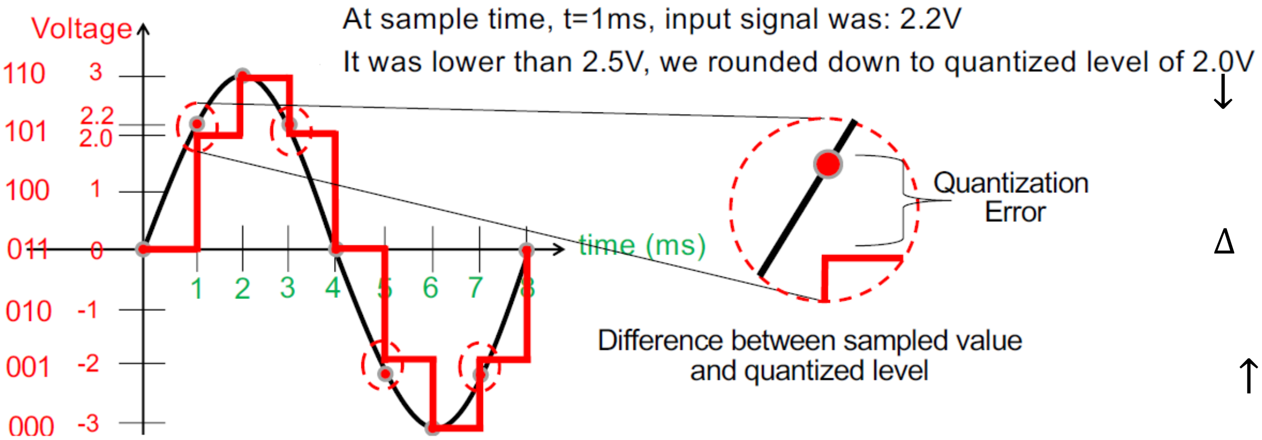


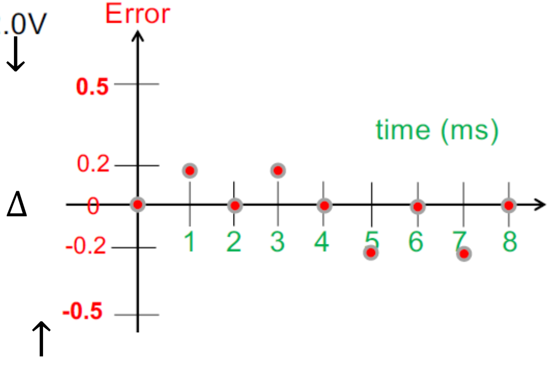
$$\Delta = resolution(quantization\ step) = \frac{v_{max} - V_{min}}{L-1}$$

$$L = Number\ of\ quantization\ levels$$

$$Number\ of\ bits\ required\ to\ each\ level\ (b) \geq \log_2 L$$

Quantization error





Mean-square quantization error: $\frac{\Delta^2}{12}$

$$\Delta = resolution(quantization\ step) = \frac{v_{max} - V_{min}}{L-1}$$

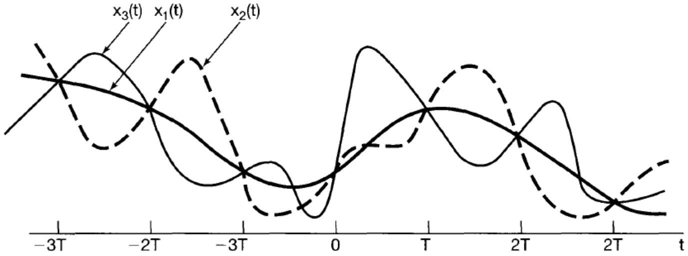
Sampling theorem

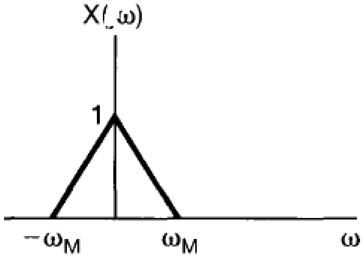
- **A continuous-time signal**
  - Can be completely represented by and recoverable from knowledge of its values, or *samples*, *at* **points equally spaced in time** - *sampling theorem*

✓ If a signal is band limited -i.e., if its Fourier transform is zero outside a finite band of frequencies

✓ If the samples are taken sufficiently close together in relation to the highest (2x) frequency present in the signal,

-- then the samples *uniquely specify* the signal, and we **can reconstruct it perfectly**





Example: Consider the analog signal

$$x_a(t) = 3\cos 2\,000\pi t + 5\sin 6\,000\pi t + 10\cos 1\,2000\pi t$$

(a) What is the *Nyquist /sampling rate required* for this signal to reconstruct at receiver?

The frequency existing in the analog signal are

$$\cos 2\,000\pi t \to \omega = 2\pi F_1 = 2000\pi \to F_1 = \frac{2000\pi}{2\pi} = 1000 = 1kHz$$

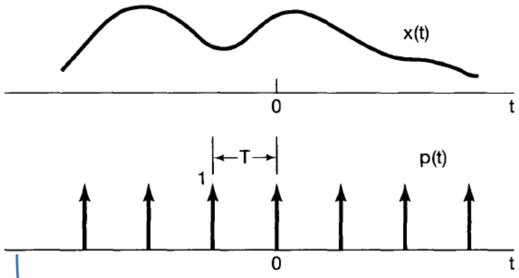
$$\sin 6\,000\pi t \to \omega = 2\pi F_2 = 6000\pi \to F_2 = \frac{6000\pi}{2\pi} = 3000 = 3kHz$$

$$\cos 1\,2000\pi t \to \omega = 2\pi F_3 = 12000\pi \to F_2 = \frac{12000\pi}{2\pi} = 6000 = 6kHz$$

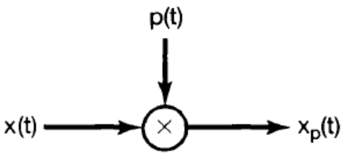
$$F_{\max} = 6kHz$$

According to sampling theorem, sampling rate should be (called Nyquist rate) $F_s > 2F_{\max} = 12kHz$

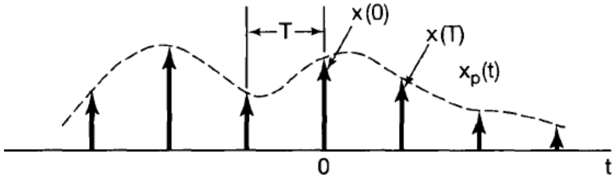
Representation of a continuous-time signal by its samples



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$x_p(t) = x(t)p(t)$$



$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

$$\boxed{x_p(t) = x(nT)}$$

In frequency domain

$$x_p(t) \overset{F.T.}{\longleftrightarrow} \frac{1}{2\pi} X(\omega) * P(\omega)$$

(using Multiplication property of FT)

$$x_p(t) = x(t)p(t) \quad \Rightarrow \quad x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

$$x_p(t) \xleftrightarrow{F.T.} \frac{1}{2\pi} X(\omega) * P(\omega)$$

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$p(t)$ - **periodic signal**, hence it can be represented by **Fourier series** $p(t) = \sum_{k=-\propto}^{\propto} a_k \, e^{j\omega_s kt}$

$; \omega_s = \dfrac{2\pi}{T}$

By definition of Fourier series, $a_k = \dfrac{1}{T}\displaystyle\int_T p(t)e^{-j\omega_s kt}\,dt$ (we have to find ak term)

Cont..

$$a_k = \frac{1}{T}\int_T p(t)\, e^{-j\omega_s kt}\, dt$$

$$= \frac{1}{T}\int_T \delta(t) e^{-j\omega_s kt}\, dt$$

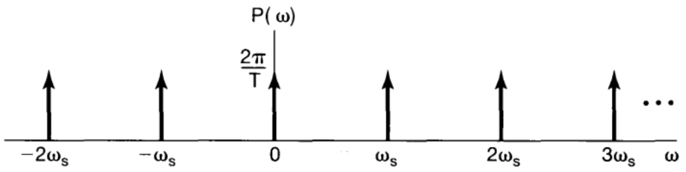
$$= \frac{1}{T}$$

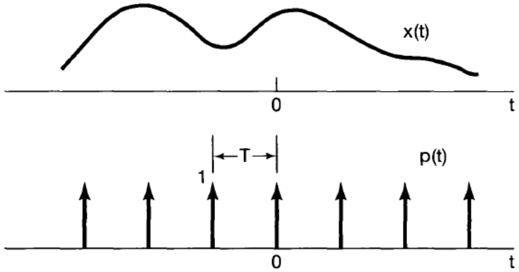
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad = \sum_{k=-\propto}^{\propto} a_k\, e^{j\omega_s kt}$$

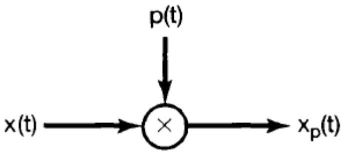
$$= \sum_{k=-\propto}^{\propto} \frac{1}{T} e^{j\omega_s kt}$$

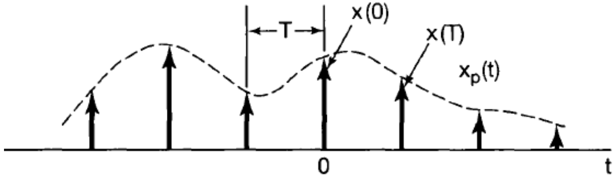
$$1.\, e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

$$p(t) \xleftrightarrow{FT} P(\omega) = \frac{2\pi}{T} \sum_{k=-\propto}^{\propto} \delta(\omega - k\omega_s)$$









$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

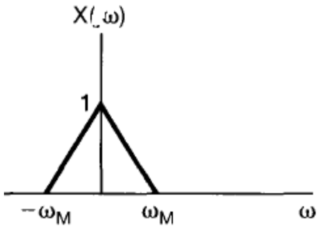
$$x_p(t) = x(t)p(t)$$

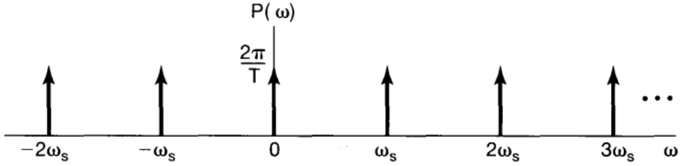
$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

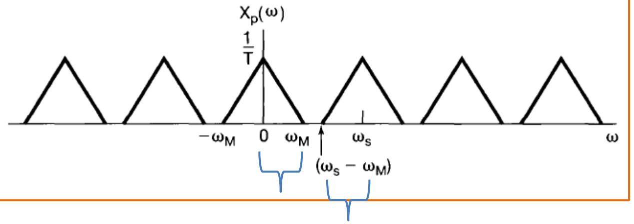
$$x(t) \overset{FT}{\longleftrightarrow} X(\omega)$$

$$p(t) \overset{FT}{\longleftrightarrow} P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$x_p(t) \overset{F.T.}{\longleftrightarrow} \frac{1}{2\pi} X(\omega) * P(\omega)$$







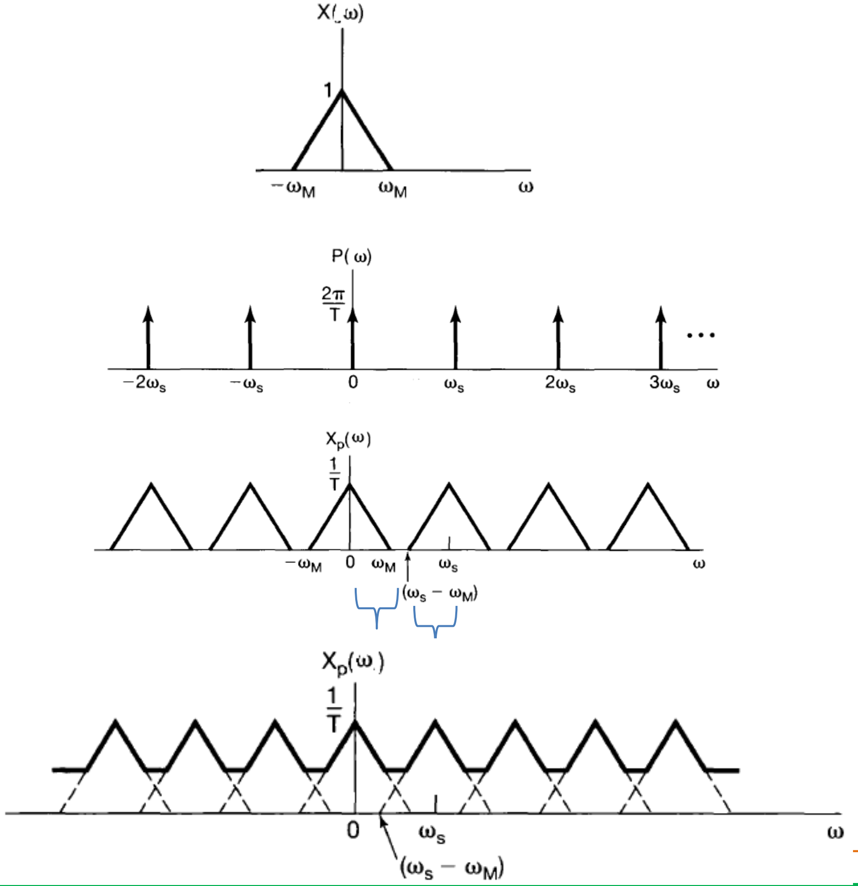
Aliasing

$$x(t) \overset{FT}{\longleftrightarrow} X(\omega)$$

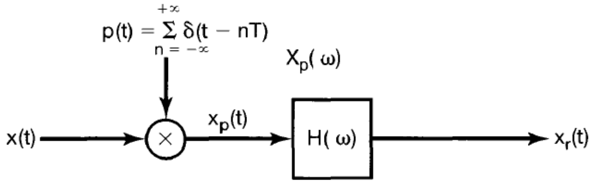
$$p(t) \overset{FT}{\longleftrightarrow} P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

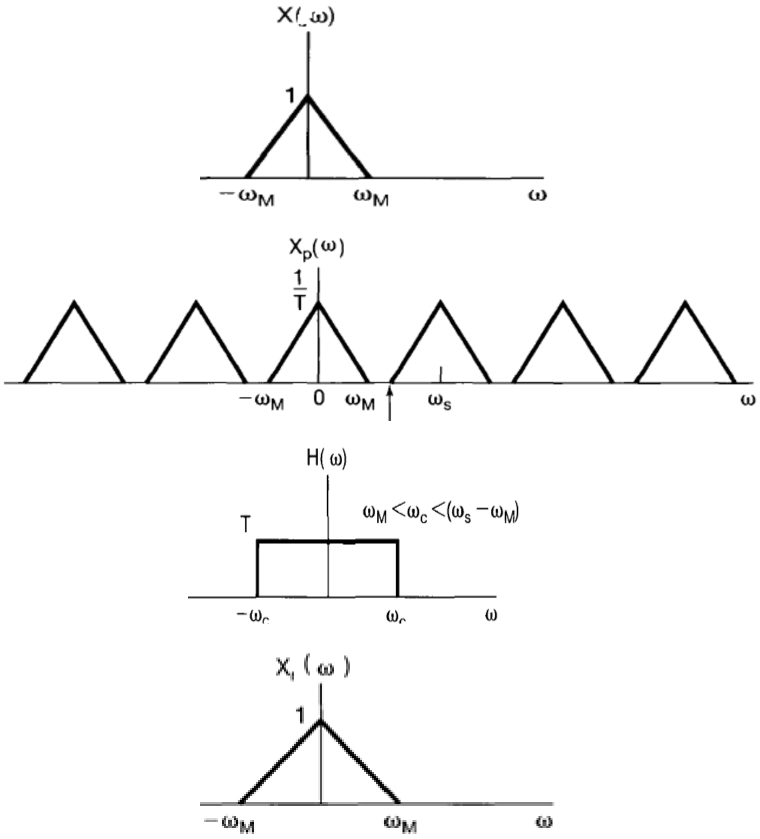
$$x_p(t) \overset{F.T.}{\longleftrightarrow} \frac{1}{2\pi} X(\omega) * P(\omega)$$

Aliasing $\omega_s < 2\omega_M$



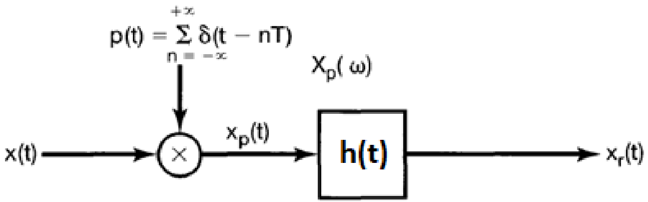
Reconstruction (frequency domain representation)





Reconstructed signal

Reconstruction (time domain)



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

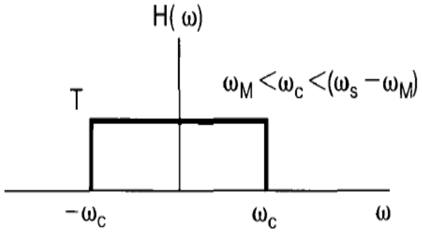
$$x(t) * \delta(t - t_0) = x(t - t_0)$$

$$x_r(t) = x_p(t) * h(t)$$

$$= \sum_{n=-\propto}^{\propto} x(nT)\delta(t - nT) * h(t)$$

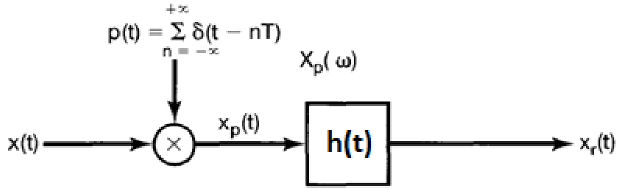
$$= \sum_{n=-\propto}^{\propto} x(nT)h(t - nT)$$

$$= \sum_{n=-\propto}^{\propto} x(nT)\frac{T\sin(\omega_c(t - nT))}{\pi(t - nT)}$$



$$\leftrightarrow h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H(\omega)e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} T.e^{j\omega t}\, d\omega = \frac{1}{2\pi}\frac{e^{j\omega t}}{jt}\Bigg|_{-\omega_c}^{\omega_c}$$

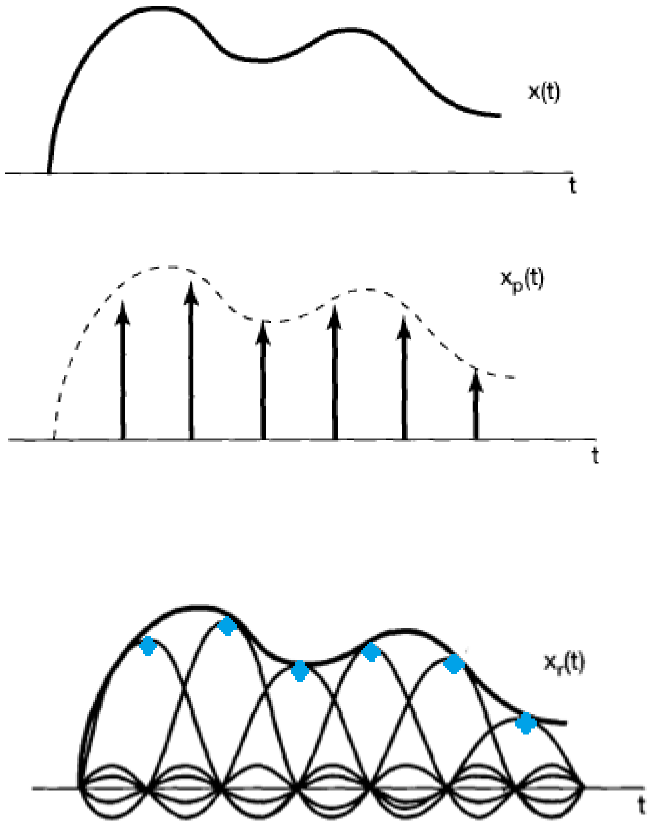
$$= \frac{T}{2\pi jt}\left[e^{j\omega_c t} - e^{-j\omega_c t}\right]\Big| = \frac{T}{\pi t}\sin(\omega_c t)$$



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

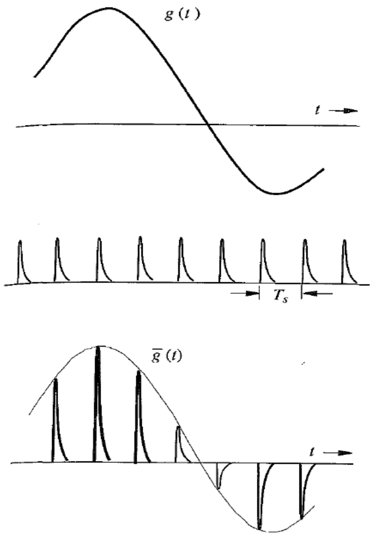
$$x_r(t) = x_p(t) * h(t)$$

$$= \sum_{n=-\propto}^{\propto} x(nT) \frac{T \sin(\omega_c(t - nT))}{\pi(t - nT)}$$

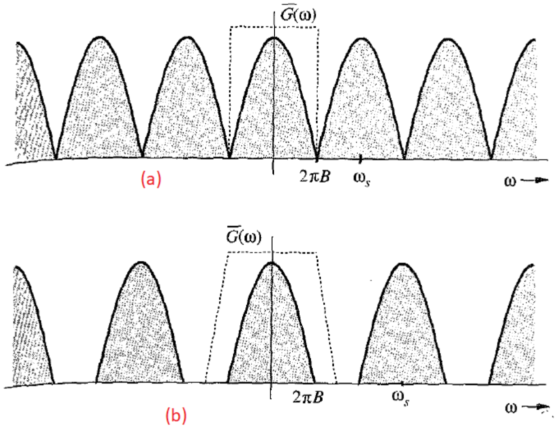


Practical difficulties in signal reconstruction

- Signal may not bandlimited → Antialiasing Filter
- Impulse train is physically nonexistence



- Spectra of sampled signal (a) at Nyquist rate $(F_s > 2 \times f_{max})$
  (b) Above the Nyquist rate



- Non ideal low pass-filter → gradual cut-off characteristics

- Let's analog signal $x(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t$

  (a) What will be the discrete-time signal after sampling at $F_s = 5000Hz$?

  (b) What will be the reconstructed analog signal from sample with ideal interpolation?

(a)

$$x(n) = x(nT)|_{t=nT}$$
$$= x(\frac{n}{F_s})$$
$$= 3\cos 2\pi\left(\frac{1000}{5000}\right)n + 5\sin 2\pi\left(\frac{3000}{5000}\right)n + 10\,cos 2\pi\left(\frac{6000}{5000}\right)n$$
$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(\frac{3}{5}\right)n + 10\,cos 2\pi\left(\frac{6}{5}\right)n$$
$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(1-\frac{2}{5}\right)n + 10\,cos 2\pi\left(1+\frac{1}{5}\right)n$$
$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(-\frac{2}{5}\right)n + 10\,cos 2\pi\left(\frac{1}{5}\right)n$$
$$= 13\cos 2\pi\left(\frac{1}{5}\right)n - 5\sin 2\pi\left(\frac{2}{5}\right)n$$

$$x(t) = A cos\,(2\pi F_0 t + \theta)\ -\ \textit{Continuous}$$
$$x(n) = x(nT)|_{t=nT} = A\cos(2\pi F_0 \,.\, nT_s)\ -\textit{Discrete}$$
$$= A\cos(\frac{2\pi F_0 n}{F_s} + \theta)$$
$$= A\cos(2\pi f_0 n + \theta)$$

$f_0 = \frac{F_0}{F_s}$ (relative freq.)

$$-\pi \le \omega = 2\pi f_0 \le \pi$$
$$-\frac{1}{2} \le f_0 \le \frac{1}{2}$$

- In the sampled signal, only frequency 1 kHz and 2 kHz. The recover analog signal will be

$$y(t) = 13\cos 2\pi \times 1000t \ - 5\sin 2\pi \times 2000t$$

This is distorted version of original signal → happened due to aliasing effect

Q2. A signal $m(t)$ band-limited to $3\ kHz$ is sampled at rate $33\frac{1}{3}\%$ higher than the Nyquist rate. The maximum acceptable error in the sample (maximum quantization error) is 0.5% of the peak amplitude m_p. Determine the minimum channel bandwidth required to transmit the encoded binary signal.

Sampling rate or Nyquist rate = $2\times fm = 2\times 3000 = 6000\ Hz$; $\left(\frac{samples}{second}\right)$

$The\ required\ sampling\ rate = 6000 + \ 6000\times\ 33\frac{1}{3}\%$ =8000 Hz

Maximum quantization error $= \frac{\Delta}{2} = 0.5\frac{m_p}{100}$

$$\rightarrow \frac{1}{2}\times\frac{2m_p}{L-1} = 0.5\frac{m_p}{100}$$

$$\boxed{L = 201}$$

$we\ know,$

$$\Delta = \frac{v_{max}-v_{min}}{L-1} = \frac{m_p-(-m_p)}{L-1} = \frac{2m_p}{L-1}$$

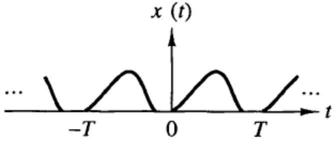
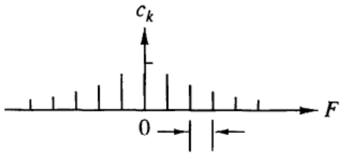
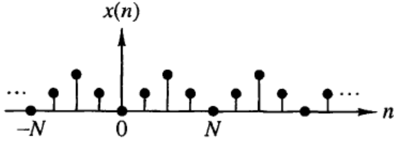
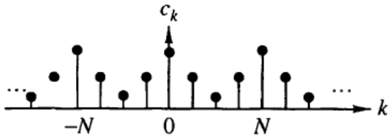
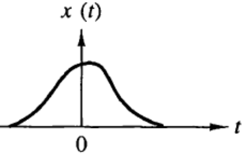
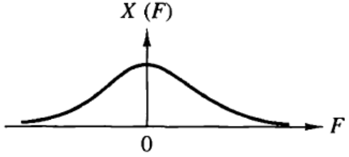
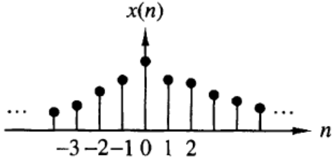
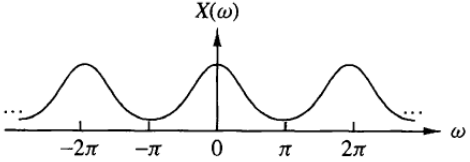
$$we\ need\ (b)\ \geq\ \log_2 L \ \boxed{= \log_2 201 = 8\frac{bits}{sample}}$$

We need to transmit total (c) = $8000Hz \left(\frac{sample}{second}\right) \times 8 \left(\frac{bits}{sample}\right) = 64,000\ bits/second$

The minimum channel transmission bandwidth $= \frac{C}{2} = 32\,000\ Hz$

$assuming,$
$we\ can\ transmit\ 2\ bit\ per\ second$

Summary on Fourier-Series and Transform

	Time-domain	Frequency-domain	Time-domain	Frequency-domain
Fourier-series				
	$F_0 = \frac{1}{T}$ $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$ ⇐	⇒ $c_k = \frac{1}{T}\int_T x(t)e^{-j2\pi kF_0 t}\,dt$	$x(n) = x(n+N)$ $x(n) = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$ ⇐	⇒ $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn}$ $c_k = c_{k+N}$
	Continuous -**periodic**	Discrete -aperiodic	Discrete -**periodic**	Discrete -**periodic**
Fourier-transform				
	$x(t) = \frac{1}{2\pi}\int_{-A\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$ ⇒ $x(t) = \int_{-\infty}^{\infty} X(F)e^{j2\pi Ft}\,dF$ ⇐	$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}\,dt$ $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$	$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\,d\omega$ ⇐	⇒ $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$ $X(\omega) = X(\omega + 2\pi)$
	Continuous -aperiodic	Continuous -aperiodic	Discrete -aperiodic	Continuous -***periodic***

Fourier-series discrete-time signal (periodic)

Harmonically Related Complex Exponentials

$$\emptyset_k(n) = e^{jk\omega_0 n}, \quad k = 0, \pm 1, \pm 2, \ldots \qquad \omega_0 = \frac{2\pi}{N}$$

; N fundamental period **(integer)**

$$k = 0 \rightarrow \emptyset_0(n) = e^{j.0.\omega_0 n} = 1$$

$$k = 1 \rightarrow \emptyset_1(n) = e^{j.1.\omega_0 n}$$

$$k = 2 \rightarrow \emptyset_2(n) = e^{j.2.\omega_0 n}$$

$$\vdots$$

$$k = N \rightarrow \emptyset_2(N) = e^{j.N.\omega_0 n} = e^{j\,N.(\omega_0 = \frac{2\pi}{N})n} = 1$$

when k is changed by any integer multiple of N

Since ***N integer***, therefore **discrete-time signal** → can **content most N frequency** components

$$x(n) = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}$$

(synthesis equation)

$\boldsymbol{a_k}$ are called the

Fourier series coefficients discrete − time signal (**DTFS**)

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-jk\frac{2\pi}{N}n}$$

(Analysis equation of DTFS)

Derivation:

$$x(n) = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$$

$$\rightarrow x(n)e^{-jk\left(\frac{2\pi}{N}\right)l} = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n}.\, e^{-jk\left(\frac{2\pi}{N}\right)l}$$

(multiplying both side $e^{-jl\left(\frac{2\pi}{N}\right)n}$)

$$\rightarrow \sum_{n=0}^{N-1} x(n)e^{-jl\left(\frac{2\pi}{N}\right)n} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n}.\, e^{-jl\left(\frac{2\pi}{N}\right)n}$$

$$\rightarrow \sum_{n=0}^{N-1} x(n)e^{-jl\left(\frac{2\pi}{N}\right)n} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} a_k e^{j\left(\frac{2\pi}{N}\right)(k-l)n}$$

$$\sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)(k-l)n} = \sum_{n=0}^{N-1} \cos(k-l)\frac{2\pi}{N}n + j\sin(k-l)\frac{2\pi}{N} = \begin{cases} N, k-l = 0, \pm N, \pm 2N, .. \\ 0, else \end{cases}$$

$$\rightarrow \sum_{n=0}^{N-1} x(n)e^{-jl\left(\frac{2\pi}{N}\right)n} = a_l N$$

$$\rightarrow \boldsymbol{a_l} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-jl\left(\frac{2\pi}{N}\right)n}$$

$$x(n) = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}$$

(synthesis equation)

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-jk\frac{2\pi}{N}n}$$

(Analysis equation of DTFS)

Convergence of Discrete Fourier Series

Since, discrete Fourier series is a finite series, there are no convergence issues with discrete Fourier series

a_k are called the

Fourier series coefficients of *discrete time signal* (**DTFS**)

$$\rightarrow a_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\,(k+N)\left(\frac{2\pi}{N}\right)n} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\,(k)\left(\frac{2\pi}{N}\right)n} \,.\, e^{-j2\pi n} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\,(k)\left(\frac{2\pi}{N}\right)n} = a_k$$

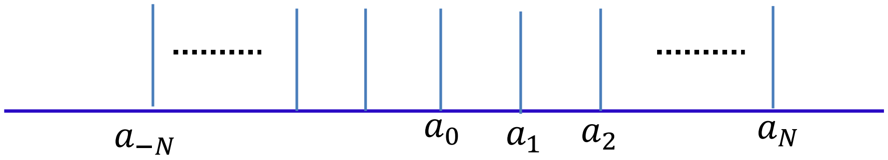
a_k coefficients are periodic with fundamental period N

$$a_k = a_{k+N}$$

$$a_k = a_{k+rN}$$

$r = 1 \rightarrow right - side$

$r = -1 \rightarrow left - side$



$$a_0 = a_N = a_{-N}$$

$$a_1 = a_{1+N} = a_{-N+1}$$

- Evaluate the DTFS coefficients for the following periodic (N) signal,

$$x(n) = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos(\frac{4\pi}{N}n + \frac{\pi}{2})$$

$$= 1 + \frac{1}{2j}[e^{j\left(\frac{2\pi}{N}\right)n} - e^{-j\left(\frac{2\pi}{N}\right)n}] + \frac{3}{2}[e^{j\left(\frac{2\pi}{N}\right)n} + e^{-j\left(\frac{2\pi}{N}\right)n}] + ½\,[e^{j\left(\frac{4\pi n}{N}+\pi/2\right)} + e^{-j\left(\frac{4\pi n}{N}+\frac{\pi}{2}\right)}],\ Eq(1)$$

$$= 1 + \left(\frac{3}{2} + \frac{1}{2j}\right)e^{j\left(\frac{2\pi}{N}\right)n} + (\frac{3}{2} - \frac{1}{2j})e^{-j\left(\frac{2\pi}{N}\right)n} + \left(\frac{1}{2}\,e^{j\left(\frac{\pi}{2}\right)}\right)e^{j2\left(\frac{2\pi}{N}\right)n} + \left(\frac{1}{2}\,e^{-j\left(\frac{\pi}{2}\right)}\right)e^{-j\left(\frac{4\pi n}{N}\right)n}$$

Comparing two equations

From DTFS expression,

$$x(n) = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \quad = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}$$

$$= a_0 + a_1 e^{j\frac{2\pi}{N}n} + a_2 e^{j\frac{4\pi}{N}n} + \ldots..$$

Using relataion (periodic): $a_k = a_{k+N}$

$$a_{-1} = a_{N-1} = \frac{3}{2} - \frac{1}{2j}$$

$$a_{-2} = a_{N-2} = \frac{1}{2j}$$

$$a_0 = 1, a_1 = \frac{3}{2} + \frac{1}{2j}, a_2 = -\frac{1}{2j}$$

$$a_{-k} = a_k^{\;*}$$

- Determine the spectra of the signal with DTFS

$(a)x(n) = \cos\sqrt{2}\pi n$ $(b)x(n) = \cos\frac{\pi n}{3}$

(a) $\omega_0 = \frac{2\pi}{N}$ $N = \frac{2}{\sqrt{2}}; not\ a\ rational\ number\ \rightarrow no\ periodic\ signal\ \rightarrow can't\ be\ express$ as Fourier-series

(b) $\omega_0 = \frac{2\pi}{N},$ $N = \frac{2\pi}{\omega_0} = 2\pi \times \frac{1}{\frac{\pi}{3}} = 6$

Using definition of DTFS:

$$x(n) = \sum_{\{k=0\}}^{\{5\}} a_k e^{jk\omega_0 n} = a_0 + a_1 e^{\frac{j2\pi}{6}.1.n} + a_2 e^{\frac{j2\pi}{6}.2.n} + a_3 e^{\frac{j2\pi}{6}.3n} + a_4 e^{\frac{j2\pi}{6}.4n} + a_5 e^{\frac{j2\pi}{6}.5n} \dots,\quad k = 0,1,\dots 5 \quad (1)$$

comparing

Expanding original x(n) $= \frac{1}{2}\left[e^{\frac{j\pi n}{3}} + e^{-\frac{j\pi n}{3}}\right]$

$$= \frac{1}{2}\left[e^{\frac{j2\pi n}{6}} + e^{-\frac{j2\pi n}{6}}\right]$$

$$= \frac{1}{2}\left[e^{\frac{j2\pi n}{6}} + e^{\frac{j2\pi(5-6)}{6}n}\right] = \frac{1}{2}\left[e^{\frac{j2\pi n}{6}} + e^{\frac{j2\pi(5)}{6}n}\right] - -(2)$$

$a_0 = 0, a_1 = \frac{1}{2}, a_5 = \frac{1}{2}$
$a_2 = 0, a_3 = 0, a_4 = 0$

Discrete-time Fourier-series (DTFS) $x(n) = sin\omega_0 n$

Using Euler's relation $x(n) = \frac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right]$ $\quad where\ \omega_0 = \frac{2\pi}{N}$

$$= \frac{1}{2j} e^{\frac{j2\pi}{N}n} - \frac{1}{2j} e^{-\frac{j2\pi}{N}n} \quad \ldots\ldots.\ \boldsymbol{Eq(1)}$$

comparing

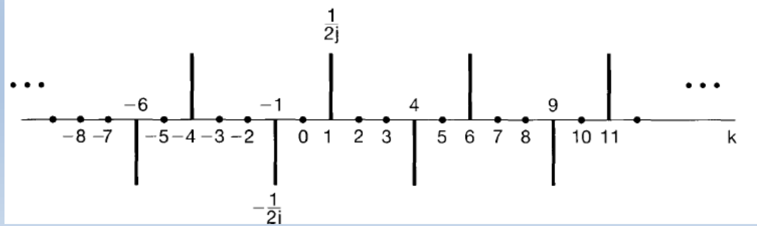
Using synthesis equation:

$$x(n) = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = a_0 e^{\frac{j0.2\pi}{N}n} + a_1 e^{j.\frac{1.2\pi}{N}n} + a_2 e^{\frac{j2.2\pi}{N}n} + \ldots \ \boldsymbol{Eq(2)}$$

$$a_1 = \frac{1}{2j}, a_{-1} = -\frac{1}{2j}$$

Using relation, $\boldsymbol{a_k = a_{k+rN}}$

$For\ N = 5$



- Parseval's theorem

$$If\ x_1(n) \overset{DTFT}{\longleftrightarrow} X_1(\omega)$$

$$x_2(n) \overset{DTFT}{\longleftrightarrow} X_2(\omega)$$

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)\, d\omega$$

R.H.S

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega \quad = \frac{1}{2\pi}\int_{2\pi}\left[\sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n}\right]X_2^*(\omega)d\omega$$

$$= \sum_{n=-\infty}^{\infty} x_1(n)\ \frac{1}{2\pi}\int_{2\pi} X_2^*(\omega)\ e^{-j\omega n}d\omega$$

$$= \sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n)$$

Using Synthesis equation

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$$

$$x^*(n) = \left(\frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\right)^* d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(\omega)e^{-j\omega n}d\omega$$

If $x_1(n) = x_2(n) = x(n)$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\omega)|^2 d\omega$$

(called $|X(\omega)|^2$ Energy density spectrum)

- Multiplication

$$If\ x_1(n) \overset{DTFT}{\longleftrightarrow} X_1(\omega)$$

$$x_2(n) \overset{DTFT}{\longleftrightarrow} X_2(\omega)$$

$$x_1(n)x_2(n) \overset{DTFT}{\longleftrightarrow} X_3(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$$

- Differentiation

$$If\ x(n) \overset{DTFT}{\longleftrightarrow} X(\omega)$$

$$nx(n) \overset{DTFT}{\longleftrightarrow} j\frac{dX(\omega)}{d\omega}$$

Analysis equation

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$\frac{dX(\omega)}{d\omega} = \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n)\frac{d}{d\omega}e^{-j\omega n}$$

$$= -j\sum_{n=-\infty}^{\infty} nx(n)\, e^{-j\omega n} = -j\ DTFT[nx(n)]$$

Properties of the Fourier Transform for Discrete-Time Signals

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$
		$= X_1(\omega)X_2^*(\omega)$
		[if $x_2(n)$ is real]
Wiener–Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in		
the frequency domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	

- Determine the energy density spectrum of the signal $x(n) = a^n u(n), -1 < a < 1$

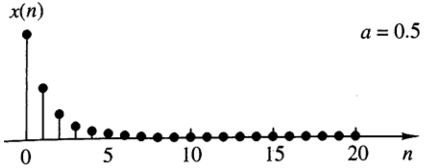
Checking $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$ whether x(n) is absolutely summable $\rightarrow$ Fourier transform exist?

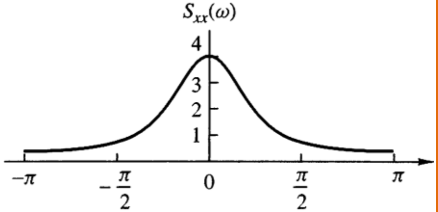
$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=-\infty}^{\infty} |a^n u(n)| = \sum_{n=0}^{\infty} |a^n| = \frac{1}{1-|a|} < \infty \ (as\ |a| < 1) \qquad (Fourier - transform\ exist)$$

$$x(n) \overset{DTFT}{\longleftrightarrow} X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = \frac{1}{1-ae^{-j\omega}}$$

Energy density spectrum

$$s_{xx}(\omega) = |X(\omega)|^2$$
$$= X(\omega)X^*(\omega)$$
$$= \frac{1}{(1-ae^{-j\omega})(1-ae^{j\omega})}$$
$$= \frac{1}{(1-2a\cos\omega + a^2)}$$





- Determine the convolution of the sequence $x_1(n) = x_2(n) = \{1, \underset{\uparrow}{1}, 1\}$

Using Fourier-transform $X_1(\omega) = X_2(\omega) = 1 + 2\cos\omega$

$1 + \cos 2\theta = 2\cos^2\theta$

$Let, y(n) = x_1(n) * x_2(n)$ $\quad Y(\omega) \xleftrightarrow{DTFT} X_1(\omega)X_2(\omega) = (1 + 2\cos\omega)^2$

$$= 1 + 2.2\cos\omega + 4\cos^2\omega$$

$$= 1 + 2.2\cos\omega + 2(1 + \cos 2\omega)$$

$$= 3 + 4\cos\omega + + 2\cos 2\omega$$

$$Y(\omega) = 3 + 4 \times \frac{1}{2}\left[e^{j\omega} + e^{-j\omega}\right] + 2 \times \frac{1}{2}\left[e^{j2\omega} + e^{-j2\omega}\right]$$

$$= 3 + 2\left[e^{j\omega} + e^{-j\omega}\right] + \left[e^{j2\omega} + e^{-j2\omega}\right]$$

$$= e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega} \quad --Eq(1)$$

From definition of Fourier-transform, we can write

$$Y(\omega) = \sum_{n=-\infty}^{\infty} y(n)e^{-j\omega n} \quad = ..+y(-2)e^{j2\omega} + y(-1)e^{j\omega} + y(0)e^{j\omega.0} + y(1)e^{-j\omega} + y(2)e^{-j2\omega} \quad --Eq(2)$$

$Comparing\ Eq(1) \&\ (2)$ $\quad y(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$

- Determine the spectrum of impulse response for the LTI system expressed as

$$y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$$

$$x(n) \overset{DTFT}{\longleftrightarrow} X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n+1) \overset{DTFT}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} x(n+1)e^{-j\omega n}$$

$$= e^{j\omega} \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Applying Fourier-transform both side of equation

$$Y(\omega) = \frac{1}{3}[e^{j\omega}X(\omega) + X(\omega) + e^{-j\omega}X(\omega)]$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{3}[e^{j\omega} + 1 + e^{-j\omega}]$$

$$= \frac{1}{3}(1 + 2\cos\omega) \quad --Eq(1)$$

From definition of Fourier-transform, we can write

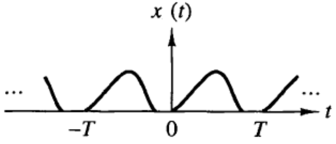
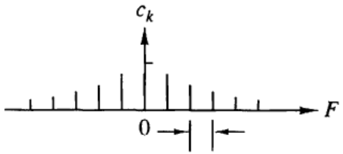
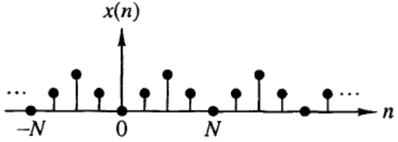
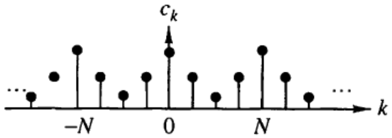
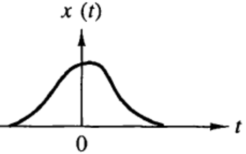
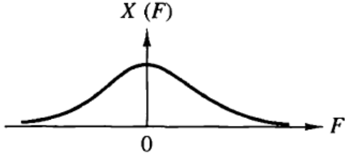
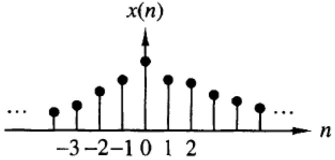
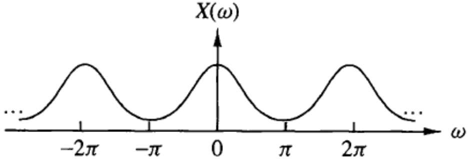
$$\mathrm{H}(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$

$$= ..+h(-1)e^{j\omega} + h(0)e^{j\omega.0} + h(1)e^{-j\omega} - Eq(2)$$

Comparing Eq(1)& (2)

$$\mathrm{h}(n) = \{\frac{1}{3}, \underset{\uparrow}{\frac{1}{3}}, \frac{1}{3}\}$$

Summary on Fourier-Series and Transform

	Time-domain	Frequency-domain	Time-domain	Frequency-domain
Fourier-series				
	$F_0 = \frac{1}{T}$ ⇒	$c_k = \frac{1}{T}\int_T x(t)e^{-j2\pi kF_0 t}\,dt$	$x(n) = x(n+N)$ ⇒	$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn}$
	$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$	⇐	$x(n) = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$	⇐ $c_k = c_{k+N}$
	Continuous -**periodic**	Discrete -aperiodic	Discrete -**periodic**	Discrete -**periodic**
Fourier-transform				
	$x(t) = \frac{1}{2\pi}\int_{-A\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$ ⇒	$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}\,dt$	⇒	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$
	$x(t) = \int_{-\infty}^{\infty} X(F)e^{j2\pi Ft}\,dF$ ⇐	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$	$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\,d\omega$ ⇐	$X(\omega) = X(\omega + 2\pi)$
	Continuous -aperiodic	Continuous -aperiodic	Discrete -aperiodic	Continuous -***periodic***

Revisited: eigenfunctions concept

Using the eigen function concept of continuous **LTI system**, we can write

$$(input)\ x_1(t) = a_1 e^{s_1 t} \rightarrow (output)\ y_1(t) = e^{s_1 t}\, a_1\, H(s_1)$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

For discrete LTI system

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$Let, (input) x(n) = A\, e^{j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} h(k)Ae^{j\omega(n-k)}$$

$$= A \sum_{k=-\infty}^{\infty} \left[h(k)e^{j\omega.-k}\right]e^{j\omega n} \quad = AH(\omega)e^{j\omega n}$$

$$y(n) = AH(\omega)e^{j\omega n} = A|H(\omega)|e^{j\measuredangle H(\omega)}.e^{j\omega n}$$

- Determine the output of the system with impulse response $h(n)$ and input $x(n)$

$$h(n) = \left(\frac{1}{2}\right)^n u(n) \qquad x(n) = Ae^{\frac{j\pi n}{2}}$$

With Fourier-transform

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n)e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n}$$

$$= 1 + \frac{1}{2}e^{-j\omega} + \left(\frac{1}{2}e^{-j2\omega}\right)^2 + ..\infty$$

$$= \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

From eigen function concept, $y(n) = AH(\omega)e^{j\omega n} = A|H(\omega)|e^{j\measuredangle H(\omega)}.e^{j\omega n}$

where, $(input)\ x(n) = A\ e^{j\omega n}$

$Given\ x(n) = Ae^{j\left(\frac{\pi}{2}\right)n}, \rightarrow \boldsymbol{\omega = \frac{\pi}{2}}$

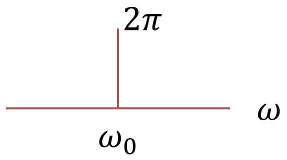
→ Now we can write $y(n) = AH\left(\frac{\pi}{2}\right) e^{\frac{j\pi}{2}n}$ $= A\left(\frac{2}{\sqrt{5}}e^{-j26.6^0}\right)e^{\frac{j\pi}{2}n}$

$$H\left(\frac{\pi}{2}\right) = \frac{1}{1-\frac{1}{2}e^{-\frac{j\pi}{2}}} = \frac{1}{1-\frac{1}{2}(\cos\frac{\pi}{2}+jsin(-\frac{\pi}{2}))} = \frac{1}{1+\frac{j1}{2}} = \frac{1-\frac{j}{2}}{1+\frac{1}{4}} = \frac{4}{5} - \frac{2j}{5} = \frac{2}{\sqrt{5}}e^{-j26.5}$$

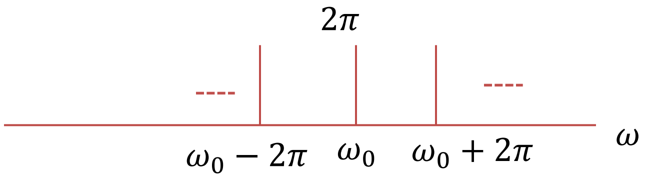
Determine DTFT of $e^{j\omega_0 n}$

Since, DTFT is periodic i.e. $X(\omega) = X(\omega + 2\pi)$

We know $e^{j\omega_0 t} \overset{FT}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$



Therefore, Fourier-transform of $e^{j\omega_0 n}$ should **have impulse** at $\omega_0, \omega_0 \pm 2\pi, \omega_0 \pm 4\pi. \ldots$



$$e^{j\omega_0 n} \overset{DTFT}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

With inverse DTFT

$$DTFT^{-1}[e^{j\omega_0 n}] = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)\, e^{j\omega n} d\omega$$

Within "interval of 2π" – *impulse train only* "ONE" impulse

$= e^{j\omega_0 n}$

- Determine the DTFT of $x(n) = \cos \omega_0 n$

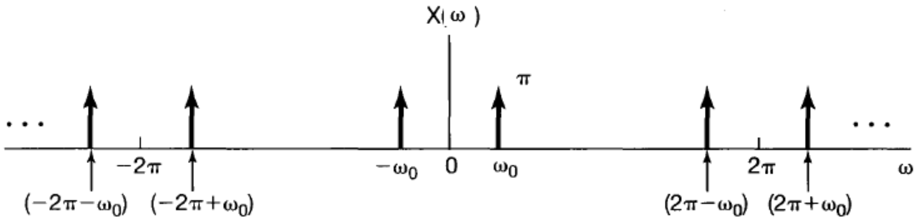
Using Euler's relation, $x(n) = \frac{1}{2}[e^{j\omega_0 n} + e^{-j\omega_0 n}]$

$$e^{j\omega_0 n} \overset{DTFT}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

$$DTFT[x(n)] \overset{DTFT}{\longleftrightarrow} \frac{1}{2} DTFT[e^{j\omega_0 n}] + \frac{1}{2} DTFT[e^{-j\omega_0 n}]$$

$$\overset{DTFT}{\longleftrightarrow} \frac{1}{2}\sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l) + \frac{1}{2}\sum_{l=-\infty}^{\infty} 2\pi\delta(\omega + \omega_0 - 2\pi l)$$

$$\overset{DTFT}{\longleftrightarrow} \pi\sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l) + \pi\sum_{l=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi l)$$



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- **Determine the DTFT** of the **DTFS signal** (periodic signal)

$$x(n) = \sum_{k=0}^{N-1} a_k e^{jk\omega n} = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$$

$\textbf{or}$ $notation$ $k=\langle N\rangle$

$$a_k = a_{k+rN}$$

Expanding $x(n)$

$$x(n) = a_0 + a_1 e^{j\left(\frac{2\pi}{N}\right)n} + a_2 e^{j2\left(\frac{2\pi}{N}\right)n} + \cdots + a_{N-1} e^{j(N-1)\left(\frac{2\pi}{N}\right)n}$$

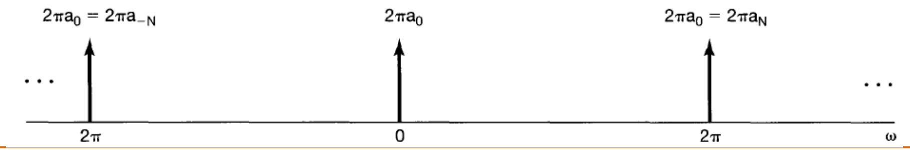
$$= a_0 e^{j.0.n} + a_1 e^{j\left(\frac{2\pi}{N}\right)n} + a_2 e^{j2\left(\frac{2\pi}{N}\right)n} + \cdots + a_{N-1} e^{j(N-1)\left(\frac{2\pi}{N}\right)n}$$

$$\boxed{e^{j\omega_0 n} \overset{DTFT}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)}$$

$$l = 0 \qquad l = 1 \qquad l = 2$$

$$a_0 e^{j.0.n} \overset{DTFT}{\longleftrightarrow} a_0 \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi l) = 2\pi a_0[\ldots + \delta(\omega) + \delta(\omega - 2\pi) + \delta(\omega - 2.2\pi) + ..]$$

$$a_k = a_{k+rN}$$



$$a_0 = a_N$$

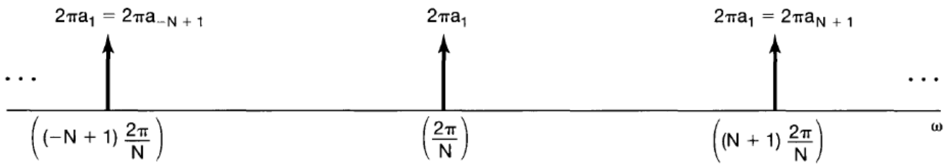
$$a_0 = a_{-N}$$

$$x(n) = a_0 e^{j.0.n} + a_1 e^{j\left(\frac{2\pi}{N}\right)n} + a_2 e^{j2\left(\frac{2\pi}{N}\right)n} + \cdots + a_{N-1} e^{j(N-1)\left(\frac{2\pi}{N}\right)n}$$

$$e^{j\omega_0 n} \overset{DTFT}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

$l = 0 \quad l = 1 \quad l = 2$

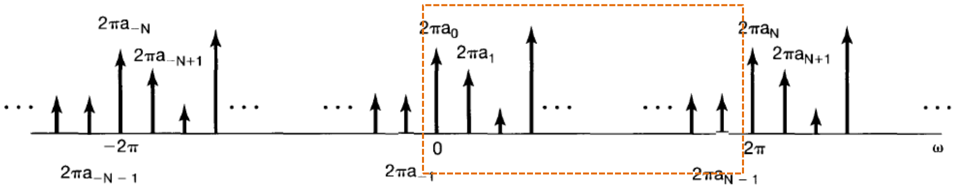
$$a_1 e^{j.\frac{2\pi}{N}.n} \overset{DTFT}{\longleftrightarrow} a_1 \sum_{l=-\infty}^{\infty} 2\pi\delta\left(\omega - \frac{2\pi}{N} - 2\pi l\right) = 2\pi a_1[\ldots + \delta\left(\omega - \frac{2\pi}{N}\right) + \delta\left(\omega - \frac{2\pi}{N} - 2\pi\right) + \delta\left(\omega - \frac{2\pi}{N} - 2.2\pi\right) + ..]$$



$a_k = a_{k+rN}$

$a_1 = a_{1+N}$

$a_1 = a_{1-N}$



$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$

$$X(\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta\left(\omega - \frac{2\pi}{N}k\right)$$

BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$

Signal	Fourier Transform
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$
$x[n] \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π
$\delta[n]$	1
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$

Property	DTFS	CTFS	DTFT	CTFT
Synthesis	$x[n] = \sum_{k=<N>} a_k e^{jk\Omega_0 n}$	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$
Analysis	$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\Omega_0 n}$	$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$	$X(e^{j\Omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\Omega n}$	$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
Linearity	$\alpha x[n] + \beta y[n] \leftrightarrow \alpha a_k + \beta b_k$	$\alpha x(t) + \beta y(t) \leftrightarrow \alpha a_k + \beta b_k$	$\alpha x[n] + \beta y[n] \leftrightarrow \alpha X(e^{j\Omega}) + \beta Y(e^{j\Omega})$	$\alpha x(t) + \beta y(t) \leftrightarrow \alpha X(j\omega) + \beta Y(j\omega)$
Time Shifting	$x[n - n_0] \leftrightarrow a_k e^{-j2\pi n_0 k/N}$	$x(t - t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$	$x[n - n_0] \leftrightarrow e^{-j\Omega n_0} X(e^{j\Omega})$	$x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)$
Frequency Shift	$x[n] e^{j2\pi mn/N} \leftrightarrow a_{k-m}$	$x(t) e^{jm\omega_0 t} \leftrightarrow a_{k-m}$	$x[n] e^{j\Omega_0 n} \leftrightarrow X(e^{j(\Omega - \Omega_0) n})$	$x(t) e^{j\omega_0 t} \leftrightarrow X(j(\omega - \omega_0))$
Conjugation	$x^*[n] \leftrightarrow a^*_{-k}$	$x^*(t) \leftrightarrow a^*_{-k}$	$x^*[n] \leftrightarrow X^*(e^{-j\Omega})$	$x^*(t) \leftrightarrow X^*(-j\omega)$
Time Reversal	$x[-n] \leftrightarrow a_{-k}$	$x(-t) \leftrightarrow a_{-k}$	$x[-n] \leftrightarrow X(e^{-j\Omega})$	$x(-t) \leftrightarrow X(-j\omega)$
Convolution	$\sum_{r=0}^{N-1} x[r] y[n - r] \leftrightarrow N a_k b_k$	$\int_T x(\tau) y(t - \tau) d\tau \leftrightarrow T a_k b_k$	$x[n] * y[n] \leftrightarrow X(e^{j\Omega}) Y(e^{j\Omega})$	$x(t) * y(t) \leftrightarrow X(j\omega) Y(j\omega)$
Multiplication	$x[n] y[n] \leftrightarrow \sum_{r=0}^{N-1} a_r b_{k-r}$	$x(t) y(t) \leftrightarrow a_k * b_k$	$x[n] y[n] \leftrightarrow \frac{1}{2\pi} \int_{2\pi} X(e^{j\theta}) Y(e^{j(\Omega - \theta)}) d\theta$	$x(t) y(t) \leftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$
First Difference/ Derivative	$x[n] - x[n-1] \leftrightarrow (1 - e^{-j2\pi k/N}) a_k$	$\frac{dx(t)}{dt} \leftrightarrow jk\omega_0 a_k$	$x[n] - x[n-1] \leftrightarrow (1 - e^{-j\Omega}) X(e^{j\Omega})$	$\frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega)$
Running Sum/ Integration	$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{a_k}{1 - e^{-j2\pi k/N}}$	$\int_{-\infty}^{t} x(\tau) d\tau \leftrightarrow \frac{a_k}{jk\omega_0}$	$\sum_{k=-\infty}^{n} x[k] \leftrightarrow \frac{X(e^{j\Omega})}{1 - e^{-j\Omega}} + \pi X(e^{j0}) \delta(\Omega)$	$\int_{-\infty}^{t} x(\tau) d\tau \leftrightarrow \frac{X(j\omega)}{j\omega} + \pi X(j0) \delta(\omega)$
Parseval's Relation	$\frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2 = \sum_{k=0}^{N-1} \lvert a_k \rvert^2$	$\frac{1}{T} \int_T \lvert x(t) \rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k \rvert^2$	$\sum_{n=-\infty}^{\infty} \lvert x[n] \rvert^2 = \frac{1}{2\pi} \int_{2\pi} \lvert X(e^{j\Omega}) \rvert^2 d\Omega$	$\int_{-\infty}^{\infty} \lvert x(t) \rvert^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(j\omega) \rvert^2 d\omega$
Real and even signals	Real and even in frequency domain			
Real and odd signals	Purely imaginary and odd in frequency domain			

Thank you!