Random Variable :-

Discrete R.V — DRV / CRV — continuous R.V

Pmf: ① $P(x) \geq 0$

② $\Sigma P(x) = 1$

Pdf: ① $P(x) \geq 0$

② $\int_{-\infty}^{\infty} f(x) = 1$

• R.V is a real valued f^n which assign a real number to each sample point in the sample space.

Eg: Tossing a fair coin thrice then,

Sample space $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

X : No. of heads

$x(S_1) = 3$

$x(S_2) = x(S_3) = x(S_4) = 2$

$x(S_5) = x(S_6) = x(S_7) = 1$

$x(S_8) = 0$

1) DRV : A R.V which takes finite or at most countable no. of ~~variables~~ values is called discrete R.V.

Eg: ① no. of heads obtained when two coins are tossed.

② no. of defective items in a bag.

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Probability distribution

X(No. of head)	0	1	2	3
P	1/8	3/8	3/8	1/8

Ex Four bad oranges are mixed with 16 bad oranges. Find probability distribution of the no. of bad oranges. in a drawn of two oranges.

Soln:

X(No. of bad oranges)	0	1	2	3	4
P	$\frac{^{16}C_2}{^{20}C_2}$	$\frac{^{16}C_1 \cdot {^4C_1}}{^{20}C_2}$	$\frac{^{4}C_2}{^{20}C_2}$	0	0

16 good
4 bad

Probability mass f^n (P.m.f) :-

Let X be a ~~R.V~~ D.R.V s.t $P(X=x) = P_i$, then P_i is said to be probability mass function (P.m.f) if it satisfy the following condition:

(1) $P_i \geq 0$

(2) $\Sigma P_i = 1$

Ex - S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Probability dist.

X(No. of heads)	0	1	2	3
P	1/8	3/8	3/8	1/8

Distribution function -

Let X be a d.R.V, then its discrete distribution f^n or commutative distribution f^n (c.d.f) is defined as
$f(x) = \Sigma P_i = P(x \le x_j)$

$$f(x) = \begin{cases} 1/8 & x \le 0 \\ 4/8 & x \le 1 \\ 7/8 & x \le 2 \\ 1 & x \le 3 \end{cases}$$

Q) Given the following probability distribution -

x	0	1	2	3	4	5	6	7
$p(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find (1) K (2) $P(X<6)$, $P(X \ge 6)$, $P(0<X<5)$

(3) Distribution f^n

(4) if $P(X \le c) > \frac{1}{2}$ find value of c

(5) ~~Fn~~ $P\left(\frac{1.5 < x < 4.5}{x > 2}\right)$

Solⁿ: if $p(x)$ = Pm.f

$\Sigma P(x) = 1$

$\Rightarrow 0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

$\Rightarrow 10K^2 + 9K - 1 = 0 \Rightarrow 10K^2 + 10K - K - 1 = 0$

$\begin{array}{r} 1.00 \\ 0.19 \\ \hline 0.81 \end{array}$

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$\Rightarrow \quad 10k(k+1) - 1(k+1) = 0$

$\Rightarrow \quad k = -1$ not possible | $k = 1/10$

can't be -ve.

so,

x	0	1	2	3	4	5	6	7
$P(x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

(2) $\cancel{P(x \geq 6)}\ P(x<6)$

$= P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

or $\quad 1 - P(x \geq 6)$

$1 - [P(6) + P(7)] = 1 - [0.19]$

$= 0.81$

$P(X \geq 6)$

$= P(6) + P(7) = 0.19$

$P(0 < X < 5)$

$= P(1) + P(2) + P(3) + P(4)$

$= 0.1 + 0.2 + 0.2 + 0.3 = 0.8$

(3)

$$P(x) = \begin{cases} 0 & x \leq 0 \\ 0.1 & x \leq 1 \\ 0.3 & x \leq 2 \\ 0.5 & x \leq 3 \\ 0.8 & x \leq 4 \\ 0.81 & x \leq 5 \\ 0.83 & x \leq 6 \\ 1 & x \leq 7 \end{cases}$$

(4) for $x \geq 4$ $p(x) = 0.8 > 1/2$

$\therefore c = 4$

(5) $P\left(\dfrac{1.5 < x < 4.5}{x > 2}\right)$

$\left[P\left(\dfrac{A}{B}\right) = \cancel{\dfrac{P(A \cap B)}{P}} \quad \dfrac{P(A \cap B)}{P(B)} \right]$

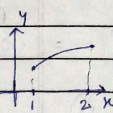
So,

$$\frac{P(1.5 < x < 4.5) \cap P(x > 2)}{P(x>2)} = \frac{P(2,3,4) \cap P(3,4,5,6,7)}{P(3,4,5,6,7)}$$

$$= \frac{P(3,4)}{P(x>2)} = \frac{P(3) + P(4)}{1 - P(x \leq 2)} = \frac{0.5}{1 - 0.3} = \frac{0.5}{0.7} = 5/7$$

Continuous Random Variable :-

* A R.V which can take infinite no. of values in an interval is known as CRV.



Eg: i) The weight of a group of individuals.
(ii) Height of a group of individuals.
(iii) Price of house

Probability density function (Pdf) -

• A f^n $f(x)$ is pdf if -

(1) $f(x) \geq 0 \quad -\infty < x < \infty$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

it is also known as density function

Eg: if x is a CRV with the following pdf

$$f(x) = \begin{cases} \alpha(2x - x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

find (1) α (2) $p(x > 1)$

Soln: By def^n of pdf $\int_{-\infty}^{\infty} f(x)\,dx = 1$

So, $\int_{-1}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx = 1$

$\Rightarrow 0 + \int_{0}^{2} f(x)\,dx + \int 0 = 1$

$\Rightarrow \int_{0}^{2} \alpha(2x - x^2)\,dx = 1 \Rightarrow \alpha\left[x^2\Big|_0^2 - \dfrac{x^3}{3}\Big|_0^2 \right] = 1$

$\Rightarrow \alpha\left[(4 - 0) - \dfrac{8}{3}\right] = 1$

$\Rightarrow \alpha\left(4 - \dfrac{8}{3}\right) = 1$

$\Rightarrow \alpha\left(\dfrac{4}{3}\right) = 1$

$\Rightarrow \alpha = 3/4$

$P(x > 1) = \int_{1}^{\infty} f(x)\,dx$

$= \int_{1}^{2} \alpha(2x - x^2)\,dx + \cancel{\int_{2}^{\infty} f(x)\,dx}^{\,0}$

$= \dfrac{3}{4}\left[x^2\Big|_1^2 - \dfrac{x^3}{3}\Big|_1^2 \right] = \dfrac{3}{4}\left(3 - \dfrac{7}{3}\right)$

$= \dfrac{3}{4}\left(\dfrac{2}{3}\right) = 1/2$

Q) A R.V X has density f^n

$$f(x) = \begin{cases} kx^2 & -3 < x < 3 \\ 0 & \text{Otherwise} \end{cases}$$

find K, $P(1 \le x \le 2)$, $P(x \le 2)$, $P(x > 1)$

Solⁿ: By the defⁿ of pdf -

$$\int_{-\infty}^{\infty} f(x) \cdot dx = 1$$

$$\Rightarrow \int_{-3}^{3} kx^2 \cdot dx = 1 \Rightarrow \left. \frac{kx^3}{3} \right|_{-3}^{3} = 1$$

$$\Rightarrow k [9 + 9] = 18k = 1 \Rightarrow k = 1/18$$

$P(1 \le x \le 2)$

$$\int_{1}^{2} \frac{1}{18} \cdot x^2 \cdot dx = \frac{1}{18} \int_{1}^{2} x^2 \cdot dx = \frac{1}{18} \cdot \left. \frac{x^3}{3} \right|_{1}^{2} \cancel{\frac{1}{18}\left(9 - \frac{1}{3}\right)}$$

$$\cancel{= \frac{1}{18}\left(\frac{26}{3}\right) = \frac{13}{27}}$$

$$= \frac{1}{18}\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{7}{3} \times \frac{1}{18} = \frac{7}{54}$$

$P(x \le 2)$

$$\cancel{\int_{-\infty}^{-3} f(x) \cdot dx} + \int_{-3}^{2} f(x) \cdot dx = \frac{1}{18} \left. \frac{x^3}{3} \right|_{-3}^{2}$$

$$= \frac{1}{18}\left(\frac{8}{3} + \frac{27}{3}\right) = \frac{1}{18}\left(\frac{35}{3}\right) = \frac{35}{54}$$

$P(x > 1)$

$$\int_{1}^{3} f(x) \cdot dx = \frac{1}{18} \int_{1}^{3} x^2 \cdot dx = \frac{1}{18} \cdot \left. \frac{x^3}{3} \right|_{1}^{3} = \frac{1}{18}\left(\frac{27}{3} - \frac{1}{3}\right)$$

$$= \frac{1}{18}\left(\frac{26}{3}\right) = \frac{13}{27}$$

- Commulative distribution function (c.d.f)

continuous

Let X be continuous random variable having pdf $f(x)$ then $F_x(x)$ will be a continuous distribution function of x if

$$F_x(x) = P(X \le x) = \int_{-\infty}^{x} f(x) \cdot dx$$

this is also known as commulative distribution function.

$\begin{array}{r} 1.00 \\ 2.25 \\ \hline 1.75 \end{array}$ $\begin{array}{r} 1.000 \\ 0.875 \\ \hline 0.125 \end{array}$

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Note:- Relation b/w distribution function and density function

$$\frac{d\,F(x)}{dx} = f(x)$$

Eg: The probability density function of random variable x is

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ 2-x & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(1) find $P(x \ge 1.5)$

(2) find commulative distribution function

Soln: $P(x \ge 1.5) = \int_{1.5}^{\infty} f(x)\cdot dx = \int_{1.5}^{2} (2-x)\cdot dx$

$$= 2x\Big|_{1.5}^{2} - \frac{1}{2}x^2\Big|_{1.5}^{2}$$

$$= 2(0.5) - \frac{1}{2}(4-2.25) = 1 - \frac{1}{2}(1.75)$$

$$= 1 - \frac{\cancel{175}\ 87.5}{\cancel{200}\ 100} = 1 - 0.875 = 0.125$$

(2) $x \le 0$

$$P(X \le x) = \int_{-\infty}^{x} f(x)\cdot dx = 0$$

$0 \le x < 1$

$$F(x) = \int_{-\infty}^{0} f(x)\cdot dx + \int_{0}^{x} f(x)\cdot dx$$

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$$F(x) = 0 + \int_{0}^{x} \cancel{(2-x)\cdot dx}\ x\cdot dx = 0 + \frac{x^2}{2}$$

$1 < x < 2$

$$\int_{0}^{1} x\cdot dx + \int_{1}^{x} (2-x)\cdot dx = \cancel{2x\Big|_1^2 - \frac{x^2}{2}\Big|_1^2 \quad \frac{1}{2} + 2(x-1) - \frac{x^2}{2}}$$

$$= \cancel{2(1) - \left(2 - \frac{1}{2}\right) = 2 - \frac{3}{2} = 1/2}$$

$$= \frac{x^2}{2}\Big|_{0}^{1} + \left(2x - \frac{x^2}{2}\right)_{1}^{x} = \frac{1}{2} + \left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{2} - 2 + \frac{1}{2} + 2x - \frac{x^2}{2}$$

$x > 2$

$$F(x) = \int_{-\infty}^{0} f(x)\cdot dx + \int_{0}^{1} f(x)\cdot dx + \int_{1}^{2} f(x)\cdot dx + \int_{2}^{\infty} f(x)\cdot dx$$

$$= 0 + \frac{x^2}{2}\Big|_{0}^{1} + \left(2x - \frac{x^2}{2}\right)_{1}^{2} + 0$$

$$= \frac{1}{2} + \left[(4-2) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + \left[2 - \frac{3}{2}\right] = \frac{1}{2} + \frac{1}{2} = 1$$

$$f_X(x) = \begin{cases} 0 & -\infty < x < 0 \\ x^2/2 & x < 1 \\ -1 + 2x - x^2/2 & x \le 2 \end{cases}$$

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Mathematical Expectation -

Let X be any R.V & $\phi(x)$ be any function of X. Then expectation of $\phi(x)$ is denoted by $E(\phi(x))$ & is defined by

$E(\phi(x))$

D.R.V

$\sum_x \phi(x) \cdot P(x)$ ↓ pmf

C.R.V

$\int_{-\infty}^{\infty} \phi(x) \cdot f(x) \cdot dx$ ↓ pdf

If $\phi(x) = x$,

$\boxed{\text{mean}} \Rightarrow$ $\bar{x}$

$E(x)$ D.R.V $E(x) = \sum_x x P(x)$

C.R.V $E(x) = \int_{-\infty}^{\infty} x f(x) \cdot dx$

Variance $= E(x^2) - (E(x))^2$

Proof: $E(x - \bar{x})^2 = E(x^2 - 2x\bar{x} + \bar{x}^2)$

$= E(x^2) - 2\bar{x}E(x) + \bar{x}^2$

$= E(x^2) - 2\bar{x}^2 + \bar{x}^2$

$= E(x^2) - \bar{x}^2$

$= E(x^2) - (E(x))^2$

Anish →

Suc. p | No. = x | n = tot

Un: $1-p$ | Un: $n-x$

${}^nC_x\, p^x \cdot (1-p)^{n-x}$

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Q) Find mean and variance of the probability distribution give by the following table -

x	1	2	3	4	5
$P(x)$	0.2	0.35	0.25	0.15	0.05

$E(x) = \sum x \cdot P(x) = 0.2 + 0.7 + 0.75 + 0.6 + 0.25$

mean $= 2.5$

$E(x^2) = \sum x^2 P(x) = 0.2 + 1.4 + 2.25 + 2.4 + 1.25 = 7.5$

$Var(x) = \sigma^2 x = E(x^2) - (E(x))^2$

$= 7.5 - (2.5)^2 = 7.5 - 6.25$

$= 1.25$

Q) Thirteen cards are drawn simultaneously from a pack of 52 cards. If ace count 1 and face card 10 and other according to their domination. find the expectation of total score in 13 cards.

Soln:

x	1	2	3	4	5	6	7	8	9	10	10	10	10
$P(x)$	$\frac{4}{52}$	$\frac{4}{52}$	$\frac{4}{52}$	$\frac{4}{52}$	"	"	"	"	"	"	"	"	"

$\frac{4}{52} \downarrow 1/13$

$E(x) = \sum x P(x) = \frac{1}{13}[1+2+3+4+5+6+7+8+9+10+10+10+10]$

$= 85/13$

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Q). A continuous R.V X has density function given by

$$f(x) = \begin{cases} 2 \cdot e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

find expected value & variance of X

Solⁿ:

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)$$

$x > 0$

$$\int_{0}^{\infty} x \cdot 2 \cdot e^{-2x} \cdot dx = 2 \int_{0}^{\infty} x \cdot e^{-2x} \cdot dx$$

Variance

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) \cdot dx = \int_{0}^{\infty} x^2 \cdot (2e^{-2x}) \cdot dx$$

$$Var. = E(x^2) - (E(x))^2$$

Binomial Distribution :

1. All the trials are independent
2. Number (n) of trial is finite
3. The probability of successes is same for each trial.

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$$P(x) = {}^nC_x \, p^x (1-p)^{n-x}$$

$n \to$ no. of trials.
$x \to$ successful trial.
$p \to$ probability of successful event.

- Show that $P(x)$ is pmf.

$\to$ for pmf : ① $P(x) \geq 0$
② $\sum_{x=0}^{n} P(x) = 1$

$$\sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} {}^nC_x \, p^x (1-p)^{n-x}$$

$$= (1-p)^n + {}^nC_1 \, p^1 (1-p)^{n-1} + \cdots {}^nC_n \, p^n$$

$$= (p + 1 - p)^n = 1$$

Hence, proved.

Find moment generating function of Binomial distribution

$$MGF = M_x(t) = E(e^{xt})$$

$$= \sum_{x=0}^{n} e^{xt} P(x) = \sum_{x=0}^{n} e^{xt} \, {}^nC_x \, p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (p \cdot e^t)^x \cdot (1-p)^{n-x}$$

$$\Downarrow$$

$$(p \cdot e^t + 1 - p)^n$$

So,

$$M_x(t) = (p e^t + (1-p))^n$$

$$\begin{array}{r} 1.00 \\ 0.65 \\ \hline 0.35 \end{array}$$

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Characteristic function -

$\phi_x(t) = E(e^{ixt})$

$= \left[P.e^{it} + (1-p)\right]^n$

Probability generating function -

$Z_x(t) = E(z^x)$

$= \sum_{x=0}^{n} z^x . P(x) = \sum_{x=0}^{n} z^x \, {}^nC_x \, p^x (1-p)^{n-x}$

$= \sum_{x=0}^{n} {}^nC_x (z.p)^x . (1-p)^{n-x}$

$= (zp + (1-p))^n$

Mean & variance of Binomial distribution -

$\boxed{\text{mean} = E(x) = np}$

Variance $=$ ~~E(x^2)~~ $\sigma x^2 = E(x^2) - (E(x))^2$

$= (n^2p^2 + npq) - (np)^2$

$= npq$

here, $q = 1-p$

i.e

$\boxed{\sigma x^2 = np(1-p)}$

2) The probability that man aged 60 will live upto 70 is 0.65 out of 10 men, now aged 60 find probability -

1. at least 7 will live upto 70
2. Exactly 9 will live upto 70
3. At most 9 will live upto 70

Soln: $n = 10$ $\quad p = 0.65$ $\quad 1-p = 0.35$

$P(x) = {}^{10}C_x \, 0.65^x (0.35)^{n-x}$

① $P\{x \geq 7\}$ ~~$= 1 - P\{x < 7\}$~~

$P\{x \geq 7\} = P(7) + P(8) + P(9) + P(10)$

$= {}^{10}C_7 (0.65)^7 (0.35)^3 + {}^{10}C_8 (0.65)^8 . (0.35)^2 + {}^{10}C_9 (0.65)^9 (0.35)$

$+ {}^{10}C_{10} (0.65)^{10}$

$= 0.5139$

② $P\{x = 9\} = {}^{10}C_9 (0.65)^9 . (0.35)^1 = 0.0725$

③ $P\{x \leq 9\} = 1 - P\{x > 9\}$

$= 1 - P\{x = 10\}$

$= 1 - {}^{10}C_{10} (0.65)^{10} (0.35)^0 = 0.9865$

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Q) Out of 800 families with 5 children each. How many families would be expected to have ① 3 boys ② 5 girls. ③ either 2 or 3 boys ④ at least 2 girls.

Solⁿ: ~~n=800~~ N=800 , n=5

$P(\text{Boys}) = 1/2$

$1-p = 1 - 1/2 = 1/2$
↓
Probab. of girls

$P(x) = {}^nC_x \, p^x (1-p)^{n-x}$

${}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x} = {}^5C_x \left(\frac{1}{2}\right)^5 = {}^5C_x \, 1/32$

① $P(3) = {}^5C_3 \, 1/32 = \frac{5\times4}{2}\left(\frac{1}{32}\right)$ No. of fam. expected $= 800 \times \frac{10}{32} = 250$

② Probab. of 5 girls = Probab. of 0 boys $= P(0)$

i.e ${}^5C_0 (1/32) = 1/32$

No. of families with 5 G = $\frac{1}{32} \times 800$

③ $P(2) + P(3)$

$= {}^5C_2\left(\frac{1}{32}\right) + {}^5C_3\left(\frac{1}{32}\right) = \frac{1}{32}\left[20\right] = \frac{10}{16} = 5/8$

No. of families $= 800\left(\frac{5}{8}\right) = 500$

④ ~~$P(X \geq 2) = 1 - P(x<2)$~~
~~$= 1 - P(x=1)$~~
~~$= 1 - {}^5C_1 (1/32) = 1 - \frac{5}{32} = 27/32$~~

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④ Probab. of at least 2 girls : 2G 3G 4G 5G
3B 2B 1B 0B
↓
i.e we have to find
$P(0) + P(1) + P(2) + P(3)$
OR
$1 - [P(4) + P(5)]$

$P(4) = {}^5C_4 \cdot 1/32 = 5/32$

$P(5) = 1/32$

so,
$1 - (6/32) = 26/32$

No. of families $= \frac{26}{32}(800)$

Q) 4 coins are tossed 100 times and following were obtained. Fit a binomial distribution for data & calculate theoretical frequency.

No. of head (x)	freq. (f)	xf
0	5	0
1	29	29
2	36	72
3	25	75
4	5	20
	$\Sigma f = 100$	$\Sigma xf = 196$

$$np = \bar{x} = \frac{\Sigma xf}{\Sigma f} = \frac{196}{100} = 1.96$$

$$4P = 1.96$$
$$P = \frac{1.96}{4} = 0.49$$

$$1 - p = 0.51$$

$$P(x) = {}^nC_x\, p^x (1-p)^{n-x}$$
$$= {}^4C_x (0.49)^x (0.51)^{4-x}$$

Total freq = 100

			$100\,P(x)$
$P(0)$	${}^4C_0 (0.49)^0 (0.51)^4$	$= 0.0676$	~~6.76 ≈ 6~~ 6.76 ≈ 6
$P(1)$	${}^4C_1 (0.49)^1 (0.51)^3$	$= 0.2544$	25.44 ≈ 26
$P(2)$	${}^4C_2 (0.49)^2 (0.51)^2$	$= 0.3747$	37.47 ≈ 37
$P(3)$	${}^4C_3 (0.49)^3 (0.51)^1$	$= 0.2400$	24.00
$P(4)$	${}^4C_4 (0.49)^4 (0.51)^0$	0.05765	5.76 ≈ 6
~~P(5)~~			100

Poisson Distribution:-

A discrete R.V X which has the following probability mass function $P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, 2, \ldots \infty$ is called poisson variable & its distribution is called poisson distribution

Q) Prove that poisson distribution is a limiting case of Binomial distribution under following condition:-

(1) $n \to \infty$

(2) $p \to 0$

(3) $np = \lambda$ (finite)

Proof: We know that, $P(x) = {}^nC_x\, p^x (1-p)^{n-x}$

$$\lim_{n\to\infty} P(x)$$

$$= \lim_{n\to\infty} {}^nC_x\, p^x (1-p)^{n-x} = \lim_{n\to\infty} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1)\ldots(n-(x-1))\cancel{(n-x)!}}{x!\cancel{(n-x)!}} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$\left[\text{From } (3),\ np = \lambda,\ p = \lambda/n\right]$

$$= \lim_{n\to\infty} \frac{\cancel{n^x}\left[\left(1-\frac{1}{n}\right)\ldots\left(1-\left(\frac{x-1}{n}\right)\right)\right]}{x!} \frac{\lambda^x}{\cancel{n^x}} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{1}{x!} \lambda^x \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} e^{-\lambda} = \frac{\lambda^x e^{-\lambda}}{x!}$$

Mean & Variance of Poisson Distribution :-

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \infty$$

$$\text{mean} = E(x) = \sum_x \cancel{p(x)}\, x P(x)$$

$$= \sum_x x \cdot \frac{\lambda^x e^{-\lambda}}{x!}$$

Normal Distribution

A continuous R-V which has the following pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

$$-\infty < \mu < 0$$

$$\sigma > 0$$

is called normal variable & its distribution is called normal distribution & is denoted by $\boxed{X \sim N(\mu, \sigma)}$

Mean & Variance of normal distribution :-

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$$

* $\boxed{E(x) = \mu}$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx = \boxed{\mu^2 + \sigma^2}$$

* $\text{Variance} = E(x^2) - (E(x))^2$

$$= \mu^2 + \sigma^2 - \mu^2$$

$$= \sigma^2$$

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Moment generating function of Normal Distribution :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$

$$M_x(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} f(x)\cdot dx$$

$$= \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{xt} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$$

let $\frac{x-\mu}{\sigma} = z$

$\frac{dx}{\sigma} = dz$

$dx = \sigma \cdot dz$

so,

$$\frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(\mu+\sigma z)t} \cdot e^{-z^2/2} \cdot \cancel{\sigma}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu t} \cdot e^{\sigma z t} \cdot e^{-z^2/2} \cdot dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} \cdot e^{\sigma^2 t^2/2} \cdot dz$$

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$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} \cdot dz$$

$\frac{1}{2}(z-\sigma t)^2 = \theta$

$(z-\sigma t)\cdot dz = d\theta$

$dz = \frac{d\theta}{z-\sigma t}$

$$\cancel{= \frac{2 e^{\mu t + \sigma^2 t^2/2}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-1/2}}$$

$$M_x(t) = \frac{2 e^{\mu t + \sigma^2 t^2/2}}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{e^{-\theta}}{\sqrt{2\theta}} \cdot d\theta$$

$dz = \frac{d\theta}{\sqrt{2\theta}}$

$$= \frac{2 e^{\mu t + \sigma^2 t^2/2}}{2\sqrt{\pi}} \int_{0}^{\infty} \theta^{-1/2} \cdot e^{-\theta} \cdot d\theta$$

$$\boxed{M_x(t) = e^{\mu t + \sigma^2 t^2/2}}$$

MGF of standard normal variable z.

$z = \frac{x-\mu}{\sigma}$

$$M_z(t) = E(e^{zt}) = E\left(e^{\left(\frac{x-\mu}{\sigma}\right)t}\right)$$

$$= E\left(e^{xt/\sigma} \cdot e^{-\mu t/\sigma}\right) = e^{-\mu t/\sigma} E\left(e^{xt/\sigma}\right)$$

$$= e^{-\mu t/\sigma} M_x(t/\sigma) = e^{-\mu t/\sigma} \cdot e$$

$$M_z(t) = e^{t^2/2}$$

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Mean & Variance by MGF -

$M_x(t) = \cancel{e^{\mu t}}\ e^{\mu t + \sigma^2 t^2/2}$

$E(x) = \frac{d}{dt}\left(M_x(t)\right)_{t=0}$

$= \frac{d}{dt}\left(e^{\mu t + \sigma^2 t^2/2}\right)\Big|_{t=0}$

$= e^{\mu t + \sigma^2 t^2/2}\left(\mu + \sigma^2 t\right)\Big|_{t=0}$

$= \cancel{e^{\mu t}(\mu t)}\ \mu$

$\boxed{E(x) = \mu}$

$E(x^2) = \frac{d^2}{dt^2}\left(M_x(t)\right)\Big|_{t=0}$

$= \frac{d}{dt}\left(e^{\mu t + \sigma^2 t^2/2}(\mu + \sigma^2 t)\right)\Big|_{t=0}$

$= e^{\mu t + \sigma^2 t^2/2}\left(\mu + \sigma^2 t\right)^2 + \cancel{\mu}\, e^{\mu t + \sigma^2 t^2/2}\left(\cancel{\mu}\sigma^2\right)\Big|_{t=0}$

$= \mu^2 + \cancel{\mu}\sigma^2$

Variance $= E(x^2) - (E(x))^2$

$= \cancel{\mu^2} + \cancel{\mu}\sigma^2 - \cancel{\mu^2} = \cancel{\mu}\sigma^2$

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Moments :-

Moment about origin	Moment about mean	Moment about any other point
$\mu_r' = E(x-0)^r$	$\mu_r = E(x-\bar{x})^r$	$\mu_r'' = E(x-A)^r$
$\mu_0' = E(x-0)^0 = 1$	$\mu_0 = E(x-\bar{x})^0 = 1$	$\mu_1'' = E(x-A)$
$\boxed{\mu_1' = E(x) = \text{mean}}$	$\mu_1 = E(x-\bar{x}) = E(x) - E(\bar{x}) = \bar{x} - \bar{x} = 0$	$\mu_2'' = E(x-A)^2$
$\mu_2' = E(x^2)$ — DRV: $\sum x^2 P(x)$; CRV: $\int x^2 P(x)$		$\mu_3'' = E(x-A)^3$
$\mu_3' = E(x^3)$		$\mu_4'' = E(x-A)^4$
$\mu_4' = E(x^4)$		

$\mu_2 = E(x-\bar{x})^2$

$= E(x^2 - 2x\bar{x} + \bar{x}^2)$

$= E(x^2) - (E(x))^2$

= variance.

$\boxed{\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3}$