Random Variable :-

DRV — CRV continuous R.V

discrete R.V

Pmf: ① $p(x) \geq 0$

② $\sum p(x) = 1$

Pdf: ① $p(x) \geq 0$

② $\int_{-\infty}^{\infty} f(x) = 1$

• R.V is a real valued f^n which assign a real number to each sample point in the sample space.

Eg: Tossing a fair coin thrice then,

Samplespace $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

X : No. of heads

$x(S_1) = 3$

$x(S_2) = x(S_3) = x(S_4) = 2$

$x(S_5) = x(S_6) = x(S_7) = 1$

$x(S_8) = 0$

1) RV : A R.V which takes finite or at most countable no. of ~~variates~~ values is called discrete R.V.

Eg: ① no. of heads obtained when two coins are tossed.

② no. of defective items in a bag.

Probability distribution

X(No. of head)	0	1	2	3
P	1/8	3/8	3/8	1/8

Ex Four bad oranges are mixed with 16 bad oranges. Find probability distribution of the no. of bad oranges in a drawn of two oranges.

Soln:

X(No. of bad oranges)	0	1	2	3	4
P	$\frac{^{16}C_2}{^{20}C_2}$	$\frac{^{16}C_1\,^{4}C_1}{^{20}C_2}$	$\frac{^{4}C_2}{^{20}C_2}$	0	0

16 good
4 bad

Probability mass f^n (Pmf) :

Let X be a ~~R.V~~ D.R.V s.t $P(X = x) = P_i$, then P_i is said to be probability mass function (P.m.f) if it satisfy the following condition:

① $P_i \geq 0$

② $\sum P_i = 1$

Ex - $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Probability dist.

X(No. of heads)	0	1	2	3
P	1/8	3/8	3/8	1/8

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Distribution function -

Let x be a d.R.V, then its discrete distribution f^n or commutative distribution f^n (c.d.f) is defined as
$f(x) = \Sigma P_i = P(x \le x_j)$

$$f(x) = \begin{cases} 1/8 & x \le 0 \\ 4/8 & x \le 1 \\ 7/8 & x \le 2 \\ 1 & x \le 3 \end{cases}$$

Q) Given the following probability distribution -

x	0	1	2	3	4	5	6	7
$p(x)$	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

find (1) K (2) $P(x<6)$, $P(x \ge 6)$, $P(0<x<5)$

(3) Distribution f^n

(4) if $P(x \le c) > \frac{1}{2}$ find value of c

(5) $P\left(\frac{1.5 < x < 4.5}{x > 2}\right)$

Solⁿ: if $p(x)$ = Pm.f

$\Sigma P(x) = 1$

$\Rightarrow 0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

$\Rightarrow 10K^2 + 9K - 1 = 0 \Rightarrow 10K^2 + 10K - K - 1 = 0$

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$\frac{0.19}{0.81}$

$\Rightarrow 10K(K+1) - 1(K+1) = 0$

$\Rightarrow K = -1$ not possible can't be -ve. | $K = 1/10$

So,

x	0	1	2	3	4	5	6	7
$p(x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

(2) ~~$P(x \ge 6)$~~ $P(x<6)$

$= P(0)+P(1)+P(2)+P(3)+P(4)+P(5)$

or $1 - P(x \ge 6)$

$1 - [P(6)+P(7)] = 1 - [0.19]$

$= 0.81$

$P(x \ge 6)$

$= P(6)+P(7) = 0.19$

$P(0<x<5)$

$= P(1)+P(2)+P(3)+P(4)$

$= 0.1 + 0.2 + 0.2 + 0.3 = 0.8$

(3)
$$P(x) = \begin{cases} 0 & x \le 0 \\ 0.1 & x \le 1 \\ 0.3 & x \le 2 \\ 0.5 & x \le 3 \\ 0.8 & x \le 4 \\ 0.81 & x \le 5 \\ 0.83 & x \le 6 \\ 1 & x \le 7 \end{cases}$$

(4) for $x \geq 4$ $P(x) = 0.8 > 1/2$

$\therefore c = 4$

(5) $P\left(\frac{1.5 < x < 4.5}{x > 2}\right)$

$$\left[P\left(\frac{A}{B}\right) = \cancel{P\left(\frac{A \cap B}{P}\right)} \quad \frac{P(A \cap B)}{P(B)} \right]$$

So,

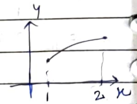
$$\frac{P(1.5 < x < 4.5) \cap P(x > 2)}{P(x > 2)} = \frac{P(2,3,4) \cap P(3,4,5,6,7)}{P(3,4,5,6,7)}$$

$$= \frac{P(3,4)}{P(x > 2)} = \frac{P(3) + P(4)}{1 - P(x \leq 2)} = \frac{0.5}{1 - 0.3} = \frac{0.5}{0.7} = 5/7$$

Continuous Random Variable :-

* A R.V which can take infinite no. of values in an interval is known as CRV.

Eg: i) The weight of a group of individuals.
(ii) Height of a group of individuals.
(iii) Price of house



Probability density function (Pdf) -

• A f^n $f(x)$ is pdf if -

① $f(x) \geq 0 \quad -\infty < x < \infty$

② $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

it is also known as density function.

Eg: if X is a CRV with the following pdf

$$f(x) = \begin{cases} \alpha(2x - x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

find ① α ② $P(x > 1)$

Solⁿ: By defⁿ of pdf $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

So,

$$\int_{-1}^{0} f(x) \cdot dx + \int_{0}^{2} f(x) \cdot dx + \int_{2}^{\infty} f(x) \cdot dx = 1$$

$$\Rightarrow 0 + \int_{0}^{2} f(x) \cdot dx + \int 0 = 1$$

$$\Rightarrow \int_{0}^{2} \alpha(2x - x^2) \cdot dx = 1 \Rightarrow \alpha\left[x^2 \Big|_0^2 - \frac{x^3}{3}\Big|_0^2 \right] = 1$$

$$\Rightarrow \alpha\left[(4 - 0) - \frac{8}{3} \right] = 1$$

$$\Rightarrow \alpha\left(4 - \frac{8}{3}\right) = 1$$

$$\Rightarrow \alpha\left(\frac{4}{3}\right) = 1$$

$$\Rightarrow \alpha = 3/4$$

$$P(x > 1) = \int_{1}^{\infty} f(x) \cdot dx$$

$$= \int_{1}^{2} \alpha(2x - x^2) \cdot dx + \cancel{\int_{2}^{\infty} f(x) \cdot dx}^{\,0}$$

$$= \frac{3}{4}\left[x^2 \Big|_1^2 - \frac{x^3}{3}\Big|_1^2 \right] = \frac{3}{4}\left[3 - \frac{7}{3} \right]$$

$$= \frac{3}{4}\left(\frac{2}{3}\right) = 1/2$$

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Q) A R.V X has density f^n

$$f(x) = \begin{cases} kx^2 & -3 < x < 3 \\ 0 & \text{Otherwise} \end{cases}$$

find K, $P(1 \le x \le 2)$, $P(x \le 2)$, $P(x > 1)$

Soln: By the defn of pdf -

$$\int_{-\infty}^{\infty} f(x)\cdot dx = 1$$

$$\Rightarrow \int_{-3}^{3} kx^2 \cdot dx = 1 \Rightarrow k\frac{x^3}{3}\Big|_{-3}^{3} = 1$$

$$\Rightarrow k\left[9+9\right] = 1 \Rightarrow k = 1/18$$

$P(1 \le x \le 2)$

$$\int_{1}^{2} \frac{1}{18}\cdot x^2 \cdot dx = \frac{1}{18}\int_{1}^{2} x^2 \cdot dx = \frac{1}{18}\cdot\frac{x^3}{3}\Big|_{1}^{2} \cancel{= \frac{1}{18}\left(9-\frac{1}{3}\right)}$$

$$\cancel{= \frac{1}{18}\left(\frac{26}{3}\right) = \frac{13}{27}} \quad = \frac{1}{18}\left(\frac{8}{3}-\frac{1}{3}\right) = \frac{7}{3}\times\frac{1}{18} = \frac{7}{54}$$

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$P(x \le 2)$

$$\cancel{\int_{-\infty}^{-3} f(x)\,dx} + \int_{-3}^{2} f(x)\cdot dx = \frac{1}{18}\cdot\frac{x^3}{3}\Big|_{-3}^{2}$$

$$= \frac{1}{18}\left(\frac{8}{3}+\frac{27}{3}\right) = \frac{1}{18}\left(\frac{35}{3}\right) = \frac{35}{54}$$

$P(x > 1)$

$$\int_{1}^{3} f(x)\cdot dx = \frac{1}{18}\int_{1}^{3} x^2\cdot dx = \frac{1}{18}\cdot\frac{x^3}{3}\Big|_{1}^{3} = \frac{1}{18}\left(\frac{27}{3}-\frac{1}{3}\right)$$

$$= \frac{1}{18}\left(\frac{26}{3}\right) = \frac{13}{27}$$

- Commulative distribution function (c.d.f)
  continuous

Let X be continuous random variable having pdf $f(x)$ then $F_x(x)$ will be a continuous distribution function of x if

$$F_x(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\cdot dx$$

this is also known as commulative distribution function.

$1.00 - 2.25 = 1.75$ $\qquad$ $1.000 - 0.875 = 0.125$

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Note:- Relation b/w distribution function and density function

$$\frac{d\,F(x)}{dx} = f(x)$$

Eg: The probability density function of random variable x is

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ 2-x & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

① find $P(x \ge 1.5)$

② find commulative distribution function

Sol^n:

$$P(x \ge 1.5) = \int_{1.5}^{\infty} f(x)\cdot dx = \int_{1.5}^{2} (2-x)\cdot dx$$

$$= 2x\Big|_{1.5}^{2} - \frac{1}{2}x^2\Big|_{1.5}^{2}$$

$$= 2(0.5) - \frac{1}{2}(4 - 2.25) = 1 - \frac{1}{2}(1.75)$$

$$= 1 - \frac{\cancel{175}\ 87.5}{\cancel{200}\ 100} = 1 - 0.875 = 0.125$$

② $x \le 0$

$$P(X \le x) = \int_{-\infty}^{x} f(x)\cdot dx = 0$$

$0 \le x < 1$

$$F(x) = \int_{-\infty}^{0} f(x)\cdot dx + \int_{0}^{x} f(x)\cdot dx$$

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$$F(x) = 0 + \int_{0}^{x} \cancel{(2-x)\cdot dx}\ x\cdot dx = 0 + \frac{x^2}{2}$$

$1 < x < 2$

$$\int_{0}^{1} x\cdot dx + \int_{1}^{x} (2-x)\cdot dx = \cancel{2x\Big|_{1}^{2} - \frac{x^2}{2}\Big|_{1}^{2} = \frac{2}{2} + 2(x-1) - \frac{x^2}{2}}$$

$$\cancel{= 2(1) - \left(2 - \frac{1}{2}\right) = 2 - \frac{3}{2} = \frac{1}{2}}$$

$$= \frac{x^2}{2}\Big|_{0}^{1} + \left(2x - \frac{x^2}{2}\right)\Big|_{1}^{x} = \frac{1}{2} + \left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{2} - 2 + \frac{1}{2} + 2x - \frac{x^2}{2}$$

$x > 2$

$$F(x) = \int_{-\infty}^{0} f(x)\cdot dx + \int_{0}^{1} f(x)\cdot dx + \int_{1}^{2} f(x)\cdot dx + \int_{2}^{\infty} f(x)\cdot dx$$

$$= 0 + \frac{x^2}{2}\Big|_{0}^{1} + \left(2x - \frac{x^2}{2}\right)\Big|_{1}^{2} + 0$$

$$= \frac{1}{2} + \left[(4-2) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + \left[2 - \frac{3}{2}\right] = \frac{1}{2} + \frac{1}{2} = 1$$

$$f_X(x) = \begin{cases} 0 & -\infty < x < 0 \\ x^2/2 & x < 1 \\ -1 + 2x - x^2/2 & x \le 2 \end{cases}$$

Mathematical Expectation -

Let x be any R.V & $\phi(x)$ be any function of x. Then expectation of $\phi(x)$ is denoted by $E(\phi(x))$ & is defined by

$E(\phi(x))$

D.R.V: $\sum_x \phi(x) \cdot p(x)$ → pmf

C.R.V: $\int_{-\infty}^{\infty} \phi(x) \cdot f(x) \cdot dx$ → pdf

If $\phi(x) = x$,

$E(x)$ D. R.V: $E(x) = \sum_x x\, P(x)$

C.R.V: $E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$

$\boxed{\text{mean}} \Rightarrow$ $\bar{x}$

Variance $= E(x^2) - (E(x))^2$

Proof: $E(x - \bar{x})^2 = E(x^2 - 2x\bar{x} + \bar{x}^2)$

$= E(x^2) - 2\bar{x}E(x) + \bar{x}^2$

$= E(x^2) - 2\bar{x}^2 + \bar{x}^2$

$= E(x^2) - \bar{x}^2$

$= E(x^2) - (E(x))^2$

Anish → Succ: p Uns: $1-p$ No.: x Un: $n-x$ n = total

${}^nC_x\, p^x\, (1-p)^{n-x}$

Q) Find mean and variance of the probability distribution give by the following table -

x	1	2	3	4	5
$P(x)$	0.2	0.35	0.25	0.15	0.05

$E(x) = \sum x \cdot P(x) = 0.2 + 0.7 + 0.75 + 0.6 + 0.25$

mean $= 2.5$

$E(x^2) = \sum x^2 p(x) = 0.2 + 1.4 + 2.25 + 2.4 + 1.25 = 7.5$

$Var(x) = \sigma^2 x = E(x^2) - (E(x))^2$

$= 7.5 - (2.5)^2 = 7.5 - 6.25$

$= 1.25$

Q) Thirteen cards are drawn simultaneously from a pack of 52 cards. If ace count 1 and face card 10 and other according to their domination. find the expectation of total score in 13 cards.

Soln:

x	1	2	3	4	5	6	7	8	9	10	10	10	10
$P(x)$	$\frac{4}{52}$	$\frac{1}{52}$	$\frac{4}{52}$	$\frac{1}{52}$	"	"	"	"	"	"	"	"	"

$\frac{4}{52} \to 1/13$

$E(x) = \sum x P(x) = \frac{1}{13}[1+2+3+4+5+6+7+8+9+10+10+10+10]$

$= 85/13$

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Q). A continuous R.V X has density function given by

$$f(x) = \begin{cases} 2 \cdot e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

find expected value & variance of X

Sol^n:

~~Mean~~ $E(x) = \int_{-\infty}^{\infty} x \cdot f(x)$

$x > 0$

$$\int_{0}^{\infty} x \cdot 2 \cdot e^{-2x} \cdot dx = 2 \int_{0}^{\infty} x \cdot e^{-2x} \cdot dx$$

Variance

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) \cdot dx = \int_{0}^{\infty} x^2 \cdot (2e^{-2x}) \cdot dx$$

$\text{Var.} = E(x^2) - (E(x))^2$

Binomial Distribution :

1. All the trials are independent
2. Number (n) of trial is finite
3. The probability of successes is same for each trial.

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$P(x) = {}^nC_x \, p^x (1-p)^{n-x}$

$n \rightarrow$ no. of trials.
$x \rightarrow$ successful trial.
$p \rightarrow$ probability of successful event.

- Show that $P(x)$ is pmf.

$\rightarrow$ for pmf : (1) $p(x) \geq 0$
(2) $\sum_{x=0}^{n} P(x) = 1$

$$\sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} {}^nC_x \, p^x (1-p)^{n-x}$$

$$= (1-p)^n + {}^nC_1 \, p^1 (1-p)^{n-1} + \dots {}^nC_n \, p^n$$

$$= (p + 1 - p)^n = 1$$

Hence, proved.

Find moment generating function of Binomial distribution

$\text{MGF} = M_x(t) = E(e^{xt})$

$$= \sum_{x=0}^{n} e^{xt} p(x) = \sum_{x=0}^{n} e^{xt} \, {}^nC_x \, p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (p \cdot e^t)^x \cdot (1-p)^{n-x}$$

$$\Downarrow$$

$$(p \cdot e^t + 1 - p)^n$$

So,

$$M_x(t) = (p e^t + (1-p))^n$$

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Characteristic function -

$\phi_x(t) = E(e^{ixt})$

$= [p.e^{it} + (1-p)]^n$

Probability generating function -

$Z_x(t) = E(z^x)$

$= \sum_{x=0}^{n} z^x . p(x) = \sum_{x=0}^{n} z^x \, {}^nC_x p^x (1-p)^{n-x}$

$= \sum_{x=0}^{n} {}^nC_x (z.p)^x . (1-p)^{n-x}$

$= (zp + (1-p))^n$

Mean & variance of Binomial distribution -

$\boxed{\text{Mean} = E(x) = np}$

Variance $=$ $\cancel{E(x^2)}$ $\sigma x^2 = E(x^2) - (E(x))^2$

$= (n^2p^2 + npq) - (np)^2$

$= npq$

here, $q = 1-p$

i.e

$\boxed{\sigma x^2 = np(1-p)}$

1.00
0.65
0.35

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Q) The probability that man aged 60 will live upto 70 is 0.65 out of 10 men, now aged 60 find probability -

(1) at least 7 will live upto 70
(2) Exactly 9 will live upto 70
(3) At most 9 will live upto 70

Soln: $n = 10$ $p = 0.65$ $1-p = 0.35$

$p(x) = {}^{10}C_x \, 0.65^x (0.35)^{n-x}$

(1) $p\{x \geq 7\}$ $\cancel{= 1 - p\{x < 7\}}$

$p\{x \geq 7\} = p(7) + p(8) + p(9) + p(10)$

$= {}^{10}C_7 (0.65)^7 (0.35)^3 + {}^{10}C_8 (0.65)^8 (0.35)^2 + {}^{10}C_9 (0.65)^9 (0.35)$

$+ {}^{10}C_{10} (0.65)^{10}$

$= 0.5139$

(2) $p\{x = 9\} = {}^{10}C_9 (0.65)^9 . (0.35)^1 = 0.0725$

(3) $p\{x \leq 9\} = 1 - p\{x > 9\}$

$= 1 - p\{x = 10\}$

$= 1 - {}^{10}C_{10} (0.65)^{10} (0.35)^0 = 0.9865$

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Q) Out of 800 families with 5 children each. How many families would be expected to have ① 3 boys ② 5 girls. ③ either 2 or 3 boys ④ at least 2 girls.

Solⁿ: ~~n=800~~ N=800 , n=5

$P(\text{Boys}) = 1/2$

$1-p = \cancel{q} \; 1-1/2 = 1/2$
↓
Probab. of girls

$P(x) = {}^nC_x \, p^x (1-p)^{n-x}$

${}^5C_x \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{5-x} = {}^5C_x \left(\frac{1}{2}\right)^5 = {}^5C_x \; 1/32$

① $P(3) = {}^5C_3 \; 1/32 = \frac{5\times4}{2}\left(\frac{1}{32}\right)$ No. of fam. expected $= 800 \times \frac{10}{32} = 250$

② ~~P(5)~~ Probab. of 5 girls = Probab. of 0 boys = $P(0)$

i.e ${}^5C_0 (1/32) = 1/32$

No. of families with 5 gr $= \frac{1}{32} \times 800$

③ $P(2) + P(3)$
$= {}^5C_2\left(\frac{1}{32}\right) + {}^5C_3\left(\frac{1}{32}\right) = \frac{1}{32}\left[20\right] = \frac{10}{16} = 5/8$

No. of families $: 800\left(\frac{5}{8}\right) : 500$

④ ~~$P(X \geq 2) = 1 - P(x < 2)$~~
~~$= 1 - P(x=1)$~~
~~$= 1 - {}^5C_1 (1/32) = 1 - \frac{5}{32} = 27/32$~~

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④ Probab. of at least 2 girls : 2G 3G 4G 5G
3B 2B 1B 0B
↓
i.e we have to find
$P(0) + P(1) + P(2) + P(3)$
OR
$1 - [P(4) + P(5)]$

$P(4) = {}^5C_4 \cdot 1/32 = 5/32$

$P(5) = 1/32$

so,
$1 - (6/32) = 26/32$

No. of families $= \frac{26}{32}(800)$

Q) 4 coins are tossed 100 times and following were obtained. Fit a binomial distribution for data & calculate theoretical frequency.

No. of head(x)	freq.(f)	xf
0	5	0
1	29	29
2	36	72
3	25	75
4	5	20
	$\Sigma f = 100$	$\Sigma xf = 196$

$np = \bar{x} = \frac{\Sigma xf}{\Sigma f} = \frac{196}{100} = 1.96$

$4p = 1.96$

$p = \frac{1.96}{4} = 0.49$

$1 - p = 0.51$

$P(x) = {}^nC_x \, p^x (1-p)^{n-x}$

$= {}^4C_x \, (0.49)^x (0.51)^{4-x}$

Total freq = 100

			$100\,P(x)$
$P(0)$	${}^4C_0 (0.49)^0 (0.51)^4$	$= 0.0676$	~~67.6 ≈ 6~~ 6.76 ≈ 6
$P(1)$	${}^4C_1 (0.49)^1 (0.51)^3$	$= 0.2544$	25.44 ≈ 26
$P(2)$	${}^4C_2 (0.49)^2 (0.51)^2$	$= 0.3747$	37.47 ≈ 37
$P(3)$	${}^4C_3 (0.49)^3 (0.51)^1$	$= 0.2400$	24.00
$P(4)$	${}^4C_4 (0.49)^4 (0.51)^0$	0.05765	5.76 ≈ 6
~~$P(5)$~~			100

Poisson Distribution:-

A discrete R.V X which has the following probability mass function $P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, 2, \ldots \infty$ is called poisson variable & its distribution is called poisson distribution.

Q) Prove that poisson distribution is a limiting case of Binomial distribution under following condition:-

① $n \to \infty$

② $p \to 0$

③ $np = \lambda$ (finite)

Proof: We know that, $P(x) = {}^nC_x \, p^x (1-p)^{n-x}$

$$\lim_{n\to\infty} P(x)$$

$$= \lim_{n\to\infty} {}^nC_x \, p^x (1-p)^{n-x} = \lim_{n\to\infty} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n(n-1)\ldots(n-(x-1))\cancel{(n-x)!}}{x!\cancel{(n-x)!}} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

[From ③, $np = \lambda$, $p = \lambda/n$]

$$= \lim_{n\to\infty} \frac{\cancel{n^x}\left[\left(1 - \frac{1}{n}\right)\ldots\left(1 - \left(\frac{x-1}{n}\right)\right)\right]}{x!} \frac{\lambda^x}{\cancel{n^x}} \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{1}{x!} \lambda^x \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} e^{-\lambda} = \frac{\lambda^x e^{-\lambda}}{x!}$$

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Mean & Variance of Poisson Distribution -

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \dots \infty$$

$$\text{mean} = E(x) = \sum_x \cancel{P(x)}\, x P(x)$$

$$= \sum_x x \cdot \frac{\lambda^x e^{-\lambda}}{x!}$$

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Normal Distribution

A continuous R.V which has the following pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2 \left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$
$$-\infty < \mu < \infty$$
$$\sigma > 0$$

is called normal variable & its distribution is called normal distribution & is denoted by $\boxed{X \sim N(\mu, \sigma)}$

Mean & Variance of normal distribution -

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$E(x) = \int_{-\infty}^{\infty} x\, f(x) \cdot dx = \int x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$$

* $\boxed{E(x) = \mu}$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx = \boxed{\mu^2 + \sigma^2}$$

* $$\text{Variance} = E(x^2) - (E(x))^2$$
$$= \mu^2 + \sigma^2 - \mu^2$$
$$= \sigma^2$$

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Moment Generating function of Normal Distribution :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

$$M_x(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{xt} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$$

let $\frac{x-\mu}{\sigma} = z$

$\frac{dx}{\sigma} = dz$

So,

$dx = \sigma \cdot dz$

$$\frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(\mu+\sigma z)t} \cdot e^{-z^2/2} \cdot \cancel{\sigma}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu t} \cdot e^{\sigma z t} \cdot e^{-z^2/2} \cdot dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} \cdot e^{\sigma^2 t^2/2} \cdot dz$$

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$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} \cdot dz$$

$\frac{1}{2}(z-\sigma t)^2 = \theta$

$(z-\sigma t)\cdot dz = d\theta$

$$\cancel{= \frac{2e^{\mu t + \sigma^2 t^2/2}}{\sqrt{2\pi}} \int_0^{\infty} e^{-1/2}}$$

$dz = \frac{d\theta}{z-\sigma t}$

$$M_x(t) = \frac{2e^{\mu t + \sigma^2 t^2/2}}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{-\theta} \cdot d\theta}{2\theta}$$

$dz = \frac{d\theta}{\sqrt{2\theta}}$

$$= \frac{2e^{\mu t + \sigma^2 t^2/2}}{2\sqrt{\pi}} \int_0^{\infty} \theta^{-1/2} \cdot e^{-\theta} \cdot d\theta$$

$$\boxed{M_x(t) = e^{\mu t + \sigma^2 t^2/2}}$$

MGF of standard normal variable z.

$z = \cancel{\mu\,\sigma}\ \frac{x-\mu}{\sigma}$

$$M_z(t) = E(e^{zt}) = E\left(e^{\left(\frac{x-\mu}{\sigma}\right)t}\right)$$

$$= E\left(e^{xt/\sigma} \cdot e^{-\mu t/\sigma}\right) = e^{-\mu t/\sigma} E\left(e^{xt/\sigma}\right)$$

$$= e^{-\mu t/\sigma} M_x(t/\sigma) = e^{-\mu t/\sigma} \cdot e$$

$$M_z(t) = e^{-t^2/2}$$

Mean & Variance by MGF :-

$M_X(t) = \cancel{e^{\mu t}} \; e^{\mu t + \sigma^2 t^2/2}$

$E(x) = \frac{d}{dt}\left(M_X(t)\right)_{t=0}$

$= \frac{d}{dt}\left(e^{\mu t + \sigma^2 t^2/2}\right)\Big|_{t=0}$

$= e^{\mu t + \sigma^2 t^2/2}\left(\mu + \sigma^2 t\right)\Big|_{t=0}$

$= \cancel{e^{\mu t}}\,\cancel{(\mu)}\;\mu$

$\boxed{E(x) = \mu}$

$E(x^2) = \frac{d^2}{dt^2}\left(M_X(t)\right)\Big|_{t=0}$

$= \frac{d}{dt}\left(e^{\mu t + \sigma^2 t^2/2}(\mu + \sigma^2 t)\right)\Big|_{t=0}$

$= e^{\mu t + \sigma^2 t^2/2}(\mu + \sigma^2 t)^2 + \cancel{\mu}\, e^{\mu t + \sigma^2 t^2/2}(\cancel{\mu}\,\sigma^2)\Big|_{t=0}$

$= \mu^2 + \cancel{\mu}\,\sigma^2$

Variance $= E(x^2) - (E(x))^2$

$= \cancel{\mu^2} + \cancel{\mu}\,\sigma^2 - \cancel{\mu^2} = \cancel{\mu}\,\sigma^2$

Moments :-

Moment about origin

$\mu_r' = E(x-0)^r$

$\mu_0' = E(x-0)^0 = 1$

$\boxed{\mu_1' = E(x) = \text{mean}}$

$\mu_2' = E(x^2) \xrightarrow{DRV} \sum x^2 P(x)$, $\xrightarrow{CRV} \int x^2 P(x)$

$\mu_3' = E(x^3)$

$\mu_4' = E(x^4)$

Moment about mean

$\mu_r = E(x-\bar{x})^r$

$\mu_0 = E(x-\bar{x})^0 = 1$

$\mu_1 = E(x-\bar{x})$
$= E(x) - E(\bar{x})$
$= \bar{x} - \bar{x}$
$= 0$

$\mu_2 = E(x-\bar{x})^2$
$= E(x^2 - 2x\bar{x} + \bar{x}^2)$
$= E(x^2) - (E(x))^2$
$=$ Variance.

$\boxed{\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3}$

Moment about any other point

$\mu_r'' = E(x-A)^r$

$\mu_1'' = E(x-A)$

$\mu_2'' = E(x-A)^2$

$\mu_3'' = E(x-A)^3$

$\mu_4'' = E(x-A)^4$