ASSIGNMENT-1

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Q1 i) TP: Product of a non-zero rational number and an irrational number is irrational.

~~Assu~~ Assumption: Let the product be rational.

Say, a is a rational number,
b is an irrational number,

$a = \frac{p}{q}$, $p, q \neq 0$ and coprimes. $p, q \in \mathbb{Z}$

$\cancel{a \times b =} \frac{pq}{b}$ &

Let $ab = \frac{m}{n}$, $m, n \neq 0$, $m, n \in \mathbb{Z}$

$\frac{p}{q} b = \frac{m}{n}$

$b = \frac{mq}{pn} \Rightarrow b$ is rational

But b is an irrational number

Therefore, our assumption was incorrect.
(Proved by contradiction)

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(1) TP: If $m, n \in \mathbb{Z}$ and mn is even, m is even or n is even.

Assumption: both m and n are odd

($\neg q \rightarrow \neg p$ instead of $p \rightarrow q$)

Let $m = 2k_1 + 1$, $k_1 \in \mathbb{Z}$

and $n = 2k_2 + 1$, $k_2 \in \mathbb{Z}$

$$mn = (2k_1+1)(2k_2+1) = 4k_1k_2 + 2k_1 + 2k_2 + 1$$

$$mn = 2(2k_1k_2 + k_1 + k_2) + 1$$

$$mn = 2A + 1$$

where $A = 2k_1k_2 + k_1 + k_2$, thus $A \in \mathbb{Z}$

$\Rightarrow$ mn is odd, contradictory to the given statement.

(Proved by contraposition)

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Q2

Say
$P(x)$: x failed in Mathematics
$Q(x)$: x attended every class
$R(x)$: x submitted assignment every week

Premises given -

$$P(Ramesh) \wedge Q(Ramesh)$$

$$\forall x\,[R(x) \longrightarrow \neg P(x)]$$

$$\forall x \neg R(x) \longrightarrow \exists x \neg R(x)$$

To conclude: $\neg \forall x\, R(x)$

1)	$\cancel{\forall x}\ P(Ramesh) \wedge Q(Ramesh)$	Premise
2)	$P(Ramesh)$	Simplification (1)
3)	$\exists x\, P(x)$	~~Universal~~ Existential generalisation (2)
4)	$\exists x \neg(\neg P(x)) \equiv \neg \forall x\, (\neg P(x))$	
5)	$\forall x\, R(x) \longrightarrow \forall x\, (\neg P(x))$	Premise
6)	$\neg \forall x\, R(x)$	Universal Modus tollens (4) and (5)

$\frac{3}{4}$

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Q3 Premises: $\forall x\,[P(x) \to Q(x)]$, $\forall x\,[Q(x) \to R(x)]$,

$\neg R(a)$, for some a in domain.

Conclude: $\neg P(a)$

1) $\forall x\,(Q(x) \to R(x))$ premise
2) $\neg R(a)$ premise
3) $\neg Q(a)$ Universal modus tollens (1) and (2)
4) $\forall x\,(P(x) \to Q(x))$ Premise
5) $\neg P(a)$ Universal modus tollens (3) and (4)

Second part:
All premises same ~~except~~ but $R(a)$ is true.
Therefore, $Q(a)$ can be either true or false.
$\Rightarrow$ $P(a)$ can be either true or false,
cannot conclude any particular value

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Q4 Say any $c = a + ib$

where both $a, b \in \mathbb{R}$

If $|\mathbb{R}| = 2^{\aleph_0}$

then

$$|\mathbb{C}| = |\mathbb{R}| \times |\mathbb{R}|$$

$$= 2^{\aleph_0} \times 2^{\aleph_0}$$

$$= 2^{2\aleph_0} = 2^{\aleph_0} = |\mathbb{R}|$$

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Q5 A: Integers divisible by 5 but not by 7, is same as
A: Integers divisible by 5 but not by 35.

Since $A \subseteq \mathbb{Z}$, A is countable.

Say $A = \{a_1, a_2, \ldots\}$

If $a_1 = 5$, $a_2 = 10$, $a_3 = 15$
$\in \mathbb{Z}$ $\in \mathbb{Z}$ $\in \mathbb{Z}$

$a_4 = 20$, $a_5 = 25$, $a_6 = 30$, $\underline{a_7 = 40}$
$\in \mathbb{Z}$ $\in \mathbb{Z}$ $\in \mathbb{Z}$ $\in \mathbb{Z}$

$\Downarrow$

$a_n = a_{n-6} + 35$

Therefore, the function of a_n is always strictly increasing, making it a one-one and an onto function, describing a one-one correspondence to the set of natural numbers.

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Q6 given $a_1 = 1$, $a_2 = 2$, $a_3 = 3$

$$a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

TP → $a_n < 2^n$

Basis step: $a_1 = 1 < 2^1$

$a_2 = 2 < 2^2$

$a_3 = 3 < 2^3$

$a_4 = a_3 + a_2 + a_1 = 6 < 2^4$

All 4 expressions true

Inductive step: Assuming the statement is true for all n=1 to n=k,

TP → $a_{k+1} < 2^{k+1}$

$a_{k+1} = a_k + a_{k-1} + a_{k-2}$

$\Rightarrow a_{k+1} < 2^k + 2^{k-1} + 2^{k-2}$

$\Rightarrow a_{k+1} < 2^{k-2}(\cancel{2^k}\, 2^2 + 2 + 1) \Rightarrow a_{k+1} < 2^{k-2}(7)$

$\Rightarrow a_{k+1} < 2^{k-2}(8-1) \Rightarrow a_{k+1} < 2^{k+1} - 2^{k-2}$

$\Rightarrow \boxed{a_{k+1} < 2^{k+1}}$ Proved using strong induction