EMBEDDING INCOMPLETE LATIN SQUARES

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An *incomplete* $n\times n$ *latin square* based on $1, \cdots, n$ is a square array of n rows and n columns such that a subset of the n^2 places are occupied by integers from the set $1, \cdots, n$ and such that no integer occurs more than once in the same row or column. We solve in this paper, the following problem. For each n, what is the smallest t such that any incomplete $n\times n$ latin square can be embedded in a $t\times t$ or smaller latin square. It is shown that, for $n\geqq 4$, $t=2n$.

Applications of this result yield the following properties of loops. An incomplete loop containing n elements can be embedded in a loop containing $2n$ elements. An infinite loop defined by a finite set of generators and relations has a finite homomorphic image of order t, for every $t\geqq k$, where k is a positive integer determined by the generators and relations of the loop.

The above result on the embedding of incomplete latin squares is a simple consequence of the following theorem, due to H. J. Ryser [7], giving the conditions under which an $r\times s$ latin rectangle based on $1, \cdots, n$ may be extended to an $n\times n$ latin square based on $1, \cdots, n$.

THEOREM 1. *Let $N(i)$ be the number of times the integer i occurs in the latin rectangle. A necessary and sufficient condition that the $r\times s$ latin rectangle can be extended to an $n\times n$ latin square is that $N(i)\geqq r+s-n$, for $i=1, \cdots, n$.*

We begin by exhibiting, for each $n\geqq 4$, an incomplete $n\times n$ latin square which cannot be embedded in a latin square of order less than $2n$. Consider the incomplete $n\times n$ latin square, which has, except for the 2nd and 3rd columns in the $(n-1)$th row and the 2nd column in the nth row, $i+j-1 \pmod n$ as the element in the ith row and jth column. The element in the $(n-1)$th row and 2nd column is 1 and the other two places are unoccupied.

Assume that this incomplete latin square S is embedded in a $t\times t$ latin square T, based on $1, \cdots, t$. The unoccupied places in S cannot be occupied in T by elements in $\{1, \cdots, n\}$ and so $t\geqq n+1$. The element $n+1$ is in each of the first $n-2$ rows of T and hence T contains at least $n-2$ columns in addition to the n columns of S. That is, $t\geqq 2n-2$. Let S' be S with the unoccupied places filled in with the elements they occupy in T. Then S' may be considered as an $n\times n$ latin rectangle based on $1, \cdots, t$ and we may apply Theorem 1. If $t < 2n$, then $n + n - t > 0$ and so $N(i)$ is positive. Hence every integer in $\{1, \cdots, t\}$ occurs at least once in S'. Since S' contains at most two elements from $\{n+1, n+2, \cdots, t\}$ and $t\geqq 2n-2$, this is a contradiction if $n>4$. If $n=4$, we obtain a similar contradiction for $t=7$. If $n=4$ and $t=6$, then $N(i)>2$. But in this case, either both unoccupied places in S contain the same element in S' so that one element in $\{1, \cdots, 6\}$ does not occur in S' or the two places are occupied by 5, 6, each of which then occurs only once in S'. Again we have a contradiction. Thus S cannot be embedded in a latin square of order $<2n$.

For the cases $n=2$, 3, not covered in the above discussion, it is easy to verify (i) an incomplete 2×2 latin square can be embedded in a 2×2 or 3×3 latin square and there is an incomplete 2×2 latin square which cannot be embedded in a 2×2 latin square, (ii) an incomplete 3×3 latin square can be embedded in a 3×3, 4×4 or 5×5 latin square and there is an incomplete 3×3 latin square which cannot be embedded in a 3×3 or 4×4 latin square.

We now prove that any incomplete $n\times n$ latin square S can be embedded in a $t\times t$ latin square T for any $t\geqq 2n$. Let A be a latin square of order $t-n$, where $t\geqq 2n$, based on the integers $n+1, n+2, \cdots, t$. Let S' be the $n\times n$ latin rectangle based on $1, \cdots, t$, obtained from S by filling in, for all i, j, an unoccupied place in the ith row and jth column of S with the element in the ith and jth column of A. Since $n+n-t\leqq 0$, the condition of Theorem 1 is satisfied and we conclude that S' can be extended to a $t\times t$ latin square based on $1, \cdots, t$. That is, S can be embedded in a $t\times t$ latin square.

THEOREM 2.* *For any n, an incomplete $n\times n$ latin square can be embedded in a $t\times t$ latin square, for any $t\geqq 2n$.*

There are immediate applications of Theorem 2 to the theory of loops and quasigroups. A *quasigroup* is a set closed with respect to a binary operation ab, such that for any pair of elements a, b, there are unique solutions of the equations $ax=b$, $ya=b$. A *loop* is a quasigroup with a unit element. An *incomplete quasigroup* is a set with a multiplication defined for some pairs such that if equations $ax=b$ or $ya=b$ have solutions, then these solutions are unique. An *incomplete loop* is an incomplete quasigroup with a unit element and the product of this unit with every element is defined. A full discussion of these ideas may be found in [1]. The multiplication tables of finite quasigroups and loops are latin squares and of finite incomplete quasigroups or loops are incomplete latin squares. Thus the following is an immediate consequence of Theorem 2.

COROLLARY. *An incomplete loop (quasigroup) containing n elements can be embedded in a loop (quasigroup) containing t elements, for any $t\geqq 2n$.*

In this deduction from Theorem 2, we cannot say in general that the elements of the incomplete loop I generate the loop L in which it is embedded. In the special cases, $t=2n$, $2n+1$ this is so (provided I is not a loop) for then the elements of I generate at least one element not in I and a loop of order $2n$ or $2n+1$ cannot contain a proper subloop of order greater than n. We can use these special cases to extend the corollary to the case where the incomplete loop does generate the loop in which it is embedded.

Let I be an incomplete loop containing n elements and with at least one product, say ab, undefined. Let I_p be the incomplete loop containing as elements the union of the set of elements in I and a set $\{c_1, \cdots, c_p\}$ of elements not in I. The multiplications defined in I_p consist of those defined in I and the products:

* I have recently been informed that a proof of this has been obtained independently by S. K. Stein.

$$ab = c_1, \qquad c_i^2 = c_{i+1}, \qquad ec_i = c_ie = c_i,$$

where e is the unit element of I.

Now I_p contains I and is generated by I. Hence any loop in which I_p is embedded and which is generated by I_p will also contain I and be generated by I. By the remarks following the above corollary, I_p can be embedded in a loop containing either $2(n+p)$ or $2(n+p)+1$ elements and I_p generates this loop.

A similar analysis gives corresponding results for quasigroups and so we may now state a stronger result than that of the first corollary.

SECOND COROLLARY TO THEOREM 2. *An incomplete loop (quasigroup) containing n elements can be embedded in a loop (quasigroup) of order t which is generated by the incomplete loop (quasigroup), for any* $t \geqq 2n$.

If L is an infinite loop defined by a finite set of generators and relations, then a finite set of generators and relations may be found for L such that these generators and relations form an incomplete loop I ([1], [3]). Furthermore, any embedding of I in a loop which I generates is a homomorphic image of L ([3]). Thus, from Corollary 2, we obtain the following theorem. (A similar theorem holds also for quasigroups.)

THEOREM 3. *An infinite loop defined by a finite set of generators and relations has a finite homomorphic image of order t, for any* $t \geqq k$, *where k is a positive integer determined by the generators and relations of the loop.*

This theorem may be compared with the example in [2] of a finitely generated and related infinite loop with uncountably many nonisomorphic homomorphic images of countably infinite order and with the example due to G. Higman [5] of a finitely generated and related group with no nontrivial finite homomorphic images.

We conclude by mentioning a few problems. Clearly some incomplete $n \times n$ latin squares can be embedded in latin squares of order less than $2n$. What are conditions for an incomplete $n \times n$ latin square to be embeddable in an $n \times n$ latin square? In particular, can an $n \times n$ incomplete latin square with at most $n-1$ places occupied be completed to an $n \times n$ latin square? It is easy to see that there are incomplete latin square with n places occupied which cannot be so completed. Can a pair of incomplete $n \times n$ latin squares which are orthogonal (insofar as the condition for orthogonality applies to the incomplete squares) be embedded respectively in a pair of $t \times t$ orthogonal latin squares and if so, what is the smallest t for each n? The ideas which are probably needed to attack these problems are discussed in detail in [6]. This paper also contains a proof of Theorem 1. A related, but presumably far more difficult, question concerns the embedding of partial planes in the sense of M. Hall [4]. Can any finite partial projective plane be embedded in a finite projective plane and if so, what is the minimal embedding? It is shown in [4] that any partial projective plane can be embedded in a projective plane but the construction given there always results in an infinite containing plane.

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CONGRUENCES FOR THE NUMBER OF n-gons FORMED BY n LINES

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1. Let g_n denote the number of polygons of n sides (including degenerate cases) formed by a network of n lines. Robinson [3] showed that g_n satisfies the recurrence

$$g_{n+1} = ng_n + \tfrac{1}{2}n(n-1)g_{n-2} \qquad (n \geqq 2), \tag{1.1}$$

where $g_1 = g_2 = 0$, $g_3 = 1$; it is convenient to define $g_0 = 1$. The writer [1] showed that if m is an arbitrary positive integer then

$$g_{n+m} \equiv g_n g_m \pmod{m_0} \qquad (n \geqq 0), \tag{1.2}$$

where $m_0 = m$ or $\frac{1}{2}m$ according as m is odd or even; more generally it was proved that

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} g_{n+sm} g_{(r-s)m} \equiv 0 \pmod{m_0^{[(r+1)/2]}} \qquad (n \geqq 0); \tag{1.3}$$

where $[x]$ denotes the greatest integer $\leqq x$.

In the present note we obtain the following results. Let m_1 denote the largest odd divisor of m. Then first we have

$$g_m \equiv (-1)^m 2^{-m} \pmod{m_1}, \tag{1.4}$$

$$g_{n+m} \equiv (-1)^m 2^{-m} g_n \pmod{m_1}. \tag{1.5}$$

Next if we put