Graphs and Combinatorics (2016) 32:241–256
DOI 10.1007/s00373-015-1571-0

ORIGINAL PAPER

Completing Partial Latin Squares with Blocks of Non-empty Cells

Jaromy Kuhl[1] · **Michael W. Schroeder**[2]

Received: 10 December 2014 / Revised: 7 April 2015 / Published online: 23 April 2015

Abstract In this paper we develop two methods for completing partial latin squares and prove the following. Let A be a partial latin square of order nr in which all non-empty cells occur in at most $n-1$ $r \times r$ squares. If $t_1, \ldots, t_m$ are positive integers for which $n \geqslant t_1^2 + t_2^2 + \cdots + t_m^2 + 1$ and if A is the union of m subsquares each with order rt_i, then A can be completed. We additionally show that if $n \geqslant r+1$ and A is the union of n identical $r \times r$ squares with disjoint rows and columns, then A can be completed. For smaller values of n we show that a completion does not always exist.

Keywords Partial · Latin square · Completion

1 Introduction

Let n be a positive integer and S be a finite symbol set. If A is an $n \times n$ array in which each cell contains a subset of S, then we call A an *array over* S. We use $[n] = \{1, 2, \ldots, n\}$ to index the rows and columns of A and $A(i, j)$ to denote the subset of symbols in the (i, j) cell of A. If the symbols are arranged so that each occurs at most once in each column, then A is called *column-latin*. *Row-latin* is defined analogously.

✉ Jaromy Kuhl
jkuhl@uwf.edu

Michael W. Schroeder
schroederm@marshall.edu

1 Department of Mathematics and Statistics, University of West Florida, 11000 University Parkway, Pensacola, FL 32514, USA

2 Department of Mathematics, Marshall University, 1 John Marshall Drive, Huntington, WV 25755, USA

Let r be a positive integer and S have cardinality nr. We say that A is a *r-semi partial latin square of order n*, or $A \in \mathrm{PLS}_r(n)$, if $|A(i, j)| \leqslant r$ for each $\{i, j\} \subseteq [n]$ and A is both column- and row-latin. If $|A(i, j)| = r$ for each $\{i, j\} \subseteq [n]$, A is called a *r-semi latin square*, or $A \in \mathrm{LS}_r(n)$. The sets of all partial latin squares and latin squares of order n are $\mathrm{PLS}(n) = \mathrm{PLS}_1(n)$ and $\mathrm{LS}(n) = \mathrm{LS}_1(n)$ respectively. For an integer i and set S, let $i + S = \{i + s \mid s \in S\}$. For an $nr \times nr$ array A and $\{i, j\} \subseteq [n]$, let A_{ij} denote the $r \times r$ sub-array of A at the intersection of rows $(i - 1)r + [r]$ and columns $(j - 1)r + [r]$.

In this paper we develop two methods of completing partial latin squares, and consequently, prove two cases of the following conjecture [3].

Conjecture 1 *All partial latin squares of order nr with symbols occurring in at most $n - 1$ disjoint $r \times r$ squares are completable.*

Conjecture 1 is a possible generalization of a theorem famously known as the Evans Conjecture [4].

Theorem 1 *If $A \in \mathrm{PLS}(n)$ with at most $n-1$ non-empty cells, then A can be completed to a latin square of order n.*

Proofs of Theorem 1 were given independently by Häggkvist for $n \geqslant 1111$ [5], and by Anderson and Hilton [1] and Smetaniuk [11] for all n. The upper bound on the number of non-empty cells is sharp. In [1], the authors determine all incompletable partial latin squares of order n with exactly n non-empty cells. One such square has symbol 1 in the first k diagonal cells and symbols $2, 3, \ldots, n - k + 1$ in the last $n - k$ cells of column $k + 1$. We name this incompletable partial latin square $B_{k,n}$, where $k < n$.

Let $P \in \mathrm{PLS}(n)$ and S_n be the symmetric group acting on $[n]$. For $\theta = (\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$, we use $\theta(P) \in \mathrm{PLS}(n)$ to denote the array in which the rows, columns, and symbols of P are permuted according to α, β, and γ respectively. The mapping θ is called an *isotopism*, and P and $\theta(P)$ are said to be *isotopic*. Each partial latin square can be written as a subset of ordered triples of $[n] \times [n] \times [n]$, where $(i, j, k) \in P$ if $k \in P(i, j)$. A *conjugate* of P is an array in which the coordinates of each triple of P are uniformly permuted. There are six, not necessarily distinct, conjugates of P. The *main class* of P is the set of all partial latin squares that are isotopic to some conjugate of P. The following theorem is Anderson and Hilton's answer to the Evans conjecture [1].

Theorem 2 *If $P \in \mathrm{PLS}(n)$ with exactly n non-empty cells, then P is completable if and only if P is not in the main class of $B_{k,n}$ for each k, $1 \leqslant k < n$.*

In [9], Theorem 1 is generalized to elements of $\mathrm{PLS}_r(n)$.

Theorem 3 *If $A \in \mathrm{PLS}_r(n)$ with at most $n - 1$ non-empty cells, then A can be completed to an element of $\mathrm{LS}_r(n)$.*

The next two theorems are applications of Theorem 3 and special cases of Conjecture 1. By uniform we mean that each fixed $r \times r$ square is a filled A_{ij} for some $\{i, j\} \subseteq [n]$.

Theorem 4 *If $A \in \mathrm{PLS}(nr)$ is uniform with symbols appearing in at most $(n-1)$ $r \times r$ squares filled as latin squares, then A can be completed.*

If we remove the uniformity and latin conditions of Theorem 4 and insist that the fixed $r \times r$ squares are column-disjoint, it has been shown that some of these partial latin squares are completable [9].

Theorem 5 *Let $A \in \mathrm{PLS}(nr)$ with symbols appearing in at most $(n-1)$ $r \times r$ squares filled in distinct columns. There is a permutation of the symbols appearing in each row of each fixed $r \times r$ square such that the resulting element of $\mathrm{PLS}(nr)$ is completable.*

It is also known that Conjecture 1 is true for $n = 3$ [3] ($n = 1$ is trivially true and $n = 2$ is an application of Ryser's Theorem [10]). Furthermore, if we assume that n is odd and only the sub-arrays A_{ii} contain non-empty cells, then the following cases are known [8].

Theorem 6 *If $A \in \mathrm{PLS}(nr)$ with $\frac{n+1}{2}$ filled $r \times r$ squares along the main diagonal, then A can be completed.*

Theorem 7 *If $A \in \mathrm{PLS}(nr)$ with $n-1$ filled $r \times r$ squares along the main diagonal and $r \geqslant n-2$, then A can be completed.*

In Sect. 2 we introduce lifts and contractions to show that if symbols occur in at most $n-1$ disjoint sub-arrays A_{ij} and if the sub-arrays together form latin squares, then a completion exists. This result generalizes Theorem 3. In Sect. 3 we introduce permissible sets to show that when all of the filled cells occur in the block diagonal $\{A_{ii} \mid i \in [n]\}$ and the blocks are identical, then a completion exists when $n \geqslant r+1$. This confirms Conjecture 1 when the $n-1$ $r \times r$ squares are identical, and column and row disjoint.

2 Lifts and Contractions

Let a, r, and n be positive integers. A *balanced partition* of a set is a partition in which each part has the same cardinality. Let S be a symbol set of cardinality anr^2 and let $\mathscr{S} = \{S_1, S_2, \dots, S_{anr}\}$ be a balanced partition of S. Let $L \in \mathrm{PLS}_{ar^2}(n)$ over S and $A \in \mathrm{PLS}_a(nr)$ over $[anr]$. The array A is a *lift* of L (or L is a *contraction* of A) if for all $\{i, j\} \subseteq [n]$ and $\sigma \in [anr]$, $|L(i, j) \cap S_\sigma| = t$ if and only if symbol σ appears t times in the sub-array A_{ij}. Arrays A and L are called a *lift/contraction pair*. See Fig. 1. We use the word contraction because in the bipartite graph representation of A, L is obtained from A by a series of vertex contractions (or identifications), and substituting each occurrence of edge color σ for a color in S_σ. Lifts and contractions are similar to the disentanglements and amalgamations found in [7] respectively, with the primary difference being the symbol sets on which each are formed. The existence of lifts of r-semi latin squares for each integer r follows from a result of Hilton [7], which we discuss below.

Let $P = (p_1, \dots, p_u)$ and $Q = (q_1, \dots, q_v)$ be partitions of n. Let S be a symbol set of order n and $\mathscr{S} = \{S_1, S_2, \dots, S_w\}$ be a partition S. The *reduction modulo*

$1\,\overline{1}\,2\,3$	$4\,\overline{4}\,5\,6$	$\overline{2}\,\overline{3}\,\overline{5}\,\overline{6}$
$\overline{2}\,\overline{3}\,4\,\overline{5}$	$1\,2\,3\,\overline{6}$	$\overline{1}\,\overline{4}\,5\,6$
$\overline{4}\,5\,6\,\overline{6}$	$\overline{1}\,\overline{3}\,\overline{2}\,\overline{5}$	$1\,2\,3\,4$

L

1	2	4	5	3	6
3	1	6	4	2	5
2	3	1	6	5	4
5	4	2	3	6	1
4	6	5	2	1	3
6	5	3	1	4	2

A

Fig. 1 Members of $\mathrm{LS}_4(3)$ and $\mathrm{LS}(6)$ which are a lift/contraction pair, where $r = 4, n = 3$, and $S_\sigma = \{\sigma, \overline{\sigma}\}$ for $\sigma \in [6]$

$(P, Q, \mathscr{S})$ of a latin square $L \in \mathrm{LS}(n)$ over S is obtained from L by amalgamating rows $p_1 + \cdots + p_{i-1} + [p_i]$, columns $q_1 + \cdots + q_{j-1} + [q_j]$, and symbols of S_k, where $1 \leqslant i \leqslant u$, $1 \leqslant j \leqslant v$, and $1 \leqslant k \leqslant w$. Thus, the reduction modulo $(P, Q, \mathscr{S})$ is a $u \times v$ array over $[w]$ such that $k \in [w]$ appears t times in cell (i, j) if and only if symbols from the set S_k appear exactly t times in the corresponding sub-array of P. In Fig. 1, L (with bars removed) is a reduction modulo $(P, Q, \mathscr{S})$ of A in which $P = Q = (2, 2, 2)$ and $\mathscr{S}$ is the partition of $S = [6]$ into singletons.

Let C be a $u \times v$ array over $[w]$ in which symbols may be repeated in a cell. Let ρ_λ and c_μ denote the number of symbols which occur in row λ and column μ, respectively, and let τ_ν be the number of occurrences of $\nu \in [w]$ in C. Then C is called an *outline rectangle with parameter* n if for some integer n the following properties are met for each $\lambda \in [u]$, $\mu \in [v]$, and $\nu \in [w]$:

1. n divides each of ρ_λ, c_μ, and τ_ν;
2. cell (λ, μ) contains $\frac{\rho_\lambda c_\mu}{n^2}$ symbols (including repetitions); and
3. the number of times ν appears in row λ (column μ) is $\frac{\rho_\lambda \tau_\nu}{n^2}$ $(\frac{c_\mu \tau_\nu}{n^2})$;

Again, by removing the bars in Fig. 1, L is an outline rectangle with parameter 6, $\rho_\lambda = c_\mu = 12$ and $\tau_\nu = 6$ for each $\{\lambda, \mu\} \subseteq [3]$ and $\nu \in [6]$. Hilton proved the result below in [7].

Theorem 8 *To each outline rectangle C with parameter n, there is a latin square L of order n and partitions P, Q, and $\mathscr{S}$ such that C is the reduction of L modulo $(P, Q, \mathscr{S})$.*

The existence of lifts of semi-latin squares which are latin squares follows from Theorem 8. Let $L \in \mathrm{LS}_{r^2}(n)$ over S and $\mathscr{S} = \{S_1, \ldots, S_{nr}\}$ be a balanced partition of S. Construct L' from L by replacing each occurrence of a symbol in S_i with i for each $i \in [nr]$. Then L' is an outline rectangle with parameter nr. So there exists $M \in \mathrm{LS}(nr)$ for which L' is its reduction modulo $(P, Q, \mathscr{S})$, where $P = Q = (r, r, \ldots, r)$. It follows that M is a lift of L. Thus, we have part (a) of the following result when $a = 1$.

Theorem 9

(a) *For each $L \in \mathrm{LS}_{ar^2}(n)$, there exists $A \in \mathrm{LS}_a(nr)$ such that A is a lift of L.*

2	46			1	58
14	2			67	$3z$
$5z$	$1v$				
38	$5z$				
$9x$	78	$1z$	26	35	$4v$
67	39	48	$5v$	$2z$	$1x$

A

$1\,2\,\overline{2}\,4\,\overline{4}\,\overline{6}$		$\overline{1}\,\overline{3}\,\overline{5}\,6\,\overline{7}\,\overline{8}\,z$
$\overline{1}\,3\,5\,\overline{5}\,\overline{8}\,v\,z\,\overline{z}$		
$\overline{3}\,6\,7\,\overline{7}\,8\,9\,\overline{9}\,x$	$\overline{1}\,2\,4\,\overline{5}\,\overline{6}\,\overline{8}\,\overline{v}\,z$	$1\,\overline{2}\,3\,\overline{4}\,5\,v\,\overline{x}\,\overline{z}$

L

Fig. 2 Members of $\mathrm{PLS}_2(6)$ and $\mathrm{PLS}_8(3)$ which are a lift/contraction pair, where $a = 2$, $r = 2$, $n = 3$, and $S_\sigma = \{\sigma, \overline{\sigma}\}$ for $\sigma \in [9] \cup \{v, x, z\}$

(b) *For each* $A \in \mathrm{LS}_a(nr)$*, there exists* $L \in \mathrm{LS}_{ar^2}(n)$ *such that* L *is a contraction of* A*.*

The proof of Theorem 8 can be extended to the case $a > 1$ in Theorem 9(a). We prove Theorem 9(b) formally as a special case of a more general result since it is not readily implied from [7]. For a given $A \in \mathrm{PLS}_a(nr)$, we wish to find a contraction $L \in \mathrm{PLS}_{ar^2}(n)$ of A. See Fig. 2.

Lemma 1 *Every array of* $\mathrm{PLS}_a(nr)$ *is the lift of some array of* $\mathrm{PLS}_{ar^2}(n)$*.*

Proof Let $A \in \mathrm{PLS}_a(nr)$ over $[anr]$ and let $\mathscr{S} = \{S_1, \ldots, S_{anr}\}$ be a balanced partition of a symbol set S of cardinality anr^2. Define G as the bipartite graph with vertex partition $R = \{r_1, \ldots, r_n\}$ and $C = \{c_1, \ldots, c_n\}$, and $(r_i, c_j) \in E(G)$ with multiplicity matching the number of filled cells in A_{ij}. For each $\sigma \in [anr]$, let M_σ be the edge set such that for all $\{i, j\} \subseteq [n]$, $(r_i, c_j) \in M_\sigma$ with multiplicity t if and only if σ appears exactly t times in A_{ij}.

Let B_σ denote the bipartite graph induced on M_σ for each $\sigma \in [anr]$. See Fig. 3. Since $A \in \mathrm{PLS}_a(nr)$, the degree of each vertex in each B_σ is at most $|S_\sigma| = r$. Hence for each $\sigma \in [anr]$, there exists a proper edge-coloring $f_\sigma : M_\sigma \to S_\sigma$. Let L be an $n \times n$ array such that for each $\{i, j\} \subseteq [n]$,

$$L(i, j) = \{f_\sigma(e) \mid \sigma \in [nr],\ e \in M_\sigma, \quad \text{and} \quad e = (i, j)\}.$$

By this construction, $L \in \mathrm{PLS}_{ar^2}(n)$ is a contraction of A. □

Let $A \in \mathrm{PLS}(n)$ over S. We write A as a subset of $[n] \times [n] \times S$, where $(i, j, k) \in A$ if $k \in A(i, j)$. We prove the following case of Conjecture 1, which is a generalization of Theorem 4.

Theorem 10 *Let* $A \in \mathrm{PLS}(nr)$ *with symbols appearing in at most* $n - 1$ *sub-arrays* A_{ij}*. Furthermore, symbols are arranged such that if symbol* z *occurs in a cell of* A_{ij}*, then* z *occurs* r *times in the rows and columns occupied by* A_{ij}*. Then* A *can be completed.*

Proof Without loss of generality, we assume that A has $n-1$ disjoint $r \times r$ sub-arrays $B_1, \ldots, B_{n-1}$ which satisfy the hypotheses of Theorem 10. See Fig. 4a for an example.

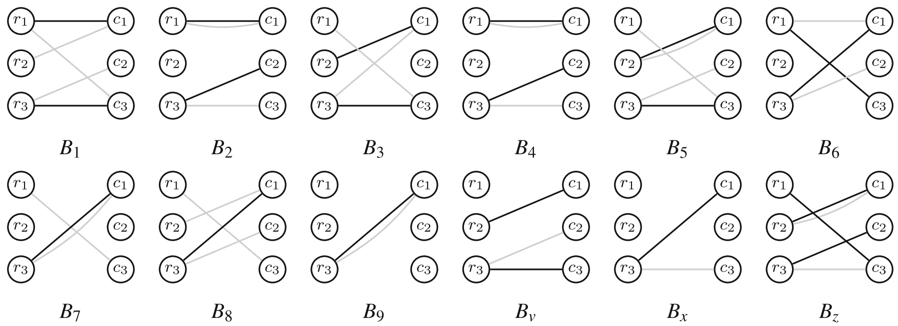


Fig. 3 The graphs of B_σ from Lemma 1 applied to A in Fig. 2. *Black* (*gray*) edges of B_σ correspond to $\sigma \in S_\sigma$ ($\overline{\sigma} \in S_\sigma$)

(a)

1	2	3	4						
3	4	1	2						
		4	3	2	1				
		2	1	4	3				
4	3			1	2				
2	1			3	4				
						1	2		
						2	1		
								1	2
								2	1

(b)

1	2	3	4						
2	3	4	1						
3	4	1	2						
4	1	2	3						
				5	6				
				6	5				
						7	8		
						8	7		
								9	10
								10	9

Fig. 4 **a** A completable PLS by Theorem 10 for which Ryser's Theorem does not apply [$r = 2$, $n = 9$, padded with 8 empty rows/columns]. **b** A completable PLS by Corollary 1 [$t_1 = 2$, $t_2 = 1$, padded by at least 6 empty rows/columns]

Let S be a symbol set of size nr^2 and let $\{S_1, \ldots, S_{nr}\}$ be a balanced partition of S. From Lemma 1, there is a contraction $L' \in \mathrm{PLS}_{r^2}(n)$ of A in which $|L'(i, j) \cap S_\sigma| = t$ if and only if symbol σ occurs t times in A_{ij}. Since there are exactly $n - 1$ non-empty cells in L', by Theorem 3, L' has a completion. Let L denote a completion of L'.

By Theorem 9(a), L can be lifted to a latin square A' of order nr. Let $B'_1, \ldots, B'_{n-1}$ denote the $r \times r$ sub-arrays of A' corresponding to $B_1, \ldots, B_{n-1}$ in A. Note that if $B_i = B'_i$ for each i, then A' is a completion of A. Let $A'' = A' \backslash \{B'_1, \ldots, B'_{n-1}\}$. If $(x, y, z) \in B_i$, then symbol z does not occur in the rows and columns of A'' occupied by B_i. Therefore $A'' \cup \{B_1, \ldots, B_{n-1}\}$ is a completion of A. □

We show a case of Theorem 10 not covered by Theorem 4. Let $L \in \mathrm{LS}(n)$. We say that $M \subset L$ is a *subsquare* of order q if $M \in \mathrm{LS}(q)$ after permuting rows, columns, and symbols of L. Two subsquares are *disjoint* if they have no rows, columns, or symbols in common.

1	2	4
2	5	6
4	3	5

A

1	2	4			
2	5	6			
4	3	5			
			1	2	4
			2	5	6
			4	3	5

$2A$

1	2	4	5	6	3
2	5	6	3	4	1
4	3	5	6	1	2
5	6	3	1	2	4
3	4	1	2	5	6
6	1	2	4	3	5

Completion of $2A$

Fig. 5 Arrays $A \in \mathrm{LS}(3; [6])$ and $2A \in \mathrm{PLS}(6)$, and a completion of $2A$

Corollary 1 *Let $t_1, \ldots, t_m$ be positive integers for which $n \geqslant t_1^2 + t_2^2 + \cdots + t_m^2 + 1$ and let $A \in \mathrm{PLS}(nr)$ be the union of m subsquares of orders $rt_1, rt_2, \ldots, rt_m$. Then A can be completed.*

Proof Each subsquare can be partitioned into t_i^2 filled $r \times r$ square sub-arrays, making A the union of at most $n-1$ filled $r \times r$ square subarrays, satisfying the hypotheses of Theorem 11. □

It is worth noting that Corollary 1 includes the case of m disjoint subsquares. This adds to the literature on the existence of latin squares with disjoint subsquares. See Chapter 4 of [2].

3 Identical Blocks

Let n and r be positive integers and Σ be a symbol set of cardinality nr. Let $\mathrm{LS}(r; \Sigma)$ denote the set of column- and row-latin arrays of order r in which each cell is filled with an element of Σ. Let $A \in \mathrm{LS}(r; \Sigma)$. Denote $nA \in \mathrm{PLS}(nr)$ as the partial latin square in which the sub-array at rows and columns $(i-1)r + [r]$ is A for each $i \in [n]$, and all other cells are empty. See Fig. 5. In this section we examine when nA can and cannot be completed. A completion clearly exists when $r = 1$, and so we assume that $r \geqslant 2$.

In the construction below we show that if $n \leqslant r - 1$, there exists $A \in \mathrm{LS}(r; \Sigma)$ for which nA is not completable. When then introduce some definitions and lemmas to show that if $n \geqslant r + 1$, then nA is always completable. Lastly, we comment on the case $n = r$.

Construction 1 Let $A' \in \mathrm{LS}(r-1)$ and define $A \in \mathrm{LS}(r; [2r-1])$ as

$$A = A' \cup \{(i, r, 2r-i), (r, i, 2r-i) \mid i \in [r]\}.$$

See Fig. 6. In nA, each symbol from $[r-1]$ occurs $n(r-1)$ times. Let B denote the $n \times n$ sub-array of nA induced by rows and columns $\{r, 2r, \ldots, nr\}$. To complete nA, each symbol from $[r-1]$ must occur n times in B. Observe that symbol r already occurs n times in B. Therefore, $n^2 \geqslant r^2$ and so $n \geqslant r$. Thus, if $n \leqslant r - 1$, nA can not be completed.

1	2	5
2	1	4
5	4	3

1	2	3	7
2	3	1	6
3	1	2	5
7	6	5	4

1	2	3	4	9
2	3	4	1	8
3	4	1	2	7
4	1	2	3	6
9	8	7	6	5

Fig. 6 Arrays A as constructed in Construction 1 with $r = 3, 4, 5$

3.1 Permissible Sets

Let $I \subseteq \Sigma$ with $|I| \leqslant r$. Define $A(I) \in \mathrm{PLS}(r)$ over I and $C(I) \subseteq [r] \times [r]$ as

$$\begin{aligned} A(I) &= \{(i, j, s) \in A \mid s \in I\} \quad \text{and} \\ C(I) &= \{(i, j) \in A \mid (i, j, s) \in A \quad \text{and} \quad s \in I\}. \end{aligned} \tag{1}$$

We say that $A(I)$ is the *partial latin square from* A *induced by* I, and $C(I)$ is the set of locations of the filled cells in $A(I)$. Furthermore, I is A-*permissible* (or simply *permissible*) if $A(I)$ is completable over some r-subset of Σ containing I. A partition $\mathscr{S}$ of Σ is A-*permissible* if each part of $\mathscr{S}$ is permissible. Note that a permissible set has cardinality at most r. For a symbol $\alpha \in \Sigma$, let σ_α denote the number of occurrences of α in A, and similarly let $\sigma(I)$ denote the number of occurrences of symbols from I in A.

Lemma 2 *Let* $A \in \mathrm{LS}(r; \Sigma)$ *and* $|\Sigma| = nr$. *If there exists a permissible, balanced partition* $\{S_1, S_2, \ldots, S_n\}$ *of* Σ, *then* nA *is completable over* Σ.

Proof For each $k \in [n]$, let B_k be the completion of $A(S_k)$. For each $k \in [n]$ and $l \in [r]$, define $B_{kl} \in \mathrm{PLS}(r)$ over S_k and $R_l \in \mathrm{LS}(r; \Sigma)$ as

$$B_{kl} = \{(i, j, s) \in B_k \mid (i, j) \in C(S_l)\} \quad \text{and} \quad R_l = \bigcup_{m=1}^{n} B_{mm'},$$

where $m + l = m' \bmod n$. Let B be the $nr \times nr$ array in which for every $\{i, j\} \subseteq [n]$, the sub-array at rows $(i-1)r + [r]$ and columns $(j-1)r + [r]$ is R_k, where $i + k = j \bmod n$. Observe that each R_n is a copy of A. Therefore $B \in \mathrm{LS}(nr)$ and B is a completion of nA. □

Example 1 Let $n = 2$, $r = 3$, and A be the 3×3 array over [6] given in Fig. 5. Let $S_1 = \{1, 2, 4\}$ and $S_2 = \{3, 5, 6\}$. Note $A(S_1)$ and $A(S_2)$ are both completable, say to B_1 and B_2. From B_1 and B_2 we derive four partial latin squares B_{11}, B_{12}, B_{21}, B_{22} and construct R_1 and R_2 (and $R_2 = A$). See Fig. 7. This gives the completion of $2A$ seen in Fig. 5.

Let $\Sigma_A \subseteq \Sigma$ be the set of symbols which occur in A.

1	2	4
2		
4		

$A(S_1) = B_{11}$

1	2	4
2	4	1
4	1	2

B_1

	4	1
	1	2

B_{12}

	5	6
	3	5

$A(S_2) = B_{22}$

5	6	3
3	5	6
6	3	5

B_2

5	6	3
3		
6		

B_{21}

5	6	3
3	4	1
6	1	2

R_1

Fig. 7 Arrays used in Example 1

Observation 1 Suppose Σ_A has a partition $\mathscr{S} = \{S_1, S_2, \ldots, S_t\}$ into A-permissible sets. We say that Σ_A has an *A-permissible partition with t parts*. If $n \geqslant t$, there exists a partition $\mathscr{S}' = \{S_1', S_2', \ldots, S_n'\}$ of Σ such that $|S_i| = r$ for each $i \in [n]$, and $S_i \subseteq S_i'$ for each $i \in [t]$. By its construction, any such partition $\mathscr{S}'$ is A-permissible. Therefore, nA is completable if Σ_A has a partition into at most n A-permissible sets.

Note that Σ_A, with A as defined in Construction 1, does have an A-permissible partition

$$\{\{x\} \mid x \in [r-1]\} \cup \{[2r-1]\backslash[r-1]\}$$

into r parts. So if $n \geqslant r$, then nA is completable. In Sect. 4 we conjecture that the condition $n \geqslant r$ is sufficient. In the meantime, we assume that $n \geqslant r+1$ and begin with observations and lemmas on completing partial latin squares.

3.2 Completion Lemmas and Observations

Unless otherwise specified, we assume that $r \geqslant 2$ and $n \geqslant r+1$.

Observation 2 (a) A partial latin square over one distinct symbol is completable.
(b) Let $P \in \mathrm{PLS}(r)$ over [2]. Suppose that $\sigma_1 = r-1$ and $\sigma_2 \neq r-1$. Suppose further that row $i \in [r]$ and column $j \in [r]$ do not contain symbol 1. Then P is completable if and only if $(i, j, 2) \notin P$.
(c) Let $P \in \mathrm{PLS}(r)$ over [2]. Suppose that $\sigma_1 \neq r-1$ and $\sigma_2 \neq r-1$. Then P is completable.

Lemma 3 *Let $A \in \mathrm{LS}(r; \Sigma)$ with $|\Sigma| = nr$ and $T = \{i \in \Sigma_A \mid \sigma_i = r-1\}$. Then either*

(a) *there exist $Y \subseteq \Sigma_A\backslash T$ and a bijection $\tau : Y \to T$ so that $\{y, \tau(y)\}$ is A-permissible for every $y \in Y$, or*
(b) *nA is completable.*

Proof Assume that (a) does not hold. We will show that (b) follows. Observe that $|T| \leqslant r+1$, and we first consider the case $|T| = r+1$. Then $\sigma(T) = r^2 - 1$, so there exists $\alpha \in \Sigma_A$ with $\sigma_\alpha = 1$, and $\Sigma_A = T \cup \{\alpha\}$. Let β be any symbol of T which shares a row or column with α. By Observations 2(a) and (b), $\{\alpha, \beta\}$ is permissible and

$$\{\{\alpha, \beta\}\} \cup \{\{x\} \mid x \in T \quad \text{and} \quad x \neq \beta\}$$

is an A-permissible partition of Σ_A with $r+1$ parts. So nA is completable by Observation 1.

Now suppose that $|T| \leqslant r$. Let $\mathscr{Y}$ be the set of subsets of $\Sigma_A \backslash T$ such that for each $Y' \in \mathscr{Y}$, there exists an injection $\tau' : Y' \to T$ such that $\{y, \tau'(y)\}$ is A-permissible for each $y \in Y'$. Let $\overline{Y} \in \mathscr{Y}$ be a set of maximal cardinality with injection $\overline{\tau}$. By our assumption, $\overline{\tau}$ is not a bijection. Let $x \in T \backslash \overline{\tau}(Y)$.

Without loss of generality, assume that x appears in the first $r-1$ rows and columns of A, and let $(r, r, \alpha) \in A$. Let $\beta \in \Sigma_A$ and $\beta \neq \alpha$. If $\beta \notin T$, then $\{x, \beta\}$ is A-permissible by Observation 2(b). Then $\{\beta\} \cup \overline{Y} \in \mathscr{Y}$ and hence $\beta \in \overline{Y}$ by the maximality of $\overline{Y}$. Therefore $\Sigma_A = T \cup \overline{Y} \cup \{\alpha\}$. If $\alpha \notin T \cup \overline{Y}$, then

$$\{\{\alpha\}\} \cup \{\{y, \overline{\tau}(y)\} \mid y \in \overline{Y}\} \cup \{\{x\} \mid x \in T \quad \text{and} \quad x \notin \overline{\tau}(\overline{Y})\}$$

is an A-permissible partition of Σ_A with at most $r+1$ parts. Otherwise $\alpha \in T \cup \overline{Y}$ and

$$\{\{y, \overline{\tau}(y)\} \mid y \in \overline{Y}\} \cup \{\{x\} \mid x \in T \quad \text{and} \quad x \notin \overline{\tau}(\overline{Y})\}$$

is an A-permissible partition of Σ_A with at most r parts. Thus nA is completable by Observation 1. □

Lemma 4 *Let* $I \subseteq \{i \in \Sigma_A \mid \sigma_i = 2 \textit{ or } 3\}$. *Then either*

(a) $\sigma(I) \leqslant r - 1$,
(b) *there exists* $I' \subseteq I$ *such that* $\sigma(I') \in \{r, r+1\}$, *or*
(c) *there exists* $I' \subseteq I$ *such that* $\sigma(I') = r + 2$ *and* $\sigma_\alpha = 3$ *for all* $\alpha \in I'$.

Proof Assume all three statements are false; that is assume that $\sigma(I) > r-1$, and for every $I' \subseteq I$, $\sigma(I') \notin \{r, r+1\}$ and either $\sigma(I') \neq r+2$ or $\sigma_\alpha = 2$ for some $\alpha \in I'$. Observe that this implies $\sigma(I) > r+1$. Let $J \subseteq I$ be a subset of minimum occurrence in A such that $\sigma(J) > r+1$.

Suppose that $\sigma(J) = r+2$. Then $\sigma_\alpha = 2$ for some $\alpha \in J$. Then $\sigma(J \backslash \{\alpha\}) = \sigma(J) - 2 = r$, which is a contradiction.

Suppose now that $\sigma(J) \geqslant r+3$. Let $\alpha \in J$ and $J' = J \backslash \{\alpha\}$. Then $\sigma(J') \geqslant r$. If $\sigma(J') \geqslant r+2$, then J is not minimal, which is a contradiction. So $\sigma(J') \in \{r, r+1\}$, which is also a contradiction. The result follows. □

Lemma 5 *Let* $I \subseteq \{\alpha \in \Sigma_A \mid \sigma_\alpha = 1\}$. *Then either* $|I| < 2r - 1$ *or there exists a permissible* r*-subset* $I' \subseteq I$.

Proof Suppose that $|I| \geqslant 2r-1$. Let I'' be any r-subset of I. If I'' is permissible, then we are done. If not, by Theorem 2, $A(I'')$ has a row R and column C such that their intersection is empty and all filled cells belong to R or C. Without loss of generality, let R contain at least 2 symbols α and β. There are $2r-2$ cells in the disjoint union of R and C, and hence there is some symbol $\gamma \in I$ which occurs outside of the disjoint union. Again, by Theorem 2, either $I'' \cup \{\gamma\} \backslash \{\alpha\}$ or $I'' \cup \{\gamma\} \backslash \{\beta\}$ is permissible. □

Let $P \in \mathrm{PLS}(r)$. Suppose that symbol $\alpha \in [r]$ occurs in P and that $\sigma_\alpha = k$. By permuting the rows and columns of P, we may assume that α occurs in the rows and columns indexed by $[k]$. We use G_α to denote the bipartite graph with vertex parts $R = C = \{k+1, \dots, r\}$ and edge set $\{(i, j) \mid P(i, j) = \emptyset\}$. If G_α contains a perfect matching M, we place symbol α in cell (x, y) of P if and only if $(x, y) \in M$. We say that symbol α can be *completed* in P. From Hall's Theorem, if each vertex in G_α has degree at least $\frac{r-k}{2}$, then M exists.

Lemma 6 *Let $t \leqslant r$ be a nonnegative integer and let $P \in \mathrm{PLS}(r)$ over $[t]$. Suppose that $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_t$.*

(a) *If $t \leqslant \frac{r-\sigma_1}{2} + 1$, then P is completable.*
(b) *If $r > 5$, $\sigma_1 = \sigma_t = 2$, and $t \leqslant \frac{r+1}{2}$, then P is completable.*
(c) *If $r > 6$, $\sigma_1 = 3$, $\sigma_2 = \sigma_t = 2$, and $t \leqslant \frac{r}{2}$, then P is completable.*

Proof Since symbols $t+1, \dots, n$ do not occur in P, we need only to show that symbols from $[t]$ can be completed. For each k, $0 \leqslant k \leqslant t$, we wish to find $P_k \in \mathrm{PLS}(r)$ such that $P \subseteq P_k$ and symbols from $[k]$ are completed in P_k. By convention, $P_0 = P$.

Suppose that $t \leqslant \frac{r-\sigma_1}{2} + 1$, $1 \leqslant k \leqslant t$, and P_{k-1} exists. We must show that P_k exists. The degree of each vertex of G_k is

$$d \geqslant r - \sigma_k - (t-1) \geqslant r - \sigma_k - \left(\tfrac{r-\sigma_1}{2}\right) \geqslant \frac{r-\sigma_k}{2}.$$

Thus G_k has a perfect matching and hence P_k exists.

Suppose that $r > 5$ and $\sigma_1 = \sigma_t = 2$. If $t < \frac{r+1}{2}$, then the result follows from (a). Therefore, we assume that $t = \frac{r+1}{2}$. The degree of each vertex of G_k is

$$d \geqslant r - 2 - (t-1) = r - 2 - \left(\frac{r-1}{2}\right) = \frac{r-3}{2}.$$

There is no perfect matching in G_k if there is a subset $S \subseteq R$ in which $|S| < |N(S)|$. Suppose that such a set S exists. Certainly $|N(S)| \geqslant \frac{r-3}{2}$. If $|S| > \frac{r-1}{2}$, there is a vertex $i \in C \backslash N(S)$ with $d(i) \leqslant |R \backslash S| < \frac{r-3}{2}$. Thus $|S| = |C \backslash N(S)| = t - 1 = \frac{r-1}{2}$ and $|N(S)| = \frac{r-3}{2}$.

Let M denote the sub-array of P_{k-1} at the intersection of S and $C \backslash N(S)$. Since there are no edges between S and $C \backslash N(S)$ in G_k, the cells in M are each filled. Since $t > 3$ and $(t-1)^2 \leqslant (k-1)(t-1) + 2(t-k)$ (the number of filled cells in M is at most the number of filled cells in the rows indexed by S), it follows that $k = t$.

Observe that M is a subsquare of P_{t-1} over $[t-1]$. Furthermore, the filled cells of P_{t-1} in rows from S or columns from $C \backslash N(S)$ are precisely M.

Let $\alpha \in [t-1]$. The *fixed occurrences* of α are the original two occurrences of α in P. Since $t > 3$, there are unfixed occurrences of α both in M and outside of S and $C \backslash N(S)$, say $(i_1, j_1, \alpha) \in M$ and $(i_2, j_2, \alpha) \in P_{t-1} \backslash M$, respectively.

Observe that cells (i_1, j_2) and (i_2, j_1) are empty in P_{t-1}. Thus

$$P' = (P_{t-1} \backslash \{(i_1, j_1, \alpha), (i_2, j_2, \alpha)\}) \cup \{(i_1, j_2, \alpha), (i_2, j_1, \alpha)\} \in \mathrm{PLS}(r)$$

has no subsquare of order $t-1$. Therefore the graph G_t formed from P' now contains a perfect matching. It follows that symbol t can be completed in P', and such a completion contains P.

Lastly, suppose that $r > 6$, $\sigma_1 = 3$, $\sigma_2 = \sigma_t = 2$, and $t \leqslant \frac{r}{2}$. If $t < \frac{r}{2}$, then the result follows from part (a). Therefore, we assume that $t = \frac{r}{2}$. We begin by first finding P_1. The degree of each vertex of G_1 is

$$d \geqslant r - 3 - (t-1) = r - 3 - \left(\frac{r-2}{2}\right) \geqslant \frac{r-4}{2}.$$

There is no perfect matching in G_1 if there is a subset $S \subseteq R$ in which $|S| = \frac{r-2}{2}$ and $|N(S)| = \frac{r-4}{2}$. This implies that there is a subsquare of order $\frac{r-2}{2}$ in P. However, since $\sum_{i=2}^{t} \sigma_i = r-2$, $\frac{(r-2)^2}{4} \leqslant r-2$ implies that $r \leqslant 6$. Thus, no such subsquare exists and there is a perfect matching in G_1, which implies that symbol 1 can be completed and therefore P_1 exists.

Consider now symbol $k > 1$. The degree of each vertex of G_k is

$$d \geqslant r - 2 - (t-1) = r - 2 - \left(\frac{r-2}{2}\right) = \frac{r-2}{2}.$$

Thus G_k has a perfect matching and thus P_k exists, where $2 \leqslant k \leqslant t$. □

Let $L \in \mathrm{LS}(2)$ and both $P \in \mathrm{PLS}(5)$ over $[5]$ and $Q \in \mathrm{PLS}(6)$ over $[6]$ be defined as

$$P = L \cup \{(3,3,3), (4,4,3)\} \quad \text{and}$$
$$Q = L \cup \{(3,3,3), (4,4,3), (5,5,3)\}.$$

Observe that P and Q are both incompletable and satisfy the conditions in Lemma 6(b) and (c), respectively, excluding the condition on r. Therefore the inequalities for r in the lemma are sharp.

Observation 3 Let $P \in \mathrm{PLS}(r)$ over $[t]$.

(a) Suppose $4 \leqslant \sigma_\alpha \leqslant \frac{r}{2}$ for each $\alpha \in [t]$ and $t \leqslant \lfloor \frac{r}{4} + 1 \rfloor$. Then $t \leqslant \frac{r - \frac{r}{2}}{2} + 1$, and hence by Lemma 6 (a), P is completable.
(b) Suppose $2 \leqslant \sigma_\alpha \leqslant 3$ for each $\alpha \in [t]$ and $t \leqslant \lfloor \frac{r-1}{2} \rfloor$. Suppose that at least two symbols have occurrence 3. By Lemma 6(a), P is completable.

(c) Suppose $2 \leqslant \sigma_\alpha \leqslant 3$ for each $\alpha \in [t]$ and $t \leqslant \lfloor \frac{r}{2} \rfloor$. Suppose there is exactly one symbol with occurrence 3. By Lemma 6(c), P is completable.
(d) Suppose $\sigma_\alpha = 2$ for each $\alpha \in [t]$ and $t \leqslant \lfloor \frac{r+1}{2} \rfloor$. By Lemma 6(b), P is completable.

3.3 Main Result

Let $r \geqslant 2, n \geqslant r+1$ and $A \in \mathrm{LS}(r; \Sigma)$, with $|\Sigma| = nr$. If $2 \leqslant r \leqslant 6$, it can be shown through tedious case-analysis that there exists an A-permissible partition of Σ_A into at most $r+1$ parts. So we proceed with $r \geqslant 7$.

Lemma 7 *If $r \geqslant 7$, $n \geqslant r+1$, and $A \in \mathrm{LS}(r; \Sigma)$ with $|\Sigma| = nr$, then nA is completable over Σ.*

Proof Let $T = \{\alpha \in \Sigma \mid \sigma_\alpha = r-1\}$. By Lemma 3, either nA is immediately completable, or there exist $Y \subseteq \Sigma_A \backslash T$ and τ for which $|T| = |Y|$ and $\{y, \tau(y)\}$ is permissible for each $y \in Y$. We assume the latter, and let $\mathscr{T} = \{\{y, \tau(y)\} \mid y \in Y\}$. We partition the remaining symbols in $\Sigma_A \backslash (T \cup Y)$ into the following sets:

$$\begin{aligned}
S &= \{\alpha \in \Sigma_A \backslash (T \cup Y) \mid \sigma_\alpha = r\},\\
Q &= \{\alpha \in \Sigma_A \backslash (T \cup Y) \mid \tfrac{r}{2} < \sigma_\alpha \leqslant r-2\},\\
P &= \{\alpha \in \Sigma_A \backslash (T \cup Y) \mid 4 \leqslant \sigma_\alpha \leqslant \tfrac{r}{2}\},\\
Z &= \{\alpha \in \Sigma_A \backslash (T \cup Y) \mid 2 \leqslant \sigma_\alpha \leqslant 3\}, \text{ and}\\
K &= \{\alpha \in \Sigma_A \backslash (T \cup Y) \mid \sigma_\alpha = 1\}.
\end{aligned}$$

We begin by partitioning each of the above sets into as many A-permissible sets as possible, each with occurrence at least r. We then combine any remaining symbols into A-permissible sets to achieve an A-permissible partition of Σ_A into at most $r+1$ parts.

Since $A(S)$ is completable, S is A-permissible. Let $\mathscr{S} = \{S\}$ if S is nonempty, and let $\mathscr{S}$ be empty otherwise. Thus (perhaps vacuously) every set in $\mathscr{S}$ has occurrence at least r.

Let $q = |Q| \bmod 2$. Let Q' be a q-subset of Q and $\mathscr{Q}$ a partition of $Q \backslash Q'$ into 2-subsets. By Observation 2(c), each part of $\mathscr{Q}$ is A-permissible with occurrence at least $r+1$. If $q = 1$, by Observation 2(a), Q' is A-permissible.

Let $p = |P| \bmod (\lfloor \frac{r}{4} + 1 \rfloor)$. Let P' be a p-subset of P and let $\mathscr{P}$ be a partition of $P \backslash P'$ into $(\lfloor \frac{r}{4} + 1 \rfloor)$-subsets. By Observation 3(a), each part of $\mathscr{P}$ is A-permissible with occurrence at least $r+1$, and by Theorem 1, P' is A-permissible.

By an inductive application of Lemma 4, there exists $Z' \subseteq Z$ and a partition $\mathscr{Z}$ of $Z \backslash Z'$ such that $\sigma(Z') \leqslant r-1$ and $\sigma(I) \in \{r, r+1, r+2\}$ for each $I \in \mathscr{Z}$, and either $\sigma(I) \in \{r, r+1\}$ or $\sigma(I) = r+2$ with $\sigma_\alpha = 3$ for every $\alpha \in I$. By Observations 3(b), (c), and (d), each $I \in \mathscr{Z}$ is A-permissible with occurrence at least $r+1$, and by Theorem 1, Z' is A-permissible.

By an inductive application of Lemma 5, there exists $K' \subseteq K$ and a partition $\mathscr{K}$ of $K \backslash K'$ such that each part of $\mathscr{K}$ is an A-permissible r-subset (and hence has occurrence r) and $|K'| < 2r-1$.

Define $\mathscr{A} = \mathscr{T} \cup \mathscr{S} \cup \mathscr{P} \cup \mathscr{Q} \cup \mathscr{Z} \cup \mathscr{K}$, which is a partition of $\Sigma_A \setminus (Q' \cup P' \cup Z' \cup K')$. Observe that $\sigma(I) \geqslant r$ for each $I \in \mathscr{A}$. Therefore $\sigma(Q' \cup P' \cup Z' \cup K') \leqslant r^2 - r|\mathscr{A}| = (r - |\mathscr{A}|)r$. We show that the remaining symbols in $Q' \cup P' \cup Z' \cup K'$ can be partitioned into A-permissible sets to produce an A-permissible partition of Σ_A into at most $r + 1$ parts.

Suppose that $|\mathscr{A}| \leqslant r - 4$. Partition K' into K_1' and K_2' so that $|K_i'| \leqslant r - 1$ for $i \in [2]$. By Theorem 1, K_i' is A-permissible for $i \in [2]$. Therefore $\mathscr{A} \cup \{Q', P', Z', K_1', K_2'\}$ is an A-permissible partition with at most $r + 1$ parts.

Suppose that $|\mathscr{A}| = r - 3$. Then $\sigma(Q' \cup P' \cup Z' \cup K') \leqslant 3r$. Partition K' into $K_{Q'}$, $K_{P'}$, $K_{Z'}$ and K'' so that $|K_I| \leqslant r - 1 - \sigma(I)$ for each $I \in \{Q', P', Z'\}$ and $|K''| \leqslant 3 \leqslant r - 1$. Define $Q'' = Q' \cup K_{Q'}$, $P'' = P' \cup K_{P'}$, and $Z'' = Z' \cup K_{Z'}$. By Theorem 1, Q'', P'', Z'', and K'' are each A-permissible. Hence $\mathscr{A} \cup \{Q'', P'', Z'', K''\}$ is an A-permissible partition with $r + 1$ parts.

Suppose that $|\mathscr{A}| = r - 2$. Then $\sigma(Q' \cup P' \cup Z' \cup K') \leqslant 2r$. Partition K' into $K_{Q'}$, $K_{P'}$, and $K_{Z'}$ with $|K_I| \leqslant r - 1 - \sigma(I)$ for each $I \in \{Q', P', Z'\}$, which is possible since $r \geqslant 3$. Define Q'', P'', and Z'' as before, and again by Theorem 1, each are A-permissible. Hence $\mathscr{A} \cup \{Q'', P'', Z''\}$ is an A-permissible partition with $r + 1$ parts.

Suppose that $|\mathscr{A}| = r - 1$. Then $\sigma(Q' \cup P' \cup Z' \cup K') \leqslant r$. If $Q' \cup P' \cup Z' \cup K'$ is empty, then $\mathscr{A}$ is an A-permissible partition of Σ_A with $r - 1$ parts. Otherwise, let $\alpha \in Q' \cup P' \cup Z' \cup K'$. Then by Observation 2(a) and Theorem 1, $\{\alpha\}$ and $(Q' \cup P' \cup Z' \cup K') \setminus \{\alpha\}$ are both A-permissible. Hence $\mathscr{A} \cup \{\{\alpha\}, (Q' \cup P' \cup Z' \cup K') \setminus \{\alpha\}\}$ is an A-permissible partition with $r + 1$ parts.

Suppose that $|\mathscr{A}| \geqslant r$. Since $\sigma(I) \geqslant r$ for each $I \in \mathscr{A}$, it follows that $|\mathscr{A}| = r$ and $\sigma(I) = r$ for each $I \in \mathscr{A}$. Therefore $\mathscr{A}$ is a partition of Σ_A which is A-permissible. $\square$

We summarize these results with the theorem below.

Theorem 11 *Let n and r be positive. If $n \geqslant r + 1$, then for every $A \in \mathrm{LS}(r; [nr])$, nA is completable. If $n \leqslant r - 1$, then there exists $A \in \mathrm{LS}(r; [nr])$ for which nA is not completable.*

4 Concluding Remarks

Theorem 11 does not include the case when $n = r$. For every example of A we investigated, there did exist an A-permissible partition of Σ_A into at most r parts. However, there are several special cases that arise when forming the partition, some of which seem difficult to reconcile with larger values of r. This leads us to the following conjecture.

Conjecture 2 *Let r and n be positive integers. Then nA is completable for every $A \in \mathrm{LS}(r; [nr])$ if and only if $n \geqslant r$.*

We end with an example illustrating that the converse of Lemma 2 does not hold—that there exists an array $A \in \mathrm{LS}(r; [nr])$ for which nA is completable and $[nr]$ has no A-permissible partition.

1	4	6	3
5	2	4	1
4	5	3	2
3	1	2	4

A

1			
	2		1
			2
	1	2	

B_1

1			3
			1
		3	
3	1		

B_2

			3
	2		
		3	2
3		2	

B_3

1	4	6	3	2	7	8	5
5	2	4	1	8	3	7	6
4	5	3	2	7	6	1	8
3	1	2	4	6	8	5	7
2	7	8	5	1	4	6	3
8	3	7	6	5	2	4	1
7	6	1	8	4	5	3	2
6	8	5	7	3	1	2	4

A completion of $2A$

Fig. 8 The arrays from Example 2

1			3
	2		1
		3	2
3	1	2	

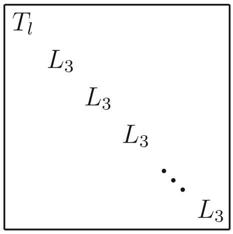
T_4

1				3
	2		1	
		3		2
		2	3	1
3	1		2	

T_5

1			3		
	2			1	
		3			2
3			1	2	
	1		2		3
		2		3	1

T_6



L

Fig. 9 Arrays for Construction 2

12			3
	23		1
		13	2
3	1	2	

M

12	47	68	35
58	23	47	16
47	56	13	28
36	18	25	47

M'

1	4	6	3
5	2	4	1
4	5	3	2
3	1	2	4

A

2	7	8	5
8	3	7	6
7	6	1	8
6	8	5	7

B

Fig. 10 The arrays from Construction 2 applied to the example in Fig. 6

Example 2 Let $A \in \mathrm{LS}(4; [8])$ be the array in Fig. 8. In a balanced partition $\{S_1, S_2\}$ of [8], at least two elements of $\{1, 2, 3\}$ belong to the same part, say S_1. Hence at least one of the arrays B_1, B_2, and B_3 is contained in $A(S_1)$. Observe that B_1, B_2, and B_3 are not completable. Therefore no balanced bipartition of [8] will induce completable partial latin squares. However, $2A$ does have a completion. See Fig. 8.

We generalize the example above to an infinite family of completable partial latin squares of the form $2A$ in which there is no permissible bipartition of $[2r]$.

Construction 2 Let $r \geqslant 4$ be given, and let $r = 3k + l$, where $l \in \{4, 5, 6\}$. Let $L_3 \in \mathrm{LS}(3)$ over [3]. Define T_l and $L \in \mathrm{PLS}(r)$ over $[r]$ as given in Fig. 9. Observe that for all $A \in \mathrm{LS}(r; [2r])$, if $L \subseteq A$, then by a similar argument used in Example 2, no balanced bipartition of $[2r]$ will induce completable partial latin squares from A.

Let $M \in \mathrm{PLS}_2(r)$ over $[2r]$ be $M = L \cup \{(1, 1, 2), (2, 2, 3), (3, 3, 1)\}$. Observe that symbols 1, 2, and 3 each occur r times in M. Using Hall's Theorem for bipartite multi-graphs [6], M can be completed to a 2-semi latin square $M' \in \mathrm{LS}_2(r)$. Let $A, B \in \mathrm{LS}(r; [2r])$ such that $A \cup B = M'$ and $L \subseteq A$. Observe that $2A$ has a completion (put copies of B in the off-diagonal squares), but A does not satisfy the hypotheses of Lemma 2 (Fig. 10).

References

1. Anderson, L.D., Hilton, A.J.W.: Thanks Evans!. Proc. Lond. Math. Soc. **47**, 507–522 (1983)
2. Dénes, J., Keedwell, A.: Latin Squares, Annals of Discrete Mathematics. New developments in the theory and applications, vol. 46. With contributions by G.B. Belyavskaya, A.E. Brouwer, T. Evans, K. Heinrich, C.C. Lindner, and D.A. Preece. With a foreword by Paul Erdős. North-Holland Publishing Co., Amsterdam (1991)
3. Denley, T., Häggkvist, R.: Completing some partial Latin squares. Eur. J. Comb. **21**, 877–880 (1995)
4. Evans, T.: Embedding incomplete Latin squares. Am. Math. Mon. **67**, 958–961 (1960)
5. Häggkvist, R.: A solution to the Evans conjecture for Latin squares of large size. Colloqu. Math. Soc. Janos Bolyai **18**, 405–513 (1978)
6. Hall, P.: On representatives of subsets. J. Lond. Math. Soc. **10**, 26–30 (1935)
7. Hilton, A.J.W.: The reconstruction of latin squares with applications to school timetables and experimental design. Math. Progr. Study **13**, 68–77 (1980)
8. Öhman, L.-D.: A note on completing Latin squares, Australas. J. Comb. **45**, 117–123 (2009)
9. Kuhl, J., Denley, T.: On a generalization of the Evans conjecture. Discret. Math. **308**, 4763–4767 (2008)
10. Ryser, H.J.: A combinatorial theorem with an application to Latin squares. Proc. Am. Math. Soc. **2**, 550–552 (1951)
11. Smetaniuk, B.: A new construction for latin squares I. Proof of the Evans conjecture. Ars Comb. **11**, 155–172 (1981)