THANK EVANS!

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ABSTRACT

In this paper we give a complete solution to the conjecture of Evans made in 1960 that if $n-1$ cells of an $n \times n$ matrix are preassigned with no element repeated in any row or column then the remaining n^2-n+1 cells can be filled so as to produce a latin square. We in fact prove the stronger statement that n cells can be preassigned except in certain cases which we specify.

1. *Introduction*

A *partial latin square* of side n on the symbols $\sigma_1, \ldots, \sigma_n$ is an $n \times n$ matrix of cells in which each cell either remains empty or contains one of the symbols $\sigma_1, \ldots, \sigma_n$, and, furthermore, no symbol occurs twice in any row or twice in any column. It is a (complete) latin square if there are no empty cells.

In 1960 Evans [7] conjectured that if a partial latin square of side n had $n-1$ cells occupied then the remaining cells could be filled in so as to produce a latin square.

The number $n-1$ is best possible, for if x ($\geqslant 1$) cells of some row and $(n-x)$ ($\geqslant 1$) cells of some column are filled with distinct symbols, the cell common to the row and column being avoided, as in Fig. 1, then there is no symbol which can be placed in the common cell. Similarly if x ($\geqslant 1$) cells of some row (column) are filled with distinct symbols, and a further symbol, which does not occur in that row (column), occurs in each of the columns (rows) through the remaining $(n-x)$ ($\geqslant 1$) cells of the row (column), as in Fig. 1, then this further symbol cannot be placed anywhere in the row. If a partial latin square of side n with at most n non-empty cells is one of the types just described we call it *bad*; otherwise we call it *good.*

The three partial latin squares of Fig. 1 illustrate the well-known concept of *conjugate* partial latin squares. If P is a partial latin square of side n, we may form a partial latin square P' of side n by the following rule: whenever cell (i, j) of P contains symbol σ_k, place symbol σ_j in cell (i, k) of P', and leave all other cells of P' empty. Then P' is one type of *conjugate* of P; we say it is obtained from P by *interchanging columns and symbols.* In the same way, other partial latin squares can be formed from P by other permutations of rows, columns, and symbols; we call them all conjugates of P (so one conjugate of P is simply its transpose).

Obviously, a partial latin square of side n can be completed to a latin square of side n if and only if all its conjugates can.

In this light it can be said that there is only one type of bad square, because the three squares in Fig. 1 are conjugates of each other. Here Type 1 is obtained from Type 2 by interchanging columns and symbols, and Type 3 is obtained from Type 1 by the permutation (rows, columns, symbols) $\rightarrow$ (columns, symbols, rows). The remaining permutations give squares of the same three types.

We shall frequently use the idea of conjugate partial latin squares in the following.

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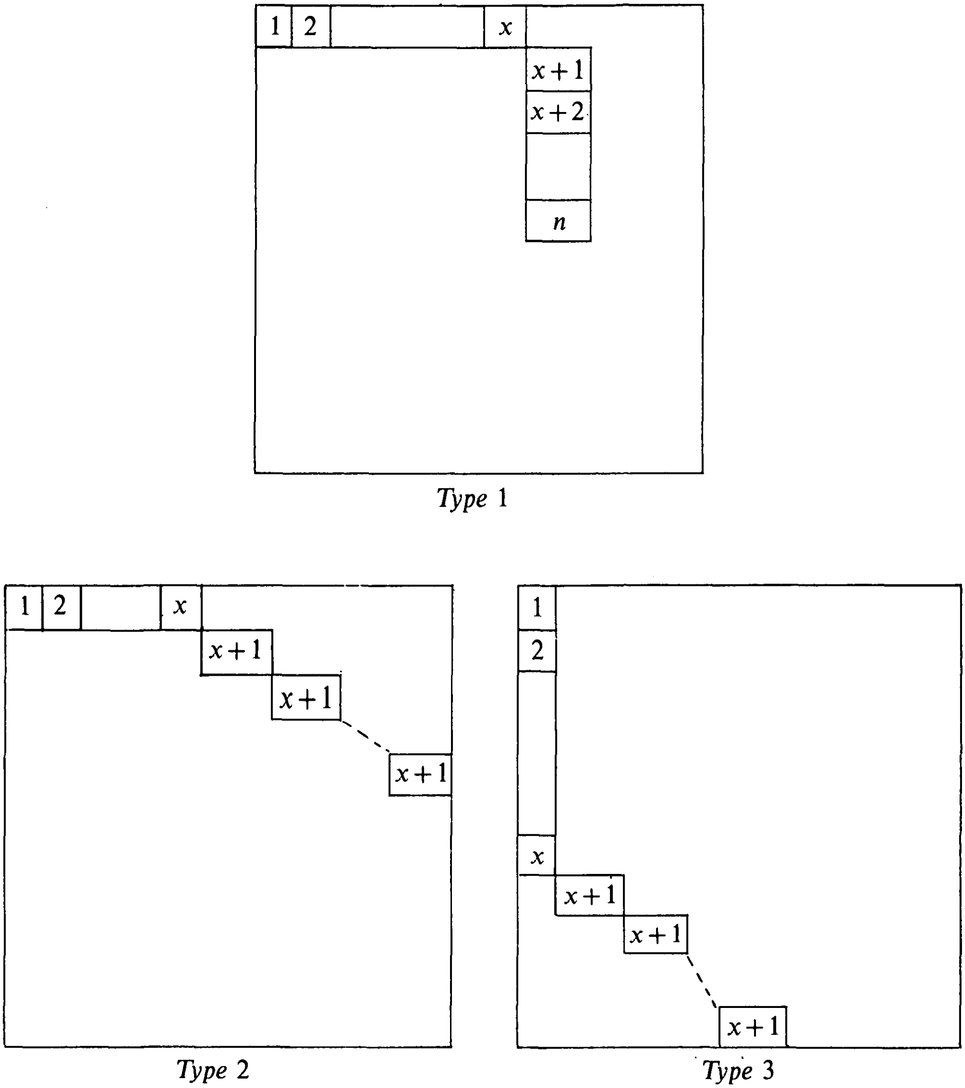

FIG. 1. Three ways of filling n cells so that the remaining cells cannot be filled so as to produce a latin square.

We in fact prove a stronger version of Evans's conjecture to the effect that, apart from the three types of partial latin square of Fig. 1, if n cells are preassigned then P can be completed.

MAIN THEOREM. *Let $n \geqslant 1$. Let a good partial latin square P of side n have at most n cells filled, the remainder being empty. Then P can be completed to form a latin square of side n.*

In fact Klarner told the second author in 1970 that he conjectured this was true; he independently posed the same conjecture as Evans in [**11**].

When we started work on this a lot of progress had already been made towards proving Evans's conjecture. First Marica and Schonheim [**14**] showed that the conjecture was true if all the preassigned cells were on the diagonal; later Chang [**5**] proved the Main Theorem in the case when all preassigned cells are on the diagonal.

Lindner [13] proved the conjecture in the case when n is even and the preassigned cells all lie in the top $\frac{1}{2}n$ rows. This was improved to $\frac{1}{2}(n+3)$ for any n by Crittenden and Eynden [6] and then to $\frac{1}{2}(n+5)$ by Wells [18], who used it in showing that the Evans conjecture is true for $n \leqslant 11$. Brandt [3] improved Lindner's result still further by showing that a partial latin square with $n-1$ non-empty cells all occurring in some $\frac{21}{20}\lfloor\frac{1}{2}n\rceil$ ($\lfloor x\rfloor$ and $\lceil x\rceil$ will denote greatest integer not greater than x and the least integer not less than x respectively) rows can be completed to form a latin square.

Finally, we should note that Häggkvist, in [8], improved all these results considerably for most values of n to $n-2\sqrt{n-1}$ rows, and then used this to obtain a proof of Evans's conjecture for all $n \geqslant 1111$.

In 1979 the result of this paper was announced and an outline of a proof was given at the 7th British Combinatorics Conference at Cambridge. We also learned then that Häggkvist had a complete proof of Evans's conjecture. In 1980 we learned that Smetaniuk had a proof of it, now published [16]. The proof we obtained in 1979 was circulated widely in typescript form, but we have since discovered an error in it—however, it is valid for some values of n, including $n = c2^r$, with $1 \leqslant c \leqslant 7$ and $r \geqslant 1$. The proof here of the strengthened conjecture is essentially different from our original one. Recently Damerell announced a proof of the strengthened conjecture, based on Smetaniuk's argument.

The paper is arranged in the following sections:

1. Introduction
2. A refinement of Lindner's theorem
3. Proof of the strengthened conjecture in some special cases (including the case when all symbols are distinct)
4. The induction step

Finally, we state precisely some known results which we use.

The first is P. Hall's theorem in the following formulation:

THEOREM 1.1 (P. Hall [10]). *A bipartite graph G with vertex classes A and B has a set of $|A|$ independent edges if and only if for every subset S of A,*

$$|S| \leqslant |N(S)|,$$

where $N(S)$ is the set of neighbours of vertices in S.

An $r \times s$ latin rectangle R on the symbols $\sigma_1, \ldots, \sigma_n$ is an $r \times s$ matrix of cells such that each cell contains one symbol and each symbol occurs at most once in each row and at most once in each column. Let $N_R(\sigma_i)$ be the number of times σ_i occurs in R.

THEOREM 1.2 (Ryser [15]). *An $r \times s$ latin rectangle R on the symbols $\sigma_1, \ldots, \sigma_n$ may be completed to form a latin rectangle of side n if and only if*

$$N_R(\sigma_i) \geqslant r+s-n \quad (1 \leqslant i \leqslant n).$$

This theorem has the following corollary.

THEOREM 1.3 (M. Hall [9]). *A partial latin square such that all cells in the first r rows are occupied, all other cells being unoccupied, can be completed to form a latin square.*

If G is a finite graph without loops, let $\Delta(G)$ denote the maximum degree of G and let $X'(G)$ denote the chromatic index (or edge-chromatic number) of G, that is the smallest integer j so that the edges of G can be coloured with j colours in such a way that no two edges which are incident with the same vertex receive the same colour.

THEOREM 1.4 (König [12]). *If G is a bipartite graph then*

$$X'(G) = \Delta(G).$$

LEMMA 1.5 (Häggkvist [8]). *Let G be a regular bipartite graph of degree m with $2n$ vertices. Let B_1 be a set of b_1 independent edges of G, and let B_2 be a set of b_2 edges of G disjoint from B_1. If $m-b_1 \geqslant \frac{1}{2}(n-1)$ and $b_2 \leqslant m-b_1-1$, then G contains a 1-factor F such that $B_1 \leqslant F$ and $F \cap B_2 = \varnothing$.*

Finally, we shall use the following simple graph-theoretic result which we state without proof (it can be proved easily using Theorem 1.1).

LEMMA 1.6. *Let G be a $K_{r,r}$ with r edges deleted ($r \geqslant 1$). Then G has a 1-factor unless the deleted edges are all incident with the same vertex.*

2. *A refinement of Lindner's theorem*

If S is a row, column, or other submatrix of a partial latin square, we define the weight $|S|$ of S to be the number of preassigned cells in S. We call S *empty* if it has no preassigned cells, and *full* if it has no empty cells. Further, if σ is any symbol, we let $S(\sigma)$ denote the set of cells from S containing the symbol σ.

THEOREM 2.1. *Let P be a good partial latin square of side n with n cells occupied, let the weight of row i of P be r_i and assume that $r_1 \geqslant r_2 \geqslant \dots \geqslant r_n = 0$. Then the first $\lfloor \frac{1}{2}(n+1) \rfloor$ rows of P can be completed.*

Proof. We identify P with a bipartite graph $K_{n,n}$, where one class of vertices corresponds to the symbols of P, the other to the columns of P: let

$$B_i = \{e \in E(K_{n,n}) \mid e = \{\sigma_k, c_l\} \text{ and } \sigma_k \text{ is preassigned in cell } (i,l) \text{ of } P\};$$

then B_i consists of independent edges, and extending B_i to a 1-factor of $K_{n,n}$ corresponds to assigning symbols to each cell of row i of P. Note that $|B_i| = r_i$.

First, let G be obtained from $K_{n,n}$ by deleting all end-vertices of edges of B_1. Then $G = K_{n-r_1,n-r_1}$. Put $B = (B_2 \cup B_3 \cup \dots \cup B_n) \cap E(G)$. Now, $|B| \leqslant n-r_1$, and since P is good, B does not consist of $n-r_1$ edges incident with the same vertex of G. By Lemma 1.6, G contains a 1-factor F' not intersecting B. Then $F_1 = B_1 \cup F'$ is a 1-factor of $K_{n,n}$ extending B_1.

Now suppose that we have a sequence of graphs $G_0, G_1, \dots, G_p$, where $G_0 = K_{n,n}$ and, for $r \geqslant 1$, $G_r = G_{r-1} - F_r$ and F_r is a 1-factor of G_{r-1} containing B_r and disjoint from $B_{r+1}, \dots, B_n$. By the first part of the argument, such a sequence exists with $p = 1$. Let $p < \lfloor \frac{1}{2}(n+1) \rfloor$. We want to extend the sequence by finding a 1-factor F_{p+1} of G_p containing B_{p+1} and disjoint from $B_{p+2}, \dots, B_n$.

We apply Lemma 1.5 with $G = G_p$, $b_1 = r_{p+1}$, $b_2 = \sum_{i=p+2}^{n} r_i$, and $m = n-p$. Since $r_n = 0$ we have that $r_1 \geqslant 2$ and so

$$b_1 + b_2 = \sum_{i=p+1}^{n} r_i \leqslant n-(p+1) = m-1,$$

and hence the inequality $b_2 \leqslant m-b_1-1$ of Lemma 1.5 is satisfied. We consider the other inequality of the lemma, that $m-b_1 \geqslant \frac{1}{2}(n-1)$, which in our case becomes

$$r_{p+1} \leqslant \tfrac{1}{2}(n+1)-p.$$

Since $p < \lfloor\frac{1}{2}(n+1)\rfloor$ this is true if $r_{p+1} \leqslant 1$. Assume that $r_{p+1} \geqslant 2$. Then

$$r_{p+1} \leqslant \tfrac{1}{2}(r_p+r_{p+1}) \leqslant \tfrac{1}{2}\left(n-\sum_{i=1}^{p-1} r_i\right) \leqslant \tfrac{1}{2}(n-2(p-1)) = \tfrac{1}{2}(n+2)-p,$$

with strict inequality unless one of the following holds:

I. $p = 1$ and $r_1 = r_2 = \frac{1}{2}n$;

II. $\sum_{i=1}^{p+1} r_i = n$ and $r_1 = r_2 = \dots = r_{p+1} = 2$ (so $p = \frac{1}{2}n-1$).

So, by Lemma 1.5 we can find the required 1-factor except in these two cases. In either case $B_{p+2} = \dots = B_n = \varnothing$, so we only have to prove that B_{p+1} can be extended to a 1-factor. We treat the two cases separately.

Case I. Here G_p is just $K_{n,n}$ with a 1-factor deleted. Let G' be obtained from G_p by deleting all end-vertices of edges of B_{p+1}. We must show that G' has a 1-factor; but G' is just $K_{\frac{1}{2}n,\frac{1}{2}n}$ with some or all edges of a 1-factor deleted, so this follows from Lemma 1.6 (since $\frac{1}{2}n > 1$, because $p < \lfloor\frac{1}{2}(n+1)\rfloor$).

Case II. In this case G_p is regular of degree $\frac{1}{2}n+1$. Let G'' be obtained from G_p by deleting the two independent edges of B_{p+1}; we must show that G'' has a 1-factor. We apply Theorem 1.1 (Hall's theorem). Let S be a non-empty set of vertices from one class of vertices of G''. We must prove that $|S| \leqslant |N(S)|$. Since each vertex of S has degree at least $(\frac{1}{2}n+1)-1$, this is obvious if $|S| \leqslant \frac{1}{2}n$. Let $|S| > \frac{1}{2}n$. Then each vertex from the other class is joined to a vertex of S (it has degree at least $\frac{1}{2}n$), and so $|N(S)| = n \geqslant |S|$.

Thus Theorem 2.1 is proved.

From this we deduce a strengthening of Lindner's theorem:

COROLLARY 2.2. *For $n \geqslant 1$, let P be a good partial latin square of side n with n cells occupied, and let all of the occupied cells which are not in the final column lie in the top $\lfloor\frac{1}{2}(n+1)\rfloor$ rows. Then P can be completed to form a latin square of side n.*

Proof. By Theorem 2.1, the first $\lfloor\frac{1}{2}(n+1\rfloor$ rows can be completed (if P contains a preassigned symbol in each row, we argue on P^{T}). By Theorem 1.3, it is possible to add to these $n-\lfloor\frac{1}{2}(n+1)\rfloor$ full rows so as to produce a latin square. By ordering these rows in accordance with the preassigned cells in the first column we obtain a completion of P.

COROLLARY 2.3. *For $n \geqslant 1$, let P be a good partial latin square of side n with n cells occupied, and let all the occupied cells be in the top $\lceil\frac{1}{2}(n+2)\rceil$ rows. Then P can be completed to form a latin square of side n.*

3. *Proof of the strengthened conjecture in some special cases (including the case when all symbols are distinct)*

An $r \times s$ matrix in which the only cells which are occupied are in the set $\{(i,i)\colon 1 \leqslant i \leqslant \min(r,s)\}$ is called *diagonal*.

LEMMA 3.1. *Let T be a partial $n \times n$ latin square on the symbols $\sigma_1, \ldots, \sigma_n$ of the form*

$$T = \begin{pmatrix} R & A \\ B & D \end{pmatrix},$$

where R is $r \times s$, the first s columns of T are full, and D is diagonal. For $1 \leqslant i \leqslant n$, let $f(\sigma_i)$ be the number of times σ_i occurs in D. Then

$$N_R(\sigma_i) \geqslant r+s-n+f(\sigma_i) \quad (1 \leqslant i \leqslant n).$$

Proof. A symbol σ_i occurs exactly s times in the first s columns, and it can occur at most $n-r-f(\sigma_i)$ times in the last $n-r$ rows of these columns, so it must occur at least $s-(n-r-f(\sigma_i))$ times in R.

Theorem 3.2 is concerned with the case when R is full and A and B are both empty.

THEOREM 3.2. *Let $n \geqslant r \geqslant 0$ and $n \geqslant s \geqslant 0$. Let R be an $r \times s$ latin rectangle and let D be an $(n-r) \times (n-s)$ diagonal matrix, both on the symbols $\sigma_1, \ldots, \sigma_n$. Let $T = \begin{pmatrix} R & 0 \\ 0 & D \end{pmatrix}$ and let D have an empty row, or an empty column, or both. For $1 \leqslant i \leqslant n$, let $f(\sigma_i)$ be the number of times σ_i occurs in D. Then T can be completed if and only if*

$$N_R(\sigma_i) \geqslant r+s-n+f(\sigma_i) \quad (1 \leqslant i \leqslant n). \tag{1}$$

If $r = s$ and D does not have an empty row or column, then (1) is necessary but no longer sufficient, and it is an interesting and seemingly very difficult problem to find necessary and sufficient conditions. There is a long section on this problem in [**1**].

Proof of Theorem 3.2. The necessity follows from Lemma 3.1. We prove the sufficiency.

If $r = s$ then the result is proved by Andersen, Häggkvist, Hilton, and Poucher in [**2**]. Suppose now that $r \neq s$. By symmetry we may assume that $r > s$. If $r = n$ then we merely have to fill the last $n-s$ columns, which is clearly possible by M. Hall's theorem (Theorem 1.3). If $r = n-1$ then each symbol satisfies Ryser's condition (Theorem 1.2) so R can be completed. If $\sum_{i=1}^{n} f(\sigma_i) < n-r = 1$ then there is no further condition on any symbol. If $\sum_{i=1}^{n} f(\sigma_i) = n-r = 1$ then there is a particular symbol, say σ_0, which is required to be placed in cell $(n, s+1)$. But for σ_0 the condition of the theorem reduces to the fact that σ_0 occurs at least s times in R, and this implies that in any latin square which embeds R, σ_0 occurs in one of the cells $(n, s+1)$, $(n, s+2), \ldots, (n, n)$. We can therefore require that σ_0 occurs in cell $(n, s+1)$. Since $s < r$, from now on, therefore, we may assume that $n-s > n-r \geqslant 2$. We also assume that $\sum_{i=1}^{n} f(\sigma_i) > 1$, for otherwise the result follows from Ryser's theorem (Theorem 1.2).

We show that we can extend R to an $(r+1) \times (s+1)$ rectangle R' and modify f to get a function f' such that the corresponding embedding of R' gives the required embedding of R and such that R' and f' satisfy the assumption of the theorem with $r+1$ and $s+1$ instead of r and s. We call a symbol σ *marginal* if $N_R(\sigma) = r+s-n+f(\sigma)$ and *nearly marginal* if $N_R(\sigma) = r+s-n+f(\sigma)+1$. Then each marginal symbol must occur twice in the extra row and column of R' (unless it occurs in cell $(r+1, s+1)$), and each nearly marginal symbol must occur at least once.

Form a bipartite graph M_ρ with vertex classes $\{\rho_1, \ldots, \rho_r\}$ and $\{\sigma'_1, \ldots, \sigma'_n\}$ with one edge joining ρ_k and σ'_l if and only if σ_l does not occur in the kth row of R. Each ρ_k has degree $n-s$, and each σ_l' has degree at most $r-(r+s-n+f(\sigma_l)) = n-s-f(\sigma_l)$. Now let M_ρ^d be obtained from M_ρ by adding a vertex ρ_d joined to each σ_l' by exactly $f(\sigma_l)$ edges. Then M_ρ^d is bipartite with maximum degree $n-s$, and this maximum is attained at all vertices σ_l' corresponding to marginal symbols σ_l.

Next form a bipartite graph M_c with vertex classes $\{c_1, \ldots, c_s\}$ and $\{\sigma_1'', \ldots, \sigma_n''\}$ with one edge joining c_k and σ_l' if and only if σ_l does not occur in the kth column of R. Each c_k has degree $n-r$ and each σ_l'' has degree at most $s-(r+s-n+f(\sigma_l)) = n-r-f(\sigma_l)$. Now let M_c^d be obtained from M_c by adding a vertex c_d joined to each σ_l'' by exactly $f(\sigma_l)$ edges. Then M_c^d is bipartite with maximum degree $n-r$, and this maximum is attained at all vertices σ_l'' corresponding to marginal symbols σ_l.

Finally, let the bipartite graph G' be obtained from $M_s^d \cup M_c^d$ by adding a new vertex σ_i^* for each nearly marginal symbol σ_i and joining σ_i^* by a single edge to each of σ_i' and σ_i''. Then G is depicted in Fig. 2. In G, ρ_d and c_d have degree at most $n-r$ because $\sum_{i=1}^n f(\sigma_i) \leqslant n-r$, and if σ_i is a marginal or nearly marginal symbol then the degree of σ_i' is $n-s$ and the degree of σ_i'' is $n-r$.

We now proceed to obtain an edge-colouring of G with $n-s$ colours satisfying the following:

(a) $r-s$ of the colours, say $\kappa_1, \ldots, \kappa_{r-s}$ are not used in M_c^d;

(b) for some symbol, say σ_d, the edges $\rho_d\sigma_d'$ and $c_d\sigma_d''$ are in G and have the same colour, say a;

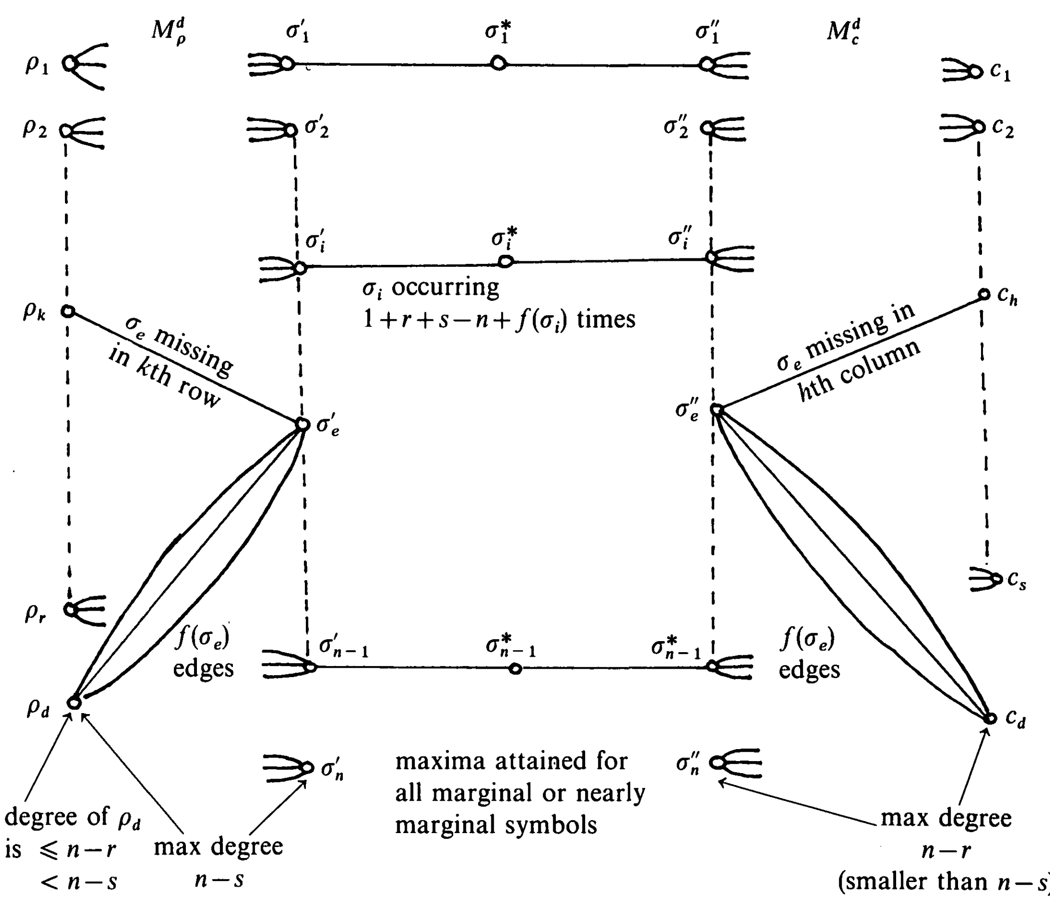

FIG. 2

(c) no colour appears twice on any vertex, except possibly on some σ_i^*, in which case the colour is not a.

Given such an edge-colouring, we get R' by placing symbol σ_d in cell $(r+1, s+1)$, for each edge $\rho_j\sigma_i'$ of M_ρ^d coloured a we place σ_i in cell $(j, s+1)$, and for each edge $\sigma_i''c_k$ of M_c^d coloured a we place σ_i in cell $(r+1, k)$.

The colour a is not one of $\kappa_1, \ldots, \kappa_{r-s}$ so it occurs on every vertex ρ_i and every vertex c_i; thus we have indeed obtained an $(r+1)\times(s+1)$ latin rectangle R'. Let $f'(\sigma_d) = f(\sigma_d)-1$ and $f'(\sigma_i) = f(\sigma_i)$ for $\sigma_i \neq \sigma_d$. Then $\sum_{i=1}^{n} f'(\sigma_i) \leqslant n-r-1$ and we have only to verify that each symbol σ_i occurs at least $(r+1)+(s+1)-n+f'(\sigma_i)$ times in R'. Apart from σ_d, this corresponds, as previously remarked, to securing that each marginal symbol is used twice and each nearly marginal symbol at least once. But this is true, because the colour a occurs on both the corresponding vertices σ' and σ'' by (a), and not in both cases on an edge to σ^* (by (c)). Also, σ_d occurs at least

$$1+r+s-n+f(\sigma_d) = (r+1)+(s+1)-n+f'(\sigma_d)$$

times in R'. So R' will be as required.

We now show how to obtain the edge-colouring satisfying (a)–(c). Theorem 1.4 is applied twice. An edge-colouring is called *proper* if no colour appears twice on any vertex.

Colour the graph $G\backslash M_c^d$ with $n-s$ colours and let $\kappa_1, \ldots, \kappa_{r-s}$ be colours which are not used on the vertex ρ_d. Let G^* be the subgraph of G obtained from G by removing all those edges of $G\backslash M_c^d$ which were coloured $\kappa_1, \ldots, \kappa_{r-s}$. Then G^* has maximum degree $(n-s)-(r-s) = n-r$.

Colour G^* with $n-r$ colours other than $\kappa_1, \ldots, \kappa_{r-s}$. Then let G be edge-coloured with each edge receiving the colour it received in G^* if it is in G^*, or the colour from $\{\kappa_1, \ldots, \kappa_{r-s}\}$ it received before if it is not in G^*. Let a be a colour used on an edge on ρ_d and let σ_d' be the vertex joined to ρ_d by the edge coloured a. Let the colour of the edge joining the vertex σ_d'' to c_d be b. If $b = a$, we are finished.

Now suppose that $a \neq b$. If b is on an edge on ρ_d, let P be the path, one end of which is the vertex ρ_d, coloured alternately κ_1 and b and maximal if it contains no edge $\sigma_i'\sigma_i^*$, or with the edge $\sigma_i\sigma_i^*$ as the last edge, if such an edge is contained in P. Now interchange the colours on P. Then a proper edge colouring is obtained except for the possibility that one of the vertices σ_i^*, say $\sigma_{i_0}^*$, may have both edges on it coloured b. Henceforth we shall assume that ρ_d has no edge on it coloured b.

Now let Q be a maximal path or circuit containing the edge $\sigma_i''c_d$ coloured a and b. If Q is a circuit then Q cannot contain ρ_d or $\sigma_{i_0}^*$. In that case, interchange the colours on Q and use the edges coloured a to fill cells $(j, s+1)$ $(j = 1, \ldots, r+1)$ and $(r+1, k)$ $(k = 1, \ldots, s+1)$ as described above. If Q is not a circuit then express Q as the union of two paths Q_1 and Q_2: Q_1 is the path with first vertex c_d and the first edge coloured a (if such an edge exists; otherwise $Q_1 = c_d$), and Q_2 is the path with first vertex c_d and first edge coloured b. If Q_2 contains one or more of the vertices σ_i^*, let Q_2^* be the path obtained from Q_2 starting at c_d and stopping at the first such vertex, say $\sigma_{i_1}^*$. The last edge in Q_2^* would then be coloured a, so $i_1 \neq i_0$. If there is no vertex σ_i^* let $Q_2^* = Q_2$. The path Q_1 could contain the vertex $\sigma_{i_0}^*$. If $\sigma_{i_0}^*$ is in Q_1 then let Q_1^* be the path obtained from Q by starting at c_d and stopping at $\sigma_{i_0}^*$; otherwise let $Q_1^* = Q_1$. The last vertex of Q_1^* cannot be ρ_d because if the last vertex of Q_1^* is in the set $\{\rho_1, \ldots, \rho_r, \rho_d\}$ then the last edge would have to be coloured b; but ρ_d now has no edge coloured b on it. Now interchange the colours on Q_1^* and Q_2^*. Then a proper edge-colouring will be produced except for the possibility that each of σ_{i_0}'' or $\sigma_{i_1}^*$ may have two edges coloured

b on it, the edges $\rho_d\sigma'_d$ and $c_d\sigma''_d$ are both coloured a, and no edge of $G\setminus M_p^d$ is coloured with any of $\kappa_1, \ldots, \kappa_{r-s}$.

Thus in all cases the edge-colouring satisfies (a)–(c). This proves Theorem 3.2.

THEOREM 3.3. *Let $n \geqslant r \geqslant s$ and let $s \leqslant \lfloor\frac{1}{2}(n+1)\rfloor$. Let P be a good partial latin square of side n and let*

$$P = \begin{pmatrix} R & 0 \\ 0 & D \end{pmatrix},$$

where R is $r \times s$ and D is diagonal with at least one empty column. Then P can be completed.

Proof. By Theorem 2.1 with rows replaced by columns, the first s columns of P can be completed (we may assume that R contains no empty columns). By Lemma 3.1, this implies that $N_R(\sigma_i) \geqslant r+s-n+f(\sigma_i)$ $(1 \leqslant i \leqslant n)$. By Theorem 3.2, P can be completed.

THEOREM 3.4. *Let P be a good partial latin square of side n with at most n cells preassigned. Suppose no row contains more than one preassigned cell. Then P can be completed to form a latin square of side n.*

Proof. For each $n \geqslant 1$, with $n \neq 2$, there are latin squares of side n with all the diagonal elements distinct. Therefore for $n \neq 2$, if all the preassigned symbols are distinct and there is a preassigned symbol in each row and in each column then the latin square can be completed.

Now suppose that P is good and *either* that not all the preassigned symbols are distinct *or* that there is at least one column with no preassigned symbols in it. By interchanging columns and symbols if necessary, there is no loss of generality in supposing that there is a column with no preassigned symbols in it.

Suppose that t columns altogether have preassigned symbols in them. By Corollary 2.3, if $t \leqslant \lceil\frac{1}{2}(n+2)\rceil$ then we may complete the latin square. Therefore suppose $\lceil\frac{1}{2}(n+2)\rceil < t \leqslant n-1$. If there are exactly v columns each containing only one symbol then

$$2(t-v)+v \leqslant n.$$

From this we obtain the inequalities

$$v \geqslant 2\lceil\tfrac{1}{2}(n+3)\rceil - n \geqslant 3 \quad \text{and} \quad t-v \leqslant \lfloor\tfrac{1}{2}(n-v)\rfloor \leqslant \lfloor\tfrac{1}{2}(n-3)\rfloor.$$

We may therefore permute rows and columns so that the preassigned cells have the pattern indicated in Fig. 2 with $r = n-v$ and $s = t-v$. P has the form $\begin{pmatrix} R & 0 \\ 0 & D \end{pmatrix}$, where R is $(n-v)\times(t-v)$. All the preassigned cells lie in R or in the lower diagonal outside R, and as $n-v > t-v$, D has an empty column. Since $t-v \leqslant \lfloor\frac{1}{2}(n+1)\rfloor$, P can be completed by Theorem 3.3. This proves Theorem 3.4.

Corollary 2.2 and Theorem 3.4 give sufficient conditions for a partial latin square to be completable. Naturally, a partial latin square can also be completed if any of its conjugates satisfy one of these conditions, for then the conjugate can be completed and so can the partial latin square.

We shall say that a good partial latin square satisfies the DL-condition (Distinct or Lindner condition) if one of the following holds:

(i) all preassigned cells are in different rows;
(ii) all preassigned symbols are distinct;
(iii) all preassigned symbols are in different columns;
(iv) all preassigned cells outside a given column occur in at most $\lceil \frac{1}{2}n \rceil$ rows;
(v) all preassigned symbols different from a given symbol occur in at most $\lceil \frac{1}{2}n \rceil$ rows;
(vi) all preassigned symbols outside a given row belong to a set of at most $\lceil \frac{1}{2}n \rceil$ symbols;
(vii) all preassigned symbols outside a given column belong to a set of at most $\lceil \frac{1}{2}n \rceil$ symbols;
(viii) all preassigned cells outside a given row occur in at most $\lceil \frac{1}{2}n \rceil$ columns;
(ix) all preassigned symbols different from a given symbol occur in at most $\lceil \frac{1}{2}n \rceil$ columns.

Then we can summarize the consequences of Corollary 2.2 and Theorem 3.4 as follows:

THEOREM 3.5. *A good partial latin square of side n with at most n cells preassigned and satisfying the DL-condition can be completed to a latin square of side n.*

We remark that, in particular, a good partial latin square satisfies the DL-condition if all preassigned cells are in $\lceil \frac{1}{2}(n+2) \rceil$ rows or $\lceil \frac{1}{2}(n+2) \rceil$ columns or belong to a set of $\lceil \frac{1}{2}(n+2) \rceil$ symbols.

A further corollary is:

COROLLARY 3.6. *The strengthened Evans conjecture is true for $n \leqslant 7$.*

Proof. If $n \leqslant 5$, a good partial latin square of side n with at most n non-empty cells satisfies Corollary 2.3 unless $n = 4$ or $n = 5$ and all symbols are in distinct rows. But then (i) is satisfied.

If $n = 6$ or 7 then it satisfies Corollary 2.3 (applied to the columns as well as the rows), or it satisfies (i) or (iii), or it has 5 (or 6 respectively) rows and columns containing preassigned symbols. But then it can be completed using Theorem 3.2.

For the remainder of this paper we shall assume that $n \geqslant 8$.

4. *The induction step*

In this section we prove our main theorem by induction on the side of the partial latin square. Let P be a good partial latin square of side n with n cells preassigned. We explain the idea of the proof. First we delete a preassigned cell, a row, and a column, and complete the resulting square on $n-1$ symbols by induction. Call a preassigned cell *diagonal* if it is the sole preassigned cell in its row and its column. Let R be the latin rectangle obtained from the completion by deleting all rows and columns which contain no preassigned cells or contain only diagonal preassigned cells. By replacing some symbols of R by the nth symbol we make the conditions of Theorem 3.2 be satisfied and so we can complete the modified R to a latin square of side n.

To be able to replace the correct number of symbols of R we need the following lemma. The idea of the proof is taken from a paper by Brouwer, de Vries, and Wieringa [4] (the idea in the paper by Woolbright [19] is similar). A *partial transversal* of a latin rectangle is a set of cells in distinct rows, in distinct columns, and containing distinct symbols. First we prove a theorem and two lemmas.

THEOREM 4.1. *Let R be an $x \times y$ latin rectangle and assume that R contains p forbidden cells with at least one in each column, where*

$$(p+1-x)(p+1-y) > p \quad \text{and} \quad \tfrac{1}{2}(x+y) \leqslant p+1.$$

Then R has a partial transversal of length $t \geqslant x+y-p$ avoiding the forbidden cells.

Proof. Let R be on the symbols $\{\sigma_1, ..., \sigma_n\}$ and consider a partial transversal of R avoiding the forbidden cells and of maximum length t; we may assume that it contains symbols $\sigma_1, ..., \sigma_t$ in cells $(1, 1), ..., (t, t)$. The conditions imply that $t \leqslant p$, $x \leqslant p$, and $y \leqslant p$, so if $t = x$ or $t = y$, we are finished. Assume that $t < x \leqslant n$ and $t < y \leqslant n$. By the maximality, no symbol from $\{\sigma_{t+1}, ..., \sigma_n\}$ can occur in rows $t+1, ..., x$ of columns $t+1, ..., y$ except in forbidden cells.

Let

$$\begin{aligned} A_0 &= \varnothing, \\ A_j &= \{i \in \{1, ..., x\} \mid \text{cell } (i, t+j) \text{ is not forbidden and its entry } \sigma_l \text{ has} \\ &\qquad l \in A_{j-1} \cup \{t+1, ..., n\}\} \quad (1 \leqslant j \leqslant y-t). \end{aligned}$$

Now define an oriented graph G on vertices

$$\bigcup_{j=1}^{y-t} (A_j \times \{t+j\})$$

and edges

$$\{((a, t+j), (b, t+k)) \mid j < k \text{ and the entry of cell } (b,\ t+k) \text{ is } \sigma_a\}.$$

We claim

LEMMA 4.2. *For all j, with $1 \leqslant j \leqslant y-t$,*

$$\{t+1, ..., x\} \cap A_j = \varnothing. \tag{1}$$

Proof. Suppose that Lemma 4.2 is false. Then G contains a vertex $(a_j,\ t+j)$ for some j, with $1 \leqslant j \leqslant y-t$, where $a_j \in \{t+1, ..., x\}$. By the definitions of G and $A_0, ..., A_{y-t}$, G must contain a directed path ending in $(a_j, t+j)$ and starting in a vertex $(a_k, t+k)$, where $k < j$ and cell $(a_k, t+k)$ has an entry from $\{\sigma_{t+1}, ..., \sigma_n\}$.

Let $(g_0, t+i_0)$, $(g_1, t+i_1), ..., (g_l, t+i_l)$ be the shortest directed path of G with the property that the entry of cell $(g_0, t+i_0)$ belongs to $\{\sigma_{t+1}, ..., \sigma_n\}$ and $g_l \in \{t+1, ..., x\}$.

Then the cells

$$\begin{aligned} &(g_k, t+i_k), \quad \text{with } 0 \leqslant k \leqslant l, \quad \text{and} \\ &(j, j), \quad \text{with } 1 \leqslant j \leqslant t, j \neq g_k \text{ for } 0 \leqslant k \leqslant l-1, \end{aligned}$$

form a partial transversal of length $t+1$, contradicting the maximality of t. We show that the cells do indeed form a partial transversal.

(i) The cells are in R and, by definition, are not forbidden.

(ii) The cells are in distinct rows, because if $g_h = g_k$, with $h < k$, then either $k = l$ and $(g_0, t+i_0), ..., (g_h, t+i_h)$ is a shorter path, or $k < l$ in which case the entry of cell

$(g_{k+1}, t+i_{k+1})$ is $\sigma_{g_k} = \sigma_{g_h}$, so that

$$(g_0, t+i_0), \ldots, (g_h, t+i_h), (g_{k+1}, t+i_{k+1}), \ldots, (g_l, t+i_l)$$

is a shorter path.

(iii) The cells are obviously in distinct columns.

(iv) The cells have distinct entries: for $g_0, \ldots, g_{l-1}$ are at most t by the shortest path property. Also cells containing $\sigma_{g_0}, \ldots, \sigma_{g_{l-1}}$ are deleted from the first partial transversal but replaced by others containing the same symbols, and cell $(g_0, t+i_0)$ has entry, say σ_l, with $l > t$.

This proves Lemma 4.2.

We now prove that Lemma 4.2 implies that $t \geqslant x+y-p$. Let p_j be the number of forbidden cells in column j, for $1 \leqslant j \leqslant y$. Then $p_j \geqslant 1$ by assumption. Rewriting the definition of A_j we have, for $1 \leqslant j \leqslant y-t$,

$$\begin{aligned} A_j = \{i \in \{1, \ldots, x\} \mid {} & \text{cell } (i, t+j) \text{ is not forbidden and} \\ & \text{its entry is not in } \{\sigma_l \mid l \in \{1, \ldots, t\} \backslash A_{j-1}\}\}. \end{aligned}$$

Therefore, by Lemma 4.2,

$$|A_j| \geqslant x-(t-|A_{j-1}|)-p_{t+j} \quad (1 \leqslant j \leqslant y-t).$$

It follows that

$$|A_{y-t}| \geqslant (y-t)(x-t)-\sum_{i=1}^{y-t} p_{t+i} = (y-t)(x-t)-\left(p-\sum_{j=1}^{t} p_j\right).$$

Here $\sum_{j=1}^{t} p_j \geqslant t$ and, as $|A_{y-t}| \leqslant t$ by Lemma 4.2, we get

$$(y-t)(x-t)-p+t \leqslant t,$$

and so $-t(x+y)+t^2 \leqslant p-xy$. Therefore

$$\begin{aligned} 0 &\leqslant p-(t-x)(t-y) \\ &< (p+1-x)(p+1-y)-(t-x)(t-y) \\ &= (p+1-t)(p+1+t-x-y). \end{aligned}$$

This proves Theorem 4.1, since the first factor is positive.

We need two more lemmas before we can prove the main theorem.

LEMMA 4.3. *Let R be an $r \times s$ matrix containing q preassigned cells of which none are diagonal, and containing no empty row or column. Then*

$$r+s \leqslant \lfloor 3q/2 \rfloor.$$

Proof. Let r_2 be the number of rows of weight at least 2. Then

$$2r_2+(r-r_2) \leqslant q,$$

so $r \leqslant q-r_2$. The $r-r_2$ cells outside these rows must occur in columns with additional preassigned cells. Therefore each column containing only such cells must contain at least two of them, and so

$$s \leqslant q-(r-r_2)+\tfrac{1}{2}(r-r_2)$$

giving $2s+r \leqslant 2q+r_2$. Finally, we have

$$2(r+s) = r+(2s+r) \leqslant (q-r_2)+(2q+r_2) = 3q,$$

which proves the lemma.

LEMMA 4.4. *Let P be a good partial latin square of side n on n symbols with n preassigned cells and with the property that neither P nor any of its conjugates have any diagonal preassigned cells. Let $r(s)$ be the number of rows (columns, respectively) containing preassigned cells and let t be the number of distinct symbols. Then*

$$\min(r+s, s+t, t+r) \leqslant 4n/3.$$

Proof. Let $R(S)$ be the set of cells in rows (columns, respectively) of weight at least 2, and let T be the set of cells containing symbols of weight at least 2. Let

$$x = |(R \cap T)\backslash S|, \quad y = |(R \cap S)\backslash T|,$$

$$z = |(T \cap S)\backslash R|, \quad w = |R \cap S \cap T|.$$

Then

$$n = x+y+z+w, \quad |R| = x+y+w,$$

and

$$r \leqslant z+\tfrac{1}{2}|R| \leqslant \tfrac{1}{2}(n+z).$$

Similarly, $s \leqslant \frac{1}{2}(n+x)$ and $t \leqslant \frac{1}{2}(n+y)$. Therefore

$$(r+s)+(s+t)+(t+r) \leqslant x+y+z+3n \leqslant 4n.$$

Therefore $\min(r+s, s+t, t+r) \leqslant 4n/3$.

Proof of the Main Theorem. Let P be a good partial latin square of side n on symbols $\sigma_1, ..., \sigma_n$ with n preassigned cells (if a partial latin square of side n has less than n preassignments, then more can be added so as to make it good with n preassigned cells). By Corollary 3.6, the theorem is true for $n \leqslant 7$, so we assume that $n \geqslant 8$.

By Theorem 3.5, we may assume that none of the DL-conditions (i)–(ix) are satisfied. We may assume that the number of rows of P containing preassigned cells is at least as great as the number of columns containing preassigned cells. Let P be arranged in the form $\begin{pmatrix} R & 0 \\ 0 & D \end{pmatrix}$, where all the preassigned cells are in R, an $r \times s$ matrix with $r \geqslant s$ and with no empty rows or columns, or in D, a diagonal matrix. By the DL-condition, we may assume that the last row of P is empty, the last column of P is empty, and that the symbol σ_n is not preassigned in P. We also assume that $s > \lfloor \frac{1}{2}(n+1) \rfloor$, because otherwise P can be completed by Theorem 3.3. We now consider two slightly different cases.

Case I: P, or a conjugate of P, contains a diagonal preassignment. By conjugating if necessary, we may assume that P has a diagonal preassignment. For each symbol σ_i, let $f(\sigma_i)$ be the number of times that σ_i occurs on the diagonal of P. Let σ_l be a symbol with $f(\sigma_l) \geqslant 1$. Let P' be obtained from P by deleting the last row, the last column, and a diagonal preassignment of the symbol σ_l. We may consider P' to be a partial latin square of side $n-1$ on the $n-1$ symbols $\sigma_1, ..., \sigma_{n-1}$ with $n-1$ cells preassigned.

Further, we can assume that P' is good, because if P' were bad of Type I then all preassigned cells of P outside one row would be in two columns, so P could be completed by DL-condition (viii); similar arguments apply for Type 2 and Type 3.

By the induction hypothesis, we can complete P' to a latin square of side $n-1$. By Lemma 3.1, in P' we have

$$N_R(\sigma_i) \geqslant r+s-n+1+f(\sigma_i) \quad \text{for } i \neq l, \text{ with } 1 \leqslant i \leqslant n-1,$$

and

$$N_R(\sigma_l) \geqslant r+s-n+1+(f(\sigma_l)-1).$$

We now disregard what is outside R; we wish to extend a modified version R_m of R to a completion of P. To do that we must have, by Theorem 3.2,

$$N_{R_m}(\sigma_i) \geqslant r+s-n+f(\sigma_i) \quad \text{for all } i, \text{ with } 1 \leqslant i \leqslant n.$$

We note that this holds, in R, for $i = l$. It holds for $i \neq l$, with $1 \leqslant i \leqslant n-1$, even if we delete one occurrence of a symbol σ_i. And finally we must make the symbol σ_n occur at least $r+s-n$ times in R_m.

We apply Theorem 4.1 to R with the preassignments as forbidden cells. The number of these is at most $n-1$, and we choose additional forbidden cells arbitrarily so as to have exactly $n-1$ forbidden cells; this is possible as $r \geqslant s > \lfloor\frac{1}{2}(n+1)\rfloor \geqslant 2$, so that the total number rs of cells satisfies $rs \geqslant 2\lfloor\frac{1}{2}(n+1)\rfloor > n-1$. The matrix R has a forbidden cell in each column, and the numerical conditions of Theorem 4.1 are satisfied, because, by Lemma 4.3, above,

$$r+s \leqslant \lfloor 3(n-1)/2 \rfloor \leqslant \tfrac{1}{2}(3n-3),$$

so

$$\tfrac{1}{2}(r+s) \leqslant \tfrac{3}{4}n-\tfrac{3}{4} \leqslant n = (n-1)+1.$$

Also if $n-r \geqslant 4$ then $s \leqslant \frac{1}{2}(r+s) \leqslant \frac{3}{4}n-\frac{3}{4}$, so $n-s \geqslant \frac{1}{4}n+\frac{3}{4}$, and so

$$((n-1)+1-r)((n-1)+1-s) \geqslant 4(\tfrac{1}{4}n+\tfrac{3}{4}) = n+3 > n-1.$$

If $n-r = 3$ then $s \leqslant \frac{1}{2}(3n-3)-r = \frac{1}{2}n+\frac{3}{2}$, so $n-s \geqslant \frac{1}{2}n-\frac{3}{2}$ and so

$$((n-1)+1-r)((n-1)+1-s) \geqslant 3(\tfrac{1}{2}n-\tfrac{3}{2}) > n-1$$

as $n \geqslant 8$. Finally, if $n-r \leqslant 2$, then $s \leqslant \frac{1}{2}(3n-3)-r \leqslant \frac{1}{2}n+\frac{1}{2}$, which contradicts $s > \lfloor\frac{1}{2}(n+1)\rfloor$.

Theorem 4.1 now ensures the existence of a partial transversal in R of length at least $r+s-n+1$ avoiding all preassignments. Replacing all entries of the partial transversal, except possibly the symbol σ_l, by the symbol σ_n, we obtain the R_m we want, and this can now be completed by Theorem 3.2 to obtain a completion of P.

Case II: neither P nor any of its conjugates contains a diagonal preassignment. In this case we may replace the condition $s > \lfloor\frac{1}{2}(n+1)\rfloor$ by $s > \lceil\frac{1}{2}(n+2)\rceil$, because if $s \leqslant \lceil\frac{1}{2}(n+2)\rceil$ then P can be completed by Corollary 2.3 applied to columns (this is, in effect, one case of DL-condition (viii), which has been excluded). By Lemma 4.4 we may assume that $r+s \leqslant 4n/3$.

By DL-condition (vi) or (vii) we can assume that there is a symbol, say σ_1, which is preassigned exactly once in P; without loss of generality, we may assume it is assigned to cell $(1, 1)$. Let P' be obtained from P by deleting the last row and the last column and removing σ_1 from cell $(1, 1)$. Then P' is a partial latin square of side $n-1$ on the set $\{\sigma_2, ..., \sigma_n\}$ of $n-1$ symbols. Again, it is easy to see that we can assume that P' is good.

By the induction hypothesis, we can complete P' to a latin square of side $n-1$. By Theorem 1.2 we have

$$N_R(\sigma_i) \geqslant r+s-n+1 \quad \text{for all } i \neq j, \text{ with } 2 \leqslant i \leqslant n.$$

What we need is (and now Theorem 1.2 suffices)

$$N_R(\sigma_i) \geqslant r+s-n \quad \text{for all } i, \text{ with } 1 \leqslant i \leqslant n.$$

Hence we must make the missing symbol σ_1 occur $r+s-n$ times, and we can delete any other symbol once.

Let σ_k be the symbol placed in cell (1, 1). Replace this σ_k of cell (1, 1) by σ_1: then we cannot delete any further occurrences of the symbol σ_k. We now need $r+s-n-1$ additional occurrences of σ_1.

We look first for a partial transversal in the $(r-1)\times(s-1)$ rectangle R' obtained from R by deleting row 1 and column 1. Since neither P nor any of its conjugates has a diagonal preassignment, there must be another preassigned cell in column 1 of R and yet another in row 1 of R. Therefore there are at most $n-3$ preassigned cells in R'. If a preassigned cell (different from the one containing σ_1) in the row deleted from R is the sole preassigned cell in its column, that column will have no preassigned cell in R'. If the number of preassigned cells in R' is $n-\mu$, then there are at most $\mu-2$ such columns: we now choose $\mu-2$ additional cells which, together with the preassigned cells, form $n-2$ forbidden cells of R', and we choose at least one such cell in each column of R' with no preassignments. Then R' has a forbidden cell in each column. We apply Theorem 4.1. We have

$$\tfrac{1}{2}((r-1)+(s-1)) = \tfrac{1}{2}(r+s)-1 \leqslant \tfrac{2}{3}n-1 \leqslant n-1 = (n-2)+1,$$

and if $n-r \geqslant 3$, then $s \leqslant \frac{1}{2}(r+s) \leqslant \frac{2}{3}n$, so $n-s \geqslant \frac{1}{3}n$ and so

$$((n-2)+1-(r-1))((n-2)+1-(s-1)) \geqslant 3(\tfrac{1}{3}n) = n > n-2.$$

If $n-r \leqslant 2$, then, since $n \geqslant 6$, $s \leqslant \lfloor 4n/3 \rfloor - r \leqslant \lceil \frac{1}{2}(n+2) \rceil$, so the case where $n-r \leqslant 2$ does not arise here. Thus, when the DL-conditions are not satisfied, Theorem 4.1 gives the existence of a partial transversal of length

$$(r-1)+(s-1)-(n-2) = r+s-n,$$

and so, replacing each symbol of the transversal, except possibly the symbol σ_k, by the symbol σ_1, we obtain the required R_m and thus a completion of P (by Theorem 1.2).

This finishes Case II and so the proof of the main theorem.

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