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Avoiding partial Latin squares and intricacy

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Abstract

In this paper we consider the following problem: Given a partial $n \times n$ latin square P on symbols $1, 2, \ldots, n$, is it possible to find an $n \times n$ latin square L on the same symbols which differs from P in every cell? In other words, is P avoidable? We show that all $2k \times 2k$ partial latin squares for $k \geqslant 2$ are avoidable and give some results on odd partial latin squares. We also use these results to show that the intricacy of avoiding partial latin squares is two and of avoiding more general arrays is at most three.

1. Introduction

A *partial* $n \times n$ *latin square* on $1, 2, \ldots, n$ is an array of n rows and n columns, filled with the symbols $1, 2, \ldots, n$ in such a way that every cell contains at most one symbol and every symbol appears at most once in every row and column. It is a *latin square* if there are no empty cells. The integer n is called the *order* of the latin square. A latin square is said to be in *standard form* if its first row and column contain the elements in their natural order. More information on latin squares can be found in [3].

An array is said to be *avoidable* if an $n \times n$ latin square, on the same symbols, can be found which differs from the array in every cell. Otherwise an array is said to be *unavoidable*.

In 1989 Häggkvist asked the general question: Which $n \times n$ arrays are avoidable? It is fairly easy to find some particular infinite families which are not avoidable [7]. The first published avoidable family is due to Häggkvist [4] in 1989 where he proved the following result.

Theorem 1.1 (Häggkvist [4]). *Let $n = 2^k$ and let P be a partial $n \times n$ column-latin square on $1, 2, \ldots, n$ with empty last column. Then there exists an $n \times n$ latin square L, on the same symbols, which differs from P in every cell.*

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In [1] we have classified all $n \times n$ chessboard squares which are avoidable. A *chessboard square* is an array with cells coloured in the form of a chessboard with at most one entry per black cell and no entries in the white cells.

Theorem 1.2 (Chetwynd and Rhodes [1]). *Let $k \geqslant 2$ and let C be a $2k \times 2k$ chessboard square on symbols $1, 2, \ldots, 2k$ in which any black cell contains at most one symbol. Then C is avoidable. Let $k \geqslant 3$ and let C be a $(2k-1) \times (2k-1)$ chessboard square on symbols $1, 2, \ldots, 2k-1$ in which all corner cells are white and each black cell contains at most one symbol. Then C is avoidable.*

In this paper we first consider partial latin squares and we can immediately show that all completable partial latin squares are avoidable. We show that there is a 2×2 partial latin square and a 3×3 partial latin square which are unavoidable but for $k \geqslant 2$ we prove that all $2k \times 2k$ and $3k \times 3k$ partial latin squares are avoidable. Also we show that for $k \geqslant 4$, any partial $(2k-1) \times (2k-1)$ latin square, with one empty row and column, is avoidable.

We then explain how the idea of intricacy can be applied to the problem of avoiding $n \times n$ arrays. We show that the intricacy of avoiding partial latin squares is two and of avoiding more general arrays is at most three. We now describe the general problem of intricacy.

Many combinatorial construction problems, though easy to formulate, can often be quite intricate. For example, on trying to construct an $n \times n$ latin square by filling in the cells one at a time, checking at each stage that no symbol has been repeated in any row or column, we will often reach a point where the partial latin square we have obtained cannot be completed to a latin square. We call this problem *intricate*, its *intricacy* being the least integer k such that any partial $n \times n$ latin square that cannot be completed to an $n \times n$ latin square can be partitioned into k or fewer parts, each of which can be extended to an $n \times n$ latin square. To partition a partial $n \times n$ latin square P into k partial $n \times n$ latin squares $P_1, P_2, \ldots, P_k$ means to fill in some cells of each P_t in such a way that if cell (i,j) of P is non-empty, then its entry appears in cell (i,j) of one of the P_t's, and all non-empty cells of the P_t's are obtained in this way.

The notion of intricacy was introduced by Daykin and Häggkvist, who conjectured that the intricacy of completing partial latin squares is 2, and posed the problem in [2]. Applications of the concept of intricacy to other combinatorial problems may be found in [6].

Every construction problem has a domain, a set of partial structures and a set of goal structures. For example, the latin square completion problem mentioned above is a construction problem whose domain is the set of cells of an $n \times n$ matrix together with a set of n symbols; the partial structures and goal structures of this problem are the set of all partial $n \times n$ latin squares and the set of all $n \times n$ latin squares respectively. Any partial structures that are subsets of goal structures are called *extensible*. A construction problem in which every partial structure is extensible is said to be *simple*; otherwise it is *intricate*. We say that a problem is *fair* if every singleton of its domain is extensible. Thus, the latin square problem is fair since any partial $n \times n$ latin square with only one entry can certainly be completed to an $n \times n$ latin square.

Given a fair construction problem, its intricacy is the smallest positive integer k such that every partial structure can be partitioned into k or fewer extensible structures. Clearly, a construction problem has intricacy 1 if and only if it is simple, and any simple problem is fair. An unfair problem, though obviously intricate, does not have a numerical value of intricacy assigned to it.

It was shown in [6] that the intricacy of the original latin square problem is 2, 3 or 4, but the problem remains in this state of partial solution. In this paper we shall consider the intricacy of a different problem, the problem of avoiding partial latin squares. The following result is used several times and we state it as a lemma.

Lemma 1.3. *Let m, k be positive integers and let S be a set of mk objects. Then the number of different ordered partitions of S into k sets of m objects is*

$$(mk)!/(m!)^k.$$

We also make use of the following theorem, concerning systems of distinct representatives, due to Hall [5].

Theorem 1.4 (Hall [5]). *Let E be a non-empty finite set and let $S_1, S_2, \ldots, S_m$ be non-empty subsets of E. Then $S_1, S_2, \ldots, S_m$ have a system of distinct representatives if and only if the union of any k of the subsets S_i contains at least k elements, for each $k : 1 \leqslant k \leqslant m$.*

2. Avoiding partial latin squares

Clearly, a partial 1×1 latin square consists either of one empty cell and is thus completable and hence avoidable or has one symbol and is not avoidable using that same symbol. From now on we shall assume all our partial latin squares are $n \times n$ where $n \geqslant 2$.

Our first theorem shows that the set of all unavoidable partial $n \times n$ latin squares is a subset of the set of all those partial $n \times n$ latin squares which cannot be completed to an $n \times n$ latin square.

Theorem 2.1. *Let P be a partial $n \times n$ latin square on $1, 2, \ldots, n$ which can be completed to an $n \times n$ latin square $n \geqslant 2$. Then P is avoidable.*

Proof. We create an $n \times n$ latin square which differs from P in every cell by completing P to an $n \times n$ latin square L on $1, 2, \ldots, n$ and then permuting the symbols in L in such a way that no symbol remains in its original position. □

1	
	2

Fig. 1.

1	2	
2	1	3
	3	1

Fig. 2.

a	b	
b	a	c
	c	a

Fig. 3.

We next consider partial latin squares of small order. The partial 2×2 latin square shown in Fig. 1 is clearly not completable and is also unavoidable. Rhodes [7] shows that the only unavoidable partial 3×3 latin square is of the form of the square of Fig. 2. In other words, up to permutations of rows, columns and symbols, there is only one unavoidable partial 3×3 latin square. We now use the fact that there is essentially only one unavoidable 3×3 partial latin square to give a whole class of avoidable partial latin squares.

Theorem 2.2. *Let $k \geqslant 2$ and let P be a partial $3k \times 3k$ latin square on symbols $1, 2, \ldots, 3k$. Then P is avoidable.*

Proof. We show that it is possible to construct a $3k \times 3k$ latin square L which differs from P in every cell. We divide P into k^2 3×3 subsquares and label each subsquare with one of the symbols $X_1, X_2, \ldots, X_k$ in the form of a latin square. Let S be the set of symbols $\{1, 2, \ldots, 3k\}$. We show that there exists an ordered partition $S_1, S_2, \ldots, S_k$ of S into k sets of three symbols in such a way that, for each $i : 1 \leqslant i \leqslant k$, S_i is a suitable set of three symbols for any 3×3 subsquare of P which is labelled X_i.

Consider a particular 3×3 subsquare X of P. Now, if X is unavoidable using a particular set of three symbols chosen from S, say $\{a, b, c\}$, then it must contain a partial 3×3 latin square of the form of the square of Fig. 3 (the only unavoidable partial 3×3 latin square).

Suppose X contains the square of Fig. 3. Then we must consider the possible entries in the remaining two cells of X. Since P is a partial latin square there can be no further entries of any of the symbols a, b or c. Thus X will have the form of one of the four squares of Fig. 4.

If X has the form of square (1) of Fig. 4 then the only set of three symbols which is unsuitable for subsquare X is the set $\{a, b, c\}$. Similarly, if X has the form of

(1)

a	b	
b	a	c
	c	a

(2)

a	b	d
b	a	c
	c	a

(3)

a	b	d
b	a	c
d	c	a

(4)

a	b	d
b	a	c
e	c	a

Fig. 4.

square (2) or square (4) of Fig. 4 then it is avoidable using any set of three symbols, chosen from S, with the exception of the set $\{a,b,c\}$. In the case of square (3) of Fig. 4, however, the elements of each of the sets $\{a,b,c\}$, $\{a,b,d\}$ and $\{a,c,d\}$ form subsquares in X of the form of the square of Fig. 3 and these three sets are thus unsuitable for subsquare X of our partial latin square P. Therefore, for any 3×3 subsquare of our partial latin square P there are at most three unsuitable sets of three symbols.

We now show that a suitable ordered partition $S_1, S_2, \ldots, S_k$ of S exists. For each $i : 1 \leqslant i \leqslant k$, there are k 3×3 subsquares of P labelled X_i. Let B_i be the set of all triplets which are unsuitable for any of the 3×3 subsquares of P which are labelled X_i. Then $|B_i| \leqslant 3k$, for each $i : 1 \leqslant i \leqslant k$.

By Lemma 1.3, the number of ordered partitions of $3k$ symbols into k triplets is $(3k)!/(3!)^k$. Of these, we must exclude all those partitions in which $S_i \in B_i$ for any $i : 1 \leqslant i \leqslant k$. The number of ordered partitions which have a particular triplet of symbols in a particular position is $[3(k-1)]!/(3!)^{k-1}$. We must, therefore, exclude as many as $[3(k-1)]!3k^2/(3!)^{k-1}$ partitions, since $|B_1| + |B_2| + \cdots + |B_k| \leqslant 3k^2$. Thus, if N is the number of suitable ordered partitions of S, we have

$$\begin{aligned} N &\geqslant \frac{(3k)!}{(3!)^k} - \frac{[3(k-1)]!3k^2}{(3!)^{k-1}} \\ &= \frac{[3(k-1)]!}{(3!)^{k-1}} \left\{ \frac{3k(3k-1)(3k-2)}{3!} - 3k^2 \right\}. \end{aligned}$$

Now $N \geqslant 1$ whenever $3k(3k-1)(3k-2) > 18k^2$, that is, for all $k \geqslant 2$. So, for $k \geqslant 2$, there exists an ordered partition $S_1, S_2, \ldots, S_k$ of S into k triplets in such a way that, for each $i : 1 \leqslant i \leqslant k$, the triplet S_i is suitable for every 3×3 subsquare of P which is labelled X_i. Thus, for any $i : 1 \leqslant i \leqslant k$, given any 3×3 subsquare X (of P) that is labelled X_i, there exists a 3×3 latin square, on the set of symbols S_i, which differs from X in every cell. By repeating this process for each of the 3×3 subsquares of P, we find k^2 3×3 latin squares that together make up a $3k \times 3k$ latin square L, which differs from P in every cell. □

We have a similar result for partial $2k \times 2k$ latin squares.

Theorem 2.3. *Let $k \geqslant 2$ and let P be a partial $2k \times 2k$ latin square on symbols $1, 2, \ldots, 2k$. Then P is avoidable.*

Proof. If P has only one empty cell then it can be completed to a $2k \times 2k$ latin square and is thus avoidable. We therefore assume that P has at least two empty cells.

We show that it is possible to construct a $2k \times 2k$ latin square L which differs from P in every cell. We divide P into k^2 2×2 subsquares and label each subsquare with one of the symbols $X_1, X_2, \ldots, X_k$ in the form of a latin square. Let S be the set of symbols $\{1, 2, \ldots, 2k\}$. We show that there exists an ordered partition $S_1, S_2, \ldots, S_k$ of S into k pairs of symbols in such a way that, for all i, the pair S_i is suitable for any 2×2 subsquare of P labelled X_i.

Consider a particular 2×2 subsquare X of P and let D be some diagonal of X. If D contains only one non-empty cell or has two entries of the same symbol, say a, then these entries do not result in any pairs of symbols being unsuitable for X, since any pair which includes a can be placed in X by putting a in the two cells not in D. If, however, D contains two different symbols, say a and b, then the pair $\{a, b\}$ is unsuitable for X since neither a nor b can be put in the cells of the diagonal D. We call a diagonal *bad* if it contains two different entries. Otherwise we call it *good*. Each bad diagonal in subsquare X will result in one pair of symbols being unsuitable for X.

Let B be the set of all bad diagonals appearing in any 2×2 subsquare of P. We show that either $|B| < 2k^2 - k$ or that it is possible to rearrange the rows and columns of P to obtain a new square for which this condition is satisfied. We shall then show that, provided $|B| < 2k^2 - k$, there will exist a suitable ordered partition of S.

Let E be the number of empty cells in P. We consider three cases:

1. $E \geqslant 2k + 1$,
2. $2 \leqslant E \leqslant 2k - 1$,
3. $E = 2k$.

Case 1: In this case, the total number of non-empty cells in P is at most $4k^2 - 2k - 1$. Thus, since each two non-empty cells together may form one bad diagonal, the set B of all bad diagonals of P satisfies the condition

$$\begin{aligned} |B| &\leqslant 2k^2 - k - 1 \\ &< 2k^2 - k, \end{aligned}$$

as required.

Case 2: If case 2 holds then the total number of non-empty cells in P is strictly greater than $2k(2k - 1)$ so there must be a symbol, say s, which occurs exactly $2k$ times in P. We can therefore rearrange the rows of P so that symbol s appears in each main diagonal cell, thus creating at least k good diagonals. The two or more empty cells of P will create at least another one good diagonal so we have

$$\begin{aligned} |B| &\leqslant \frac{4k^2 - 2}{2} - k \\ &= 2k^2 - k - 1 \\ &< 2k^2 - k. \end{aligned}$$

Case 3: Finally, in case 3 the total number of non-empty cells in P is $4k^2 - 2k$. Thus there is at least one symbol, say s, which occurs $2k - 1$ times (or more) in P. We can therefore rearrange the rows of P so that all $2k - 1$ entries of symbol s appear in the main diagonal cells. It then holds that

$$\begin{aligned} |B| &\leqslant \frac{4k^2 - 2k}{2} - (k-1) \\ &= 2k^2 - 2k + 1 \\ &< 2k^2 - k, \end{aligned}$$

as required.

We now show that a suitable ordered partition $S_1, S_2, \ldots, S_k$ of S exists. Let B_i be the set of all pairs of symbols which are unsuitable for any of the 2×2 subsquares of P which are labelled X_i. Then we have

$$|B_1| + |B_2| + \cdots + |B_k| \leqslant |B|.$$

Now, by Lemma 1.3, the number of ordered partitions of $2k$ symbols into k pairs is $(2k)!/(2!)^k$. Of these we must exclude all those partitions in which $S_i \in B_i$, for any i, $1 \leqslant i \leqslant k$. The number of ordered partitions of S which have a particular pair of symbols in a particular position is $[2(k-1)]!/(2!)^{k-1}$. We must therefore exclude at most $|B|[2(k-1)]!/(2!)^{k-1}$ partitions. Thus, if N is the number of suitable partitions of S, we have

$$\begin{aligned} N &\geqslant \frac{(2k)!}{(2!)^k} - |B| \frac{[2(k-1)]!}{(2!)^{k-1}} \\ &= \frac{[2(k-1)]!}{(2!)^{k-1}} \{k(2k-1) - |B|\}. \end{aligned}$$

Now $N \geqslant 1$ if $|B| < 2k^2 - k$, as required. Therefore, there exists a suitable ordered partition of S and we can find k^2 2×2 latin squares that together make up a $2k \times 2k$ latin square L, which differs from P in every cell. □

Next we consider partial latin squares of odd order. We shall give a theorem which shows that, for $k \geqslant 4$, any partial $(2k-1) \times (2k-1)$ latin square, on $1, 2, \ldots, 2k-1$, with one empty row and column, is avoidable. First we need to establish the following lemma.

Lemma 2.4. *Let $m \geqslant 1$ and let P be a partial $2m \times 2m$ latin square on $1, 2, \ldots, 2m$. Then it is possible to permute the rows of P to form a new partial latin square P' in such a way that any symbol appears at most $m+1$ times in the main diagonal of P'.*

Proof. We show that the rows of P can be rearranged, to form a new partial $2m \times 2m$ latin square P', in such a way that any symbols appearing in the main diagonal cells

$(1,1),(2,2),\ldots,(m,m)$ of P' are distinct and thus that no symbol appears more than $m+1$ times in the entire main diagonal of P'.

Suppose that any symbols appearing in cells $(1,1),(2,2),\ldots,(i-1,i-1)$ are distinct and consider cell (i,i). By permuting rows $i,i+1,\ldots,2m$ there are $2m-(i-1) = 2m-i+1$ possible choices of symbols (or empty cells) for cell (i,i). At most $i-1$ of these symbols appear in cells $(1,1),(2,2),\ldots,(i-1,i-1)$, so there are at least $2m-i+1-(i-1) = 2m-2i+2$ possible choices for cell (i,i) which are distinct from the main diagonal entries in rows $1,2,\ldots,i-1$. Thus, if $2m-2i+2>0$, that is if $i \leqslant m$, we can ensure that either cell (i,i) of P' is empty or it contains a symbol distinct from any appearing in cells $(1,1),(2,2),\ldots,(i-1,i-1)$. $\square$

The next result is proved in [1].

Lemma 2.5. *Let A be an $n\times n$ array on $1,2,\ldots,n$. Let A' be an $(n+1)\times(n+1)$ array which has A as its upper left $n\times n$ subsquare and in which, for each $i: 1\leqslant i\leqslant n$, cells $(i,n+1)$ and $(n+1,i)$ both contain the same symbol (if any) as the diagonal cell (i,i) of A. Now, if there exists an $(n+1)\times(n+1)$ latin square L' on $1,2,\ldots,n+1$ which differs from A' in every cell, which has symbol $n+1$ in each main diagonal cell and in which, for each $i: 1\leqslant i\leqslant n$, cells $(i,n+1)$ and $(n+1,i)$ contain the same symbol, then A is avoidable using symbols $1,2,\ldots,n$.*

We shall also make use of the following simple result.

Lemma 2.6. *For any $n\geqslant 4$, there exists an $n\times n$ latin square, on $1,2,\ldots,n$, which has symbol n in each main diagonal cell, and has the entries of the last row in the same order as the last column.*

We now prove our result on partial latin squares of odd order.

Theorem 2.7. *Let $k\geqslant 4$ and let P be a partial $(2k-1)\times(2k-1)$ latin square on symbols $1,2,\ldots,2k-1$ with empty last row and column. Then P is avoidable.*

Proof. By Lemma 2.4 we can rearrange the first $2k-2$ rows of P to form a new partial $(2k-1)\times(2k-1)$ latin square Q in which each symbol appears at most k times in the main diagonal. Let $M(Q)$ be the maximum number of entries of any one symbol in the main diagonal of Q. Then $M(Q)\leqslant k$; we consider two cases:

1. $M(Q)\leqslant k-1$,
2. $M(Q)=k$.

In each of cases 1 and 2 we shall rearrange the diagonal entries in cells $(1,1)$, $(2,2),\ldots,(2k-2,2k-2)$ of Q in a certain way. Cells (i,i) and (j,j) may be permuted by permuting rows i and j and columns i and j.

Case 1: In case 1 we rearrange the diagonal cells of Q in such a way that, for each $i: 1\leqslant i\leqslant 2k-3$, if cells (i,i) and $(i+1,i+1)$ are both non-empty then they do not contain the same symbol.

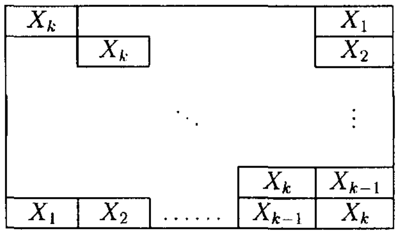

Fig. 5.

Case 2: In case 2 we rearrange the diagonal cells of Q so that if cells (i,i) and $(i+1,i+1)$ both contain the same symbol then $i = 2k-3$. In this case, we permute rows $2k-1$ and $2k-2$ of Q so that the whole of row $2k-2$ (and thus cell $(2k-2,2k-2)$) becomes empty.

In either of the above cases we are left with a new partial latin square R in which no two consecutive main diagonal cells contain the same symbol. We show that R (and hence P) is avoidable.

Let T be a $2k \times 2k$ array which has R as its upper left $(2k-1) \times (2k-1)$ subsquare and in which, for each $i : 1 \leqslant i \leqslant 2k-2$, cells $(i,2k)$ and $(2k,i)$ both contain the same symbol (if any) as the diagonal cell (i,i) of R. We divide T into k^2 2×2 subsquares and label each subsquare with one of the symbols $X_1, X_2, \ldots, X_k$ in the form of a $k \times k$ latin square in such a way that the k main diagonal subsquares each have symbol X_k and the kth row and column each have the symbols $X_1, X_2, \ldots, X_k$ in that order, as shown in Fig. 5. Such a latin square exists for $k \geqslant 4$, by Lemma 2.6. Since R is a partial latin square and by the way the diagonal cells of R have been rearranged, it is ensured that no row or column of any 2×2 subsquare of T contains two entries of the same symbol.

We show that there exists a $2k \times 2k$ latin square L on $1,2,\ldots,2k$ which differs from T in every cell, has symbol $2k$ in each main diagonal cell and in which, for each $i : 1 \leqslant i \leqslant 2k-1$, cells $(i,2k)$ and $(2k,i)$ both contain the same symbol.

Symbol $2k$ is suitable for each main diagonal cell of each subsquare of T which is labelled X_k, since symbol $2k$ does not appear anywhere in T. We must also find a symbol which is suitable for each anti-diagonal cell of each subsquare of T which is labelled X_k. Since the lower right 2×2 subsquare of T labelled X_k is completely empty in each of cases 1 and 2 above, there is a total of at most $2k-2$ non-empty anti-diagonal cells in all the subsquares labelled X_k. Therefore, there remains at least one symbol which is suitable for these cells; we may assume without loss of generality that symbol $2k-1$ is chosen.

Let S be the set of symbols $\{1,2,\ldots,2k-2\}$. We show that there exists an ordered partition $S_1, S_2, \ldots, S_{k-1}$ of S, into $k-1$ pairs of symbols, such that each pair S_i contains symbols which are suitable for any 2×2 subsquare of T labelled X_i.

Consider a particular 2×2 subsquare X (of T) which is labelled with one of the symbols $X_1, X_2, \ldots, X_{k-1}$ and let D be one of the two diagonals of X. If D contains two different symbols, say a and b, then the pair $\{a,b\}$ is unsuitable for X since neither a nor b can be put in the cells of the diagonal D. Recall that we call a diagonal of a 2×2 subsquare *bad* if it contains two different entries. Each bad diagonal in subsquare X will result in one pair of symbols being unsuitable for X.

Let B be the set of all bad diagonals appearing in any of the subsquares of T which are labelled $X_1, X_2, \ldots, X_{k-1}$ and, for each $i: 1 \leqslant i \leqslant k-1$, let B_i be the set of all pairs of symbols which are unsuitable for any 2×2 subsquare of T labelled X_i. Then

$$|B_1| + |B_2| + \cdots + |B_{k-1}| \leqslant |B|.$$

We consider $|B|$ in each of cases 1 and 2 above.

Case 1: In case 1, row $2k-1$ and column $2k-1$ of T are both empty. Thus, for each $i: 1 \leqslant i \leqslant k-1$, there are two 2×2 subsquares of T labelled X_i which contain no bad diagonals, namely, the two subsquares labelled X_i which intersect row $2k-1$ and column $2k-1$ respectively. Therefore, in case 1 we have

$$\begin{aligned}|B| &\leqslant 2k(k-1) - 2[2(k-1)]\\ &= 2k^2 - 6k + 4.\end{aligned}$$

Case 2: Similarly, in case 2, column $2k-1$ and rows $2k-2$ and $2k$ of T are both empty. This results in $3k-4$ subsquares, each labelled with one of the symbols $X_1, X_2, \ldots, X_{k-1}$, having no bad diagonals. Thus, in case 2 we have

$$\begin{aligned}|B| &\leqslant 2k(k-1) - 2(3k-4)\\ &= 2k^2 - 8k + 8.\end{aligned}$$

Hence for $k \geqslant 4$ we have $B \leqslant 2k^2 - 6k + 4$. We now show that a suitable partition, $S_1, S_2, \ldots, S_{k-1}$, of S exists. By Lemma 1.3, the number of ordered partitions of $2k-2$ symbols into $k-1$ pairs is $(2k-2)!/(2!)^{k-1}$. Of these, we must exclude all those partitions for which $S_i \in B_i$ for any $i: 1 \leqslant i \leqslant k-1$. The number of ordered partitions of S which have a particular pair of symbols in a particular position is $[2(k-2)]!/(2!)^{k-2}$. We must therefore exclude at most $|B|\,[2(k-2)]!/(2!)^{k-2}$ partitions.

Thus, if N is the number of suitable ordered partitions of S, we have

$$\begin{aligned}N &\geqslant \frac{(2k-2)!}{(2!)^{k-1}} - |B|\frac{[2(k-2)]!}{(2!)^{k-2}}\\ &= \frac{[2(k-2)]!}{(2!)^{k-2}}\left\{\frac{(2k-2)(2k-3)}{(2!)} - |B|\right\}\end{aligned}$$

and

$$\begin{aligned}\frac{(2k-2)(2k-3)}{(2!)} - |B| &\geqslant (k-1)(2k-3) - (2k^2 - 6k + 6)\\ &= k-1.\end{aligned}$$

So we have $N \geqslant 1$ if $k-1 > 0$. Since $k \geqslant 4$ the required ordered partition $S_1, S_2, \ldots, S_k$ of S exists. Thus, there exist k^2 2×2 latin squares that together make up a $2k \times 2k$

latin square L on $1,2,\dots,2k$ which differs from T in every cell and satisfies the conditions of Lemma 2.5. It follows that R (and thus P) is avoidable using symbols $1,2,\dots,2k-1$. $\square$

The following lemma on avoiding partial latin squares is needed in the next section.

Lemma 2.8. *Let $n>2$ and let P be a partial $n\times n$ latin square on $1,2,\dots,n$ whose only non-empty cells appear in the union of one row and one column. Then P is avoidable.*

Proof. We may suppose, without loss of generality, that all the non-empty cells of P appear in the first row and column. Let the cells of P be labelled with the symbols $s_1,s_2,\dots,s_n$ in the form of any standard form $n\times n$ latin square. Let $S=\{1,2,\dots,n\}$ and, for $i:1\leqslant i\leqslant n$, define S_i to be the subset of S which contains symbol k if no cell of P which is labelled s_i contains symbol k. Then any system of distinct representatives of $S_1,S_2,\dots,S_n$ gives rise to an $n\times n$ latin square L, which differs from P in every cell, by placing the chosen representative of S_i in all those cells of P which are labelled s_i.

We show, using Hall's Theorem (Theorem 1.4), that such a system of distinct representatives exists. We have $|S_1|\geqslant n-1$ and $|S_i|\geqslant n-2$ for all $i:2\leqslant i\leqslant n$. Thus, if $m\leqslant n-2$ then any selection of m S_i's will certainly contain at least m distinct elements since $|S_i|\geqslant n-2$ for all $i:1\leqslant i\leqslant n$. Each symbol of S appears at most twice in P and so is a member of at least $n-2$ of the S_i's. Consequently, if $m\geqslant n-1$, the union of any m S_i's will contain every element of S. Thus, for every $m:1\leqslant m\leqslant n$, the union of any m S_i's contains at least m distinct elements and it follows, by Hall's Theorem, that $S_1,S_2,\dots,S_n$ have a system of distinct representatives, as required. $\square$

We finish this section with the following conjecture:

Conjecture 2.9. Let $k\geqslant 3$ and let P be a partial $(2k-1)\times(2k-1)$ latin square on symbols $1,2,\dots,2k-1$. Then P is avoidable.

3. Intricacy of avoiding latin squares

Our first result gives the precise value of the intricacy of avoiding partial latin squares.

Theorem 3.1. *For $n>1$, the intricacy of avoiding partial $n\times n$ latin squares is 2.*

Proof. We show that, for all $n>1$, any partial $n\times n$ latin square P on $1,2,\dots,n$ is either avoidable, using symbols $1,2,\dots,n$, or can be partitioned into two partial $n\times n$ latin squares, each of which is avoidable using $1,2,\dots,n$.

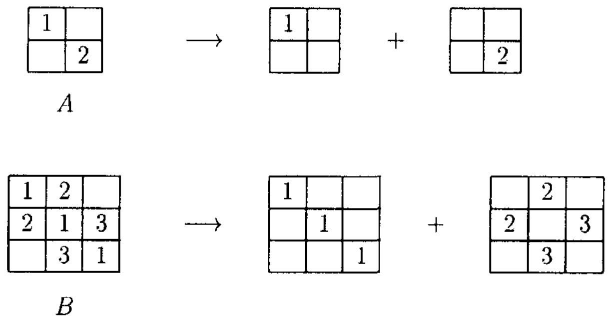

Fig. 6.

We begin by considering some small cases. It was shown in [1] that there is only one unavoidable partial 2×2 latin square, namely that of the form of square A in Fig. 6. Similarly, any unavoidable partial 3×3 latin square has the form of square B of the same figure. In this figure we show how each of squares A and B can be partitioned into two avoidable partial latin squares.

If $k > 1$ and $n = 2k$ then any partial $n \times n$ latin square is avoidable, by Theorem 2.3. So it remains to consider the partial $n \times n$ latin squares of odd order (with $n \geqslant 5$). We show that any such partial latin square can always be partitioned into two avoidable partial latin squares. We deal first with the case where $n \geqslant 7$, leaving the case $n = 5$ to be considered separately.

Let $k \geqslant 4$ and suppose $n = 2k - 1$. Then, by Theorem 2.7, any partial $n \times n$ latin square on $1, 2, \ldots, n$ having empty last row and column is avoidable. Thus, if we partition P into two partial $n \times n$ latin squares Q and R, where Q contains all those entries in the intersection of the first $n - 1$ rows and columns of P, and R contains all the entries from the last row and column, then both Q and R are avoidable, by Theorem 2.7 and Lemma 2.8, respectively.

Finally, we consider the case $n = 5$. Let $M(P)$ be the maximum number of entries of any one symbol in P. Then clearly $1 \leqslant M(P) \leqslant 5$. We consider three cases:

1. $M(P) = 5$,
2. $M(P) = 4$,
3. $M(P) \leqslant 3$.

We may assume without loss of generality that, in each of the above cases, symbol 1 appears $M(P)$ times in P and that all entries of symbol 1 appear in the main diagonal cells of P.

Cases 1 and 2: In cases 1 and 2, P may be partitioned into two partial 5×5 latin squares Q and R, as shown in Fig. 7. Square R is of the form of a 5×5 chessboard square of the type described in Theorem 1.2 and is, as such, avoidable. We show that Q is also avoidable.

Label the cells of Q with the symbols x_1, x_2, x_3, x_4, x_5 in the form of the latin square shown in Fig. 8.

1	*	*	*	*
*	1	*	*	*
*	*	1	*	*
*	*	*	1	*
*	*	*	*	*

P

$\longrightarrow$

1	*	*		
*	1	*		
*	*	1		
			1	*
			*	*

Q

$+$

			*	*
			*	*
			*	*
*	*	*		
*	*	*		

R

Fig. 7.

x_1	x_2	x_3	x_4	x_5
x_5	x_1	x_2	x_3	x_4
x_4	x_5	x_1	x_2	x_3
x_3	x_4	x_5	x_1	x_2
x_2	x_3	x_4	x_5	x_1

Q

Fig. 8.

For each $i : 1 \leqslant i \leqslant 5$, let X_i be the set of symbols which are each suitable for any cell of Q labelled x_i. Then any system of distinct representatives of X_1, X_2, X_3, X_4, X_5 gives rise to a 5×5 latin square L, which differs from Q in every cell, by placing the chosen representative of X_i in every cell of P which is labelled x_i.

We show that, for each $m : 1 \leqslant m \leqslant 5$, any m of the X_i's have between them at least m distinct elements; it then follows, by Hall's Theorem, that X_1, X_2, X_3, X_4, X_5 have a system of distinct representatives.

By considering the non-empty cells of Q we see that $|X_1| \geqslant 3$, $|X_2| \geqslant 2$, $|X_3| \geqslant 4$, $|X_4| \geqslant 4$ and $|X_5| \geqslant 2$. It is then easy to see that, for $m \leqslant 4$, any selection of m of the X_i's will contain at least one X_i for which $|X_i| \geqslant m$ and thus that between them they will have at least m distinct elements.

Taking X_1, X_2, X_3, X_4, X_5 together, we consider cases 1 and 2 separately. In case 1, there are certainly five distinct elements in this union since symbol 1 appears in each of X_2, X_3, X_4, X_5 and symbols 2, 3, 4 and 5 each appear in X_1. In case 2, each symbol appears at most four times in Q and consequently appears in at least one X_i, so again the union will contain every symbol.

Case 3: Finally, if case 3 occurs, no symbol appears more than three times in P. In this case we partition P into two partial latin squares V and W, with V containing all the entries of symbols 1, 2 and 3 and W containing the entries of symbols 4 and 5. We show that V is avoidable; then W will be avoidable by a similar argument.

Again we label the cells of V with the symbols x_1, x_2, x_3, x_4, x_5 in the form of the latin square of Fig. 8. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$. For each symbol $k : 1 \leqslant k \leqslant 5$ define the set Y_k to be the subset of X in which x_i is present if no cell of V labelled x_i contains symbol k.

Recall that all entries of symbol 1 appear in the main diagonal of V and that V contains no entries of symbols 4 or 5. Thus we have $|Y_1| \geqslant 4$, $|Y_2| \geqslant 2$, $|Y_3| \geqslant 2$, $|Y_4| = 5$ and $|Y_5| = 5$. It is easy to see that any selection of m Y_k's contains at least one Y_k for which $|Y_k| \geqslant m$ and hence has at least m distinct elements. Then, by Hall's Theorem, Y_1, Y_2, Y_3, Y_4, Y_5 have a system of distinct representatives which gives rise to a 5×5 latin square, which differs from V in every cell, by taking the chosen representative of each Y_k and placing symbol k in each cell of V having that label.

We have thus shown that, for all $n > 1$, any partial $n \times n$ latin square on $1, 2, \ldots, n$ is either avoidable or can be partitioned into two avoidable partial $n \times n$ latin squares; by the definition of intricacy, it follows that, for $n > 1$, the problem of avoiding partial $n \times n$ latin squares has intricacy 2. □

A natural progression at this point seems to be to consider the intricacy, for $n > 1$, of avoiding $n \times n$ arrays, having at most one entry per cell, which are not necessarily latin. Again the problem is a fair construction problem and it is certainly intricate as it is easy to find examples of $n \times n$ arrays on $1, 2, \ldots, n$ which are not latin and which are unavoidable using symbols $1, 2, \ldots, n$.

We have a theorem which shows that the intricacy of the problem is either 2 or 3.

Theorem 3.2. *For $n > 1$, let k be the intricacy of avoiding $n \times n$ arrays on $1, 2, \ldots, n$, in which every cell contains at most one symbol. Then $2 \leqslant k \leqslant 3$.*

Proof. We show that, for $n > 1$, any unavoidable $n \times n$ array on $1, 2, \ldots, n$ can be partitioned into three or fewer $n \times n$ squares on $1, 2, \ldots, n$, each of which is avoidable using symbols $1, 2, \ldots, n$.

We begin by considering the case $n = 2$. Now, any unavoidable 2×2 array on $1, 2$ will either be of the form of or contained in one of squares A, B, C or D of Fig. 9. Each of these squares can be partitioned into two avoidable 2×2 squares, as shown in the figure.

Similarly, any 3×3 square on $1, 2, 3$ can be partitioned into three 3×3 squares as shown in Fig. 10. Each of these three squares will be of the form of one of the three partial 3×3 squares of Fig. 11, each of which is avoidable.

If $k \geqslant 2$ and $n = 2k$ then any unavoidable $n \times n$ square on $1, 2, \ldots, n$ may be partitioned into two $n \times n$ chessboard squares, each of which is avoidable by Theorem 1.2. Fig. 12 shows, for example, how a 4×4 square may be partitioned in this way.

It now remains to consider the case $n = 2k - 1$, for $k \geqslant 3$. Let F be a $(2k-1) \times (2k-1)$ square, on $1, 2, \ldots, 2k - 1$, and divide F into five blocks, A, B, C, D and E, as shown in Fig. 13. We partition F into three squares G, H and J, containing blocks A and E, blocks C and D and block B respectively, then show that each of squares G, H and J is avoidable.

Odd-order chessboard squares are of two basic types, namely those with white corner cells and those with black. From Theorem 1.2 we know that for $k \geqslant 3$, any $(2k - 1) \times (2k - 1)$ chessboard square with white corner cells (with any entries in

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & \\ \hline & 1 \\ \hline \end{array} + \begin{array}{|c|c|}\hline & 1 \\ \hline 1 & \\ \hline \end{array}$$

A

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 2 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & \\ \hline & \\ \hline \end{array} + \begin{array}{|c|c|}\hline & 1 \\ \hline 1 & 2 \\ \hline \end{array}$$

B

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & \\ \hline 2 & \\ \hline \end{array} + \begin{array}{|c|c|}\hline & 1 \\ \hline & 2 \\ \hline \end{array}$$

C

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline & \\ \hline \end{array} + \begin{array}{|c|c|}\hline & \\ \hline 1 & 2 \\ \hline \end{array}$$

D

Fig. 9.

$$\begin{array}{|c|c|c|}\hline * & * & * \\ \hline * & * & * \\ \hline * & * & * \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline * & & \\ \hline & * & \\ \hline & & * \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline & * & \\ \hline & & * \\ \hline * & & \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline & & * \\ \hline * & & \\ \hline & * & \\ \hline \end{array}$$

Fig. 10.

$$\begin{array}{|c|c|c|}\hline 1 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & & \\ \hline & 1 & \\ \hline & & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & & \\ \hline & 2 & \\ \hline & & 3 \\ \hline \end{array}$$

Fig. 11.

$$\begin{array}{|c|c|c|c|}\hline * & * & * & * \\ \hline * & * & * & * \\ \hline * & * & * & * \\ \hline * & * & * & * \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline * & & * & \\ \hline & * & & * \\ \hline * & & * & \\ \hline & * & & * \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline & * & & * \\ \hline * & & * & \\ \hline & * & & * \\ \hline * & & * & \\ \hline \end{array}$$

Fig. 12.

black cells) is avoidable. Furthermore, it is easy to see that the rows and columns of such a chessboard square can be rearranged in such a way that the black cells (and thus any entries) are contained within two independent blocks of sizes $k \times (k-1)$ and $(k-1) \times k$.

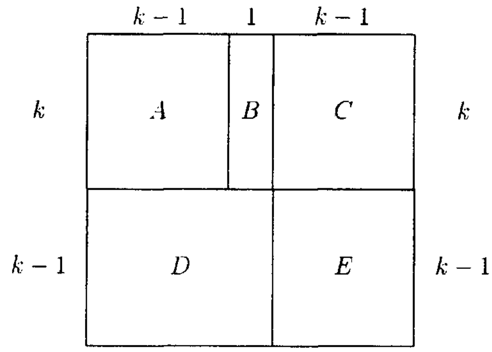

Fig. 13.

For each of squares G, H and J, it is possible to find at most two such independent blocks, of sizes $k \times (k-1)$ and $(k-1) \times k$, which between them contain all the non-empty cells of the square. Therefore, each of squares G, H and J is of the form of a chessboard square, of the type described, and is thus avoidable.

We have thus shown that, for $n > 1$, any $n \times n$ array on $1, 2, \ldots, n$ is either avoidable or can be partitioned into three or less avoidable arrays; it follows that the intricacy of avoiding $n \times n$ arrays on $1, 2, \ldots, n$ is either 2 or 3. □

Conjecture 3.3. For $n > 1$, the intricacy of avoiding $n \times n$ arrays, on symbols $1, 2, \ldots, n$, in which every cell contains at most one symbol, is 2.

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