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A NOTE ON LATIN SQUARES WITH RESTRICTED SUPPORT

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The purpose of this note is to give a simple theorem which hopefully will inspire some reader to more profound explorations. First we give some definitions. A *partial $n \times n$ column-latin square L* on $1, 2, \ldots, n$ is an $n \times n$ array filled with the symbols $1, 2, \ldots, n$ in such a way that every cell contains at most one symbol, and every symbol occurs at most once in every column. The array L is a *latin square* if, in addition, every symbol occurs exactly once in every row and column.

Theorem. *Let $n = 2^k$ and let L be a partial $n \times n$ column-latin square on $1, 2, \ldots, n$ with empty last column. Then there exists an $n \times n$ latin square A on the same symbols which differs from L in every cell.*

Proof. We use induction on k. The theorem is obviously true when $k = 0$. Assume that the theorem has been proved for order m and let $n = 2m$. By rearranging rows if necessary (and filling in some empty cells perhaps), we may assume that the mth column of L has the entries $1, 2, \ldots, 2m$ in that order. If we suppress the symbols $1, 2, \ldots, m$ in the upper left $m \times m$ quadrant B and the lower right $m \times m$ quadrant E in L, we find ourselves with a pair of partial column-latin squares H and I on $m + 1, m + 2, \ldots, 2m$ which both have empty last columns. Therefore we can find a pair of latin squares F and G on $m + 1, m + 2, \ldots, 2m$, without any entries in common with H and I respectively, and certainly not with B and E either. Similarly, by suppressing the symbols $m + 1, m + 2, \ldots, 2m$ in the upper right $m \times m$ quadrant C and lower left $m \times m$ quadrant D in L, and applying the theorem, we find a pair of latin squares J and K on the symbols $1, 2, \ldots, m$, which fit into the upper right and lower left corner of L respectively, without any entries in common with C and D. Together F, J, G and K make up A. $\square$

The theorem is not valid for every n as seen by example below.

$$L = \begin{matrix} 1 & 1 & * \\ 3 & 2 & * \\ 2 & 3 & * \end{matrix}$$

However, this is likely to be the only exception. In general perhaps the following is true for some positive constant c, which could be as large as $\frac{1}{3}$, say.

Conjecture. *Let L be an $n \times n$ array of m-sets from a set of symbols* $1, 2, \ldots, n$ *where every symbol is used at most $m \leqslant cn$ times in each row and column. Then there exists an $n \times n$ latin square A on* $1, 2, \ldots n$ *with entries in the complement of L.*

A positive answer could have some impact on the following question.

Dinitz' problem.

Given an $m \times m$ array of m-sets, is it always possible to choose one element from each set, keeping the chosen elements distinct in every row and column?

For some related material see the references.

References

[1] B. Bollobás and A.J. Harris, List-colourings of graphs, Graphs and Combinatorics 1 (1985) 115–127.
[2] A.G. Chetwynd and R. Häggkvist, A note on list-colourings, to appear in J. Graph Theory.
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