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Some partial Latin cubes and their completions

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ABSTRACT

It is well known that all $n \times n$ partial Latin squares with at most $n-1$ entries are completable. Our intent is to extend this well known statement to partial Latin cubes. We show that if an $n \times n \times n$ partial Latin cube contains at most $n - 1$ entries, no two of which occupy the same row, then the partial Latin cube is completable. Also included in this paper is the problem of completing $2 \times n \times n$ partial Latin boxes with at most $n - 1$ entries. Given certain sufficient conditions, we show when such partial Latin boxes are completable and then extendable to a deeper Latin box.

1. Introduction

One statement that is central to the theory of partial Latin squares is Smetaniuk's theorem [11]; a confirmation of the famous Evans conjecture (1960).

Theorem 1 (*Smetaniuk [11]*)**.** *Every partial Latin square of order n with at most $n - 1$ entries can be completed.*

There have been many attempts at generalizing Theorem 1. Some of these do so by generalizing the Latin square structure and others by generalizing the Evans condition; that is, the condition of having at most $n - 1$ entries. If at least n entries is assumed, then it is necessary to insist on some structure for the filled cells. Clearly this is the case as, in general, $n - 1$ entries are best possible (see Fig. 1). In this section we present two generalizations of Theorem 1 (one of them is a conjecture) and in Section 2 we propose a third generalization.

Let $r \geq 1$ be a positive integer. An r-semi Latin square of order n is an $n \times n$ array of nr symbols such that each symbol appears once in each row and column and each cell contains r symbols. It is assumed that a partial r-semi Latin square does not contain cells with fewer than r symbols. The first

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1	2	3	4	
				5

1				
2				
3				
4				
	5			

1				
	1			
		1		
			1	
				2

Fig. 1. Incompletable partial Latin squares of order 5 with 5 entries.

generalization [8] states that the Evans condition is sufficient for completing partial r-semi Latin squares.

Theorem 2. *Every partial r-semi Latin square of order n with at most $n-1$ cells filled can be completed.*

Prior to Theorem 2, Häggkvist (1979) proposed a conjecture for partial Latin squares of order nr that, if true, generalizes the Evans condition to $(n-1)r^2$ entries.

Conjecture 1. *Every partial Latin square of order nr with at most $n-1$ disjoint $r \times r$ squares filled can be completed.*

Conjecture 1 was confirmed for $n = 3$ [6]. The most recent work on Conjecture 1 [8] gives added conditions to the $n-1$ disjoint $r \times r$ squares.

Theorem 3. *Let P be a partial Latin square of order nr with at most $n-1$ disjoint $r \times r$ squares filled as Latin squares such that if two $r \times r$ squares share a column, then they share r columns. Then P can be completed.*

Theorem 4. *If P is a partial Latin square of order nr with at most $n-1$ disjoint $r \times r$ squares filled in distinct columns, then there is a permutation of the symbols appearing in each row of each fixed $r \times r$ square so that P can be completed.*

In this paper we consider the problem of completing partial Latin cubes and, more generally, partial Latin boxes that comply with the Evans condition. A *partial Latin cube* P of order n is a $n \times n \times n$ array of n symbols such that each $n \times n$ sub array is a partial Latin square. The set of n symbols is $[n] = \{1, 2, \ldots, n\}$ unless otherwise stated. If each of the $n \times n$ sub arrays are Latin squares, then P is called a *Latin cube*. In what follows, we will make use of the following conventions. Coordinates 2 and 3 of an $n \times n \times n$ array refer to row and column respectively and coordinate 1 refers to the horizontal layer that coordinates 2 and 3 occupy (horizontal layers are labeled in increasing order from the bottom up). For $i, j, k, l \in [n]$, we say that $(i, j, k, l) \in P$ if and only if cell (j, k) in horizontal layer i contains symbol l. The 4-tuples in P with $i \in [n]$ in the first coordinate make up the ith horizontal partial Latin square which we denote by P_i. For symbol l appearing in cell (j, k) of P_i, we write $(j, k, l) \in P_i$. Similarly, the 4-tuples in P with $j \in [n]$ in the second coordinate refer to a vertical partial Latin square as well as those with $k \in [n]$ in the third coordinate. A *fiber* is a column of a vertical partial Latin square in P.

2. The three-dimensional Evans conjecture

In this section we consider a 3-dimensional version of the Evans conjecture.

Conjecture 2. *If P is a partial Latin cube of order n with at most $n-1$ entries, then P can be completed to a Latin cube of order n.*

It should be noted that Conjecture 2, if true, is best possible as it is best possible for partial Latin squares. Indeed, if n entries appear in P_1 forbidding P_1 to be completed (see Fig. 1), then clearly P can not be completed. Supposing that we require n entries to appear in more than one layer, there still remain incompletable partial Latin cubes (see Fig. 2).

The following theorem by Denley and Öhman [7] is the strongest known result towards a confirmation of Conjecture 2.

Theorem 5. *Let P be a partial Latin cube all of whose at most $n-1$ entries have either first coordinate 1 or third coordinate 1. Then P is completable.*

1	2	3	4	

				5

Fig. 2. Two layers of an incompletable partial Latin cube of order 5.

According to the hypothesis of Theorem 5, all entries appear either in P_1 or in the first column of P_i for $i \in [n]$. We wish to allow a more scattered distribution of entries. Towards this end we present a lemma that describes a technique for constructing Latin cubes from Latin squares.

Lemma 1. *Let L_1 and L_2 be Latin squares of order n on the symbol set X. The set given by $\{(i, j, k, l) : (j, k, x) \in L_1$ and $(i, l, x) \in L_2$ for some $x \in X\}$ is a Latin cube of order n.*

The construction in Lemma 1 is known in the literature. For example, in Section 5 of [10] the authors use this construction to compose quasigroups. It is easy to see that the set given in Lemma 1 is a Latin cube Q by observing that symbol l in layer i of Q covers one and only one symbol in L_1. Furthermore, symbol l will cover each of the symbols in L_1.

With the use of Lemma 1, we can now present our main result.

Theorem 6. *Let P be a partial Latin cube of order n with at most $n - 1$ entries such that all filled cells appear in distinct fibers. Then P can be completed to a Latin cube of order n.*

Proof. Without loss of generality we may assume that P is a partial Latin cube of order n with exactly $n - 1$ entries such that all filled cells appear in distinct fibers. Let $X = \{(l, i) :$ symbol l appears in layer i of $P\}$. Certainly $|X| = n - 1$. Let $X' = X \cup \{(a, b)\}$ where $(a, b) \notin X$.

We construct two partial Latin squares on the symbol set X as follows. Let P_1 be an $n \times n$ partial array on X where symbol $(l, i) \in X$ appears in cell (j, k) if and only if $(i, j, k, l) \in P$. Since all filled cells in P appear in distinct fibers, P_1 is a partial Latin square. Since $|X| = n - 1$, P_1 contains $n - 1$ entries. By Theorem 1, P_1 completes to a Latin square L_1 of order n on the symbol set X'.

Let P_2 be an $n \times n$ partial array on X where symbol $(l, i) \in X$ appears in cell (i, l). Certainly P_2 is a partial Latin square and certainly P_2 contains $n - 1$ entries. By Theorem 1, P_2 completes to a Latin square L_2 of order n on the symbol set X'.

We now construct an $n \times n \times n$ array P' such that $(i, j, k, l) \in P'$ if and only if there is an $x \in X$ such that $(j, k, x) \in L_1$ and $(i, l, x) \in L_2$. By Lemma 1, P' is a Latin cube and clearly P is a subset of P'. $\square$

3. Completing partial Latin boxes

In this section we consider the problem of completing partial $2 \times n \times n$ Latin boxes. Our consideration is in the spirit of the previous section; given the Evans condition, is a $2 \times n \times n$ partial Latin box completable?

3.1. Avoiding multiple partial Latin squares

Let P be a partial Latin square of order n. P is called *avoidable* if, for every set of n symbols, there is a Latin square L of order n such that $(j, k, i) \notin L$ whenever $(j, k, i) \in P$ for $i, j, k \in [n]$. Recent work of Öhman and Cavenagh [1,2] and Kuhl and Denley [9] along with seminal work by Chetwynd and Rhodes [3] shows that all partial Latin squares of order at least 4 are avoidable.

Theorem 7. *Every partial Latin square of order $k \geq 4$ is avoidable.*

It is necessary that $k \geq 4$ as there are unavoidable partial Latin squares of orders 2 and 3. As shown by Chetwynd and Rhodes [3], Fig. 3 contains all unavoidable partial Latin squares of orders 2 and 3 up to isotopisms.

Avoidability is a natural topic of interest when completing partial Latin boxes or extending Latin boxes. Indeed, for extending a $k \times n \times n$ Latin box P to a $l \times n \times n$ Latin box for $k < l \leq n$, we are

1	
	2

1	2	3
3		1
2	1	

Fig. 3. Unavoidable partial Latin squares of order 2 and 3.

required to find $l - k$ Latin squares that not only avoid one another but also avoid each of the original k Latin squares in P. Thus, it is helpful to know not only that partial Latin squares can be avoided, but also that partial Latin squares can be avoided simultaneously. Two theorems in this vein are the following, due to Chetwynd and Rhodes [4] and Cutler and Öhman [5] respectively.

Theorem 8. *Let $k > 3240$ and let F be an array of order $4k$ on $\{1, 2, \ldots, 4k\}$ in which every cell contains at most two symbols and every symbol appears at most twice in every row and column. Then F is avoidable.*

Theorem 9. *Let $m \in \mathbb{N}$. There exists a constant $c = c(m)$ such that if $k > c = c(m)$ and if F is an array of order $2mk$ on $[2mk]$ in which every cell contains at most m symbols and every symbol appears at most m times in every row and column, then F is avoidable.*

Theorem 8 implies that two partial Latin squares of order $4k$ can be avoided simultaneously for $k > 3240$ and Theorem 9 implies that m partial Latin squares of order $2mk$ can be avoided simultaneously for k big enough. In this section we continue efforts being made on avoiding multiple partial Latin squares.

As stated earlier, Fig. 3 shows unavoidable partial Latin squares of orders 2 and 3. In the proofs that follow, it is possible that we will encounter these partial Latin squares. If there is no Latin square on $\{a, b\}$ avoiding P, a partial Latin square of order 2, then the symbols a and b are said to form a *bad diagonal* in P. If there is no Latin square on $\{a, b, c\}$ avoiding P, a partial Latin square of order 3, then the symbols a, b, and c are said to form a *bad configuration* in P. An important note for the following proofs is that if P is a partial Latin square of order 2, then P contains at most one bad diagonal on $\{a, b\}$. Similarly, if P is a partial Latin square of order 3, then P contains at most one bad configuration on $\{a, b, c\}$.

There is another kind of configuration on the symbol set $\{a, b, c\}$ that we wish to categorize as bad and it is found in the following lemma [9].

Lemma 2. *Let A be a partial Latin square of order 3 on the symbol set $\{a, b, c\}$ such that*

1. *cell $(3, 3)$ is empty,*
2. *$\{(3, i, a), (j, 3, a)\} \nsubseteq A$ for any $i, j \in [3]$, and*
3. *$\{(3, i, b), (j, 3, b)\} \nsubseteq A$ for any $i, j \in [3]$.*

If c does not appear in the upper left 2×2 subsquare of A, then there is a Latin square L of order 3 avoiding A such that $(3, 3, c) \in L$ unless A contains one of the following partial Latin squares.

a	b	
b	a	

b	a	
a	b	

If P is a partial Latin square of order 3 on $\{a, b, c\}$ with the conditions given in Lemma 2 and if there is no Latin square L on $\{a, b, c\}$ avoiding P such that $(3, 3, c) \in L$, then the symbols a, b, and c are said to form a *conditionally bad configuration*. Readily observed from Lemma 2 is that there is only one conditionally bad configuration on $\{a, b, c\}$ in P. Up to isotopisms, Fig. 4 contains the partial Latin squares that contain a bad diagonal on $\{a, b\}$, a bad configuration on $\{a, b, c\}$, and a conditionally bad configuration on $\{a, b, c\}$.

Implied from the next lemma is when 3 partial Latin squares of order 4 (2 of which contain one and only one entry in cell (4, 4)) can be avoided simultaneously.

a	b	
b	a	

b	a	
a	b	

a	
	b

a	b	c
c		a
b	a	

a	b	
b	a	

Fig. 4. Partial Latin squares with a bad diagonal, a bad configuration, and a bad conditional configuration respectively.

Lemma 3. *Let $i \in [4]$. Let P be a partial Latin square of order 4 such that $(4, 4, i) \notin P$ and i appears only in the last row and column of P. Then there is a Latin square L that avoids P with $(4, 4, i) \in L$.*

Proof. Without loss of generality suppose that $i = 1$. Then symbol 1 appears only in the last row and column of P. Let Y_{11}, Y_{12}, Y_{21}, and Y_{22} denote the upper left, upper right, lower left, and lower right 2×2 subsquares of P respectively. By permuting the first three rows and columns of P, we may assume that symbol 4 appears only in Y_{11} and Y_{22}.

Let Q be a Latin square of order 2 on the symbol set $\{X_1, X_2\}$ with X_1 on the leading diagonal. Let S_1, S_2 be a partition of the symbol set [4] such that $|S_1| = |S_2| = 2$ and $1 \in S_1$. If there are Latin squares of order 2 on the symbol set S_i avoiding the 2×2 subsquares of P corresponding to X_i for $i \in [2]$, then the partition S_1, S_2 is called a good partition. The existence of a good partition implies that there is a Latin square of order 4 on the symbol set [4] avoiding P. We need only to show the existence of a good partition and that the implied Latin square avoiding P is allowed to contain $(4, 4, 1)$.

Suppose that $(3, 4, 2), (4, 3, 2) \notin P$. Then there is no bad diagonal in Y_{11} and Y_{22} on $\{1, 2\}$. Since symbol 4 appears only in Y_{11} and Y_{22}, there is no bad diagonal in Y_{12} and Y_{21} on $\{3, 4\}$. Therefore $S_1 = \{1, 2\}$ and $S_2 = \{3, 4\}$ is a good partition of [4] and clearly the Latin square of order 2 on S_1 avoiding Y_{22} is allowed to contain $(2, 2, 1)$. Thus the Latin square avoiding P is allowed to contain $(4, 4, 1)$. Similarly, if $(3, 4, 3), (4, 3, 3) \notin P$, then $S_1 = \{1, 3\}$ and $S_2 = \{2, 4\}$ is a good partition and the Latin square of order 2 on S_1 avoiding Y_{22} is allowed to contain $(2, 2, 1)$.

We may therefore assume that $(3, 4, 2) \in P$ and $(4, 3, 3) \in P$. Note that if $(3, 4, 3) \in P$ and $(4, 3, 2) \in P$, then we may simply interchange symbol 2 with symbol 3. In this situation, at least one of cells $(4, 1)$ and $(4, 2)$ of P does not contain symbol 2. If $(4, 1, 2) \notin P$, then interchange columns 1 and 3 and then rows 1 and 3. If $(4, 2, 2) \notin P$, then interchange columns 2 and 3 and then rows 2 and 3. In this way, symbol 4 remains outside Y_{12} and Y_{21} and either $(3, 4, 2), (4, 3, 2) \notin P$ or $(3, 4, 3), (4, 3, 3) \notin P$. $\square$

Implied from the next theorem is when $k-1$ partial Latin squares of order k ($k-2$ of which contain one and only one entry in cell (k, k)) can be avoided simultaneously.

Theorem 10. *Let $i \in [4]$. Let $k \geq 9$ be a positive integer such that $k \not\equiv 2 \bmod 3$ and let P be a partial Latin square of order k such that $(k, k, i) \notin P$. Then there is a Latin square L that avoids P with $(k, k, i) \in L$.*

Proof. Let P be a partial Latin square of order k. We break the subsequent proof into two cases: $k = 3q$ and $k = 3q + 1$ for $q \geq 3$.

Case a. $k = 3q$ for $q \geq 3$

Without loss of generality we may assume that $i = 1$. We may assume that the symbols, if any, in cells $(3q, 3q-2)$, $(3q, 3q-1)$, $(3q-2, 3q)$, and $(3q-1, 3q)$ of P are distinct. Let Q be a Latin square of order q on the symbol set $\{X_1, X_2, \ldots, X_q\}$ such that $(q, q, X_q) \in Q$. Let Y denote the 3×3 subsquare of P corresponding to X_q in cell (q, q) of Q. We may further assume that Y does not contain symbol 1.

Let $S_1, \ldots, S_q$ be a partition of $[3q]$ such that $|S_i| = 3$ for $i \in [q]$ and $1 \in S_q$. If, for each $i \in [q]$, there are Latin squares of order 3 on the symbol set S_i avoiding the 3×3 subsquares in P corresponding to X_i, then we call the partition $S_1, \ldots, S_q$ a good partition. It follows that if there is a good partition of $[3q]$ and the Latin square on S_q avoiding Y is allowed to contain $(3, 3, 1)$, then P can be avoided by a Latin square L such that $(3q, 3q, 1) \in L$.

There are, in total,

$$\frac{(3q-1)!}{2(3!)^{q-1}}$$

partitions of $[3q]$ with $1 \in S_q$. Consider one of these partitions $S_1, \ldots, S_q$. It is possible that S_i forms a bad configuration in a 3×3 subsquare of P corresponding to X_i. It is also possible that S_q forms a conditionally bad configuration in Y. Recall that $1 \in S_q$, 1 does not appear in Y, and that cells $(3q, 3q-2)$, $(3q, 3q-1)$, $(3q-2, 3q)$, and $(3q-1, 3q)$ in P contain distinct symbols, if they contain symbols. By Lemma 3, $S_q = \{1, a, b\}$ forms a conditionally bad configuration if Y contains

a	b	
b	a	

If S_i forms a bad configuration in P, then we will subtract from the total number of partitions those that contain S_i. If S_q forms a conditionally bad configuration in Y, then we will subtract from the total number of partitions those that contain S_q. It follows that any leftover partitions are good partitions.

There are at most

$$\frac{(3q-4)!}{2(3!)^{q-2}}$$

partitions containing a fixed S_i for some $i \in [q]$. Let B be the set of all 3-tuples of symbols that form a bad configuration in the 3×3 subsquares of P or a conditionally bad configuration in Y. As discussed earlier, each 3×3 subsquare of P contains at most one bad configuration of symbols and Y contains at most one conditionally bad configuration of symbols. It follows that $|B| \leq q^2$. Then

$$\frac{(3q-1)!}{2(3!)^{q-1}} - |B|\frac{(3q-4)!}{2(3!)^{q-2}} \geq \frac{(3q-1)!}{2(3!)^{q-1}} - q^2\frac{(3q-4)!}{2(3!)^{q-2}} > 0$$

provided

$$\frac{(3q-1)(3q-2)(3q-3)}{6} - q^2 > 0.$$

Thus, for $q \geq 2$, there is a partition $S_1, \ldots, S_q$ such that $S_i \notin B$ for each i and so there exists a good partition of $[3q]$. By Lemma 2, P can be avoided by a Latin square containing $(3q, 3q, 1)$.

Case b. $k = 3q + 1$ for $q \geq 2$

Without loss of generality suppose that $i = 3q + 1$. Let P' be the partial Latin square formed from P by removing the last row and last column. Let Q be a Latin square of order q on the symbol set $\{X_1, \ldots, X_q\}$ such that $(i, i, X_i) \in Q$ for $i \in [q]$. For each $i \in [q]$, we use Y_i to denote the 3×3 subsquare in P' corresponding to X_i in cell (i, i) of Q. We may assume that Y_i does not contain symbol $3q + 1$ for each $i \in [q]$. From Case a, there is a good partition of $[3q]$. Thus there is a Latin square L' on $[3q]$ that avoids P'. In what follows, our goal is to use L' to build a Latin square L of order $3q + 1$ such that L avoids P and $(3q + 1, 3q + 1, 3q + 1) \in L$.

For each $i \in [q]$, let C_i denote the 3×1 rectangle in column $3q + 1$ of P occupying the same rows as Y_i and let R_i denote the 1×3 rectangle in row $3q + 1$ of P occupying the same columns as Y_i. For $i \in [q]$, let Z_i denote the partial Latin square of order 4 formed by joining together Y_i, C_i, R_i, and cell $(3q + 1, 3q + 1)$. Because symbol $3q + 1$ appears only in the last row or column of Z_i for each i, by Lemma 3, there is a Latin square L_i on the symbol set $\{S_i, 3q+1\}$ avoiding Z_i such that $(4, 4, 3q+1) \in L_i$ for each i.

We use L' and L_i for each i to build a Latin square of order $3q+1$ with $(3q+1, 3q+1, 3q+1) \in L$. Let L be an array of order $3q+1$ on the symbol set $[3q+1]$. L is filled in the following way where $r \equiv r' \bmod 3$ and $s \equiv s' \bmod 3$.

1. If (r, s) does not correspond to a cell in Z_i for $i \in [q]$ and $(r, s, l) \in L'$, then $(r, s, l) \in L$.
2. If (r, s) corresponds to a cell in Y_i for $i \in [q]$ and $(r', s', l) \in L_i$, then $(r, s, l) \in L$.
3. If (r, s) corresponds to a cell in C_i for $i \in [q]$ and $(r', 4, l) \in L_i$, then $(r, 3q+1, l) \in L$.
4. If (r, s) corresponds to a cell in R_i for $i \in [q]$ and $(4, s', l) \in L_i$, then $(3q+1, s, l) \in L$.
5. $(3q+1, 3q+1, 3q+1) \in L$.

The array L is a Latin square since L' is a Latin square and L_i is a Latin square on $S_i \cup \{3q+1\}$ with $(4, 4, 3q+1) \in L_i$ for each i. Furthermore, L avoids P since L' avoids P' and L_i avoids Z_i for each i. $\square$

Absent from Theorem 10 is the case $k = 3q+2$. However, we believe this case holds for $k \geq 9$ as well. In fact, we believe Theorem 10 holds for $k \geq 4$. Thus we give the following conjecture.

Conjecture 3. *Let $k \geq 4$ be a positive integer and let P be a partial Latin square of order k such that $(k, k, i) \notin P$ for some $i \in [k]$. Then there is a Latin square L that avoids P with $(k, k, i) \in L$.*

3.2. Completing and extending partial Latin boxes

With the use of Theorem 10, we can now prove when a partial Latin box consisting of two partial layers can be completed and then extended to a deeper Latin box.

Theorem 11. *Let $k \geq 9$ such that $k \not\equiv 2 \bmod 3$. Let P be a partial Latin square of order kn with at most $n-1$ entries and let Q be a partial Latin square of order kn that avoids P. Then P can be completed to avoid Q.*

Proof. Let $k \geq 9$ be a positive integer such that $k \not\equiv 2 \bmod 3$ and let P be a partial Latin square of order kn with precisely $n-1$ entries. Let Q be a partial Latin square that avoids P. Furthermore, let $(i, j)_k$ denote the $k \times k$ subsquare of P with rows $ik+1, \ldots, (i+1)k$ and columns $jk+1, \ldots, (j+1)k$ for $0 \leq i, j \leq n-1$. There are permutations of the rows, columns, and symbols of P such that $(i, j)_k$ contains at most one symbol from $[n]$ for all i and j, and $(i, j)_k$ and $(a, b)_k$ do not contain the same symbol if $a = i$ or $b = j$. We perform these permutations on both P and Q so that Q continues to avoid P.

Let P' denote the partial Latin square of order n on $\{X_1, \ldots, X_n\}$ such that $(i, j, X_l) \in P'$ if and only if $(i, j)_k$ contains symbol l. Then clearly P' contains $n-1$ entries and so, by Theorem 1, P' can be completed. Let $S_1, S_2, \ldots, S_n$ be a partition of $[kn]$ such that $|S_i| = k$ and $i \in S_i$ for each i. By Theorems 7 and 10, for each i and each appearance of X_i in P', there is a Latin square of order k on S_i completing the $k \times k$ subsquare in P corresponding to X_i and avoiding the $k \times k$ partial Latin square corresponding to X_i in Q. Thus P can be completed to avoid Q. $\square$

Theorem 12. *Let $k \geq 9$ such that $k \not\equiv 2 \bmod 3$. Let P be a partial $2 \times nk \times nk$ Latin box with at most $nk-1$ entries of which at most $n-1$ appear in P_2. Then P can be completed and then extended to a $(n+1) \times nk \times nk$ Latin box.*

Proof. Suppose that P contains at most $nk-1$ entries and that of these entries at most $n-1$ appear in P_2. We begin by completing P_1 and we do so by first filling the empty cells in P_1 that share a fiber with a fixed entry in P_2. Clearly these cells can be filled appropriately as there are at most $nk-1$ entries in P. P_1 now contains at most $nk-1$ entries and so P_1 is completable by Theorem 1. Since P_2 contains at most $n-1$ entries, P_2 can be completed to avoid P_1 by Theorem 11. Hence P is completable.

According to the proof of Theorem 11 there is a Latin square P' of order n on the symbol set $\{X_1, \ldots, X_n\}$ and a partition of $[nk]$, $S_1, \ldots, S_n$, such that $(i, j, X_l) \in P'$ provided there is a $k \times k$ Latin square on S_l in the (i, j)th $k \times k$ subsquare of P_2. Let Q be a Latin cube of order n on the symbol set $\{X_1, \ldots, X_n\}$ such that the first layer is P'. Let $Q_2, \ldots, Q_n$ be the remaining layers of Q. Consider Q_m for $2 \leq m \leq n$. We construct a Latin square L_m of order nk by placing a Latin square of order k on S_l in the (i, j)th $k \times k$ subsquare that corresponds to entry $(i, j, X_l) \in Q_m$. Certainly L_m avoids P_2 by virtue of Q being a Latin cube. By Theorem 7 the n^2 Latin squares of order k making up L_m can be chosen such that L_m avoids P_1. Thus the Latin squares $L_2, \ldots, L_n$ extend P to an $(n+1) \times nk \times nk$ Latin box. $\square$

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