Monoidal Categories and Multicategories

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Abstract

Monoidal categories are categories equipped with a notion of tensor product, encoded as a bifunctor. Multicategories are generalisations of categories where morphisms have finite lists of domains. Hermida (2000) showed that the category of monoidal categories is equivalent to the category of representable multicategories (those with a canonical factorisation method). Leinster (2003) reformulated this result but did not publish a proof. We examine monoidal categories and multicategories, culminating in a reconstruction of Leinster's proof. We provide an overview of the necessary basics of category theory, definitions and examples of monoidal categories and multicategories, and a thorough treatment of the proof.

1 Introduction

Category theory is the study of high-level mathematical structure. By abstracting away the details and instead looking at patterns across seemingly distinct constructions, it enables us to discover new and insightful perspectives on mathematics as a whole.

Categories have objects, morphisms (relationships from one to object to another) and a method of composing morphisms, subject to identity and associativity laws. Maps between categories, known as functors, let us study how relationships change in different contexts. In particular, they highlight which categories encode the same information in different ways, i.e. which categories are equivalent.

First proposed by Eilenberg and Mac Lane (1945), category theory has since come to dominate the landscape of pure mathematics. With this growth, the objects of study have diversified. In this document, we examine two of these specialised constructions: monoidal categories and multicategories.

A monoidal category is a category equipped with tensor product. That is, we can combine any two objects A and B to obtain another object $A \otimes B$, and similarly for morphisms, such that the tensor is a functor and satisfies certain coherence conditions (Barr and Wells, 1999). One notable example is the category of vector spaces (over a fixed field) and linear maps. Here, $\otimes$ takes its more familiar form as tensor product in the sense of linear algebra. Monoidal categories are strikingly common, with many applications in physics and computer science (Baez and Stay, 2010).

Multicategories meanwhile have morphisms with finite lists $A_1 \dots A_n$ of domains instead of only one (Leinster, 2003). We therefore view these multimorphisms as having multiple inputs and one output. A motivating example is the multicategory of vector spaces (over a fixed field) and multilinear maps. Multicategories have had considerable influence in logic and linguistics (Lambek, 1989).

As shown above, we can thus turn vector spaces and multi/tensored linear maps into two different categorical constructions containing the same information. Surely then these contexts must be related?

In fact, this is only the case for a certain type of multicategory. Hermida (2000) showed in his Theorem 9.8 that the category of monoidal categories **MonCat** is equivalent to the category of so-called representable multicategories, **RMulticat**. While every monoidal category is a multicategory in disguise, the converse is not true.

Leinster (2003) restructured this result in his Theorem 3.3.4, suggesting an alternative and more concrete proof method. However, he did not publish details of this proof. It appears no complete proof other than Hermida's highly abstract treatment has since appeared in the literature, despite frequent references to the result (see Pisani (2013) or Bourke and Lack (2018) for instance).

In this document, we seek to reconstruct Leinster's proof of Hermida's result (which we will consequently name the Leinster-Hermida Equivalence Theorem). We begin by covering the basics of category theory for uninitiated readers. We then introduce monoidal categories, focusing on unbiased monoidal categories, the generalisation Leinster used. We then describe multicategories and representable multicategories. Finally, with all these tools at our disposal, we prove the Leinster-Hermida Equivalence Theorem.

Given the stated ubiquity of both monoidal categories and multicategories, this result is significant. It provides a new perspective for those accustomed to working with only one definition, a perspective that may clear the way for future discoveries.

We use the convention $\mathbb{N} = \{1, 2, 3 \dots\}$. The natural numbers including zero will thus be denoted $\mathbb{N}_0 = \{0, 1, 2 \dots\}$. A list $A_1 \dots A_n$ with $n = 0$ is taken to be empty. We assume the axiom of choice.

This document is typeset at 10pt font in LaTeX and runs to approximately 40 pages without diagrams.

2 Basic Concepts in Category Theory

We dedicate this section to outlining the fundamental concepts in category theory we will use in our investigation. Anyone familiar with category theory may skip this section, with the understanding that some notational conventions may differ from those they are familiar with.

We begin with three essential concepts:

- Categories, the setting for category theory.
- Functors, maps between categories.
- Natural transformations, maps between functors.

Our definitions are standard, though we adopt and adapt the syntax of Leinster (2014).

2.1 Categories

The fundamental construction in category theory is, unsurprisingly, a category. In the most basic sense, a category is a context for doing mathematics. Almost all mathematical settings have:

- Objects with particular structure.
- Directed connections between objects which preserve that structure.
- A method of placing two such structure-preserving connections end-to-end to produce another, such that this operation is associative and has identities.

A category neatly encapsulates all these properties into 'a bird's eye view of mathematics' (Leinster, 2014, p. 1). Category theory zooms out from the specific properties of any one context, instead studying how contexts themselves interact.

Definition 2.1.1 (Category)

A *category* $\mathcal{C}$ consists of:

- A class $\text{ob}(\mathcal{C})$ called the class of *objects* of $\mathcal{C}$.
- For all $A, B \in \text{ob}(\mathcal{C})$, a class $\mathcal{C}(A, B)$ called the *homset* of *morphisms* from A to B.
- For all $A, B, C \in \text{ob}(\mathcal{C})$, a function $\circ : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$, $(g, f) \mapsto g \circ f$ called *composition*.
- For all $A \in \text{ob}(\mathcal{C})$, a morphism $1_A \in \mathcal{C}(A, A)$ called the *identity* on A.

Composition is subject to the following conditions $\forall A, B, C, D \in \text{ob}(\mathcal{C})$:

- For all morphisms $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$ and $h \in \mathcal{C}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$. That is, composition is associative.
- For all morphisms $f \in \mathcal{C}(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.

Throughout this document, we shall use the notation $A \in \mathcal{C}$ to mean $A \in \text{ob}(\mathcal{C})$. We may also write $f : A \to B$ to mean $f \in \mathcal{C}(A, B)$.

Remark 2.1.2

The use of the term *class* above may present some difficulties. Strictly speaking, in general we cannot define $\text{ob}(\mathcal{C})$ or $\mathcal{C}(A, B)$ to be sets, due to issues with naive set theory (Leinster, 2014, pp. 73 – 77). For instance, it is not possible to take $\text{ob}(\mathcal{C})$ to be the set of all sets, as we might innocently do in Example 2.1.3(d) below. The axioms of set theory forbid this, as such a construction immediately invokes Russell's paradox (does the set containing all sets not containing themselves contain itself?) and similar obstacles. We must be delicate and use classes instead to avoid these problems.

For the most part, we will ignore this distinction. The explicit categories we encounter will be *locally small* (for any two objects, the class of morphisms between them is in fact a set) and general constructions are assumed to share this property.

A class that is a genuine set is called a *small set*. The set of all small sets is well-defined, so we shall consider specific categories whose classes of objects contain only small sets, i.e. are themselves sets. Further, we can talk about an entire category being *small* if both $\text{ob}(\mathcal{C})$ and the class of all morphisms in $\mathcal{C}$ – that is, the union over every homset – are genuine sets (Leinster, 2003, p. 4). This will allow us to build the category of small categories (see Definition 2.4.5) without issue.

Example 2.1.3

For illustrative purposes, we list some simple example categories.

(a) Let $\text{ob}(\mathcal{C}) = \{*\}$ and $\mathcal{C}(*, *) = \{1_*\}$. That is, $\mathcal{C}$ has one object and one morphism. Then $\mathcal{C}$ is a category, called the *trivial category*.

(b) For any $\text{ob}(\mathcal{C})$ and any $A, B \in \mathcal{C}$ with $A \neq B$, let $\mathcal{C}(A, A) = \{1_A\}$, $\mathcal{C}(B, B) = \{1_B\}$ and $\mathcal{C}(A, B) = \mathcal{C}(B, A) = \varnothing$. This is the *discrete category* on $\text{ob}(\mathcal{C})$. In a discrete category, each object is only connected to itself.

(c) Let $\text{ob}(\mathcal{C}) = \{A, B\}$ and let the non-identity morphisms be given by $\mathcal{C}(A, B) = \{f\}$ only, so $\mathcal{C}(B, A) = \varnothing$. We then call $\mathcal{C}$ the *arrow category*, as it has only a single non-trivial "arrow", namely $f : A \to B$.

(d) The category **Set** has all (small) sets as objects and set functions as morphisms. Composition is regular function composition, which is associative.

(e) Similarly, **Grp** consists of all groups and group homomorphisms. The composition of two homomorphisms is itself a homomorphism, so we have a well-defined composition in this category. Associativity again follows from the associativity of function composition, while the identity on any group is trivially a homomorphism.

(f) Entirely analogous are the categories **Mon** of monoids and monoid homomorphisms, **Top** of topological spaces and continuous maps, $\mathbf{Top}_*$ of topological spaces with a base point and base point preserving continuous maps, and $\mathbf{Vect}_{\mathbb{K}}$ of vector spaces (over a fixed field $\mathbb{K}$) and linear maps.

(g) Let $\text{ob}(\mathcal{C}) = \mathbb{R}$ and for any $x, y \in \mathbb{R}$, define a unique morphism from x to y if and only if $x \leqslant y$. This satisfies the axioms of a category – identities and composition follow from the reflexivity and transitivity of $\leqslant$ respectively. Denote this category by $(\mathbb{R}, \leqslant)$.

(h) For any topological space X, its fundamental groupoid $\Pi_1(X)$ forms a category. Objects are points in X, and for $x, y \in X$, a morphism from x to y is an equivalence class of paths from x to y (where equivalence is derived from homotopy). We define composition to be path concatenation, which is preserved under homotopy and hence gives a well-defined composition operation (Spivak, 2014, pp. 259–260).

It should be noted that many of these are sets with extra structure and functions which respect that structure (**Set**, **Grp**, **Top**, etc.). Such constructions are arguably the most common and most intuitive categorical construction, but the other examples listed above demonstrate that there are many other possibilities.

Definition 2.1.4 (Commutative Diagram)

Let $\mathcal{C}$ be a category. For any objects of $\mathcal{C}$ and any morphisms between those objects, we may represent their structure by means of a directed graph. Nodes are objects and edges are morphisms with arrows indicating direction.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

We say such a diagram *commutes* (i.e. is a *commutative diagram*) if, for any two objects in the diagram, all composite morphisms between them are equal. That is, any two ways of following arrows to traverse from one object to the other are equal, where we compose morphisms in (reverse) order as we go.

The diagram above, for instance, commutes if and only if $h = g \circ f$.

Example 2.1.5

Consider the arrow category defined in Example 2.1.3(c). We may represent the entire category with the following diagram.

$$\begin{tikzcd} A \arrow[loop left, "1_A"] \arrow[r, "f"] & B \arrow[loop right, "1_B"] \end{tikzcd}$$

Since f is the only non-identity morphism here, by the properties of the identities 1_A and 1_B, the diagram commutes. The identities therefore provide little interesting information, and we usually omit them from all diagrams. We thus have the following simplification.

$$A \xrightarrow{f} B$$

The name arrow category is consequently very appropriate, as is the notation $f : A \to B$.

Definition 2.1.6 (Isomorphism)

Let $\mathcal{C}$ be a category, let $A, B \in \mathcal{C}$, and let $f : A \to B$ be a morphism. We say f is an *isomorphism* if $\exists g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. We call g the *inverse* of f.

In other words, f is an isomorphism if and only if the diagram below commutes.

$$\begin{tikzcd} A \arrow[loop left, "1_A"] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \arrow[loop right, "1_B"] \end{tikzcd}$$

For $A, B \in \mathcal{C}$, if $\exists f \in \mathcal{C}(A, B)$ an isomorphism, we say A and B are *isomorphic*, written $A \cong B$.

Proposition 2.1.7

Let $\mathcal{C}$ be a category and let $A, B \in \mathcal{C}$. Let $f : A \to B$ be an isomorphism with inverse $g : B \to A$. If $h : B \to A$ is such that $h \circ f = 1_A$ or $f \circ h = 1_B$, then $h = g$.

In other words, inverses are unique, and right or left inverses of isomorphisms are actual inverses.

Proof. First suppose $f \circ h = 1_B$. By definition,

$$\begin{aligned} g &= g \circ 1_B \text{ by identity} \\ &= g \circ (f \circ h) \\ &= (g \circ f) \circ h \text{ by associativity} \\ &= 1_A \circ h \\ &= h. \end{aligned}$$

The argument in the case $h \circ f = 1_A$ is similar. $\square$

Remark 2.1.8

The notion of isomorphism is prevalent in mathematics. A group homomorphism is an isomorphism if and only if it has an inverse which is also a group homomorphism. The same is true for linear maps, ring homomorphisms, and so on.

In these settings, we can show any set-theoretic inverse must have the same properties, e.g. the set-theoretic inverse of a group homomorphism is a group homomorphism. This is not true for all categories. In **Top** for example, the set-theoretic inverse of a continuous map need not be continuous. But a category-theoretic inverse always retains the defining property, since it is a morphism in the same category. Thus, an isomorphism in **Top** is a continuous map with continuous inverse.

Isomorphisms in **Set** meanwhile are simply bijections, as the setting requires no extra structure for morphisms. We might therefore write $A \cong B$ if two sets are in bijective correspondence.

Isomorphism captures the idea that objects are functionally the same, where the meaning of *functionally* depends on the working category. If one cares only for set functions, any relabelling of a set retains the same properties. This is why $(A \times B) \times C \cong A \times (B \times C)$ in **Set** (where $\times$ is cartesian product) and we can treat these as the same set, even though $A \times (B \times C) \neq (A \times B) \times C$ in a strict sense. We return to this idea later in our discussion of monoidal categories.

Definition 2.1.9 (Subcategory)

Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ has:

- Objects $\mathrm{ob}(\mathcal{D}) \subseteq \mathrm{ob}(\mathcal{C})$.
- For all $A, B \in \mathcal{D}$, homset $\mathcal{D}(A, B) \subseteq \mathcal{C}(A, B)$ such that $\mathcal{D}$ contains all identities from $\mathcal{C}$, and is closed under the composition induced from $\mathcal{C}$.

A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called *full* if $\mathcal{D}(A, B) = \mathcal{C}(A, B)\ \forall A, B \in \mathcal{D}$.

Definition 2.1.10 (Product Category)

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Their *product category* $\mathcal{C} \times \mathcal{D}$ has:

- Objects $\mathrm{ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{D})$.
- For all $A, A' \in \mathcal{C}$ and $B, B' \in \mathcal{D}$, homset $(\mathcal{C} \times \mathcal{D})((A, B), (A', B')) = \mathcal{C}(A, A') \times \mathcal{D}(B, B')$. That is, morphisms are pairs $(f, f') : A \times B \to A' \times B'$ for $f : A \to A'$ and $f' : B \to B'$.
- Composition $(g', g) \circ (f', f) = (g' \circ f', g \circ f)$ for all f, f', g, g' where $g \circ f$ and $g' \circ f'$ are defined.
- Identities $1_{A \times B} = (1_A, 1_B)\ \forall A \in \mathcal{C},\ \forall B \in \mathcal{D}$.

Here, $\times$ denotes the cartesian product of sets.

Remark 2.1.11

The notion of subcategory comes naturally from comparison with subset, subgroup, etc., and product categories are similarly analogous to cartesian products. One can easily verify both satisfy the definition of category.

We may do the obvious thing and extend the definition of product category, obtaining product categories of the form $\mathcal{C}_1 \times \mathcal{C}_2 \times \ldots \times \mathcal{C}_n$ for any $n \in \mathbb{N}$. Such product functors will lead the way to monoidal categories.

2.2 Functors

Functors are maps between categories, and can be thought of as a change in context or perspective. From one angle, we may value one property of an object above all others, but applying a functor rotates our worldview, asking us to value a different property.

Definition 2.2.1 (Functor)

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- A function $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, $A \mapsto F(A)$.
- For all $A, B \in \mathcal{C}$, a function $\mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$, $f \mapsto F(f)$.

The function on morphisms is subject to the following conditions $\forall A, B, C \in \mathcal{C}$:

- If $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$, then $F(g \circ f) = F(g) \circ F(f)$. That is, F preserves composition.
- Similarly, $F(1_A) = 1_{F(A)}$, so F preserves identities.

The above conditions are called the *functorality axioms*.

In some contexts, we may write FA for $F(A)$ and Ff for $F(f)$.

Example 2.2.2

As before, we provide some illustrative examples, some simple, some more involved.

(a) The *forgetful functor* between groups and sets is $U : \mathbf{Grp} \to \mathbf{Set}$ sending a group $(G, *)$ to its underlying set G and a homomorphism to its underlying set function. Notice U does nothing to these objects except forget they have the extra properties required to sit in **Grp**.

(b) Any category whose objects are specialised sets and whose morphisms are specialised set functions admits a similar forgetful functor into **Set**, such as $\mathbf{Top} \to \mathbf{Set}$ or $\mathbf{Mon} \to \mathbf{Set}$.

(c) Let A be a set. The *Kleene closure* of A is A^* consisting of the set of finite strings $a_1 \ldots a_n$ where characters are elements of A. Defining the empty string $\varnothing$ to be the identity, A^* becomes a monoid with concatenation as the binary operation (Barr and Wells, 1999, p. 25).

    Define a functor $F : \mathbf{Set} \to \mathbf{Mon}$ as follows. For $A \in \mathbf{Set}$, define $F(A) = A^*$. For a morphism $f \in \mathbf{Set}(A, B)$, define $F(f) \in \mathbf{Mon}(A, B)$ by giving its effect on elements as $F(f)(a_1 \ldots a_n) = f(a_1) \ldots f(a_n)$ $\forall a_1 \ldots a_n \in A$ and $F(f)(\varnothing) = \varnothing$. Then $F(A)$ is a monoid and $F(f)$ is a monoid homomorphism. Further, F is a functor, known as the *free monoid functor*.

(d) We can construct the free group functor $F : \textbf{Set} \to \textbf{Grp}$ by extending A^* to include artificial inverses a^{-1} $\forall a \in A$. These do not correspond to any existing elements – instead, they are defined to formally realise $aa^{-1} = a^{-1}a = \varnothing$ whenever such a string appears as a substring of an element of A^*. With the addition of such inverses, A^* satisfies the axioms of a group, so the functor is well-defined (Leinster, 2014, p. 19).

(e) Further free functor constructions exist for $\textbf{Vect}_{\mathbb{K}}$ and so on. In fact, all finite dimensional vector spaces are free vector spaces over their basis vectors by definition.

(f) The map $\pi_1 : \textbf{Top}_* \to \textbf{Grp}$ sending topological space (X, x) to its fundamental group $\pi_1(X, x)$ and $p : (X, x) \to (Y, y)$ to its pushforward map $p_* : \pi_1(X, x) \to \pi_1(Y, y)$ is a functor.

(g) Consider $(\mathbb{R}, \leqslant)$ from Example 2.1.3(g). Define $(\mathcal{P}(\mathbb{R}), \subseteq)$ to be the category where objects are subsets of $\mathbb{R}$, with a unique morphism from A to B if and only if $A \subseteq B$. Then we can define a functor $F : (\mathbb{R}, \leqslant) \to (\mathcal{P}(\mathbb{R}), \subseteq)$ by $F(x) = (-\infty, x)$ for $x \in \mathbb{R}$. If there exists a morphism $f : x \to y$, then $x \leqslant y$, so $F(x) \subseteq F(y)$. Thus, there exists a unique morphism $g : F(x) \to F(y)$. So we can define $F(f) = g$ to specify the functor's effect on morphisms.

Remark 2.2.3

For any category $\mathcal{C}$, there is an identity functor $1_{\mathcal{C}} : F \to F$, which does nothing to objects and morphisms.

Functors may also be composed. The composition of $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ is $G \circ F : \mathcal{C} \to \mathcal{E}$, which sends $A \in \mathcal{C}$ to $G(F(A)) \in \mathcal{E}$ and $f : A \to B$ to $G(F(f)) : G(F(A)) \to G(F(B))$. Here we see the utility in writing functor application without brackets, i.e. $GFf : GFA \to GFB$ in this case.

Consequently, functors act like morphisms in a category, and under certain conditions, they are. We return to this idea later.

Proposition 2.2.4

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, let $A, B \in \mathcal{C}$ and let $f : A \to B$ be an isomorphism with inverse $g : B \to A$. Then $F(f)$ is an isomorphism with inverse $F(g)$, i.e. $F(A) \cong F(B)$.

In particular, functors preserve isomorphisms and inverses.

Proof. By functorality,

$$\begin{aligned} 1_{F(A)} &= F(1_A) \\ &= F(g \circ f) \\ &= F(g) \circ F(f). \end{aligned}$$

Similarly, $1_{F(B)} = F(f) \circ F(g)$, so $F(f)$ is an isomorphism with inverse $F(g)$. $\square$

Definition 2.2.5 (Faithful, Full and Essentially Surjective)

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

- If $\mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$, $f \mapsto F(f)$ is injective $\forall A, B \in \mathcal{C}$, then we say F is *faithful*.
- If $\mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$, $f \mapsto F(f)$ is surjective $\forall A, B \in \mathcal{C}$, then we say F is *full*.
- If $\forall B \in \mathcal{D}$, $\exists A \in \mathcal{C}$ such that $F(A) \cong B$, then we say F is *essentially surjective on objects*.

Remark 2.2.6

Like many things in category theory, functors – and particularly their fullness and faithfulness – become clearer in the context of commutative diagrams.

Suppose we have objects $A, B, C \in \mathcal{C}$ arranged as shown in the left commutative diagram. In particular, $h = g \circ f_1 = g \circ f_2$, where we assume $f_1 \neq f_2$. What happens to the diagram when we apply a functor $F : \mathcal{C} \to \mathcal{D}$?

$$\begin{tikzcd}
A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[dr, "h"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F(A) \arrow[r, shift left, "F(f_1)"] \arrow[r, shift right, "F(f_2)"'] \arrow[dr, "F(h)"'] & F(B) \arrow[d, "F(g)"] \\
& F(C)
\end{tikzcd}$$

By the functorality axioms, $F(h) = F(g) \circ F(f_1) = F(g) \circ F(f_2)$. Identities are also implicitly preserved. The functor F therefore preserves commutativity of diagrams and the representation above is valid.

But our new diagram may contain information that is incomplete, redundant, or even both. Fullness and faithfulness can resolve this issue.

If F is not faithful, it may be that $F(f_1) = F(f_2)$. If F is not full, there may be new morphisms between $F(A)$, $F(B)$ and $F(C)$ that have no analogue in $\mathcal{C}$. In either case, we do not have an accurate picture of the situation in $\mathcal{D}$. We have duplicate morphisms, missing morphisms, or both.

However, if F is full and faithful, this cannot happen. The network of relationships between A, B and C maps exactly onto the network between $F(A)$, $F(B)$ and $F(C)$, i.e. the diagram on the right is the simplest complete description of morphisms between these objects. What those relationships *are* exactly may have changed – hence the use of different categories $\mathcal{C}$ and $\mathcal{D}$ with potentially different types of morphism – but we know their arrangement.

Yet our picture of the whole category is incomplete. We know the situation for the full subcategory $\mathcal{D}'$ with objects $F(A)$, $F(B)$ and $F(C)$, but have no idea about other objects and their relationships. Even if we apply F to every object in $\mathcal{C}$, we may have only a small window into $\mathcal{D}$.

We need a functor that is in addition essentially surjective. This provides an equivalence of categories, which we will explore later.

2.3 Natural Transformations

While functors are maps between categories, natural transformations go one further: maps between functors. They 'compare different transformations' between categories (Spivak, 2014, p. 143).

Natural transformations might initially seem obscure, but they are in fact fundamental to much of category theory. Eilenberg and Mac Lane (1945) sought to understand natural transformations in their seminal paper. Defining natural transformations required defining functors, and defining functors required defining categories. Category theory was thus born as a side-effect of studying natural transformations.

Definition 2.3.1 (Natural Transformation)

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\alpha : F \to G$ consists of, $\forall A \in \mathcal{C}$, a morphism $\alpha_A : F(A) \to G(A)$ in $\mathcal{D}$, such that $\forall B \in \mathcal{C}$ and $\forall f \in \mathcal{C}(A, B)$, the following diagram commutes.

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

The morphisms $\{\alpha_A : A \in \mathcal{C}\}$ are called the *components* of α. The above diagram is called the *naturality square* for α.

For notational convenience, we may sometimes specify a natural transformation by giving only its components, omitting a precise definition of the functors it maps between.

Example 2.3.2

Natural transformations are intimidating. We therefore provide a few in-depth examples.

(a) Let $n \in \mathbb{N}$ and let $\mathcal{C}_n$ denote the discrete category whose objects are $1, 2 \ldots n$. Define functors $\sigma : \mathcal{C}_n \to \mathcal{C}_n$ and $\tau : \mathcal{C}_n \to \mathcal{C}_n$ to be some permutations of $\{1, 2 \ldots n\}$. Note that this fully describes their effect on morphisms too, as they must send identities to identities, but identities are the only morphisms in this category. Then there is a natural transformation $\alpha : \sigma \to \tau$ if and only if $\sigma = \tau$.

This is because we require some $\alpha_i : \sigma(i) \to \tau(i)$ for each $i \in \{1 \ldots n\}$, but since the category is discrete, this is the case only when $\sigma(i) = \tau(i)$ and $\alpha_i = 1_i$. In such a case, all naturality squares commute – the only morphisms we could take to form the squares are identities, which automatically provide commutativity.

(b) Let **CRing** denote the category of commutative rings and ring homomorphisms. Consider commutative rings $R, S \in$ **CRing** and a ring homomorphism $f : R \to S$.

Let $n \in \mathbb{N}$. The sets $M_n(R)$ and $M_n(S)$ of $n \times n$ matrices with entries in R and S respectively are monoids under composition. This defines a functor M_n : **CRing** $\to$ **Mon** into the category of monoids.

But every commutative ring R can be viewed as a monoid under multiplication by ignoring its addition operation. So consider U : **CRing** $\to$ **Mon**, the forgetful functor sending a commutative ring to its underlying monoid.

We can thus ask: how are M_n and U related? One immediate example is the determinant function $\det_R : M_n(R) \to U(R)$. Since $\det(AB) = \det(A)\det(B)$, we see $\det_R$ is a monoid homomorphism, hence a morphism in **Mon**.

Further, the following naturality square commutes.

$$\begin{tikzcd} M_n(R) \arrow[r, "M_n(f)"] \arrow[d, "\det_R"'] & M_n(S) \arrow[d, "\det_S"] \\ U(R) \arrow[r, "U(f)"'] & U(S) \end{tikzcd}$$

Here, $M_n(f)$ is the monoid homomorphism applying f to each entry of a matrix, and $U(f)$ is the underlying monoid homomorphism of the ring homomorphism f, i.e. f itself.

To verify commutativity, let $(a_{ij}) \in M_n(R)$. Then, using the sum formula for the determinant of a matrix, we have

$$\begin{aligned}
(\det_S \circ M_n(f))(a_{ij}) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) f\left(a_{1\sigma(1)}\right) \cdots f\left(a_{n\sigma(n)}\right) \\
&= f\left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}\right) \quad \text{as } f \text{ is a ring homomorphism} \\
&= (f \circ \det_R)(a_{ij}) \\
&= (U(f) \circ \det_R)(a_{ij}).
\end{aligned}$$

Thus, $\det_S \circ M_n(f) = U(f) \circ \det_R$ as required. From this commutativity, we deduce that the family $\{\det_R : R \in \mathbf{CRing}\}$ defines a natural transformation $\det : M_n \to U$ (Leinster, 2014, pp. 29 – 30).

(c) Let $L : \mathbf{Set} \to \mathbf{Set}$ be the functor sending a set X to the set of all lists with entries in X. Then $L \circ L$ is also a functor, sending X to $L(L(X))$, the set of lists of lists with entries in X (Spivak, 2014, p. 144).

We can flatten a list of lists into a single list by concatenating its constituent lists in order. For example, $[[1,2],[1],[3,2]]$ flattens to $[1,2,1,3,2]$. Flattening like this is in fact a natural transformation. We can flatten the lists then apply a given function, or apply the function and then flatten, and obtain the same result in each case.

For each $X, Y \in \mathbf{Set}$ and set function $f : X \to Y$, we therefore have the following commutative diagram, where $\alpha_X : L(L(X)) \to L(X)$ is the flatten operation.

$$\begin{tikzcd}
L(L(X)) \arrow[r, "L(L(f))"] \arrow[d, "\alpha_X"'] & L(L(Y)) \arrow[d, "\alpha_Y"] \\
L(X) \arrow[r, "L(f)"] & L(Y)
\end{tikzcd}$$

Here, $L(f)$ applies f to each element in a list, and $L(L(f))$ applies f to each element in each list in a list. For instance, if $f : \mathbb{N} \to \mathbb{N}$, $f(n) = n^2$, then the top-right and left-bottom routes through the diagram respectively:

- Map $[[1,2],[1],[3,2]] \mapsto [[1,4],[1],[9,4]] \mapsto [1,4,1,9,4]$.
- Map $[[1,2],[1],[3,2]] \mapsto [1,2,1,3,2] \mapsto [1,4,1,9,4]$.

(d) In differential geometry, exterior derivative is a map $d : \Omega^k(U) \to \Omega^{k+1}(U)$ for $k \in \mathbb{N}$ and $\Omega^k(U)$ the space of differential k-forms on open $U \subseteq \mathbb{R}^n$. For $\phi : U \to V$ smooth, there is a pullback map $\phi^* : \Omega^k(V) \to \Omega^k(U)$ for any $k \in \mathbb{N}$, and exterior derivative commutes with pullback. We therefore obtain the following commuting square.

$$\begin{tikzcd}
\Omega^k(V) \arrow[r, "\phi^*"] \arrow[d, "d"'] & \Omega^k(U) \arrow[d, "d"] \\
\Omega^{k+1}(V) \arrow[r, "\phi^*"] & \Omega^{k+1}(U)
\end{tikzcd}$$

Noting that Ω^k and Ω^{k+1} are both (contravariant) functors, with their effect on morphisms given by pullback, we see that exterior derivative is a natural transformation. *Contravariant* here means these functors switch the order of composition (we have not defined this formally as it is irrelevant elsewhere in this document).

Definition 2.3.3 (Natural Isomorphism)

A natural transformation $\alpha : F \to G$ is a *natural isomorphism* if all its component morphisms are isomorphisms. We then say F and G are *naturally isomorphic*, written $F \cong G$.

Given naturally isomorphic functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$, we say $F(A) \cong G(A)$ *naturally* in A (or the isomorphism is *natural* in A) $\forall A \in \mathcal{C}$.

Remark 2.3.4

The notation $F \cong G$ suggests F and G are isomorphic in some category. This is indeed the case, and we will explore this shortly. Such a description of a natural isomorphism is often given as the definition. These definitions are equivalent, as outlined in Lemma 2.4.3 below.

2.4 Equivalence of Categories

With a plethora of categories now available to us, we may ask which categories share certain properties. Isomorphisms within a category suggest certain objects are functionally the same, and similar notions exist for categories themselves. There is added nuance here however, as both *isomorphism* and *equivalence* are valid notions of rough equality for categories.

Equivalence of categories is the weaker of the two, but is more common and arguably more useful. Isomorphism is often too restrictive, like requiring two strings of text to be the same only if they are written in the same font, when all we really care about are the characters (Spivak, 2014, p. 160).

Definition 2.4.1 (Functor Category)

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *functor category* $[\mathcal{C}, \mathcal{D}]$ (also written $\mathcal{D}^{\mathcal{C}}$) has:

- Functors $F : \mathcal{C} \to \mathcal{D}$ as objects.
- For all $F, G \in [\mathcal{C}, \mathcal{D}]$, the homset $[\mathcal{C}, \mathcal{D}](F, G)$ is the set of natural transformations $F \to G$.

Remark 2.4.2

To verify $[\mathcal{C}, \mathcal{D}]$ forms a category, it suffices to notice that natural transformations $\alpha : F \to G$ and $\beta : G \to H$ may be composed via composition of their components, i.e. $(\alpha \circ \beta)_A = \alpha_A \circ \beta_A \ \forall A \in \mathcal{C}$, and that there is an identity natural transformation $1_F : F \to F$ with $(1_F)_A = 1_{F(A)} \ \forall A \in \mathcal{C}$.

Composing as above is called vertical composition of natural transformations. We may also compose natural transformations horizontally, though this is unnecessary in our analysis.

Lemma 2.4.3

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors, and let $\alpha : F \to G$ be a natural transformation. Then α is a natural isomorphism if and only if it is an isomorphism in the category $[\mathcal{C}, \mathcal{D}]$.

Proof. First note that α is a morphism in $[\mathcal{C}, \mathcal{D}]$ by definition, so it remains to show that α is a natural isomorphism if and only if it has an inverse morphism in $[\mathcal{C}, \mathcal{D}]$.

Suppose α is a natural isomorphism and let $A \in \mathcal{C}$. Then $\alpha_A : F(A) \to G(A)$ is an isomorphism. Thus, $\exists \beta_A : G(A) \to F(A)$ such that $\alpha_A \circ \beta_A = 1_{G(A)}$ and $\beta_A \circ \alpha_A = 1_{F(A)}$.

By this result and the naturality of α, the following square commutes $\forall A, B \in \mathcal{C}$.

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, shift right, "\alpha_A"'] & F(B) \arrow[d, shift left, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"] \arrow[u, shift right, "\beta_A"'] & G(B) \arrow[u, shift left, "\beta_B"]
\end{tikzcd}$$

Ignoring the α components, we see that we have the naturality square for $\beta : G \to F$ with components $\{\beta_A : A \in \mathcal{C}\}$. So β defined as such is a natural transformation. We thus have a morphism β in $[\mathcal{C}, \mathcal{D}]$ with $\alpha \circ \beta = 1_F$ and $\beta \circ \alpha = 1_G$, so α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$.

The reverse implication is similar. □

Remark 2.4.4

Writing $F \cong G$ for functors therefore conveys multiple meanings, all of which coincide.

We would like to do the same for categories. Unfortunately, we run into size issues again here. The prospective category of categories falls prey to Russell's paradox as its class of objects would contain the category itself, so such a naive construction is impossible.

We can nevertheless define a substitute, but this comes at the cost of breaking the connection between different notions of isomorphism.

Definition 2.4.5 (Category of Categories)

Denote by **Cat** the category where objects are (small) categories and morphisms are functors.

Definition 2.4.6 (Isomorphism of Categories)

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic categories*, written $\mathcal{C} \cong \mathcal{D}$, if $\exists F : \mathcal{C} \to \mathcal{D}$ and $\exists G : \mathcal{D} \to \mathcal{C}$ functors such that $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$.

Remark 2.4.7

If $\mathcal{C}, \mathcal{D} \in \mathbf{Cat}$, then $\mathcal{C} \cong \mathcal{D}$ has two meanings. Namely, the definition above, and that $\mathcal{C}$ and $\mathcal{D}$ are isomorphic as objects in **Cat**. Since **Cat** has functors as morphisms, both definitions clearly agree in this case.

The problem is that if $\mathcal{C} \notin \mathbf{Cat}$ or $\mathcal{D} \notin \mathbf{Cat}$, i.e. if either is not small, only one definition applies. Since there is no category containing both $\mathcal{C}$ and $\mathcal{D}$, strictly speaking, there cannot be morphisms between them, hence no isomorphisms. So we are unable to say $\mathcal{C}$ and $\mathcal{D}$ are isomorphic in the general categorical sense, and must use the specialised definition instead.

We might avoid this by working in **CAT**, a more delicate construction that allows us to talk about the category of all categories (Barr and Wells, 1999, p. 65). We ignore such a construction in this document.

Definition 2.4.8 (Equivalence of Categories)

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *equivalence* $(F, G, \eta, \varepsilon)$ between $\mathcal{C}$ and $\mathcal{D}$ consists of:

- Two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$.
- Two natural isomorphisms $\eta : 1_{\mathcal{C}} \to G \circ F$ and $\varepsilon : F \circ G \to 1_{\mathcal{D}}$.

We call the functor F an *equivalence* with *pseudo-inverse* G.

If an equivalence between $\mathcal{C}$ and $\mathcal{D}$ exists, we say $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*, written $\mathcal{C} \simeq \mathcal{D}$.

Remark 2.4.9

Notice that isomorphism requires $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$, whereas equivalence relaxes this requirement to $G \circ F \cong 1_{\mathcal{C}}$ and $F \circ G \cong 1_{\mathcal{D}}$.

Further, an equivalence $F : \mathcal{C} \to \mathcal{D}$ does not require a *unique* pseudo-inverse. There may be several, even an infinite amount. The axiom of choice (for any set X of non-empty sets, there exists a map out of X that picks one element from each set) comes to our aid here. If we know a pseudo-inverse must exist but lack an explicit description, the axiom of choice lets us pick a general pseudo-inverse, even with infinitely many to choose from. In this document, we therefore assume the axiom of choice to allow for such possibilities.

Equivalence of categories captures the idea of two categories containing *essentially* the same information. Results about one category transfer to the other, and proving two categories are equivalent hints at a fundamental connection between what may have initially seemed disparate contexts. Isomorphism also captures the similarity of categories, so it can be hard to mentally separate the two definitions.

Often it is much easier – and much more intuitive – to use an alternative characterisation of equivalence, which we proceed to state (but not prove).

Proposition 2.4.10

A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is full, faithful and essentially surjective on objects.

Proof. See proofs such as those of Mac Lane (1998, pp. 93 – 94) and Perrone (2021, pp. 62 – 64). The proof is too long for us to reproduce here. □

Remark 2.4.11

Recall the setup from Remark 2.2.6, with the diagrams below in $\mathcal{C}$ and $\mathcal{D}$ respectively.

$$\begin{tikzcd}
A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[dr, "h"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F(A) \arrow[r, shift left, "F(f_1)"] \arrow[r, shift right, "F(f_2)"'] \arrow[dr, "F(h)"'] & F(B) \arrow[d, "F(g)"] \\
& F(C)
\end{tikzcd}$$

If the left diagram represents all of $\mathcal{C}$, an isomorphism $\mathcal{C} \cong \mathcal{D}$ via F would imply the right diagram is all of $\mathcal{D}$.

Equivalence meanwhile is almost as informative, but not quite. If $\mathcal{C} \simeq \mathcal{D}$ with F the equivalence functor, then there may be extra objects in $\mathcal{D}$, but we know they must be isomorphic to something already in our diagram by the essential surjectivity of F. Suppose we discovered $D \in \mathcal{D}$ with $D \neq F(A)\ \forall A \in \mathcal{C}$. Then we know we would have a situation like the one given below.

$$\begin{tikzcd}
D \arrow[r, shift left, "\alpha"] & F(A) \arrow[l, shift left, "\alpha^{-1}"] \arrow[r, shift left, "F(f_1)"] \arrow[r, shift right, "F(f_2)"'] \arrow[dr, "F(h)"'] & F(B) \arrow[d, "F(g)"] \\
& & F(C)
\end{tikzcd}$$

Here, $\alpha : D \to F(A)$ is an isomorphism in $\mathcal{D}$ with inverse α^{-1}. Thus, $D \cong F(A)$.

The situation may be more complicated if there are extra morphisms between D and $F(A)$, or between D and itself. However, composing with the isomorphisms α or α^{-1} shows that these correspond exactly with morphisms $F(A) \to F(A)$, which by faithfulness and fullness of F, we know we have already dealt with.

So is there really any new information here? While we have a new object D, it acts exactly like $F(A)$. Informally, we can subsume D into $F(A)$ and consider the diagram without D to be the whole category. This is the power of equivalence of categories.

Example 2.4.12

We provide some simple and more complex examples of equivalent categories.

(a) Any category is trivially equivalent to itself via the identity functor and identity natural isomorphism, both self-inverses.

(b) Any isomorphism of categories is an equivalence of categories. If $\mathcal{C} \cong \mathcal{D}$ via functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, then $G \circ F = 1_{\mathcal{C}}$, so $\eta : 1_{\mathcal{C}} \to G \circ F$ would be just the identity natural transformation, and the same for ε.

(c) The *skeleton* of a category $\mathcal{C}$ is a category $\mathcal{D}$ such that $\mathcal{C} \simeq \mathcal{D}$, and $\forall A, B \in \mathcal{D}$, if $A \cong B$ then $A = B$ (Spivak, 2014, p. 164). In other words, $\mathcal{D}$ is $\mathcal{C}$ with all extraneous objects removed, as we constructed in Remark 2.4.9.

(d) Let **FSet** denote the category of all finite sets, whose morphisms are set functions. Then $\forall A \in \mathbf{FSet}$, we have $|A| = n$ for some $n \in \mathbb{N}_0$. Denoting by S_n the set $\{1 \ldots n\}$ for $n \geqslant 1$ and $S_0 = \varnothing$, we have $A \cong S_n$ in **FSet** since isomorphism here refers simply to bijective correspondence. Thus, $\mathbf{FSet} \simeq \mathcal{C}$ where $\mathcal{C}$ has $\mathrm{ob}(\mathcal{C}) = \{S_n : n \in \mathbb{N}\}$ and all set functions between these as morphisms. In fact, $\mathcal{C}$ is the skeleton of **FSet** (Mac Lane, 1998, p. 93).

(d) Though it requires more machinery to state in detail than we have defined, one famous equivalence is the duality between algebra and geometry (Leinster, 2014, pp. 35 – 36).

Proposition 2.4.13

Equivalence of categories is transitive. That is, if $\mathcal{C} \simeq \mathcal{D}$ and $\mathcal{D} \simeq \mathcal{E}$, then $\mathcal{C} \simeq \mathcal{E}$.

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be the equivalence functors. By Proposition 2.4.10, both are full, faithful and essentially surjective. It follows from the definitions that their composition also has these properties, so reversing the result tells us that $G \circ F$ is an equivalence. $\square$

Remark 2.4.14

All the equivalences we have met in this section describe a fundamental similarity between the categories in question. Equivalent categories encode the same information in different ways.

We are thus led to ask: can we do the same for the categories of monoidal categories and multicategories? Are these merely two ways of looking at the same thing?

3 Monoidal Categories

We now introduce and motivate monoidal categories, providing informative examples before highlighting key results which will aid our investigation of their connection to multicategories.

Monoidal categories feature a binary tensor product, which acts like a monoid homomorphism, up to isomorphism. We can generalise and consider unbiased monoidal categories, with n-ary tensor products for any $n \in \mathbb{N}_0$. These will be central to our investigation.

Content here is adapted from Barr and Wells (1999) and Leinster (2003) unless stated otherwise.

3.1 Basic Monoidal Categories

Monoidal categories imitate the properties of an algebraic structure called a *monoid*. A monoid is a set M equipped with a binary operation $M \times M \to M$ that is associative and has an identity element. The comparison to groups is obvious – a monoid is a group which might not have inverses.

The binary operation in monoidal categories is the tensor product, encoded as a functor from the product category (or *bifunctor*) $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. This functor must satisfy certain coherence axioms.

Definition 3.1.1 (Monoidal Category)

A *monoidal category* $(\mathcal{C}, \otimes, I, \alpha, \rho, \lambda)$ consists of:

- A category $\mathcal{C}$.
- A functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product functor*. We write $\otimes(A, B) = A \otimes B$ and $\otimes(f, g) = f \otimes g$ for objects $A, B \in \mathcal{C}$ and morphisms f and g in $\mathcal{C}$.
- An object $I \in \mathcal{C}$ called the *unit*.
- A natural isomorphism α called the *associator*, with components $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C \ \forall A, B, C \in \mathcal{C}$.
- Natural isomorphisms ρ and λ called the right and left *unitors* respectively, with components $\rho_A : A \otimes I \to A$ and $\lambda_A : I \otimes A \to A$ respectively $\forall A \in \mathcal{C}$.

The isomorphisms are subject to the condition that the following diagrams commute $\forall A, B, C, D \in \mathcal{C}$.

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "1_A \otimes \lambda_B"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes 1_B"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
& A \otimes (B \otimes (C \otimes D)) \arrow[dl, "1_A \otimes \alpha_{B,C,D}"'] \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\
A \otimes ((B \otimes C) \otimes D) \arrow[d, "\alpha_{A,B\otimes C,D}"'] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A\otimes B,C,D}"] \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B,C} \otimes 1_D"] & & ((A \otimes B) \otimes C) \otimes D
\end{tikzcd}$$

These diagrams are called the *triangle identity* and *pentagon identity* respectively. Together, they are called the *coherence conditions for monoidal categories*.

Remark 3.1.2

The definition of monoidal categories appears highly technical. Much of this is due to the complexity of the functor and natural transformation notation, and unwrapping this may aid our understanding.

That the tensor product maps $(A, B) \in \mathcal{C} \times \mathcal{C}$ to $A \otimes B \in \mathcal{C}$ means that a monoidal category is closed under tensor product. That is, given $A, B \in \mathcal{C}$, we know there is another object $A \otimes B \in \mathcal{C}$, and the functor $\otimes$ gives a prescription for obtaining it.

The various isomorphisms then impose restrictions on how the outputs of this tensor product relate to one another and the original objects. For example, the right unitor says $A \otimes I \cong A$ for any $A \in \mathcal{C}$. That is, $A \otimes I$ is practically identical to A, but not necessarily equal.

The element I therefore acts like the identity in a monoid. The left and right unitors are analogous to the fact that $a \cdot e = a = e \cdot a$ for a two-sided identity e in a monoid. However, this is only true *up to natural isomorphism.*

Similarly, the associator says $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ for any $A, B, C \in \mathcal{C}$. This is how we would expect an associative binary operation to behave, with the same twist – the objects might not be equal, they merely need to be naturally isomorphic. The isomorphisms can be identities in some cases (see Example 3.1.4(e)) but not in general.

This does however beg the question: why the diagrams? If monoids are characterised by an associative binary operation with identities and we have such an operation, why do we require further conditions on how the outputs of the tensor functor relate?

We first assure ourselves that the diagrams are saying sensible things. Consider the isomorphism labelled $1_A \otimes \lambda_B$ in the triangle identity. The functor $\otimes$ takes a pair of morphisms 1_A and λ_B to their tensor product $1_A \otimes \lambda_B : A \otimes (I \otimes B) \to A \otimes B$. We can think of this situation via the following "decomposition".

$$
\begin{tikzcd}
A \arrow[d, "1_A"] & I \otimes B \arrow[d, "\lambda_B"] \\
A & B
\end{tikzcd}
$$

Here we have split $A \otimes (I \otimes B)$ and $A \otimes B$ according to their outermost tensor product. We see the identity 1_A does nothing to A, while the left unitor is an isomorphism $I \otimes B \to B$ by definition. Putting the diagrams back together by applying $\otimes$ to all objects and morphisms, we have the isomorphism $A \otimes (I \otimes B) \cong A \otimes B$. This makes sense in the context of identities in monoids, as we do indeed have $a \cdot (e \cdot b) = a \cdot b$ after cancelling the identity e.

The pentagon identity is similar, and breaking down its morphisms provides the same insight. Consider the morphism $1_A \otimes \alpha_{B,C,D}$.

$$
\begin{tikzcd}
A \arrow[d, "1_A"] & B \otimes (C \otimes D) \arrow[d, "\alpha_{B,C,D}"] \\
A & (B \otimes C) \otimes D
\end{tikzcd}
$$

Again, these individual morphisms make sense. One is the identity and the other is the associator, which by definition establishes the isomorphism $B \otimes (C \otimes D) \cong (B \otimes C) \otimes D$. Pairing them together using $\otimes$ gives the top left morphism found in the pentagon identity. The other morphisms may be approached similarly.

The need for the coherence conditions becomes apparent when considering the different orders in which one can rearrange an expression in a monoid. If e is the identity in a monoid, we can reduce the element $a \cdot (e \cdot b)$ to find it is equal to $a \cdot b$ via at least two routes:

- We could first use associativity, then cancel the identity. That is, $a \cdot (e \cdot b) = (a \cdot e) \cdot b = a \cdot b$.
- We could cancel the identity directly, so $a \cdot (e \cdot b) = a \cdot b$.

In a monoid, these methods always produce the same result. However, without the coherence conditions, the same is not necessarily true in a monoidal category, as we do not work with strict equality in general.

Compare the above derivation with the triangle identity. Commutativity of the diagram asserts that rearranging by associativity followed by cancelling the identity is equivalent to simply cancelling the identity. This is exactly the result we want.

The triangle identity therefore deals with a few ways we could rearrange two objects and an identity. The pentagon identity is the same for four-fold associativity, insisting that any two re-bracketing routes which arrive at the same object are equal.

In general, given an object $A \in \mathcal{C}$ for $\mathcal{C}$ a monoidal category, if A is an iterated composition of other elements under the tensor product, there will be many ways to rearrange it using associativity and identities. But Mac Lane (1998, pp. 165 – 170) showed that the two coherence conditions we already have will always be enough, and are the simplest possible such conditions.

No matter the complexity of A, we can always relate two rearrangement routes of A with the same destination via repeated use of the triangle and pentagon identities. In more memorable if slightly inaccurate terms, ‘every diagram commutes’ (Mac Lane, 1998, p. 165).

Proposition 3.1.3 (Tensor Interchange Law)

Let $(\mathcal{C}, \otimes, I, \alpha, \rho, \lambda)$ be a monoidal category with composition $\circ$ of morphisms in category $\mathcal{C}$.

- Let $A, B, C, A', B', C' \in \mathcal{C}$ and let $f : A \to B$, $g : B \to C$, $f' : A' \to B'$ and $g' : B' \to C'$ be morphisms. Then $(g' \circ f') \otimes (g \circ f) = (g' \otimes g) \circ (f' \otimes f)$.
- Let $A, B \in \mathcal{C}$. Then $1_A \otimes 1_B = 1_{A \otimes B}$.

Proof. By the functorality of $\otimes$ and the definition of composition in a product category,

$$\begin{aligned}
(g' \circ f) \otimes (g \circ f) &= \otimes((g' \circ f'), (g \circ f)) \\
&= \otimes((g', g) \circ (f', f)) \\
&= \otimes(g', g) \circ \otimes(f', f) \\
&= (g' \otimes g) \circ (f' \otimes f).
\end{aligned}$$

Similarly, as $(1_A, 1_B)$ is an identity in $\mathcal{C} \times \mathcal{C}$, preservation of identities says $\otimes(1_A, 1_B) = 1_{A \otimes B}$. □

Example 3.1.4

As we discuss shortly, monoidal categories are surprisingly prevalent. We examine a few simple examples.

(a) The category **Set** of sets and set functions is monoidal. The tensor product is cartesian product, since for $A, B \in$ **Set**, we know $A \times B \in$ **Set**.

Notice $A \times (B \times C)$ and $(A \times B) \times C$ are not equal, as the former consists of elements of the form $(a, (b, c))$ for $a \in A$, $b \in B$ and $c \in C$, while elements of the latter are $((a, b), c)$. However, since we can form the well-defined bijection $(a, (b, c)) \mapsto ((a, b), c)$, and since isomorphism in **Set** is simply bijection, we have $A \times (B \times C) \cong (A \times B) \times C$ in **Set**. Thus, we have an obvious associator.

The unit in **Set** can be any singleton set, so we pick $\{1\}$ for simplicity. Since $\{1\} \times A$ has the same cardinality as A for any $A \in \mathbf{Set}$, $\{1\} \times A \cong A$ provides the left unitor isomorphism, and the right unitor is similar.

The triangle and pentagon identities then hold by applying these bijections as needed.

(b) The category **Cat** of small categories and functors is monoidal (Mac Lane, 1998, p. 171). We take the categorical product $\mathcal{C} \times \mathcal{D}$ to be the tensor product, and the trivial category (Example 2.1.3(a)) as the unit.

(c) Let M be a monoid. We may regard M as a category, and unsurprisingly, it is a monoidal category. Objects are elements of M, the only morphisms are identities (so this is a discrete category) and tensor product is the binary operation on the monoid (Mac Lane, 1998, p. 162).

(d) Let $\mathbb{K}$ be a field. The category $\mathbf{Vect}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$ is a monoidal category (Leinster, 2003, p. 82). In fact, tensor product in this category is tensor product of vector spaces from linear algebra, motivating our choice of nomenclature. The unit is the field $\mathbb{K}$ itself.

(e) If the coherence isomorphisms are in fact identities, i.e. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ and $I \otimes A = A = A \otimes I$, then we say the category is a *strict monoidal category.* All coherence diagrams automatically commute and we need not check them.

One example of a strict monoidal category is the category $[\mathcal{C}, \mathcal{C}]$ of *endofunctors* on a category $\mathcal{C}$. Define tensor product to be functor composition, which is associative and has identity $1_{\mathcal{C}}$ which we can take to be the unit.

Every monoidal category is equivalent to a strict monoidal category (Trujillo, 2020, pp. 63 – 128). This is in fact equivalent to the statement that the pentagon and triangle identities are necessary and sufficient for all diagrams in a monoidal category built from associators and units to commute. This fact means we can be lax with bracketing in monoidal categories, since there is a strict monoidal category without brackets behind the scenes.

However, there is *not* an equivalence between the category of monoidal categories and the category of strict monoidal categories. We would require a proof of isomorphism between strict and non-strict monoidal categories, not just the equivalence we do have. This is not true in general (Etingof et al., 2015, p. 38).

Remark 3.1.5

Many categories we encounter regularly in mathematics are monoidal. Indeed, **Set** and **Cat** above are common enough to make monoidal categories worthy of study by themselves. Any category with finite products and a terminal object is monoidal (finite products means we may take products of objects and obtain unique projections, while terminal objects are objects with a unique morphism from every object).

More surprisingly, some have proposed monoidal categories as the setting for investigations in non-mathematical disciplines. Cartesian closed categories are a special kind of monoidal category (Barr and Wells, 1999, pp. 405 – 406) and have applications in logic and computation theory. The Curry-Howard-Lambek Correspondence states that cartesian closed categories provide an abstract context for proof theory and the lambda calculus model of computation (Brogi, 2021).

Baez and Stay (2010) go even further, arguing that monoidal categories (and in particular, closed symmetric monoidal categories) are the "Rosetta Stone" between physics, topology, logic and computation. Even Feynman diagrams have underlying monoidal categories, demonstrating the enormous scope of category theory to illuminate complex systems in all disciplines.

Definition 3.1.6 (Monoidal Functor)

Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathcal{D}, \odot, J, \beta, \mu, \sigma)$ be monoidal categories. A *monoidal functor* $(F, \phi) : \mathcal{C} \to \mathcal{D}$ consists of:

- A functor $F : \mathcal{C} \to \mathcal{D}$.
- A natural isomorphism ϕ with components $\phi_{A,B} : F(A) \odot F(B) \to F(A \otimes B)$ $\forall A, B \in \mathcal{C}$.
- An isomorphism $\phi_0 : J \to F(I)$.

The morphisms $\phi_{A,B}$ and ϕ_0, called the *coherence morphisms*, are subject to the condition that the following diagrams commute $\forall A, B, C \in \mathcal{C}$.

$$\begin{tikzcd}
(FA \odot FB) \odot FC \arrow[d, "\phi_{A,B} \odot 1_{FC}"'] & & FA \odot (FB \odot FC) \arrow[ll, "\beta_{FA,FB,FC}"'] \arrow[d, "1_{FA} \odot \phi_{B,C}"] \\
F(A \otimes B) \odot FC \arrow[d, "\phi_{A \otimes B, C}"'] & & FA \odot F(B \otimes C) \arrow[d, "\phi_{A, B \otimes C}"] \\
F((A \otimes B) \otimes C) & & F(A \otimes (B \otimes C)) \arrow[ll, "F\alpha_{A,B,C}"]
\end{tikzcd}$$

$$\begin{tikzcd}
J \odot FA \arrow[r, "\phi_0 \odot 1_{FA}"] \arrow[d, "\mu_{FA}"'] & FI \odot FA \arrow[d, "\phi_{I,A}"] \\
FA & F(I \otimes A) \arrow[l, "F\lambda_A"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \odot J \arrow[r, "1_{FA} \odot \phi_0"] \arrow[d, "\sigma_{FA}"'] & FA \odot FI \arrow[d, "\phi_{A,I}"] \\
FA & F(A \otimes I) \arrow[l, "F\rho_A"]
\end{tikzcd}$$

We call these diagrams the *hexagon identity* and *square identities* respectively, collectively known as the *coherence conditions for monoidal functors*.

Remark 3.1.7

Throughout the literature, we find varying terminology for the above construction. Leinster (2003) describes it as a *weak monoidal functor*, compared to *lax monoidal functors* which do not require the coherence morphisms to be isomorphisms. Schauenburg (2001) and others call it simply a *monoidal functor*, omitting the non-invertible case. Others still, such as Trujillo (2020), describe it as a *strong monoidal functor*, using weak monoidal functor to mean Leinster's lax monoidal functor. These inconsistencies likely arise due to different perspectives – the "default" will depend on the level of abstraction one usually uses when dealing with monoidal categories

Since we only care for constructions involving isomorphisms in our investigation, we opt for the simplest approach, defining monoidal functors as above, and no other variations. We thus hope to avoid this mess of contradictory terminology.

Though the definition appears highly complicated, breaking it down reveals that monoidal functors are simply functors which preserve monoidal structure, up to isomorphism. The components $\phi_{A,B}$ of the natural transformation ϕ impose the restriction that the functor respects tensor product, and ϕ_0 insists further that it respects units. Thus, if we think of a monoidal category as a monoid, a monoidal functor is a monoid homomorphism, i.e. a function between monoids which respects composition and identities.

The commutative diagrams therefore play a similar role to those in the definition of a monoidal category, and we can view them through the lens of monoid homomorphisms. Given a monoid homomorphism $f : (M, \cdot) \to (N, *)$ and $a, b, c \in M$, we can consider multiple ways of manipulating $f(a) * f(b) * f(c)$ to reach the expression $f(a \cdot b \cdot c)$.

- First use the homomorphism property on the left, so $f(a) * f(b) * f(c) = f(a \cdot b) * f(c)$. Then use the homomorphism property again to obtain $f(a \cdot b) * f(c) = f(a \cdot b \cdot c)$.
- Conversely, first reduce on the right, then reduce on the outside. So $f(a) * f(b) * f(c) = f(a) * f(b \cdot c) = f(a \cdot b \cdot c)$.

As before, we want both these routes to be equal (as "morphisms" $f(a) * f(b) * f(c) \to f(a \cdot b \cdot c)$). The hexagon identity ensures this, with respect to associativity when needed. The square identities play similar roles for cancellation by units on the left and right.

Definition 3.1.8 (Category of Monoidal Categories)

Denote by **MonCat** the category of (small) monoidal categories and monoidal functors.

Remark 3.1.9

We verify **MonCat** is well-defined by noting that the composition of two monoidal functors is a monoidal functor (similar to composing homomorphisms to produce a homomorphism in **Mon**), and that there is an identity monoidal functor obtained by taking all relevant morphisms to be identities.

Since Leinster (2003) uses the term *weak monoidal functor* to refer to our monoidal functors, his notation differs slightly here, preferring $\mathbf{MonCat}_{\mathrm{wk}}$. Similarly, he opts for $\mathbf{UMonCat}_{\mathrm{wk}}$ for unbiased monoidal categories (see Definition 3.2.1). Again, we ignore other types of monoidal functors, so choose **MonCat** and **UMonCat** in this document.

Example 3.1.10

We include examples of monoidal functors for illustrative purposes.

(a) From Example 3.1.4(a), we see **Set** is a monoidal category, and Example 3.1.4(d) tells us $\mathbf{Vect}_{\mathbb{K}}$ for a field $\mathbb{K}$ is also a monoidal category. Meanwhile, Example 2.2.2(e) gave us a functor $F : \mathbf{Set} \to \mathbf{Vect}_{\mathbb{K}}$ which constructs the free vector space $F(S)$ on a set S.

This functor is monoidal. To see this, it suffices to notice two facts:

- There is an isomorphism of vector spaces $F(X \times Y) \cong F(X) \otimes F(Y)$, where $\times$ is cartesian product (tensor product in **Set**) and $\otimes$ is tensor product in $\mathbf{Vect}_{\mathbb{K}}$.
- There is an isomorphism of vector spaces $F(\{1\}) \cong \mathbb{K}$. Recall $\{1\}$ and $\mathbb{K}$ were units in **Set** and $\mathbf{Vect}_{\mathbb{K}}$ respectively.

Deconstructing these isomorphisms provides the requisite natural isomorphisms for a monoidal functor (Fuller, 2017, p. 14).

However, the forgetful functor $U : \mathbf{Vect}_{\mathbb{K}} \to \mathbf{Set}$ is not monoidal according to our definition. If we drop the requirement for the coherence morphisms to be isomorphisms though, we do obtain a lax monoidal functor as per Leinster's (2003) definition.

(b) Recall the category of pointed topological spaces and base point preserving continuous maps $\mathbf{Top}_*$ from Example 2.1.3(f). For spaces $(X, x), (Y, y) \in \mathbf{Top}_*$, define the tensor product by $(X, x) \otimes (Y, y) = (X \times Y, (x, y))$. Define the unit to be the trivial space $(\{*\}, *)$. One can verify this makes $\mathbf{Top}_*$ into a monoidal category.

Also recall $\mathbf{Grp}$ from Example 2.1.3(e). We may similarly add a monoidal structure by defining $G \otimes H = G \times H$, the cartesian product of groups with component-wise operations. The unit in this case is the trivial group $\{e\}$.

As we saw in Example 2.2.2(f), the map $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ sending a space to its fundamental group and a map to its pushforward is a functor. In fact, it is a monoidal functor (Trujillo, 2020, pp. 59 – 60). This means $\pi_1((X \times Y), (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y)$ and $\pi_1(\{*\}, *) \cong e$. Thus, not only can we make a group out of every topological space, but we can do it in a way that respects products and trivial objects (up to natural isomorphism, of course).

3.2 Unbiased Monoidal Categories

Unbiased monoidal categories are, in some sense, a natural extension of monoidal categories. While regular monoidal categories have only bifunctors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (and a "zero" functor picking out the unit), unbiased monoidal categories have functors $\mathcal{C}^n \to \mathcal{C}$ for any $n \in \mathbb{N}_0$. Here, $\mathcal{C}^n$ is understood to mean $\mathcal{C} \times \cdots \times \mathcal{C}$ with n copies of $\mathcal{C}$ when $n > 0$, and a functor $\mathcal{C}^0 \to \mathcal{C}$ corresponds to picking out a fixed object of $\mathcal{C}$ (manipulations for the zero case are therefore trivial and we ignore them).

Our need for unbiased monoidal categories stems from Leinster (2003), who introduces them as a tool for proving the equivalence of monoidal categories and representable multicategories. Hermida (2000) does not use them explicitly, preferring to rely on strict monoidal categories and the fact that any monoidal category is equivalent to a strict one (see Example 3.1.4(e)).

Definition 3.2.1 (Unbiased Monoidal Category)

An *unbiased monoidal category* $(\mathcal{C}, \otimes_n, \gamma, \iota)$ consists of:

- A category $\mathcal{C}$.
- For each $n \in \mathbb{N}_0$, a functor $\otimes_n : \mathcal{C}^n \to \mathcal{C}$, called the *n-fold tensor product functor*, written $\otimes_n(A_1 \dots A_n) = (A_1 \otimes \cdots \otimes A_n)$ for objects and $\otimes_n(f_1 \dots f_n) = (f_1 \otimes \cdots \otimes f_n)$ for morphisms.
- For all $n, m_i \in \mathbb{N}_0$ and any double sequence of objects $A_{1,1} \dots A_{1,m_1} \dots A_{n,1} \dots A_{n,m_n} \in \mathcal{C}$, an isomorphism

$$\begin{aligned}\gamma_{(A_{1,1} \dots A_{1,m_1}) \dots (A_{n,1} \dots A_{n,m_n})} : ((A_{1,1} \otimes \cdots \otimes A_{1,m_1}) \otimes \cdots \otimes (A_{n,1} \otimes \cdots \otimes A_{n,m_n})) \\ \to (A_{1,1} \otimes \cdots \otimes A_{1,m_1} \otimes \cdots \otimes A_{n,1} \otimes \cdots \otimes A_{n,m_n})\end{aligned}$$

  which is natural in each of the $A_{i,j}$.
- A natural isomorphism $\iota : 1_{\mathcal{C}} \to \otimes_1$ with components $\iota_A : A \to (A)\ \forall A \in \mathcal{C}$. Note (A) here denotes application of the 1-fold tensor $\otimes_1$ to A.

These are subject to the condition that the following diagrams commute for any sequence of objects $B_1 \dots B_n \in \mathcal{C}$, and any triple sequence of objects $(((A_{i,j,k})_{k=1}^{p_{i,j}})_{j=1}^{m_i})_{i=1}^{n}$ for $A_{i,j,k} \in \mathcal{C}$ and indices $n, m_i, p_{i,j} \in \mathbb{N}_0$.

$$
\begin{tikzcd}[column sep=small]
& \begin{matrix}(((A_{1,1,1}\otimes\cdots\otimes A_{1,1,p_{1,1}})\otimes\cdots\otimes(A_{1,m_1,1}\otimes\cdots\otimes A_{1,m_1,p_{1,m_1}}))\otimes\cdots \\ \otimes((A_{n,1,1}\otimes\cdots\otimes A_{n,1,p_{n,1}})\otimes\cdots\otimes(A_{n,m_n,1}\otimes\cdots\otimes A_{n,m_n,p_{n,m_n}})))\end{matrix} \arrow[dl, "(\gamma_{S_1}\otimes\cdots\otimes\gamma_{S_n})"'] \arrow[dr, "\gamma_{S'}"] & \\
\begin{matrix}((A_{1,1,1}\otimes\cdots\otimes A_{1,m_1,p_{1,m_1}})\otimes\cdots \\ \otimes(A_{n,1,1}\otimes\cdots\otimes A_{n,m_n,p_{n,m_n}}))\end{matrix} \arrow[dr, "\gamma_S"'] & & \begin{matrix}((A_{1,1,1}\otimes\cdots\otimes A_{1,1,p_{1,1}})\otimes\cdots \\ \otimes(A_{n,m_n,1}\otimes\cdots\otimes A_{n,m_n,p_{n,m_n}}))\end{matrix} \arrow[dl, "\gamma_{S''}"] \\
& (A_{1,1,1}\otimes\cdots\otimes A_{n,m_n,p_{n,m_n}}) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& (B_1\otimes\cdots\otimes B_n) \arrow[dl, "(\iota_{B_1}\otimes\cdots\otimes\iota_{B_n})"'] \arrow[dr, "\iota_{(B_1\otimes\cdots\otimes B_n)}"] \arrow[dd, "1_{(B_1\otimes\cdots\otimes B_n)}"] & \\
((B_1)\otimes\cdots\otimes(B_n)) \arrow[dr, "\gamma_{(B_1)\ldots(B_n)}"'] & & ((B_1\otimes\cdots\otimes B_n)) \arrow[dl, "\gamma_{(B_1\ldots B_n)}"] \\
& (B_1\otimes\cdots\otimes B_n) &
\end{tikzcd}
$$

The double sequences S_i, S, S', S'' for $i \in \mathbb{N}_0$ are defined as

$$
\begin{aligned}
S_i &= (A_{i,1,1}\ldots A_{i,1,p_{i,1}})\ldots(A_{i,m_i,1}\ldots A_{i,m_i,p_{i,m_i}}),\\
S &= (A_{1,1,1}\ldots A_{1,m_1,p_{1,m_1}})\ldots(A_{n,1,1}\ldots A_{n,m_n,p_{n,m_n}}),\\
S' &= ((A_{1,1,1}\otimes\cdots\otimes A_{1,1,p_{1,1}})\ldots(A_{1,m_1,1}\otimes\cdots\otimes A_{1,m_1,p_{1,m_1}})\\
&\qquad\ldots(A_{n,1,1}\otimes\cdots\otimes A_{n,1,p_{n,1}})\ldots(A_{n,m_n,1}\otimes\cdots\otimes A_{n,m_n,p_{n,m_n}})),\\
S'' &= (A_{1,1,1}\ldots A_{1,1,p_{1,1}})\ldots(A_{n,m_n,1}\ldots A_{n,m_n,p_{n,m_n}}).
\end{aligned}
$$

We call these diagrams the *associativity diamond* and *identity diamond* respectively. Together, these form the *coherence conditions for unbiased monoidal categories*.

We refer to a monoidal category that is not unbiased (i.e. has only a 2-fold tensor product) as *biased*.

Remark 3.2.2

While the diagrams look horrid, this is merely a side-effect of being precise with the generality of the sequences involved. In the words of Leinster (2003, p. 68), 'the bark of the associativity axiom is far worse than its bite'.

We can summarise the information they encode much more succinctly: given two objects which are the same except for bracketing, any two ways to move brackets from one arrangement to the other are the same.

Consider first the case for the associativity diamond. Suppose we had an object such as $((A \otimes B \otimes C) \otimes D) \in \mathcal{C}$. Although the outer bracketing here is technically required as per the definition of the n-fold tensor (otherwise we could not distinguish an object from the 1-fold tensor applied to that object), we will often omit it for brevity.

How might we re-bracket $(A \otimes B \otimes C) \otimes D$ to arrive at, say, $A \otimes (B \otimes C) \otimes D$? Focusing in on the bottom two rows of the associativity diamond tells us. We first apply $\gamma_{(A,B,C),(D)}$ to remove brackets entirely, obtaining $A \otimes B \otimes C \otimes D$. Then, since the γ components are invertible, we can apply $\gamma_{(A)(B,C)(D)}{}^{-1}$ to add brackets back in, now with a different arrangement. The commutativity of the associativity diamond ensures that there is a unique reassociation isomorphism between different bracketings in all such cases.

Meanwhile, the identity diamond ensures that applying the 1-fold tensor to an object, and applying the 1-fold tensor to each object in a tensor product, induce isomorphisms, and that these isomorphisms interact consistently. Intuitively, γ removes brackets, while ι adds them, so we expect either compatible method of composing their components to do nothing overall, i.e. to be the identity.

It is useful here to make the comparison to ordinary monoidal categories. In that case, we had objects like $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$ which were isomorphic. Here, we again have these two, but also an explicit object $A \otimes B \otimes C$, all three isomorphic. In monoidal categories, we treat $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$ as if they are the same, and call that imagined object $A \otimes B \otimes C$. In unbiased monoidal categories, such an object actually exists.

Readers familiar with the concept of a *monad* may recognise the above diagrams as defining an algebra for a particular 2-monad (Leinster, 2003, pp. 69 – 70). Though framing unbiased monoidal categories in terms of monads might cut notational clutter, this would require more theoretical background. We restrict ourselves to the familiar (if clunky) world without them.

Example 3.2.3

The category **Set** may be viewed as a biased monoidal category, as in Example 3.1.4(a), but is also an unbiased monoidal category. This is because we can take the cartesian product of any number of sets, not just two.

For $A, B, C \in$ **Set**, there are objects $(A \times B) \times C, A \times (B \times C), A \times B \times C \in$ **Set**, for instance. As before, these are not equal – elements are tuples $((a, b), c)$, $(a, (b, c))$ and (a, b, c) respectively – but there is a canonical bijection between any two of them. In this case, the bijection would send $((a, b), c) \mapsto (a, b, c)$ or $((a, b), c) \mapsto (a, (b, c))$, to give a few examples. The structure of cartesian product means no matter the order in which we apply it, we can recover the same information, expressed by the bijection. Hence, these sets are isomorphic in **Set**.

This is why, in practice, we can always reduce to the case $A \times B \times C$ and deal with elements of the form (a, b, c). While grouping the cartesian product in different ways technically produces different sets, we can ignore this.

A classic example from the realm of physics is spacetime. Though spacetime is a 4D object, we intuitively think of it as the pairing of 3D space and 1D time. But $\mathbb{R}^4 \cong \mathbb{R}^3 \times \mathbb{R}$ in **Set**, so as far as any calculations we perform are concerned, either view is acceptable. The coherence conditions for the unbiased monoidal category **Set** handle all the implicit re-bracketing in the background. We can work with elements (x, y, z, t) or $((x, y, z), t)$ as we see fit.

Proposition 3.2.4 (Unbiased Tensor Interchange Law)

Let $(\mathcal{C}, \otimes_n, \gamma, \iota)$ be an unbiased monoidal category with composition $\circ$ of morphisms in category $\mathcal{C}$.

- Let $n \in \mathbb{N}_0$, let $A_i, B_i, C_i \in \mathcal{C}$ and let $f_i : A_i \to B_i$ and $g_i : B_i \to C_i$ be morphisms $\forall i = 1 \dots n$. Then $(g_1 \circ f_1) \otimes \cdots \otimes (g_n \circ f_n) = (g_1 \otimes \cdots \otimes g_n) \circ (f_1 \otimes \cdots \otimes f_n)$.
- Let $n \in \mathbb{N}_0$ and let $A_1 \dots A_n \in \mathcal{C}$. Then $1_{A_1} \otimes \cdots \otimes 1_{A_n} = 1_{A_1 \otimes \cdots \otimes A_n}$.

Proof. Exactly as in Proposition 3.1.3, now with n-fold product morphisms. $\square$

Definition 3.2.5 (Unbiased Monoidal Functor)

Let $(\mathcal{C}, \otimes_n, \gamma, \iota)$ and $(\mathcal{D}, \odot_n, \delta, \kappa)$ be unbiased monoidal categories. An *unbiased monoidal functor* $(F, \pi) : \mathcal{C} \to \mathcal{D}$ consists of:

- A functor $F : \mathcal{C} \to \mathcal{D}$.
- For every $n \in \mathbb{N}_0$ and every sequence of objects $A_1 \dots A_n \in \mathcal{C}$, a coherence isomorphism $\pi_{A_1 \dots A_n} : (FA_1 \odot \cdots \odot FA_n) \to F(A_1 \otimes \cdots \otimes A_n)$ which is natural in each A_i.

These are subject to the conditions that $\forall A \in \mathcal{C}$, $\forall n, m_i \in \mathbb{N}_0$ and for each double sequence $A_{1,1} \dots A_{1,m_1} \dots A_{n,1} \dots A_{n,m_n} \in \mathcal{C}$, the following diagrams commute.

$$\begin{tikzcd}
((FA_{1,1} \odot \cdots \odot FA_{1,m_1}) \odot \cdots \odot (FA_{n,1} \odot \cdots \odot FA_{n,m_n})) \arrow[r, "\delta_{(FA_{1,1} \dots FA_{1,m_1}) \dots (FA_{n,1} \dots FA_{n,m_n})}"] \arrow[d, "(\pi_{A_{1,1} \dots A_{1,m_1}} \odot \cdots \odot \pi_{A_{n,1} \dots A_{n,m_n}})"] & (FA_{1,1} \odot \cdots \odot FA_{n,m_n}) \arrow[dd, "\pi_{A_{1,1} \dots A_{n,m_n}}"] \\
(F(A_{1,1} \otimes \cdots \otimes A_{1,m_1}) \odot \cdots \odot F(A_{n,1} \otimes \cdots \otimes A_{n,m_n})) \arrow[d, "\pi_{(A_{1,1} \otimes \cdots \otimes A_{1,m_1}) \dots (A_{n,1} \otimes \cdots \otimes A_{n,m_n})}"] & \\
F((A_{1,1} \otimes \cdots \otimes A_{1,m_1}) \otimes \cdots \otimes (A_{n,1} \otimes \cdots \otimes A_{n,m_n})) \arrow[r, "F\gamma_{(A_{1,1} \dots A_{1,m_1}) \dots (A_{n,1} \dots A_{n,m_n})}"'] & F(A_{1,1} \otimes \cdots \otimes A_{n,m_n})
\end{tikzcd}$$

$$\begin{tikzcd}
& FA \arrow[dl, "\kappa_{FA}"'] \arrow[dr, "F\iota_A"] & \\
(FA) \arrow[rr, "\pi_A"'] & & F(A)
\end{tikzcd}$$

We call these diagrams the *pentagon identity for monoidal functors* and *triangle identity for monoidal functors* respectively, together the *coherence conditions for unbiased monoidal functors.*

Remark 3.2.6

Though the connection is less obvious than before, the coherence conditions for unbiased monoidal functors once again ensure that they preserve unbiased monoidal structure. Compare with biased monoidal functors, and in particular Remark 3.1.7 – in both cases, the functor preserves the tensor up to isomorphism, and the coherence conditions mean any two ways of rearranging tensored objects to arrive at the same result are equal.

Further, there is clearly an identity unbiased monoidal functor. Showing that the composition of two unbiased monoidal functors produces another one requires more work. Since doing so is necessary to define the category of unbiased monoidal categories, which will ultimately help us prove the equivalence between categories of monoidal categories and representable multicategories, we proceed to state and prove some of this result.

Proposition 3.2.7

Let $(\mathcal{C}, \otimes_n, \gamma, \iota)$, $(\mathcal{D}, \odot_n, \delta, \kappa)$ and $(\mathcal{E}, \circledast_n, \varepsilon, \lambda)$ be unbiased monoidal categories and let $(F, \pi) : \mathcal{C} \to \mathcal{D}$ and $(G, \rho) : \mathcal{D} \to \mathcal{E}$ be unbiased monoidal functors. Then $(G \circ F, G\pi \circ \rho) : \mathcal{C} \to \mathcal{E}$ is an unbiased monoidal functor, where the components of $G\pi \circ \rho$ are $(G\pi \circ \rho)_{A_1 \ldots A_n} = G\pi_{A_1 \ldots A_n} \circ \rho_{FA_1 \ldots FA_n}$.

Proof. The only subtle point here is showing that the $(G\pi \circ \rho)_{A_1 \ldots A_n}$ are indeed the coherence isomorphisms in $\mathcal{E}$ for $G \circ F$.

We verify the triangle identity. The following diagrams commute in $\mathcal{D}$ and $\mathcal{E}$ respectively by definition $\forall A \in \mathcal{C}$ and $\forall B \in \mathcal{D}$.

$$\begin{tikzcd} & FA \arrow[dl, "\kappa_{FA}"'] \arrow[dr, "F\iota_A"] & \\ (FA) \arrow[rr, "\pi_A"'] & & F(A) \end{tikzcd} \qquad \begin{tikzcd} & GB \arrow[dl, "\lambda_{GB}"'] \arrow[dr, "G\kappa_B"] & \\ (GB) \arrow[rr, "\rho_B"'] & & G(B) \end{tikzcd}$$

Applying G to everything in the first diagram retains its commutativity (see Remark 2.2.6). So let $B = FA$ for $A \in \mathcal{C}$ and we can glue the resulting diagrams together to see that the following diagram commutes.

$$\begin{tikzcd} & GFA \arrow[dl, "\lambda_{GFA}"'] \arrow[d, "G\kappa_{FA}"] \arrow[dr, "GF\iota_A"] & \\ (GFA) \arrow[r, "\rho_{FA}"'] & G(FA) \arrow[r, "G\pi_A"'] & GF(A) \end{tikzcd}$$

The outer morphisms then provide the triangle identity for $(G \circ F, G\pi \circ \rho)$.

The strategy for the pentagon identity is similar, with the addition of naturality squares for π and ρ. As it is much more notationally heavy, we omit it. $\square$

Definition 3.2.8 (Category of Unbiased Monoidal Categories)

Denote by **UMonCat** the category of (small) unbiased monoidal categories and unbiased monoidal functors.

Theorem 3.2.9

There is an equivalence of categories **MonCat** $\simeq$ **UMonCat**.

Proof. See Corollary 3.2.5 in Leinster (2003, p. 81) and the preceding results. $\square$

Remark 3.2.10

Ideally, we would like to prove the above result here to complete the chain of equivalences required on the journey towards our main theorem. But the proof is simply too long and relies on too many new concepts for us to cover succinctly. Though, as Leinster did provide this proof – unlike the equivalence of unbiased monoidal categories and representable multicategories – this is not a great loss for our mission to fill in the gaps.

We have thus examined two types of monoidal categories (biased and unbiased), and Theorem 3.2.9 shows that these are really two sides of the same coin. The world of unbiased monoidal categories is the world of regular monoidal categories viewed through a different lens, and vice versa.

From here on out, we deal only with unbiased monoidal categories, as these possess the necessary tools that will allow us to build underlying multicategories.

4 Multicategories

Following our discussion of monoidal categories, we introduce the second central concept of study, multicategories. We provide definitions and simple examples.

Multicategories are one generalisation of categories. Morphisms in categories have one "input" and one "output", while multimorphisms in multicategories have many inputs and one output.

We also study representable multicategories, those where multimorphisms have well-behaved "factorisations". As we shall see in the following section, the category of monoidal categories is not equivalent to the category of all multicategories, only the subcategory of representable multicategories.

In this section, we adapt definitions and results from Leinster (2003) and Hermida (2000) unless stated otherwise.

4.1 Basic Multicategories

Categories are very general structures, encoding the vast majority of mathematical constructions we encounter in algebra and related fields. We can however generalise them further to encompass even more contexts and concepts.

Multicategories are one such approach. In multicategories, we no longer restrict ourselves to considering morphisms with only one domain. Instead, we construct multimorphisms with finite lists of domains. Composition, identities and commutative diagrams are then the natural extensions of these concepts in regular categories.

One may ask whether it would be mathematically sound to consider generalised categories whose morphism-like constructions have multiple domains *and* multiple codomains. Indeed it is. These have names such as *polycategory* or others (Leinster, 2003, pp. 212 – 215). These lie outside the scope of our investigation.

Note we use terms *domain* and *input* interchangeably here, and similarly for *codomain* and *output*.

Definition 4.1.1 (Multicategory)

A *multicategory* $\mathcal{C}$ consists of:

- A class $\mathrm{ob}(\mathcal{C})$ called the class of *objects* of $\mathcal{C}$.
- For all $n \in \mathbb{N}_0$ and all $A_1 \dots A_n, B \in \mathrm{ob}(\mathcal{C})$, a class $\mathcal{C}(A_1 \dots A_n; B)$ called the *homset* of *multimorphisms* from $A_1 \dots A_n$ to B. For $f \in \mathcal{C}(A_1 \dots A_n; B)$, we call n the *arity* of f.
- For all $n, m_i \in \mathbb{N}_0$ and all $A_{i,j}, B_i, C \in \mathrm{ob}(\mathcal{C})$, a function

$$\begin{aligned}\circ : \mathcal{C}(B_1 \dots B_n; C) \times \mathcal{C}(A_{1,1} \dots A_{1,m_1}; B_1) \times \cdots \times \mathcal{C}(A_{n,1} \dots A_{n,m_n}; B_n)\\ \to \mathcal{C}(A_{1,1} \dots A_{1,m_1} \dots A_{n,1} \dots A_{n,m_n}; C)\end{aligned}$$

  sending $(f, f_1 \dots f_n) \mapsto f \circ (f_1 \dots f_n)$, called *composition*.
- For all $A \in \mathrm{ob}(\mathcal{C})$, a morphism $1_A \in \mathcal{C}(A; A)$ called the *identity* on A.

Composition is subject to the following for all compatible multimorphisms $f, f_i, f_{i,j}$, i.e. for all such multimorphisms where the following compositions are defined.

- Composition is associative, that is,

$$
\begin{aligned}
&f \circ (f_1 \circ (f_{1,1} \dots f_{1,m_1}) \dots f_n \circ (f_{n,1} \dots f_{n,m_n})) \\
&= (f \circ (f_1 \dots f_n)) \circ (f_{1,1} \dots f_{1,m_1} \dots f_{n,1} \dots f_{n,m_n}).
\end{aligned}
$$

- For all $f \in \mathcal{C}(A_1 \dots A_n; B)$, we have $f \circ (1_{A_1} \dots 1_{A_n}) = f = 1_B \circ f$.

Throughout this document, we shall use the notation $A \in \mathcal{C}$ to mean $A \in \mathrm{ob}(\mathcal{C})$. We may write $f : A_1 \dots A_n \to B$ to mean $f \in \mathcal{C}(A_1 \dots A_n; B)$. We may also write $f \circ f_1$ instead of $f \circ (f_1)$ when f has arity $n = 1$.

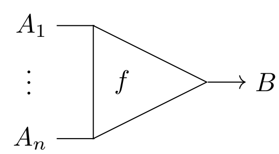
Remark 4.1.2

As with monoidal categories, notation obfuscates the key ideas of multicategories. Commutative diagrams – or rather, their analogue in multicategories – come to our rescue here.

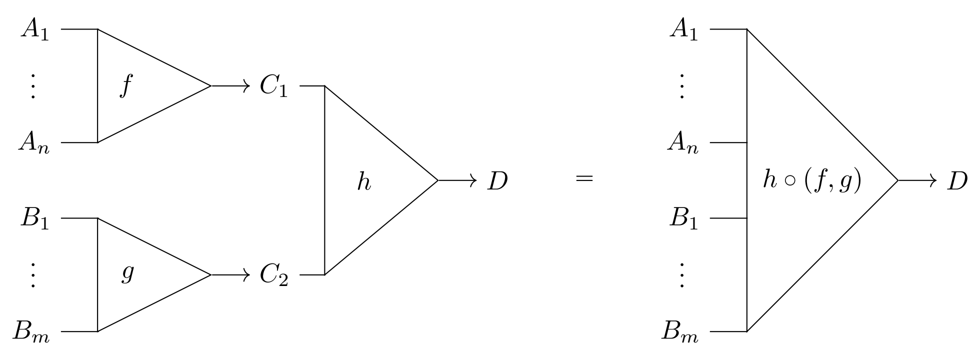
Notice first that there is nothing special about the objects in a multicategory. We simply have a class $\mathrm{ob}(\mathcal{C})$, which in practice, we again consider as a set. The difficulty arises from the connections between objects, multimorphisms.

To generalise commutative diagrams, we depict multimorphisms not as arrows, but triangles. Compare a morphism $f : A \to B$ (in an ordinary category) to a multimorphism $f : A_1 \dots A_n \to B$ below.

$$A \xrightarrow{f} B$$



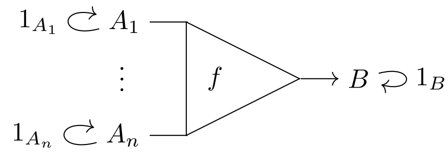
Composing multimorphisms then becomes a matter of sticking triangles end to end in reasonable ways. To be more specific, consider the diagram on the left below.



Here, we have multimorphisms $f : A_1 \dots A_n \to C_1$, $g : B_1 \dots B_m \to C_2$ and $h : C_1, C_2 \to D$, and we can thus produce the composite multimorphism, written $h \circ (f, g) : A_1 \dots A_n, B_1 \dots B_m \to D$ and displayed on the right.

The above diagrams show the case when h has two domains, but we can clearly extend this notion in general if h has domains $C_1 \dots C_m$ and there are multimorphisms $f_1 \dots f_m$ with $C_1 \dots C_m$ as their respective codomains. This completely defines composition.

What do identities look like in multicategories? These are multimorphisms with only one domain and a matching codomain, so they look exactly like identities in an ordinary category.



The defining property of the identity means any diagram of the form shown above must commute. Commutativity of a multicategory diagram is the extension of commutativity for ordinary diagrams – all multimorphisms in the diagram which share all domains and share codomain are the same.

Associativity is similarly analogous to associativity in a regular category.

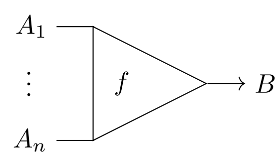
Further, a multimorphism might be an isomorphism if it has a two-sided inverse, just as in a category. Crucially though, this can only occur if the multimorphism has one input, i.e. looks like an ordinary morphism. We say multimorphisms with arity 1 are *unary* (some texts may instead refer to these as *linear*). Identities and isomorphisms are necessarily unary.

Example 4.1.3

It is easy to create simple examples of multicategories.

(a) Any category is a multicategory. Multimorphisms in general take any number of inputs, but if every multimorphism in a given multicategory is unary, these multimorphisms are in fact morphisms in a standard category. The identity and associativity constraints for multicategories reduce to those in categories.

(b) Let $n \in \mathbb{N}$, define $\text{ob}(\mathcal{C}) = \{A_1 \dots A_n, B\}$, and let the non-identity multimorphisms be given by $\mathcal{C}(A_1 \dots A_n; B) = \{f\}$ only, so all other homsets are empty. That is, $f : A_1 \dots A_n \to B$ is the only non-identity multimorphism. We then call $\mathcal{C}$ the *n-arrow multicategory.*



Note the resemblance to the arrow category in Example 2.1.3(c). We have simply generalised the notion of a single morphism, and can similarly represent the entire category in one diagram.

(c) An *operad* is a multicategory with one object (Leinster, 2003, p. 33). That is, $\text{ob}(\mathcal{C}) = \{*\}$ and multimorphisms are $f : * \dots * \to *$ for any number of copies of $*$ as domains. These have a wide variety of applications. Note that some authors use the term operad (*coloured operad* in particular) to mean multicategory, and vice versa.

(d) Diagrams in multicategories are reminiscent of logic gates, and here we find one important use for operads. Take a multicategory with one object, the set $\{0, 1\}$. Define multimorphisms $\{0, 1\} \dots \{0, 1\} \to \{0, 1\}$ to be set functions $\{0, 1\} \times \cdots \times \{0, 1\} \to \{0, 1\}$. Define the multimorphism $\text{NAND} : \{0, 1\}, \{0, 1\} \to \{0, 1\}$ according to the following table.

x	y	NAND(x, y)
0	0	1
0	1	1
1	0	1
1	1	0

Thus, this morphism is the logical NAND operation. We may also define AND, OR, XOR, etc. as two-input multimorphisms, and NOT as a one-input multimorphism. We then obtain logical identities by composition, e.g. AND = NOT ∘ NAND = NAND ∘ (NOT, NOT).

A more delicate construction allows us to build all logical operations out of NAND gates, as is the case in actual computer circuitry (Spivak, 2013, pp. 11 – 12). As it stands, we have no way of duplicating inputs, so cannot use this fact in the above encoding.

(e) Let $\overline{\mathbf{Set}}$ be the multicategory whose objects are (small) sets and whose multimorphisms are functions from cartesian products. That is, a multimorphism $f : X_1 \ldots X_n \to X$ is specified by an ordinary set function $f : X_1 \times \cdots \times X_n \to X$. Identities in $\overline{\mathbf{Set}}$ are the same as those in **Set**, and associativity once again follows from the associativity of normal function composition.

(f) Multicategories have applications in deductive logic and even linguistics. They act as a bridge between the worlds of algebra and semantics, especially in computer science (Lambek, 1989).

Definition 4.1.4 (Multifunctor)

Let $\mathcal{C}$ and $\mathcal{D}$ be multicategories. A *multifunctor* $F : \mathcal{C} \to \mathcal{D}$ consists of:

- A function $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, $A \mapsto F(A)$.
- For all $A_1 \ldots A_n, B \in \mathcal{C}$, a function $\mathcal{C}(A_1 \ldots A_n; B) \to \mathcal{D}(F(A_1) \ldots F(A_n); F(B))$, $f \mapsto F(f)$.

The function on multimorphisms is then subject to the following conditions $\forall n, m_i \in \mathbb{N}_0$ and $\forall A_{1,1} \ldots A_{n,m_n}, B_1 \ldots B_n, C \in \mathcal{C}$:

- If $f \in \mathcal{C}(B_1 \ldots B_n; C)$ and $f_i \in \mathcal{C}(A_{i,1} \ldots A_{i,m_i}; B_i)$ for $i = 1 \ldots n$, then $F(f \circ (f_1 \ldots f_n)) = F(f) \circ (F(f_1) \ldots F(f_n))$. That is, F preserves composition.
- Similarly, $F(1_A) = 1_{F(A)}$, so F preserves identities.

The above conditions are called the *multifunctorality axioms*.

In some contexts, we may write FA for $F(A)$ and Ff for $F(f)$.

Remark 4.1.5

Just as multicategories generalise categories, multifunctors generalise functors. All of the above notation amounts to saying that multifunctors preserve composition and identities in multicategories.

Note that Leinster (2003) calls such constructions *maps of multicategories*, though we opt for multifunctor for extra clarity, as maps of multicategories may be confused with multimorphisms (which Leinster calls *arrows*).

Further, multifunctors are not to be confused with *bifunctors* and so on. The latter are functors from products of categories, rather than maps between multicategories.

As is to be expected by now, the composition of two multifunctors is a multifunctor, and there is an identity multifunctor. We can thus define our next important category.

Definition 4.1.6 (Category of Multicategories)

Denote by **Multicat** the category of (small) multicategories and multifunctors.

4.2 Representable Multicategories

Certain multicategories possess extra properties, and we will be interested in those in which multimorphisms have canonical decompositions for any grouping of inputs. These are called representable multicategories. As we shall see, representable multicategories are simply monoidal categories in disguise, and vice versa.

Hermida (2000, pp. 192 – 193) introduced representable multicategories to the literature. Leinster (2003, p. 84) expanded on the definitions, and we adapt his conventions.

Definition 4.2.1 (Representable Multicategory)

Let $\mathcal{C}$ be a multicategory. A *representation* (R, u) of $\mathcal{C}$ consists of, for all $n \in \mathbb{N}_0$ and $A_1 \dots A_n \in \mathcal{C}$:

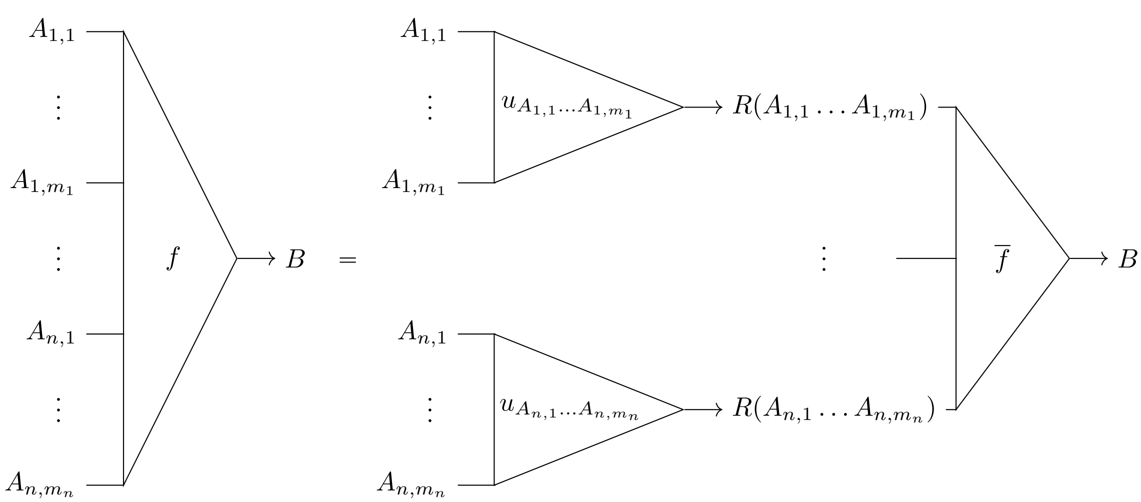
- An object $R(A_1 \dots A_n) \in \mathcal{C}$.
- A multimorphism $u_{A_1 \dots A_n} : A_1 \dots A_n \to R(A_1 \dots A_n)$.

These are such that, for all objects $A_{1,1} \dots A_{1,m_1} \dots A_{n,1} \dots A_{n,m_n}, B \in \mathcal{C}$ and any multimorphism $f : A_{1,1} \dots A_{n,m_n} \to B$, there is a unique $\overline{f} : R(A_{1,1} \dots A_{1,m_1}) \dots R(A_{n,1} \dots A_{n,m_n}) \to B$ such that $\overline{f} \circ (u_{A_{1,1} \dots A_{1,m_1}} \dots u_{A_{n,1} \dots A_{n,m_n}}) = f$.

A multicategory is *representable* if such a representation exists.

Remark 4.2.2

This is better illustrated with a diagram. Suppose we have any multimorphism which is given by $f : A_{1,1} \dots A_{1,m_1} \dots A_{n,1} \dots A_{n,m_n} \to B$, where we have separated its list of inputs into arbitrary sub-lists (with all sub-lists containing consecutive objects in their original order). This situation is depicted on the left below.



If the multicategory is representable, we can factorise f using the objects and multimorphisms specified by the representation. For each choice of sub-lists, there then exists a unique $\overline{f}$ such that the composite on the right is equal to f, i.e. the two diagrams above correspond to the same multimorphism.

The $R(A_1 \dots A_n)$ and $u_{A_1 \dots A_n}$ therefore provide standard factorisations for any multimorphism, so (R, u) provides crucial information about the multicategory at large.

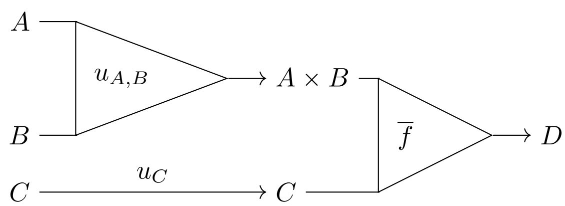
Example 4.2.3

Understanding which multicategories are or are not representable is vitally helpful.

(a) Recall $\overline{\mathbf{Set}}$ from Example 4.1.3(e). This multicategory is representable.

What exactly is this representation? It may not be surprising to learn that an object $R(X_1 \dots X_n)$ is $X_1 \times \cdots \times X_n$, and a multimorphism $u_{X_1 \dots X_n} : X_1 \dots X_n \to R(X_1 \dots X_n)$ is a set function identity. Given a multimorphism f, we then find $\overline{f}$ is a multimorphism out of a certain bracketing of $X_1 \times \cdots \times X_n$ whose values are uniquely determined by f.

For instance, let $A, B, C, D \in \overline{\mathbf{Set}}$ and let $f : A, B, C \to D$ be a multimorphism. If we grouped A, B, C as (A, B) and (C), we would have the following situation.



First notice that $u_{A,B}$, as a set function, is the identity on $A \times B$. The subtlety is that it is *not* the identity multimorphism on $A \times B$ – as far as the multicategory structure is concerned, it combines two separate objects into one. The situation for u_C is simpler, as this is necessarily the multicategorical identity.

Then $\overline{f} : A \times B, C \to D$ is the set function $(A \times B) \times C \to D$ sending $((a, b), c) \mapsto f(a, b, c)$. Since we require $\overline{f} \circ (u_{A,B}, u_C) = f$, there is no other way to define such an $\overline{f}$, so it is unique.

(b) Any unbiased monoidal category gives rise to a representable multicategory in a similar way to $\overline{\mathbf{Set}}$ above. The central point is that tensored objects are effectively the same no matter how we have bracketed them. Though the element $((a, b), c) \in (A \times B) \times C$ is not per se equal to $(a, b, c) \in A \times B \times C$, it still contains all the necessary information to construct a unique $\overline{f}$ that does the same job as f.

It takes much more work to make this precise, and we dedicate much of the next section to constructing the underlying multicategory of a general unbiased monoidal category.

(b) Consider the n-arrow multicategory from Example 4.1.3(b). If $n = 1$ or $n = 2$, then this multicategory is representable, but if $n \geqslant 3$, it is not.

If $n = 1$, then we clearly cannot re-group the inputs of $f : A \to B$, so we need only note the trivial case where we take $R(A) = A$ and $u_A = 1_A$ to obtain $\overline{f} = f$. If $n = 2$, then for $f : A_1, A_2 \to B$, any grouping of A_1, A_2 produces either the whole list or sub-lists of length one. The representation multimorphisms are then f itself or identities respectively.

But for $n = 3$, there are no multimorphisms of arity 2, only 1 (identities) or 3 (the single non-trivial $f : A_1, A_2, A_3 \to B$). So if we group the inputs of f as (A_1, A_2) and (A_3), there does not exist some u_{A_1, A_2} required for a representation. The same occurs for $n > 3$.

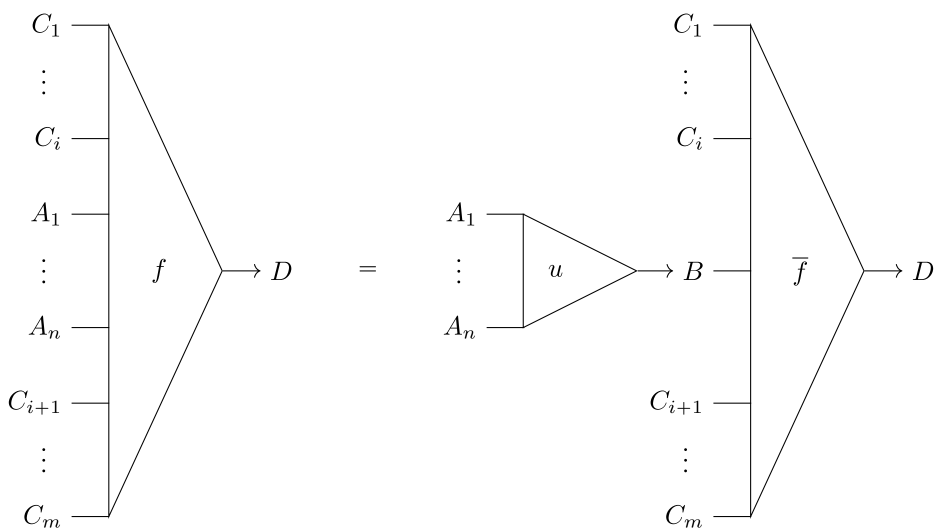
Definition 4.2.4 (Universal Multimorphism)

Let $\mathcal{C}$ be a multicategory and let $A_1 \dots A_n, B \in \mathcal{C}$. We say that a multimorphism $u : A_1 \dots A_n \to B$ is *universal* if $\forall C_1 \dots C_m, D \in \mathcal{C}$, $\forall i = 0 \dots m$ and for any $f : C_1 \dots C_i, A_1 \dots A_n, C_{i+1} \dots C_m \to D$, there exists $\overline{f} : C_1 \dots C_i, B, C_{i+1} \dots C_m \to D$ unique such that $\overline{f} \circ (1_{C_1} \dots 1_{C_i}, u, 1_{C_{i+1}} \dots 1_{C_m}) = f$.

Write $\overline{f} \circ_i u$ to denote $\overline{f} \circ (1_{C_1} \dots 1_{C_i}, u, 1_{C_{i+1}} \dots 1_{C_m})$.

Remark 4.2.5

The following diagrams may help illuminate the situation. Suppose we are given a multimorphism $f : C_1 \dots C_i, A_1 \dots A_n, C_{i+1} \dots C_m \to D$, displayed on the left. If we have a universal multimorphism $u : A_1 \dots A_n \to B$, then there exists a unique $\overline{f}$ such that the composite on the right is equal to f.



A universal multimorphism allows us to "condense" inputs from other multimorphisms and retain the same properties. In effect, the existence of $u : A_1 \dots A_n \to B$ tells us that encoding $A_1 \dots A_n$ as separate objects in the multicategory is redundant – everything we would want to know about how they interact with other objects can be found by studying B.

Representable multicategories present a more extreme version of universality, which we make explicit below. In essence, having a list of separate input objects at all is redundant, as every multimorphism can be characterised by a multimorphism with a single input.

Proposition 4.2.6

Let $\mathcal{C}$ be a multicategory with representation (R, u). Then $u_{A_1 \dots A_n}$ is universal $\forall A_1 \dots A_n \in \mathcal{C}$.

Proof. Since (R, u) is a representation of $\mathcal{C}$, for any $C_1 \dots C_m, D \in \mathcal{C}$, any $i = 0 \dots m$ and any $g : C_1 \dots C_i, A_1 \dots A_n, C_{i+1} \dots C_m \to D$, there exist $R(C_i)$ and u_{C_i} which give a unique $\overline{g}$ such that

$$\begin{aligned} g &= \overline{g} \circ (u_{C_1} \dots u_{C_i}, u_{A_1 \dots A_n}, u_{C_{i+1}} \dots u_{C_m}) \\ &= \overline{g} \circ (u_{C_1} \circ 1_{C_1} \dots u_{C_i} \circ 1_{C_i}, 1_{R(A_1 \dots A_n)} \circ u_{A_1 \dots A_n}, u_{C_{i+1}} \circ 1_{C_{i+1}} \dots u_{C_m} \circ 1_{C_m}) \\ &= \overline{g} \circ (u_{C_1} \dots u_{C_i}, 1_{R(A_1 \dots A_n)}, u_{C_{i+1}} \dots u_{C_m}) \circ (1_{C_1} \dots 1_{C_i}, u_{A_1 \dots A_n}, 1_{C_{i+1}} \dots 1_{C_m}) \text{ by associativity} \\ &\quad \text{in the multicategory.} \end{aligned}$$

Taking $\overline{g}^* = \overline{g} \circ (u_{C_1} \dots u_{C_i}, 1_{R(A_1 \dots A_n)}, u_{C_{i+1}} \dots u_{C_m})$, we therefore have $\overline{g}^*$, unique by the uniqueness of $\overline{g}$, such that $g = \overline{g}^* \circ_i u_{A_1 \dots A_n}$. That is, $u_{A_1 \dots A_n}$ is universal. ☐

Corollary 4.2.7 (Universal Factorisation)

In a multicategory with representation (R, u), every multimorphism $f : A_1 \dots A_n \to B$ admits a *universal factorisation* $f = \overline{f} \circ u_{A_1 \dots A_n}$, where $\overline{f} : R(A_1 \dots A_n) \to B$ is unary and unique.

Proof. By Proposition 4.2.6 above, $\exists \overline{f} : R(A_1 \dots A_n) \to B$ unique such that $\overline{f} \circ u_{A_1 \dots A_n} = f$. ☐

Proposition 4.2.8

Let $\mathcal{C}$ be a multicategory. Then universal multimorphisms in $\mathcal{C}$ are closed under composition. That is, if u and u_i for $i = 1 \dots n$ are universal, then $u \circ (u_1 \dots u_n)$ is universal.

Proof. For ease of notation, leave objects implicit. Let f have list of domains including the lists of domains of $u_1 \dots u_n$ beginning at the $(i+1)$th position (for $i \geqslant 0$). Then by universality of u_1, there exists a unique $\overline{f_1}$ such that $f = \overline{f_1} \circ_i u_1$. Inductively, we find unique $\overline{f_1} \dots \overline{f_n}$ such that

$$\begin{aligned} f &= \overline{f_1} \circ_i u_1 \\ &= (\overline{f_2} \circ_{i+1} u_2) \circ_i u_1 \\ &= \cdots \\ &= ((\overline{f_n} \circ_{i+n-1} u_n) \circ_{i+n-2} \cdots) \circ_i u_1. \end{aligned}$$

Since the domains of $\overline{f_n}$ now include those of u from the $(i+1)$th position, by universality of u we have unique $\overline{f}$ with $f = (((\overline{f} \circ_i u) \circ_{i+n-1} u_n) \circ_{i+n-2} \cdots) \circ_i u_1 = \overline{f} \circ_i (u \circ (u_1 \dots u_n))$ by associativity. □

Proposition 4.2.9

Let $u : A_1 \dots A_n \to B$ and $v : A_1 \dots A_n \to C$ be multimorphisms in multicategory $\mathcal{C}$. Suppose u is universal, so $v = \overline{v} \circ u$ for unique $\overline{v} : B \to C$. Then v is universal if and only if $\overline{v}$ is an isomorphism.

Proof. First suppose v is universal. Then $u = \overline{u} \circ v$ for unique $\overline{u} : B \to C$. Thus, $u = \overline{u} \circ \overline{v} \circ u$. By the universality of u, we see $\overline{u} \circ \overline{v}$ is the unique multimorphism with this property. But notice 1_B also satisfies $u = 1_B \circ u$, so in fact $\overline{u} \circ \overline{v} = 1_B$. The reverse argument for v, which is universal by assumption, shows $\overline{v} \circ \overline{u} = 1_C$. Thus, $\overline{v}$ is an isomorphism with inverse $\overline{u}$.

Now suppose $\overline{v}$ is an isomorphism, so $\overline{v}^{-1} : C \to B$ exists, and from $v = \overline{v} \circ u$ we have $\overline{v}^{-1} \circ v = u$. By the universality of u, for any compatible f and i, there exists a unique $\overline{f}$ such that

$$\begin{aligned} f &= \overline{f} \circ_i u \\ &= \overline{f} \circ_i (\overline{v}^{-1} \circ v) \\ &= (\overline{f} \circ_i \overline{v}^{-1}) \circ_i v \text{ by associativity in the multicategory.} \end{aligned}$$

Since $\overline{f}$ and $\overline{v}^{-1}$ are unique, v is thus also universal. □

Definition 4.2.10 (Universal-Preserving Multifunctor)

Let $\mathcal{C}$ and $\mathcal{D}$ be multicategories. A multifunctor $F : \mathcal{C} \to \mathcal{D}$ is *universal-preserving* if u universal in $\mathcal{C}$ implies that $F(u)$ is universal in $\mathcal{D}$.

Definition 4.2.11 (Category of Representable Multicategories)

Denote by **RMulticat** the category of representable multicategories and universal-preserving multifunctors.

Remark 4.2.12

The identity multifunctor is universal-preserving, as is any composition of universal-preserving multifunctors, so **RMulticat** is a category. By Proposition 4.2.6, a universal-preserving multifunctor keeps all the necessary structure to preserve the representation, so it makes sense for these to be the morphisms in **RMulticat**. In fact, they are precisely the analogues to monoidal functors.

With **RMulticat** defined, all the tools are now in place for us to make explicit the connection between monoidal categories and multicategories. Examples in this section already suggest how we might go about this for individual cases, but showing **MonCat** $\simeq$ **RMulticat**, i.e. that the entire categories share similar structure, requires substantially more work.

5 The Leinster-Hermida Equivalence Theorem

Our discussion centres on the supposed equivalence between the category of monoidal categories and the category of representable multicategories. We now proceed to state and prove this result.

5.1 Statement and Discussion

While the connection between monoidal categories and multicategories may have been known for some time, the specific result we study is originally due to Hermida (2000). Hermida introduces representable multicategories before exhibiting the desired equivalence in his Theorem 9.8.

Leinster (2003) expands upon this result via a different route, culminating in his Theorem 3.3.4. A proof did not appear in print. We seek to reconstruct it. Though our statement of what we shall call the *Leinster-Hermida Equivalence Theorem* does not match Leinster's version exactly, the conclusion is the same, and we inevitably pass through some of Leinster's auxiliary results.

Theorem 5.1.1 (Leinster-Hermida Equivalence Theorem)

There is an equivalence of categories $\mathbf{MonCat} \simeq \mathbf{RMulticat}$.

Remark 5.1.2

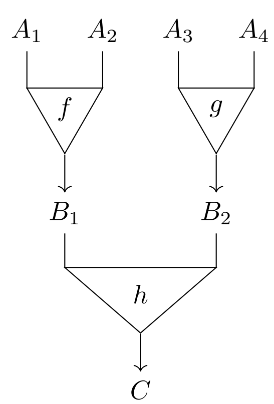
Why should we expect this to be true?

Consider the motivating example of vector spaces (over a fixed field) and linear maps. As noted in Example 3.1.4(d), the category $\mathbf{Vect}_{\mathbb{K}}$ is monoidal when equipped with the usual tensor product of vector spaces. We can also construct the multicategory $\overline{\mathbf{Vect}_{\mathbb{K}}}$ of vector spaces (over a fixed field) and multilinear maps (that is, maps which are linear in each entry). Fundamentally though, these are two different encodings of the same information.

We therefore expect to find some canonical translation between these contexts. The above theorem asserts that $\overline{\mathbf{Vect}_{\mathbb{K}}}$ is isomorphic to the underlying multicategory of the monoidal category $\mathbf{Vect}_{\mathbb{K}}$. That is, $\overline{\mathbf{Vect}_{\mathbb{K}}}$ (or a functionally identical version of it) is hiding underneath $\mathbf{Vect}_{\mathbb{K}}$ if we know how to look for it.

Importantly however, there is *not* an equivalence between **MonCat** and the entirety of **Multicat**. If this were the case, multicategories would arguably be a redundant notion. The theorem also tells us there are many multicategories which do not lie beneath monoidal categories. In a sense, multicategories are more general structures than monoidal categories.

One way to see why representability is the key property is, as always, to examine a few diagrams.

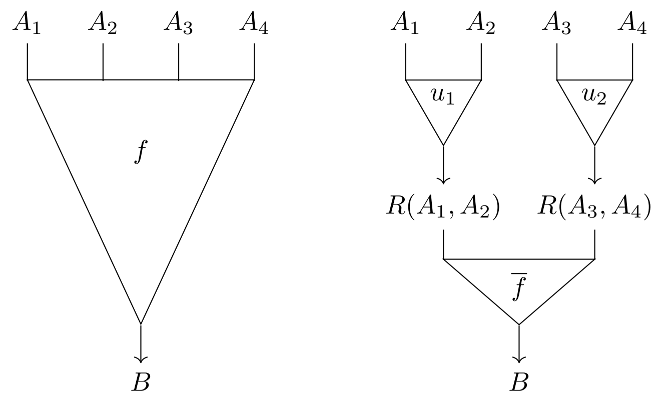


$$\begin{tikzcd}
A_1 \otimes A_2 \otimes A_3 \otimes A_4 \arrow[d, "\gamma^{-1}_{(A_1,A_2),(A_3,A_4)}"] \\
(A_1 \otimes A_2) \otimes (A_3 \otimes A_4) \arrow[d, "f \otimes g"] \\
B_1 \otimes B_2 \arrow[d, "h"] \\
C
\end{tikzcd}$$

Multimorphisms in the underlying multicategory correspond exactly with morphisms out of tensor products. But notice inputs of the composite $h \circ (f, g) : A_1, A_2, A_3, A_4 \to C$ are not grouped in any particular way, even though constructing it via the monoidal category relies on grouping the inputs of f and g separately. Happily, if we work in an unbiased monoidal category (which, by Theorem 3.2.9, is a biased monoidal category in disguise), there exists an object $A_1 \otimes A_2 \otimes A_3 \otimes A_4$. Since the category provides us with the natural isomorphism γ, we can define composition to implicitly bracket this object as shown above, ensuring $h \circ (f, g)$ makes sense.

However, in an unbiased monoidal category, the object $A_1 \otimes A_2 \otimes A_3 \otimes A_4$ is isomorphic to any other bracketing. Since the list of inputs for a multimorphism is unbiased, such multimorphisms must therefore in some sense be invariant under regrouping of their inputs. Representations provide exactly this flexibility.

Consider the case for $f : A_1, A_2, A_3, A_4 \to B$ below, where we have written $u_1 = u_{A_1,A_2}$ and $u_2 = u_{A_3,A_4}$ for ease of notation. Representability asserts that f is for example equal to the composite multimorphism on the right.



By our definition of composition in the underlying multicategory, this is equivalent to requiring the following diagram to commute in the unbiased monoidal category.

$$\begin{tikzcd}
A_1 \otimes A_2 \otimes A_3 \otimes A_4 \arrow[r, "f"] \arrow[d, "\gamma^{-1}_{(A_1,A_2),(A_3,A_4)}"'] & B \\
(A_1 \otimes A_2) \otimes (A_3 \otimes A_4) \arrow[r, "u_1 \otimes u_2"'] & R(A_1, A_2) \otimes R(A_3, A_4) \arrow[u, "\overline{f}"']
\end{tikzcd}$$

As we shall see, the structure of an unbiased monoidal category is necessary and sufficient to ensure all such diagrams commute, no matter the bracketing, provided we define γ or (R, u) properly depending on the direction.

That the categories **MonCat** and **RMulticat** are equivalent tells us even more. Not only are monoidal categories and representable multicategories in correspondence, but so are the relations between them, monoidal functors and multifunctors respectively. The structure-preserving connections between representable multicategories are exactly those which preserve the structure of monoidal categories. The contexts as a whole are remarkably similar.

The equivalence **MonCat** $\simeq$ **RMulticat** hints at an insightful conclusion. As noted in Remark 3.1.5, monoidal categories are everywhere. But the complexity of their definition – particularly the unwieldy diagrams – can be an obstacle. We can instead use the arguably much simpler definition of representable multicategories, safe in the knowledge that these are exactly the same thing.

5.2 Proof

At a high level, we proceed with the proof in several stages:

- We construct the underlying multicategory functor $V : \mathbf{UMonCat} \to \mathbf{Multicat}$ and show it is well-defined (definitions and results up to Lemma 5.2.5).
- We show a multicategory $\mathcal{C}$ is representable if and only if it is isomorphic to $V(\mathcal{D})$ for some unbiased monoidal category $\mathcal{D}$ (Lemma 5.2.6 and Lemma 5.2.7). So V is essentially surjective.
- We show the image of V lies in **RMulticat** (Lemma 5.2.8), and it induces a full and faithful functor $V' : \mathbf{UMonCat} \to \mathbf{RMulticat}$ (Lemma 5.2.9 and Lemma 5.2.10).
- As V' is faithful, full and essentially surjective, it is an equivalence $\mathbf{UMonCat} \simeq \mathbf{RMulticat}$.
- Together with $\mathbf{MonCat} \simeq \mathbf{UMonCat}$ (Theorem 3.2.9), this gives the final result.

This is the approach outlined by Leinster (2003). Hermida (2000) instead directly constructs functors $\mathbf{MonCat} \to \mathbf{RMulticat}$ and $\mathbf{RMulticat} \to \mathbf{MonCat}$, then shows they are pseudo-inverses.

Throughout this proof, we avoid confusion by always denoting tensor product with outer bracketing, i.e. writing $(A_1 \otimes \cdots \otimes A_n)$ rather than $A_1 \otimes \cdots \otimes A_n$. We also write FA and Ff rather than $F(A)$ and Ff for application of functors (other than V) and multifunctors. Note in particular then that $F(A)$ denotes F applied to the 1-fold tensor of A.

Definition 5.2.1 (Underlying Multicategory)

Let $(\mathcal{C}, \otimes_n, \gamma, \iota)$ be an unbiased monoidal category. We construct its *underlying multicategory*, denoted $V(\mathcal{C})$, as follows.

- Objects are the same as in $\mathcal{C}$. That is, $\mathrm{ob}(V(\mathcal{C})) = \mathrm{ob}(\mathcal{C})$.
- Let $A_1 \dots A_n \in V(\mathcal{C})$. A multimorphism $f : A_1 \dots A_n \to B$ in $V(\mathcal{C})$ is precisely a morphism $f : (A_1 \otimes \cdots \otimes A_n) \to B$ in $\mathcal{C}$. That is, $V(\mathcal{C})(A_1 \dots A_n; B) = \mathcal{C}((A_1 \otimes \cdots \otimes A_n), B)$.
- Let $n, m_i \in \mathbb{N}_0$ and let $A_{i,j}, B_i, C \in \mathrm{ob}(V(\mathcal{C}))$. Let $f \in V(\mathcal{C})(B_1 \dots B_n; C)$ and $\forall i = 1 \dots n$ let $f_i \in V(\mathcal{C})(A_{i,1} \dots A_{i,m_i}; B_i)$. Their composition $f \circ (f_1 \dots f_n) \in V(\mathcal{C})(A_{1,1} \dots A_{n,m_n}; C)$ is defined to be

  $$f \circ (f_1 \dots f_n) = f \cdot (f_1 \otimes \cdots \otimes f_n) \cdot \gamma^{-1}_{(A_{1,1} \dots A_{1,m_1}) \dots (A_{n,1} \dots A_{n,m_n})}$$

  where $\cdot$ is composition in $\mathcal{C}$.

  Equivalently, $\circ$ is defined so that all diagrams of the following form commute in $\mathcal{C}$.

  $$\begin{tikzcd}
  (A_{1,1} \otimes \cdots \otimes A_{1,m_1} \otimes \cdots \otimes A_{n,1} \otimes \cdots \otimes A_{n,m_n}) \arrow[r, "f \circ (f_1 \dots f_n)"] \arrow[d, "\gamma^{-1}_{(A_{1,1} \dots A_{1,m_1}) \dots (A_{n,1} \dots A_{n,m_n})}"'] & C \\
  ((A_{1,1} \otimes \cdots \otimes A_{1,m_1}) \otimes \cdots \otimes (A_{n,1} \otimes \cdots \otimes A_{n,m_n})) \arrow[r, "(f_1 \otimes \cdots \otimes f_n)"'] & (B_1 \otimes \cdots \otimes B_n) \arrow[u, "f"']
  \end{tikzcd}$$

- For all $A \in \mathrm{ob}(\mathcal{C})$, the identity on A is $\iota_A^{-1} : (A) \to A$. Denote this by $\mathbb{1}_A : A \to A$ to make clear the distinction between monoidal and multicategory identities.

Lemma 5.2.2

The underlying multicategory $V(\mathcal{C})$ of an unbiased monoidal category $(\mathcal{C}, \otimes_n, \gamma, \iota)$ is well-defined.

Proof. First, as the components of γ and ι are isomorphisms, inverses exist. The commutative diagram in the definition thus shows composition is well-defined, as the domains and codomains of successive morphisms match and $f \circ (f_1 \dots f_n)$ is indeed a multimorphism $A_{1,1} \dots A_{n,m_n} \to B$.

Further, by the definition of the multimorphisms, a multimorphism $A \to A$ in $V(\mathcal{C})$ corresponds to a morphism $(A) \to A$ in $\mathcal{C}$. So identities $\mathbb{1}_A$ are well-defined multimorphisms.

Note that composition of morphisms in $\mathcal{C}$ is associative, so we implicitly rearrange brackets for compositions in $\mathcal{C}$ below. Denote this composition by $\cdot$ to contrast with composition $\circ$ in $V(\mathcal{C})$.

Since ι is a natural isomorphism, we can consider its naturality square for any $f : (A_1 \otimes \cdots \otimes A_n) \to B$.

$$\begin{tikzcd}
(A_1 \otimes \cdots \otimes A_n) \arrow[r, "f"] \arrow[d, "\iota_{(A_1 \otimes \cdots \otimes A_n)}"'] & B \arrow[d, "\iota_B"] \\
((A_1 \otimes \cdots \otimes A_n)) \arrow[r, "(f)"] & (B)
\end{tikzcd}$$

Inverting all isomorphisms above and in the identity diamond (see Definition 3.2.1) and gluing the resulting diagrams together, we then see the following diagram must also commute.

$$\begin{tikzcd}
((A_1) \otimes \cdots \otimes (A_n)) \arrow[r, "(\iota_{A_1} \otimes \cdots \otimes \iota_{A_n})^{-1}"] & (A_1 \otimes \cdots \otimes A_n) \arrow[r, "f"] & B \\
(A_1 \otimes \cdots \otimes A_n) \arrow[u, "\gamma^{-1}_{(A_1)\dots(A_n)}"] \arrow[ur, "1_{(A_1 \otimes \cdots \otimes A_n)}"'] \arrow[r, "\gamma^{-1}_{(A_1 \dots A_n)}"'] & ((A_1 \otimes \cdots \otimes A_n)) \arrow[u, "\iota^{-1}_{(A_1 \otimes \cdots \otimes A_n)}"'] \arrow[r, "(f)"] & (B) \arrow[u, "\iota_B^{-1}"']
\end{tikzcd}$$

Since $\otimes_n$ is a functor and hence preserves inverses, we have $(\iota_{A_1} \otimes \cdots \otimes \iota_{A_n})^{-1} = (\iota_{A_1}^{-1} \otimes \cdots \otimes \iota_{A_n}^{-1})$. Substituting this into the above diagram and expanding by definition of composition, we see that

$$\begin{aligned}
\mathbb{1}_B \circ f &= \iota_B^{-1} \cdot (f) \cdot \gamma^{-1}_{(A_1 \dots A_n)} \\
&= f \cdot 1_{(A_1 \otimes \cdots \otimes A_n)} \\
&= f \cdot (\iota_{A_1}^{-1} \otimes \cdots \otimes \iota_{A_n}^{-1}) \cdot \gamma^{-1}_{(A_1) \dots (A_n)} \\
&= f \circ (\mathbb{1}_{A_1} \dots \mathbb{1}_{A_n}).
\end{aligned}$$

From $f \cdot 1_{(A_1 \otimes \cdots \otimes A_n)} = f$, we see our candidate identities indeed act as identities in the multicategory.

Let $A_{i,j,k}, B_{i,j}, C_i, D \in V(\mathcal{C})$, and let $f, f_1 \dots f_n, f_{1,1} \dots f_{1,m_1} \dots f_{n,1} \dots f_{n,m_n}$ be multimorphisms in $V(\mathcal{C})$ such that the required compositions make sense. Recall the object sequences S, S', S'', S_i in Definition 3.2.1 and define $S''' = (B_{1,1} \dots B_{1,m_1}) \dots (B_{n,1} \dots B_{n,m_n})$.

Consider the naturality square induced by the components of γ (for ease of notation, we have omitted all but the first bracket for tensored objects).

$$\begin{tikzcd}
((A_{1,1,1} \otimes \cdots \otimes A_{1,1,p_{1,1}}) \otimes \cdots) \arrow[d, "(f_{1,1} \otimes \cdots \otimes f_{n,m_n})"'] & (((A_{1,1,1} \otimes \cdots) \otimes \cdots \otimes (A_{1,m_1,1} \otimes \cdots)) \otimes \cdots) \arrow[l, "\gamma_{S'}"'] \arrow[d, "((f_{1,1} \otimes \cdots \otimes f_{1,m_1}) \otimes \cdots \otimes (f_{n,1} \otimes \cdots \otimes f_{n,m_n}))"] \\
(B_{1,1} \otimes \cdots \otimes B_{n,m_n}) & ((B_{1,1} \otimes \cdots \otimes B_{1,m_1}) \otimes \cdots) \arrow[l, "\gamma_{S'''}"]
\end{tikzcd}$$

As we did for ι, we may invert all isomorphisms above. Inverting all isomorphisms in the associativity diamond and combining, the following diagram (with $f \cdot (f_1 \otimes \cdots \otimes f_n)$ appended) commutes.

$$\begin{tikzcd}
(A_{1,1,1} \otimes \cdots \otimes A_{n,m_n,p_{n,m_n}}) \arrow[r, "\gamma_S^{-1}"] \arrow[d, "\gamma_{S''}^{-1}"'] & ((A_{1,1,1} \otimes \cdots \otimes A_{1,m_1,p_{1,m_1}}) \otimes \cdots) \arrow[d, "(\gamma_{S_1} \otimes \cdots \otimes \gamma_{S_n})^{-1}"] \\
((A_{1,1,1} \otimes \cdots \otimes A_{1,1,p_{1,1}}) \otimes \cdots) \arrow[r, "\gamma_{S'}^{-1}"] \arrow[d, "(f_{1,1} \otimes \cdots \otimes f_{n,m_n})"'] & (((A_{1,1,1} \otimes \cdots) \otimes \cdots \otimes (A_{1,m_1,1} \otimes \cdots)) \otimes \cdots) \arrow[d, "((f_{1,1} \otimes \cdots \otimes f_{1,m_1}) \otimes \cdots \otimes (f_{n,1} \otimes \cdots \otimes f_{n,m_n}))"] \\
(B_{1,1} \otimes \cdots \otimes B_{n,m_n}) \arrow[r, "\gamma_{S'''}^{-1}"'] & ((B_{1,1} \otimes \cdots \otimes B_{1,m_1}) \otimes \cdots) \arrow[d, "(f_1 \otimes \cdots \otimes f_n)"] \\
D & (C_1 \otimes \cdots \otimes C_n) \arrow[l, "f"]
\end{tikzcd}$$

Again, the functorality of the tensor gives $(\gamma_{S_1} \otimes \cdots \otimes \gamma_{S_n})^{-1} = (\gamma_{S_1}^{-1} \otimes \cdots \otimes \gamma_{S_n}^{-1})$. We thus apply the unbiased tensor interchange law (Proposition 3.2.4) twice and find the following diagram commutes.

$$\begin{tikzcd}
((A_{1,1,1} \otimes \cdots \otimes A_{1,m_1,p_{1,m_1}}) \otimes \cdots) \arrow[d, "(\gamma_{S_1} \otimes \cdots \otimes \gamma_{S_n})^{-1}"'] \arrow[dddd, bend left=60, "((f_1 \cdot (f_{1,1} \otimes \cdots \otimes f_{1,m_1}) \cdot \gamma_{S_1}^{-1}) \otimes \cdots \otimes (f_n \cdot (f_{n,1} \otimes \cdots \otimes f_{n,m_n}) \cdot \gamma_{S_n}^{-1}))"] \\
(((A_{1,1,1} \otimes \cdots) \otimes \cdots \otimes (A_{1,m_1,1} \otimes \cdots)) \otimes \cdots) \arrow[dd, "((f_{1,1} \otimes \cdots \otimes f_{1,m_1}) \otimes \cdots \otimes (f_{n,1} \otimes \cdots \otimes f_{n,m_n}))"'] \\
\\
((B_{1,1} \otimes \cdots \otimes B_{1,m_1}) \otimes \cdots) \arrow[d, "(f_1 \otimes \cdots \otimes f_n)"'] \\
(C_1 \otimes \cdots \otimes C_n)
\end{tikzcd}$$

Applying our definition of multicategory composition, we can thus combine the diagrams to read off

$$\begin{aligned}
&(f \circ (f_1 \dots f_n)) \circ (f_{1,1} \dots f_{1,m_1} \dots f_{n,1} \dots f_{n,m_n}) \\
&\quad = f \cdot (f_1 \otimes \cdots \otimes f_n) \cdot \gamma_{S'''}^{-1} \cdot (f_{1,1} \otimes \cdots \otimes f_{n,m_n}) \cdot \gamma_{S''}^{-1} \\
&\quad = f \cdot (f_1 \otimes \cdots \otimes f_n) \cdot ((f_{1,1} \otimes \cdots \otimes f_{1,m_1}) \otimes \cdots \otimes (f_{n,1} \otimes \cdots \otimes f_{n,m_n})) \cdot (\gamma_{S_1} \otimes \cdots \otimes \gamma_{S_n})^{-1} \cdot \gamma_S^{-1} \\
&\quad = f \cdot ((f_1 \cdot (f_{1,1} \otimes \cdots \otimes f_{1,m_1}) \cdot \gamma_{S_1}^{-1}) \otimes \cdots \otimes (f_n \cdot (f_{n,1} \otimes \cdots \otimes f_{n,m_n}) \cdot \gamma_{S_n}^{-1})) \cdot \gamma_S^{-1} \\
&\quad = f \circ (f_1 \circ (f_{1,1} \dots f_{1,m_1}) \dots f_n \circ (f_{n,1} \dots f_{n,m_n})).
\end{aligned}$$

So composition is associative and preserves identities, making $V(\mathcal{C})$ a well-defined multicategory. $\square$

Definition 5.2.3 (Underlying Multifunctor)

Let $(\mathcal{C}, \otimes_n, \gamma, \iota)$ and $(\mathcal{D}, \odot_n, \delta, \kappa)$ be unbiased monoidal categories and let $(F, \pi) : \mathcal{C} \to \mathcal{D}$ be an unbiased monoidal functor. We construct its *underlying multifunctor* $V(F, \pi) : V(\mathcal{C}) \to V(\mathcal{D})$ as follows.

- The function $\mathrm{ob}(V(\mathcal{C})) \to \mathrm{ob}(V(\mathcal{D}))$ sends an object A to FA.
- For $A_1 \dots A_n, B \in V(\mathcal{C})$, define the function $V(\mathcal{C})(A_1 \dots A_n; B) \to V(\mathcal{D})(FA_1 \dots FA_n; FB)$ by $V(F, \pi)(f) = Ff \cdot \pi_{A_1 \dots A_n}$, where $\cdot$ is composition in $\mathcal{D}$.

Lemma 5.2.4

The underlying multifunctor $V(F,\pi)$ of an unbiased monoidal functor $(F,\pi) : \mathcal{C} \to \mathcal{D}$ is well-defined.

Proof. Write $\circ$ for composition in $V(\mathcal{C})$ and $V(\mathcal{D})$, and $\cdot$ for composition in $\mathcal{C}$ and $\mathcal{D}$.

Note first that, as $\mathrm{ob}(V(\mathcal{C})) = \mathrm{ob}(\mathcal{C})$, $\mathrm{ob}(V(\mathcal{D})) = \mathrm{ob}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$, the function on objects is well-defined. Any $f : A_1 \dots A_n \to B$ in $V(\mathcal{C})$ is $f : (A_1 \otimes \cdots \otimes A_n) \to B$ in $\mathcal{C}$, so we may apply F to obtain a morphism in $\mathcal{D}$. Also, since $\pi_{A_1 \dots A_n} : (FA_1 \odot \cdots \odot FA_n) \to F(A_1 \otimes \cdots \otimes A_n)$ in $\mathcal{D}$, we deduce that $\pi_{A_1 \dots A_n} : FA_1 \dots FA_n \to F(A_1 \otimes \cdots \otimes A_n)$ in $V(\mathcal{D})$, so $\pi_{A_1 \dots A_n}$ and hence $Ff \cdot \pi_{A_1 \dots A_n}$ are multimorphisms in $V(\mathcal{D})$. So the function on multimorphisms is well-defined.

Consider the identity $\mathbb{1}_A : A \to A$ in $V(\mathcal{C})$. As $\mathbb{1}_A = \iota_A^{-1}$, we see

$$\begin{aligned} V(F,\pi)(\mathbb{1}_A) &= F\iota_A^{-1} \cdot \pi_A \\ &= (F\iota_A)^{-1} \cdot \pi_A \text{ as } F \text{ preserves inverses} \\ &= \iota_{FA}^{-1} \cdot \pi_A^{-1} \cdot \pi_A \text{ by the triangle identity} \\ &= \iota_{FA}^{-1}. \end{aligned}$$

So $V(F,\pi)(\mathbb{1}_A) = \mathbb{1}_{FA}$ and $V(F,\pi)$ preserves identities.

Now let $f : B_1 \dots B_n \to C$ and $f_i : A_{i,1} \to A_{i,m_i}$ be multimorphisms in $V(\mathcal{C})$ $\forall i = 1 \dots n$. For ease of notation, define the sequences for $i \in \mathbb{N}_0$ by

$$\begin{aligned} T_i &= (A_{i,1} \dots A_{i,m_i}) \\ T &= (A_{1,1} \otimes \cdots \otimes A_{1,m_1}) \dots (A_{n,1} \otimes \cdots \otimes A_{n,m_n}), \\ T' &= (A_{1,1} \dots A_{1,m_1}) \dots (A_{n,1} \dots A_{n,m_n}) \\ T'' &= (A_{1,1} \dots A_{n,m_n}). \end{aligned}$$

Consider the naturality square induced by π (with tensors abridged as before).

$$\begin{tikzcd} (F(A_{1,1} \otimes \cdots \otimes A_{1,m_n}) \odot \cdots) \arrow[r, "(Ff_1 \odot \cdots \odot Ff_n)"] \arrow[d, "\pi_T"'] & (FB_1 \odot \cdots \odot FB_n) \arrow[d, "\pi_{B_1 \dots B_n}"] \\ F((A_{1,1} \otimes \cdots \otimes A_{1,m_1}) \otimes \cdots) \arrow[r, "F(f_1 \otimes \cdots \otimes f_n)"'] & F(B_1 \otimes \cdots \otimes B_n) \end{tikzcd}$$

Combining with the pentagon identity for monoidal functors (see Definition 3.2.5) and using the unbiased tensor interchange law, we see the following diagram commutes.

$$\begin{tikzcd} & (FA_{1,1} \odot \cdots \odot FA_{n,m_n}) \arrow[dddd, bend left, "\pi_{T''}"] \\ & ((FA_{1,1} \odot \cdots \odot FA_{1,m_1}) \odot \cdots) \arrow[u, "\delta_{T'}"'] \arrow[dl, "((Ff_1 \cdot \pi_{T_1}) \odot \cdots \odot (Ff_n \cdot \pi_{T_n}))"'] \arrow[d, "(\pi_{T_1} \odot \cdots \odot \pi_{T_n})"] \\ (FB_1 \odot \cdots \odot FB_n) \arrow[d, "\pi_{B_1 \dots B_n}"'] & (F(A_{1,1} \otimes \cdots \otimes A_{1,m_n}) \odot \cdots) \arrow[l, "(Ff_1 \odot \cdots \odot Ff_n)"] \arrow[d, "\pi_T"'] \\ F(B_1 \otimes \cdots \otimes B_n) & F((A_{1,1} \otimes \cdots \otimes A_{1,m_n}) \otimes \cdots) \arrow[l, "F(f_1 \otimes \cdots \otimes f_n)"] \arrow[d, "F\gamma_{T'}"'] \\ & F(A_{1,1} \otimes \cdots \otimes A_{n,m_n}) \end{tikzcd}$$

Using the fact that components of γ, δ and π are isomorphisms, hence invertible, we thus read off

$$\begin{aligned}
V(F,\pi)(f \circ (f_1 \dots f_n)) &= F(f \cdot (f_1 \otimes \dots \otimes f_n) \cdot \gamma_{T'}^{-1}) \cdot \pi_{T''} \text{ by definition of composition in } V(\mathcal{C}) \\
&= Ff \cdot F(f_1 \otimes \dots \otimes f_n) \cdot F\gamma_{T'}^{-1} \cdot \pi_{T''} \text{ by functorality} \\
&= Ff \cdot F(f_1 \otimes \dots \otimes f_n) \cdot (F\gamma_{T'})^{-1} \cdot \pi_{T''} \text{ since } F \text{ preserves inverses} \\
&= Ff \cdot \pi_{B_1 \dots B_n} \cdot ((Ff_1 \cdot \pi_{T_1}) \odot \dots \odot (Ff_1 \cdot \pi_{T_n})) \cdot \delta_{T'}^{-1} \\
&= V(F,\pi)(f) \cdot ((Ff_1 \cdot \pi_{T_1}) \odot \dots \odot (Ff_1 \cdot \pi_{T_n})) \cdot \delta_{T'}^{-1} \\
&= V(F,\pi)(f) \circ (V(F,\pi)(f_1) \dots V(F,\pi)(f_n)) \text{ by composition in } V(\mathcal{D}).
\end{aligned}$$

Thus, $V(F,\pi)$ also preserves composition, making it a well-defined multifunctor. □

Lemma 5.2.5

Define V : **UMonCat** → **Multicat** by sending unbiased monoidal categories to their underlying multicategories and unbiased monoidal functors to their underlying multifunctors. Then V is a well-defined functor.

Proof. We have shown the functions on objects and morphisms for V : **UMonCat** → **Multicat** are well-defined, so it remains to prove that V preserves identities and composition.

Let $(1_\mathcal{C}, \pi) : \mathcal{C} \to \mathcal{C}$ be the identity unbiased monoidal functor. Then all components $\pi_{A_1 \dots A_n}$ are identities $1_{(A_1 \otimes \dots \otimes A_n)}$. Let $f \in V(\mathcal{C})(A_1 \dots A_n; B)$, so $f : (A_1 \otimes \dots \otimes A_n) \to B$ in $\mathcal{C}$. Then

$$\begin{aligned}
V(1_\mathcal{C}, \pi)(f) &= 1_\mathcal{C} f \cdot \pi_{A_1 \dots A_n} \\
&= f \cdot 1_{(A_1 \otimes \dots \otimes A_n)} \\
&= f.
\end{aligned}$$

Also, $V(1_\mathcal{C}, \pi)(A) = A$ by definition $\forall A \in V(\mathcal{C})$. So $V(1_\mathcal{C}, \pi)$ is the identity multifunctor and V preserves identities.

Now let $(F,\pi) : \mathcal{C} \to \mathcal{D}$ and $(G,\rho) : \mathcal{D} \to \mathcal{E}$ be unbiased monoidal functors. Let $\cdot$ denote composition in monoidal categories and let $\circ$ denote functor composition. First, the composition of (F,π) and (G,ρ) is $(G \circ F, G\pi \cdot \rho)$ (see Proposition 3.2.7). Thus, $\forall A \in V(\mathcal{C})$, we have $(V(G,\rho) \circ V(F,\pi))(A) = GFA = V(G \circ F, G\pi \cdot \rho)(A)$, so the functors agree on objects.

Now consider a morphism $f \in V(\mathcal{C})(A_1 \dots A_n; B)$. We have

$$\begin{aligned}
(V(G,\rho) \circ V(F,\pi))(f) &= V(G,\rho)(Ff \cdot \pi_{A_1 \dots A_n}) \\
&= G(Ff \cdot \pi_{A_1 \dots A_n}) \cdot \rho_{FA_1 \dots FA_n} \\
&= GFf \cdot G\pi_{A_1 \dots A_n} \cdot \rho_{FA_1 \dots FA_n} \text{ by the functorality of } G \\
&= V(G \circ F, G\pi \cdot \rho)(f) \text{ by definition of } G\pi \cdot \rho \text{ (see Proposition 3.2.7).}
\end{aligned}$$

Thus, $V(G,\rho) \circ V(F,\pi) = V(G \circ F, G\pi \cdot \rho)$, and V preserves composition. It is therefore a functor. □

Lemma 5.2.6

Let $\mathcal{C}$ be a multicategory. If $\mathcal{C} \cong V(\mathcal{D})$ for some unbiased monoidal category $(\mathcal{D}, \otimes_n, \gamma, \iota)$, then $\mathcal{C}$ is representable.

Proof. By assumption, we have multifunctors $F : \mathcal{C} \to V(\mathcal{D})$ and $G : V(\mathcal{D}) \to \mathcal{C}$ such that $G \circ F = 1_\mathcal{C}$ and $F \circ G = 1_{V(\mathcal{D})}$.

Let $A_1 \dots A_n \in \mathcal{C}$, so $A_1 \dots A_n \in V(\mathcal{C})$. Then $FA_1 \dots FA_n \in V(\mathcal{D})$, i.e. $FA_1 \dots FA_n \in \mathcal{D}$. Since $\mathcal{D}$ is an unbiased monoidal category, $\exists (FA_1 \otimes \dots \otimes FA_n) \in \mathcal{D}$. So we can define the components of a representation (R, u) of $\mathcal{C}$ as follows for any $A_1 \dots A_n \in \mathcal{C}$.

- Define $R(A_1 \dots A_n) = G(FA_1 \otimes \cdots \otimes FA_n)$.
- Define $u_{A_1 \dots A_n} = G1_{(FA_1 \otimes \cdots \otimes FA_n)}$.

Now let $f : A_{1,1} \dots A_{n,m_n} \to B$ be a multimorphism in $\mathcal{C}$ and define an object sequence by $S = (FA_{1,1} \dots FA_{1,m_1}) \dots (FA_{n,1} \dots FA_{n,m_n})$. Let $\overline{f} = G(Ff \cdot \gamma_S)$ where $\cdot$ is composition in $\mathcal{D}$. Then, with $\circ$ composition in $V(\mathcal{D})$, we have

$$\begin{aligned}
&\overline{f} \circ (u_{A_{1,1} \dots A_{1,m_1}} \dots u_{A_{n,1} \dots A_{n,m_n}}) \\
&\quad = G(Ff \cdot \gamma_S) \circ \left(G1_{(FA_{1,1} \otimes \cdots \otimes FA_{1,m_1})} \dots G1_{(FA_{n,1} \otimes \cdots \otimes FA_{n,m_n})} \right) \\
&\quad = G\left(Ff \cdot \gamma_S \cdot \left(1_{(FA_{1,1} \otimes \cdots \otimes FA_{1,m_1})} \otimes \cdots \otimes 1_{(FA_{n,1} \otimes \cdots \otimes FA_{n,m_n})} \right) \cdot \gamma_S^{-1} \right) \text{ by the} \\
&\qquad \text{multifunctorality of } G \text{ and definition of composition in } V(\mathcal{D}) \\
&\quad = G\left(Ff \cdot \gamma_S \cdot 1_{(FA_{1,1} \otimes \cdots \otimes FA_{n,m_n})} \cdot \gamma_S^{-1} \right) \text{ as the tensor preserves identities} \\
&\quad = G(Ff \cdot \gamma_S \cdot \gamma_S^{-1}) \\
&\quad = GFf \\
&\quad = f.
\end{aligned}$$

We thus have existence of $\overline{f}$. It remains to prove uniqueness. Suppose there exists an $\overline{f}$ satisfying the representability equation with the given objects and multimorphisms. Then applying F shows

$$\begin{aligned}
Ff &= F(\overline{f} \circ (u_{A_{1,1} \dots A_{1,m_1}} \dots u_{A_{n,1} \dots A_{n,m_n}})) \\
&= F\overline{f} \circ \left(FG1_{(FA_{1,1} \otimes \cdots \otimes FA_{1,m_1})} \dots FG1_{(FA_{n,1} \otimes \cdots \otimes FA_{n,m_n})} \right) \text{ by multifunctorality of } F \\
&= F\overline{f} \cdot \left(1_{(FA_{1,1} \otimes \cdots \otimes FA_{1,m_1})} \otimes \cdots \otimes 1_{(FA_{n,1} \otimes \cdots \otimes FA_{n,m_n})} \right) \cdot \gamma_S^{-1} \text{ by composition in } V(\mathcal{D}) \\
&= F\overline{f} \cdot \gamma_S^{-1} \text{ as the tensor preserves identities.}
\end{aligned}$$

Composing with γ_S and applying G then gives $\overline{f} = G(Ff \cdot \gamma_S)$, so in fact $\overline{f}$ is uniquely determined. Hence, $\mathcal{C}$ is representable. $\square$

Lemma 5.2.7

Let $\mathcal{C}$ be a representable multicategory. Then $\mathcal{C} \cong V(\mathcal{D})$ for some unbiased monoidal category $(\mathcal{D}, \otimes_n, \gamma, \iota)$.

Proof. Suppose $\mathcal{C}$ is representable with representation (R, u). We seek to construct $(\mathcal{D}, \otimes_n, \gamma, \iota)$ such that $V(\mathcal{D}) \cong \mathcal{C}$.

First, by Corollary 4.2.7, since $\mathcal{C}$ is representable, we can write any multimorphism f in $\mathcal{C}$ as $f = \overline{f} \circ u_{A_1 \dots A_n}$ for a unique $\overline{f}$. For the remainder of this proof, $\overline{f}$ denotes this unary multimorphism obtained in the universal factorisation of a given f.

Define $\mathrm{ob}(\mathcal{D}) = \mathrm{ob}(\mathcal{C})$. For $A, B \in \mathcal{D}$, define $\mathcal{D}(A, B) = \mathcal{C}(A; B)$. That is, morphisms in $\mathcal{D}$ are unary multimorphisms in $\mathcal{C}$. Composition in $\mathcal{D}$ is then composition in $\mathcal{C}$, restricted to unary homsets. The identity and associativity axioms thus follow immediately, making $\mathcal{D}$ a well-defined category.

Define the unbiased tensor by $(A_1 \otimes \cdots \otimes A_n) = R(A_1 \dots A_n)$ for objects. For morphisms $f_i : A_i \to B_i$ $\forall i = 1 \dots n$, define $(f_1 \otimes \cdots \otimes f_n) = \overline{u_{B_1 \dots B_n} \circ (f_1 \dots f_n)}$ to be the tensor's effect on morphisms. Since $u_{B_1 \dots B_n} \circ (f_1 \dots f_n)$ is a multimorphism $A_1 \dots A_n \to R(B_1 \dots B_n)$, the universal factorisation tells us $(f_1 \otimes \cdots \otimes f_n) : R(A_1 \dots A_n) \to R(B_1 \dots B_n)$ has the desired domain and codomain (and is in particular unary, so in $\mathcal{D}$).

We must verify $\otimes_n$ is a functor. Let $A_1 \dots A_n \in \mathcal{D}$. Notice that the universal factorisation of $u_{A_1 \dots A_n}$ is $u_{A_1 \dots A_n} = \overline{u_{A_1 \dots A_n}} \circ u_{A_1 \dots A_n}$ for unique $\overline{u_{A_1 \dots A_n}}$. But the identity satisfies this equation, so by uniqueness, we have $\overline{u_{A_1 \dots A_n}} = 1_{R(A_1 \dots A_n)}$ and hence

$$\begin{aligned}
(1_{A_1} \otimes \cdots \otimes 1_{A_n}) &= \overline{u_{A_1 \dots A_n} \circ (1_{A_1} \dots 1_{A_n})} \\
&= \overline{u_{A_1 \dots A_n}} \\
&= 1_{R(A_1 \dots A_n)} \\
&= 1_{(A_1 \otimes \cdots \otimes A_n)}.
\end{aligned}$$

Now let $f_i : A_i \to B_i$ and $g_i : B_i \to C_i$ be morphisms in $\mathcal{D}$ for $i = 1 \dots n$. We then have

$$\begin{aligned}
\otimes_n((g_1 \dots g_n) \circ (f_1 \dots f_n)) &= \overline{u_{C_1 \dots C_n} \circ (g_1 \dots g_n) \circ (f_1 \dots f_n)}, \\
\otimes_n(g_1 \dots g_n) \circ \otimes_n(f_1 \dots f_n) &= \overline{u_{C_1 \dots C_n} \circ (g_1 \dots g_n)} \circ \overline{u_{B_1 \dots B_n} \circ (f_1 \dots f_n)}.
\end{aligned}$$

Composing the latter with $u_{A_1 \dots A_n}$ and using associativity to reverse universal factorisations yields

$$\begin{aligned}
\overline{u_{C_1 \dots C_n} \circ (g_1 \dots g_n)} \circ \overline{u_{B_1 \dots B_n} \circ (f_1 \dots f_n)} \circ u_{A_1 \dots A_n} &= \overline{u_{C_1 \dots C_n} \circ (g_1 \dots g_n)} \circ u_{B_1 \dots B_n} \circ (f_1 \dots f_n) \\
&= u_{C_1 \dots C_n} \circ (g_1 \dots g_n) \circ (f_1 \dots f_n).
\end{aligned}$$

Thus, both $\otimes_n((g_1 \dots g_n) \circ (f_1 \dots f_n))$ and $\otimes_n(g_1 \dots g_n) \circ \otimes_n(f_1 \dots f_n)$ satisfy the universal factorisation property for $u_{C_1 \dots C_n} \circ (g_1 \dots g_n) \circ (f_1 \dots f_n)$. So by uniqueness, they are equal, and the tensor preserves composition. Thus, $\otimes_n$ is a well-defined functor $\mathcal{D}^n \to \mathcal{D}$.

Since $R(A) = (A)$ and $u_A : A \to R(A)$ in the multicategory, we may define $\iota : A \to (A)$ in the monoidal category by $\iota_A = u_A$. By universality, $1_A = \overline{1_A} \circ u_A = \overline{1_A} \circ \iota_A$.

Further, 1_A is universal. Indeed, $\overline{f} \circ (1_{C_1} \dots 1_{C_i}, 1_A, 1_{C_{i+1}} \dots C_m) = f$ uniquely determines $\overline{f} = f$ by the identity axiom. By Proposition 4.2.9, $\overline{1_A}$ is an isomorphism, and as we have shown ι_A is its right inverse, ι_A must also be its left inverse (Proposition 2.1.7). So ι_A is also an isomorphism with inverse $\iota_A^{-1} = \overline{1_A}$.

By definition of the tensor and ι, the following diagrams commute for any $f : (A_1 \otimes \cdots \otimes A_n) \to B$.

$$\begin{tikzcd}
(A_1 \otimes \cdots \otimes A_n) \arrow[d, "\iota_{(A_1 \otimes \cdots \otimes A_n)}"'] \arrow[dr, "\overline{u_B \circ f} \circ u_{A_1 \dots A_n}"] & \\
((A_1 \otimes \cdots \otimes A_n)) \arrow[r, "(f)"'] & (B)
\end{tikzcd}
\qquad
\begin{tikzcd}
(A_1 \otimes \cdots \otimes A_n) \arrow[r, "f"] \arrow[dr, "u_B \circ f"'] & B \arrow[d, "\iota_B"] \\
& (B)
\end{tikzcd}$$

Notice however that $u_B \circ f = \overline{u_B \circ f} \circ u_{A_1 \dots A_n}$ by definition of the universal factorisation, so in fact we can combine the above diagrams to produce the naturality square for ι. Thus, ι is a natural isomorphism.

Let $A_{1,1} \dots A_{1,m_1} \dots A_{n,1} \dots A_{n,m_n} \in \mathcal{D}$. Consider the representation multimorphism components $u_{R(A_{1,1} \dots A_{1,m_1}) \dots R(A_{n,1} \dots A_{n,m_n})}$ and $u_{A_{i,1} \dots A_{i,m_i}}$ $\forall i = 1 \dots n$. Denote these by u and u_i respectively.

By Proposition 4.2.6, u and $u_1 \dots u_n$ are universal. By Proposition 4.2.8, we therefore also know that $u \circ (u_1 \dots u_n)$ is universal.

Let $u' = u_{A_{1,1} \dots A_{1,m_1} \dots A_{n,1} \dots A_{n,m_n}}$. Universal factorisation gives $u \circ (u_1 \dots u_n) = \overline{u \circ (u_1 \dots u_n)} \circ u'$. Since $u \circ (u_1 \dots u_n)$ is universal, $\overline{u \circ (u_1 \dots u_n)}$ is an isomorphism by Proposition 4.2.9, so its inverse exists.

Define $\gamma_{(A_{1,1}\dots A_{1,m_1})\dots(A_{n,1}\dots A_{n,m_n})} = \overline{u \circ (u_1 \dots u_n)}^{-1}$. By definition of the tensor as $(A_1 \otimes \cdots \otimes A_n) = R(A_1 \dots A_n)$, the components of γ are isomorphisms with the required domain and codomain to be the coherence isomorphisms for the unbiased monoidal category. Naturality of γ follows much as it does for ι, but with significantly more notational headache. We therefore omit it.

Let $B_1 \dots B_n \in \mathcal{D}$. Working through our definitions yields

$$
\begin{aligned}
(\iota_{B_1} \otimes \cdots \otimes \iota_{B_n}) &= \overline{u_{R(B_1)\dots R(B_n)} \circ (u_{B_1} \dots u_{B_n})}, \\
\gamma_{(B_1)\dots(B_n)} &= \overline{u_{R(B_1)\dots R(B_n)} \circ (u_{B_1} \dots u_{B_n})}^{-1}, \\
\iota_{(B_1 \otimes \cdots \otimes B_n)} &= u_{R(B_1 \dots B_n)}, \\
\gamma_{(B_1 \dots B_n)} &= \overline{u_{R(B_1 \dots B_n)} \circ u_{B_1 \dots B_n}}^{-1}.
\end{aligned}
$$

By universal factorisation, we can factorise $u_{R(B_1 \dots B_n)} \circ u_{B_1 \dots B_n} = \overline{u_{R(B_1 \dots B_n)} \circ u_{B_1 \dots B_n}} \circ u_{B_1 \dots B_n}$. But notice $u_{R(B_1 \dots B_n)}$ clearly satisfies this equation in place of $\overline{u_{R(B_1 \dots B_n)} \circ u_{B_1 \dots B_n}}$, so by uniqueness, they are equal. Hence, $\gamma_{(B_1 \dots B_n)} = u^{-1}_{R(B_1 \dots B_n)}$.

We therefore have the following commutative diagram.

$$
\begin{tikzcd}
 & (B_1 \otimes \cdots \otimes B_n) \arrow[dl, "\overline{u_{R(B_1)\dots R(B_n)} \circ (u_{B_1} \dots u_{B_n})}"'] \arrow[dr, "u_{R(B_1 \dots B_n)}"] \arrow[dd, "1_{(B_1 \otimes \cdots \otimes B_n)}"] & \\
((B_1) \otimes \cdots \otimes (B_n)) \arrow[dr, "\overline{u_{R(B_1)\dots R(B_n)} \circ (u_{B_1} \dots u_{B_n})}^{-1}"'] & & ((B_1 \otimes \cdots \otimes B_n)) \arrow[dl, "u^{-1}_{R(B_1 \dots B_n)}"] \\
 & (B_1 \otimes \cdots \otimes B_n) &
\end{tikzcd}
$$

Substituting our definitions back in shows that ι and γ satisfy the identity diamond.

The associativity diamond is similar. Again, as it is much more notationally heavy, we omit it for the sake of brevity.

Thus, $(\mathcal{D}, \otimes_n, \gamma, \iota)$ is a well-defined unbiased monoidal category. It remains to show that $V(\mathcal{D}) \cong \mathcal{C}$ in **Multicat**.

Consider $A_1 \dots A_n, B \in V(\mathcal{D})$. We have, by various definitions,

$$
\begin{aligned}
V(\mathcal{D})(A_1 \dots A_n; B) &= \mathcal{D}((A_1 \otimes \cdots \otimes A_n), B) \\
&= \mathcal{C}(R(A_1 \dots A_n); B).
\end{aligned}
$$

The map $\mathcal{C}(A_1 \dots A_n; B) \to \mathcal{C}(R(A_1 \dots A_n); B)$, $f \mapsto \overline{f}$ is well-defined by the existence and uniqueness of universal factorisation. If $\overline{f} = \overline{g}$, then $f = \overline{f} \circ u_{A_1 \dots A_n} = \overline{g} \circ u_{A_1 \dots A_n} = g$, so this map is injective. It is also surjective, since for any $g \in \mathcal{C}(R(A_1 \dots A_n); B)$, we can simply define $f = g \circ u_{A_1 \dots A_n} \in \mathcal{C}(A_1 \dots A_n; B)$ and we obtain $g = \overline{f}$. Consequently, $f \mapsto \overline{f}$ is a bijection, and we have an isomorphism $V(\mathcal{D})(A_1 \dots A_n; B) \cong \mathcal{C}(A_1 \dots A_n; B)$ in **Set**.

We may therefore define $F : \mathcal{C} \to V(\mathcal{D})$ by $FA = A \ \forall A \in \mathcal{C}$ and $Ff = \overline{f}$ for all morphisms in $\mathcal{C}$. This is well-defined on objects as $\mathrm{ob}(V(\mathcal{D})) = \mathrm{ob}(\mathcal{D}) = \mathrm{ob}(\mathcal{C})$, and well-defined on morphisms by the isomorphism above. We may similarly define $G : V(\mathcal{D}) \to \mathcal{C}$ by $GA = A \ \forall A \in V(\mathcal{D})$ and $G\overline{f} = f$ unambiguously for the same reasons.

Clearly $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{V(\mathcal{D})}$, so to exhibit an isomorphism $\mathcal{C} \cong V(\mathcal{D})$, it remains to prove that F and G are multifunctors.

First, for any $A \in \mathcal{C}$ and identity 1_A in $\mathcal{C}$ (equivalently an identity in $\mathcal{D}$), we have already shown in our definition of ι that $\overline{1_A}$ is an isomorphism with inverse ι_A. Thus, $F1_A = \overline{1_A} = \iota_A^{-1} = \mathbb{1}_A$ is the identity on $FA = A$ in $V(\mathcal{D})$. So F preserves identities.

Now let $f_i : A_{i,1} \dots A_{i,m_i} \to B_i$ for $i = 1 \dots n$ and $f : B_1 \dots B_n \to C$ be multimorphisms in $\mathcal{C}$. Define the sequence $S = (A_{1,1} \dots A_{1,m_1}) \dots (A_{n,1} \dots A_{n,m_n})$. Writing $\cdot$ for composition in $\mathcal{C}$ and $\mathcal{D}$ (by definition the same operation), and $\circ$ for composition in $V(\mathcal{D})$, we have

$$\begin{aligned}
Ff \circ (Ff_1 \dots Ff_n) &= \overline{f} \cdot (\overline{f_1} \otimes \dots \otimes \overline{f_n}) \cdot \gamma_S^{-1} \text{ by definition of multicategory composition} \\
&= \overline{f} \cdot \overline{u_{B_1 \dots B_n} \cdot (\overline{f_1} \dots \overline{f_n})} \cdot \gamma_S^{-1} \\
&= \overline{f} \cdot \overline{u_{B_1 \dots B_n} \cdot (\overline{f_1} \dots \overline{f_n})} \cdot \overline{u \cdot (u_1 \dots u_n)} \text{ with } u, u_1 \dots u_n \text{ as defined with } \gamma.
\end{aligned}$$

Also recall $u' = u_{A_{1,1} \dots A_{n,k_n}}$ from the definition of γ. Composing the above with u' and repeatedly reducing universal factorisations in $\mathcal{C}$ yields

$$\begin{aligned}
(Ff \circ (Ff_1 \dots Ff_n)) \cdot u' &= \overline{f} \cdot \overline{u_{B_1 \dots B_n} \cdot (\overline{f_1} \dots \overline{f_n})} \cdot \overline{u \cdot (u_1 \dots u_n)} \cdot u' \\
&= \overline{f} \cdot \overline{u_{B_1 \dots B_n} \cdot (\overline{f_1} \dots \overline{f_n})} \cdot u \cdot (u_1 \dots u_n) \\
&= \overline{f} \cdot u_{B_1 \dots B_n} \cdot (\overline{f_1} \dots \overline{f_n}) \cdot (u_1 \dots u_n) \\
&= \overline{f} \cdot u_{B_1 \dots B_n} \cdot (\overline{f_1} \cdot u_1 \dots \overline{f_n} \cdot u_n) \text{ by associativity in the multicategory} \\
&= \overline{f} \cdot u_{B_1 \dots B_n} \cdot (f_1 \dots f_n) \\
&= f \cdot (f_1 \dots f_n).
\end{aligned}$$

Notice $F(f \cdot (f_1 \dots f_n)) = \overline{f \cdot (f_1 \dots f_n)}$ also satisfies $f \cdot (f_1 \dots f_n) = \overline{f \cdot (f_1 \dots f_n)} \cdot u'$. Uniqueness of the universal factorisation then tells us that $F(f \cdot (f_1 \dots f_n)) = Ff \circ (Ff_1 \dots Ff_n)$. Thus, F preserves composition.

Since F preserves identities and composition, it is a well-defined multifunctor. Similar calculations show the same is true for G.

Hence, if $\mathcal{C}$ is representable, we can construct an unbiased monoidal category $(\mathcal{D}, \otimes_n, \gamma, \iota)$ and multifunctors $F : \mathcal{C} \to V(\mathcal{D})$ and $G : V(\mathcal{D}) \to \mathcal{C}$ such that $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{V(\mathcal{D})}$. In other words, $\mathcal{C} \cong V(\mathcal{D})$. ☐

Lemma 5.2.8

Define a functor $V' : \mathbf{UMonCat} \to \mathbf{RMulticat}$ by $V'(\mathcal{C}) = V(\mathcal{C})$ and $V'(F, \pi) = V(F, \pi)$. Then V' is well-defined. In particular, $V'(F, \pi)$ is universal-preserving.

Proof. First note that for $(\mathcal{C}, \otimes_n, \gamma, \iota) \in \mathbf{UMonCat}$, $V(\mathcal{C})$ is representable by Lemma 5.2.6. So $V'(\mathcal{C})$ is an object in $\mathbf{RMulticat}$.

Let $(\mathcal{C}, \otimes_n, \gamma, \iota), (\mathcal{D}, \odot_n, \delta, \kappa) \in \mathbf{UMonCat}$ and let $(F, \pi) : \mathcal{C} \to \mathcal{D}$ be an unbiased monoidal functor. Denote composition in $\mathcal{C}$ and $\mathcal{D}$ by $\cdot$, and in $V(\mathcal{C})$ and $V(\mathcal{D})$ by $\circ$. Let $u : A_1 \dots A_n \to B$ be a universal in $V(\mathcal{C})$, so by definition of the underlying multicategory, $u : (A_1 \otimes \dots \otimes A_n) \to B$ in $\mathcal{C}$. We aim to show that $V(F, \pi)(u) = Fu \cdot \pi_{A_1 \dots A_n}$ is universal in $V(\mathcal{D})$.

First, by definition of u universal, $\exists \overline{1_{(A_1 \otimes \dots \otimes A_n)}} : B \to (A_1 \otimes \dots \otimes A_n)$ unique such that $1_{(A_1 \otimes \dots \otimes A_n)} = \overline{1_{(A_1 \otimes \dots \otimes A_n)}} \circ u$. Denote this by $\overline{1}$ for notational convenience. Note that $\overline{1} : (B) \to (A_1 \otimes \dots \otimes A_n)$ as a morphism in $\mathcal{C}$.

Let $f : C_1 \dots C_i, FA_1 \dots FA_n \dots C_{i+1} \dots C_m \to D$ be a multimorphism in $V(\mathcal{D})$. Define the sequences $S = (C_1)\dots(C_i),(FA_1 \dots FA_n),(C_{i+1})\dots(C_m)$ and $T = A_1 \dots A_n$ for notational convenience. Now define $\overline{f} = f \cdot \delta_S \cdot (\kappa_{C_1} \odot \dots \odot \kappa_{C_i} \odot (\pi_T^{-1} \cdot F\overline{1} \cdot F\iota_B) \odot \kappa_{C_{i+1}} \odot \dots \odot \kappa_{C_m})$. This satisfies

$$\begin{aligned}
&\overline{f} \circ (\mathbb{1}_{C_1} \dots \mathbb{1}_{C_i}, Fu \cdot \pi_T, \mathbb{1}_{C_{i+1}} \dots \mathbb{1}_{C_m}) \\
&\qquad = \overline{f} \cdot (\kappa_{C_1}^{-1} \odot \dots \odot \kappa_{C_i}^{-1} \odot (Fu \cdot \pi_T) \odot \kappa_{C_{i+1}}^{-1} \odot \dots \odot \kappa_{C_m}^{-1}) \cdot \delta_S^{-1} \text{ by definition of} \\
&\qquad\quad \text{multicategory composition and since } \mathbb{1}_{C_j} = \kappa_{C_j}^{-1}\ \forall j = 1 \dots m \\
&\qquad = f \cdot \delta_S \cdot (\kappa_{C_1} \odot \dots \odot \kappa_{C_i} \odot (\pi_T^{-1} \cdot F\overline{1} \cdot F\iota_B) \odot \kappa_{C_{i+1}} \odot \dots \odot \kappa_{C_m}) \\
&\qquad\quad \cdot (\kappa_{C_1}^{-1} \odot \dots \odot \kappa_{C_i}^{-1} \odot (Fu \cdot \pi_T) \odot \kappa_{C_{i+1}}^{-1} \odot \dots \odot \kappa_{C_m}^{-1}) \cdot \delta_S^{-1} \text{ by definition of } \overline{f} \\
&\qquad = f \cdot \delta_S \cdot (1_{C_1} \odot \dots \odot 1_{C_i} \odot (\pi_T^{-1} \cdot F\overline{1} \cdot F\iota_B \cdot Fu \cdot \pi_T) \odot 1_{C_{i+1}} \dots \odot 1_{C_m}) \cdot \delta_S^{-1} \\
&\qquad\quad \text{by the unbiased tensor interchange law.}
\end{aligned}$$

In particular, notice that

$$\begin{aligned}
\pi_T^{-1} \cdot F\overline{1} \cdot F\iota_B \cdot Fu \cdot \pi_T &= \pi_T^{-1} \cdot F(\overline{1} \cdot \iota_B \cdot u) \cdot \pi_T \text{ by functorality} \\
&= \pi_T^{-1} \cdot F(\overline{1} \cdot (u) \cdot \iota_{(A_1 \otimes \dots \otimes A_n)}) \cdot \pi_T \text{ by naturality of } \iota \\
&= \pi_T^{-1} \cdot F(\overline{1} \cdot (u) \cdot \gamma_{(A_1 \dots A_n)}^{-1}) \cdot \pi_T \text{ by the identity diamond} \\
&= \pi_T^{-1} \cdot F(\overline{1} \circ u) \cdot \pi_T \text{ by definition of multicategory composition} \\
&= \pi_T^{-1} \cdot F1_{(A_1 \otimes \dots \otimes A_n)} \cdot \pi_T \text{ by definition of } \overline{1} \\
&= \pi_T^{-1} \cdot 1_{F(A_1 \otimes \dots \otimes A_n)} \cdot \pi_T \text{ by functorality} \\
&= \pi_T^{-1} \cdot \pi_T \\
&= 1_{(FA_1 \odot \dots \odot FA_n)}.
\end{aligned}$$

As the tensor functor preserves identities, we substitute into the above to find

$$\begin{aligned}
f &= f \cdot \delta_S \cdot \delta_S^{-1} \\
&= f \cdot \delta_S \cdot 1_{((C_1) \odot \dots \odot (C_i) \odot (FA_1 \odot \dots \odot FA_n) \odot (C_{i+1}) \odot \dots \odot (C_m))} \cdot \delta_S^{-1} \\
&= f \cdot \delta_S \cdot (1_{C_1} \odot \dots \odot 1_{C_i} \odot (\pi_T^{-1} \cdot F\overline{1} \cdot F\iota_B \cdot Fu \cdot \pi_T) \odot 1_{C_{i+1}} \dots \odot 1_{C_m}) \cdot \delta_S^{-1} \\
&= \overline{f} \circ (\mathbb{1}_{C_1} \dots \mathbb{1}_{C_i}, Fu \cdot \pi_{A_1 \dots A_n}, \mathbb{1}_{C_{i+1}} \dots \mathbb{1}_{C_m}) \text{ by the calculations above} \\
&= \overline{f} \circ_i (Fu \cdot \pi_T).
\end{aligned}$$

We have existence of such an $\overline{f}$. Universality of u ensures it is unique. Thus, $Fu \cdot \pi_{A_1 \dots A_n}$ is universal. So $V'(F, \pi)$ preserves universals. As V' inherits functorality from V, it is a well-defined functor **UMonCat** → **RMulticat**. □

Lemma 5.2.9

The functor V' : **UMonCat** → **RMulticat** is full.

Proof. Let $(\mathcal{C}, \otimes_n, \gamma, \iota), (\mathcal{D}, \odot_n, \delta, \kappa) \in$ **UMonCat**. We seek to show that every universal-preserving multifunctor $G : V(\mathcal{C}) \to V(\mathcal{D})$ is $V(F, \pi)$ for some unbiased monoidal functor $(F, \pi) : \mathcal{C} \to \mathcal{D}$.

Write $\cdot$ for composition in $\mathcal{C}$ and $\mathcal{D}$, and $\circ$ for composition in $V(\mathcal{C})$ and $V(\mathcal{D})$. Let $G : V(\mathcal{C}) \to V(\mathcal{D})$ be a universal-preserving multifunctor. Define an unbiased monoidal functor $(F, \pi) : \mathcal{C} \to \mathcal{D}$ as follows.

- For $A \in \mathcal{C}$, define $FA = GA$.
- For $f : A \to B$ a morphism in $\mathcal{C}$, define $Ff = G(f \cdot \iota_A^{-1}) \cdot \kappa_{FA}$.
- For $A_1 \dots A_n \in \mathcal{C}$, define $\pi_{A_1 \dots A_n} = G1_{(A_1 \otimes \dots \otimes A_n)}$.

Notice first that $1_{(A_1 \otimes \dots \otimes A_n)} : A_1 \dots A_n \to (A_1 \otimes \dots \otimes A_n)$ is a multimorphism in $V(\mathcal{C})$, so we may apply G to find $G1_{(A_1 \otimes \dots \otimes A_n)} : FA_1 \dots FA_n \to F(A_1 \otimes \dots \otimes A_n)$ in $V(\mathcal{D})$. By definition of multimorphisms in the underlying multicategory, we thus have $\pi_{A_1 \dots A_n} : (FA_1 \odot \dots \odot FA_n) \to F(A_1 \otimes \dots \otimes A_n)$, so each component of π has the correct domain and codomain.

Similarly, from $f \cdot \iota_A^{-1} : (A) \to B$ in $\mathcal{C}$, we see $f \cdot \iota_A^{-1}$ is a well-defined multimorphism $A \to B$ in $V(\mathcal{C})$. Consequently, $G(f \cdot \iota_A^{-1}) : FA \to FB$ in $V(\mathcal{D})$, equivalently $G(f \cdot \iota_A^{-1}) : (FA) \to FB$ in $\mathcal{D}$. Composing with $\kappa_{FA} : FA \to (FA)$ then gives $Ff : FA \to FB$ as required.

We now show F is a functor. Firstly, for $A \in \mathcal{C}$,

$$\begin{aligned}
F1_A &= G(1_A \cdot \iota_A^{-1}) \cdot \kappa_{FA} \\
&= G\iota_A^{-1} \cdot \kappa_{FA} \\
&= G\mathbb{1}_A \cdot \kappa_{FA} \\
&= \mathbb{1}_{FA} \cdot \kappa_{FA} \text{ since } G \text{ preserves multicategory identities} \\
&= \kappa_{FA}^{-1} \cdot \kappa_{FA} \\
&= 1_{FA}.
\end{aligned}$$

Furthermore, for $f : A \to B$ and $g : B \to C$ morphisms in $\mathcal{C}$, we have

$$\begin{aligned}
Fg \cdot Ff &= G(g \cdot \iota_B^{-1}) \cdot \kappa_{FB} \cdot G(f \cdot \iota_A^{-1}) \cdot \kappa_{FA} \\
&= G(g \cdot \iota_B^{-1}) \cdot (G(f \cdot \iota_A^{-1})) \cdot \kappa_{(FA)} \cdot \kappa_{FA} \text{ by naturality of } \kappa \\
&= G(g \cdot \iota_B^{-1}) \cdot (G(f \cdot \iota_A^{-1})) \cdot \delta_{(FA)}^{-1} \cdot \kappa_{FA} \text{ by the identity diamond} \\
&= (G(g \cdot \iota_B^{-1}) \circ G(f \cdot \iota_A^{-1})) \cdot \kappa_{FA} \text{ by definition of multicategory composition} \\
&= G((g \cdot \iota_B^{-1}) \circ (f \cdot \iota_A^{-1})) \cdot \kappa_{FA} \text{ since } G \text{ preserves composition} \\
&= G(g \cdot \iota_B^{-1} \cdot (f \cdot \iota_A^{-1}) \cdot \gamma_{(A)}^{-1}) \cdot \kappa_{FA} \text{ by definition of multicategory composition} \\
&= G(g \cdot f \cdot \iota_A^{-1} \cdot \iota_{(A)}^{-1} \cdot \gamma_{(A)}^{-1}) \cdot \kappa_{FA} \text{ by naturality of } \iota \\
&= G(g \cdot f \cdot \iota_A^{-1}) \cdot \kappa_{FA} \text{ by the identity diamond} \\
&= F(g \cdot f).
\end{aligned}$$

Thus, F preserves composition and identities, making it a functor $\mathcal{C} \to \mathcal{D}$.

Next, we show the $\pi_{A_1 \dots A_n}$ are isomorphisms. Let $A_1 \dots A_n \in \mathcal{C}$. Note that $1_{(A_1 \otimes \dots \otimes A_n)}$, as a multimorphism $A_1 \dots A_n \to (A_1 \otimes \dots \otimes A_n)$ in $V(\mathcal{C})$, is universal. To see this, recall the proof of Lemma 5.2.6. Taking F and G to be the identity functors yields $u_{A_1 \dots A_n} = 1_{(A_1 \otimes \dots \otimes A_n)}$ as the representation multimorphisms for $V(\mathcal{C})$. By Proposition 4.2.6, these are universal.

As G preserves universals, $\pi_{A_1 \dots A_n} = G1_{(A_1 \otimes \dots \otimes A_n)}$ is also universal. Thus, in particular, since $1_{(FA_1 \odot \dots \odot FA_n)} : FA_1 \dots FA_n \to (FA_1 \odot \dots \odot FA_n)$ as a multimorphism in $V(\mathcal{D})$, there exists a unique multimorphism $\overline{1} : F(A_1 \otimes \dots \otimes A_n) \to (FA_1 \odot \dots \odot FA_n)$ such that $1_{(FA_1 \odot \dots \odot FA_n)} = \overline{1} \circ \pi_{A_1 \dots A_n}$. Notice that $\overline{1} : (F(A_1 \otimes \dots \otimes A_n)) \to (FA_1 \odot \dots \odot FA_n)$ as a morphism in $\mathcal{D}$.

By Proposition 4.2.9, since $1_{(FA_1 \odot \dots \odot FA_n)}$ is itself universal, $\overline{1}$ is an isomorphism in the multicategory sense. That is, $\exists \overline{1}^{-1} : (FA_1 \odot \dots \odot FA_n) \to F(A_1 \otimes \dots \otimes A_n)$ a unique multimorphism such that $\mathbb{1}_{(FA_1 \odot \dots \odot FA_n)} = \overline{1} \circ \overline{1}^{-1}$. As above, $\overline{1}^{-1} : ((FA_1 \odot \dots \odot FA_n)) \to F(A_1 \otimes \dots \otimes A_n)$ as a morphism.

By our definition of composition in $V(\mathcal{D})$, the two squares on the left below commute. The two squares on the right meanwhile are naturality squares for κ, and the top and bottom triangles are immediate consequences of the identity diamond. So the entire diagram below commutes.

$$\begin{tikzcd}
(FA_1 \odot \cdots \odot FA_n) \arrow[r, "\delta^{-1}_{(FA_1 \ldots FA_n)}"] \arrow[rr, bend left, "1_{(FA_1 \odot \cdots \odot FA_n)}"] \arrow[d, shift right, "1_{(FA_1 \odot \cdots \odot FA_n)}"'] \arrow[d, shift left, "\overline{1} \circ \pi_{A_1 \ldots A_n}"] & ((FA_1 \odot \cdots \odot FA_n)) \arrow[d, "(\pi_{A_1 \ldots A_n})"] & (FA_1 \odot \cdots \odot FA_n) \arrow[l, "\kappa_{(FA_1 \odot \cdots \odot FA_n)}"'] \arrow[d, "\pi_{A_1 \ldots A_n}"] \\
(FA_1 \odot \cdots \odot FA_n) & (F(A_1 \otimes \cdots \otimes A_n)) \arrow[l, "\overline{1}"] & F(A_1 \otimes \cdots \otimes A_n) \arrow[l, "\kappa_{F(A_1 \otimes \cdots \otimes A_n)}"] \\
((FA_1 \odot \cdots \odot FA_n)) \arrow[u, shift left, "\mathbb{1}_{(FA_1 \odot \cdots \odot FA_n)}"] \arrow[u, shift right, "\overline{1} \circ \overline{1}^{-1}"'] \arrow[r, "\delta^{-1}_{((FA_1 \ldots FA_n))}"'] \arrow[rr, bend right, "1_{((FA_1 \odot \cdots \odot FA_n))}"'] & (((FA_1 \odot \cdots \odot FA_n))) \arrow[u, "(\overline{1}^{-1})"'] & ((FA_1 \odot \cdots \odot FA_n)) \arrow[l, "\kappa_{((FA_1 \odot \cdots \odot FA_n))}"] \arrow[u, "\overline{1}^{-1}"']
\end{tikzcd}$$

Recall that $\mathbb{1}_B = \kappa_B^{-1}$ for all $B \in \mathcal{D}$. Observe that taking $\overline{1}^{-1} = \pi_{A_1 \ldots A_n} \cdot \kappa^{-1}_{(FA_1 \odot \cdots \odot FA_n)}$ therefore makes the diagram commute. Since such a $\overline{1}^{-1}$ was unique, the diagram shows that $\pi_{A_1 \ldots A_n} = \overline{1}^{-1} \cdot \kappa_{(FA_1 \odot \cdots \odot FA_n)}$ has a two-sided inverse $\overline{1} \cdot \kappa_{F(A_1 \otimes \cdots \otimes A_n)}$ and is hence an isomorphism in $\mathcal{D}$ as required.

Now let $A_{1,1} \ldots A_{n,m_n}, B_1 \ldots B_n \in \mathcal{C}$. Define sequences

$$\begin{aligned}
S &= (A_{1,1} \otimes \cdots \otimes A_{1,m_1}) \ldots (A_{n,1} \otimes \cdots \otimes A_{n,m_n}), \\
T &= F(A_{1,1} \otimes \cdots \otimes A_{1,k_1}) \ldots F(A_{n,1} \otimes \cdots \otimes A_{n,m_n}).
\end{aligned}$$

It can be shown (with significant notational difficulty) that our definitions, together with the unbiased tensor interchange law and identity diamond, imply that the following diagrams both commute (again, we write only the first tensor in each object).

$$\begin{tikzcd}
(F(A_{1,1} \otimes \cdots \otimes A_{1,m_n}) \odot \cdots) \arrow[r, "(Ff_1 \odot \cdots \odot Ff_n)"] \arrow[dr, "G(f_1 \otimes \cdots \otimes f_n) \cdot \delta_T"'] & (FB_1 \odot \cdots \odot FB_n) \arrow[d, "\pi_{B_1 \ldots B_n}"] \\
& F(B_1 \otimes \cdots \otimes B_n)
\end{tikzcd}$$

$$\begin{tikzcd}
(F(A_{1,1} \otimes \cdots \otimes A_{1,m_n}) \odot \cdots) \arrow[d, "\pi_S"'] \arrow[dr, "G(f_1 \otimes \cdots \otimes f_n) \cdot \delta_T"] & \\
F((A_{1,1} \otimes \cdots \otimes A_{1,m_1}) \otimes \cdots) \arrow[r, "F(f_1 \otimes \cdots \otimes f_n)"'] & F(B_1 \otimes \cdots \otimes B_n)
\end{tikzcd}$$

Gluing these along the shared morphism produces the naturality square for π. Thus, π is natural in each component as desired.

Finally, we verify the coherence conditions for (F, π). By definition,

$$\begin{aligned} F\iota_A &= G(\iota_A \cdot \iota_A^{-1}) \cdot \kappa_{FA} \\ &= G1_{(A)} \cdot \kappa_{FA} \\ &= \pi_A \cdot \kappa_{FA}. \end{aligned}$$

Thus, π_A and $F\iota_A$ are such that the triangle identity for unbiased monoidal functors commutes.

$$\begin{tikzcd} & FA \arrow[dl, "\kappa_{FA}"'] \arrow[dr, "F\iota_A"] & \\ (FA) \arrow[rr, "\pi_A"'] \arrow[rr, bend right, "G(\iota_A \cdot \iota_A^{-1})"'] & & F(A) \end{tikzcd}$$

The pentagon identity for unbiased monoidal functors also commutes, but due to the unworkability of rendering its notation succinctly, we omit it. The necessary manipulations are conceptually straightforward.

From a universal-preserving multifunctor $G : V(\mathcal{C}) \to V(\mathcal{D})$, we have thus constructed an unbiased monoidal functor $(F, \pi) : \mathcal{C} \to \mathcal{D}$. To prove V' is full, it remains to show that $V'(F, \pi) = G$.

Objects are trivial as $FA = GA\ \forall A \in \mathcal{C}$ by definition. We need not check $V'(F, \pi)(f) = Gf$ for all morphisms f in $\mathcal{C}$, since G is not defined unless f is a multimorphism, i.e. a morphism out of a tensor. So let $f : (A_1 \otimes \cdots \otimes A_n) \to B$.

Since $f : A_1 \ldots A_n \to B$ as a multimorphism, we have $Gf : FA_1 \ldots FA_1 \to FB$ in $V(\mathcal{D})$, i.e. $Gf : (FA_1 \odot \cdots \odot FA_n) \to FB$ as a morphism in $\mathcal{D}$. By definition, $Ff : F(A_1 \otimes \cdots \otimes A_n) \to FB$ and $\pi_{A_1 \ldots A_n} : (FA_1 \odot \cdots \odot FA_n) \to F(A_1 \otimes \cdots \otimes A_n)$, giving $Ff \cdot \pi_{A_1 \ldots A_n} : (FA_1 \odot \cdots \odot FA_n) \to FB$. So the domain and codomain of Gf and $V'(F, \pi)(f)$ match.

We can thus expand our definition to find

$$\begin{aligned} V'(F, \pi)(f) &= Ff \cdot \pi_{A_1 \ldots A_n} \\ &= G\left(f \cdot \iota^{-1}_{(A_1 \otimes \cdots \otimes A_n)}\right) \cdot \kappa_{F(A_1 \otimes \cdots \otimes A_n)} \cdot \pi_{A_1 \ldots A_n} \\ &= G\left(f \cdot \iota^{-1}_{(A_1 \otimes \cdots \otimes A_n)}\right) \cdot (\pi_{A_1 \ldots A_n}) \cdot \kappa_{(FA_1 \odot \cdots \odot FA_n)} \text{ by naturality of } \kappa \\ &= G\left(f \cdot \iota^{-1}_{(A_1 \otimes \cdots \otimes A_n)}\right) \cdot (\pi_{A_1 \ldots A_n}) \cdot \delta^{-1}_{(FA_1 \ldots FA_n)} \text{ by the identity diamond} \\ &= G\left(f \cdot \iota^{-1}_{(A_1 \otimes \cdots \otimes A_n)}\right) \circ \pi_{A_1 \ldots A_n} \text{ by definition of composition in } V(\mathcal{D}) \\ &= G\left(f \cdot \iota^{-1}_{(A_1 \otimes \cdots \otimes A_n)}\right) \circ G1_{(A_1 \otimes \cdots \otimes A_n)} \\ &= G\left(\left(f \cdot \iota^{-1}_{(A_1 \otimes \cdots \otimes A_n)}\right) \circ 1_{A_1 \otimes \cdots \otimes A_n}\right) \text{ since } G \text{ preserves multicategory composition} \\ &= G\left(f \cdot \iota^{-1}_{(A_1 \otimes \cdots \otimes A_n)} \cdot \left(1_{(A_1 \otimes \cdots \otimes A_n)}\right) \cdot \gamma^{-1}_{(A_1 \ldots A_n)}\right) \text{ by definition of composition in } V(\mathcal{C}) \\ &= G\left(f \cdot \iota^{-1}_{(A_1 \otimes \cdots \otimes A_n)} \cdot \gamma^{-1}_{(A_1 \ldots A_n)}\right) \text{ since } \otimes_n \text{ preserves identities} \\ &= Gf \text{ by the identity diamond.} \end{aligned}$$

Thus, for all multimorphisms $f : A_1 \ldots A_n \to B$ in $V(\mathcal{C})$, we have that $V'(F, \pi)(f) = Gf$, giving $V'(F, \pi) = G$. This shows V' is surjective on the homsets in **RMulticat**, i.e. it is full. $\square$

Lemma 5.2.10

The functor $V' : \mathbf{UMonCat} \to \mathbf{RMulticat}$ is faithful.

Proof. Let $(\mathcal{C}, \otimes_n, \gamma, \iota), (\mathcal{D}, \odot_n, \delta, \kappa) \in \mathbf{UMonCat}$ and let $(F, \pi), (F', \pi') : \mathcal{C} \to \mathcal{D}$ be unbiased monoidal functors. Suppose $V'(F, \pi) = V'(F', \pi')$. Then $FA = F'A \ \forall A \in \mathcal{C}$. We therefore write FA for all objects below. Also write $\cdot$ for monoidal composition.

For any multimorphism $f : A_1 \ldots A_n \to B$ in $V(\mathcal{C})$, we have $V(F, \pi)(f) = V(F', \pi')(f)$. This is in particular true for $1_{(A_1 \otimes \cdots \otimes A_n)}$, which is a multimorphism $A_1 \ldots A_n \to (A_1 \otimes \cdots \otimes A_n)$ by definition of $V(\mathcal{C})$. Thus,

$$
\begin{aligned}
\pi_{A_1 \ldots A_n} &= 1_{F(A_1 \otimes \cdots \otimes A_n)} \cdot \pi_{A_1 \ldots A_n} \\
&= F1_{(A_1 \otimes \cdots \otimes A_n)} \cdot \pi_{A_1 \ldots A_n} \text{ since functors preserve identities} \\
&= V(F, \pi)\left(1_{(A_1 \otimes \cdots \otimes A_n)}\right) \\
&= V(F', \pi')\left(1_{(A_1 \otimes \cdots \otimes A_n)}\right) \\
&= F'1_{(A_1 \otimes \cdots \otimes A_n)} \cdot \pi'_{A_1 \ldots A_n} \\
&= 1_{F(A_1 \otimes \cdots \otimes A_n)} \cdot \pi'_{A_1 \ldots A_n} \text{ since functors preserve identities} \\
&= \pi'_{A_1 \ldots A_n}.
\end{aligned}
$$

Since all components of π and π' are therefore equal, we have $\pi = \pi'$. From our assumption that $Ff \cdot \pi_{A_1 \ldots A_n} = F'f \cdot \pi'_{A_1 \ldots A_n}$, and knowing components of $\pi = \pi'$ are invertible, we recover $Ff = F'f$.

Thus, F and F' have the same effect on all objects and morphisms, so $F = F'$. Hence, $V'(F, \pi) = V'(F', \pi')$ implies $(F, \pi) = (F', \pi')$, so V' is faithful. $\square$

Proof (Leinster-Hermida Equivalence Theorem). We have a functor $V : \mathbf{UMonCat} \to \mathbf{Multicat}$, which is well-defined by Lemma 5.2.5. Its image lies in the subcategory $\mathbf{RMulticat}$ by Lemma 5.2.8, and it is full and faithful with this restriction by Lemma 5.2.9 and Lemma 5.2.10.

By Lemma 5.2.7, every $\mathcal{C} \in \mathbf{RMulticat}$ is isomorphic to $V(\mathcal{D})$ for some $(\mathcal{D}, \otimes_n, \gamma, \iota) \in \mathbf{UMonCat}$. But this is an isomorphism in $\mathbf{Multicat}$. We need to show it is also an isomorphism in $\mathbf{RMulticat}$, i.e. that multifunctors $F : \mathcal{C} \to V(\mathcal{D})$ and $G : V(\mathcal{D}) \to \mathcal{C}$, where $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{V(\mathcal{D})}$, are universal-preserving.

Note that, since G and F are inverses, they must induce bijections on homsets and bijections on objects. So the function $\mathcal{C}(A_1 \ldots A_n; B) \to V(\mathcal{D})(FA_1 \ldots FA_n; FB)$ is a bijection $\forall A_1 \ldots A_n, B \in \mathcal{C}$, and $\forall A \in V(\mathcal{D})$, there exists a unique $GA \in \mathcal{C}$ such that $A = FGA$.

Let $u : A_1 \ldots A_n \to B$ be universal in $\mathcal{C}$. Let $g : C_1 \ldots C_i, FA_1 \ldots FA_n, C_{i+1} \ldots C_m \to D$ be a multimorphism in $V(\mathcal{D})$. Then $g = Ff$ for unique $f : GC_1 \ldots GC_i, A_1 \ldots A_n, GC_{i+1} \ldots GC_m \to GD$ in $\mathcal{C}$ by the bijections above. Since u is universal, $f = \overline{f} \circ_i u$ for unique $\overline{f}$. Multifunctorality gives

$$
\begin{aligned}
g &= Ff \\
&= F(\overline{f} \circ_i u) \\
&= F\overline{f} \circ_i Fu.
\end{aligned}
$$

Since $\overline{f}$ was unique, the above bijections tell us $F\overline{f}$ is unique, so Fu is universal. Thus, F preserves universals. A mirrored argument shows the same is true for G.

We thus have $\mathcal{C} \cong V(\mathcal{D})$ with isomorphism now in $\mathbf{RMulticat}$, so V' is essentially surjective on objects. As V' is full and faithful, Proposition 2.4.10 gives an equivalence $\mathbf{UMonCat} \simeq \mathbf{RMulticat}$.

By Theorem 3.2.9, we thus have a chain of equivalences $\mathbf{MonCat} \simeq \mathbf{UMonCat} \simeq \mathbf{RMulticat}$, and transitivity of equivalence (Proposition 2.4.13) gives the final result. $\square$

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