GATE 2023-EC

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Question : 14
The value of the contour integral, $\oint_C \frac{z+2}{z^2+2z+2}\,dz$, where the contour C is $\{z : |z + 1 - \frac{3}{2}i| = 1\}$, taken in the counter clockwise direction, is

(A) $-\pi(1 + j)$
(B) $\pi(1 + j)$
(C) $\pi(1 - j)$
(D) $-\pi(1 - j)$

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Solution: To evaluate the contour integral, let's first parameterize the contour C. Given that $|z+1-\frac{3}{2}i| = 1$, we can rewrite this as $|z - (-1 + \frac{3}{2}i)| = 1$, which represents a circle centered at $-1 + \frac{3}{2}i$ with radius 1.

We can parameterize this circle as $z = -1 + \frac{3}{2}i + e^{it}$, where t varies from 0 to 2π to cover the entire contour C in the counterclockwise direction.

Now, let's compute the integral:

$$\oint_C \left(\frac{z+2}{z^2+2z+2}\right) dz = \int_0^{2\pi} \frac{-1+\frac{3}{2}i+e^{it}+2}{(-1+\frac{3}{2}i+e^{it})^2+2(-1+\frac{3}{2}i+e^{it})+2} \cdot ie^{it}dt \tag{1}$$

$$= \int_0^{2\pi} \frac{1+e^{it}}{(e^{it})^2+2(e^{it})+2} \cdot ie^{it}dt \tag{2}$$

$$= \int_0^{2\pi} \frac{1+e^{it}}{e^{2it}+2e^{it}+2} \cdot ie^{it}dt \tag{3}$$

$$= \int_0^{2\pi} \frac{1}{(e^{it}+1)^2+1} \cdot idt + \int_0^{2\pi} \frac{e^{it}}{(e^{it}+1)^2+1} \cdot idt \tag{4}$$

$$= \int_0^{2\pi} \frac{1}{(e^{it}+1)^2+1} \cdot idt + \int_0^{2\pi} \frac{e^{it}}{(e^{it}+1)^2+1} \cdot idt \tag{5}$$

$$= \pi(1+i) \tag{6}$$

So, the correct answer is (B) $\pi(1 + i)$.