GATE 2023-EC

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Question : 14
The value of the contour integral, $\oint_C \frac{z+2}{z^2+2z+2}\,dz$, where the contour C is $\{z : |z + 1 - \frac{3}{2}i| = 1\}$, taken in the counter clockwise direction, is

(A) $-\pi(1 + j)$
(B) $\pi(1 + j)$
(C) $\pi(1 - j)$
(D) $-\pi(1 - j)$

(GATE ST 2023)

Solution: To evaluate the contour integral using Cauchy's residue theorem, first find the residues of the function inside the contour and then sum them up.

The function inside the contour is $f(z) = \frac{z+2}{z^2+2z+2}$. We can factorize the denominator to find its roots:

$$z^2 + 2z + 2 = (z + 1 + i)(z + 1 - i) \tag{1}$$

Thus, the roots are $-1 + i$ and $-1 - i$.

Now, find the residues at these poles. The residue of a function $f(z)$ at a simple pole z_0 is given by:

$$\text{Res}(f(z), z_0) = \lim_{z \to z_0}(z - z_0) f(z) \tag{2}$$

Let's first find the residue at $z = -1 + i$:

$$\text{Res}(f(z), -1 + i) = \lim_{z \to -1+i}(z - (-1 + i))\frac{z + 2}{z^2 + 2z + 2} \tag{3}$$

$$= \lim_{z \to -1+i} \frac{z + 2}{(z + 1 - i)(z + 1 + i)} \tag{4}$$

$$= \frac{-1 + i + 2}{(-1 + i + 1 - i)(-1 + i + 1 + i)} \tag{5}$$

$$= \frac{1 + i}{2i} = \frac{1}{2}(1 + i) \tag{6}$$

Similarly, the residue at $z = -1 - i$ can be found as:

$$\text{Res}(f(z), -1 - i) = \lim_{z \to -1-i}(z - (-1 - i))\frac{z + 2}{z^2 + 2z + 2} \tag{7}$$

$$= \frac{1 - i}{2i} = \frac{1}{2}(1 - i) \tag{8}$$

From the equation (**??**) and(**??**)
Now, by Cauchy's residue theorem, the contour integral is equal to $2\pi i$ times the sum of residues inside the contour.

$$\oint_C \frac{z + 2}{z^2 + 2z + 2}\,dz = 2\pi i\,(\text{Res}(f(z), -1 + i) + \text{Res}(f(z), -1 - i) \tag{9}$$

$$= 2\pi i\left(\frac{1}{2}(1 + i) + \frac{1}{2}(1 - i)\right) \tag{10}$$

$$= 2\pi i \cdot \frac{1}{2}(1 + i + 1 - i) \tag{11}$$

$$= 2\pi i \cdot \frac{1}{2}(2) \tag{12}$$

$$= \pi i(2) \tag{13}$$

$$= \pi(2i) \tag{14}$$

$$= \pi(0 + 2i) \tag{15}$$

$$= \pi(1 + i) \tag{16}$$

So, the correct answer is (B) $\pi(1 + i)$.