

Postulates of quantum mechanics with dual-complex numbers

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1 Introduction

2 Dual numbers

Extending real numbers with an imaginary unit i with $i^2 = -1$ gives rise to the complex numbers. Complex numbers form a field and are used in quantum mechanics to represent physical systems and their evolution using complex linear algebra. The set of complex numbers is $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = -1\}$. If a number $z \in \mathbb{C}$ has the form $z = a + bi$ we denote a the real part and b the imaginary part. We can define the functions Re and Im as follows

$$\text{Re} : \mathbb{C} \rightarrow \mathbb{R} : a + bi \rightarrow a \quad (1) \quad \text{Im} : \mathbb{C} \rightarrow \mathbb{R} : a + bi \rightarrow b \quad (2) \quad \forall z \in \mathbb{C}, z = \text{Re}(z) + \text{Im}(z)i \quad (3)$$

Dual numbers \mathbb{D} were first introduced by Clifford (1871) and follow a similar idea. They are generated by introducing an imaginary unit ϵ with $\epsilon^2 = 0$ to the reals. ϵ is commuting in multiplication and addition with reals. To avoid confusion with complex, we call the real part of a dual the significant part (Sig) and its imaginary part the infinitesimal part (Inf).

$$\text{Sig} : \mathbb{D} \rightarrow \mathbb{R} : a + b\epsilon \rightarrow a \quad (4) \quad \text{Inf} : \mathbb{D} \rightarrow \mathbb{R} : a + b\epsilon \rightarrow b \quad (5) \quad \forall d \in \mathbb{D}, d = \text{Sig}(d) + \text{Inf}(d)\epsilon \quad (6)$$

Extending the real numbers with both i and ϵ gives rise to dual-complex numbers \mathbb{DC} . We can see the complex, dual and dual-complex numbers as quotient rings:

$$\mathbb{C} \approx \mathbb{R}[i]/\langle i^2 + 1 \rangle \quad (7) \quad \mathbb{D} \approx \mathbb{R}[\epsilon]/\langle \epsilon^2 \rangle \quad (8) \quad \mathbb{DC} \approx \mathbb{C}[\epsilon]/\langle \epsilon^2 \rangle \quad (9)$$

Alternatively, dual-complex (resp. dual) numbers can be seen as the subalgebra of the matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \quad (10)$$

Where $a, b \in \mathbb{C}$ (resp. $a, b \in \mathbb{R}$). Dual and dual-complex numbers are used in automatic differentiation as they allow to “cut” the Taylor expansion from the second term. Baydin et al. (2018); Revels et al. (2016)

$$f(z + \epsilon) = f(z_0) + \epsilon f'(z_0) + \epsilon^2 f''(z_0)/2 + \dots = f(z_0) + \epsilon f'(z_0). \quad (11)$$

Higher-order versions of dual-numbers were developed to deal with higher-order derivative. See Behr et al. (2019); Szirmay-Kalos (2021).

Ring operations and inversion on dual-complex numbers

Addition and multiplication are defined as follows:

$$(z_1 + t_1\epsilon) + (z_2 + t_2\epsilon) = (z_1 + z_2) + (t_1 + t_2)\epsilon \quad (12)$$

$$(z_1 + t_1\epsilon)(z_2 + t_2\epsilon) = (z_1 z_2) + (t_1 z_2 + z_1 t_2)\epsilon \quad (13)$$

Notice how with the infinitesimal unit $z_2 t_2 \epsilon^2$ vanish into oblivion when with the imaginary unit it becomes real and negative.

A dual-complex is called infinitesimal when its significant part is zero. The set of infinitesimal dual-complex numbers is denoted $\mathcal{O}_\epsilon = \{0 + t\epsilon | t \in \mathbb{C}\} = \epsilon\mathbb{C}$. A number that is not infinitesimal is called appreciable. We can observe three cases when dividing $w_1 = z_1 + t_1\epsilon$ by $w_2 = z_2 + t_2\epsilon$:

1. When w_2 is appreciable, i.e. $z_2 \neq 0$. In this case, the inverse of w_2 exists and division is defined as

$$\frac{z_1 + t_1\epsilon}{z_2 + t_2\epsilon} = \frac{(z_1 + t_1\epsilon)(z_2 - t_2\epsilon)}{(z_2 + t_2\epsilon)(z_2 - t_2\epsilon)} = \frac{z_1 z_2 - z_1 t_2\epsilon + t_1 z_2\epsilon}{z_2^2} = \frac{z_1}{z_2} + \frac{z_2 t_1 - z_1 t_2}{z_2^2} \epsilon \quad (14)$$

2. $z_2 = 0$ and we divide the infinitesimal part ($z_1 = 0$). The usual constraint that $w' = \frac{w_1}{w_2} \iff w'w_2 = w_1$ leaves us (infinitely) many solutions. Let's say that we want to find $w' = z' + t'\epsilon$, the quotient of w_1 and w_2 .

$$z' + t'\epsilon = \frac{t_1\epsilon}{t_2\epsilon} \iff z't_2\epsilon + t't_2\epsilon^2 = t_1\epsilon \iff z' = t_1/t_2 \quad (15)$$

There is no constraint on t' . However, a natural definition would be $t' = 0$, implying $\frac{t_1\epsilon}{t_2\epsilon} = \frac{t_1}{t_2}$. This is only a convenience definition as w_2 has no inverse.

3. $z_2 = 0$ and we divide the (non-zero) non-dual part z_1 . In this case we have no solution because $z' + t'\epsilon = \frac{z_1}{t_2\epsilon} \iff z't_2\epsilon = z_1$ which is impossible (by hypothesis $z', t_2, z_1 \in \mathbb{C}$).

Division between w_1 and w_2 is therefore defined for w_1 infinitesimal or w_2 appreciable. Exponentiation and rooting of dual-complex are given by

Proposition 1. $(z + t\epsilon)^n = z^n + nz^{n-1}t\epsilon$

Proof. \mathbb{DC} is a commutative ring and therefore the binomial theorem holds.

$$(z + t\epsilon)^n = \sum_{k=0}^n \binom{n}{k} z^{n-k} (t\epsilon)^k = z^n + nz^{n-1}t\epsilon + (t\epsilon)^2 \sum_{k=2}^n \binom{n}{k} z^{n-k} (t\epsilon)^{k-2} = z^n + nz^{n-1}t\epsilon \quad (16)$$

□

Corollary 2. $\sqrt[n]{z + t\epsilon} = \sqrt[n]{z} + (\frac{t}{n(\sqrt[n]{z})^{n-1}})\epsilon$ for $z \neq 0$.

Automatic differentiation

Dual numbers are useful for automatic differentiation. Given an analytic complex-to-complex function $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)z^n}{n!}$ on an open set D , we extend it by defining \hat{f} on $D + \mathcal{O}_\epsilon$ as:

$$\hat{f}(w) \rightarrow \sum_{n=0}^{\infty} \frac{f^{(n)}(0)w^n}{n!} \quad (17)$$

Messelmi (2015) proves we can always extend a complex analytic to complex-dual numbers like this. The usefulness of dual numbers for differentiation comes from the following result:

Proposition 3 (Automatic differentiation). *Given $f(z)$, a function from complex to complex which is polymorphic on an open set D , there exists a unique polymorphic function $\hat{f}(w)$ defined on $D + \mathcal{O}_\epsilon$ that agrees with $f(z)$ for every $w \in D$. This function \hat{f} is given by*

$$\hat{f}(z + t\epsilon) = f(z) + zf'(z)\epsilon \quad (18)$$

for $z \in D$ and $t \in \mathbb{R}$.

Proof. The following proof being simplified and only meant for intuition, see Messelmi (2015) for a full formal proof. Given $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!}$ and $\hat{f}(z + t\epsilon)$ agreeing with f for complex values and polymorphic on $D + \mathcal{O}_\epsilon$, we have, by proposition 1,

$$\hat{f}(z + t\epsilon) = \sum_{n=0}^{\infty} \frac{\hat{f}^{(n)}(z_0)((z-z_0) + t\epsilon)^n}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)((z-z_0)^n + n(z-z_0)^{n-1}t\epsilon)}{n!} \quad (19)$$

Which can be rewritten as

$$\begin{aligned}
\hat{f}(z + t\epsilon) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z - z_0)^n}{n!} + \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)n(z - z_0)^{n-1}t\epsilon}{n!} \\
&= f(z) + t\epsilon \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z - z_0)^n}{n!} \right)' \\
&= f(z) + t\epsilon f'(z).
\end{aligned} \tag{20}$$

□

Corollary 4. *Dual-complex exponential, logarithm, sine and cosine are given by*

$$\exp(z + t\epsilon) = \exp(z)(1 + t\epsilon) \tag{21} \quad \sin(z + t\epsilon) = \sin(z) + \cos(z)t\epsilon \tag{23}$$

$$\log(z + t\epsilon) = \log(z) + \frac{t}{z}\epsilon \tag{22} \quad \cos(z + t\epsilon) = \cos(z) - \sin(z)t\epsilon \tag{24}$$

3 Linear algebra over dual-complex numbers

In this section, we recall known and introduce new properties of \mathbb{D} and \mathbb{DC} required to show the consistency of our extension of quantum mechanics' postulates to dual-complex numbers in section 4.

We first introduce in section 3.1 an ordering on dual numbers that will be needed to define a set of valid probabilities. In section 3.2, we motivate a choice of conjugation and norm on \mathbb{DC} , establish some properties about them before recalling results about diagonalization of dual-complex Hermitian operators. Finally, we establish that dual-complex unitary operators are also diagonalizable and establish their relation to dual-complex skew-Hermitian operators in the sense that they are the exponential and logarithm of each other.

Ordering dual and dual-complex numbers

We know that while \mathbb{R} can be ordered consistently with the field operations, this is not the case for \mathbb{C} . Indeed if \mathbb{C} could be totally ordered we would have either $i \geq 0$ or $-i \geq 0$ and then $(\pm i)^2 = -1 \geq 0$. Orderability of \mathbb{D} depends on the definition of an ordered ring, we choose the usual one: a binary relation \leq such that for all a, b in the ring $a \leq b \implies a + c \leq b + c$ and $0 \leq a, 0 \leq b \implies 0 \leq ab$. [Fcuhs \(1963\)](#) It turns out that there is a very natural lexicographic ordering for \mathbb{D} .

Proposition 5. *\mathbb{D} can be totally ordered w.r. to \mathbb{R} 's natural ordering and this order is unique up to $\pm\epsilon > 0$.*

Proof. Suppose $\leq_{\mathbb{D}}$ is a total ring order on \mathbb{D} such that for any $a, b \in \mathbb{R}$, $a \leq b \iff a \leq_{\mathbb{D}} b$. Since $\mathbb{R}[\epsilon] \approx \mathbb{R}[-\epsilon]$ we say wlog that $\epsilon \geq 0$. Now aiming at contradiction, suppose that for some $a \in \mathbb{R}^+$ we have $a \leq \epsilon$. Then, $a^2 \leq \epsilon^2 = 0$ which is a contradiction, implying that for every $a \in \mathbb{R}^+$, $a \geq \epsilon$. This is equivalent to say that $\leq_{\mathbb{D}}$ is a lexicographic order, i.e.

$$a_1 + b_1\epsilon \leq_{\mathbb{D}} a_2 + b_2\epsilon \iff a_1 < a_2 \vee (a_1 = a_2 \wedge b_1 \leq b_2) \tag{25}$$

This ordering is total and it is easily seen that it is a ring ordering. □

Linear algebra

Conjugation for hypercomplex numbers of dimension 2 and imaginary unit X is given by $\overline{a + bX} = a - bX$. For dual-complex, we use the complex conjugation as in ?. We quickly justify this choice by looking at alternative definitions.

[Messelmi \(2015\)](#) suggests four different possible conjugations of interest for us

$$\bar{w} = \bar{z} + \bar{t}\epsilon \tag{26} \quad \tilde{w} = z - t\epsilon \tag{27} \quad \widetilde{\bar{w}} = \bar{z} - \bar{t}\epsilon \tag{28} \quad w^* = \bar{z}(1 - t\epsilon/z) \tag{29}$$

w^* is the only possible conjugation that satisfies both $w^*w \in \mathbb{R}$ and $\text{Re}(\text{Sig}(w)) = \text{Re}(\text{Sig}(w^*))$. It is, however, non-linear. Our purpose being to extend quantum mechanics with a unit for infinitesimal variations, this conjugation is unfortunate. The choice of \bar{w} over \tilde{w} and $\widetilde{\bar{w}}$ is motivated by the idea that we can accept a dual norm as ϵ represents an infinitesimal variation of it, but not a complex norm.

Definition 6. For $n \in \mathbb{N}$, \mathbb{DC}^n represents the module of vectors of length n over \mathbb{DC} equipped with an inner-product:

$$\langle \psi | \phi \rangle = \sum_k \overline{\psi_k} \phi_k \quad (30)$$

The norm $|w|$ of $w = z + t\epsilon$ is defined as $|w| = \sqrt{w^* w}$. We can observe the following properties

$$\begin{aligned} \overline{w_1 + w_2} &= \overline{w_1} + \overline{w_2} & (31) \quad \overline{w_1 w_2} &= \overline{w_1} \overline{w_2} & (34) \quad \overline{\overline{w}} &= w & (37) \\ w + \overline{w} &= 2 \operatorname{Re}(z) + 2 \operatorname{Re}(t)\epsilon & (32) \quad \overline{w} w &= |z|^2 + 2 \operatorname{Re}(z \bar{t}) \in \mathbb{D} & (35) \\ |w| = 0 &\iff z = 0 & (33) \quad |w| &\geq 0 & (36) \quad |w| = \sqrt{\overline{w} w} = |z| + \frac{\operatorname{Re}(z \bar{t})}{|z|} & (38) \end{aligned}$$

In particular equations ?? and ?? entails the following propositions.

Proposition 7. For every $|\psi_\epsilon\rangle = |\psi\rangle + \epsilon |\phi\rangle \in \mathbb{DC}^n$, $\langle \psi_\epsilon | \psi_\epsilon \rangle = \langle \psi | \psi \rangle + 2 \operatorname{Re}(\langle \psi | \phi \rangle) \epsilon$ and therefore

$$\| |\psi_\epsilon\rangle \| = \sqrt{\langle \psi_\epsilon | \psi_\epsilon \rangle} = \| |\psi\rangle \| + \frac{\operatorname{Re}(\langle \psi | \phi \rangle)}{\| |\psi\rangle \|} \epsilon. \quad (39)$$

Proposition 8. For any $|\psi\rangle \in \mathbb{DC}^n$, $\| |\psi\rangle \| = \sum_{k=1}^n \psi_k^* \psi_k \geq 0$.

A vector in \mathbb{D}^n or \mathbb{DC}^n is called appreciable when its norm is non-zero or equivalently when at least one of its entries is appreciable. Two vectors are called orthogonal when their inner product is zero and appreciably orthogonal when their inner product is infinitesimal.

We define the adjoint as the transpose conjugate, i.e. $A^\dagger = (A^T)^*$. An operator H is Hermitian if it is self-adjoint, i.e. $H^\dagger = H$ and skew-Hermitian if $H^\dagger = -H$. An operator is Hermitian (resp. skew-Hermitian) if and only if its significant and infinitesimal are Hermitian (resp. skew-Hermitian) and H is Hermitian if and only if iH is skew Hermitian.

An operator U is unitary when $U^\dagger U = I$. The following proposition can be derived from this definition.

Proposition 9. The following propositions are equivalent for U_ϵ a linear operator:

- a U_ϵ is unitary, i.e. $U_\epsilon^\dagger U_\epsilon = I$.
- b $U_\epsilon U_\epsilon^\dagger = I$.
- c U_ϵ preserves the inner product, i.e. $(U_\epsilon |\psi\rangle)^\dagger (U_\epsilon |\phi\rangle) = \langle \psi | \phi \rangle$ and therefore the norm $\| U_\epsilon |\psi\rangle \| = \| |\psi\rangle \|$.
- d Rows of U_ϵ form an orthonormal basis.
- e Columns of U_ϵ form an orthonormal basis.
- f $U_\epsilon = U - i\epsilon UH$ where U is a complex unitary and H is a complex Hermitian.

Proof. For point equivalence between point (f) and (a), let $U_\epsilon = A + B\epsilon$.

$$\begin{aligned} U_\epsilon^\dagger U_\epsilon = I &\iff A^\dagger A + \epsilon A^\dagger B + \epsilon B^\dagger A = I \\ &\iff \begin{cases} A^\dagger A = I \\ A^\dagger B + B^\dagger A = 0 \end{cases} \\ &\iff \begin{cases} U := A \text{ is a unitary.} \\ U^\dagger B = -(U^\dagger B)^\dagger \end{cases} \end{aligned} \quad (40)$$

The second condition can be restated as

$$\begin{aligned} U^\dagger B &= -(U^\dagger B)^\dagger \\ &\iff H := iU^\dagger B \text{ is Hermitian.} \\ &\iff B = -iUH \text{ and } H \text{ is Hermitian.} \end{aligned} \quad (41)$$

Points from (a) to (e) are easily seen to be equivalent by following a reasoning analogous to the one for complex unitary operators which conclude our proof. \square

In the rest of this section, we prove that the exponential of dual-complex skew-Hermitian are unitary and the logarithms of a dual-complex unitary are skew-Hermitian. We first introduce the exponential of a dual-complex operator, then recall a known result about dual-complex Hermitian diagonalization and finally introduce two new results about dual-complex unitary operators.

Proposition 10. *The dual-complex matrix exponential always exists.*

Proof. For scalar exponentiation in ??, we used the dual-complex extension from ?? which works thanks to binomial theorem and, therefore, commutativity. Commutativity is notoriously not a property of linear operators in general. Without commutativity, we get

$$\exp(A + B\epsilon) = \sum_{n=0}^{\infty} \frac{1}{n!} (A + B\epsilon)^n = \sum_{n=0}^{\infty} \frac{A^n}{n!} + \epsilon \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} A^k B A^{n-k} \quad (42)$$

The significant part is the exponential of the complex matrix A which is known to always converges. The infinitesimal part also converges as

$$\begin{aligned} \left\| \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} A^k B A^{n-k} \right\| &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \|A^k\| \|B\| \|A^{n-k}\| \\ &\leq \|B\| \sum_{n=0}^{\infty} \frac{2^n \|A\|^n}{n!} \\ &= \|B\| \exp(2\|A\|) \end{aligned} \quad (43)$$

□

A proof identical to that of ? for the case of dual numbers can be used to show that the exponential of dual-complex matrix $A + B\epsilon$ is $\exp(A) + \epsilon L_{\exp}(A, B)$ where $L_{\exp}(A, B)$ is the Fréchet derivative of the exponential function of A in the direction of B .

Proposition 11 (Qi and Luo (2021)). *Suppose H is a Hermitian operator. Then there exists an orthonormal dual-complex basis $|j_{\epsilon}\rangle$ and n dual eigenvalues λ_j of H such that $H = \sum_j \lambda_j |j_{\epsilon}\rangle \langle j_{\epsilon}|$.*

We now establish that dual-complex unitary operators, just like dual-complex Hermitian operators, are diagonalizable and deduce that their logarithms are Hermitian.

Proposition 12 (Spectral theorem for dual-complex unitary operators). *A unitary operator is unitarily diagonalizable, i.e. for every dual-complex unitary operator U_{ϵ} , there is a dual-complex orthonormal basis $|j_{\epsilon}\rangle$ with associated dual-complex eigenvalues λ_j such that $U_{\epsilon} = \sum_j \lambda_j |j_{\epsilon}\rangle \langle j_{\epsilon}|$. These eigenvalues are of the form $e^{i\theta_j} + \epsilon e^{i\theta_j} \mu_j$ where θ_j and μ_j are real numbers.*

Proof. Let $U_{\epsilon} = U + i\epsilon UH$ be a dual-complex unitary where $U = \exp(iH_0)$ is a complex unitary operator and H is a Hermitian operator. We are going to build a diagonalizing basis for U_{ϵ} from a basis of $H + \epsilon H_0$.

By proposition ??, we have that $H_0 + \epsilon H$ is diagonalizable. Let $|j_{\epsilon}\rangle = |j_1\rangle + \epsilon |j_2\rangle$ be an orthonormal basis in which $H_0 + \epsilon H$ is diagonal. Then, $|j_1\rangle$ also forms an orthonormal basis, and we can uniquely write $H = \sum_{jk} h_{jk} |j_{\epsilon}\rangle \langle k_{\epsilon}|$ where $h_{jk}^* = -h_{kj}$. Since the eigenvalues of $H_0 + \epsilon H$ are of the form $\theta_j + \epsilon \mu_j$ with $\theta_j, \mu_j \in \mathbb{R}$ we have

$$H_0 |j_2\rangle + H |j_1\rangle = \theta_j |j_2\rangle + \mu_j |j_1\rangle \iff (H_0 - \theta_j I) |j_2\rangle = (\mu_j I - H) |j_1\rangle \quad (44)$$

This implies for each $|k_1\rangle$

$$\langle k_1 | (H_0 - \theta_j I) |j_2\rangle = \langle k_1 | (\mu_j I - H) |j_1\rangle \iff (\theta_k - \theta_j) \langle k_1 | j_2\rangle = -h_{kj} \quad (45)$$

Which implies that for each $j \neq k$ such that $\theta_j = \theta_k$, $h_{jk} = 0$.

We will now be able to prove the following claim: we can always construct an orthonormal basis for U_{ϵ} , whose significant part is $|j_1\rangle$ and with associated eigenvectors whose significant part is $\lambda_j = e^{i\theta_j}$. We need to find vectors $|j_3\rangle$ and complex values σ_j such that

$$(U - \lambda_j I) |j_3\rangle = (\sigma_j I - iUH) |j_1\rangle \quad (46)$$

Where $\lambda_j = e^{i\theta_j}$. Since $U = \sum_k \lambda_k |k_1\rangle \langle k_1|$, the left side must necessarily be orthogonal to $|j_1\rangle$ which means

$$\langle j_1 | (\sigma_j I - iUH) | j_1 \rangle = 0 \iff \sigma_j = i\lambda_j h_{jj} \quad (47)$$

Let $|j_3\rangle = \sum_j \alpha_{jk} |k_1\rangle$, or, equivalently, let $\alpha_{jk} = \langle k_1 | j_3 \rangle$. We find for $k \neq j$

$$\langle k_1 | (U - \lambda_j I) | j_3 \rangle = \langle k_1 | (\sigma_j I - iUH) | j_1 \rangle \iff (\lambda_k - \lambda_j) \alpha_{jk} = -i\lambda_j h_{kj} \quad (48)$$

When $\lambda_j \neq \lambda_k$. When $\lambda_j = \lambda_k$, we showed that $h_{jk} = 0$ and any value of α_{jk} in \mathbb{C} is a solution. Since we always have α_{jk} satisfying ??, we have a basis. Consider we choose that $\alpha_{jj} = 0$ and $\alpha_{jk} = 0$ when $\lambda_j = \lambda_k$. It happens that the basis $|j_U\rangle := |j_1\rangle + \epsilon |j_3\rangle$ is orthonormal. Indeed

$$\begin{aligned} \alpha_{jk}^* &= (\lambda_j \lambda_k) \frac{-i\lambda_j^* h_{jk}}{\lambda_k - \lambda_j} = \frac{-i\lambda_k h_{jk}}{\lambda_k - \lambda_j} = -\alpha_{kj} \\ \implies \langle j_U | k_U \rangle &= \langle j_1 | k_1 \rangle + \epsilon \langle j_3 | k_1 \rangle + \epsilon \langle j_1 | k_3 \rangle = \delta_{jk} + \epsilon(\alpha_{kj}^* + \alpha_{jk}) = \delta_{jk} \end{aligned}$$

We now showed that $|j_U\rangle$ is an orthonormal basis and by construction we have $\theta_j, \sigma_j \in \mathbb{R}$ and

$$\begin{aligned} \sum_j (\lambda_j + \epsilon \sigma_j) |j_U\rangle \langle j_U| &= \left(\sum_j e^{i\theta_j} |j_1\rangle \langle j_1| \right) + \epsilon \left(\sum_j e^{i\theta_j} |j_3\rangle \langle j_3| + \sigma_j |j_1\rangle \langle j_1| \right) \\ &= U + i\epsilon UH \end{aligned} \quad (49)$$

□

Thanks to the diagonalization of unitary operators, we can establish the following result.

Proposition 13. *The exponential of Hermitian linear operator is a unitary. Moreover, every unitary U_ϵ is of the form $\exp(iH)$ where H is Hermitian. H is unique modulo 2π .*

Proof. To prove the first statement, let H be a Hermitian operator. Then we have

$$\exp(iH)^\dagger = \left(\sum_{n=0}^{\infty} \frac{(iH)^n}{n!} \right)^\dagger = \sum_{n=0}^{\infty} \frac{(-iH^\dagger)^n}{n!} = \exp(-iH) \quad (50)$$

This shows us that the adjoint of the exponential of iH is its inverse:

$$\exp(iH)^\dagger \exp(iH) = \exp(-iH) \exp(iH) = \exp(-iH + iH) = \exp(0) = I \quad (51)$$

For the second statement, let $U_\epsilon = \sum_j (\lambda_j + \epsilon \sigma_j) |j_U\rangle \langle j_U|$ be a dual-complex unitary operator diagonalized as in proposition ??. We can write $\lambda_j + \epsilon \sigma_j = e^{i\theta_j} (1 + i\mu_j)$ where $\theta_j, \mu_j \in \mathbb{R}$. Therefore $\log(U_\epsilon) = \sum_j \log(\lambda_j + \epsilon \sigma_j) |j_U\rangle \langle j_U| = \sum_j (i\theta_j + \epsilon i\mu_j) |j_U\rangle \langle j_U|$ which is indeed skew-Hermitian. □

Corollary 14. *Let H_1 and H_2 be complex Hermitian matrices, $L_{\exp}(iH_1, iH_2)$ is of the form iUH where U is unitary and H Hermitian.*

Another result is required to extend the measurement postulate. We say the operator E is appreciably positive if $\langle \psi | E | \psi \rangle$ positive and appreciable or exactly zero for every vector $|\psi\rangle$.

Proposition 15. *For every dual-complex linear operator M_ϵ , $M_\epsilon^\dagger M_\epsilon$ is appreciably semipositive.*

Proof. Let $|\psi\rangle$ be an arbitrary vector and $|\phi\rangle = \sum_j \phi_j |j\rangle = M_\epsilon |\psi\rangle$ for an orthonormal basis $|j\rangle$. Then $\langle \psi | M_\epsilon^\dagger M_\epsilon | \psi \rangle = \langle \phi | \phi \rangle = \sum_j \phi_j^* \phi_j$. From equations 28, 30 and 31 combined, we have that for all j , $\phi_j^* \phi_j$ is either appreciable or exactly zero and always greater or equal to zero. It follows that $M_\epsilon^\dagger M_\epsilon$ is appreciably semipositive. □

4 Postulates of quantum theory over dual-complex numbers

In this section we introduce an extension of quantum mechanics postulate to allow for dual-complex amplitudes and operators. We replace the usual notion of state by that of an “ ϵ -state”, a dual-complex unit vector in the dual-complex extension of a Hilbert-space. We extend the usual notions of unitary evolution and Hermitian observables by allowing dual-complex unitary “ ϵ -evolutions” and dual-complex Hermitian “ ϵ -observables”. These notions will allow us to study linear approximation of quantum states, which we will call “ ϵ -quantum states.”.

We present a suggestion of extension of the usual postulates of quantum theory, then justify them quickly.

Postulate 1. An ϵ -state vector is a unit dual-complex vector. At every time, a dual-complex quantum system is described by an ϵ -state.

Postulate 2. An operator U represents a valid ϵ -evolution if and only if U is unitary. The evolution of a closed ϵ -quantum system is described by an ϵ -evolution.

Postulate 3. A set of operators $\{M_m\}$ is an ϵ -measurement operator collection if it is bound by the completeness relation. The evolution of an open ϵ -quantum system is described by such a collection. The outcome m appears with the dual probability $p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$. The resulting ϵ -state is

$$|\psi'\rangle = \frac{M_m |\psi\rangle}{\sqrt{p(m)}} \quad (52)$$

Postulate 4. Given two ϵ -states representing two systems, the composite system is represented by the tensor product of these two ϵ -states.

State space

Postulate 1 aims at capturing the notion of the extension of a state vector to the dual-complex domain. The goal being to take a linear approximation of a system according to a certain parameter, we start at time zero with a usual complex unit vector that we evolve according to postulates 2 and 3, and so our states will always keep a norm equal to 1.

If we choose to start from a dual-complex vector, since the infinitesimal part of the vector represents the derivative of the significant part according to some parameter, equation ?? tells us that the norm is one if and only if the derivation operator is skew-Hermitian.

Evolution

This postulate is motivated by the fact that given an analytic path of unitary $U(z)$, its dual-complex extension will be unitary (which is easily seen from the fact that the derivative $U(z)$ is of the form $iU(z)H(z)$ where $H(z)$ is Hermitian). ϵ -evolutions represent therefore a first-order approximation of traditional unitary evolutions.

In traditional quantum mechanics, this postulate is equivalently expressed through the Schrödinger equation. From proposition ??, we can restate our extension of postulate 2 as

Postulate 2’. The evolution of a closed system is described by the Schrödinger equation,

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle \quad (53)$$

Where $|\psi\rangle$ is an ϵ -state, H a dual-complex Hermitian linear operator and \hbar is the Planck’s constant.

Measurement

Similar to postulate 2, we extend general measurements with dual-complex operators by taking a collection of analytic measurement operators and extend them into \mathbb{DC} . It can easily be seen that since $\sum_m M_m^\dagger M_m = I$, their derivatives \dot{M}_m is such that $\sum_m M_m^\dagger \dot{M}_m$ is skew-Hermitian. It can also be seen that for an ϵ -quantum measurement operators collection $\{M_m + \epsilon N_m\}$, the completeness relation $\sum_m (M_m + \epsilon N_m)^\dagger (M_m + \epsilon N_m)$ can be restated as

$$(i) \sum_m M_m^\dagger M_m = I \quad (54)$$

$$(ii) \sum_m M_m^\dagger N_m \text{ is skew-Hermitian.} \quad (55)$$

The traditional postulate for measurements also defines what are the resulting states and with which probability. To ensure our extension doesn't create unwanted behaviors, we must check that probabilities are positive and sum to one. We must also check that the resulting state is well defined as it is renormalized by a probability that could maybe be infinitesimal. Fortunately, such behaviors do not arise, as we show in the next section (??).

In particular, as in traditional quantum theory, we can see that when we have exactly one measurement operator, it is a unitary, which means that the extended postulate 2 can be reduced to the postulate 3.

Composite systems

Composing systems is done in the traditional way, using tensor products. Again the postulate preserves the unit norm. If we have two ϵ -states $|\psi\rangle$ and $|\phi\rangle$, their norm will be $\sqrt{\langle\psi|\psi\rangle\langle\phi|\phi\rangle} = 1$.

Density operators formalism

We use the usual notion of density operators, which is easily seen to be compatible with the extended postulates. An ϵ -density operator ρ is a dual-complex, unit-trace, positive operator. As in the traditional case, applying a unitary U to ρ yields $U\rho U^\dagger$ and measuring ρ with $\{M_m\}$ yields $\sum_m M_m \rho M_m^\dagger$.

5 Consistency of the extended postulates

In this section, we show that the evolution of a system preserves the norm of the state vector (self-consistency) and that our postulates provide a first-order approximation of quantum systems.

Self-consistency

Mettre un théorème qui dit que les ϵ -states sont stables sous les opérations physiques admises par ϵ -evolution et ϵ -measurement. Probabilités somment à 1 et sont positives.

Proposition 16. *A system that starts from an ϵ -states and is evolved by ϵ -evolutions and ϵ -measurements is described by an ϵ -state at all time, i.e. if there's a sequence $|\psi_0\rangle, |\psi_1\rangle, \dots, |\psi_k\rangle$ such that $|\psi_0\rangle$ is an ϵ -state and $|\psi_{j+1}\rangle$ was obtained either by applying an ϵ -unitary or an ϵ -measurement to $|\psi_j\rangle$, every $|\psi_j\rangle$ is an ϵ -state. Moreover, probabilities from any ϵ -measurement are either zero or appreciably positive and sum to one.*

Proof. By proposition ??, a dual-complex unitary preserves the norm. For measurements, from ?? norm preservation is given by

$$\left\| \frac{M_m |\psi\rangle}{\sqrt{p(m)}} \right\| = \left\| \frac{M_m |\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \right\| = \frac{\|M_m |\psi\rangle\|}{\|M_m \psi\|} = 1. \quad (56)$$

Probabilities sum to one since $\sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle = \langle\psi|(\sum_m M_m^\dagger M_m)|\psi\rangle = \langle\psi|I|\psi\rangle = 1$ and are either zero or appreciably positive by proposition ??. \square

Consistency with traditional quantum theory

paramétrer les unitaires avec des parenthèses

Écrire la réciproque + rajouter des définitions pour tout faire dans la preuve de la proposition

Definition 17. *We introduce the dual-complex extension of a quantum operation and the correction of dual-complex quantum operation.*

- Let $\mathcal{E}_z(\rho)$ be a quantum operation analytic in z whose operator-sum representation is, for some basis $|e_k\rangle$ and unitary operator U_z ,

$$\mathcal{E}_z(\rho) = \text{tr}_B(U_z[\rho \otimes |e_0\rangle\langle e_0|]U_z^\dagger) \quad (57)$$

Let $\dot{\mathcal{E}}_z = \frac{d\mathcal{E}_z}{dz}$. We call \mathcal{E} 's dual-complex extension the dual-complex operation

$$\mathcal{F}_{z+\dot{z}\epsilon}(\rho) = \mathcal{E}_z(\rho) + \dot{z}\epsilon\dot{\mathcal{E}}_z(\rho) \quad (58)$$

- Let $\mathcal{F}_w(\rho)$ be a dual-complex extension of a quantum operation $\mathcal{E}_z(\rho)$ as in ?? where $w = z + \dot{z}\epsilon$. Then \mathcal{F}_w has the form

$$\mathcal{F}_w(\rho) = \text{tr}_B(V_w[\rho \otimes |e_0\rangle\langle e_0|]V_w^\dagger) \quad (59)$$

where $V_w = U_z + \epsilon\dot{z}\dot{U}_z$, $\dot{U}_z = \frac{dU_z}{dz} = iU_zH$ and H is Hermitian. $\tilde{V}_w(h)$ is the complex correction of V_w with parameter $h \in \mathbb{C}$, defined as

$$\tilde{V}_w(h) = U_z + i\dot{z}hU_zH + h^2 \sum_{n=0}^{\infty} \frac{1}{n!} (iH)^{n+2} h^n \quad (60)$$

$\tilde{\mathcal{F}}_w(\rho, h)$ is the complex correction of $\mathcal{F}_w(\rho, h)$ with parameter $h \in \mathbb{C}$, defined as

$$\tilde{F}_w(\rho, h) = \text{tr}_B(\tilde{V}_w(h)[\rho \otimes |e_0\rangle\langle e_0|]\tilde{V}_w(h)^\dagger) \quad (61)$$

Proposition 18. Let $\mathcal{E}_z(\rho)$ and $\mathcal{F}_w(\rho)$ be a quantum operation and its dual-complex extension. For any parameter $h \in \mathbb{C}$, the correction $\tilde{F}_w(\rho, h)$ is a valid quantum operation and $\tilde{F}_{z+\dot{z}\epsilon}(\rho, h) = \mathcal{E}_z(\rho) + O(h^2)$.

Proof. First, observe that $\tilde{V}_w(h)$ is a unitary

$$\tilde{V}_w(h) = U_z + i\dot{z}hU_zH + h^2 \sum_{n=0}^{\infty} \frac{1}{n!} (iH)^{n+2} h^n = U_z \left(\sum_{n=0}^{\infty} \frac{1}{n!} (iHh)^n \right) = U_z \exp(iHh). \quad (62)$$

It immediately follows that ?? is a quantum operation. For the second claim, we can see that

$$\begin{aligned} \mathcal{E}_{z+\dot{z}h}(\rho) &= \mathcal{E}_z(\rho) + \dot{z}h\dot{\mathcal{E}}_z(\rho) + O(h^2) \\ \tilde{F}_{z+\dot{z}\epsilon}(\rho, h) &= \text{tr}_B \left(\tilde{V}_{z+\dot{z}\epsilon}(h) [\rho \otimes |e_0\rangle\langle e_0|] \tilde{V}_{z+\dot{z}\epsilon}(h)^\dagger \right) \\ &= \text{tr}_B \left(\left(U_z + \dot{z}h\dot{U}_z + O(h^2) \right) [\rho \otimes |e_0\rangle\langle e_0|] \left(U_z + \dot{z}h\dot{U}_z + O(h^2) \right)^\dagger \right) \\ &= \text{tr}_B \left(U_z [\rho \otimes |e_0\rangle\langle e_0|] U_z^\dagger \right) \\ &\quad + \text{tr}_B \left(U_z [\rho \otimes |e_0\rangle\langle e_0|] (\dot{z}h\dot{U}_z)^\dagger + (\dot{z}h\dot{U}_z) [\rho \otimes |e_0\rangle\langle e_0|] U_z^\dagger \right) + O(h^2) \\ &= \mathcal{E}_z(\rho) + \dot{z}h\dot{\mathcal{E}}_z(\rho) + O(h^2) \end{aligned}$$

□

6 Application example: Quantum Cellular Automata

Rappeler le contexte

Ajouter la figure

In this section, we introduce an example of analytic quantum systems from ?, present their dual-complex extension and go back to traditional quantum systems.

The Dirac QCA is

The Quantum Cellular Automata is defined as in fig 1. All gates are identical and defined by

$$W_h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -is & c & 0 \\ 0 & c & -is & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (64)$$

where $s = \sin(mh)$ and $c = \cos(mh)$. W_h is approximated by

$$W_\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -im\epsilon & 1 & 0 \\ 0 & 1 & -im\epsilon & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (65)$$

The associated gate is

$$W'_h = e^h \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-im} & 0 \\ 0 & e^{-im} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (66)$$

Montrer la limite, faire la limite vers l'équation de Dirac grâce à l'AD

Montrer que c'est strictement Lorentz covariant au sens de Lorentz discrete covariance du papier + faire le lien entre manière discrète de montrer que l'équation continue est covariante (au sens continu).

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