assignment2(vision)

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1 discretization

$E[u_c] = \sum_{i,j} \Omega[i,j](u_c[i,j] - g_c[i,j])^2 + \sum_{i,j} \lambda|\nabla u_c[i,j]|^2$

Using forward difference:

$|\nabla u_c[i,j]| = \sqrt{(\nabla u_{cx}[i,j]^2 + \nabla u_{cy}[i,j]^2))} = \sqrt{(u_c[i+1,j] - u_c[i,j])^2 + (u_c[i,j+1] - u_c[i,j])^2)}$

Thus:

$|\nabla u_c[i,j]|^2 = (u_x[i,j]^2 + u_y[i,j]^2) = (u_c[i+1,j] - u_c[i,j])^2 + (u_c[i,j+1] - u_c[i,j])^2$

Finally:

$$E_c[i,j] = \sum_{i,j=1}^{N} \Omega[i,j](u_c[i,j] - g_c[i,j])2 \qquad (1)$$

$$+ \sum_{i,j=1}^{N-1} \lambda((u_c[i+1,j] - u_c[i,j])^2 + (u_c[i,j+1] - u_c[i,j])^2) \qquad (2)$$

(we have used a forward difference in the second term. Thus, the upper bound should be $N-1$ to compensate for terms that include $i+1$ and $j+1$)

2 Gradient calculation

when taking the derivative of $E_c[i,j]$,the presence of the second part above (2) requires us to carry out this calculation for different states including $\forall i,j \in [2, N-1]$ and boundaries. First, we write the derivative of the regularization term by introducing τ.

$$\frac{\partial E_C[i,j]}{\partial u[i,j]} = 2\Omega[i,j](u_c[i,j] - g_c[i,j]) + \frac{\partial ||\nabla u||^2}{\partial u[i,j]}$$

we can use this equation:

$$\frac{\partial||\nabla u||^2}{\partial u[i,j]} = \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i-1,j]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]}$$

where:

$$\tau[i,j] = (u_c[i+1,j] - u_c[i,j])^2 + (u_c[i,j+1] - u_c[i,j])^2$$

$$\frac{\partial \tau[i,j]}{\partial u[i,j]} = -2(u_c[i+1,j] - u_c[i,j]) - 2(u_c[i,j+1] - u_c[i,j])$$
$$= 4u_c[i,j] - 2(u_c[i+1,j] + u_c[i,j+1])$$

$$\tau[i-1,j] = (u_c[i,j] - u_c[i-1,j])^2 + (u_c[i-1,j+1] - u_c[i-1,j])^2$$

$$\frac{\partial \tau[i-1,j]}{\partial u[i,j]} = 2(u_c[i,j] - u_c[i-1,j])$$

$$\tau[i,j-1] = (u_c[i+1,j-1] - u_c[i,j-1])^2 + (u_c[i,j] - u_c[i,j-1])^2$$

$$\frac{\partial \tau[i,j-1]}{\partial u[i,j]} = 2(u_c[i,j] - u_c[i,j-1])$$

Therefore:

$$\forall i,j \in [2, N-1]:$$

$$\nabla E[i,j] = 2\Omega[i,j](u_c[i,j] - g_c[i,j]) + \frac{\partial \tau[i,j]}{\partial u[i,j]} + \frac{\partial \tau[i-1,j]}{\partial u[i,j]} + \frac{\partial \tau[i,j-1]}{\partial u[i,j]}$$

Replacing the derivatives of τ :

$$\nabla E[i,j] = 2\Omega[i,j](u_c[i,j] - g_c[i,j])$$
$$-\,2(u_c[i+1,j] - u_c[i,j]) - 2(u_c[i,j+1] - u_c[i,j])$$
$$+\,2(u_c[i,j] - u_c[i-1,j]) + 2(u_c[i,j] - u_c[i,j-1])$$

$boundaries : on boundaries we replace the$ i,j $in the \nabla E$ using the three expressions we made with τ.
It is also worth mentioning that we remove those expressions that have N+1 or 0 in their indices.

i,j=1

$$\nabla E_[1,1] = -2(u[2,1] - u[1,1]) - 2(u[1,2] - u[1,1]) + 2\Omega(u_c[1,1] - g_c[1,1])$$

i,j=N

$$\nabla E[N,N] = 2(u_c[N,N] - u_c[N-1,N]) + 2(u_c[N,N] - u_c[N,N-1])$$
$$+\,2\Omega(u_c[N,N] - g_c[N,N])$$

i=1 , j $\in [2, N-1]$

$$\nabla E[1,j] = -2(u_c[2,j] - u_c[1,j]) - 2(u_c[1,j+1] - u_c[1,j]) + 2(u_c[1,j] - u_c[1,j-1]) + 2\Omega(u_c[1,j] - g_c[1,j])$$

$i = N, j \in [2, N-1]$

$$\nabla E[N,j] = -2(u_c[N,j+1] - u_c[N,j]) + 2(u_c[N,j] - u_c[N-1,j])$$
$$+\,2(u_c[N,j] - u_c[N,j-1]) + 2\Omega(u_c[N,j] - g_c[N,j])$$

i $\in [2, N-1], j = 1$

$$\nabla E[i,1] = -2(u_c[i+1,1-]u_c[i,1]) - 2(u_c[i,2] - u_c[i,1])$$
$$+\,2(u_c[i,1] - u_c[i-1,1]) + 2\Omega(u_c[i,1] - g_c[i,1])$$

i $\in [1, N-1], j = N$

$$\nabla E[i,N] = -2(u_c[i+1,N] - u_c[i,N]) - 2(u_c[i,N] - u_c[i-1,N])$$
$$+\,2(u_c[i,N] - u_c[i,N-1]) + 2\Omega(u_c[i,N] - g_c[i,N])$$

i=1,j=N

$$\nabla E[1,N] = -2(u_c[2,N] - u_c[1,N])$$
$$+\,2(u_c[1,N] - u_c[1,N-1]) + 2\Omega(u_c[1,N] - g_c[1,N])$$

$i = N, j = 1$

$$\nabla E[N,1] = -2(u_c[N,2] - u_c[N,1]) \\ + 2(u_c[N,1] - u_c[N-1,1]) + 2\Omega(u_c[N,1] - g_c[N,1])$$

we repeat the above steps for the two other channels as their u and g values are different and result in different gradient value. we do the dradient descent steps for ecach channel seperately to update the u values and finaly stack the results to build the final three-channel inpainted image.