Non-stationary Poisson processes

In this section, we consider Poisson processes with a rate function $\lambda = \lambda(t)$, which varies with time. Such processes are called ***non-stationary*** or ***non-homogeneous*** Poisson processes. If $\{N(t): t \geq 0\}$ is a Poisson process with a rate function $\lambda(t)$, then an increment, $N(t) - N(s)$, giving the number of events in the time interval $(s, t]$, has a Poisson distribution with parameter $\mu = \int_s^t \lambda(u)\,du$ and increments over disjoint intervals are independent random variables.

- For example, during rush hours, the arrivals/departures of vehicles into/out of Manhattan is at a higher rate than at (say) 2:00AM.
- To accommodate this, we can allow the rate $\lambda = \lambda(t)$ to be a deterministic function of time t.
- For example,
  - consider time in hours and suppose $\lambda(t) = 100$ per hour except during the time interval (morning rush hour) $(8, 9)$
  - when $\lambda(t) = 200$,
  - that is $\lambda(t) = 200,\ t \in (8, 9),\ \lambda(t) = 100,\ t \notin (8, 9)$

In such a case, for a given rate function $\lambda(s)$, the expected number of arrivals by time t is thus given by

$$m(t) \stackrel{\text{def}}{=} E(N(t)) = \int_0^t \lambda(s)ds. \tag{1}$$

For a compound such process such as buses arriving: If independently each bus holds a random number of passengers (generically denoted by B) with some probability mass function $P(k) = P(B = k),\ k \geq 0$, and mean $E(B)$. Letting $B_1, B_2, \ldots$ denote the iid sequential bus sizes, the number of passengers to arrive by time t, $X(t)$ is given by

$$X(t) = \sum_{n=1}^{N(t)} B_n, \tag{2}$$

where $N(t)$ is the counting process for the non-stationary Poisson process; $N(t) =$ the number of buses to arrive by time t. This is known as a *compound* or *batch* non-stationary Poisson arrival process. We have $E(X(t)) = E(N(t))E(B) = m(t)E(B)$.

We have already learned how to simulate a stationary Poisson process up to any desired time t, and next we will learn how to do so for a non-stationary Poisson process.

1.1 The non-stationary case: Thinning

In general the function $\lambda(t)$ is called the *intensity* of the Poisson process, and the following holds:

For each $t > 0$, the counting random variable $N(t)$ is Poisson distributed with mean

$$m(t) = \int_0^t \lambda(s)ds.$$

$$\begin{aligned} E(N(t)) &= m(t) \\ P(N(t) = k) &= e^{-m(t)} \frac{m(t)^k}{k!},\ k \geq 0. \end{aligned}$$

More generally, the increment $N(t+h) - N(t)$ has a Poisson distribution with mean $m(t+h) - m(t) = \int_t^{t+h} \lambda(s)ds$.

Example 13

Demands on a first aid facility in a certain location occur according to a non-stationary Poisson process with the following rate function:

$$\lambda(t) = \begin{cases} 3t, & 0 \le t \le 1, \\ 3, & 1 \le t \le 2, \\ 5-t, & 2 \le t \le 5, \end{cases}$$

where t is measured in hours from the opening time of the facility. Find the probability that:

(a) two demands occur in the first hour of operation,

(b) two demands occur in the second hour of operation,

(c) one demand occurs in the last three hours of operation.

Solution

(a) The mean for the first hour of operation is $\mu = \int_0^1 3t\,dt = 1\frac{1}{2}$.

Hence, $P\{(N(1) - N(0)) = 2\} = \dfrac{(3/2)^2 e^{-3/2}}{2!} = 0.251.$

(b) The mean for the second hour of operation is 3. It follows that the number of events in the time interval $(1, 2]$ has the Poisson distribution with mean 3. Therefore,

$$P\{(N(2) - N(1)) = 2\} = \frac{3^2 e^{-2}}{2!} = 0.609.$$

(c) The mean for the last three hours of operation is

$$\mu = \int_2^5 (5-t)dt = \left[5t - \frac{1}{2}t^2\right]_2^5 = 4.5.$$

Therefore,

$$P\{(N(5) - N(2)) = 1\} = \frac{4.5e^{-4.5}}{1!} = 0.05.$$

Example 14

In Example 13, find the probability that two demands occur in the first two hours of operation.

Solution

The mean for the first two hours of operation is

$$\mu = \int_0^1 3t dt + \int_1^2 3dt = 4.5.$$

Therefore, $N(2)-N(0)$ has the Poisson distribution with mean 4.5. Thus,

$$P\{N(2) - N(0) = 2\} = \frac{(4.5)^2 e^{-4.5}}{2!} = 0.1125.$$

Example 15

Demands on a first aid facility in a certain location occur according to a non-homogeneous Poisson process with the following rate function

$$\lambda(t)=\begin{cases}2t, & 0\le t\le 1,\\ 2, & 1\le t\le 2,\\ 4-t, & 2\le t\le 4,\end{cases}$$

where t is measured in hours from the opening time of the facility. Find the probability that two demands occur

(a) in the first two hours of operation,

(b) in the second two hours of operation.

Solution

(a) We wish to find $P\{N(2)-N(0)=2\}$. The mean for the first hour of operation is

$$\mu = \int_0^1 2t\,dt = \left[t^2\right]_0^1 = 1.$$

The mean for the second hour of operation is 2. Thus, the mean for the first two hours of operation is $1+2=3$. $N(2)-N(0)$ has the Poisson distribution with mean 3. Thus,

$$P\{N(2)-N(0)=2\} = \tfrac{(3)^2 e^{-3}}{2!} = 0.2240.$$

(b) We wish to find $P\{N(4)-N(2)=2\}$

The mean for the second 2 hours of operation is

$$\mu = \int_2^4 (4-t)\,dt = \left[4t-\tfrac{1}{2}t^2\right]_2^4 = 2.$$

$N(4)-N(2)$ has the Poisson distribution with mean 2. Thus,

$$P\{N(4)-N(2)=2\} = \frac{2^2 e^{-2}}{2!} = 0.2707.$$