Poisson distribution

If events happen independently of each other, with average number of events in some fixed interval λ, then the distribution of the number of events k in that interval is Poisson.

A random variable X has the Poisson distribution with parameter $\lambda(> 0)$ if

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad (k = 0,1,2, \dots)$$

If you are interested in the derivation, see the notes.

Examples of possible Poisson distributions

1) Number of messages arriving at a telecommunications system in a day

2) Number of flaws in a metre of fibre optic cable

3) Number of radio-active particles detected in a given time

4) Number of photons arriving at a CCD pixel in some exposure time (e.g. astronomy observations)

Sum of Poisson variables

If X is Poisson with average number λ_{X} and Y is Poisson with average number λ_Y

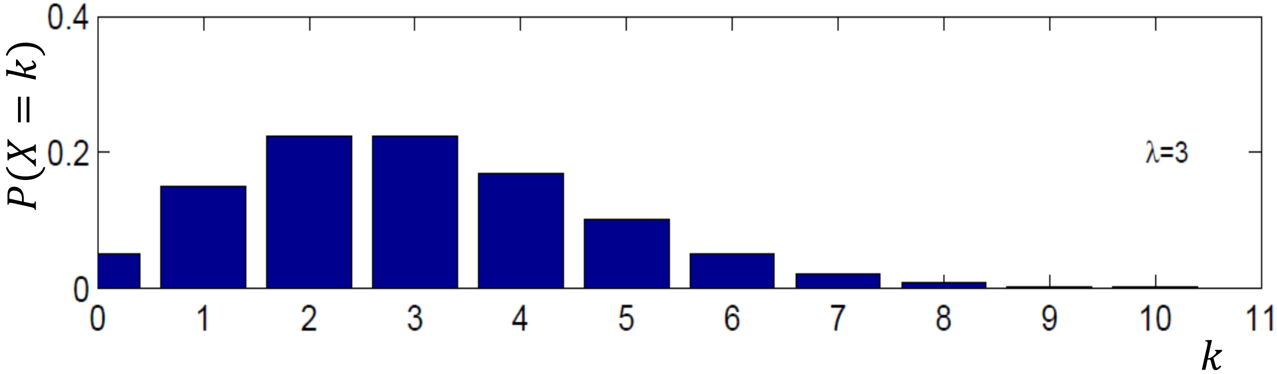
Then $X + Y$ is Poisson with average number $\lambda_X + \lambda_Y$

The probability of events per unit time does not have to be constant for the total number of events to be Poisson – can split up the total into a sum of the number of events in smaller intervals.

Example: On average lightning kills three people each year in the UK, $\lambda = 3$. What is the probability that only one person is killed this year?

Answer:

Assuming these are independent random events, the number of people killed in a given year therefore has a Poisson distribution:



Let the random variable X be the number of people killed in a year.

Poisson distribution $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$ with $\lambda = 3$

$$\Rightarrow P(X = 1) = \frac{e^{-3}3^1}{1!} \approx 0.15$$

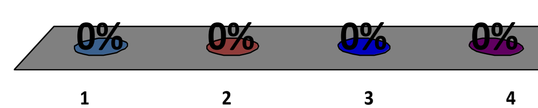
Poisson distribution

Suppose that trucks arrive at a receiving dock with an average arrival rate of 3 per hour. What is the probability exactly 5 trucks will arrive in a two-hour period?

Question from Derek Bruff

1. $\dfrac{e^{-3}3^5}{5!}$
2. $\dfrac{e^{-3}3^{2.5}}{2.5!}$
3. $\dfrac{e^{-5}5^6}{6!}$
4. $\dfrac{e^{-6}6^5}{5!}$

Reminder: $P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$



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Countdown

Poisson distribution

Suppose that trucks arrive at a receiving dock with an average arrival rate of 3 per hour. What is the probability exactly 5 trucks will arrive in a two-hour period?

In two hours mean number is $\lambda = 2 \times 3 = 6$.

$$P(X = k = 5) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-6}6^5}{5!}$$

***Example:* Telecommunications**

Messages arrive at a switching centre at random and at an average rate of 1.2 per second.

(a) Find the probability of 5 messages arriving in a 2-sec interval.

(b) For how long can the operation of the centre be interrupted, if the probability of losing one or more messages is to be no more than 0.05?

Answer:

Times of arrivals form a Poisson process, rate $\nu = 1.2/\text{sec}$.

(a) Let *Y* = number of messages arriving in a 2-sec interval.

Then *Y* ~ Poisson, mean number $\lambda = \nu t = 1.2 \times 2 = 2.4$

$$P(Y = k = 5) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-2.4}2.4^5}{5!} = 0.060$$

Question: (b) For how long can the operation of the centre be interrupted, if the probability of losing one or more messages is to be no more than 0.05?

Answer:

(b) Let the required time = t seconds. Average rate of arrival is 1.2/second.

Let k = number of messages in t seconds, so that

$$k \sim \text{Poisson, with } \lambda = 1.2 \times t = 1.2t$$

Want P(At least one message) = $P(k \geq 1) = 1 - P(k = 0) \leq 0.05$

$$P(k = 0) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-1.2t}(1.2t)^0}{0!} = e^{-1.2t}$$

$$\Rightarrow 1 - e^{-1.2t} \leq 0.05$$

$$\Rightarrow \quad -e^{-1.2t} \leq 0.05 - 1$$

$$\Rightarrow e^{-1.2t} \geq 0.95$$

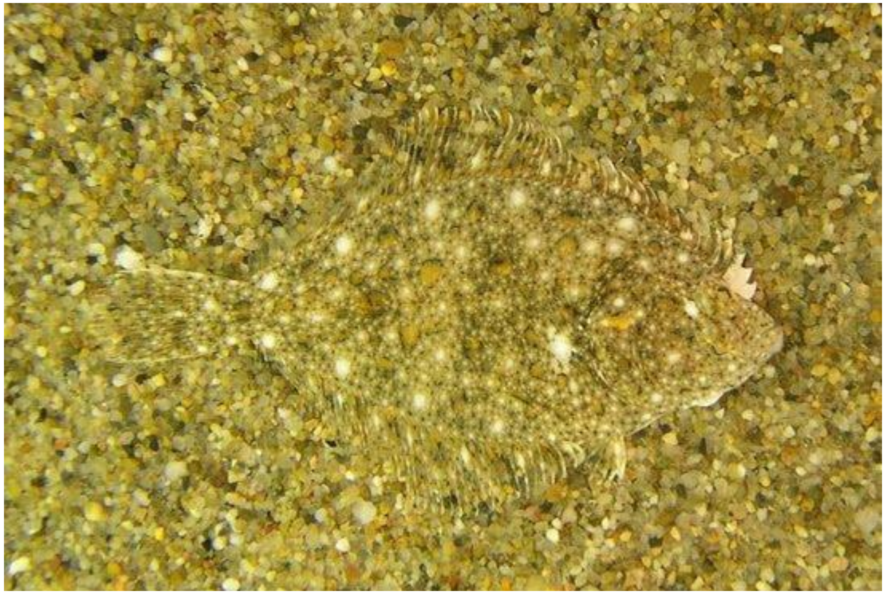
$$\Rightarrow -1.2t \geq \ln(0.95) = -0.05129$$

$$\boxed{\Rightarrow t \leq 0.043 \text{ seconds}}$$

Poisson or not?

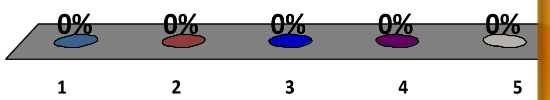
Which of the following are likely to be well modelled by a Poisson distribution?

(can click more than one)



Can you tell what is fish, or will you flounder?

1. Number of duds found when I test four components
2. The number of heart attacks in Brighton each year
3. The number of planes landing at Heathrow between 8 and 9am
4. The number of cars getting punctures on the M1 each year
5. Number of people in the UK flooded out of their home in July



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Countdown

Are they Poisson? *Answers*:

Number of duds found when I test four components

- NO: this is Binomial
(it is not the number of independent random events in a continuous interval)

The number of heart attacks in Brighton each year

- YES: large population, no obvious correlations between heart attacks in different people

The number of planes landing at Heathrow between 8 and 9am

- NO: 8-9am is rush hour, planes land **regularly** to land as many as possible (1-2 a minute) – they do not land at random times or they would hit each other!

The number of cars getting punctures on the M1 each year

- YES (roughly): If punctures are due to tires randomly wearing thin, then expect punctures to happen independently at random
  But: may not all be independent, e.g. if there is broken glass in one lane

Number of people in the UK flooded out of their home in July

- NO: floodings of different homes not at all independent; usually a small number of floods each flood many homes at once, $P(\text{flooded}|\text{next door flooded}) \gg P(\text{flooded})$

Poisson Distribution Summary

Describes discrete random variable that is the number of *independent* and *randomly occurring* events, with mean number λ. Probability of k such events is

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Mean and variance: $\mu = \sigma^2 = \lambda$

The sum of Poisson variables $\sum X_i$ is also Poisson, with average number $\sum_i \lambda_i$

Approximation to Binomial for large n and small p:

if $X \sim B(n, p)$ then $P(X = k) \approx \frac{e^{-\lambda}\lambda^k}{k!}$ where $\lambda = np$