Chapter 2
Queuing system and Markov Chains

1. Queuing system: Introduction

Most systems of interest in a simulation study contain a process in which there is a demand for services. The system can service entities at a rate which is greater than the rate at which entities arrives. The entities are then said to join waiting line. The line where the entities or customers wait is generally known as queue. The combination of all entities in system being served and being waiting for services will be called a queuing system. The general diagram of queuing system can be shown as a queuing system involves customers arriving at a constant or variable time rate for service at a service station. Customers can be students waiting for registration in college, airplane queuing for landing at airfield, or jobs waiting in machines shop. If the customer after arriving can enter the service center, it is good, otherwise they have to wait for the service and form a queue i.e. waiting line. They remain in queue till they are provided the service. Sometimes queue being too long, they will leave the queue and go, it results a loss of customer. Customers are to be serviced at a constant or variable rate before they leave the service station.

2. Characteristics or elements of queuing system

In order to model queuing systems, we first need to be a bit more precise about what constitutes a queuing system. The three basic elements common to all queuing systems are:

1. Arrival Process or patterns
2. Service process or patterns
3. Queuing discipline

a) Arrival Process or patterns

Any queuing system must work on something − customers, parts, patients, orders, etc. We generally called them as entities or customers. Before entities can be processed or subjected to waiting, they must first enter the system. Depending on the environment, entities can arrive smoothly or in an unpredictable fashion. They can arrive one at a time or in clumps (e.g., bus loads or batches). They can arrive independently or according to some kind of correlation. A special arrival process, which is highly useful for modeling purposes, is the Markov arrival process. Both of these names refer to the situation where entities arrive one at a time and the times between arrivals are exponential random variables. This type of arrival process is memoryless, which means that the likelihood of an arrival within the next t minutes is the same no matter how long it has been since the last arrival.

Examples where this occurs are phone calls arriving at an exchange, customers arriving at a fast food restaurant, hits on a web site, and many others.

b) Service Process

Once entities have entered the system they must be served. The physical meaning of "service" depends on the system. Customers may go through the checkout process. Parts may go through machining. Patients may go through medical treatment. Orders may be filled. And so on. From a modeling standpoint, the operational characteristics of service matter more than the physical characteristics. Specifically, we care about whether service times are long or short, and whether they are regular or highly variable. We care about whether entities are processed in first-come-first-serve (FCFS) order or according to some kind of priority rule. We care about whether entities are serviced by a single server or by multiple servers working in parallel etc.

Markov Service Process

A special service process is the Markov service process, in which entities are processed one at a time in FCFS order and service times are independent and exponential. As with the case of Markov arrivals, a Markov service process is memoryless, which means that the expected time until an entity is finished remains constant regardless of how long it has been in service. For example, in a Markov service process would imply that the additional time required resolving a caller's problem is 15 minutes, no matter how long the technician has already spent talking to the customer. While this may seem unlikely, it does occur when the distribution of service times looks like the case shown in Figure 1. This depicts a case where the average service time is 15 minutes, but many customers require calls much shorter than 15 minutes (e.g., to be reminded of a password or basic procedures) while a few customers require significantly more than 15 minutes (e.g., to perform complex diagnostics or problem resolution). Simply knowing how long a customer has been in service doesn't tell us enough about what kind of problem the customer has to predict how much more time will be required.

c) Queuing Discipline:

The third required component of a queuing system is a queue, in which entities wait for service. The number of customer can wait in a line is called system capacity. The simplest case is an unlimited queue which can accommodate any number of customers. It is called system with unlimited capacity. But many systems (e.g., phone exchanges, web servers, call centers), have limits on the number of entities that can be in queue at any given time.

Arrivals that come when the queue is full are rejected (e.g., customers get a busy signal when trying to dial into a call center). Even if the system doesn't have a strict limit on the queue size, the logical ordering of customer in a waiting line is called Queuing discipline and it determines which customer will be chosen for service. We may say that queuing discipline is a rule to choose the customer for service from the

waiting line.
The queuing discipline includes:

a) FIFO (First in First out): According to this rule, Service is offered on the basis of arrival time of customer. The customer who comes first will get the service first. So in other word the customer who get the service next will be determine on the basis of longest waiting time.

b) Last in First out (LIFO): It is usually abbreviated as LIFO, occurs when service is next offered to the customer that arrived recently or which have waiting time least. In the crowded train the passenger getting in or out from the train is an example of LIFO.

c) Service in Random order (SIRO): it means that a random choice is made between all waiting customers at the time service is offered i.e. a customer is picked up randomly forms the waiting queue for the service.

d) Shortest processing time First (SPT): it means that the customer with shortest service time will be chosen first for the service i.e. the shortest service time customer will get the priority in the selection process.

e) Priority: a special number is assigned to each customer in the waiting line and it is called priority. Then according to this number, the customer is chosen for service.

Queuing Behavior

Customers may balk at joining the queue when it is too long (e.g., cars pass up a drive through restaurant if there are too many cars already waiting). It is called balking. Customer may also exit the system due to impatience (e.g., customers kept waiting too long at a bank decide to leave without service) or perishability (e.g., samples waiting for testing at a lab spoil after some time period). It is called reneging. When there is more than one line forming for the same service or server, the action of moving customer from one line to another line because they think that they have chosen slow line. It is called Jockeying.

3) Queuing Notations (or KENDALL'S NOTATION)

We will be frequently using notation for queuing system, called Kendall's notation, i.e A/B/c/N/K, where, A, B, c, N, K respectively indicate arrival pattern, service pattern, number of servers, system capacity, and Calling population.

The symbols used for the probability distribution for inter arrival time, and service time are, D for deterministic, M for exponential (or Markov) and Ek for Erlang.

If the capacity Y is not specified, it is taken as infinity, and if calling population is not specified, it is assumed unlimited or infinite

Example

a) $M/D/2/5/\infty$ stands for a queuing system having exponential arrival times, deterministic service time, 2 servers, capacity of 5 customers, and infinite population.
b) If notation is given as M/D/2 means exponential arrival time, deterministic service time, 2 servers, infinite service capacity, and infinite population.

4) Single server queuing system

For the case of simplicity, we will assume for the time being, that there is single queue and only one server serving the customers. We make the following assumptions.

- **First-in, First-out (FIFO):** Service is provided on the first come, first served basis.
- **Random**: Arrivals of customers is completely random but at a certain arrival rate.
- **Steady state**: The queuing system is at a steady state condition.

The above conditions are very ideal conditions for any queuing system and assumptions are made to model the situation mathematically. First condition only means irrespective of customer, one who comes first is attended first and no priority is given to anyone.

5) Poison arrival Patterns

Second condition says that arrival of a customer is completely random. This means that an arrival can occur at any time and the time of next arrival is independent of the previous arrival. With this assumption it is possible to show that the distribution of the inter-arrival time is exponential. This is equivalent to saying that the number of arrivals per unit time is a random variable with a Poisson's distribution. This distribution is used when chances of occurrence of an event out of a large sample is small.

That is if X = number of arrivals per unit time, then, probability distribution function of arrival is given as

$$f(x) = \Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \begin{cases} x = 0, 1, 2, \ldots \\ \lambda > 0 \end{cases}$$

$$E(X) = \lambda$$

Where λ. is the average number of arrivals per unit time (1/ τ), E(X) is the expected number, and x is the number of customers per unit time. This pattern of arrival is called Poisson's arrival pattern. τ is inter arrival time.

Illustrative example

In a single pump service station, vehicles arrive for fueling with an average of 5 minutes between arrivals. If an hour is taken as unit of time, cars arrive according to Poison's process with an average of

λ= 12 cars/hr.

The distribution of the number of arrivals per hour is,

$$f(x) = \Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-12}12^x}{x!}, \begin{cases} x = 0, 1, 2, .. \\ \lambda > 0 \end{cases}$$

5) Measure of Queues

We have already defined the mean inter arrival time Ta and the mean service time Ts and the corresponding rates;

Arrival rate λ=1/Ta (T$_a$ id denoted by tou (τ))

Service rate μ=1/Ts

The following measures are used in the analysis of queue system

Traffic intensity

The ratio of the mean service time to the mean inter arrival time is called traffic intensity.

I.e. u= λ"Ts or u=Ts/Ta

If there is any balking or reneging, not all arriving entities get served. It is necessary therefore to distinguish between actual arrival rate and the arrival rate of entities that get served.

Here λ" denoted the all arrivals including balking or reneging.

Server utilization

It consists of only the arrival that gets served. It is denoted by and defined as

= λTs= λ/ μ (server utilization for single server).

This is also the average number of customers in the service facility.
Thus probability of finding service counter free is
$(1 - \rho)$
That is there are zero customers in the service facility.

6) Concept of Multi-server Queue

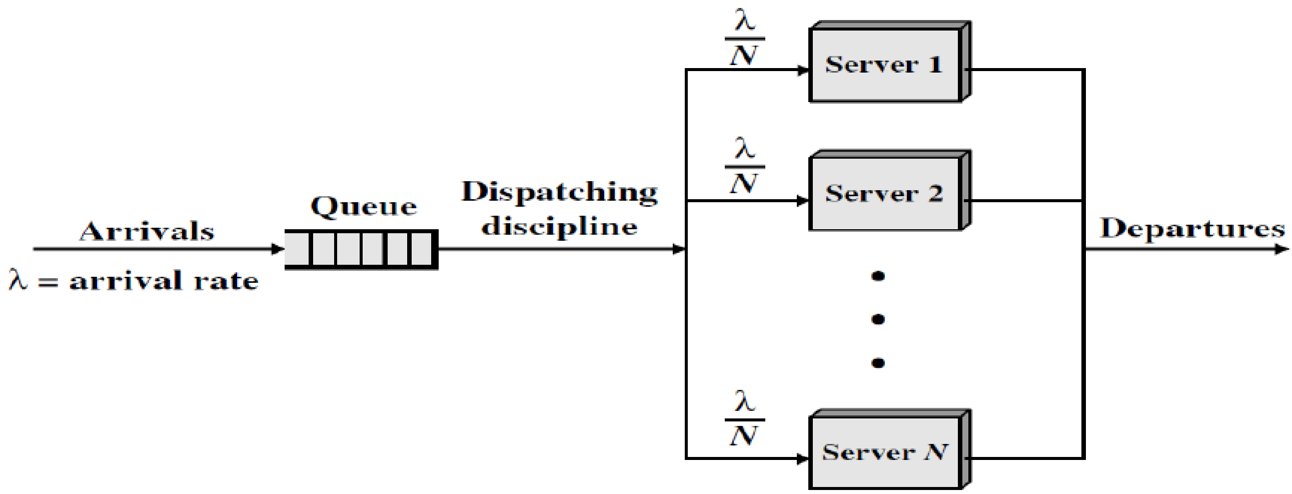
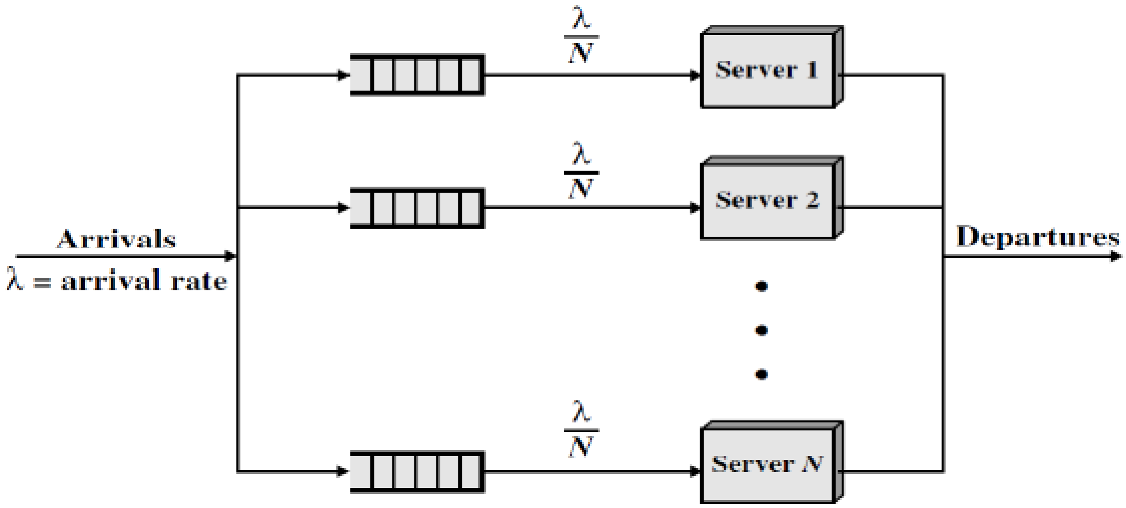


Figure shows a generalization of the simple model we have been discussing for multiple servers, all sharing a common queue. If an item arrives and at least one server is available, then the item is immediately dispatched to that server. It is assumed that all servers are identical; thus, if more than one server is available, it makes no difference which server is chosen for the item. If all servers are busy, a queue begins to form. As soon as one server becomes free, an item is dispatched from the queue using the dispatching discipline in force. The key characteristics typically chosen for the multi-server queue correspond to those for the single-server queue. That is, we assume an infinite population and an infinite queue size, with a single infinite queue shared among all servers. Unless otherwise stated, the dispatching discipline is FIFO. For the multi-server case, if all servers are assumed identical, the selection of a particular server for a waiting item has no effect on service time.

The total server utilization in case of Multi-server queue for N server system is

$$\rho = \lambda / c\mu$$

Where μ is the service rate and λ is the arrival rate.
There is another concept which is called multiple single server queue system as shown below



7) Some notation or Formula used to Measure the different parameter of queue

Two principal measures of queuing system are;

a) The mean number of customers waiting and

b) The mean time the customer spend waiting

Bothe these quantities may refer to the total number of entities in the system, those waiting and those being served or they may refer only to customer in the waiting line.

Average number of customers in the System $\overline{L_S} = \frac{\rho}{1-\rho} = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda}$

Average number of customers in the Queue $\overline{L_Q}$

= Average number of customers in the System − Server Utilization

$$= \overline{L_S} - \frac{\lambda}{\mu} = \frac{\lambda}{\mu-\lambda} - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

Average waiting time in the System $\overline{W_S} = \frac{Average\ number\ of\ customer\ in\ the\ system}{Mean\ arrival\ rate}$

$$= \frac{\overline{L_S}}{\lambda} = \frac{\frac{\lambda}{\mu-\lambda}}{\lambda} = \frac{1}{\mu-\lambda}$$

Average waiting time in the Queue $\overline{W_Q} = \frac{Average\ number\ of\ customer\ in\ the\ Queue}{Mean\ arrival\ rate}$

$$= \frac{\bar{L}_Q}{\lambda} = \frac{\frac{\lambda^2}{\mu(\mu-\lambda)}}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$$

Example

At the ticket counter of football stadium, people come in queue and purchase tickets. Arrival rate of customers is 1/min. It takes at the average 20 seconds to purchase the ticket.

(a) If a sport fan arrives 2 minutes before the game starts and if he takes exactly 1.5 minutes to reach the correct seat after he purchases a ticket, can the sport fan expects to be seated for the kick-off?

Solution:

(a) A minute is used as unit of time. Since ticket is disbursed in 20 seconds, this means, three customers enter the stadium per minute, that is service rate is 3 per minute.

Therefore,

λ = 1 arrival/min

μ = 3 arrivals/min

$\bar{W}_S$ = waiting time in the system=1/(μ- λ)=0.5 minutes

The average time to get the ticket plus the time to reach the correct seat is 2 minutes exactly, so the sports fan can expect to be seated for the kick-off.

Example2

Customers arrive in a bank according to a Poisson's process with mean inter arrival time of 10 minutes. Customers spend an average of 5 minutes on the single available counter, and leave.

(a) What is the probability that a customer will not have to wait at the counter?

(b)What is the expected number of customers in the bank?

(c) How much time can a customer expect to spend in the bank?

Solution:

We will take an hour as the unit of time. Thus,

λ = 6 customers/hour,

μ = 12 customers/hour.

The customer will not have to wait if there are no customers in the bank. Thus,

$P_0 = 1 - \lambda/\mu = 1 - 6/12 = 0.5$

Expected numbers of customers in the bank are given by

$\bar{L}_S = \lambda / (\mu - \lambda) = 6/6 = 1$

Expected time to be spent in the bank is given by

$\bar{W}_S = 1/(\mu - \lambda) = 1/(12-6) = 1/6$ hour = 10 minutes.

8) Markov Chains and its applications

a) Markov chains and Markov Process

Important classes of stochastic processes are Markov chains and Markov processes. A Markov chain is a discrete-time process for which the future behavior, given the past and the present, only depends on the present and not on the past. A Markov process is the continuous-time version of a Markov chain. Many queuing models are in fact Markov processes. This chapter gives a short introduction to Markov chains and Markov processes focusing on those characteristics that are needed for the modeling and analysis of queuing problems.

A Markov chain

A Markov chain, named after Andrey Markov, is a mathematical system that undergoes transitions from one state to another, between a finite or countable number of possible states. It is a random process characterized as memoryless: the next state depends only on the current state and not on the sequence of events that preceded it. This specific kind of "memorylessness" is called the Markov property. Markov chains have many applications as statistical models of real-world processes.

Formally Definition of Markov Chain

A Markov chain is a sequence of random variables X1, X2, X3, ... with the Markov property, namely that, given the present state, the future and past states are independent.

$$\Pr(X_{n+1} = x | X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \Pr(X_{n+1} = x | X_n = x_n)$$

Example; A simple whether model (Land of OZ Example)

The probabilities of weather conditions (modeled as either rainy or sunny), given the weather on the preceding day, can be represented by a transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$$

The matrix P represents the weather model in which a sunny day is 90% likely to be followed by another sunny day, and a rainy day is 50% likely to be followed by another rainy day. The columns can be labeled "sunny" and "rainy" respectively, and the rows can be labeled in the same order.

Notice that the rows of P sum to 1: This is because P is a stochastic matrix.

The weather on day 0 is known to be sunny. This is represented by a vector in which the "sunny" entry is 100%, and the "rainy" entry is 0%:

$$\mathbf{x}^{(0)} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The weather on day 1 can be predicted by:

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} P = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix}$$

The weather on day 2 can be predicted in the same way:

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} P = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.86 & 0.14 \end{bmatrix}$$

Or

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} P = \mathbf{x}^{(0)} P^2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}^2 = \begin{bmatrix} 0.86 & 0.14 \end{bmatrix}$$

In general

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} P = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.86 & 0.14 \end{bmatrix}$$

General rules for day *n* are:

$$\mathbf{x}^{(n)} = \mathbf{x}^{(n-1)} P$$
$$\mathbf{x}^{(n)} = \mathbf{x}^{(0)} P^n$$

b) Markov chain or process Applications

Physics

Markovian systems appear extensively in thermodynamics and statistical mechanics, whenever probabilities are used to represent unknown or unmodelled details of the system, if it can be assumed that the dynamics are time-invariant, and that no relevant history need be considered which is not already included in the state description.

Queuing theory

Markov chains are the basis for the analytical treatment of queues (queuing theory). Agner Krarup Erlang initiated the subject in 1917. This makes them critical for optimizing the performance of telecommunications networks, where messages must often compete for limited resources (such as bandwidth).

Internet applications
The Page Rank of a webpage as used by Google is defined by a Markov chain. It is the probability to be at page i in the stationary distribution on the following Markov chain on all (known) web pages

Statistics
Markov chain methods have also become very important for generating sequences of random numbers to accurately reflect very complicated desired probability distributions, via a process called Markov chain Monte Carlo (MCMC) And many more.

9) Differential and partial deferential equations

Continuous model
When continuous system is modeled mathematically, the variables of model representing the attribute of system are controlled by continuous functions. The distributed lag model is an example of a continuous model. Since in continuous system, the relationship between variables describe the rate at which the value of variable change, these system consist of differential equations.

Continuous system simulation uses the notation of differential equation to represent the change on the basic parameter of the system with respect to time. Hence the Mathematical model for continuous system simulation is usually represented by differential and partial differential equations.

Differential Equations
An example of a linear differential equation with constant coefficients to describe the wheel suspension system of an automobile can be given as

$$M\ddot{x} + Dx + K\dot{x} = KF($$

Here the dependent variable x appears together with first and second derivatives single dot and double dot respectively.
The simple differential equation can model the simplest continuous system and they can have one or more linear differential equation with constant coefficients. It is then often possible to solve the model without using simulation technique i.e. we can solve such equations using analytical methods as (we have done in Numerical methods)
However when non linearity involves into the model, it may be impossible or at least

very difficult to solve such model without simulation.

Partial Differential Equations

When more than one independent variable occurs in a differential equation the equation is said to be partial differential equations. It can involve the derivatives of the same dependent variable with respect to each of the independent variable.

Differential equations both linear and nonlinear occur frequently in scientific and engineering studies. The reason for this is that most physical and chemical process involves rates of change, which require differential equation to represent their mathematical descriptions. Since partial deferential equation can also represent a growth rate, continuous model can also be applied to the problems of a social or economic nature.

Analog Computer

Before the invention of digital computer, there existed devices whose behavior is equivalent of mathematical operation such as addition, subtraction, integration etc. Putting together these device in a manner specified by a mathematical model or equation of a system, allowed us to simulate the system. Some devices have been created for simulation continuous system and called analog computer or differential analyzer.

Digital analog simulators

To avoid the disadvantages of analog computers, many digital computer programming language have been written to produce digital-analog simulators. They allow or facilitate a continuous model to be programmed on a digital computer in essentially the same way as it is solved on analog computer. The language contains micro instructions that carry the action of addition, integration and sign changer. A program is written to link together these micro instructions in the same way as operational amplifiers are connected in analog computer. Since more powerful digital computer and programming language have been developed for this purpose of simulating continuous system on digital computer, the digital-analog simulators are now in extensive use.

Some important Questions

Long questions

1) What do you mean by Queuing system? Explain the characteristics of Queuing

system with example.
OR
Define the queuing system. Explain the elements of queuing system with example.
2) Explain about the Poison arrival process and Service process with example
Short questions
1) Define a Markov chains and its application
OR
What are the key features of Markov chains?
2) Explain about the server utilization and Traffic intensity.
3) What do you mean by multi server queues?
4) What are the Kendall notations of queuing system?
OR
5) Explain about the Queuing Discipline and behaviors.
6) Explain about the uses of differential equations in simulations.