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Data Driven Verification of Positive Invariant Sets for Discrete, Nonlinear Systems

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Abstract

Invariant sets are essential when establishing safety of nonlinear systems. However, certifying the existence of a positive invariant set for a nonlinear model is difficult and often requires knowledge of the system's dynamic model. This paper presents a data driven method to certify a positive invariant set for an unknown, discrete, nonlinear system. A triangulation of a subset of the state space is used to query data points. Then, a convex optimization problem is used to create a continuous piecewise affine (CPA) function that fulfills the criteria of the Extended Invariant Set Principle by leveraging an inequality error bound that uses the system's Lipschitz constant. Numerical results demonstrate the program's ability to certify positive invariant sets from sampled data.

Keywords: LaSalle's invariance principle, stability of nonlinear systems, invariant sets, data driven

1. Introduction

A set is invariant for a dynamical system if trajectories starting in the set remain there in perpetuity. Invariant sets are important for certifying the safety of linear and nonlinear systems, as feasible invariant sets are regions where constraints can be guaranteed to be obeyed.

Determining an invariant set for nonlinear systems is a longstanding challenge in control theory and has resulted in methods such as synthesizing Lyapunov functions of different function forms, modelling nonlinear systems as DC functions, and graph theory based methods (Blanchini (1999);Fiacchini et al. (2010);Decardi-Nelson and Liu (2021)). However, these methods often require a known dynamic model of the system, which may be difficult to determine for more complex, nonlinear systems. While robust invariant set synthesis methods guarantee invariance of a set when there are uncertainties in the dynamic model, it is appealing to further verify the existence of an invariant set through model-free methods that directly use data sampled from the system.

Recently, there has been a focus on directly verifying or synthesizing the invariant set of an unknown dynamical system from data (Wang and Jungers (2020);Korda (2020);Jin et al. (2023)). Direct verification and synthesis methods often rely on assumptions about certain characteristics of the dynamic system, such as Lipschitz continuity (Jin et al. (2023)) or being Borel measurable (Korda (2020); Wang and Jungers (2020)), to relate the sampled data to the entirety of the region of interest in the state space.

This paper presents a data driven method to certify the existence of a positive invariant set for an unknown discrete, nonlinear autonomous system using an extension of LaSalle's Invariant Set Principle (Alberto et al. (2007)). Similar to discrete Lyapunov methods, LaSalle's Principle requires the existence of a scalar function, V, over a region of the state space that obeys a non-increase condition ($V(\mathbf{x}^+) - V(\mathbf{x}) \leq 0$) over the positive invariant set. Even for a known, nonlinear system, the task of finding this function can be extremely difficult and may require leveraging specific aspects of the model to determine the function form.

Continuous piecewise affine (CPA) functions have previously been used to synthesize Lyapunov functions over a triangulated region of the state space for discrete and continuous systems (Giesl and Hafstein (2014a);Li et al. (2015);Hafstein (2018);Giesl and Hafstein (2014b)) and even synthesize control Lyapunov functions (CLFs) (Lavaei and Bridgeman (2023b)). The structure of a CPA function across a triangulation is appealing, as the CPA function is easily parameterized, because it affine across each simplex in the triangulation. The CPA function can then be represented by its value at each vertex point in the triangulation. Moreover, an error term has been developed that guarantees constraints enforced at the vertices are also applied across the simplicies of the triangulation (Giesl and Hafstein (2014a);Baier et al. (2012)) – making it possible to synthesize CPA functions that obey varied constraints. Even for nonlinear dynamics and non-convex constraints, this can be accomplished through a series of semidefinite programs (Lavaei and Bridgeman (2023a)).

In this paper, a CPA function over a triangulated region of an unknown system's state space is synthesized through a convex optimization problem and is used to confirm a positive invariant set from data samples. First, a triangulation over some region of the state space is used to create a data set of sampled pairs, $\{\mathbf{x}^+, \mathbf{x}\}$. Then, a convex optimization problem uses the data to synthesize a CPA function that obeys the Extended Invariance Principle across the invariant set. Rather than using a system model, this method only requires Lipschitz continuity of the unknown dynamic system. Therefore, it can be applied to linear and nonlinear discrete systems, so long as the system is Lipschitz continuous. Overall, this method provides a means to verify the existence of a known positive invariant set when only data is available from the dynamic system.

2. Preliminaries

The interior, boundary, and closure of the set $\Omega \subset \mathbb{R}^n$ are denoted as Ω^o, $\delta\Omega$, and $\bar{\Omega}$, respectively. The notation $\mathfrak{R}^n$ denotes the set of all compact subsets $\Omega \subset \mathbb{R}^n$ satisfying i) Ω^o is connected and contains the origin and ii) $\Omega = \bar{\Omega}^o$. Scalars, vectors, and matrices are denoted as x, $\mathbf{x}$, and $\mathbf{X}$, respectively. The notation $\mathbb{Z}_a^b$ ($\mathbb{Z}_{\bar{a}}^{\bar{b}}$) denotes the set of integers between a and b inclusive (exclusive).

2.1. Invariant Sets

Invariant sets are often used to confirm the safety of an autonomous dynamical system, as they can ensure that the entirety of a system's trajectory will remain in a subset of the state space. Positive invariant sets may also be defined as contractive invariant sets, where the safe set may contract by some factor, γ, when the dynamic system is applied to it.

Definition 1 *Positive Invariant Set (Alberto et al. (2007)) The set $\mathcal{S} \subseteq \Omega$ is positive invariant for mapping $T : \mathbb{R}^n \to \mathbb{R}^n$, if $T(\mathcal{S}) \subset \mathcal{S}$. For any $\mathbf{x}_0 \in \mathcal{S}$, $T^n(\mathbf{x}_0) \in \mathcal{S}$ for $n \geq 0$.*

LaSalle created a means with which to determine if a set is invariant using a scalar Lyapunov-like function for both continuous and discrete systems (LaSalle (1960);LaSalle (2012)). While originally the principle required that the Lyapunov-like function must maintain the non-increase condition for all states in the set, it has since been generalized to allow for regions fully enclosed in the invariant set to have increases of the Lyapunov-like function (Alberto et al. (2007)).

Theorem 2 *The Extended Invariance Principle (Alberto et al. (2007);Gabriel Filho (2004)) Consider the autonomous discrete dynamical system*

$$\mathbf{x}_{k+1} = T(\mathbf{x}_k), \quad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

and suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$ are continuous functions. Let $L \in \mathbb{R}$ be a constant such that $A_L := \{\mathbf{x} \in \mathbb{R}^n \mid V(\mathbf{x}) < L\}$ is bounded and define $\Delta V(\mathbf{x}_k) = V(T(\mathbf{x}_k)) - V(\mathbf{x}_k)$. Let $C := \{\mathbf{x} \in A_L \mid \Delta V(\mathbf{x}) > 0\}$, where $A_L \setminus C$ maintains $\Delta V(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in A_L \setminus C$. Suppose that $\sup_{\mathbf{x}\in C} V(\mathbf{x}) \leq l$ and $\sup_{\mathbf{x}\in C} \Delta V(\mathbf{x}) \leq s$. Suppose that $l + s < L$. Define $A_{l+s} := \{\mathbf{x} \in A_L \mid V(\mathbf{x}) \leq l + s\}$ and $E := \{\mathbf{x} \in \bar{A}_L \mid \Delta V(\mathbf{x}) = 0\} \cup A_{l+s}$. Let M be the largest invariant set of (1) contained in E. Then,

1. *A_L and A_{l+s} are positively invariant sets relative to (1);*
2. *solutions of (1) that enter A_{l+s} converge to the largest invariant set contained in A_{l+s};*
3. *every solution of (1) starting in A_L converges to the set M as $n \to \infty$.*

The key aspect of Theorem 2 is that A_L and A_{l+s} can enclose regions where $\Delta V > 0$. This is important for the proposed data-driven techniques because near an equilibrium point ΔV approaches zero, but the uncertainty from not having a model remains finite, making it impossible to verify $\Delta V < 0$ near the equilibrium without infinite data collection. Theorem 2 allows us to avoid this problem, as long as that equilibrium point is not on the boundary of the invariant set.

2.2. Continuous Piecewise Affine Functions

This paper uses a triangulation ($\mathcal{T}$) over a region of the dynamic system's state space to sample $\{\mathbf{x}^+, \mathbf{x}\}$ pairs. Then, a CPA function, defined on each vertex of the triangulation as $W_{\mathbf{x}}$, is determined for the triangulation using the convex optimization problem developed in this paper. In the optimization problem, constraints are formulated with an additional error bound for each vertex point of a simplex, so that the entirety of the simplex obeys the constraint, paralleling Giesl and Hafstein (2014a). The following section describes the necessary definitions and properties of the triangulation and the CPA function, so that this optimization problem can be developed.

Definition 3 *Affine independence (Giesl and Hafstein (2014b)): A collection of m vectors $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset \mathbb{R}^n$ is affinely independent if $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_m - \mathbf{x}_0$ are linearly independent.*

Definition 4 *n - simplex (Giesl and Hafstein (2014b)): A simplex, σ, is defined as the convex hull of $n+1$ affinely independent vectors, $co\{\mathbf{x}_j\}_{j=0}^n$, where each vector, $\mathbf{x}_j \in \mathbb{R}^n$, is a vertex.*

Definition 5 *Triangulation (Giesl and Hafstein (2014b)): Let $\mathcal{T} = \{\sigma_i\}_{i=1}^{m_\mathcal{T}} \in \mathfrak{R}^n$ represent a finite collection of $m_\mathcal{T}$ simplices, where the intersection of any two simplices is a face or an empty set.*

Let $\mathcal{T} = \{\sigma_i\}_{i=1}^{m_\mathcal{T}}$. Let $\{\mathbf{x}_{i,j}\}_{j=0}^n$ be the set of simplex σ_i's vertices, where $\sigma_i = \text{co}(\{\mathbf{x}_{i,j}\}_{j=0}^n)$. The vertices of the triangulation $\mathcal{T}$ of the set Ω is denoted by $\mathbb{E}_\Omega$. Likewise, all vertices of the triangulation $\mathcal{T}$ are denoted as $\mathbb{E}_\mathcal{T}$.

Lemma 6 *(Remark 9, Giesl and Hafstein (2014b)) Consider the triangulation $\mathcal{T} = \{\sigma_i\}_{i=1}^{m_\mathcal{T}}$, where $\sigma_i = co(\{\mathbf{x}_{i,j}\}_{j=0}^{n})$, and a set $\mathbf{W} = \{W_\mathbf{x}\}_{\mathbf{x}\in\mathbb{E}_\mathcal{T}} \subset \mathbb{R}$, where $W(\mathbf{x}) = W_\mathbf{x}$, $\forall \mathbf{x} \in \mathbb{E}_\mathcal{T}$. For simplex σ_i, let $\mathbf{X}_i \in \mathbb{R}^{n\times n}$ be a matrix that has $\mathbf{x}_{i,j} - \mathbf{x}_{i,0}$ as its j-th row and $\bar{W}_i \in \mathbb{R}^n$ be a vector that has $W_{\mathbf{x}_{i,j}} - W_{\mathbf{x}_{i,0}}$, as its j-th element. The function $W(\mathbf{x}) = \mathbf{x}^\top \mathbf{X}_i^{-1}\bar{W}_i$ is the unique CPA interpolation of $\mathbf{W}$ on $\mathcal{T}$ for $\mathbf{x} \in \sigma_i$.*

In Baier et al. (2012), the Lipschitz constant of a system can be used to bound the error between a function evaluated at the vertex of simplex, $\mathbf{x}_j \in \sigma$, versus the function evaluated at $\mathbf{x} \in \sigma$. This error bound is used in Sections 3.2 and 3.3 to develop constraints that ensure the non-increase condition is enforced over the entirety of a simplex.

Lemma 7 *Proposition 4.1 (Baier et al. (2012)) Let $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ be affinely independent vectors. Define $\sigma := co(\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k\})$, $c = \max_{\mathbf{x},\mathbf{y}\in\sigma} \|\mathbf{x} - \mathbf{y}\|$, and consider a convex combination $\sum_{j=0}^{n} \lambda_j \mathbf{x}_j \in \sigma$. If $g : \Omega \to \mathbb{R}$ is Lipschitz with constant L, i.e. $\|g(\mathbf{x}) - g(\mathbf{y})\| \leq L \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \Omega$, then*

$$\left| g\left(\sum_{j=0}^{n} \lambda_j \mathbf{x}_j\right) - \sum_{j=0}^{n} \lambda_j g(\mathbf{x}_j)\right| \leq Lc. \tag{2}$$

3. Main Results

This paper aims to use data samples generated from an unknown, Lipschitz continuous system to analyze whether or not a predefined set is control invariant. While the system model is unknown, it can produce N data pairs, $\{\mathbf{x}_n^+, \mathbf{x}_n\}_{n=0}^{N}$, where $\mathbf{x}_n$ is the initial state, and $\mathbf{x}_n^+$ is the state at the next time-step. The points $n = 0, \ldots, N$ need not be from the same trajectory. Instead, a triangulation of the relevant subspace is used to create query points that are sampled to create a data set.

In Section 3.2, conditions that are necessary for a CPA function to uphold Theorem 2 for a predefined set are derived. These conditions are expressed as constraints enforced at the vertex points of relevant simplices in the triangulation and depend on the system's Lipschitz constant. The Lipschitz constant of a system can be known a priori or can be found from the sampled data set using techniques available in the literature (Calliess (2017); Chakrabarty et al. (2020)). The conditions found in Section 3.2 are then used in Section 3.3 to create a convex optimization problem capable of synthesizing a CPA function that determines whether the predefined set is positive invariant.

3.1. Data Driven Triangulation

A typical CPA approach involves formulating a triangulation, $\mathcal{T} = \{\sigma_i\}_{i=1}^{m_\mathcal{T}}$, over some region of the dynamic system's state space, $\Omega \in \mathfrak{R}^n$. Then, the system model can the be applied to each vertex point in the triangulation to understand how the system evolves over the region of the state space. However, without knowledge of a system model, the triangulation is constrained to using sampled pairs as vertices. Consider that Theorem 2 requires knowledge of how the system evolves given an initial state, i.e. $\Delta T(\mathbf{x}) = T(\mathbf{x}) - \mathbf{x}$. This evolution is only known at the given data points, so the vertex points of the triangulation must be $\{\mathbf{x}_n\}_{n=0}^{N}$, so that $\{\mathbf{x}_n^+\}_{n=0}^{N}$ can give information about how the system evolves over time. Any understanding about how the system evolves over points in the state space that are not sampled relies on the Lipschitz continuity of the dynamic system.

In this paper, the system is assumed to be able to be queried, i.e. some state, $\mathbf{x}$, can be selected and then the corresponding $\mathbf{x}^+$ found. This makes it easy to formulate a triangulation scheme of the state space a priori and then gather data for each of the vertex points. The MESH2D toolbox was used to create the triangulation meshes used for query points in this paper (Engwirda (2014)).

3.2. CPA Function Conditions

By Theorem 2, a positive invariant set has a scalar function defined on it that meets the non-increase condition for all states surrounding an interior set where the non-increase condition is not required. When synthesizing the scalar function for a known invariant set, it can be difficult to create conditions that ensure the function meets this requirement because enforcing the requirement on all relevant states in the set results in an infinite number of constraints.

The unique structure of a CPA function across a triangulation with convex simplices can be exploited to provide a finite set of constraints on each relevant vertex on the CPA function, as seen in Giesl and Hafstein (2014a). The Lipschitz continuity of the unknown plant can be used to develop an error term for the non-increase condition on the scalar function. Then, if the non-increase condition with the error term is successfully enforced on the vertices of relevant simplices, the non-increase conditions holds for all states within those simplices (Giesl and Hafstein (2014a);Li et al. (2015)). The following theorem demonstrates how a CPA function can meet the conditions required of the scalar function in Theorem 2.

Theorem 8 *Consider the dynamic system*

$$\mathbf{x}_{k+1} = g(\mathbf{x}_k), \qquad \mathbf{x} \in \mathcal{X} \in \mathfrak{R}^n, \tag{3}$$

where g has Lipschitz constant $L_g > 0$. Let $\Omega \subset \mathfrak{R}^n$ be an invariant set for (3), and define $\bar{\mathcal{T}} = \{\sigma_i\}_{i=1}^{m_{\bar{\mathcal{T}}}}$ as a triangulation over Ω. For some $\hat{\mathcal{S}} \subset \mathcal{S} \subset \Omega$, define $\mathcal{T}$ as the subset of $\bar{\mathcal{T}}$ in $\mathcal{S}$ and $\widetilde{\mathcal{T}}$ as the subset of $\mathcal{T}$ in $\mathcal{S} \setminus \hat{\mathcal{S}}$, and $\widehat{\mathcal{T}}$ as the subset of $\mathcal{T}$ in $\hat{\mathcal{S}}$. Further, define $\mathbf{x} \in \mathbb{E}_{\partial\mathcal{T}}$ as the vertex points comprising $\partial\mathcal{S}$. Let there exist $b, \epsilon, s, l \in \mathbb{R}$ and a CPA function $W = \{W_\mathbf{x}\}_{\mathbf{x}\in\mathbb{E}_{\bar{\mathcal{T}}}} \in \mathbb{R}$ satisfying

$$\begin{aligned}
&W_\mathbf{x} \geq \epsilon, \quad \forall \mathbf{x} \in \mathbb{E}_{\bar{\mathcal{T}}\setminus\mathcal{T}} && (4)\\
&W_\mathbf{x} = \epsilon, \quad \forall \mathbf{x} \in \mathbb{E}_{\partial\mathcal{T}}, && (5)\\
&W_\mathbf{x} < \epsilon, \quad \forall \mathbf{x} \in \mathbb{E}_{Int(\widetilde{\mathcal{T}})}, && (6)\\
&W_\mathbf{x} < l, \quad \forall \mathbf{x} \in \mathbb{E}_{\widehat{\mathcal{T}}}, && (7)\\
&|\nabla W_i| \leq b, \quad \forall i \in \mathbb{Z}_1^{m_{\bar{\mathcal{T}}}}, && (8)\\
&W(g(\mathbf{x}_{i,j})) - W(\mathbf{x}_{i,j}) + bL_g c_i \leq 0, \quad \forall i \in \mathbb{Z}_1^{m_{\widetilde{\mathcal{T}}}}, \quad \forall j \in \mathbb{Z}_0^n, && (9)\\
&W(g(\mathbf{x}_{i,j})) - W(\mathbf{x}_{i,j}) + bL_g c_i \leq s, \quad \forall i \in \mathbb{Z}_1^{m_{\widehat{\mathcal{T}}}}, \quad \forall j \in \mathbb{Z}_0^n, && (10)\\
&l + s < \epsilon, && (11)\\
&s > 0, && (12)
\end{aligned}$$

where

$$c_i = \max_{\mathbf{x},\mathbf{y}\in\sigma_i} \|\mathbf{x} - \mathbf{y}\|. \tag{13}$$

Then, $\mathcal{S}^\circ$ is a positive invariant set.

Proof By definition, the CPA function W is continuous over Ω. Within each simplex, any point $\mathbf{x} \in \sigma_i$ can be described as a convex combination of the simplex vertex points, i.e. $\mathbf{x} = \sum_{j=0}^{n} \lambda_j \mathbf{x}_{i,j}$. Note that by convexity of linear functions, $W(\mathbf{x})$ of any $\mathbf{x} \in \sigma_i$ can be described by a convex combination of the CPA function vertex points, i.e. $W(\mathbf{x}) = \sum_{j=0}^{n} \lambda_j W(\mathbf{x}_{i,j})$. Further, for each σ_i in $\widetilde{\mathcal{T}}$, the value of the vertices $W_{\mathbf{x}_{i,j}}$ is less than or equal to ϵ by Constraints 5 and 6. Consequently,

$$W(\mathbf{x}) = \sum_{j=0}^{n} \lambda_j W(\mathbf{x}_{i,j}) < \sum_{j=0}^{n} \lambda_j \epsilon < \epsilon.$$

This inequality shows that for all relevant simplices, $W(\mathbf{x}) < \epsilon$ for all $\mathbf{x} \in \sigma_i$. Likewise, for each σ_i in $\widehat{\mathcal{T}}$, the vertices, $W_{\mathbf{x}_{i,j}}$, are constrained to be less than l, which implies $W(\mathbf{x}) < l$ for all $\mathbf{x} \in \hat{\mathcal{S}}$. For simplices at the border of $\hat{\mathcal{S}}$, some vertices may be less than l, while others are less than ϵ. The points within the simplices will always be less than some convex combination of l and ϵ. Because of Equation 12, this convex combination is always less than ϵ. Therefore, $W(\mathbf{x})$ is bounded above for all $\mathbf{x} \in \mathcal{S}$. Further, because $g(\mathbf{x})$ is Lipschitz continuous and Ω is a bounded set, $W(\mathbf{x})$ is bounded below on Ω.

The change in W for each simplex can be found using Lemma 6. The non-increase condition

$$W\left(g(\mathbf{x})\right) - W(\mathbf{x}) \leq 0,$$

must be shown to hold for all $\mathbf{x} \in \mathcal{S} \setminus \hat{\mathcal{S}}$. First, this condition will be shown to hold on each simplex, $\sigma_i \in \widetilde{\mathcal{T}}$. This section of the proof parallels that of Theorem 2.10, Case 1 in Giesl and Hafstein (2014a). However, this paper is concerned with an unknown model. Therefore, the Lipschitz constant of the unknown system L_g is used to bound the change from $W(\mathbf{x})$ to $W(g(\mathbf{x}))$ rather than the tighter model-based bound in Giesl and Hafstein (2014a).

For some $\mathbf{x} \in \sigma_i$, the non-increase constraint is equivalently expressed as

$$\sum_{j=0}^{n} \lambda_j W\left(g(\mathbf{x}_{i,j})\right) + W\left(g(\mathbf{x})\right) - \sum_{j=0}^{n} \lambda_j W\left(g(\mathbf{x}_{i,j})\right) - \sum_{j=0}^{n} \lambda_j W(\mathbf{x}_{i,j}) \leq 0.$$

The left-hand side is bounded above by

$$\sum_{j=0}^{n} \lambda_j W(g(\mathbf{x}_{i,j})) + \left| W(g(\mathbf{x})) - \sum_{j=0}^{n} \lambda_j W(g(\mathbf{x}_{i,j})) \right| - \sum_{j=0}^{n} \lambda_j W(\mathbf{x}_{i,j}).$$

Note that $W(g(\mathbf{x}_{i,j}))$ and $W(g(\mathbf{x}))$ may or may not also be in the same $n-$simplex as $\mathbf{x}$ (σ_i).

From Giesl and Hafstein (2014a), $|W(\mathbf{x}) - W(\mathbf{y})| \leq b \left\|\mathbf{x} - \mathbf{y}\right\|_\infty$, where here by Constraint 8 $|\nabla W| \leq b$. By equivalence of norms $\left\|\mathbf{x} - \mathbf{y}\right\|_\infty \leq \left\|\mathbf{x} - \mathbf{y}\right\|_2$, so $|W(\mathbf{x}) - W(\mathbf{y})| \leq b \left\|\mathbf{x} - \mathbf{y}\right\|_2$. Using this fact and Lemma 7, a final upper bound on the constraint for any $\mathbf{x} \in \sigma_i$ is

$$\sum_{j=0}^{n} \lambda_j \left(W(g(\mathbf{x}_{i,j})) - W(\mathbf{x}_{i,j}) + b L_g c_i \right) \leq 0.$$

By assumption, this equation holds for each vertex point of $\sigma_i \in \widetilde{\mathcal{T}}$. By convexity of linear inequalities, this equation also holds for all $\mathbf{x}$ in each simplex of $\widetilde{\mathcal{T}}$. Therefore, the non-increase condition holds for all $\mathbf{x} \in \mathcal{S} \backslash \hat{\mathcal{S}}$. The same argument can be used to show that $W(g(\mathbf{x})) - W(\mathbf{x}) \leq s$ for all $\mathbf{x} \in \hat{\mathcal{S}}$. Then, by Theorem 2, $\mathcal{S}^\circ$ is a positive invariant set. ■

Theorem 8 guarantees that the CPA function, W, will obey the conditions of the extended invariance principle. However, the error term introduced in Constraint 9 ($bL_g c_i$) does introduce some conservatism that limits when W can be found. For example, Theorem 8 cannot be used to define a scalar function for a purely invariant set, because if $\mathbf{x}$ and $g(\mathbf{x})$ are both on the boundary of $\mathcal{S}$, then Constraint 9 is infeasible. Therefore, Theorem 8 can only be applied to contractive sets. Likewise, if $\mathcal{S}$ contains a fixed point, there must be some area, $\hat{\mathcal{S}}$, surrounding the fixed point where the non-increase condition is not enforced. Similar to the boundary point, if $W(g(\mathbf{x})) = W(\mathbf{x})$, Constraint 9 becomes infeasible.

3.3. Optimization Problem for CPA Function Synthesis

Theorem 8 assumes that the value of the CPA function W can be found for $g(\mathbf{x})$ for all $\mathbf{x} \in \Omega$. However, in a data driven scenario, the triangulation scheme may result in a triangulation where certain a sampled point $\mathbf{x}$ results in a corresponding $\mathbf{x}^+$ outside of Ω.

The following theorem formulates a convex optimization problem to determine if a known positive invariant set is upheld using data and a data driven triangulation. The problem synthesizes a function $\tilde{W} : \mathbb{R}^n \to \mathbb{R}^n$, which is a combined CPA and quadratic function – CPA over Ω and quadratic elsewhere. Therefore, while $\tilde{W}$ is continuous over Ω, it is a discontinuous function overall. The form of $\tilde{W}$ was a design choice to ensure that when $\mathbf{x}^+$ of a data pair $\{\mathbf{x}^+, \mathbf{x}\}$ leaves Ω, it is defined. However, the quadratic element of $\tilde{W}$ ensures that the non-increase conditions (expressed as Constraints 21 and 22) are always infeasible when $\mathbf{x}^+$ leaves Ω.

Theorem 9 *Consider a dynamic system of form (3) that generates N data pairs $\{\mathbf{x}_n^+, \mathbf{x}_n\}_{n=0}^N$, where $\{\mathbf{x}_n\}_{n=0}^N \in \Omega$. Let $L_g > 0$ be the Lipschitz constant of the dynamic system. Define $\bar{\mathcal{T}} = \{\sigma_i\}_{i=1}^{m_{\bar{\mathcal{T}}}}$ as a triangulation over $\{\mathbf{x}_n\}_{n=0}^N$. Let $\hat{\mathcal{S}} \subset \mathcal{S} \subset \Omega$, where $\mathcal{S}$ is the proposed positive invariant set of the dynamic system. Define $\mathcal{T}$ as the subset of $\bar{\mathcal{T}}$ in $\mathcal{S}$, $\widetilde{\mathcal{T}}$ as the subset of $\mathcal{T}$ in $\mathcal{S} \setminus \hat{\mathcal{S}}$, and $\widehat{\mathcal{T}}$ as the subset of $\mathcal{T}$ in $\hat{\mathcal{S}}$. Further, $\mathbf{x} \in \mathbb{E}_{\partial \mathcal{T}}$ are the vertex points that comprise $\partial \mathcal{S}$. Define $\tilde{W}$ as*

$$\tilde{W} = \begin{cases} W(\mathbf{x}) & \forall \mathbf{x} \in \Omega \\ \mathbf{x}^\top \mathbf{P} \mathbf{x} + \epsilon & \forall \mathbf{x} \notin \Omega, \end{cases} \tag{14}$$

where $W = \{W_\mathbf{x}\}_{\mathbf{x} \in \mathbb{E}_{\bar{\mathcal{T}}}}$ is a CPA function and $\mathbf{P} \in \mathbb{R}^{n \times n}$ is a symmetric, positive definite matrix. Let $b, l, s, \epsilon \in \mathbb{R}$. Consider the optimization problem

$$\min_{\mathbf{W}, b, l, s, \mathbf{P}} J = 0$$

$$\mathbf{P} \succ 0, \tag{15}$$

$$W_\mathbf{x} \geq \epsilon, \quad \forall \mathbf{x} \in \mathbb{E}_{\bar{\mathcal{T}} \setminus \mathcal{T}}, \tag{16}$$

$$W_\mathbf{x} = \epsilon, \quad \forall \mathbf{x} \in \mathbb{E}_{\partial \mathcal{T}}, \tag{17}$$

$$W_\mathbf{x} < \epsilon, \quad \forall \mathbf{x} \in \mathbb{E}_{Int(\widetilde{\mathcal{T}})}, \tag{18}$$

$$W_\mathbf{x} < l, \quad \forall \mathbf{x} \in \mathbb{E}_{\widehat{\mathcal{T}}}, \tag{19}$$

$$|\nabla W_i| \leq b, \quad \forall i \in \mathbb{Z}_1^{m_{\bar{\mathcal{T}}}}, \tag{20}$$

$$\tilde{W}(\mathbf{x}_{i,j}^{+}) - \tilde{W}(\mathbf{x}_{i,j}) + bL_g c_i \le 0, \quad \forall i \in \mathbb{Z}_1^{m_{\tilde{\mathcal{T}}}}, \forall j \in \mathbb{Z}_0^n, \tag{21}$$
$$\tilde{W}(\mathbf{x}_{i,j}^{+}) - \tilde{W}(\mathbf{x}_{i,j}) + bL_g c_i \le s, \quad \forall i \in \mathbb{Z}_1^{m_{\hat{\mathcal{T}}}}, \forall j \in \mathbb{Z}_0^n, \tag{22}$$
$$l + s < \epsilon, \tag{23}$$
$$s > 0, \tag{24}$$

where c_i is defined by Equation 13. If the optimization problem is feasible, then $\mathcal{S}^\circ$ is a positive invariant set.

Proof The CPA function, W, is continuous over Ω° and locally Lipschitz over Ω° by Constraint 20. Constraints 17, 18 and 19 ensure that W is bounded above. Because the dynamic system is known to be Lipschitz continuous and the data generated is over the bounded region Ω, the samples $\{\mathbf{x}_n^+\}_{n=0}^N$ are bounded below. Therefore, $W(\mathbf{x}^+)$ and $W(\mathbf{x})$ are also bounded below.

Using the same logic as the proof of Theorem 8, Constraint 21 ensures that the non-increase condition holds for all $n-$simplices in $\mathcal{S} \setminus \hat{\mathcal{S}}$, and Constraint 22 ensures that ΔW in $\hat{\mathcal{S}}$ remains bounded for all $\mathbf{x} \in \hat{\mathcal{S}}$. If the data pair $\{\mathbf{x}^+, \mathbf{x}\}$ contains a point $\mathbf{x}^+$ that leaves Ω, then, by the definition of $\tilde{W}$, $\tilde{W}(\mathbf{x}^+)$ is positive. The non-increase condition (Constraint 21) and the condition on $\hat{\mathcal{S}}$ (Constraint 22) will always be violated for a $\mathbf{x} \in \mathcal{S}$ that has $\mathbf{x}^+ \notin \Omega$ and therefore, infeasible. Likewise, Constraint 16 does the same for $\mathbf{x}^+ \in \Omega \setminus \mathcal{S}$. Note that while b may not be valid for $\tilde{W}$ when $\mathbf{x}^+$ leaves Ω, it is irrelevant as the decrease constraints are already infeasible across the given $n-$simplex. Therefore, if the problem is feasible, $\mathcal{S}^\circ$ is positive invariant. ■

Implementation Notes: Theorem 9 is an optimization problem with strict equality and inequality constraints. For implementation, the strict equality Constraint 17 can be set before optimization rather than treating those CPA vertex points variables. Strict inequality constraints can be implemented using small buffer variables. For example, Constraint 24, $s > 0$, becomes $s \ge 1 \times 10^{-10}$.

4. Numerical Experiments

The convex optimization problem developed in Theorem 9 provides a means to determine whether or not a positive invariant set, defined a priori, is upheld through data samples. Two numerical examples are shown to demonstrate the efficacy of the program. The first example demonstrates the programs ability to ascertain the validity of a positive invariant set for a linear system where some sampled pairs $\{\mathbf{x}^+, \mathbf{x}\}$ include $\mathbf{x}^+$ points that leave the triangulated region of the state space. In the second example, the optimization problem is used to determine if the entirety of a region of the state space for a nonlinear system is positive invariant. In both cases, the triangulation of the state space is used to determine where points of the system will be sampled. If the optimization problem is unable to find a feasible solution, the triangulation is refined to produce a more dense sampled data set. If continual refinements still result in failure, then the set is assumed to not be positive invariant. Note that the program will never return a false positive; the optimization problem will never be feasible if the set is not positive invariant.

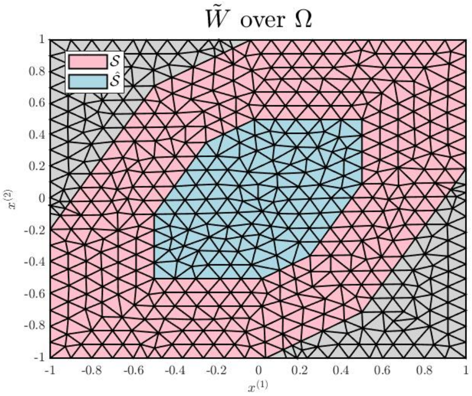
4.1. Linear System

This example considers the stable, linear system,

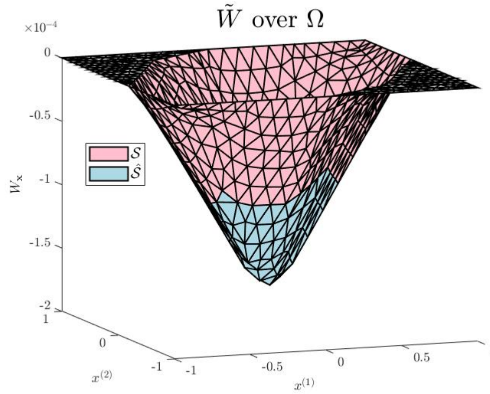
$$\mathbf{x}_{k+1} = \begin{bmatrix} 0.366 & -0.587 \\ 0.671 & -0.372 \end{bmatrix} \mathbf{x}_k. \tag{25}$$

The states were constrained to $\Omega = [-1, 1]^2$. The Lipschitz constant of the system is the spectral norm of the dynamic's matrix, $L_g = 1$, and is assumed to be known. MPT3 (Herceg et al. (2013)) was used to determine a positive invariant set, $\mathcal{S}$, that had a contraction factor of $\gamma = 0.6$. The set $\hat{\mathcal{S}}$, which is not required to have a non-increase condition, was selected to be $0.5\mathcal{S}$.

The triangulation ($\bar{\mathcal{T}}$) of Ω that allowed the optimization problem to find a feasible $\tilde{W}$ is shown in Figure 1(a). The regions $\mathcal{S}$ and $\hat{\mathcal{S}}$ are shown as well. For each vertex point of $\bar{\mathcal{T}}$, the next time step of the dynamic system (25) was determined – resulting in the data pairs, $\{\mathbf{x}_n, \mathbf{x}_n^+\}_{n=0}^{\mathbb{E}_{\bar{\mathcal{T}}}}$. The optimization problem from Theorem 9 was applied to $\bar{\mathcal{T}}$ with $\epsilon = 0$ and successfully found a CPA function $\tilde{W}$, shown in Figure 1(b). The existence of $\tilde{W}$ implies the positive invariance of $\mathcal{S}^\circ$.



(a)



(b)

Figure 1: 1(a) shows the triangulation over $\Omega = [-1, 1]^2$ and the subsets $\hat{\mathcal{S}} \subset \mathcal{S}$ within Ω. 1(b) displays the CPA function synthesized to confirm that $\mathcal{S}^\circ$ is an invariant set. Note that the non-increase condition was only enforced on the pink section of Ω, which is $\mathcal{S} \setminus \hat{\mathcal{S}}$.

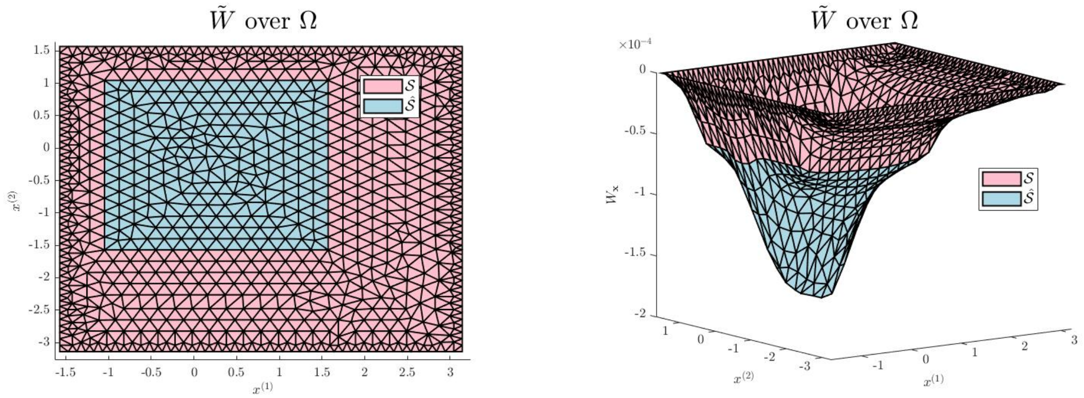
4.2. Nonlinear example

The simple discrete, nonlinear system

$$\begin{bmatrix} x_{k+1}^{(1)} \\ x_{k+1}^{(2)} \end{bmatrix} = \begin{bmatrix} 0.75\cos(x_k^{(1)}) \\ 0.15\cos(x_k^{(2)}) \end{bmatrix} \tag{26}$$

over the region $\Omega : x^{(1)} \in [-\frac{\pi}{2}, \pi]$, $x^{(2)} \in [\pi, \frac{\pi}{2}]$ was considered. This system has a Lipschitz constant $L_g = 0.765$.

Here, the entire triangulated region was known to be positively invariant, $\mathcal{S} \subseteq \Omega$, and $\hat{\mathcal{S}}$ was defined as $\hat{\mathcal{S}} : x^{(1)} \in [-\frac{\pi}{3}, \frac{\pi}{2}]\ x^{(2)} \in [-\frac{\pi}{2}, \frac{\pi}{3}]$. Figure (2($a$)) shows the triangulated region of the state space, as well as $\mathcal{S}$ and $\hat{\mathcal{S}}$. Figure 2(b) shows the synthesized function $\tilde{W}$ resulting from the optimization problem developed in Theorem 9 with $\epsilon = 0$. By successfully synthesizing $\tilde{W}$ from the sampled triangulation, Ω° is confirmed to be positively invariant.



(*a*) (*b*)

Figure 2: The triangulation of the region Ω is shown in 2(*a*). $\hat{\mathcal{S}}$ is the blue region, while $\mathcal{S}$ is the combined blue and pink region. The function $\tilde{W}$ over Ω is shown in 2(*b*). The existence of $\tilde{W}$ shows that the entirety of Ω° is a positive invariant set for (26).

5. Discussion

In this work, a convex optimization problem is developed to synthesize a CPA function that fulfills the Extended Invariant Set Principle using only data samples and the Lipschitz continuity of an unknown discrete system. A triangulation of the state space is used to prompt data samples of $\{\mathbf{x}, \mathbf{x}^{+}\}$ pairs. By using an error bound on the triangulation that leverages the Lipschitz continuity of the plant and the convexity of the $n-$simplices, a CPA function can be determined to verify the existence of a positive invariant set. A linear and nonlinear example are used to show the efficacy of the convex optimization problem developed in Theorem 9.

A limitation of the current work is the reliance on an a priori triangulation to prompt specific data samples, as this may be impractical for some systems or an impossible criteria to fulfill if data samples were gathered prior to the verification process. Moreover, the triangulation may require an unrealistically dense sampling of the state space to produce a feasible CPA function. This limitation is particularly relevant in systems with a large number of states. Higher dimensions will demand a finer triangulation scheme. The finer triangulation creates higher data requirements and requires a larger number of constraints in the optimization problem.

Future work should consider methods to decrease the density of samples in regions of the triangulation and build a triangulation from a pre-collected data set. Moreover, now that the basis to verify a positive invariant set using CPA functions has been established, verifying uncertain invariant sets and synthesis of previously unknown positive invariant sets should be addressed next.

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