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Safe Learning in Nonlinear Model Predictive Control

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Abstract

A robust Model Predictive Control algorithm is proposed for learning-based control with model represented by an affine combination of basis functions. The online optimization is formulated as a sequence of convex programming problems derived by linearizing concave components of the dynamic model. A tube-based approach ensures satisfaction of constraints on control variables and model states while avoiding conservative bounds on linearization errors. The linear dependence of the model on unknown parameters is exploited to allow safe online parameter adaptation. The resulting algorithm is recursively feasible and provides closed loop stability and performance guarantees. Numerical examples are provided to illustrate the approach.

Keywords: Nonlinear predictive control, convex programming, adaptive control, learning control.

1. Introduction

Model Predictive Control (MPC) is an optimal control strategy with well-established theoretical properties and a wide range of applications (Kouvaritakis and Cannon, 2016). The main idea is to solve an open-loop optimal control problem repeatedly online with the current state as the initial condition at a given time. Although the structure of the controlled system model may be known, it is often unavoidable that parametric model uncertainty and disturbances act on the system during closed-loop operation. Therefore a major focus of the literature, both in the context of linear and nonlinear systems, has been to ensure robustness of MPC strategies. Recently, much attention has been given to combinations of learning-based and robust MPC approaches (Hewing et al., 2020). Similarly to robust adaptive MPC algorithms, these methods inherit the main advantages of robust MPC, but use information on the controlled system that is gathered while a control task is performed to improve the quality of the system model and closed-loop control performance.

While several approaches for linear robust adaptive MPC have been proposed (Lorenzen et al., 2019; Lu et al., 2021), the general case of nonlinear system models has received relatively little attention (Köhler et al., 2021; Adetola et al., 2009). Although the approaches of Köhler et al. (2021) and Adetola et al. (2009) have strong system-theoretical properties such as recursive feasibility, they are necessarily conservative, for example because of the fixed ellipsoidal tubes that are used, and computationally intensive because they rely on the global solution of a nonconvex program online.

This paper considers an alternative adaptive MPC approach based on sequential convex approximation (Cannon et al., 2011; Doff-Sotta and Cannon, 2022). Methods based on linearizing dynamics around previously predicted trajectories have the important advantage that the optimization can be split into convex subproblems (often as variants of linear MPC), which are therefore efficiently solvable and benefit from the stability properties and robustness of linear MPC. To ensure stability and convergence, the perturbations around state and control linearization points are limited to regions within which the model approximation is meaningful and the effect of the approximation error is bounded by constructing tubes containing the predicted trajectories. However, choosing the set of allowable perturbations is based on heuristics and can become conservative (Cannon et al., 2011). In Doff-Sotta and Cannon (2022) a successive convex approximation MPC method is proposed that does not require heuristic bounds on the allowable perturbation step size, thus overcoming a major drawback of Cannon et al. (2011). The approach is extended to systems with unknown additive disturbances in Lishkova and Cannon (2024), where a worst-case linearization approach around updated seed tubes is used to obtain bounds on the predicted successor states.

In this paper we extend the theory of Doff-Sotta and Cannon (2022) and Lishkova and Cannon (2024) to the context of nonlinear systems with additive and parametric uncertainty, and we derive a robust adaptive nonlinear model predictive control algorithm using set-based parameter estimation (Lorenzen et al., 2019). The resulting MPC law is obtained by sequential convex optimization with efficiently solvable subproblems. Furthermore, the approach does not require heuristic prior bounds on the allowable perturbations in each convex subproblem. The method is recursively feasible and ensures a closed loop performance bound. Finally, the approach offers many potential applications and extensions for variants of model-learning within MPC.

2. Problem statement

This paper considers nonlinear systems subject to bounded disturbances with parameters to be learned online. The model state $x_t \in \mathcal{X}$ has the discrete-time dynamics

$$x_{t+1} = f(x_t, u_t, \theta) + w_t. \tag{1}$$

where $u_t \in \mathcal{U}$ is the control input, $\theta \in \Theta_0$ is a vector of unknown parameters, $w_t \in \mathcal{W}$ is an unknown disturbance, and t denotes the discrete time index. We assume compact polytopic sets $\mathcal{X} := \{x \in R^n : Ex \leq \mathbf{1}\}, \mathcal{U} := \{u \in R^m : Fu \leq \mathbf{1}\}$, $\Theta_0 := \{\theta \in R^p : H_\Theta \theta \leq h_0\} = \text{Co}\{\theta^v, v = 1, \ldots, n_\Theta\}$ and $\mathcal{W} := \{w \in R^{n_x} : G_w w \leq \mathbf{1}\} = \text{Co}\{w^{(j_a)}, j_a \in [1, \ldots, n_w]\}$. The function $f(x_t, u_t, \theta)$ is an affine combination of convex basis functions $\{f_i(x_t, u_t), i = 0, \ldots, p\}$

$$f(x_t, u_t, \theta) = f_0(x_t, u_t) + \sum_{i=1}^{p} \theta_i f_i(x_t, u_t). \tag{2}$$

We consider the problem of minimizing a quadratic cost, with weighting matrices $R \succ 0$ and $Q \succeq 0$,

$$\sum_{t=0}^{\infty} (\|x_t\|_Q^2 + \|u_t\|_R^2). \tag{3}$$

The parameter vector θ is assumed unknown but contained in a known polytope at time $t = 0$, i.e. $\theta \in \Theta_0$. The proposed approach can be used in combination with any parameter learning algorithm that generates at time t a polytopic parameter set Θ_t satisfying, forall $t > 0$, $\theta \in \Theta_t \subseteq \Theta_{t-1}$.

3. Recursively feasible tube construction and linearization for convex systems

To simplify explanation, in the following all parameters are assumed positive ($\theta_i \geq 0$), and each component of $f_i(x, u)$ is assumed convex. We highlight extensions to the case of sign-indefinite parameters in specific remarks in each section. We apply a successive linearization approach to the MPC optimization problem and incorporate online model learning via parameter adaptation. By exploiting convexity properties of the system description, this provides a sequence of convex subproblems that address the combined effects of parametric uncertainty and unknown disturbances.

To ensure recursive feasibility, we construct a feasible control law at the start of each iteration based on the control perturbation sequence $\mathbf{c}^0 = \{c_0^0, \ldots, c_{N-1}^0\}$ determined at the previous iteration. Similar to Lishkova and Cannon (2024) we successively construct the state tube sequence $\mathbf{X}^0 = \{X_0^0, \ldots, X_N^0\}$ where each tube element is a hyperrectangular set[1] defined by:

$$X_k^0 = \{x \in R^{n_x} : \underline{x}_k \leq x \leq \overline{x}_k\} = \mathrm{Co}\{x_k^j,\, j = 1, \ldots, v_x\}.$$

The vertices $\{x_k^j,\, j = 1, \ldots, v_x\}$ can be conveniently obtained from the bounds $\underline{x}_k$, $\overline{x}_k$ defining X_k^0, and upper bounds on the components of the successor state can be found by computing

$$[\overline{x}_{k+1}]_r = \max_{j \in [1,\ldots,v_x], v \in [1,\ldots,n_\Theta]} [f(x_k^j, Kx_k^j + c_k^0, \theta^v)]_r + [\overline{w}]_r, \tag{4}$$

where the maximum occurs on the vertices of X_k (due to convexity) and the vertices of Θ (due to the linear dependence on the parameter θ). Here $[x]_r$ represents the rth component of a vector x, and $[\overline{w}]_r = \max_{w \in \mathcal{W}} [w]_r$. Lower bounds on the state components can be constructed, once more exploiting convexity, by linearizing each component of the function f defining the model. A conventional linearization around a nominal trajectory could be used, with a backtracking line search (Nocedal and Wright, 2006) to ensure recursive feasibility. However we describe here an alternative method of ensuring feasibility by defining the linearization point for the rth component of f_i as the value of $x \in X_k^0$ at which $[f_i(x, Kx + c_k^0)]_r$ attains its minimum. The linear system obtained by linearizing at these points defines a lower bound on the nonlinear system because $f(x, Kx + c_k^0)$ is convex, and this lower bound property holds for arbitrary perturbations relative to the linearization points. The linearization points are thus obtained for each basis function $i = 0, \ldots, p$ and for each element of the state $r = 1, \ldots, n_x$ by solving a convex optimization problem:

$$x_{k,i,r}^0 = \arg \min_{\{x \in X_k^0\}} [f_i(x, Kx + c_k^0)]_r \tag{5}$$

These points are collected (for $k = 0, \ldots, N-1$) to define the linearization point sequence $\mathbf{x}_{i,r}^0 = \{x_{0,i,r}^0, \ldots, x_{N-1,i,r}^0\}$. Based on the individual linearization points for each basis function and the vertices of Θ, the overall lower bound can be defined as

$$[\underline{x}_{k+1}]_r = [f_0(x_{k,i,r}^0, Kx_{k,i,r}^0 + c_0^0)]_r + \min_{v \in [1,\ldots,n_\Theta]} \sum_{i=1}^{p} \theta_i^v [f_i(x_{k,i,r}^0, Kx_{k,i,r}^0 + c_k^0)]_r + [\underline{w}]_r. \tag{6}$$

The forward simulation procedure at the start of an iteration for constructing an initial state tube (called a *seed tube*) and the linearization points is summarized in Algorithm 1 (with the linearization matrix sequences $\mathbf{A}_i = \{A_{0,i}, \ldots, A_{N-1,i}\}$, $\mathbf{B}_i = \{B_{0,i}, \ldots, B_{N-1,i}\}$ for $i = 0, \ldots, p$).

To extend the theory to the general case we consider some $\theta_{i^*} \leq 0$ and convex $f_{i^*}(x_t, u_t)$ which results in concave $\theta_{i^*} f_{i^*}(x_t, u_t)$. The linearization point for i^* is still defined by (5), since $f_{i^*}(x_t, u_t)$ is convex. However, this point maximizes the concave function $\theta_{i^*}^v f_{i^*}(x_t, u_t)$. From (6) we therefore move the term $\theta_{i^*}^v [f_{i^*}(x_{k,i^*,r}^0, Kx_{k,i^*,r}^0 + c_k^0)]_r$ to (4) and likewise move $\theta_{i^*}^v [f_{i^*}(x_{k,i^*,r}, Kx_{k,i^*,r}+$

1. The approach can be extended to polytopic tubes of the form $X_k^0 = \{x : H_k x \leq h_k\}$ for fixed H_k and variable h_k.

$c_k^0)]_r$ from (4) to (6). For a parameter with unknown sign we can split the contribution of the basis function by considering its convex and concave contributions $\theta_{i^*} f_{i^*}(x_t, u_t) = \theta_{i_1^*} f_{i^*}(x_t, u_t) + \theta_{i_2^*} f_{i^*}(x_t, u_t)$ with $\theta_{i_1^*} \geq 0$ and $\theta_{i_2^*} \leq 0$. Therefore this case can be treated analogously. Finally, we note that the expansion (2) over convex basis functions $f_i(x, u)$ is equivalent to a difference of convex functions (DC) representation of $f(x, u, \theta)$ (Horst and Thoai, 1999; Hartman, 1959).

Algorithm 1: Forward Simulation: Linearization and Seed Tube Construction

Input : Seed control $\mathbf{c}^0$, feedback gain K and initial state x^p. Disturbance set $\mathcal{W}$ and parameter set Θ_t.
Output: Seed tube sequence $\mathbf{X}^0$, linearization matrix sequences $\mathbf{A}_i$, $\mathbf{B}_i$ and linearization point sequences $\mathbf{x}_{i,r}^0$ for $i = 0, \ldots, p$ and for $r = 1, \ldots, n_x$

```
Set X_0^0 = {x_t^p}.
for k = 0, ..., N-1 do
    for r = 1, ..., n_x do
        for i = 0, ..., p do
            Solve Problem (5) for x_{k,i,r}^0 and define u_{k,i,r}^0 = K x_{k,i,r}^0 + c_k^0.
            Compute [A_{k,i}]_r = ∇_x[f_i]_r(x_{k,i,r}^0, u_{k,i,r}^0) and [B_{k,i}]_r = ∇_u[f_i]_r(x_{k,i,r}^0, u_{k,i,r}^0) .
            Compute the tube bounds [x̄_{k+1}]_r, [x̲_{k+1}]_r for the successor stage k+1 via (4) and (6).
        end
    end
    Define X_{k+1}^0 ← {x ∈ R^{n_x}: x̲_{k+1} ≤ x ≤ x̄_{k+1}}.
end
```

4. Sequential convex programming

This section defines the subproblem for each iteration of the successive linearization MPC method. The subproblem is defined in such a way that it is robust against parameter variations of θ, disturbances w, and any introduced approximations of model (1). As already discussed in Section 3, the convexity of (1) can be exploited in the definition of upper and lower bounds on the successor state (defining a predicted state tube) in the convex optimization subproblem during successive linearization updates. First we exploit the fact that the *upper bound* of a convex function $f(x)$ over a bounded polytope X_k occurs on the vertices of X_k, and thus upper bounds can be conveniently defined using convex constraints on the corresponding optimization variables. Secondly, we construct a tight *lower bound* using the Jacobian linearization around the minimum point (5) of $[f(x, Kx + c_k^0)]_r$, namely the rth component of the convex state transition function $f(x, Kx + c_k^0)$, over the bounded polytope $x \in X_k$. The Jacobian linearization constructed around this minimum point necessarily provides a lower bound on $[f(x, Kx + c_k^0 + c)]_r$ for all $x \in X_k$ and all c such that $Kx + c_k^0 + c \in \mathcal{U}$.

The feedback law is parameterized by $u_k = Kx_k + c_k^0 + c_k$ and we consider a worst-case quadratic predicted cost in the sense that it forms an upper bound on the cost (3)

$$J(\mathbf{c^0}, \mathbf{c}, \mathbf{X}) = \sum_{k=0}^{N-1} \max_{x_k \in X_k} (\|x_k\|_Q^2 + \|Kx_k + c_k^0 + c_k\|_R^2) + \max_{x_N \in X_N} \|x_N\|_P^2 \tag{7}$$

We consider the perturbation c_k (relative to the seed perturbation c_k^0) and the variables defining the tube cross sections X_k (i.e. $\overline{x}_k$, $\underline{x}_k$) as optimization variables that are chosen to minimize the predicted cost. The optimization problem (at each iteration at time t) is then:

$$\min_{\{\mathbf{c}, \mathbf{X}\}} J(\mathbf{c^0}, \mathbf{c}, \mathbf{X}) \tag{8a}$$

subject to $X_0 = \{x_t^p\}$, and $\forall k \in \{0, \dots, N-1\},\ \forall r \in \{1, \dots, n_x\},\ \forall \theta \in \Theta_t,\ \forall x \in X_k$:

$$[\overline{x}_{k+1}]_r \geq [f_0(x, Kx + c_k^0 + c_k) + \sum_{i=1}^{p} \theta_i \{f_i(x, Kx + c_k^0 + c_k)\}]_r + [\overline{w}]_r \tag{8b}$$

$$\begin{aligned}[\underline{x}_{k+1}]_r \leq {} & [f_0(x_{k,0,r}^0, Kx_{k,0,r}^0 + c_k^0) + \sum_{i=1}^{p} \theta_i \{f_i(x_{k,i,r}^0, Kx_{k,i,r}^0 + c_k^0)\}]_r + [A_{k,0} + B_{k,0}K]_r(x - x_{k,0,r}^0) \\ & + [B_{k,0}]_r c_k + \sum_{i=1}^{p} \theta_i \{[A_{k,i} + B_{k,i}K]_r(x - x_{k,i,r}^0) + [B_{k,i}]_r c_k\} + [\underline{w}]_r \end{aligned} \tag{8c}$$

$$X_k \subseteq \mathcal{X},\ KX_k \oplus \{c_k^0 + c_k\} \subseteq \mathcal{U} \tag{8d}$$

$$X_N \subseteq \mathcal{X}_N,\ [\overline{x}_N]_r \geq [(A_N^{(j)} + B_N^{(j)}K)x]_r + [\overline{w}]_r,\ [\underline{x}_N]_r \leq [(A_N^{(j)} + B_N^{(j)}K)x]_r + [\underline{w}]_r, \forall j \tag{8e}$$

where (8b-c) bound the successor states for all permissible realizations of model uncertainty, x_t^p is the plant state at time t defining the initial tube element X_0, and analogously to $[\overline{w}]_r$ we define $[\underline{w}]_r = \min_{w \in \mathcal{W}} [w]_r$. The final tube element X_N is required to be a subset of the terminal set $\mathcal{X}_N$ (the computation of which is discussed in Section 5). The MPC law is summarised in Algorithm 2.

Algorithm 2: Convex tube MPC

Input : Seed control sequence $\mathbf{c}^0$, feedback gain K, terminal cost P, terminal set $\mathcal{X}_N$ and initial plant state x_t^p (at time t). The disturbance set $\mathcal{W}$ and parameter set Θ_t are also assumed to be known.
Output: Optimal input u_t (at time t) and updated control sequence $\mathbf{c}^0$
Obtain x^p at a given time t. Set $iter \leftarrow 1$.
while $iter \leq maxiter$ *and* $\|\mathbf{c}^*\| \geq tol$ **do**
  Execute Algorithm 1 to obtain $\mathbf{A}_i$, $\mathbf{B}_i$ and $\mathbf{x}_{i,r}^0$ for $i = 0, \dots, p$ and for $r = 1, \dots, n_x$.
  Solve (8) with $x \in \mathcal{V}(X_k)$ and $\theta \in \mathcal{V}(\Theta_t)$ to obtain the optimal perturbation $\mathbf{c}^*$ and predicted tube $\mathbf{X}^*$.
  Update the control sequence: $\mathbf{c}^\mathbf{0} \leftarrow \mathbf{c}^\mathbf{0} + \mathbf{c}^*$, $iter \leftarrow iter + 1$.
end
Implement $u_t = Kx_t^p + c_0^0$ and set $\mathbf{c}^0 \leftarrow \{c_1^0, \dots, c_{N-1}^0, 0\}$.

Problem (8) can be reformulated on the vertices $\mathcal{V}(X_k)$ of X_k (i.e. $\{x_k^j,\ j = 1, \dots, v_x\}$), for $k = 0, \dots, N-1$, and the vertices $\mathcal{V}(\Theta_t)$ of Θ_t (i.e. $\{\theta^v,\ v = 1, \dots n_\Theta\}$) due to the convexity of $f_i(x, u)$. Finally, the objective function can be expressed in epigraph form using second order cone constraints, resulting in a convex optimization subproblem. To extend the method to the general case of continuous but not necessarily convex f, we consider again $\theta_{i^*} \leq 0$ for some $i^* \in \{1, \dots, p\}$. In this case the linearization terms defined by A_{k,i^*},B_{k,i^*} and $x_{k,i^*,r}^0$ moves from (8c) to (8b), whereas the nonlinear (concave) term $\theta_{i^*}[f_{i^*}(x_k^j, Kx_k^j + c_k^0 + c_k)]_r$ moves from (8b) to (8c).

5. Terminal conditions

This section considers the construction of an ellipsoidal terminal set $\mathcal{X}_N$ and a terminal cost weight Q_N. To apply linear set invariance theory we consider a difference inclusion (LDI) approximation (e.g. Boyd et al., 1994) of the model (1). Further, we consider an affine combination of LDIs over the parameter set Θ_0 resulting in an aggregate LDI. This provides a computationally convenient means of ensuring that the terminal set is robustly invariant under the terminal control law $u = Kx$.

The system (1) with $u = Kx$ is approximated for $x \in \bar{\mathcal{X}} = \{x : |x| \leq \delta_x,\ |Kx| \leq \delta_u\}$, using

$$f(x, u, \theta) + w \in \mathrm{Co}\{(A_N^{(j_m)} + B_N^{(j_m)}K)x + w^{(j_a)},\ j_m = 1, \dots, n_M,\ j_a = 1, \dots, n_W\} \tag{9}$$

This approximation can be constructed by considering LDIs for the individual basis functions $f_i(x, Kx)$ and by combining these using all possible vertex combinations of Θ_0.

Lemma 1 *The inequality*

$$\|x\|_P^2 - \|f(x, Kx, \theta) + w\|_P^2 \geq \|x\|_Q^2 + \|Kx\|_R^2 - \beta \tag{10}$$

holds $\forall (w, \theta) \in W \times \Theta_0$, *for positive definite* P *and* $\beta \geq 0$ *if the following Linear Matrix Inequality (LMI) in variables* $S = P^{-1}$ *and* $Y = KP^{-1}$ *holds* $\forall j_m \in \{1, \ldots, n_M\}$ *and* $\forall j_a \in \{1, \ldots, n_W\}$*:*

$$\begin{bmatrix} S & 0 & (A_N^{(j_m)} S + B_N^{(j_m)} Y)^\top & S & Y^\top \\ 0 & \beta & w^{(j_a)\top} & 0 & 0 \\ * & * & S & 0 & 0 \\ * & * & * & Q^{-1} & 0 \\ * & * & * & * & R^{-1} \end{bmatrix} \succeq 0 \tag{11}$$

Proof This follows by substitution of (9) into (10) and using Schur complements. ■

Theorem 2 *The ellipsoidal terminal set* $\mathcal{X}_N := \{x : x^\top P x \leq \gamma\}$ *is positively invariant, i.e.* $f(x, Kx, \theta) + w \in \mathcal{X}_N$, $\forall (x, w, \theta) \in \mathcal{X}_N \times W \times \Theta_0$, *if* P *satisfies Lemma 1 and*

$$\gamma \geq \beta / \sigma_{\max}\big(P^{-1/2}(Q + K^\top R K)P^{-1/2}\big) \tag{12}$$

($\sigma_{\max}(A)$ *is the maximum singular value of a matrix* A*). Constraints* $\mathcal{X}_N \subseteq \mathcal{X} \cap \bar{\mathcal{X}}$ *and* $K\mathcal{X}_N \subseteq \mathcal{U}$ *additionally require, for* $i \in \{1, \ldots, n_x\}$, $j \in \{1, \ldots, n_u\}$, $k \in \{1, \ldots, n_E\}$, $l \in \{1, \ldots, n_F\}$,

$$\gamma \leq \min_{i,j,k,l} \left\{ \frac{[\delta_x]_i^2}{[P^{-1}]_{ii}}, \quad \frac{[\delta_u]_j^2}{[K]_j P^{-1} [K]_j^\top}, \quad \frac{1}{[E]_k P^{-1} [E]_k^\top}, \quad \frac{1}{[FK]_l P^{-1} [FK]_l^\top} \right\}. \tag{13}$$

Proof Condition (10) implies that $\mathcal{X}_N$ is positively invariant if $\max_{x \in \mathcal{X}_N} \|x\|_{Q+K^T RK}^2 \geq \beta$ (since this implies $\|x_t\|_P^2 \geq \|x_{t+1}\|_P^2$), which is equivalent to (12). Condition (13) follows from enforcing the constraints $x \in \bar{\mathcal{X}}$, $Kx \in \bar{\mathcal{U}}$ and $x \in \mathcal{X}$, $Kx \in \mathcal{U}$, respectively, for all $x \in \mathcal{X}_N$. ■

A procedure for computing $\mathcal{X}_N$ and P is summarized in Algorithm 3. This involves the solution of a (convex) semidefinite program (SDP). We have assumed that the terminal set is computed in Algorithm 3 using Θ_0 offline. In principle, $\mathcal{X}_N$ can be updated online using $\Theta_t \subseteq \Theta_0$, providing an improved (larger) terminal set and thus improving the performance of Algorithm 2.

Algorithm 3: Computation of terminal constraint set and terminal cost

Input : The parameters defining the LDI in (9): $A_N^{(j_m)}$, $B_N^{(j_m)}$, $w^{(j_a)}$ with associated bounds δ_x, δ_u, the cost matrices Q, R and the state and control sets $\mathcal{X}$ and $\mathcal{U}$
Output: $\mathcal{X}_N$, P, K, β, γ
Solve $(S^*, Y^*, \beta^*) = \min_{S,Y,\beta} \beta$ s.t. LMI (11). Set $P \leftarrow {S^*}^{-1}$, $K \leftarrow Y^* P$ and $\beta \leftarrow \beta^*$.
Solve $\gamma^* = \max \gamma$ s.t. (13). Set $\gamma \leftarrow \gamma^*$ and $\mathcal{X}_N \leftarrow \{x : x^\top P x \leq \gamma\}$.

6. Recursive feasibility and stability

In Section 3 we exploited the convexity of the system dynamics to obtain tight lower bounds on tubes bounding future predicted states, and we used linearization to formulate upper bounds as convex constraints on the vertices of tube cross sections. For a given sequence $\mathbf{c}^0$, Algorithm 1 generates a seed tube $\mathbf{X}^0$ originating from the current plant state x^p that is consistent with the tube constraints employed in Algorithm 2. This implies that the seed tube $\mathbf{X}^0$ and the linearizations $\mathbf{A}_i$, $\mathbf{B}_i$ generate a feasible optimization (8) since zero control perturbation ($\mathbf{c} = 0$) is necessarily feasible. Therefore Algorithm 2 must remain feasible between successive iterations. Further, the robust invariance of the terminal constraint set ensures recursive feasibility at consecutive time steps.

We demonstrate this here, starting with the nestedness of the seed tube at the start of iteration $iter + 1$ denoted $\mathbf{X}^{0,+}$ and the optimal tube obtained at the end of iteration $iter$ denoted $\mathbf{X}^*$. We denote $\mathbf{c}^{0,+}$ and $\mathbf{x}^{0,+}_{i,r}$ as the corresponding control perturbation and linearization point sequences.

Lemma 3 *For $k = 0, \dots, N-1$ Algorithm 2 generates tube cross sections which satisfy the following nestedness property: $X^{0,+}_k \subseteq X^*_k$.*

Proof This is shown by induction. We first demonstrate that: $X^{0,+}_k \subseteq X^*_k \implies X^{0,+}_{k+1} \subseteq X^*_{k+1}$. Considering the linearization points for a given basis function i, we obtain (for $r = 1, \dots, n_x$)

$$\begin{aligned}[][f_i(x^{0,+}_{k,i,r}, Kx^{0,+}_{k,i,r} + c^{0,+}_k)]_r &= \min_{x \in X^{0,+}_k} [f_i(x, Kx + c^{0,+}_k)]_r \geq \min_{x \in X^*_k} [f_i(x, Kx + c^{0,+}_k)]_r \\ &\geq \min_{x \in X^*_k} [f_i(x^0_{k,i,r}, Kx^0_{k,i,r} + c^0_k)]_r + [A_{k,i}]_r(x - x^0_{k,i,r}) + [B_{k,i}]_r c^*_k),\end{aligned}$$

which follows from the set inclusion $X^{0,+}_k \subseteq X^*_k$, the convexity of f and $\mathbf{c}^{0,+} = \mathbf{c}^0 + \mathbf{c}^*$. For $\theta_i \geq 0$ this implies that due to (8c) and (6) the linear combination of the basis functions satisfies $\underline{x}^{0,+}_{k+1} \geq \underline{x}^*_{k+1}$. Upper bounds obtained from $X^{0,+}_k \subseteq X^*_k$ and the convexity of f are given by

$$\max_{x \in X^{0,+}_k} [f_i(x, Kx + c^{0,+}_k)]_r \leq \max_{x \in X^*_k} [f_i(x, Kx + c^{0,+}_k)]_r = \max_{x \in X^*_k} [f_i(x, Kx + c^0_k + c^*_k)]_r,$$

which implies $\overline{x}^{0,+}_{k+1} \leq \overline{x}^*_{k+1}$ due to (8b) and (4) (for $\theta_i \geq 0$). Since $X^{0,+}_0 = X^*_0 = \{x^p_t\}$, the entire tube sequence is nested and this completes the proof. ■

Lemma 4 *For all $k = 0, \dots, N-1$ we obtain the recursive nestedness property of the tube cross sections for subsequent time steps of Algorithm 2, i.e. $X^{0,init}_k(t+1) \subseteq X^{*,final}_{k+1}(t)$ and $X^{0,init}_N(t+1) \subseteq \mathcal{X}_N$. (The superscripts init and final refer to the initial and final iterations.)*

Proof The last line of Algorithm 2 defines the initial seed trajectory at time $t+1$ as $c^{0,init}_k(t+1) = c^{0,final}_{k+1}(t) + c^{*,final}_{k+1}(t)$ for $k = 0, \dots, N-2$ and $c^{0,init}_{N-1}(t+1) = 0$. Therefore $X^{0,init}_k(t+1) \subseteq X^{*,final}_{k+1}(t)$, and $X^{0,init}_N(t+1) \subseteq \mathcal{X}_N$ follows from condition (8e) and the invariance of $\mathcal{X}_N$. ■

Theorem 5 *If the assumptions of Theorem 2 are satisfied and a feasible control seed trajectory $\mathbf{c^0}$ at time $t = 0$ is available, then Algorithm 2 is feasible at each iteration and for all times ($t \geq 0$)*

Proof The initial seed $\mathbf{c}^0$ is feasible by assumption at iteration 0. Furthermore, assume that at time t and iteration $iter$, Algorithm 2 returns $\mathbf{c}^*$ and the associated optimal tube $\mathbf{X}^*$ as the solution. Then due to the nestedness property of the seed and optimal tube according to Lemma 3, $\mathbf{c} = 0$ and $\mathbf{X}^{0,+}$ provide a feasible solution at iteration $iter + 1$ (with origin $X_0^{0,+} = X_0^* = \{x_t^p\}$). Further, from Lemma 4 we conclude that the solution obtained in the final iteration at time step t: $\mathbf{c}^{*,final}(t)$ with $\mathbf{X}^{*,final}(t)$ can be used to construct a feasible solution at time $t+1$: specifically $\mathbf{c}(t+1) = 0$ and $\mathbf{X}^{0,init}(t+1)$ are feasible, since $X_0^{0,init}(t+1) = x_{t+1}^p \subseteq X_1^{*,final}(t)$. ■

Lemma 6 *The optimal objective in (8) at successive iterations of Algorithm 2 satisfies* $J^{*,+} \leq J^*$.

Proof Lemma 3 implies $\mathbf{c} = 0$ and $\mathbf{X}^{0,+}$ are feasible and hence $J^{*+} \leq J(\mathbf{c}^{0,+}, 0, \mathbf{X}^{0,+})$. Lemma 3 implies $X_k^{0,+} \subseteq X_k^*$ and therefore $J(\mathbf{c}^{0,+}, 0, \mathbf{X}^{0,+}) = J(\mathbf{c}^0, \mathbf{c}^*, \mathbf{X}^{0,+}) \leq J^*$ due to the definition of the predicted cost (7) as the worst-case over the tube cross sections $\{X_k,\ k = 0, \ldots, N\}$. ■

Theorem 7 *If the offline computation described in Algorithm 3 is feasible, then the closed loop system defined by the controlled system (1) and the control law of Algorithm 2 robustly satisfies the constraints* $x_t \in \mathcal{X}$ *and* $u_t \in \mathcal{U}$. *Furthermore, the closed loop system satisfies the bound:* $\lim_{t\to\infty} \frac{1}{t} \sum_{t=0}^{t-1}(\|x_t\|_Q^2 + \|u_t\|_R^2) \leq \beta$.

Proof By Lemma 4 and Theorem 5, $\mathbf{c}(t+1) = 0$ and $\mathbf{X}^{0,init}(t+1)$ are feasible at the first iteration of Algorithm 2 at time $t+1$. This implies that $J^{*,init}(t+1) \leq J(\mathbf{c}^{0,init}(t+1), 0, \mathbf{X}^{0,init}(t+1)) \leq J(\mathbf{c}^0(t), \mathbf{c}^*(t), \mathbf{X}^*(t)) - \|x_t\|_Q^2 - \|u_t\|_R^2 + \beta$ due to Lemma 1. Lemma 6 implies that $J^{*,final}(t+1) \leq J^{*,init}(t+1)$ and therefore we obtain $J^{*,final}(t+1) \leq J^{*,final}(t) - \|x_t\|_Q^2 - \|u_t\|_R^2 + \beta$. The finiteness of the optimal predicted cost implies the bound provided on the average stage cost. ■

To extend the theory to the general case we consider again some $\theta_{i^*} \leq 0$. Since, as previously discussed, the linearization terms defined by A_{k,i^*},B_{k,i^*} and $x^0_{k,i^*,r}$ move from (8c) to (8b), whereas the nonlinear (concave) term $\theta_{i^*}[f_{i^*}(x_k, Kx_k + c_k^0 + c_k)]_r$ moves from (8b) to (8c), it follows that the extension of the proofs of Lemma 3 and 4 is straightforward.

7. Example method for online model learning

For a set-based parameter estimation scheme (Lorenzen et al., 2019), we rewrite the model (1) as:

$$x_{t+1} = D_t\theta + d_t + w \tag{14}$$

with $D_t(x_t, u_t) = [f_1(x_t, u_t), \ldots, f_p(x_t, u_t)]$ and $d_t(x_t, u_t) = f_0(x_t, u_t)$. Let $\Theta_t := \{\theta \in \mathbb{R}^p : H_\theta\theta \leq h_t\}\ \forall t \geq 0$ be a fixed complexity parameter set (Lu et al., 2021), where H_θ is fixed and h_t is updated at each time t to obtain an improved set estimate. For an estimation horizon of length N_Θ we determine Θ_{t+1} from the intersection of Θ_t with unfalsified parameter sets over window of N_Θ past time steps. The largest such intersection can be computed by solving a linear program for each row i of H_Θ:

$$[h_{t+1}]_i = \max_{\theta\in\Theta_t} [H_\Theta]_i\theta \quad \text{subject to } \theta \in \Theta_t,\ x_{t+1-l} - D_{t-l}\theta - d_{t-l} \in \mathcal{W}, \forall l \in \{0, \ldots, N_\Theta - 1\}$$

and updating the vertex representation $\Theta_t = \mathrm{Co}\{\theta^v, v = 1, ..., n_\Theta\}$. In Lu et al. (2021) it is shown that the estimated parameter set satisfies $\Theta_{t+1} \subseteq \Theta_t \subseteq \cdots \subseteq \Theta_0$, and under specific conditions that ensure persistency of excitation, the parameter set Θ_t converges to the true parameter θ^*.

8. Simulation results

Example 1: Online model learning. Consider the 2nd order nonlinear system $x_{t+1} = f_0(x_t, u_t) + \theta_1 f_1(x_t, u_t) + w_t$ with $x_t \in \mathbb{R}^2$, $u_t \in \mathcal{U} := \{u \in \mathbb{R} : -1 \leq u \leq 1\}$, $\theta_1 \in \Theta_0 := \{\theta \in \mathbb{R} : 0 \leq \theta \leq 0.05\}$ with true value $\theta_1^* = 0.0375$, and $w_t \in \mathcal{W} := \{w \in \mathbb{R}^2 : -0.05 \leq [w]_i \leq 0.05,\ i = 1, 2\}$. The known part of the model is given by $f_0(x, u) = Ax + Bu$ with $A = \left[\begin{smallmatrix} 0.7 & 1 \\ 0 & 0.7 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and the unknown part is $\theta_1 f_1(x, u) = \theta_1 [x_1^2 \;\; x_2^2]^\top$. The cost weights are $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $R = 50$, the prediction horizon length is $N = 10$, and the simulation total duration is $N_{sim} = 15$ steps. We execute Algorithm 3 based on the LDI bounds $\delta_{x_{1,2}} \leq 1.5$ and obtain $P = \left[\begin{smallmatrix} 3.84 & 9.43 \\ 9.43 & 54.22 \end{smallmatrix}\right]$, $K = [-0.157 \; -0.948]$, $\gamma = 3.65$, and $\beta = 0.006$. Θ_t is updated using a parameter set estimation scheme as described in Section 7 with $H_\Theta = [1 \;\; -1]^\top$, $h_0 = [0.05 \;\; 0]^\top$, estimation window length $N_\Theta = 5$, and $d_t(x_t, u_t) = f_0(x_t, u_t)$, $D_t(x_t, u_t) = [f_1(x_t, u_t)]$. We apply Algorithm 2 with $tol = 10^{-8}$, using (Gurobi Optimization, LLC, 2023) to solve problem (8), requiring on average 3.8 iterations with an average CPU time of 71ms per iteration (AMD Ryzen Pro 7 @ 1.7 GHz). The closed-loop state and control trajectories with w_t uniformly distributed on $\mathcal{W}$ and the evolution of the parameter-set estimate Θ_t are shown in Figure 1.

Example 2: Coupled tank problem. Consider the following model of two connected water tanks[2]

$$[x_{t+1}]_1 = [x_t]_1 - \delta \frac{A_1}{A} \sqrt{2g[x_t]_1} + \delta \frac{k_p}{A} u_t, \tag{15a}$$

$$[x_{t+1}]_2 = [x_t]_2 - \delta \frac{A_2}{A} \sqrt{2g[x_t]_2} + \delta \frac{A_1}{A} \sqrt{2g[x_t]_1}. \tag{15b}$$

Here $[x_t]_1$, $[x_t]_2$ are the depths of fluid in each tank, with $x_t \in \mathcal{X} := \{x \in \mathbb{R}^2 : 0\,\text{cm} < [x]_i \leq 30\,\text{cm},\ i = 1, 2\}$, and $u_t \in \mathcal{U} := \{u \in \mathbb{R} : 0\,\text{V} \leq u \leq 24\,\text{V}\}$ is the command signal to a pump connected to one of the tanks. Each tank has area $A = 15.2\,\text{cm}^2$, the outflow orifice areas are $(A_1, A_2) = (0.13, 0.14)\,\text{cm}^2$, $g = 981\,\text{cm}\,\text{s}^{-2}$ is gravitational acceleration, $k_p = 3.3\,\text{cm}^3\text{s}^{-1}\text{V}^{-1}$ is the pump gain and the sampling interval is $\delta = 1\,\text{s}$. The dynamics in (15) are learnt as a weighted sum of convex radial basis functions (RBF) $x_{t+1} = \theta_0 + \sum_i^p \theta_i f_i(z_t)$ with the multiquadric kernel $f_i(z_t) = (1 + \|z_t - c_i\|^2)^{1/2}$ where $z_t = [x_t^\top \;\; u_t]^\top$ and $c_i \in \mathbb{R}^3$ are precomputed centers. The sign of each parameter θ_i is not constrained; hence the dynamics can be learned as a difference of convex functions (DC), thus providing sufficient representation power for smooth system dynamics (Hartman, 1959). The DC decomposition is $x_{t+1} = g(z_t) - h(z_t)$, where $g(z_t) = \theta_0 + \sum_i^p f_i(z_t) \max\{0, \theta_i\}$ and $h(z_t) = \sum_i^p f_i(z_t) \max\{0, -\theta_i\}$ are convex. The DC decomposition with $p = 49$ and uniformly spaced centers is shown in Fig. 2 (left) for the $[x_{t+1}]_2$ dynamics. Training on 10^3 random samples results in a mean absolute error (MAE) of $[0.095 \;\; 0.044]^\top\,\text{cm}\,\text{s}^{-1}$. The control problem is to drive the system to a reference state $x_k^r = [(A_2/A_1)^2 h_r \;\; h_r]^\top$ with $h_r = 15\,\text{cm}$. We take $Q = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $R = 0.1$, and the MPC prediction horizon has $N = 50$ time steps.

Three cases are considered: (i) a nominal problem in which the MPC algorithm experiences modelling errors due to the RBF approximation error and an additional i.i.d. Gaussian disturbance with standard deviation equal to the MAE; (ii) a perturbed problem in which the flow rate between the tanks in (i) is reduced by 20% after training the RBF; (iii) the same perturbed problem with 20% flow-rate decrease between tanks and online retraining of the RBF parameters. We apply Algorithm 2 with conventional linearisation and solve problem (8) using (MOSEK ApS, 2021),

2. MATLAB code for Example 2 is available at `https://github.com/martindoff/Radial-basis-TMPC`.

requiring on average $0.62\,\mathrm{s}$ per time-step.[3] The cost evaluated along closed loop system trajectories increases by 30% from scenario (i) to (ii), but the suboptimality is reduced to 17% when the RBF parameters are re-trained. The closed loop responses for each case are compared in Fig. 2.

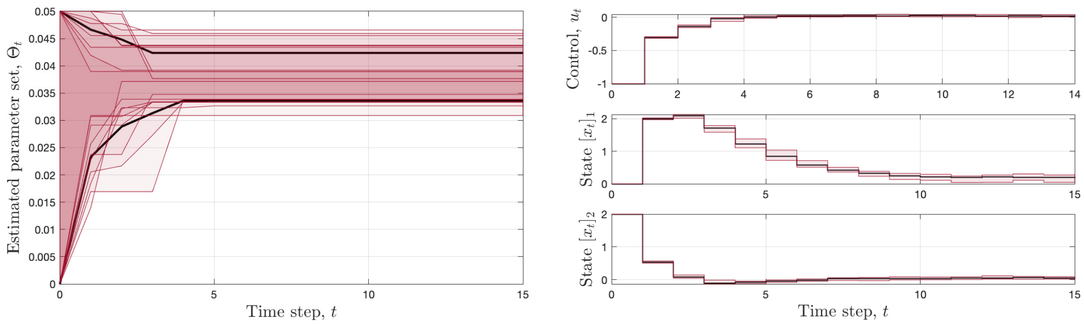

Figure 1: Closed loop responses for 10 simulations of Example 1. Left: Estimated parameter set. Right: control and state trajectories; mean values (black lines) and variation (red shading).

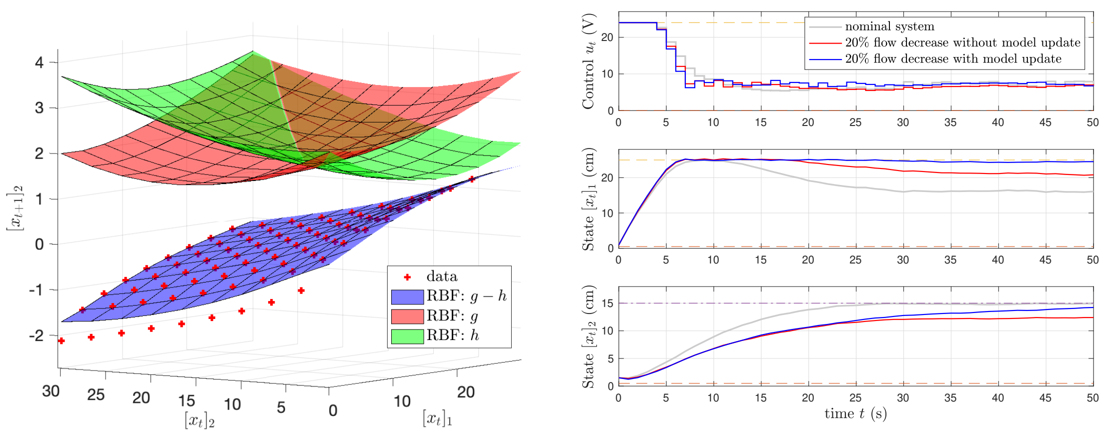

Figure 2: DC decomposition of the model (15b) and responses for Example 2. Left: $f = g - h$ (blue) with g (red) and h (green) fitted to data. Right: closed loop responses for MPC with a nominal model approximation and with a 20% decrease in flow between the tanks, with and without re-training.

9. Conclusions

This paper introduces a convex programming framework for robust nonlinear MPC with bounded additive disturbances and online estimation of model parameters. The approach is applicable to feedforward neural network models with convex activation functions, allowing the linear parameters of the output layer to be estimated online. The method is based on sequential convex approximation, allowing efficient implementation, and the online optimization can be terminated after an arbitrary number of iterations without compromising stability or constraint satisfaction. Guarantees of recursive feasibility and closed-loop performance bounds are derived. Further work will address the scalability of the approach with respect to the state dimension by replacing the hyperrectangular tube cross sections in the vertex formulation of Algorithm 2 with simplex cross sections.

3. Computation may be reduced significantly using a bespoke first order solver, as discussed in Doff-Sotta et al. (2022).

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