Proceedings of Machine Learning Research vol 242:1–13, 2024 6th Annual Conference on Learning for Dynamics and Control

Strengthened stability analysis of discrete-time Lurie systems involving ReLU neural networks

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Editors: A. Abate, K. Margellos, A. Papachristodoulou

Abstract

This paper addresses the stability analysis of a discrete-time (DT) Lurie system featuring a static repeated ReLU nonlinearity. Such systems often arise in the analysis of recurrent neural networks and other neural feedback loops. Custom quadratic constraints, satisfied by the repeated ReLU, are employed to strengthen the standard DT Circle and DT Popov Criteria for this specific Lurie system. The criteria can be expressed as a set of linear matrix inequalities (LMIs) with less restrictive conditions on the matrix variables. It is further shown that if the Lurie system under consideration has a unique equilibrium point at the origin, then this equilibrium point is in fact globally stable or unstable, meaning that local stability analysis will provide no additional benefit. Numerical examples demonstrate that the strengthened criteria achieve a desirable balance between reduced conservatism and complexity when compared to existing criteria.

Keywords: Lyapunov methods, neural networks, semi-definite programming, stability of nonlinear systems

1. Introduction

This paper considers the dichotomy between *model-based* and *learning-based* control design and analysis. Model-based approaches use white box system models and control policies which accommodate performance certificates. Learning-based methods, like reinforcement learning, use expressive black box models and control policies which are more challenging to verify performance certificates for, making them unsuitable for safety critical systems. To bridge this gap, the paper focuses on stability analysis for systems with neural network (NN) involvement. This approach certifies the performance of systems using learning-based NN models and control policies, combining the strengths of both paradigms: performance certification from model-based methods as well as expressive models and policies from learning-based methods.

Literature: Numerous systems incorporating NNs can be modelled as discrete-time (DT) Lurie systems, Figure 1. In this context, the nonlinearity, $\Phi(\cdot)$, is a vector function that encompasses all the individual scalar NN activation functions. For instance, a DT linear time-invariant (LTI) system interconnected with a feed-forward NN Pauli et al. (2021) and the discretised counterpart to the rate-based recurrent neural network (RNN), introduced in Wilson and Cowan (1972) and more recently studied in Kozachkov et al. (2022). To ensure the stability of a DT Lurie system, one can employ various absolute stability criteria, including the classical DT Circle and DT Popov[1] Criteria Haddad and Bernstein (1994), alternative Lyapunov-based criteria Kapila and Haddad (1996); Park et al. (2019); Drummond and Valmorbida (2023), and Zames-Falb multipliers Turner and Drummond (2021); Ahmad et al. (2014); Carrasco et al. (2019). These criteria vary in their approach to balancing computational complexity and conservatism. The DT Circle and DT Popov Criteria offer lower complexity, while Park and Zames-Falb multipliers have higher complexity, but tend to be less conservative.

The absolute stability criteria mentioned earlier can all be formulated as semi-definite programming (SDP) problems involving linear matrix inequalities (LMIs) Boyd et al. (1994). Efficient solutions to these problems can be achieved in polynomial time using tools like MOSEK and the LMI toolbox Andersen and Andersen (2000); Balas et al. (2007). Consequently, recent research has utilized the absolute stability framework and associated SDP tools to address various challenges in NN analysis. This includes tasks such as estimating the region of attraction Yin et al. (2021a); Hashemi et al. (2021); Wang et al. (2023), synthesizing NN controllers Yin et al. (2021b); Junnarkar et al. (2022), and conducting robustness analysis Pauli et al. (2022); Fazlyab et al. (2021, 2020); Newton and Papachristodoulou (2023). The primary challenge in NN analysis lies in the typically large number of activation functions, denoted by m. This leads to increased computational complexity in absolute stability problems compared to cases with a small m. Some less conservative tools, like Zames-Falb multiplier analysis, exhibit poor scalability with m, resulting in computational issues. Conversely, simpler tools like the DT Circle Criterion become overly conservative for large m, limiting their utility in NN analysis. Few articles have tackled this problem; in continuous-time Richardson et al. (2023) enhanced the low complexity Circle and Popov Criteria for the Lurie system with repeated ReLU nonlinearity and Yin et al. (2021a) narrowed down the stability analysis scope to a local region, leveraging local properties of the nonlinearity to reduce conservatism.

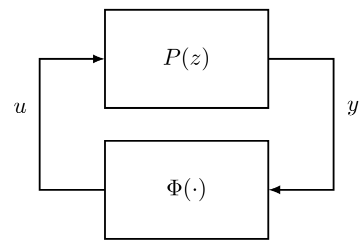


Figure 1: DT Lurie system

Contribution: This paper focuses on the *specialised DT Lurie system*, where the nonlinearity is the repeated ReLU, commonly used in deep learning. The first contribution confronts the challenge of balancing conservatism and computational complexity in absolute stability problems encountered in NN analysis. It achieves this by enhancing the low complexity DT Circle and DT Popov Criteria tailored for this specialised Lurie system (Theorem 12 and Theorem 14). The second contribution is the remarkable discovery that, under certain conditions, local stability at the origin of the specialised DT Lurie system is equivalent to global stability (Theorem 19). The consequence of this is that, if global stability is not provable, then attempts to prove local stability will be similarly futile.

2. Preliminaries

2.1. Notation

The sets of non-negative real numbers, m-dimensional vectors with non-negative elements, and square m-dimensional matrices with non-negative elements are, respectively, denoted by $\Re_{\geq 0}$, $\Re^m_{\geq 0}$, and $\Re^{m\times m}_{\geq 0}$. The set of square m-dimensional symmetric positive definite matrices is represented by $\mathcal{S}^m_+$, with the diagonal subset $\mathcal{D}^m_+ \subset \mathcal{S}^m_+$. The set of square m-dimensional matrices with non-positive off-diagonal elements is denoted by $\mathcal{Z}^m \subset \Re^{m\times m}$: these are known as *Z-matrices*. A

1. Several versions of the DT Popov Criterion exist. When the DT Popov Criterion is referenced in this article, we always mean the version presented in (Haddad and Bernstein, 1994, Theorem 4.3).

matrix M of elements m_{ij} is sometimes expressed as $M = [m_{ij}]$, a negative definite matrix is denoted by $M \prec 0$ and $He(M) := M + M'$. The space of real rational transfer function matrices, analytic in the open unit disk, is represented by $\mathcal{RH}_\infty$.

2.2. Mean Value Theorem (MVT)

Theorem 1 (MVT) *If $\phi : \Re \to \Re$ is continuous on $[a, b]$, there is a point $\epsilon \in (a, b)$ such that:*

$$\int_a^b \phi(\sigma)d\sigma = (b-a)\phi(\epsilon) \tag{1}$$

Proof: (Wrede and Spiegel, 2010, Page 100). □

Theorem 2 (MVT for slope-restricted functions) *If $\phi : \Re \to \Re$ is continuous on $[a, b]$ and slope-restricted on $[0, \mu]$, then the following inequality must hold:*

$$\int_a^b \phi(\sigma)d\sigma \leq \mu(b-a)^2 + \phi(a)(b-a) \tag{2}$$

Proof: If $\epsilon \in (a, b)$ and $\phi(\cdot)$ is slope-restricted on the same domain, then (3) must hold since $0 < \epsilon - a < b - a$. Subbing (3) into (1) results in Theorem 2.

$$0 \leq \frac{\phi(\epsilon) - \phi(a)}{\epsilon - a} \leq \mu \quad \implies \quad \phi(a) \leq \phi(\epsilon) \leq \mu(b-a) + \phi(a) \tag{3}$$

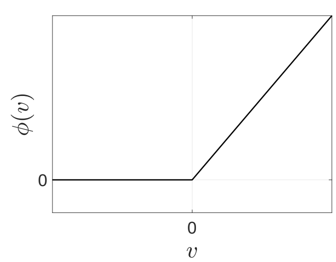
□

2.3. Properties of the ReLU function

The global stability analysis results are constructed on the observation that the ReLU function satisfies a number of properties (see Richardson et al. (2023) for more details); the properties relevant to this paper are summarised in Table 1. Although the slope-restricted property holds for many static nonlinearities, the first four properties are much less typical. In fact, the complementarity property holds for few activation functions other than ReLU.

Definition 3 (Repeated ReLU) *The ReLU function $\phi(\cdot) : \Re \to \Re_{\geq 0}$ is defined in (4) and illustrated in Figure 2. As a consequence, the repeated ReLU $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$ is defined by (5).*

$$\phi(v) := \begin{cases} v & v \geq 0 \\ 0 & v < 0 \end{cases} \tag{4}$$



$$\Phi(\cdot) := \begin{bmatrix} \phi(\cdot) \\ \vdots \\ \phi(\cdot) \end{bmatrix} \tag{5}$$

Figure 2: ReLU function

Fact 4 *The repeated ReLU $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$ can be expressed by (6) where $u(\cdot)$ is the unit step function and $U(v) = diag(u(v_1) \dots u(v_m))$.*

$$\Phi(v) = U(v)v \tag{6}$$

Proof: $\phi(v_i) = u(v_i)v_i$. Expressing this in vector form results in Fact 4. □

Table 1: Properties of the ReLU function

$\phi(v) \geq 0$	$\forall v \in \Re$	Positive
$\phi(v) - v \geq 0$	$\forall v \in \Re$	Positive complement
$\phi(v)(v - \phi(v)) = 0$	$\forall v \in \Re$	Complementarity
$\phi(\alpha v) = \alpha\phi(v)$	$\forall v \in \Re, \alpha \in \Re_{\geq 0}$	Positive homogeneity
$0 \leq \frac{\phi(\tilde{v})-\phi(v)}{\tilde{v}-v} \leq 1$	$\forall \tilde{v}, v \neq \tilde{v} \in \Re$	Slope-restricted

2.4. Positively homogenous functions

A key property of the ReLU function, and the repeated ReLU by extension, which enables one to prove the remarkable results in Section 4 is *positive homogeneity*. Several facts about positively homogenous functions (Table 1) are introduced below.

Fact 5 *If $\theta(\cdot) : \Re^m \to \Re^m$ is bijective and positively homogenous, then $\theta^{-1}(\cdot) : \Re^m \to \Re^m$ is positively homogenous too.*

Proof: Assume that $\theta^{-1}(\cdot)$ exists and is not positively homogenous, that is $\alpha\theta^{-1}(v) \neq \theta^{-1}(\alpha v)$. However, because $\theta(\cdot)$ *is* positively homogenous and bijective, it follows that for all $\alpha \in \Re_{\geq 0}$:

$$z = \theta^{-1}\Big(\frac{1}{\alpha}\theta(\alpha z)\Big) \tag{7}$$

Now, by the assumption that $\theta^{-1}(\cdot)$ is not positively homogenous, this means that:

$$z \neq \frac{1}{\alpha}\theta^{-1} \circ \theta(\alpha z) = z \tag{8}$$

Clearly this is a contradiction and hence $\theta^{-1}(\cdot)$ must be positively homogenous. $\square$

Fact 6 *Let $\theta(\cdot) : \Re^m \to \Re^m$ be defined by $\theta(v) := v - D\Phi(v)$ for some matrix $D \in \Re^{m\times m}$. If $\Phi(\cdot) : \Re^m \to \Re^m$ is positively homogenous, then so is $\theta(\cdot)$.*

Proof: $\alpha\theta(v) = \alpha v - D\alpha\Phi(v) = \alpha v - D\Phi(\alpha v) = \theta(\alpha v)$ for all $\alpha \in \Re_{\geq 0}$. $\square$

2.5. Quadratic constraints satisfied by the Repeated ReLU

Using the properties from Table 1, novel or less restrictive quadratic constraints (QCs) were constructed in (Richardson et al., 2023, Section 3) which are satisfied by the repeated ReLU. The same QCs are leveraged in Section 3 of this work, so are restated without proof in Facts 7-8. In summary, Fact 7 leverages the positive, positive complement and complementarity properties to derive a less restrictive variation of the sector-bounded QC and Fact 8 uses the positive property to derive a novel QC. Fact 9 is the well known slope-restricted QC which is also satisfied by the Repeated ReLU; the proof of this can also be found in (Richardson et al., 2023, Section 3).

Fact 7 (Sector-like QC) *Let $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$ be the repeated ReLU. If $\mathbf{V} \in \mathcal{Z}^m$ then the following QC holds:*

$$\Phi(v)'\mathbf{V}\Big(v - \Phi(v)\Big) \geq 0 \;\; \forall\, v \in \Re^m \tag{9}$$

Fact 8 (Positivity QC) *Let $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$ be the repeated ReLU. If $\mathbf{Q}_{11} \in \Re^{m\times m}_{\geq 0}$ then the following QC holds:*

$$\Phi(v)'\mathbf{Q}_{11}\Phi(\tilde{v}) \geq 0 \quad \forall\, v, \tilde{v} \in \Re^m \tag{10}$$

Note that (10) also holds for the special case $\tilde{v} = v$.

Fact 9 (Slope-restricted QC) *Let $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$ be the repeated ReLU and $\Psi(\tilde{v}, v) := \Phi(\tilde{v}) - \Phi(v)$. If $\mathbf{W} \in \mathcal{D}^m_+$ then the following QC is satisfied:*

$$\Psi(\tilde{v}, v)'\mathbf{W}\Big(\tilde{v} - v - \Psi(\tilde{v}, v)\Big) \geq 0 \;\; \forall\, \tilde{v}, v \in \Re^m \tag{11}$$

2.6. Problem setup

Consider the DT Lurie system in Figure 1, where $P(z) \in \mathcal{RH}_\infty$ is a finite dimensional DT LTI system with state space realisation (A, B, C, D) and the static nonlinearity $\Phi(\cdot)$ is the repeated ReLU. The Lurie system is modelled by (12) with $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{m\times n}$ and $D \in \Re^{m\times m}$. As the repeated ReLU satisfies $\Phi(0) = 0$, the origin is an equilibrium point of (12).

$$x_{k+1} = Ax_k + B\Phi(y_k) \qquad y_k = Cx_k + D\Phi(y_k) \tag{12}$$

Assumption 10 (Well-posedness) *System (12) with $\Phi(\cdot)$ being the repeated ReLU is well-posed.*

Well-posedness is equivalent to the existence of a unique solution to the state space equations (12). Since $\Phi(\cdot)$ is globally Lipschitz (and differentiable almost everywhere), this is ensured if there exists a unique solution $y_k = \theta^{-1}(Cx_k)$ to $\theta(y_k) := y_k - D\Phi(y_k) = Cx_k$. A sufficient condition for this is given by Lemma 11, adapted from (Valmorbida et al., 2018, Section II).

Lemma 11 (Well-posedness) *Assumption 10 holds if there exists $\mathbf{V} \in \mathcal{D}^m_+$ such that:*

$$2\mathbf{V} - \mathbf{V}D - D'\mathbf{V} \succ 0 \tag{13}$$

In many absolute stability results (e.g., DT Circle Criterion) this LMI (13) is an intrinsic part of the stability conditions, so well-posedness is guaranteed. For instance, for the zero order hold (ZOH) discretisation of the rate-based RNN Wilson and Cowan (1972), system (12) has the form $A = e^{-\tau I}$, $B = \int_0^\tau e^{-\tau I} W d\tau$, $C = I$, $D = 0$ and hence is, trivially, well-posed. A comprehensive examination of well-posedness is outside the paper's scope; it is enough to state that numerous DT systems incorporating NNs inherently exhibit well-posedness. Refer to Richardson et al. (2023) for more discussion of this.

Throughout the remainder of the paper, Assumption 10 is assumed to hold. This implies a solution $y_k = \theta^{-1}(Cx_k)$ exists to (12); furthermore the solution is positively homogenous by Fact 5 and Fact 6, which relies on the positive homogeneity of the repeated ReLU.

3. Global stability analysis

This section applies the DT counterpart of the Barbashin-Krasovskii Theorem Khalil (2002) to derive two LMIs which verify the origin of the specialised Lurie system (12) is globally asymptotically stable (GAS). The main results are constructed using quadratic and Lurie-type Lyapunov candidates, as is respectively the case in the DT Circle and DT Popov Criteria. The QCs introduced in Section 2.5 are leveraged within the proofs.

Theorem 12 (DT Circle-like Criterion) *Consider the DT Lurie system (12) with $\Phi(\cdot)$ the repeated ReLU. Let Assumption 10 be satisfied. If there exists $\mathbf{P} \in \mathcal{S}_+^n$, $\mathbf{V} \in \mathcal{Z}^m$, and $\mathbf{Q}_{11} \in \Re_{\geq 0}^{m\times m}$ such that:*

$$\begin{bmatrix} A'\mathbf{P}A - \mathbf{P} & A'\mathbf{P}B + C'\mathbf{V}' \\ \star & B'\mathbf{P}B + He\left(\mathbf{Q}_{11} - \mathbf{V}(I-D)\right) \end{bmatrix} \prec 0 \tag{14}$$

then the origin of (12) is GAS.

Proof: Choosing a quadratic Lyapunov candidate $V_{cl}(x) = x'\mathbf{P}x$ with $\mathbf{P} \in \mathcal{S}_+^n$ and looking at the difference along the trajectories of system (12) gives:

$$\begin{aligned} \Delta V_{cl} &= \left(Ax_k + B\Phi_k\right)'\mathbf{P}\left(Ax_k + B\Phi_k\right) - x_k'\mathbf{P}x_k \\ &= x_k'(A'\mathbf{P}A - \mathbf{P})x_k + x_k'A'\mathbf{P}B\Phi_k + \Phi_k'B'\mathbf{P}Ax_k + \Phi_k'B'\mathbf{P}B\Phi_k \end{aligned} \tag{15}$$

where $\Delta V_{cl} := V_{cl}(x_{k+1}) - V_{cl}(x_k)$ and $\Phi_k := \Phi(y_k)$. Appending the sector-like QC (9) and the positivity QC (10) for the special case $\tilde{y}_k = y_k$, leads to:

$$\begin{aligned} \Delta V_{cl} \leq\ & x_k'(A'\mathbf{P}A - \mathbf{P})x_k + x_k'A'\mathbf{P}B\Phi_k + \Phi_k'B'\mathbf{P}Ax_k + \Phi_k'B'\mathbf{P}B\Phi_k \\ & + 2\Phi_k'\mathbf{V}\Big(y_k - \Phi_k\Big) + 2\Phi_k'\mathbf{Q}_{11}\Phi_k \end{aligned} \tag{16}$$

The right hand side of (16) is guaranteed to be negative definite if Theorem 12 is satisfied. This can be seen by subbing (12) in for y_k, then putting (16) into quadratic form. □

Remark 13 *Theorem 12 strengthens the DT Circle Criterion when $\Phi(\cdot)$ is the repeated ReLU. One expects Theorem 12 to verify GAS for a larger space of (A, B, C, D) matrices since the DT Circle Criterion has more restrictions on the LMI variables: $\mathbf{V} \in \mathcal{D}_+^m$ and $\mathbf{Q}_{11} = 0$.*

Theorem 14 (DT Popov-like Criterion) *Consider the Lurie system (12) with $\Phi(\cdot)$ the repeated ReLU and let $D = 0$. If there exists $\mathbf{P} \in \mathcal{S}_+^n$; $\mathbf{H} \in \Re^{m\times m}$; $\mathbf{\Lambda}, \mathbf{W} \in \mathcal{D}_+^m$; $\mathbf{V} \in \mathcal{Z}^m$; $\tilde{\mathbf{Q}}_{11}, \mathbf{Q}_{11} \in \Re_{\geq 0}^{m\times m}$ such that:*

$$\begin{bmatrix} X_{11} & X_{12} & (A-I)'C'\mathbf{H}'\mathbf{\Lambda} + C'(\mathbf{H}-I)'\mathbf{W} \\ \star & X_{22} & B'C'\mathbf{H}'\mathbf{\Lambda} + \tilde{\mathbf{Q}}_{11} \\ \star & \star & -2\mathbf{W} \end{bmatrix} \prec 0 \tag{17}$$

$$\begin{aligned} X_{11} &= A'\mathbf{P}A - \mathbf{P} + He\left((A-I)'C'\mathbf{H}'\mathbf{\Lambda}\mathbf{H}C(A-I)\right) \\ X_{12} &= A'\mathbf{P}B + C'\mathbf{V}' + (A-I)'C'\mathbf{H}'\mathbf{\Lambda} + 2(A-I)'C'\mathbf{H}'\mathbf{\Lambda}\mathbf{H}CB \\ X_{22} &= B'\mathbf{P}B + He\left(\mathbf{Q}_{11} - \mathbf{V} + \mathbf{\Lambda}\mathbf{H}CB + B'C'\mathbf{H}'\mathbf{\Lambda}\mathbf{H}CB + \tilde{\mathbf{Q}}_{11}\right) \end{aligned}$$

then the origin of (12) is GAS.

Proof: A generalised Lurie-type Lyapunov candidate (Richardson et al., 2023, Section 4) $V_{pl}(x) = x'\mathbf{P}x + 2\int_0^{\mathbf{H}y_k} \mathbf{\Lambda}\Phi(\sigma)\cdot d\sigma$ with $\mathbf{P} \in \mathcal{S}_+^n$, $\mathbf{H} \in \Re^{m\times m}$ and $\mathbf{\Lambda} \in \mathcal{D}_+^m$ was selected. Due to the integral term in the generalised Lurie-type Lyapunov candidate, Theorem 14 may only be applied to systems with $D = 0$, as is the case with the DT Popov Criterion. Now, looking at the difference along the trajectories of system (12) results in:

$$\begin{aligned} \Delta V_{pl} = 2\int_0^{\mathbf{H}y_{k+1}} \mathbf{\Lambda}\Phi(\sigma)\cdot d\sigma - 2\int_0^{\mathbf{H}y_k} \mathbf{\Lambda}\Phi(\sigma)\cdot d\sigma & \\ + \underbrace{(Ax_k + B\Phi_k)'\mathbf{P}(Ax_k + B\Phi_k) - x_k'\mathbf{P}x_k}_{=\Delta V_{cl}} & \end{aligned} \tag{18}$$

where $\Delta V_{pl} := V_{pl}(x_{k+1}) - V_{pl}(x_k)$ and $\Phi_k := \Phi(y_k)$. Since the line integrals are path independent (see (Richardson et al., 2023, Section 4)), assume the system follows a trajectory which begins at the origin, passes through $\mathbf{H}y_k$ and ends at $\mathbf{H}y_{k+1}$, without loss of generality. As a result:

$$\Delta V_{pl} = \Delta V_{cl} + 2\int_{\tilde{y}_k}^{\tilde{y}_{k+1}} \mathbf{\Lambda}\Phi(\sigma)\cdot d\sigma \tag{19}$$

where $\tilde{y}_k := \mathbf{H}y_k$. As $\mathbf{\Lambda} = \text{diag}(\lambda_1 \dots \lambda_m)$, we may express the line integral as:

$$\int_{\tilde{y}_k}^{\tilde{y}_{k+1}} \mathbf{\Lambda}\Phi(\sigma)\cdot d\sigma = \sum_{i=1}^{m} \lambda_i \int_{\tilde{y}_{k,i}}^{\tilde{y}_{k+1,i}} \phi(\sigma_i) d\sigma_i \tag{20}$$

The ReLU function, $\phi(\cdot)$, is slope-restricted on $[0,1]$; hence, Theorem 2 can be applied with $a = \tilde{y}_{k,i}$, $b = \tilde{y}_{k+1,i}$ and $\mu = 1$. Expressing the resulting summation in quadratic form and subbing into (19) leads to:

$$\Delta V_{pl} \le \Delta V_{cl} + 2\tilde{\Phi}_k'\mathbf{\Lambda}\Big(\tilde{y}_{k+1} - \tilde{y}_k\Big) + 2\Big(\tilde{y}_{k+1} - \tilde{y}_k\Big)'\mathbf{\Lambda}\Big(\tilde{y}_{k+1} - \tilde{y}_k\Big) \tag{21}$$

where $\tilde{\Phi}_k := \Phi_k(\tilde{y}_k)$. Appending the sector-like QC (9), both cases of the positivity QC (10) and the slope-restricted QC (11) gives:

$$\begin{aligned}\Delta V_{pl} \le\ & 2\tilde{\Phi}_k'\mathbf{\Lambda}\Big(\tilde{y}_{k+1} - \tilde{y}_k\Big) + 2\Big(\tilde{y}_{k+1} - \tilde{y}_k\Big)'\mathbf{\Lambda}\Big(\tilde{y}_{k+1} - \tilde{y}_k\Big) \\ & + \underbrace{\Delta V_{cl} + 2\Phi_k'\mathbf{V}\Big(y_k - \Phi_k\Big) + 2\Phi_k'\mathbf{Q}_{11}\Phi_k}_{(14)\text{ when } D=0} + 2\Phi_k'\tilde{\mathbf{Q}}_{11}\tilde{\Phi}_k + 2\Psi_k'\mathbf{W}\Big(\tilde{y}_k - y_k - \Psi_k\Big)\end{aligned} \tag{22}$$

where $\Psi_k := \Psi(\tilde{y}_k, y_k) = \tilde{\Phi}_k - \Phi_k$. Now, substituting $\tilde{\Phi}_k = \Psi_k + \Phi_k$ and replacing $\tilde{y}_{k+1}, \tilde{y}_k, y_k$ with the respective functions of x_k gives:

$$\begin{aligned}\Delta V_{pl} \le\ & 2\Big(\Psi_k + \Phi_k\Big)'\mathbf{\Lambda}\mathbf{H}C\Big((A-I)x_k + B\Phi_k\Big) + 2\Phi_k'\tilde{\mathbf{Q}}_{11}\Big(\Psi_k + \Phi_k\Big) \\ & + 2\Big((A-I)x_k + B\Phi_k\Big)'C'\mathbf{H}'\mathbf{\Lambda}\mathbf{H}C\Big((A-I)x_k + B\Phi_k\Big) + 2\Psi_k'\mathbf{W}\Big((\mathbf{H}-I)Cx_k - \Psi_k\Big) \\ & + \underbrace{\Delta V_{cl} + 2\Phi_k'\mathbf{V}\Big(y_k - \Phi_k\Big) + 2\Phi_k'\mathbf{Q}_{11}\Phi_k}_{(14)\text{ when } D=0}\end{aligned} \tag{23}$$

The right hand side of (23) is guaranteed to be negative definite if Theorem 14 is satisfied. This can be seen by substituting the associated quadratic form of LMI (14) into (23) when $D = 0$, then manipulating the full expression into quadratic form. ☐

Remark 15 *Theorem 14 contains matrix variable products which prevents the matrix inequality from being linear. Setting $\mathbf{H} = I$ is one convex relaxation which reduces (17) to an LMI. Furthermore, this relaxation forces $\Psi(\tilde{y}_k, y_k) \equiv 0$, hence (17) collapses to the (2×2) block matrix:*

$$\begin{bmatrix} A'\mathbf{P}A - \mathbf{P} + He\left((A-I)'C'\mathbf{\Lambda}C(A-I)\right) & A'\mathbf{P}B + C'\mathbf{V}' + (A-I)'C'\mathbf{\Lambda} + 2(A-I)'C'\mathbf{\Lambda}CB \\ \star & B'\mathbf{P}B + He\left(\mathbf{Q}_{11} - \mathbf{V} + \mathbf{\Lambda}CB + B'C'\mathbf{\Lambda}CB + \tilde{\mathbf{Q}}_{11}\right) \end{bmatrix} \prec 0 \tag{24}$$

Remark 16 *Remark 15 strengthens the DT Popov Criterion when $\Phi(\cdot)$ is the repeated ReLU. One expects Remark 15 to verify GAS for a larger space of (A, B, C) matrices since the DT Popov Criterion has more restrictions on the LMI variables: $\mathbf{V} \in \mathcal{D}_+^m$ and $\mathbf{Q}_{11} = \tilde{\mathbf{Q}}_{11} = 0$.*

4. Local stability is equivalent to global stability

As many properties of the ReLU function hold globally, limiting the scope of the stability analysis to a local region did not seem promising, unlike in Yin et al. (2021a). For this reason, conditions which ensured an asymptotically stable equilibrium point was, in fact, GAS were investigated.

Theorem 17 *System (12) has a unique equilibrium point at the origin if all matrices in the set $\mathcal{D}$ are full rank where:*

$$\mathcal{D} = \left\{ A - I + BU(I - DU)^{-1}C \mid U \in \mathcal{U} \right\} \tag{25}$$

and the 2^m possible permutations of U are captured by the set:

$$\mathcal{U} = \{\mathrm{diag(u_1, \dots, u_m)} \mid \ \mathrm{u_i} \in 0, 1 \textit{ and } i \in 1, 2, \dots, m\} \tag{26}$$

Proof: Any equilibrium point, x_{eq}, of (12) must satisfy $x_{k+1} - x_k = 0$. This is equivalent to:

$$0 = (A - I)x_{k,eq} + B\Phi(y_{k,eq}) \qquad y_{k,eq} = Cx_{k,eq} + D\Phi(y_{k,eq}) \tag{27}$$

Using Fact 4, the output equation of (27) may equivalently be expressed as:

$$\Big(I - DU(y_{k,eq})\Big)y_{k,eq} = Cx_{k,eq} \;\;\Leftrightarrow\;\; y_{k,eq} = \underbrace{\Big(I - DU(y_{k,eq})\Big)^{-1} C}_{:=K(y_{k,eq})} x_{k,eq} \tag{28}$$

Equation (28) does not provide the general solution to the output equation, but instead a relationship between an equilibrium state and the corresponding system output. Using Fact 4 and subbing (28) into the state equation of (27) results in a single equation which any equilibrium point must satisfy.

$$0 = (A-I)x_{k,eq} + BU(y_{k,eq})y_{k,eq} = \Big(A - I + BU(y_{k,eq})K(y_{k,eq})\Big)x_{k,eq} =: \breve{A}(y_{k,eq})x_{k,eq} \tag{29}$$

Since $U(y_{k,eq}) \in \mathcal{U}$, then the matrices $\breve{A}(y_{k,eq}) \in \mathcal{D}$. Therefore, if all matrices in $\mathcal{D}$ have full rank, the only solution to equation (29) is $x_{k,eq} = 0$. This condition is sufficient, but may be conservative as the state equations may only allow $U(y_k)$ to enter a subset of the 2^m possible permutations. $\square$

Remark 18 *As a square matrix is only full rank if it is also invertible, Theorem 17 can be verified by computing the determinant for each matrix in the set $\mathcal{D}$. If all are non-zero, the set $\mathcal{D}$ is full rank.*

Theorem 19 (LAS $\equiv$ GAS for a DT Lurie system with repeated ReLU nonlinearity) *If the origin of (12) is a unique equilibrium point and a ball of any radius r_x can be established as a region of attraction $\mathcal{B}_{rx} = \{x : ||x|| \leq r_x\}$ then, the origin is actually a GAS equilibrium point.*

Proof: Under Assumption 10, system (12) can be expressed as:

$$x_{k+1} = Ax_k + B\Phi \circ \theta^{-1}(Cx_k) \tag{30}$$

where $\theta(y_k) := y_k - D\Phi(y_k) = Cx_k$. Now assume Theorem 17 is satisfied, which guarantees the origin is a unique equilibrium point of (30). Furthermore, assume it can be established the origin of (30) has a region of attraction given by $\mathcal{B}_{rx} := \{x : ||x|| \leq r_x\}$. Now, define a positively scaled initial state $z_0 := \alpha x_0$ where $\alpha > 0$. The initial state z_0 will follow the trajectory:

$$z_k(z_0) = \alpha x_k(x_0) \tag{31}$$

which is an instance of the general solution to the scaled system:

$$\begin{aligned} z_{k+1} &= \alpha x_{k+1} = \alpha A x_k + \alpha B \Phi \circ \theta^{-1}(C x_k) \\ &= A(\alpha x_k) + B\Phi \circ \theta^{-1}\Big(C(\alpha x_k)\Big) \quad \text{(See Fact 5 and Fact 6)} \\ &= A z_k + B \Phi \circ \theta^{-1}(C z_k) \end{aligned} \tag{32}$$

As the scaled system (32) is of the same form as (30), a region of attraction $\mathcal{B}_{rz} = \{z : ||z|| \leq r_z\}$ must exist around the origin of (32). Using (31), a relationship between $\mathcal{B}_{rx}$ and $\mathcal{B}_{rz}$ can be established. First, $\mathcal{B}_{rx}$ may equivalently be expressed in terms of z:

$$\mathcal{B}_{rx} = \{x : ||x|| \leq r_x\} = \{\frac{1}{\alpha} z : \frac{1}{\alpha}||z|| \leq r_x\} \tag{33}$$

A new set may then be defined which contains the unscaled vectors, z, satisfying (33):

$$\mathcal{B}_{rz} = \{z : ||z|| \leq \alpha r_x\} \tag{34}$$

This is the region of attraction, where we have found $r_z = \alpha r_x$. As α can represent any positive scalar, one may set $\mathcal{B}_{rz} = \Re^n$. Thus, if it can be shown the origin of (30) has a region of attraction, it implies the origin of (30) is actually GAS, since (30) and (32) are equivalent. □

Remark 20 *The implications of Theorem 19 are profound: provided the origin is a unqiue equilibrium of system (30), then if global stability cannot be established, it is* futile *to attempt to establish local stability. This is somewhat counterintuitive as typically in absolute stability analysis if one cannot establish global stability, one attempts a local stability analysis. Theorem 19 implies that, for ReLU problems, this will not be fruitful.*

5. Numerical examples

The maximum series gain was used to compare the conservatism of the criteria developed in this paper against criteria with low (DT Circle and DT Popov Haddad and Bernstein (1994)) and high (Park et al. (2019)) complexity. Matlab employs the Projective method Gahinet and Nemirovski (1997) to solve the stability criteria posed as SDP problems with LMIs. The complexity of the criteria was assessed by comparing the total number of decision variables, N, given the number of floating-point operations per iteration is proportional to N^3.

5.1. Experimental setup

The configuration entailed substituting $\Phi(y_k) \to \alpha\Phi(y_k)$ in (12) with $\alpha \in \Re_{\geq 0}$, which is equivalent to replacing $B \to \alpha B$ and $D \to \alpha D$ in the LMIs of each criterion. The maximum series gain represents the highest value of α for which each criterion can verify the origin of (12) is GAS. The Nyquist gain provides an upper bound on this quantity.

Table 2 gives a list of the examples and values of (n, m) where $x \in \Re^n$ and $\Phi(\cdot) : \Re^m \to \Re^m$. Each example features a distinct state space model (A, B, C, D) from the literature. The examples were the same as in Richardson et al. (2023), but were discretised using the ZOH method with a sampling period of $T_s \in \{10^{-2}, 10^{-4}\}$ seconds. Each state space model set $D = 0$ to facilitate comparison with the DT Popov and DT Popov-like Criteria. Consult the related code[2] to see the specific examples. Finally, as the DT Popov-like Criterion was a BMI, the convex relaxation stated in Remark 15 was deployed, to avoid losing the benefits of framing problems as SDP with LMIs.

2. https://github.com/CR-Richardson/DT-Max-Series-Gain

Table 2: Maximum series gain and the number of decision variables for various criteria.

Ex	n	m	Maximum series gain (left) and number of decision variables (right) DT Circle		Theorem 12		DT Popov		Remark 15		Park		Nyquist gain
1	9	3	20.8659	48	39.4281	63	411.5889	51	3310.3797	66	450.4463	579	6666.6651
2	3	3	89.9000	9	89.9000	24	89.9000	12	89.9000	27	89.9000	159	89.9000
3	3	4	0.5236	10	0.6818	38	0.5236	14	0.6818	42	0.5236	207	0.6983
4	8	4	0.0010	40	0.0012	68	0.0010	44	0.0015	72	0.0010	572	0.0020
5	6	4	0.0813	25	0.0814	53	0.0824	29	0.0829	57	0.0845	402	0.0869
6	6	4	0.1951	25	0.3916	53	0.1951	29	0.5007	57	0.2272	402	0.8212
7	8	4	0.0967	40	0.1234	68	0.0969	44	0.1450	72	0.1037	572	0.2008
8	5	5	2.0221	20	2.0221	65	2.0221	25	2.0221	70	2.0221	395	2.0221

5.2. Discussion

Theorem 12 and Remark 15 were of similar complexity. Both had slightly higher complexity than the DT Circle and DT Popov Criteria, but significantly lower than the Park Criterion. Example 8 demonstrates a positive system wherein the multivariable Aizerman Conjecture is valid for the continuous time counterpart Drummond et al. (2022). In this scenario, all approaches attained the linear upper limit on α. In 7 out of 8 examples, Remark 15 exhibits conservatism that is either equal to or less than that of all existing criteria. In 6 out of 8 examples, Theorem 12 demonstrates conservatism that is either equal to or less than all existing criteria, albeit remaining inferior to Remark 15 in 5 out of 8 instances. However, one notable advantage of Theorem 12 is its applicability when $D \neq 0$. Finally, each system with α set less than the Nyquist gain has a unique equilibrium point. By Theorem 19, it will be futile to attempt a local stability analysis on such systems.

The limitation of these results is that $m \leq 5$: while absolute stability analysis deems this a high-dimensional nonlinearity, it does not align with the perspective in the NN literature. Nonetheless, the confidence in the performance of the strengthened criteria increases, given that both Theorem 12 and Remark 15 are less conservative than existing criteria in the majority of examples. This suggests that the enhanced criteria should fare well even in scenarios where m could be significantly larger. Additionally, both criteria exhibit superior scalability compared to Park's approach.

6. Conclusion

This work incorporates the flexibility of learning-based control within a model-based framework. The DT Circle and DT Popov Criteria are strengthened for analysing DT Lurie systems with the repeated ReLU nonlinearity. These refined criteria offer the potential for significantly lower conservatism levels compared to the standard DT Circle and DT Popov Criteria whilst maintaining their computational efficiency. Numerical examples demonstrate reduced conservatism without the high computational overhead associated with methods like the Park Criterion. Local stability was also investigated, but this resulted in conditions which show that if the Lurie system under consideration has a unique equilibrium point at the origin, then this equilibrium point is in fact globally stable or unstable, meaning that local stability analysis will provide no additional benefit. The main deficiency of the new results is their limitation to the ReLU nonlinearity. Despite this, it is hoped that these results may help bring NN based control into the domain of safety critical systems. Future work will extend these results to account for biases within the neural network and by incorporating a notion of robustness to model uncertainty. This will allow our method to account for more expressive neural networks and leverage the flexibility of learning-based approaches whilst reducing the reliance on an accurate physical model, an inherent problem of model-based control.

Acknowledgments

This work was supported in part by the Defence Science and Technology Laboratory (DSTL) and the UK Research and Innovation (UKRI) centre of Machine Intelligence for Nano-electronic Devices and Systems [EP/S024298/1].

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