Deep Neural Networks
Convolutional Networks III

Bhiksha Raj

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Outline

- Quick recap
- Back propagation through a CNN
- Modifications: Transposition, scaling, rotation and deformation invariance
- Segmentation and localization
- Some success stories
- Some advanced architectures
  - Resnet
  - Densenet

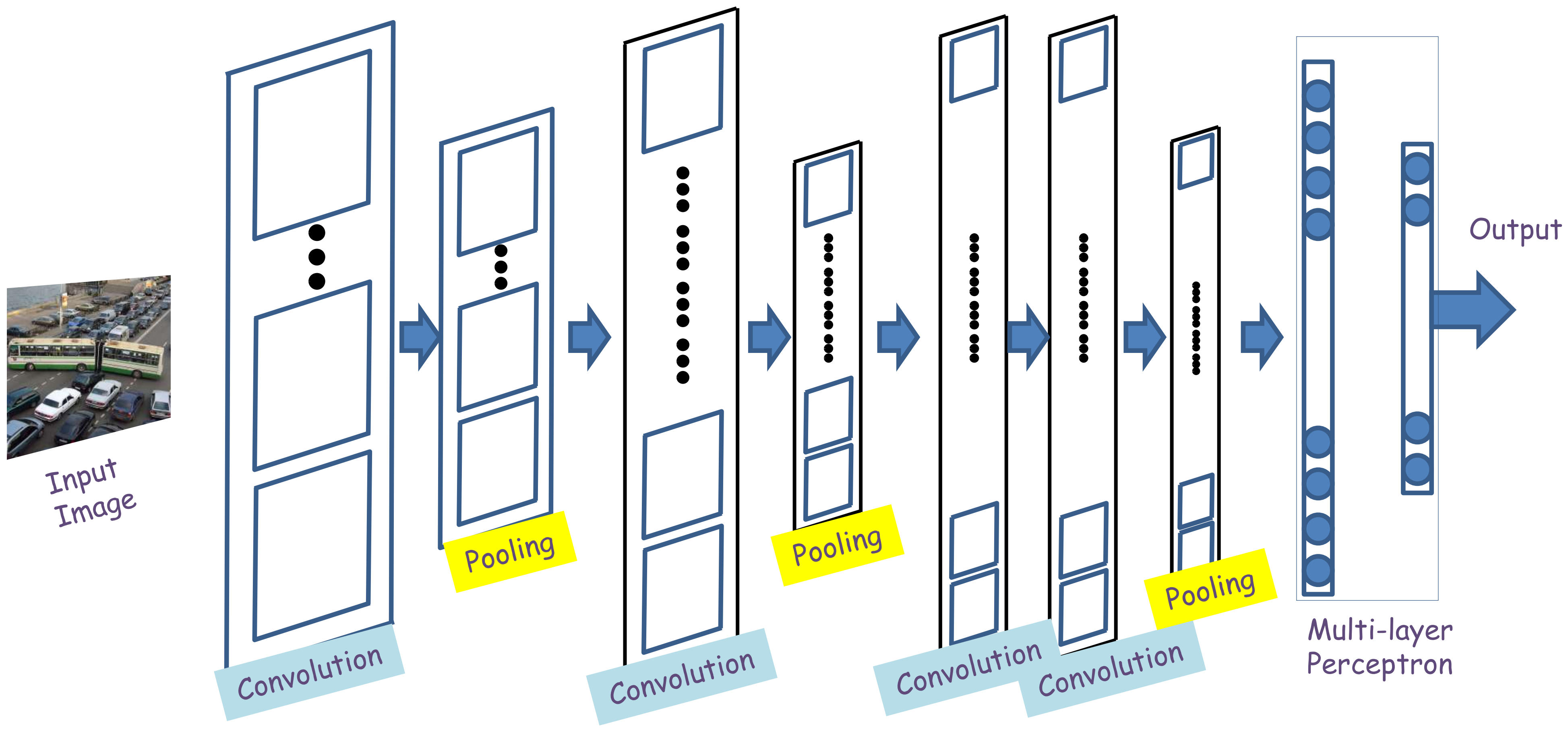
Story so far

- Pattern classification tasks such as “does this picture contain a cat”, or “does this recording include HELLO” are best performed by scanning for the target pattern

- Scanning an input with a network and combining the outcomes is equivalent to scanning with individual neurons hierarchically
  - First level neurons scan the input
  - Higher-level neurons scan the “maps” formed by lower-level neurons
  - A final “decision” unit or layer makes the final decision
  - Deformations in the input can be handled by “pooling”

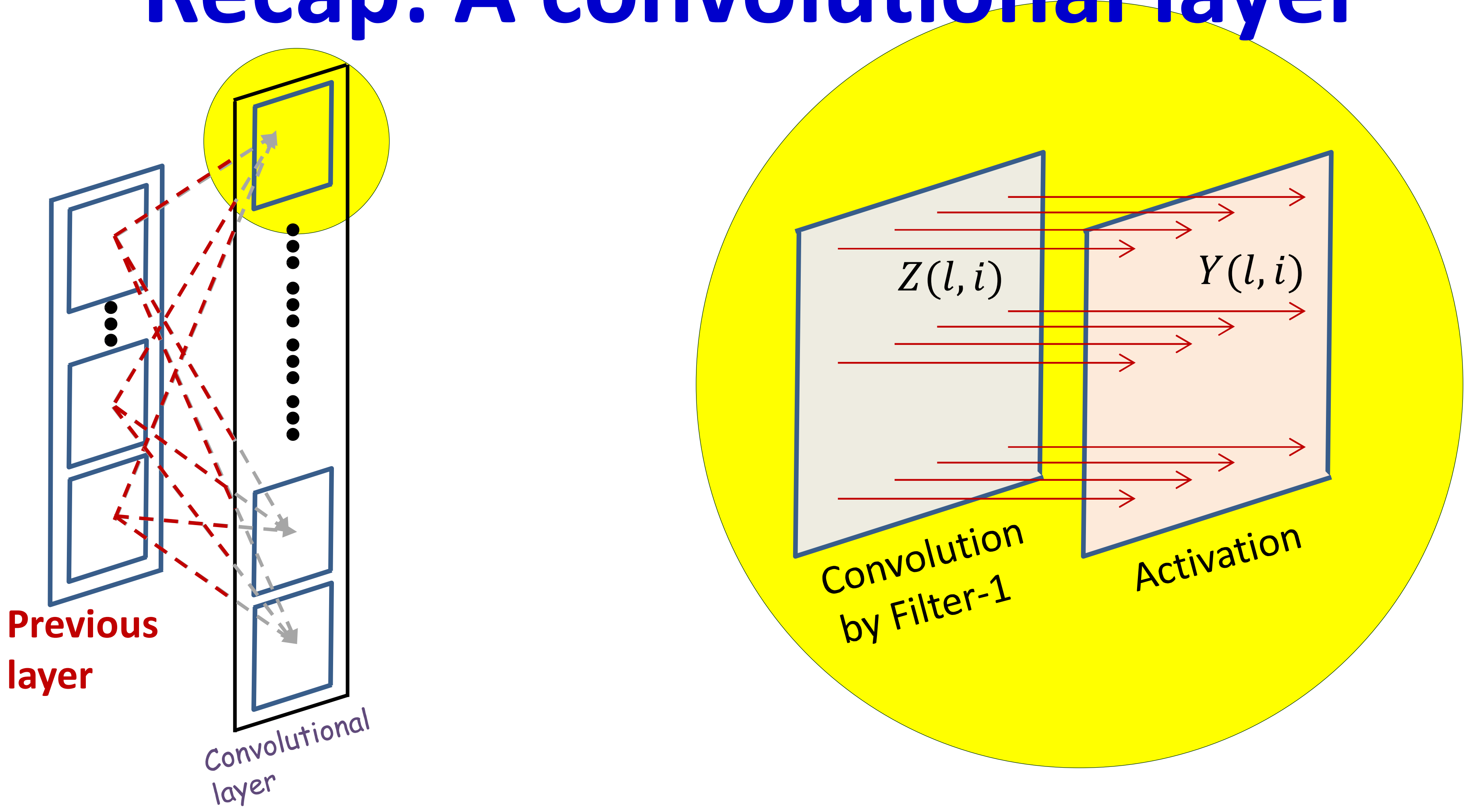
- For 2-D (or higher-dimensional) scans, the structure is called a convnet
- For 1-D scan along time, it is called a Time-delay neural network

Recap: The general architecture of a convolutional neural network



- A convolutional neural network comprises of "convolutional" and optional "pooling" layers
- Followed by an MLP with one or more layers

Recap: A convolutional layer



- The computation of each output map has two stages
  - Computing an *affine* map, by *convolution* of a *filter* (representing a pattern of weights) over maps in the previous layer
    - Each affine map has, associated with it, a ***learnable filter***
  - An *activation* that operates *point-wise* on the output of the convolution

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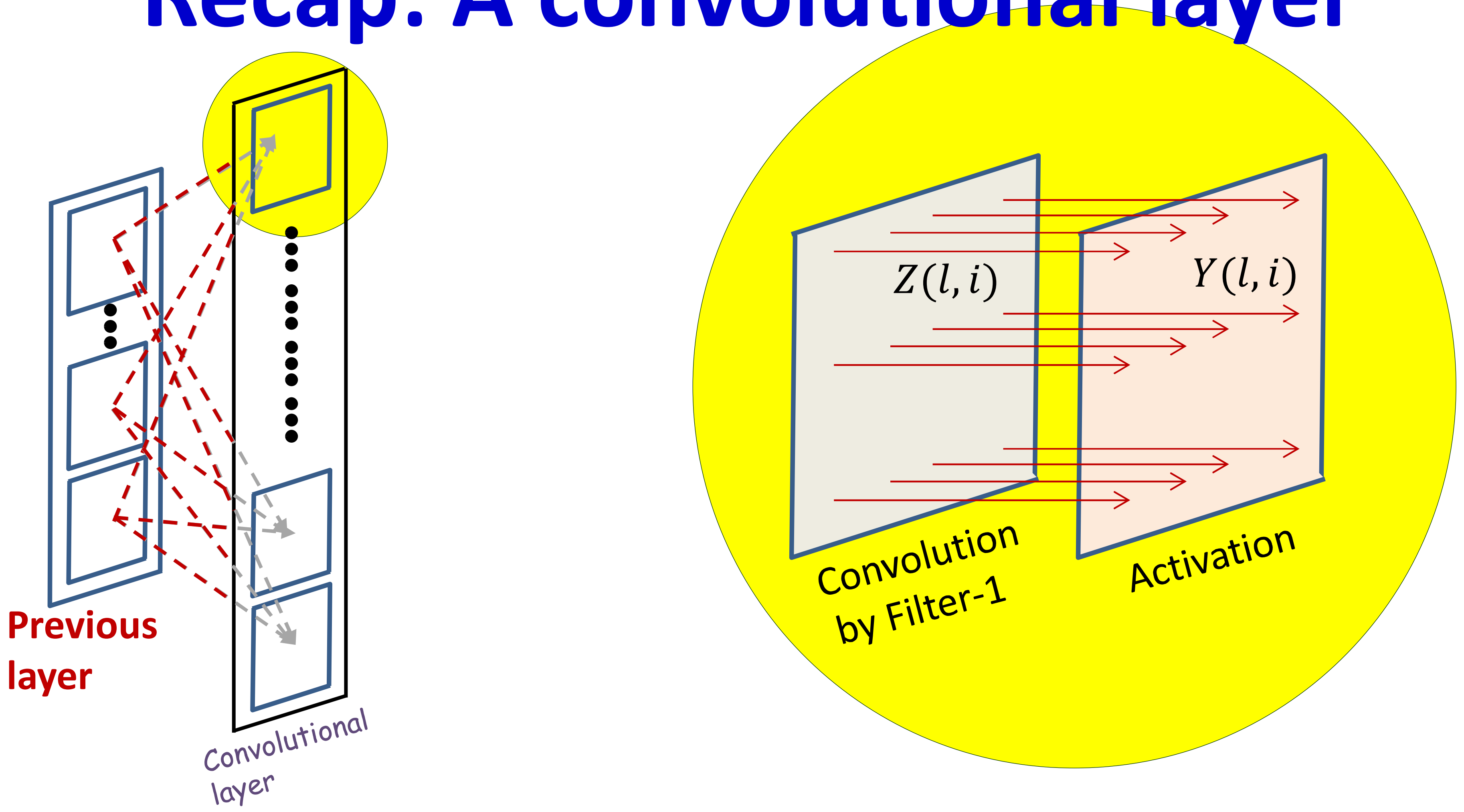

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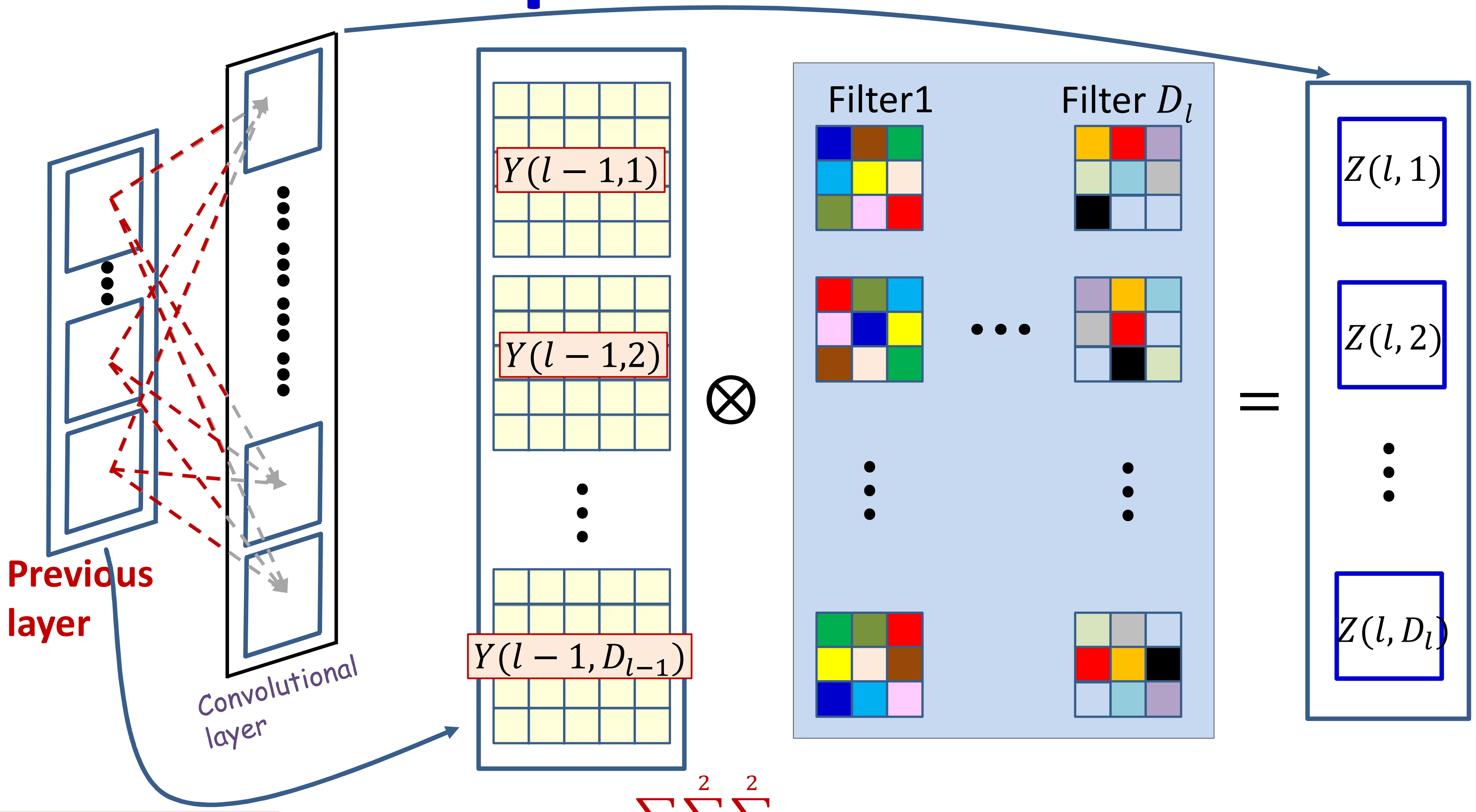
Recap: Convolution



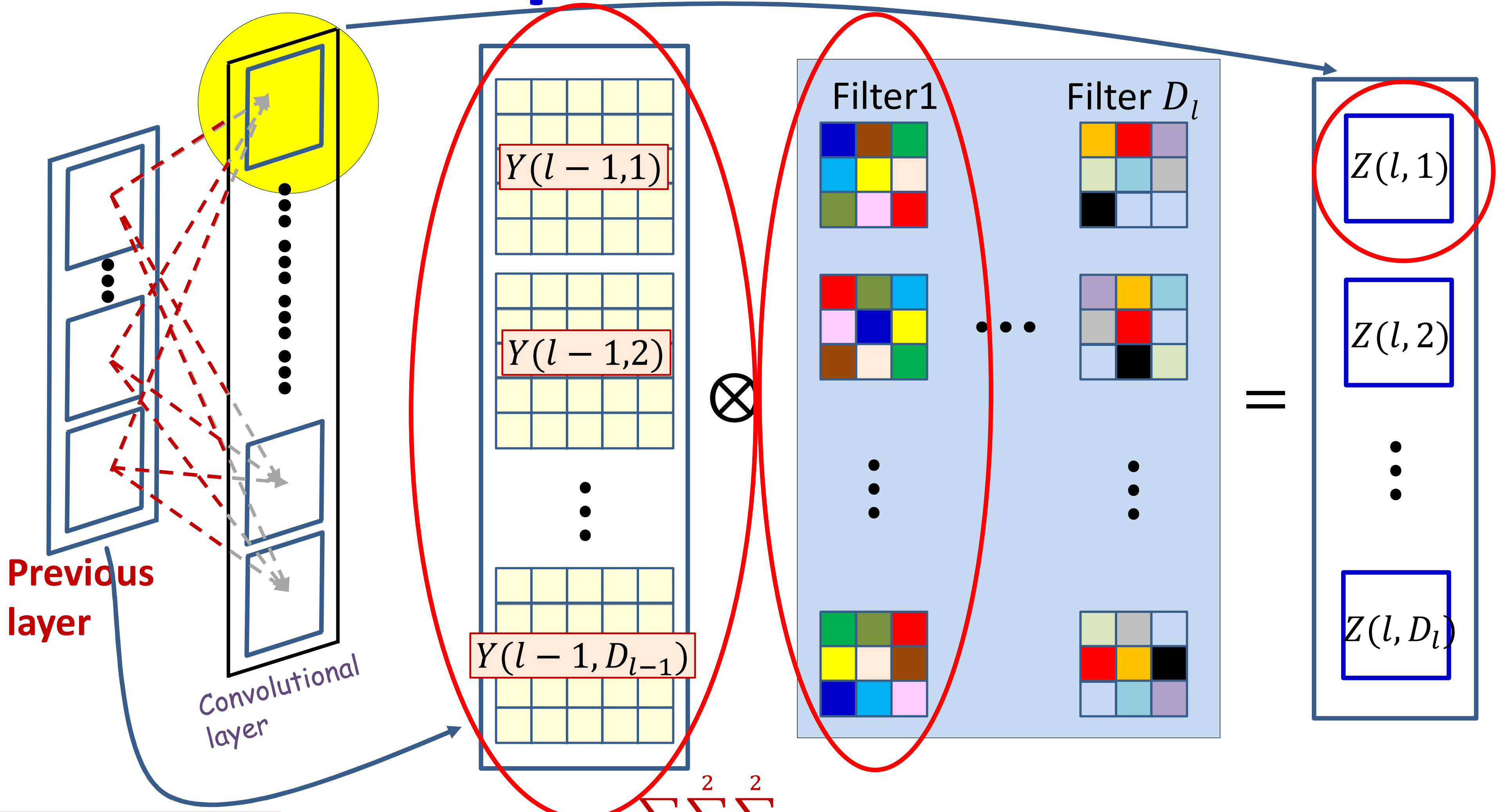
Caveat : 0-based indexing $\longrightarrow$

$$z(l,n,x,y) = \sum_{m}\sum_{i=0}^{2}\sum_{j=0}^{2} w_l(m,n,i,j)y(l-1,m,x+i,y+j) + b_l(n)$$

- Each affine output map is computed from multiple input maps simultaneously
- There are as many weights (for each output map) as

  size of the filter x *no. of maps in previous layer*

Recap: Convolution



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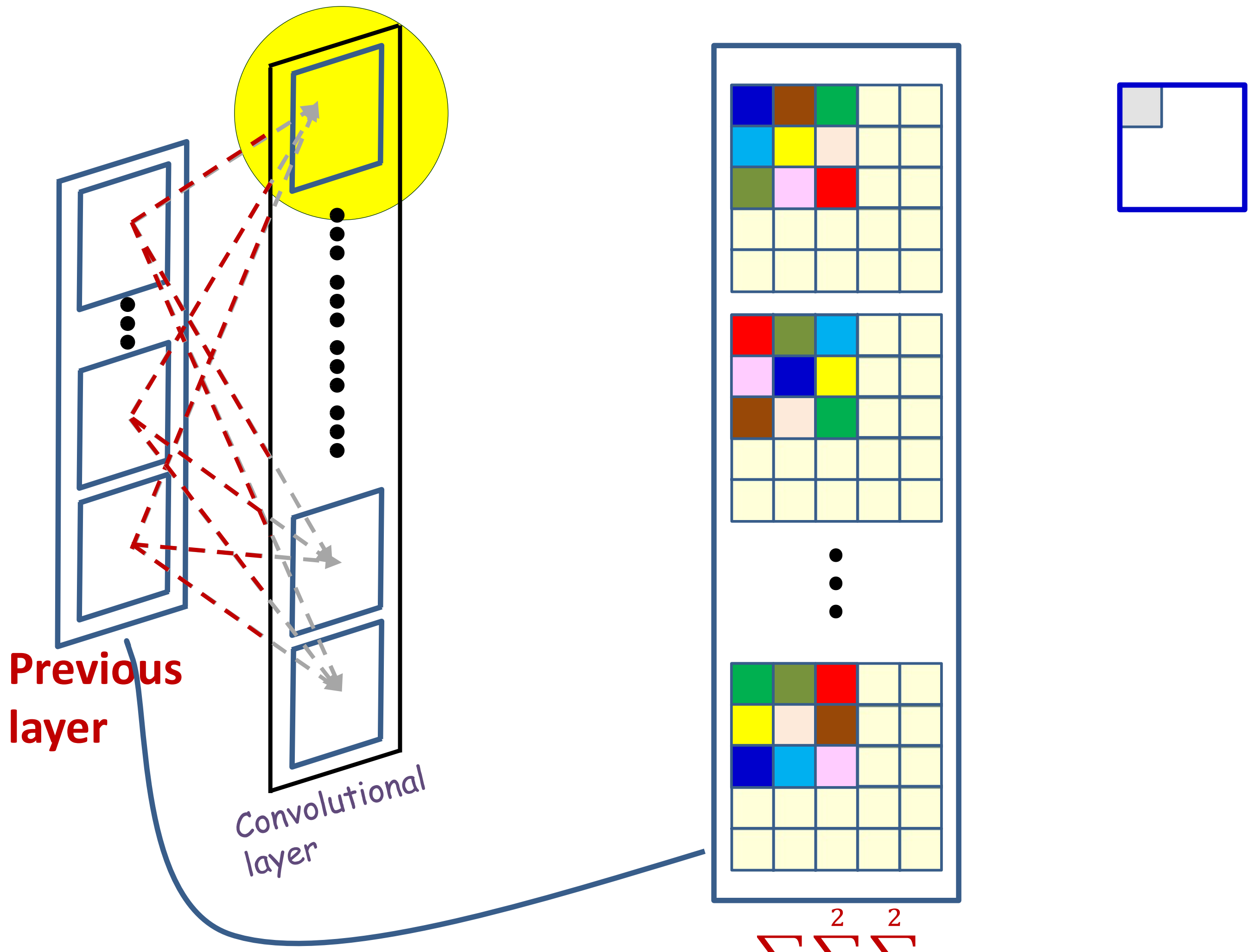

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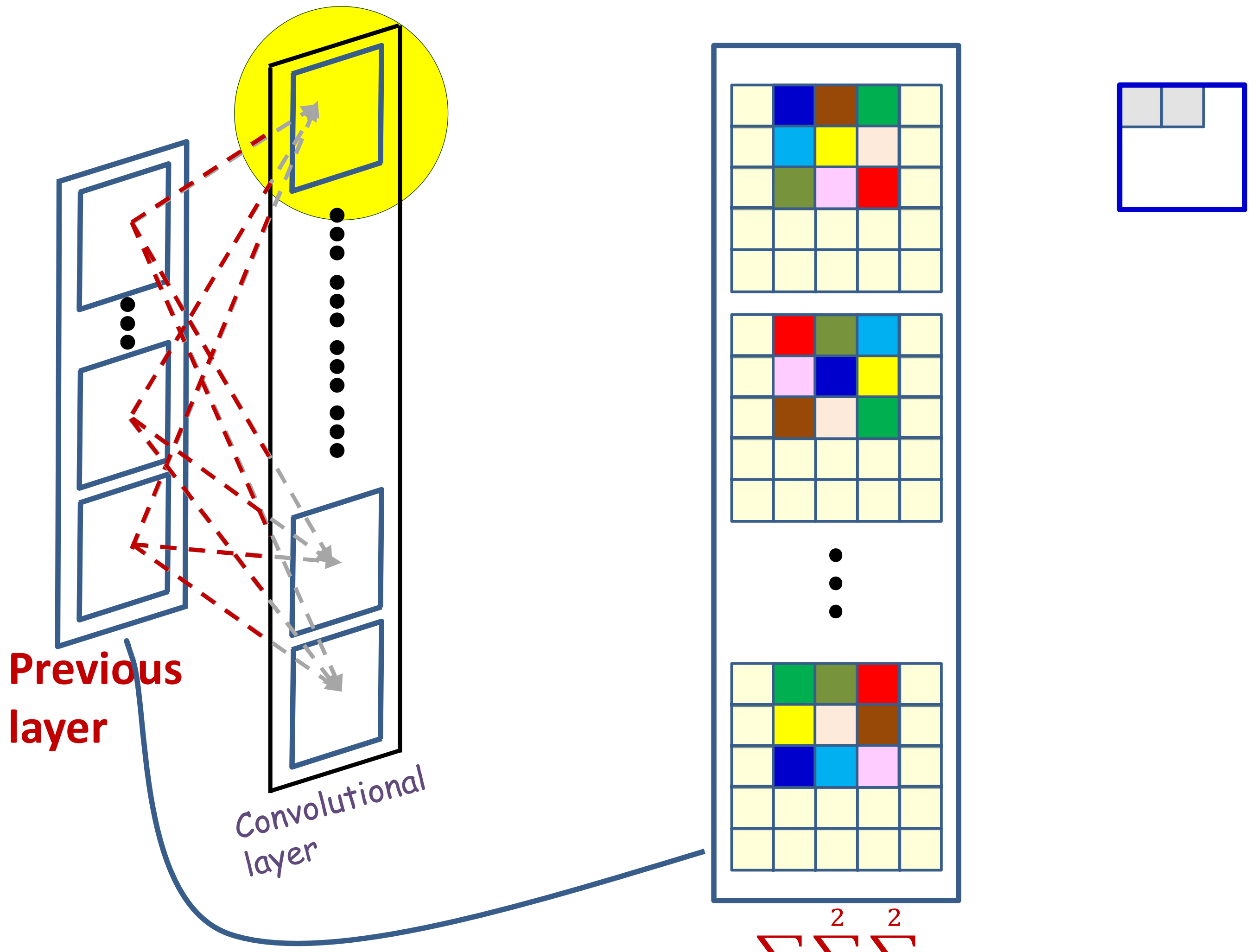

$$z(l,n,x,y) = \sum_{m}\sum_{i=0}^{2}\sum_{j=0}^{2} w_l(m,n,i,j)\,y(l-1,m,x+i,y+j) + b_l(n)$$

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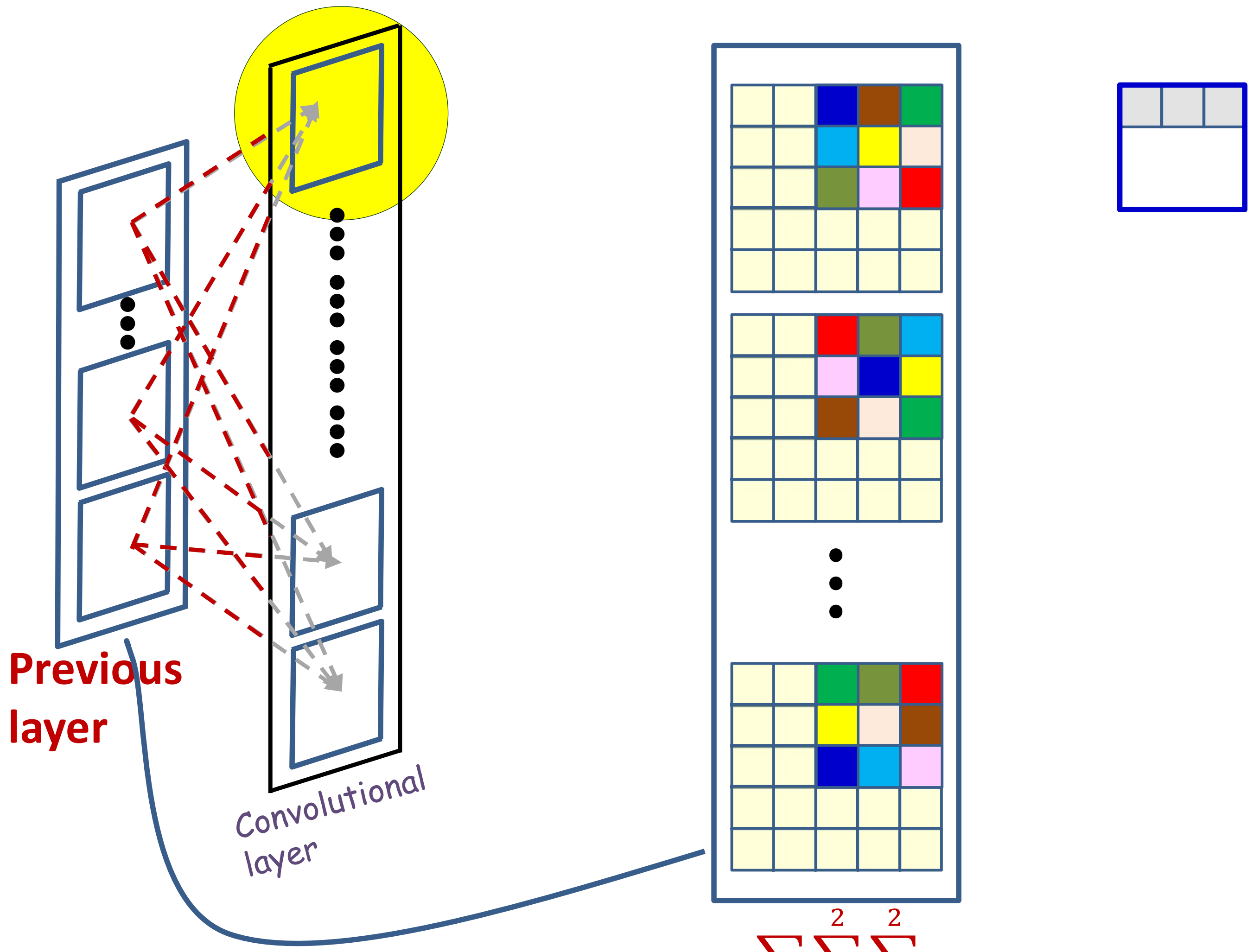

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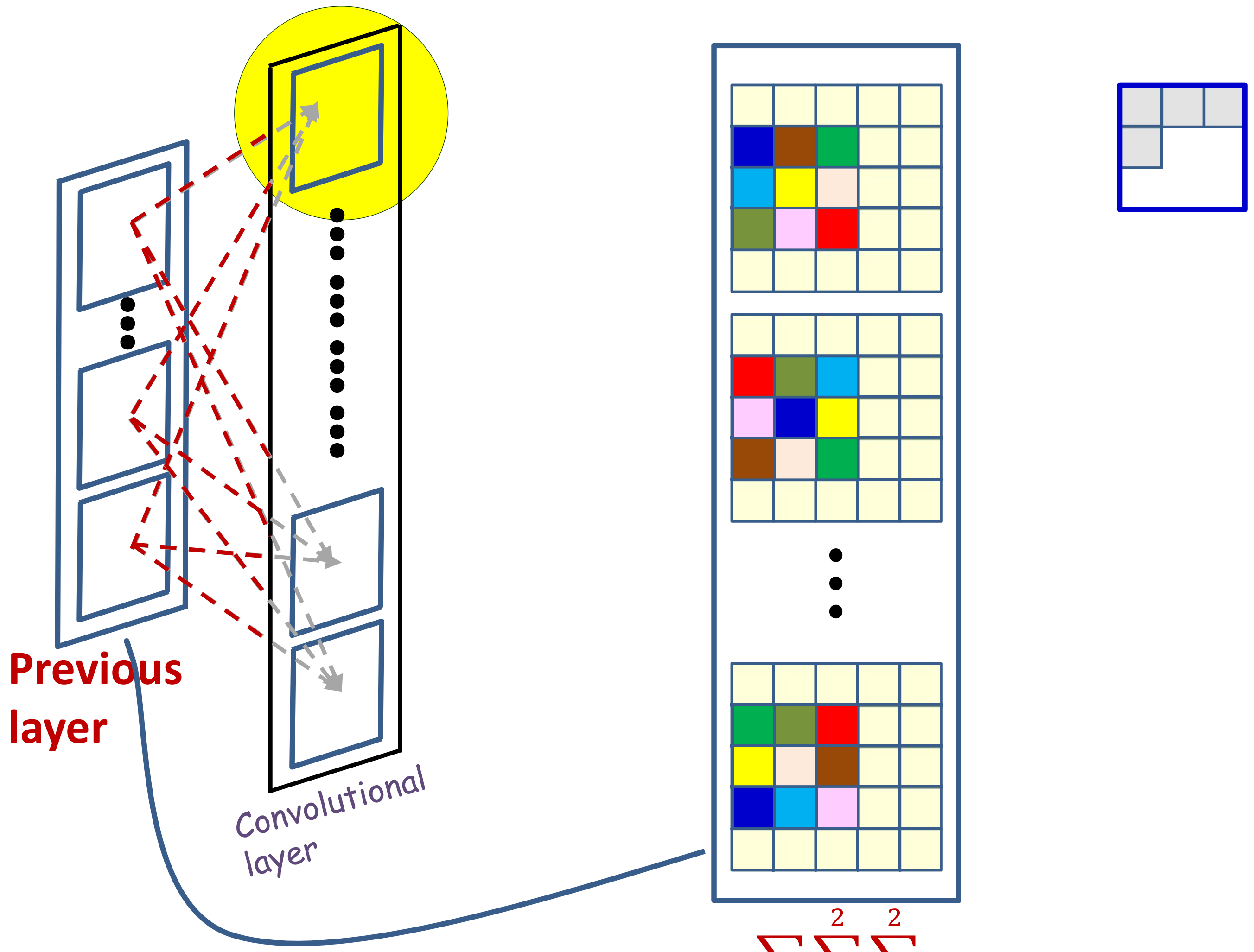

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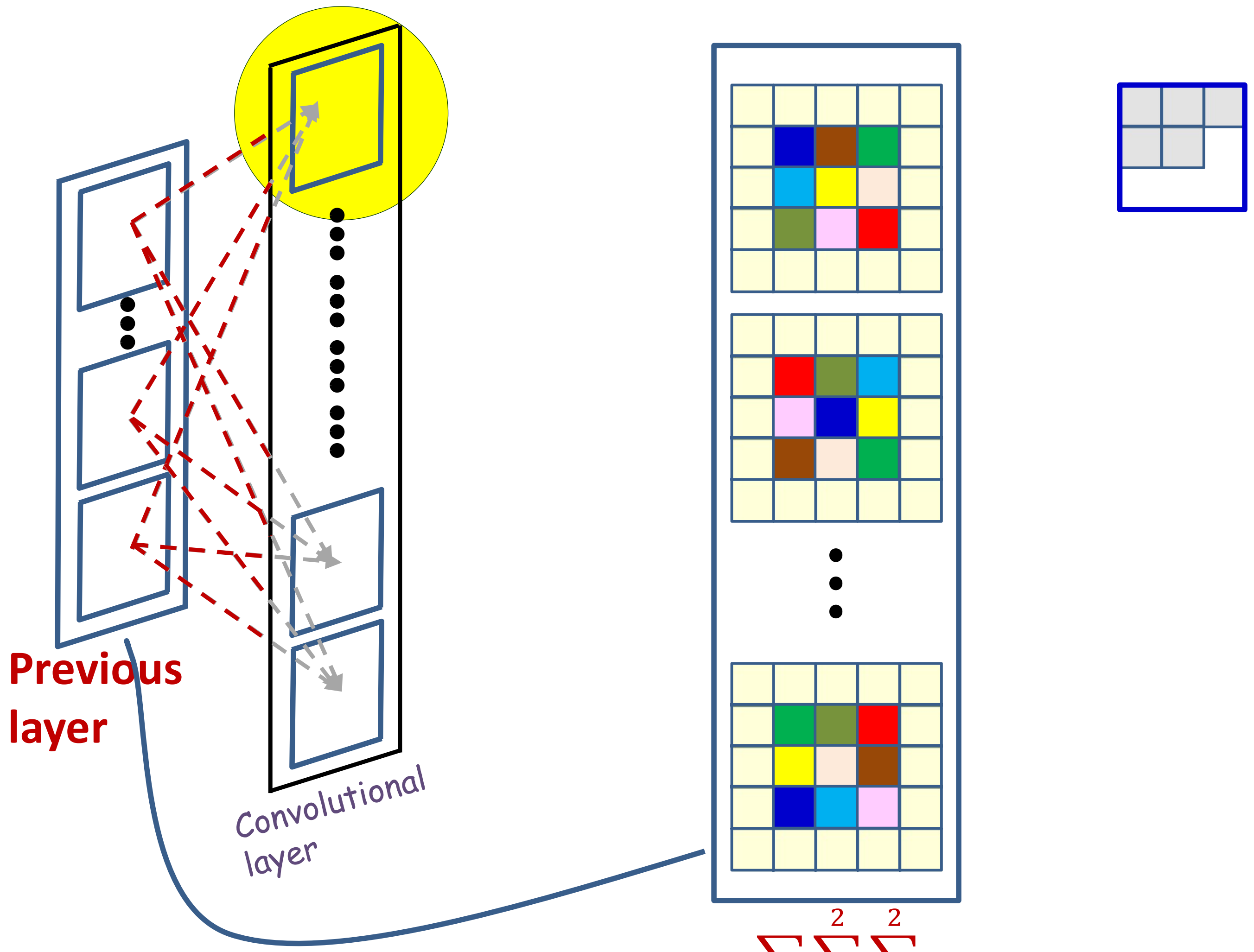

$$z(l,n,x,y) = \sum_{m} \sum_{i=0}^{2} \sum_{j=0}^{2} w_l(m,n,i,j)\, y(l-1,m,x+i,y+j) + b_l(n)$$

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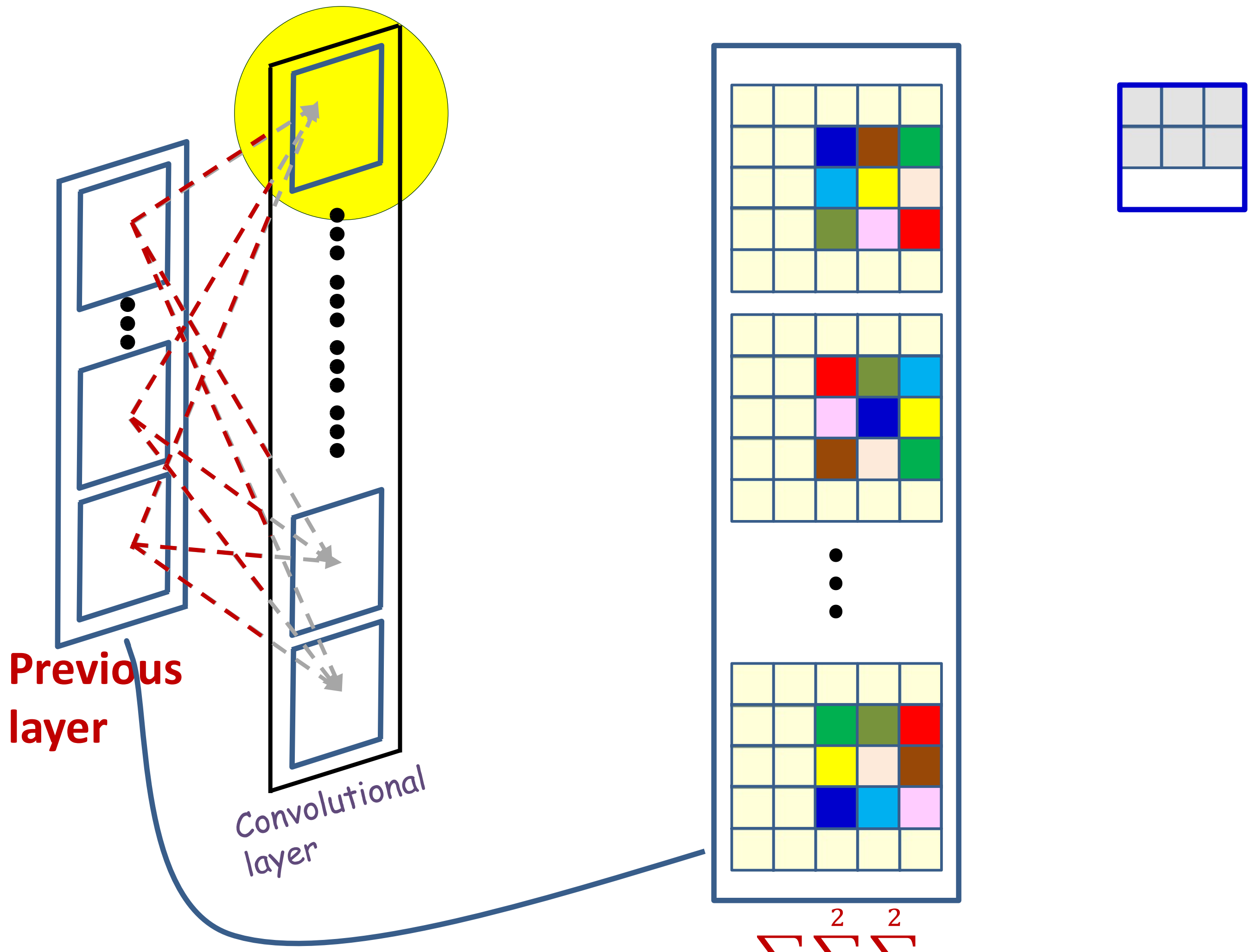

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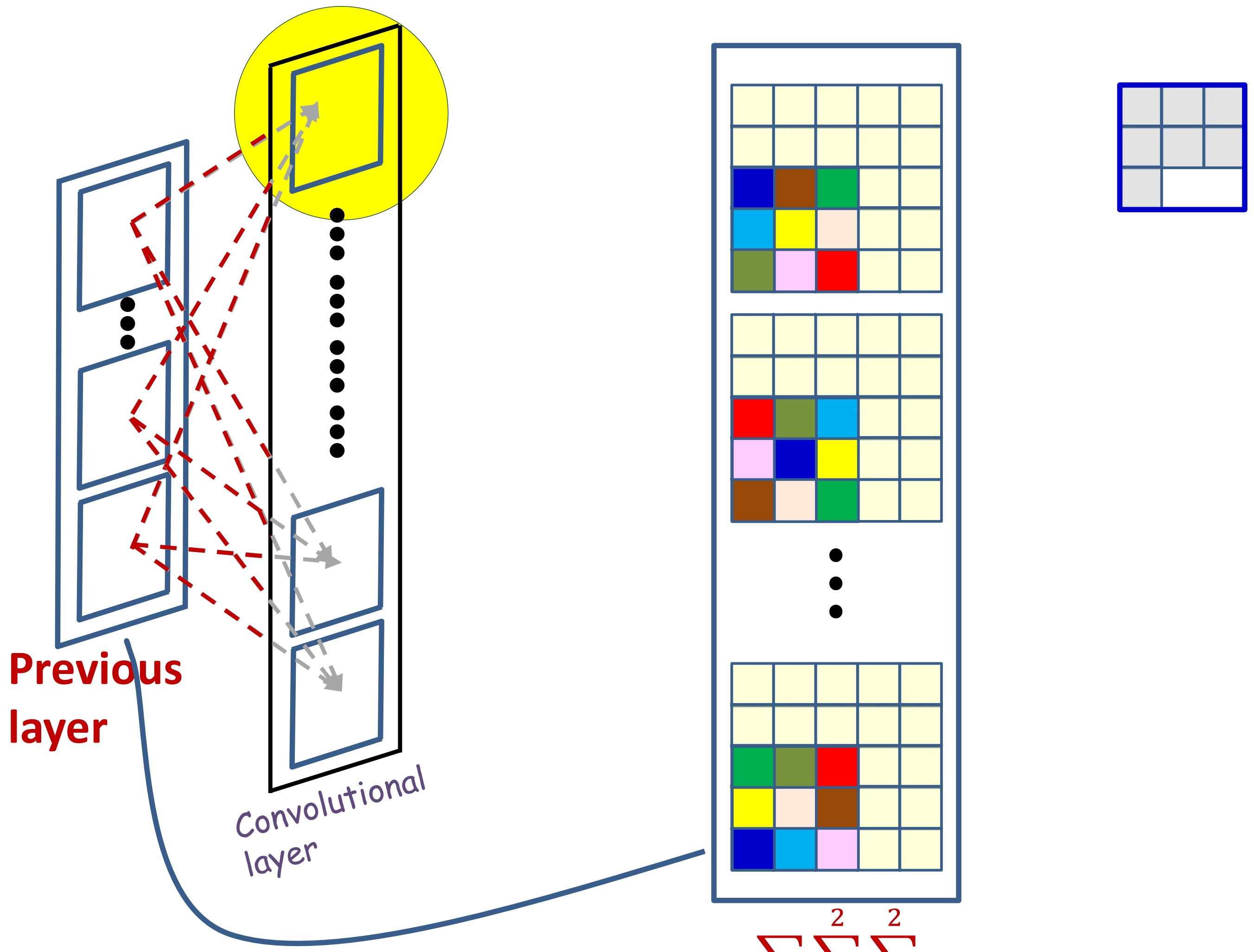

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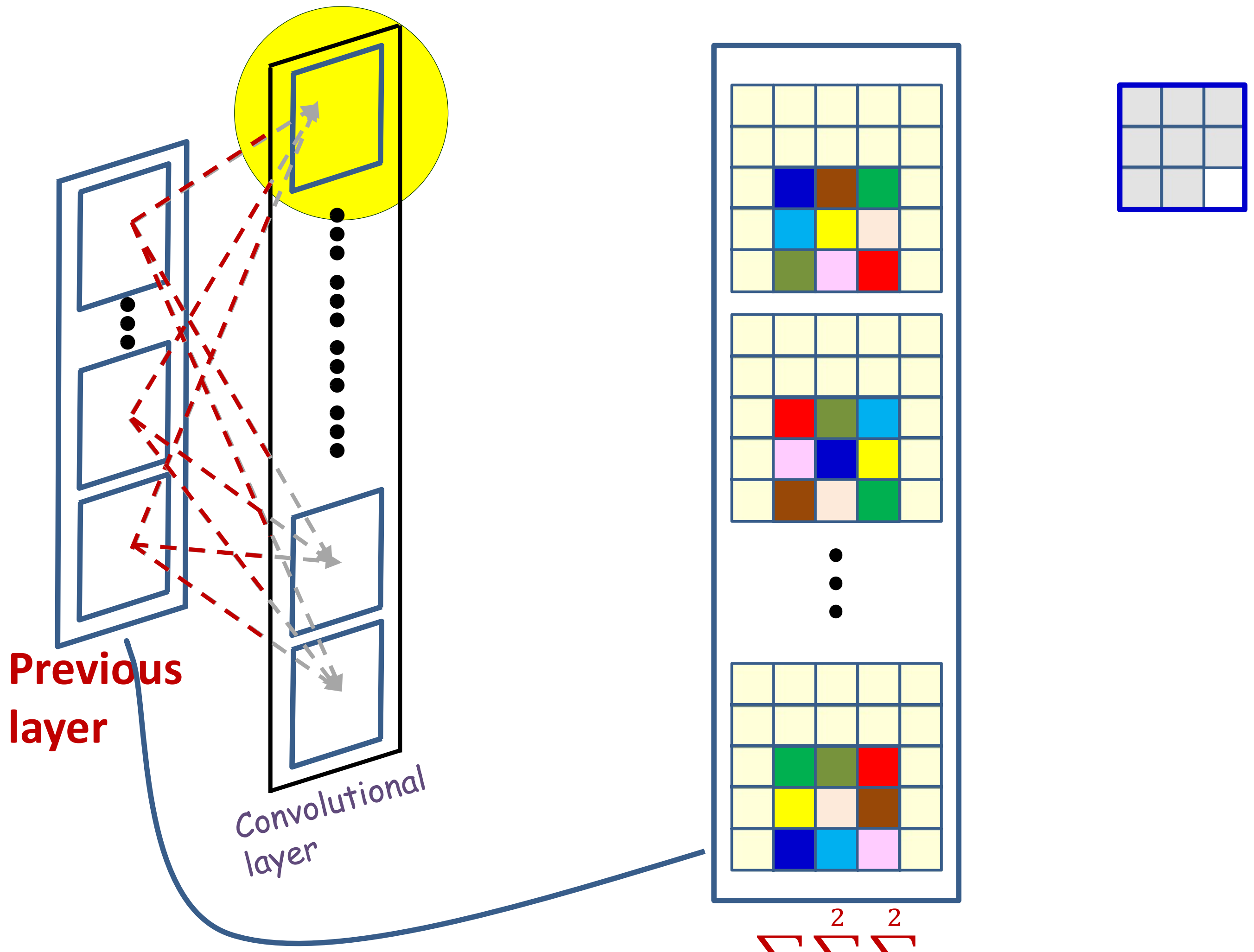

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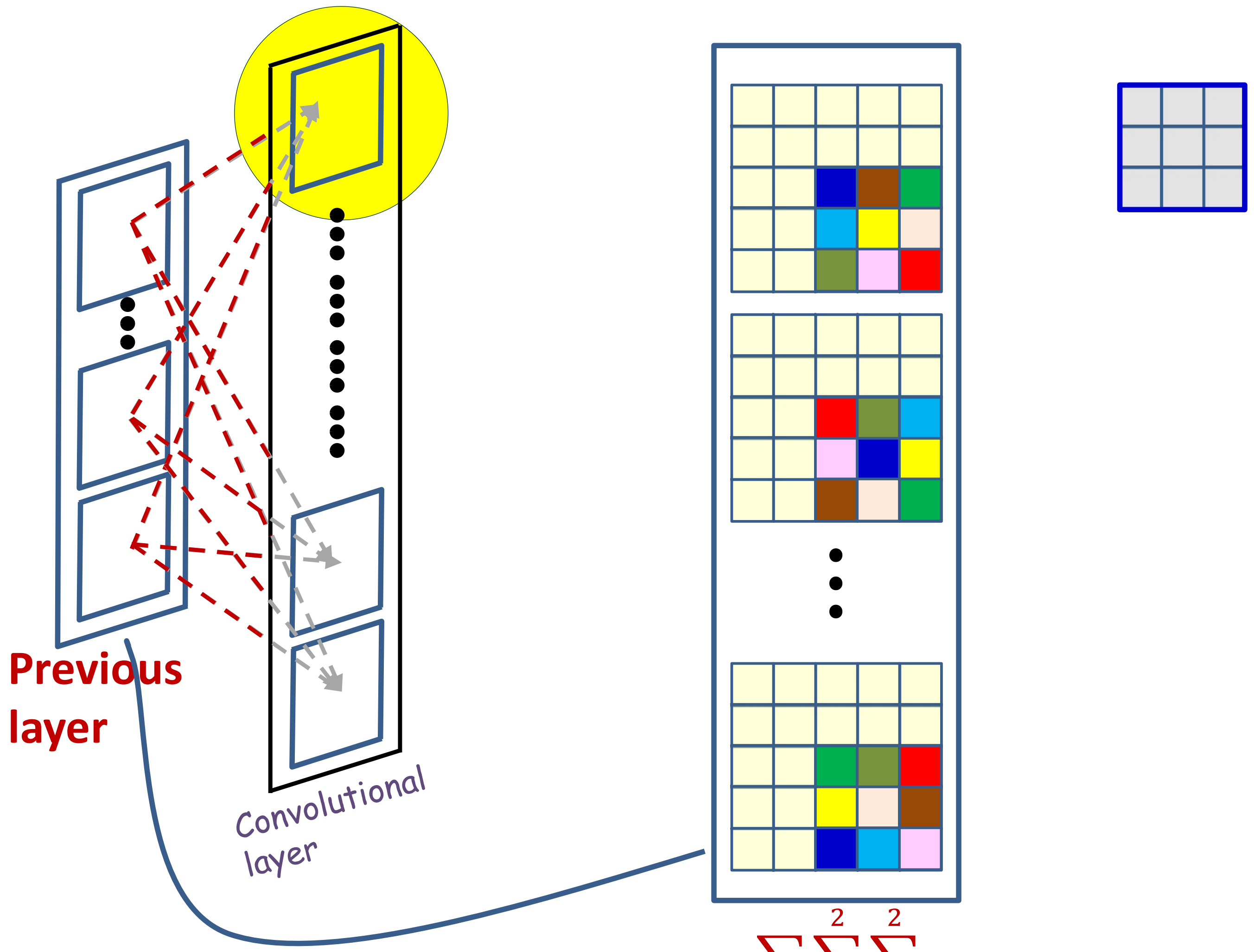

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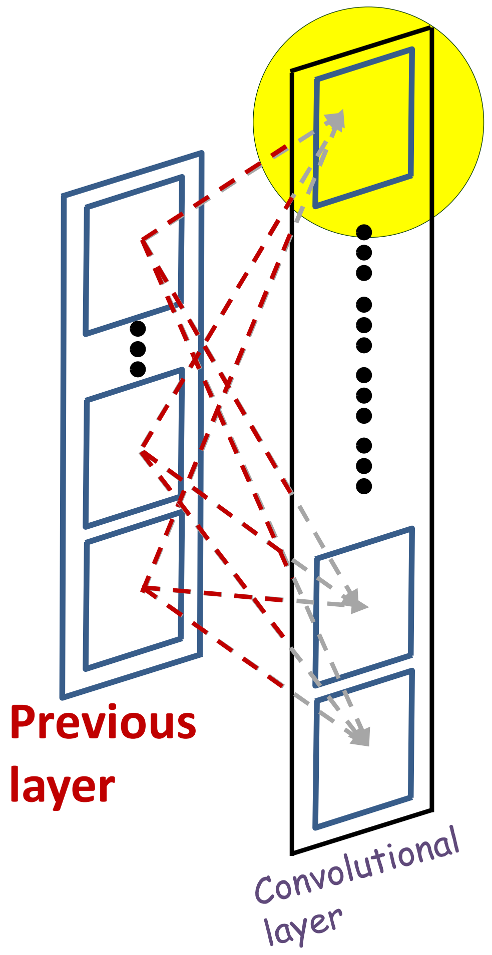
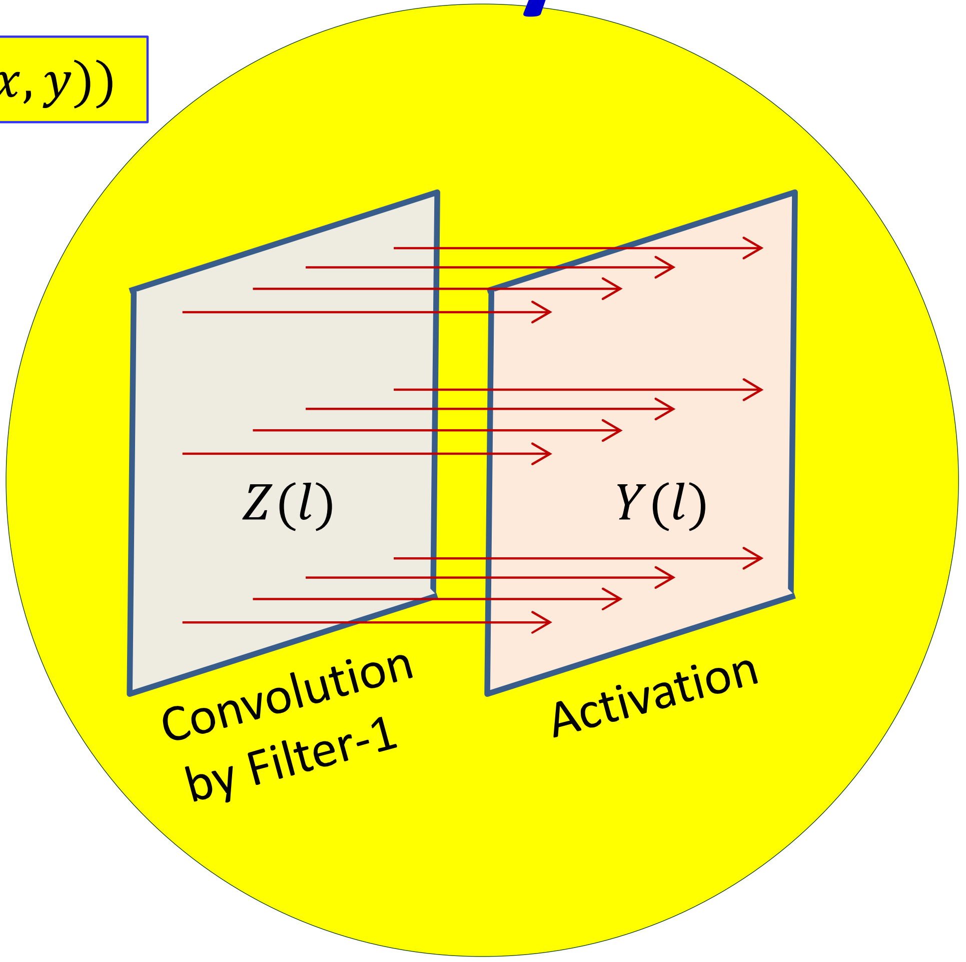
Recap: A convolutional layer



$$y(l,i,x,y) = f(z(l,i,x,y))$$



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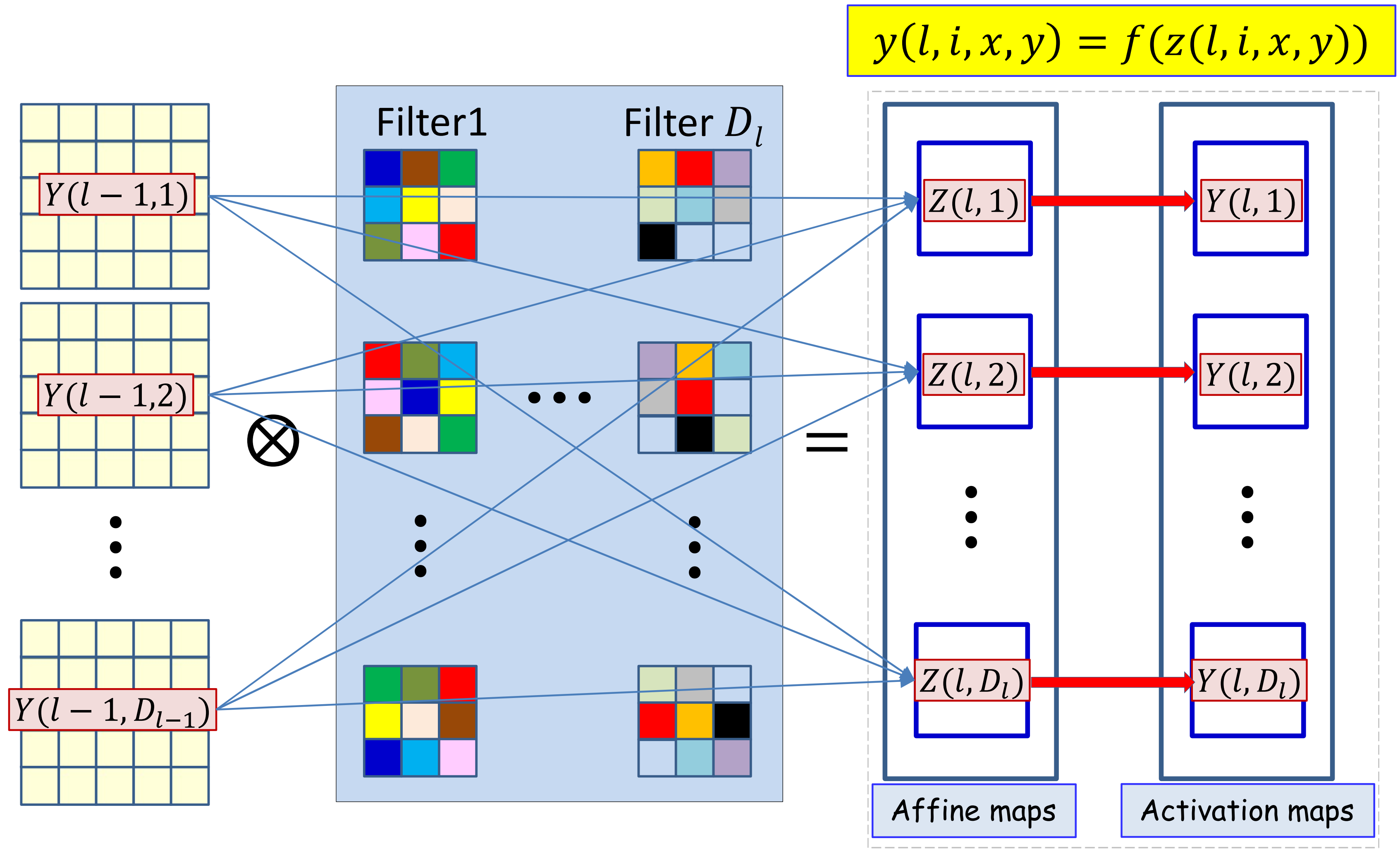
Convolution layer: A more explicit illustration



- Input maps $Y(l-1,*)$ are convolved with several filters to generate the affine maps $Z(l,*)$
  - Each filter consists of a set of square patterns of weights, with one set for each map in $Y(l-1,*)$
  - We get one affine map per filter
- A *point-wise* activation function $f(z)$ is applied to each map in $Z(l,*)$ to produce the activation maps $Y(l,*)$

Pseudocode: Vector notation

The weight W(l,j)**is a 3D D_{l-1}xK_lxK_l tensor**

```
Y(0) = Image
for l = 1:L  # layers operate on vector at (x,y)
  for x = 1:$W_{l-1}-K_l+1$
    for y = 1:$H_{l-1}-K_l+1$
      for j = 1:$D_l$
        segment = Y(l-1,:,x:x+$K_l$-1,y:y+$K_l$-1) #3D tensor
        z(l,j,x,y) = W(l,j).segment + b(l,j)#tensor prod.
        Y(l,j,x,y) = activation(z(l,j,x,y))
Y = softmax( {Y(L,:,:,:)} )
```

Pseudocode has 1-based indexing

Poll 1

Select all true statements about a convolution layer.

- The number of “channels” in any filter equals the number of input maps (output maps from the previous layer)
- The number of “channels” in any filter equals the number of output maps (affine maps output by the layer)
- The number of filters equals the number of input maps
- The number of filters equals the number of output maps

Poll 1

Select all true statements about a convolution layer.

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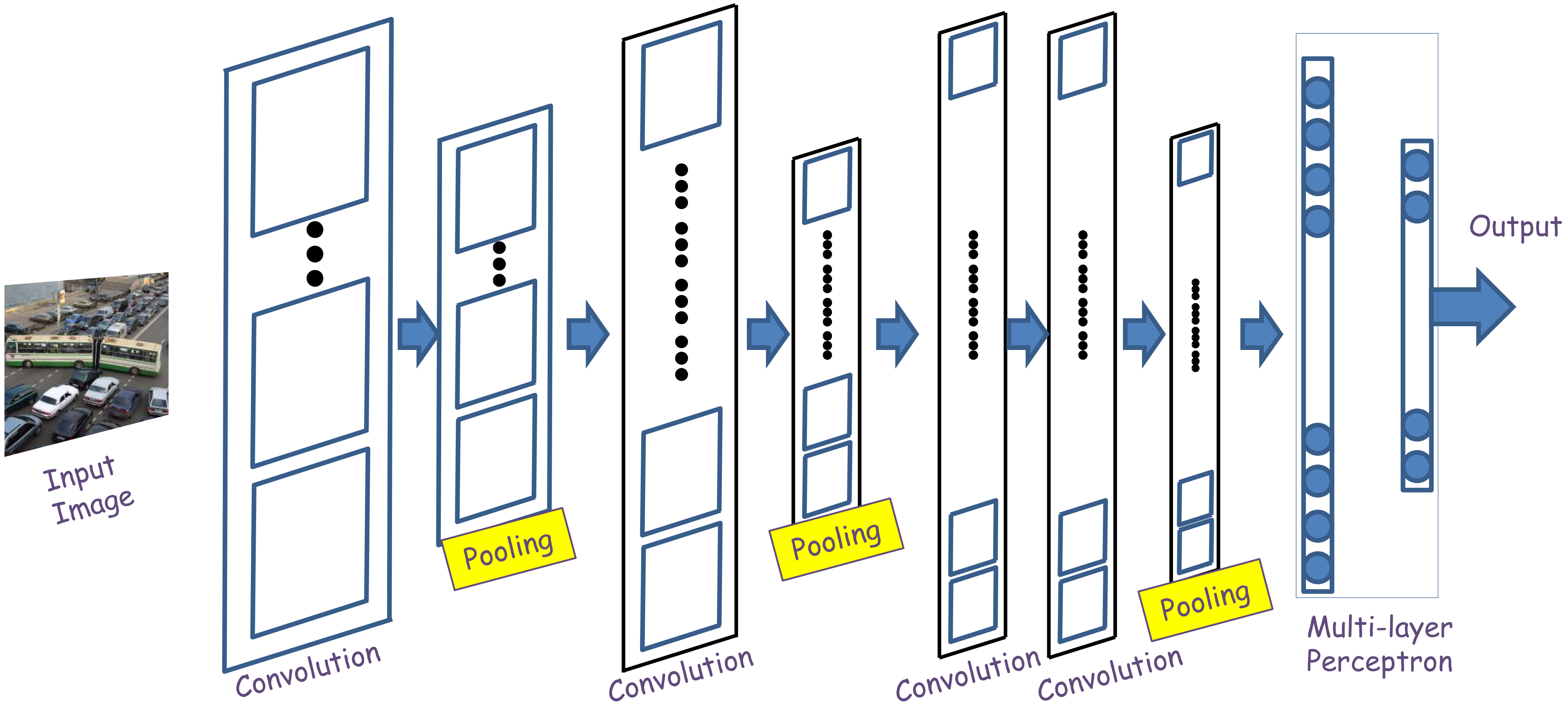
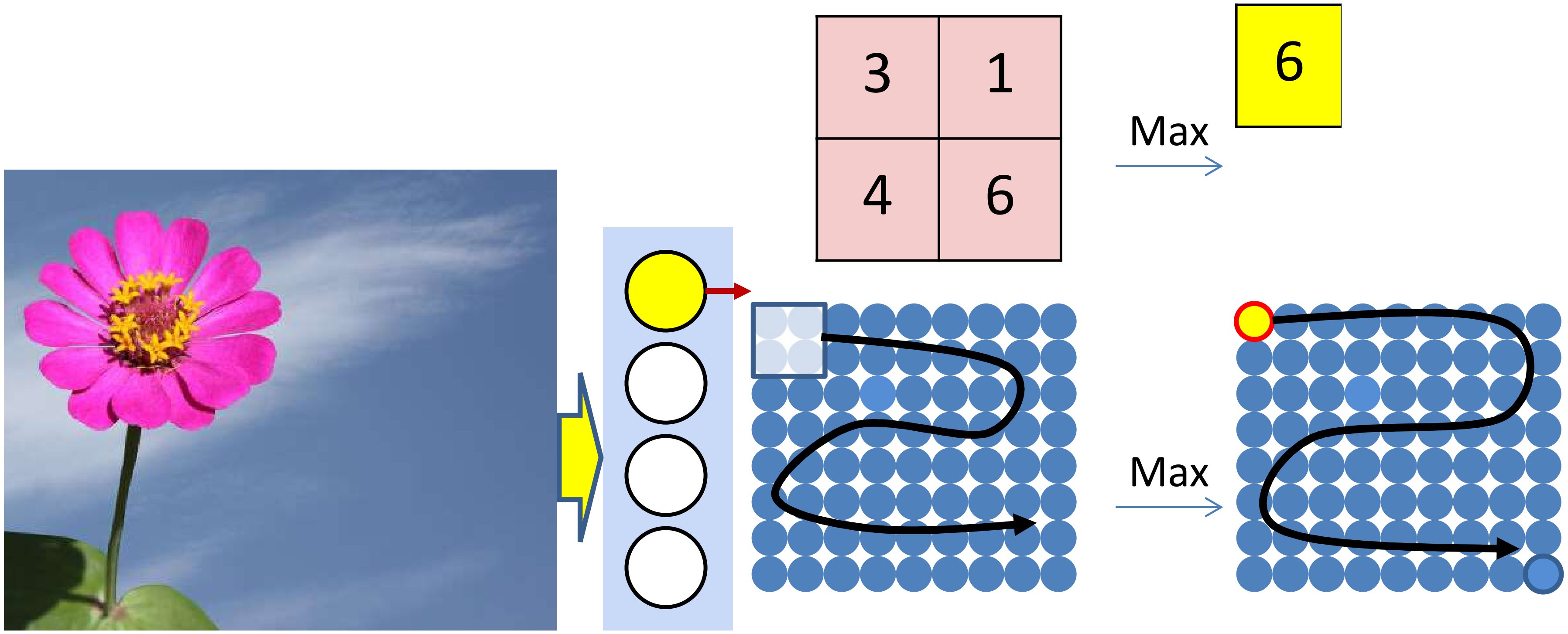
- The number of filters equals the number of input maps
- **The number of filters equals the number of output maps**

Pooling



- Convolutional (and activation) layers are followed intermittently by "pooling" layers
  - Often, they alternate with convolution, though this is not necessary

Recall: Max pooling



- Max pooling selects the largest from a pool of elements
- Pooling is performed by “scanning” the input with a “max-pooling filter”

Recap: Pooling and downsampling layer

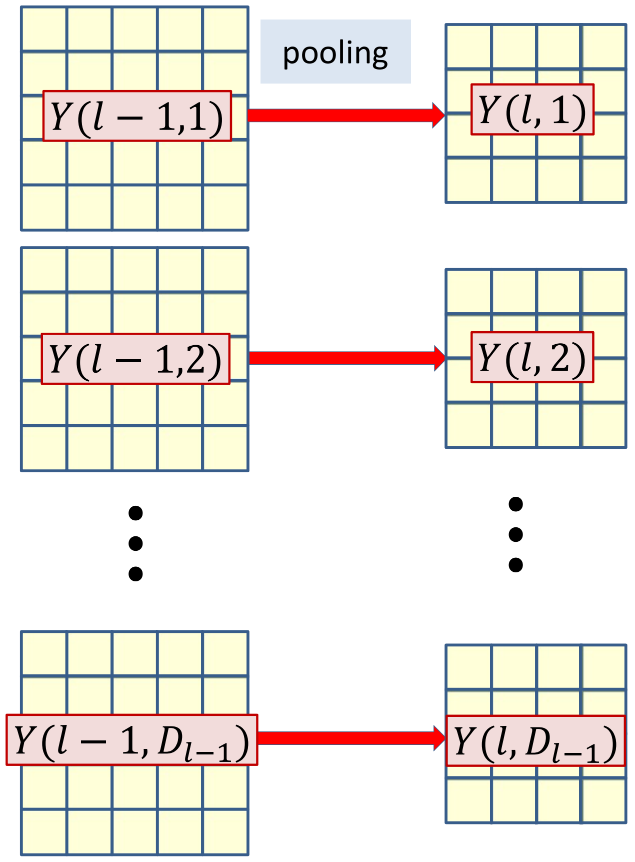


Image assumes pooling with window of size 2x2

- Input maps $Y(l-1,*)$ are operated on individually by pooling operations to produce the pooled maps $Y(l,*)$

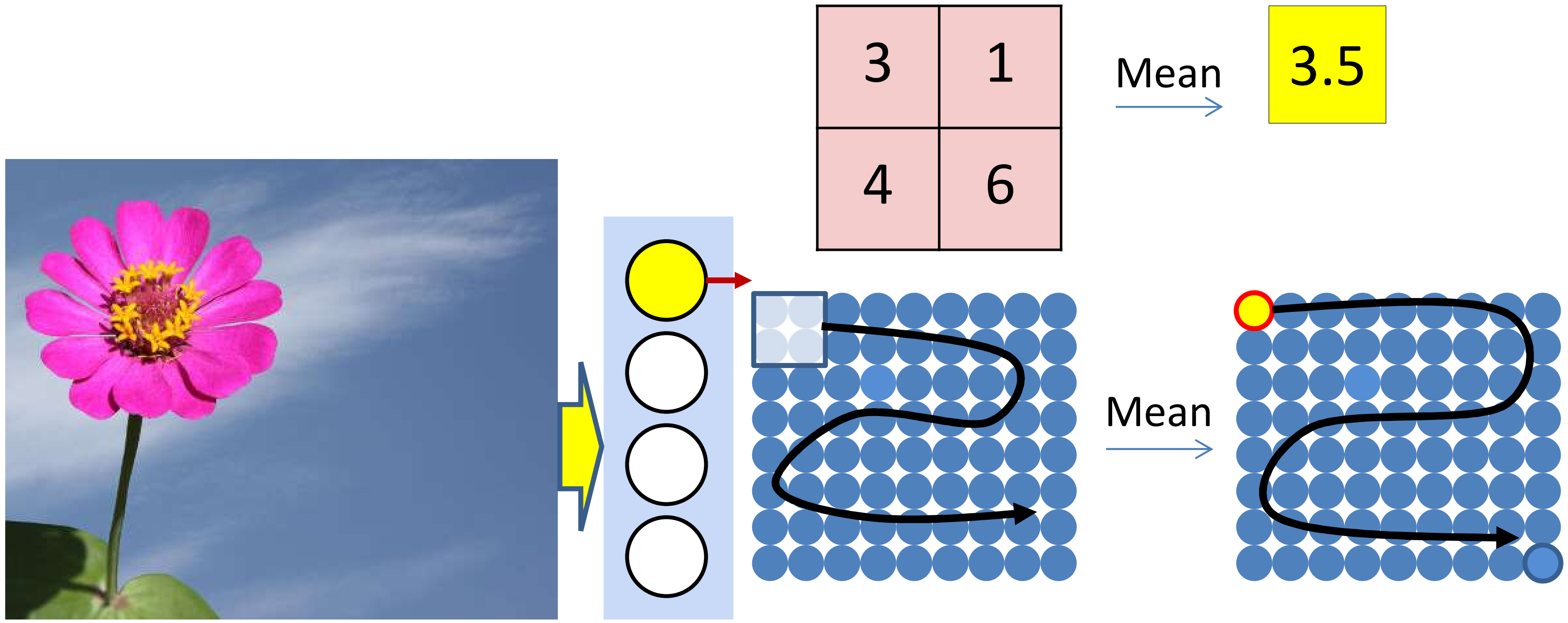
Recap: Max Pooling layer at layer *l*

a) Performed separately for every map (j).
    *) *Not* combining multiple maps within a single max operation.
b) Keeping track of location of max

Max pooling

```
for j = 1:$D_l$
  for x = 1:$W_{l-1}$-$K_l$+1
    for y = 1:$H_{l-1}$-$K_l$+1
      pidx(l,j,x,y) = maxidx(Y(l-1,j,x:x+$K_l$-1,y:y+$K_l$-1))
      u(l,j,x,y) = Y(l-1,j,pidx(l,j,m,n))
```

Recall: Mean pooling



- Mean pooling computes the *mean* of the window of values
  - As opposed to the max of max pooling

Recap: Mean Pooling layer at layer *l*

a) Performed separately for every map (j)

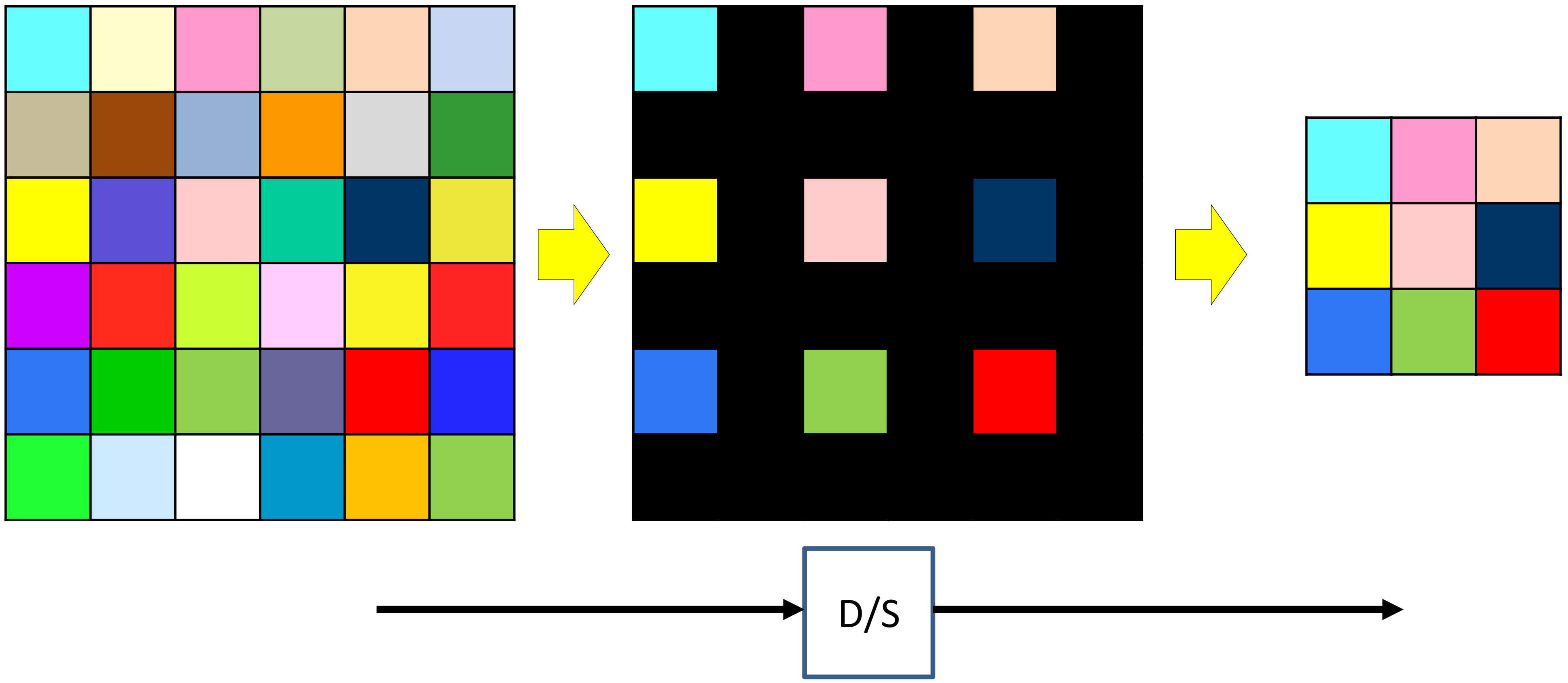
Mean pooling

```
for j = 1:$D_l$
  for x = 1:$W_{l-1}-K_l+1$
    for y = 1:$H_{l-1}-K_l+1$
      u(l,j,x,y) = mean(Y(l-1,j,x:x+$K_l$-1,y:y+$K_l$-1))
```

Recap: Resampling

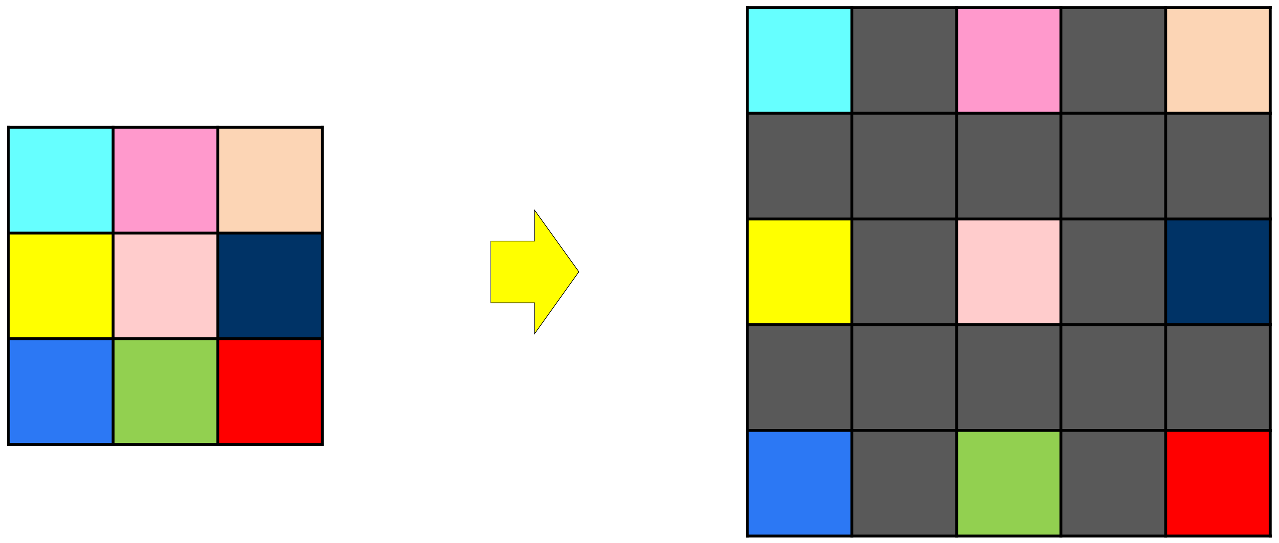
- We can also proportionately decrease or increase the size of the maps by dropping or inserting zeros
  - Downsampling: Drop S-1 rows/columns between rows/columns
    - Reduces the size of the maps by S on each side
  - Upsampling: Insert S-1 rows/columns of zeros between adjacent entries
    - Increases the size of the map by S on each side

The Downsampling Layer



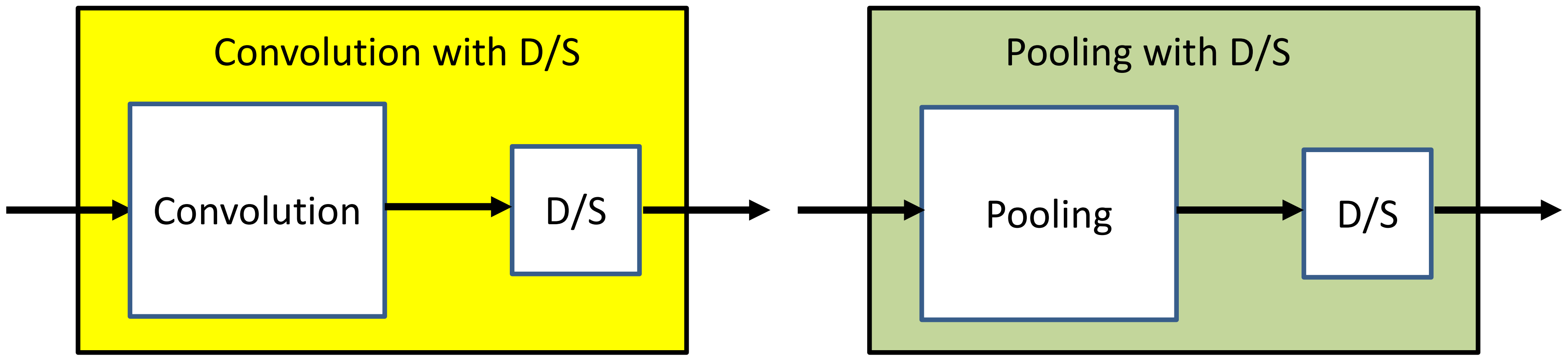
- A *downsampling* layer simply "drops" $S - 1$ of S rows and columns for every map in the layer
  - Effectively reducing the size of the map by factor S in every direction

The Upsampling Layer



- A *upsampling* (or dilation) layer simply introduces $S - 1$ rows and columns for every map in the layer
  - Effectively *increasing* the size of the map by factor S in every direction
- Used explicitly to increase the map size by a uniform factor

Downsampling in practice



- In practice, the downsampling is combined with the layers just before it by performing the operations with a stride > 1
  - Could be convolutional or pooling layers

Convolution with downsampling

The weight W(l,j) is now a 4D $D_l \times D_{l-1} \times K_l \times K_l$ tensor

The product in blue is a tensor inner product with a scalar output

```
Y(0) = Image
for l = 1:L  # layers operate on vector at (x,y)
    m = 1
    for x = 1:S:$W_{l-1}-K_l+1$
      n = 1
      for y = 1:S:$H_{l-1}-K_l+1$
         segment = Y(l-1,:,x:x+$K_l$-1,y:y+$K_l$-1) #3D tensor
         z(l,:,m,n) = W(l).segment #tensor inner prod.
         Y(l,:,m,n) = activation(z(l,:,m,n))
         n++
     m++
Y = softmax( {Y(L,:,:,:)} )
```

STRIDE

Downsampled indices

Max Pooling with Downsampling

Max pooling

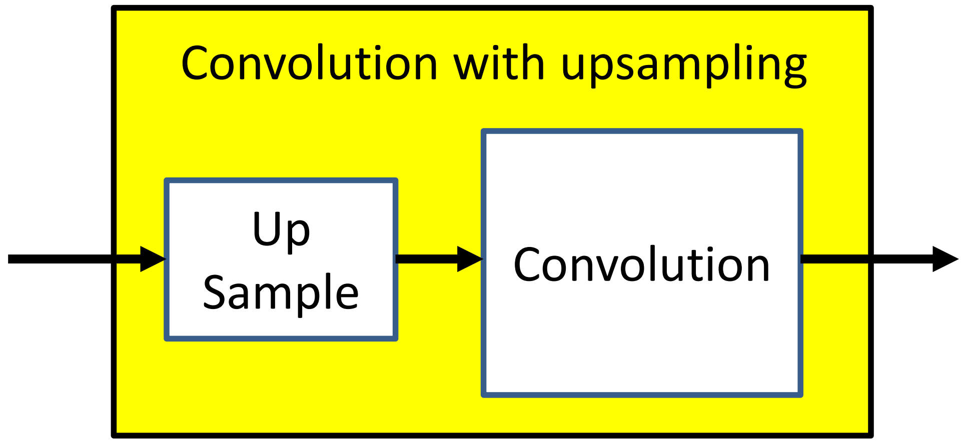
```
for j = 1:$D_l$
    m = 1
    for x = 1:stride(l):$W_{l-1}-K_l+1$
      n = 1
      for y = 1:stride(l):$H_{l-1}-K_l+1$
        pidx(l,j,m,n) = maxidx(Y(l-1,j,x:x+$K_l$-1,y:y+$K_l$-1))
        Y(l,j,m,n) = Y(l-1,j,pidx(l,j,m,n))
        n = n+1
      m = m+1
```

Mean Pooling with Downsampling

Mean pooling

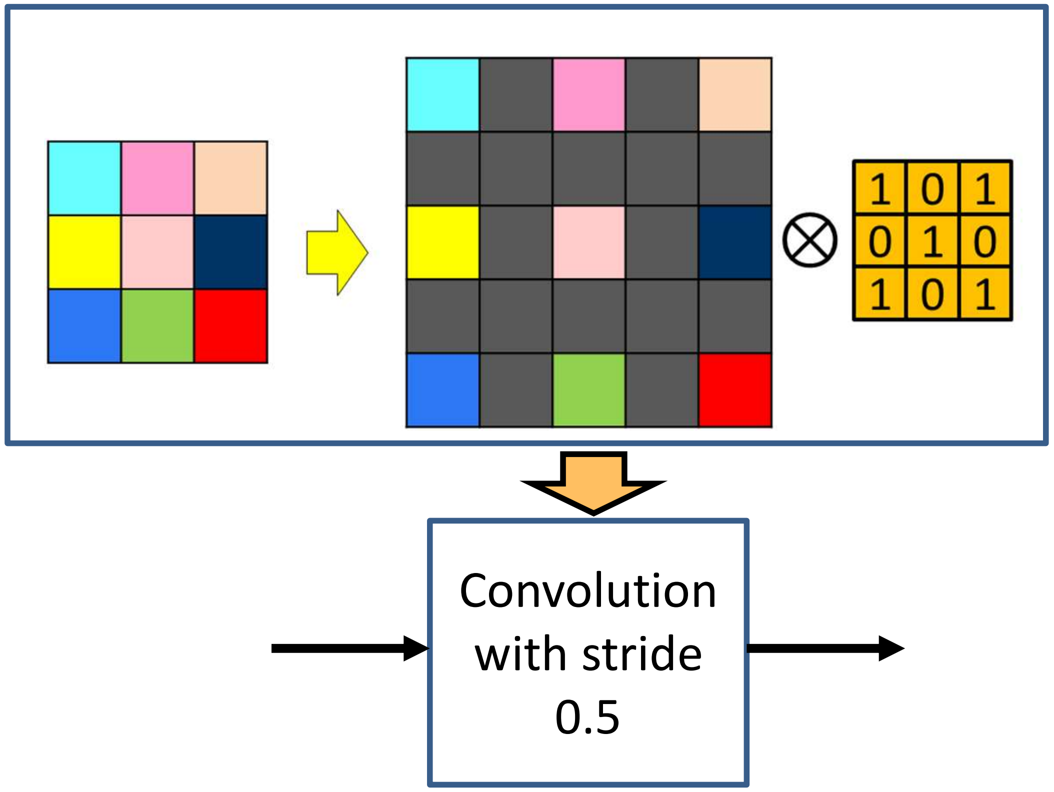
```
for j = 1:$D_l$
   m = 1
   for x = 1:stride(l):$W_{l-1}-K_l+1$
      n = 1
      for y = 1:stride(l):$H_{l-1}-K_l+1$
         Y(l,j,m,n) = mean(Y(l-1,j,x:x+$K_l$-1,y:y+$K_l$-1))
         n = n+1
      m = m+1
```

The Upsampling Layer



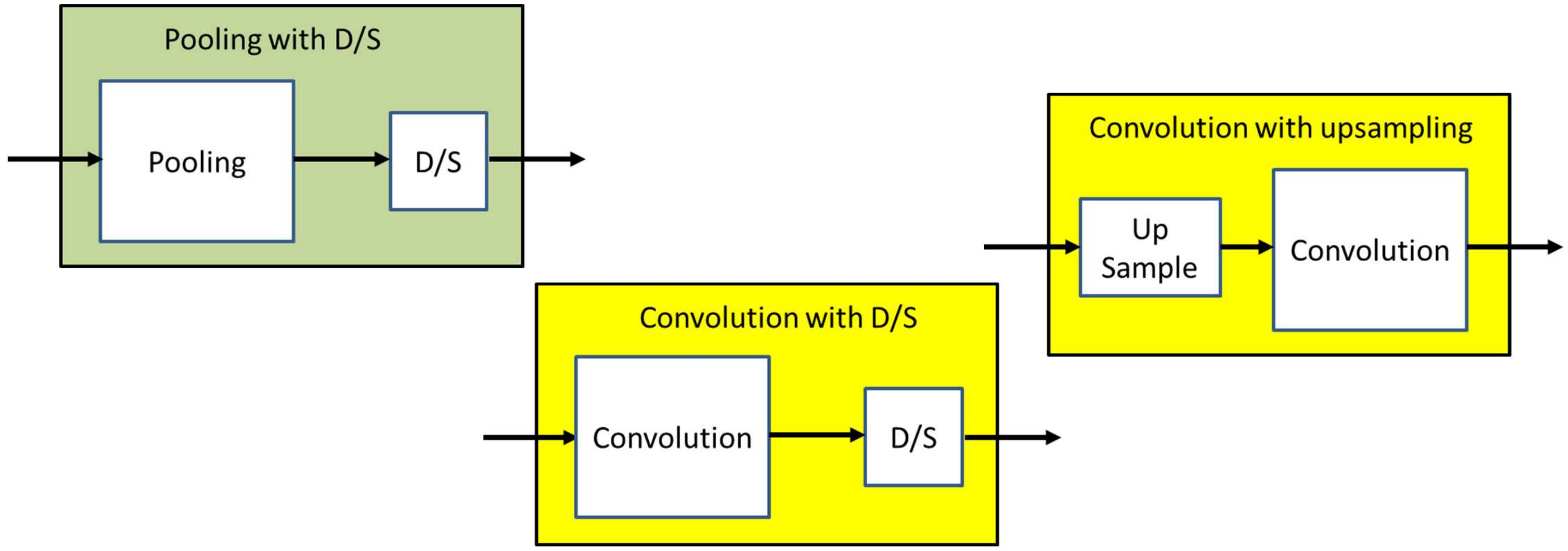
- A *upsampling* layer is generally followed by a CNN layer
  - It is not useful to follow it by a pooling layer
  - It is also not useful as the *final* layer of a CNN

The Upsampling Layer



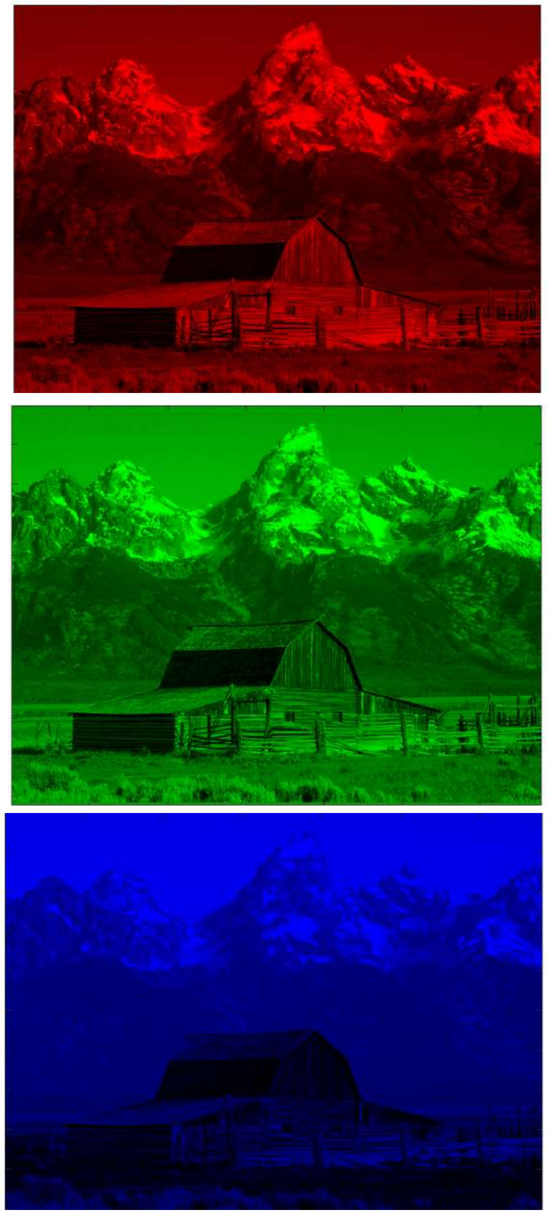
- Upsampling layers followed by a convolutional layer are also often viewed as convolving with a fractional stride
  - Upsampling by factor S is the same as striding by factor $1/S$
- Also called “transpose convolutions” for reasons we won’t get into here

* with resampling



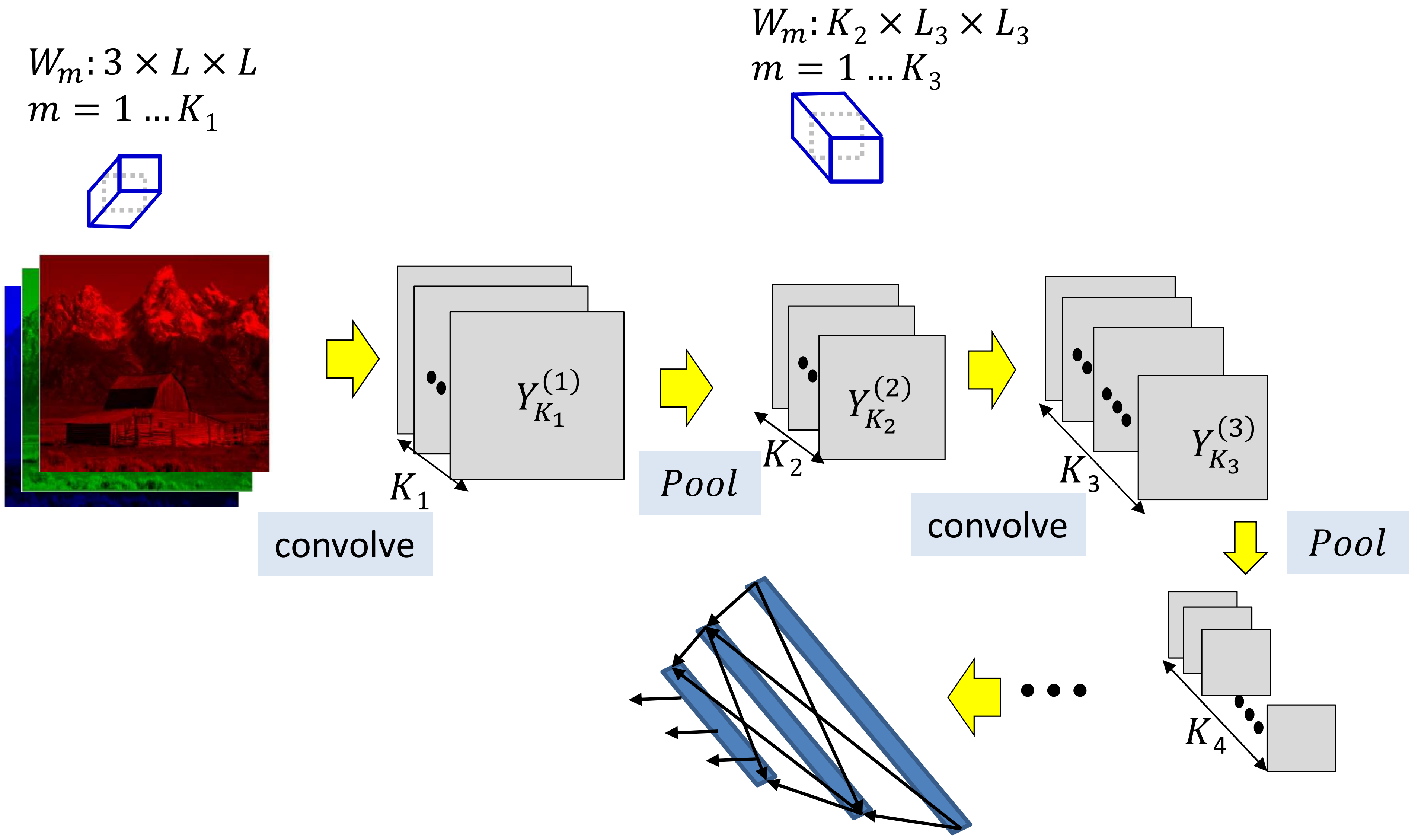
- Although the resampling operation is generally merged with convolutions or pooling (by changing their stride) in the forward pass in practical implementations…
- …It is more convenient to think of the two as separate operations in the backward pass
  - More on this later…

Recap: A CNN, end-to-end



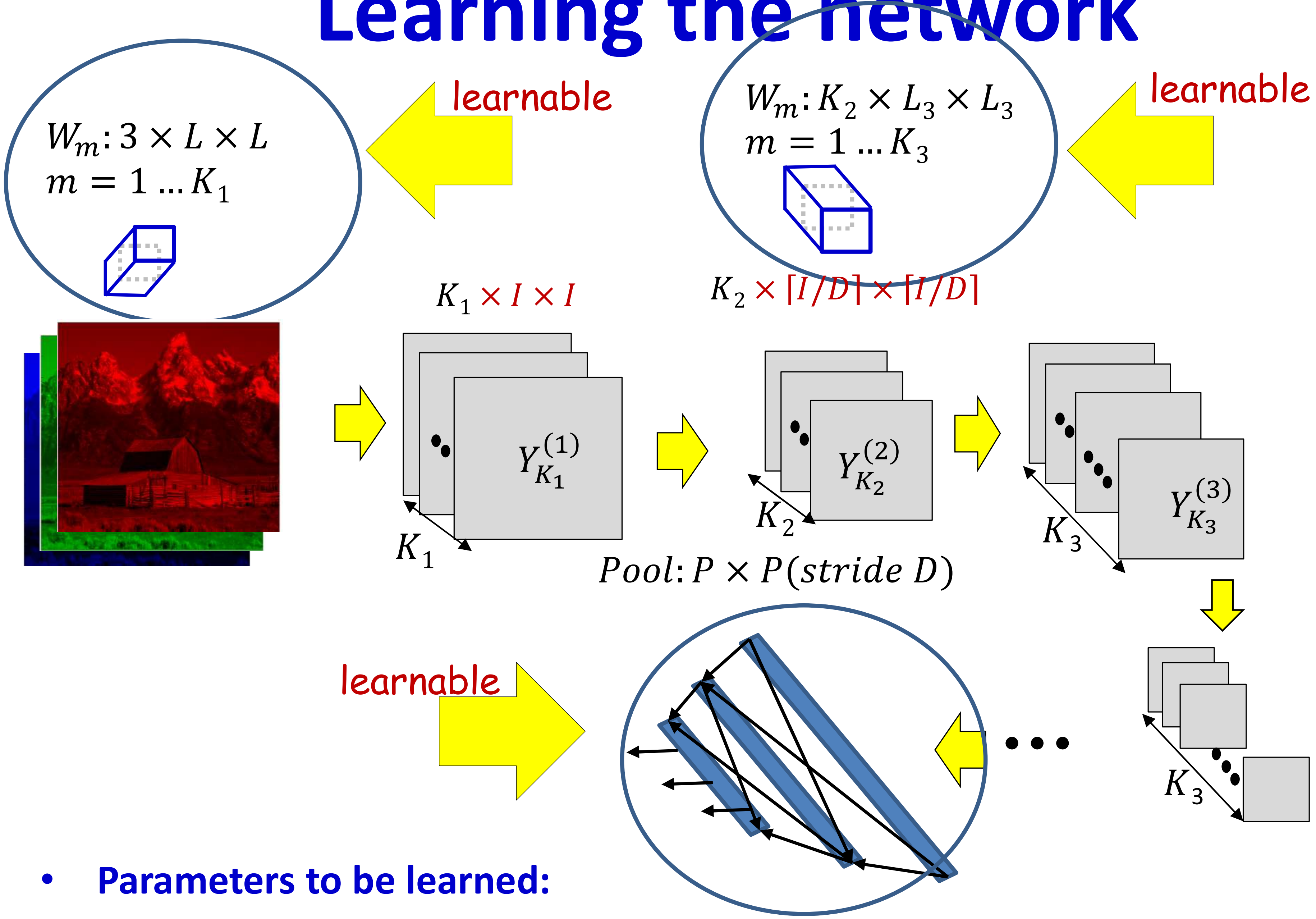
- Typical image classification task
  - Assuming maxpooling..
- Input: RBG images
  - Will assume color to be generic

Recap: A CNN, end-to-end



- Several convolutional and pooling layers.
- The output of the last layer is “flattened” and passed through an MLP

Learning the network



- **Parameters to be learned:**
  - **The weights of the neurons in the final MLP**
  - **The (weights and biases of the) filters for every *convolutional* layer**

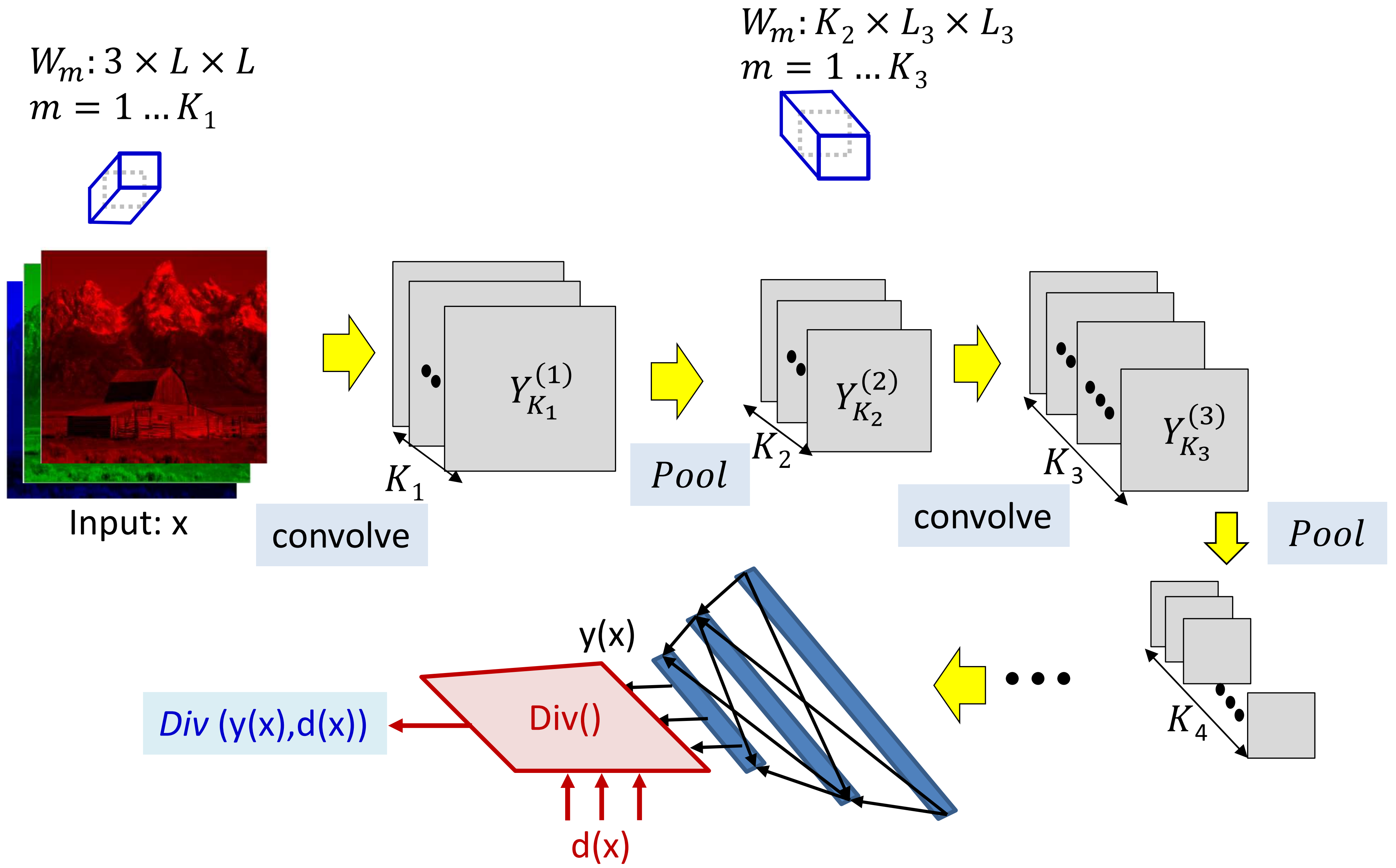
Recap: Learning the CNN

- Training is as in the case of the regular MLP
  - The *only* difference is in the *structure* of the network
- **Training examples of (Image, class) are provided**

- **Define a loss:**
  - Define a divergence between the desired output and true output of the network in response to any input
  - The loss aggregates the divergences of the training set

- **Network parameters are trained to minimize the loss**
  - Through variants of gradient descent
  - Gradients are computed through backpropagation

Defining the loss



- The loss for a single instance

Recap: Problem Setup

- Given a training set of input-output pairs $(X_1, d_1), (X_2, d_2), \ldots, (X_T, d_T)$
- The divergence on the ith instance is $div(Y_i, d_i)$
- The aggregate Loss

$$Loss = \frac{1}{T}\sum_{i=1}^{T} div(Y_i, d_i)$$

- Minimize $Loss$ w.r.t $\{W_m, b_m\}$
  – Using gradient descent

Recap: The derivative

Total training loss:

$$Loss = \frac{1}{T}\sum_i Div(Y_i, d_i)$$

- Computing the derivative

Total derivative:

$$\frac{dLoss}{dw} = \frac{1}{T}\sum_i \frac{dDiv(Y_i, d_i)}{dw}$$

Recap: The derivative

Total training loss:

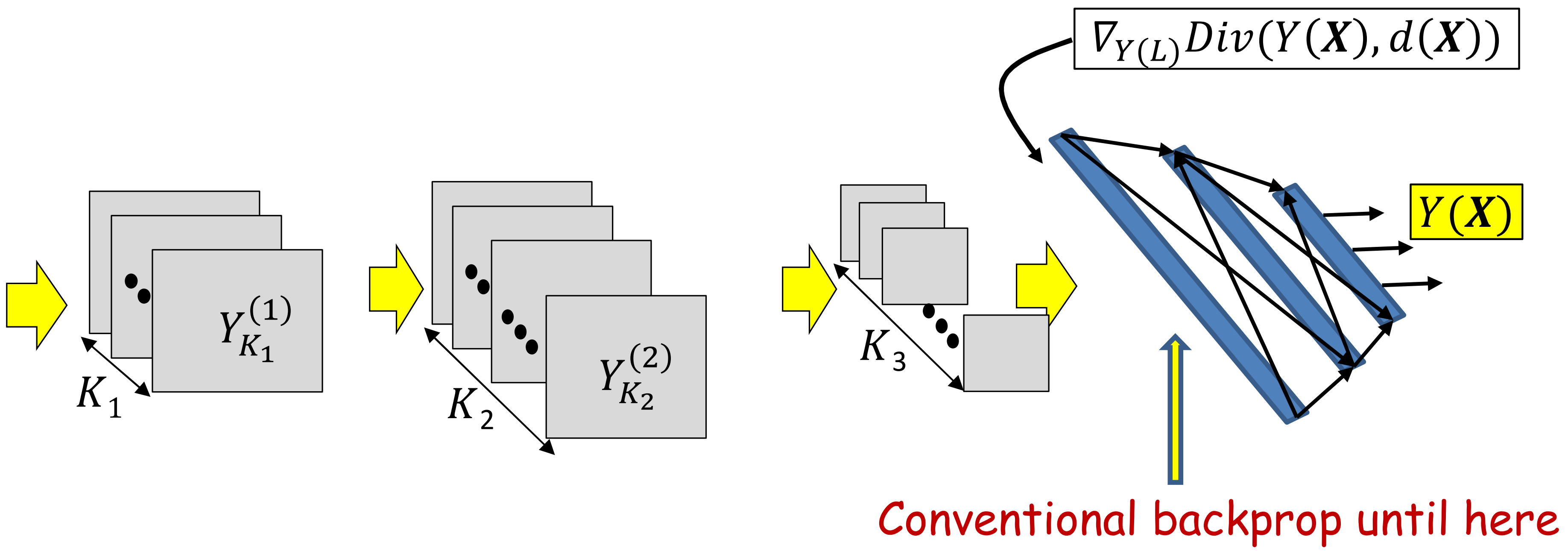
$$Loss = \frac{1}{T}\sum_i Div(Y_i, d_i)$$

- Computing the derivative

Total derivative:

$$\frac{dLoss}{dw} = \frac{1}{T}\sum_i \frac{dDiv(Y_i, d_i)}{dw}$$

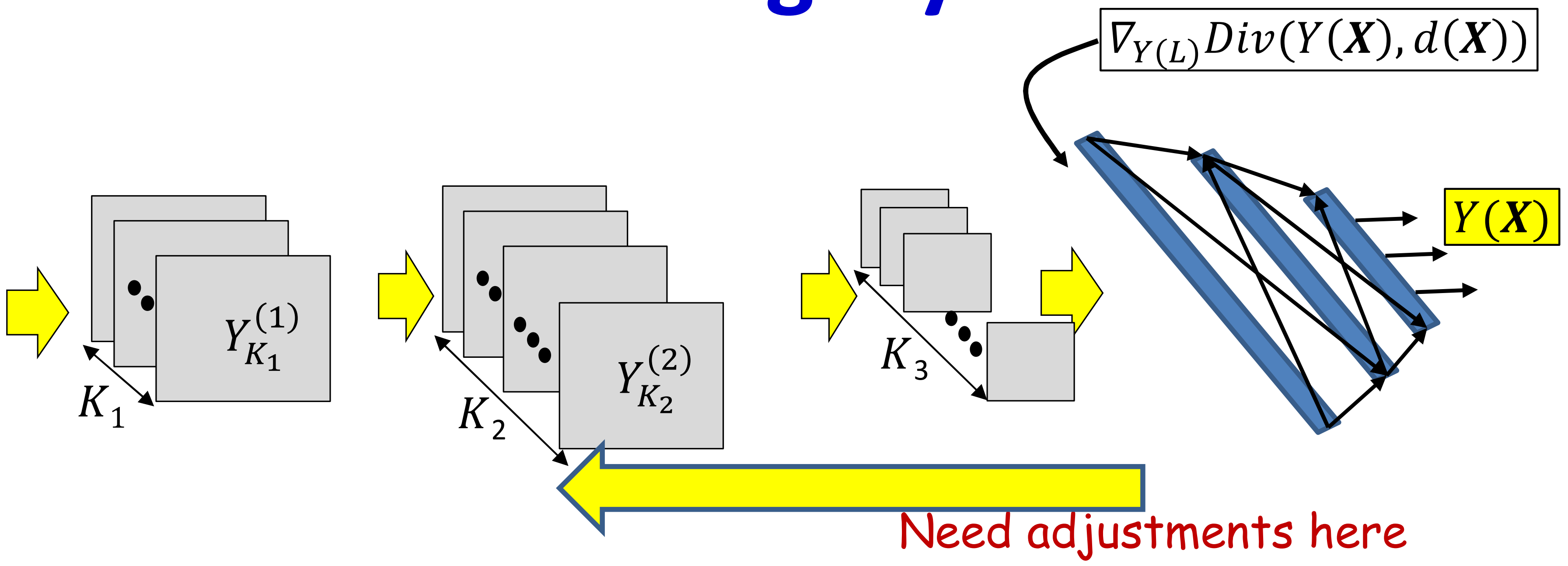
Backpropagation: Final flat layers


$\nabla_{Y(L)} Div(Y(\boldsymbol{X}), d(\boldsymbol{X}))$
$Y(\boldsymbol{X})$
$Y_{K_1}^{(1)}$
K_1
$Y_{K_2}^{(2)}$
K_2
K_3
Conventional backprop until here

- For each training instance: First, a forward pass through the net
- Then the backpropagation of the derivative of the divergence

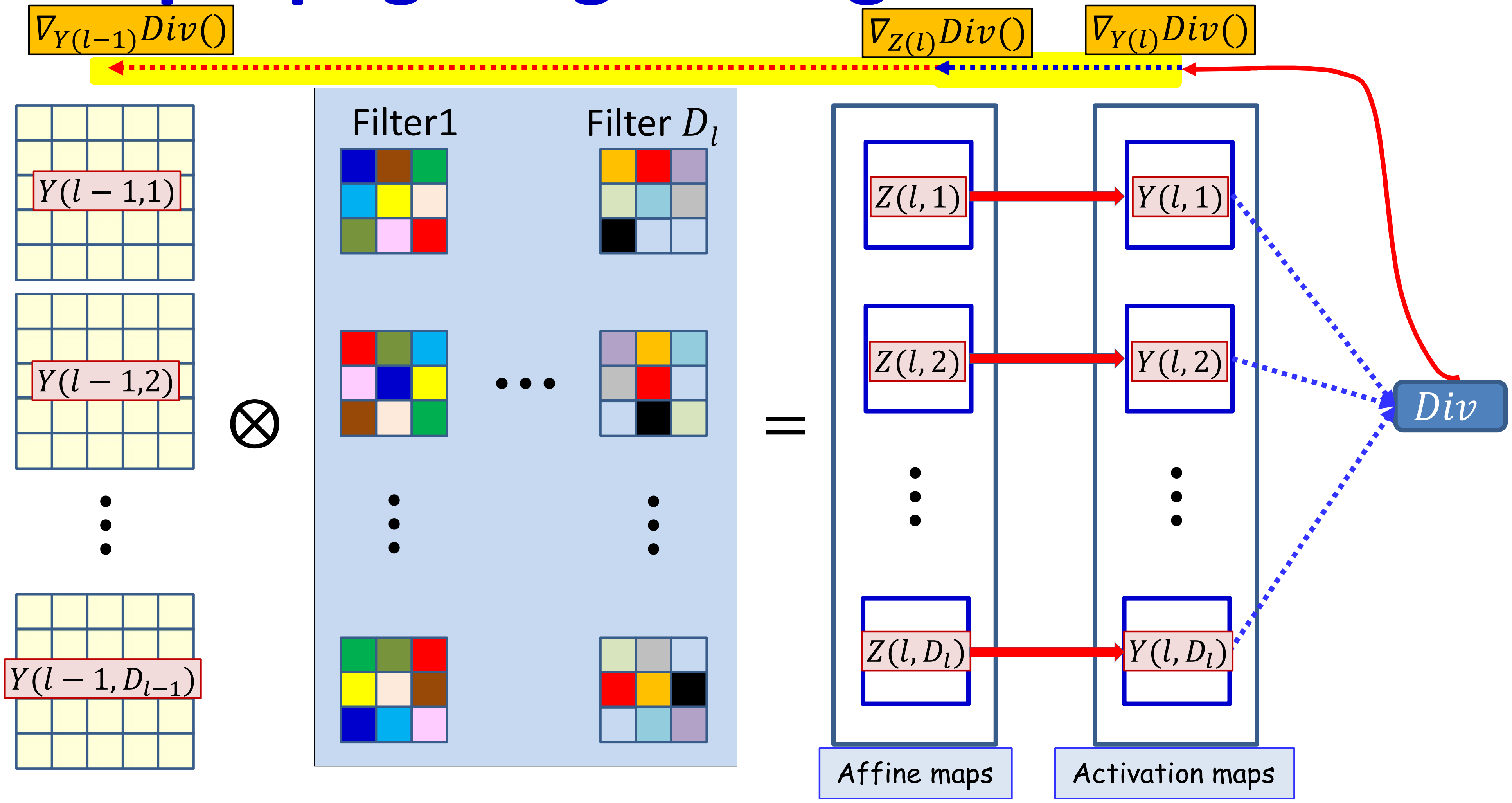
- Backpropagation continues in the usual manner until the computation of the derivative of the divergence w.r.t the inputs to the first “flat” layer
  - Important to recall: the first flat layer is only the “unrolling” of the maps from the final convolutional layer

Backpropagation: Convolutional and Pooling layers



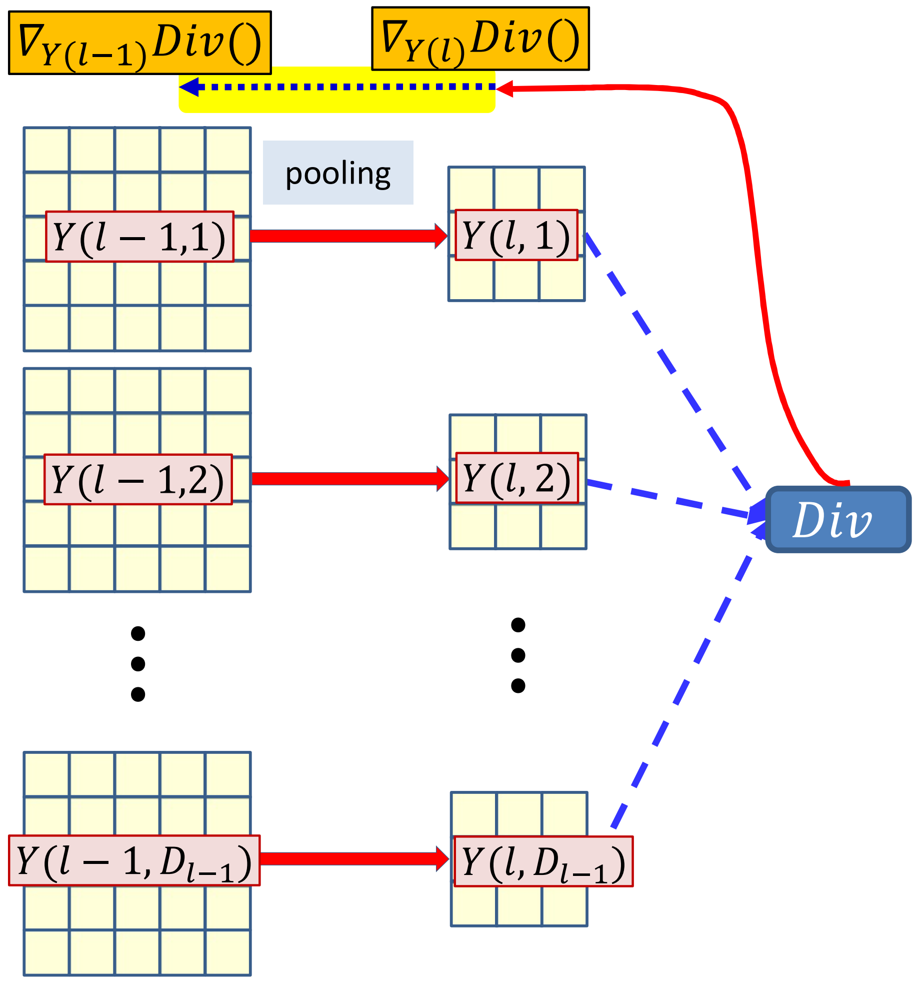
- Backpropagation from the flat MLP requires special consideration of
  - The shared computation in the convolution layers
  - The pooling layers

Backpropagating through the convolution



- **Convolution layers:**
- We already have the derivative w.r.t (all the elements of) activation map $Y(l,*)$
  - Having backpropagated it from the divergence
- We must backpropagate it through the activation to compute the derivative w.r.t. $Z(l,*)$ and further back to compute the derivative w.r.t the filters and $Y(l-1,*)$

Backprop: Pooling layer



- **Pooling layers:**
- We already have the derivative w.r.t $Y(l,*)$
  - Having backpropagated it from the divergence
- We must compute the derivative w.r.t $Y(l-1,*)$

Backpropagation: Convolutional and Pooling layers

- **Assumption:** We already have the derivatives w.r.t. the elements of the maps output by the final convolutional (or pooling) layer
  - Obtained as a result of backpropagating through the flat MLP

- **Required:**
  - **For convolutional layers**:
    - How to compute the derivatives w.r.t. the affine combination $Z(l)$ maps from the activation output maps $Y(l)$
    - How to compute the derivative w.r.t. $Y(l-1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$
  - **For pooling layers**:
    - How to compute the derivative w.r.t. $Y(l-1)$ given derivatives w.r.t. $Y(l)$

Backpropagation: Convolutional and Pooling layers

- **Assumption:** We already have the derivatives w.r.t. the elements of the maps output by the final convolutional (or pooling) layer
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- **Required:**
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    - How to compute the derivatives w.r.t. the affine combination $Z(l)$ maps from the activation output maps $Y(l)$
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  - **For pooling layers**:
    - How to compute the derivative w.r.t. $Y(l-1)$ given derivatives w.r.t. $Y(l)$

Backpropagation: Convolutional and Pooling layers

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- **Required:**
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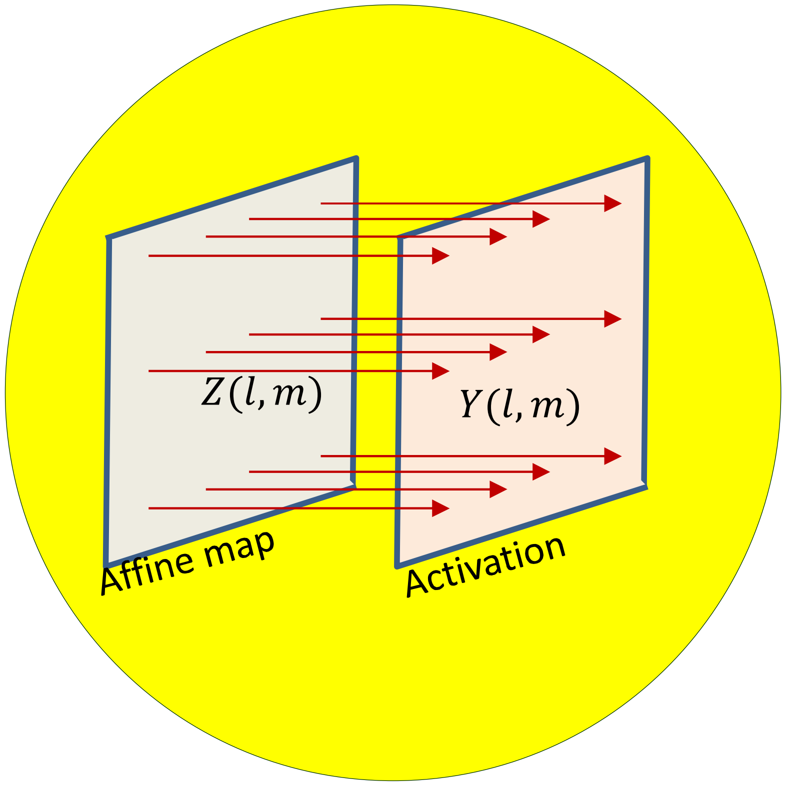
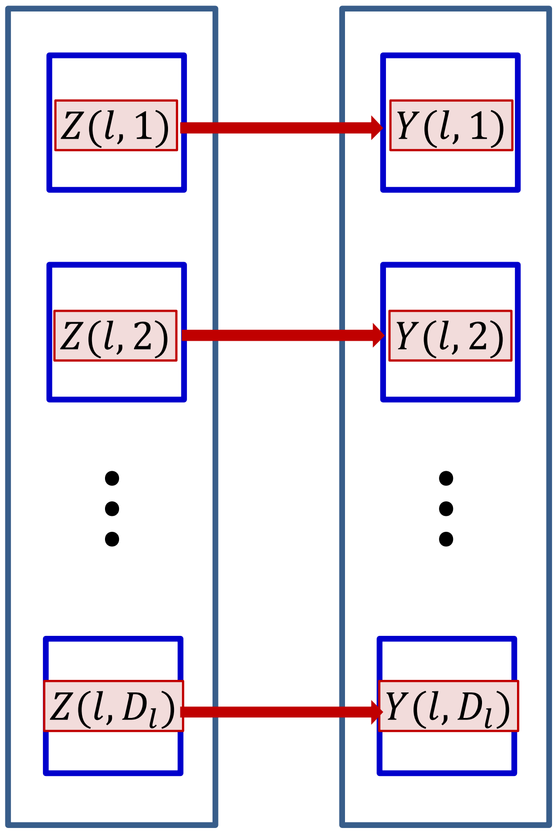
    - How to compute the derivatives w.r.t. the affine combination $Z(l)$ maps from the activation output maps $Y(l)$
    - How to compute the derivative w.r.t. $Y(l-1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$
  - **For pooling layers**:
    - How to compute the derivative w.r.t. $Y(l-1)$ given derivatives w.r.t. $Y(l)$

Backpropagating through the activation



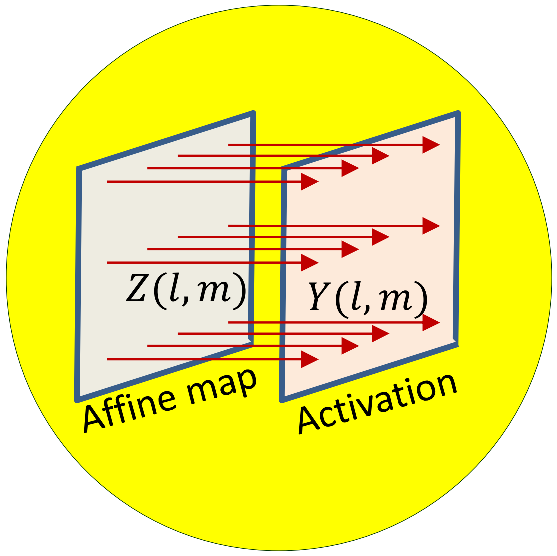


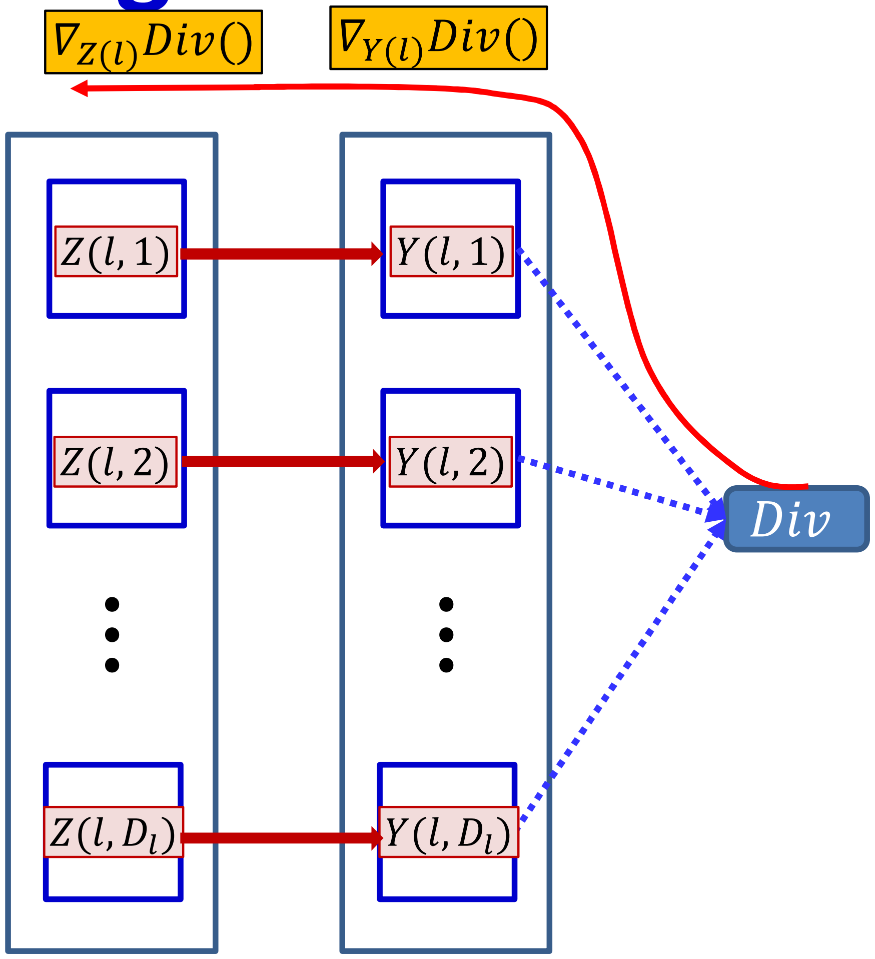
- **Forward computation:** The activation maps are obtained by point-wise application of the activation function to the affine maps

$$y(l,m,x,y) = f(z(l,m,x,y))$$

  - The affine map entries $z(l,m,x,y)$ have already been computed via convolutions over the previous layer

Backpropagating through the activation





$$y(l,m,x,y) = f(z(l,m,x,y))$$

$$\frac{dDiv}{dz(l,m,x,y)} = \frac{dDiv}{d\,y(l,m,x,y)} f'(z(l,m,x,y))$$

- **Backward computation:** For every map $Y(l,m)$ for every position (x,y), we already have the derivative of the divergence w.r.t. $y(l,m,x,y)$
  - Obtained via backpropagation
- We obtain the derivatives of the divergence w.r.t. $z(l,m,x,y)$ using the chain rule:

$$\frac{dDiv}{dz(l,m,x,y)} = \frac{dDiv}{d\,y(l,m,x,y)} f'(z(l,m,x,y))$$

  - Simple component-wise computation

Backpropagation: Convolutional and Pooling layers

- **Assumption:** We already have the derivatives w.r.t. the elements of the maps output by the final convolutional (or pooling) layer
  - Obtained as a result of backpropagating through the flat MLP

- **Required:**
  - **For convolutional layers**:
    - ✓ How to compute the derivatives w.r.t. the affine combination $Z(l)$ maps from the activation output maps $Y(l)$
    - How to compute the derivative w.r.t. $Y(l-1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$
  - **For pooling layers**:
    - How to compute the derivative w.r.t. $Y(l-1)$ given derivatives w.r.t. $Y(l)$

Backpropagating through affine map

- Forward affine computation:
  - Compute affine maps $\mathrm{z}(l, n, x, y)$ from previous layer maps $y(l-1, m, x, y)$ and filters $w_l(m, n, x, y)$

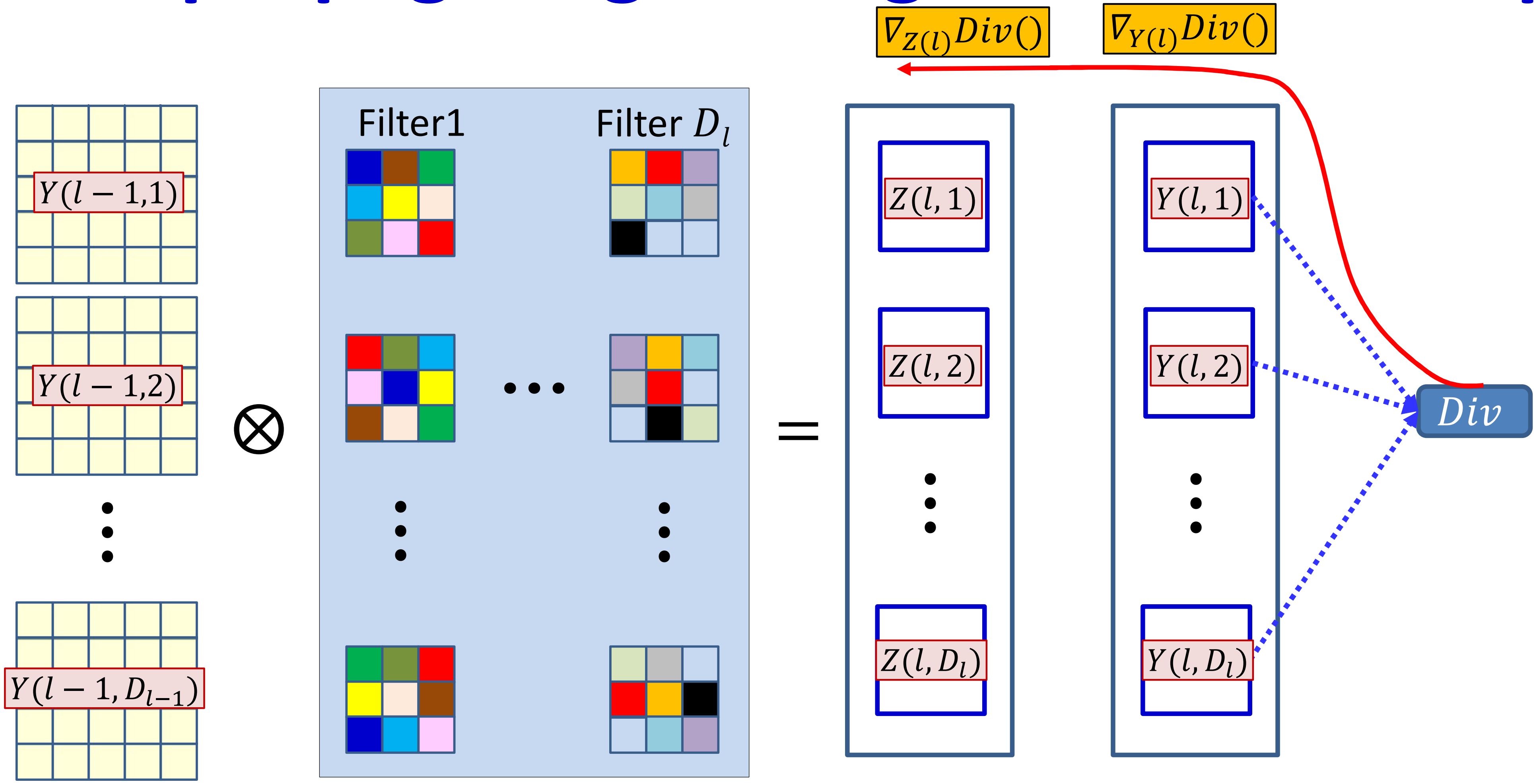
- Backpropagation: Given $\frac{dDiv}{dz(l,n,x,y)}$
  - Compute derivative w.r.t. $y(l-1, m, x, y)$
  - Compute derivative w.r.t. $w_l(m, n, x, y)$

Backpropagating through affine map

- Forward affine computation:
  - Compute affine maps $\mathrm{z}(l, n, x, y)$ from previous layer maps $y(l-1, m, x, y)$ and filters $w_l(m, n, x, y)$

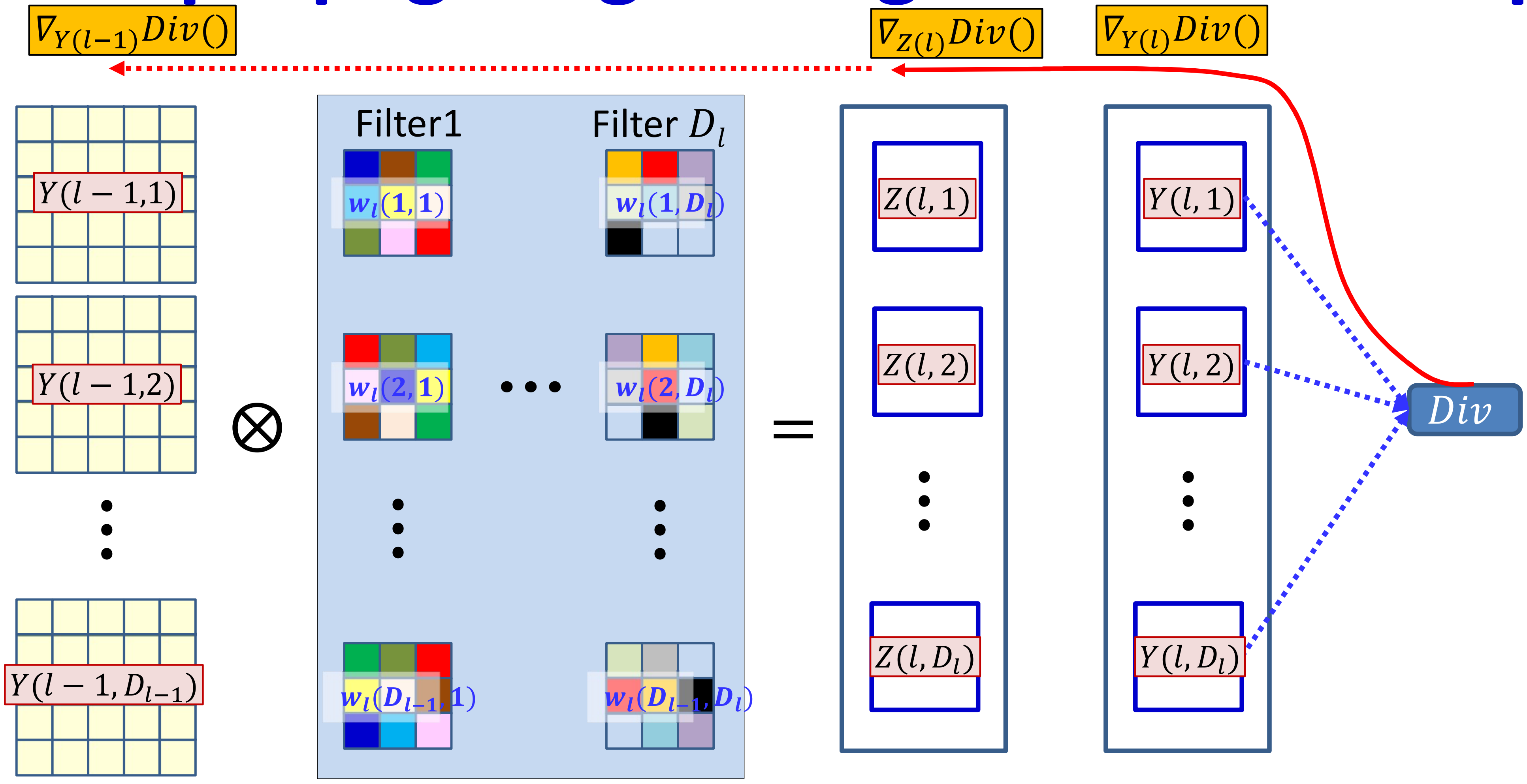
- Backpropagation: Given $\frac{dDiv}{dz(l,n,x,y)}$
  - $\boxed{\text{Compute derivative w.r.t. } y(l-1, m, x, y)}$
  - Compute derivative w.r.t. $w_l(m, n, x, y)$

Backpropagating through the affine map



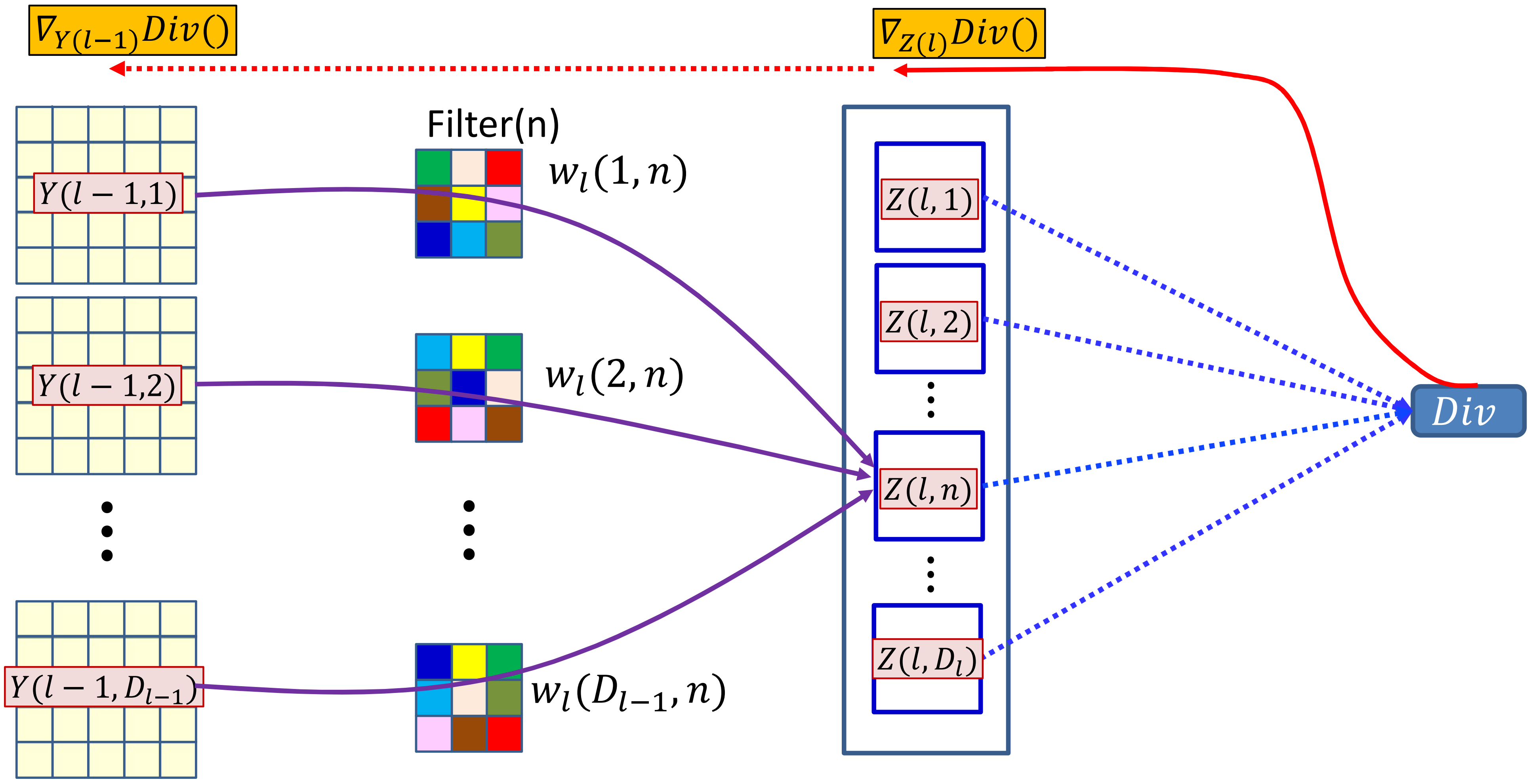
- We already have the derivative w.r.t $Z(l,*)$
  - Having backpropagated it past $Y(l,*)$

Backpropagating through the affine map



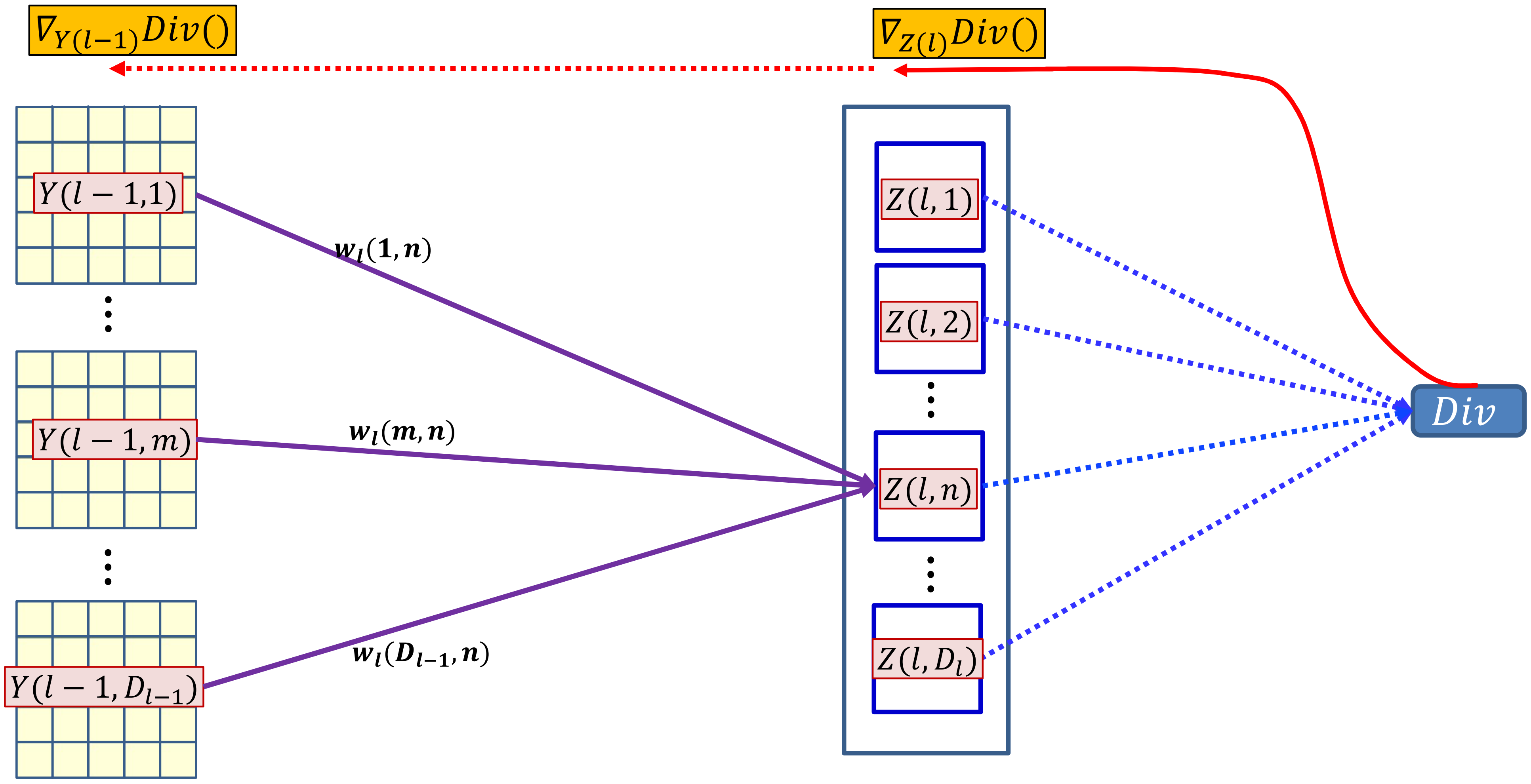
- We already have the derivative w.r.t $Z(l,*)$
  - Having backpropagated it past $Y(l,*)$
- We must compute the derivative w.r.t $Y(l-1,*)$

Dependency between Z(l,n) and Y(l-1,*)



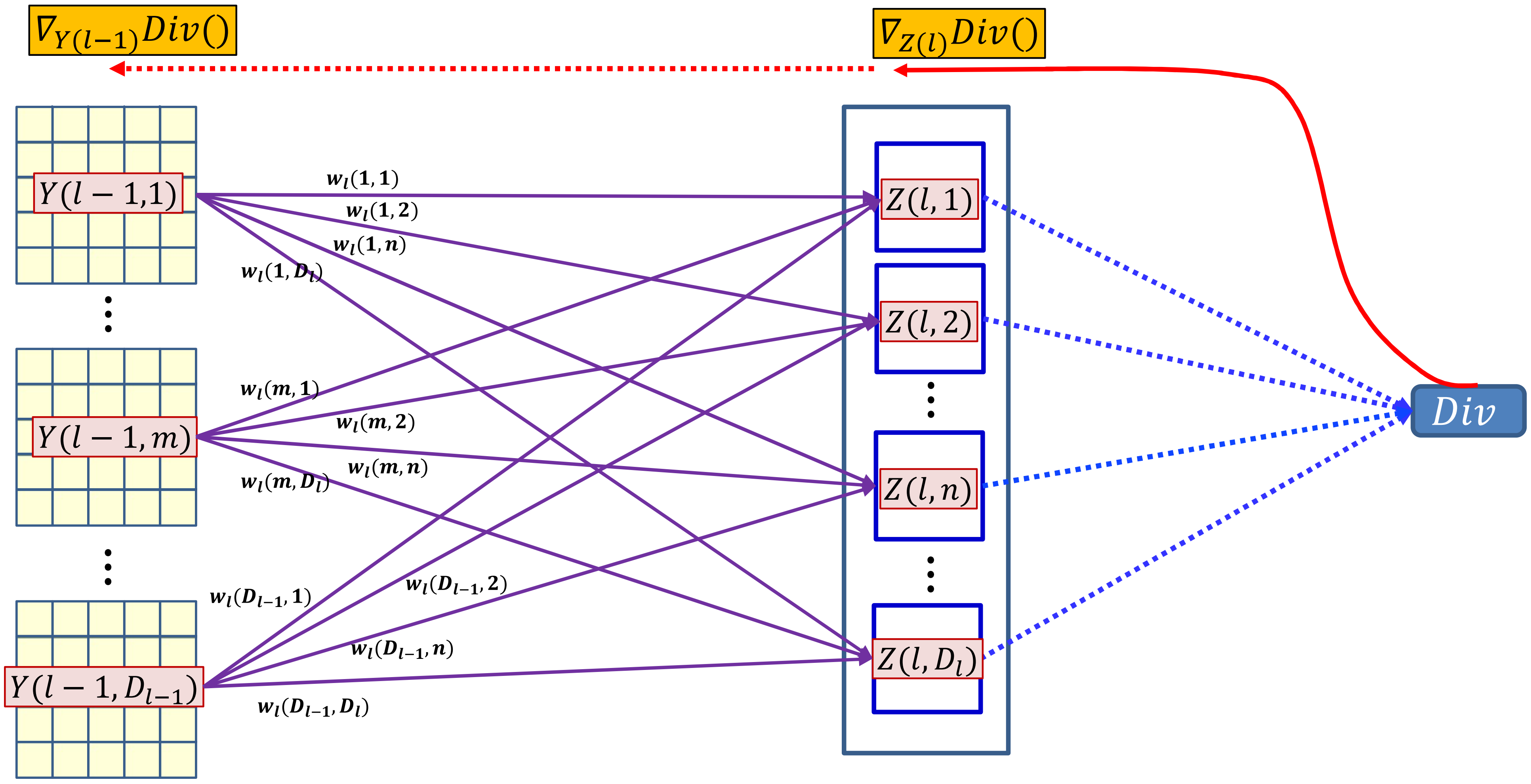
- Each $Y(l-1,m)$ map/channel influences $Z(l,n)$ through the mth "plane"(channel) of the nth filter $w_l(m,n)$

Dependency between Z(l,n) and Y(l-1,*)



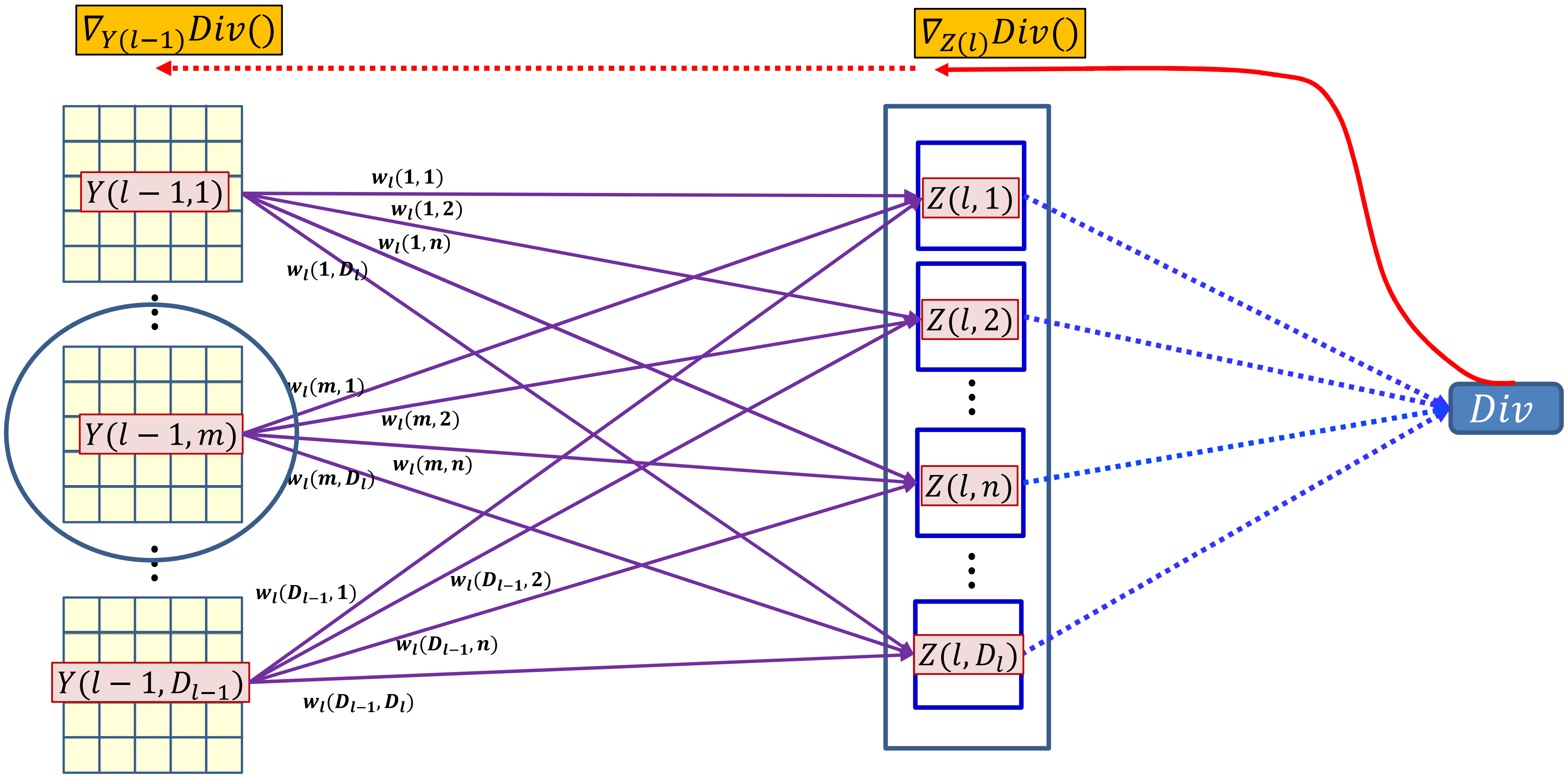
- Each $Y(l-1,m)$ map/channel influences $Z(l,n)$ through the mth "plane"(channel) of the nth filter $w_l(m,n)$

Dependency between Z(l,*) and Y(l-1,*)



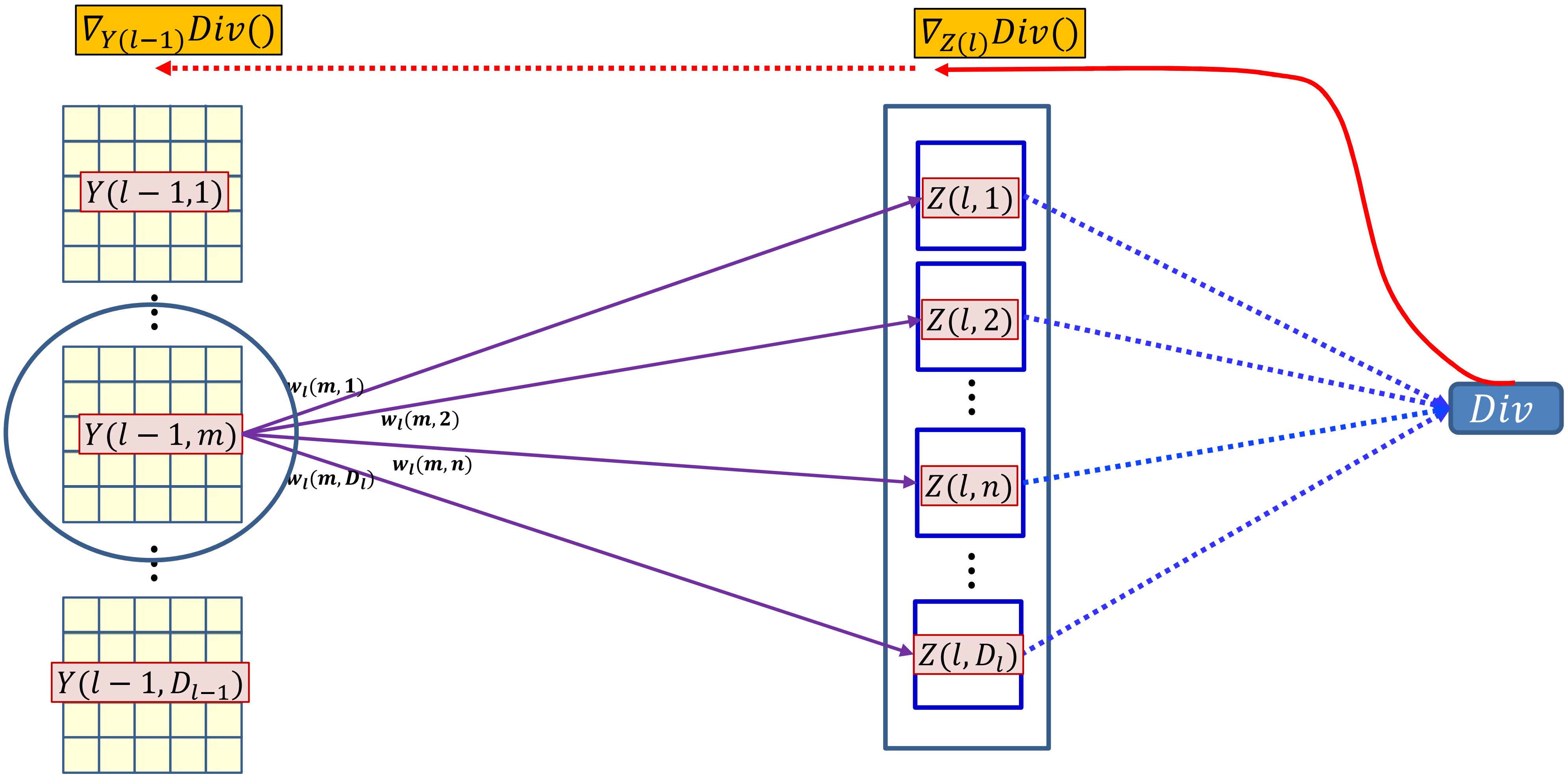
- Each $Y(l-1,m)$ map/channel influences $Z(l,n)$ through the mth "plane"(channel) of the nth filter $w_l(m,n)$

Dependency between Z(l,*) and Y(l-1,*)



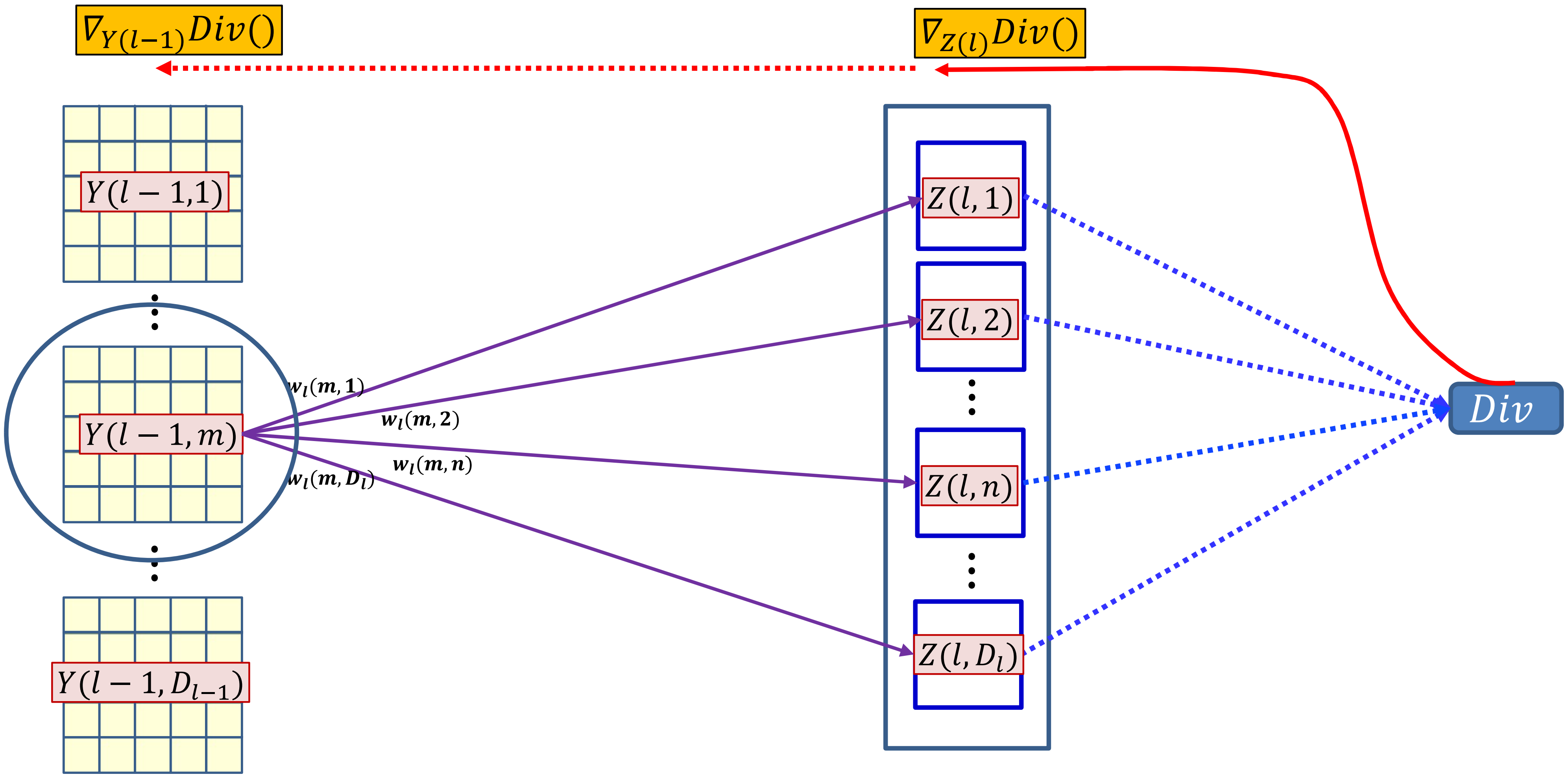
- Each $Y(l-1,m)$ map/channel influences $Z(l,n)$ through the mth "plane"(channel) of the nth filter $w_l(m,n)$

Dependency diagram for a single map



- Each $Y(l-1,m)$ map/channel influences $Z(l,n)$ through the mth "plane"(channel) of the nth filter $w_l(m,n)$
- $Y(l-1,m,*,*)$ influences the divergence through all $Z(l,n,*,*)$ maps

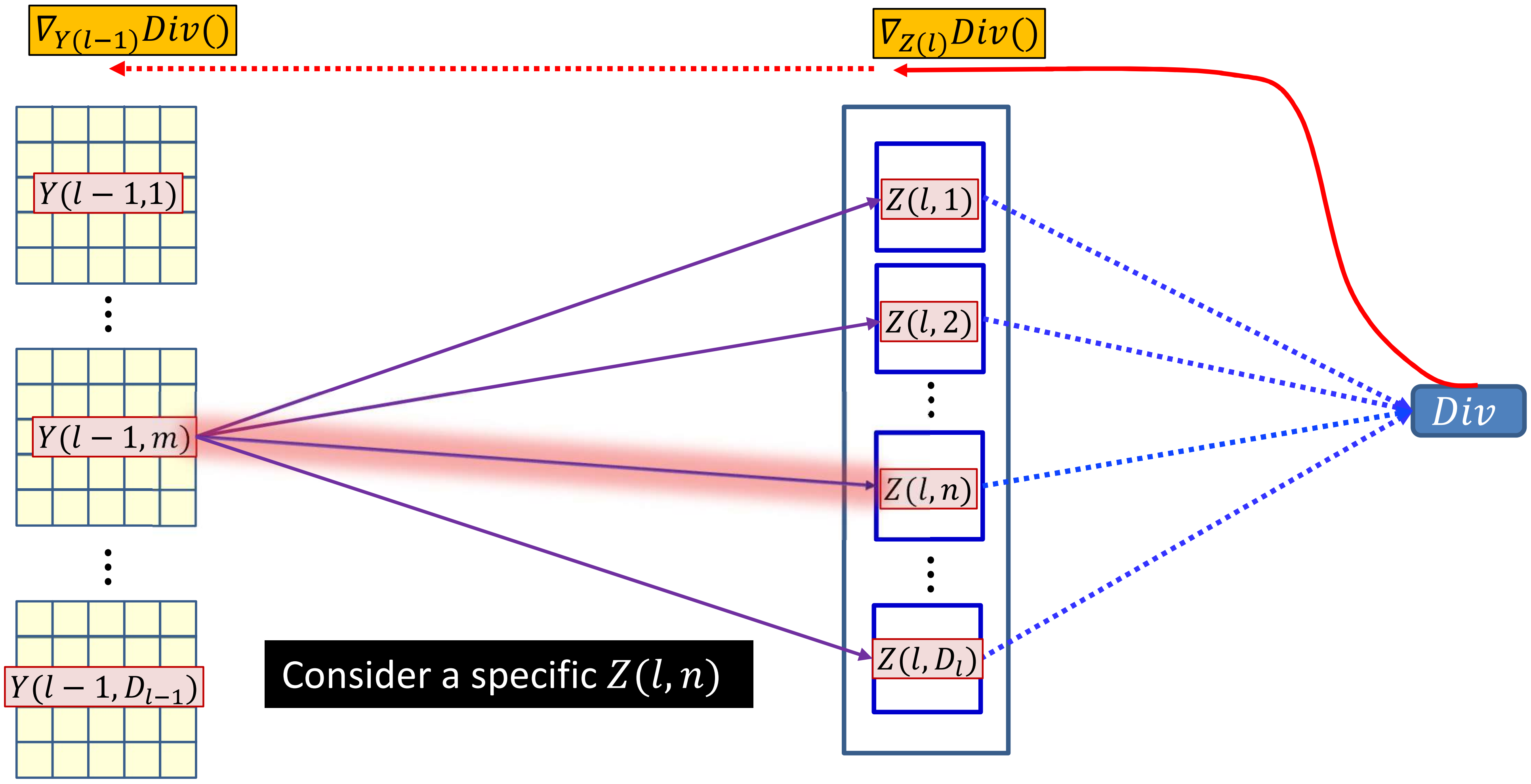
Dependency diagram for a single map



$$\nabla_{Y(l-1,m)}Div(.) = \sum_n \nabla_{Z(l,n)}Div(.)\,\underbrace{\nabla_{Y(l-1,m)}Z(l,n)}$$

- Need to compute $\nabla_{Y(l-1,m)}Z(l,n)$, the derivative of $Z(l,n)$ w.r.t. $Y(l-1,m)$ to complete the computation of the formula

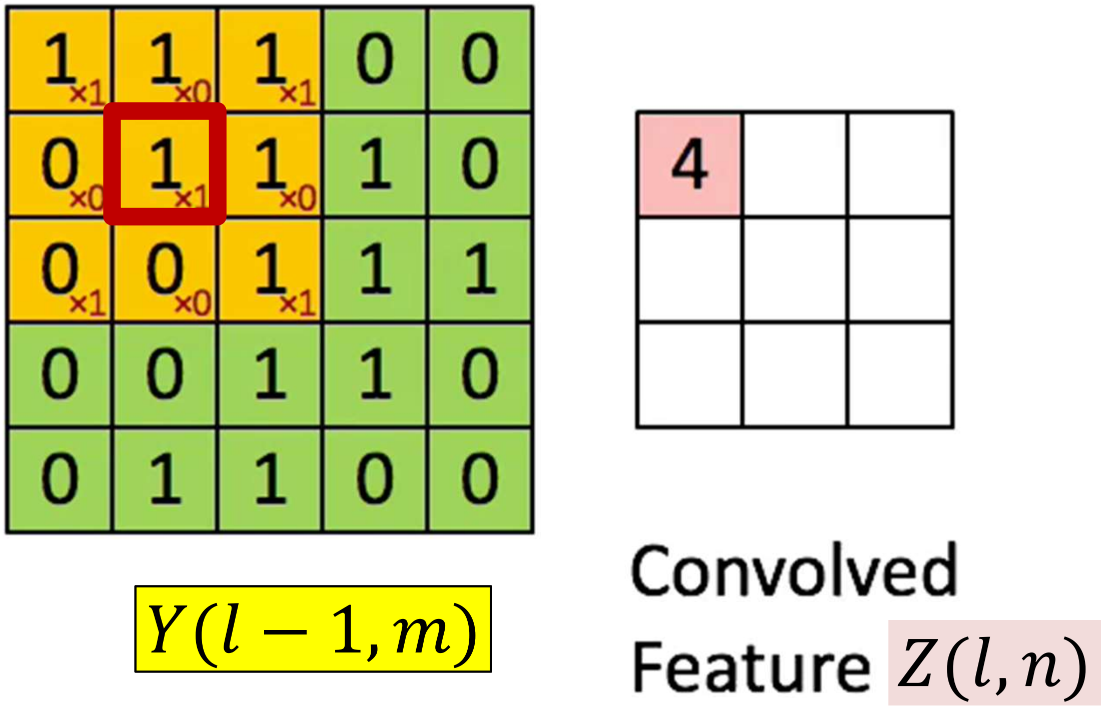
Dependency diagram for a single map



$$\nabla_{Y(l-1,m)}Div(.) = \sum_n \nabla_{Z(l,n)}Div(.)\, \underbrace{\nabla_{Y(l-1,m)}Z(l,n)}$$

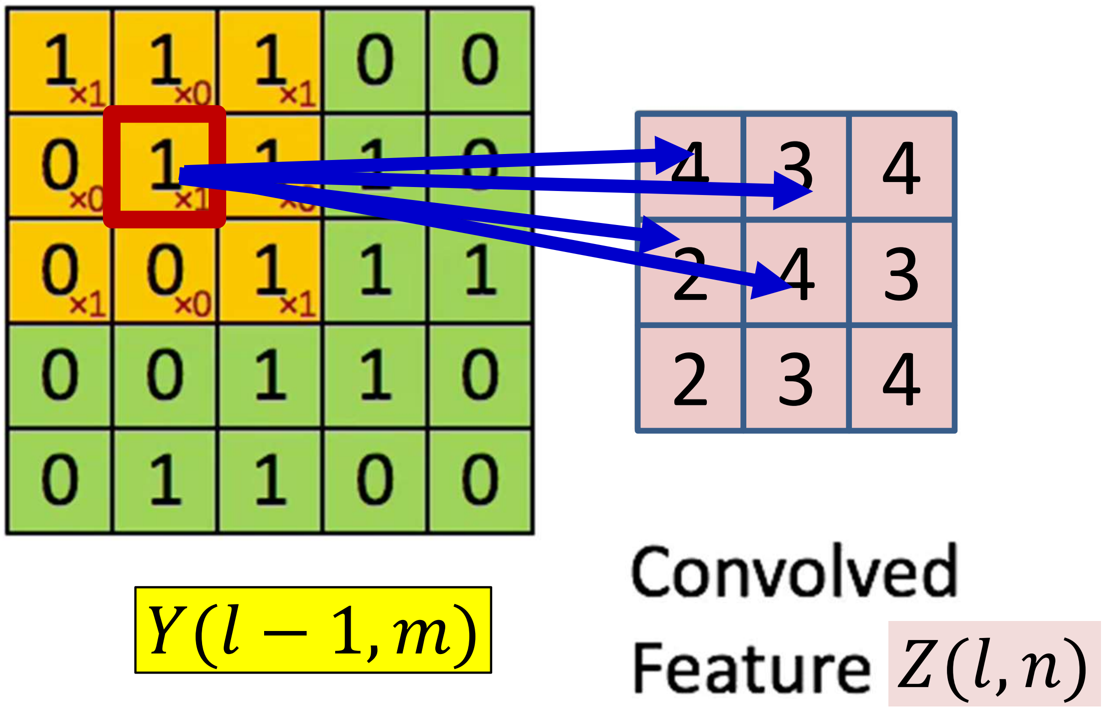
- Need to compute $\nabla_{Y(l-1,m)}Z(l,n)$, the derivative of $Z(l,n)$ w.r.t. $Y(l-1,m)$ to complete the computation of the formula

BP: Convolutional layer



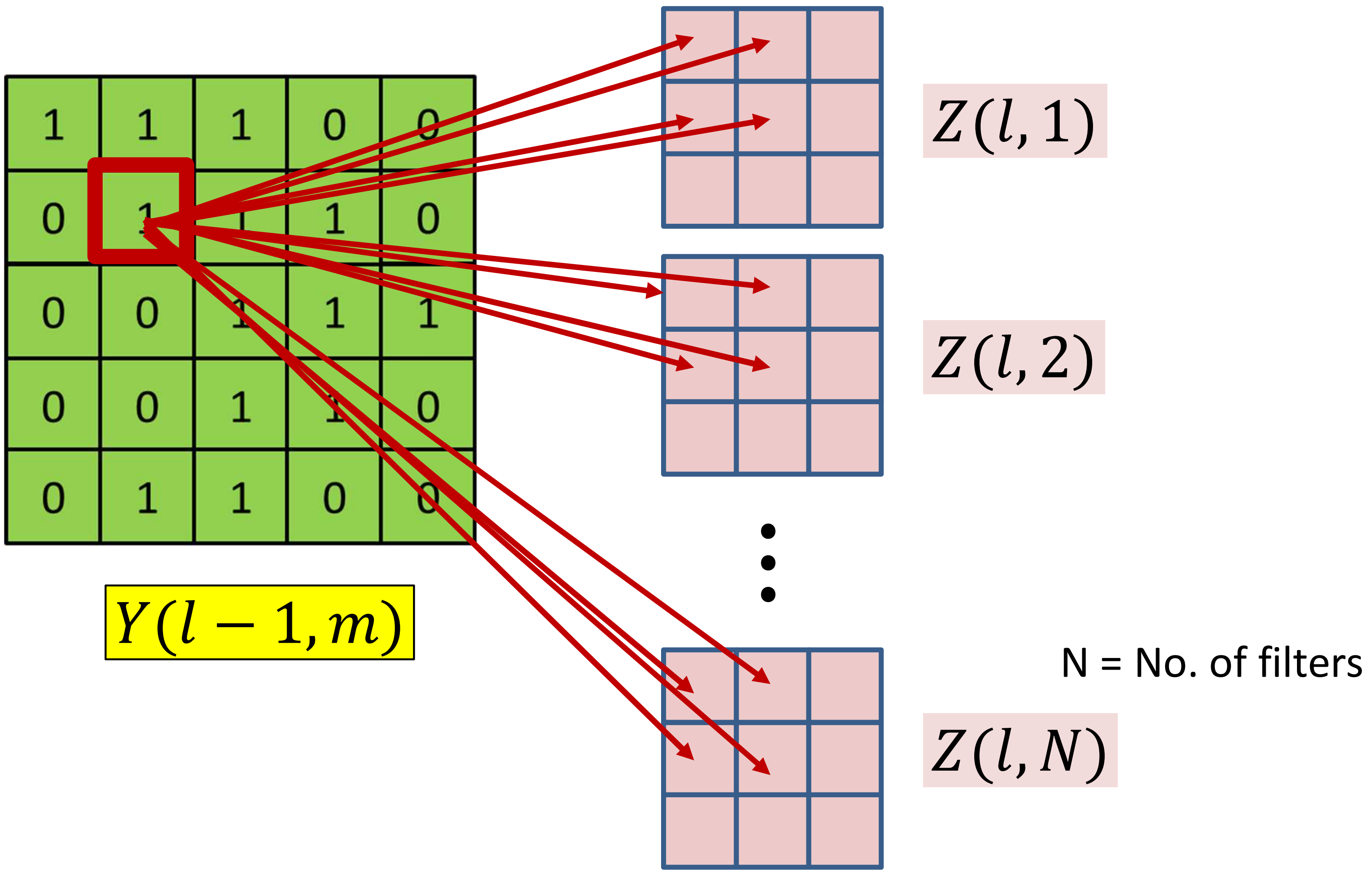
- Each $Y(l-1,m,x,y)$ affects several $z(l,n,x',y')$ terms

BP: Convolutional layer



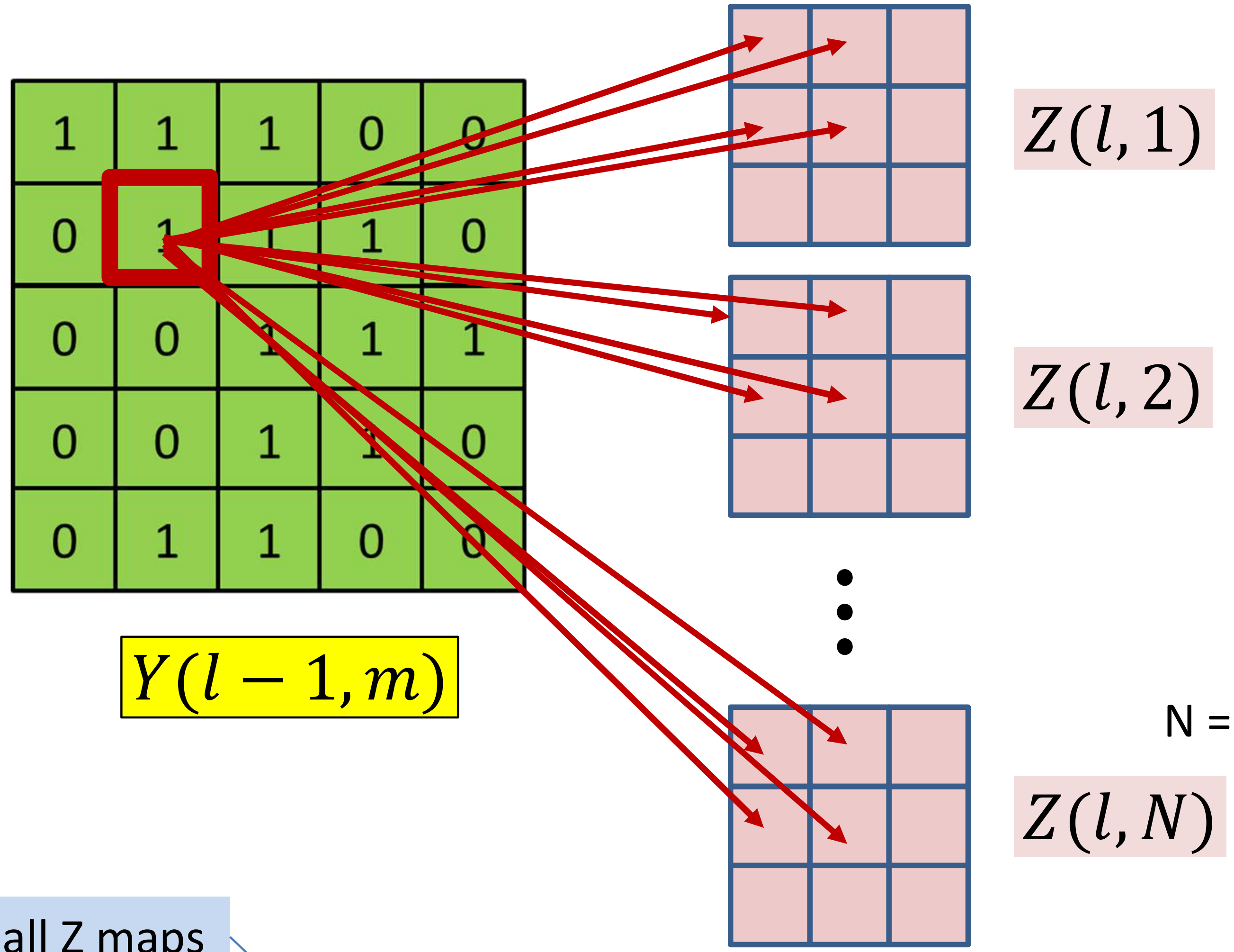
- Each $Y(l-1,m,x,y)$ affects several $z(l,n,x',y')$ terms

BP: Convolutional layer



- Each $Y(l-1,m,x,y)$ affects several $z(l,n,x',y')$ terms
  - Affects terms in *all* l th layer Z maps

BP: Convolutional layer

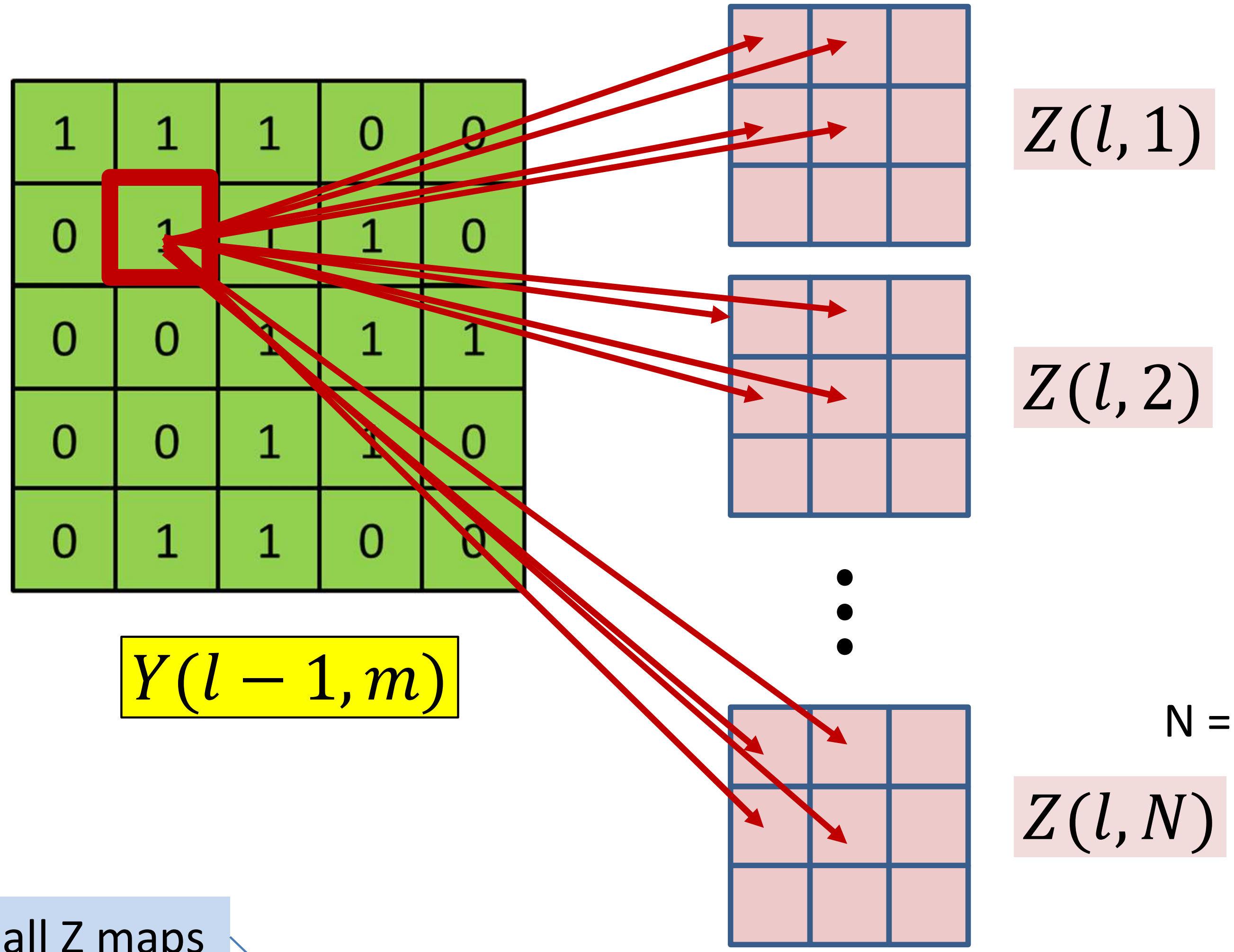


N = No. of filters

Summing over all Z maps

$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_{n}\sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')}\frac{dz(l,n,x',y')}{dY(l-1,m,x,y)}$$

BP: Convolutional layer



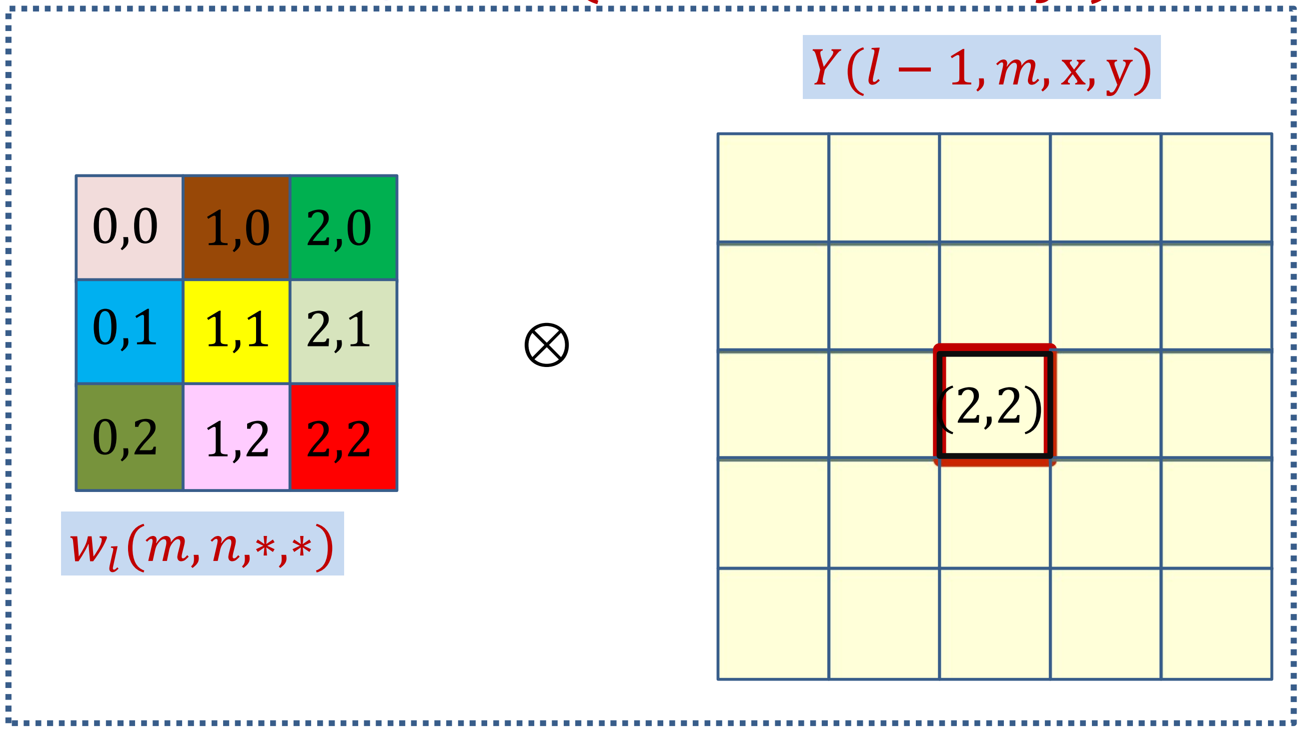
N = No. of filters

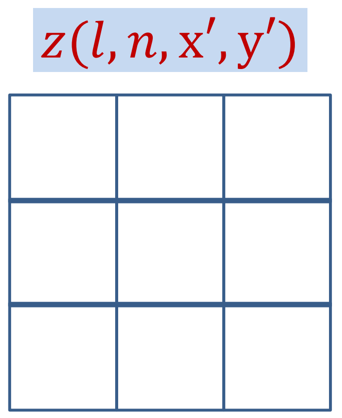
Summing over all Z maps

What is this?

$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} \frac{dz(l,n,x',y')}{dY(l-1,m,x,y)}$$

How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$

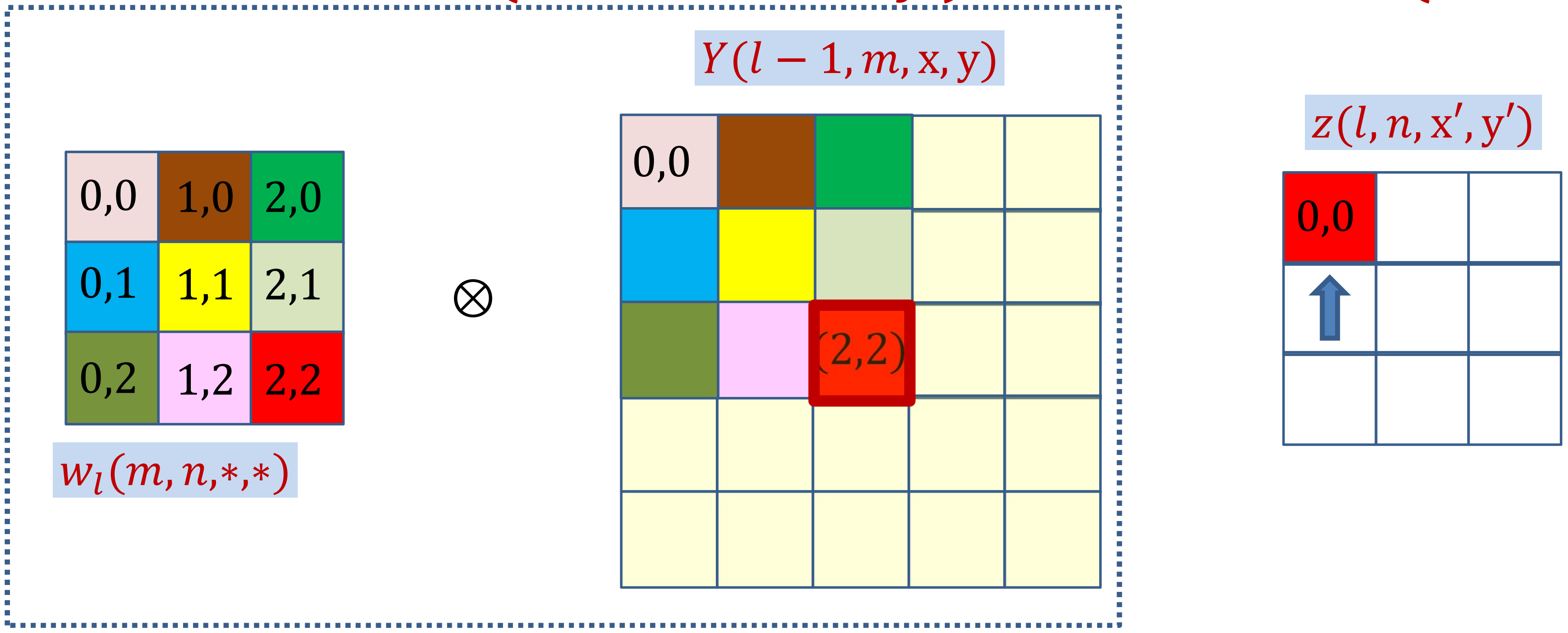




Assuming indexing begins at 0

- Compute how *each* x, y in Y influences various locations of z

How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$

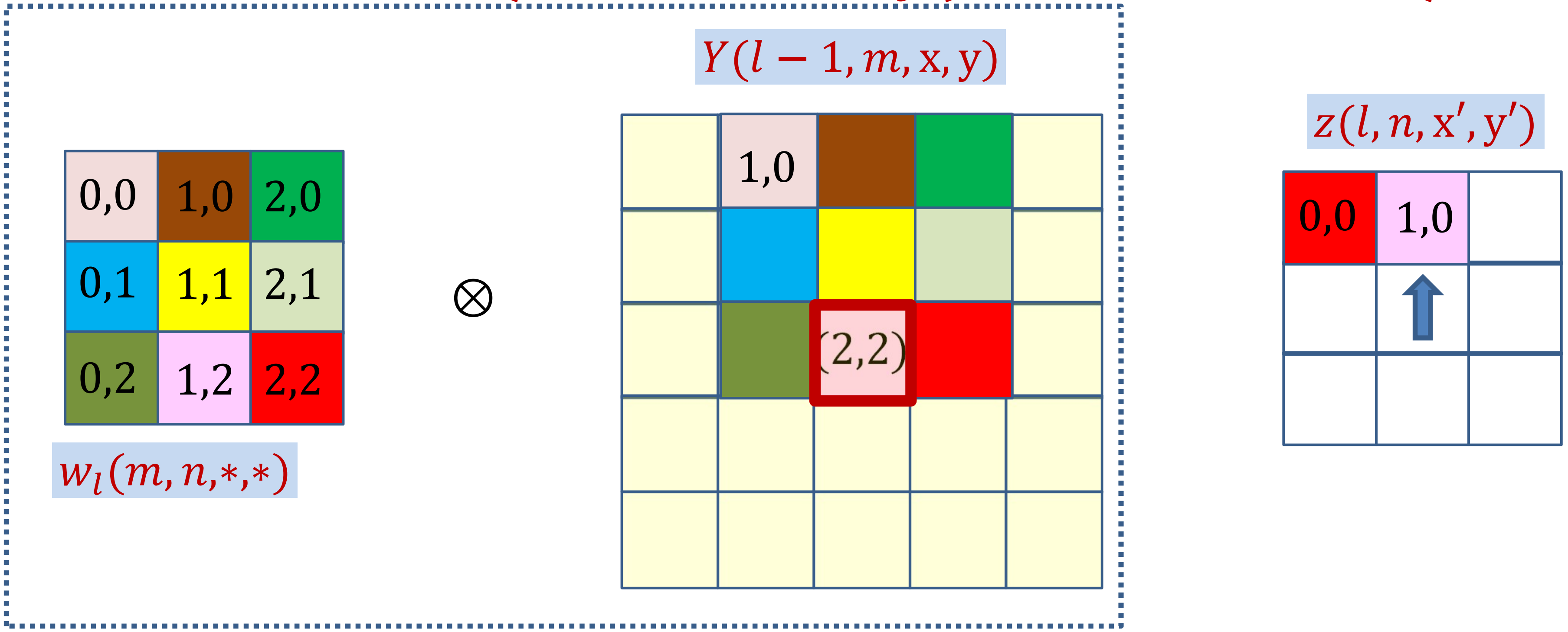


$$z(l,n,0,0) \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,2,2)$$

- **Note:** The coordinates of $z(l,n)$ and $w_l(m,n)$ sum to the coordinates of $Y(l-1,m)$

$$z(l,n,x',y') \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,2-x',2-y')$$

How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$

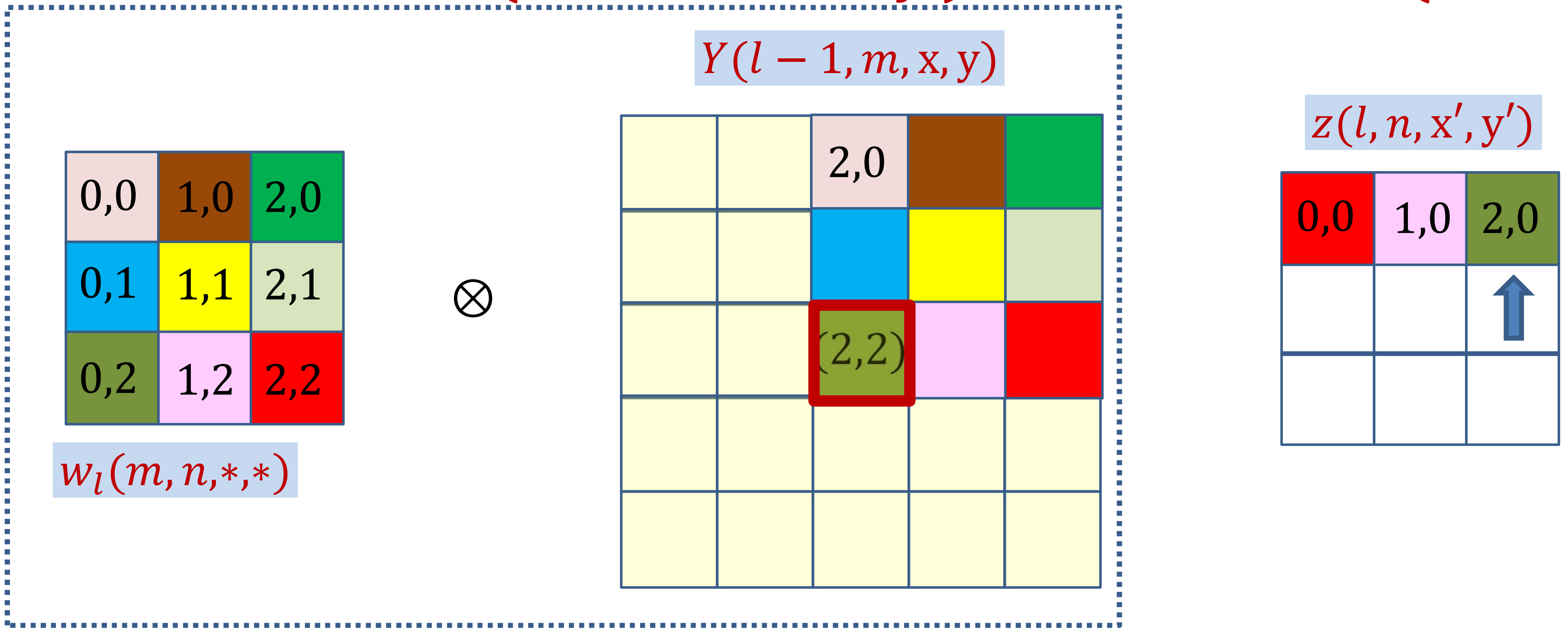


$$z(l,n,1{,}0) \mathrel{+}= Y(l-1,m,2{,}2)w_l(m,n,1{,}2)$$

- **Note:** The coordinates of $z(l,n)$ and $w_l(m,n)$ sum to the coordinates of $Y(l-1,m)$

$$z(l,n,x',y') \mathrel{+}= Y(l-1,m,2{,}2)w_l(m,n,2-x',2-y')$$

How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$

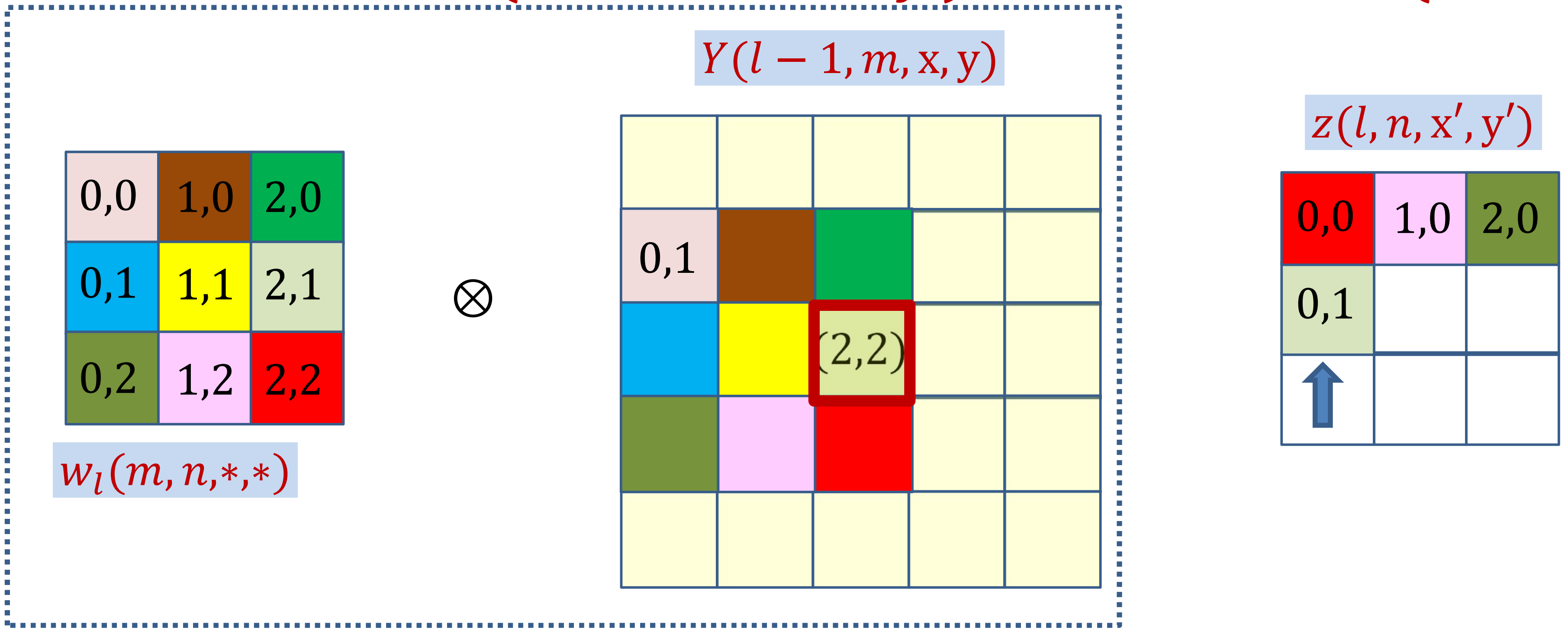


$$z(l,n,2{,}0) \mathrel{+}= Y(l-1,m,2{,}2)w_l(m,n,0{,}2)$$

- **Note:** The coordinates of $z(l,n)$ and $w_l(m,n)$ sum to the coordinates of $Y(l-1,m)$

$$z(l,n,x',y') \mathrel{+}= Y(l-1,m,2{,}2)w_l(m,n,2-x',2-y')$$

How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$

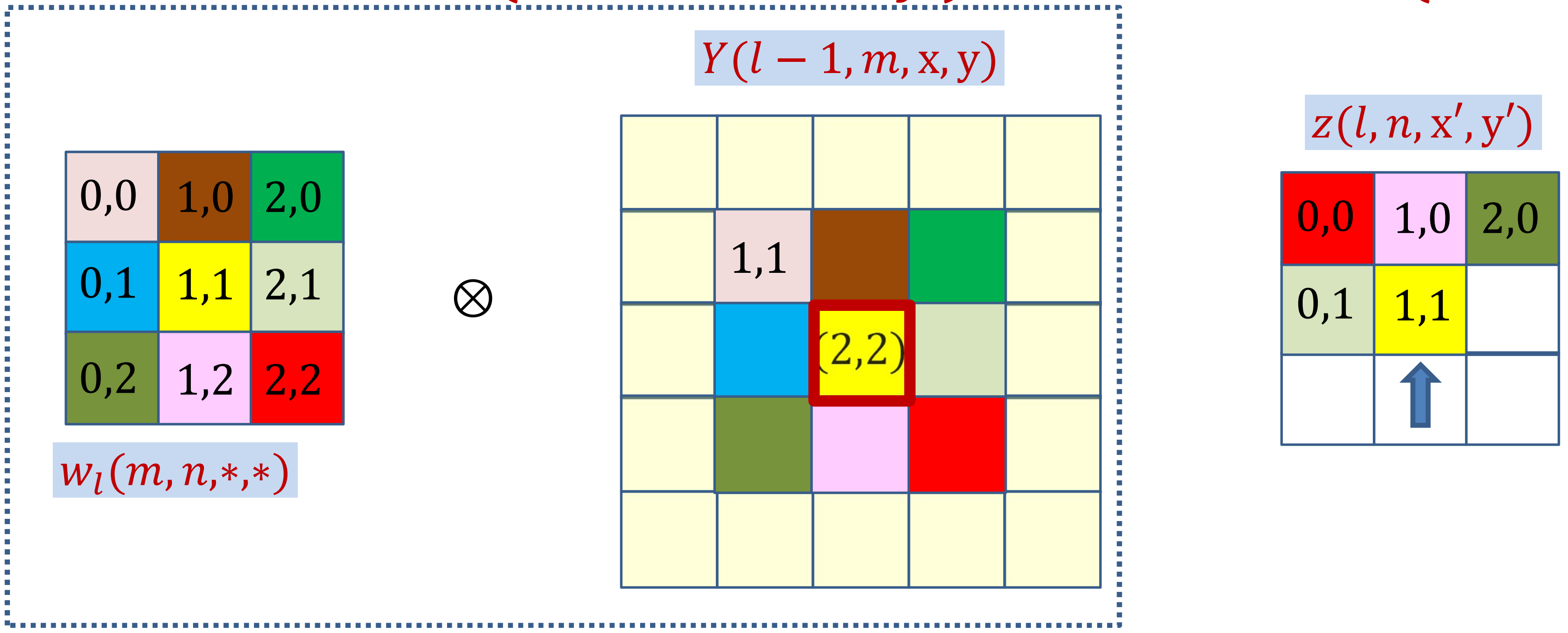


$$z(l,n,0,1) \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,2,1)$$

- **Note:** The coordinates of $z(l,n)$ and $w_l(m,n)$ sum to the coordinates of $Y(l-1,m)$

$$z(l,n,x',y') \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,2-x',2-y')$$

How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$

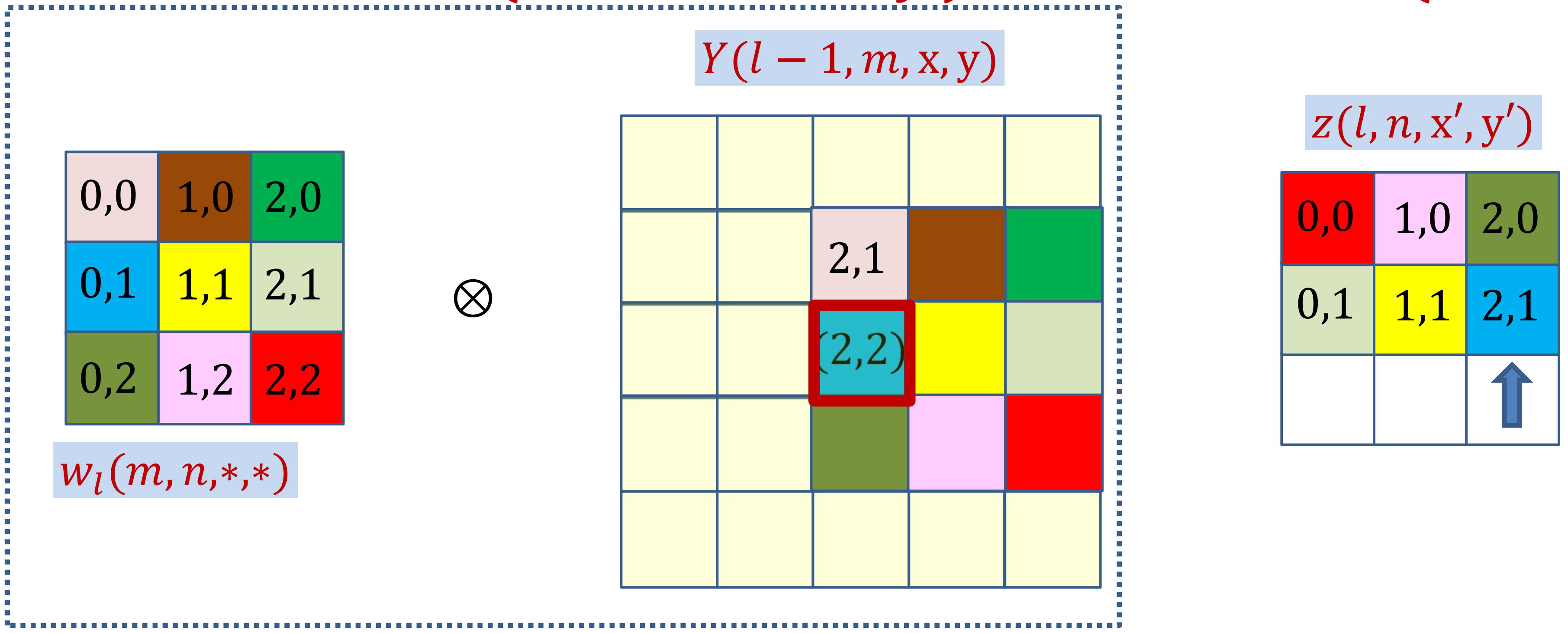


$$z(l,n,1,1) += Y(l-1,m,2,2)w_l(m,n,1,1)$$

- **Note:** The coordinates of $z(l,n)$ and $w_l(m,n)$ sum to the coordinates of $Y(l-1,m)$

$$z(l,n,x',y') += Y(l-1,m,2,2)w_l(m,n,2-x',2-y')$$

How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$

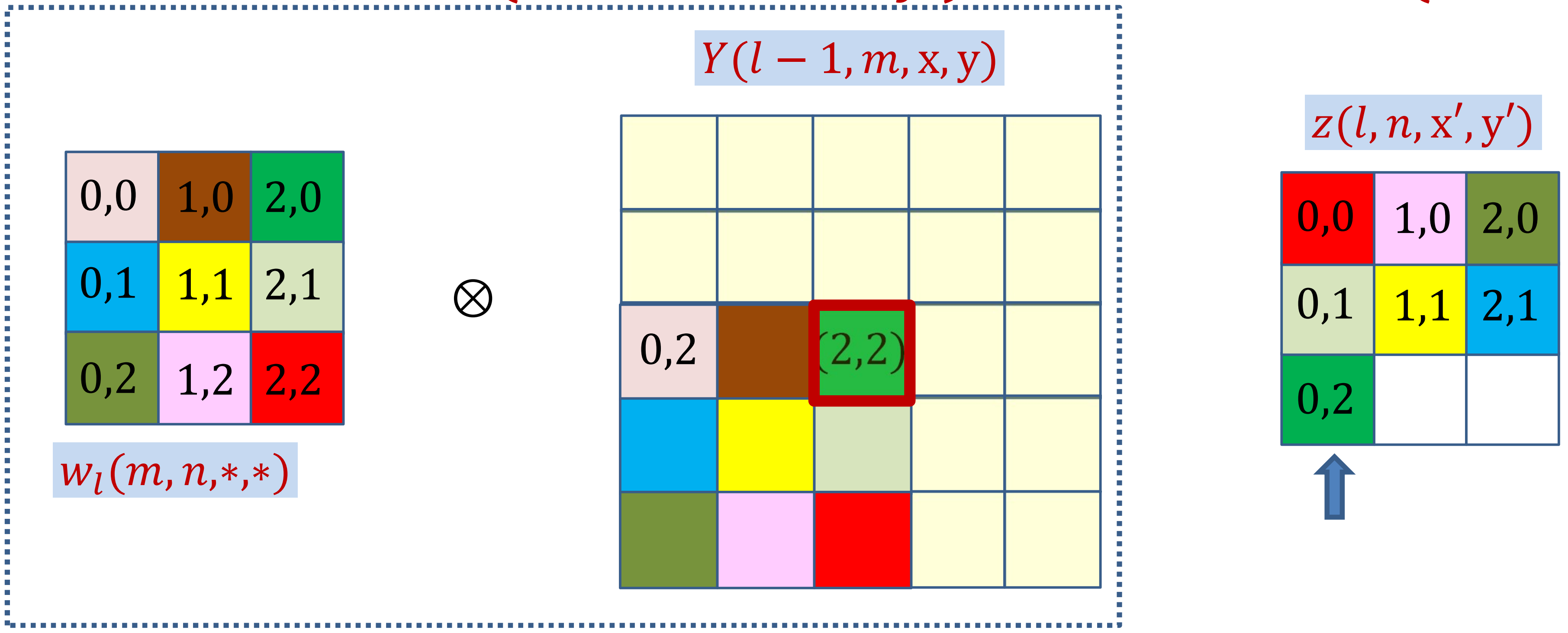


$$z(l,n,2,1) += Y(l-1,m,2,2)w_l(m,n,0,1)$$

- **Note:** The coordinates of $z(l,n)$ and $w_l(m,n)$ sum to the coordinates of $Y(l-1,m)$

$$z(l,n,x',y') += Y(l-1,m,2,2)w_l(m,n,2-x',2-y')$$

How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$

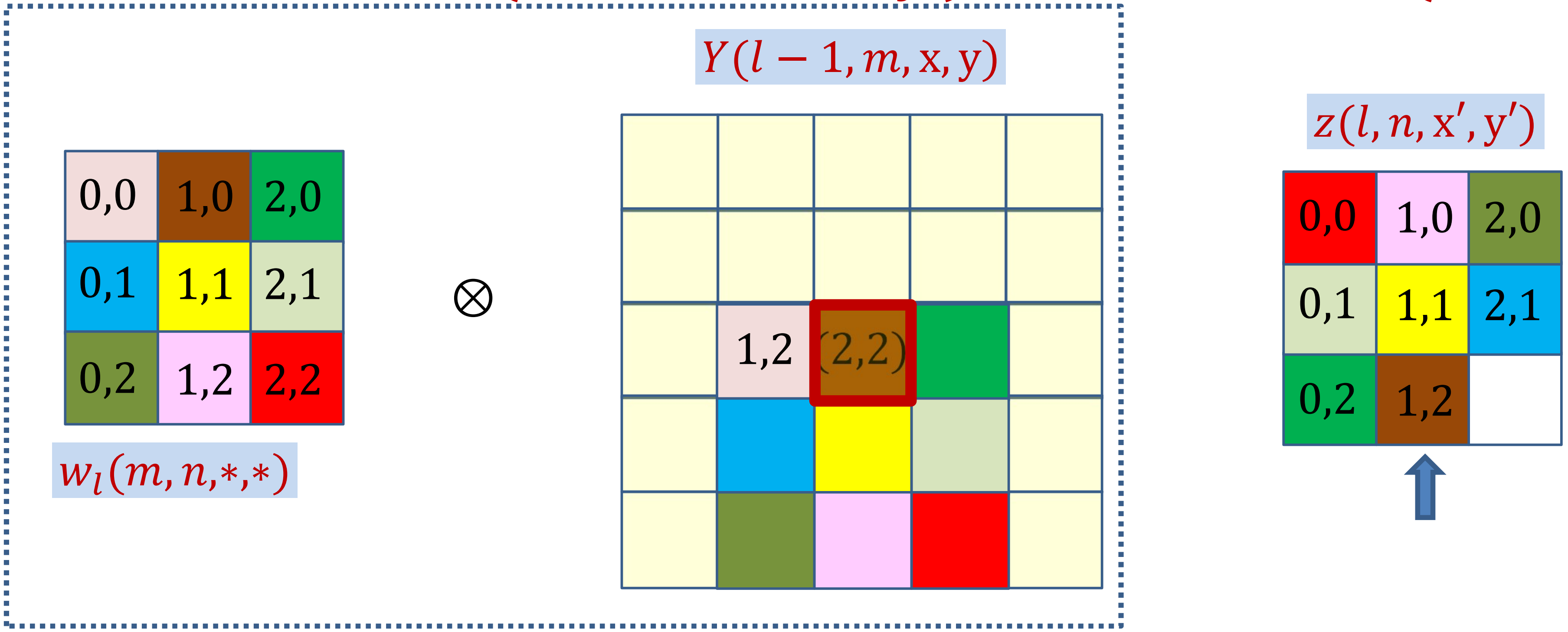


$$z(l,n,0,2) \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,2,0)$$

- **Note:** The coordinates of $z(l,n)$ and $w_l(m,n)$ sum to the coordinates of $Y(l-1,m)$

$$z(l,n,x',y') \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,2-x',2-y')$$

How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$

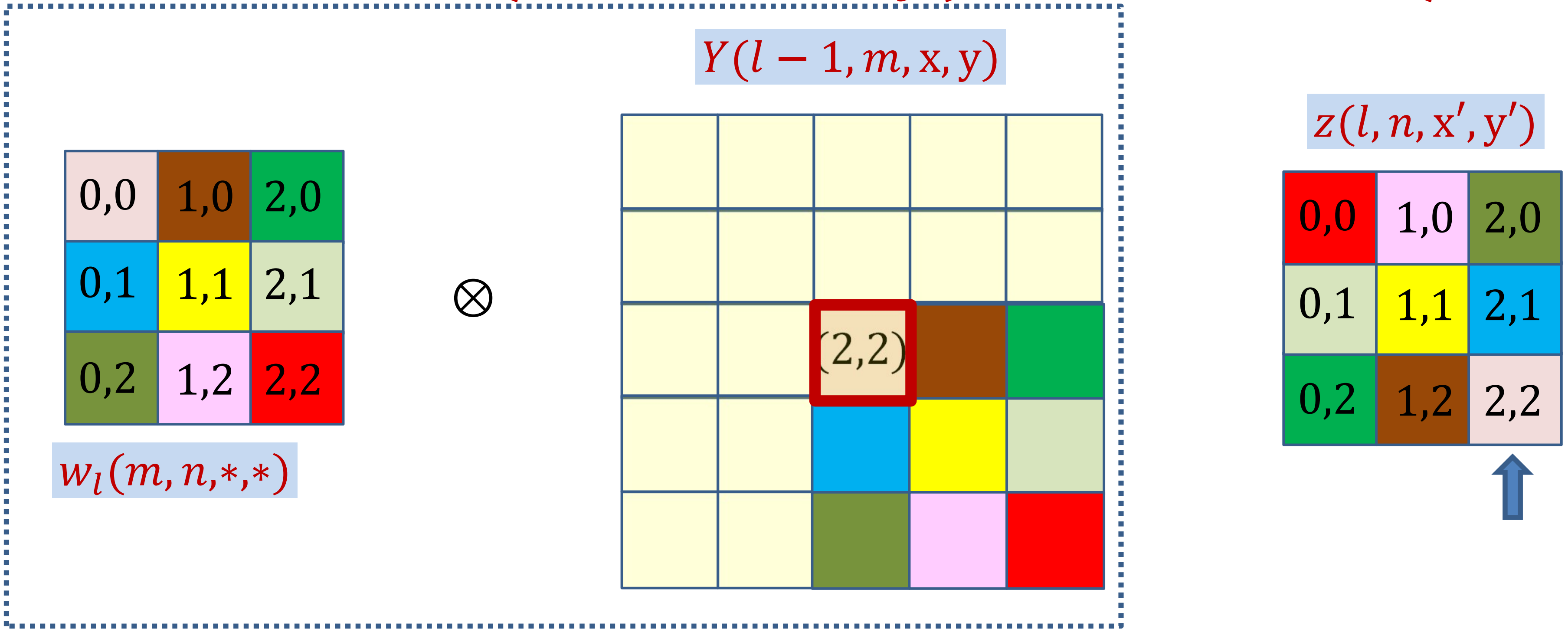


$$z(l,n,1,2) \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,2,1)$$

- **Note:** The coordinates of $z(l,n)$ and $w_l(m,n)$ sum to the coordinates of $Y(l-1,m)$

$$z(l,n,x',y') \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,2-x',2-y')$$

How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$

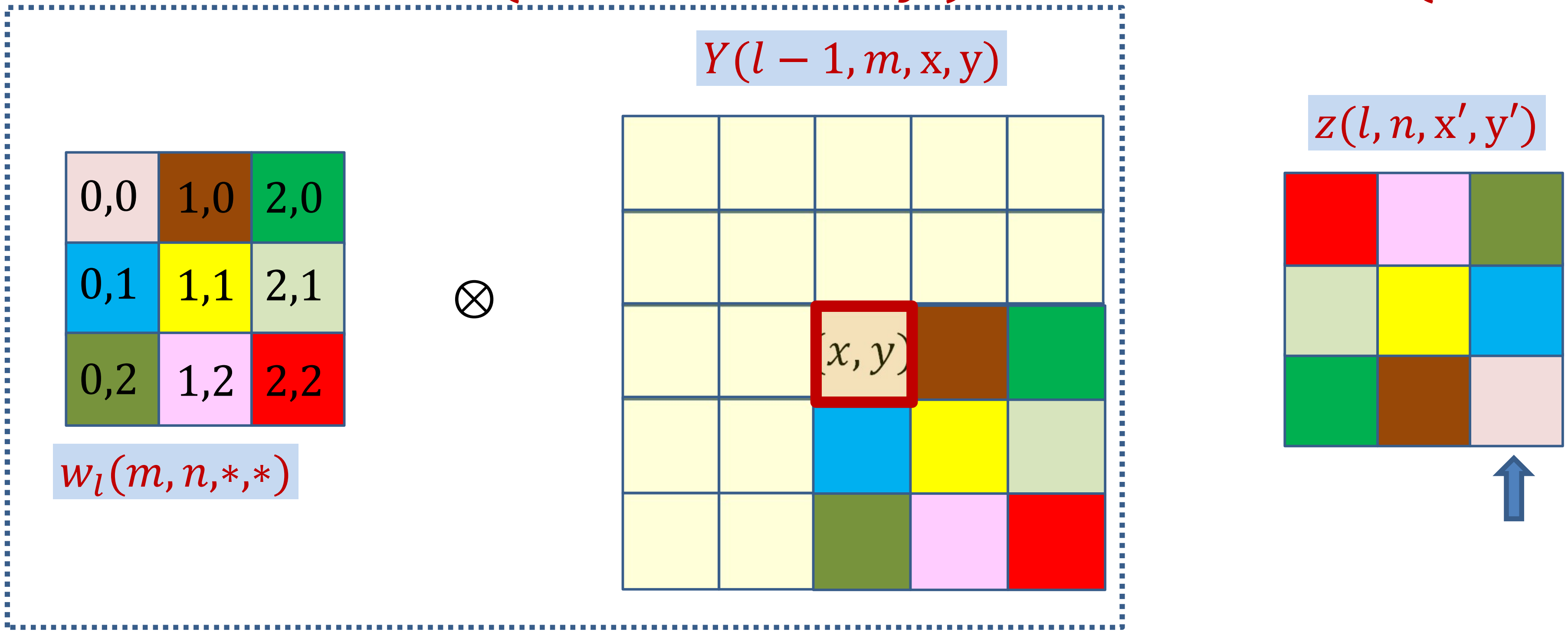


$$z(l,n,2,2) \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,0,0)$$

- **Note:** The coordinates of $z(l,n)$ and $w_l(m,n)$ sum to the coordinates of $Y(l-1,m)$

$$z(l,n,x',y') \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,2-x',2-y')$$

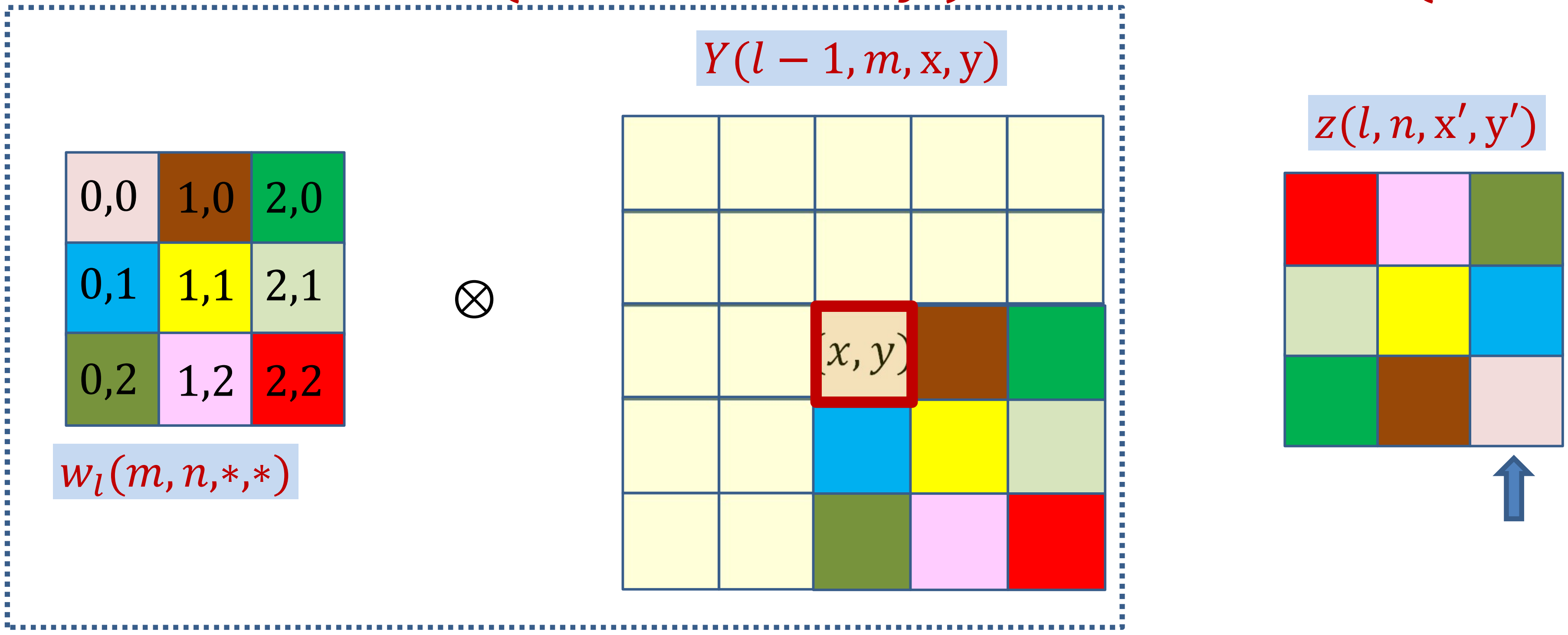
How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$



$$z(l,n,x',y') \mathrel{+}= Y(l-1,m,x,y)\,w_l(m,n,x-x',y-y')$$

- **Note:** The coordinates of $z(l,n)$ and $w_l(m,n)$ sum to the coordinates of $Y(l-1,m)$

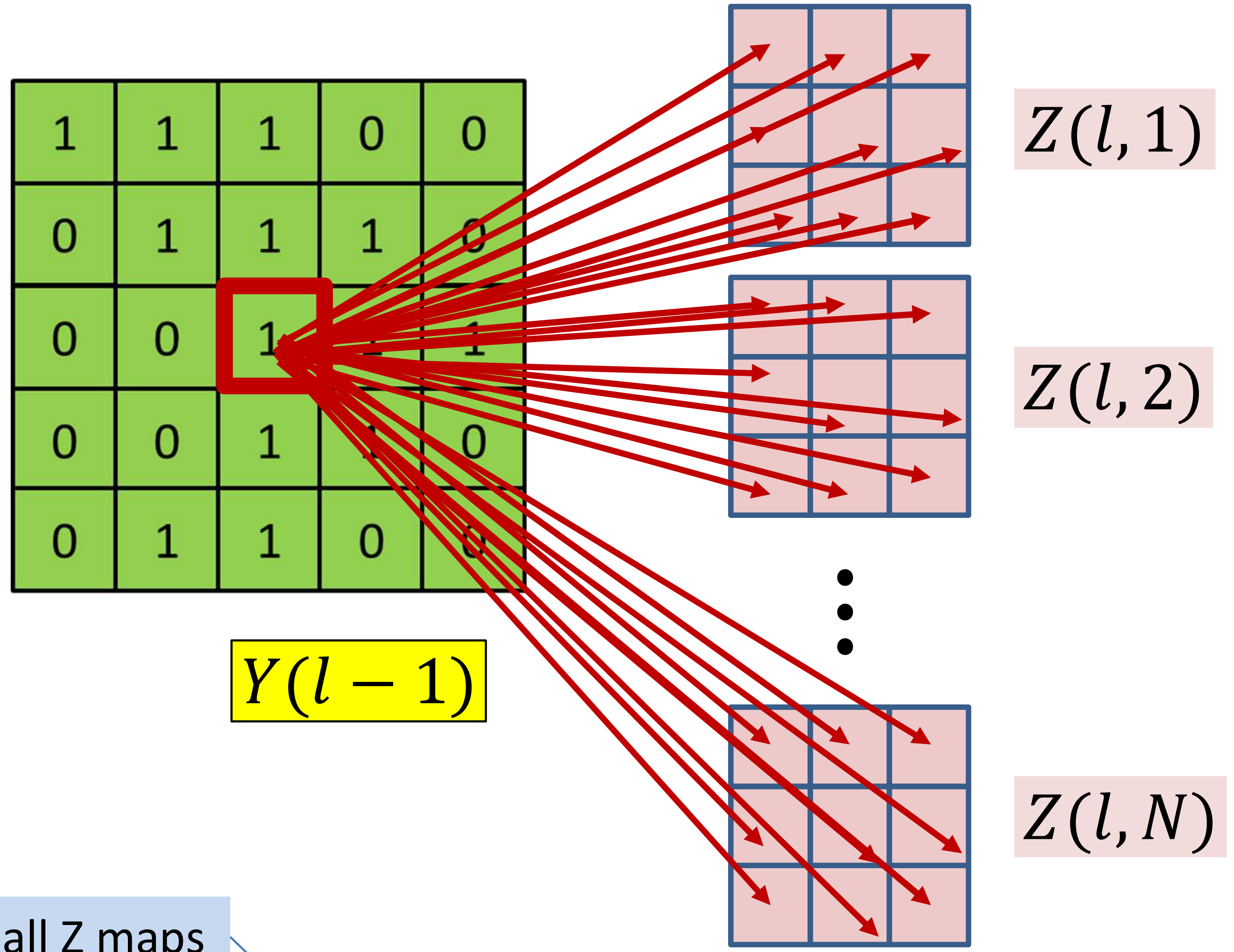
How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$



$$z(l,n,x',y') \mathrel{+}= Y(l-1,m,x,y)\,w_l(m,n,x-x',y-y')$$

$$\boxed{\frac{dz(l,n,x',y')}{dY(l-1,m,x,y)} = w_l(m,n,x-x',y-y')}$$

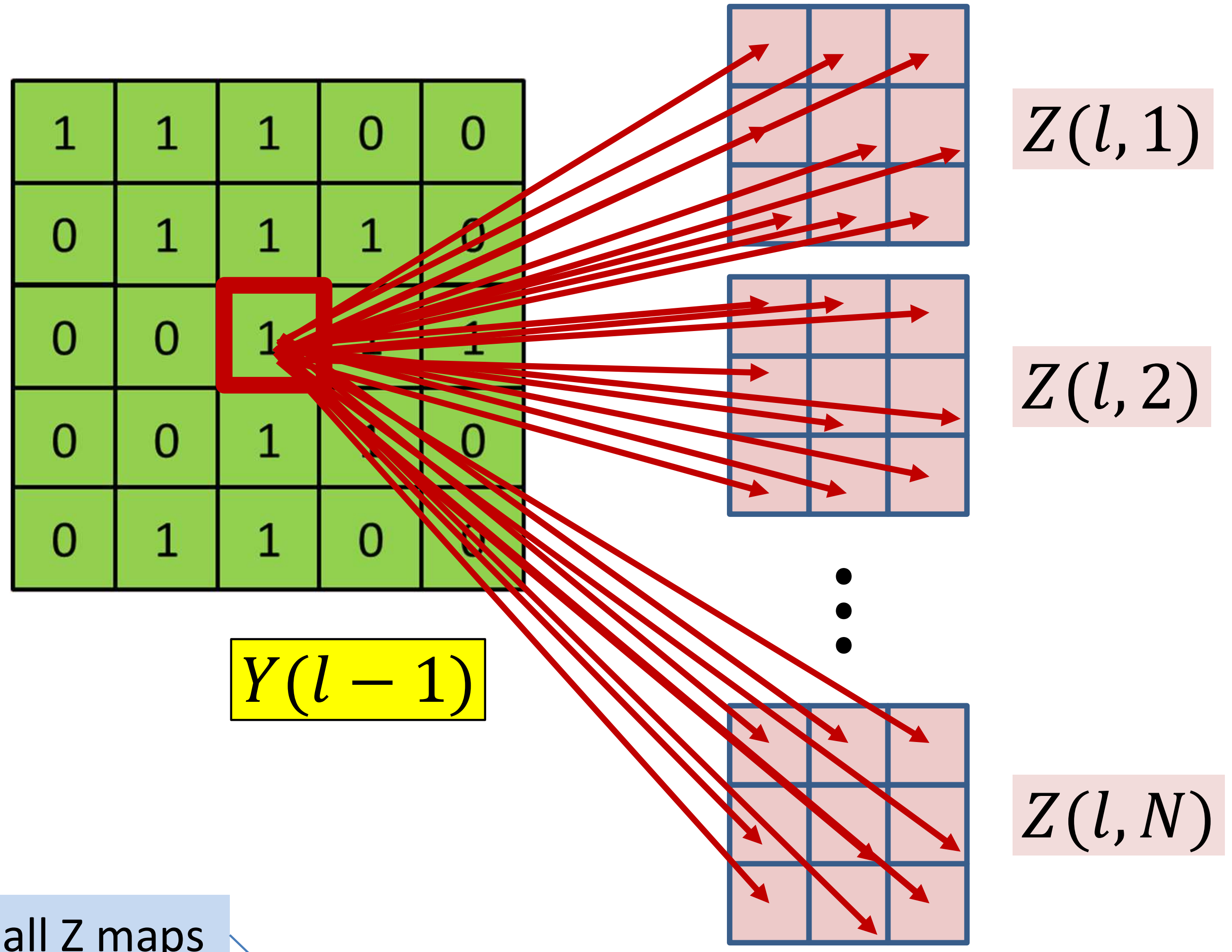
BP: Convolutional layer



Summing over all Z maps

$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} \frac{dz(l,n,x',y')}{dY(l-1,m,x,y)}$$

BP: Convolutional layer



Summing over all Z maps

$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_{n} \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$

Poll 2

In order to compute the derivative at a single affine element Y(l,m,x,y), we must consider the contributions of *every* position of *every* affine map at the next layer: True or false

- True
- False

The derivative for a single affine element Y(l,m,x,y) will require summing over every position of every Z map in the next layer: True of false

- True
- False

Poll 2

In order to compute the derivative at a single affine element Y(l,m,x,y), we must consider the contributions of *every* position of *every* affine map at the next layer: True or false

- **True**
- False

The derivative for an single affine element Y(l,m,x,y) will require summing over every position of every Z map in the next layer: True of false

- **True**
- False

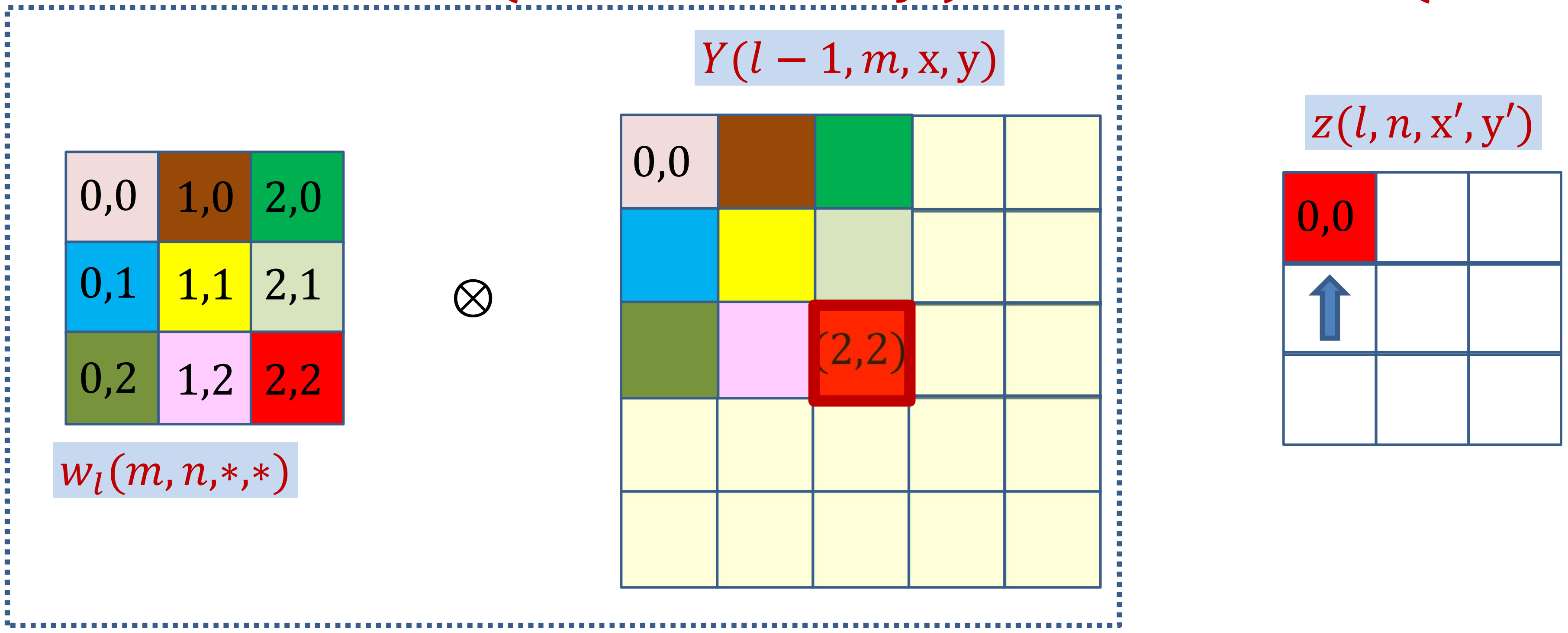
Computing derivative for $\boldsymbol{Y(l-1, m, *, *)}$

- The derivatives for every element of every map in $Y(l-1)$ by direct implementation of the formula:

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_{n} \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

- But this is actually a convolution!
  - Let's see how

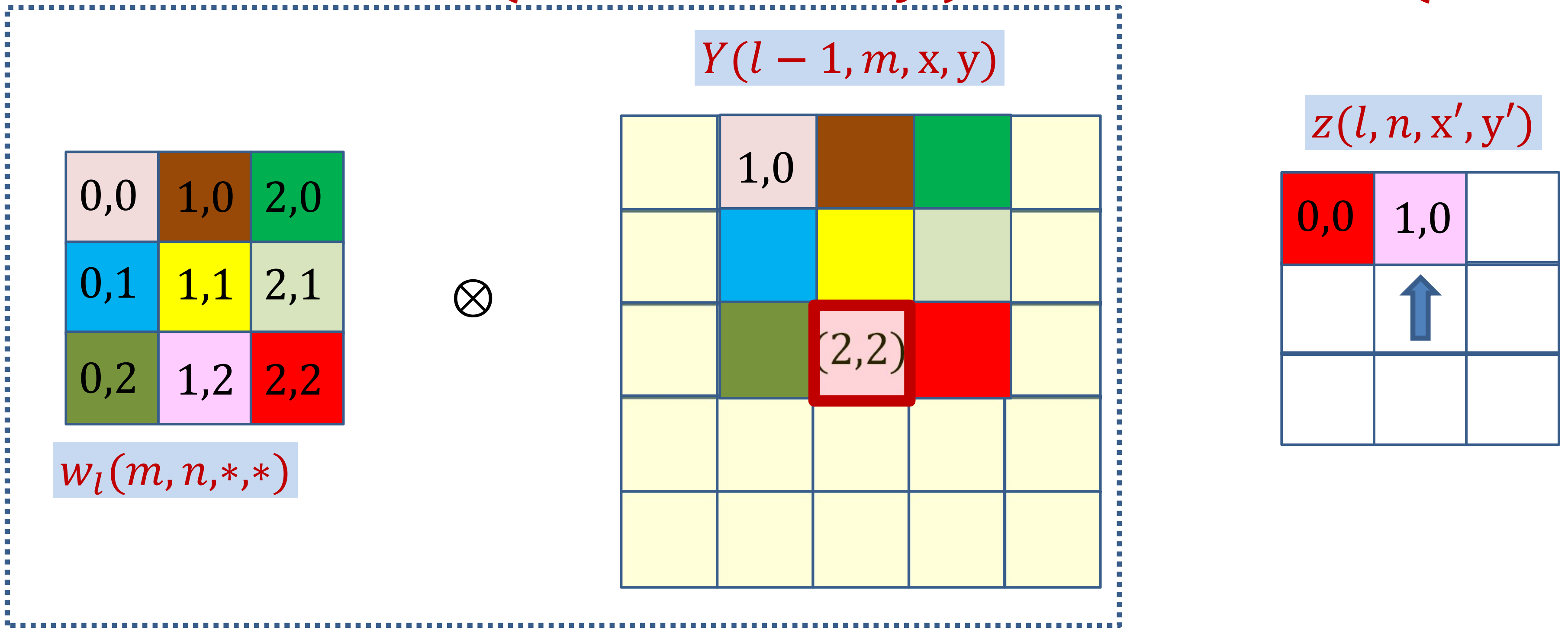
How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$



$$z(l,n,0,0) \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,2,2)$$

$$\frac{dDiv}{dY(l-1,m,2,2)} \mathrel{+}= \frac{dDiv}{dz(l,n,0,0)} w_l(m,n,2,2)$$

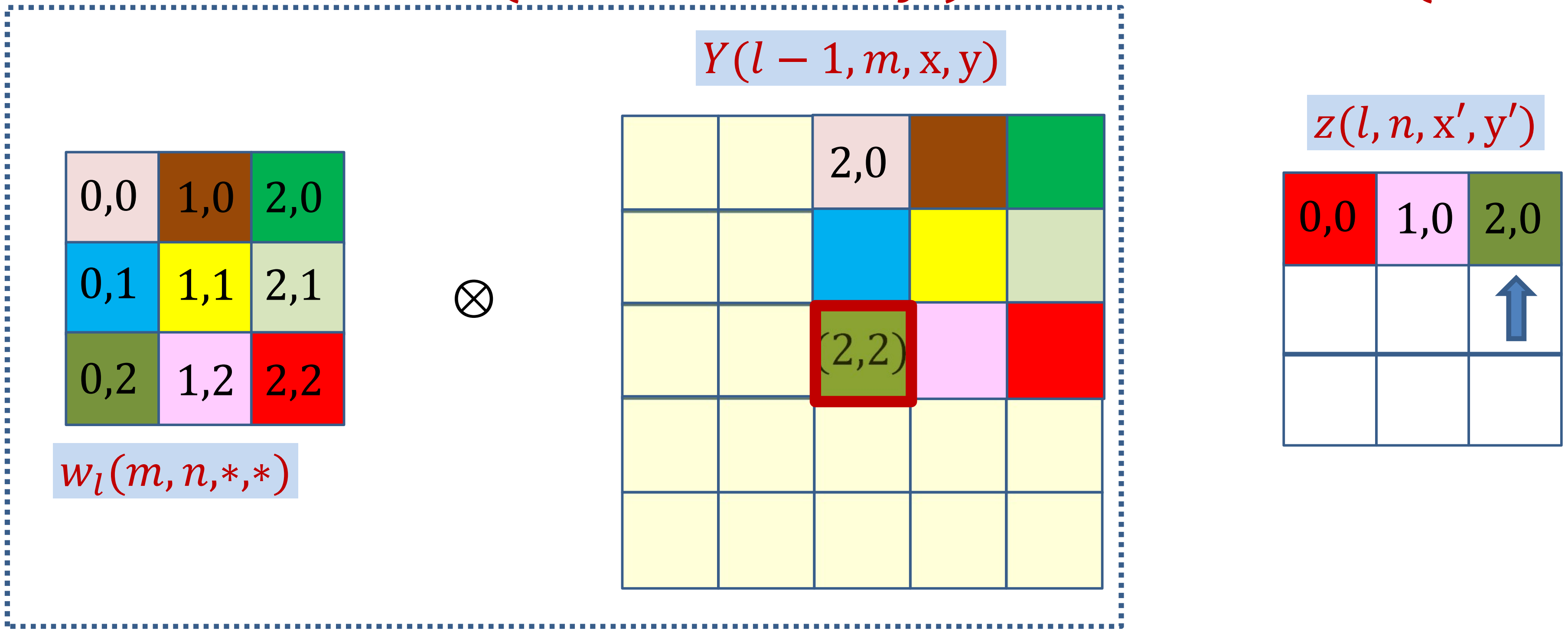
How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$



$$z(l,n,1{,}0) \mathrel{+}= Y(l-1,m,2{,}2)\,w_l(m,n,1{,}2)$$

$$\frac{dDiv}{dY(l-1,m,2{,}2)} \mathrel{+}= \frac{dDiv}{dz(l,n,1{,}0)}\,w_l(m,n,1{,}2)$$

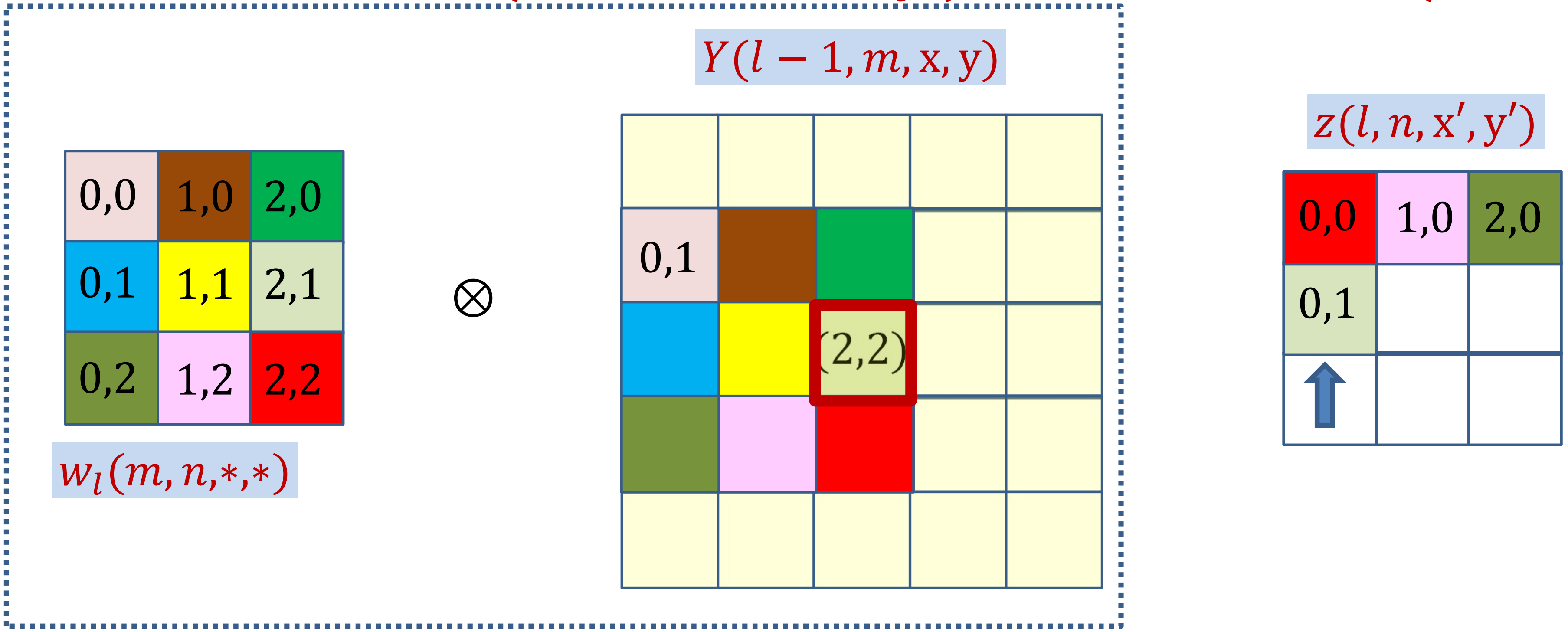
How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$



$$z(l,n,2{,}0) \mathrel{+}= Y(l-1,m,2{,}2)w_l(m,n,0{,}2)$$

$$\frac{dDiv}{dY(l-1,m,2{,}2)} \mathrel{+}= \frac{dDiv}{dz(l,n,2{,}0)} w_l(m,n,0{,}2)$$

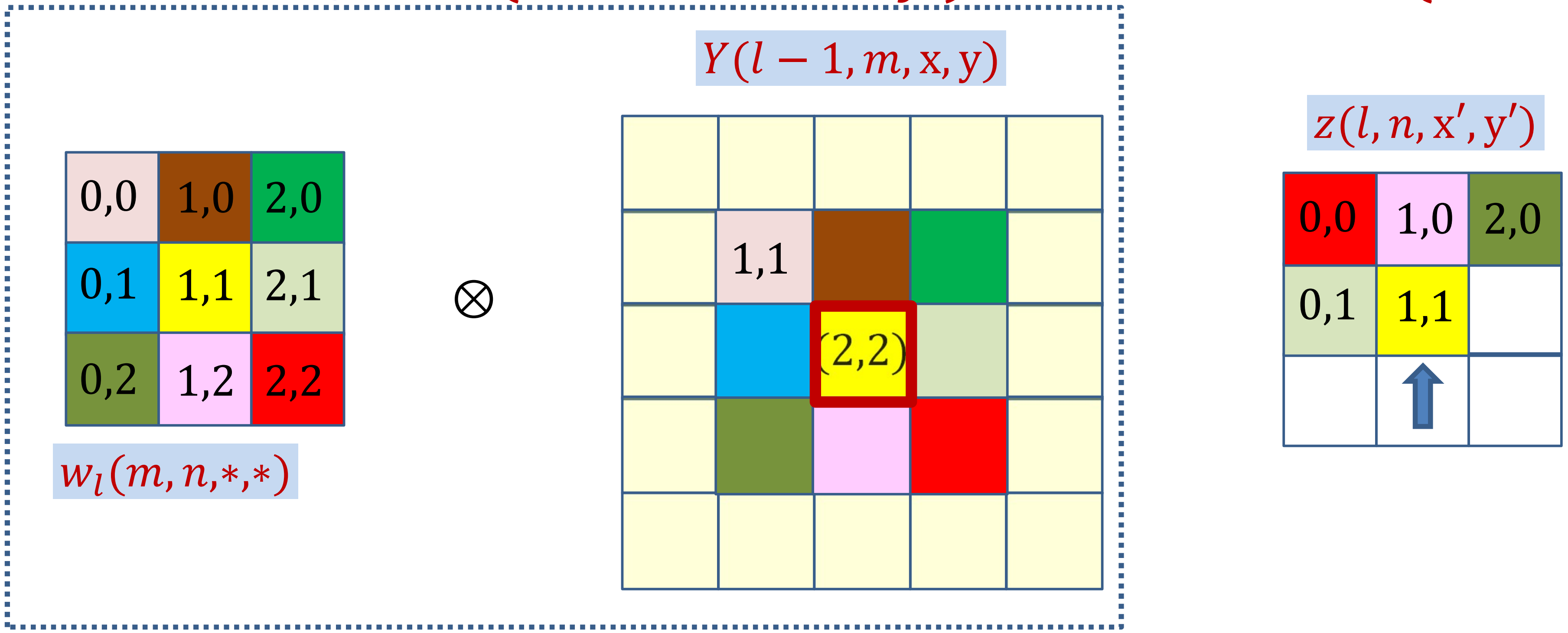
How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$



$$z(l,n,0{,}1) \mathrel{+}= Y(l-1,m,2{,}2)\,w_l(m,n,2{,}1)$$

$$\frac{dDiv}{dY(l-1,m,2{,}2)} \mathrel{+}= \frac{dDiv}{dz(l,n,0{,}1)}w_l(m,n,2{,}1)$$

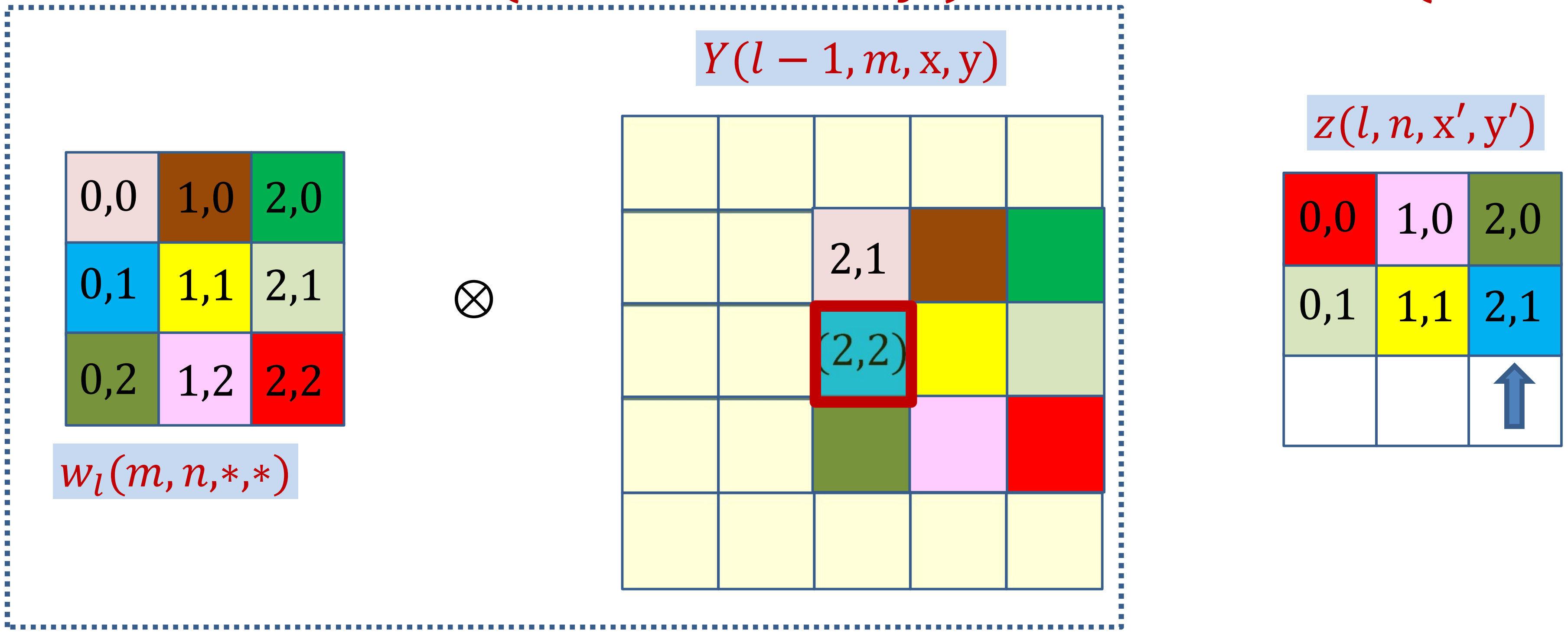
How a single $Y(l-1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 1{,}1) \mathrel{+}= Y(l-1, m, 2{,}2)\, w_l(m, n, 1{,}1)$$

$$\frac{dDiv}{dY(l-1, m, 2{,}2)} \mathrel{+}= \frac{dDiv}{dz(l, n, 1, 1)} w_l(m, n, 1{,}1)$$

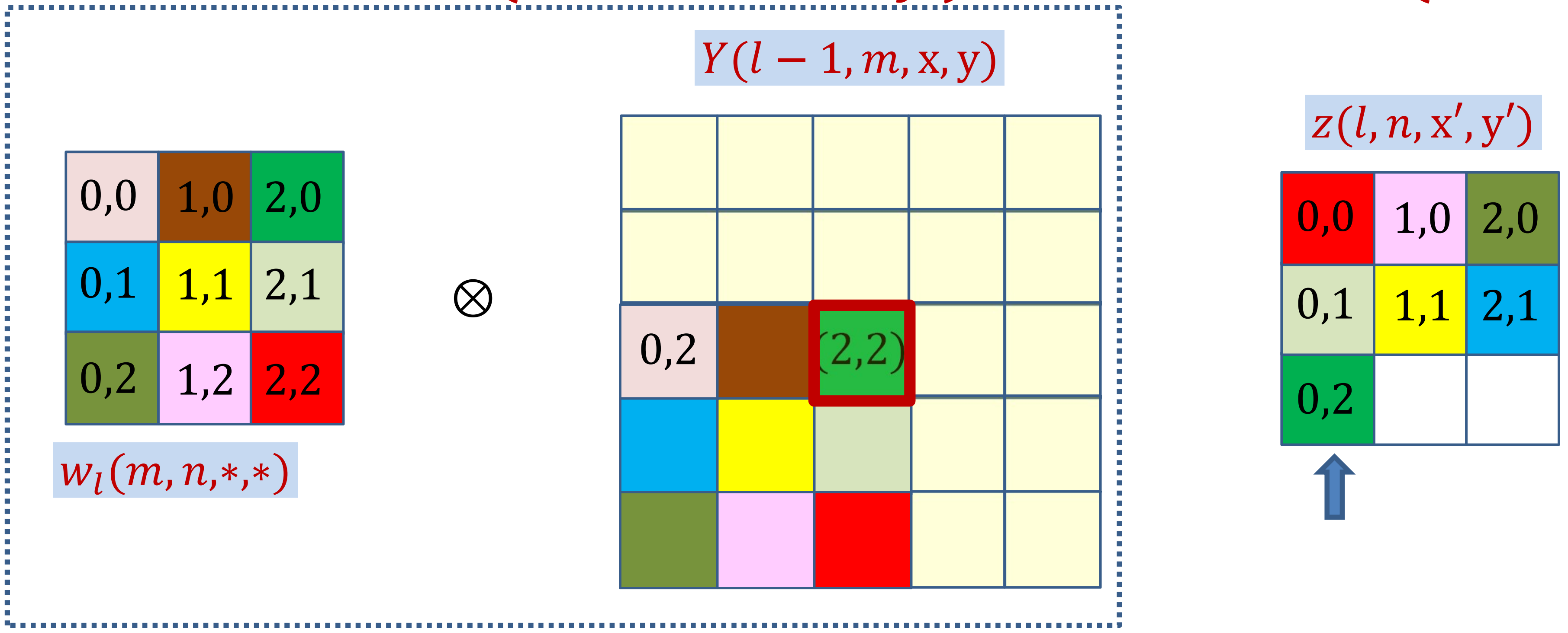
How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$



$$z(l,n,2,1) \mathrel{+}= Y(l-1,m,2,2)\,w_l(m,n,0,1)$$

$$\frac{dDiv}{dY(l-1,m,2,2)} \mathrel{+}= \frac{dDiv}{dz(l,n,2,1)}\,w_l(m,n,0,1)$$

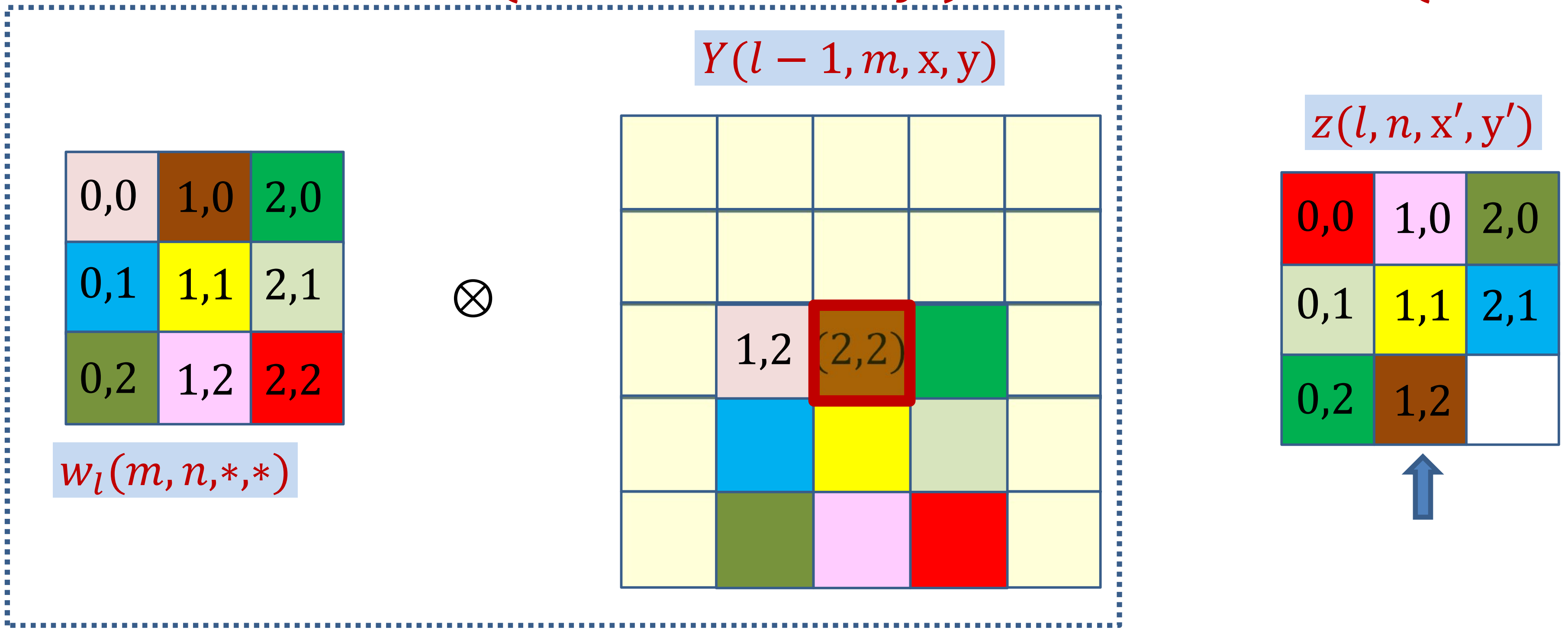
How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$



$$z(l,n,0,2) \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,2,0)$$

$$\frac{dDiv}{dY(l-1,m,2,2)} \mathrel{+}= \frac{dDiv}{dz(l,n,0,2)} w_l(m,n,2,0)$$

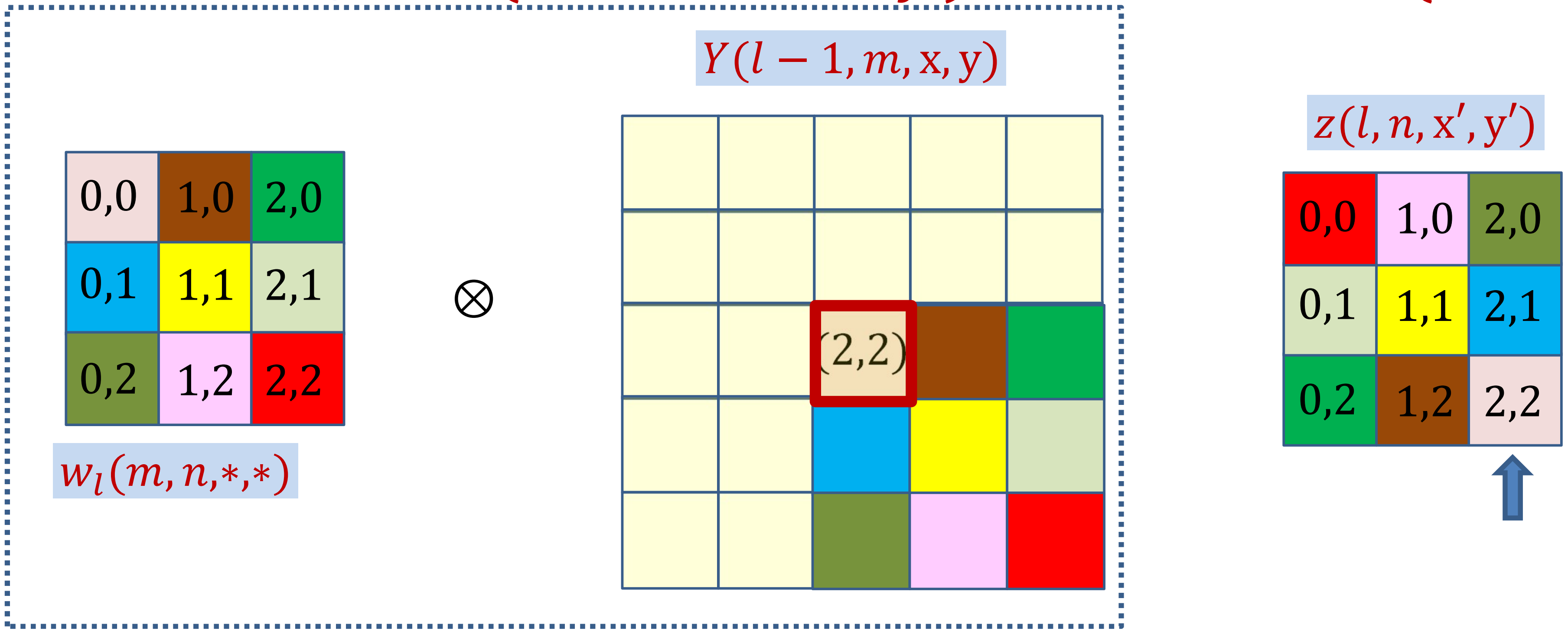
How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$



$$z(l,n,1,2) \mathrel{+}= Y(l-1,m,2,2)\,w_l(m,n,2,1)$$

$$\frac{dDiv}{dY(l-1,m,2,2)} \mathrel{+}= \frac{dDiv}{dz(l,n,1,2)}\,w_l(m,n,1,0)$$

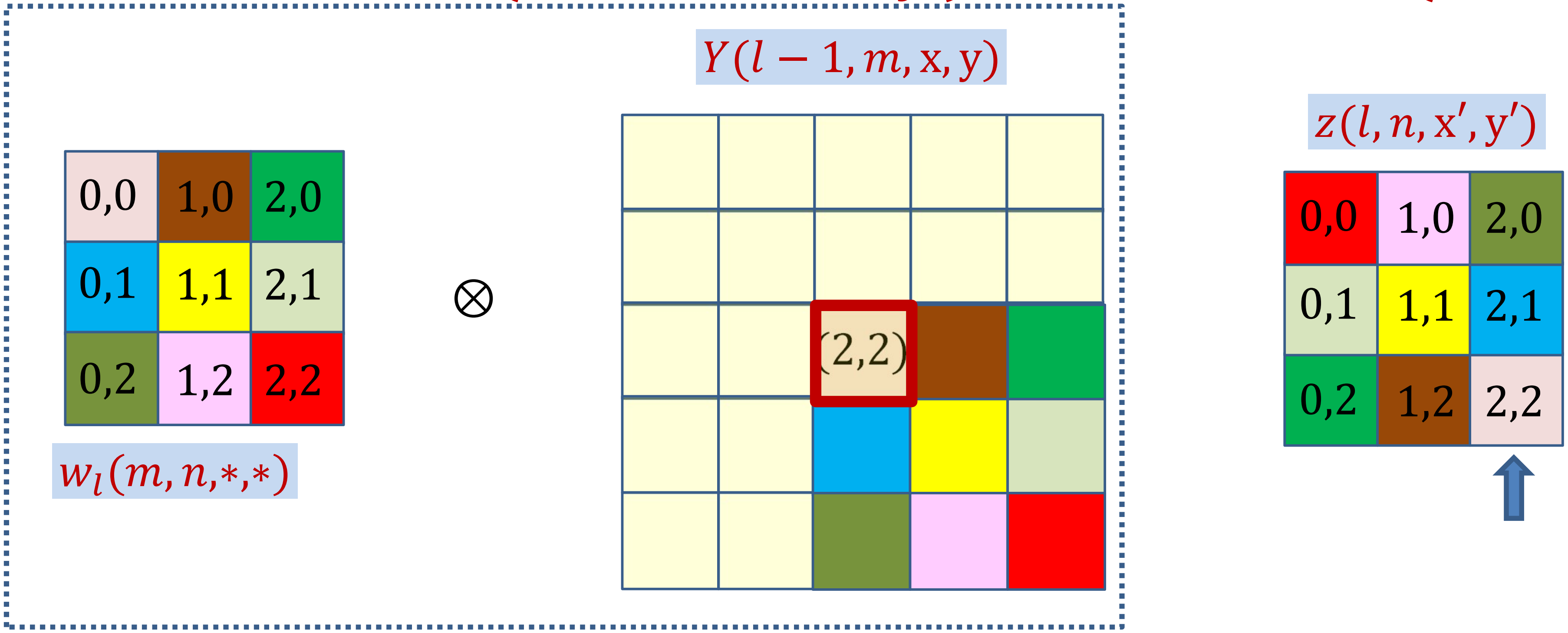
How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$



$$z(l,n,2{,}2) \mathrel{+}= Y(l-1,m,2{,}2)\,w_l(m,n,0{,}0)$$

$$\frac{dDiv}{dY(l-1,m,2{,}2)} \mathrel{+}= \frac{dDiv}{dz(l,n,2{,}2)}\,w_l(m,n,0{,}0)$$

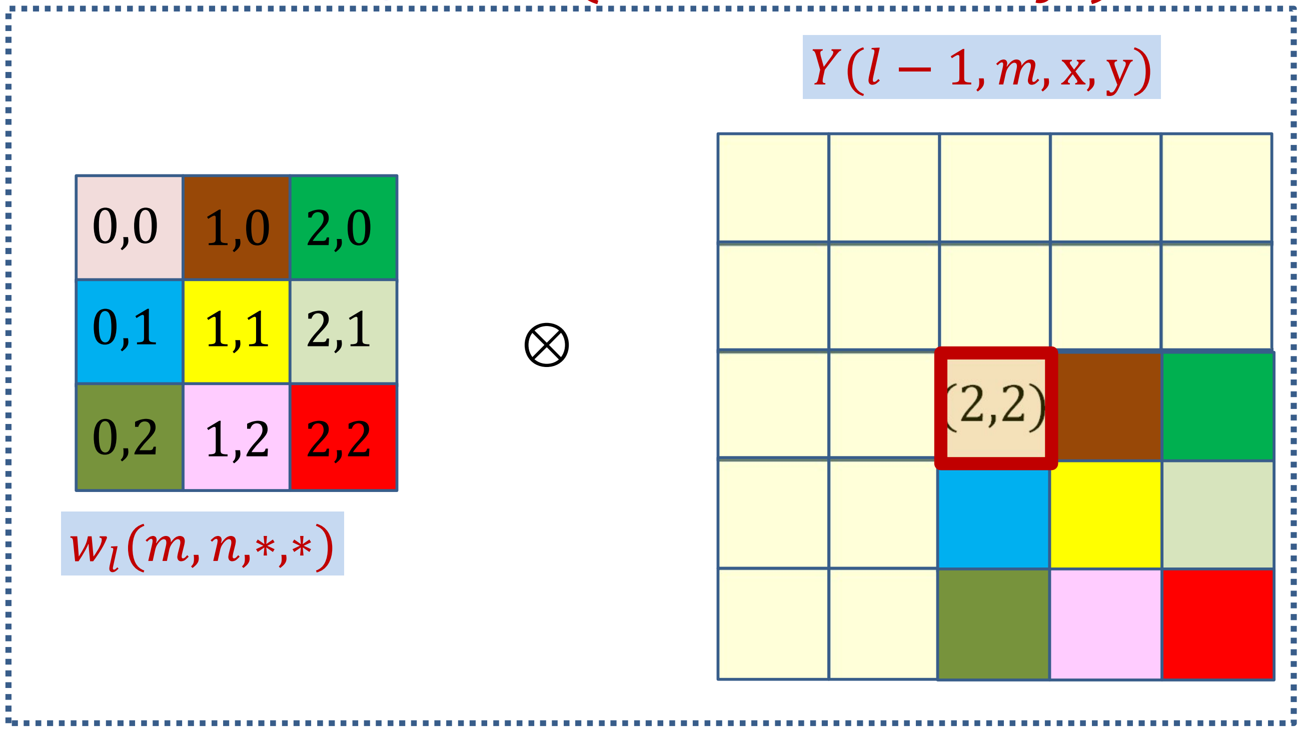
How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$

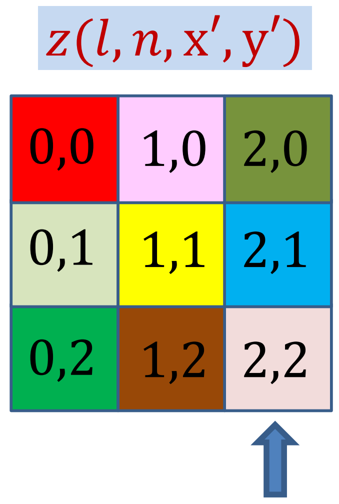


$$z(l,n,2{,}2) \mathrel{+}= Y(l-1,m,2{,}2)w_l(m,n,0{,}0)$$

$$\frac{dDiv}{dY(l-1,m,2{,}2)} = \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,2-\mathrm{x}',2-\mathrm{y}')$$

How a single $Y(l-1,m,x,y)$ influences $z(l,n,x',y')$



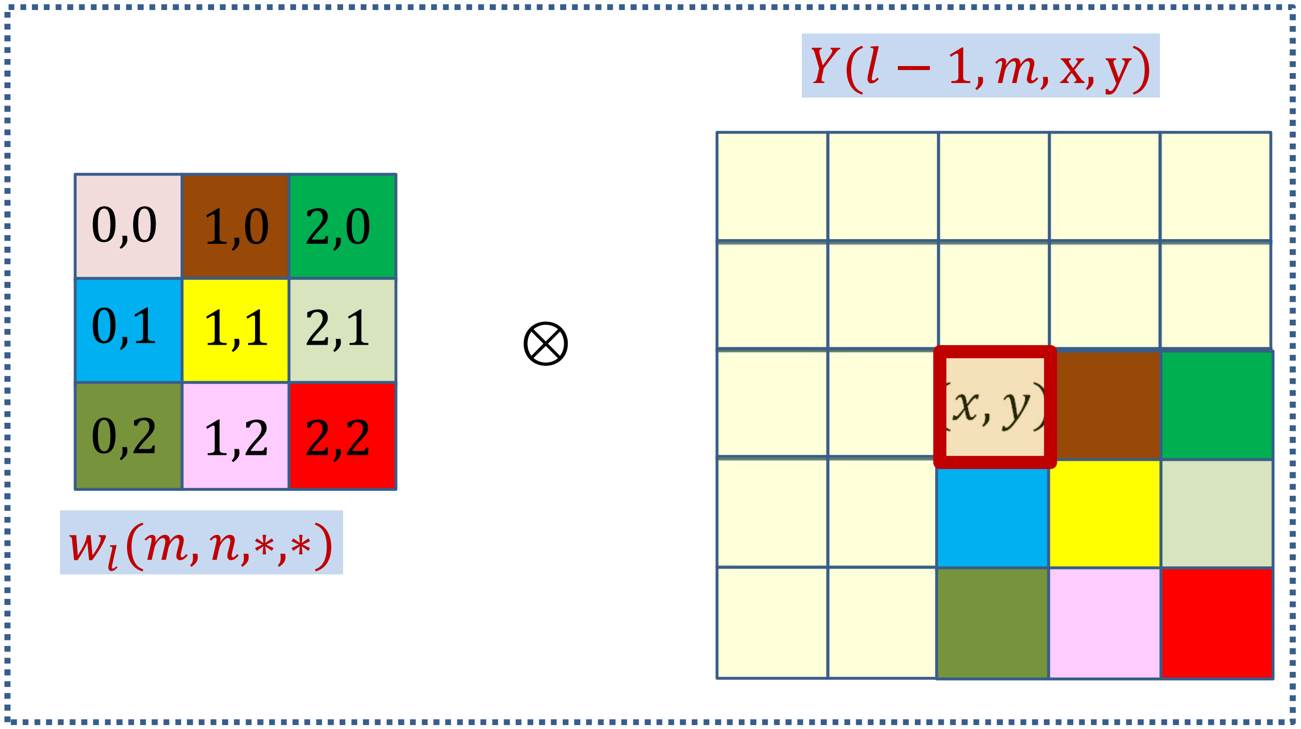


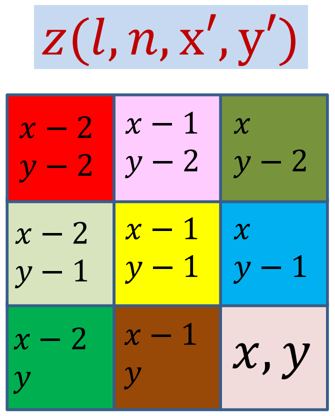
$$z(l,n,x',y') \mathrel{+}= Y(l-1,m,2,2)w_l(m,n,2-x',2-y')$$

$$\frac{dDiv}{dY(l-1,m,2,2)} = \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,2-x',2-y')$$

- The derivative at $Y(l-1,m,2,2)$ is the sum of component-wise product of the filter elements (shown by color) and the elements of the derivative at $z(l,m,.,.)$

Derivative at $Y(l-1,m,x,y)$ from a single $Z(l,n)$ map



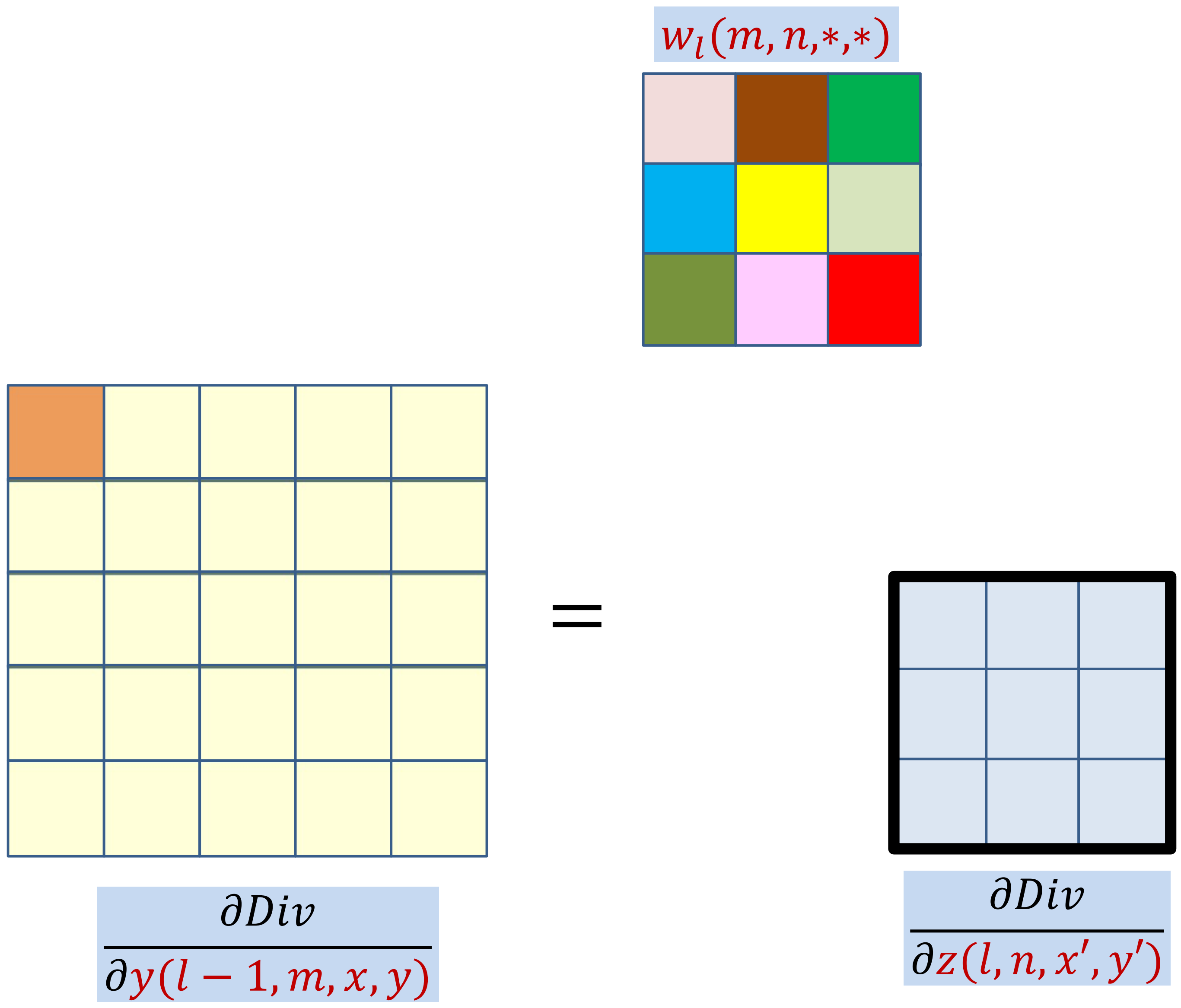


$$z(l,n,x',y') \mathrel{+}= Y(l-1,m,x,y)w_l(m,n,x-x',y-y')$$

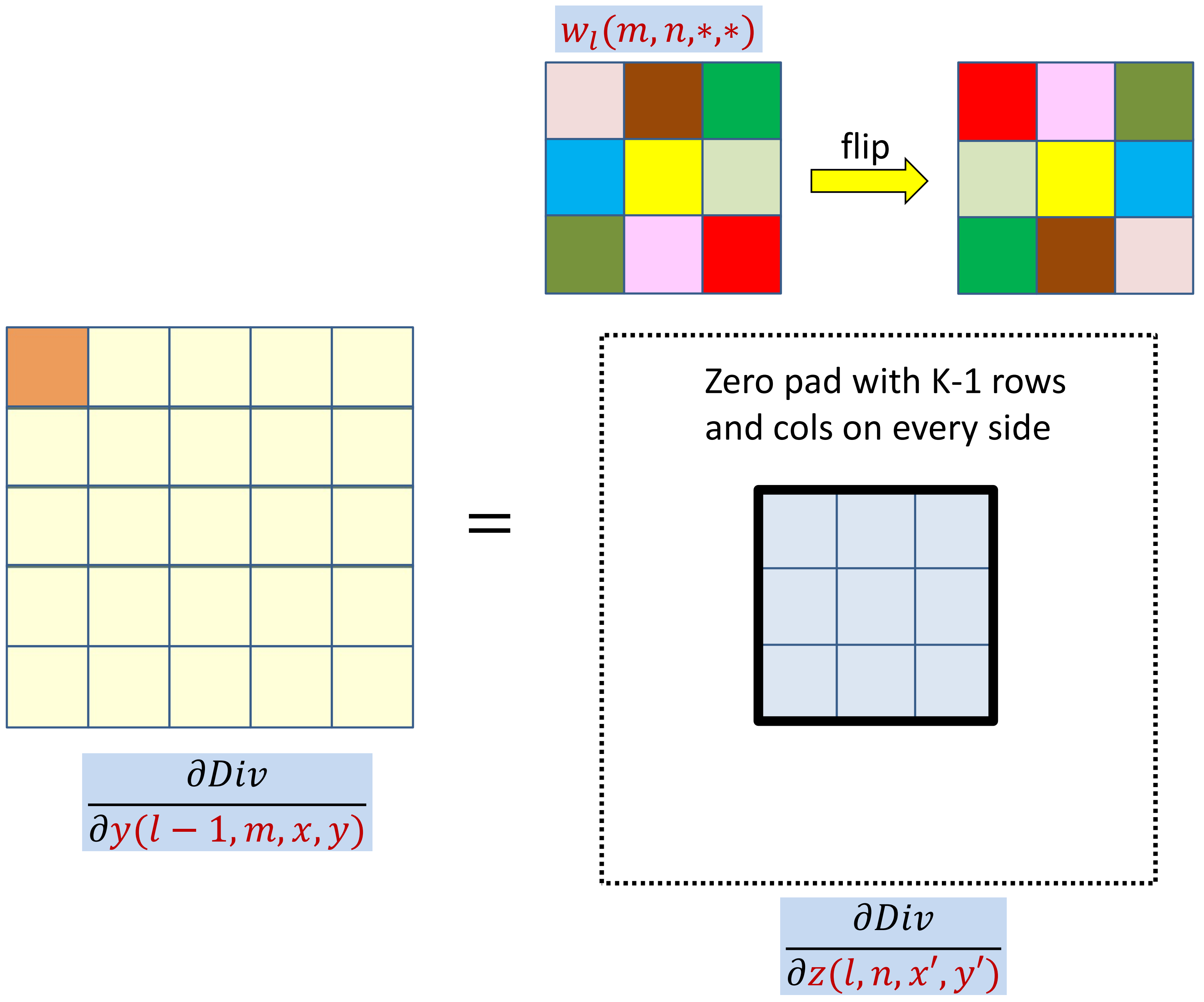
$$\boxed{\frac{dDiv}{dY(l-1,m,x,y)} \mathrel{+}= \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',x-y')}$$

Contribution of the entire nth affine map $z(l,n,*,*)$

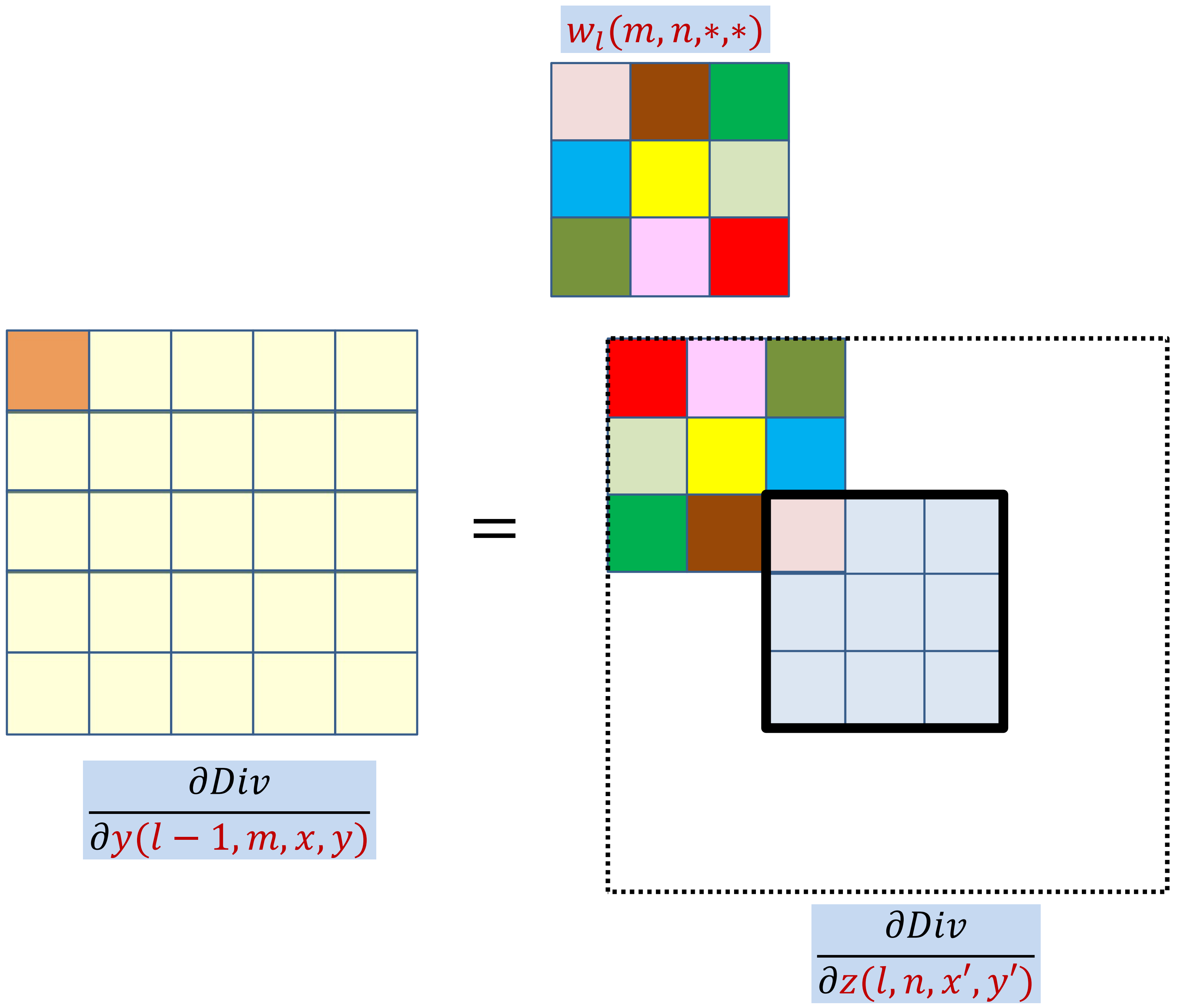
Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map


$w_l(m,n,*,*)$
$=$
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

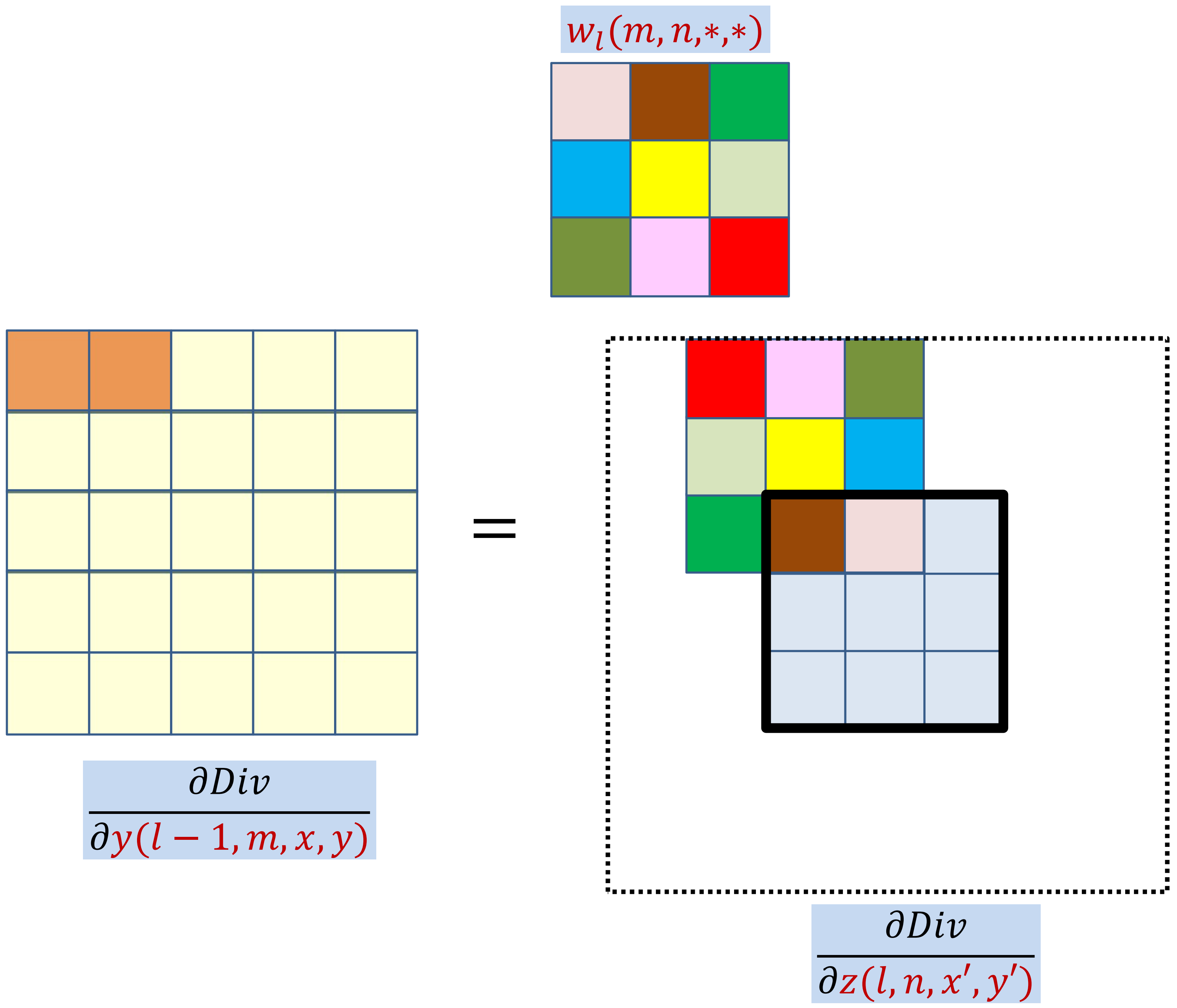
Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map


$w_l(m,n,*,*)$
flip
=
Zero pad with K-1 rows and cols on every side
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

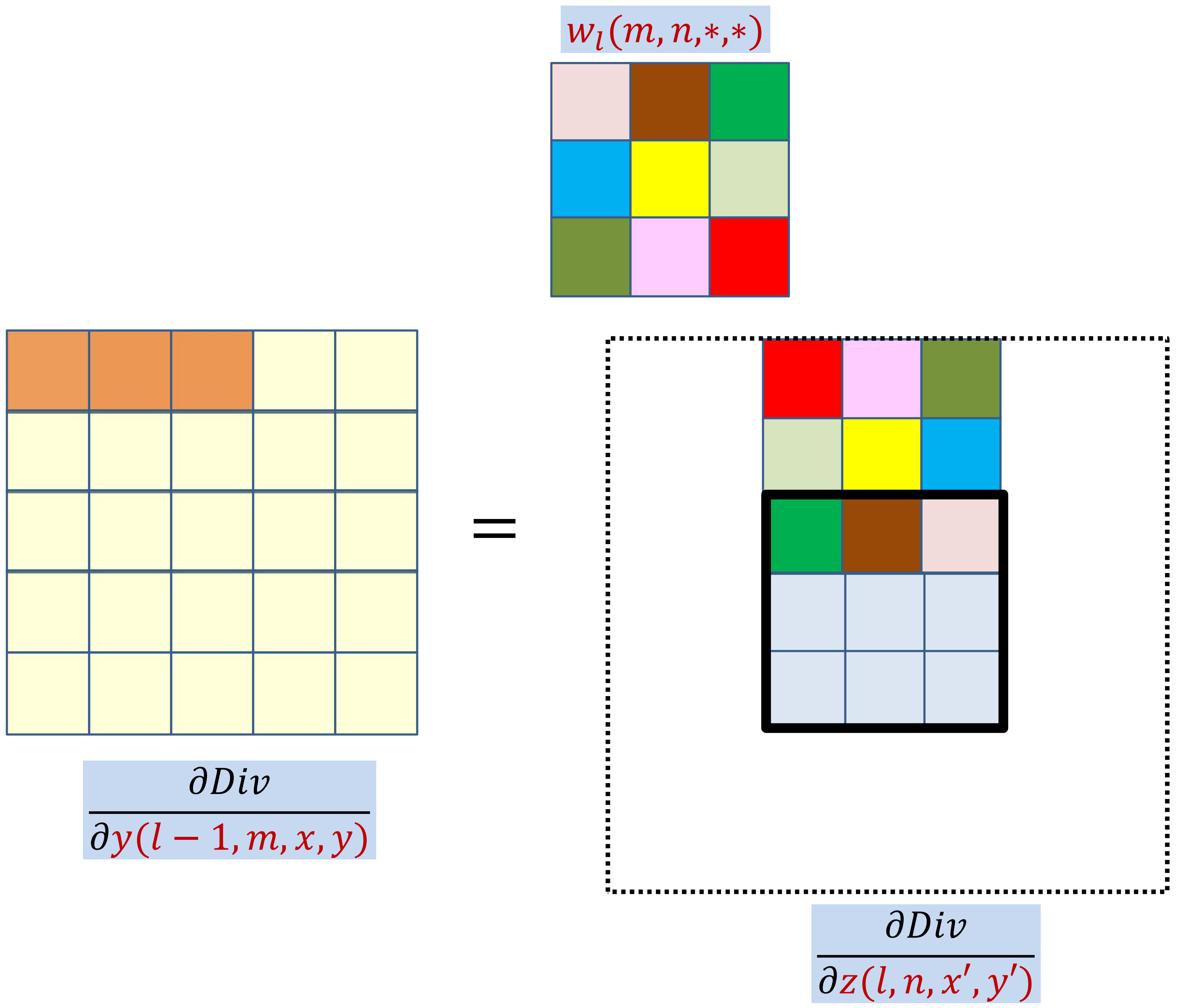
Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map


$w_l(m,n,*,*)$
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$=$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map

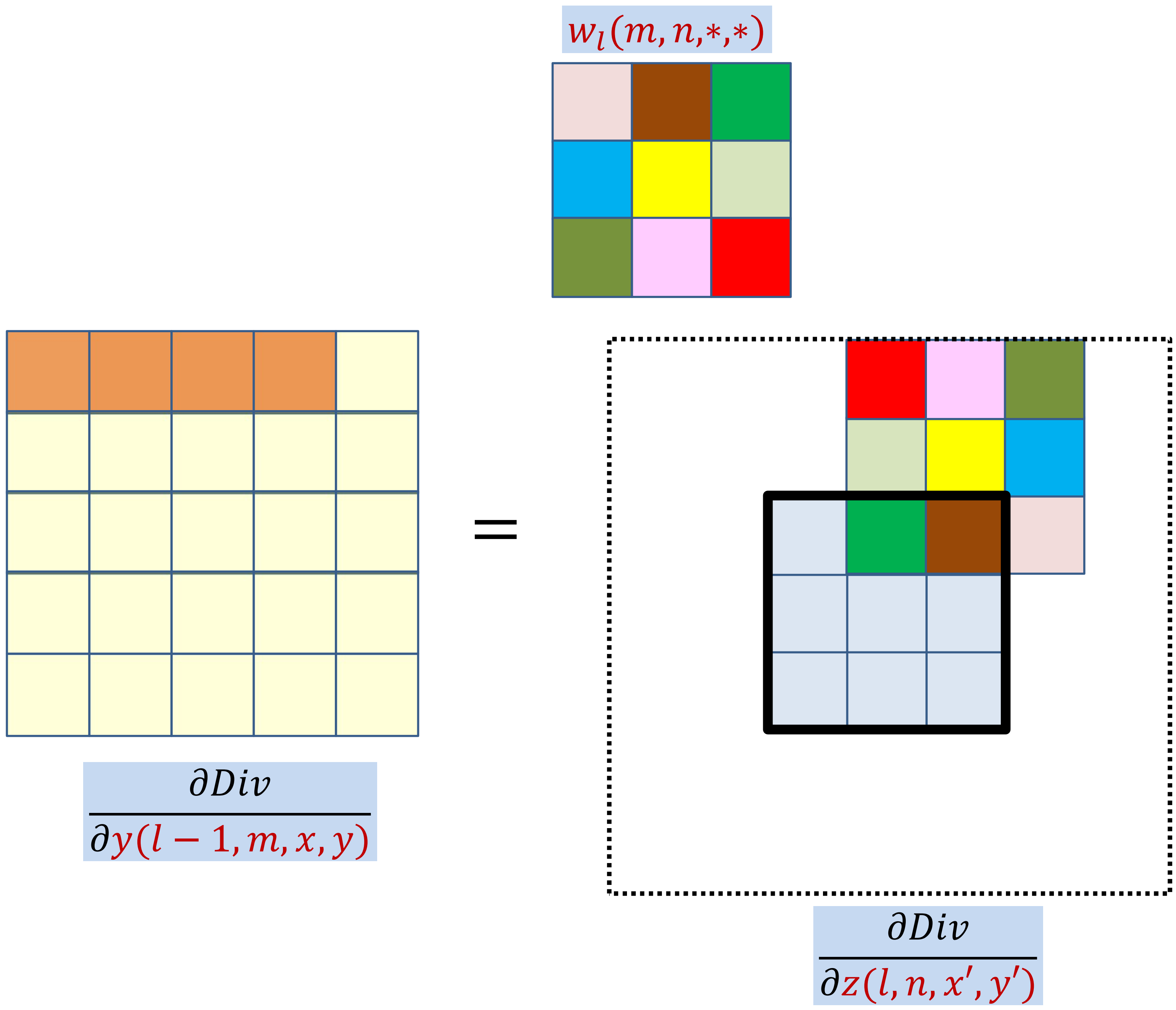

$w_l(m,n,*,*)$
$=$
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map



$w_l(m,n,*,*)$
$=$
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

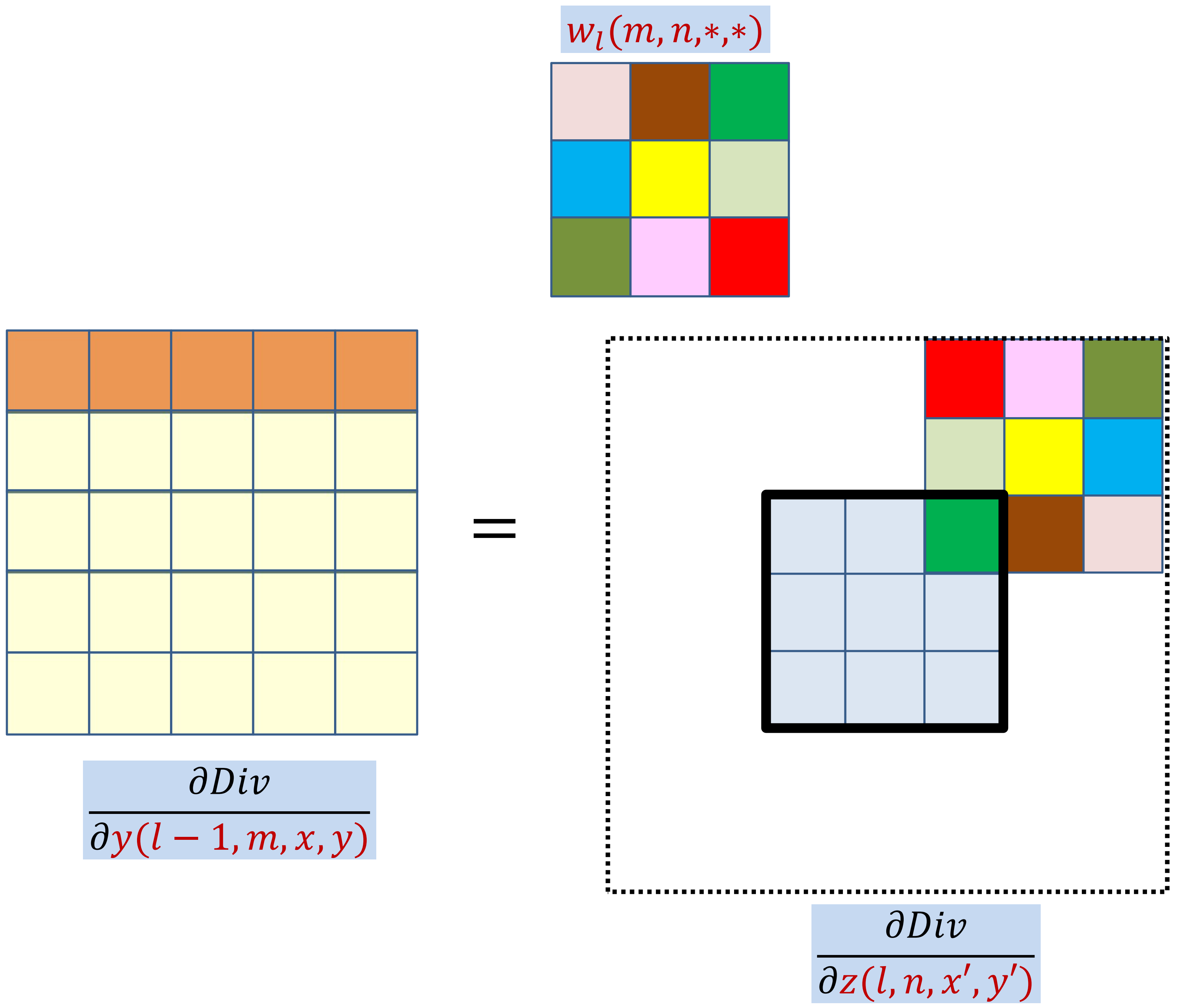
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



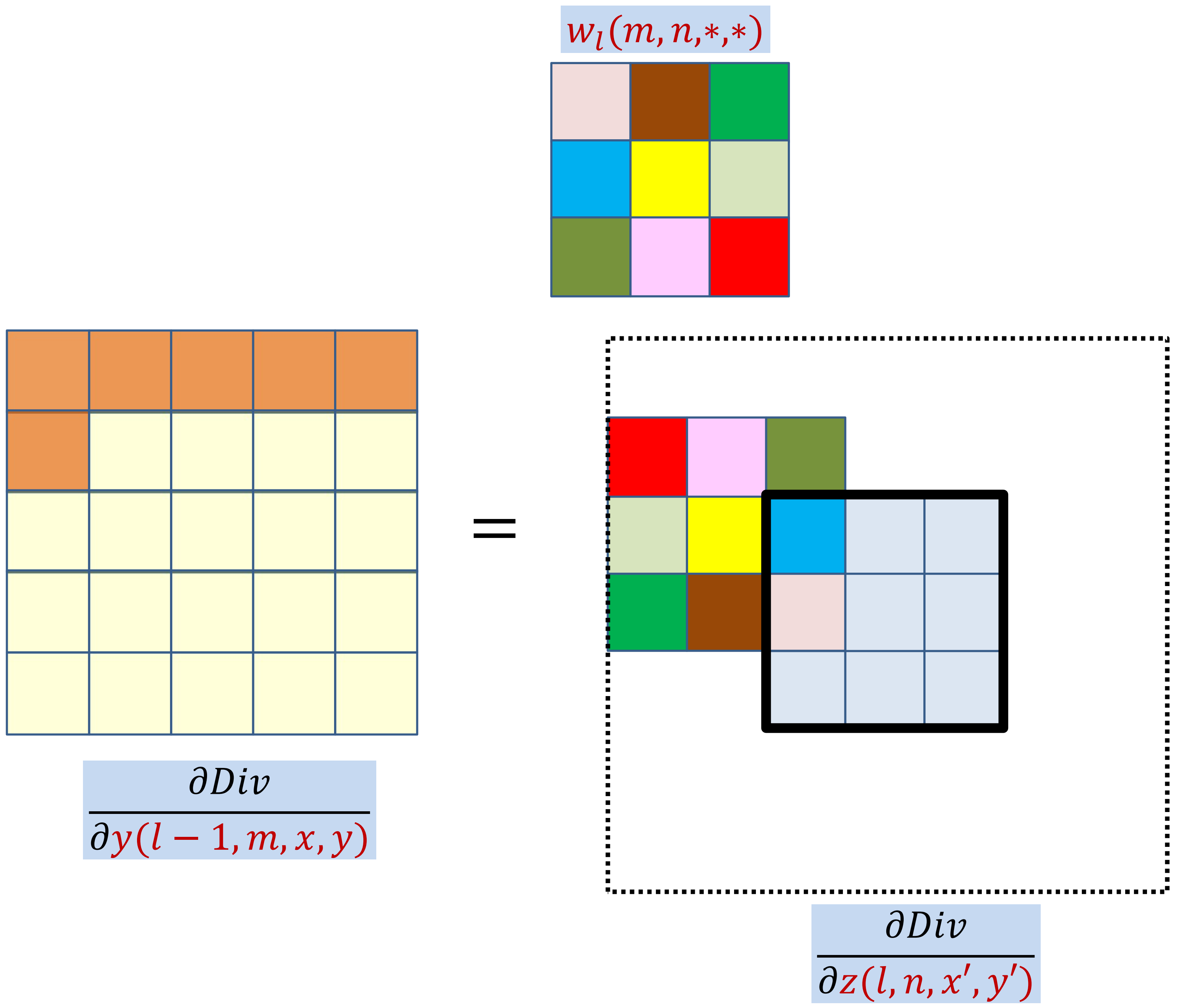
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map

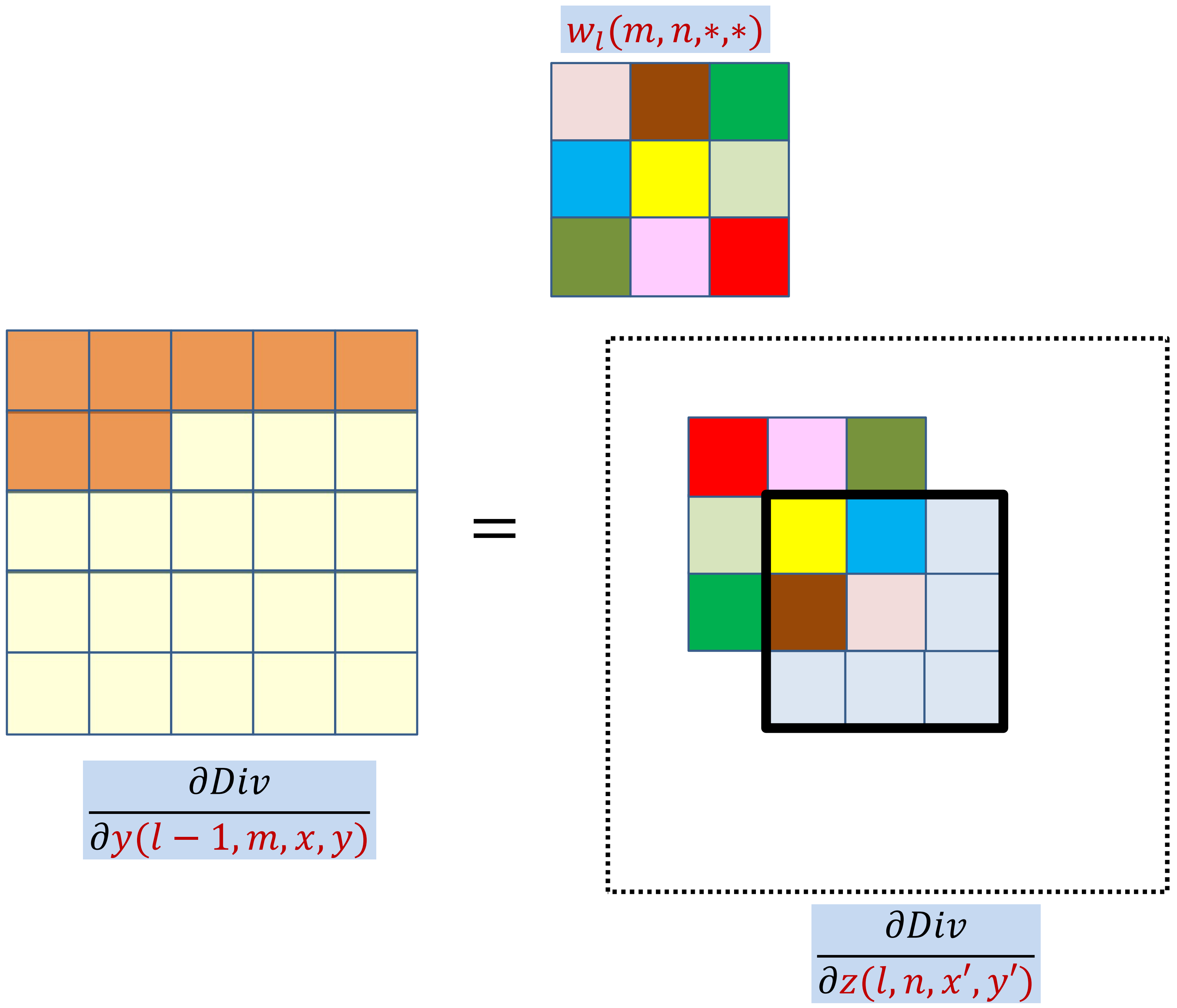

$w_l(m,n,*,*)$
$=$
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

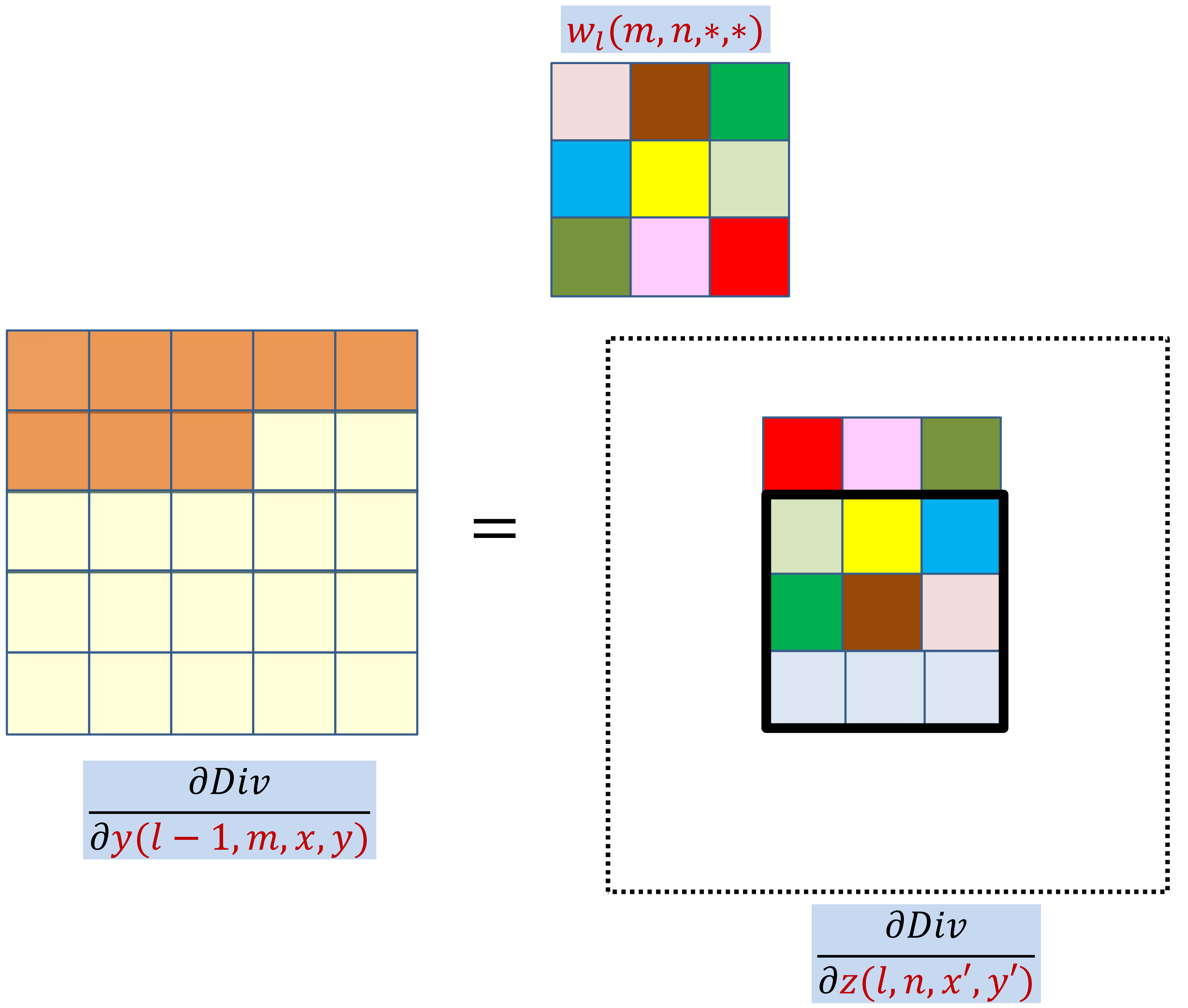


$w_l(m, n, *, *)$
$=$
$\frac{\partial Div}{\partial y(l-1, m, x, y)}$
$\frac{\partial Div}{\partial z(l, n, x', y')}$

Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map

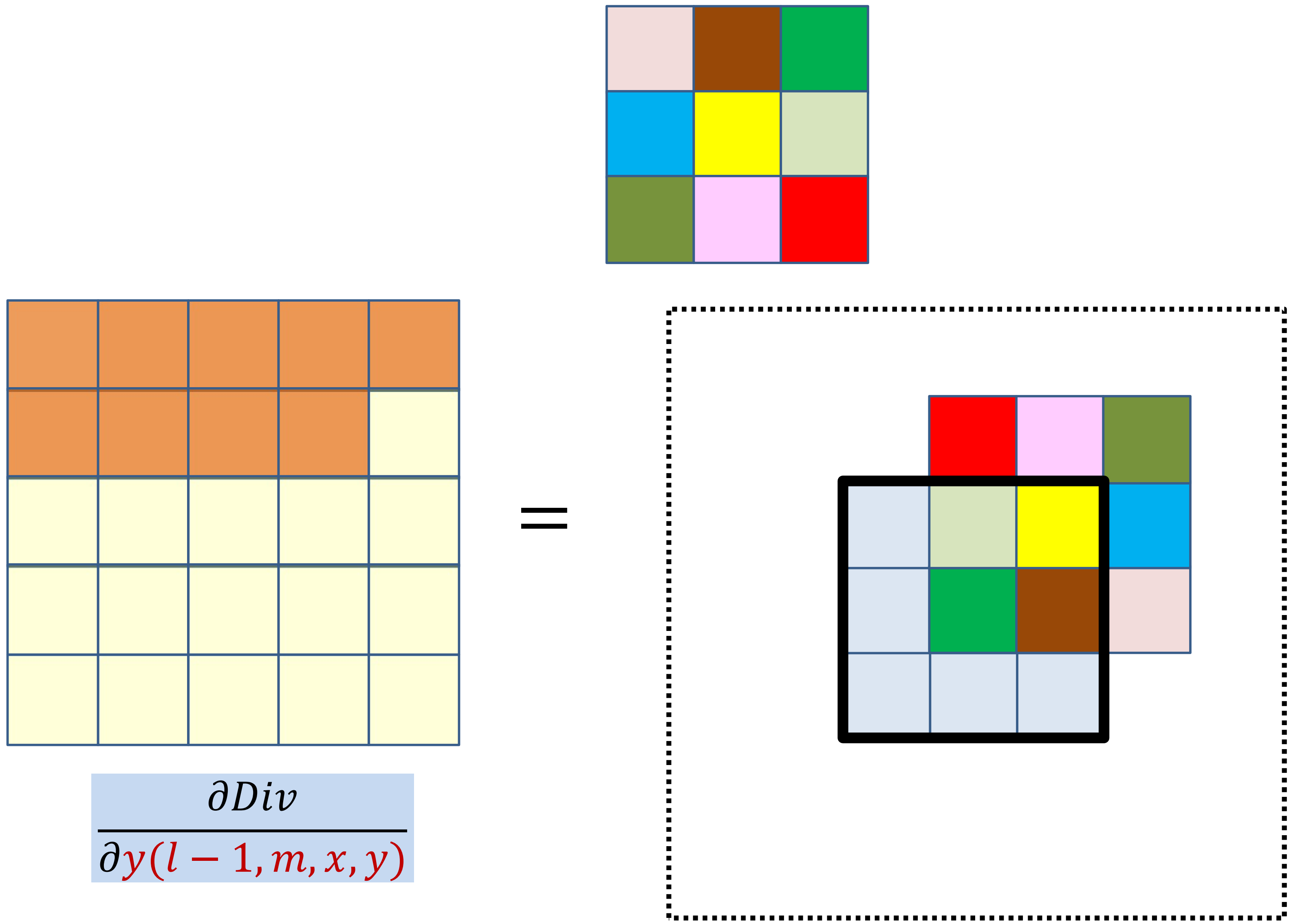

$w_l(m,n,*,*)$
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$=$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map


$w_l(m, n, *, *)$
=
$\frac{\partial Div}{\partial y(l-1, m, x, y)}$
$\frac{\partial Div}{\partial z(l, n, x', y')}$

Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map

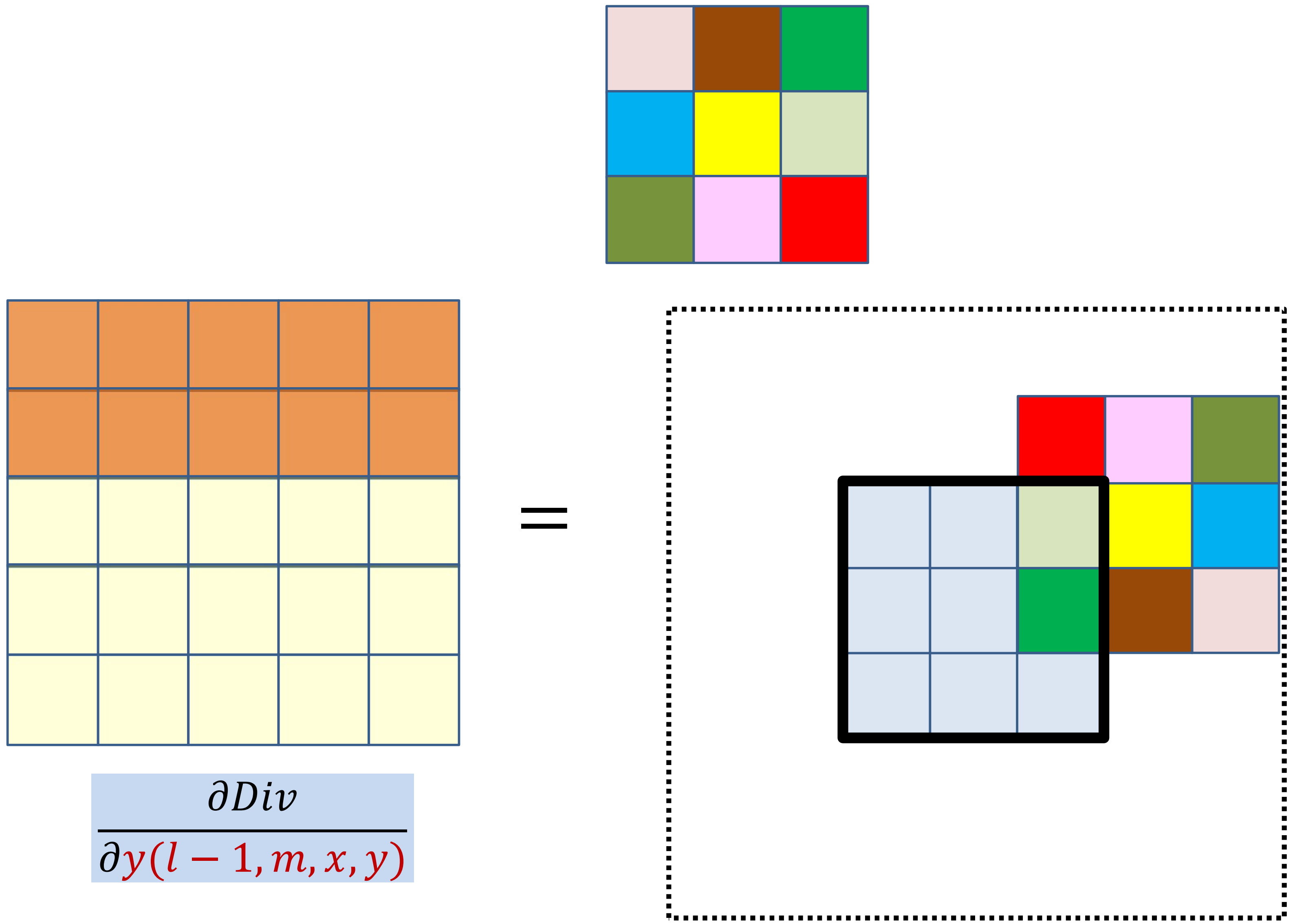
$w_l(m,n,*,*)$



$$\frac{\partial Div}{\partial y(l-1,m,x,y)} = \frac{\partial Div}{\partial z(l,n,x',y')}$$

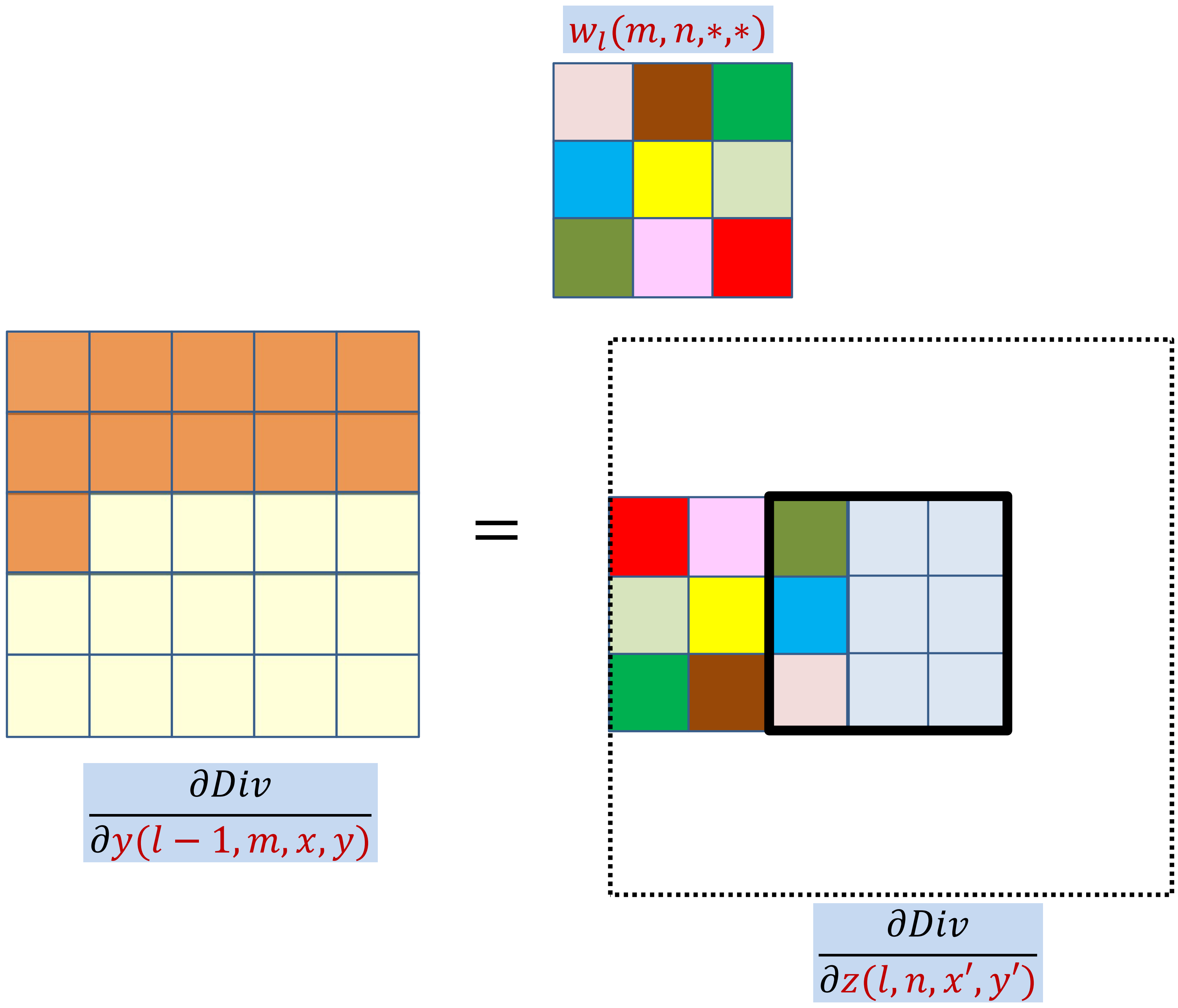
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map

$w_l(m, n, *, *)$



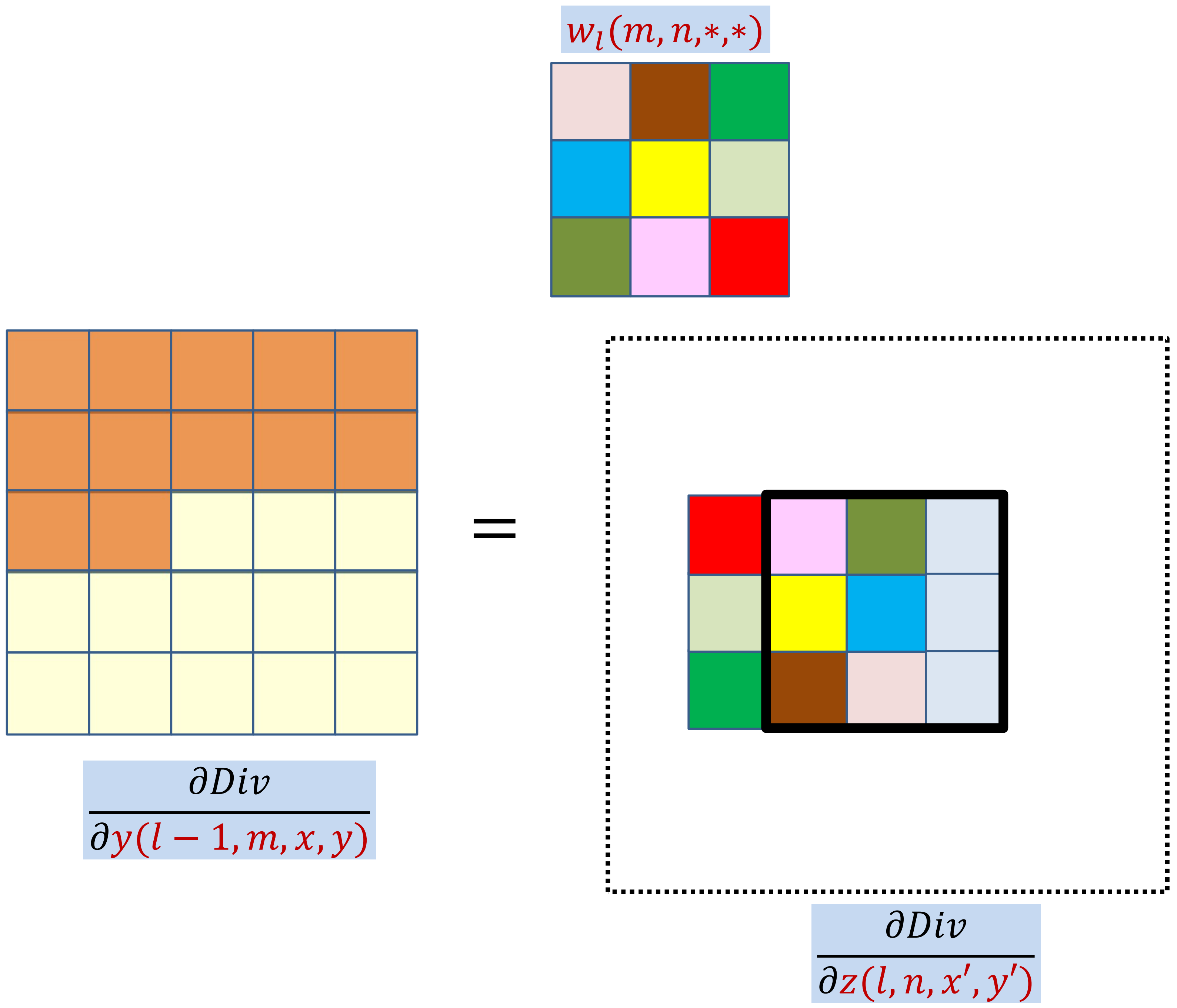
$$\frac{\partial Div}{\partial y(l-1, m, x, y)} = \frac{\partial Div}{\partial z(l, n, x', y')}$$

Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map



$w_l(m, n,*,*)$
=
$\frac{\partial Div}{\partial y(l-1, m, x, y)}$
$\frac{\partial Div}{\partial z(l, n, x', y')}$

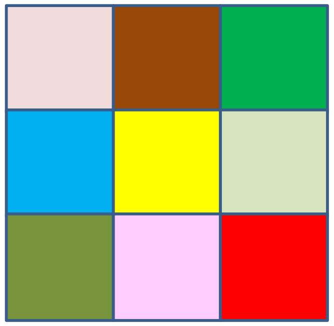
Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map

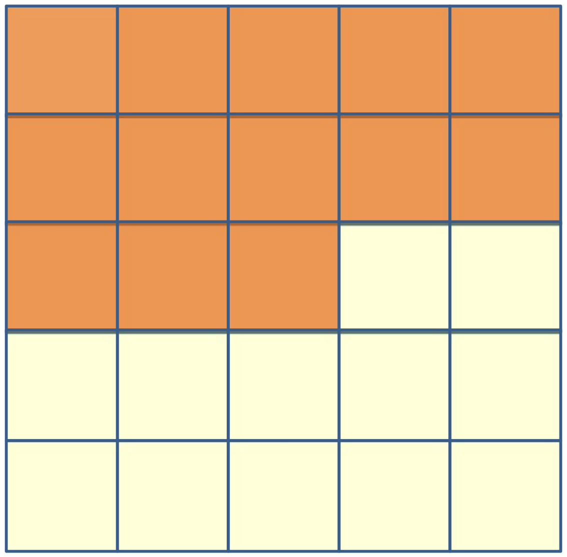


$w_l(m,n,*,*)$
$=$
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

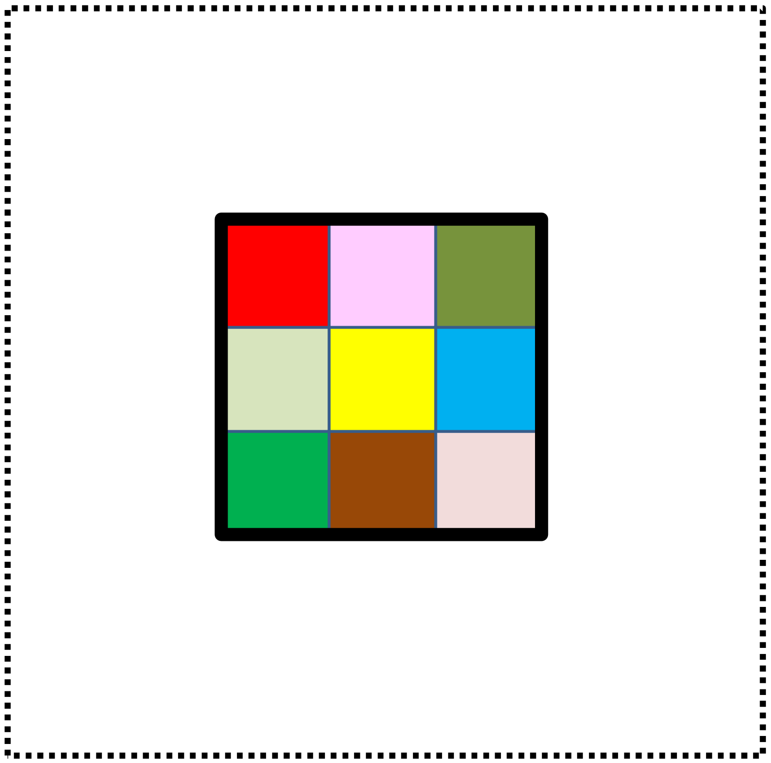
Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map

$w_l(m,n,*,*)$





$=$

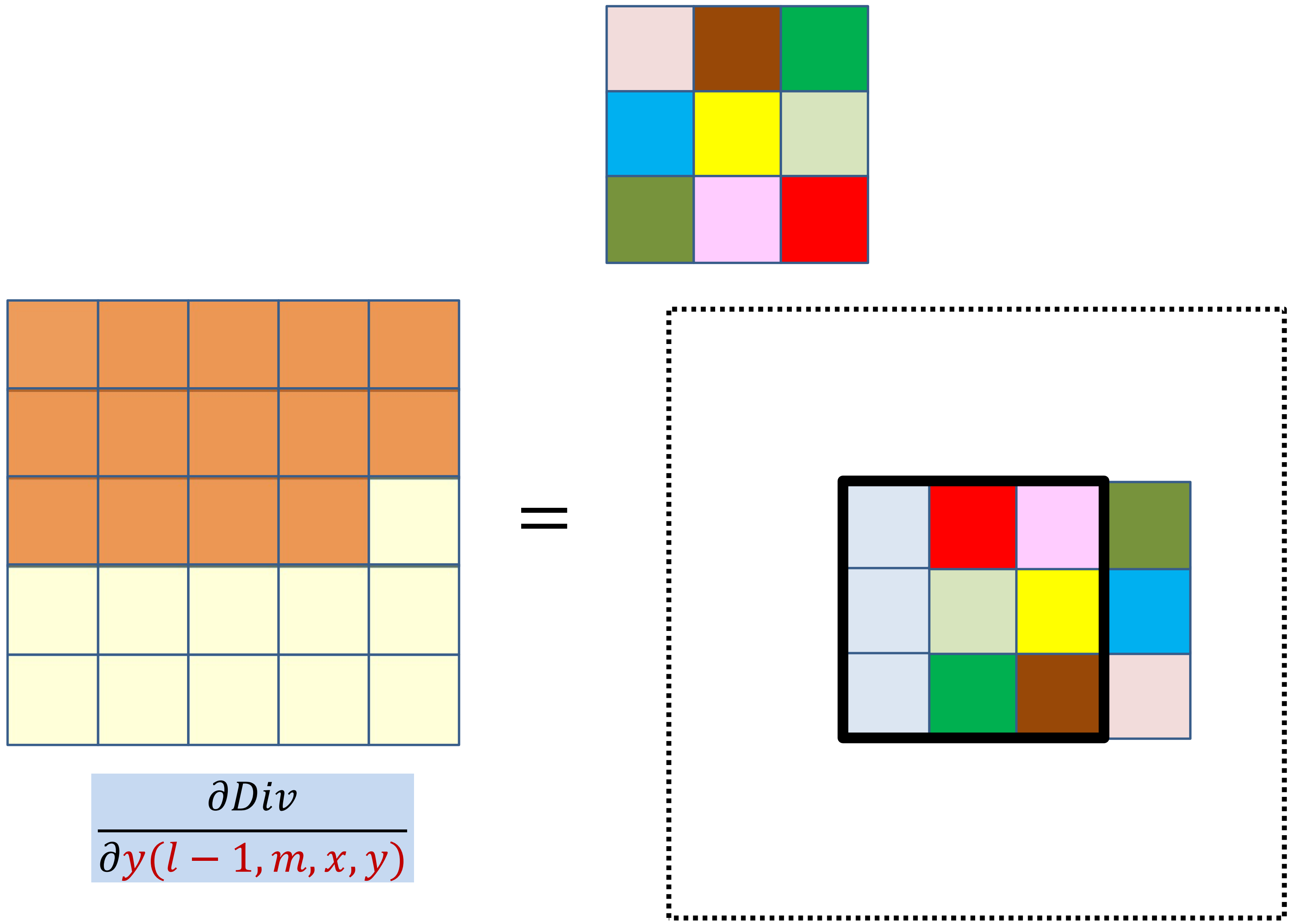


$$\frac{\partial Div}{\partial y(l-1,m,x,y)}$$

$$\frac{\partial Div}{\partial z(l,n,x',y')}$$

Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map

$w_l(m,n,*,*)$



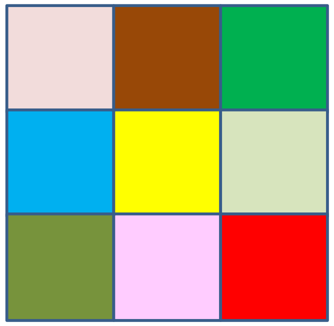
$$\frac{\partial Div}{\partial y(l-1,m,x,y)}$$

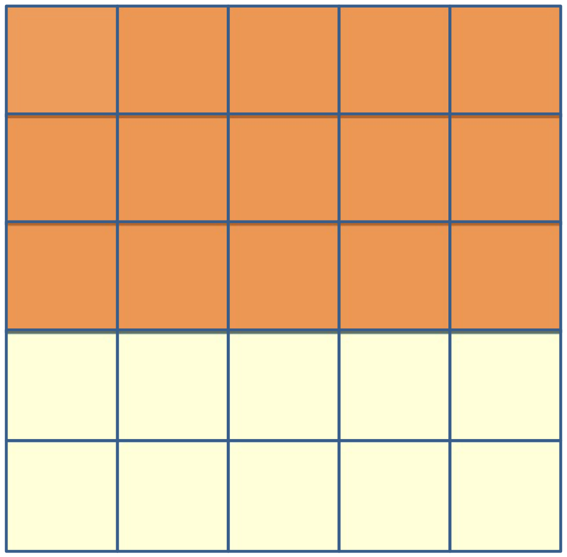
$=$

$$\frac{\partial Div}{\partial z(l,n,x',y')}$$

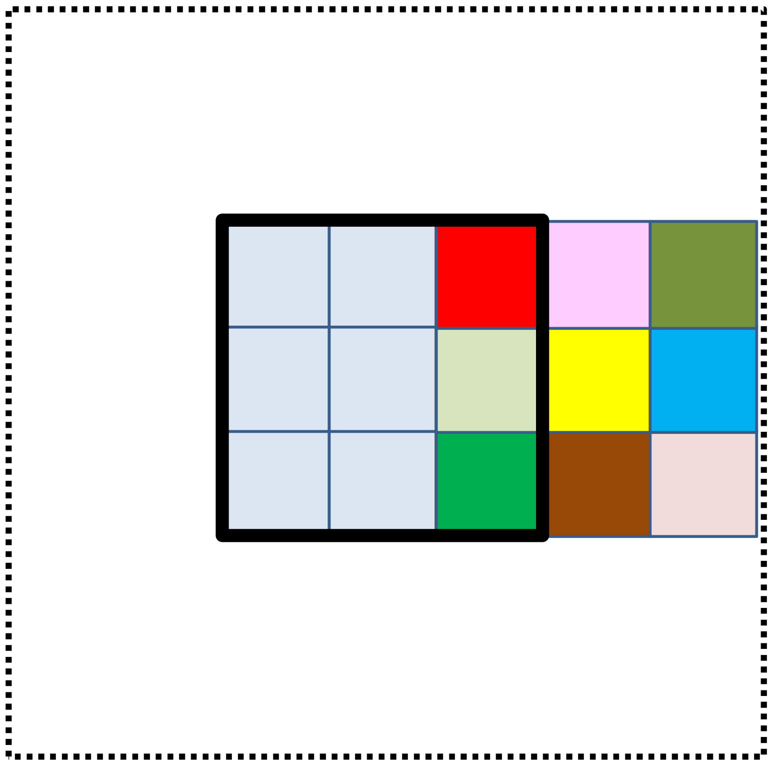
Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map

$w_l(m,n,*,*)$





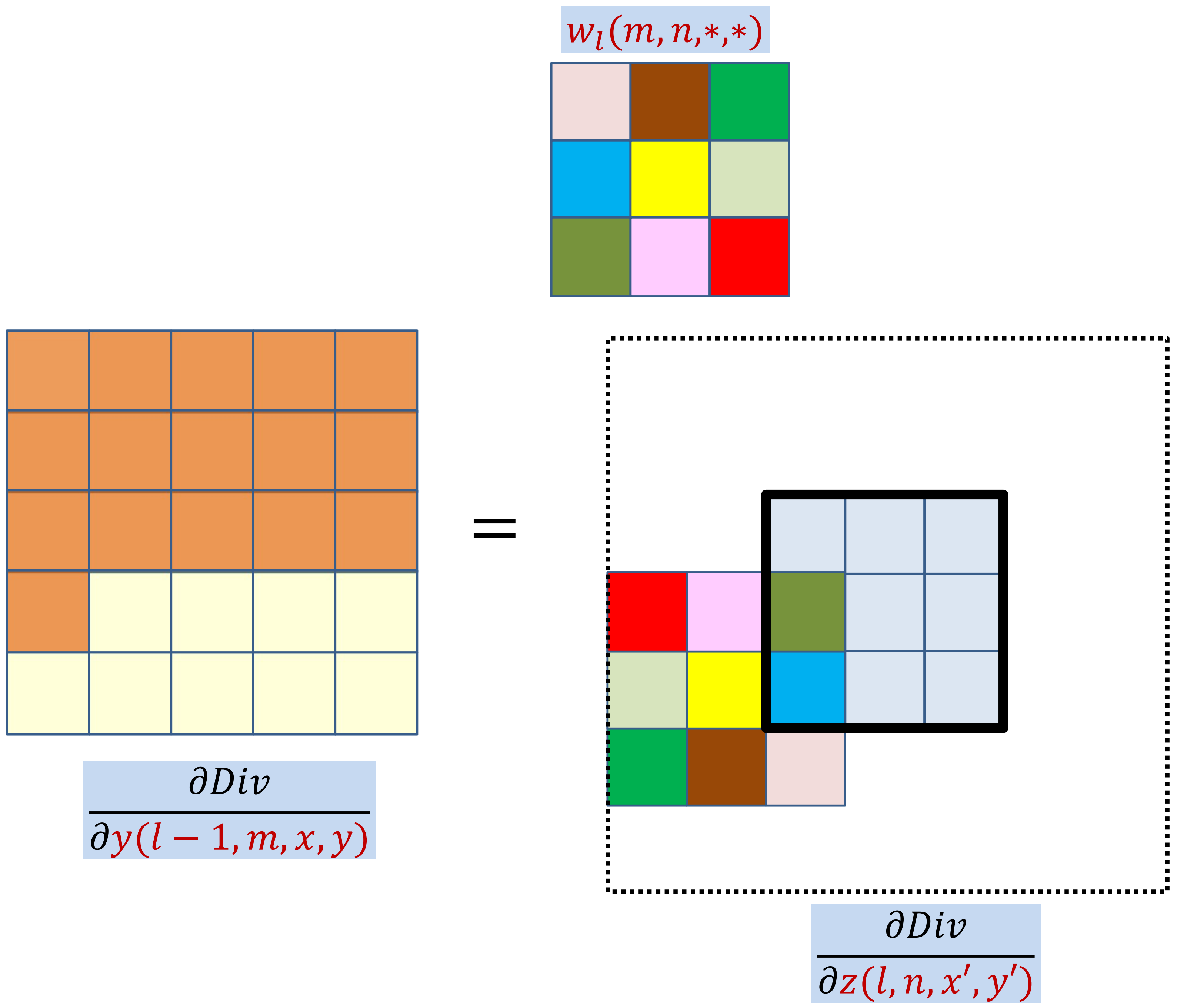
$$= $$



$$\frac{\partial Div}{\partial y(l-1,m,x,y)}$$

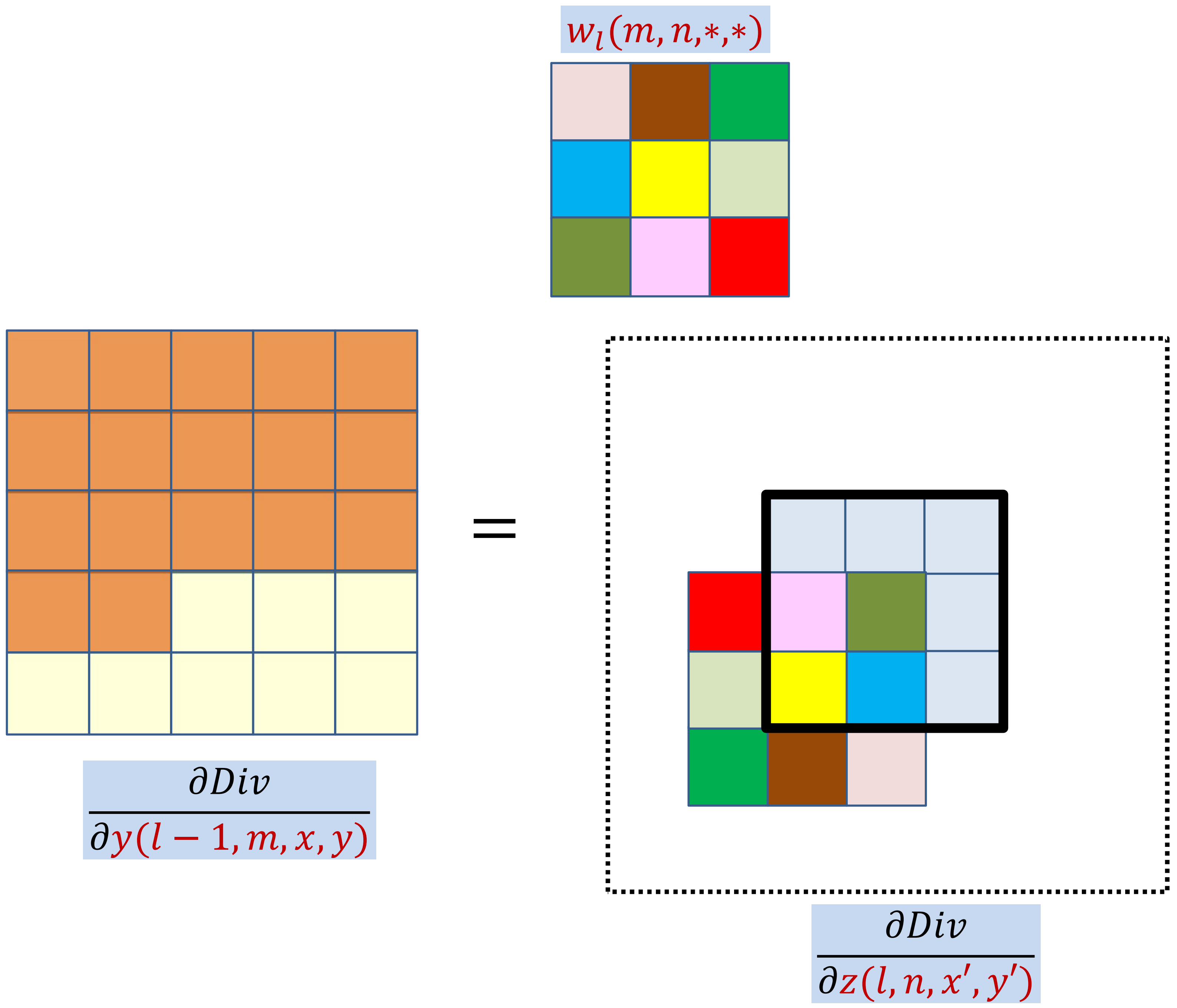
$$\frac{\partial Div}{\partial z(l,n,x',y')}$$

Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map

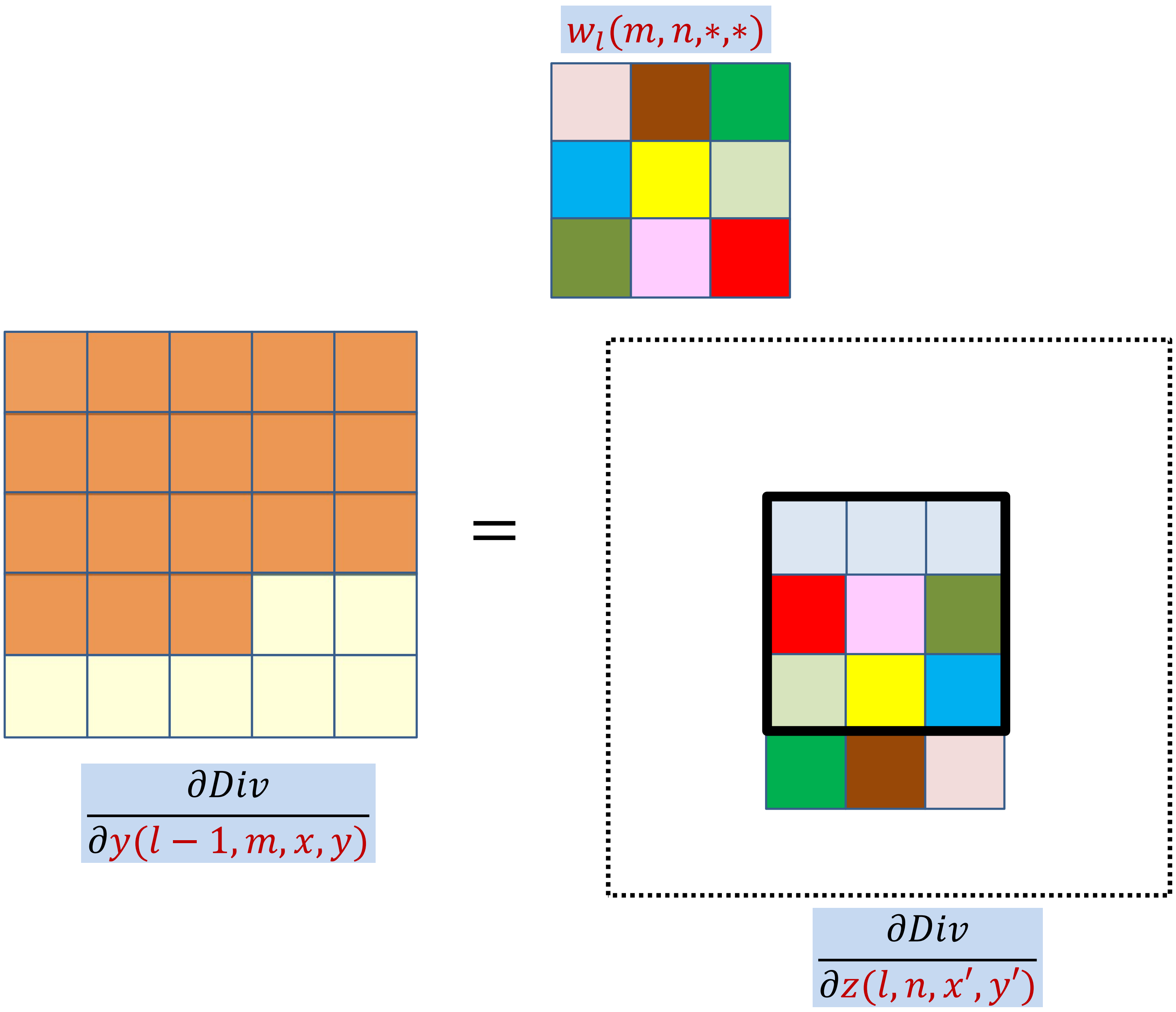


$w_l(m,n,*,*)$
$=$
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

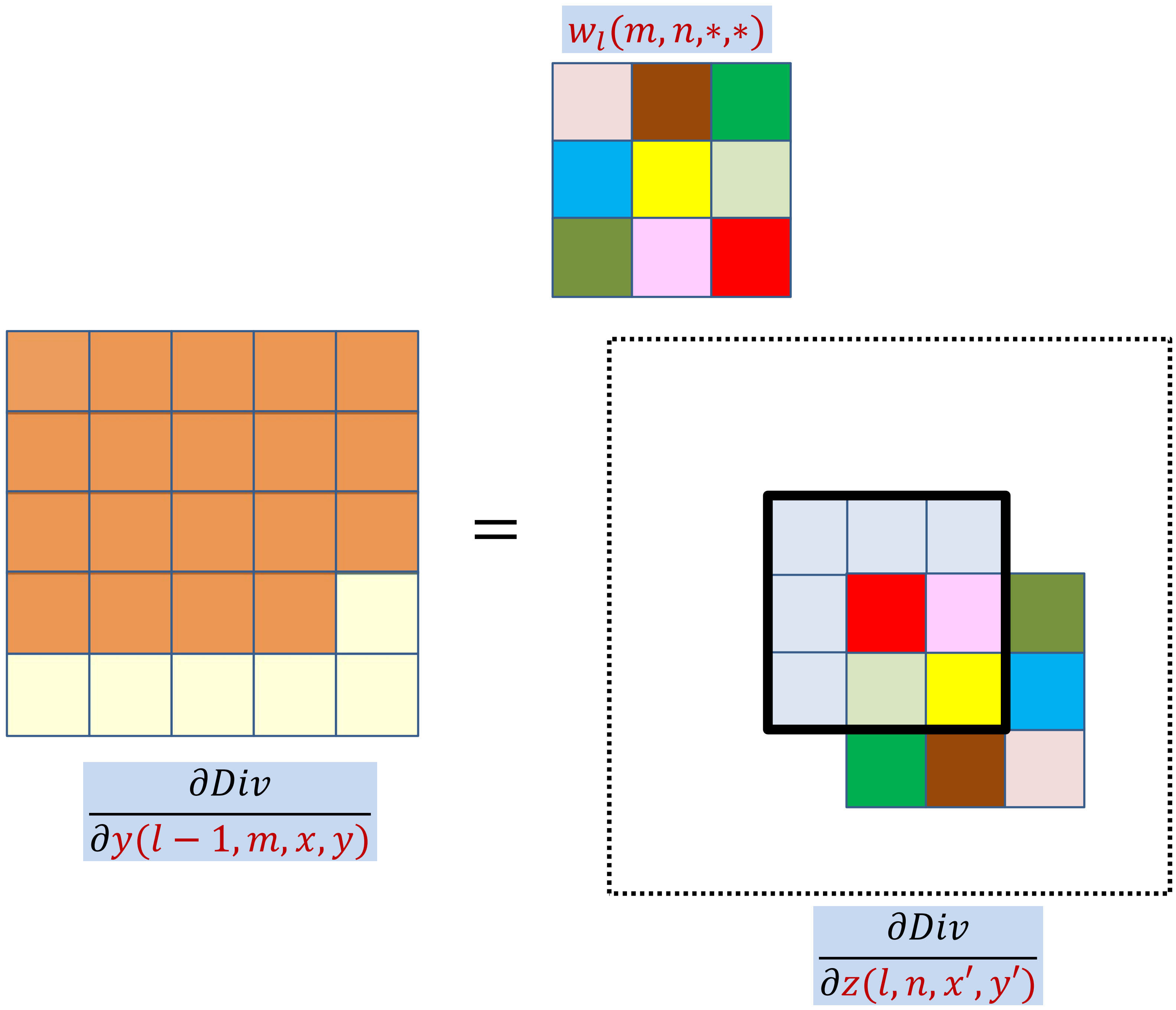
Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map


$w_l(m,n,*,*)$
$=$
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

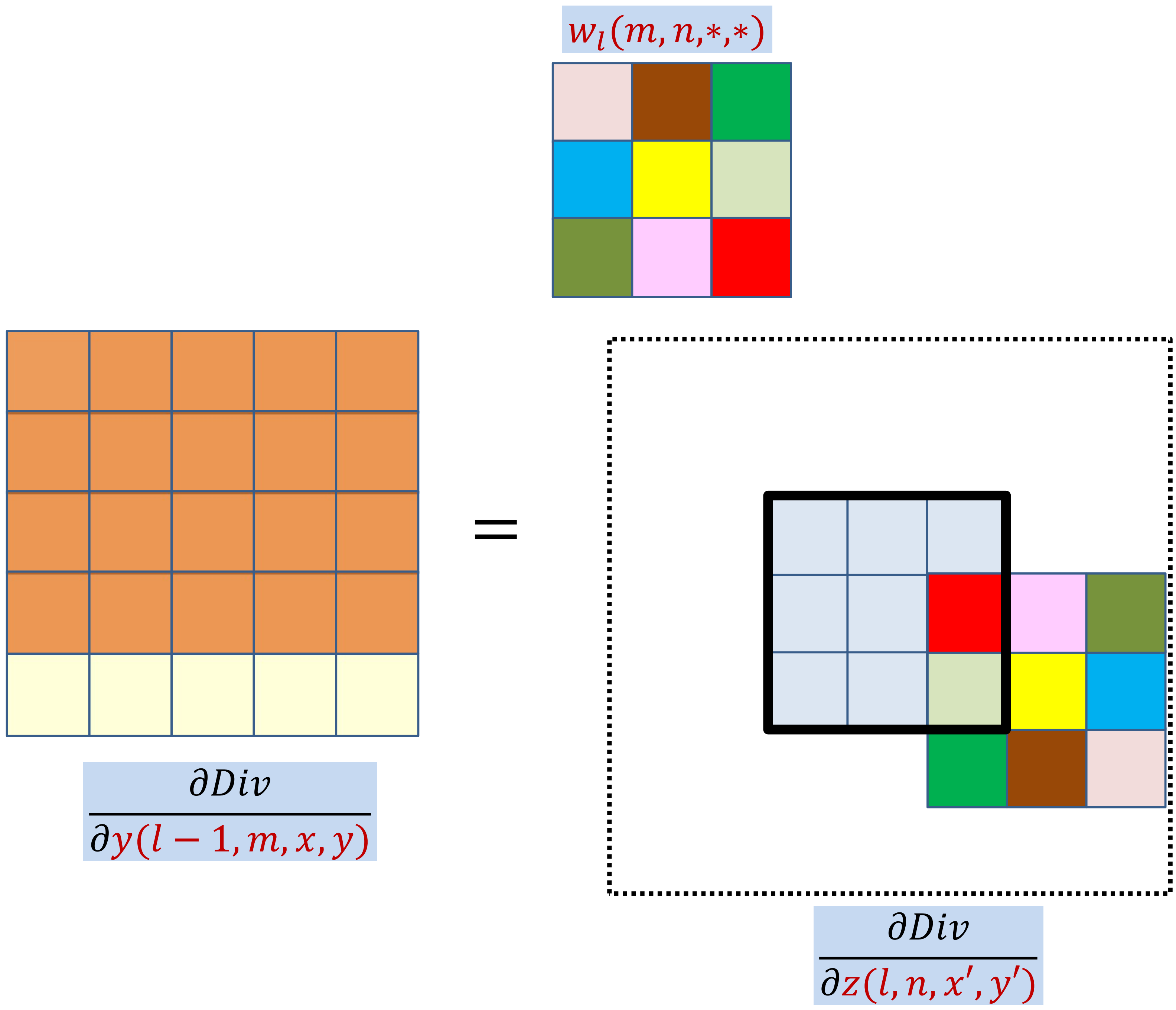
Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map


$w_l(m,n,*,*)$
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$=$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map


$w_l(m, n, *, *)$
$=$
$\frac{\partial Div}{\partial y(l-1, m, x, y)}$
$\frac{\partial Div}{\partial z(l, n, x', y')}$

Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map


$w_l(m,n,*,*)$
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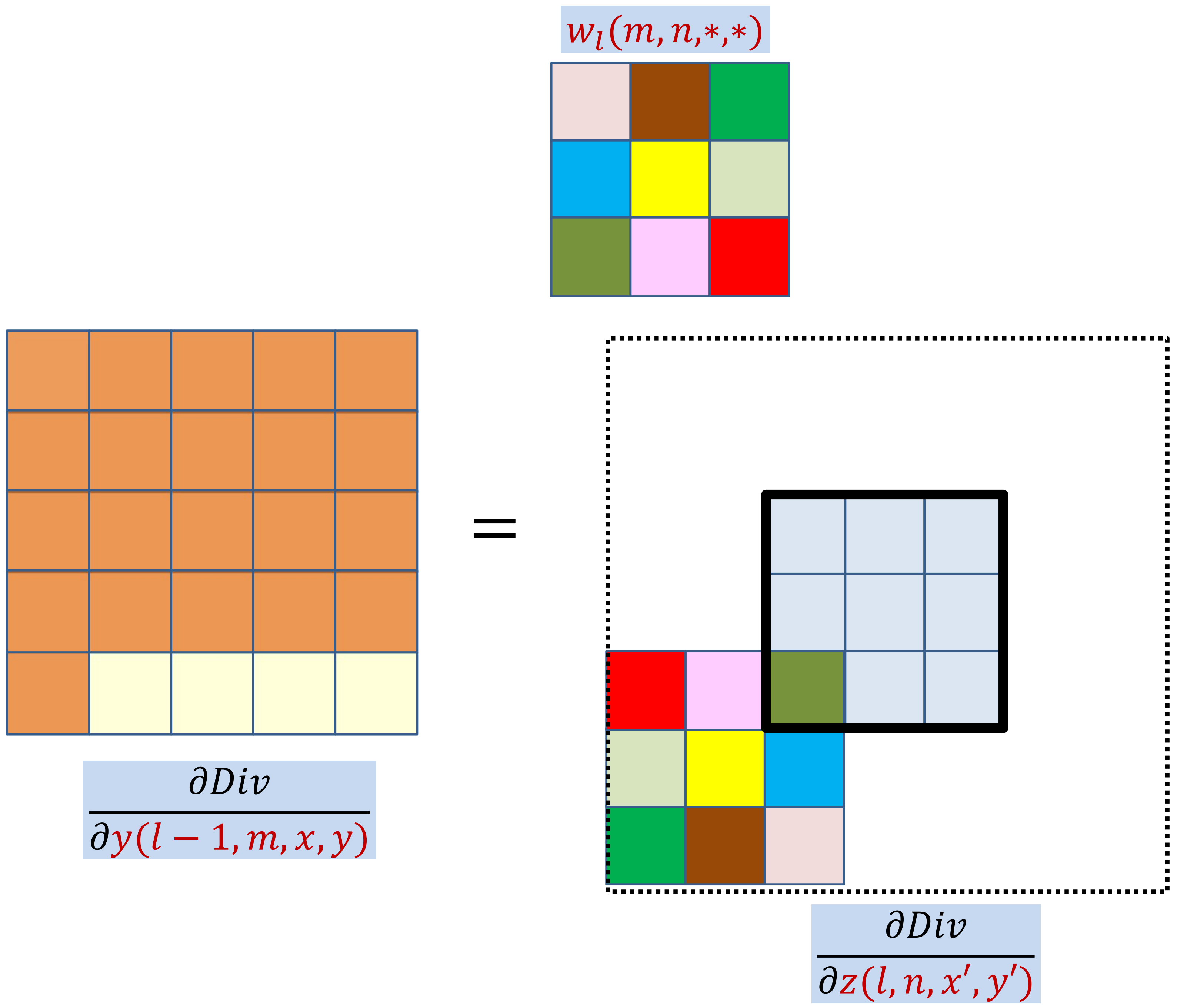

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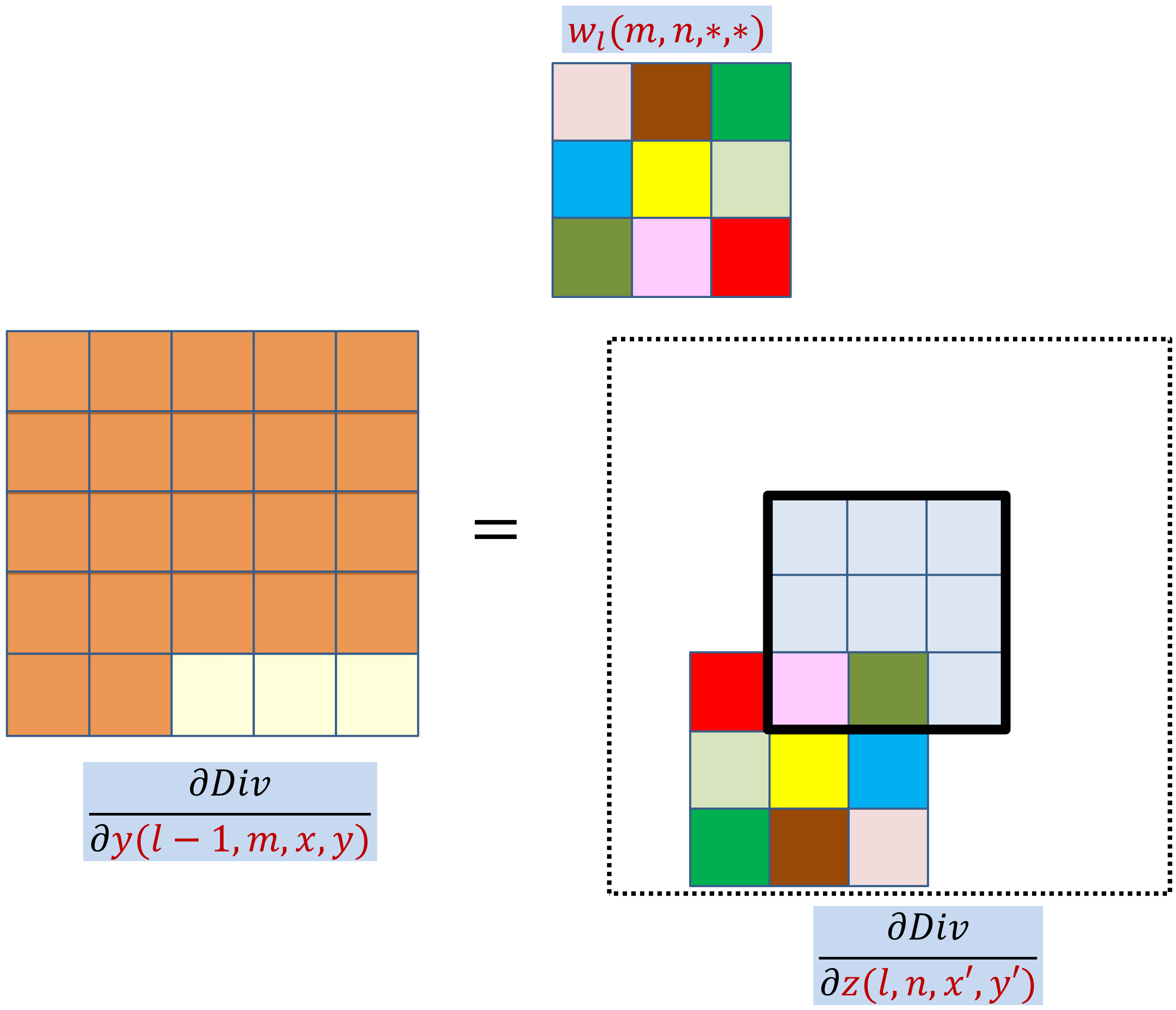
Derivative at $Y(l-1, m)$ from a single $Z(l, n)$ map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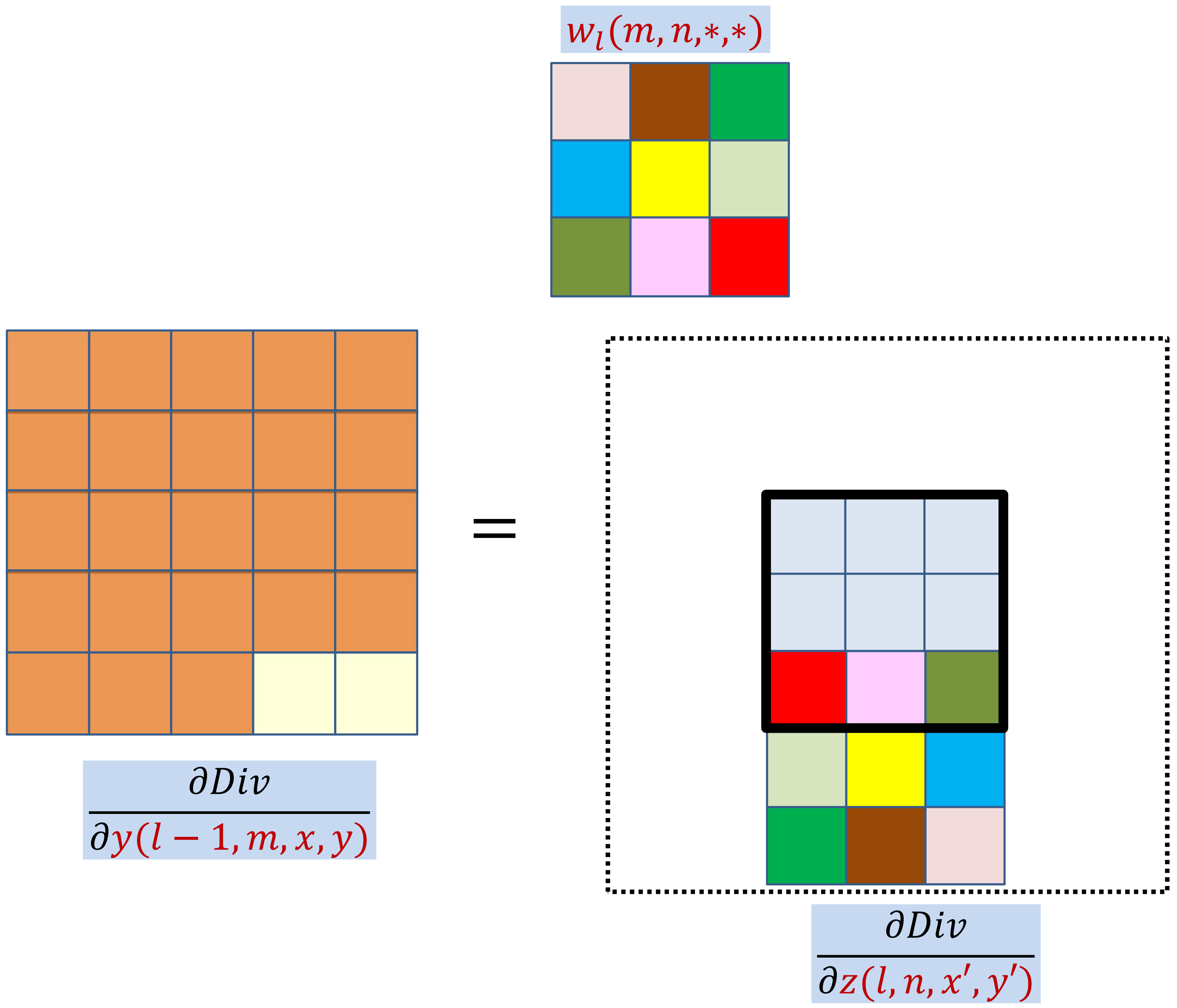
Derivative at $Y(l-1,m)$ from a single $Z(l,n)$ map



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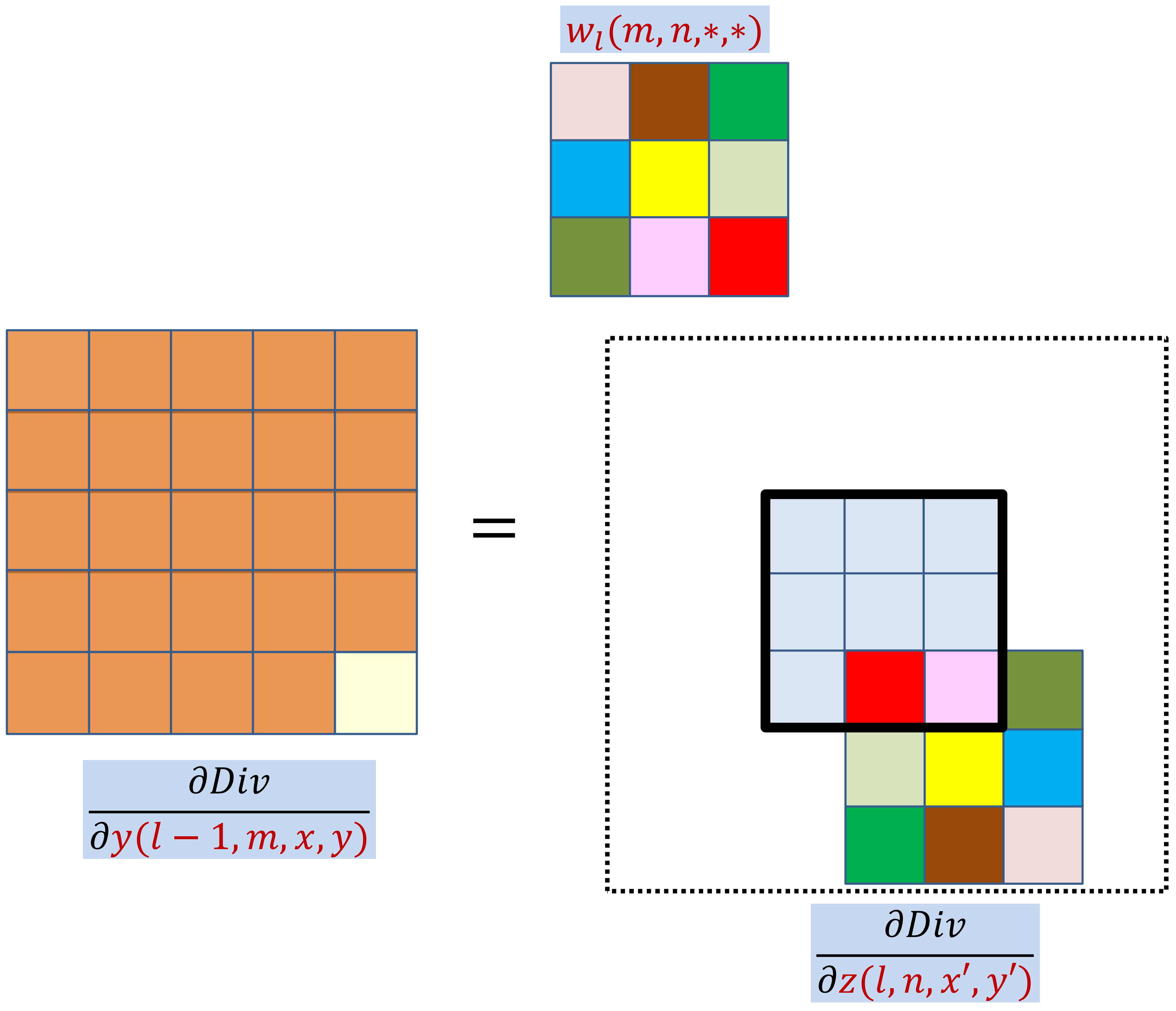

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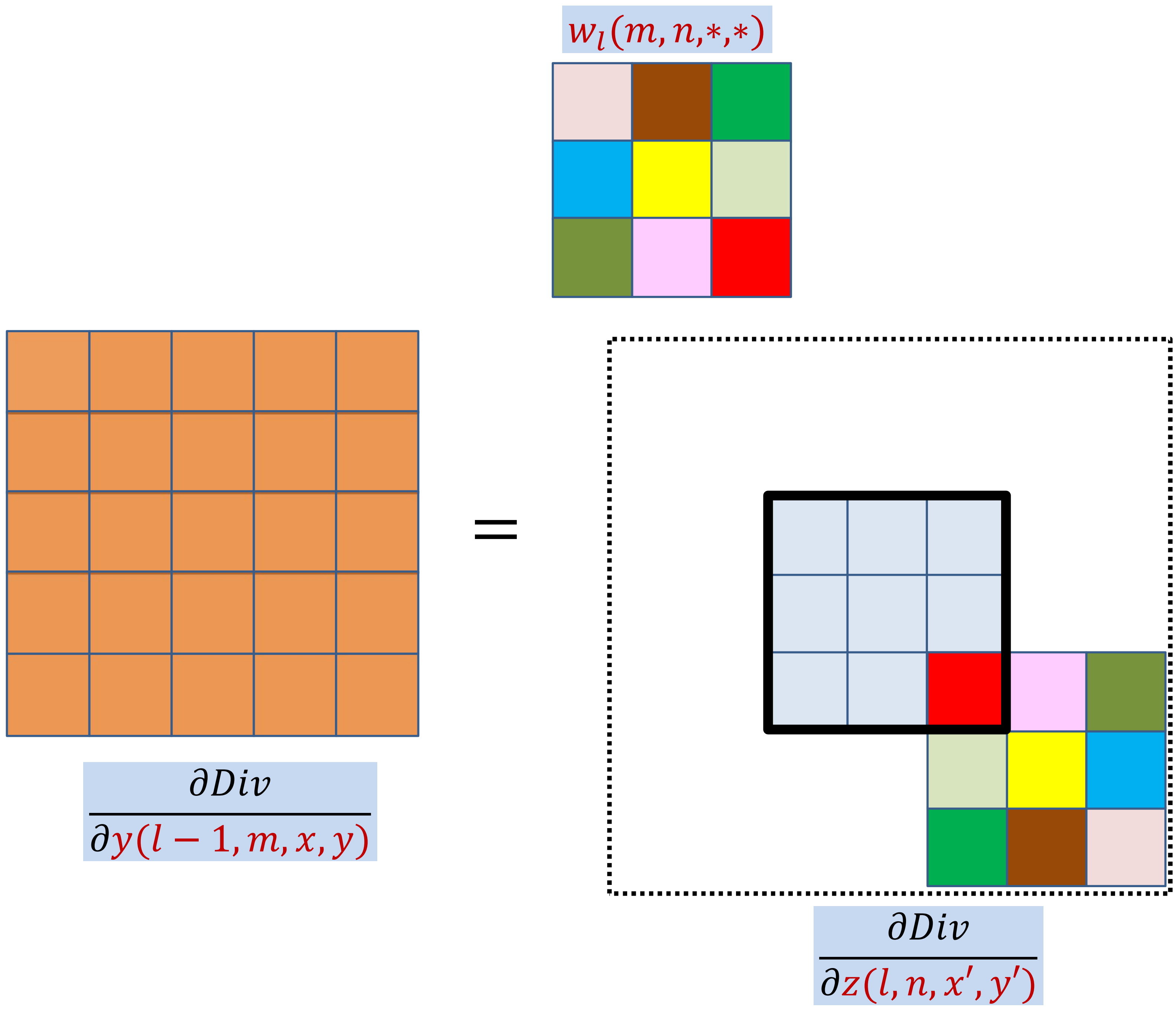
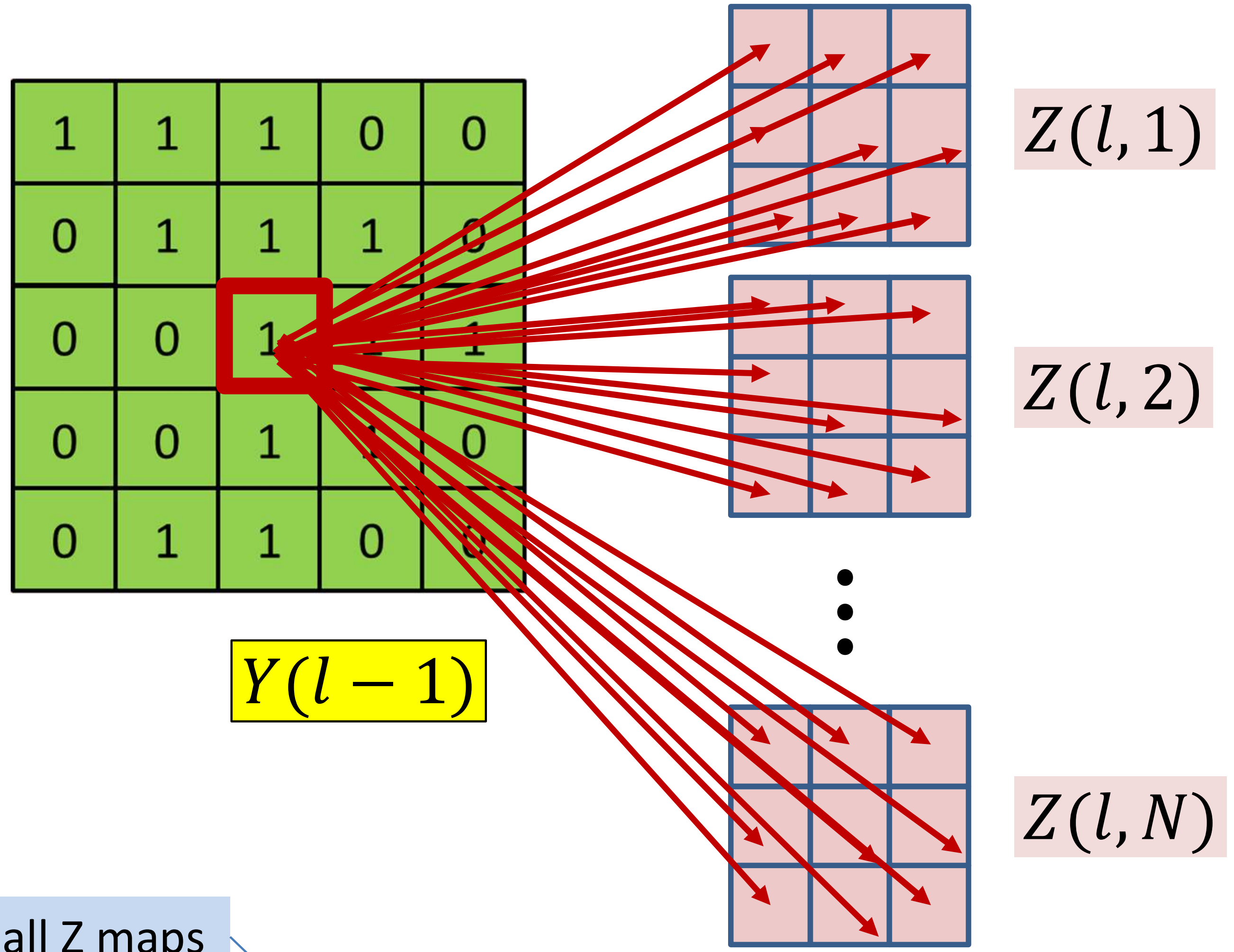


$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

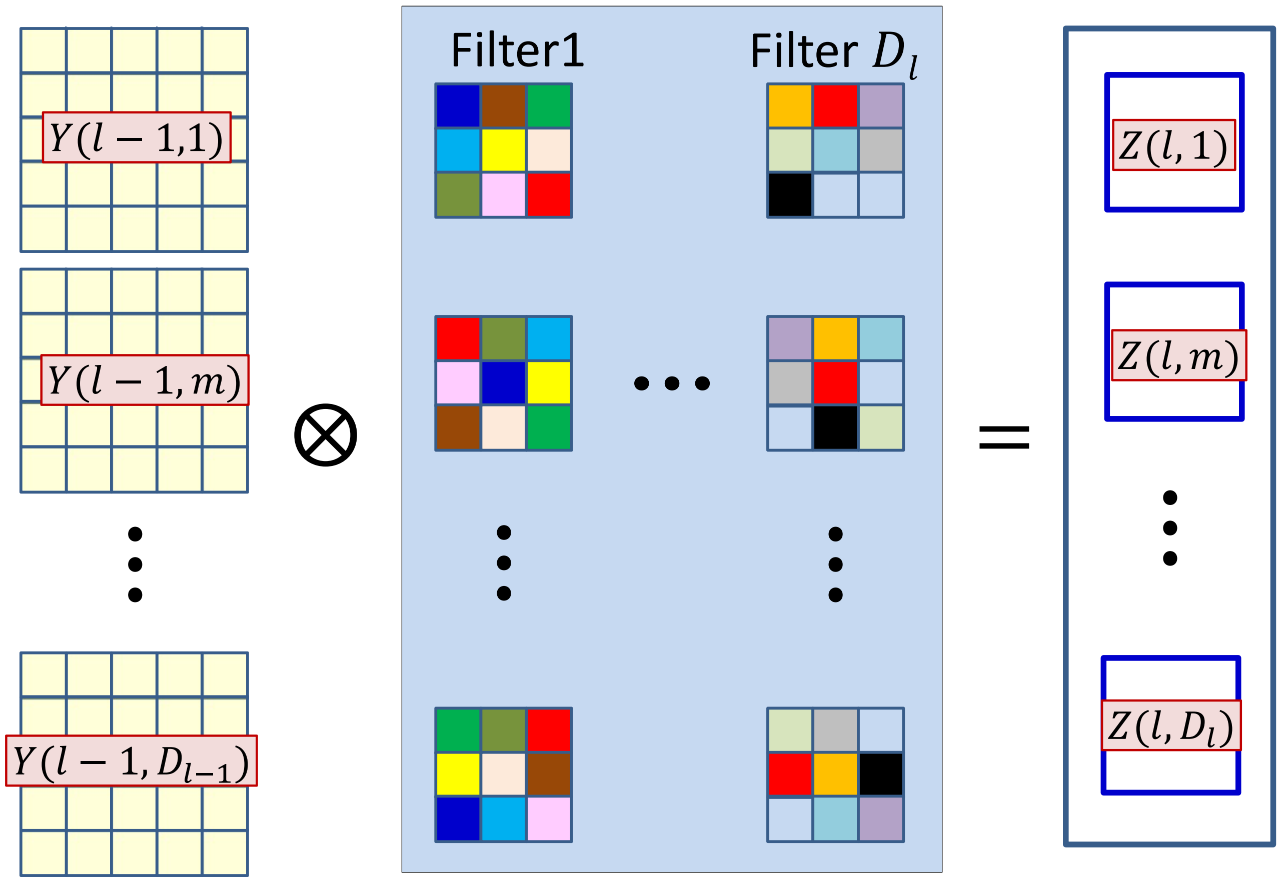
BP: Convolutional layer



Summing over all Z maps

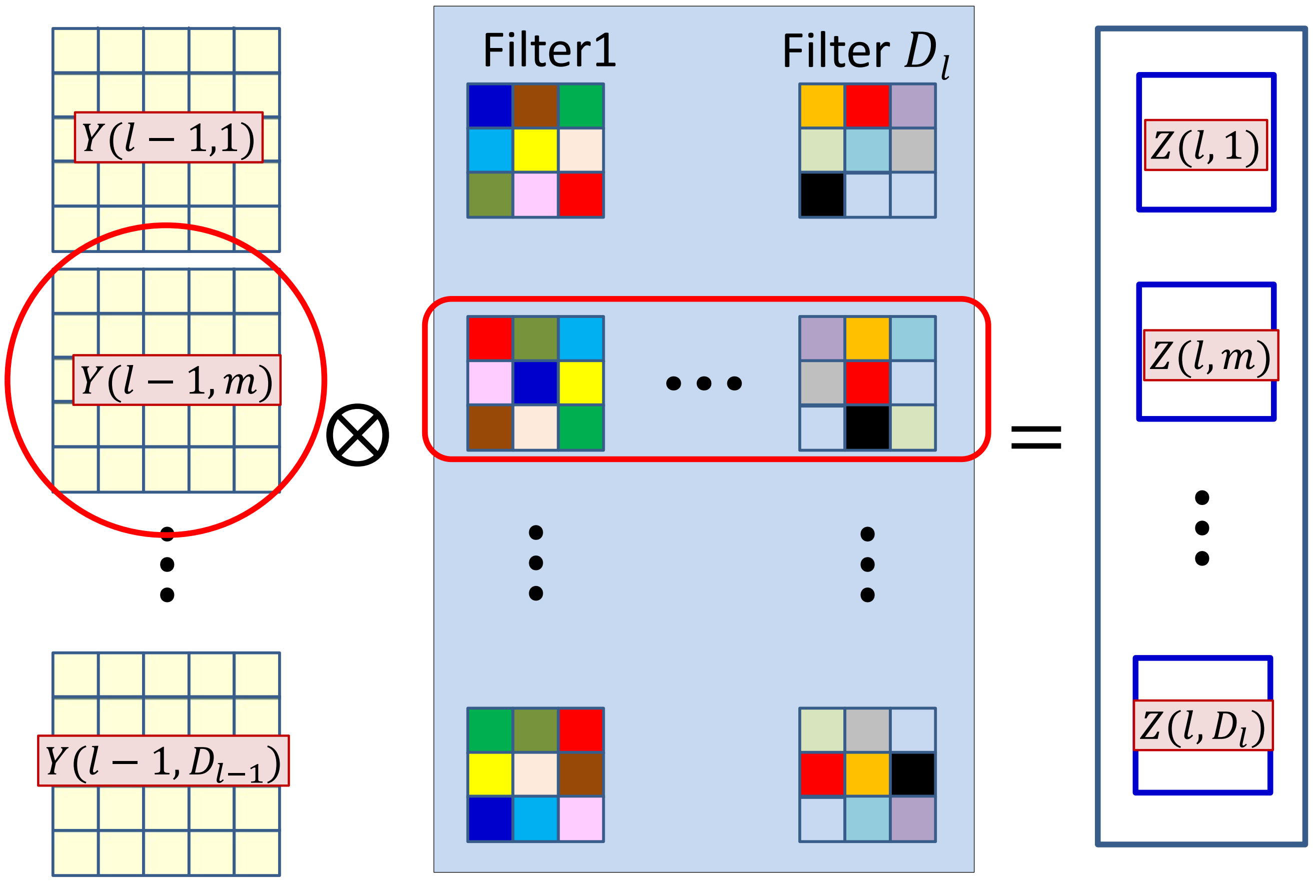
$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$

The actual convolutions

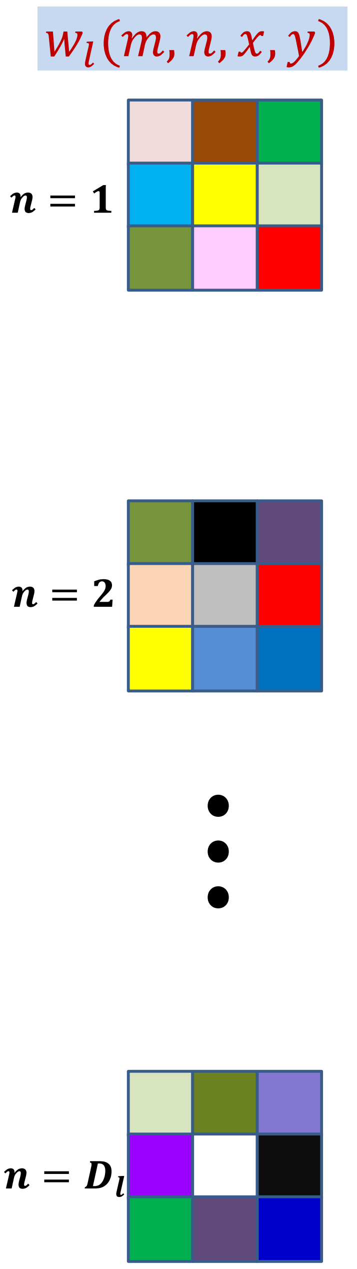


- The D_l affine maps are produced by convolving with D_l filters

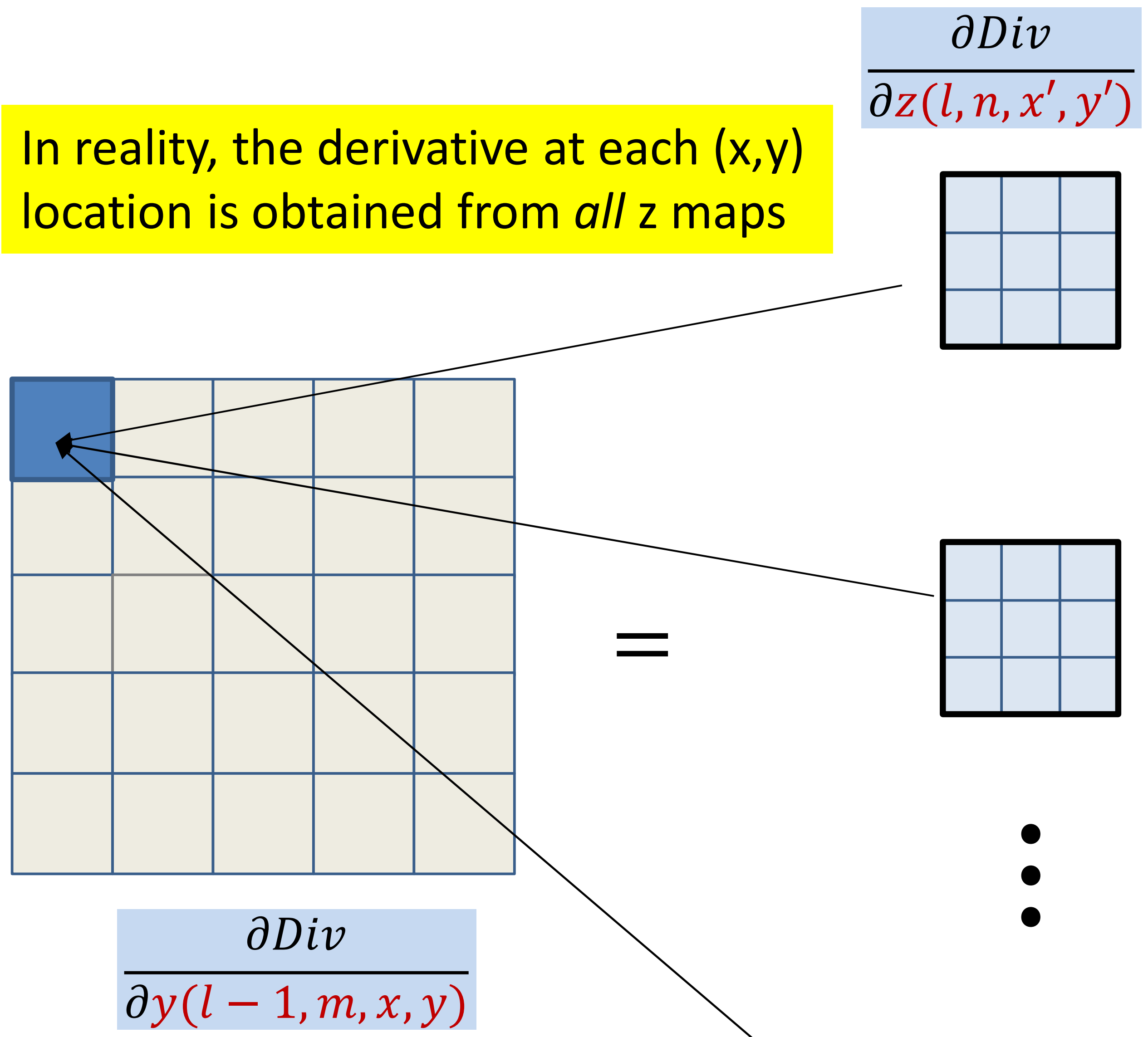
The actual convolutions

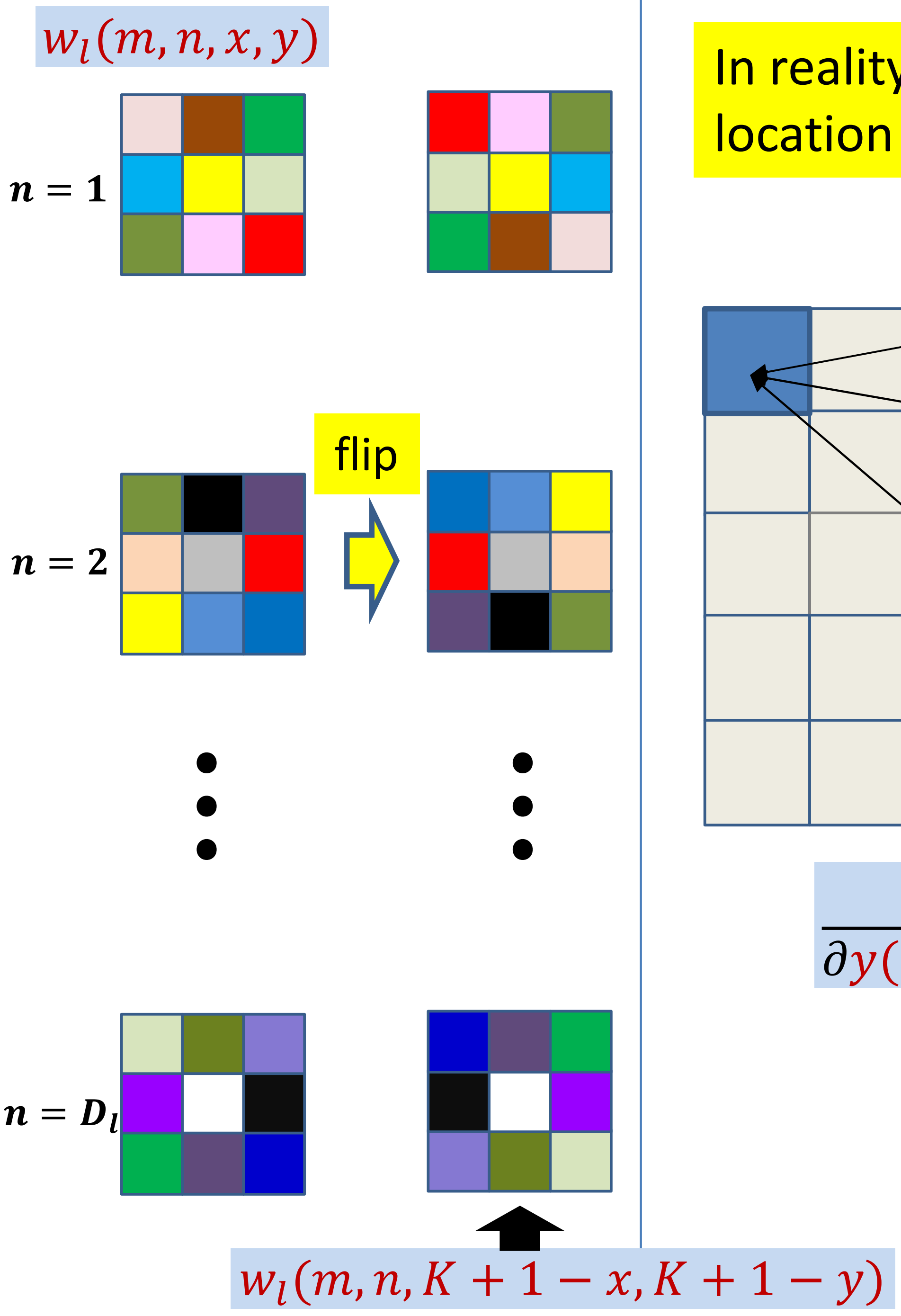


- The D_l affine maps are produced by convolving with D_l filters
- The mth Y map always convolves the mth plane of the filters
- The derivative for the mth Y map will invoke the mth plane of *all* the filters



In reality, the derivative at each (x,y) location is obtained from *all* z maps


$\frac{\partial Div}{\partial z(l,n,x',y')}$
$=$
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$



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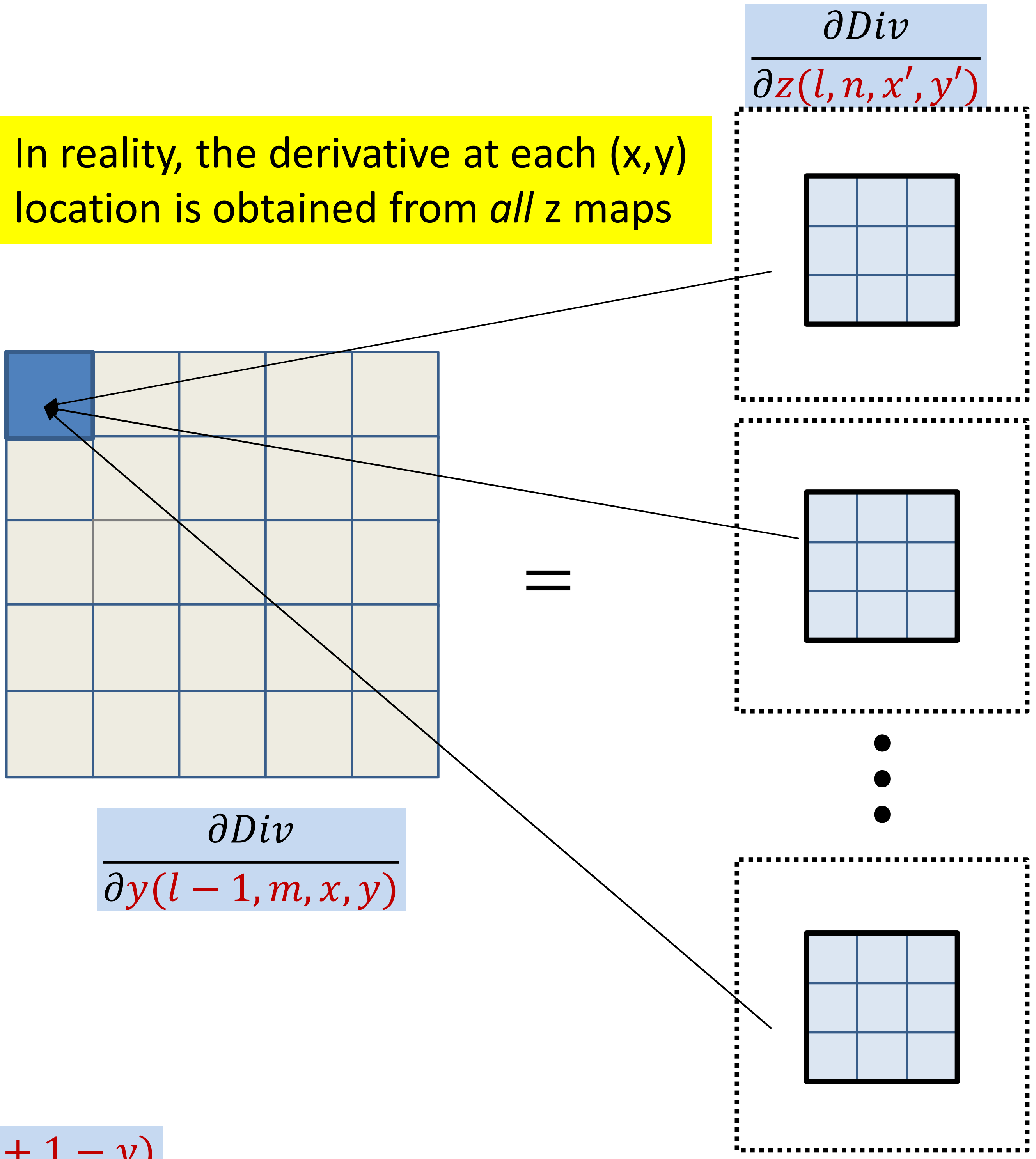

$\frac{\partial Div}{\partial z(l,n,x',y')}$
$=$
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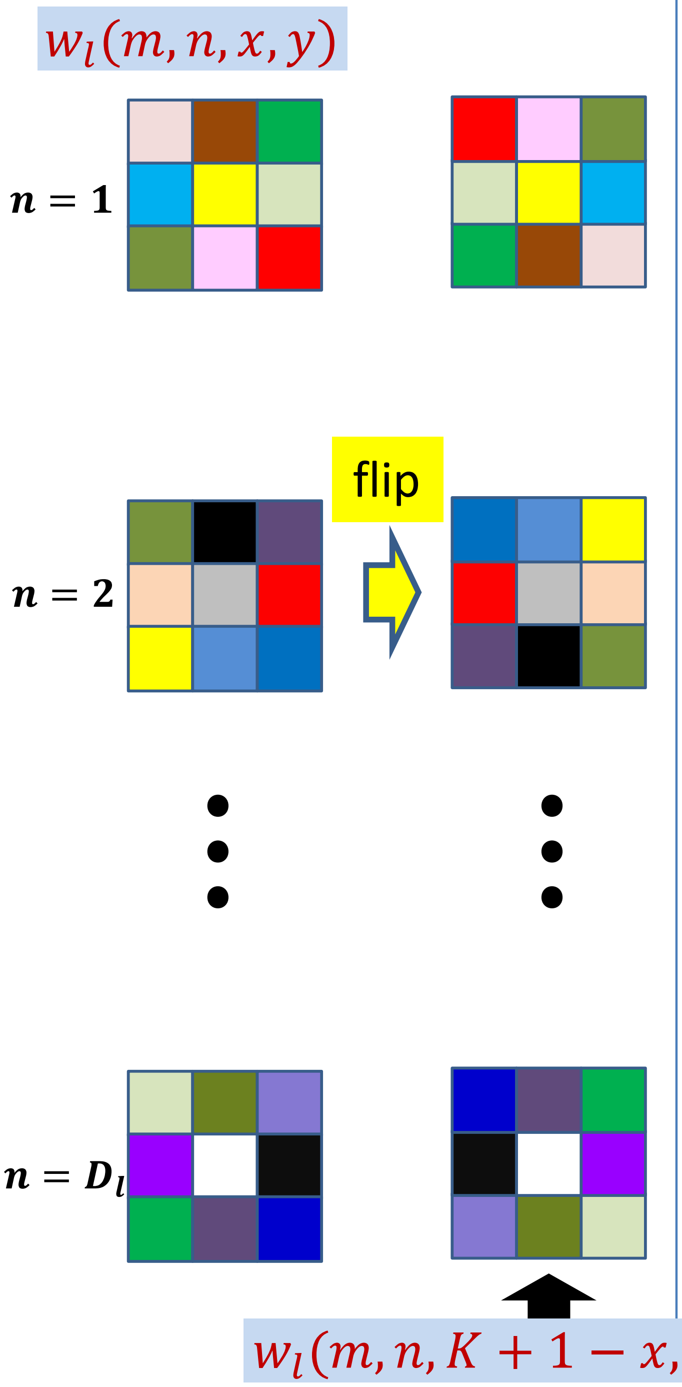
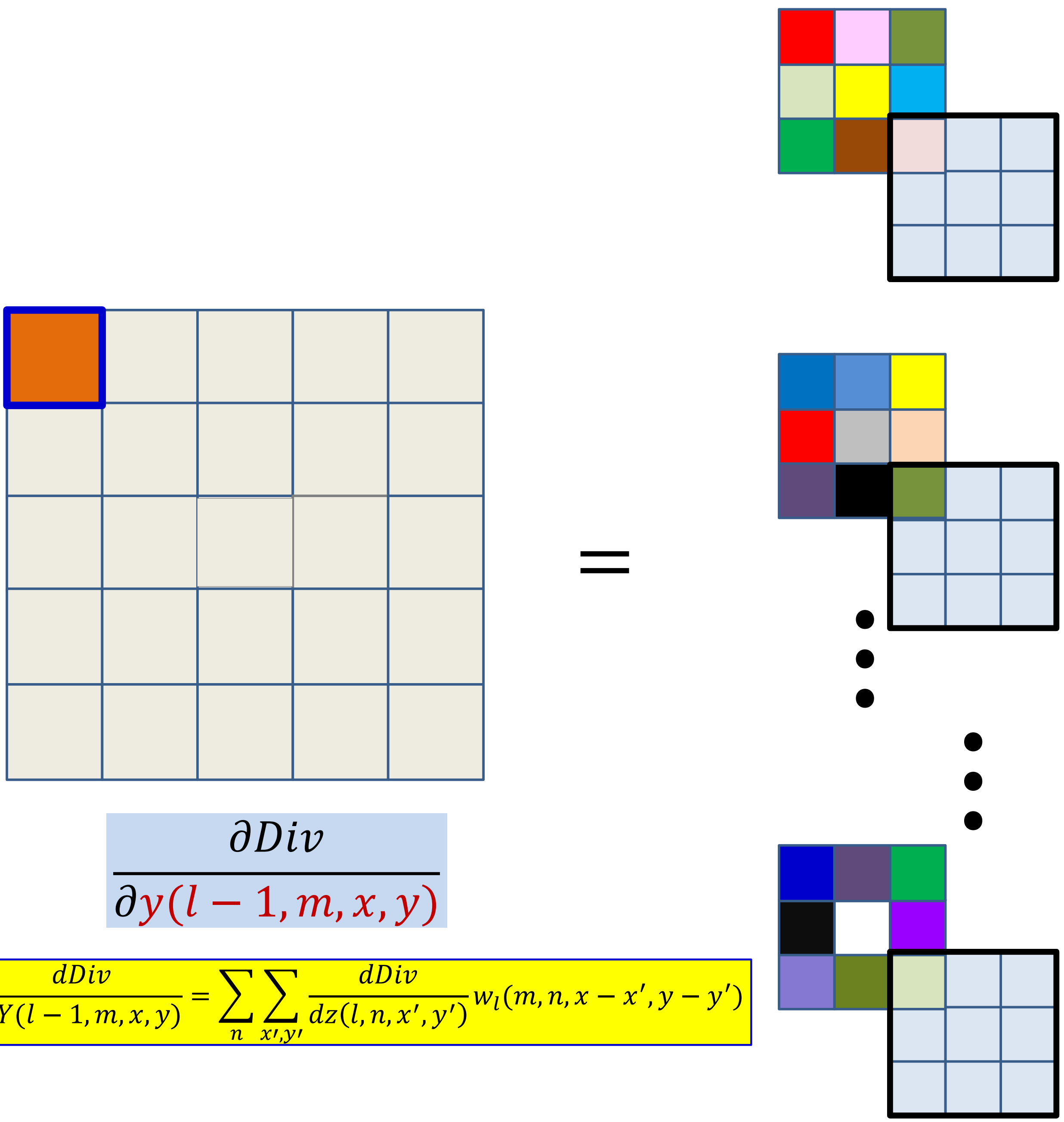
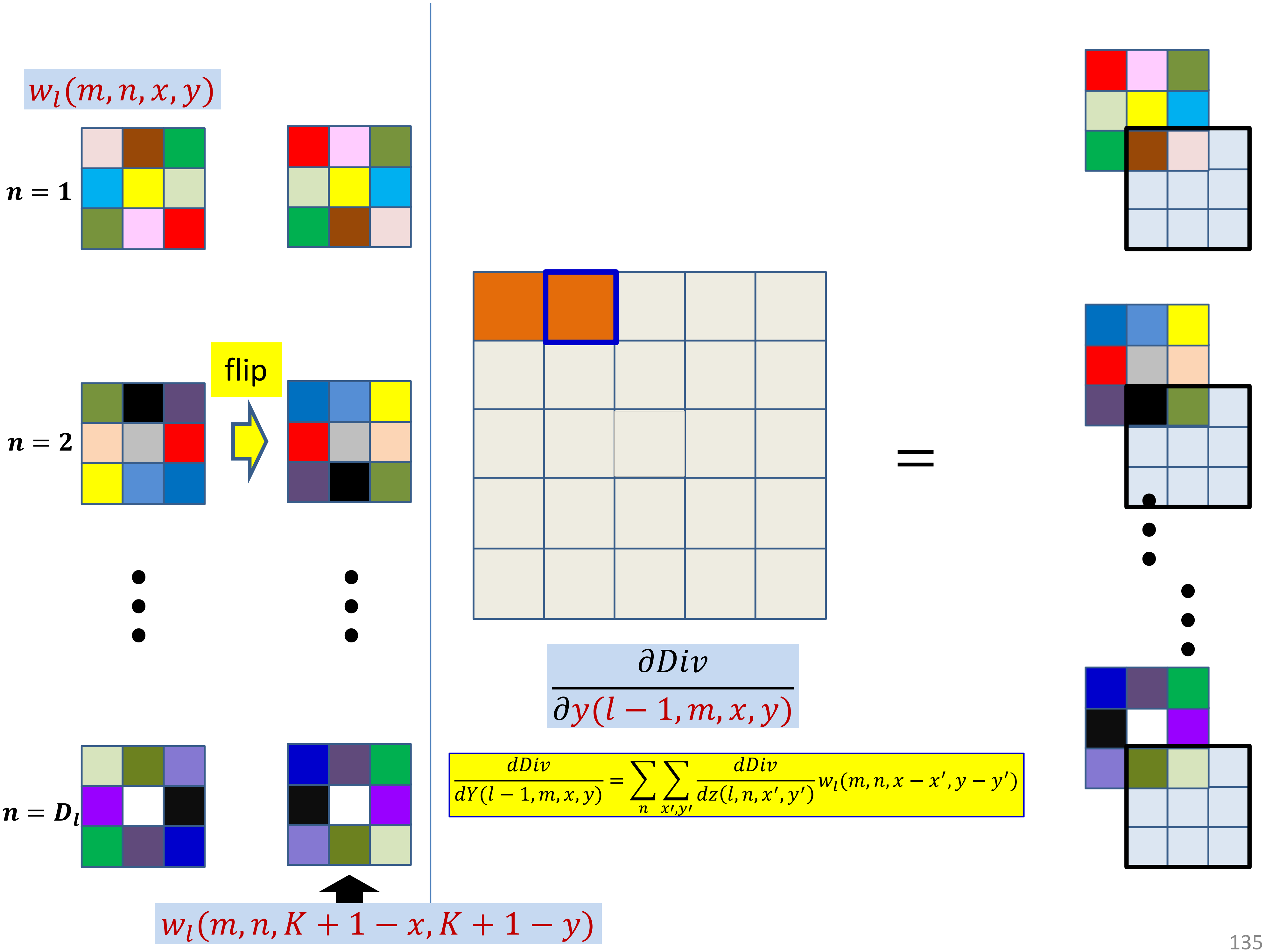




$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$



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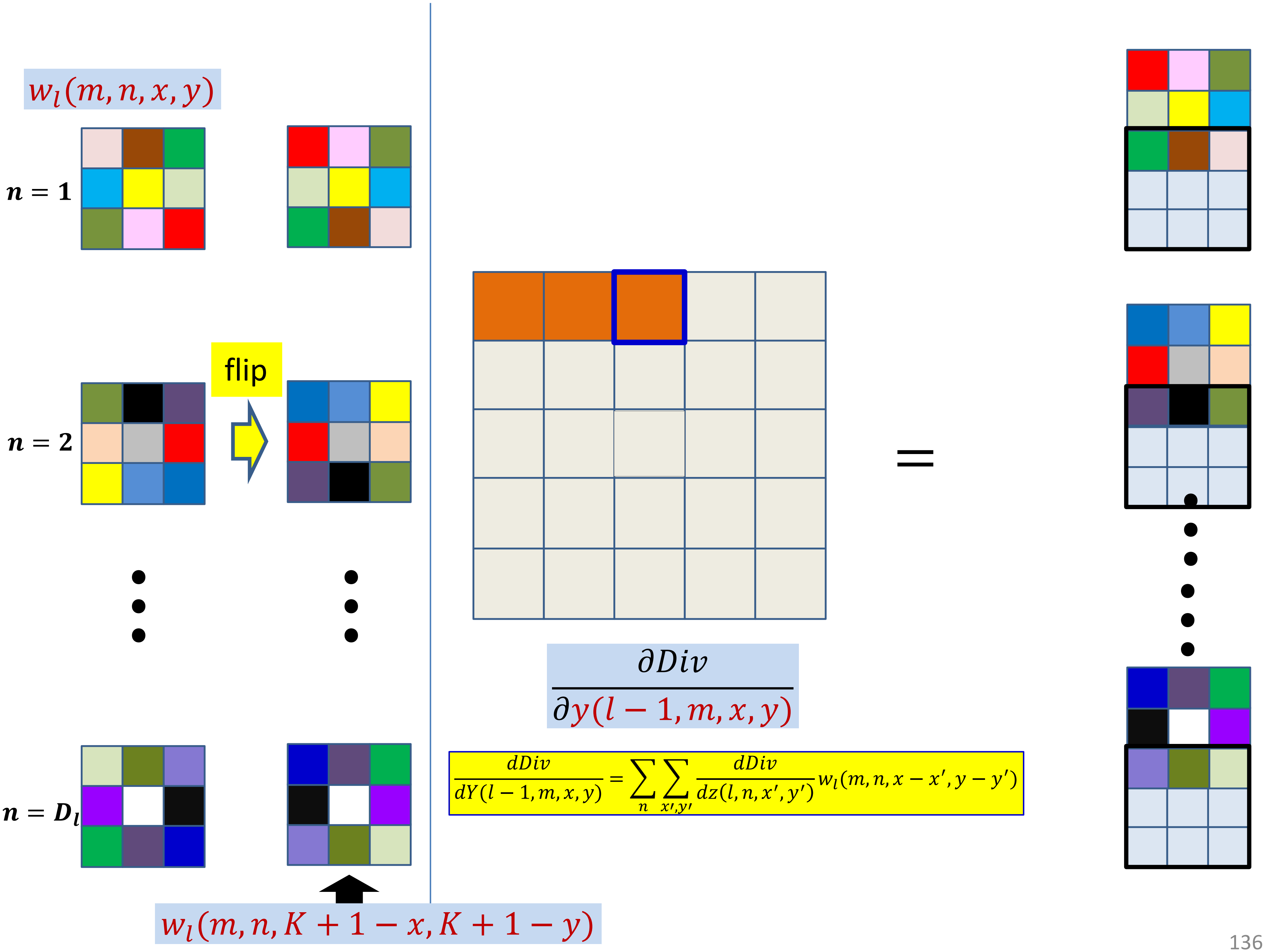

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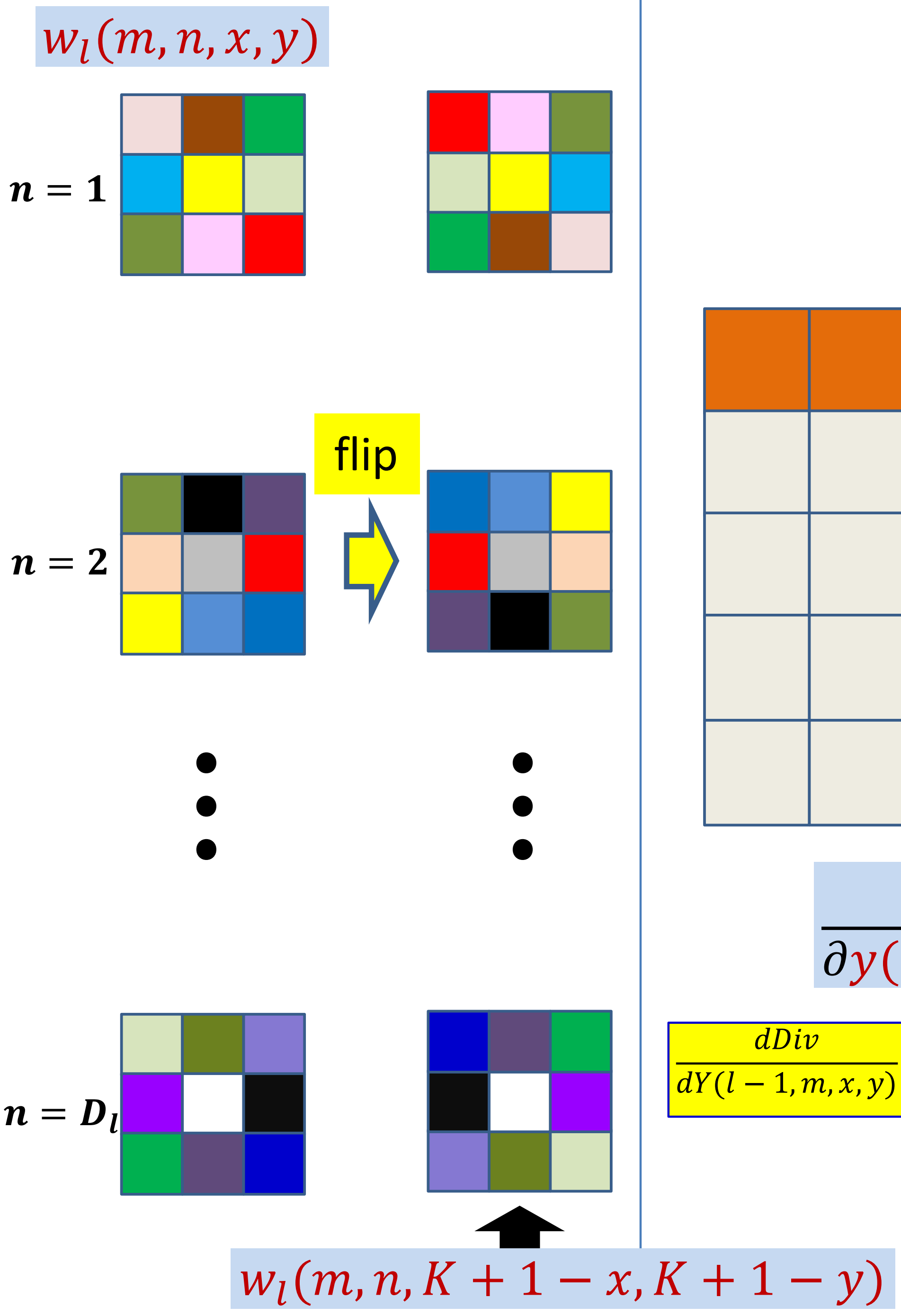
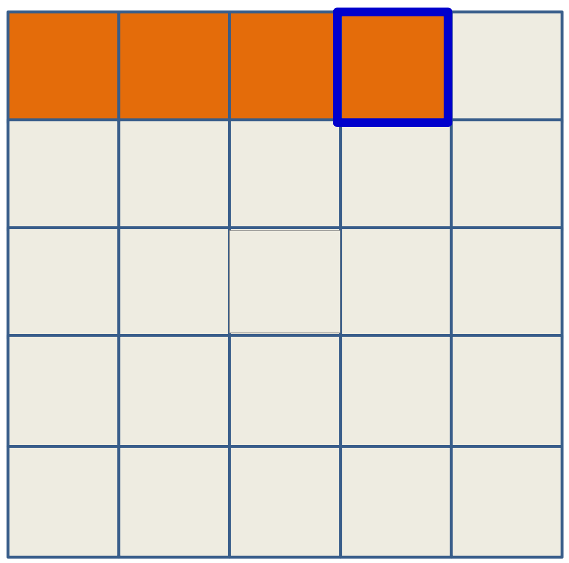
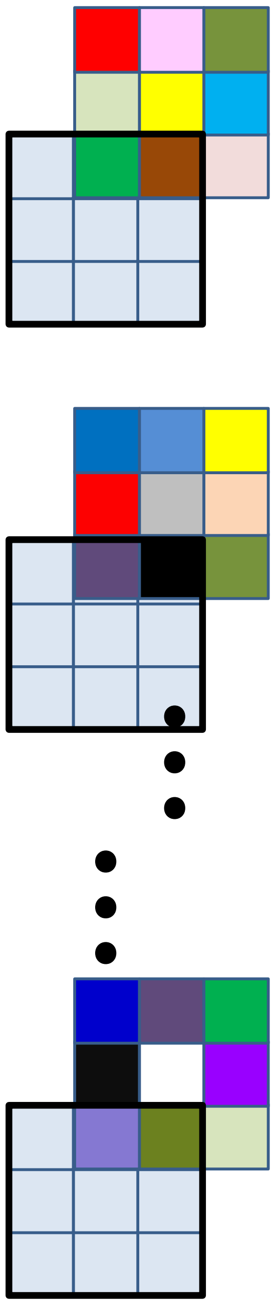


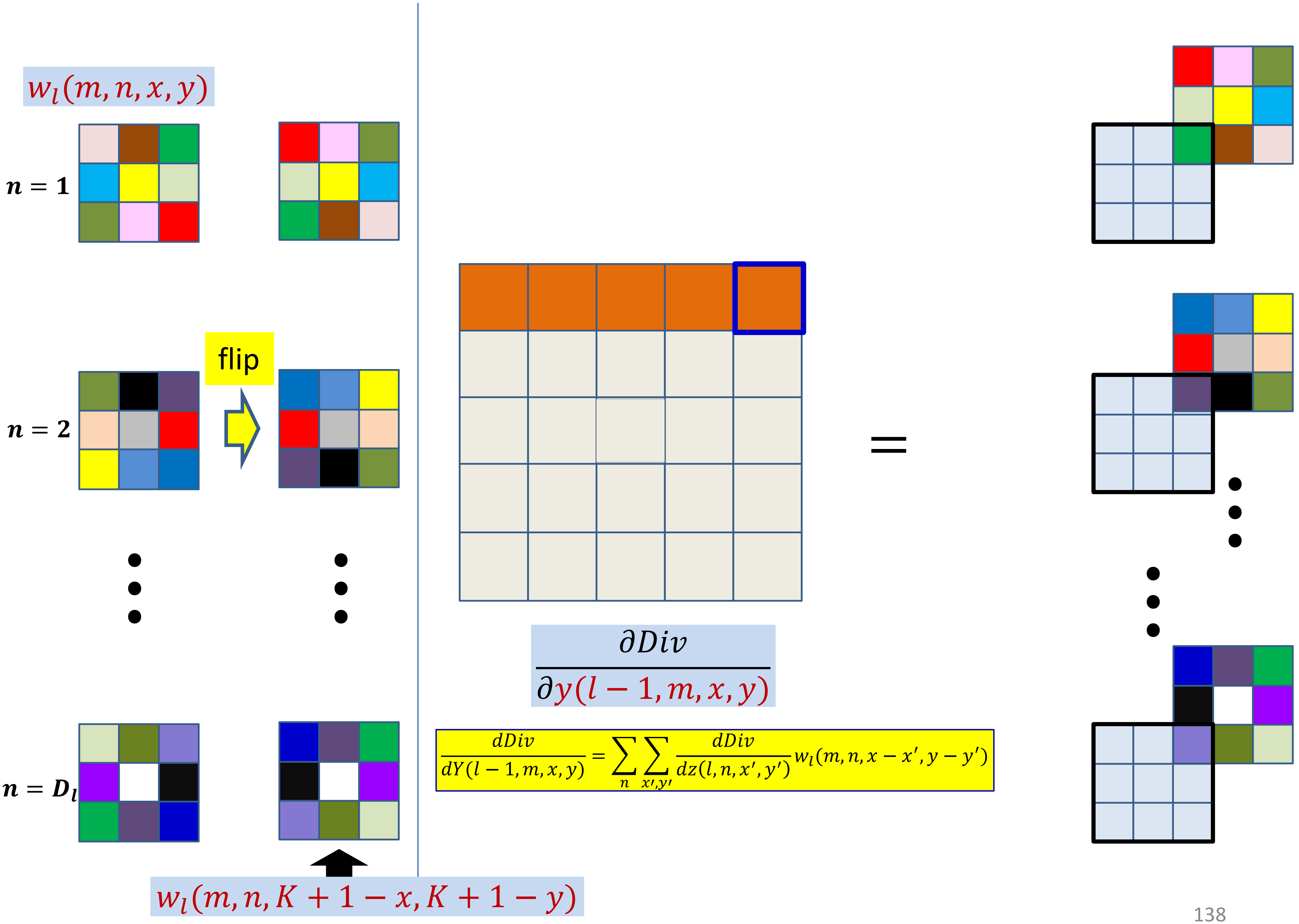


$$\frac{\partial Div}{\partial y(l-1,m,x,y)}$$

=



$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_{n}\sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$



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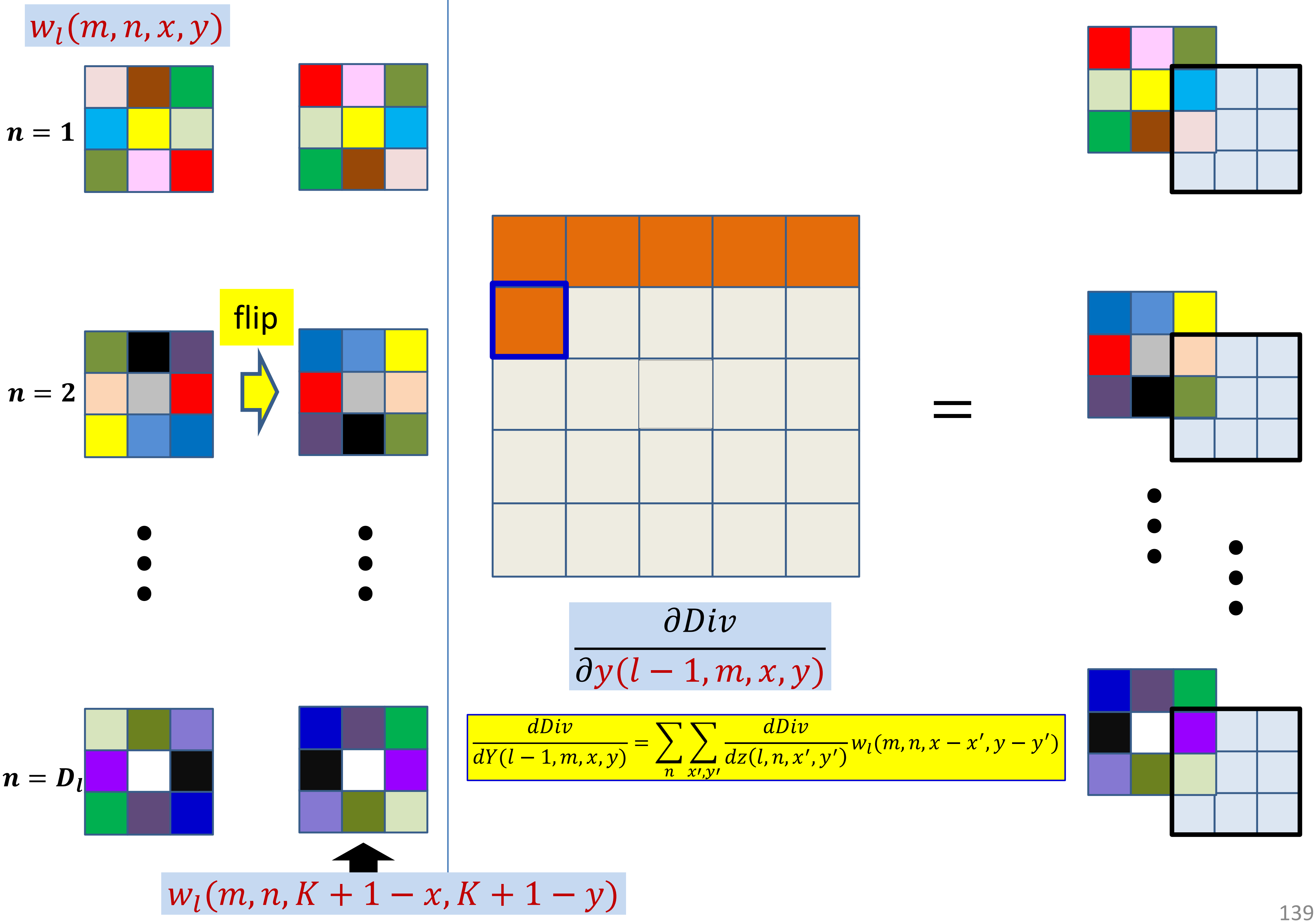

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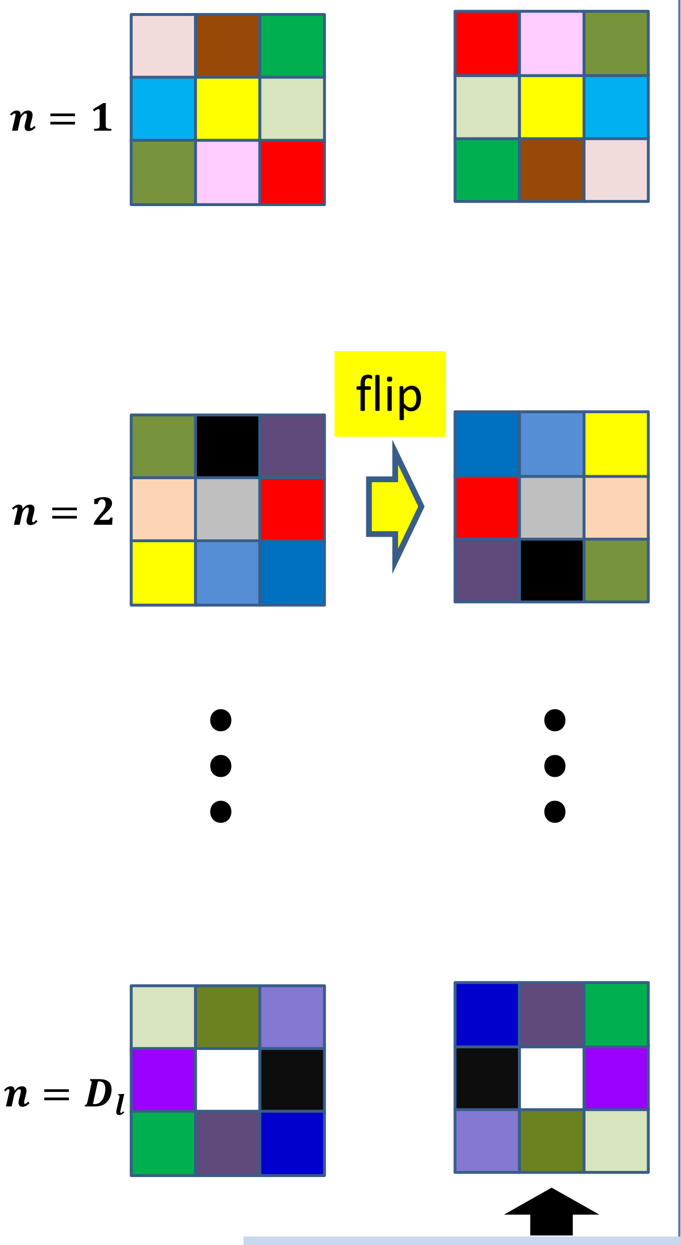
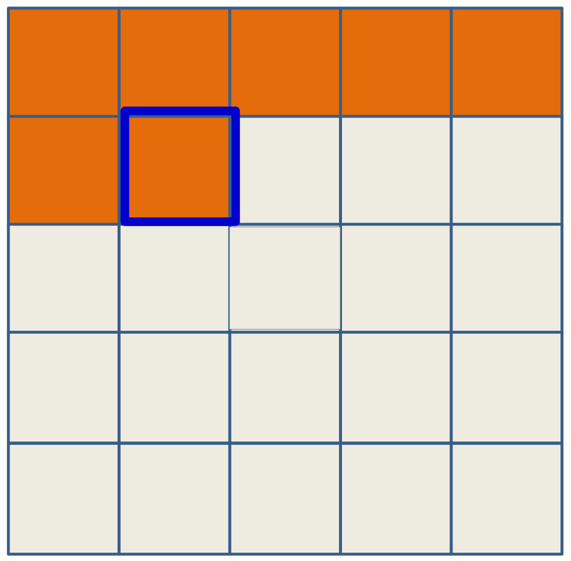
$w_l(m,n,x,y)$

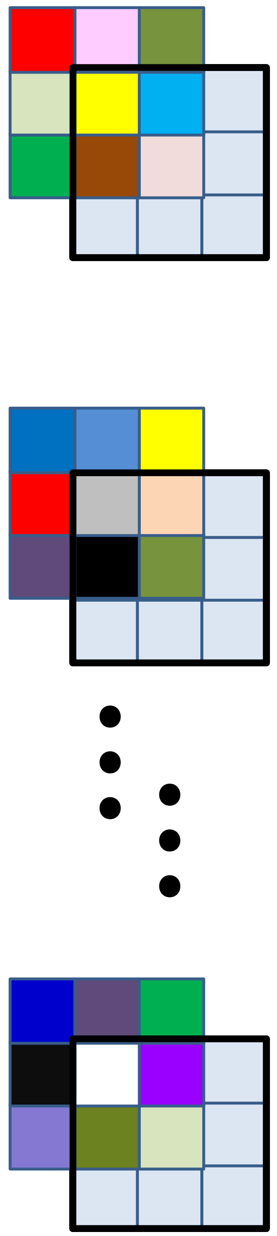


$w_l(m,n,K+1-x,K+1-y)$



$$\frac{\partial Div}{\partial y(l-1,m,x,y)}$$

$=$



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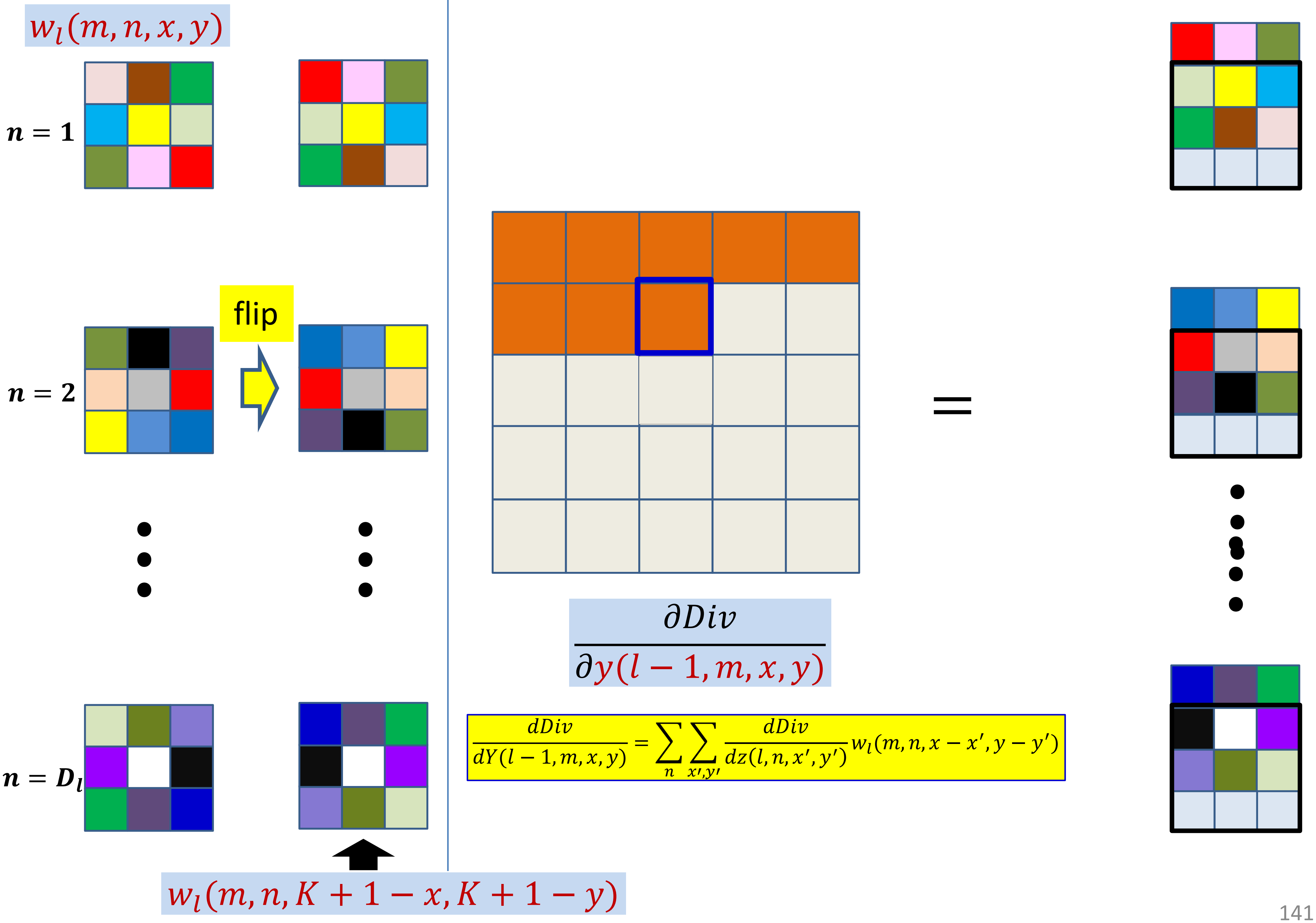

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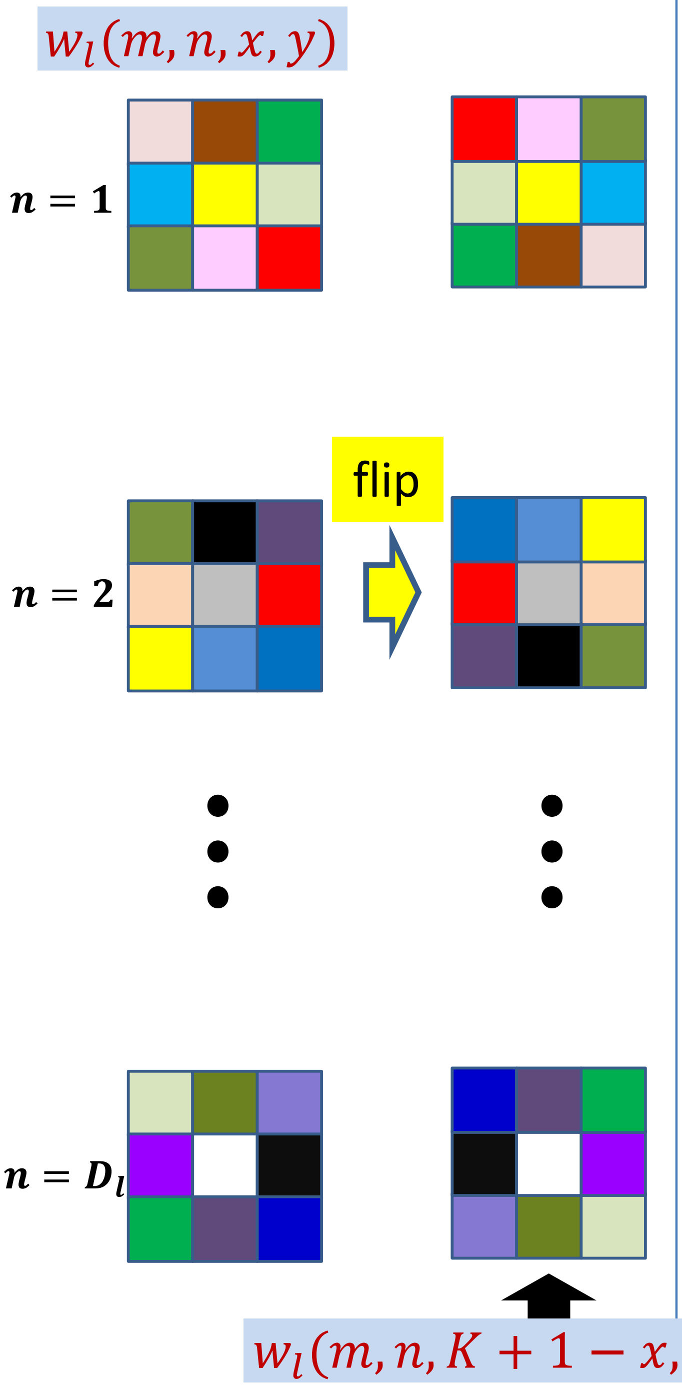

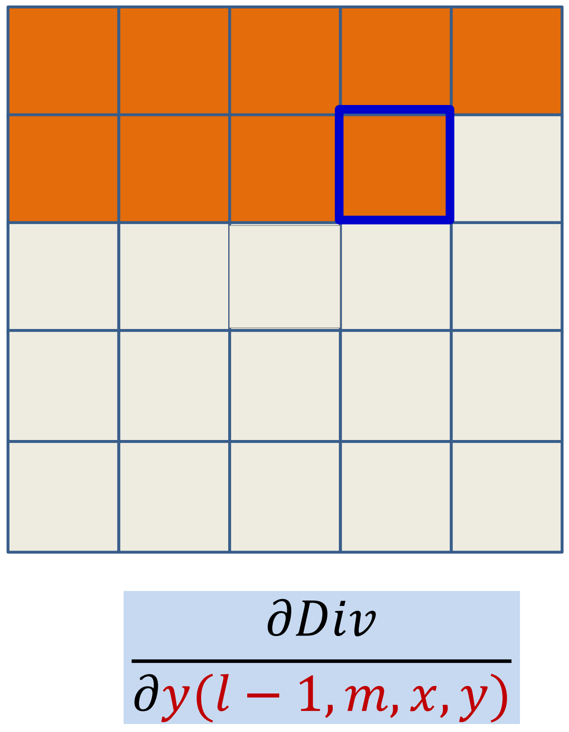

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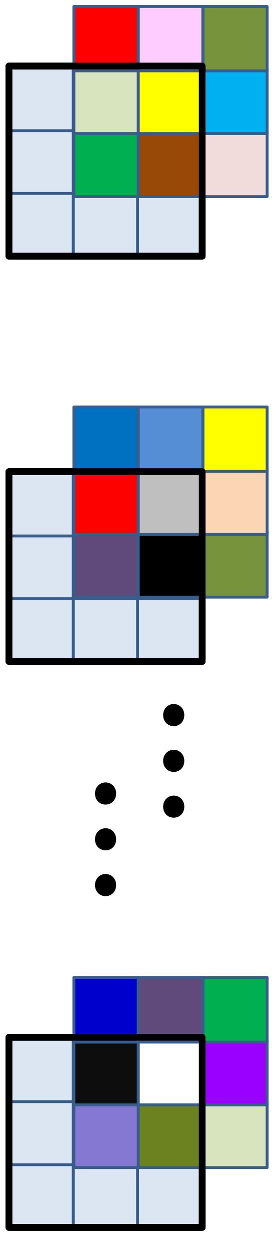

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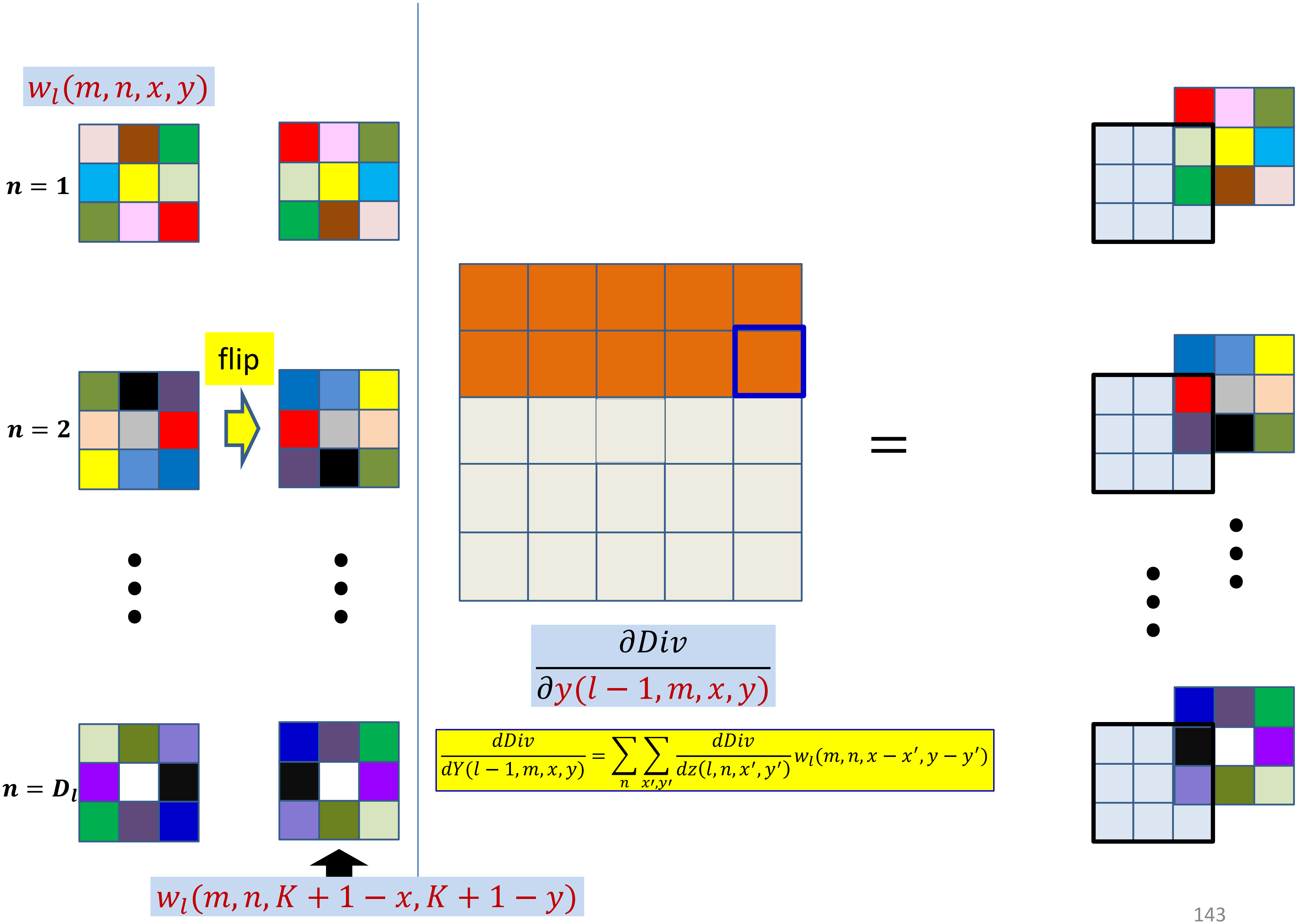
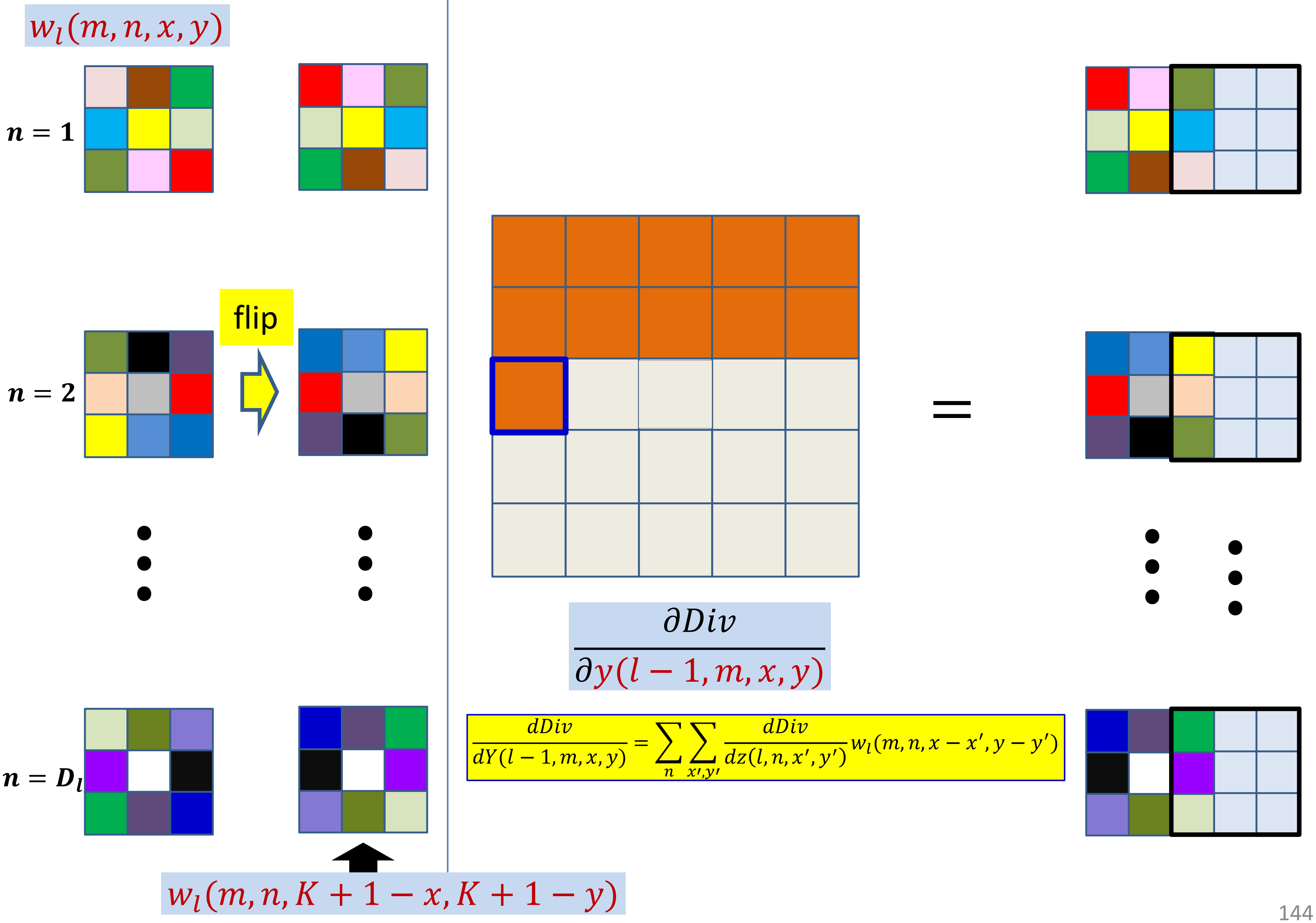


$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$



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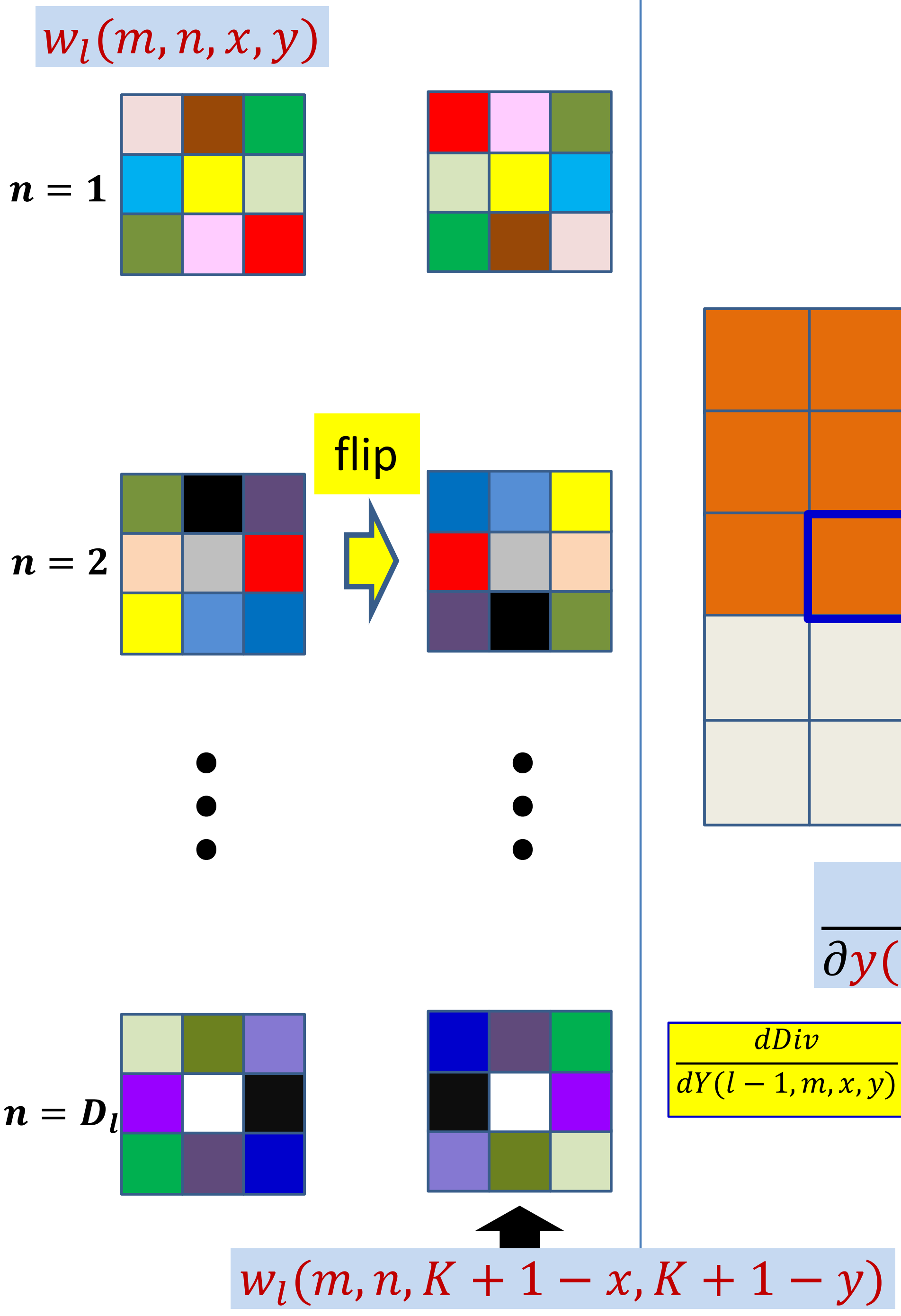
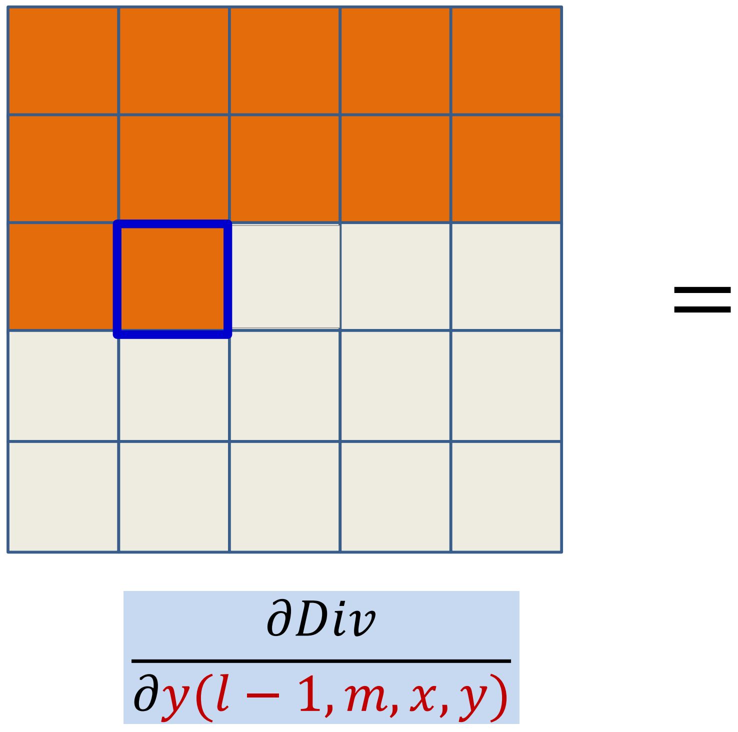
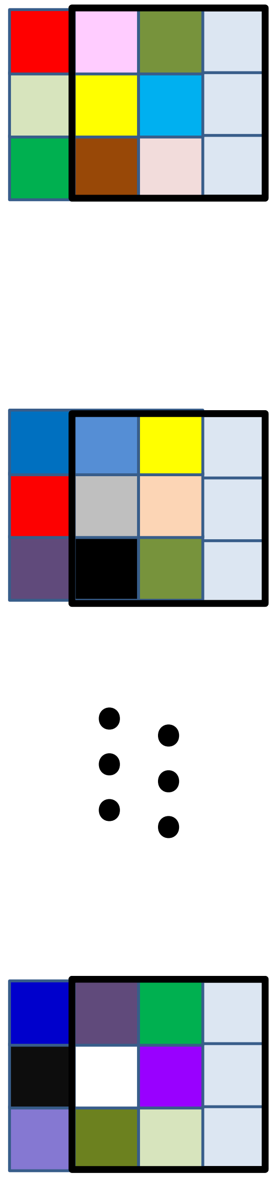
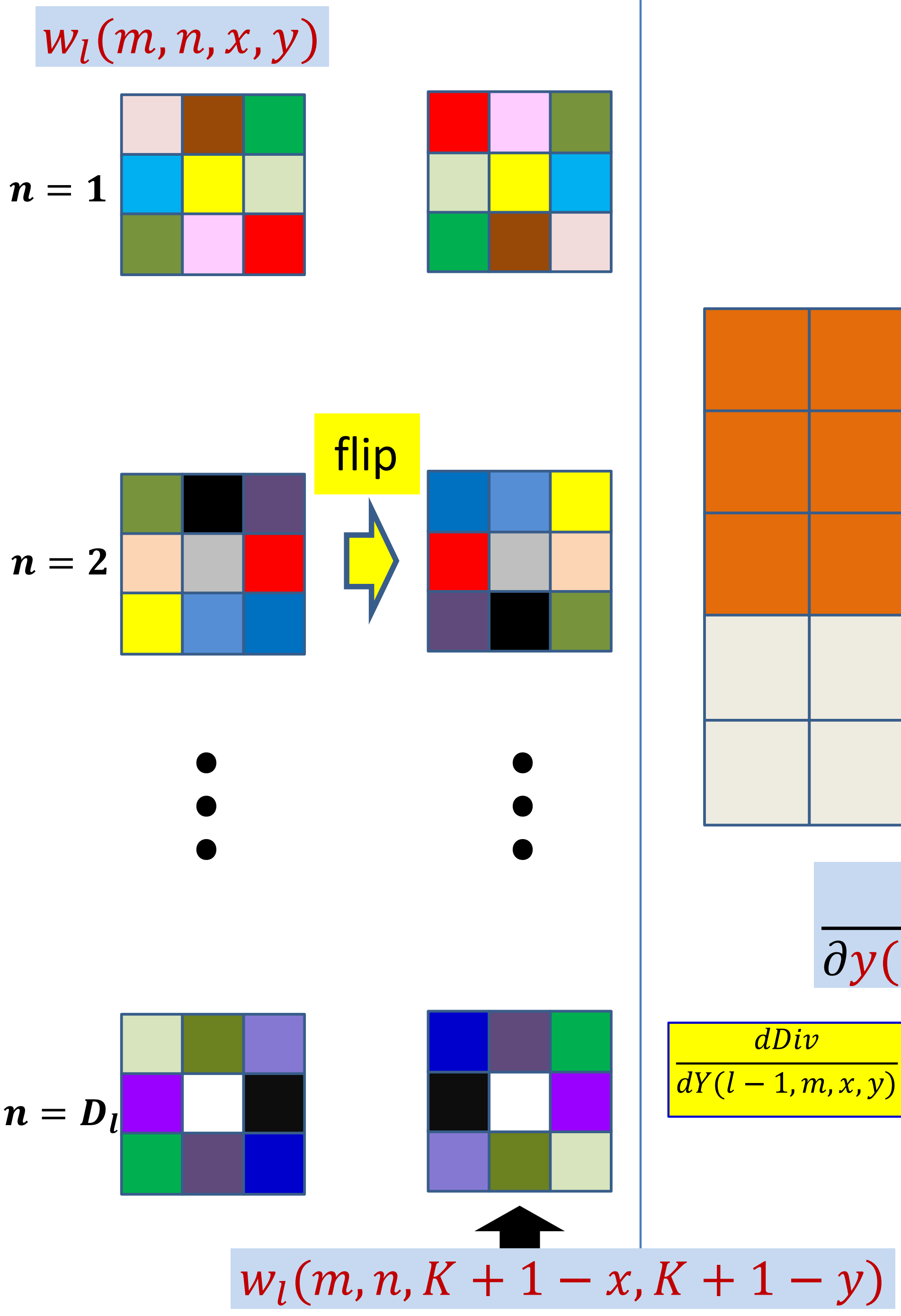
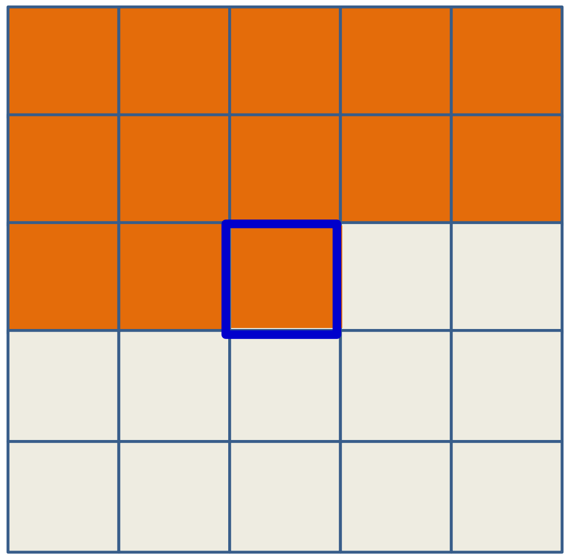






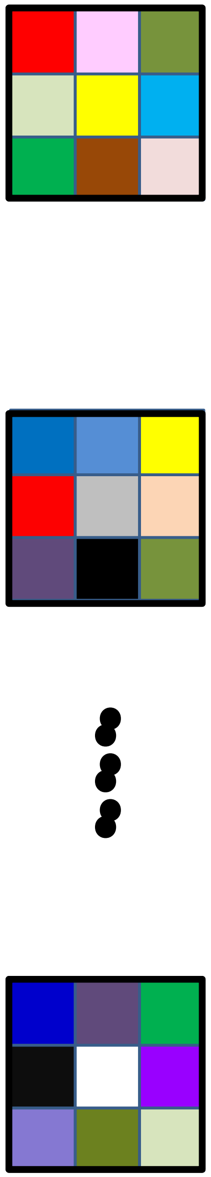
$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_{n} \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$





$$\frac{\partial Div}{\partial y(l-1,m,x,y)}$$

=



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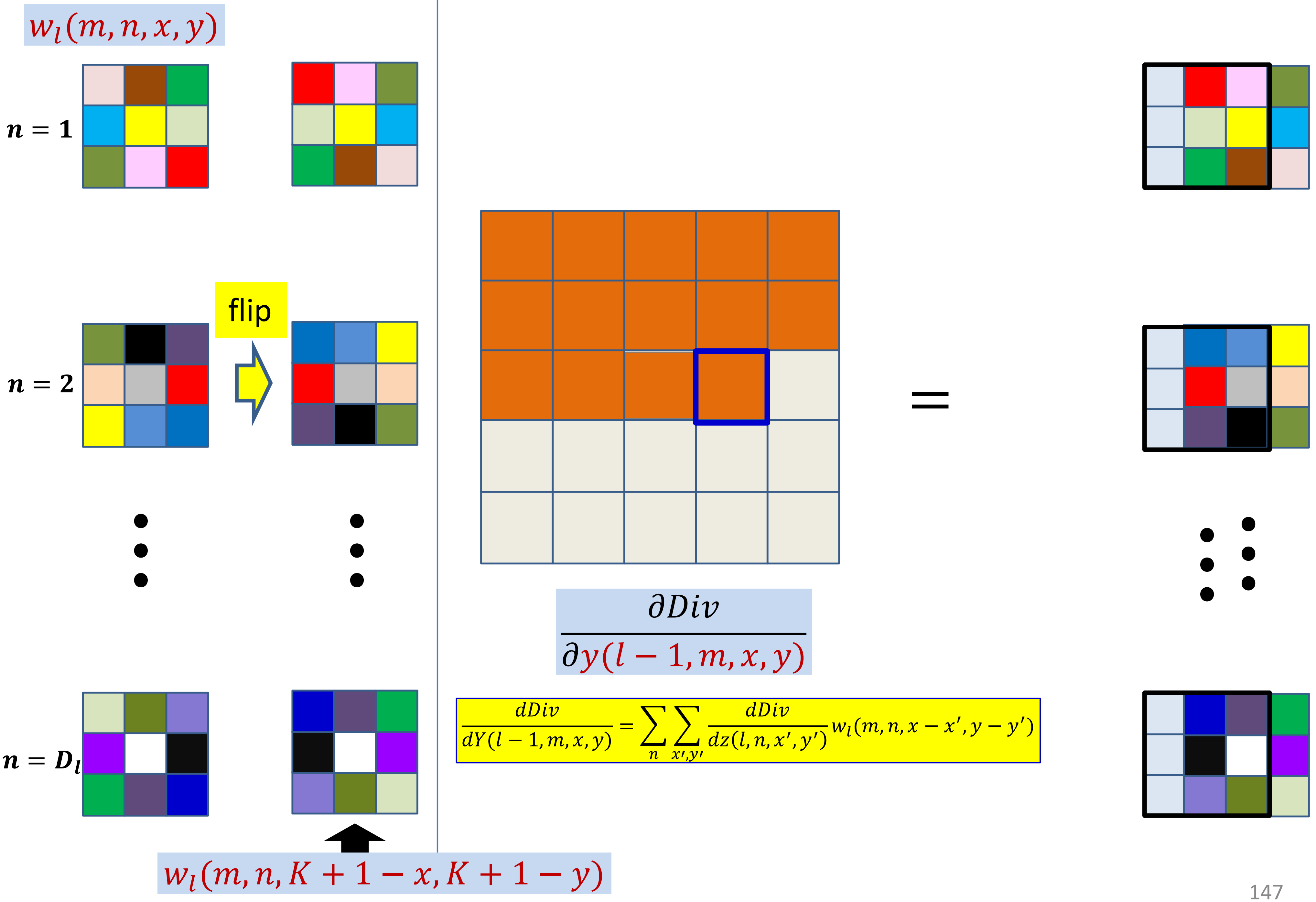

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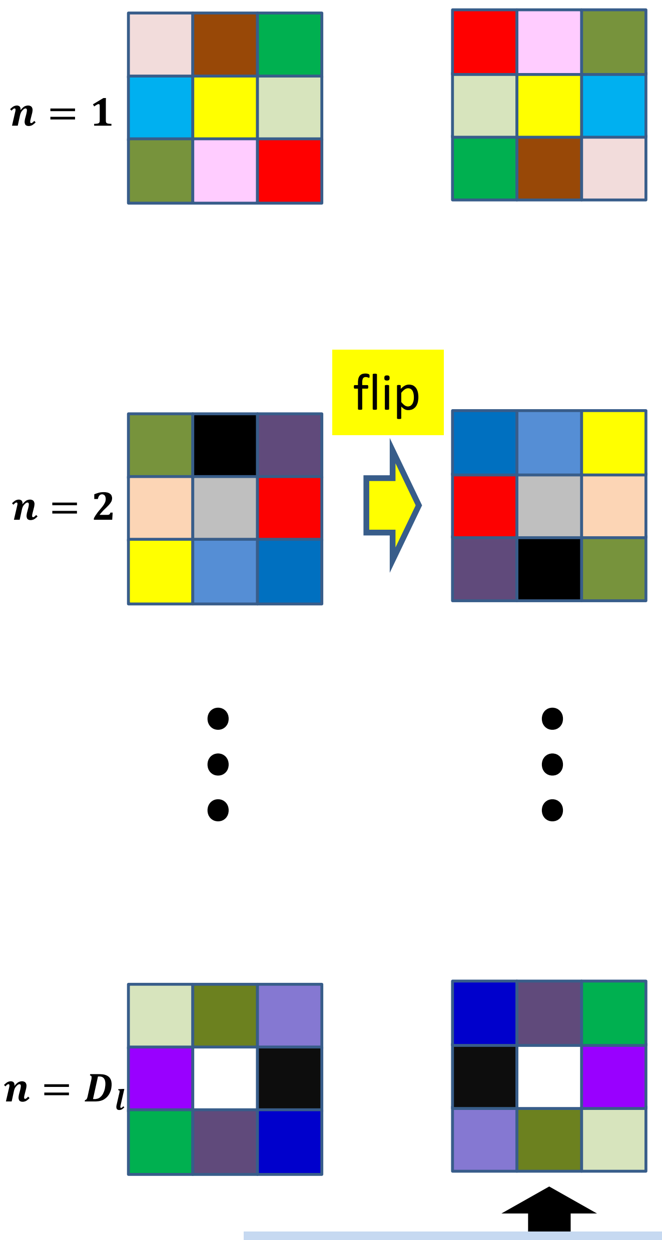
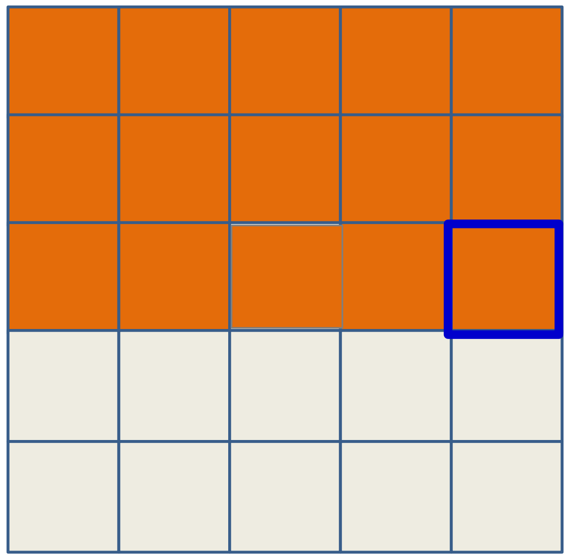
$w_l(m,n,x,y)$



$w_l(m,n,K+1-x,K+1-y)$



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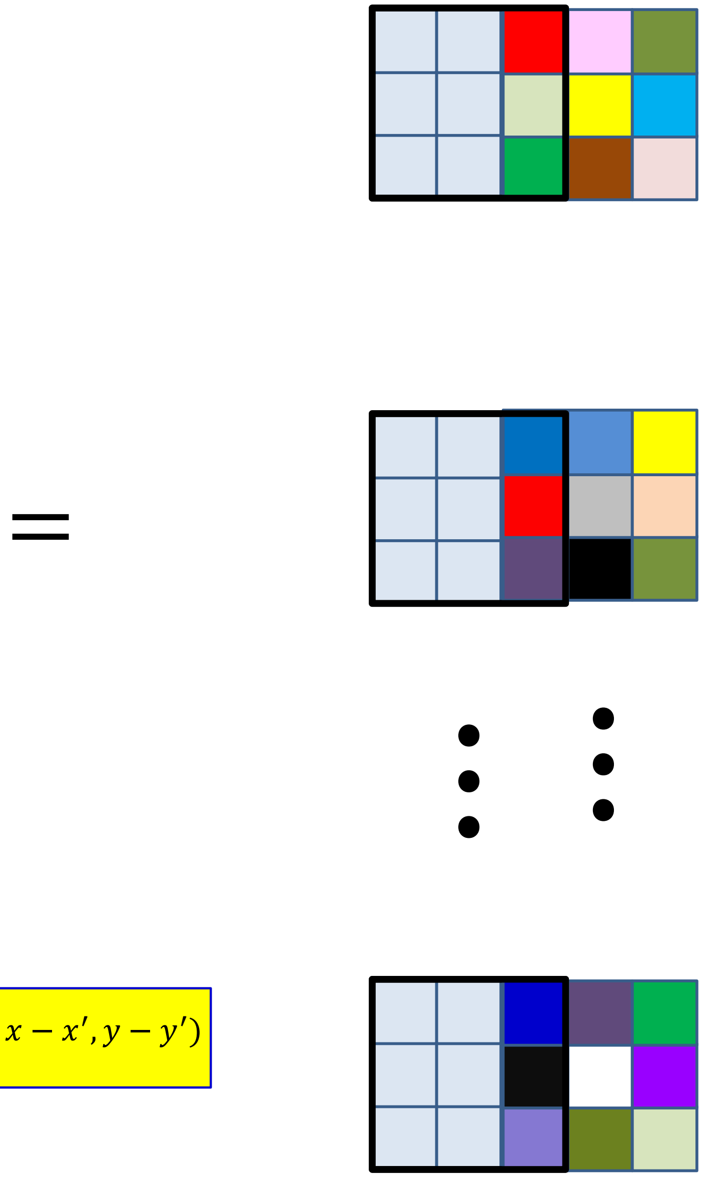

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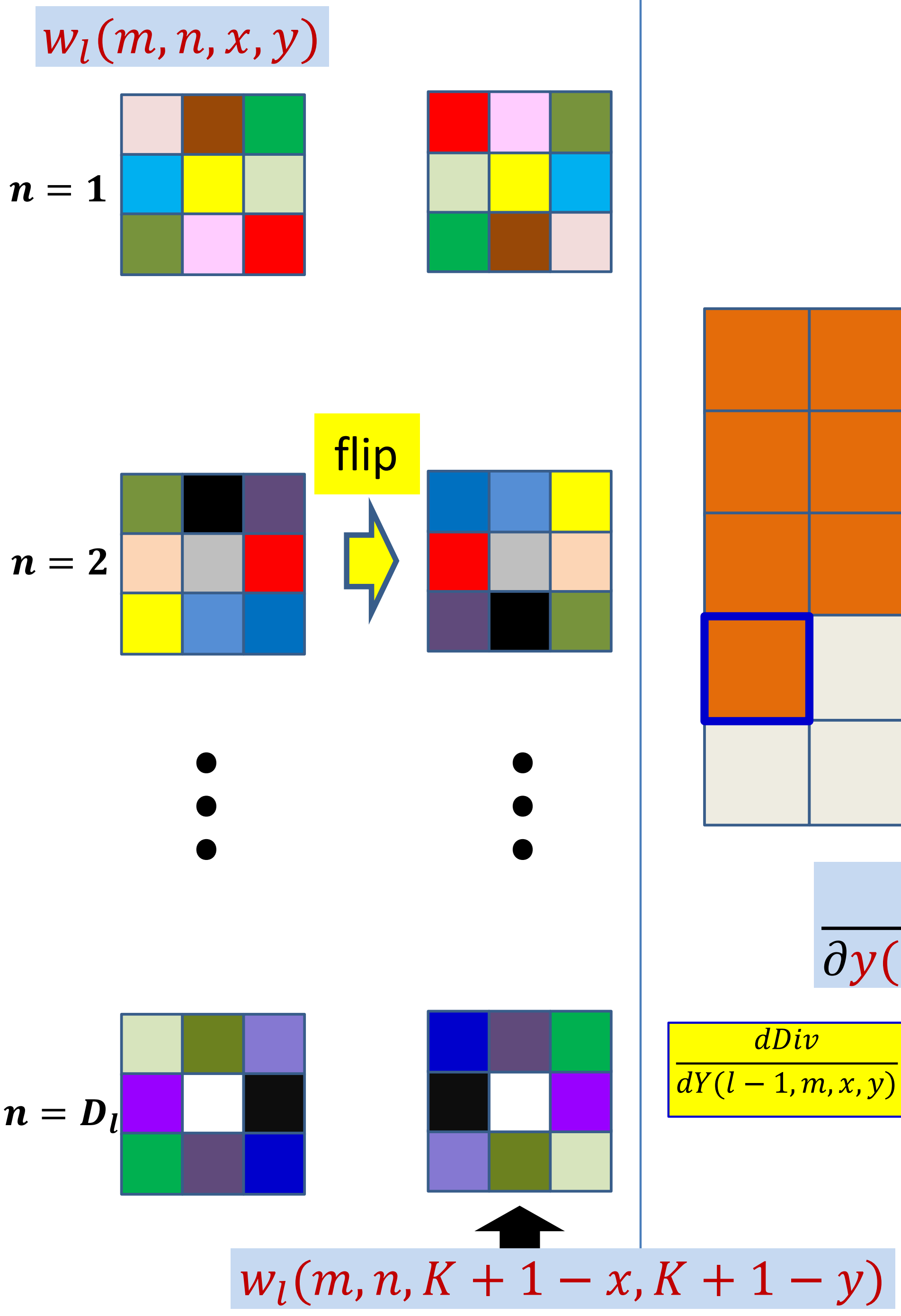
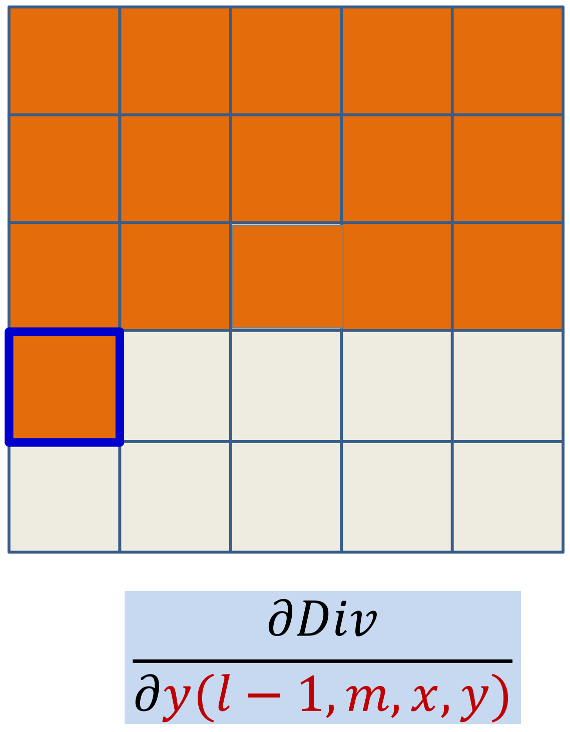
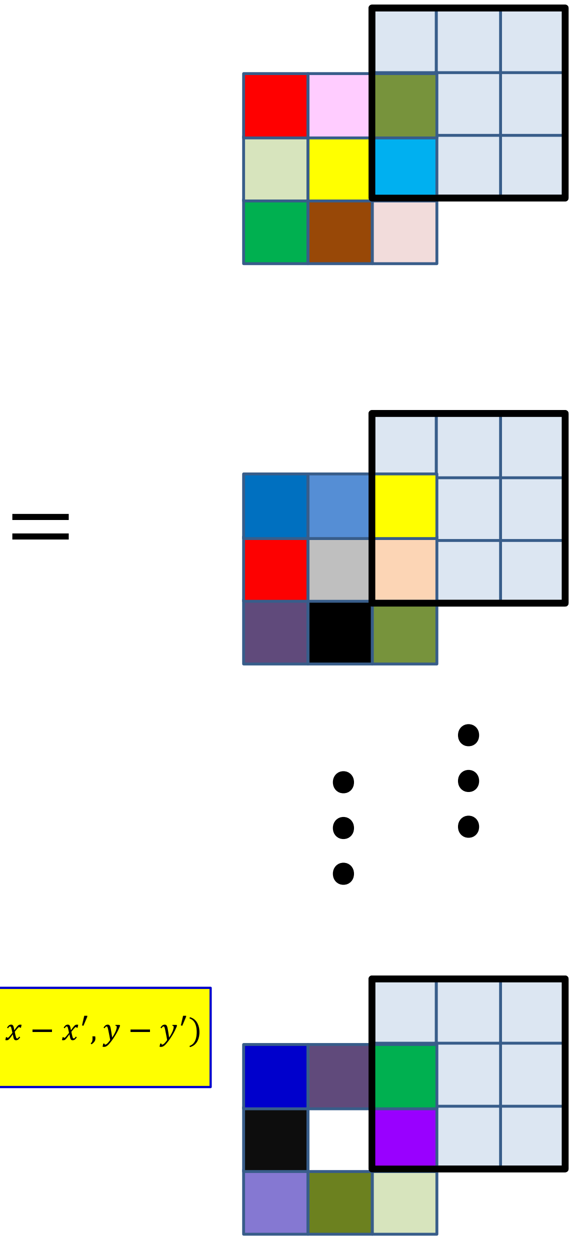
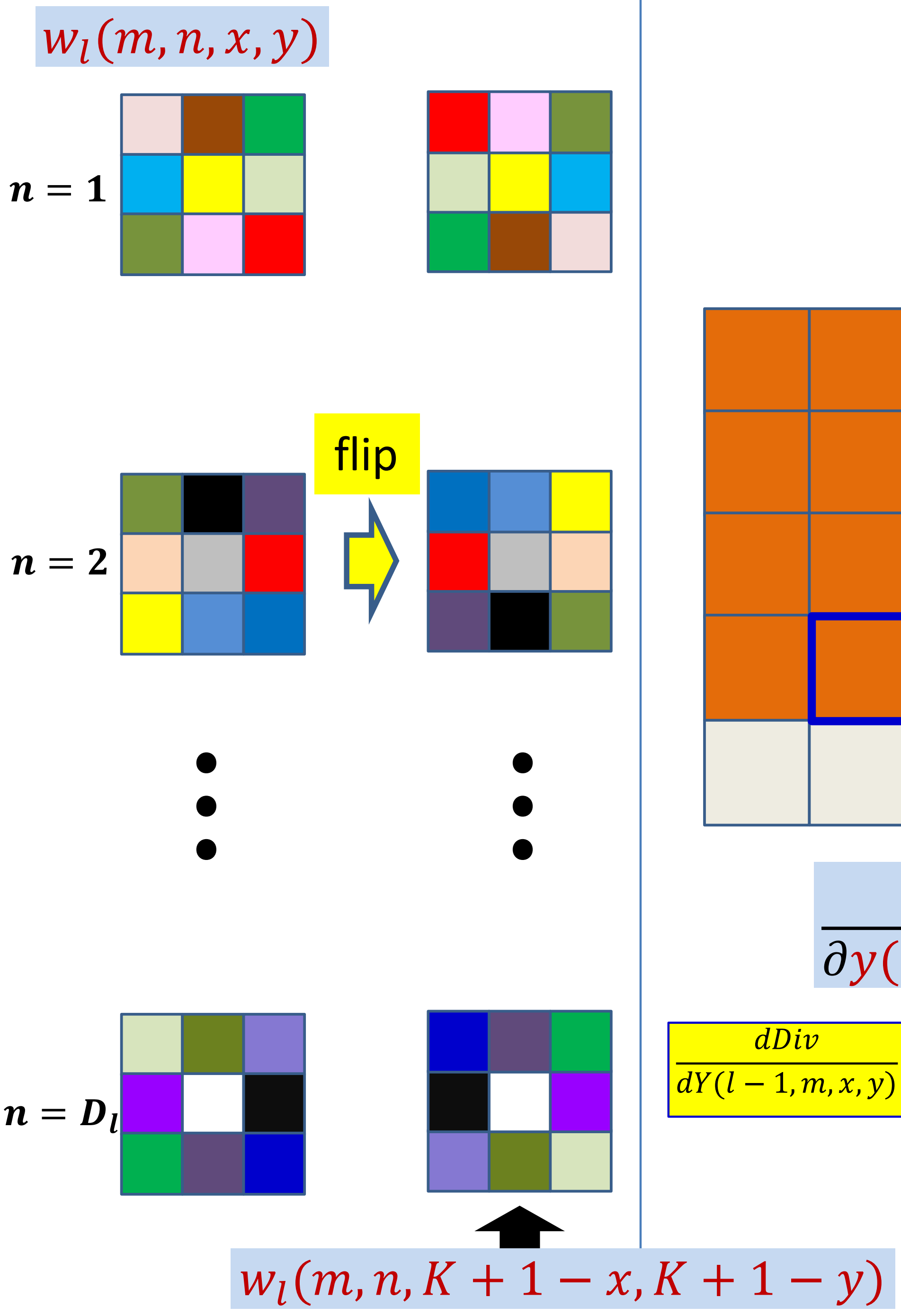
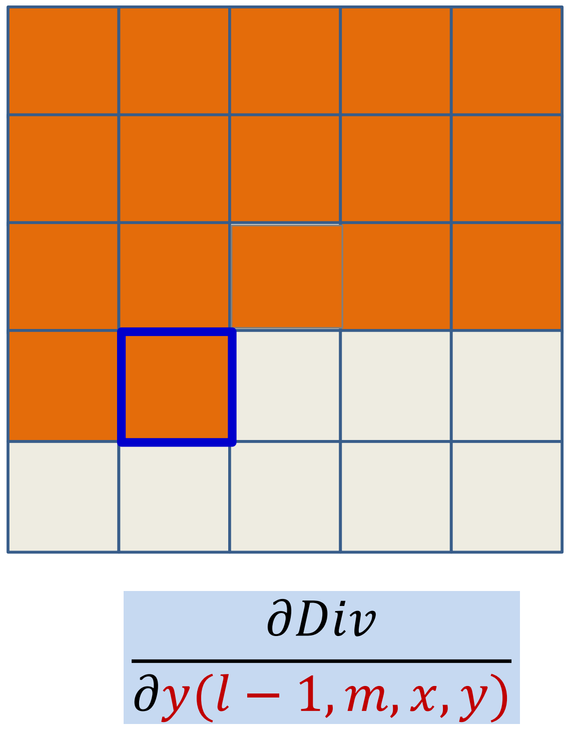






$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_{n}\sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$





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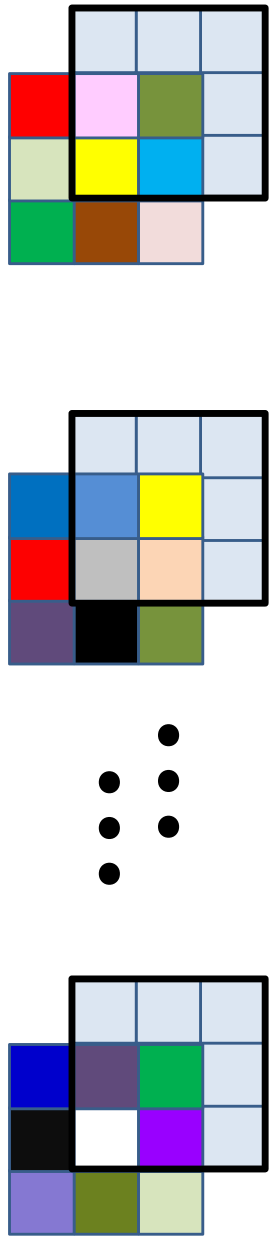
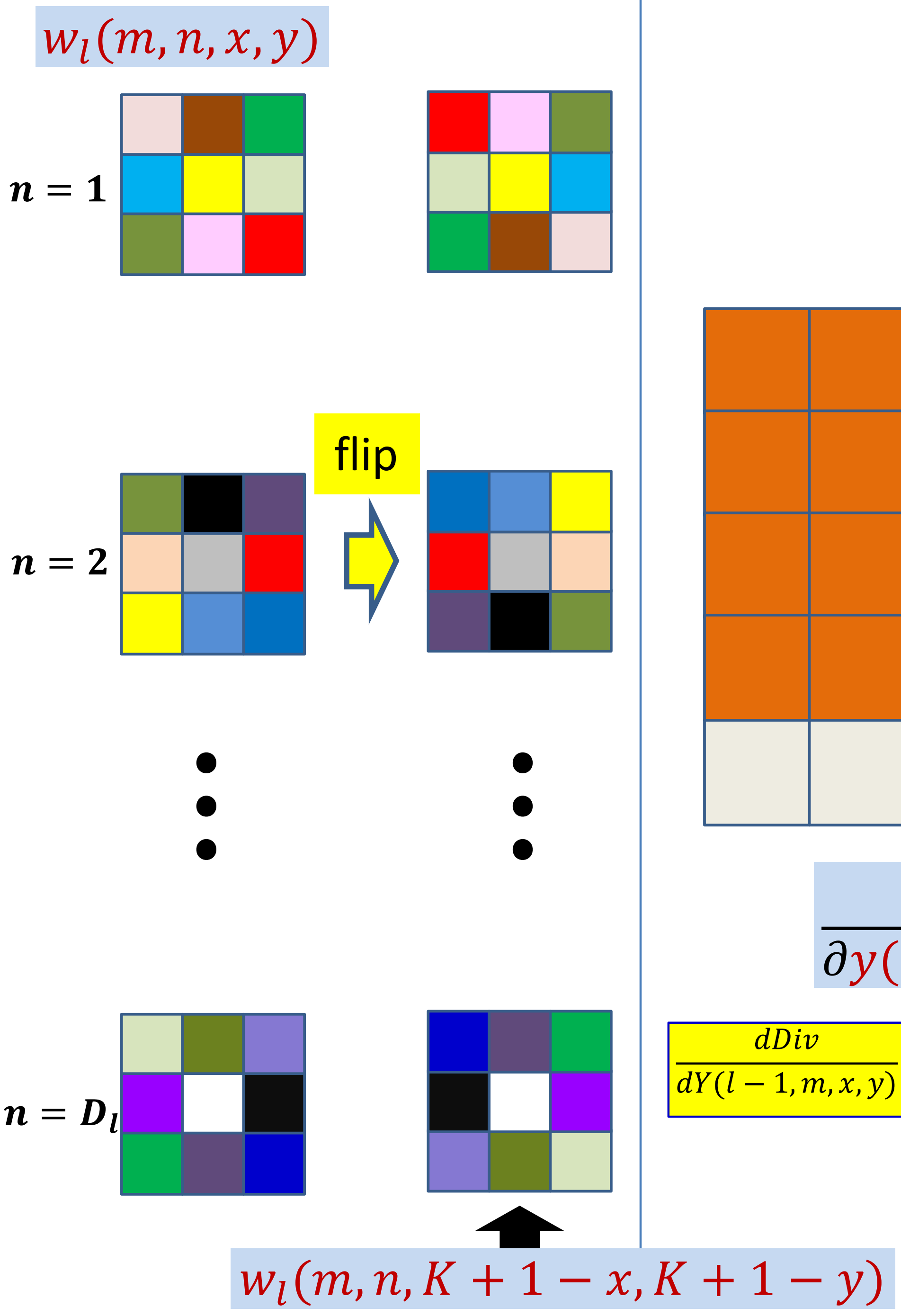
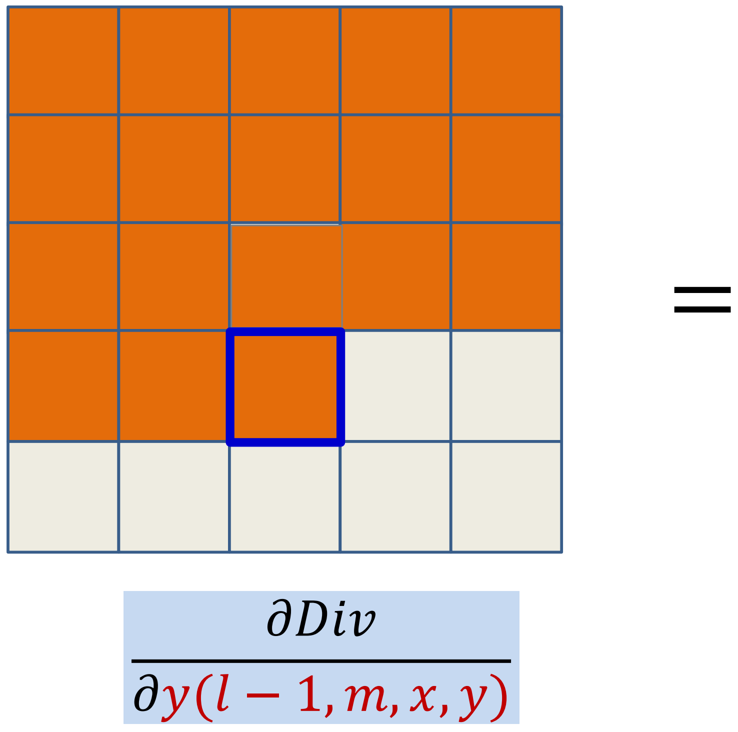
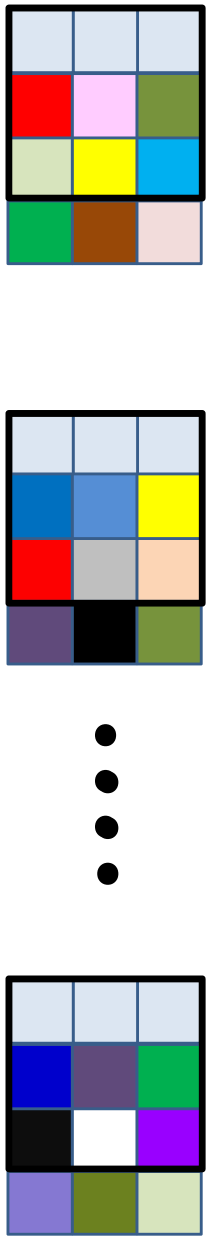


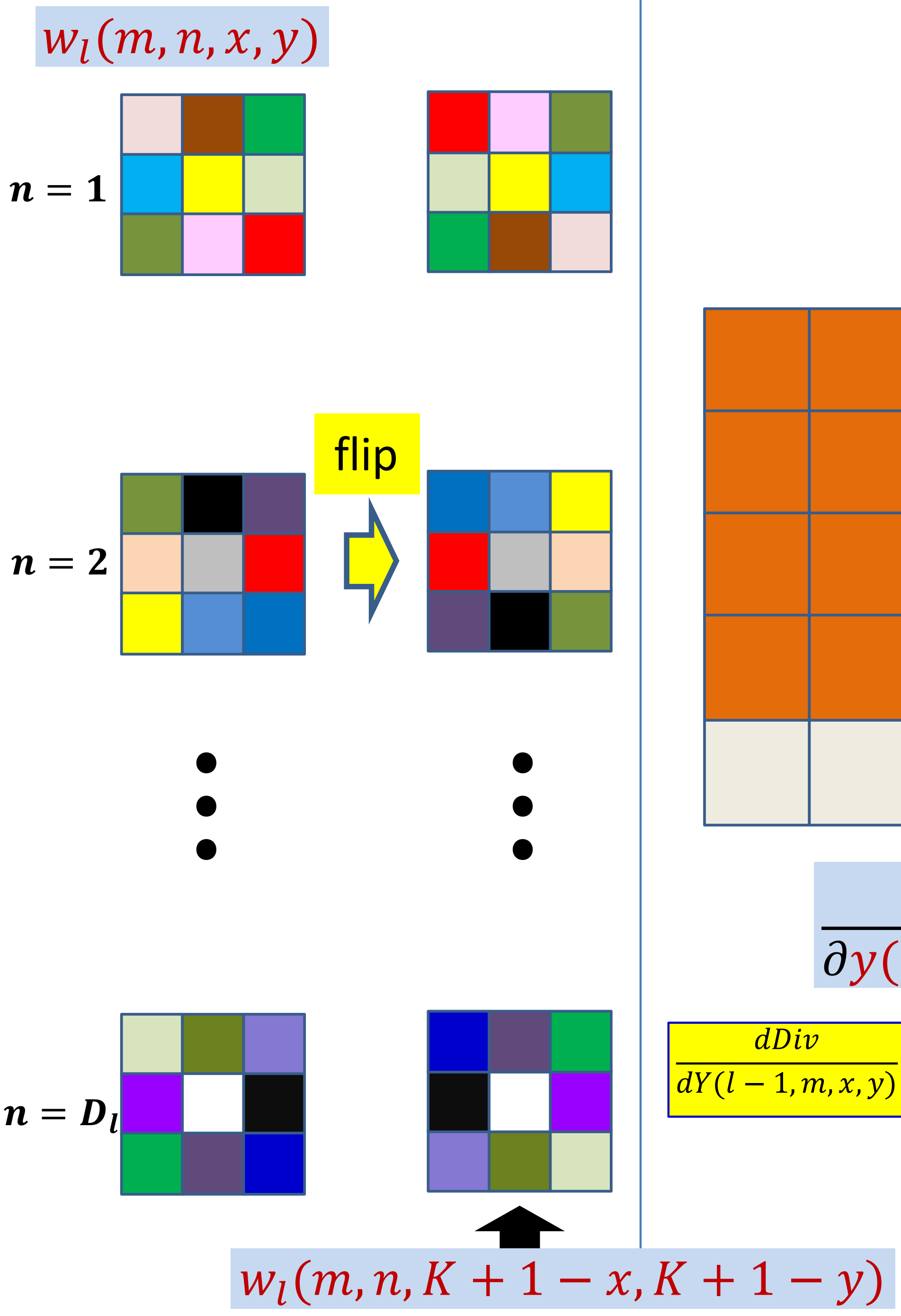
$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$

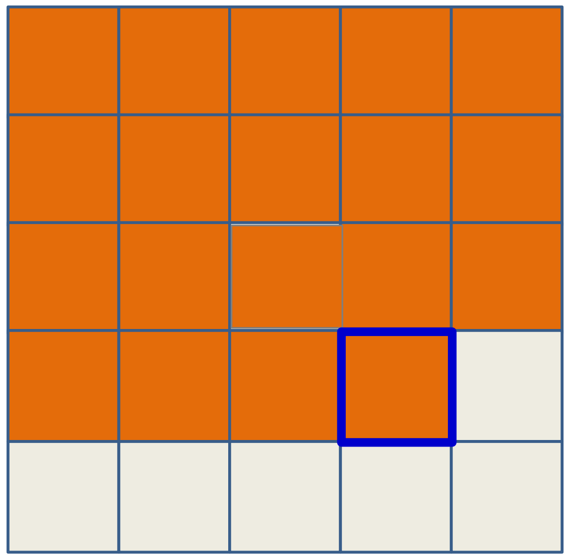






$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_{n}\sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$





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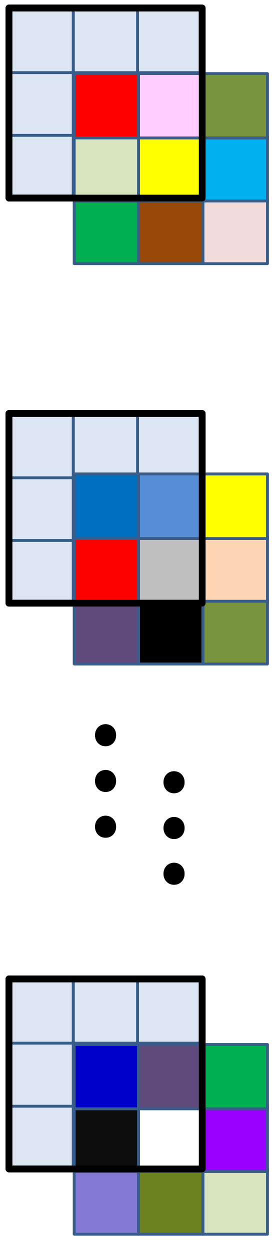

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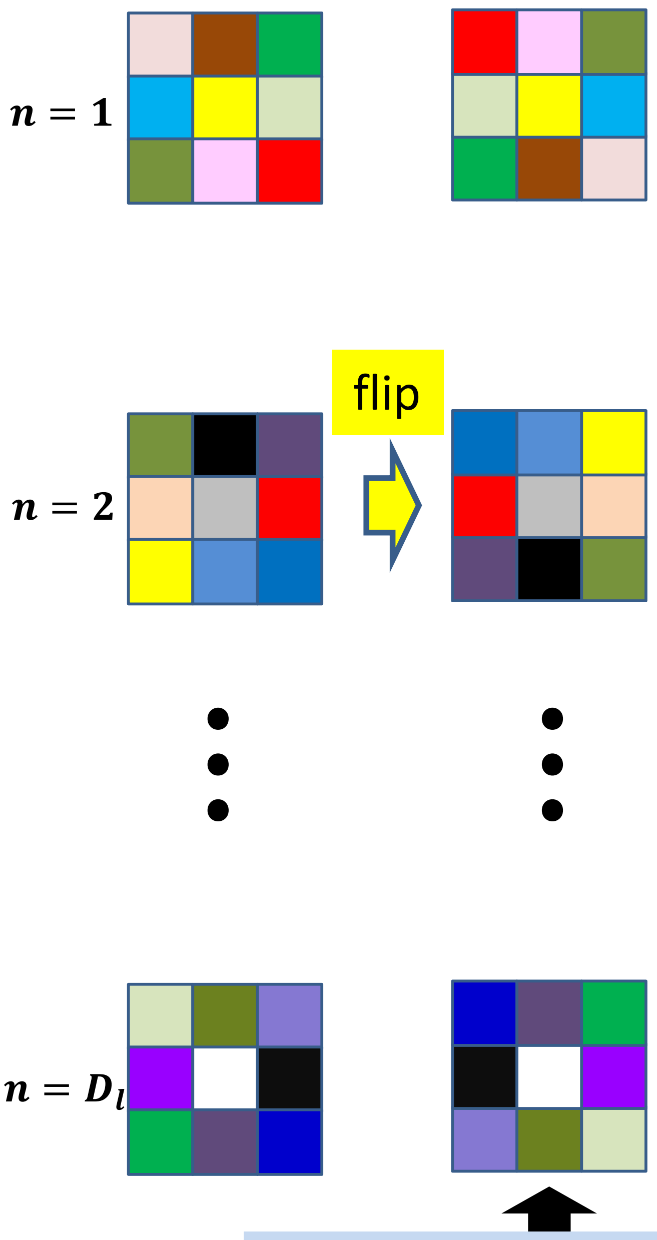
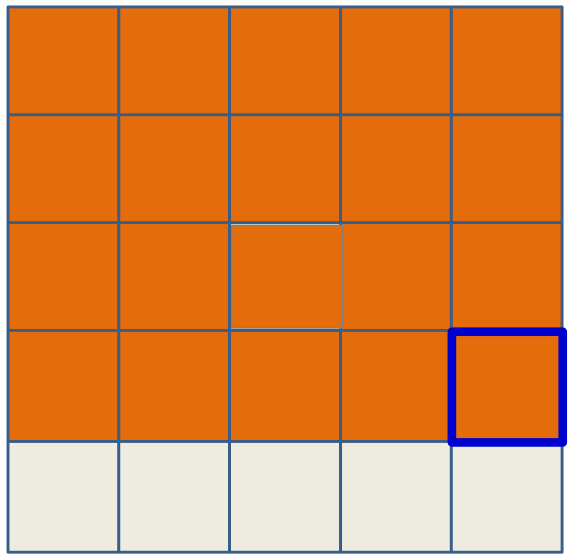
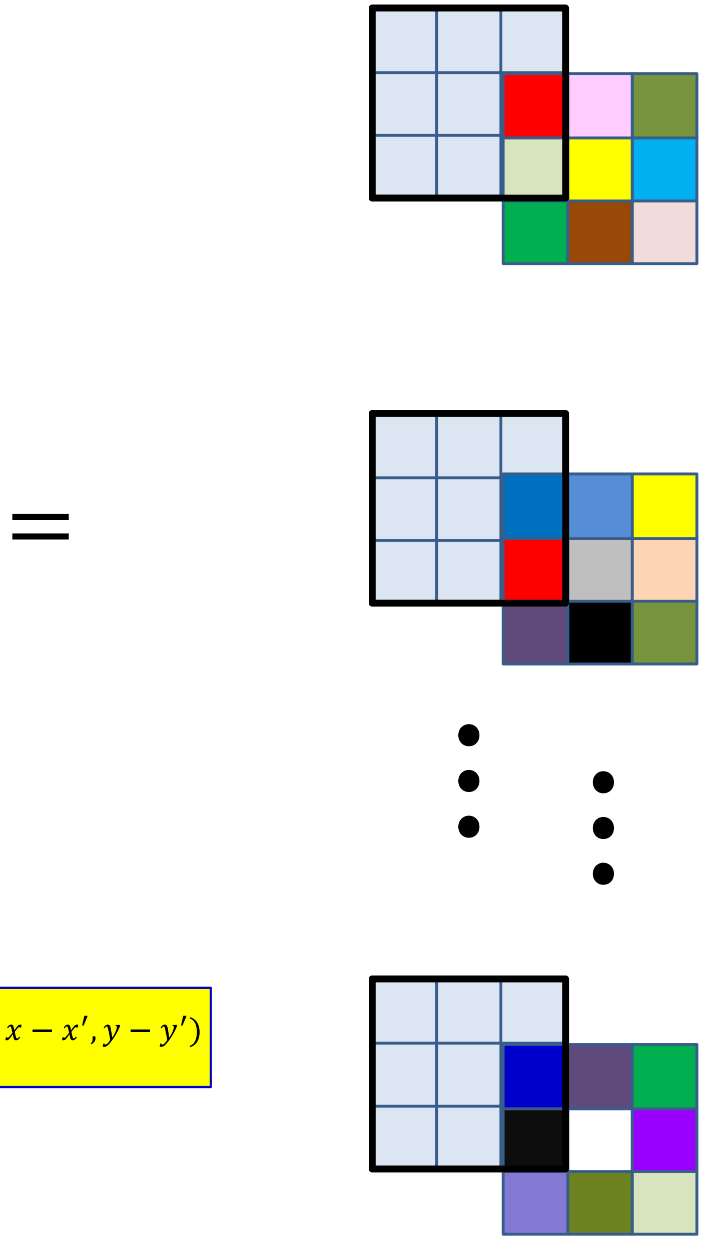
$w_l(m, n, x, y)$



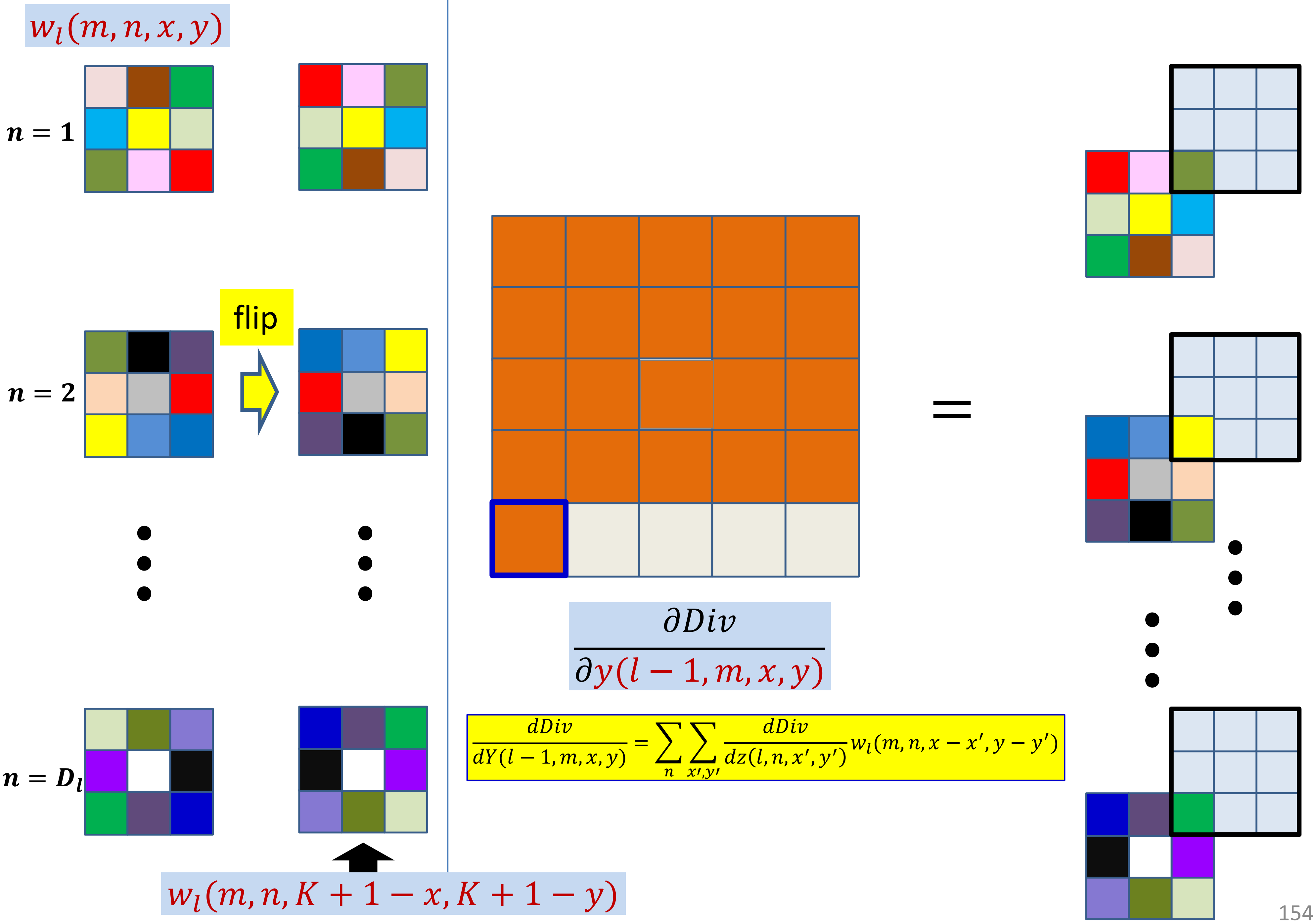
$w_l(m, n, K+1-x, K+1-y)$



$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

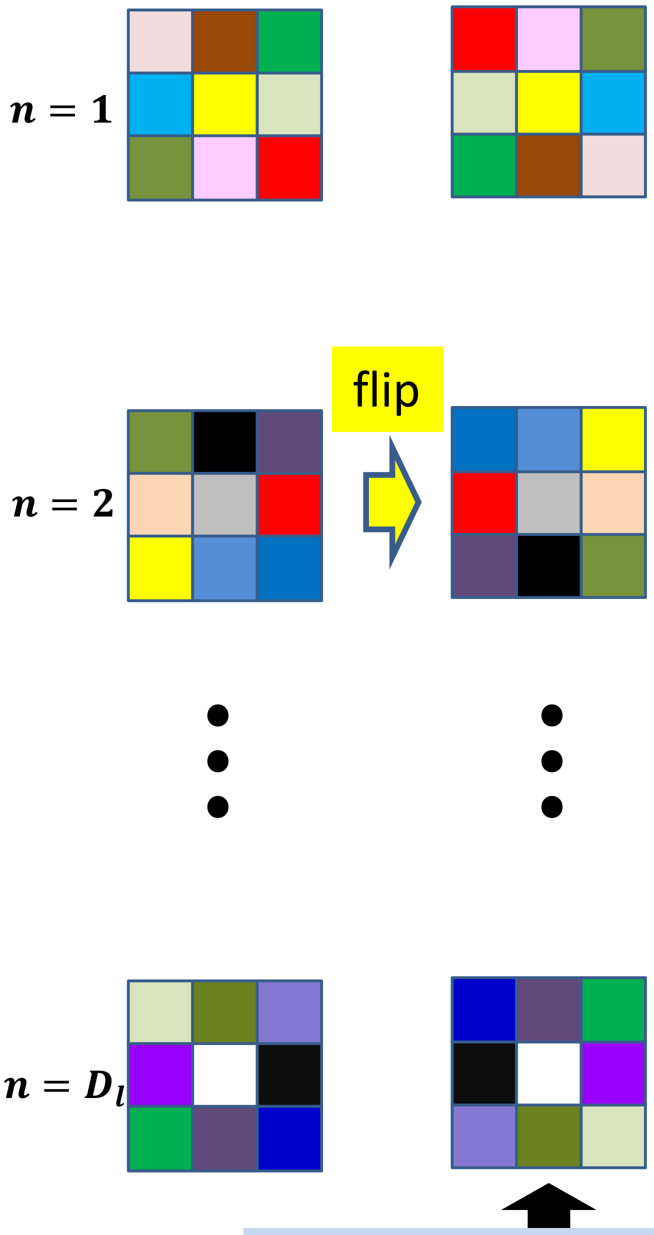


$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x-x', y-y')$$

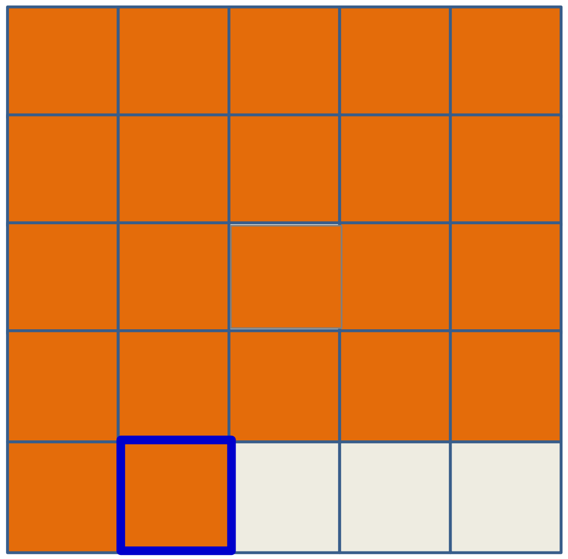


$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_{n} \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$

$w_l(m,n,x,y)$

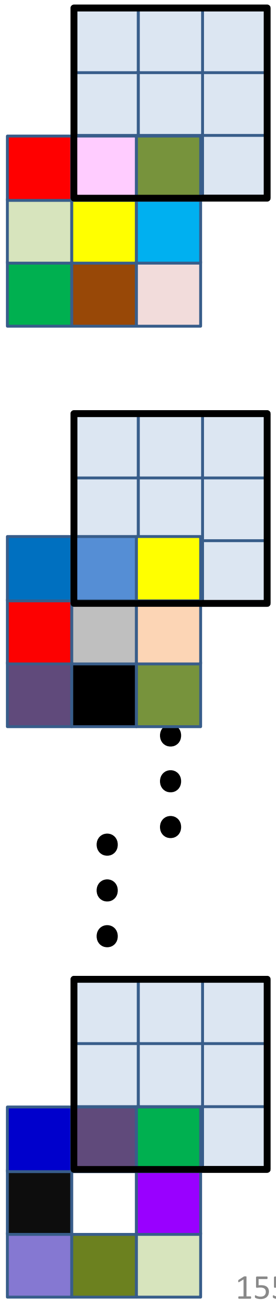


$w_l(m,n,K+1-x,K+1-y)$

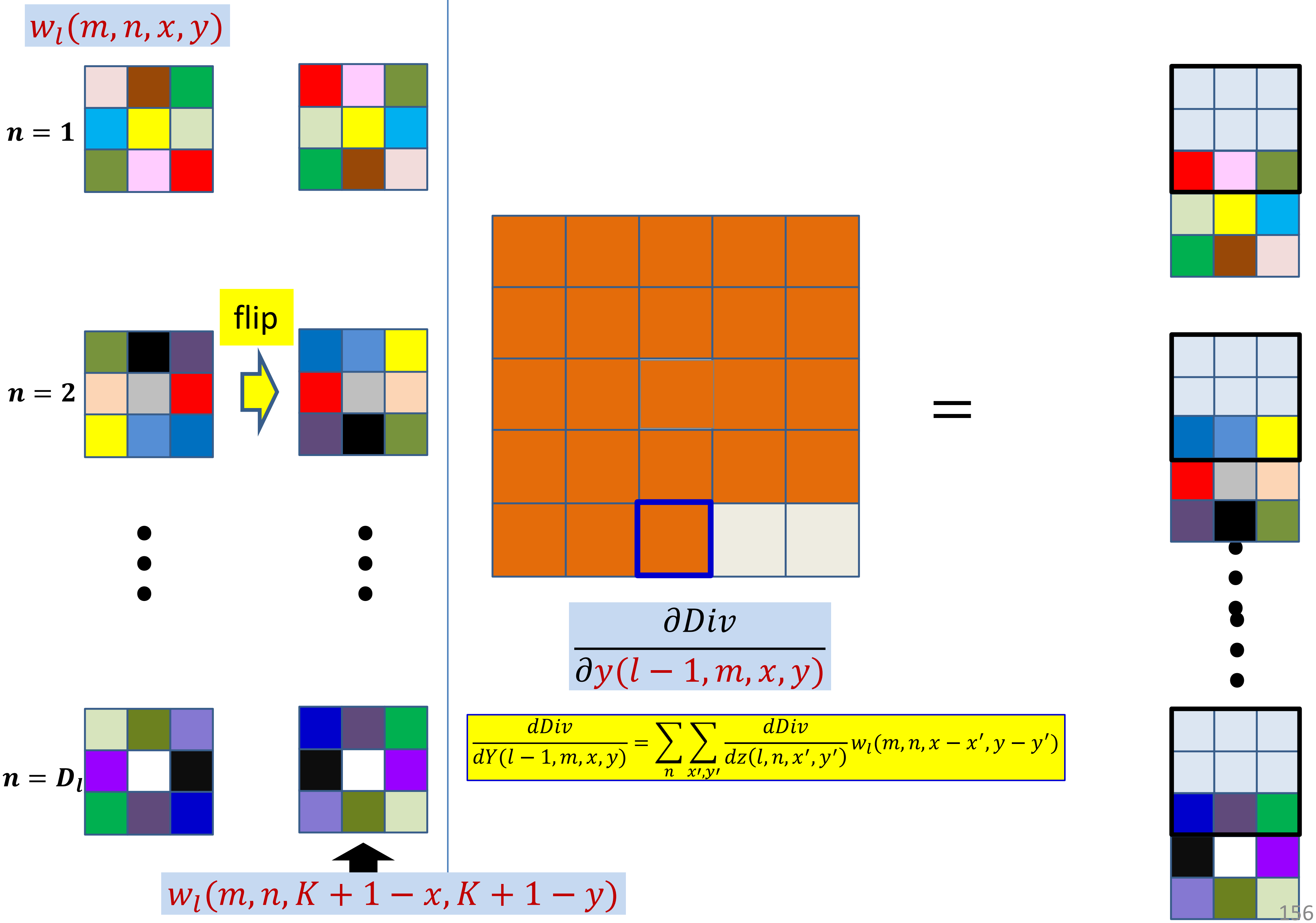


$$\frac{\partial Div}{\partial y(l-1,m,x,y)}$$

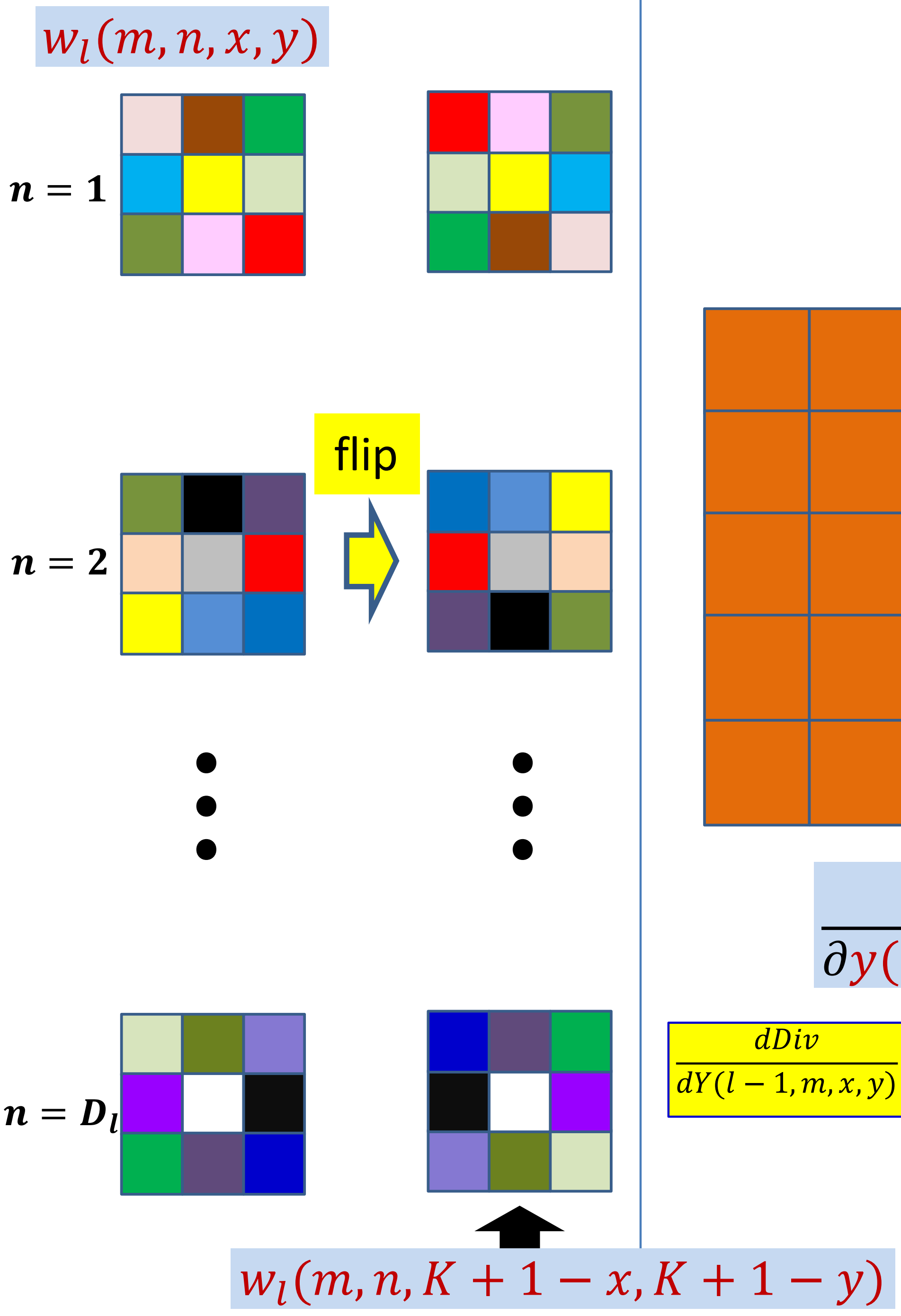
$=$

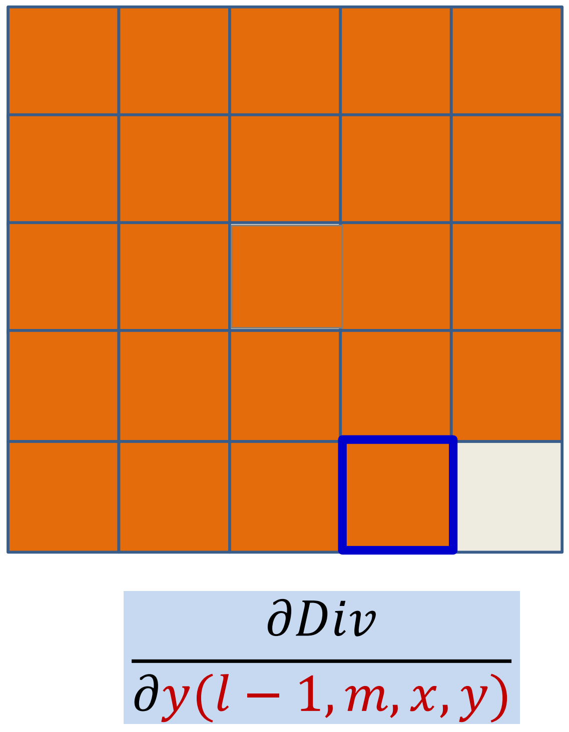


$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_{n} \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$

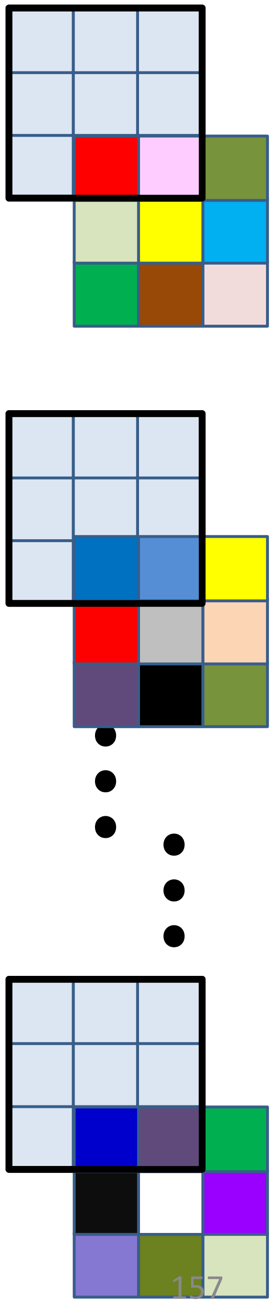


$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$

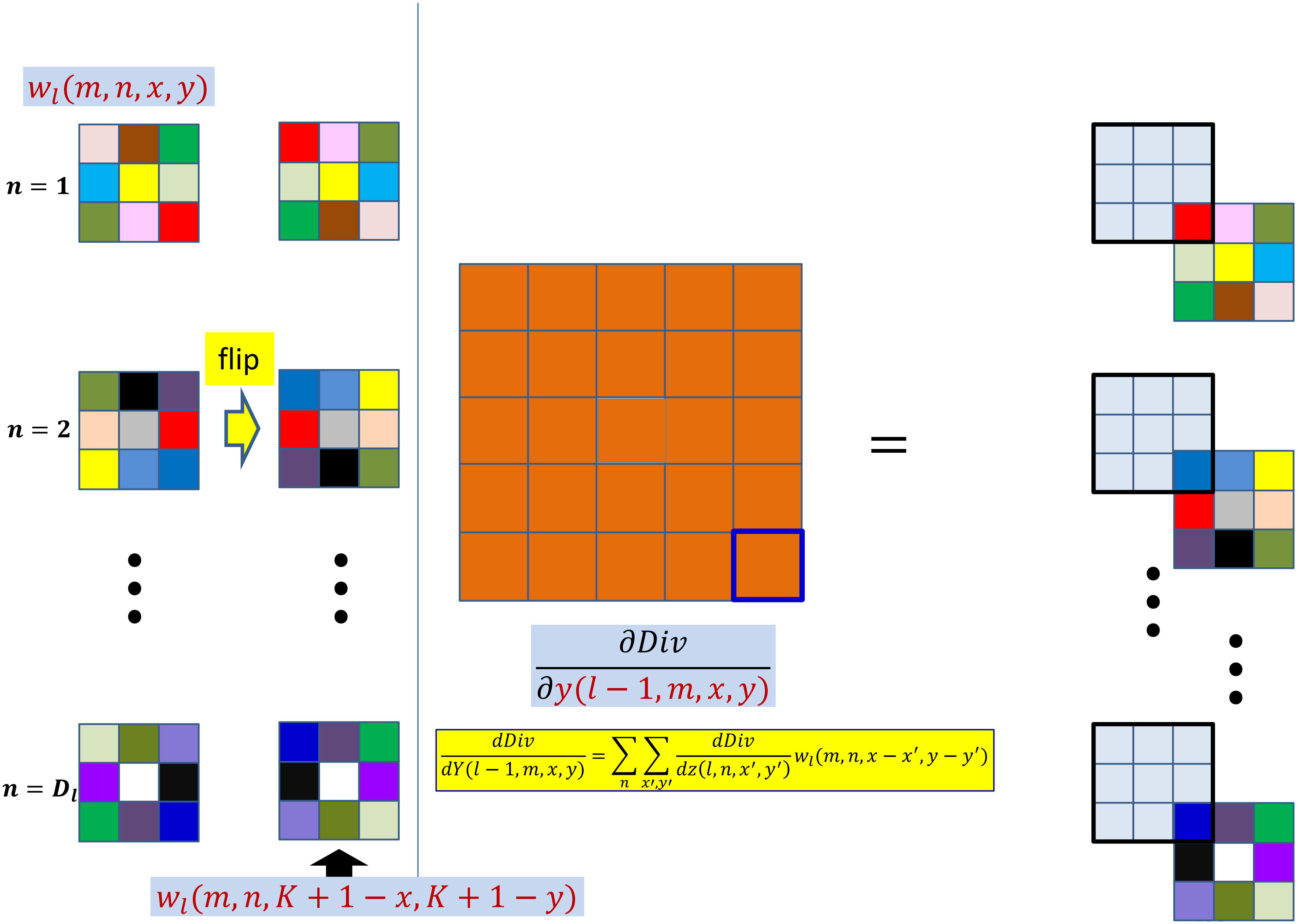




=

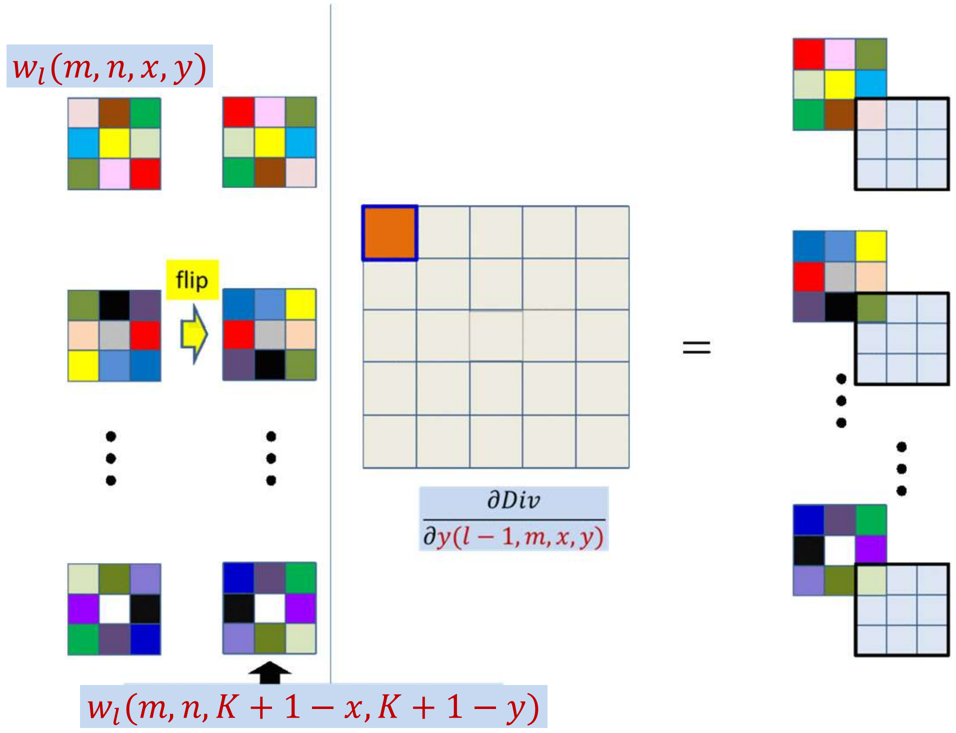


$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$



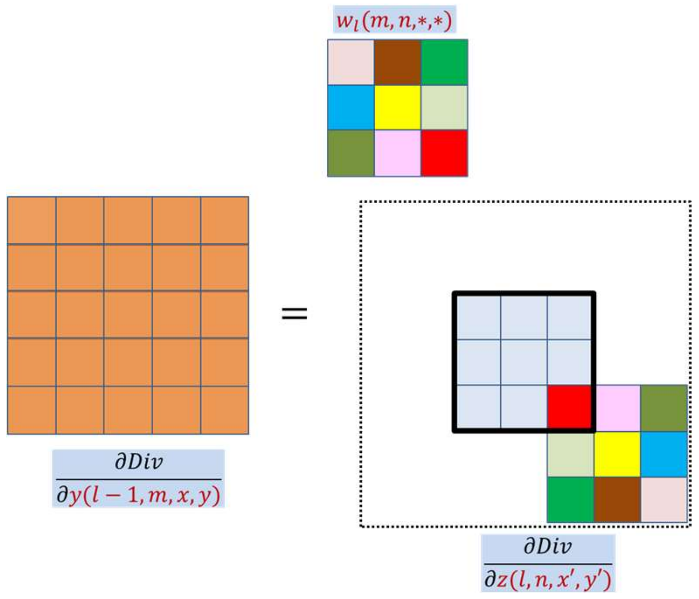
$$\frac{dDiv}{dY(l-1,m,x,y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')$$

Computing the derivative for $Y(l-1, m)$



- This is just a convolution of the zero-padded $\frac{\partial Div}{\partial z(l,n,x,y)}$ maps by the transposed and flipped filter
  - After zero padding it first with $K-1$ zeros on every side

The size of the Y-derivative map

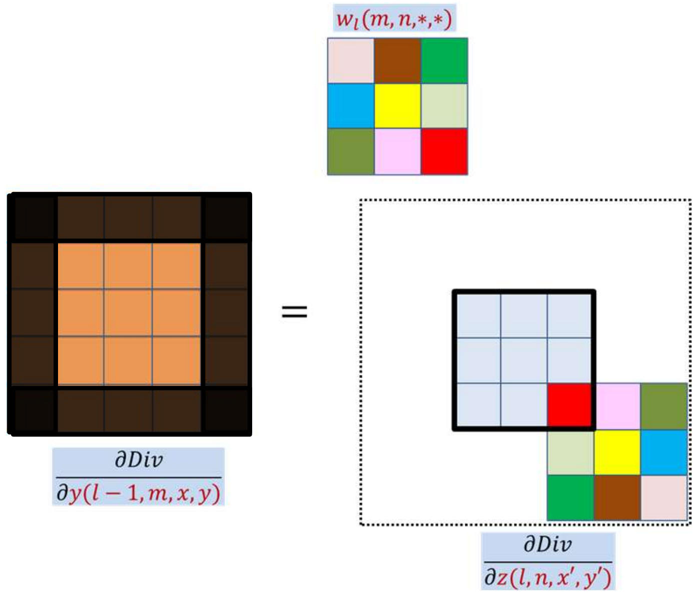

$w_l(m,n,*,*)$
$=$
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

- We continue to compute elements for the derivative Y map as long as the (flipped) filter has at least one element in the (unpadded) derivative Zmap
  - I.e. so long as the Y derivative is non-zero

- The size of the Y derivative map will be $(H + K - 1) \times (W + K - 1)$
  - H and W are heidght and width of the Zmap

- This will be the size of the actual Y map that was originally convolved

The size of the Y-derivative map


$w_l(m,n,*,*)$
=
$\frac{\partial Div}{\partial y(l-1,m,x,y)}$
$\frac{\partial Div}{\partial z(l,n,x',y')}$

- If the Y map was zero-padded in the forward pass, the derivative map will be the size of the *zero-padded* map
  – The zero padding regions must be deleted before further backprop

Poll 3

Select all statements that are true about how to compute the derivative of the divergence w.r.t lth layer activation maps by backpropagation

- To compute the derivative w.r.t. the mth activation map of the lth convolutional layer, we must select the mth “planes” of all the (l+1)th layer filters
- The selected filter planes must be flipped left-right and up-down
- They must convolve the derivative (maps) for the (l+1)th layer affine values
- The output of the convolution must be flipped back left-right and up-down

Poll 3

Select all statements that are true about how to compute the derivative of the divergence w.r.t lth layer activation maps by backpropagation

- **To compute the derivative w.r.t. the mth activation map of the lth convolutional layer, we must select the mth "planes" of all the (l+1)th layer filters**
- **The selected filter planes must be flipped left-right and up-down**
- **They must convolve the derivative (maps) for the (l+1)th layer affine values**
- The output of the convolution must be flipped back left-right and up-down

Overall algorithm for computing derivatives w.r.t. $Y(l-1)$

- Given the derivatives $\frac{dDiv}{dz(l,n,x,y)}$

- Compute derivatives using:

$$\boxed{\frac{dDiv}{dY(l-1,m,x,y)} = \sum_{n}\sum_{x',y'} \frac{dDiv}{dz(l,n,x',y')} w_l(m,n,x-x',y-y')}$$

Can be computed by convolution with flipped filter

Derivatives for a single layer l: Vector notation

```
# The weight W(l,m)is a 3D $D_{l-1}$x$K_l$x$K_l$
# Assuming dz has already been obtained via backprop

dzpad = zeros($D_l$x($H_l$+2($K_l$-1))x($W_l$+2($K_l$-1))) # zeropad
for j = 1:$D_l$
    for i = 1:$D_{l-1}$   # Transpose and flip
        Wflip(i,j,:,:) = flipLeftRight(flipUpDown(W(l,i,j,:,:)))
    dzpad(j,$K_l$:$K_l$+$H_l$-1,$K_l$:$K_l$+$W_l$-1) = dz(l,j,:,:) #center map
end

for j = 1:$D_{l-1}$
  for x = 1:$W_{l-1}$
    for y = 1:$H_{l-1}$
        segment = dzpad(:, x:x+$K_l$-1, y:y+$K_l$-1) #3D tensor
        dy(l-1,j,x,y) = Wflip.segment #tensor inner prod.
```

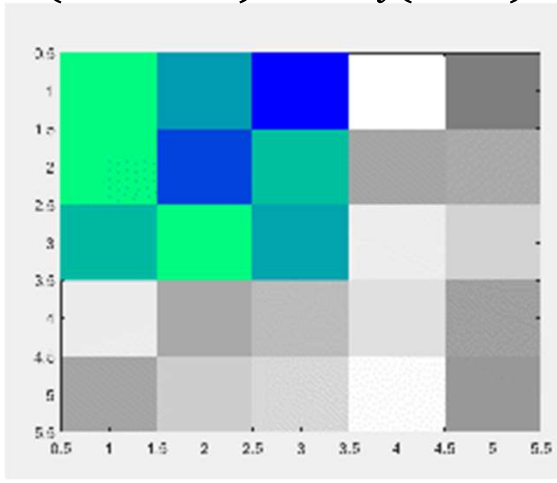
Backpropagating through affine map

- Forward affine computation:
  - Compute affine maps $\mathrm{z}(l, n, x, y)$ from previous layer maps $y(l-1, m, x, y)$ and filters $w_l(m, n, x, y)$

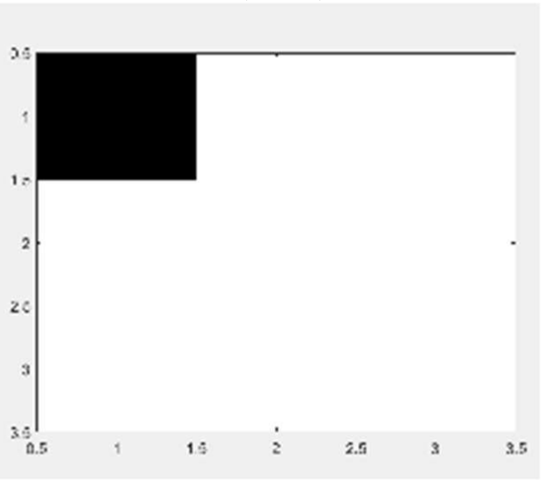
- Backpropagation: Given $\frac{dDiv}{dz(l,n,x,y)}$
  - ✓ Compute derivative w.r.t. $y(l-1, m, x, y)$
  - Compute derivative w.r.t. $w_l(m, n, x, y)$

The derivatives for the weights

$Y(l-1,m) \otimes w_l(m,n)$



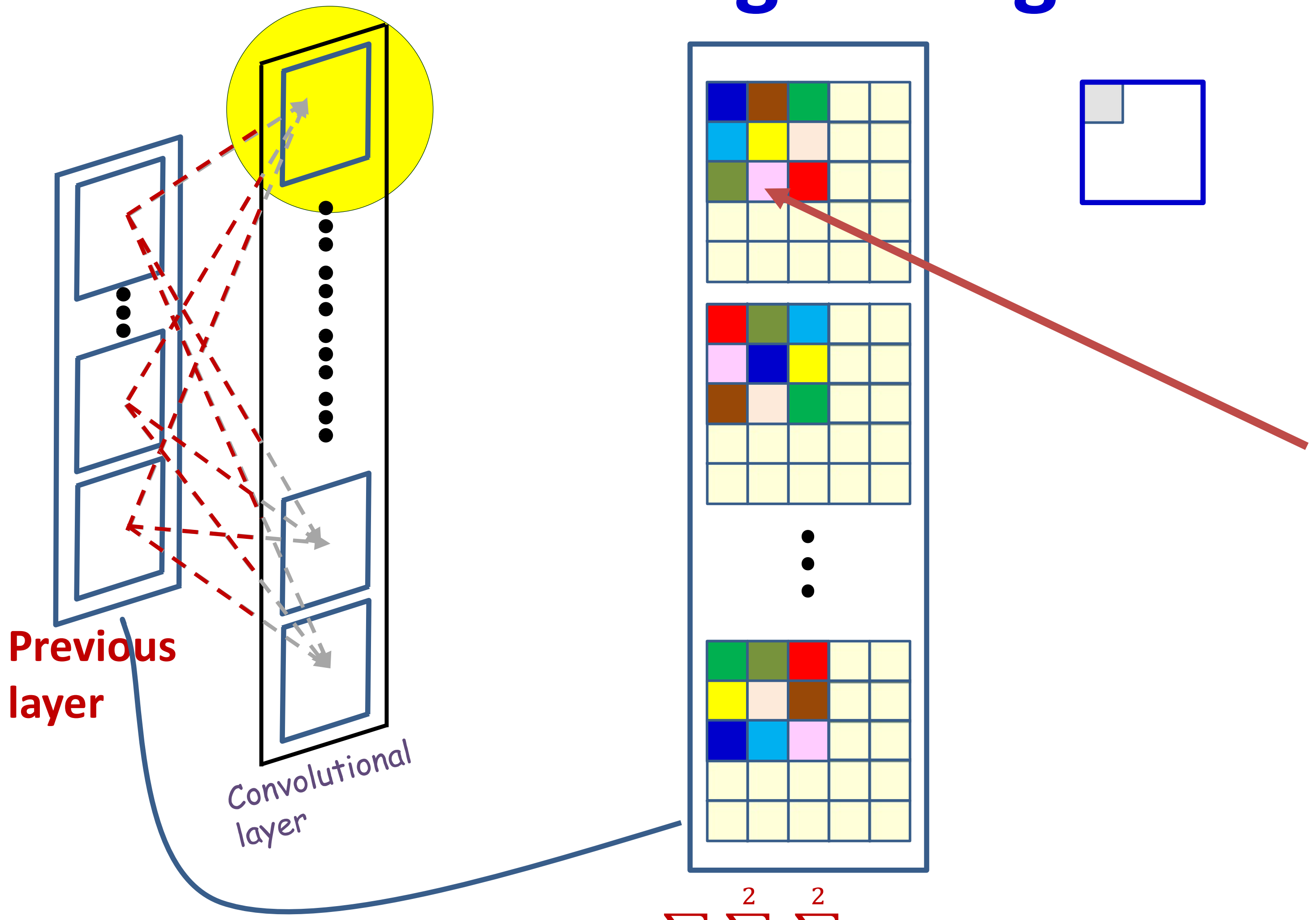
$Z(l,n)$



$$z(l,n,x,y) = \sum_m \sum_{x\prime,y\prime} w_l(m,n,x',y')y(l-1,m,x+x',y+y') + b_l(n)$$

- Each ***weight*** $w_l(m,n,x',y')$ affects several $z(l,n,x,y)$ but only within a *single* affine ($z(l,n,*,*)$) map/channel
  - And is also linked to several $y(l-1,m,x,y)$ but only within a single previous-layer output map/channel $y(l-1,m,*,*)$
    - $w_l(m,n,*,*)$ connects $y(l-1,m,*,*)$ to $z(l,n,*,*)$
  - Consider the contribution of one filter components: $w_l(m,n,i,j)$ (e.g. $w_l(m,n,1,2)$) in the above animation for illustration

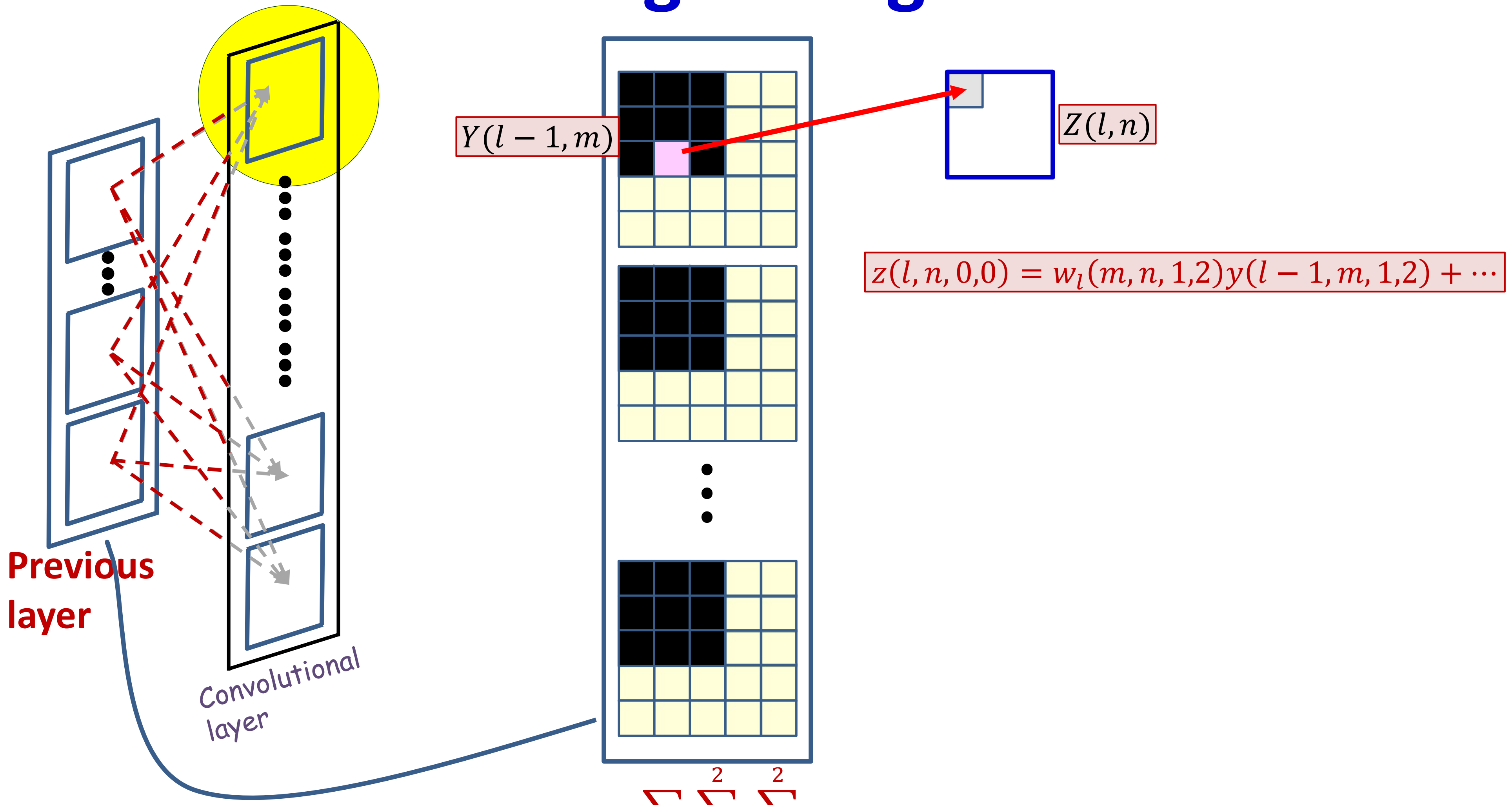
Convolution: the contribution of a single weight



$$z(l,n,x,y) = \sum_{m} \sum_{x'=0}^{2} \sum_{y'=0}^{2} w_l(m,n,x',y')y(l-1,m,x+x',y+x') + b_l(n)$$

- Each affine output is computed from multiple input maps simultaneously
- Each ***weight*** $w_l(m,n,i,j)$ affects several $z(l,n,x,y)$ within the nth output affine map

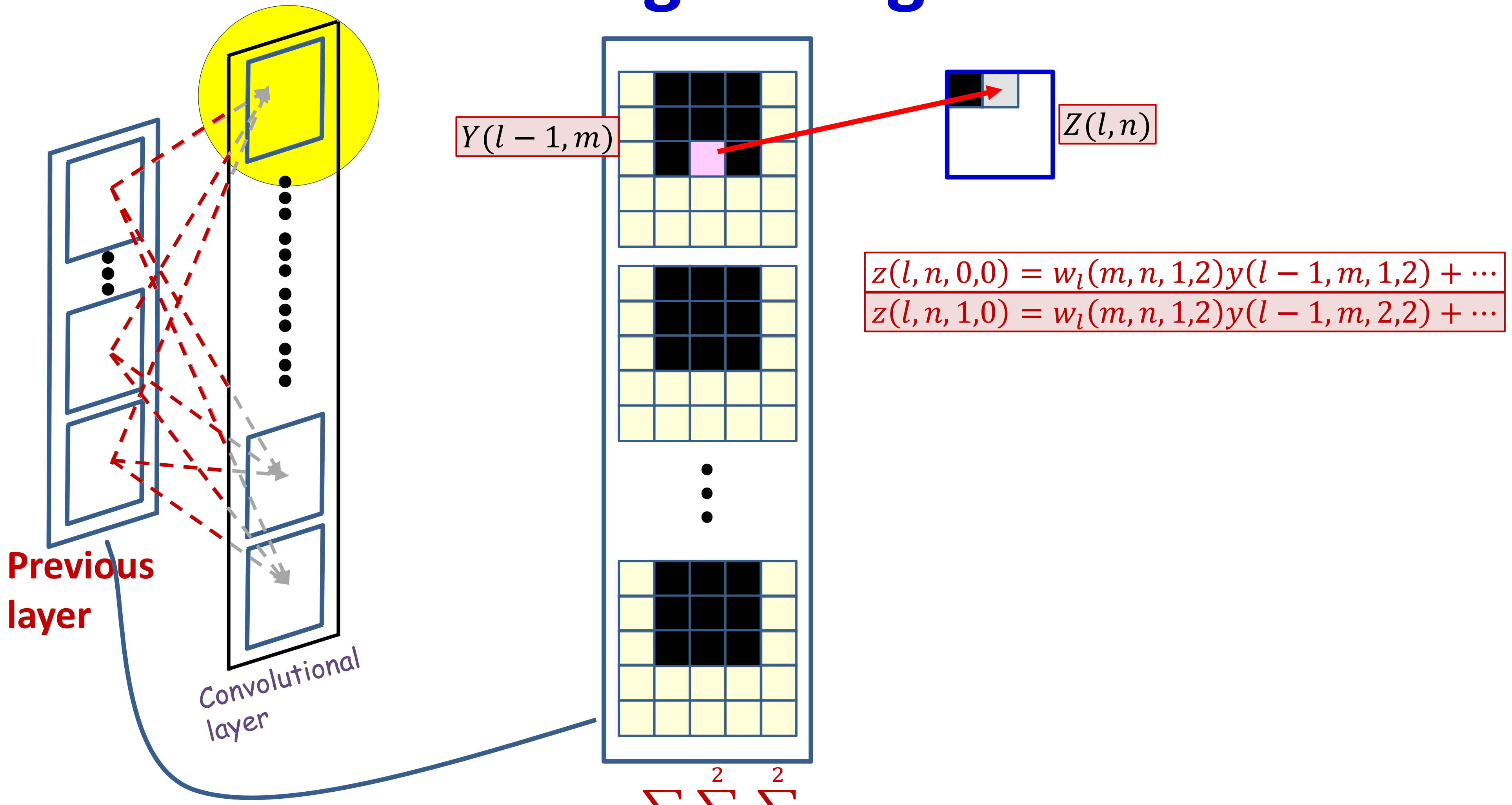
Convolution: the contribution of a single weight



$$z(l,n,x,y) = \sum_m \sum_{x'=0}^{2} \sum_{y'=0}^{2} w_l(m,n,x',y')y(l-1,m,x+x',y+x') + b_l(n)$$

- Each weight $w_l(m,n,i,j)$ affects several $z(l,n,x,y)$ in the nth output affine map
  - Consider the contribution of one filter components: e.g. $w_l(m,n,1,2)$

Convolution: the contribution of a single weight



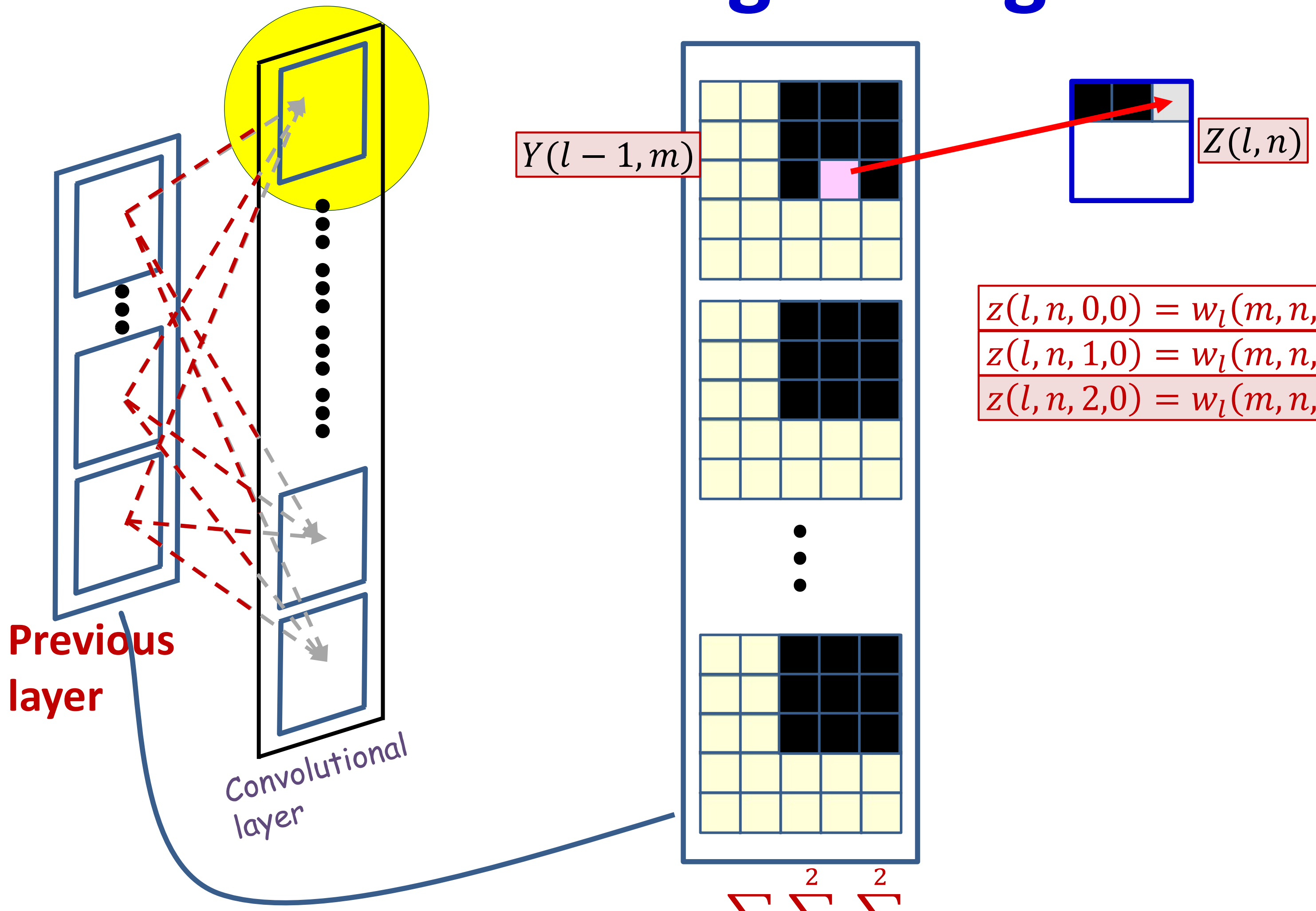
$z(l,n,0,0) = w_l(m,n,1{,}2)y(l-1,m,1{,}2) + \cdots$

$z(l,n,1,0) = w_l(m,n,1{,}2)y(l-1,m,2{,}2) + \cdots$

$$z(l,n,x,y) = \sum_m \sum_{x'=0}^{2} \sum_{y'=0}^{2} w_l(m,n,x',y')y(l-1,m,x+x',y+x') + b_l(n)$$

- Each weight $w_l(m,n,i,j)$ affects several $z(l,n,x,y)$ in the nth output affine map
  - Consider the contribution of one filter components: e.g. $w_l(m,n,1{,}2)$

Convolution: the contribution of a single weight



$z(l,n,0,0) = w_l(m,n,1,2)y(l-1,m,1,2) + \cdots$

$z(l,n,1,0) = w_l(m,n,1,2)y(l-1,m,2,2) + \cdots$

$z(l,n,2,0) = w_l(m,n,1,2)y(l-1,m,3,2) + \cdots$

$$z(l,n,x,y) = \sum_m \sum_{x'=0}^{2} \sum_{y'=0}^{2} w_l(m,n,x',y')y(l-1,m,x+x',y+x') + b_l(n)$$

- Each weight $w_l(m,n,i,j)$ affects several $z(l,n,x,y)$ in the nth output affine map
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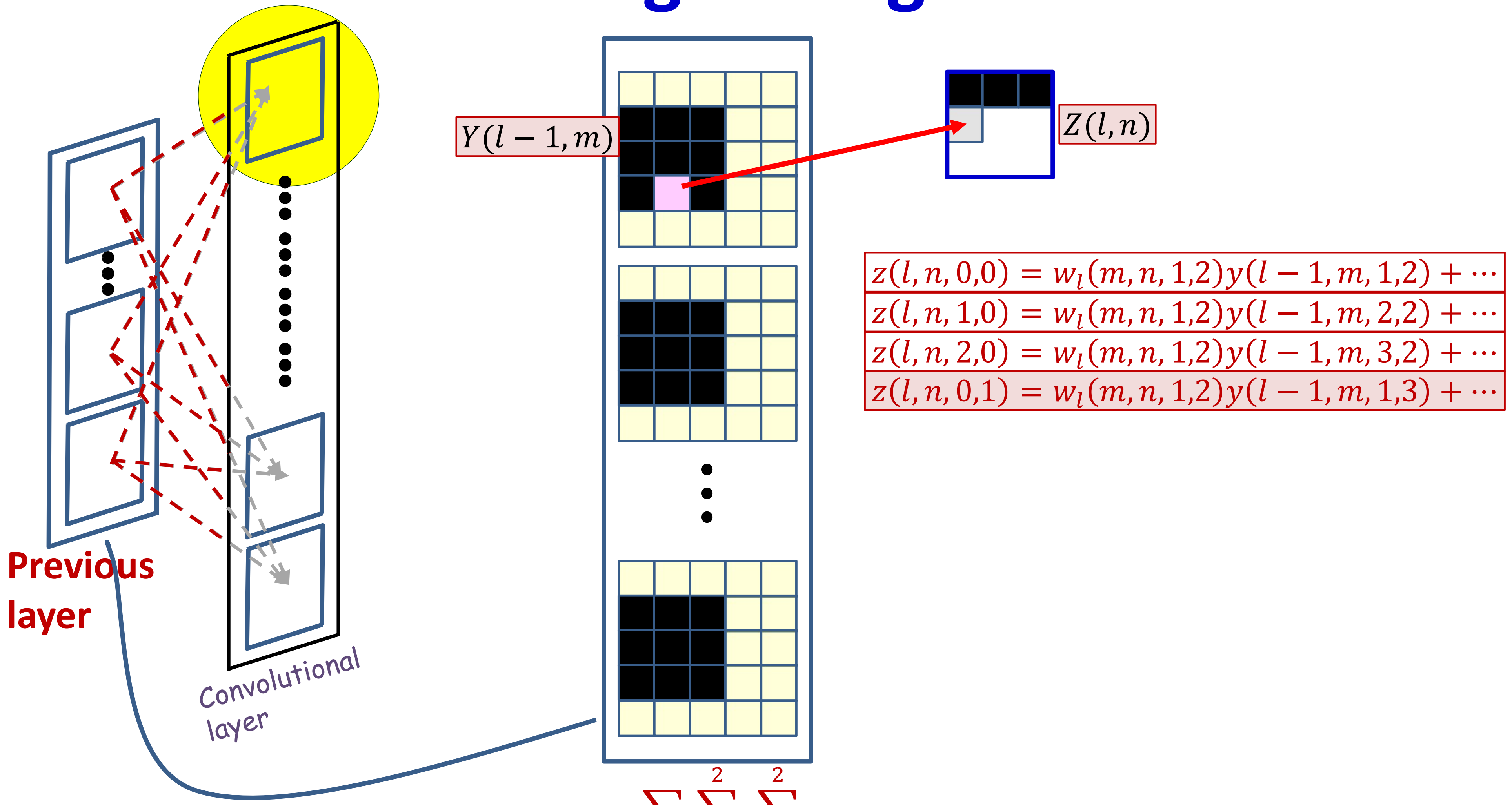
Convolution: the contribution of a single weight



$z(l,n,0,0) = w_l(m,n,1{,}2)y(l-1,m,1{,}2) + \cdots$

$z(l,n,1,0) = w_l(m,n,1{,}2)y(l-1,m,2{,}2) + \cdots$

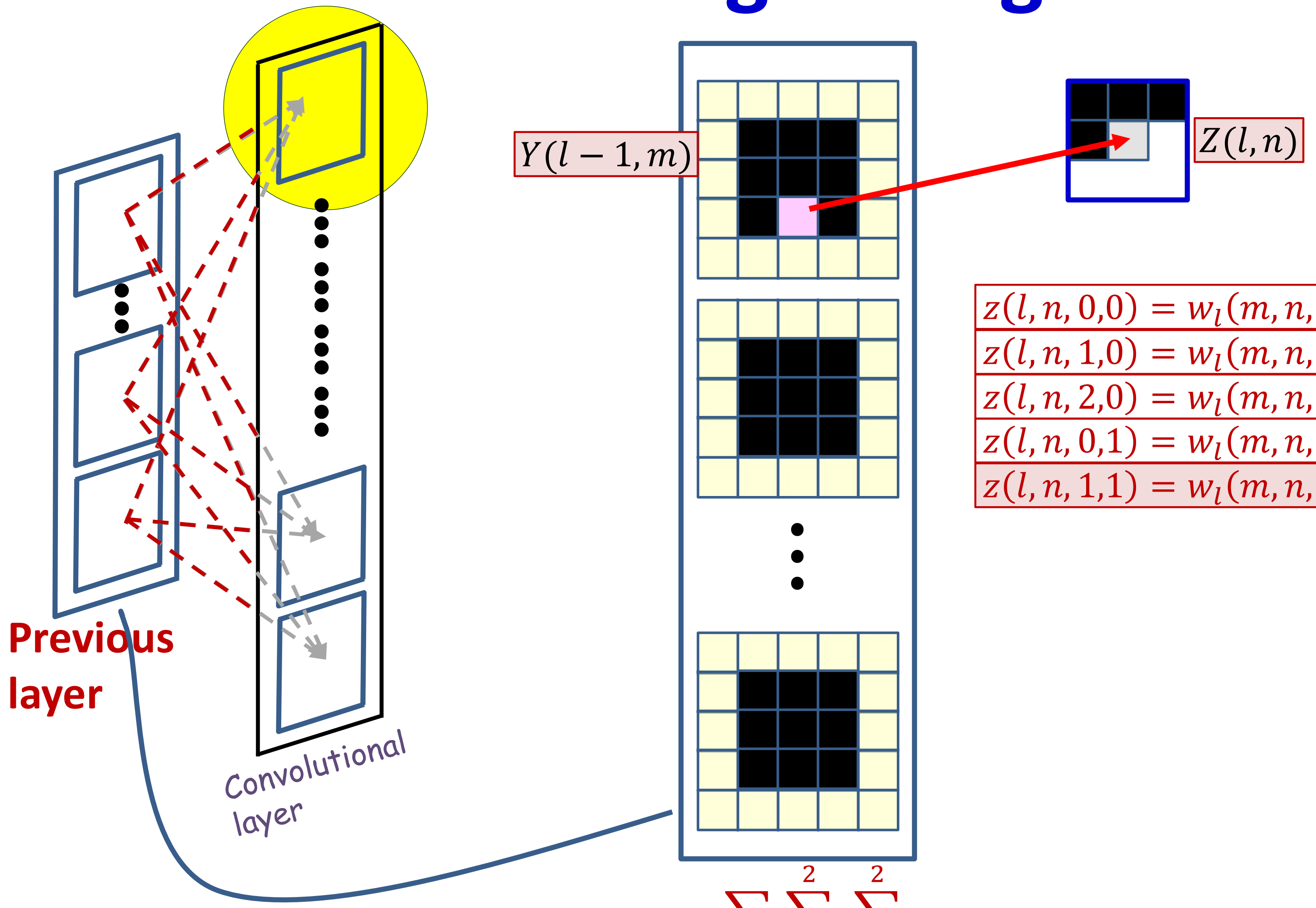
$z(l,n,2,0) = w_l(m,n,1{,}2)y(l-1,m,3{,}2) + \cdots$

$z(l,n,0,1) = w_l(m,n,1{,}2)y(l-1,m,1{,}3) + \cdots$

$$z(l,n,x,y) = \sum_m \sum_{x'=0}^{2} \sum_{y'=0}^{2} w_l(m,n,x',y')y(l-1,m,x+x',y+x') + b_l(n)$$

- Each weight $w_l(m,n,i,j)$ affects several $z(l,n,x,y)$ in the nth output affine map
  - Consider the contribution of one filter components: e.g. $w_l(m,n,1{,}2)$

Convolution: the contribution of a single weight

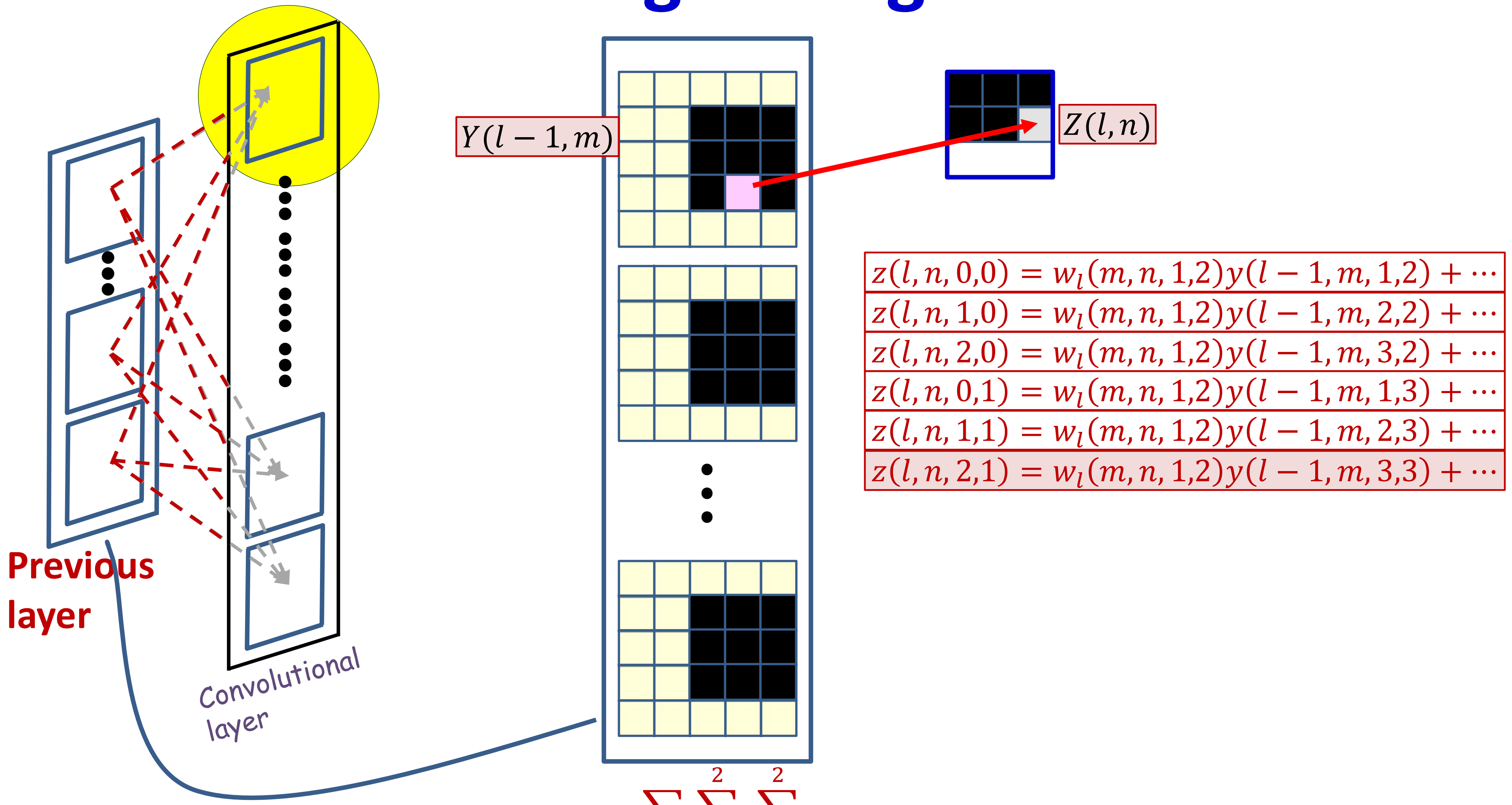


$z(l,n,0,0) = w_l(m,n,1,2)y(l-1,m,1,2) + \cdots$
$z(l,n,1,0) = w_l(m,n,1,2)y(l-1,m,2,2) + \cdots$
$z(l,n,2,0) = w_l(m,n,1,2)y(l-1,m,3,2) + \cdots$
$z(l,n,0,1) = w_l(m,n,1,2)y(l-1,m,1,3) + \cdots$
$z(l,n,1,1) = w_l(m,n,1,2)y(l-1,m,2,3) + \cdots$

$$z(l,n,x,y) = \sum_m \sum_{x'=0}^{2} \sum_{y'=0}^{2} w_l(m,n,x',y')y(l-1,m,x+x',y+x') + b_l(n)$$

- Each weight $w_l(m,n,i,j)$ affects several $z(l,n,x,y)$ in the nth output affine map
  - Consider the contribution of one filter components: e.g. $w_l(m,n,1,2)$

Convolution: the contribution of a single weight



$z(l,n,0,0) = w_l(m,n,1,2)y(l-1,m,1,2) + \cdots$

$z(l,n,1,0) = w_l(m,n,1,2)y(l-1,m,2,2) + \cdots$

$z(l,n,2,0) = w_l(m,n,1,2)y(l-1,m,3,2) + \cdots$

$z(l,n,0,1) = w_l(m,n,1,2)y(l-1,m,1,3) + \cdots$

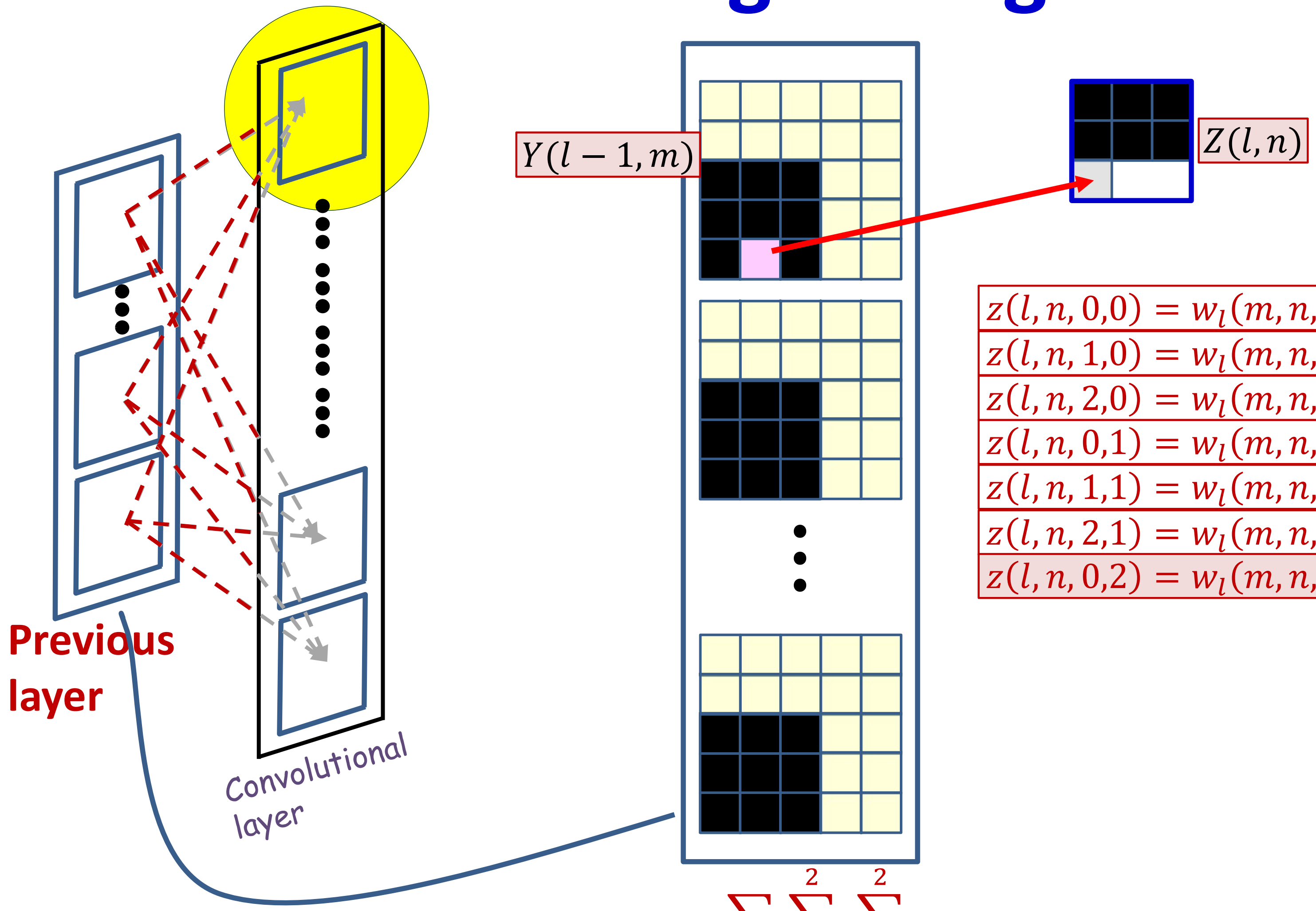
$z(l,n,1,1) = w_l(m,n,1,2)y(l-1,m,2,3) + \cdots$

$z(l,n,2,1) = w_l(m,n,1,2)y(l-1,m,3,3) + \cdots$

$$z(l,n,x,y) = \sum_{m} \sum_{x'=0}^{2} \sum_{y'=0}^{2} w_l(m,n,x',y')y(l-1,m,x+x',y+x') + b_l(n)$$

- Each weight $w_l(m,n,i,j)$ affects several $z(l,n,x,y)$ in the nth output affine map
  - Consider the contribution of one filter components: e.g. $w_l(m,n,1,2)$

Convolution: the contribution of a single weight



$z(l,n,0,0) = w_l(m,n,1,2)y(l-1,m,1,2) + \cdots$

$z(l,n,1,0) = w_l(m,n,1,2)y(l-1,m,2,2) + \cdots$

$z(l,n,2,0) = w_l(m,n,1,2)y(l-1,m,3,2) + \cdots$

$z(l,n,0,1) = w_l(m,n,1,2)y(l-1,m,1,3) + \cdots$

$z(l,n,1,1) = w_l(m,n,1,2)y(l-1,m,2,3) + \cdots$

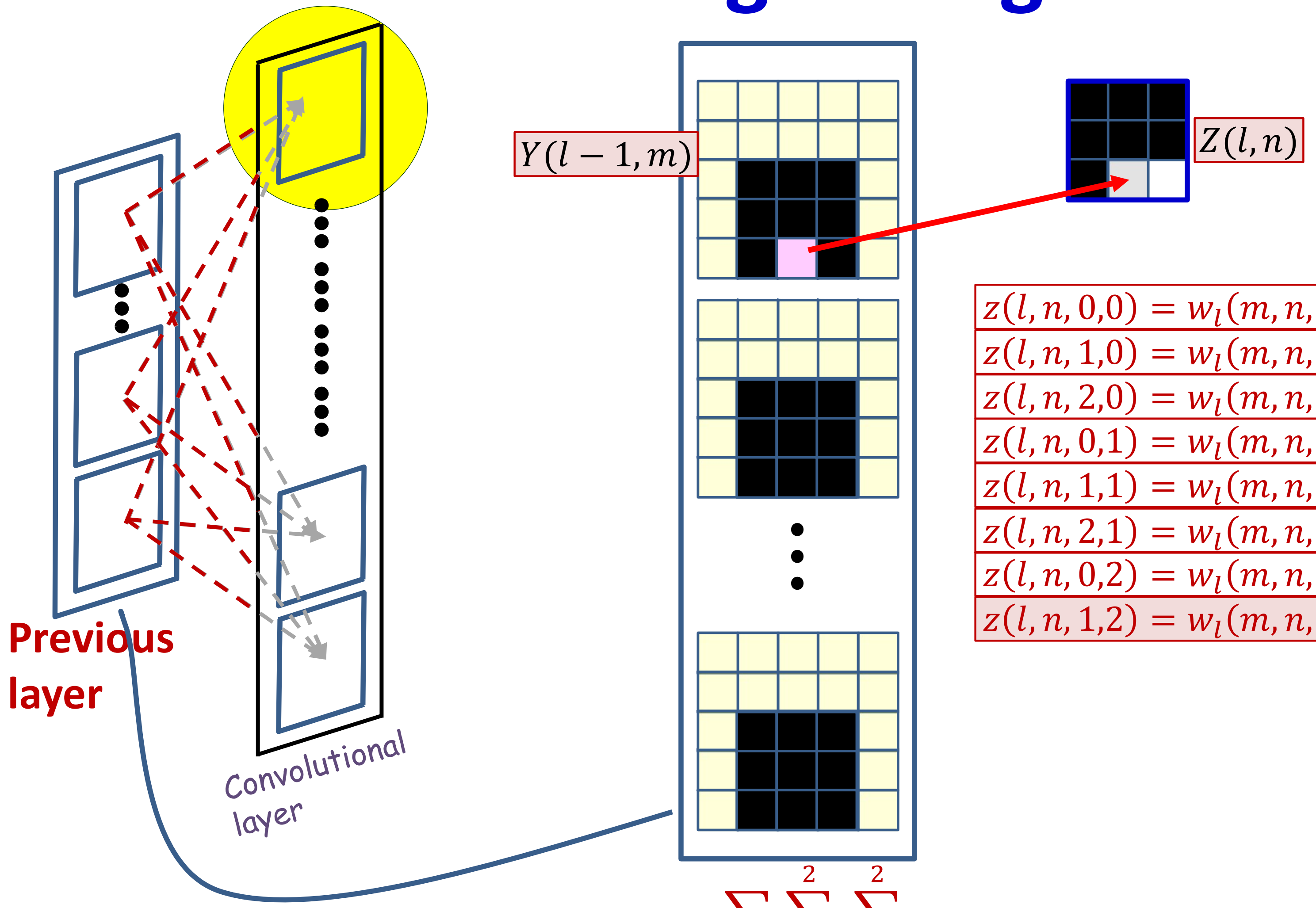
$z(l,n,2,1) = w_l(m,n,1,2)y(l-1,m,3,3) + \cdots$

$z(l,n,0,2) = w_l(m,n,1,2)y(l-1,m,1,4) + \cdots$

$$z(l,n,x,y) = \sum_m \sum_{x'=0}^{2} \sum_{y'=0}^{2} w_l(m,n,x',y')y(l-1,m,x+x',y+x') + b_l(n)$$

- Each weight $w_l(m,n,i,j)$ affects several $z(l,n,x,y)$ in the nth output affine map
  - Consider the contribution of one filter components: e.g. $w_l(m,n,1,2)$

Convolution: the contribution of a single weight



$z(l,n,0,0) = w_l(m,n,1,2)y(l-1,m,1,2) + \cdots$

$z(l,n,1,0) = w_l(m,n,1,2)y(l-1,m,2,2) + \cdots$

$z(l,n,2,0) = w_l(m,n,1,2)y(l-1,m,3,2) + \cdots$

$z(l,n,0,1) = w_l(m,n,1,2)y(l-1,m,1,3) + \cdots$

$z(l,n,1,1) = w_l(m,n,1,2)y(l-1,m,2,3) + \cdots$

$z(l,n,2,1) = w_l(m,n,1,2)y(l-1,m,3,3) + \cdots$

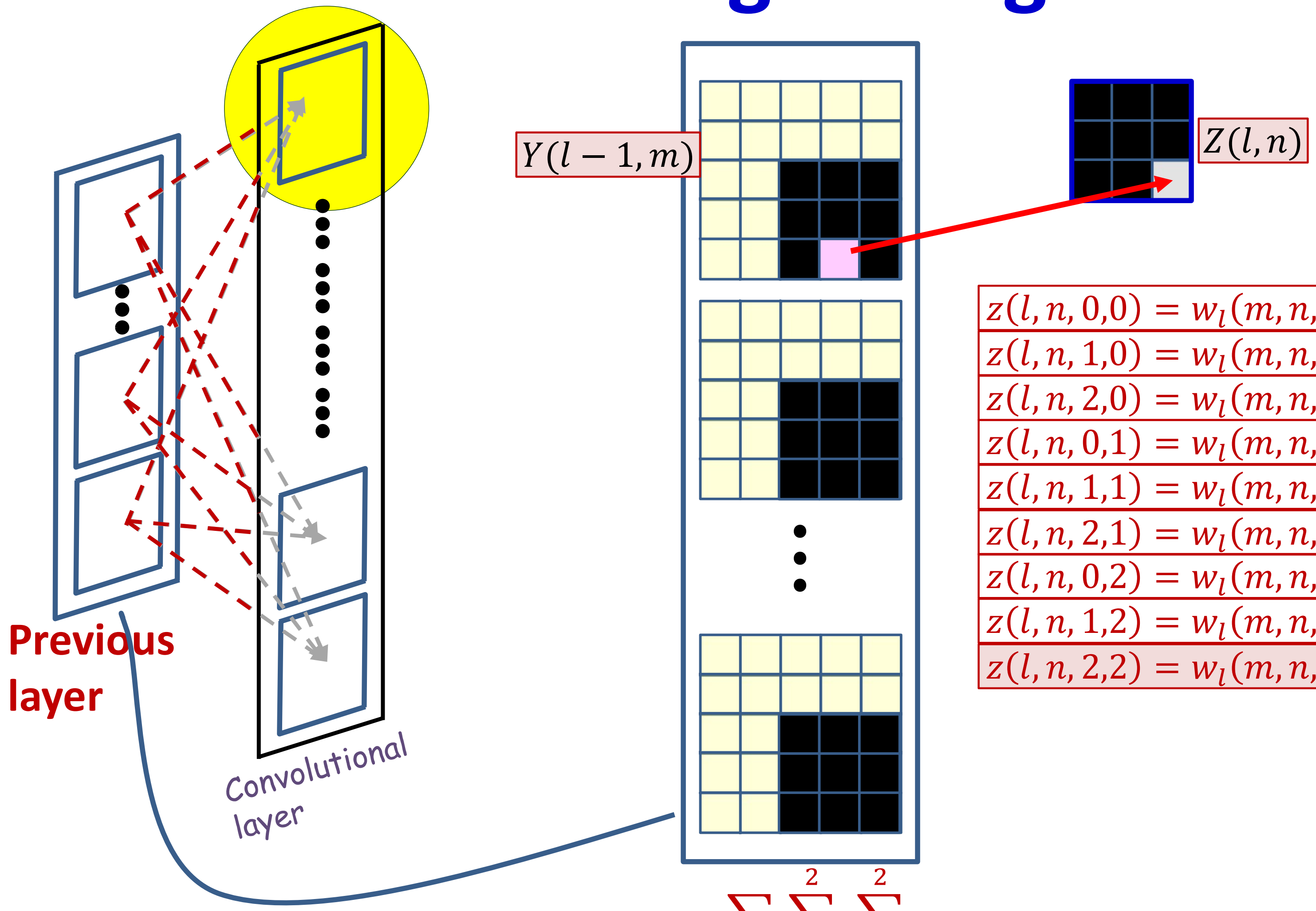
$z(l,n,0,2) = w_l(m,n,1,2)y(l-1,m,1,4) + \cdots$

$z(l,n,1,2) = w_l(m,n,1,2)y(l-1,m,2,4) + \cdots$

$$z(l,n,x,y) = \sum_m \sum_{x'=0}^{2} \sum_{y'=0}^{2} w_l(m,n,x',y')y(l-1,m,x+x',y+x') + b_l(n)$$

- Each weight $w_l(m,n,i,j)$ affects several $z(l,n,x,y)$ in the nth output affine map
  - Consider the contribution of one filter components: e.g. $w_l(m,n,1,2)$

Convolution: the contribution of a single weight



$z(l,n,0,0) = w_l(m,n,1,2)y(l-1,m,1,2) + \cdots$

$z(l,n,1,0) = w_l(m,n,1,2)y(l-1,m,2,2) + \cdots$

$z(l,n,2,0) = w_l(m,n,1,2)y(l-1,m,3,2) + \cdots$

$z(l,n,0,1) = w_l(m,n,1,2)y(l-1,m,1,3) + \cdots$

$z(l,n,1,1) = w_l(m,n,1,2)y(l-1,m,2,3) + \cdots$

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$z(l,n,0,2) = w_l(m,n,1,2)y(l-1,m,1,4) + \cdots$

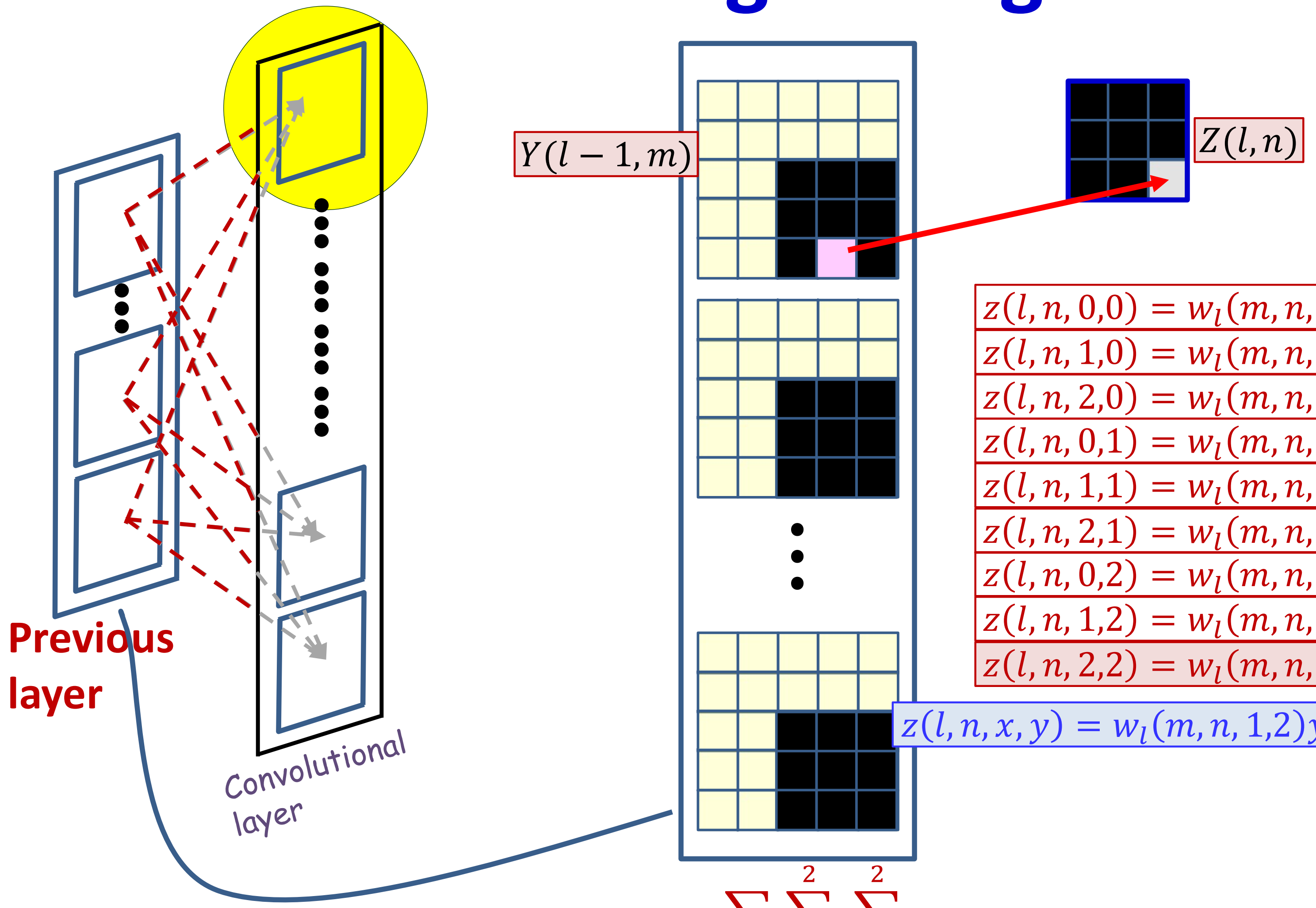
$z(l,n,1,2) = w_l(m,n,1,2)y(l-1,m,2,4) + \cdots$

$z(l,n,2,2) = w_l(m,n,1,2)y(l-1,m,3,4) + \cdots$

$$z(l,n,x,y) = \sum_{m} \sum_{x'=0}^{2} \sum_{y'=0}^{2} w_l(m,n,x',y')y(l-1,m,x+x',y+x') + b_l(n)$$

- Each weight $w_l(m,n,i,j)$ affects several $z(l,n,x,y)$ in the nth output affine map
  – Consider the contribution of one filter components: e.g. $w_l(m,n,1,2)$

Convolution: the contribution of a single weight



$$z(l,n,0,0) = w_l(m,n,1,2)y(l-1,m,1,2) + \cdots$$
$$z(l,n,1,0) = w_l(m,n,1,2)y(l-1,m,2,2) + \cdots$$
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$$z(l,n,x,y) = \sum_m \sum_{x'=0}^{2} \sum_{y'=0}^{2} w_l(m,n,x',y')y(l-1,m,x+x',y+x') + b_l(n)$$

- Each weight $w_l(m,n,i,j)$ affects several $z(l,n,x,y)$ in the nth output affine map
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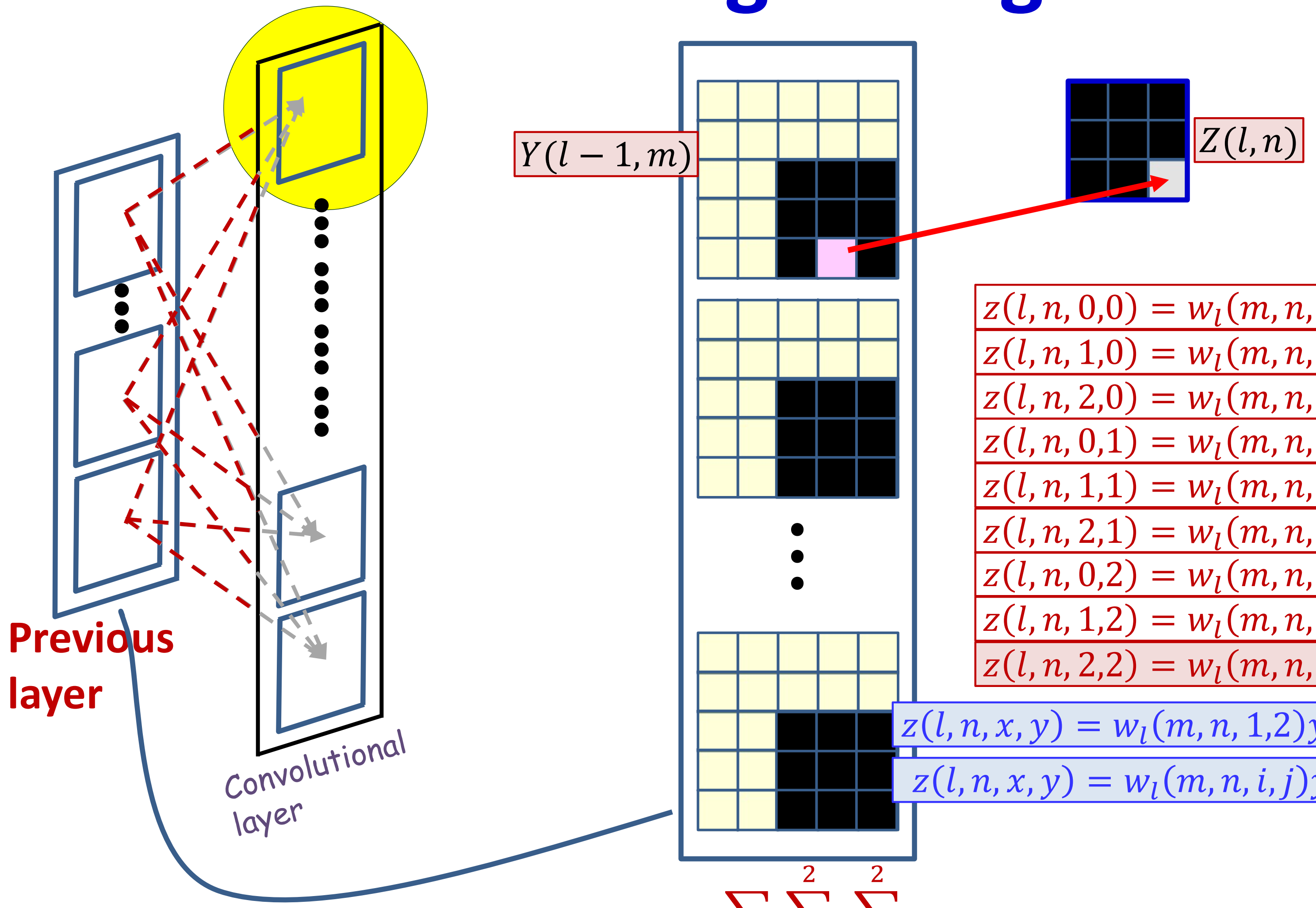
Convolution: the contribution of a single weight



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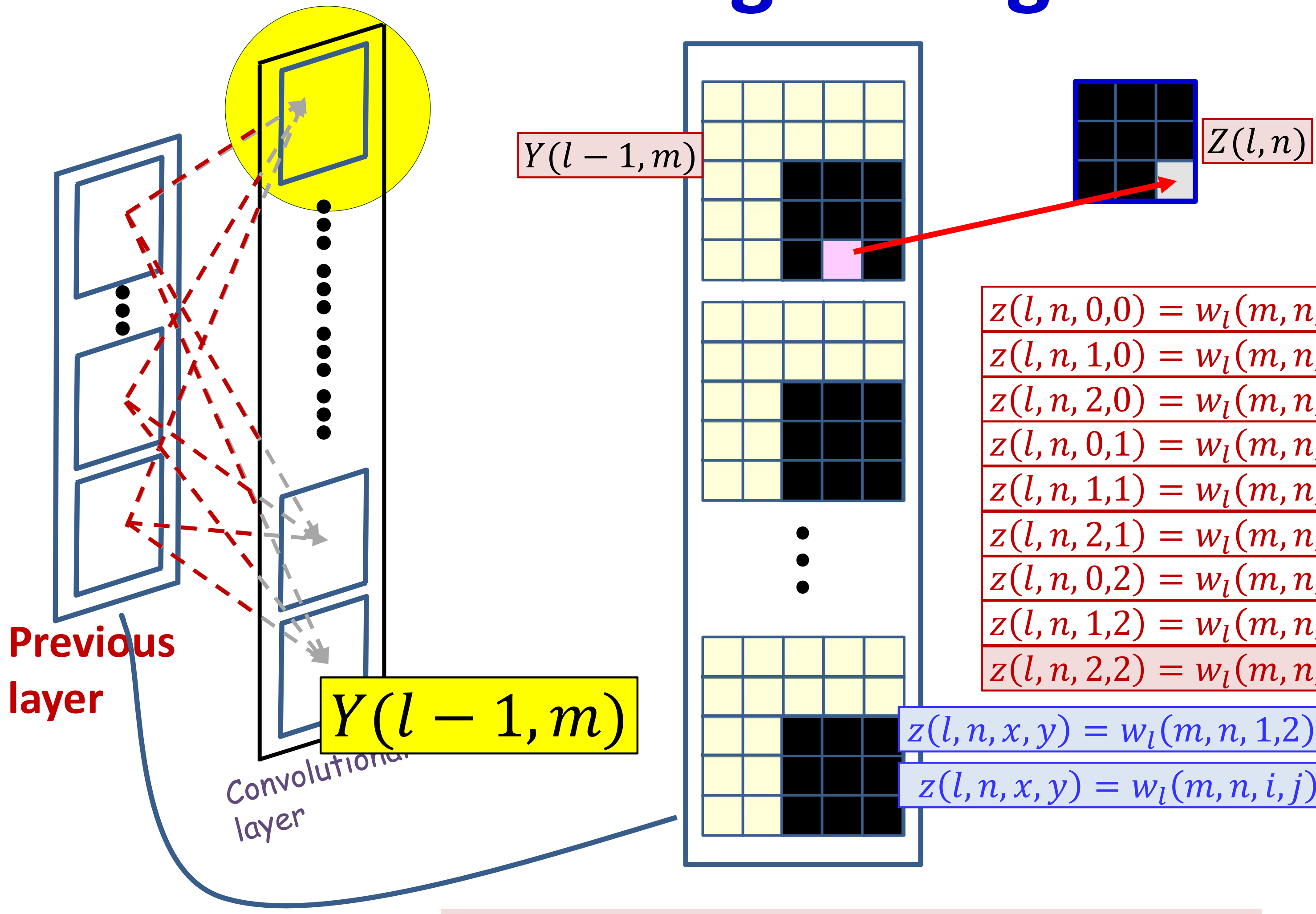

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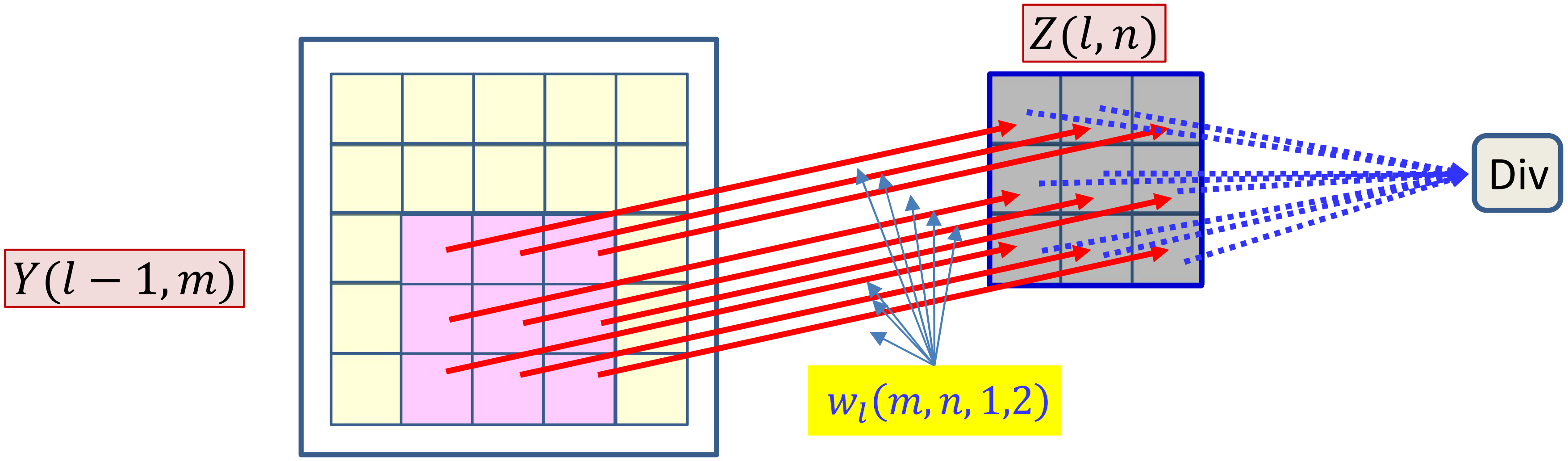
$$\frac{dz(l,n,x,y)}{dw_l(m,n,i,j)} = y(l-1,m,x+i,y+j)$$

The derivative for a single weight



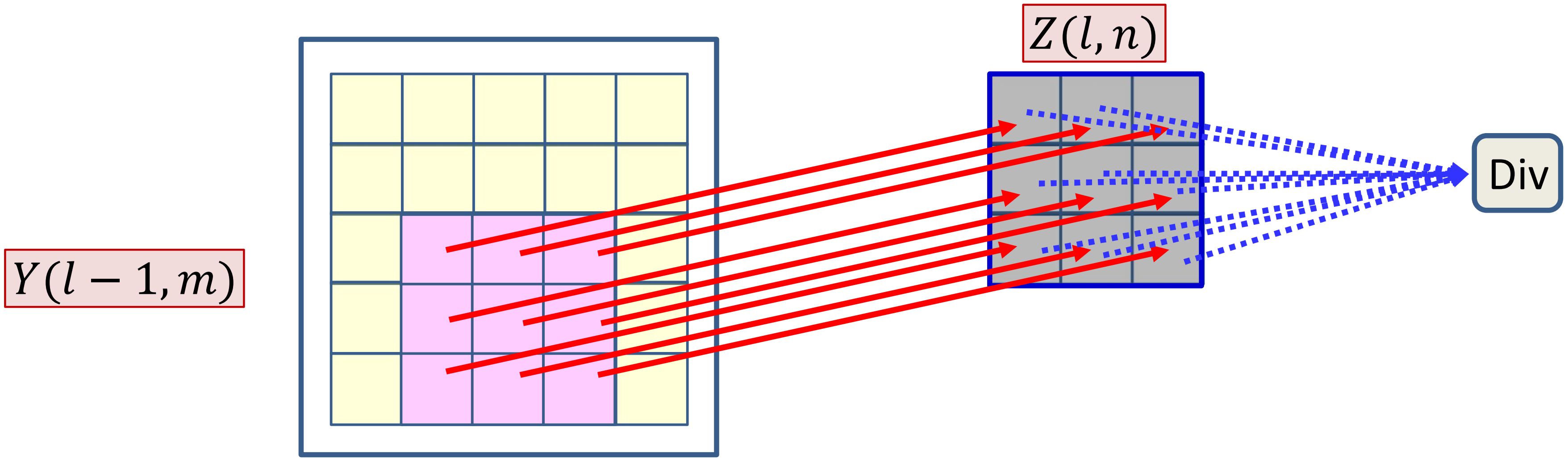
- Each filter component $w_l(m, n, i, j)$ affects several $z(l, n, x, y)$
  - The derivative of each $z(l, n, x, y)$ w.r.t. $w_l(m, n, i, j)$ is given by

$$\frac{dz(l, n, x, y)}{dw_l(m, n, i, j)} = y(l-1, m, x+i, y+j)$$

- The final divergence is influenced by *every* $z(l, n, x, y)$
- The derivative of the divergence w.r.t $w_l(m, n, i, j)$ must sum over all $z(l, n, x, y)$ terms it influences

$$\frac{dDiv}{dw_l\,(m, n, i, j)} = \sum_{x,y} \frac{dDiv}{dz(l, n, x, y)} \frac{dz(l, n, x, y)}{dw_l\,(m, n, i, j)}$$

The derivative for a single weight



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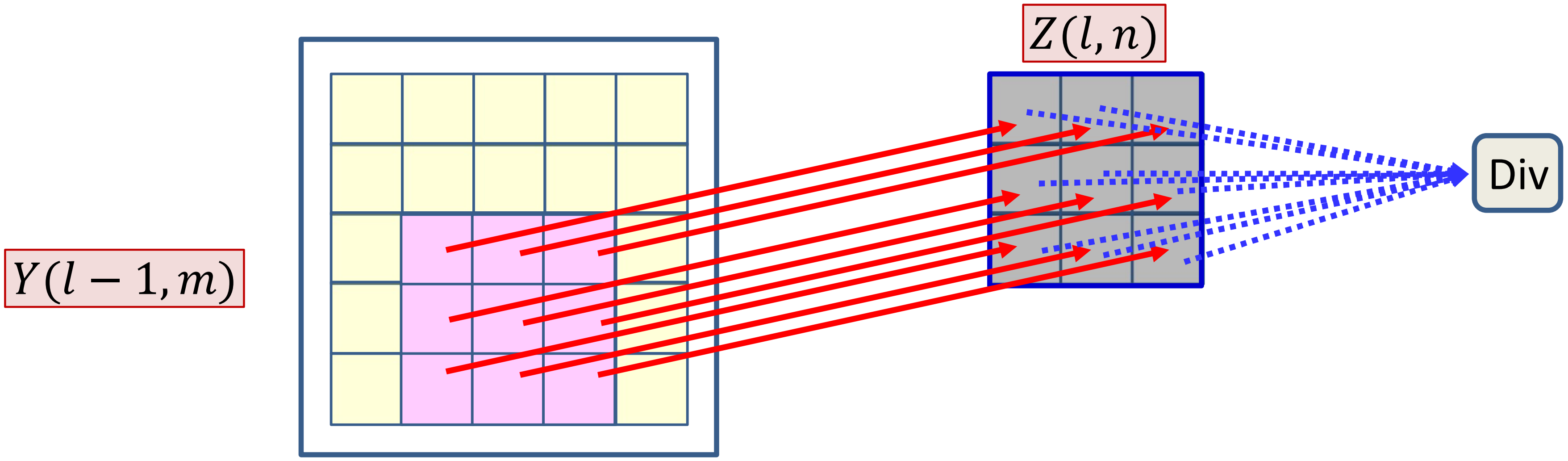
The derivative for a single weight



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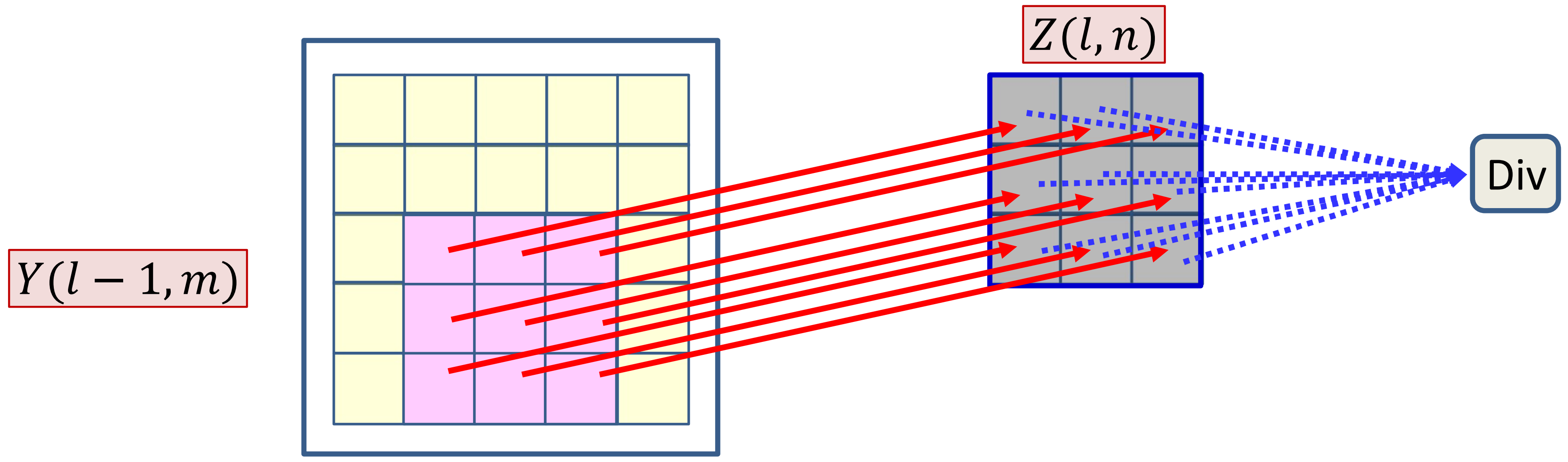
Already computed

$$\frac{dDiv}{dw_l\,(m,n,i,j)} = \sum_{x,y} \boxed{\frac{dDiv}{dz(l,n,x,y)}}\,\boxed{\frac{dz(l,n,x,y)}{dw_l\,(m,n,i,j)}}$$

The derivative for a single weight



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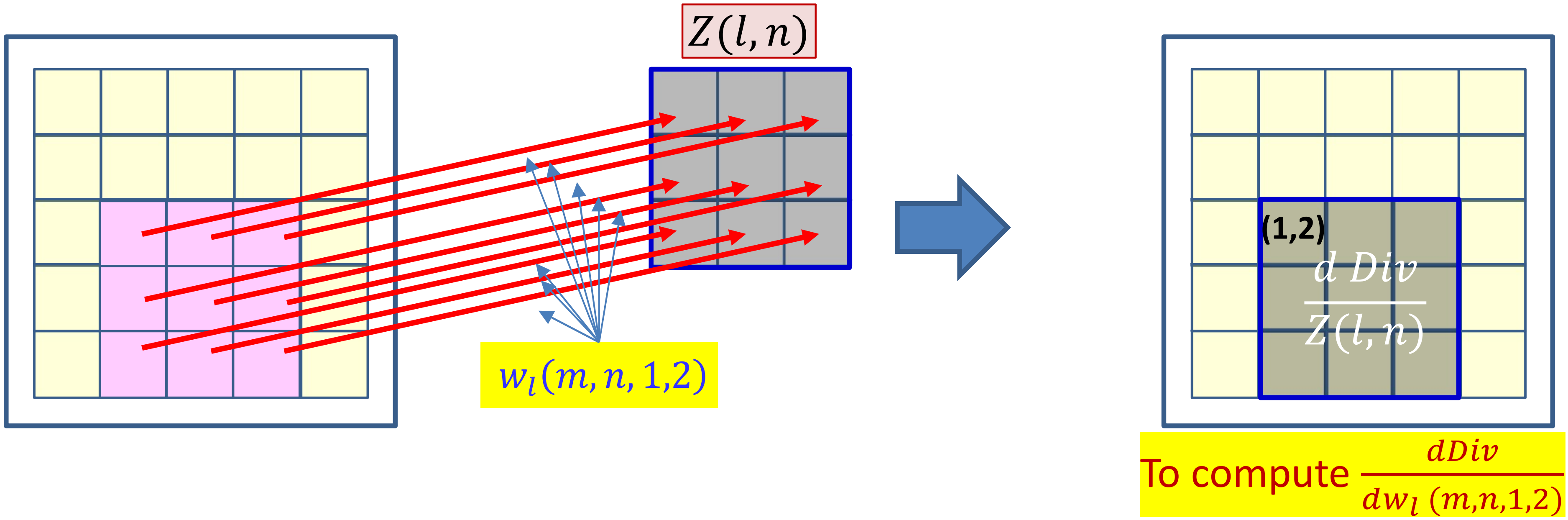
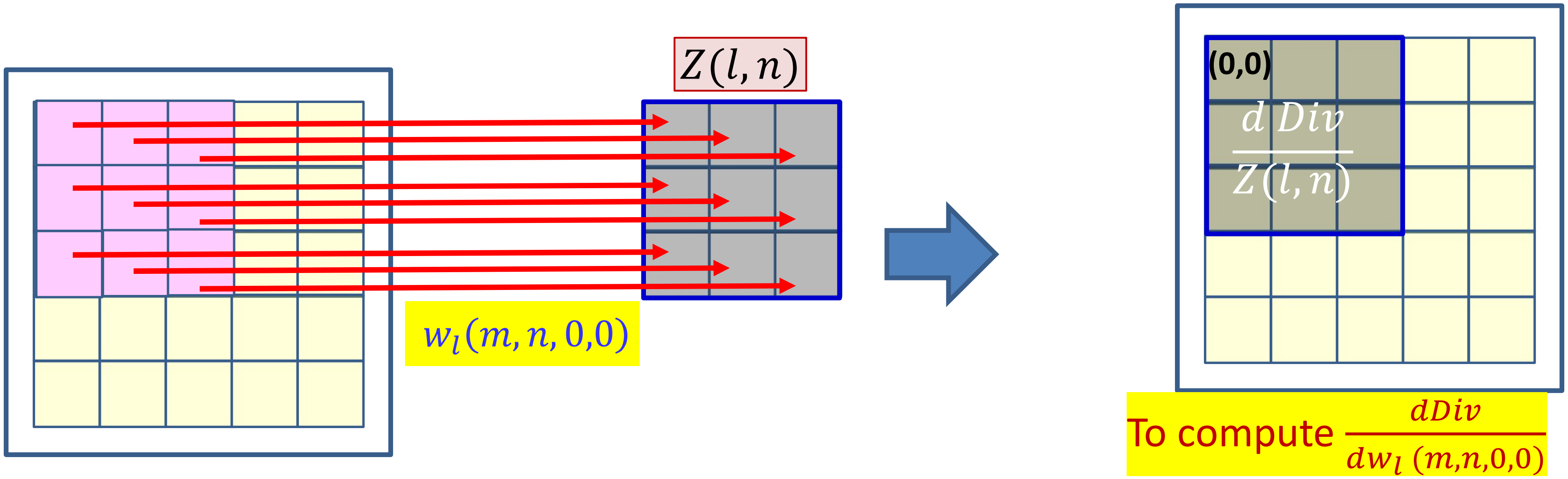
The derivative for a single weight



To compute $\frac{dDiv}{dw_l\,(m,n,1,2)}$



To compute $\frac{dDiv}{dw_l\,(m,n,0,0)}$

But this too is a convolution

$$\boxed{\frac{dDiv}{dw_l\,(m,n,i,j)} = \sum_{x,y} \frac{dDiv}{dz(l,n,x,y)} y(l-1,m,x+i,y+j)}$$

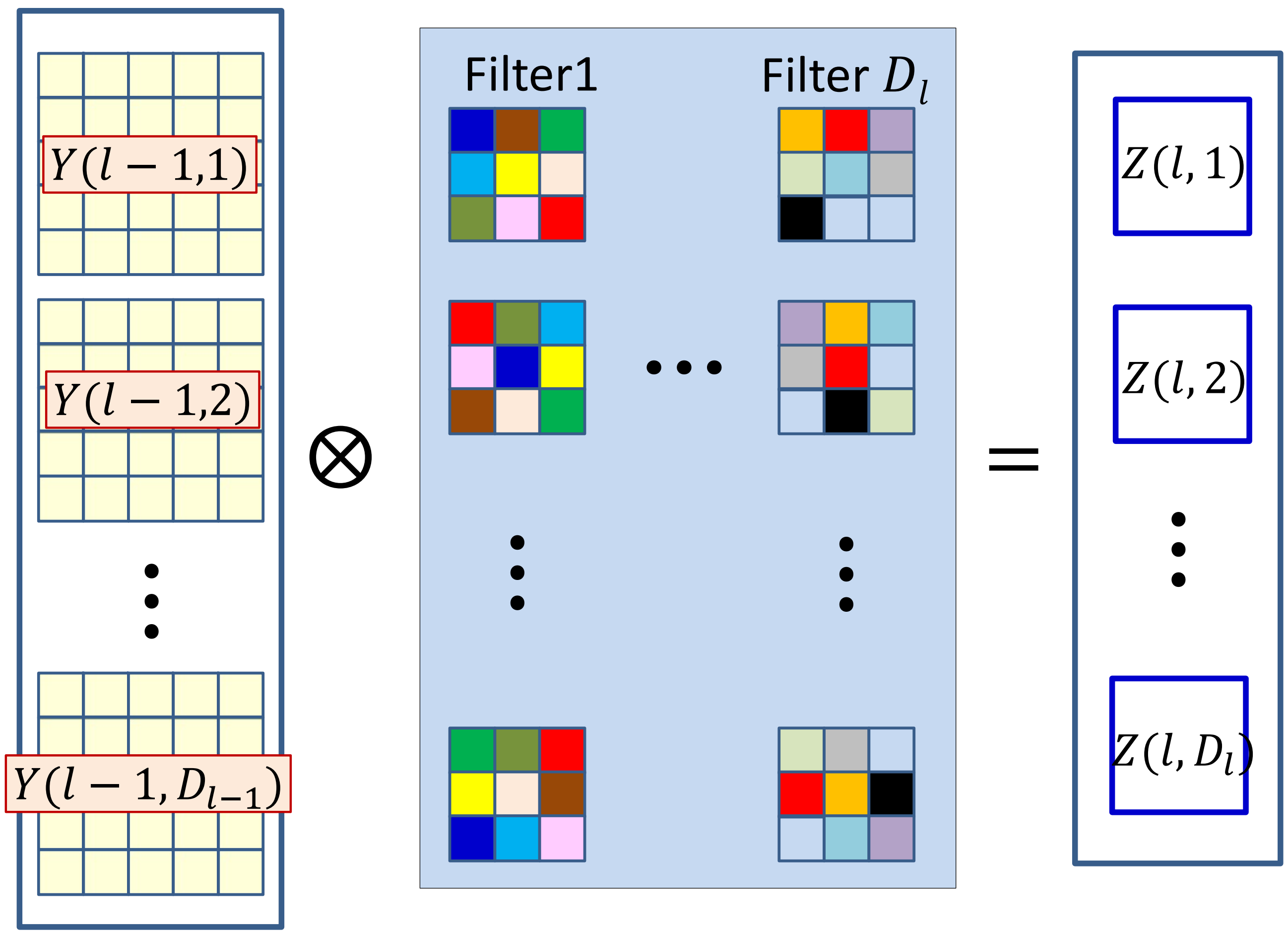
- The derivatives for all components of all filters can be computed directly from the above formula
  - To compute the derivative for $w_l\,(m,n,i,j)$, "place" the $dDiv/dz(l,n)$ map on $y(l-1,m)$ map positioned at (i,j) and compute the inner product

- In fact, it is just a convolution

$$\boxed{\frac{dDiv}{dw_l\,(m,n,i,j)} = \frac{dDiv}{dz(l,n)} \otimes y(l-1,m)}$$

- How?

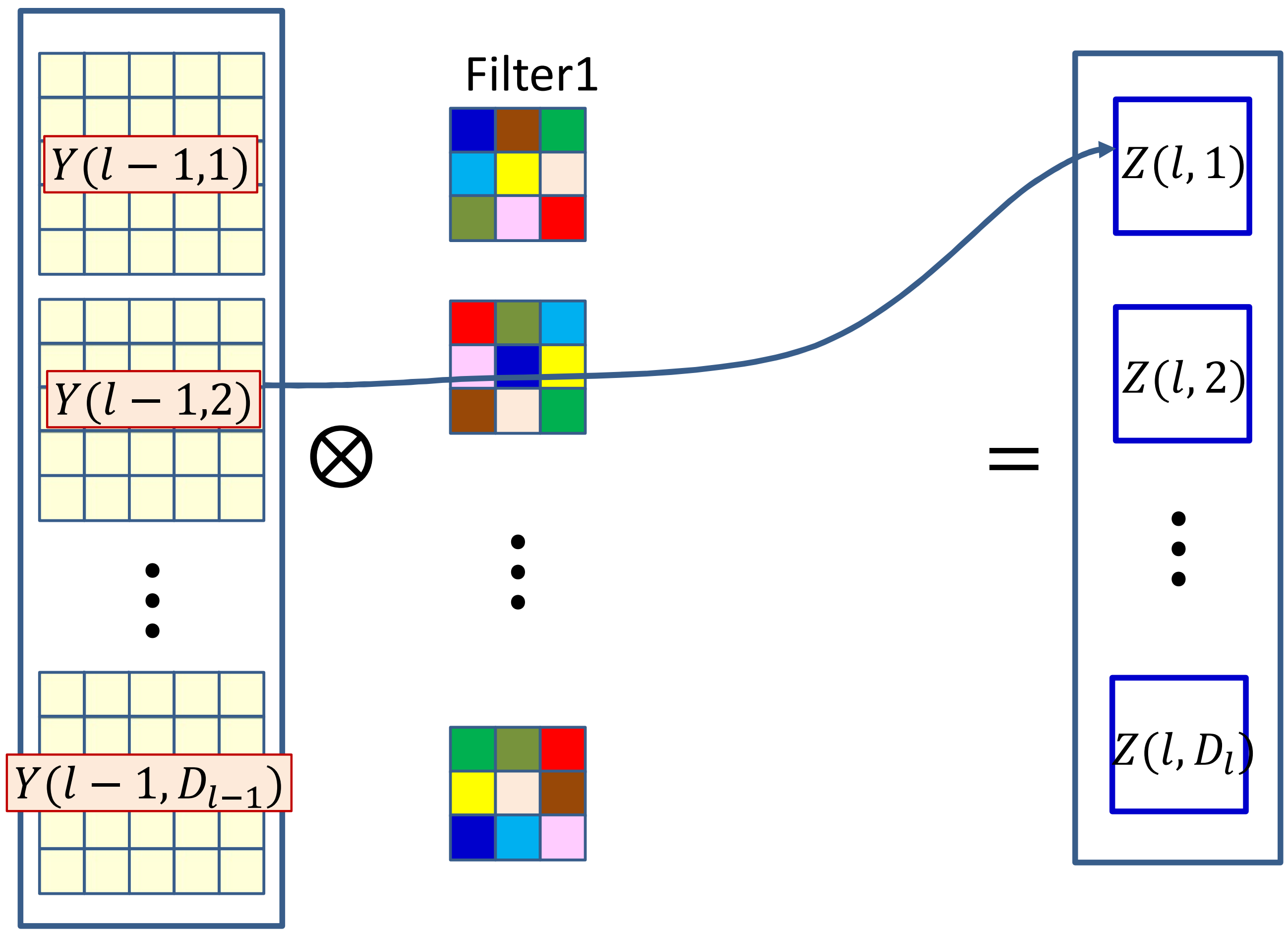
Recap: Convolution



$$z(l,n,x,y) = \sum_{m}\sum_{i=0}^{2}\sum_{j=0}^{2} w_l(m,n,i,j)\,y(l-1,m,x+i,y+j) + b_l(n)$$

- Forward computation: Each filter produces an affine map

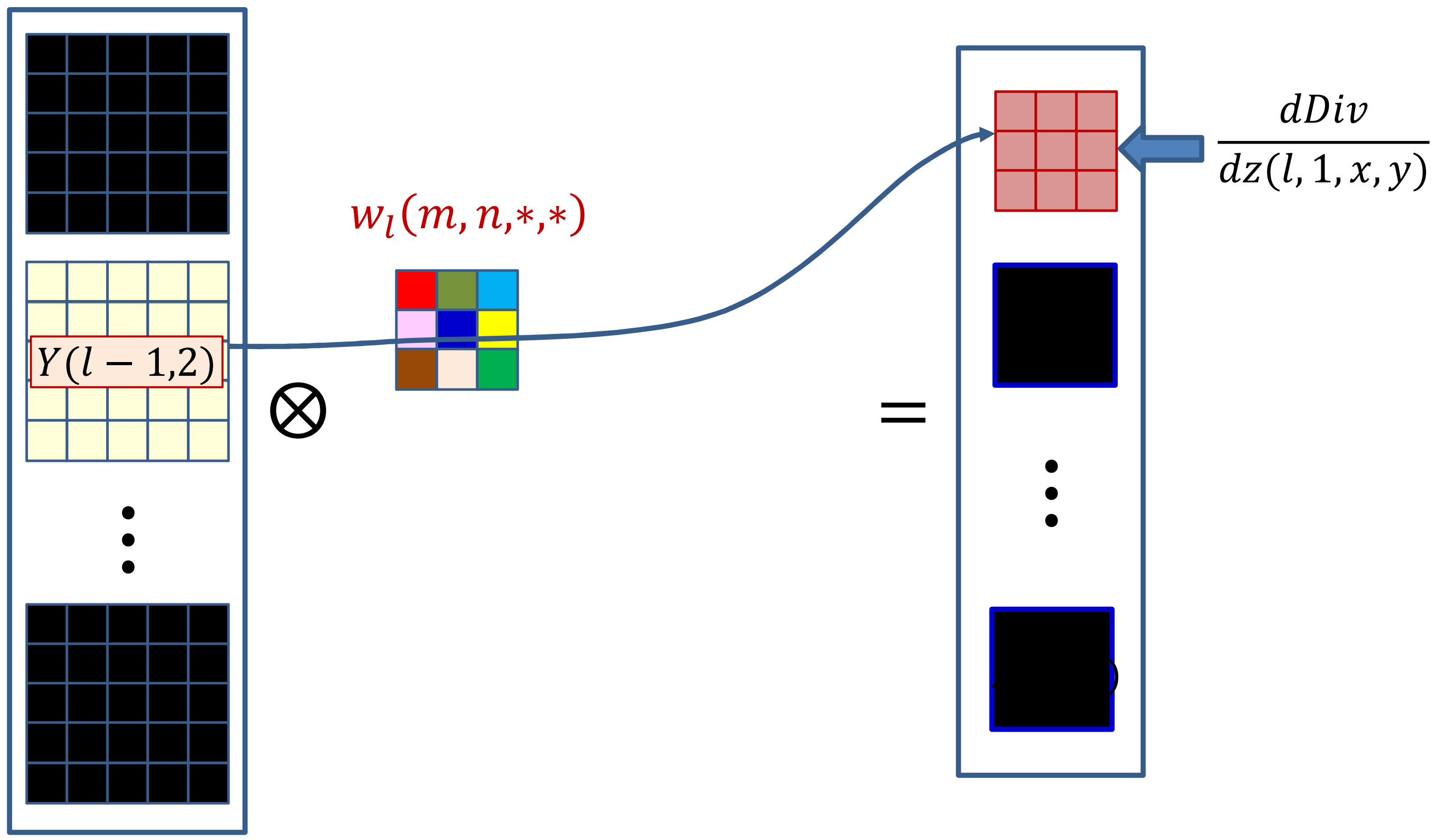
Recap: Convolution



$$z(l,n,x,y) = \sum_m \sum_{i=0}^{2} \sum_{j=0}^{2} w_l(m,n,i,j) y(l-1,m,x+i,y+j) + b_l(n)$$

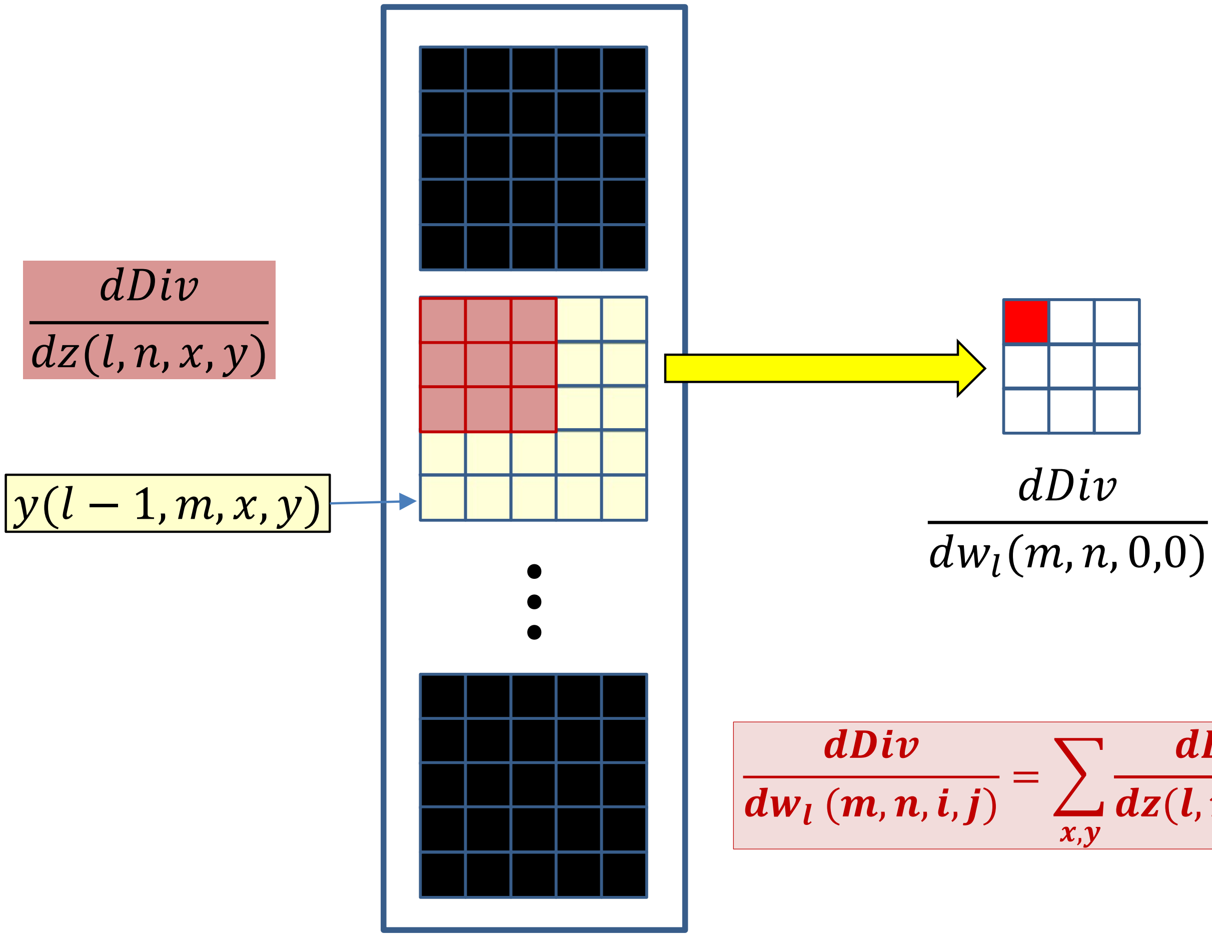
- $Y(l-1,m)$ influences $Z(l,n)$ through $w_l(m,n)$

The filter derivative



- The derivatives of the divergence w.r.t. every element of $Z(l,n)$ is known
  - Must use them to compute the derivative for $w_l(m,n,*,*)$

The filter derivative



$$\frac{dDiv}{dw_l\,(m,n,i,j)} = \sum_{x,y} \frac{dDiv}{dz(l,n,x,y)} y(l-1,m,x+i,y+j)$$

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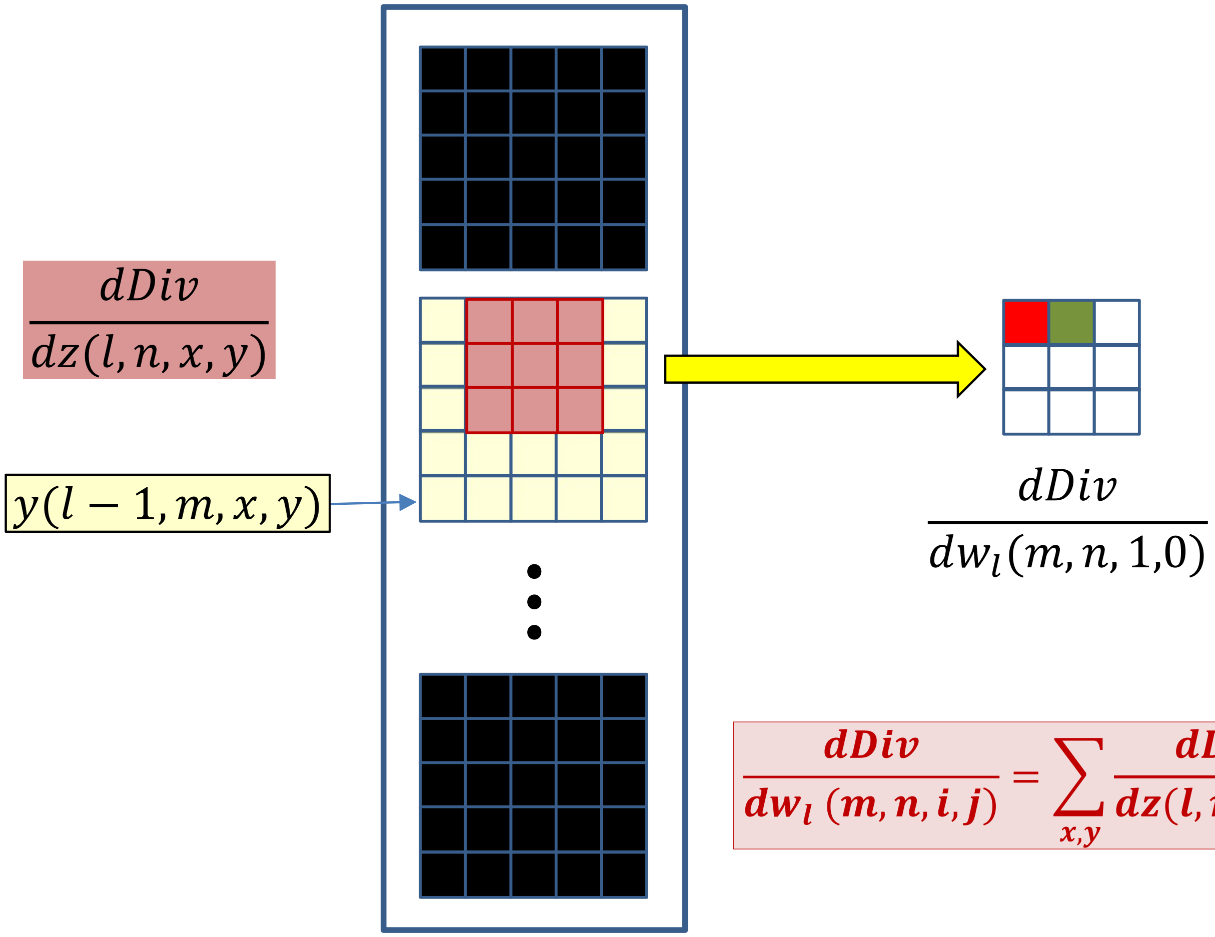

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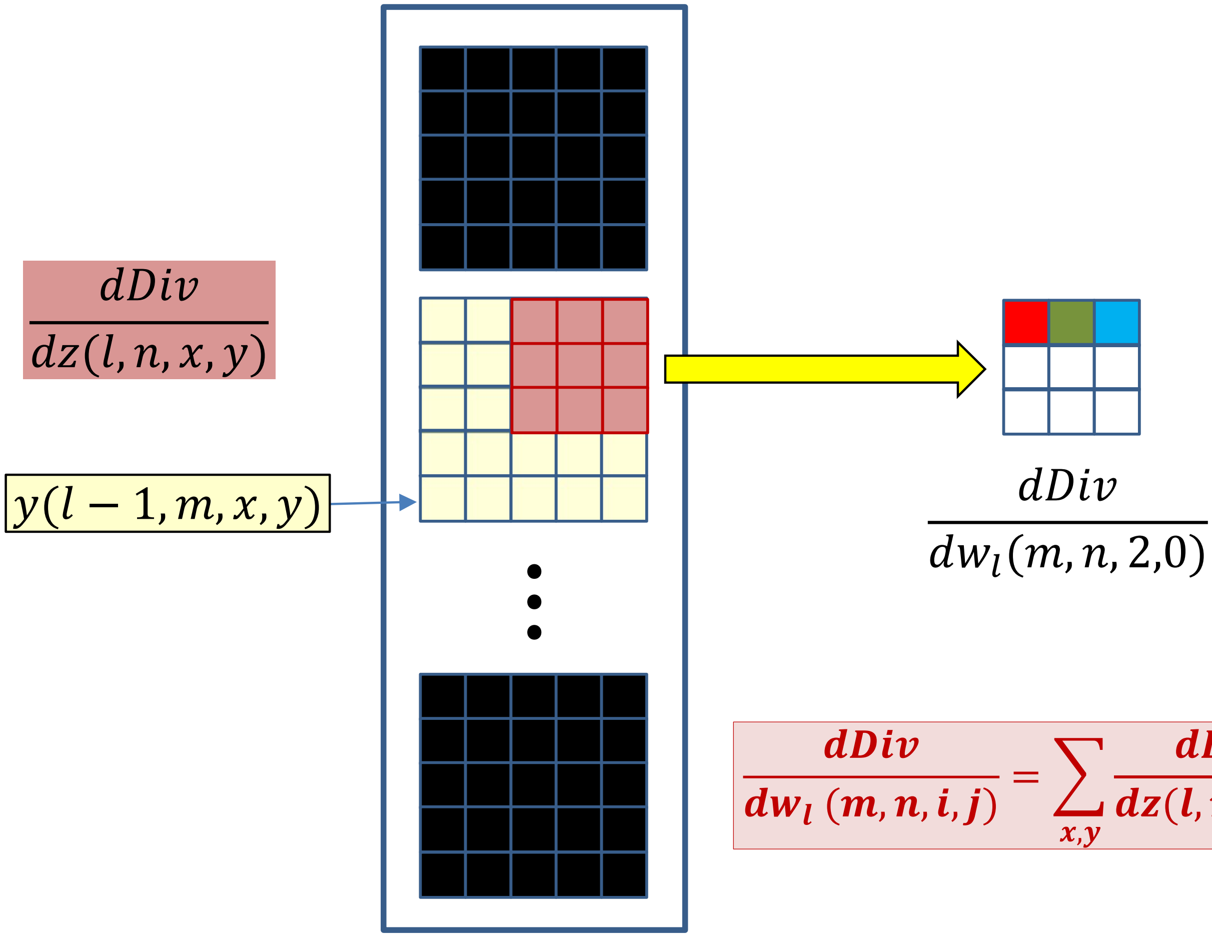

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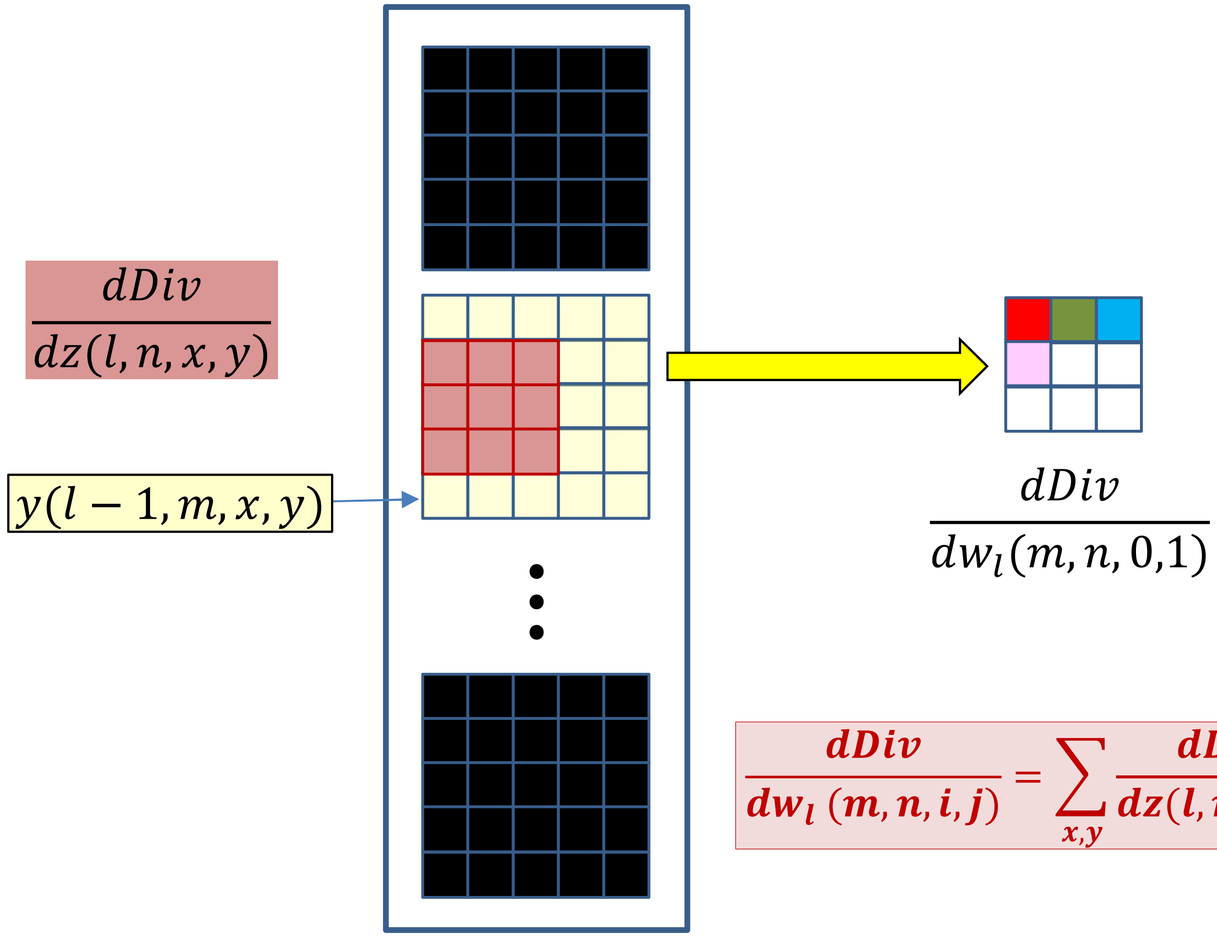

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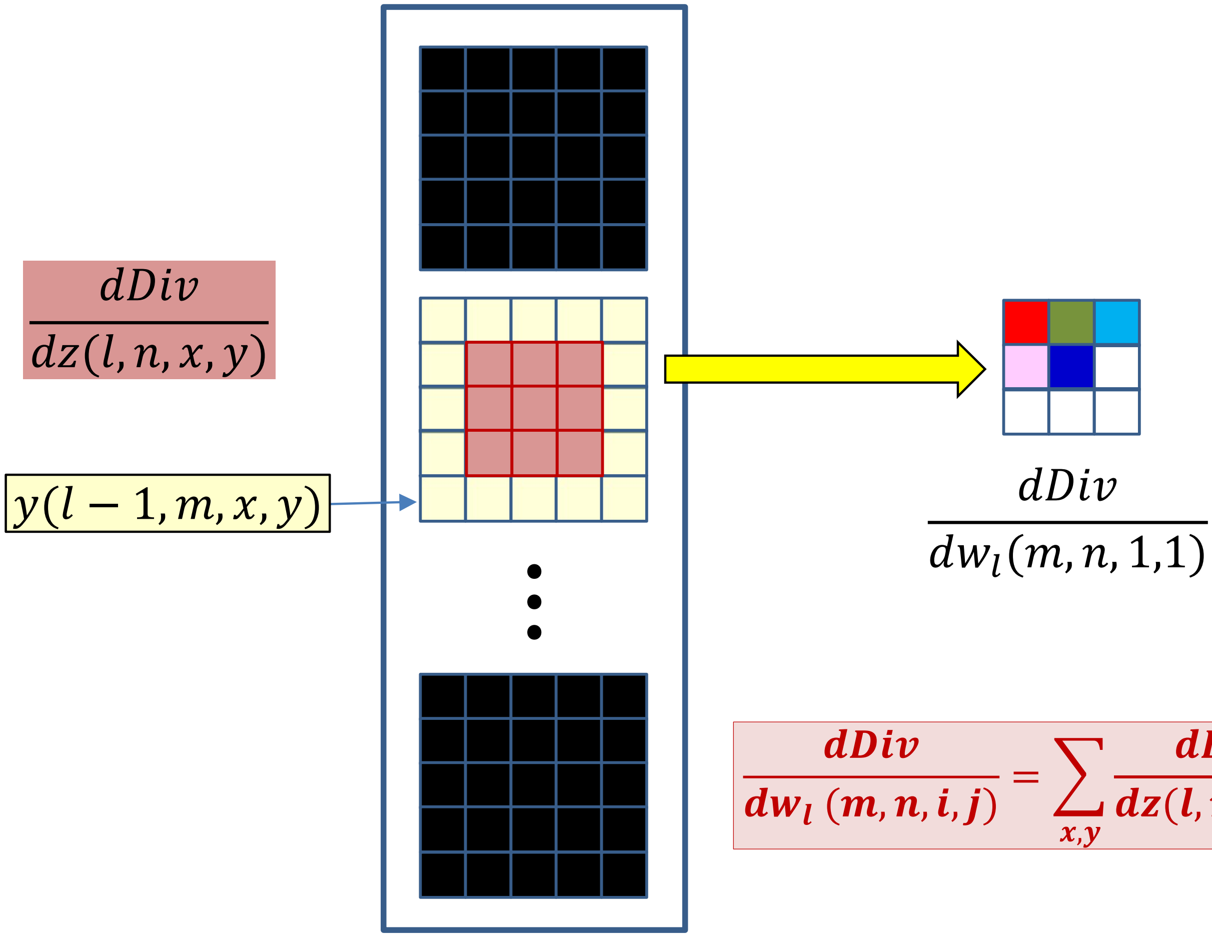

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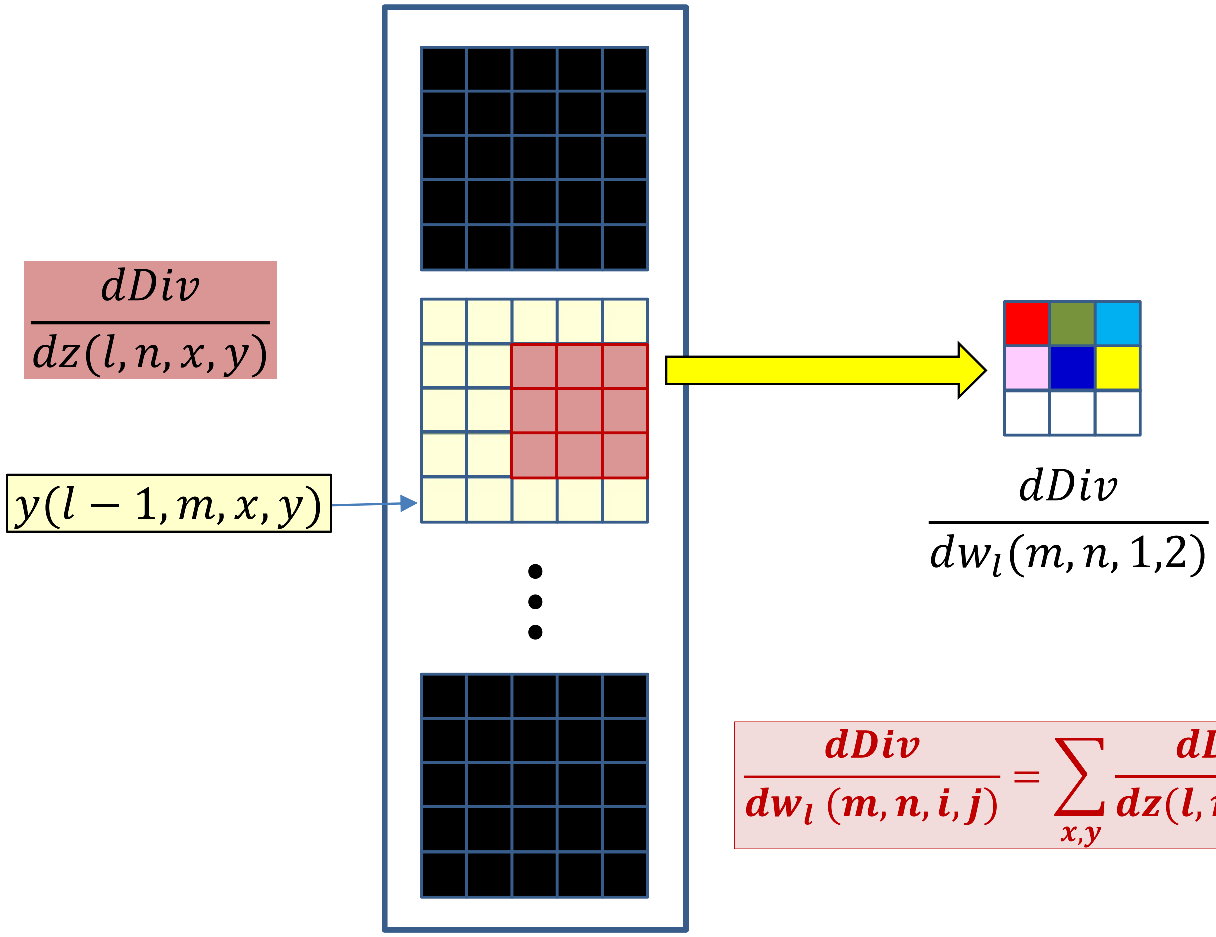

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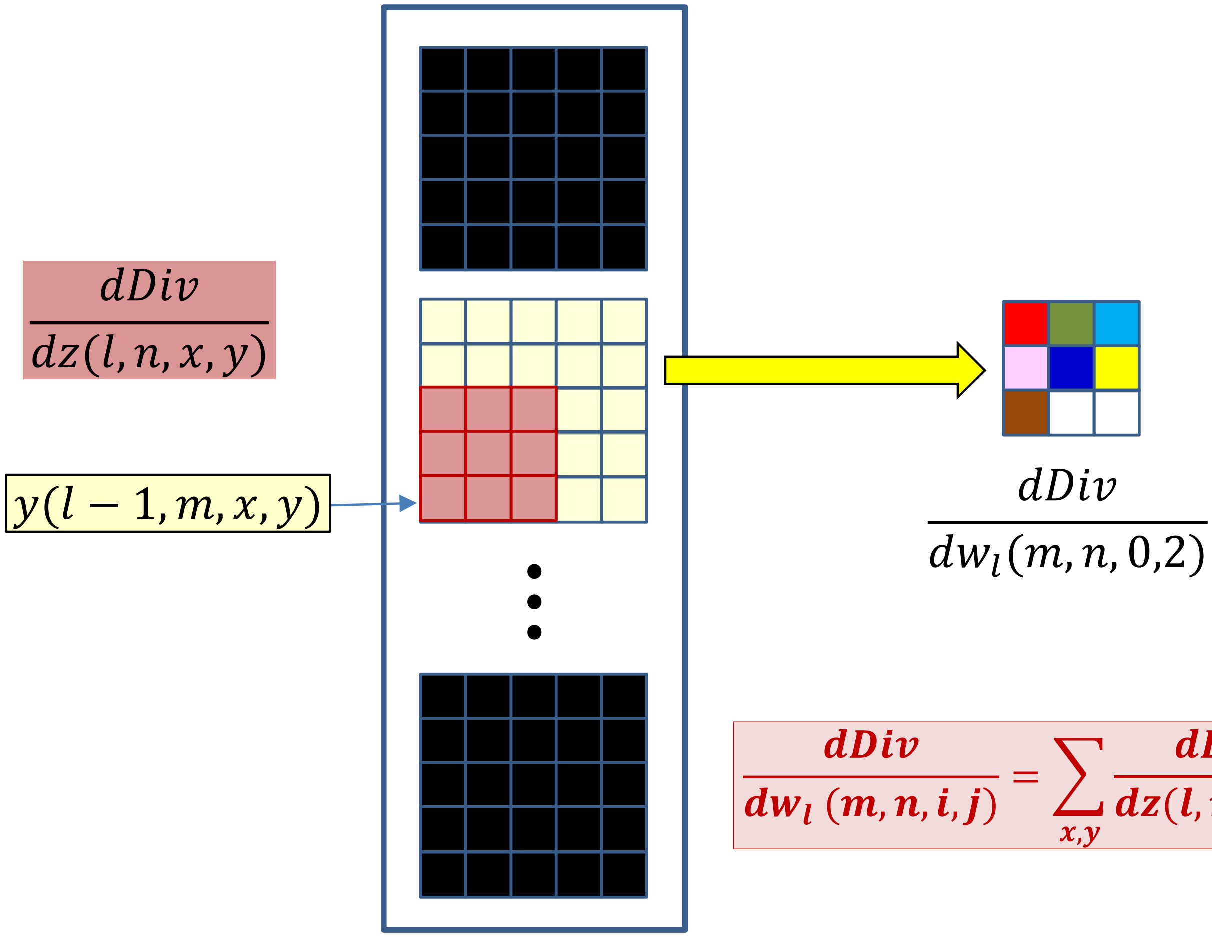

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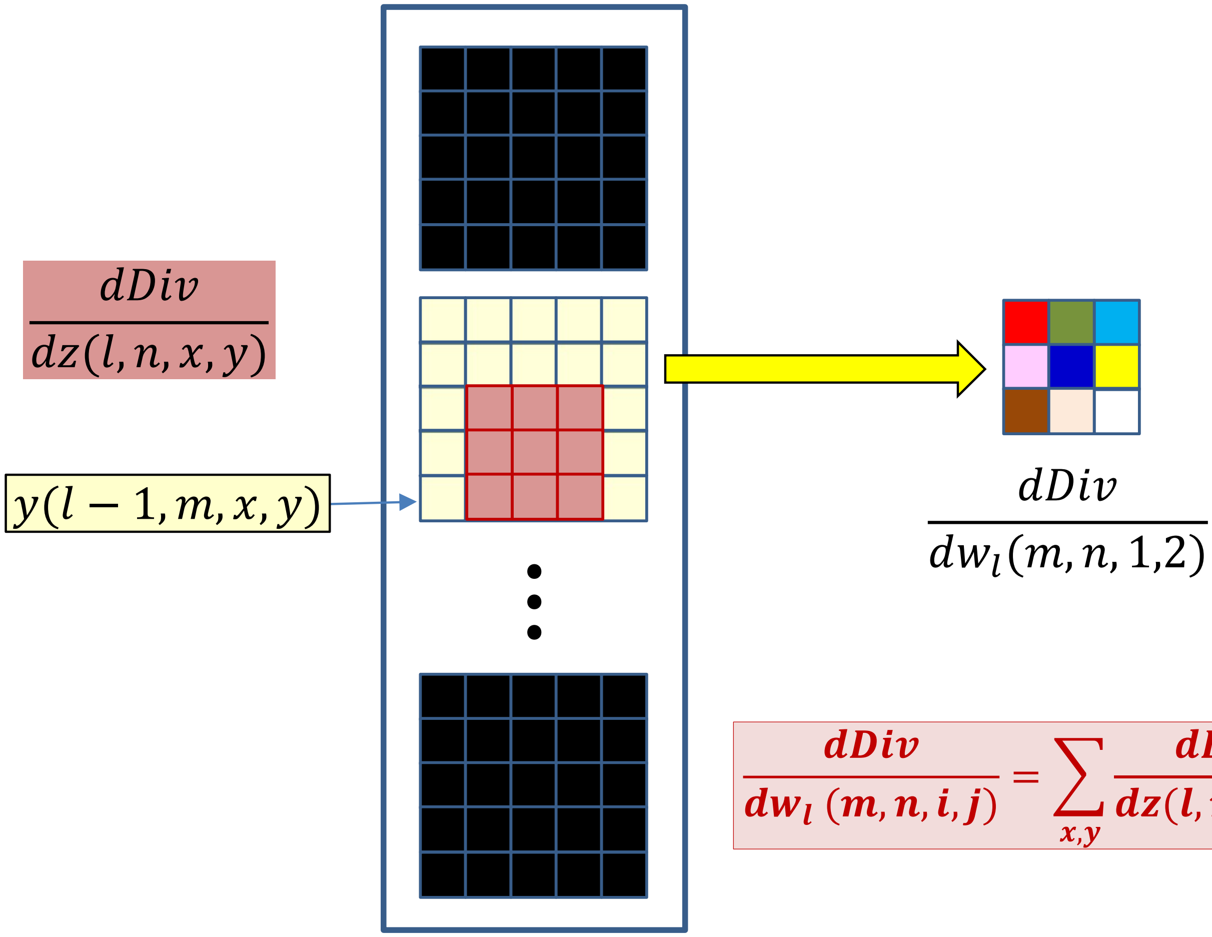

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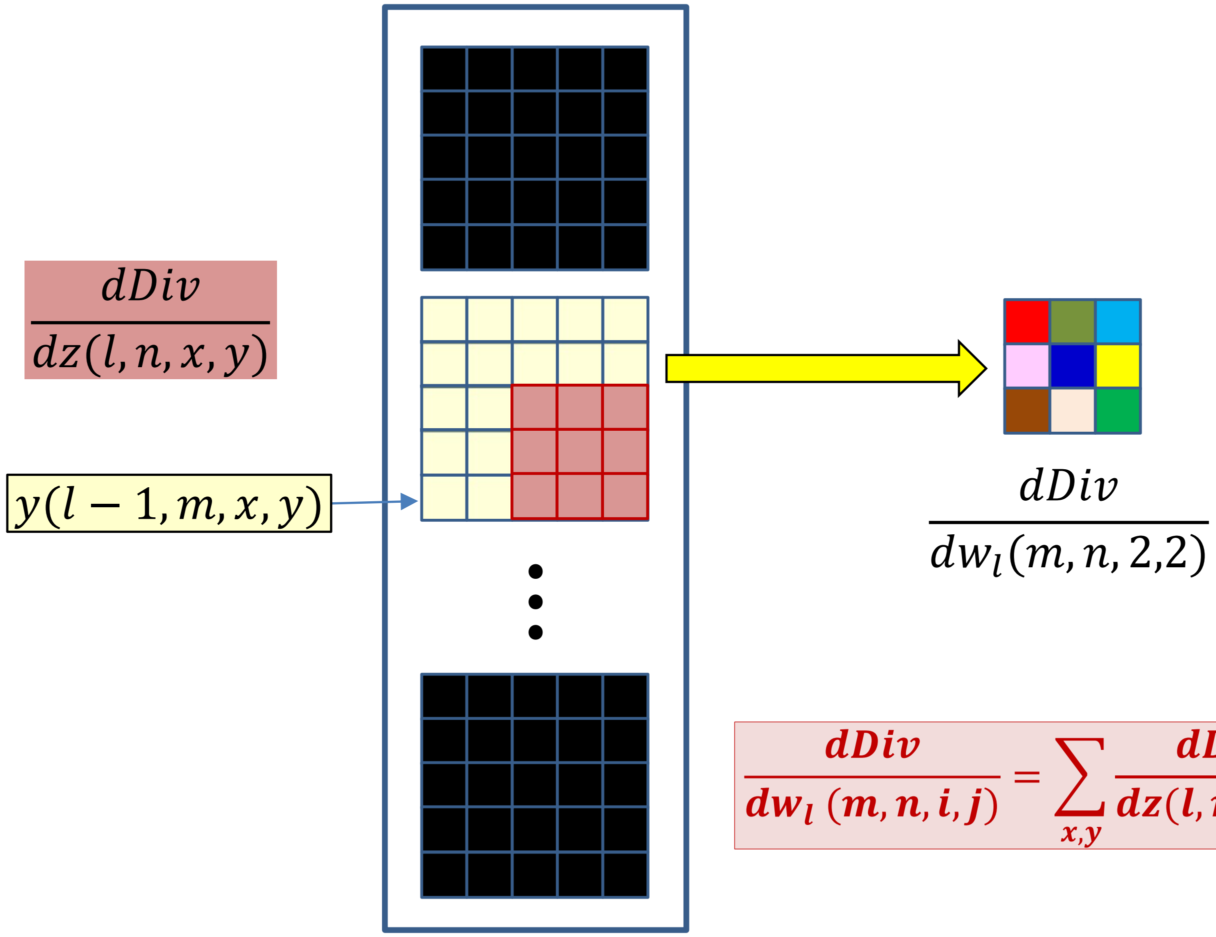

$$\frac{\boldsymbol{dDiv}}{\boldsymbol{dw_l\,(m,n,i,j)}} = \sum_{x,y} \frac{\boldsymbol{dDiv}}{\boldsymbol{dz(l,n,x,y)}} \boldsymbol{y(l-1,m,x+i,y+j)}$$

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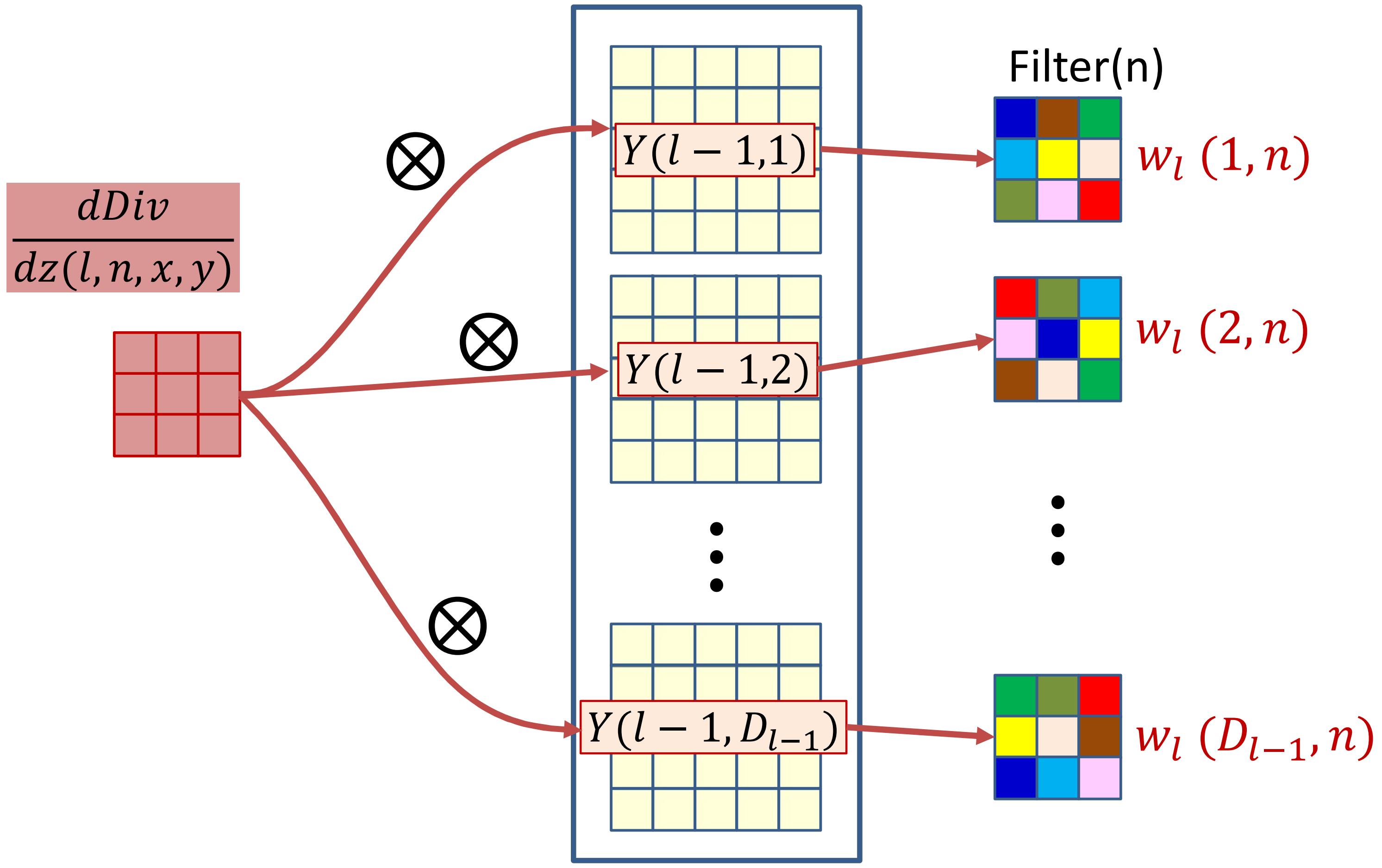
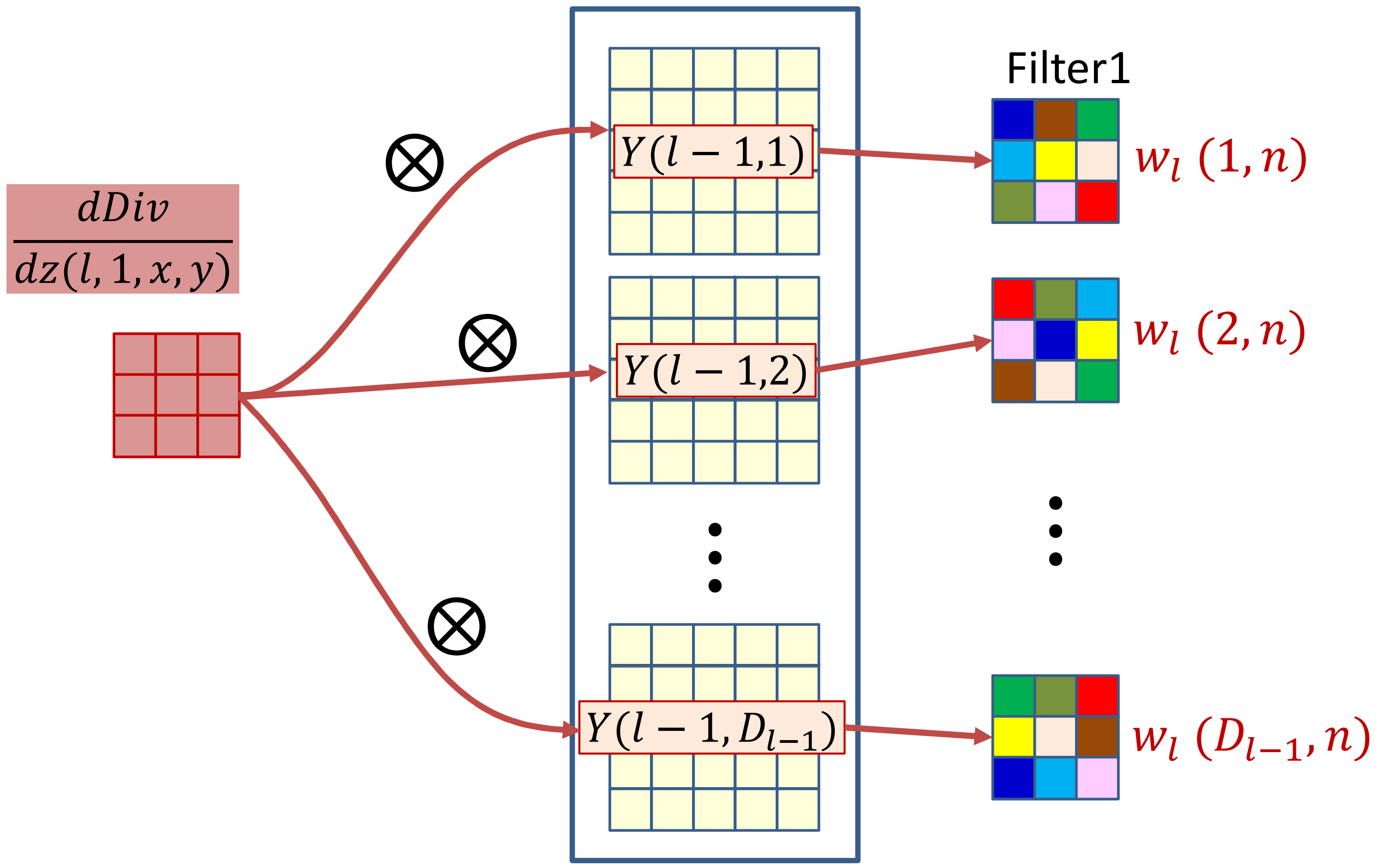


- The derivative of the nth affine map $Z(l,n)$ convolves with every output map $Y(l-1,m)$ of the $(l-1)$th layer, to get the derivative for $w_l(m,n)$, the mth “channel” of the nth filter

The filter derivative



$$\frac{dDiv}{dw_l\,(m,n,i,j)} = \sum_{x,y} \frac{dDiv}{dz(l,n,x,y)} y(l-1,m,x+i,y+j)$$

$$= \frac{dDiv}{dz(l,n)} \otimes y(l-1,m)$$

If $Y(l-1,m)$ was zero padded in the forward pass, it must be zero padded for backprop

Poll 4

Select all statements that are true about how to compute the derivative of the divergence w.r.t lth layer filters using backpropagation

- The derivative for the mth plane of the nth filter is computed by convolving the mth input (l-1th) layer map with the nth output (lth) layer affine derivative map
- The output map must be flipped left-right/up-down before convolution

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Derivatives for the filters at layer *l*: Vector notation

```
# The weight W(l,j)is a 3D $D_{l-1} x K_l x K_l$
# Assuming that derivative maps have been upsampled
#     if stride > 1
# Also assuming y map has been zero-padded if this was
#     also done in the forward pass
# The width and height of the dz map are W and H
```

```
for n = 1:$D_l$
  for x = 1:$K_l$
    for y = 1:$K_l$
      for m = 1:$D_{l-1}$
        dw(l,m,n,x,y) = dz(l,n,:,:).           #dot product
                        y(l-1,m,x:x+H-1,y:y+W-1)
```

Derivatives through a convolutional layer

- The entire process is simpler if we simply look at it through code
  - Through the reapplication of two simple rules:
- For any computation of the form

$$y = \sigma(z)$$

  - The loss derivative for z given the loss derivative of y is

$$\frac{dL}{dz} = \frac{dL}{dy}\sigma^{'(z)}$$

- For any computation in the forward pass

$$z = wy$$

  - The backward computation to compute loss derivatives for the terms on the right, given loss derivatives to the left is

$$dL/dy \mathrel{+}= wdL/dz \;;\; dL/dw \mathrel{+}= ydL/dz$$

  - Since this is "backpropgation", all computations are reversed

CNN: Forward

```
Y(0,:,:,:) = Image
for l = 1:L  # layers operate on vector at (x,y)
    for x = 1:$W_{l-1}$-$K_l$+1
        for y = 1:$H_{l-1}$-$K_l$+1
            for j = 1:$D_l$
                z(l,j,x,y) = 0
                for i = 1:$D_{l-1}$
                    for x' = 1:$K_l$
                        for y' = 1:$K_l$
                            z(l,j,x,y) += w(l,j,i,x',y')
                                          Y(l-1,i,x+x'-1,y+y'-1)
                Y(l,j,x,y) = activation(z(l,j,x,y))

Y = softmax( Y(L,:,1,1)..Y(L,:,W-K+1,H-K+1) )
```

Switching to 1-based indexing with appropriate adjustments

Backward layer l

```
dw(l) = zeros($D_l$x$D_{l-1}$x$K_l$x$K_l$)
dY(l-1) = zeros($D_{l-1}$x$W_{l-1}$x$H_{l-1}$)
for x = $W_{l-1}$-$K_l$+1:downto:1
   for y = $H_{l-1}$-$K_l$+1:downto:1
      for j = $D_l$:downto:1
         dz(l,j,x,y) = dY(l,j,x,y).f'(z(l,j,x,y))
         for i = $D_{l-1}$:downto:1
            for x' = $K_l$:downto:1
               for y' = $K_l$:downto:1
                  dY(l-1,i,x+x'-1,y+y'-1) +=
                                 w(l,j,i,x',y')dz(l,j,x,y)
                  dw(l,j,i,x',y') +=
                           dz(l,j,x,y)Y(l-1,i,x+x'-1,y+y'-1)
```

Complete Backward (no pooling)

```
dY(L) = dDiv/dY(L)
for l = L:downto:1   # Backward through layers
    dw(l) = zeros($D_l$x$D_{l-1}$x$K_l$x$K_l$)
    dY(l-1) = zeros($D_{l-1}$x$W_{l-1}$x$H_{l-1}$)
    for x = $W_{l-1}$-$K_l$+1:downto:1
        for y = $H_{l-1}$-$K_l$+1:downto:1
            for j = $D_l$:downto:1
                dz(l,j,x,y) = dY(l,j,x,y).f'(z(l,j,x,y))
                for i = $D_{l-1}$:downto:1
                    for x' = $K_l$:downto:1
                        for y' = $K_l$:downto:1
                            dY(l-1,i,x+x'-1,y+y'-1) +=
                                w(l,j,i,x',y')dz(l,j,x,y)
                            dw(l,j,i,x',y') +=
                            dz(l,j,x,y)y(l-1,i,x+x'-1,y+y'-1)
```

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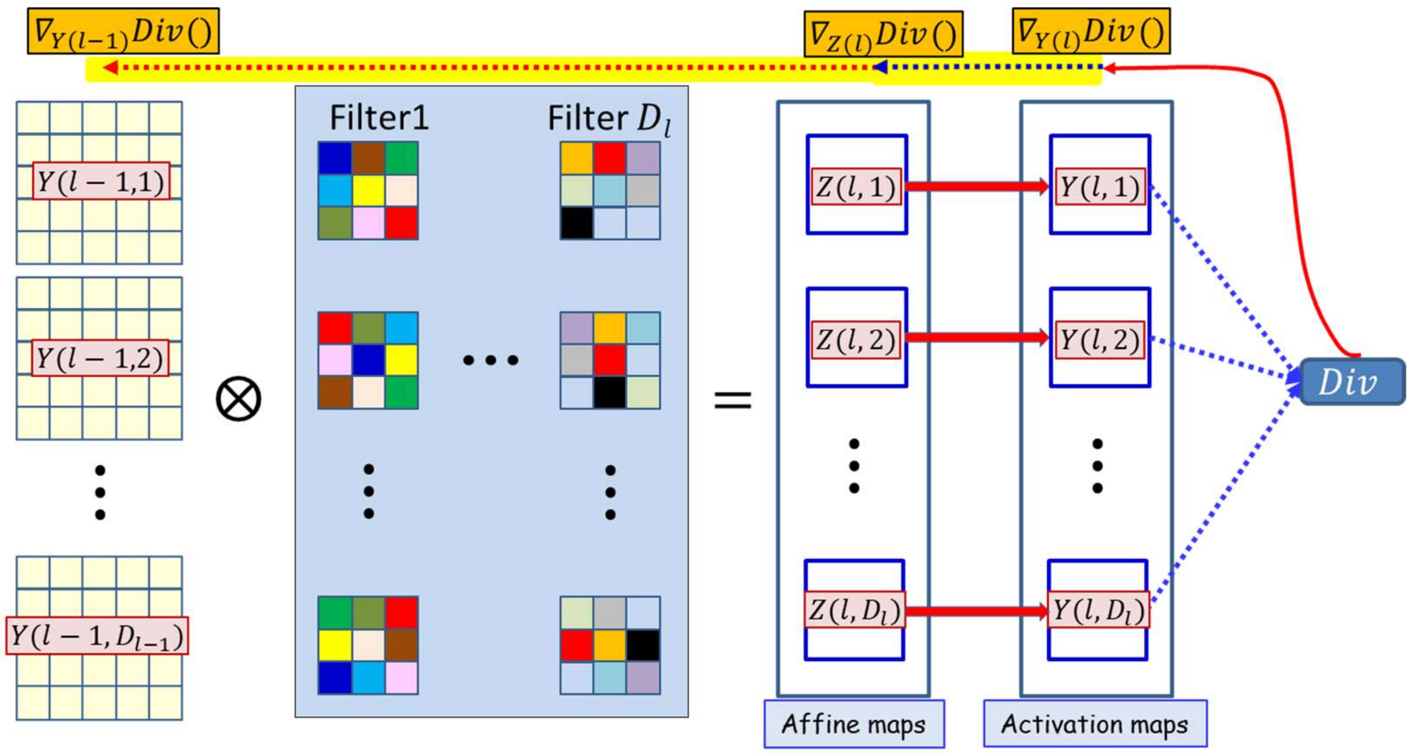
    for x = $W_{l-1}$-$K_l$+1:downto:1
        for y = $H_{l-1}$-$K_l$+1:downto:1
            for j = $D_l$:downto:1
                dz(l,j,x,y) = dY(l,j,x,y).f'(z(l,j,x,y))
                for i = $D_{l-1}$:downto:1
                    for x' = $K_l$:downto:1
                        for y' = $K_l$:downto:1
                            dY(l-1,i,x+x'-1,y+y'-1) +=
                                w(l,j,i,x',y')dz(l,j,x,y)
                            dw(l,j,i,x',y') +=
                            dz(l,j,x,y)y(l-1,i,x+x'-1,y+y'-1)
```

Multiple ways of recasting this as tensor/ vector operations.

Will not discuss here

Backpropagation: Convolutional layers



- **For convolutional layers**:

✓ How to compute the derivatives w.r.t. the affine combination $Z(l)$ maps from the activation output maps $Y(l)$

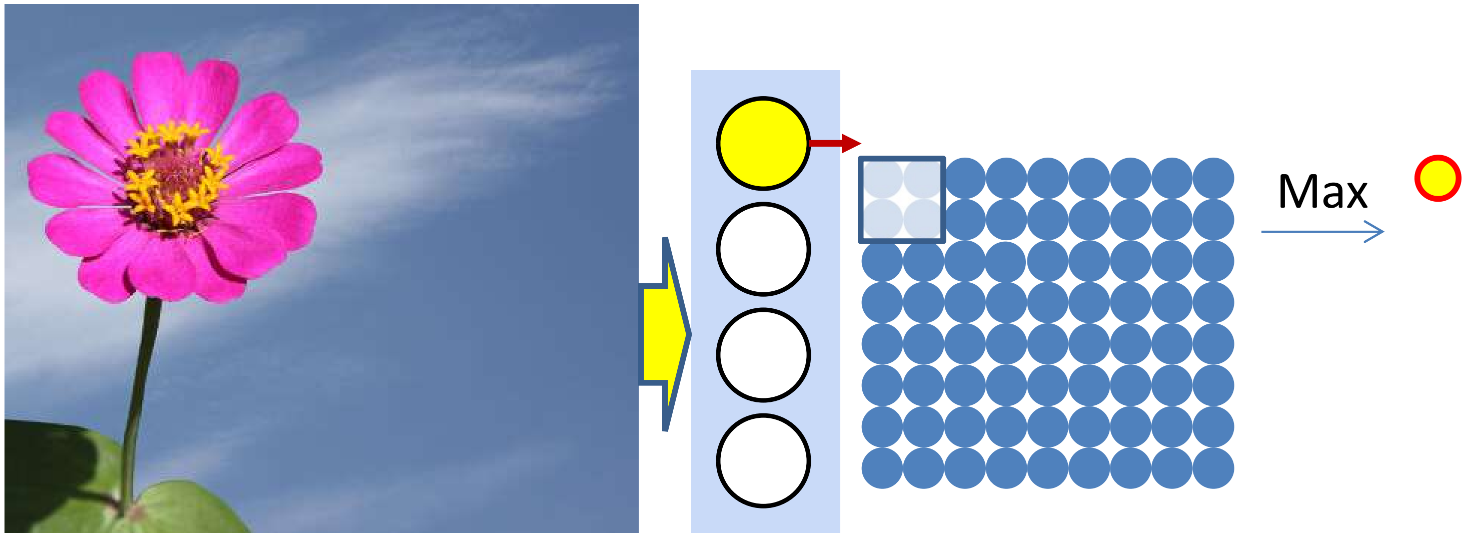
✓ How to compute the derivative w.r.t. $Y(l-1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$

Backpropagation: Convolutional and Pooling layers

- **Assumption:** We already have the derivatives w.r.t. the elements of the maps output by the final convolutional (or pooling) layer
  - Obtained as a result of backpropagating through the flat MLP

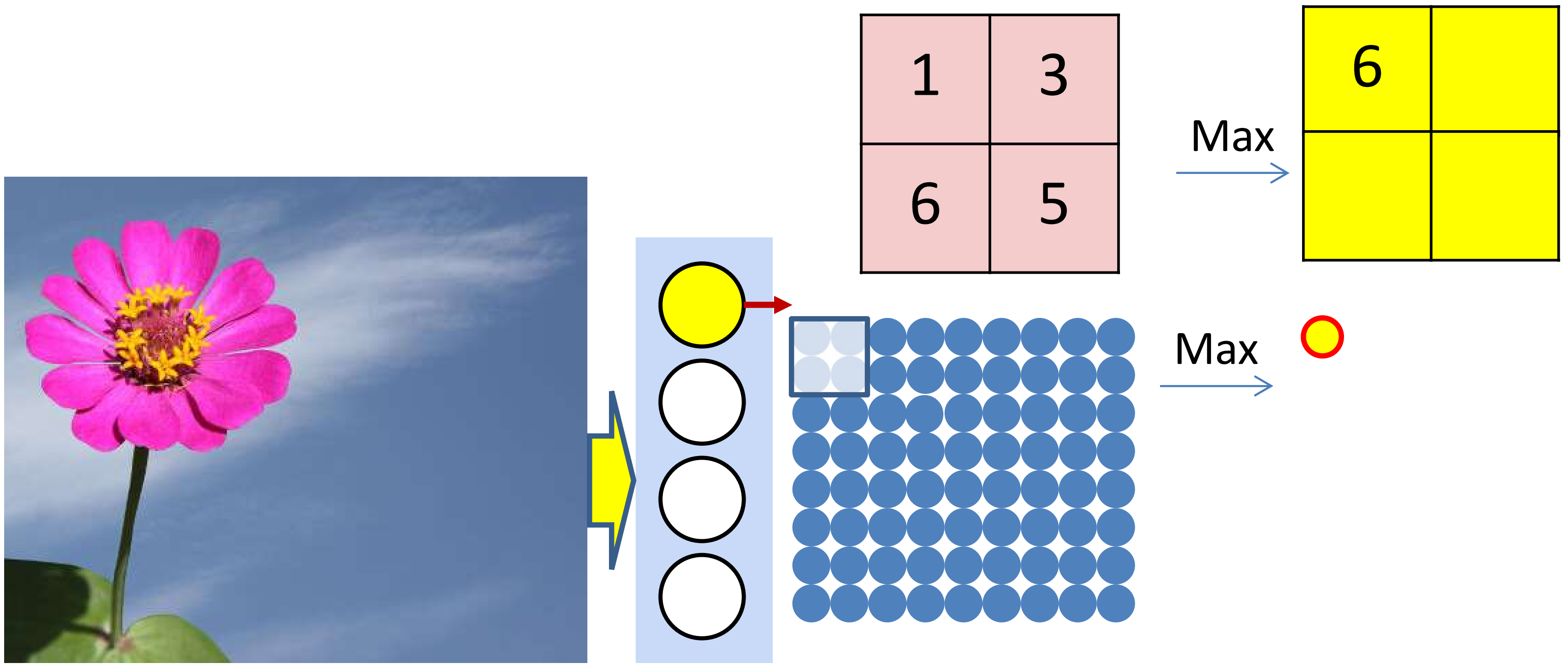
- **Required:**
  - ✓ **For convolutional layers**:
    - How to compute the derivatives w.r.t. the affine combination $Z(l)$ maps from the activation output maps $Y(l)$
    - How to compute the derivative w.r.t. $Y(l-1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$
  - **For pooling layers**:
    - How to compute the derivative w.r.t. $Y(l-1)$ given derivatives w.r.t. $Y(l)$

Pooling



- Pooling "pools" groups of values to reduce jitter-sensitivity
  - Scanning with a "pooling" filter
- The most common pooling is "Max" pooling

Max Pooling

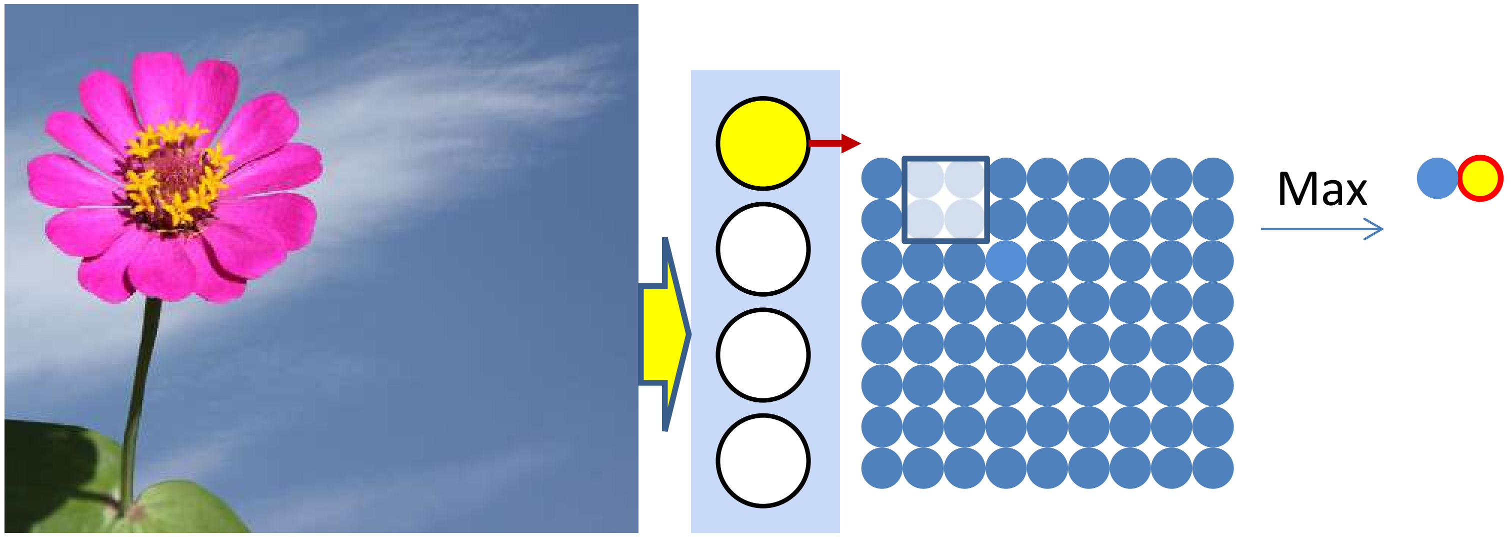


- Max pooling selects the largest from a pool of elements
- Pooling is performed by “scanning” the input

$$P(l,m,i,j) = \underset{\substack{k\in\{i,\, i+K_{lpool}-1\},\\ n\in\{j, j+K_{lpool}-1\}}}{\operatorname{argmax}} Y(l-1,m,k,n)$$

$$Y(l,m,i,j) = Y(l-1,m,P(l,m,i,j))$$

Max pooling



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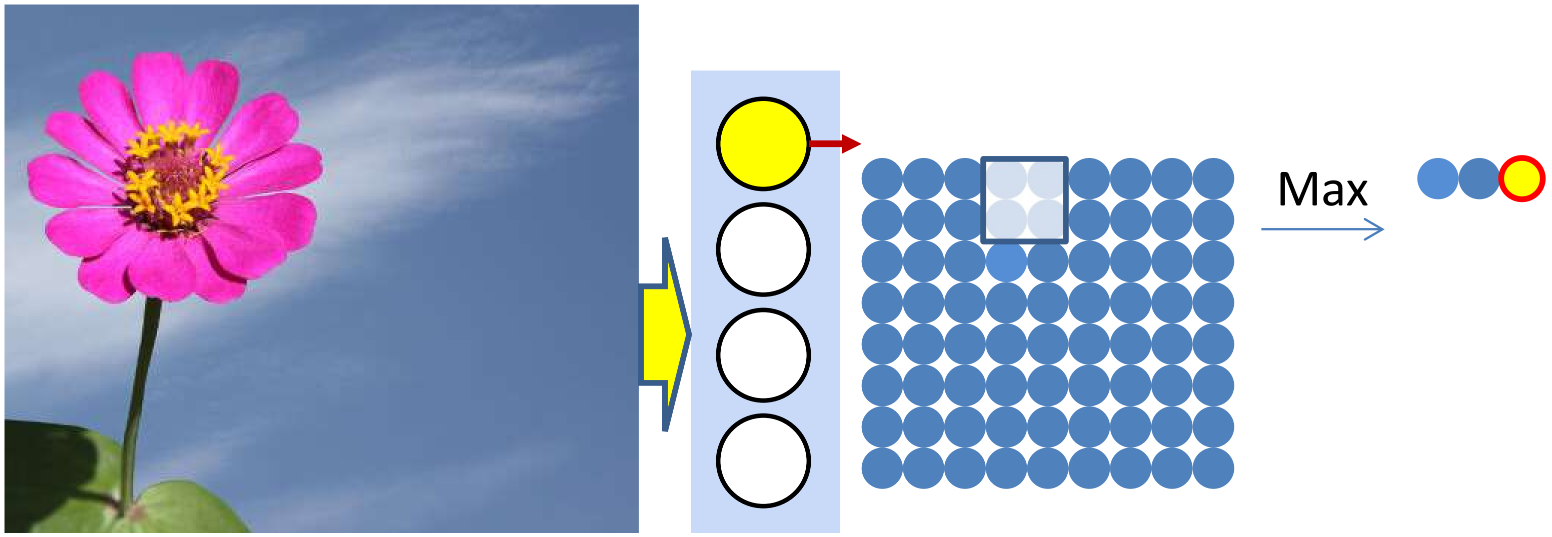
Max pooling



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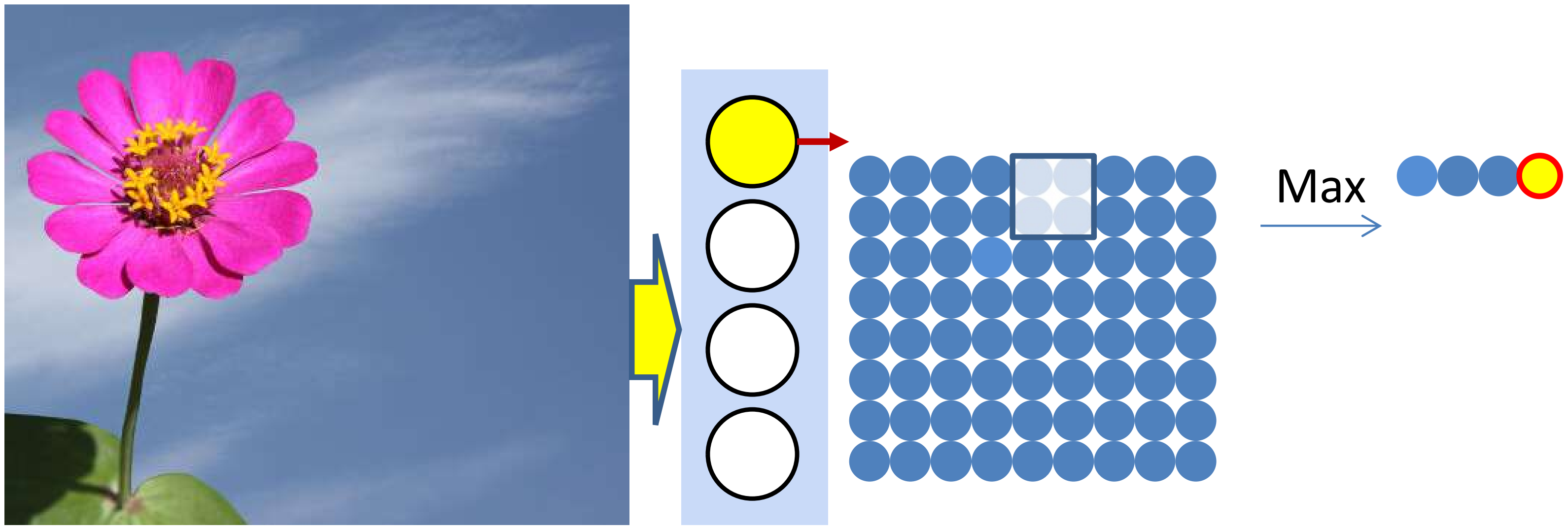
Max pooling



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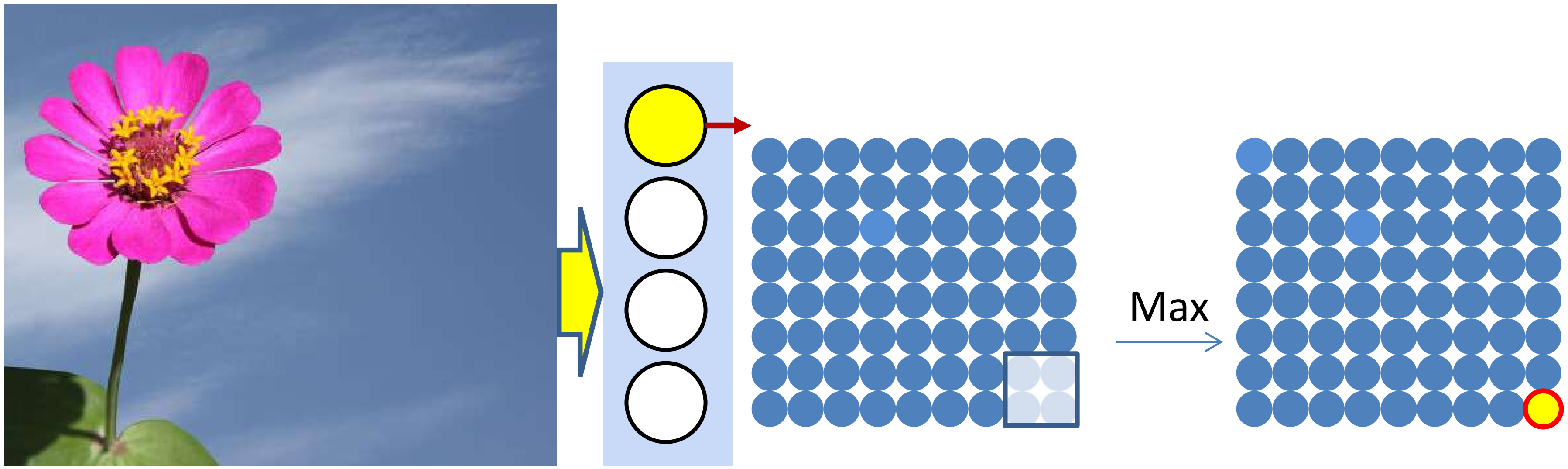
Max pooling



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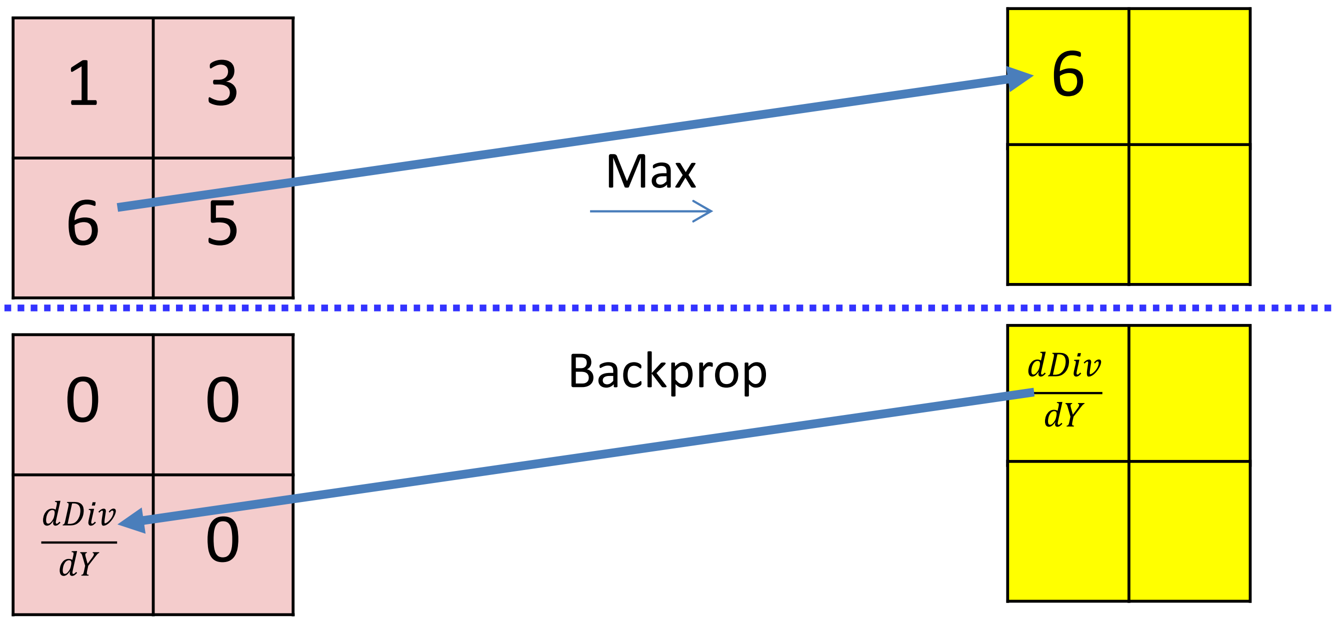
$$P(l,m,i,j) = \underset{\substack{k\in\{i,\, i+K_{lpool}-1\},\\ n\in\{j,j+K_{lpool}-1\}}}{\operatorname{argmax}} Y(l-1,m,k,n)$$

$$Y(l,m,i,j) = Y(l-1,m,P(l,m,i,j))$$

Derivative of Max pooling



$$\boxed{\frac{dDiv}{dy(l-1,m,k,l)} = \begin{cases} \frac{dDiv}{dy(l,m,i,j)} & if\ (k,l) = P(l,m,i,j) \\ 0\ otherwise & \end{cases}}$$

- Max pooling selects the largest from a pool of elements

$$P(l,m,i,j) = \underset{\substack{k\in\{i,\, i+K_{lpool}-1\},\\ n\in\{j,j+K_{lpool}-1\}}}{\operatorname{argmax}} Y(l-1,m,k,n)$$

$$y(l,m,i,j) = y(l-1,m,P(l,m,i,j))$$

Max Pooling layer at layer *l*

a) Performed separately for every map (j).
*) *Not* combining multiple maps within a single max operation.
b) Keeping track of location of max

Max pooling

```
for j = 1:$D_l$
  for x = 1:$W_{l-1}-K_l+1$
    for y = 1:$H_{l-1}-K_l+1$
      pidx(l,j,x,y) = maxidx(y(l-1,j,x:x+$K_l$-1,y:y+$K_l$-1))
      y(l,j,x,y) = y(l-1,j,pidx(l,j,x,y))
```

Derivative of max pooling layer at layer l

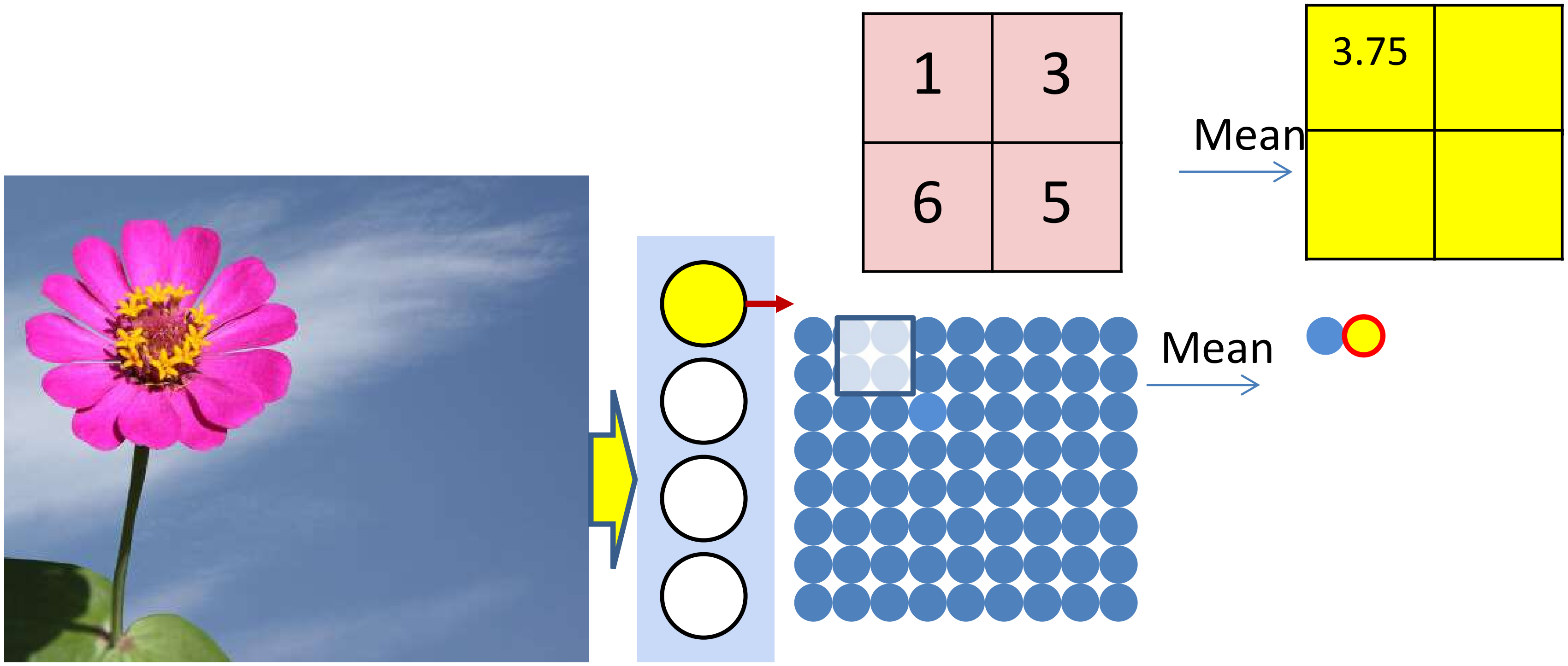
a) Performed separately for every map (j).
   *\) *Not combining multiple maps within a single max operation.*
b) Keeping track of location of max

Max pooling

```
dy(:,:,:) = zeros(D_l x W_l x H_l)
for j = 1:D_l
    for x = 1:W_l
        for y = 1:H_l
            dy(l-1,j,pidx(l,j,x,y)) += dy(l,j,x,y)
```

“+=“ because this entry may be selected in multiple adjacent overlapping windows

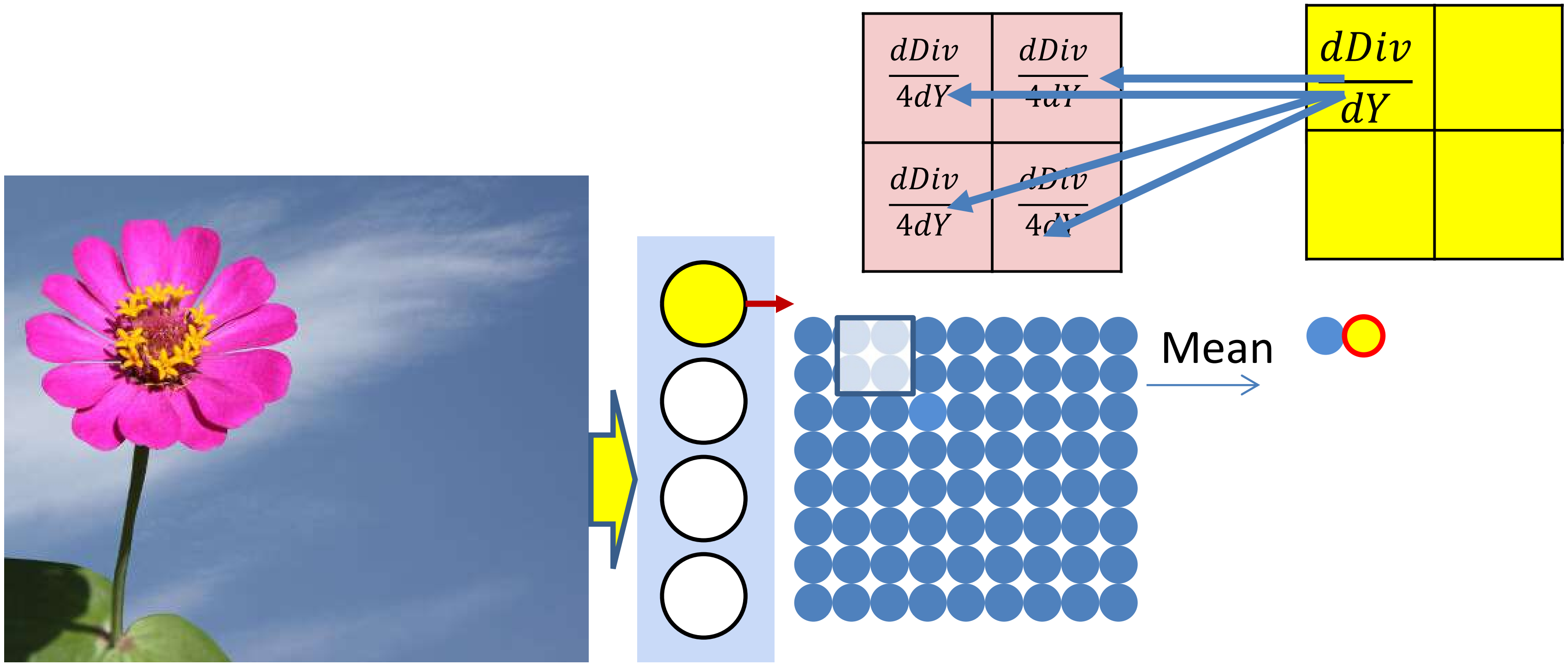
Mean pooling



- Mean pooling compute the mean of a pool of elements
- Pooling is performed by “scanning” the input

$$y(l,m,i,j) = \frac{1}{K_{lpool}^2} \sum_{\substack{k\in\{i,\, i+K_{lpool}-1\},\\ n\in\{j, j+K_{lpool}-1\}}} y(l-1,m,k,n)$$

Derivative of mean pooling



- The derivative of mean pooling is distributed over the pool

$$k \in \{i, i + K_{lpool} - 1\},\quad n \in \{j, j + K_{lpool} - 1\} \quad dy(l-1, m, k, n) \mathrel{+}= \frac{1}{K_{lpool}^2} dy(l, m, k, n)$$

Mean Pooling layer at layer *l*

Mean pooling

```
for j = 1:$D_l$  #Over the maps
    for x = 1:$W_{l-1}$-$K_l$+1 #$K_l$ = pooling kernel size
        for y = 1:$H_{l-1}$-$K_l$+1
            y(l,j,x,y) = mean(y(l-1,j,x:x+$K_l$-1,y:y+$K_l$-1))
```

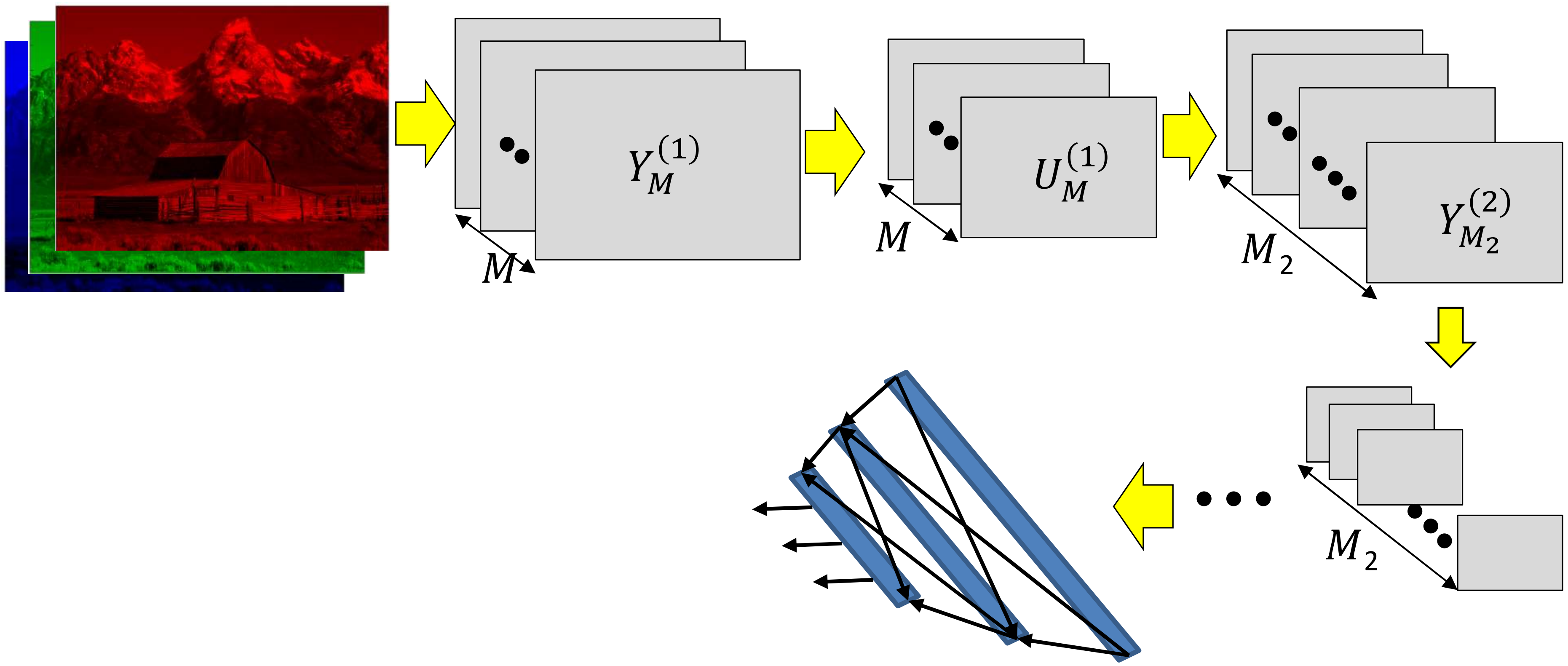
Derivative of mean pooling layer at layer l

Mean pooling

```
dy(:,:,:) = zeros($D_l$ x $W_l$ x $H_l$)
for k = 1:$D_l$
  for x = 1:$W_l$
    for y = 1:$H_l$
      for i = 1:$K_{lpool}$
        for j = 1:$K_{lpool}$
          dy(l-1,k,p,x+i,y+j) += $(1/K^2_{lpool})$dy(l,k,x,y)
```

“+=“ because adjacent windows may overlap

Learning the network



- Have shown the derivative of divergence w.r.t every intermediate output, and every free parameter (filter weights)
- Can now be embedded in gradient descent framework to learn the network
- Still missing one component… resampling
  - Next class

Story so far

- The convolutional neural network is a supervised version of a computational model of mammalian vision
- It includes
  - Convolutional layers comprising learned filters that scan the outputs of the previous layer
  - Pooling layers that operate over groups of outputs from the convolutional layer to reduce network size
- The parameters of the network can be learned through regular back propagation
  - Maxpooling layers must propagate derivatives only over the maximum element in each pool
    - Other pooling operators can use regular gradients or subgradients
  - Derivatives must sum over appropriate sets of elements to account for the fact that the network is, in fact, a shared parameter network