

# Neural Networks: Optimization Part 1

**Intro to Deep Learning, Spring 2022**

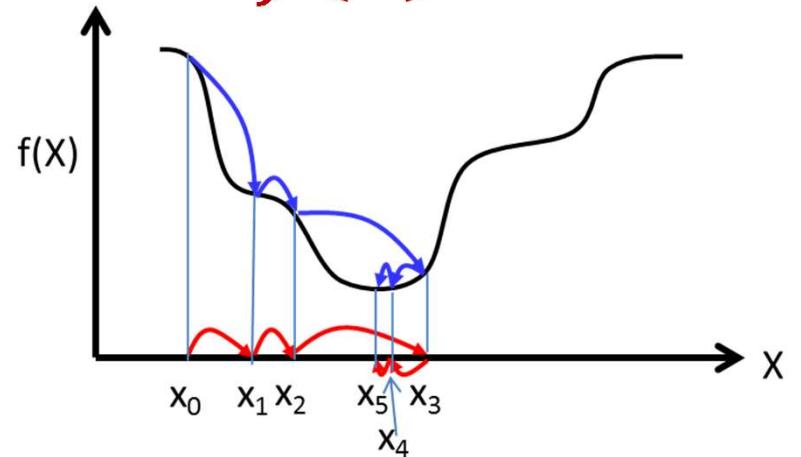
# Story so far

- Neural networks are universal approximators
  - Can model any odd thing
  - Provided they have the right architecture
- We must *train* them to approximate any function
  - Specify the architecture
  - Learn their weights and biases
- Networks are trained to minimize total “loss” on a training set
  - We do so through empirical risk minimization
- We use variants of gradient descent to do so
- The gradient of the error with respect to network parameters is computed through backpropagation



# Recap: Gradient Descent Algorithm

- In order to minimize any function  $f(x)$  w.r.t.  $x$
- Initialize:
  - $x^0$
  - $k = 0$
- Do
  - $k = k + 1$
  - $x^{k+1} = x^k - \eta \nabla_x f^T$
- while  $|f(x^k) - f(x^{k-1})| > \varepsilon$



# Recap: Training Neural Nets by Gradient Descent

Total training error:

$$Loss = \frac{1}{T} \sum_t Div(Y_t, d_t; W_1, W_2, \dots, W_K)$$

- Gradient descent algorithm:
- Initialize all weights  $W_1, W_2, \dots, W_K$
- Do:
  - For every layer  $k$ , compute:
    - $\nabla_{W_k} Loss = \frac{1}{T} \sum_t \nabla_{W_k} Div(Y_t, d_t)$
    - $W_k = W_k - \eta \nabla_{W_k} Loss^T$
- Until  $Loss$  has converged

Computed using backprop

# Neural network training algorithm

- Initialize all weights and biases ( $\mathbf{W}_1, \mathbf{b}_1, \mathbf{W}_2, \mathbf{b}_2, \dots, \mathbf{W}_N, \mathbf{b}_N$ )
  - Do:
    - $Loss = 0$
    - For all  $k$ , initialize  $\nabla_{\mathbf{W}_k} Loss = 0, \nabla_{\mathbf{b}_k} Loss = 0$
    - For all  $t = 1:T$  # Loop through training instances
      - Forward pass : Compute
        - Output  $\mathbf{Y}(\mathbf{X}_t)$ ,
        - Divergence  $\mathbf{Div}(\mathbf{Y}_t, \mathbf{d}_t)$
      - Backward pass: For all  $k$  compute:
        - $\nabla_{\mathbf{W}_k} \mathbf{Div}(\mathbf{Y}_t, \mathbf{d}_t), \nabla_{\mathbf{b}_k} \mathbf{Div}(\mathbf{Y}_t, \mathbf{d}_t)$
        - $\nabla_{\mathbf{W}_k} Loss += \nabla_{\mathbf{W}_k} \mathbf{Div}(\mathbf{Y}_t, \mathbf{d}_t); \nabla_{\mathbf{b}_k} Loss += \nabla_{\mathbf{b}_k} \mathbf{Div}(\mathbf{Y}_t, \mathbf{d}_t)$
    - For all  $k$ , update:
$$\mathbf{W}_k = \mathbf{W}_k - \frac{\eta}{T} (\nabla_{\mathbf{W}_k} Loss)^T; \quad \mathbf{b}_k = \mathbf{b}_k - \frac{\eta}{T} (\nabla_{\mathbf{b}_k} Loss)^T$$
  - Until  $Loss$  has converged
- 

# Issues

- Convergence: How well does it learn
  - And how can we improve it
- How well will it generalize (outside training data)
- What does the output really mean?
- *Etc..*

# Poll 0

Backpropagating from the  $k$ th layer, which is the derivative for the weights  $W_k$ ?

- $y_{k-1} \cdot \nabla_{z_k} \text{Div}$ : The product of the output  $y$  of the  $k - 1$ th layer and the derivative for the affine value  $z$  of the  $k$ th layer (in that order)
- $\nabla_{z_k} \text{Div} y_{k-1}$ : The product of the derivative for the affine value  $z$  at the  $k$ th layer and the output  $y$  of the  $k - 1$ th layer (in that order)
- $y_{k-1}^\top \cdot \nabla_{z_k} \text{Div}$ : The product of the transpose of the output  $y$  of the  $k - 1$ th layer and the derivative for the affine value  $z$  of the  $k$ th layer (in that order)
- $\nabla_{z_k} \text{Div} \cdot y_{k-1}^\top$ : The product of the derivative for the affine value  $z$  at the  $k$ th layer and the transpose output  $y$  of the  $k - 1$ th layer (in that order)

# Poll 0

Backpropagating from the  $k$ th layer, which is the derivative for the weights  $W_k$ ?

- $y_{k-1} \cdot \nabla_{z_k} \text{Div}$ : The product of the output  $y$  of the  $k - 1$ th layer and the derivative for the affine value  $z$  of the  $k$ th layer (in that order)
- $\nabla_{z_k} \text{Div } y_{k-1}$ : The product of the derivative for the affine value  $z$  at the  $k$ th layer and the output  $y$  of the  $k - 1$ th layer (in that order)
- $y_{k-1}^\top \cdot \nabla_{z_k} \text{Div}$ : The product of the transpose of the output  $y$  of the  $k - 1$ th layer and the derivative for the affine value  $z$  of the  $k$ th layer (in that order)
- $\nabla_{z_k} \text{Div} \cdot y_{k-1}^\top$ : The product of the derivative for the affine value  $z$  at the  $k$ th layer and the transpose output  $y$  of the  $k - 1$ th layer (in that order)

# Onward



# Onward

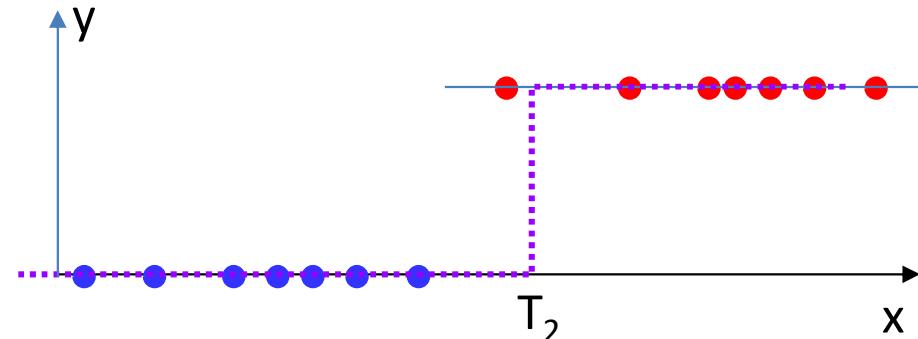
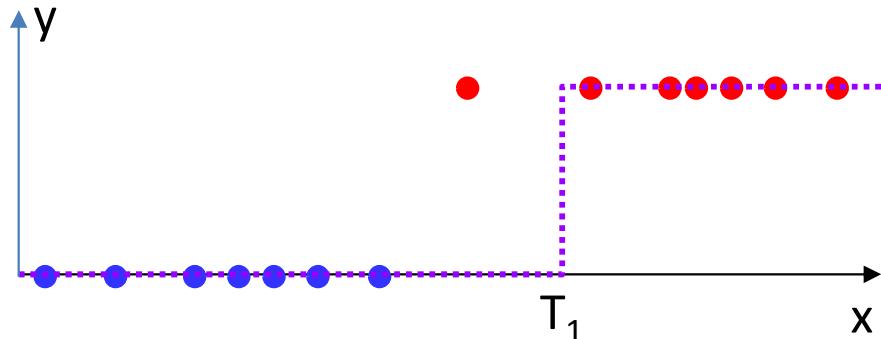
- Does backprop always work?
- Convergence of gradient descent
  - Rates, restrictions,
  - Hessians
  - Acceleration and Nestorov
  - Alternate approaches
- Modifying the approach: Stochastic gradients
- Speedup extensions: RMSprop, Adagrad

# Does backprop do the right thing?

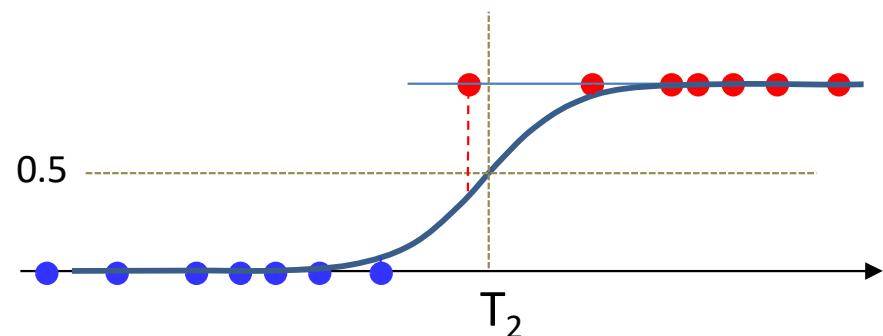
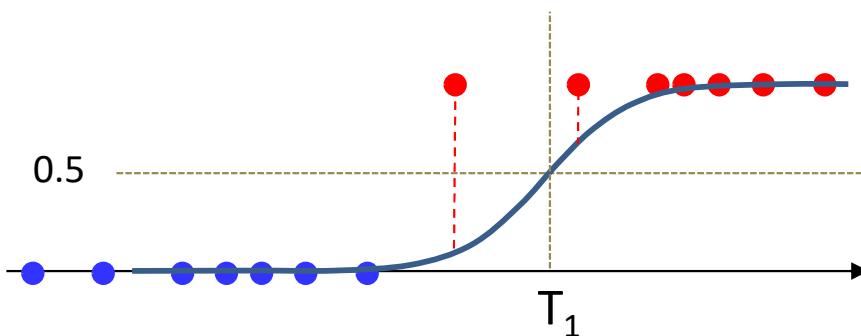
- **Is backprop always right?**
  - Assuming it actually finds the minimum of the divergence function?

(Actual question: Does gradient descent find the right solution, even when it finds the actual minimum)

# Recap: The differentiable activation



- Threshold activation: Equivalent to counting errors
  - Shifting the threshold from  $T_1$  to  $T_2$  does not change classification error
  - Does not indicate if moving the threshold left was good or not

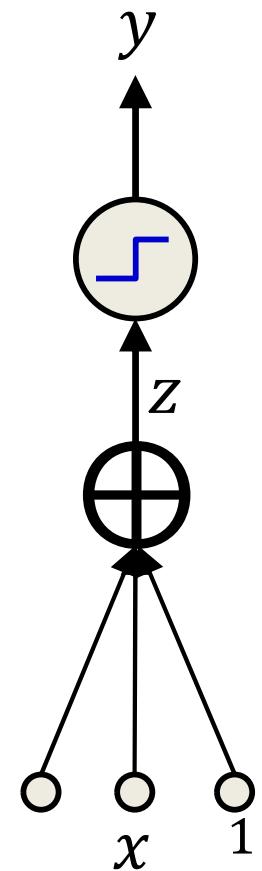
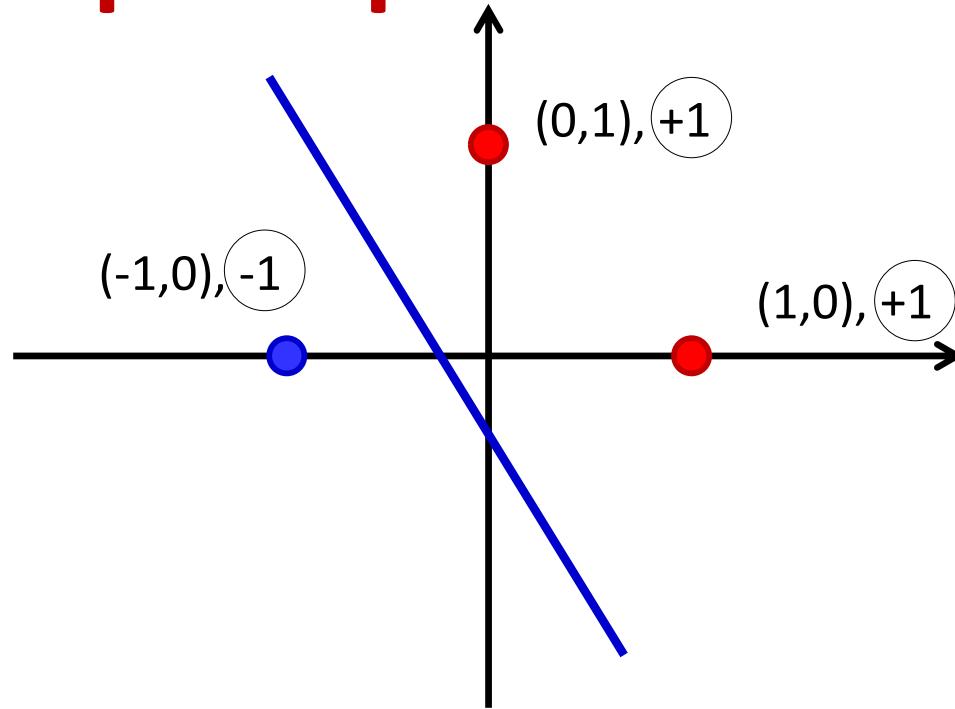


- Differentiable activation: Computes “distance to answer”
  - “Distance” == divergence
  - Perturbing the function changes this quantity,
    - Even if the classification error itself doesn’t change

# Does backprop do the right thing?

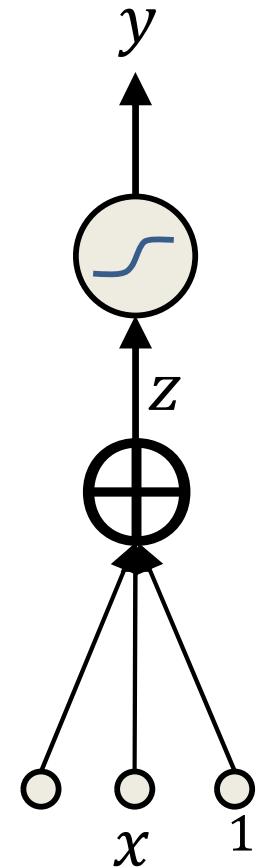
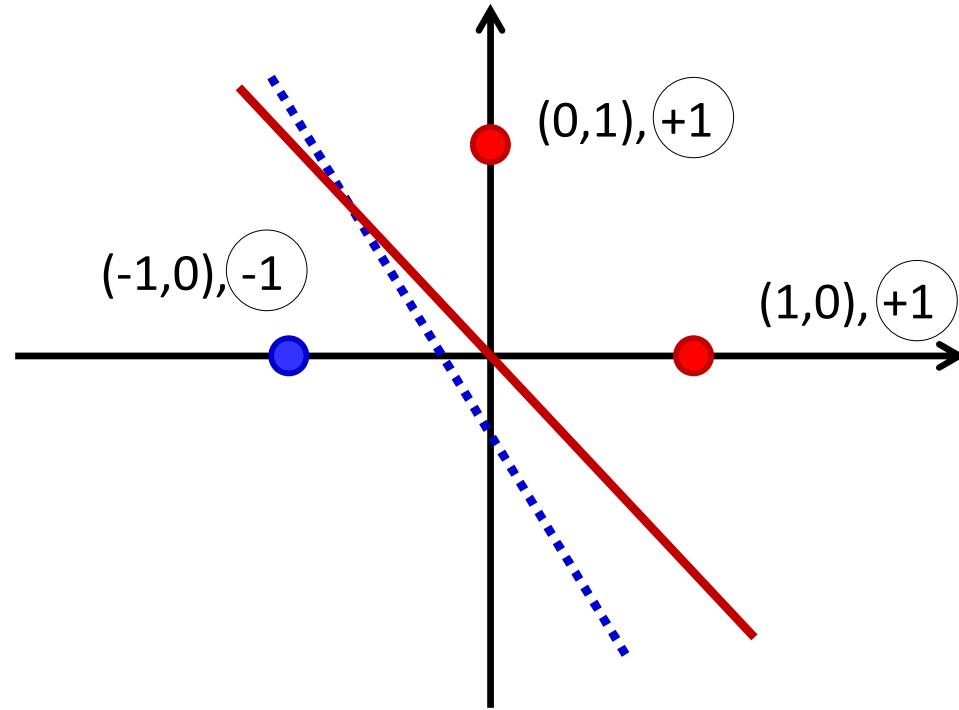
- **Is backprop always right?**
  - Assuming it actually finds the global minimum of the loss (average divergence)?
- In classification problems, the classification error is a non-differentiable function of weights
- The divergence function minimized is only a *proxy* for classification error
- Minimizing divergence may not minimize classification error

# Backprop fails to separate where perceptron succeeds



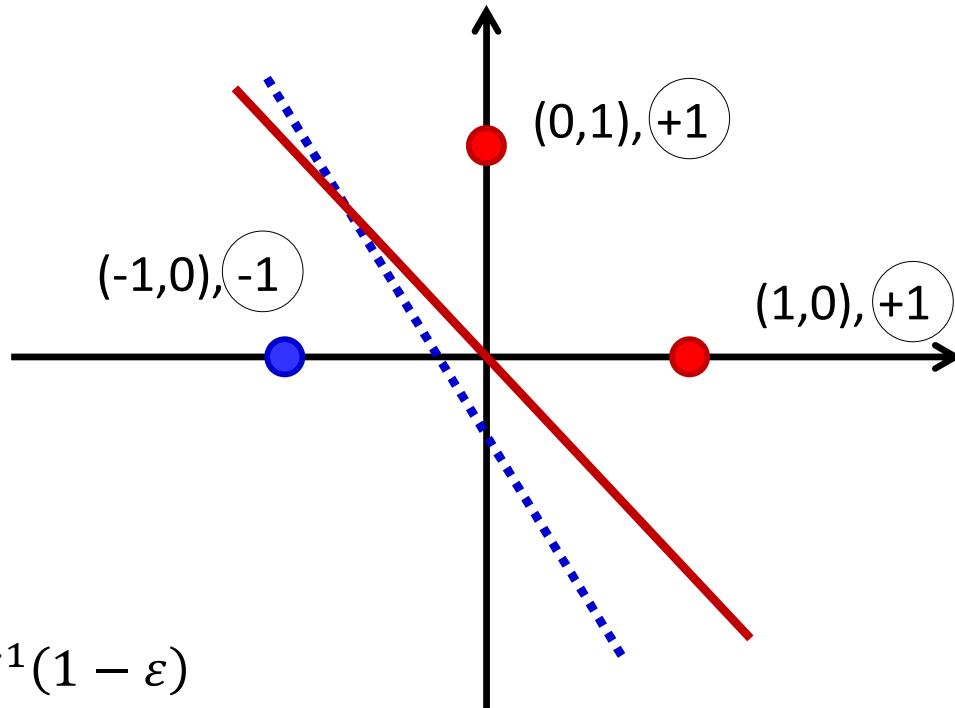
- Brady, Raghavan, Slawny, '89
- Simple problem, 3 training instances, single neuron
- Perceptron training rule trivially find a perfect solution

# Backprop vs. Perceptron



- Back propagation using logistic function and  $L_2$  divergence ( $Div = (y - d)^2$ )
- Unique minimum trivially proved to exist, backprop finds it

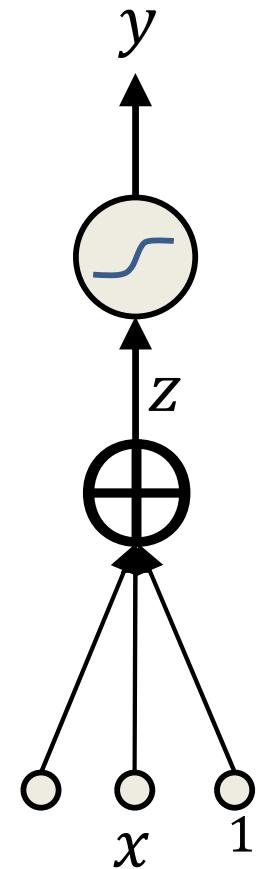
# Unique solution exists



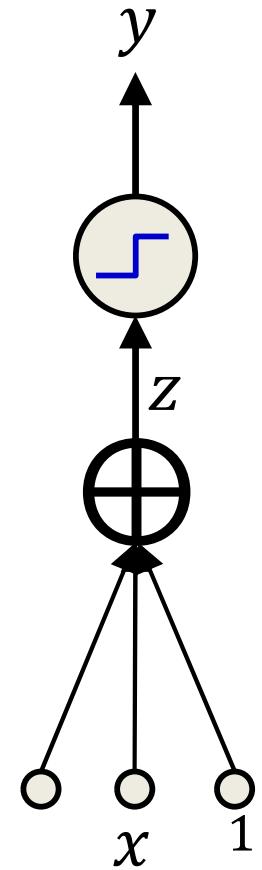
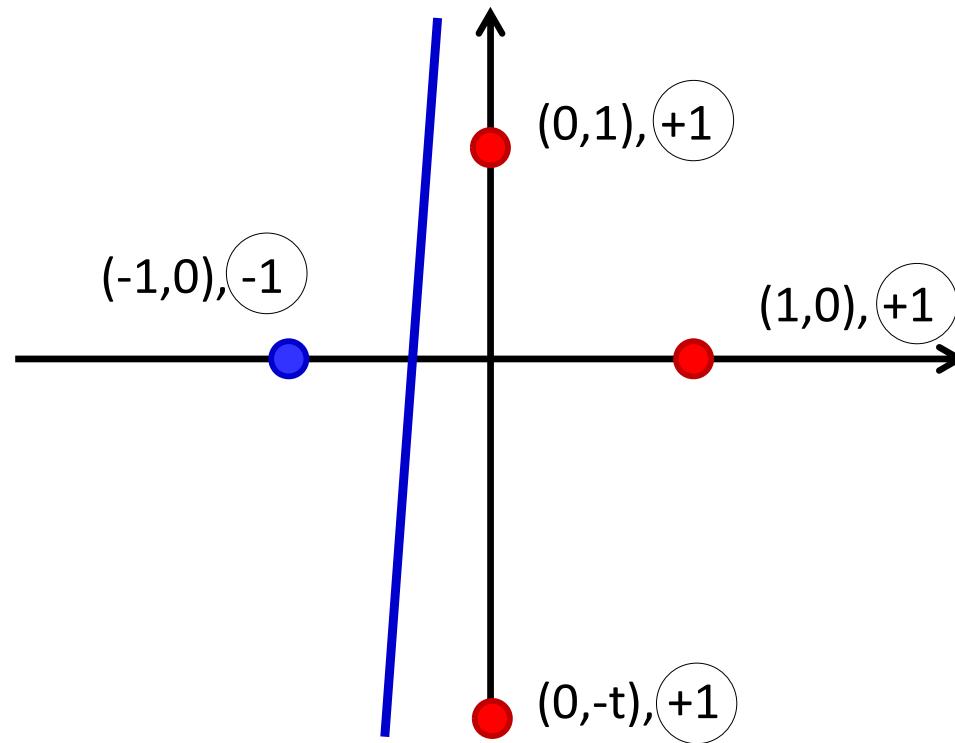
- Let  $u = f^{-1}(1 - \varepsilon)$ 
  - E.g.  $u = f^{-1}(0.99)$  representing a 99% confidence in the class
- From the three points we get three independent equations:

$$\begin{aligned} w_x \cdot 1 + w_y \cdot 0 + b &= u \\ w_x \cdot 0 + w_y \cdot 1 + b &= u \\ w_x \cdot -1 + w_y \cdot 0 + b &= -u \end{aligned}$$

- Unique solution  $(w_x = u, w_y = u, b = 0)$  exists
  - represents a unique line regardless of the value of  $u$



# Backprop vs. Perceptron

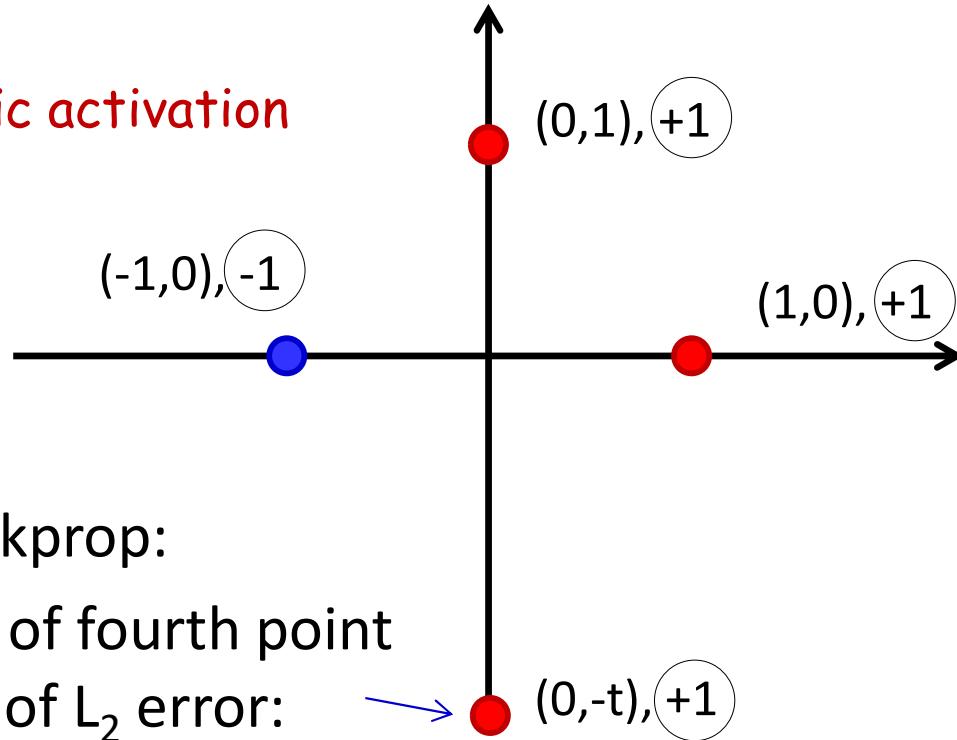


- Now add a fourth point
- $t$  is very large (point near  $-\infty$ )
- Perceptron trivially finds a solution (may take  $t^2$  iterations)

# Backprop

Notation:

$y = \sigma(z)$  = logistic activation



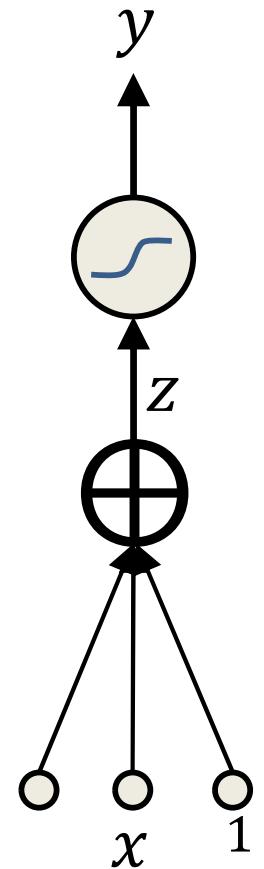
- Consider backprop:
- Contribution of fourth point to derivative of  $L_2$  error:

$$div_4 = (1 - \varepsilon - \sigma(-w_y t + b))^2$$

$$\frac{d div_4}{d w_y} = 2(1 - \varepsilon - \sigma(-w_y t + b)) \sigma'(-w_y t + b) t$$

$$\frac{d div_4}{d b} = -2(1 - \varepsilon - \sigma(-w_y t + b)) \sigma'(-w_y t + b)$$

1- $\varepsilon$  is the actual achievable value



# Backprop

Notation:

$y = \sigma(z)$  = logistic activation

$$div_4 = (1 - \varepsilon - \sigma(-w_y t + b))^2$$

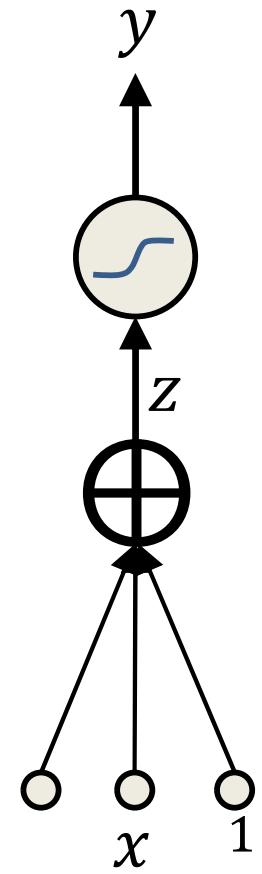
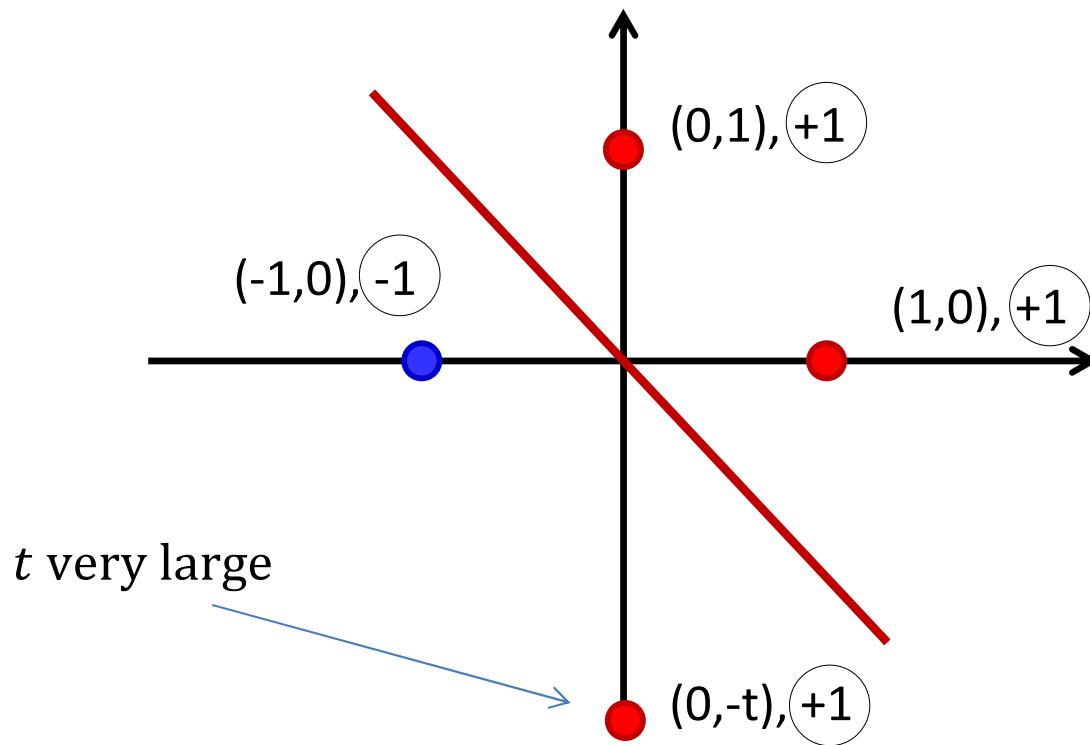
$$\frac{d div_4}{dw_y} = 2(1 - \varepsilon - \sigma(-w_y t + b))\sigma'(-w_y t + b)t$$

$$\frac{d div_4}{db} = 2(1 - \sigma(-w_y t + b))\sigma'(-w_y t + b)t$$

- For very large positive  $t$ ,  $|w_y| > \epsilon$  (where  $\mathbf{w} = [w_x, w_y, b]$ )
- $(1 - \varepsilon - \sigma(-w_y t + b)) \rightarrow 1$  as  $t \rightarrow \infty$
- $\sigma'(-w_y t + b) \rightarrow 0$  exponentially as  $t \rightarrow \infty$
- Therefore, for very large positive  $t$

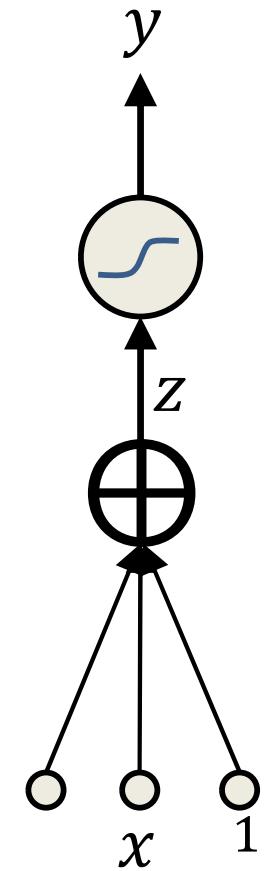
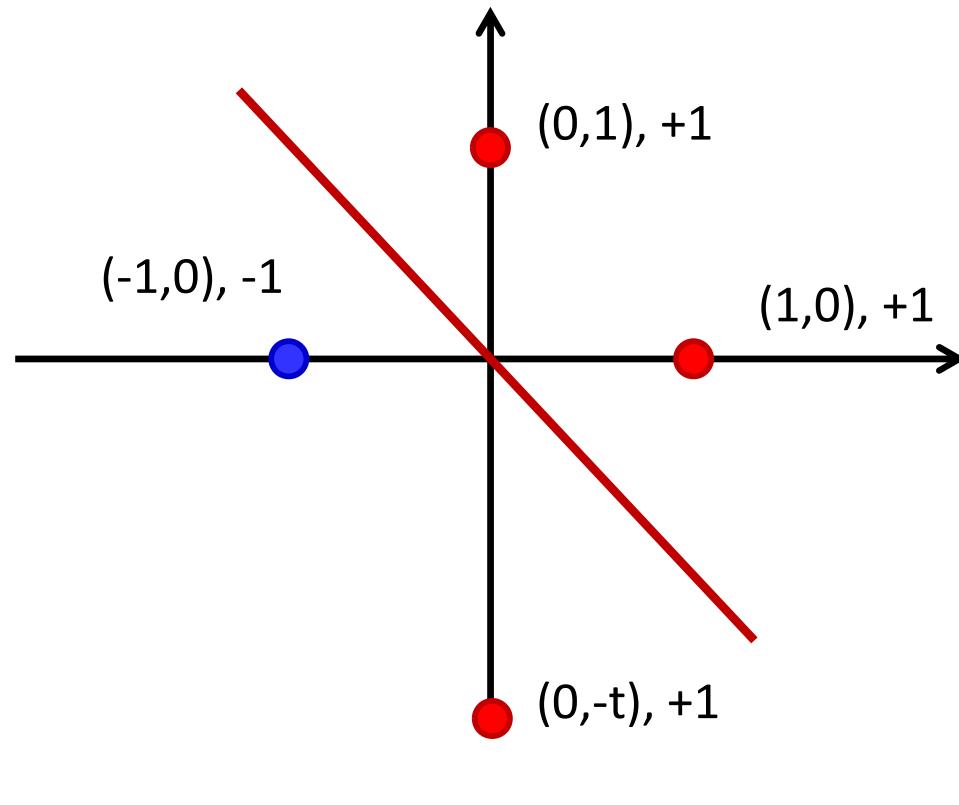
$$\frac{d div_4}{dw_y} = \frac{d div_4}{db} = 0$$

# Backprop



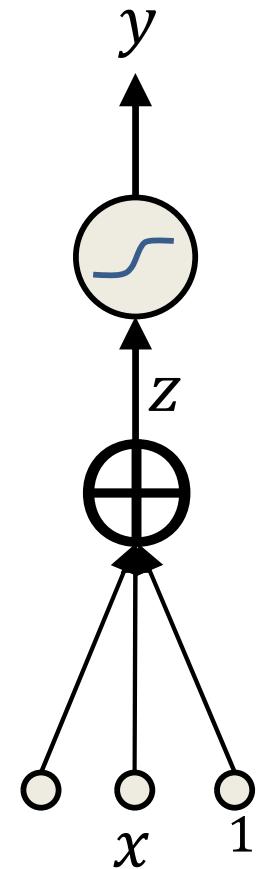
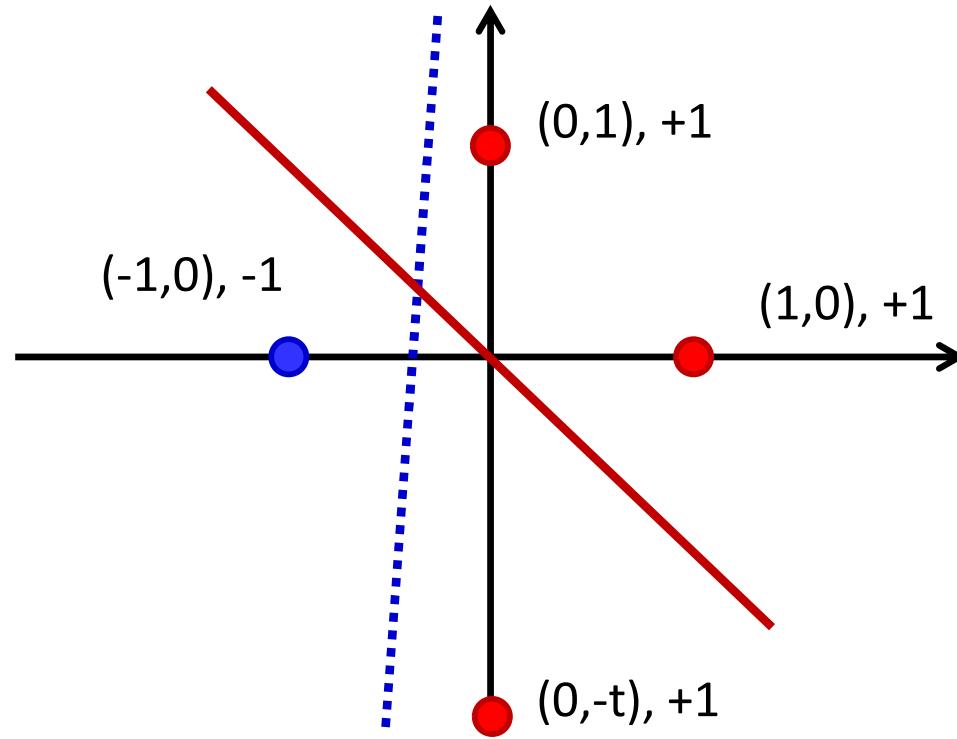
- The fourth point at  $(0, -t)$  does not change the gradient of the  $L_2$  divergence near the optimal solution for 3 points
- The optimum solution for 3 points is also a broad *local* minimum (0 gradient) for the 4-point problem!
  - Will be found by backprop nearly all the time
    - Although the global minimum with unbounded weights will separate the classes correctly

# Backprop



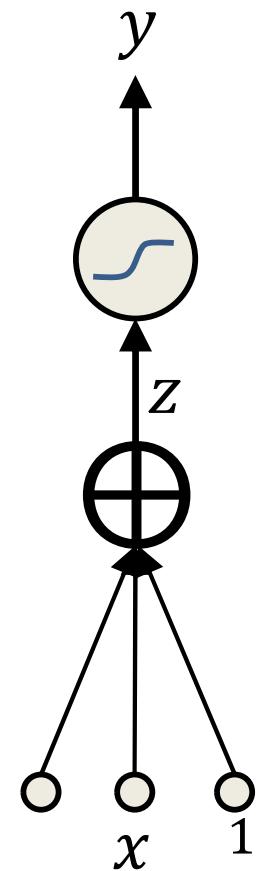
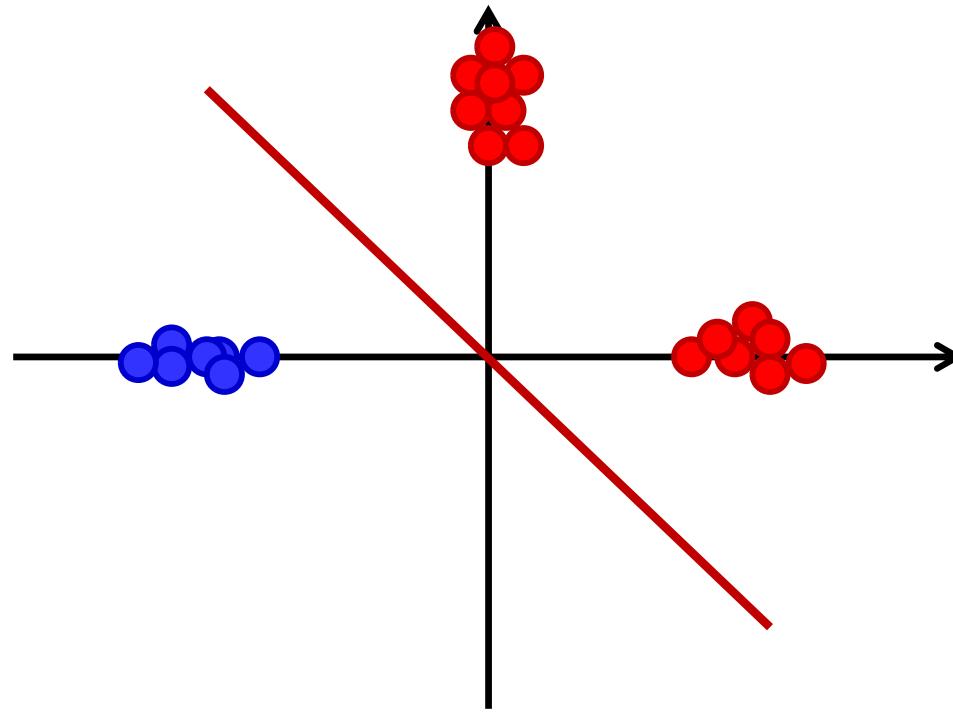
- Local optimum solution found by backprop
- Does not separate the points *even though the points are linearly separable!*

# Backprop



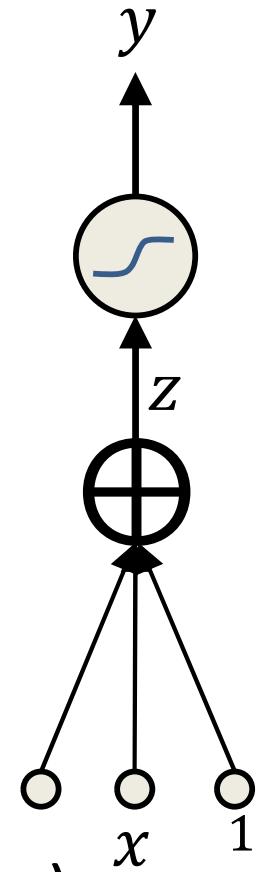
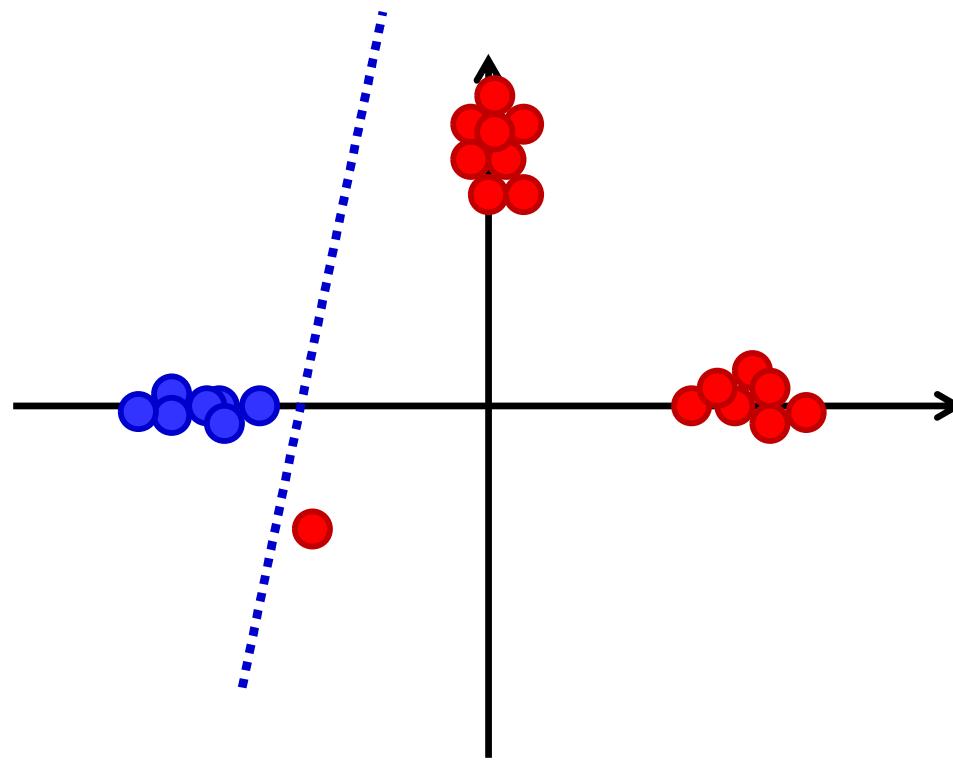
- Solution found by backprop
- Does not separate the points *even though the points are linearly separable!*
- Compare to the perceptron: *Backpropagation fails to separate where the perceptron succeeds*

# Backprop fails to separate where perceptron succeeds



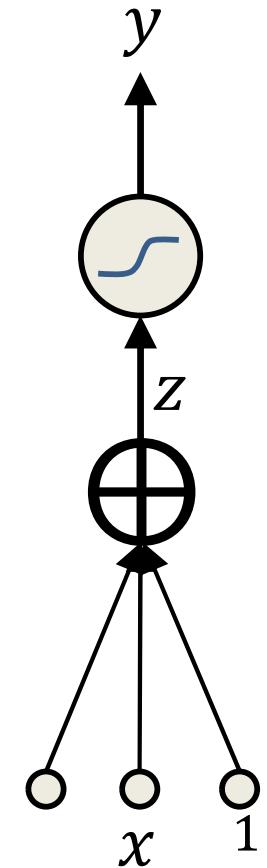
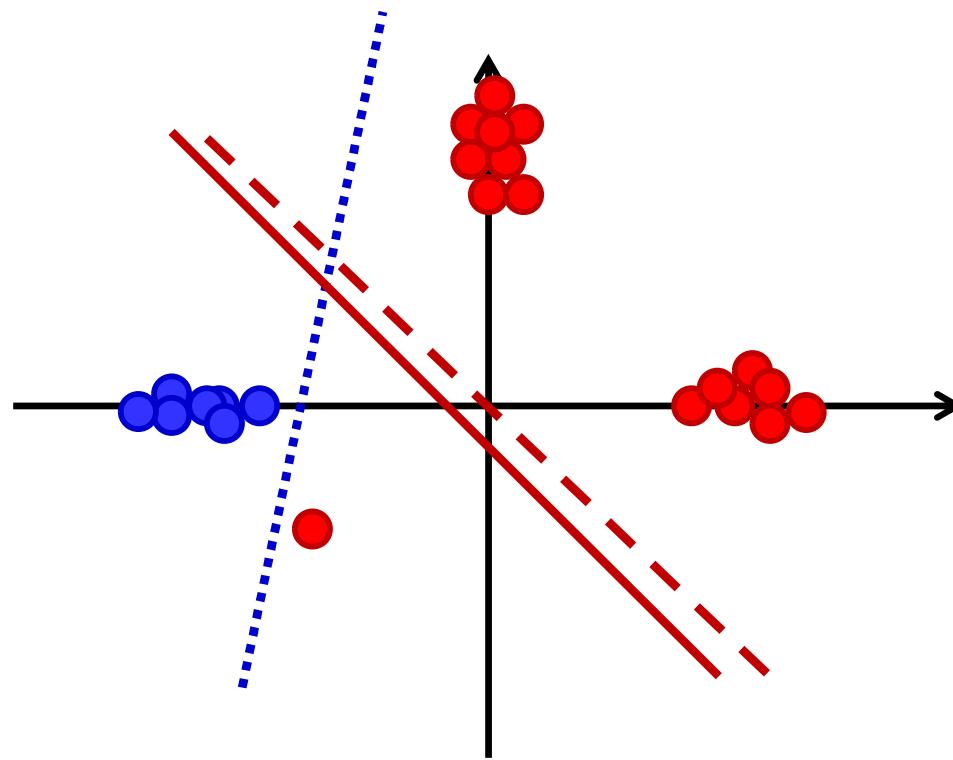
- Brady, Raghavan, Slawny, '89
- *Several* linearly separable training examples
- Simple setup: **both backprop and perceptron algorithms find solutions**

# A more complex problem



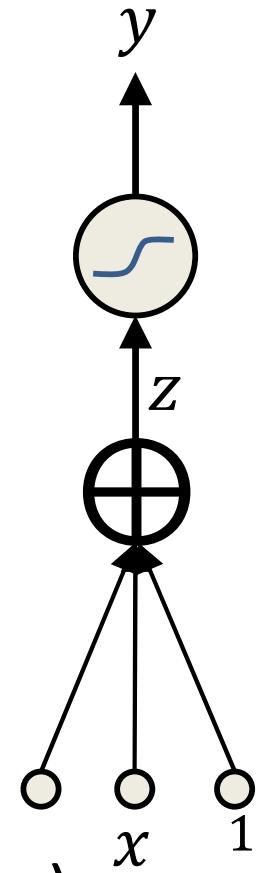
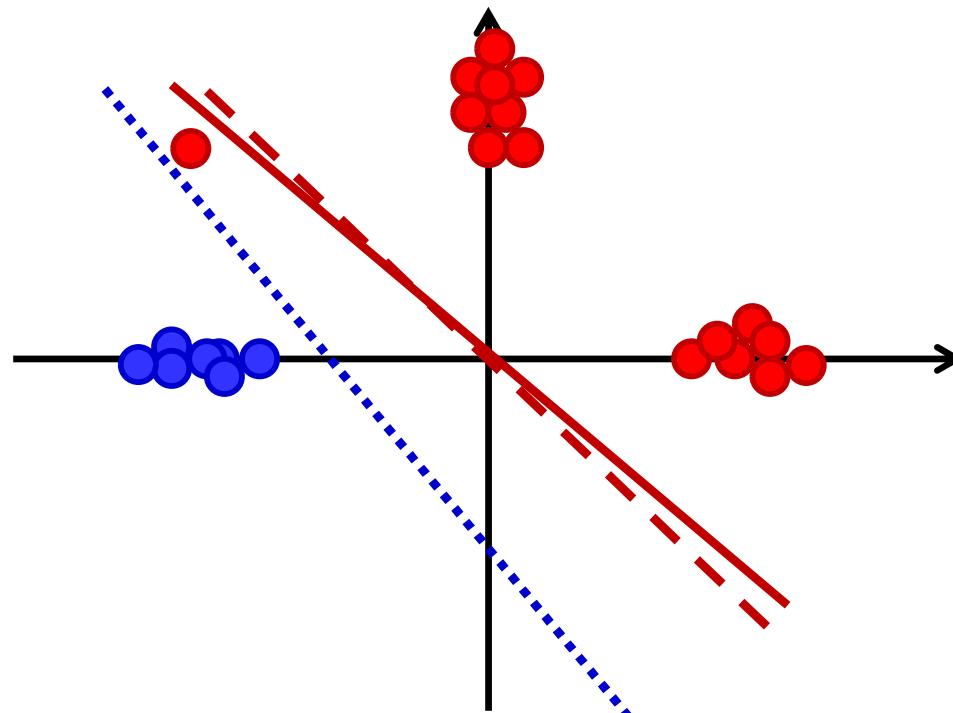
- Adding a “spoiler” (or a small number of spoilers)
  - Perceptron finds the linear separator,

# A more complex problem



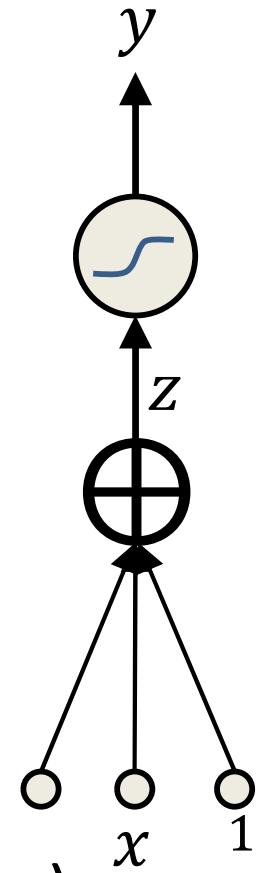
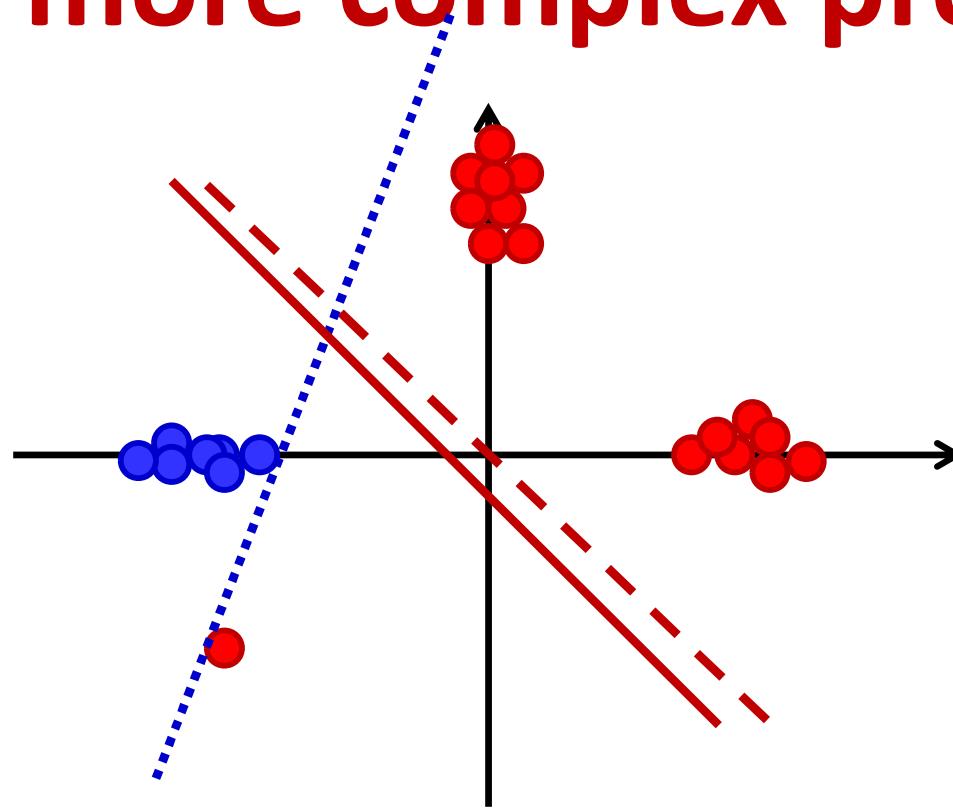
- Adding a “spoiler” (or a small number of spoilers)
  - Perceptron finds the linear separator,
  - Backprop does not find a separator
    - A single additional input does not change the loss function significantly
      - Assuming weights are constrained to be bounded

# A more complex problem



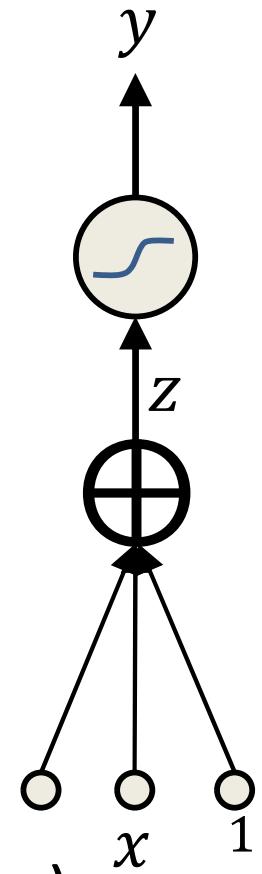
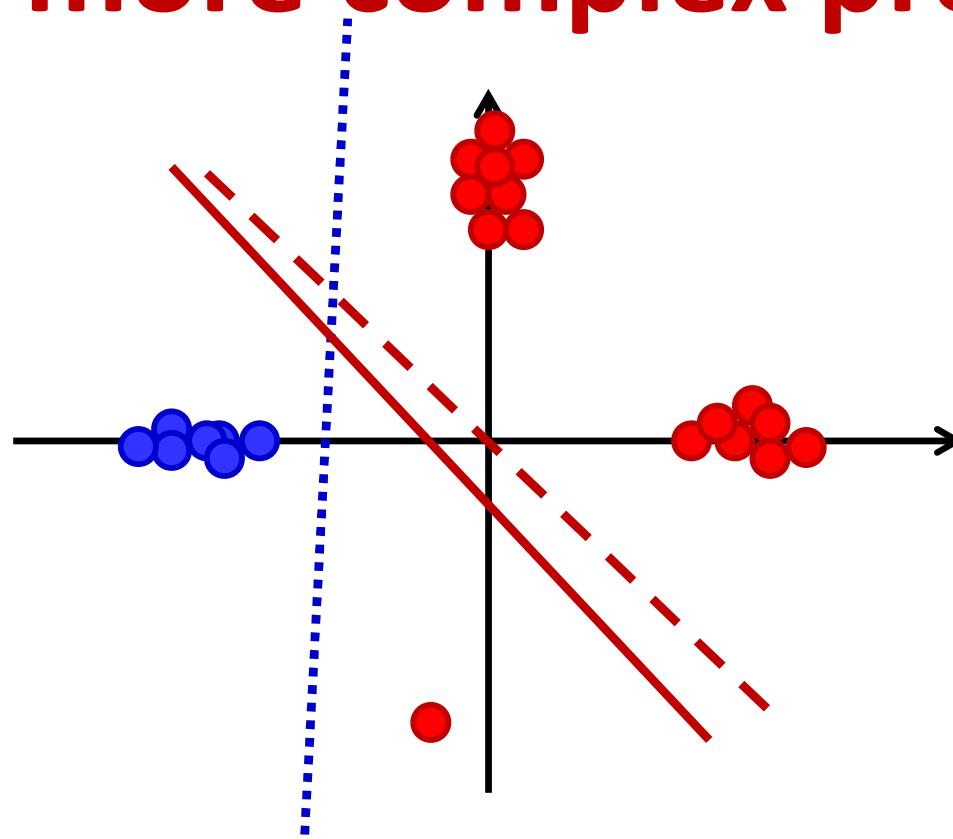
- Adding a “spoiler” (or a small number of spoilers)
  - Perceptron finds the linear separator,
  - For bounded  $w$ , backprop does not find a separator
    - A single additional input does not change the loss function significantly

# A more complex problem



- Adding a “spoiler” (or a small number of spoilers)
  - Perceptron finds the linear separator,
  - For bounded  $w$ , backprop does not find a separator
    - A single additional input does not change the loss function significantly

# A more complex problem

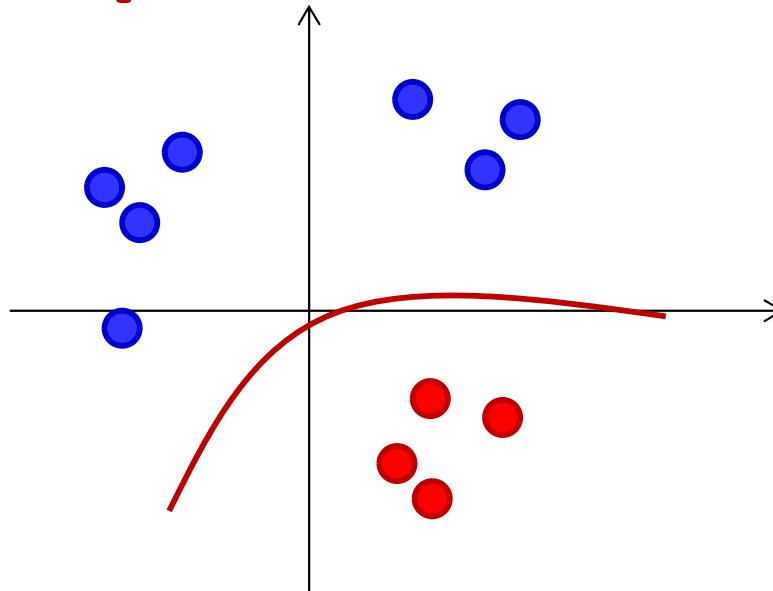
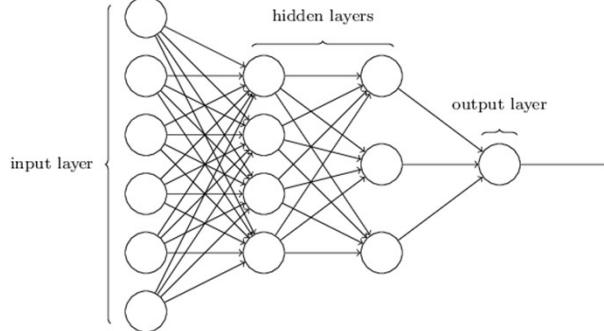


- Adding a “spoiler” (or a small number of spoilers)
  - Perceptron finds the linear separator,
  - For bounded  $w$ , backprop does not find a separator
    - A single additional input does not change the loss function significantly

# So what is happening here?

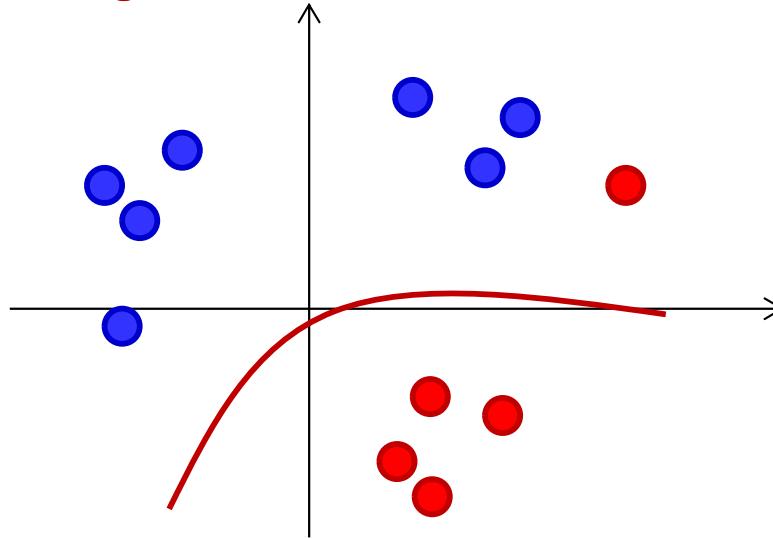
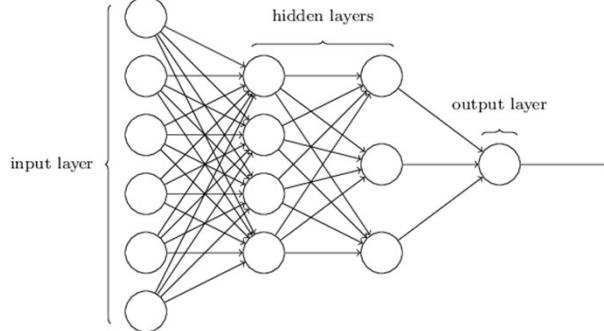
- The perceptron may change greatly upon adding just a single new training instance
  - But it fits the training data well
  - The perceptron rule has *low bias*
    - Makes no errors if possible
  - But high variance
    - Swings wildly in response to small changes to input
- Backprop is minimally changed by new training instances
  - Prefers consistency over perfection
  - It is a *low-variance* estimator, at the potential cost of bias

# Backprop fails to separate even when possible



- This is not restricted to single perceptrons
- An MLP learns non-linear decision boundaries that are determined from the entirety of the training data
- Adding a few “spoilers” will not change their behavior

# Backprop fails to separate even when possible



- This is not restricted to single perceptrons
- An MLP learns non-linear decision boundaries that are determined from the entirety of the training data
- Adding a few “spoilers” will not change their behavior

# Backpropagation: Finding the separator

- Backpropagation will often not find a separating solution *even though the solution is within the class of functions learnable by the network*
- This is because the separating solution is not a feasible optimum for the loss function
- One resulting benefit is that a backprop-trained neural network classifier has lower variance than an optimal classifier for the training data

# Poll 1

Minimizing the (differentiable) loss function will also minimize classification error, true or false

- True
- False

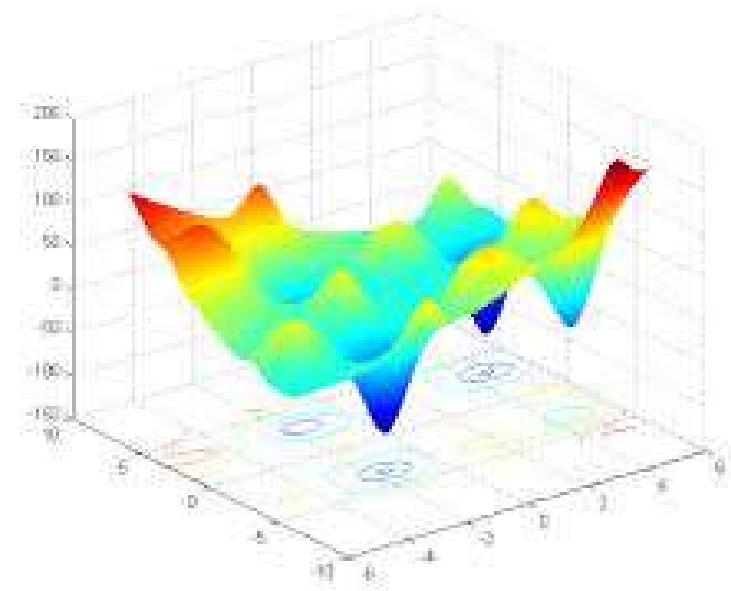
# Poll 1

Minimizing the (differentiable) loss function will also minimize classification error, true or false

- True
- False (**true**)

# The Loss Surface

- The example (and statements) earlier assumed the loss objective had a single global optimum that could be found
  - Statement about variance is assuming global optimum
- What about local optima



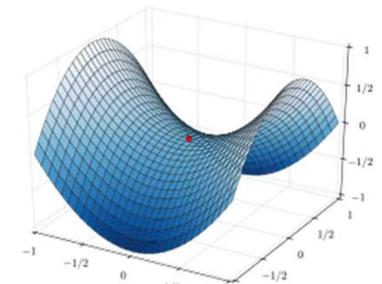
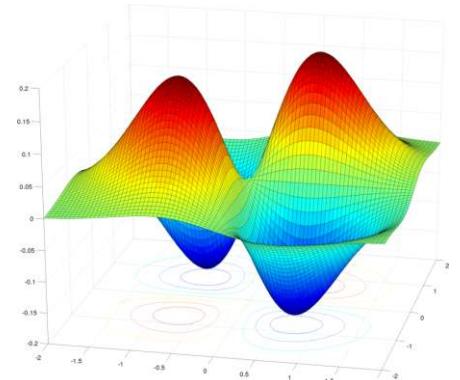
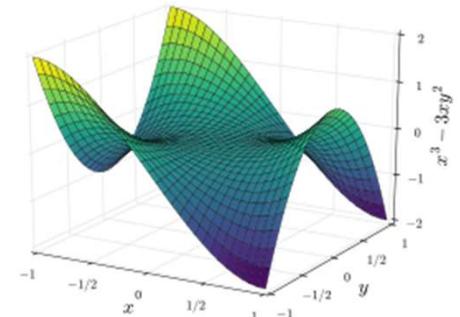
# The Loss Surface

- **Popular hypothesis:**

- In large networks, saddle points are far more common than local minima
  - Frequency of occurrence exponential in network size
- Most local minima are equivalent
  - And close to global minimum
- This is not true for small networks

- **Saddle point:** A point where

- The slope is zero
- The surface increases in some directions, but decreases in others
  - Some of the Eigenvalues of the Hessian are positive; others are negative
- Gradient descent algorithms often get “stuck” in saddle points



# The Controversial Loss Surface

- **Baldi and Hornik (89)**, “*Neural Networks and Principal Component Analysis: Learning from Examples Without Local Minima*” : An MLP with a single hidden layer has only saddle points and no local Minima
- **Dauphin et. al (2015)**, “*Identifying and attacking the saddle point problem in high-dimensional non-convex optimization*” : An exponential number of saddle points in large networks
- **Chomoranksa et. al (2015)**, “*The loss surface of multilayer networks*” : For large networks, most local minima lie in a band and are equivalent
  - Based on analysis of spin glass models
- **Swirszcz et. al. (2016)**, “*Local minima in training of deep networks*”, In networks of finite size, trained on finite data, you *can* have horrible local minima
- Watch this space...

# Story so far

- Neural nets can be trained via gradient descent that minimizes a loss function
- Backpropagation can be used to derive the derivatives of the loss
- Backprop *is not guaranteed* to find a “true” solution, even if it exists, and lies within the capacity of the network to model
  - The optimum for the loss function may not be the “true” solution
- For large networks, the loss function may have a large number of unpleasant saddle points or local minima
  - Which backpropagation may find

# Convergence

- In the discussion so far we have assumed the training arrives at a local minimum
- Does it always converge?
- How long does it take?
- Hard to analyze for an MLP, but we can look at the problem through the lens of convex optimization

# A quick tour of (convex) optimization

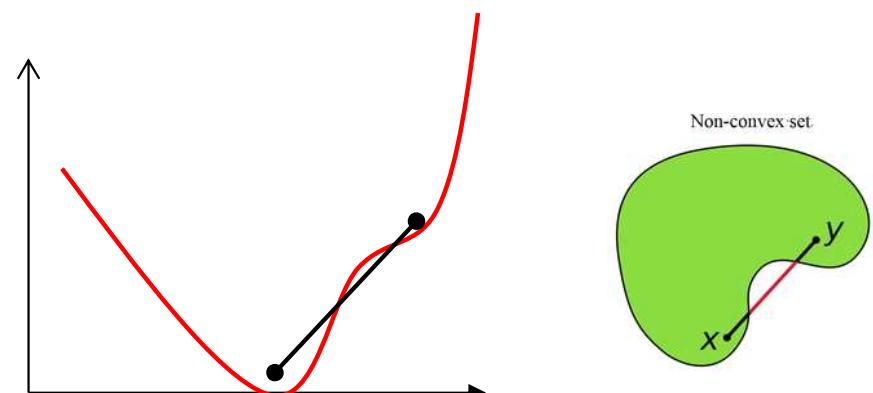
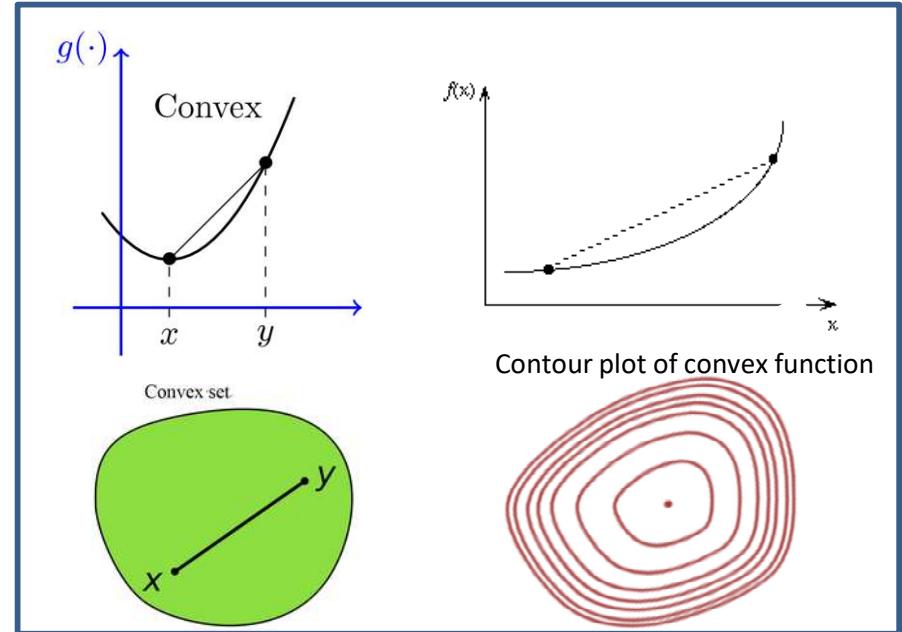


The streetlight effect is a type of observational bias where people only look for whatever they are searching by looking where it is easiest

*"I'm searching for my keys."*

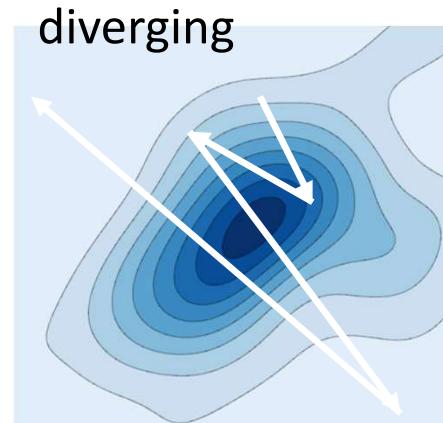
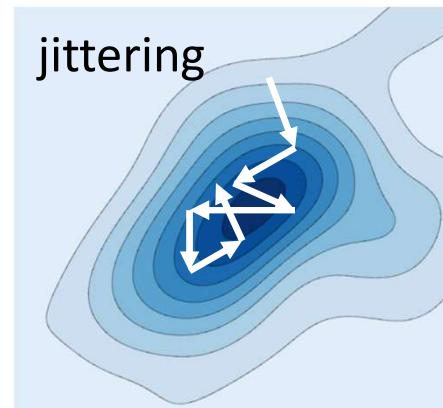
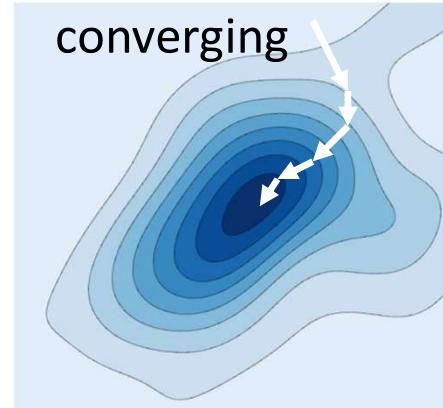
# Convex Loss Functions

- A surface is “convex” if it is continuously curving upward
  - We can connect any two points on or above the surface without intersecting it
  - Many mathematical definitions that are equivalent
- Caveat: Neural network loss surface is generally not convex
  - Streetlight effect



# Convergence of gradient descent

- An iterative algorithm is said to *converge* to a solution if the value updates arrive at a fixed point
  - Where the gradient is 0 and further updates do not change the estimate
- The algorithm may not actually converge
  - It may jitter around the local minimum
  - It may even diverge
- Conditions for convergence?



# Convergence and convergence rate

- Convergence rate: How fast the iterations arrive at the solution
- Generally quantified as

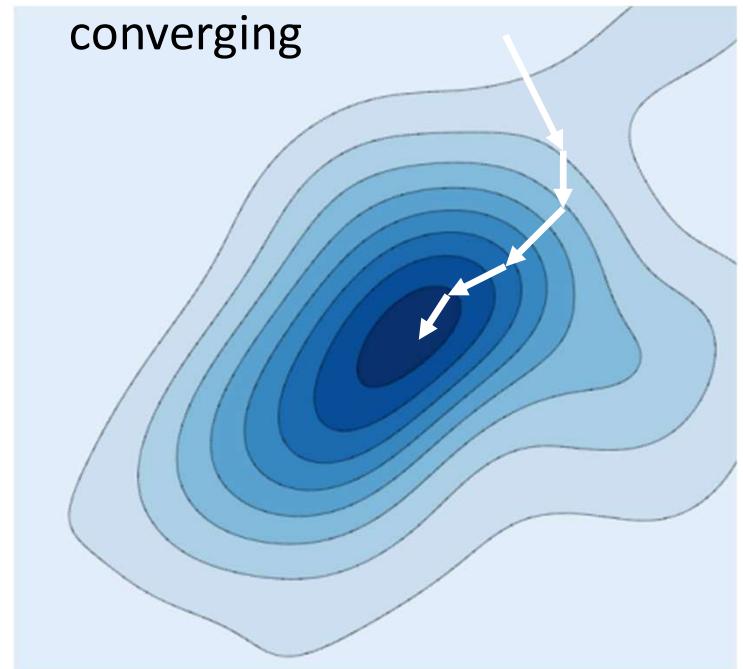
$$R = \frac{|f(x^{(k+1)}) - f(x^*)|}{|f(x^{(k)}) - f(x^*)|}$$

- $x^{(k+1)}$  is the k-th iteration
- $x^*$  is the optimal value of  $x$

- If  $R$  is a constant (or upper bounded), the convergence is *linear*

- In reality, its arriving at the solution exponentially fast

$$|f(x^{(k)}) - f(x^*)| \leq R^k |f(x^{(0)}) - f(x^*)|$$

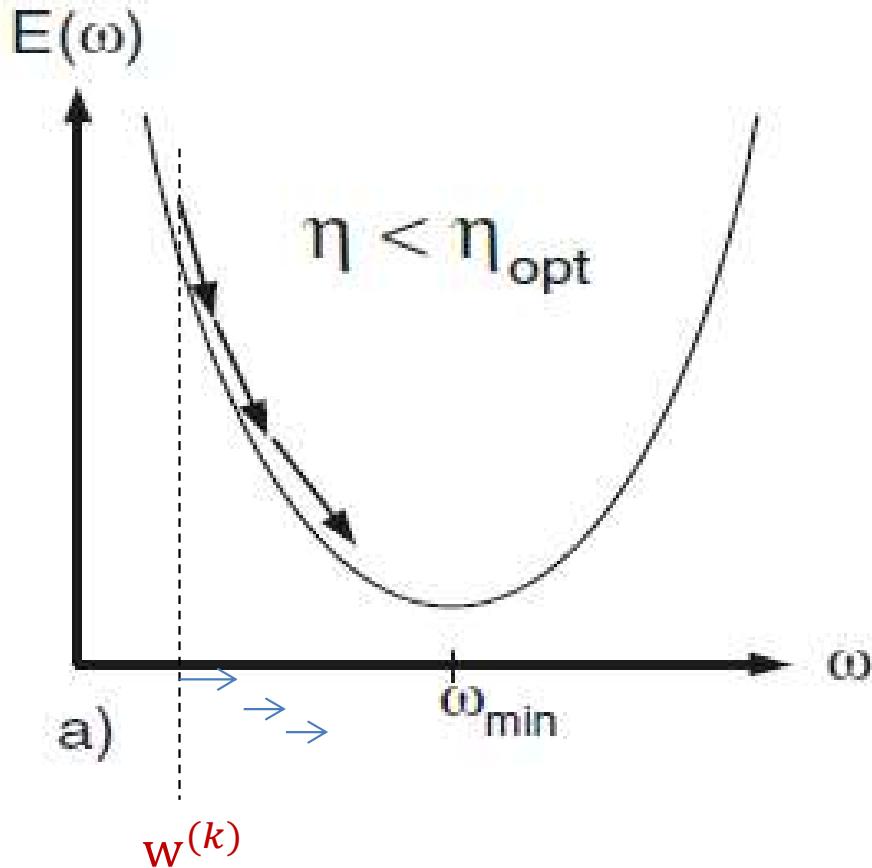


# Convergence for quadratic surfaces

$$\text{Minimize } E = \frac{1}{2} aw^2 + bw + c$$

$$w^{(k+1)} = w^{(k)} - \eta \frac{dE(w^{(k)})}{dw}$$

Gradient descent with fixed step size  $\eta$  to estimate *scalar* parameter  $w$

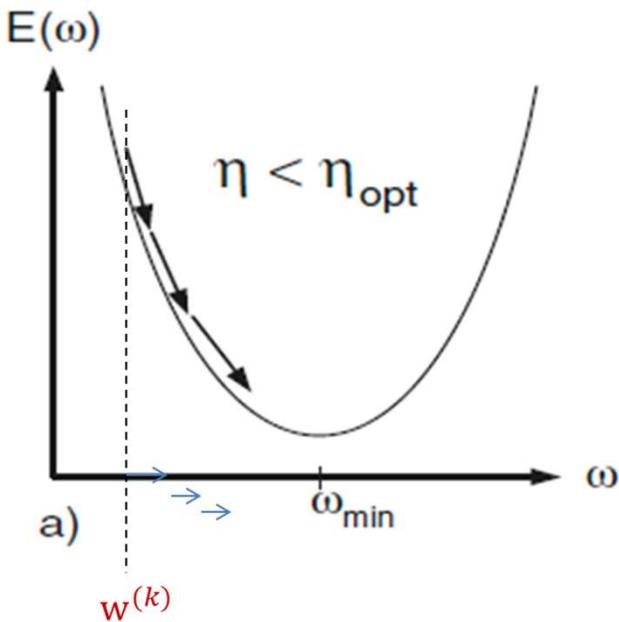


- Gradient descent to find the optimum of a quadratic, starting from  $w^{(k)}$
- Assuming fixed step size  $\eta$
- What is the optimal step size  $\eta$  to get there fastest?

# Convergence for quadratic surfaces

$$E = \frac{1}{2} aw^2 + bw + c$$

$$w^{(k+1)} = w^{(k)} - \eta \frac{dE(w^{(k)})}{dw}$$



- Any quadratic objective can be written as

$$\begin{aligned} E(w) &= E(w^{(k)}) + E'(w^{(k)})(w - w^{(k)}) \\ &\quad + \frac{1}{2} E''(w^{(k)})(w - w^{(k)})^2 \end{aligned}$$

– Taylor expansion

- Minimizing w.r.t  $w$ , we get (Newton's method)

$$w_{min} = w^{(k)} - E''(w^{(k)})^{-1} E'(w^{(k)})$$

- Note:

$$\frac{dE(w^{(k)})}{dw} = E'(w^{(k)})$$

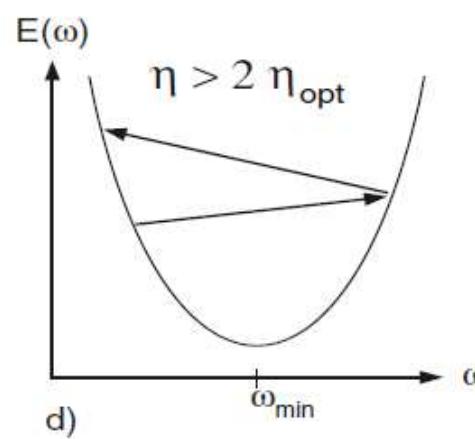
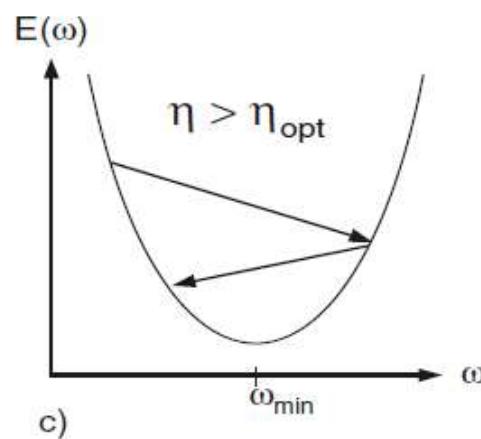
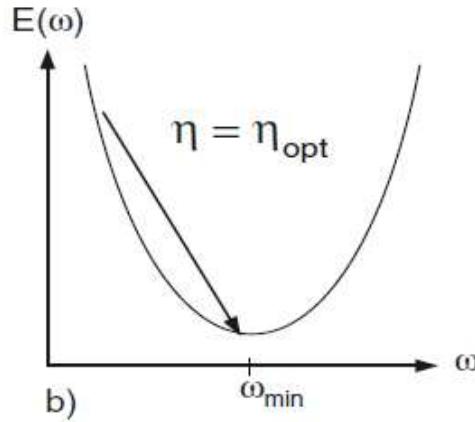
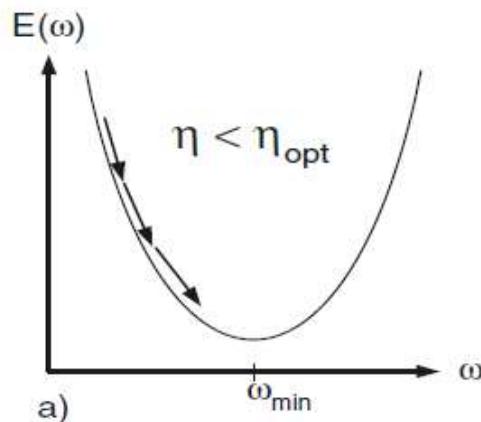
- Comparing to the gradient descent rule, we see that we can arrive at the optimum in a single step using the optimum step size

$$\eta_{opt} = E''(w^{(k)})^{-1} = a^{-1}$$

# With non-optimal step size

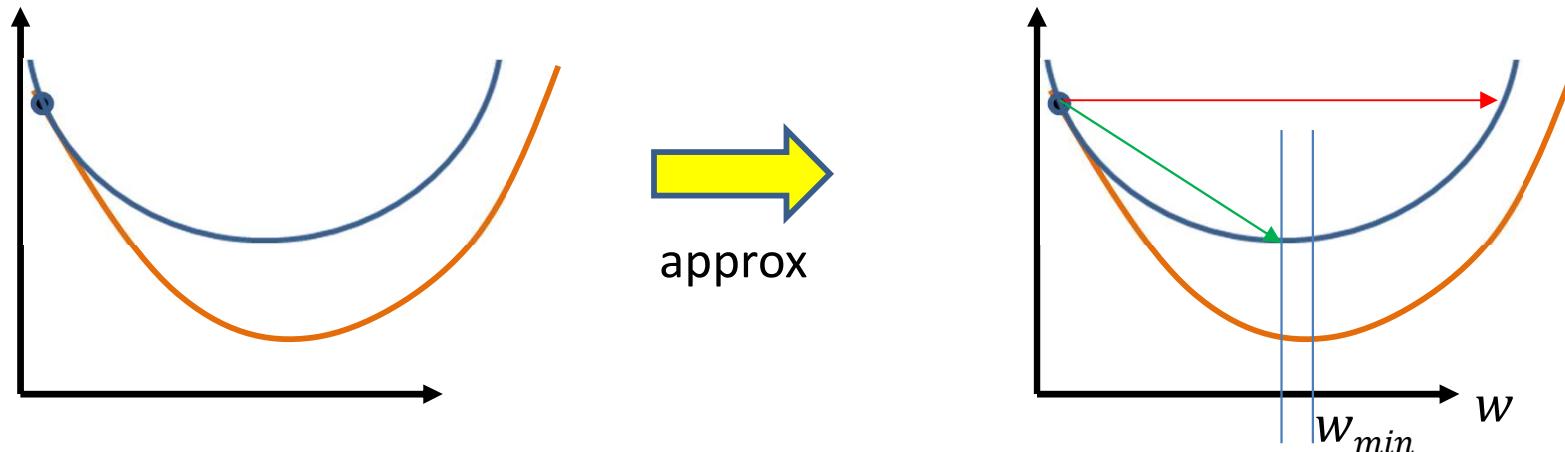
$$w^{(k+1)} = w^{(k)} - \eta \frac{dE(w^{(k)})}{dw}$$

Gradient descent with fixed step size  $\eta$  to estimate scalar parameter  $w$



- For  $\eta < \eta_{\text{opt}}$  the algorithm will converge monotonically
- For  $2\eta_{\text{opt}} > \eta > \eta_{\text{opt}}$  we have oscillating convergence
- For  $\eta > 2\eta_{\text{opt}}$  we get divergence

# For generic differentiable convex objectives



- Any differentiable convex objective  $E(w)$  can be approximated as

$$E \approx E(w^{(k)}) + (w - w^{(k)}) \frac{dE(w^{(k)})}{dw} + \frac{1}{2} (w - w^{(k)})^2 \frac{d^2 E(w^{(k)})}{dw^2} + \dots$$

– Taylor expansion

- Using the same logic as before, we get (Newton's method)

$$\eta_{opt} = \left( \frac{d^2 E(w^{(k)})}{dw^2} \right)^{-1}$$

- We can get divergence if  $\eta \geq 2\eta_{opt}$

# For functions of *multivariate* inputs

$$E = g(\mathbf{w}), \mathbf{w} \text{ is a vector } \mathbf{w} = [w_1, w_2, \dots, w_N]$$

- Consider a simple quadratic convex (paraboloid) function

$$E = \frac{1}{2} \mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{w}^T \mathbf{b} + c$$

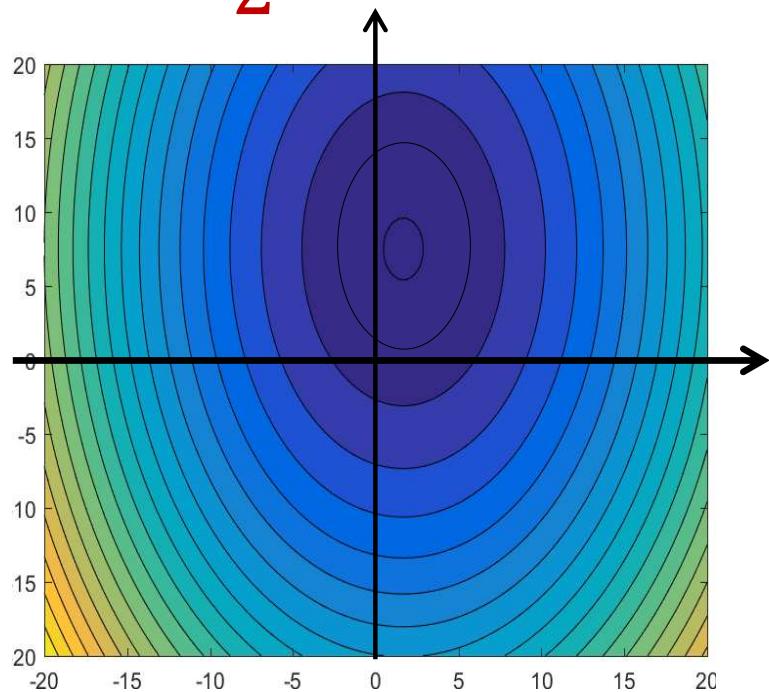
- Since  $E^T = E$  ( $E$  is scalar),  $\mathbf{A}$  can always be made symmetric
  - For strictly **convex**  $E$ ,  $\mathbf{A}$  is always positive definite, and has positive eigenvalues
- When  $\mathbf{A}$  is diagonal:

$$E = \frac{1}{2} \sum_i (a_{ii} w_i^2 + b_i w_i) + c$$

- The  $w_i$ s are *uncoupled*
- For paraboloid (*convex*)  $E$ , the  $a_{ii}$  values are all positive
- Just a sum of  $N$  independent quadratic functions

# Multivariate Quadratic with Diagonal A

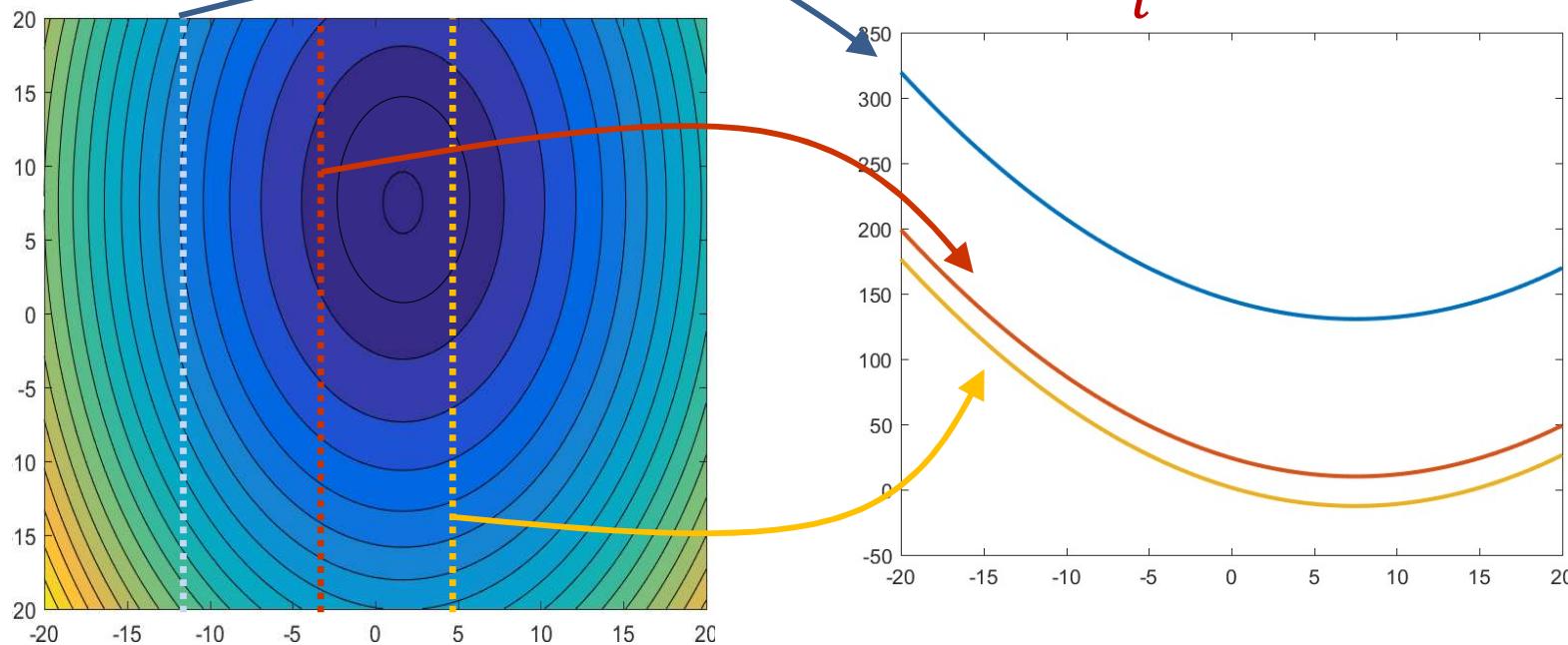
$$E = \frac{1}{2} \mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{w}^T \mathbf{b} + c = \frac{1}{2} \sum_i (a_{ii} w_i^2 + b_i w_i) + c$$



- Equal-value contours will ellipses with principal axes parallel to the spatial axes

# Multivariate Quadratic with Diagonal A

$$E = \frac{1}{2} \mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{w}^T \mathbf{b} + c = \frac{1}{2} \sum_i (a_{ii} w_i^2 + b_i w_i) + c$$

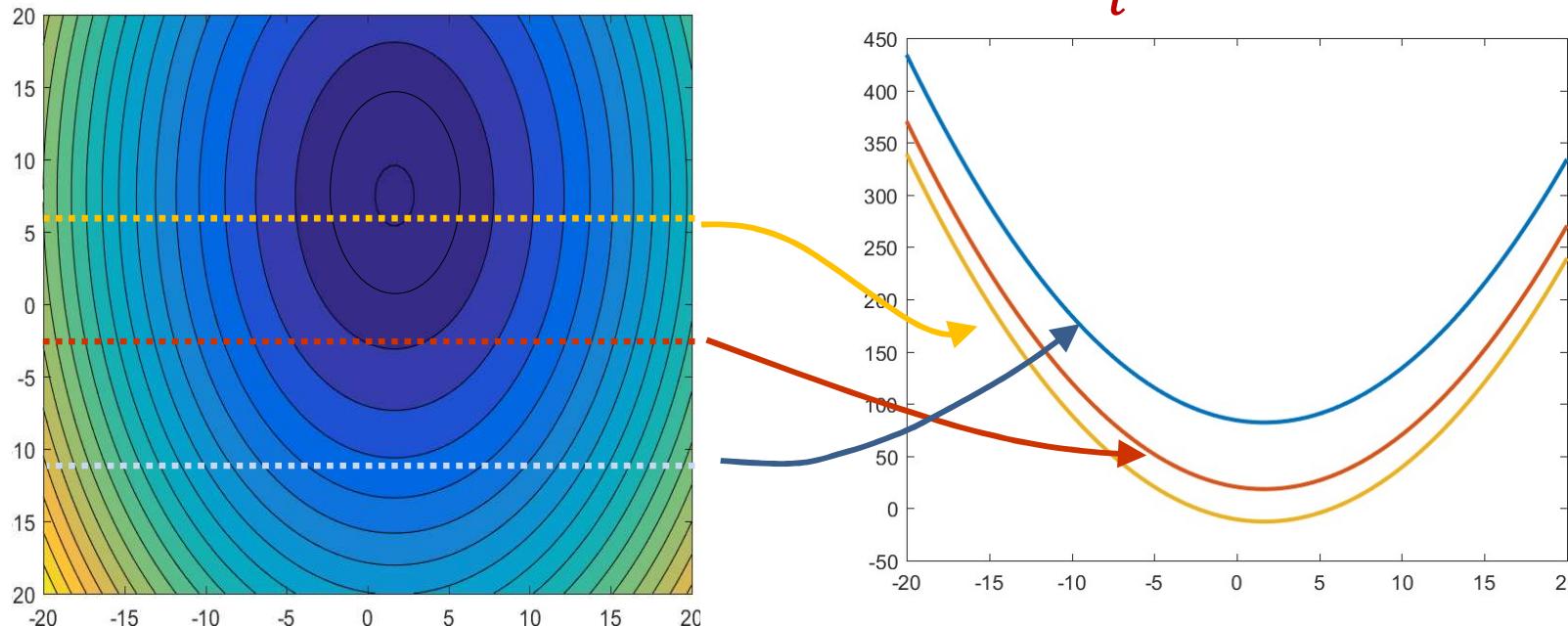


- Equal-value contours will be parallel to the axes
  - All “slices” parallel to an axis are shifted versions of one another

$$E = \frac{1}{2} a_{ii} w_i^2 + b_i w_i + c + C(\neg w_i)$$

# Multivariate Quadratic with Diagonal A

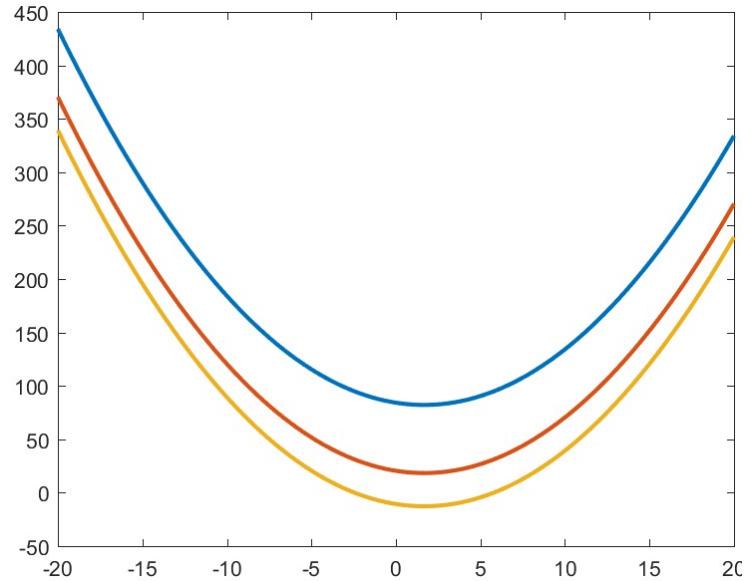
$$E = \frac{1}{2} \mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{w}^T \mathbf{b} + c = \frac{1}{2} \sum_i (a_{ii} w_i^2 + b_i w_i) + c$$



- Equal-value contours will be parallel to the axis
  - All “slices” parallel to an axis are shifted versions of one another

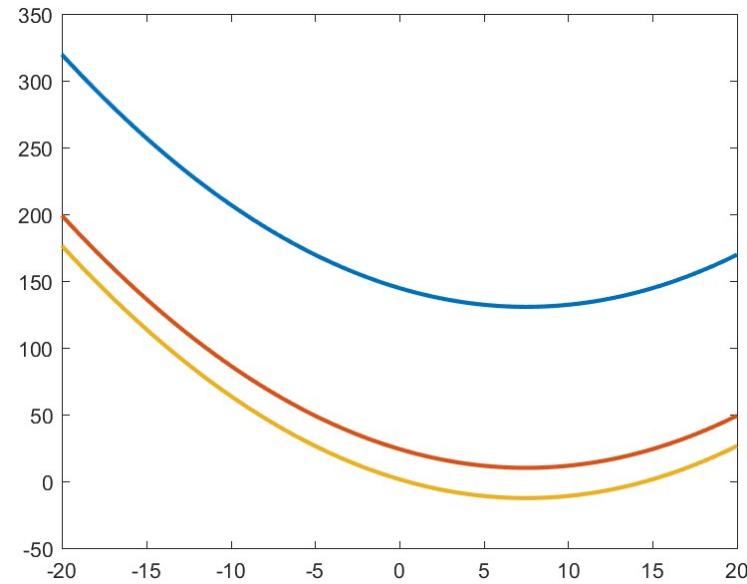
$$E = \frac{1}{2} a_{ii} w_i^2 + b_i w_i + c + C(\neg w_i)$$

# “Descents” are uncoupled



$$E = \frac{1}{2} a_{11} w_1^2 + b_1 w_1 + c + C(\neg w_1)$$

$$\eta_{1,opt} = a_{11}^{-1}$$

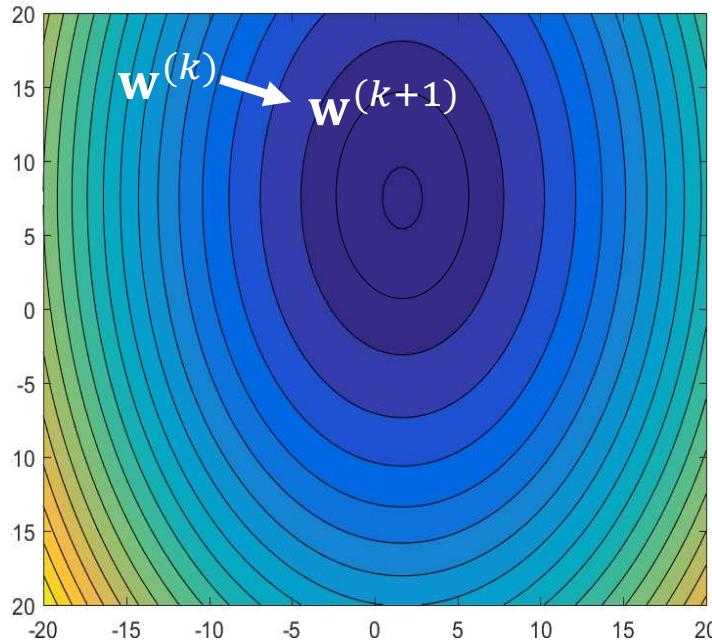


$$E = \frac{1}{2} a_{22} w_2^2 + b_2 w_2 + c + C(\neg w_2)$$

$$\eta_{2,opt} = a_{22}^{-1}$$

- The optimum of each coordinate is not affected by the other coordinates
  - i.e. we could optimize each coordinate independently
- **Note: Optimal learning rate is different for the different coordinates**

# Vector update rule



$$\mathbf{w}^{(k+1)} \leftarrow \mathbf{w}^{(k)} - \eta \nabla_{\mathbf{w}} E^{\top}$$

$$w_i^{(k+1)} = w_i^{(k)} - \eta \frac{\partial E(w_i^{(k)})}{d\partial w}$$

- Conventional vector update rules for gradient descent:  
update entire vector against direction of gradient
  - Note : Gradient is perpendicular to equal value contour
  - The same learning rate is applied to all components

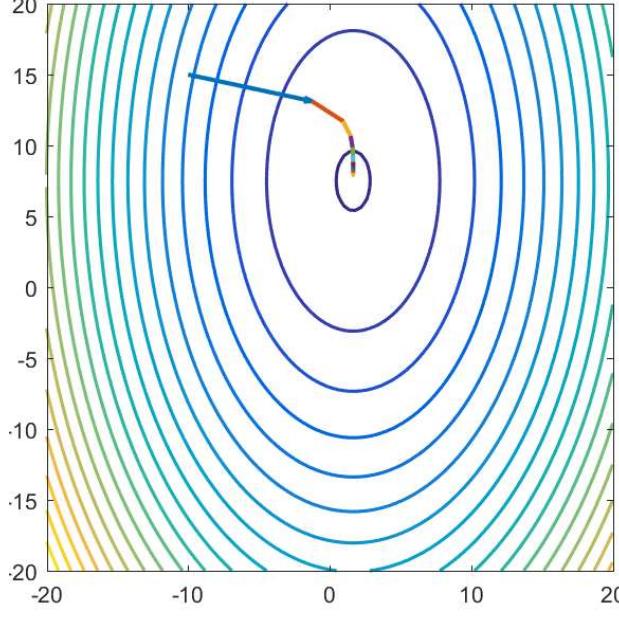
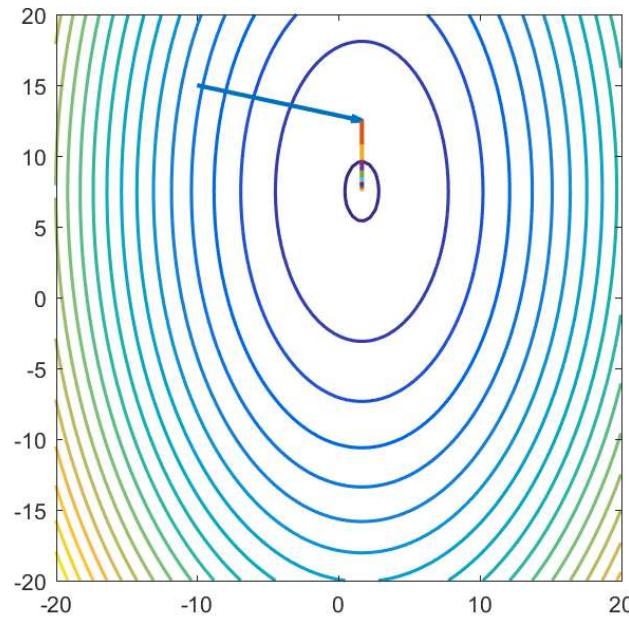
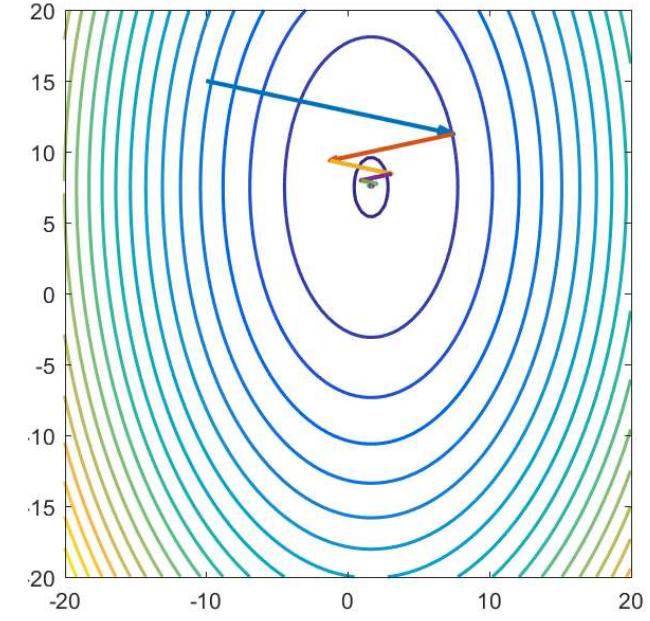
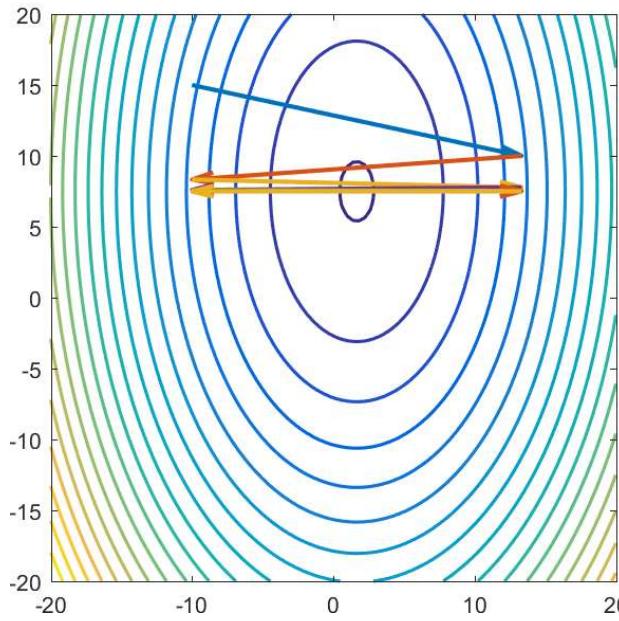
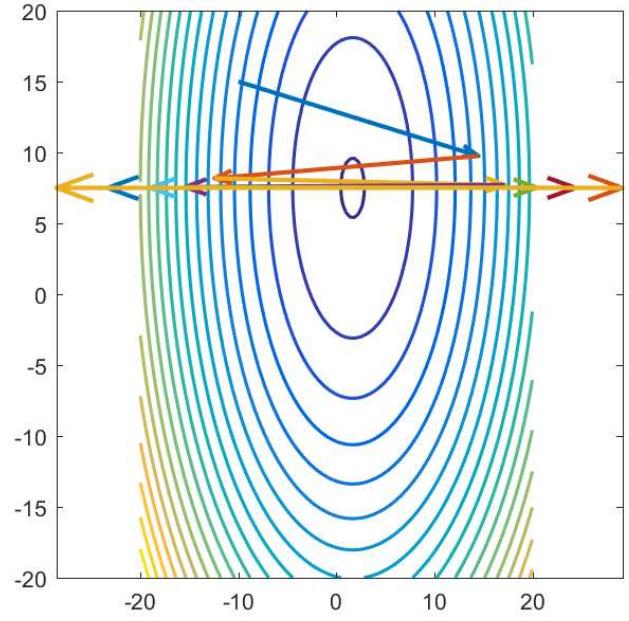
# Problem with vector update rule

$$\mathbf{w}^{(k+1)} \leftarrow \mathbf{w}^{(k)} - \eta \nabla_{\mathbf{w}} E^T$$

$$w_i^{(k+1)} = w_i^{(k)} - \eta \frac{\partial E(w_i^{(k)})}{\partial w}$$

$$\eta_{i,opt} = \left( \frac{\partial^2 E(w_i^{(k)})}{\partial w_i^2} \right)^{-1} = a_{ii}^{-1}$$

# Dependence on learning rate



- $\eta_{1,opt} = 1; \eta_{2,opt} = 0.33$
- $\eta = 2.1\eta_{2,opt}$
- $\eta = 2\eta_{2,opt}$
- $\eta = 1.5\eta_{2,opt}$
- $\eta = \eta_{2,opt}$
- $\eta = 0.75\eta_{2,opt}$

# Problem with vector update rule

$$\mathbf{w}^{(k+1)} \leftarrow \mathbf{w}^{(k)} - \eta \nabla_{\mathbf{w}} E^T$$

$$w_i^{(k+1)} = w_i^{(k)} - \eta \frac{\partial E(w_i^{(k)})}{\partial w}$$

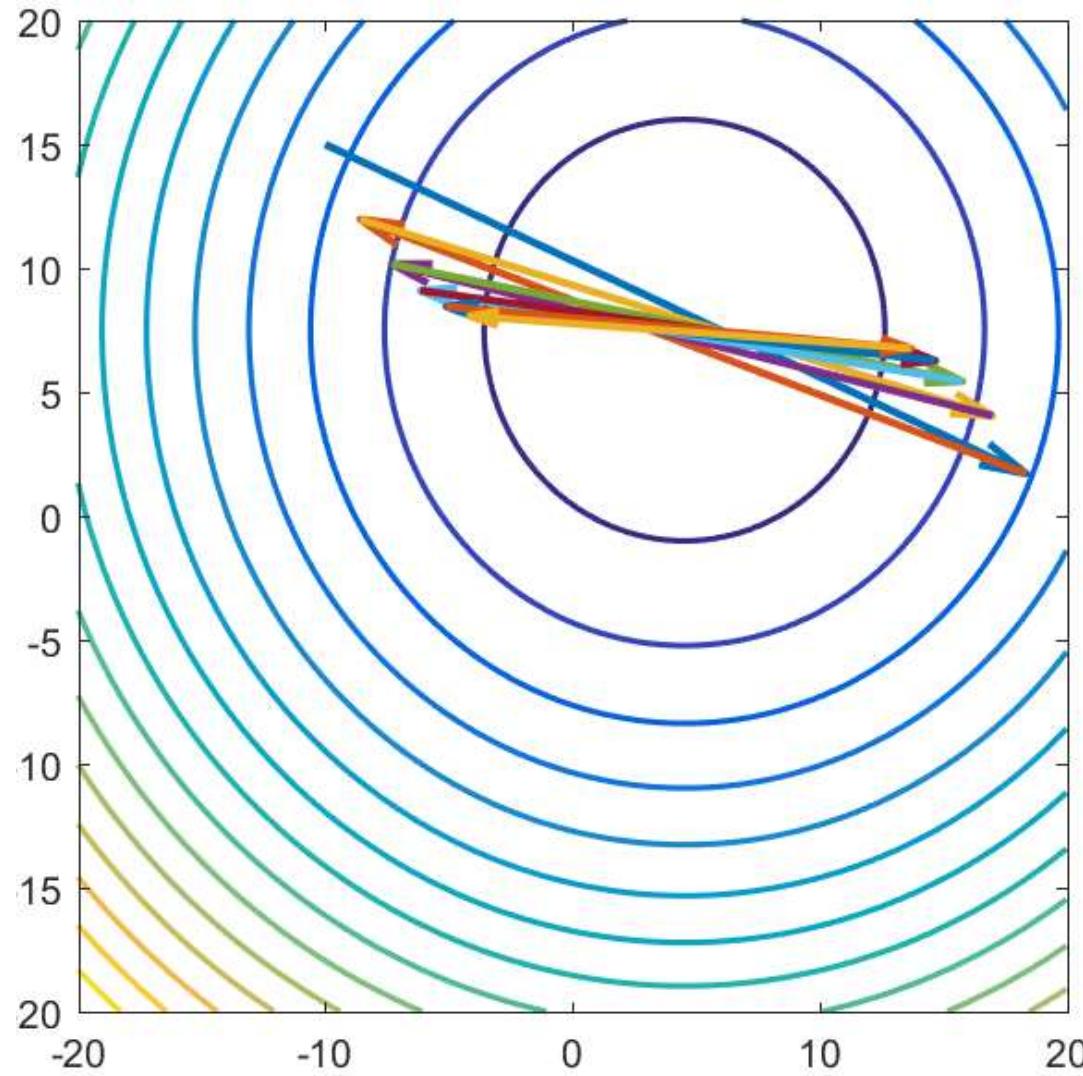
$$\eta_{i,opt} = \left( \frac{\partial^2 E(w_i^{(k)})}{\partial w_i^2} \right)^{-1} = a_{ii}^{-1}$$

- The learning rate must be lower than twice the *smallest* optimal learning rate for any component

$$\eta < 2 \min_i \eta_{i,opt}$$

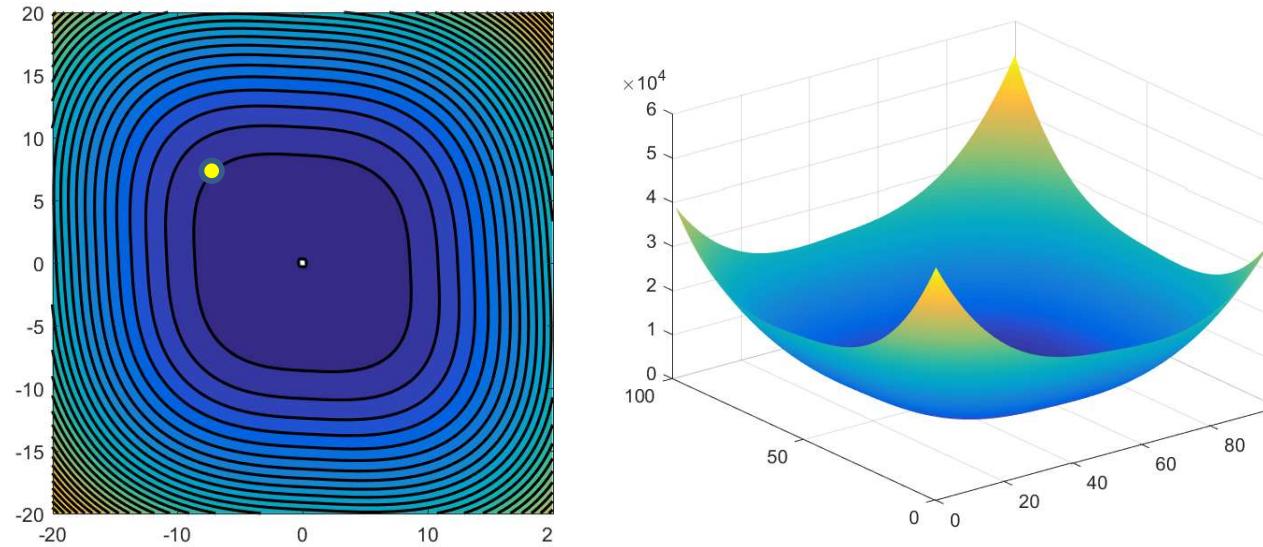
- Otherwise the learning will diverge
- This, however, makes the learning very slow
  - And will oscillate in all directions where  $\eta_{i,opt} \leq \eta < 2\eta_{i,opt}$

# Dependence on learning rate



- $\eta_{1,opt} = 1; \eta_{2,opt} = 0.91; \quad \eta = 1.9 \eta_{2,opt}$

# Generic differentiable *multivariate convex functions*



- For generic convex multivariate functions (not necessarily quadratic), we can employ quadratic Taylor series expansions and much of the analysis still applies
- Taylor expansion

$$E(\mathbf{w}) \approx E(\mathbf{w}^{(k)}) + \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})(\mathbf{w} - \mathbf{w}^{(k)}) + \frac{1}{2} (\mathbf{w} - \mathbf{w}^{(k)})^T \mathbf{H}_E(\mathbf{w}^{(k)}) (\mathbf{w} - \mathbf{w}^{(k)})$$

- The optimal step size is inversely proportional to the Eigen values of the Hessian
  - The second derivative along the orthogonal coordinates
  - For the smoothest convergence, these must all be equal

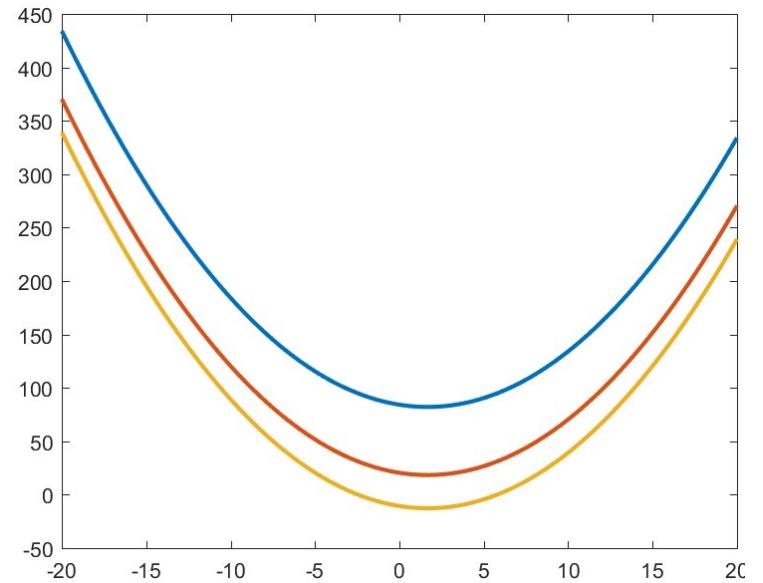
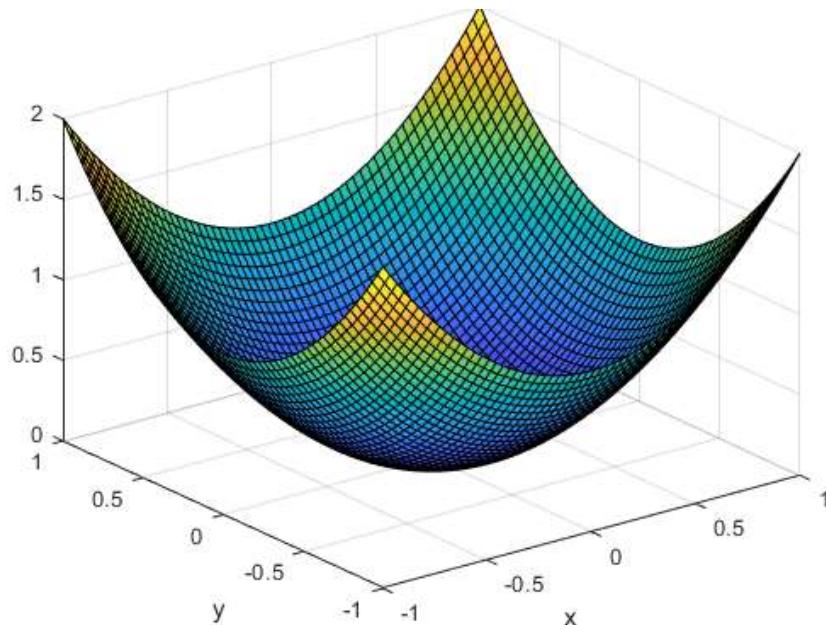
# Convergence

- Convergence behaviors become increasingly unpredictable as dimensions increase
- For the fastest convergence, ideally, the learning rate  $\eta$  must be close to both, the largest  $\eta_{i,opt}$  and the smallest  $\eta_{i,opt}$ 
  - To ensure convergence in every direction
  - Generally infeasible
- Convergence is particularly slow if  $\frac{\max_i \eta_{i,opt}}{\min_i \eta_{i,opt}}$  is large
  - The “condition” number
    - Must be close to 1.0 for fast convergence
- Following (hidden) slides discuss solutions that “normalize the space by stretching different directions differently to standardize optimal step size
  - A big topic for optimization
  - Unfortunately, infeasible for neural networks

# Comments on the quadratic

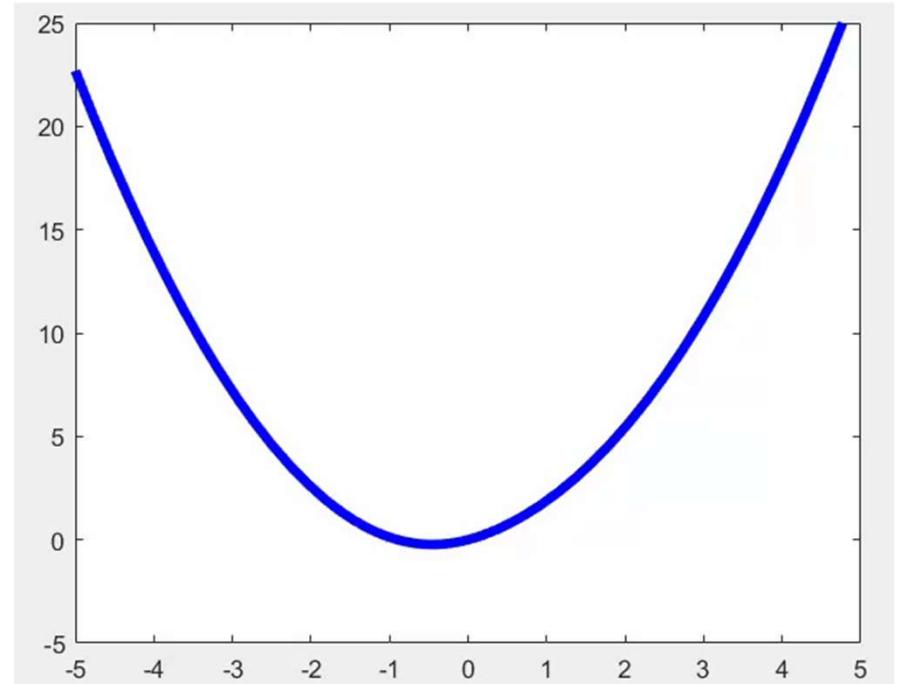
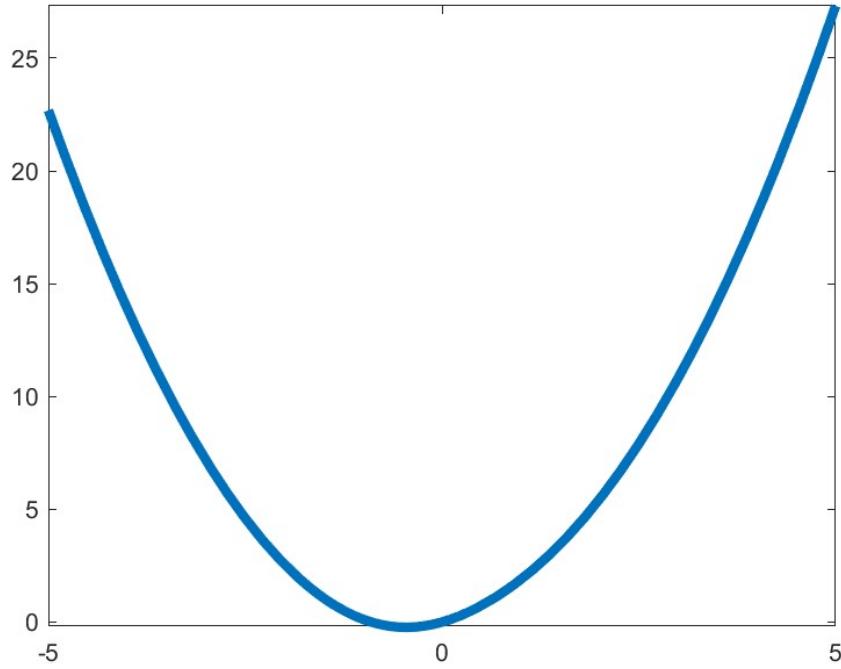
- Why are we talking about quadratics?
  - Quadratic functions form some kind of benchmark
  - Convergence of gradient descent is linear
    - Meaning it converges to solution exponentially fast
- The convergence for other kinds of functions can be viewed against this benchmark
- Actual losses will not be quadratic, but may locally have other structure
  - Local between current location and nearest local minimum
- Some examples in the following slides..
  - Strong convexity
  - Lipschitz continuity
  - Lipschitz smoothness
  - ..and how they affect convergence of gradient descent

# Quadratic convexity



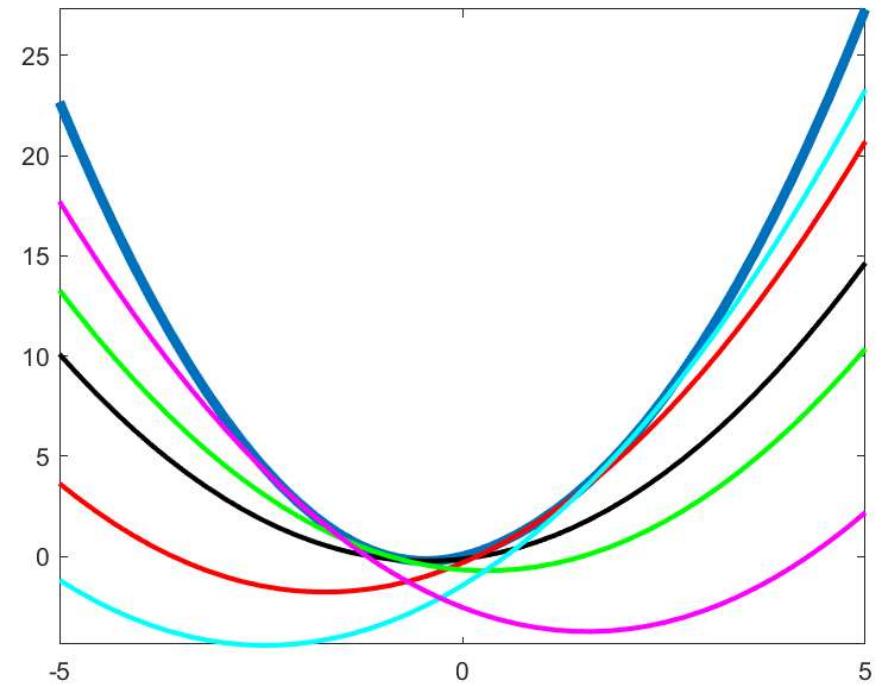
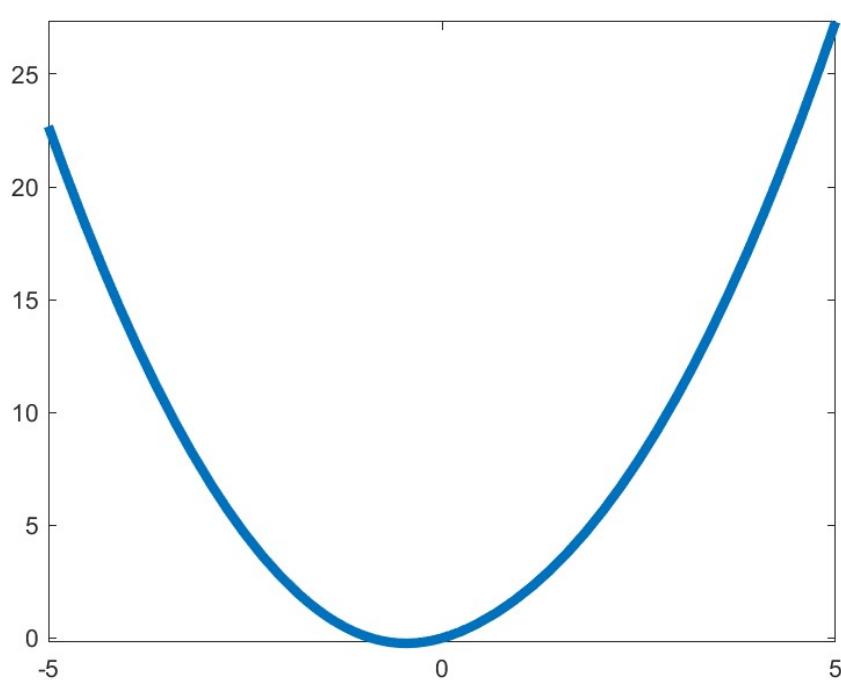
- A quadratic function has the form  $\frac{1}{2}\mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{w}^T \mathbf{b} + c$ 
  - Every “slice” is a quadratic bowl
- In some sense, the “standard” for gradient-descent based optimization
  - Others convex functions will be steeper in some regions, but flatter in others
- Gradient descent solution will have linear convergence
  - Take  $O(\log 1/\varepsilon)$  steps to get within  $\varepsilon$  of the optimal solution

# Strong convexity



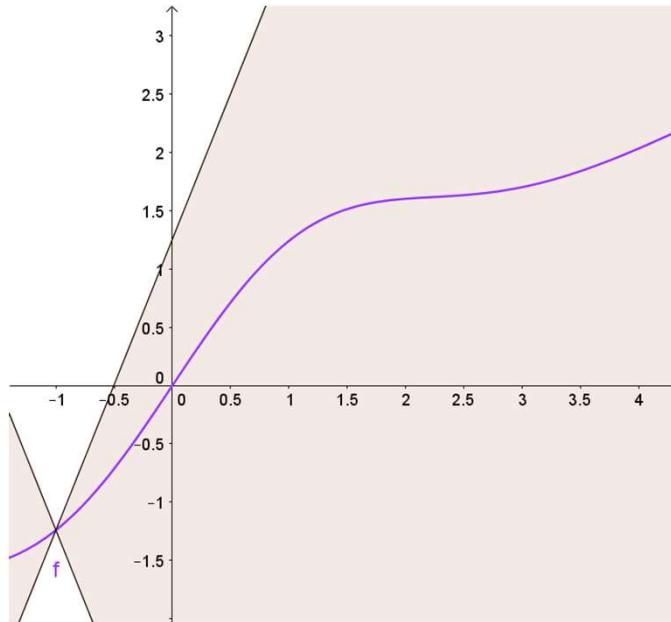
- A strongly convex function is *at least* quadratic in its convexity
  - Has a lower bound to its second derivative
- The function sits within a quadratic bowl
  - At any location, you can draw a quadratic bowl of fixed convexity (quadratic constant equal to lower bound of 2<sup>nd</sup> derivative) touching the function at that point, which contains it
- Convergence of gradient descent algorithms at least as good as that of the enclosing quadratic

# Strong convexity



- A strongly convex function is *at least* quadratic in its convexity
  - Has a lower bound to its second derivative
- The function sits within a quadratic bowl
  - At any location, you can draw a quadratic bowl of fixed convexity (quadratic constant equal to lower bound of 2<sup>nd</sup> derivative) touching the function at that point, which contains it
- Convergence of gradient descent algorithms at least as good as that of the enclosing quadratic

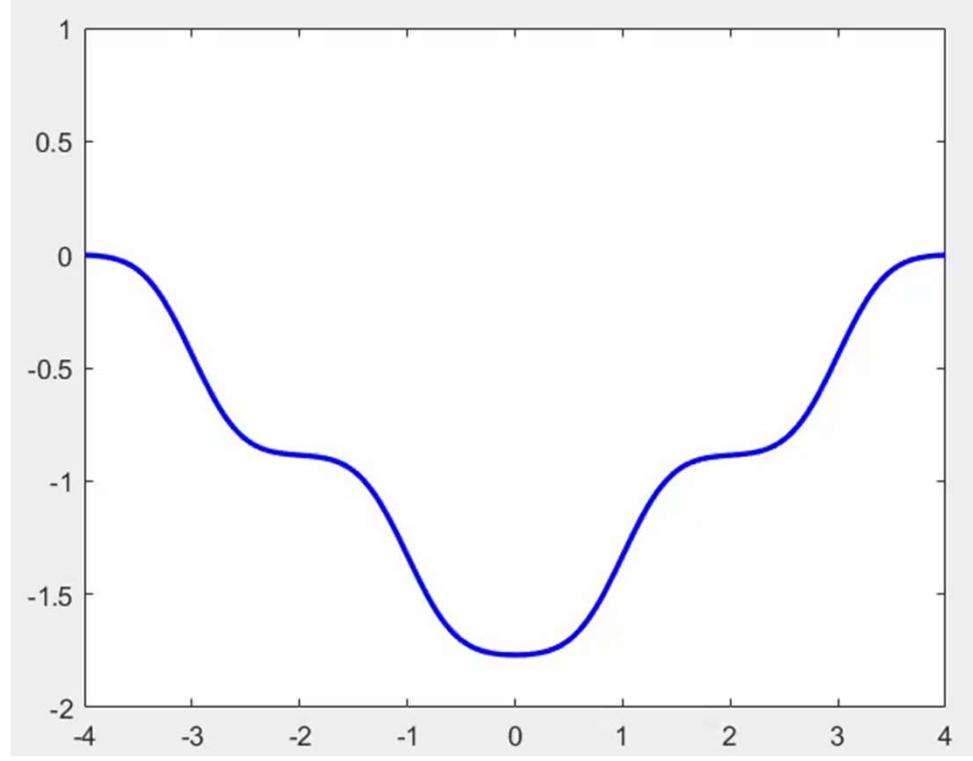
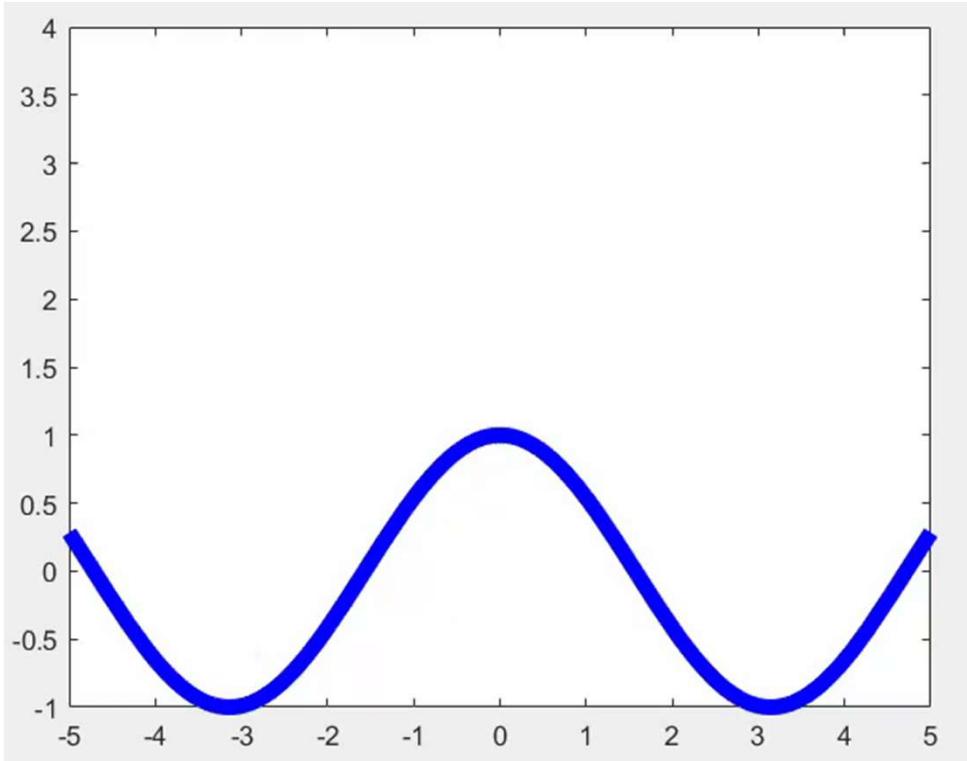
# Types of continuity



From wikipedia

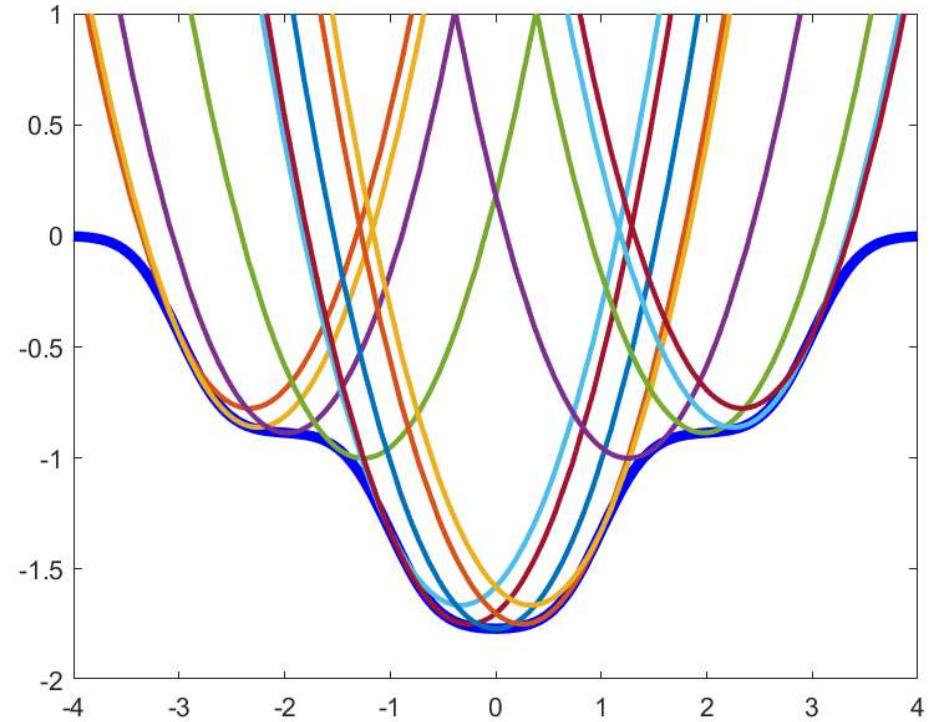
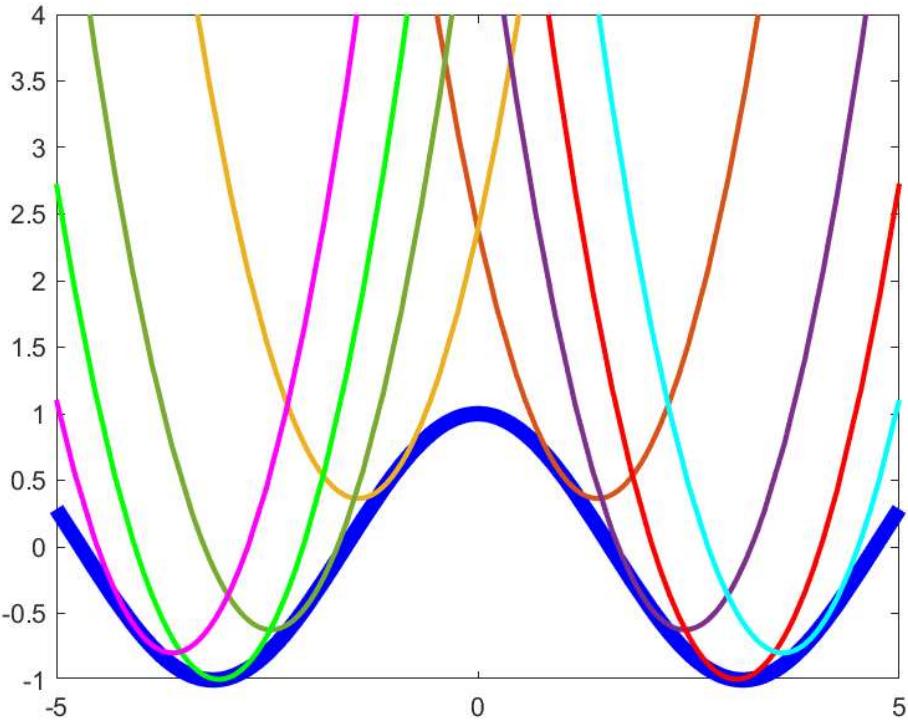
- Most functions are not strongly convex (if they are convex)
- Instead we will talk in terms of Lipschitz smoothness
- But first : a definition
- **Lipschitz continuous:** The function always lies outside a cone
  - The slope of the outer surface is the Lipschitz constant
  - $|f(x) - f(y)| \leq L|x - y|$

# Lipschitz smoothness



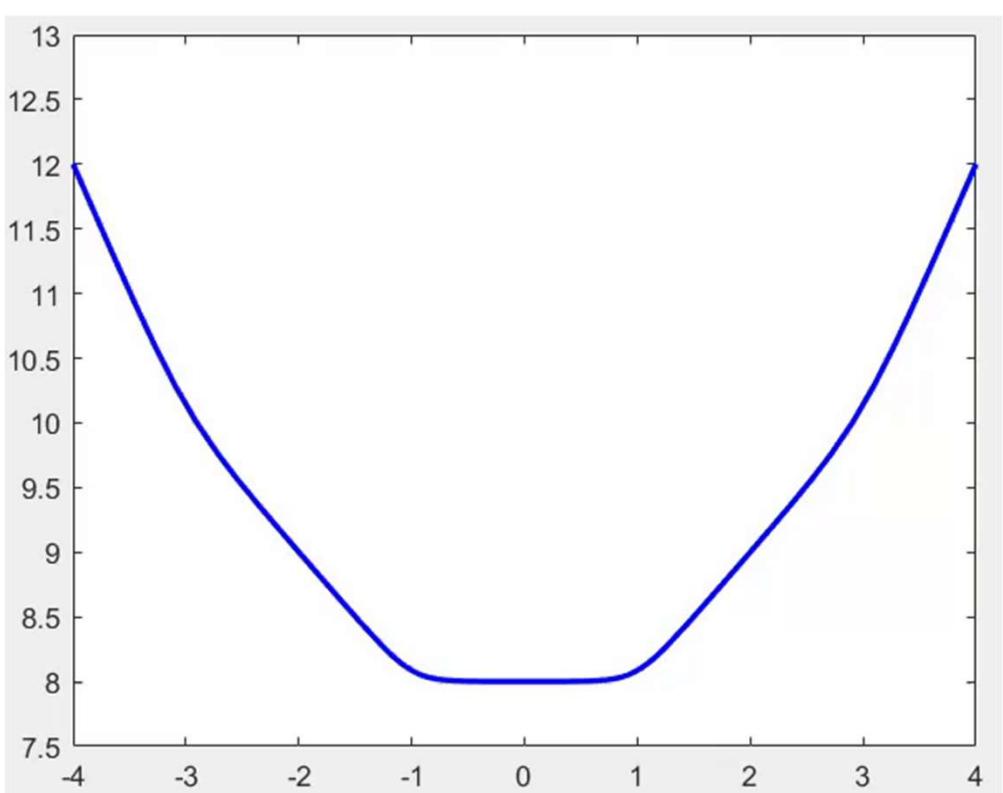
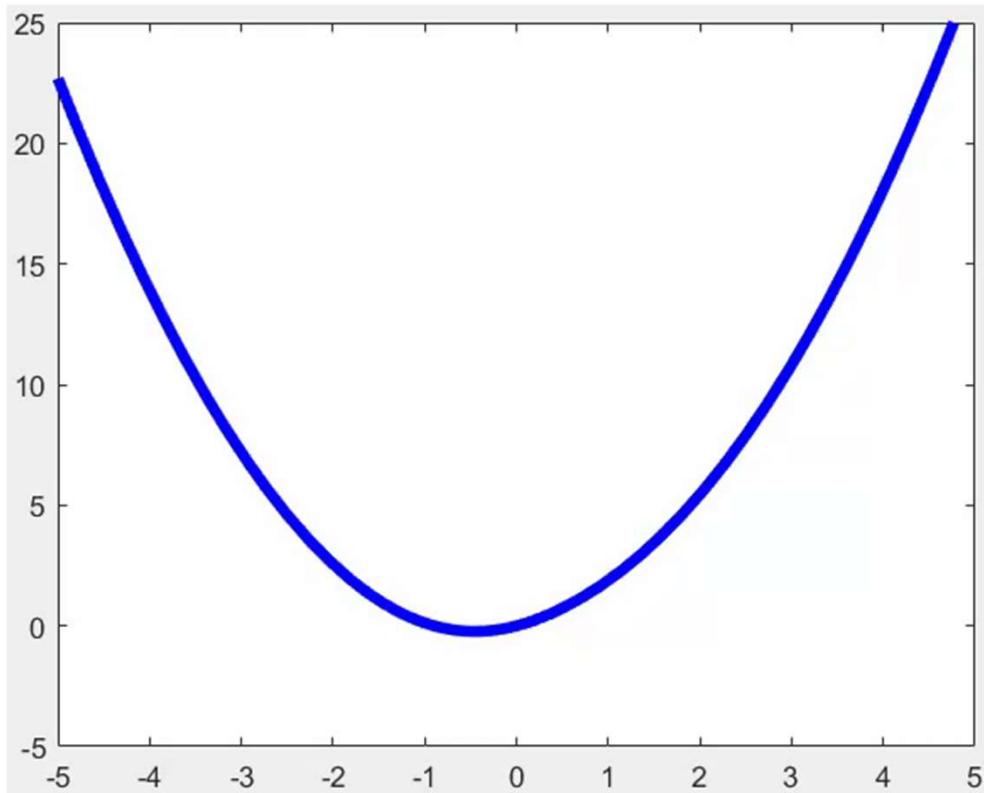
- Lipschitz smooth: The function's *derivative* is Lipschitz continuous
  - Need not be convex (or even differentiable)
  - Has an *upper bound* on second derivative (if it exists)
- Can always place a quadratic bowl of a fixed curvature within the function
  - Minimum curvature of quadratic must be  $\geq$  upper bound of second derivative of function (if it exists)

# Lipschitz smoothness



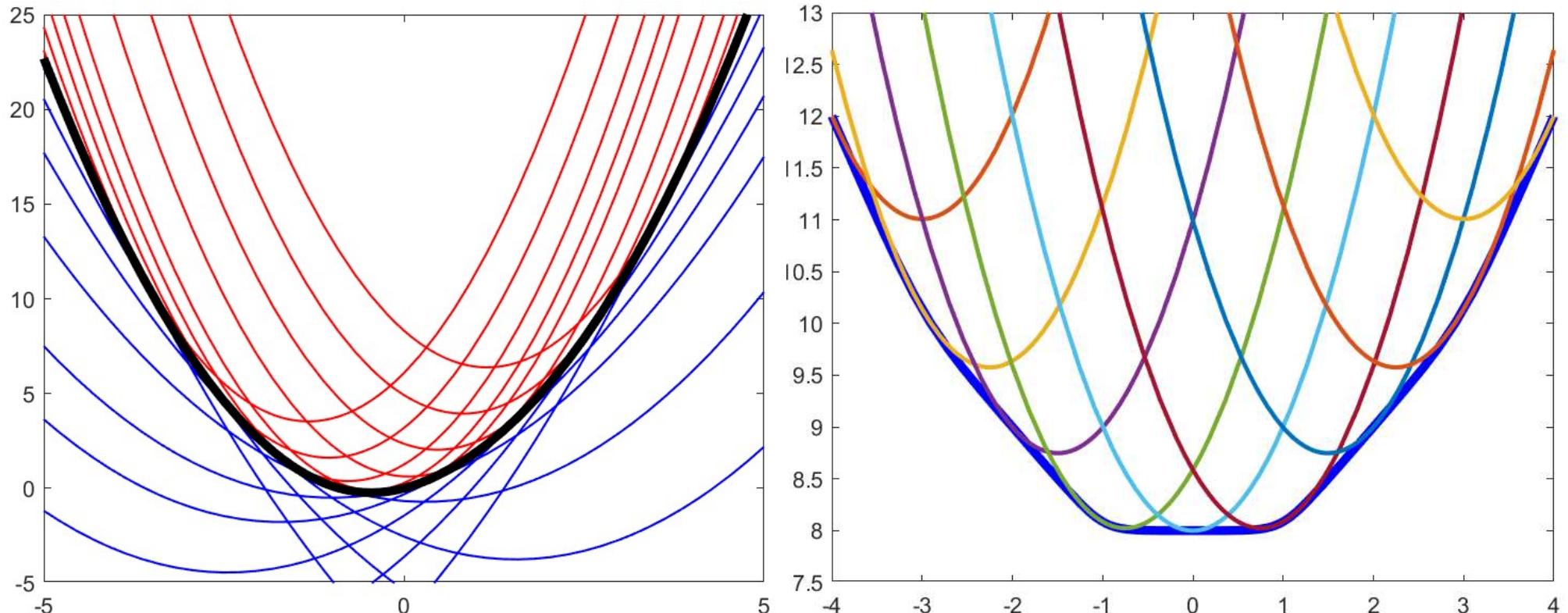
- Lipschitz smooth: The function's *derivative* is Lipschitz continuous
  - Need not be convex (or even differentiable)
  - Has an *upper bound* on second derivative (if it exists)
- Can always place a quadratic bowl of a fixed curvature within the function
  - Minimum curvature of quadratic must be  $\geq$  upper bound of second derivative of function (if it exists)

# Types of smoothness



- A function can be both strongly convex and Lipschitz smooth
  - Second derivative has upper *and* lower bounds
  - Convergence depends on curvature of strong convexity (at least linear)
- A function can be convex and Lipschitz smooth, but not strongly convex
  - Convex, but upper bound on second derivative
  - Weaker convergence guarantees, if any (at best linear)
  - This is often a reasonable assumption for the local structure of your loss function

# Types of smoothness



- A function can be both strongly convex and Lipschitz smooth
  - Second derivative has upper *and* lower bounds
  - Convergence depends on curvature of strong convexity (at least linear)
- A function can be convex and Lipschitz smooth, but not strongly convex
  - Convex, but upper bound on second derivative
  - Weaker convergence guarantees, if any (at best linear)
  - This is often a reasonable assumption for the local structure of your loss function

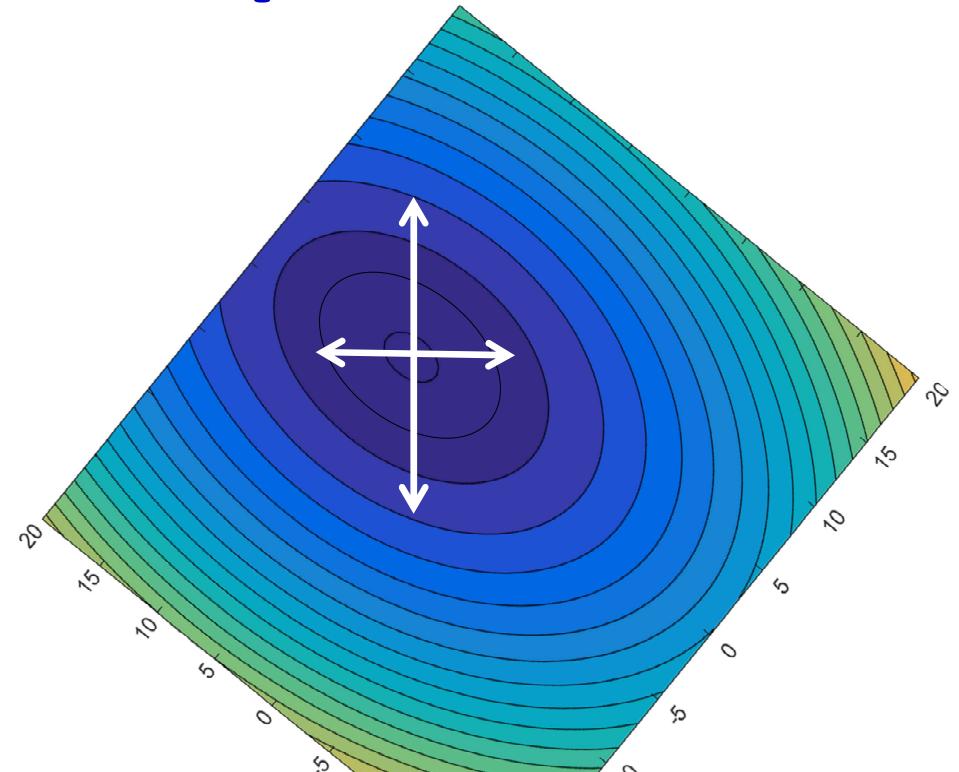
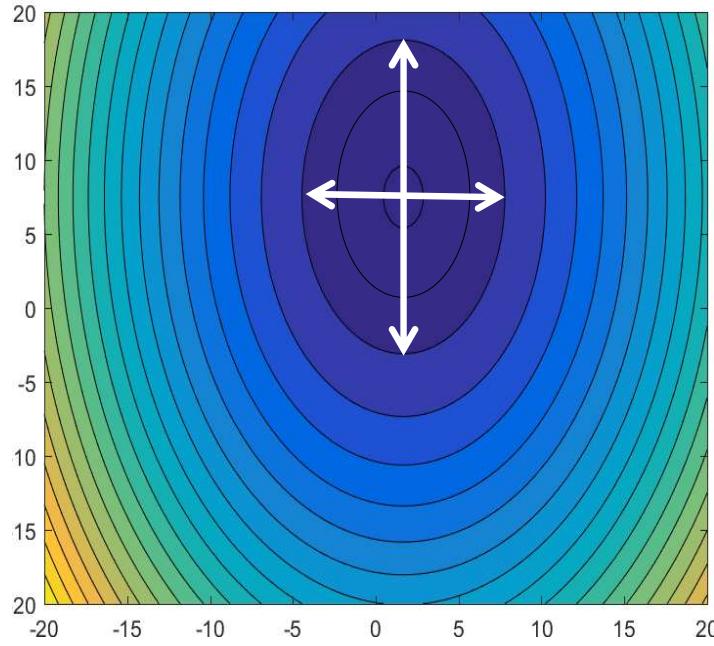
# Convergence Problems

- For quadratic (strongly) convex functions, gradient descent is exponentially fast
  - Linear convergence
    - Assuming learning rate is non-divergent
- For generic (Lipschitz Smooth) convex functions however, it is very slow
  - And inversely proportional to learning rate
$$|f(w^{(k)}) - f(w^*)| \propto \frac{1}{k} |f(w^{(0)}) - f(w^*)|$$
  - Takes  $O(1/\epsilon)$  iterations to get to within  $\epsilon$  of the solution
  - An inappropriate learning rate will destroy your happiness
- Second order methods will *locally* convert the loss function to quadratic
  - Convergence behavior will still depend on the nature of the original function
- ***Continuing with the quadratic-based explanation...***

# Convergence

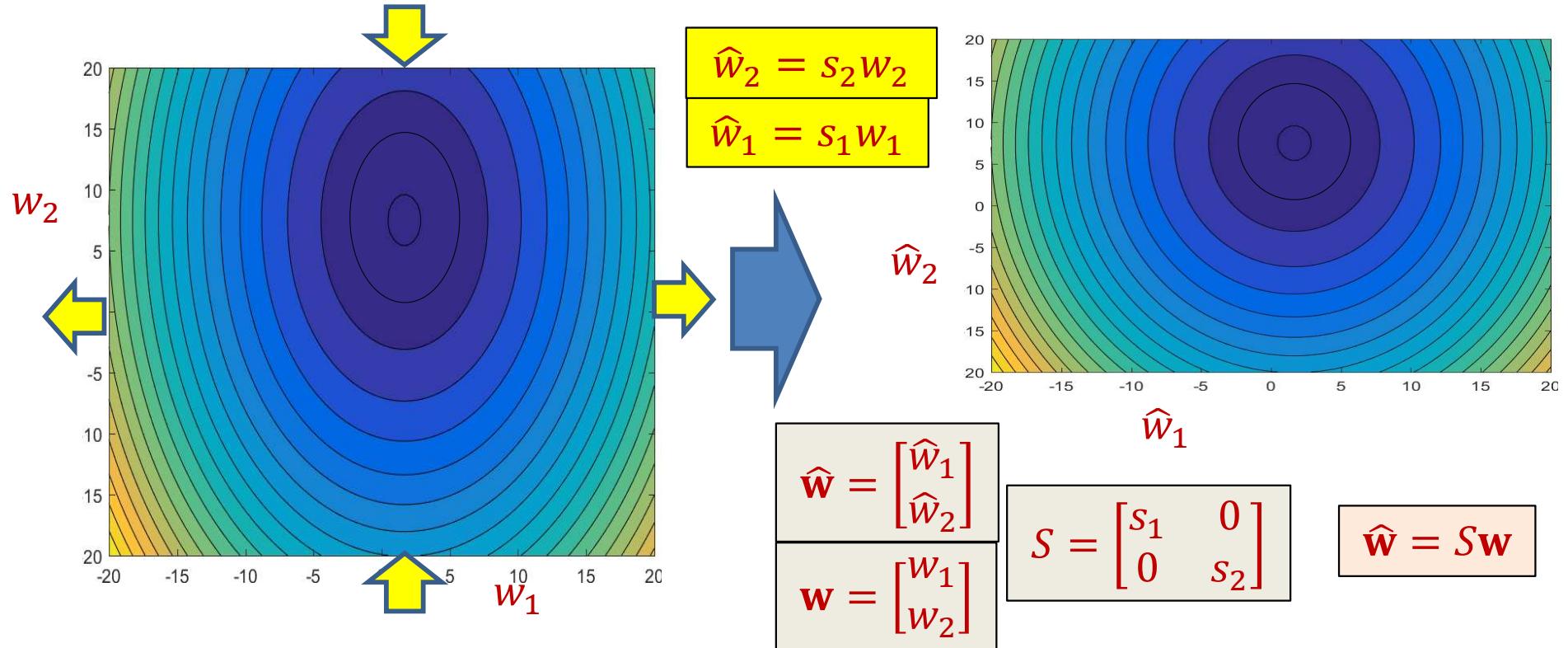
- Convergence behaviors become increasingly unpredictable as dimensions increase
- For the fastest convergence, ideally, the learning rate  $\eta$  must be close to both, the largest  $\eta_{i,opt}$  and the smallest  $\eta_{i,opt}$ 
  - To ensure convergence in every direction
  - Generally infeasible
- Convergence is particularly slow if  $\frac{\max_i \eta_{i,opt}}{\min_i \eta_{i,opt}}$  is large
  - The “condition” number is small

# One reason for the problem



- The objective function has different eccentricities in different directions
  - Resulting in different optimal learning rates for different directions
  - **The problem is more difficult when the ellipsoid is not axis aligned: the steps along the two directions are coupled!** Moving in one direction changes the gradient along the other
- Solution: *Normalize* the objective to have identical eccentricity in all directions
  - Then all of them will have identical optimal learning rates
  - Easier to find a working learning rate

# Solution: Scale the axes



- Scale (and rotate) the axes, such that all of them have identical (identity) “spread”
  - Equal-value contours are circular
  - Movement along the coordinate axes become independent
- **Note:** equation of a quadratic surface with circular equal-value contours can be written as

$$E = \frac{1}{2} \widehat{\mathbf{w}}^T \widehat{\mathbf{w}} + \widehat{\mathbf{b}}^T \widehat{\mathbf{w}} + c$$

# Scaling the axes

- Original equation:

$$E = \frac{1}{2} \mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{b}^T \mathbf{w} + c$$

- We want to find a (diagonal) scaling matrix  $\mathbf{S}$  such that

$$\mathbf{S} = \begin{bmatrix} s_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_N \end{bmatrix}, \quad \hat{\mathbf{w}} = \mathbf{S} \mathbf{w}$$

- And

$$E = \frac{1}{2} \hat{\mathbf{w}}^T \hat{\mathbf{w}} + \hat{\mathbf{b}}^T \hat{\mathbf{w}} + c$$

# Scaling the axes

- Original equation:

$$E = \frac{1}{2} \mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{b}^T \mathbf{w} + c$$

- We want to find a (diagonal) scaling matrix  $\mathbf{S}$  such that

$$\mathbf{S} = \begin{bmatrix} s_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_N \end{bmatrix}, \quad \hat{\mathbf{w}} = \mathbf{S} \mathbf{w}$$

- And

$$E = \frac{1}{2} \hat{\mathbf{w}}^T \hat{\mathbf{w}} + \hat{\mathbf{b}}^T \hat{\mathbf{w}} + c$$

By inspection:  
 $\mathbf{S} = \mathbf{A}^{0.5}$

# Scaling the axes

- We have

$$E = \frac{1}{2} \mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{b}^T \mathbf{w} + c$$

$$\hat{\mathbf{w}} = S \mathbf{w}$$

$$\begin{aligned} E &= \frac{1}{2} \hat{\mathbf{w}}^T \hat{\mathbf{w}} + \hat{\mathbf{b}}^T \hat{\mathbf{w}} + c \\ &= \frac{1}{2} \mathbf{w}^T S^T S \mathbf{w} + \hat{\mathbf{b}}^T S \mathbf{w} + c \end{aligned}$$

- Equating linear and quadratic coefficients, we get

$$S^T S = \mathbf{A}, \quad \hat{\mathbf{b}}^T S = -\mathbf{b}^T$$

- Solving:  $S = \mathbf{A}^{0.5}, \quad \hat{\mathbf{b}} = \mathbf{A}^{-0.5} \mathbf{b}$

# Scaling the axes

- We have

$$E = \frac{1}{2} \mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{b}^T \mathbf{w} + c$$

$$\hat{\mathbf{w}} = \mathbf{S} \mathbf{w}$$

$$E = \frac{1}{2} \hat{\mathbf{w}}^T \hat{\mathbf{w}} + \hat{\mathbf{b}}^T \hat{\mathbf{w}} + c$$

- Solving for  $\mathbf{S}$  we get

$$\hat{\mathbf{w}} = \mathbf{A}^{0.5} \mathbf{w}, \quad \hat{\mathbf{b}} = \mathbf{A}^{-0.5} \mathbf{b}$$

# Scaling the axes

- We have

$$E = \frac{1}{2} \mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{b}^T \mathbf{w} + c$$

$$\hat{\mathbf{w}} = \mathbf{S} \mathbf{w}$$

$$E = \frac{1}{2} \hat{\mathbf{w}}^T \hat{\mathbf{w}} + \hat{\mathbf{b}}^T \hat{\mathbf{w}} + c$$

- Solving for  $\mathbf{S}$  we get

$$\hat{\mathbf{w}} = \mathbf{A}^{0.5} \mathbf{w}, \quad \hat{\mathbf{b}} = \mathbf{A}^{-0.5} \mathbf{b}$$

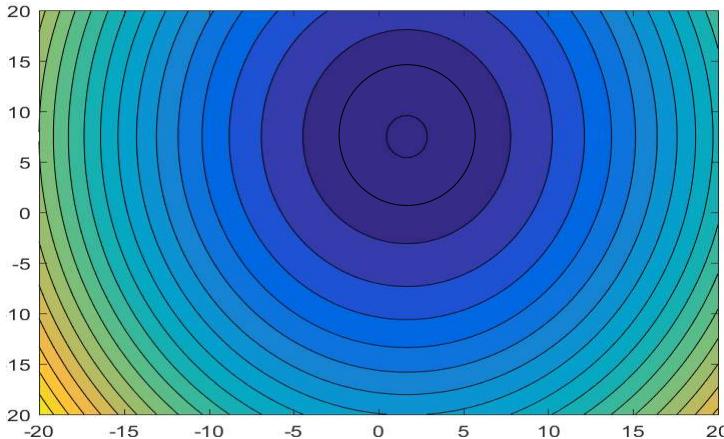
# The Inverse Square Root of A

- For *any* positive definite  $\mathbf{A}$ , we can write

$$\mathbf{A} = \mathbf{E}\boldsymbol{\Lambda}\mathbf{E}^T$$

- Eigen decomposition
  - $\mathbf{E}$  is an orthogonal matrix
  - $\boldsymbol{\Lambda}$  is a diagonal matrix of non-zero diagonal entries
- 
- Defining  $\mathbf{A}^{0.5} = \mathbf{E}\boldsymbol{\Lambda}^{0.5}\mathbf{E}^T$ 
    - Check  $(\mathbf{A}^{0.5})^T\mathbf{A}^{0.5} = \mathbf{E}\boldsymbol{\Lambda}\mathbf{E}^T = \mathbf{A}$
  
  - Defining  $\mathbf{A}^{-0.5} = \mathbf{E}\boldsymbol{\Lambda}^{-0.5}\mathbf{E}^T$ 
    - Check:  $(\mathbf{A}^{-0.5})^T\mathbf{A}^{-0.5} = \mathbf{E}\boldsymbol{\Lambda}^{-1}\mathbf{E}^T = \mathbf{A}^{-1}$

# Returning to our problem

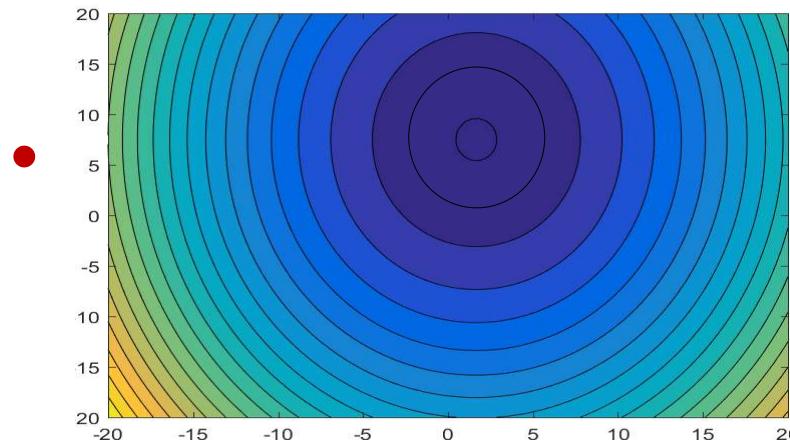


$$E = \frac{1}{2} \hat{\mathbf{w}}^T \hat{\mathbf{w}} + \hat{\mathbf{b}}^T \hat{\mathbf{w}} + c$$

- Computing the gradient, and noting that  $\mathbf{A}^{0.5}$  is symmetric, we can relate  $\nabla_{\hat{\mathbf{w}}} E$  and  $\nabla_{\mathbf{w}} E$ :

$$\begin{aligned}\nabla_{\hat{\mathbf{w}}} E &= \hat{\mathbf{w}}^T + \hat{\mathbf{b}}^T \\&= \mathbf{w}^T \mathbf{A}^{0.5} + \mathbf{b}^T \mathbf{A}^{-0.5} \\&= (\mathbf{w}^T \mathbf{A} + \mathbf{b}^T) \mathbf{A}^{-0.5} \\&= \nabla_{\mathbf{w}} E \cdot \mathbf{A}^{-0.5}\end{aligned}$$

# Returning to our problem



$$E = \frac{1}{2} \hat{\mathbf{w}}^T \hat{\mathbf{w}} + \hat{\mathbf{b}}^T \hat{\mathbf{w}} + c$$

- Gradient descent rule:

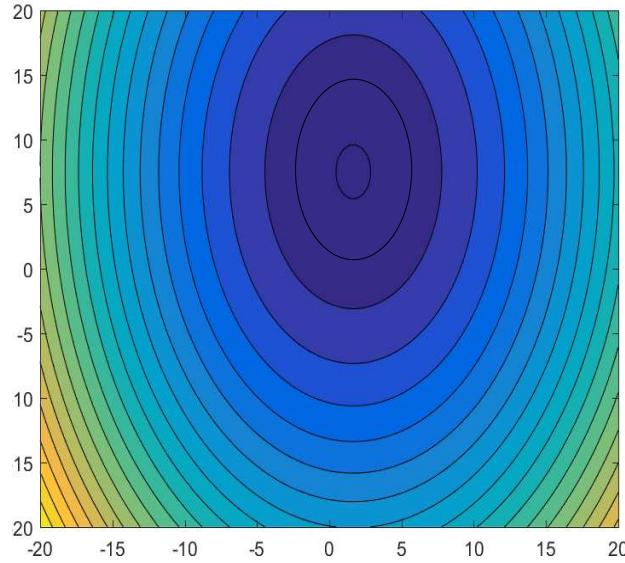
$$-\hat{\mathbf{w}}^{(k+1)} = \hat{\mathbf{w}}^{(k)} - \eta \nabla_{\hat{\mathbf{w}}} E(\hat{\mathbf{w}}^{(k)})^T$$

- Learning rate is now independent of direction

- Using  $\hat{\mathbf{w}} = \mathbf{A}^{0.5} \mathbf{w}$ , and  $\nabla_{\hat{\mathbf{w}}} E(\hat{\mathbf{w}})^T = \mathbf{A}^{-0.5} \nabla_{\mathbf{w}} E(\mathbf{w})^T$

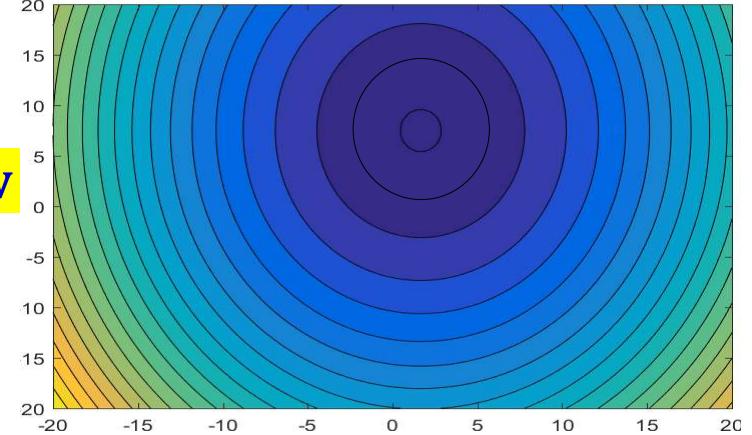
$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta \mathbf{A}^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

# Modified update rule



$$\hat{\mathbf{w}} = \mathbf{A}^{0.5} \mathbf{w}$$

$$E = \frac{1}{2} \mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{b}^T \mathbf{w} + c$$

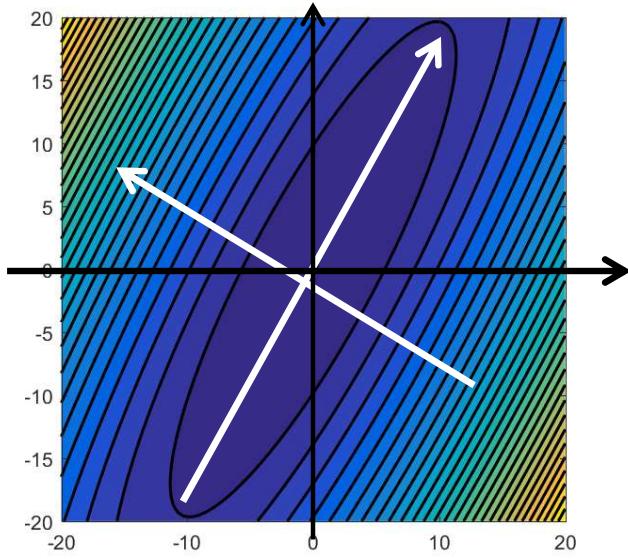


$$E = \frac{1}{2} \hat{\mathbf{w}}^T \hat{\mathbf{w}} + \hat{\mathbf{b}}^T \hat{\mathbf{w}} + c$$

- $\hat{\mathbf{w}}^{(k+1)} = \hat{\mathbf{w}}^{(k)} - \eta \nabla_{\hat{\mathbf{w}}} E(\hat{\mathbf{w}}^{(k)})^T$
- Leads to the modified gradient descent rule

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta \mathbf{A}^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

# For non-axis-aligned quadratics..

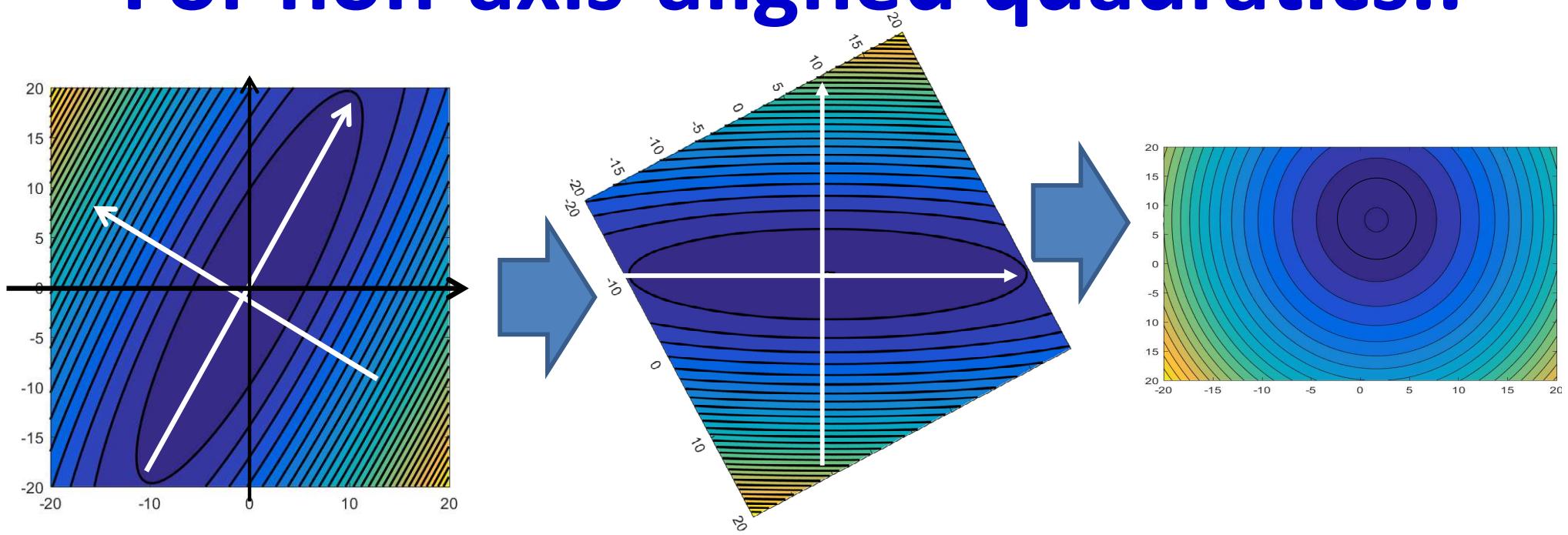


$$E = \frac{1}{2} \mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{w}^T \mathbf{b} + c$$

$$\begin{aligned} E = & \frac{1}{2} \sum_i a_{ii} w_i^2 + \sum_{i \neq j} a_{ij} w_i w_j \\ & + \sum_i b_i w_i + c \end{aligned}$$

- If  $\mathbf{A}$  is not diagonal, the contours are not axis-aligned
  - Because of the cross-terms  $a_{ij} w_i w_j$
  - The major axes of the ellipsoids are the *Eigenvectors* of  $\mathbf{A}$ , and their diameters are proportional to the Eigen values of  $\mathbf{A}$
- But this does not affect the discussion
  - This is merely a rotation of the space from the axis-aligned case
  - The component-wise optimal learning rates along the major and minor axes of the equal-contour ellipsoids will be different, causing problems
    - The optimal rates along the axes are Inversely proportional to the *eigenvalues* of  $\mathbf{A}$

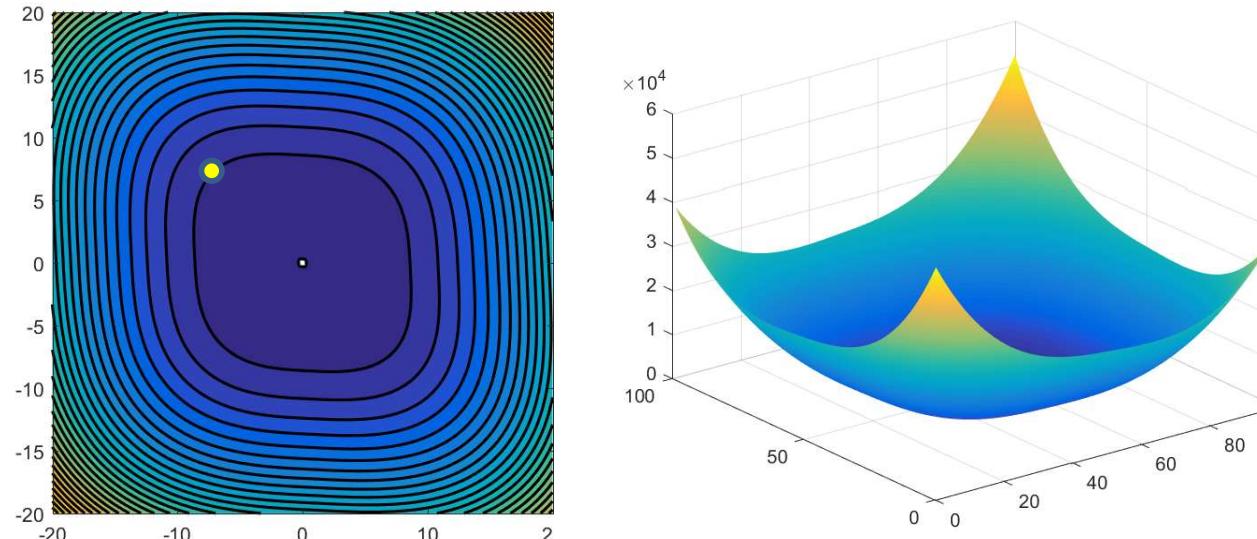
# For non-axis-aligned quadratics..



- The component-wise optimal learning rates along the major and minor axes of the contour ellipsoids will differ, causing problems
  - Inversely proportional to the *eigenvalues* of  $\mathbf{A}$
- This can be fixed as before by rotating and resizing the different directions to obtain the same *normalized* update rule as before:

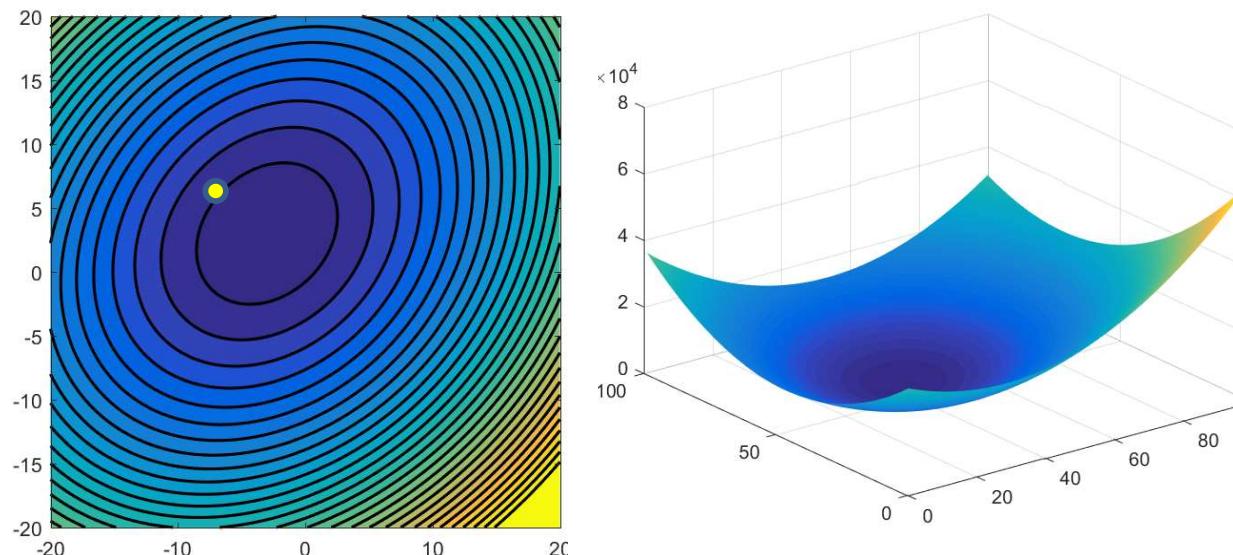
$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta \mathbf{A}^{-1} \mathbf{b}$$

# Generic differentiable *multivariate convex functions*

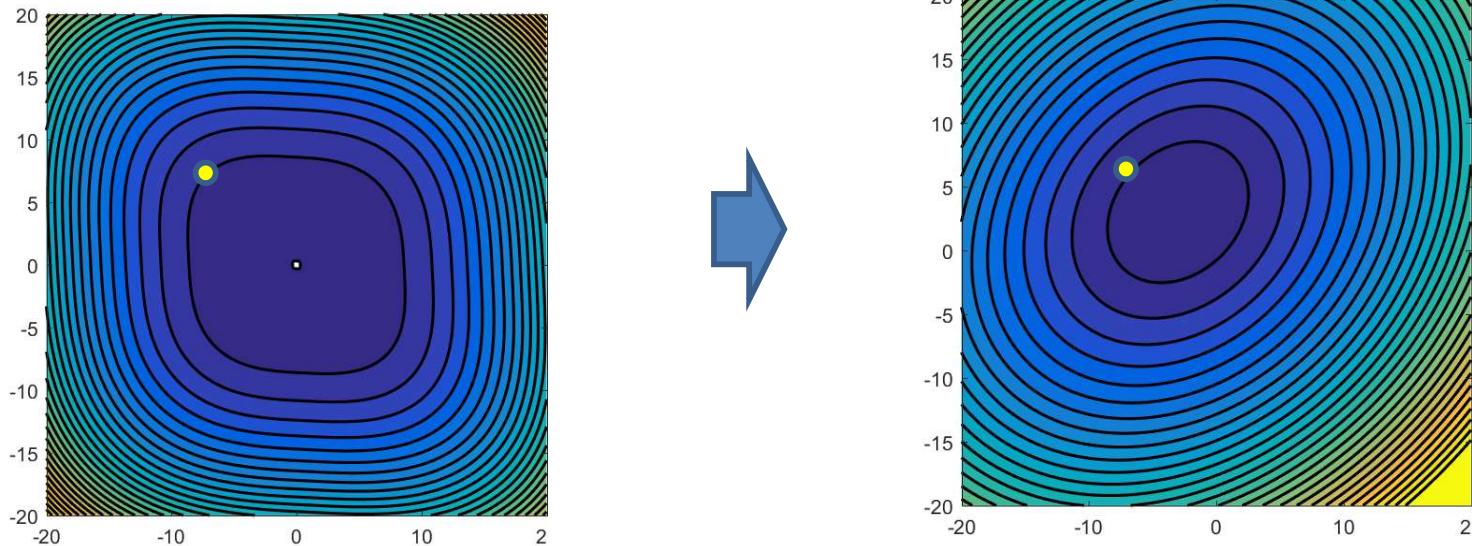


- Taylor expansion

$$E(\mathbf{w}) \approx E(\mathbf{w}^{(k)}) + \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})(\mathbf{w} - \mathbf{w}^{(k)}) + \frac{1}{2} (\mathbf{w} - \mathbf{w}^{(k)})^T H_E(\mathbf{w}^{(k)}) (\mathbf{w} - \mathbf{w}^{(k)}) + \dots$$



# Generic differentiable *multivariate convex functions*



- Taylor expansion

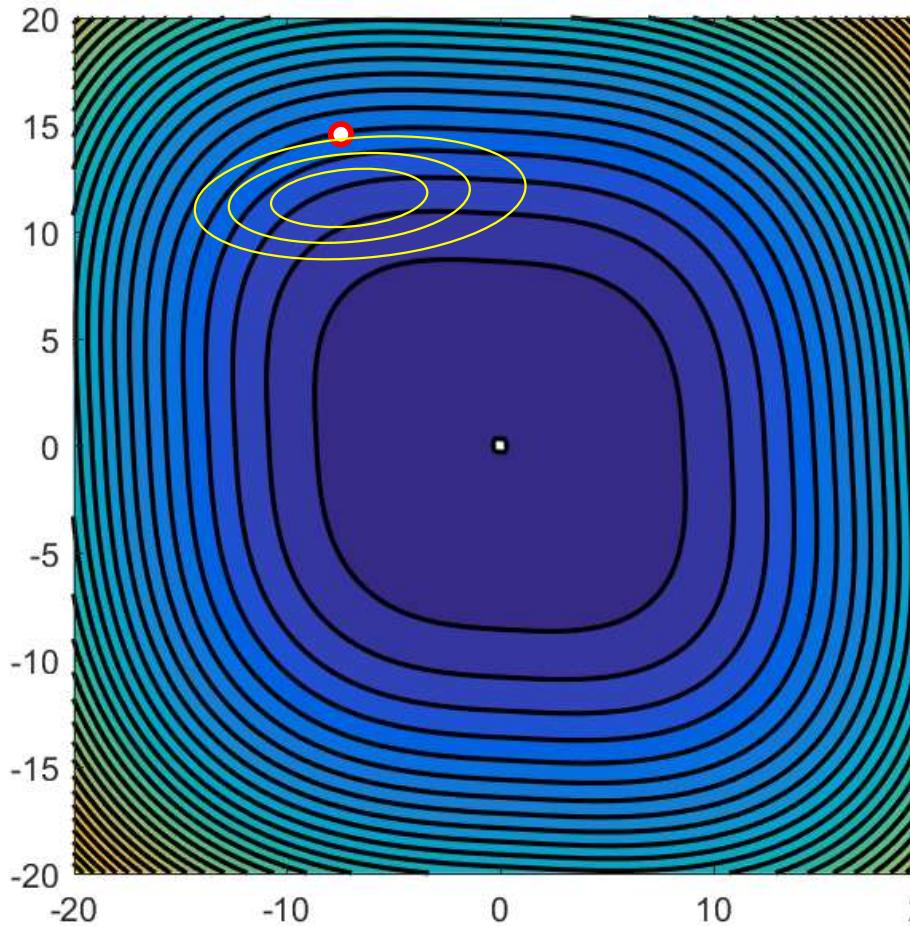
$$E(\mathbf{w}) \approx E(\mathbf{w}^{(k)}) + \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})(\mathbf{w} - \mathbf{w}^{(k)}) + \frac{1}{2} (\mathbf{w} - \mathbf{w}^{(k)})^T H_E(\mathbf{w}^{(k)}) (\mathbf{w} - \mathbf{w}^{(k)}) + \dots$$

- Note that this has the form  $\frac{1}{2} \mathbf{w}^T \mathbf{A} \mathbf{w} + \mathbf{w}^T \mathbf{b} + c$
- Using the same logic as before, we get the normalized update rule

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

- For a quadratic function, the optimal  $\eta$  is 1 (which is exactly Newton's method)
  - And should not be greater than 2!

# Minimization by Newton's method ( $\eta = 1$ )



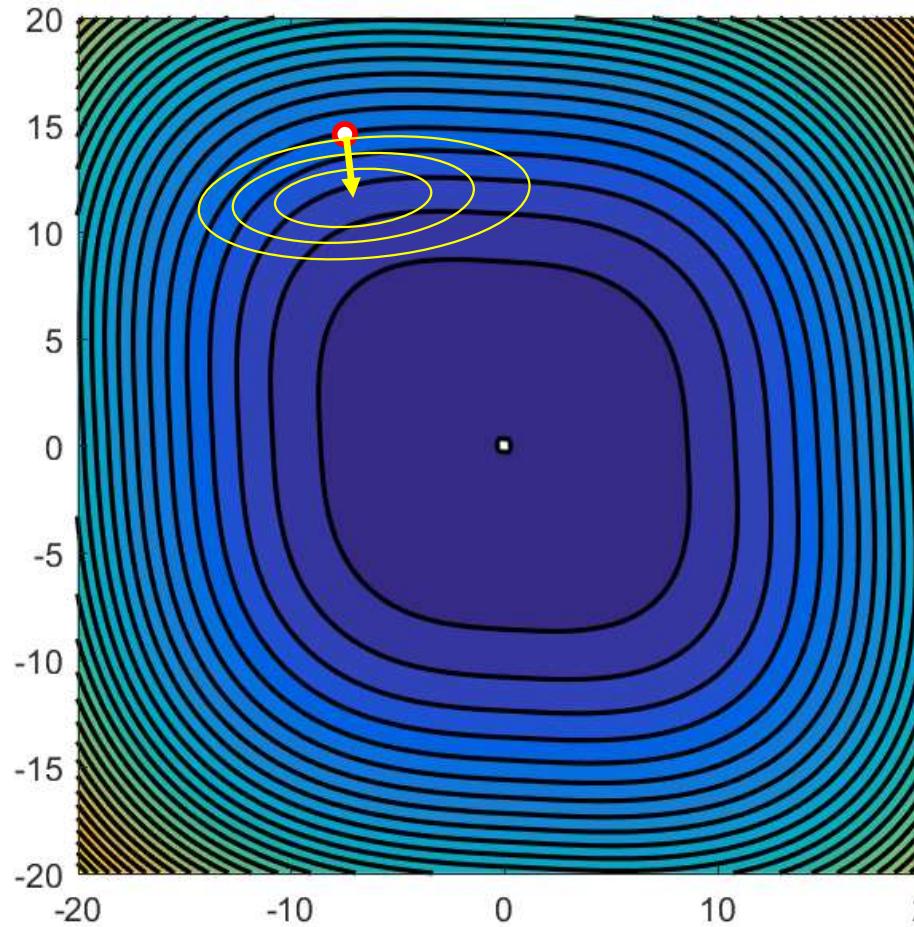
Fit a quadratic at each point and find the minimum of that quadratic

- Iterated localized optimization with quadratic approximations

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

$$-\eta = 1$$

# Minimization by Newton's method ( $\eta = 1$ )

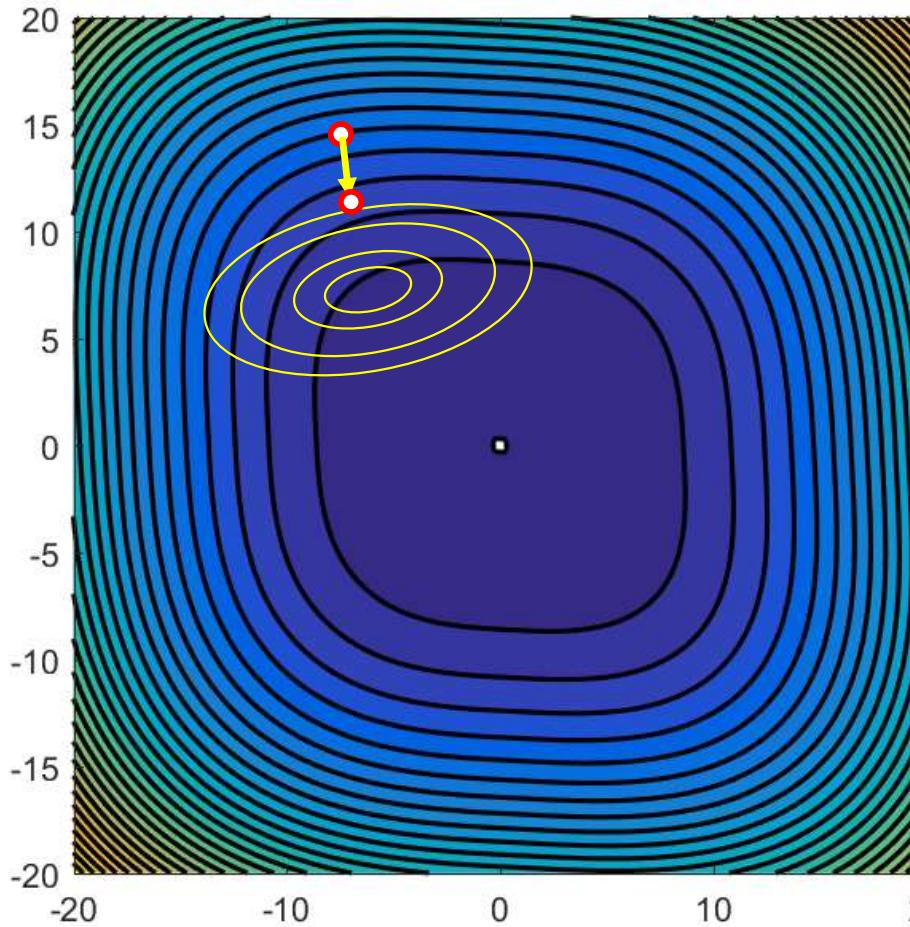


- Iterated localized optimization with quadratic approximations

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

$$-\eta = 1$$

# Minimization by Newton's method ( $\eta = 1$ )

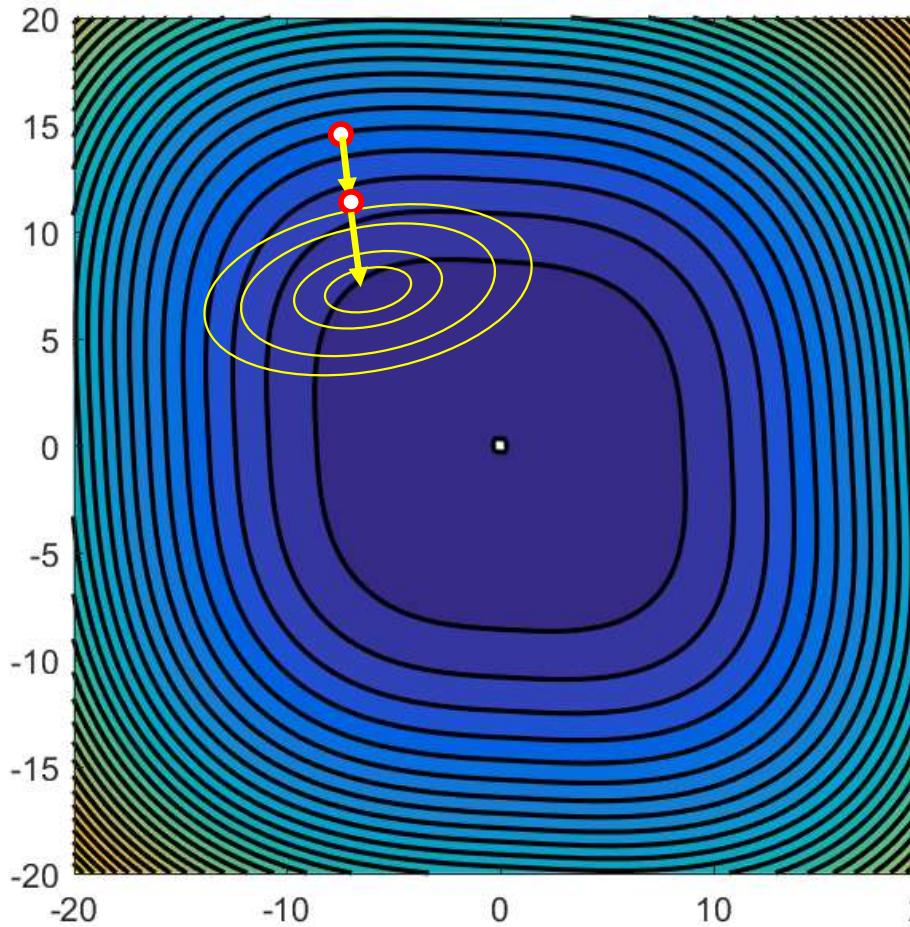


- Iterated localized optimization with quadratic approximations

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

$$-\eta = 1$$

# Minimization by Newton's method ( $\eta = 1$ )

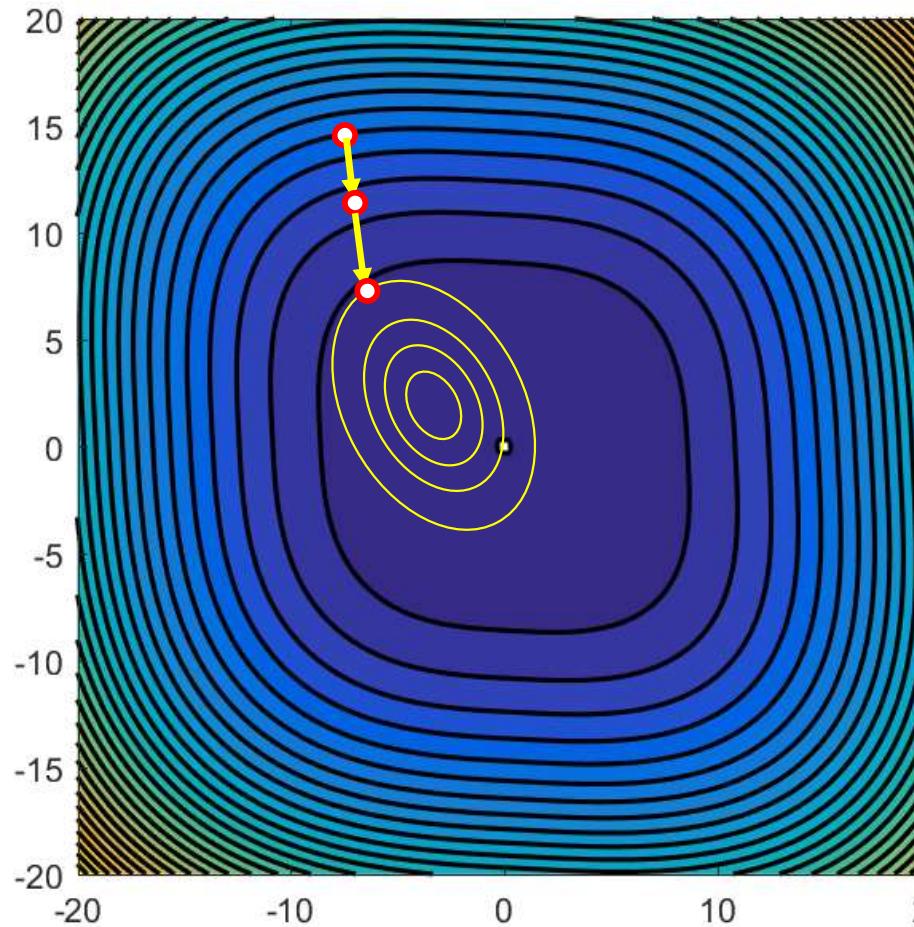


- Iterated localized optimization with quadratic approximations

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

$$-\eta = 1$$

# Minimization by Newton's method

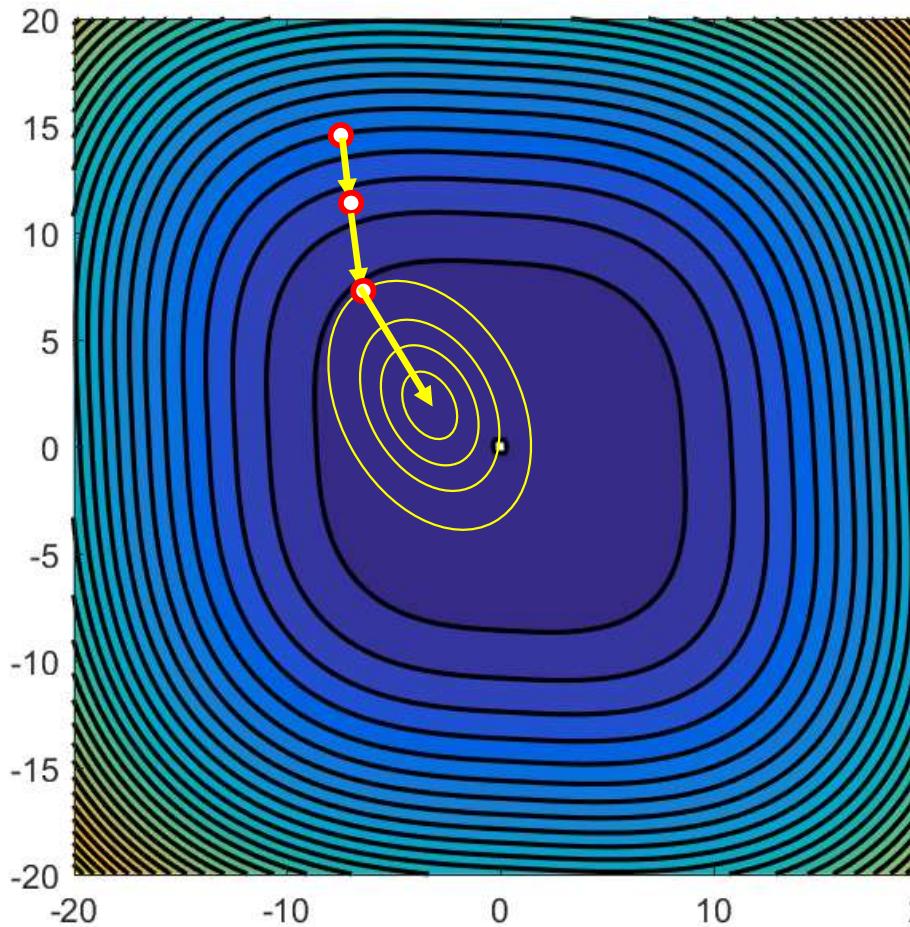


- Iterated localized optimization with quadratic approximations

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

$$-\eta = 1$$

# Minimization by Newton's method

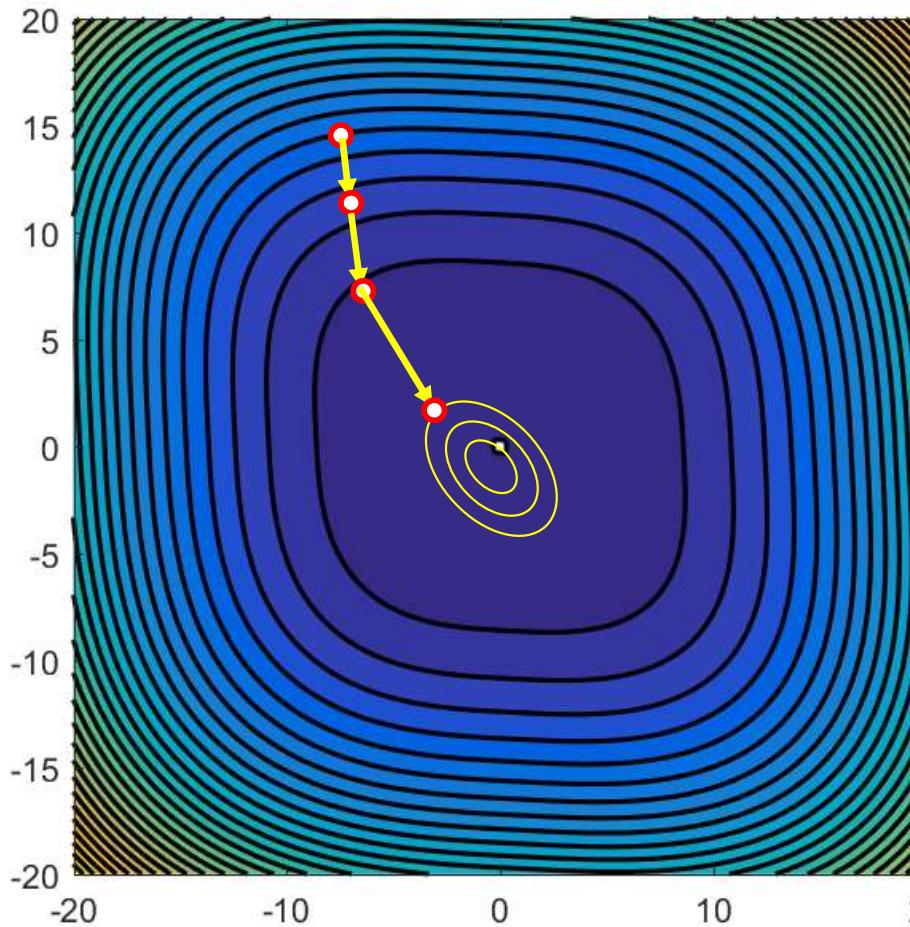


- Iterated localized optimization with quadratic approximations

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

$$-\eta = 1$$

# Minimization by Newton's method

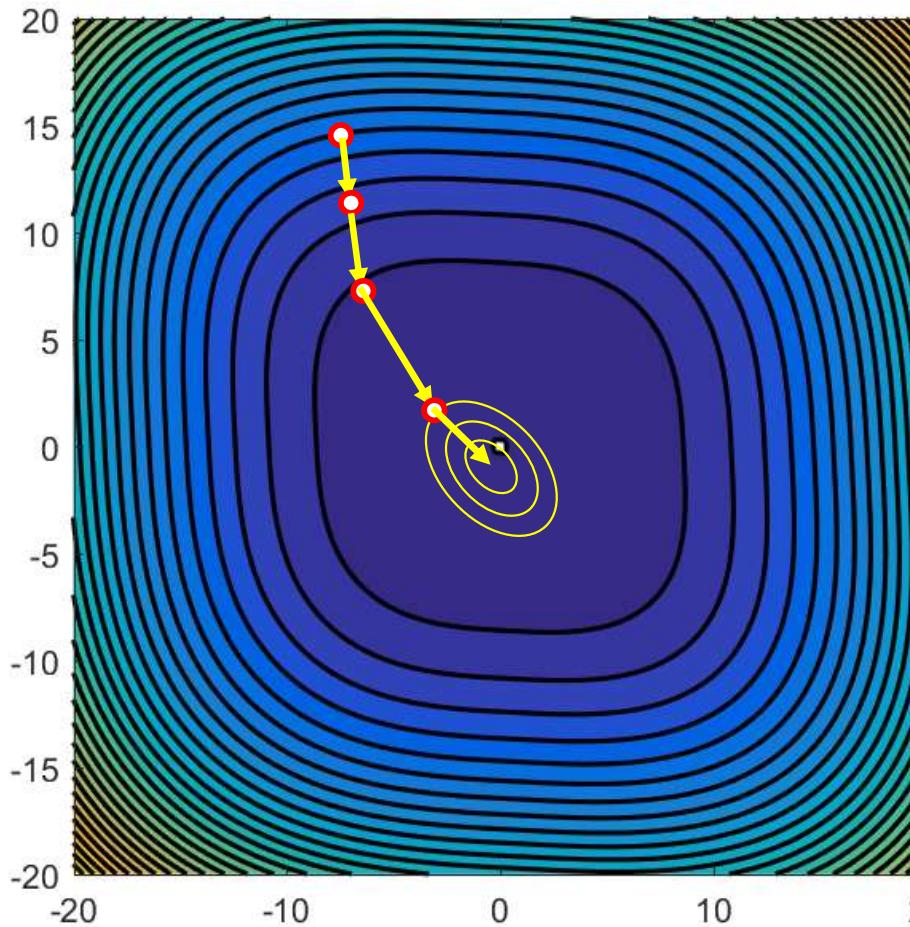


- Iterated localized optimization with quadratic approximations

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

$$-\eta = 1$$

# Minimization by Newton's method

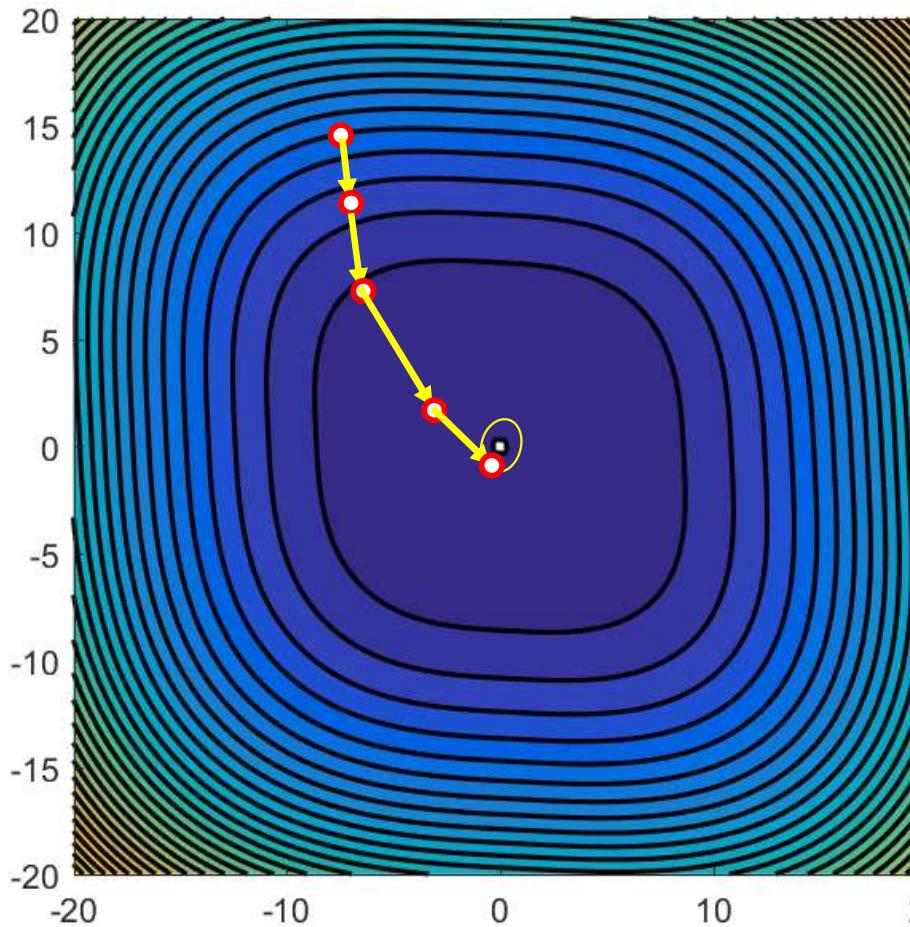


- Iterated localized optimization with quadratic approximations

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

$$-\eta = 1$$

# Minimization by Newton's method

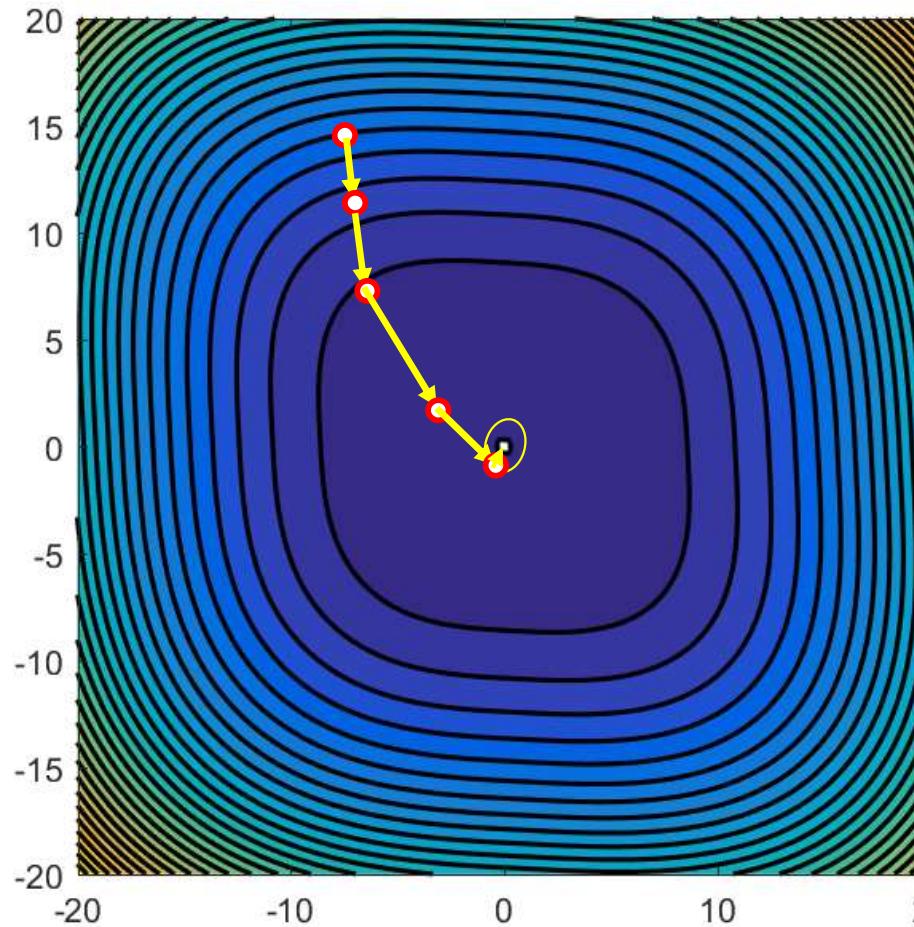


- Iterated localized optimization with quadratic approximations

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

$$-\eta = 1$$

# Minimization by Newton's method

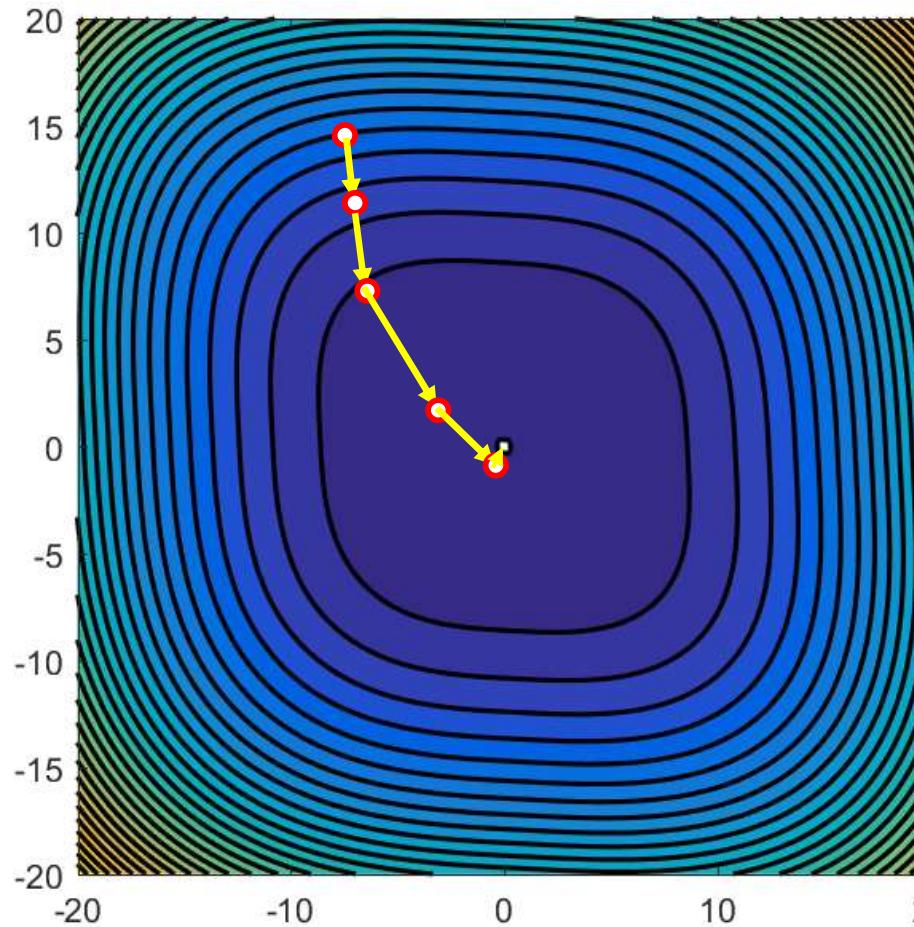


- Iterated localized optimization with quadratic approximations

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

$$-\eta = 1$$

# Minimization by Newton's method



- Iterated localized optimization with quadratic approximations

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

$$-\eta = 1$$

# Issues: 1. The Hessian

- Normalized update rule

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$

- For complex models such as neural networks, with a very large number of parameters, the Hessian  $H_E(\mathbf{w}^{(k)})$  is extremely difficult to compute
  - For a network with only 100,000 parameters, the Hessian will have  $10^{10}$  cross-derivative terms
  - And its even harder to invert, since it will be enormous

# Issues: 1. The Hessian



- For non-convex functions, the Hessian may not be positive semi-definite, in which case the algorithm can *diverge*
  - Goes away from, rather than towards the minimum

# Issues: 1. The Hessian

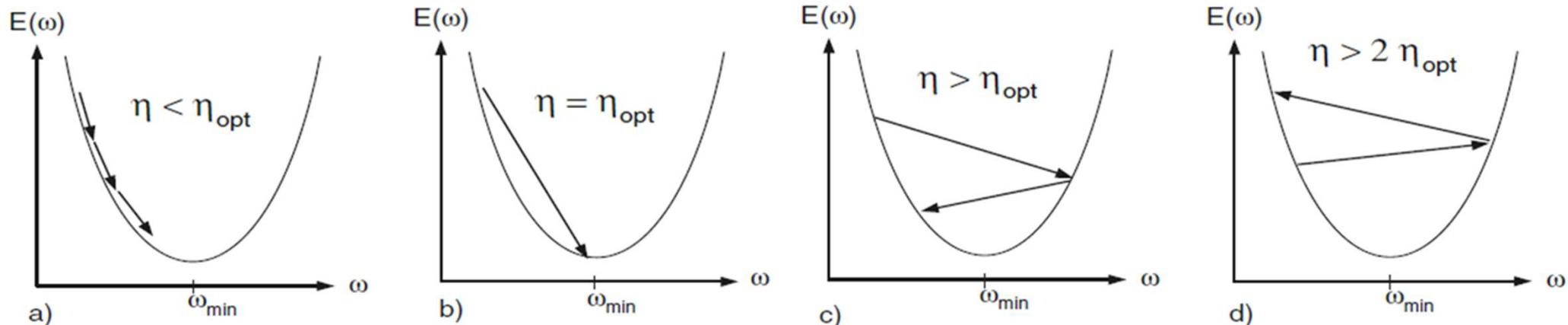


- For non-convex functions, the Hessian may not be positive semi-definite, in which case the algorithm can *diverge*
  - Goes away from, rather than towards the minimum
  - Now requires additional checks to avoid movement in directions corresponding to -ve Eigenvalues of the Hessian

# Issues: 1 – contd.

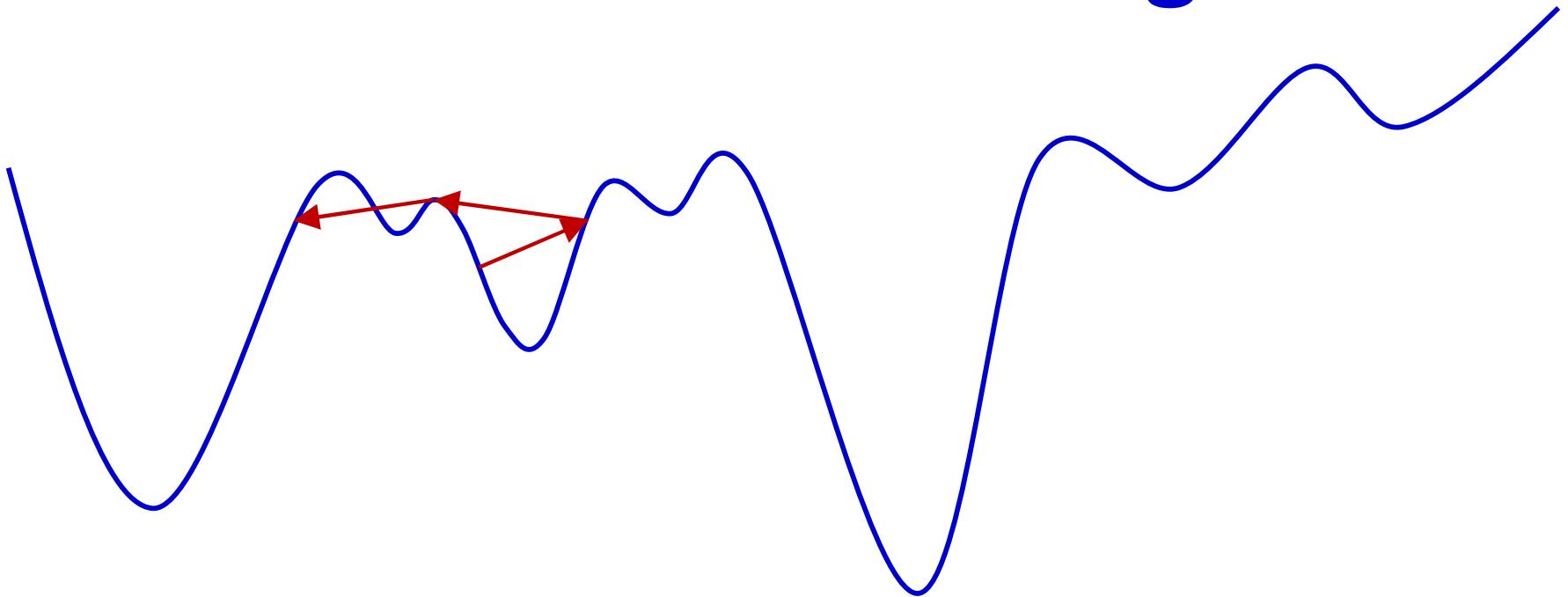
- A great many approaches have been proposed in the literature to *approximate* the Hessian in a number of ways and improve its positive definiteness
  - Boyden-Fletcher-Goldfarb-Shanno (BFGS)
    - And “low-memory” BFGS (L-BFGS)
    - Estimate Hessian from finite differences
  - Levenberg-Marquardt
    - Estimate Hessian from Jacobians
    - Diagonal load it to ensure positive definiteness
  - Other “Quasi-newton” methods
- Hessian estimates may even be *local* to a set of variables
- Not particularly popular anymore for large neural networks..

## Issues: 2. The learning rate



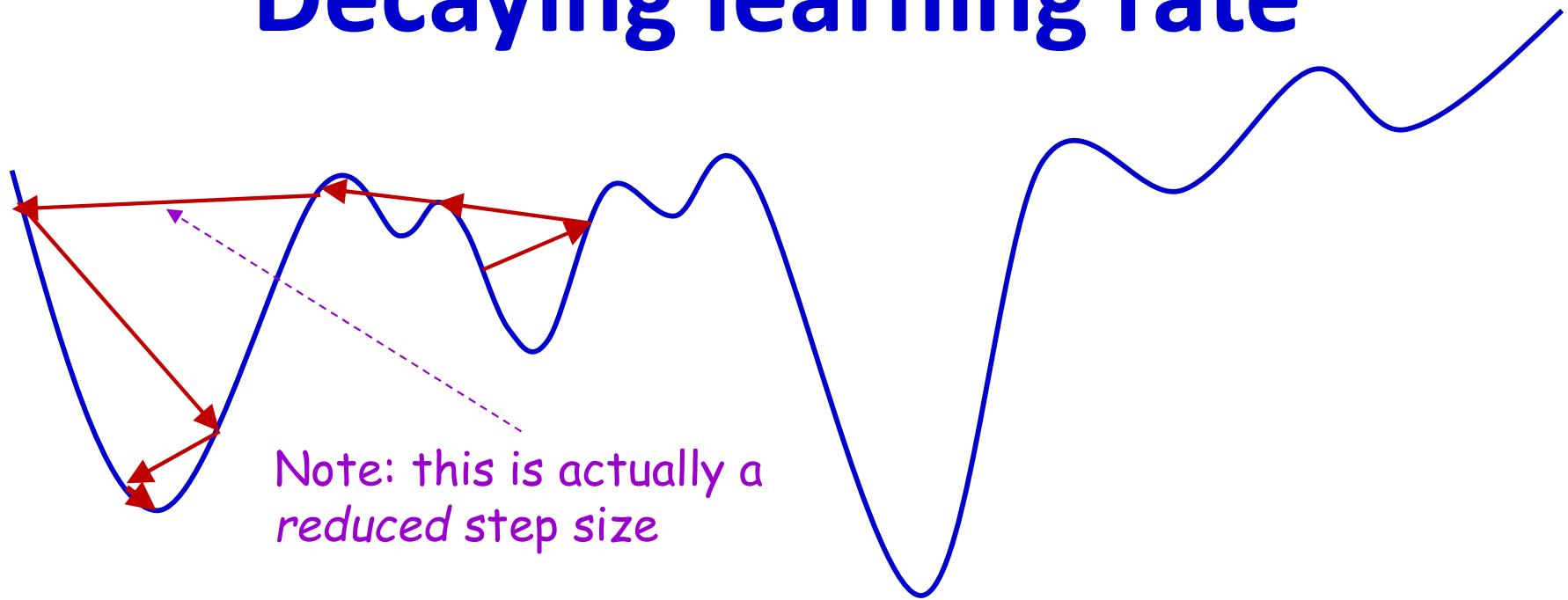
- Much of the analysis we just saw was based on trying to ensure that the step size was not so large as to cause divergence within a convex region
  - $\eta < 2\eta_{\text{opt}}$

## Issues: 2. The learning rate



- For complex models such as neural networks the loss function is often not convex
  - Having  $\eta > 2\eta_{opt}$  can actually help escape local optima
- However *always* having  $\eta > 2\eta_{opt}$  will ensure that you never ever actually find a solution

# Decaying learning rate



- Start with a large learning rate
  - Greater than 2 (assuming Hessian normalization)
  - Gradually reduce it with iterations

# Decaying learning rate

- Typical decay schedules
  - Linear decay:  $\eta_k = \frac{\eta_0}{k+1}$
  - Quadratic decay:  $\eta_k = \frac{\eta_0}{(k+1)^2}$
  - Exponential decay:  $\eta_k = \eta_0 e^{-\beta k}$ , where  $\beta > 0$
- A common approach (for nnets):
  1. Train with a fixed learning rate  $\eta$  until loss (or performance on a held-out data set) stagnates
  2.  $\eta \leftarrow \alpha\eta$ , where  $\alpha < 1$  (typically 0.1)
  3. Return to step 1 and continue training from where we left off

# Story so far : Convergence

- Gradient descent can miss obvious answers
  - And this may be a *good* thing
- Convergence issues abound
  - The loss surface has many saddle points
    - Although, perhaps, not so many bad local minima
    - Gradient descent can stagnate on saddle points
  - Vanilla gradient descent may not converge, or may converge toooooo slowly
    - The optimal learning rate for one component may be too high or too low for others

# Poll 2

Mark all true statements

- Step sizes that are greater than twice the inverse of the second derivative can cause gradient descent to diverge
- This is always a bad thing
- Gradient descent will not converge without decaying learning rates

# Poll 2

Mark all true statements

- Step sizes that are greater than twice the inverse of the second derivative can cause gradient descent to diverge (**true**)
- This is always a bad thing
- Gradient descent will not converge without decaying learning rates

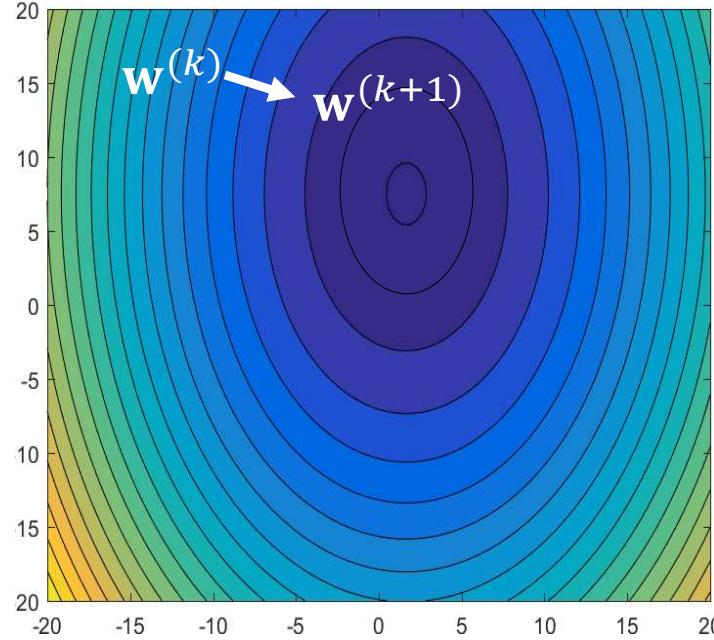
# Story so far : Second-order methods

- Second-order methods “normalize” the variation along the components to mitigate the problem of different optimal learning rates for different components
  - But this requires computation of inverses of second-order derivative matrices
  - Computationally infeasible
  - Not stable in non-convex regions of the loss surface
  - Approximate methods address these issues, but simpler solutions may be better

# Story so far : Learning rate

- Divergence-causing learning rates may not be a bad thing
  - Particularly for ugly loss functions
- *Decaying* learning rates provide good compromise between escaping poor local minima and convergence
- *Many of the convergence issues arise because we force the same learning rate on all parameters*

# Lets take a step back



$$\mathbf{w}^{(k+1)} \leftarrow \mathbf{w}^{(k)} - \eta (\nabla_{\mathbf{w}} E)^T$$

$$w_i^{(k+1)} = w_i^{(k)} - \eta \frac{dE(w_i^{(k)})}{dw}$$

- Problems arise because of requiring a fixed step size across all dimensions
  - Because steps are “tied” to the gradient
- Let’s try releasing this requirement

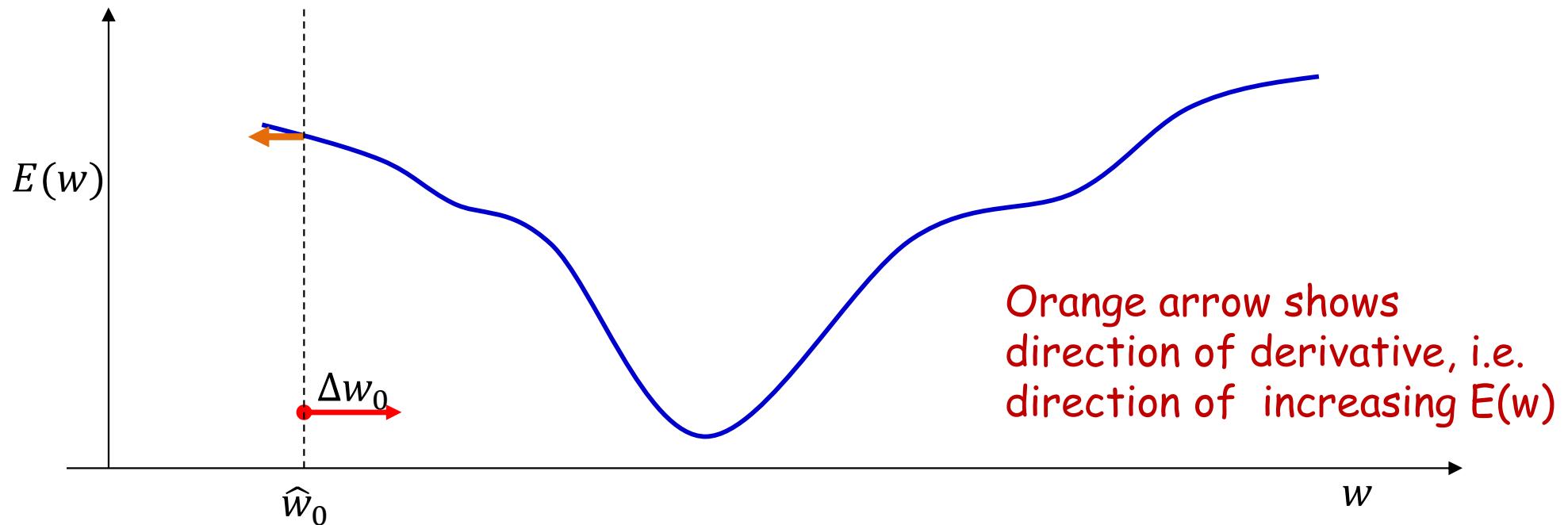
# **Derivative-*inspired* algorithms**

- Algorithms that use derivative information for trends, but do not follow them absolutely
- Rprop
- Quick prop

# RProp

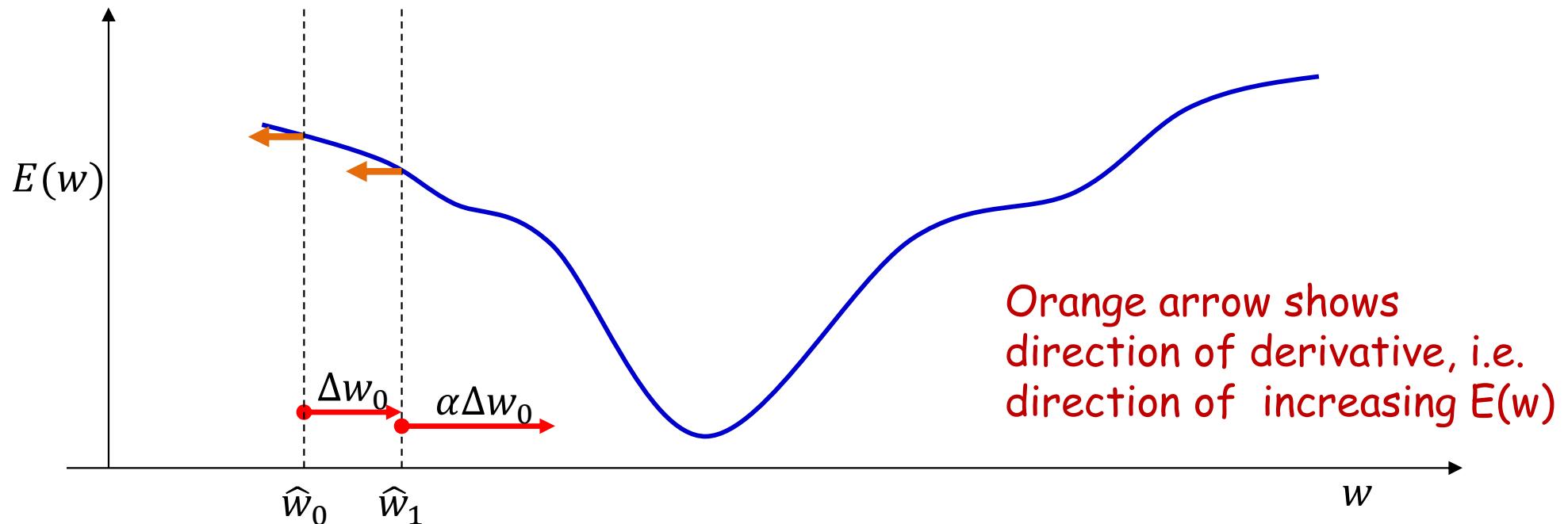
- *Resilient* propagation
- Simple algorithm, to be followed *independently* for each component
  - I.e. steps in different directions are not coupled
- At each time
  - If the derivative at the current location recommends continuing in the same direction as before (i.e. has not changed sign from earlier):
    - *increase* the step, and continue in the same direction
  - If the derivative has changed sign (i.e. we've overshot a minimum)
    - *reduce* the step and reverse direction

# Rprop



- Select an initial value  $\hat{w}$  and compute the derivative
  - Take an initial step  $\Delta w$  against the derivative
    - In the direction that reduces the function
      - $\Delta w = sign\left(\frac{dE(\hat{w})}{dw}\right) \Delta w$
      - $\hat{w} = \hat{w} - \Delta w$

# Rprop

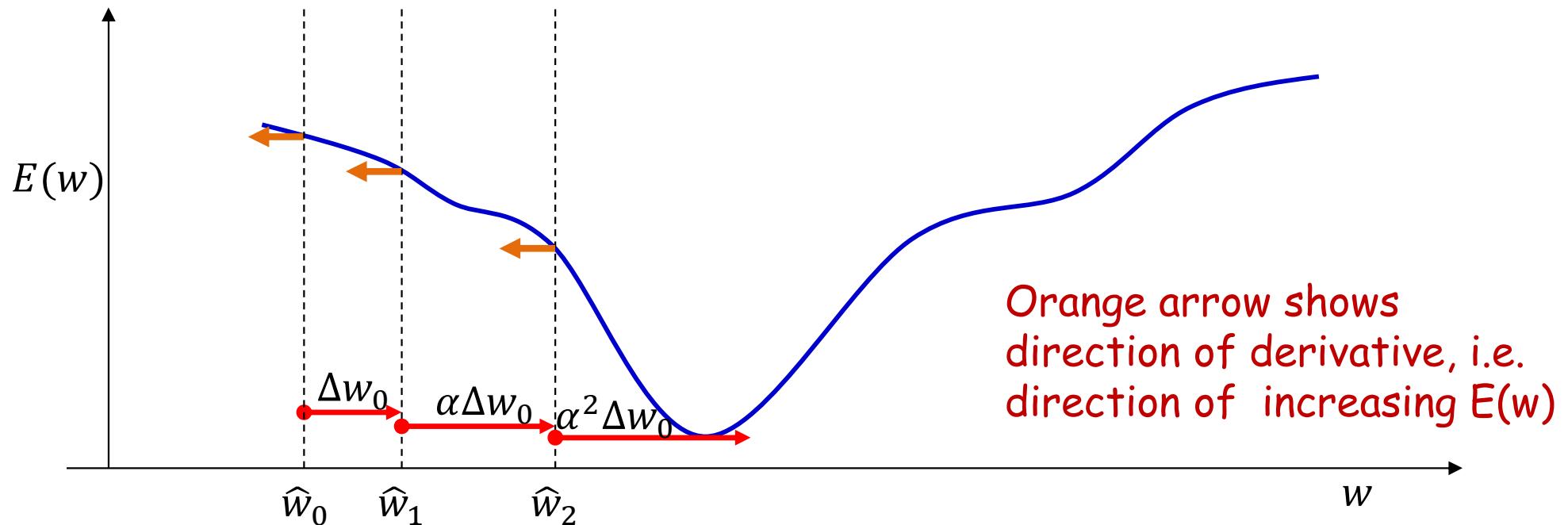


- Compute the derivative in the new location
  - If the derivative has not changed sign from the previous location, increase the step size and take a longer step

$$\alpha > 1$$

- $\Delta w = \alpha \Delta w$
- $\hat{w} = \hat{w} - \Delta w$

# Rprop

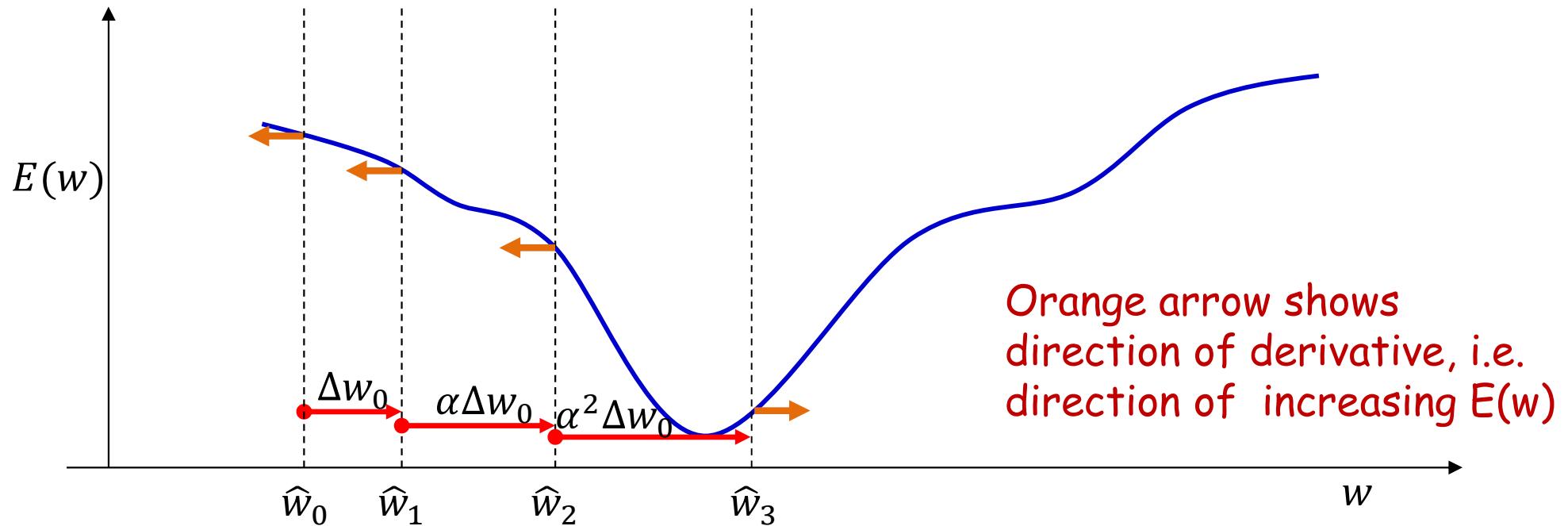


- Compute the derivative in the new location
  - If the derivative has not changed sign from the previous location, increase the step size and take a step

$\alpha > 1$

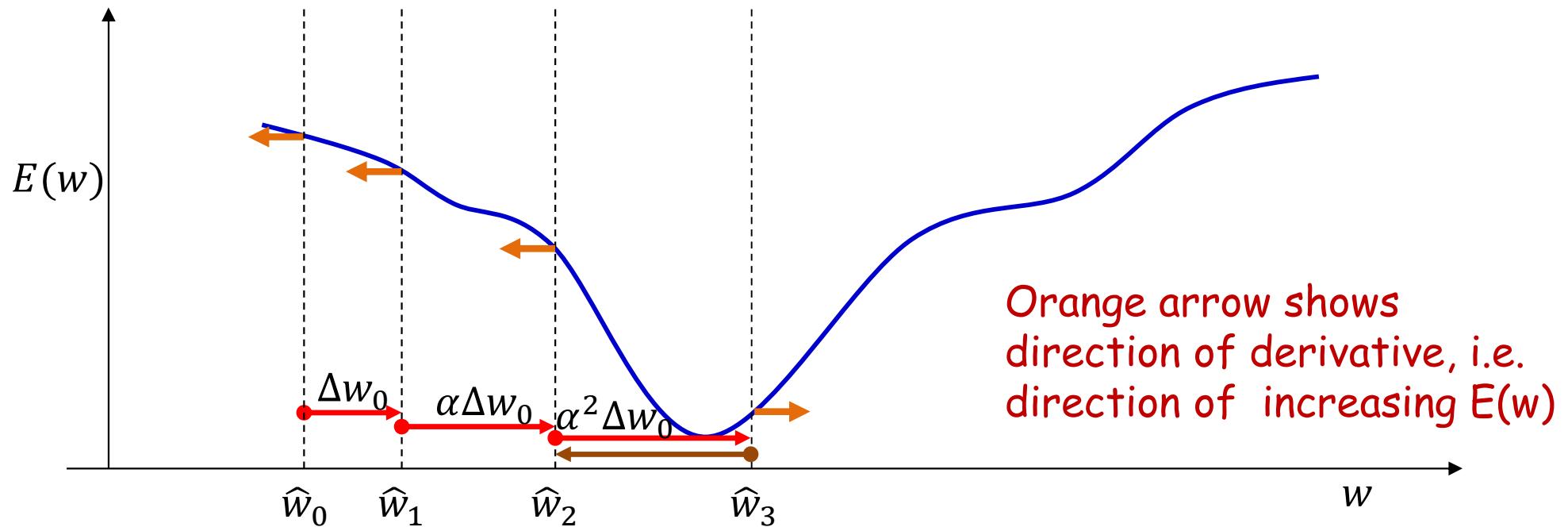
- $\Delta w = \alpha \Delta w$
- $\hat{w} = \hat{w} - \Delta w$

# Rprop



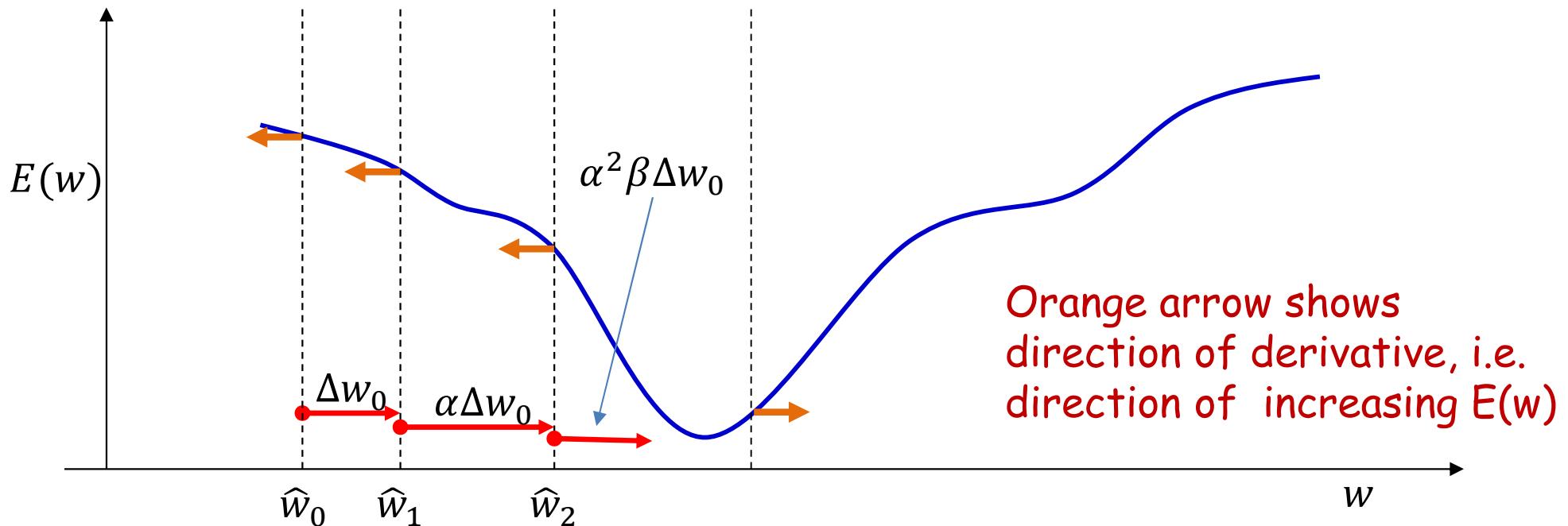
- Compute the derivative in the new location
  - If the derivative has changed sign

# Rprop



- Compute the derivative in the new location
  - If the derivative has changed sign
  - Return to the previous location
    - $\hat{w} = \hat{w} + \Delta w$

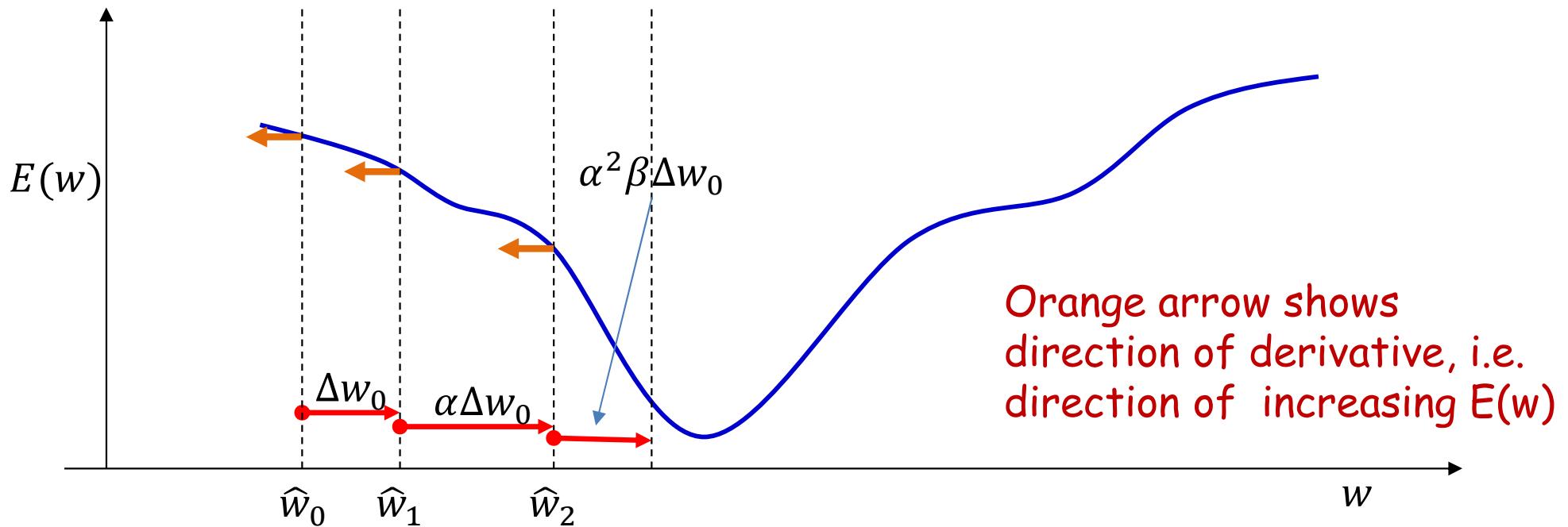
# Rprop



- Compute the derivative in the new location
  - If the derivative has changed sign
  - Return to the previous location
    - $\hat{w} = \hat{w} + \Delta w$
  - Shrink the step
    - $\Delta w = \beta \Delta w$

$\beta < 1$

# Rprop



- Compute the derivative in the new location
  - If the derivative has changed sign
  - Return to the previous location
    - $\hat{w} = \hat{w} + \Delta w$
- Shrink the step
  - $\Delta w = \beta \Delta w$
- Take the smaller step forward
  - $\hat{w} = \hat{w} - \Delta w$

$\beta < 1$

# Rprop (simplified)

- Set  $\alpha = 1.2$ ,  $\beta = 0.5$
- For each layer  $l$ , for each  $i, j$ :

- Initialize  $w_{l,i,j}$ ,  $\Delta w_{l,i,j} > 0$ ,

- $prevD(l, i, j) = \frac{dLoss(w_{l,i,j})}{dw_{l,i,j}}$

- $\Delta w_{l,i,j} = \text{sign}(prevD(l, i, j))\Delta w_{l,i,j}$

- While not converged:

- $w_{l,i,j} = w_{l,i,j} - \Delta w_{l,i,j}$

- $D(l, i, j) = \frac{dLoss(w_{l,i,j})}{dw_{l,i,j}}$

- If  $\text{sign}(prevD(l, i, j)) == \text{sign}(D(l, i, j))$ :

- $\Delta w_{l,i,j} = \min(\alpha\Delta w_{l,i,j}, \Delta_{max})$

- $prevD(l, i, j) = D(l, i, j)$

- else:

- $w_{l,i,j} = w_{l,i,j} + \Delta w_{l,i,j}$

- $\Delta w_{l,i,j} = \max(\beta\Delta w_{l,i,j}, \Delta_{min})$

Ceiling and floor on step

# Rprop (simplified)

- Set  $\alpha = 1.2$ ,  $\beta = 0.5$
- For each layer  $l$ , for each  $i, j$ :

- Initialize  $w_{l,i,j}, \Delta w_{l,i,j} > 0$ ,

- $prevD(l, i, j) = \frac{dLoss(w_{l,i,j})}{dw_{l,i,j}}$

- $\Delta w_{l,i,j} = \text{sign}(prevD(l, i, j))\Delta w_{l,i,j}$

- While not converged:

- $w_{l,i,j} = w_{l,i,j} - \Delta w_{l,i,j}$

- $D(l, i, j) = \frac{dLoss(w_{l,i,j})}{dw_{l,i,j}}$

- If  $\text{sign}(prevD(l, i, j)) == \text{sign}(D(l, i, j))$ :

- $\Delta w_{l,i,j} = \alpha \Delta w_{l,i,j}$

- $prevD(l, i, j) = D(l, i, j)$

- else:

- $w_{l,i,j} = w_{l,i,j} + \Delta w_{l,i,j}$

- $\Delta w_{l,i,j} = \beta \Delta w_{l,i,j}$

Obtained via backprop

Note: Different parameters updated independently

# RProp

- A remarkably simple first-order algorithm, that is frequently much more efficient than gradient descent.
  - And can even be competitive against some of the more advanced second-order methods
- Only makes minimal assumptions about the loss function
  - No convexity assumption

# Poll 3

The derivative of the loss w.r.t a parameter  $w$ , computed at the current estimate is positive. After taking a step (updating the parameter by a increment  $dw$ ) the sign of the derivative becomes negative. Mark all true statements

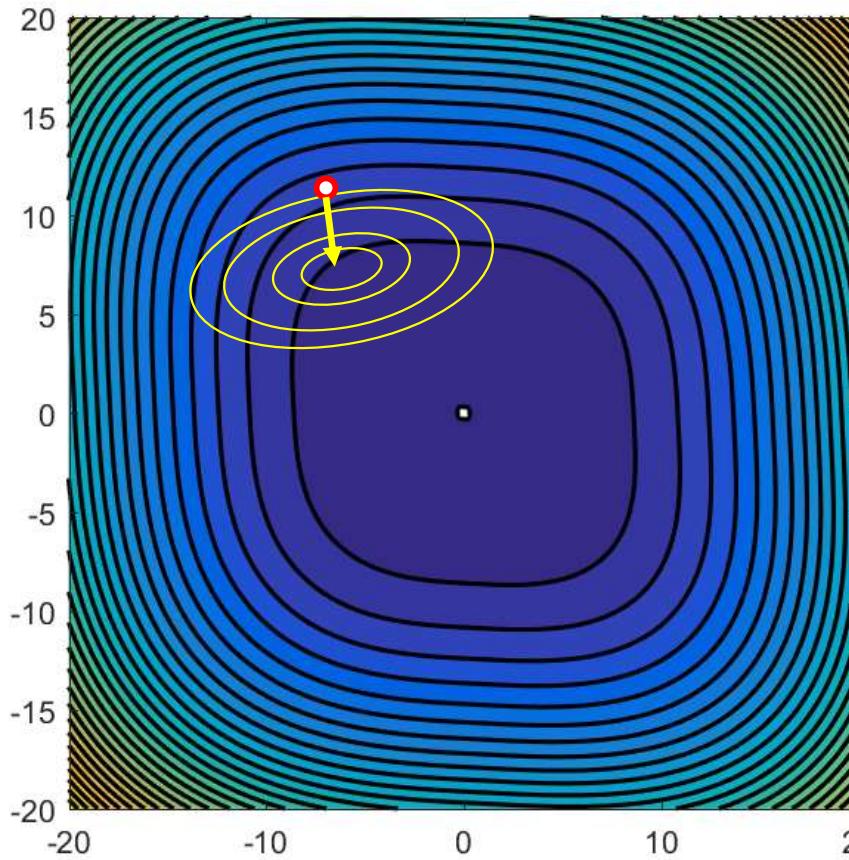
- Rprop will revert to the earlier estimate and take a smaller step
- Rprop will change direction and begin taking steps in the opposite direction

# Poll 3

The derivative of the loss w.r.t a parameter  $w$ , computed at the current estimate is positive. After taking a step (updating the parameter by a increment  $dw$ ) the sign of the derivative becomes negative. Mark all true statements

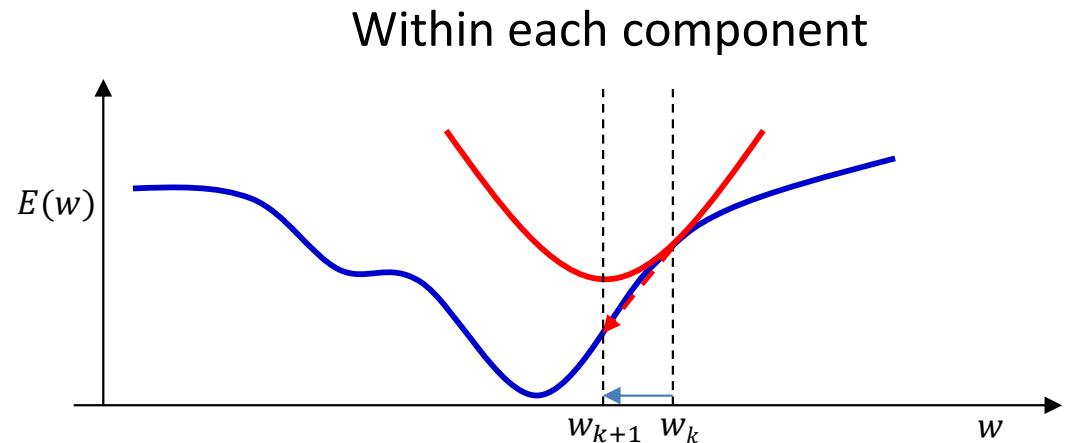
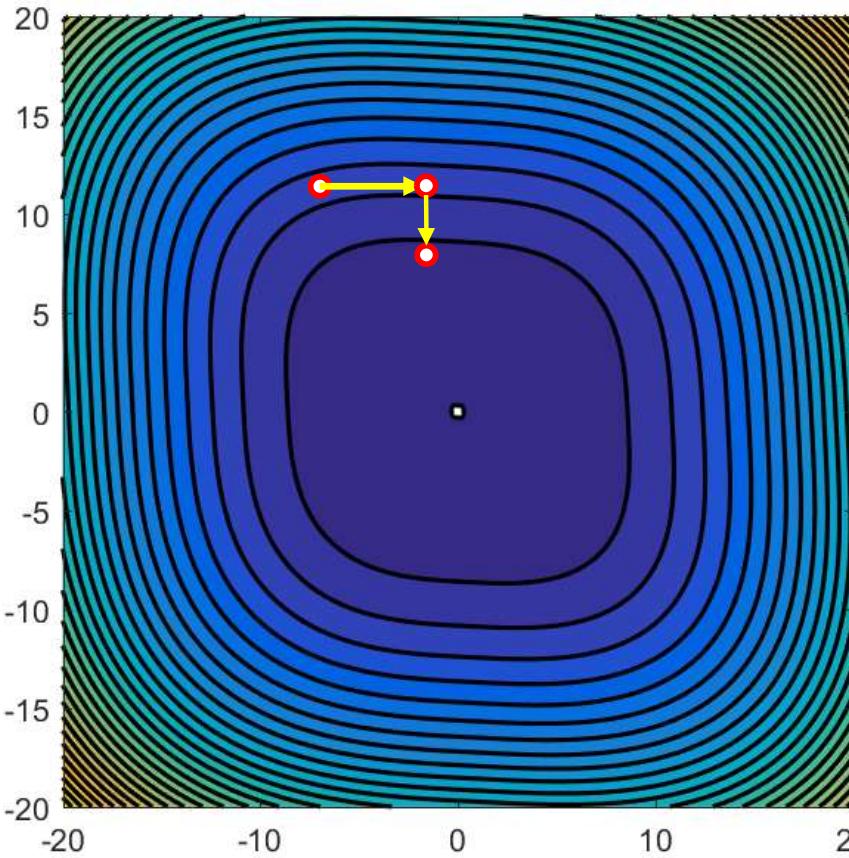
- Rprop will revert to the earlier estimate and take a smaller step (**true**)
- Rprop will change direction and begin taking steps in the opposite direction

# QuickProp



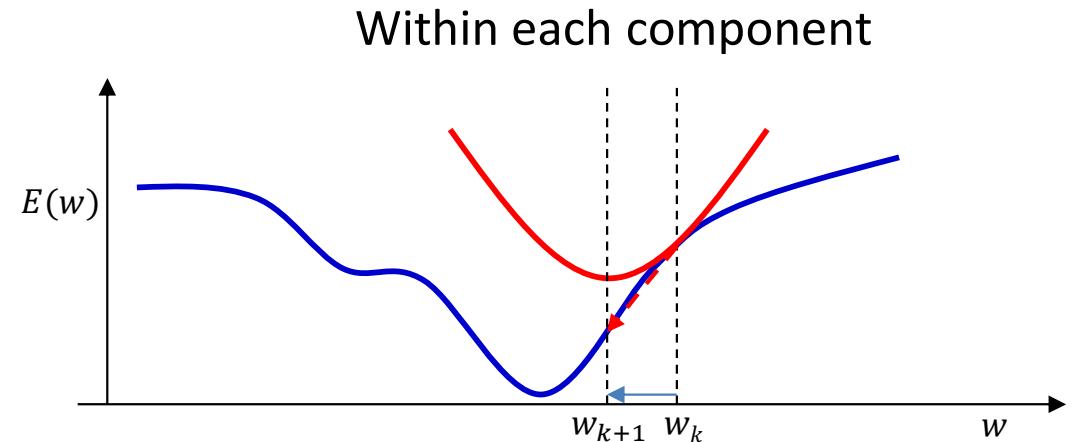
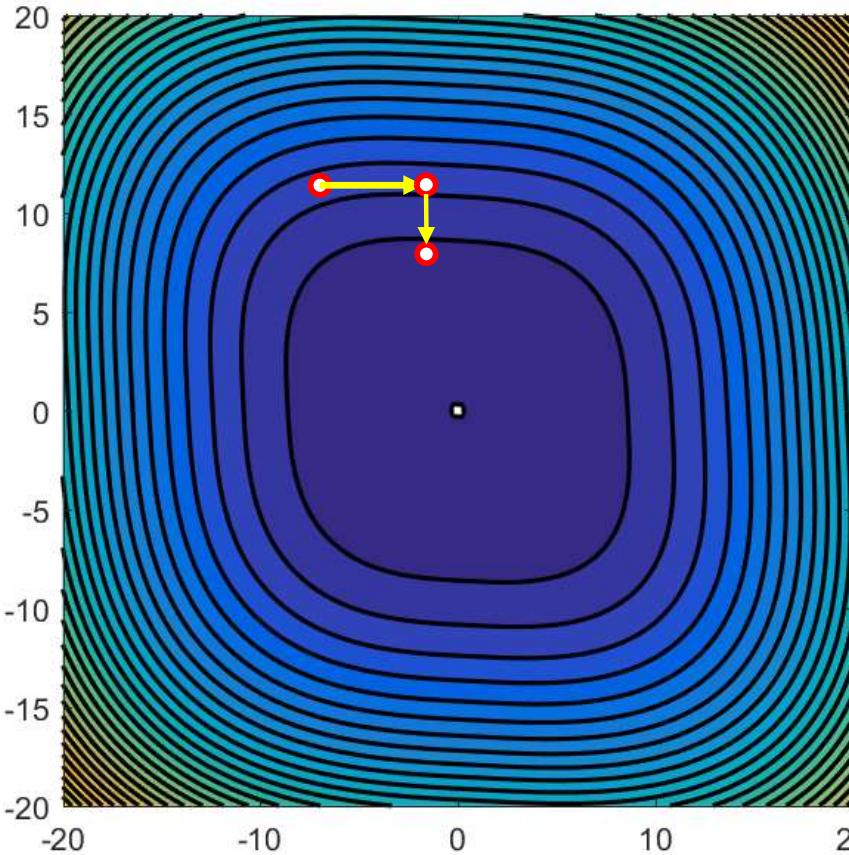
- Quickprop employs the Newton updates with two modifications
$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta H_E(\mathbf{w}^{(k)})^{-1} \nabla_{\mathbf{w}} E(\mathbf{w}^{(k)})^T$$
- But with two modifications

# QuickProp: Modification 1



- It treats each dimension independently
- For  $i = 1:N$ 
$$w_i^{k+1} = w_i^k - E''(w_i^k | w_j^k, j \neq i)^{-1} E'(w_i^k | w_j^k, j \neq i)$$
- This eliminates the need to compute and invert expensive Hessians

# QuickProp: Modification 2



- It approximates the second derivative through finite differences
  - For  $i = 1:N$
- $$w_i^{k+1} = w_i^k - D(w_i^k, w_i^{k-1})^{-1} E'(w_i^k | w_j^k, j \neq i)$$
- This eliminates the need to compute expensive double derivatives

# QuickProp

$$w^{(k+1)} = w^{(k)} - \left( \frac{E'(w^{(k)}) - E'(w^{(k-1)})}{\Delta w^{(k-1)}} \right)^{-1} E'(w^{(k)})$$

Finite-difference approximation to double derivative  
obtained assuming a quadratic  $E()$

- Updates are independent for every parameter
- For every layer  $l$ , for every connection from node  $i$  in the  $(l - 1)^{\text{th}}$  layer to node  $j$  in the  $l^{\text{th}}$  layer:

$$\Delta w_{l,ij}^{(k)} = \frac{\Delta w_{l,ij}^{(k-1)}}{Err'(w_{l,ij}^{(k)}) - Err'(w_{l,ij}^{(k-1)})} Err' \left( w_{l,ij}^{(k)} \right)$$

$$w_{l,ij}^{(k+1)} = w_{l,ij}^{(k)} - \Delta w_{l,ij}^{(k)}$$

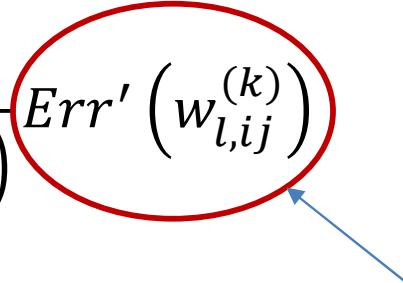
# QuickProp

$$w^{(k+1)} = w^{(k)} - \left( \frac{E'(w^{(k)}) - E'(w^{(k-1)})}{\Delta w^{(k-1)}} \right)^{-1} E'(w^{(k)})$$

Finite-difference approximation to double derivative  
obtained assuming a quadratic  $E()$

- Updates are independent for every parameter
- For every layer  $l$ , for every connection from node  $i$  in the  $(l-1)^{\text{th}}$  layer to node  $j$  in the  $l^{\text{th}}$  layer:

$$\Delta w_{l,ij}^{(k)} = \frac{\Delta w_{l,ij}^{(k-1)}}{Err'(w_{l,ij}^{(k)}) - Err'(w_{l,ij}^{(k-1)})}$$



$$w_{l,ij}^{(k+1)} = w_{l,ij}^{(k)} - \Delta w_{l,ij}^{(k)}$$

Computed using  
backprop

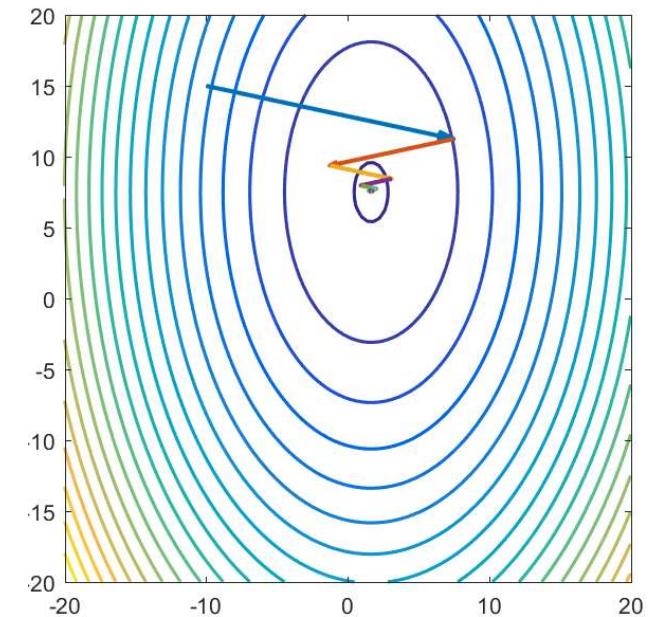
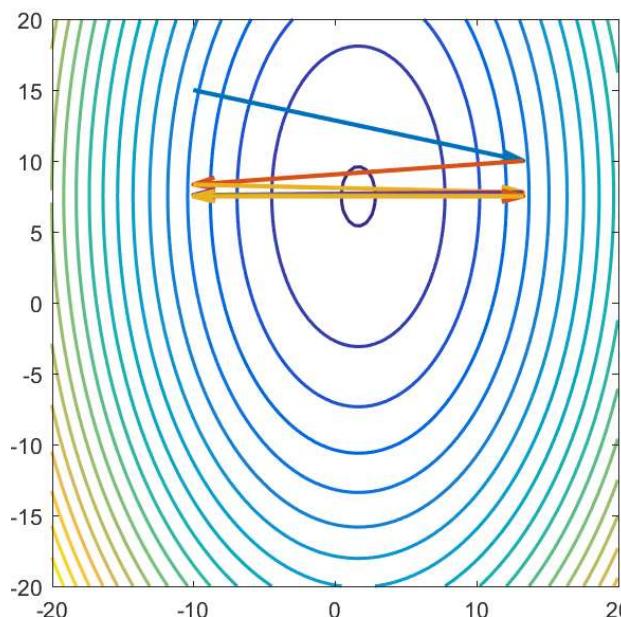
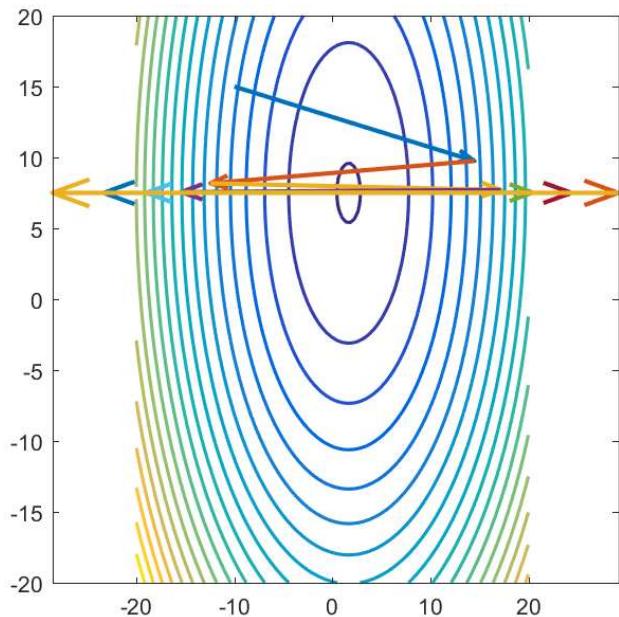
# Quickprop

- Employs Newton updates with empirically derived derivatives
- Prone to some instability for non-convex objective functions
- But is still one of the fastest training algorithms for many problems

# Story so far : Convergence

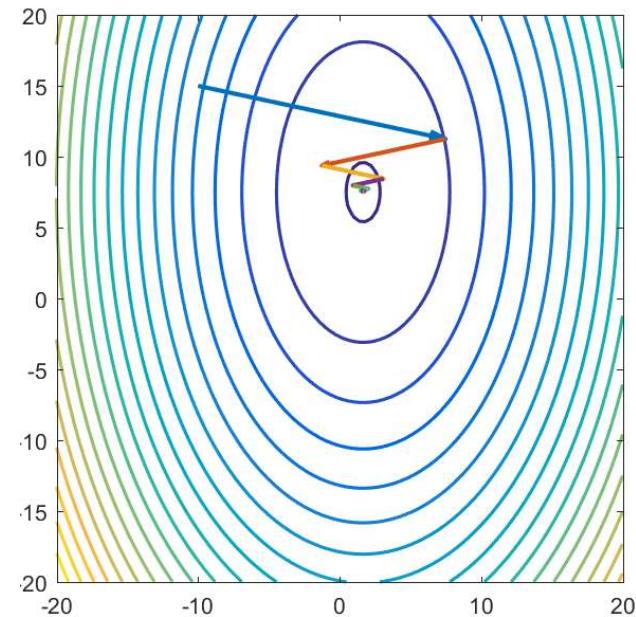
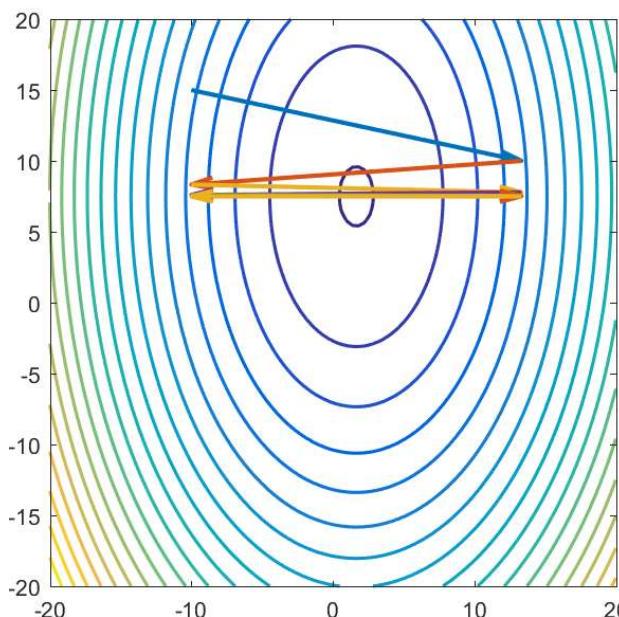
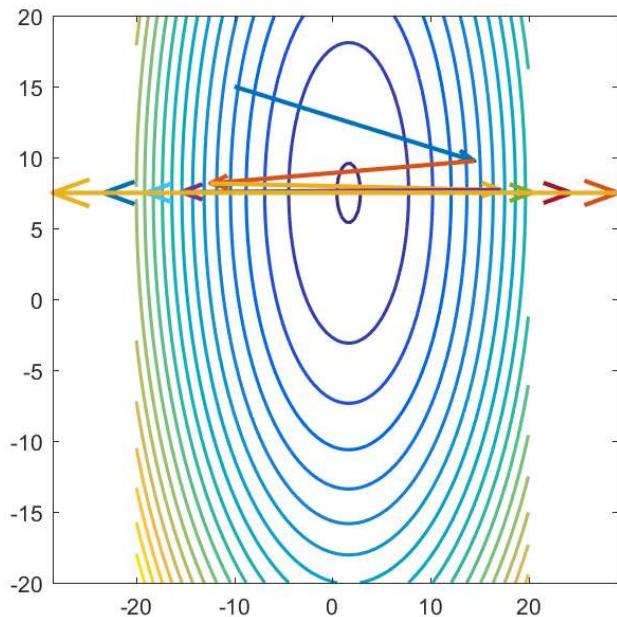
- Gradient descent can miss obvious answers
  - And this may be a *good* thing
- Vanilla gradient descent may be too slow or unstable due to the differences between the dimensions
- Second order methods can normalize the variation across dimensions, but are complex
- Adaptive or decaying learning rates can improve convergence
- Methods that decouple the dimensions can improve convergence

# A closer look at the convergence problem



- With dimension-independent learning rates, the solution will converge smoothly in some directions, but oscillate or diverge in others

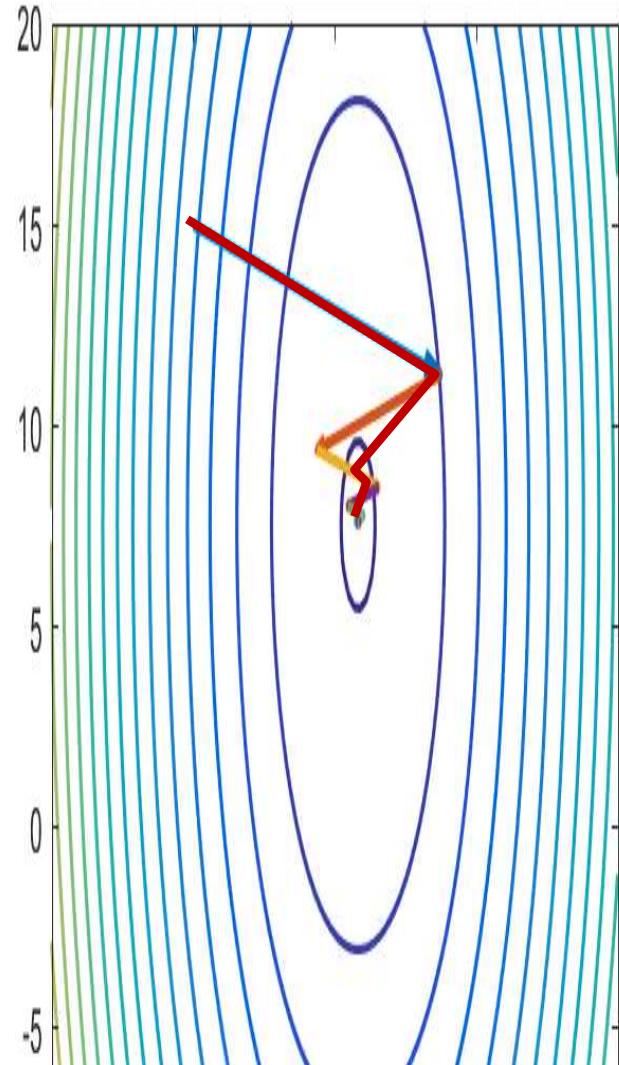
# A closer look at the convergence problem



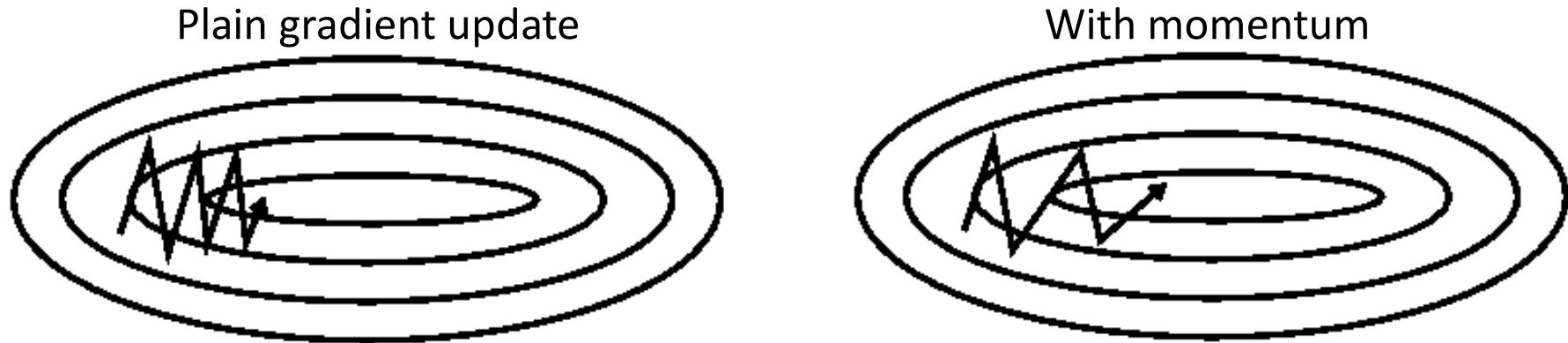
- With dimension-independent learning rates, the solution will converge smoothly in some directions, but oscillate or diverge in others
- Proposal:**
  - Keep track of oscillations
  - Emphasize steps in directions that converge smoothly
  - Shrink steps in directions that bounce around..

# The momentum methods

- Maintain a running average of all past steps
  - In directions in which the convergence is smooth, the average will have a large value
  - In directions in which the estimate swings, the positive and negative swings will cancel out in the average
- Update with the running average, rather than the current gradient



# Momentum Update



- The momentum method maintains a running average of all gradients until the *current* step

$$\Delta W^{(k)} = \beta \Delta W^{(k-1)} - \eta \nabla_W \text{Loss}(W^{(k-1)})^T$$

$$W^{(k)} = W^{(k-1)} + \Delta W^{(k)}$$

- Typical  $\beta$  value is 0.9
- The running average steps
  - Get longer in directions where gradient retains the same sign
  - Become shorter in directions where the sign keeps flipping

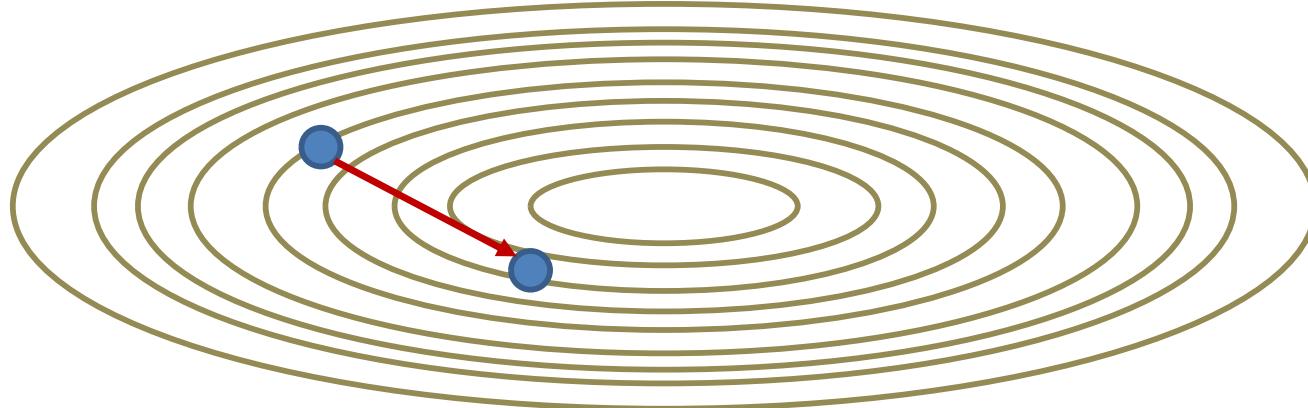
# Training by gradient descent

- Initialize all weights  $\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_K$
- Do:
  - For all  $i, j, k$ , initialize  $\nabla_{W_k} Loss = 0$
  - For all  $t = 1:T$ 
    - For every layer  $k$ :
      - Compute  $\nabla_{W_k} Div(Y_t, d_t)$
      - Compute  $\nabla_{W_k} Loss += \frac{1}{T} \nabla_{W_k} Div(Y_t, d_t)$
    - For every layer  $k$ :
$$W_k = W_k - \eta (\nabla_{W_k} Loss)^T$$
  - Until  $Loss$  has converged

# Training with momentum

- Initialize all weights  $\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_K$
- Do:
  - For all layers  $k$ , initialize  $\nabla_{W_k} Loss = 0, \Delta W_k = 0$
  - For all  $t = 1:T$ 
    - For every layer  $k$ :
      - Compute gradient  $\nabla_{W_k} \text{Div}(Y_t, d_t)$
      - $\nabla_{W_k} Loss += \frac{1}{T} \nabla_{W_k} \text{Div}(Y_t, d_t)$
    - For every layer  $k$ 
$$\Delta W_k = \beta \Delta W_k - \eta (\nabla_{W_k} Loss)^T$$
$$W_k = W_k + \Delta W_k$$
  - Until  $Loss$  has converged

# Momentum Update

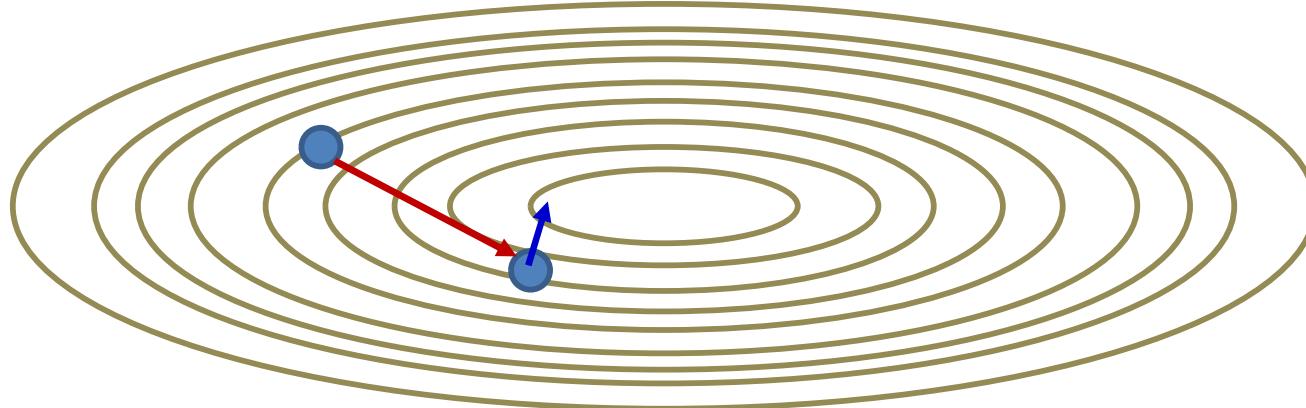


- The momentum method

$$\Delta W^{(k)} = \beta \Delta W^{(k-1)} - \eta \nabla_W \text{Loss}(W^{(k-1)})^T$$

- At any iteration, to compute the current step:

# Momentum Update

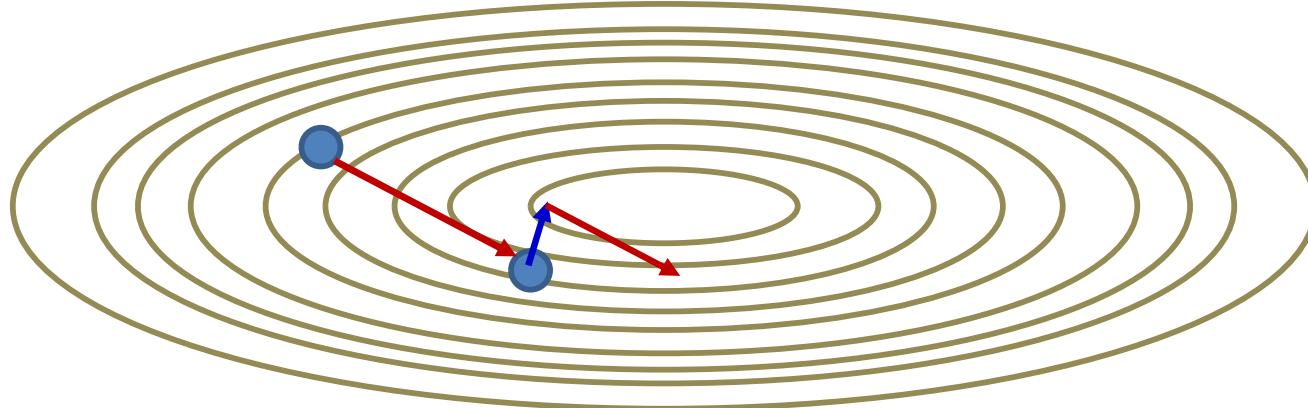


- The momentum method

$$\Delta W^{(k)} = \beta \Delta W^{(k-1)} - \eta \nabla_W \text{Loss}(W^{(k-1)})^T$$

- At any iteration, to compute the current step:
  - First computes the gradient step at the current location

# Momentum Update

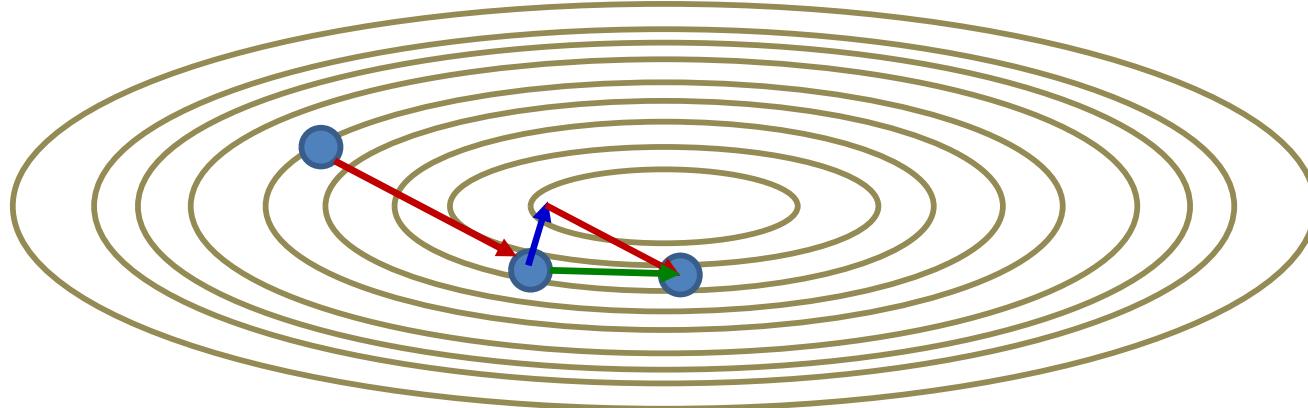


- The momentum method

$$\Delta W^{(k)} = \beta \Delta W^{(k-1)} - \eta \nabla_W Loss(W^{(k-1)})^T$$

- At any iteration, to compute the current step:
  - First computes the gradient step at the current location
  - Then adds in the scaled *previous* step
    - Which is actually a running average

# Momentum Update

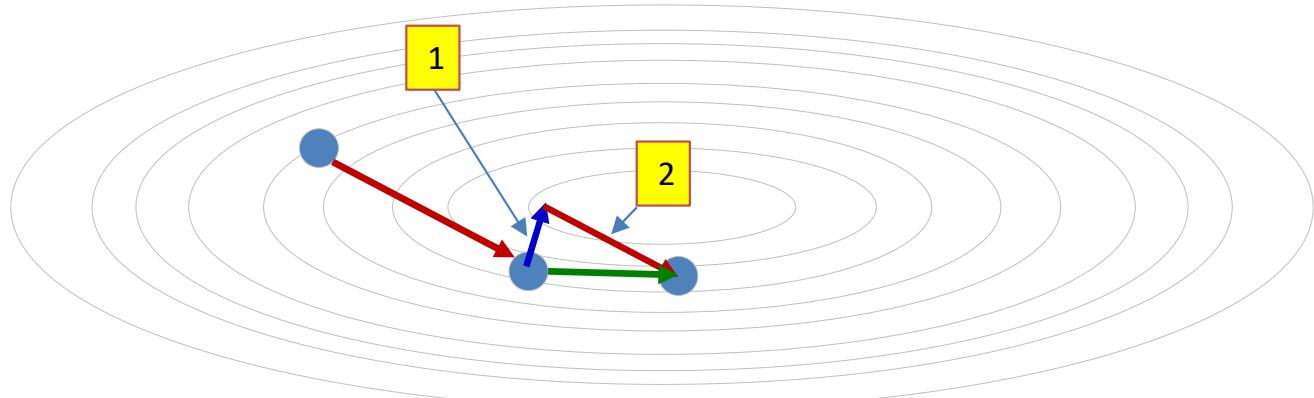


- The momentum method

$$\Delta W^{(k)} = \beta \Delta W^{(k-1)} - \eta \nabla_W Loss(W^{(k-1)})^T$$

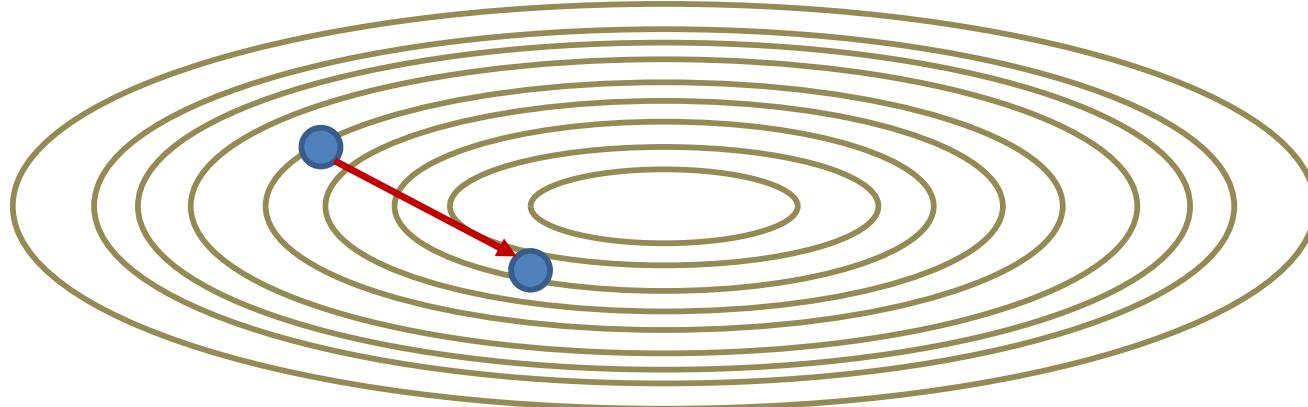
- At any iteration, to compute the current step:
  - First computes the gradient step at the current location
  - Then adds in the scaled *previous* step
    - Which is actually a running average
  - To get the final step

# Momentum update



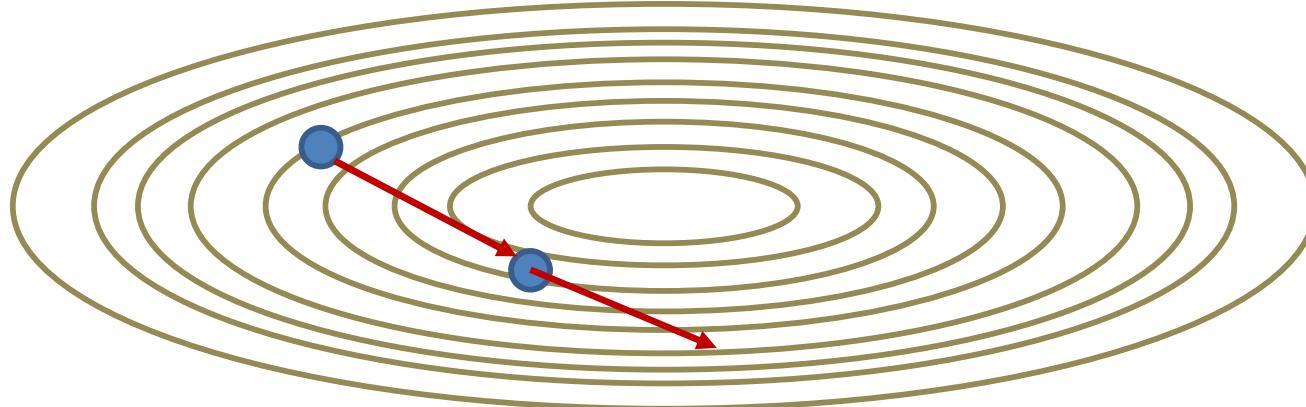
- Momentum update steps are actually computed in two stages
  - First: We take a step against the gradient at the current location
  - Second: Then we add a scaled version of the previous step
- The procedure can be made more optimal by reversing the order of operations..

# Nestorov's Accelerated Gradient



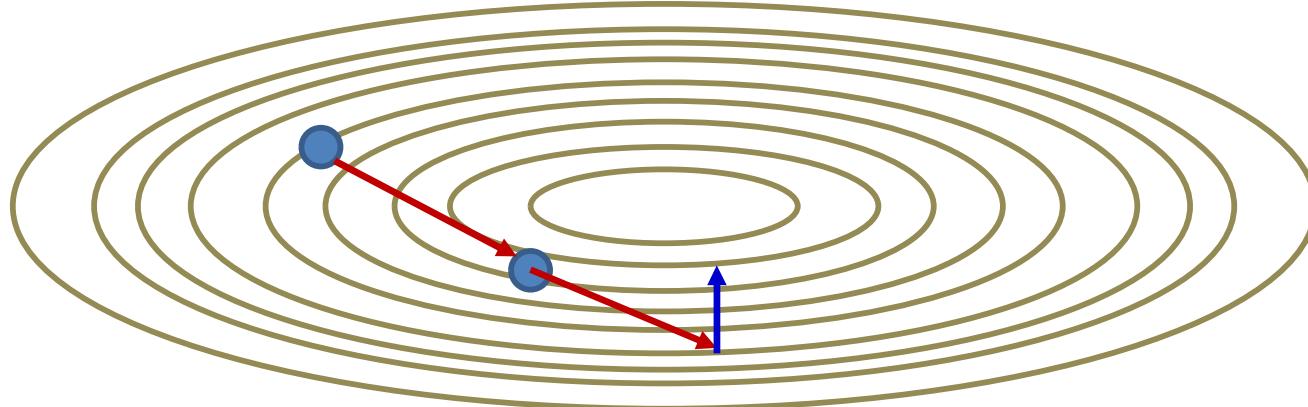
- Change the order of operations
- At any iteration, to compute the current step:

# Nestorov's Accelerated Gradient



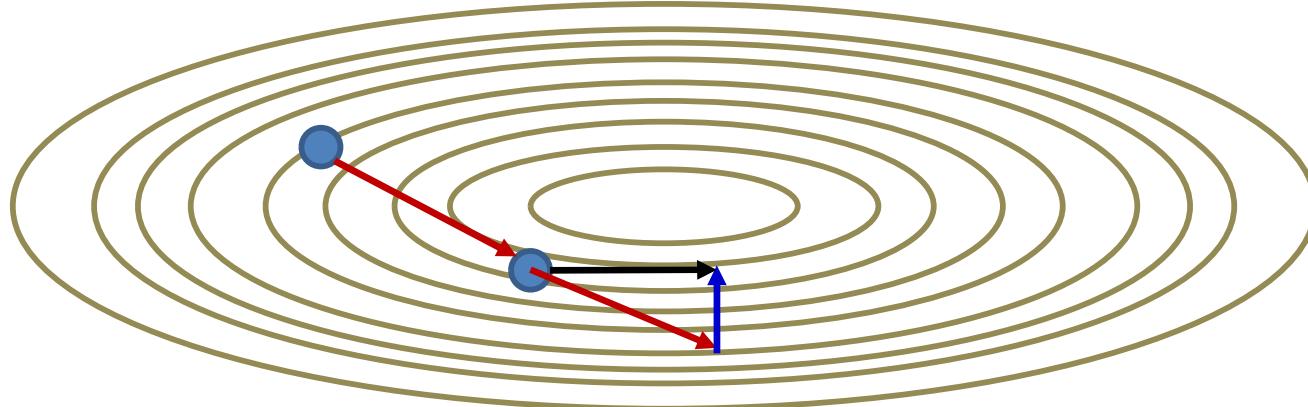
- Change the order of operations
- At any iteration, to compute the current step:
  - First extend the previous step

# Nestorov's Accelerated Gradient



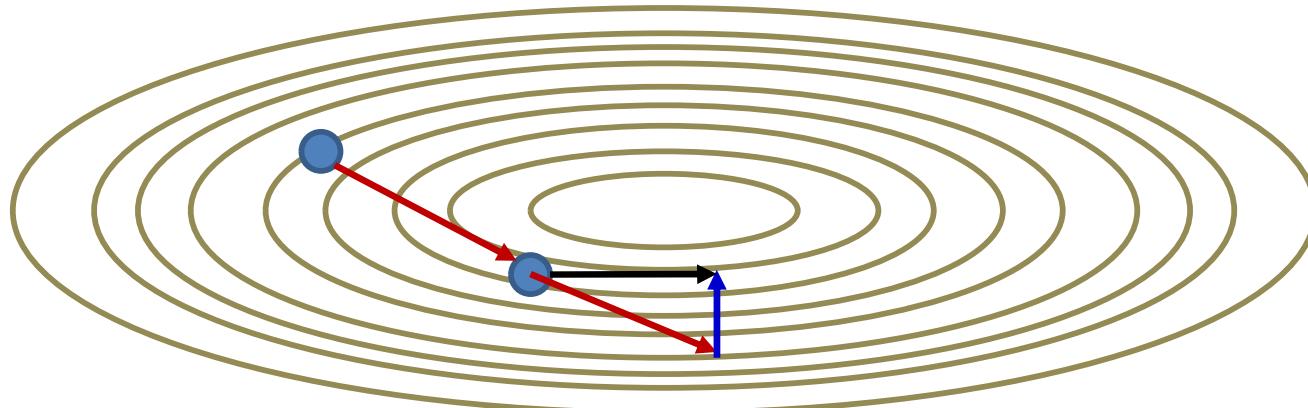
- Change the order of operations
- At any iteration, to compute the current step:
  - First extend the previous step
  - Then compute the gradient step at the resultant position

# Nestorov's Accelerated Gradient



- Change the order of operations
- At any iteration, to compute the current step:
  - First extend the previous step
  - Then compute the gradient step at the resultant position
  - Add the two to obtain the final step

# Nestorov's Accelerated Gradient

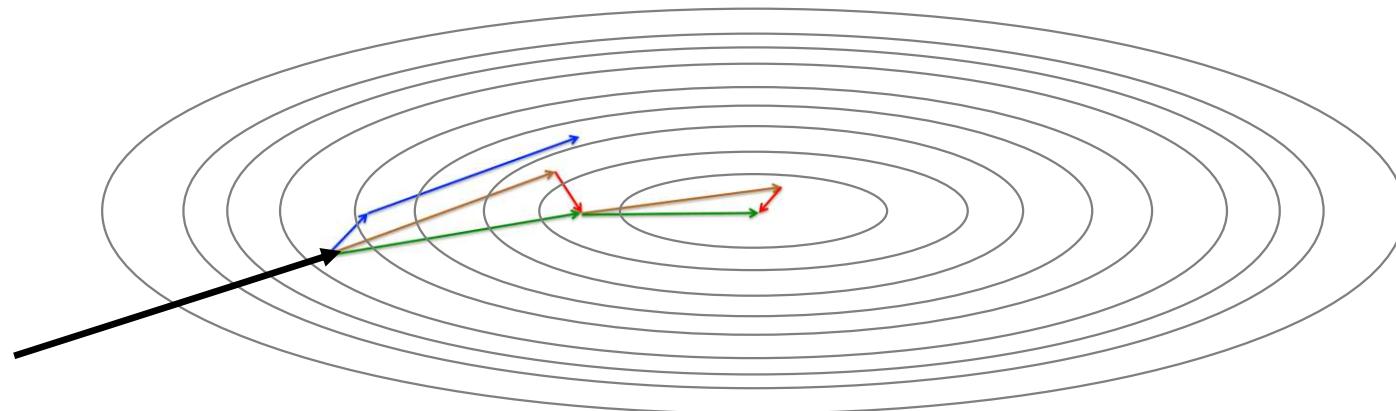


- Nestorov's method

$$\Delta W^{(k)} = \beta \Delta W^{(k-1)} - \eta \nabla_W Loss(W^{(k-1)} + \beta \Delta W^{(k-1)})^T$$

$$W^{(k)} = W^{(k-1)} + \Delta W^{(k)}$$

# Nestorov's Accelerated Gradient



- Comparison with momentum (example from Hinton)
- Converges much faster

# Training with Nestorov

- Initialize all weights  $\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_K$
- Do:
  - For all layers  $k$ , initialize  $\nabla_{\mathbf{W}_k} Loss = 0, \Delta W_k = 0$
  - For every layer  $k$ 
$$W_k = W_k + \beta \Delta W_k$$
  - For all  $t = 1:T$ 
    - For every layer  $k$ :
      - Compute gradient  $\nabla_{\mathbf{W}_k} \text{Div}(Y_t, d_t)$
      - $\nabla_{\mathbf{W}_k} Loss += \frac{1}{T} \nabla_{\mathbf{W}_k} \text{Div}(Y_t, d_t)$
    - For every layer  $k$ 
$$W_k = W_k - \eta (\nabla_{\mathbf{W}_k} Loss)^T$$
$$\Delta W_k = \beta \Delta W_k - \eta (\nabla_{\mathbf{W}_k} Loss)^T$$
  - Until  $Loss$  has converged

# Momentum and trend-based methods..

- We will return to this topic again, very soon..

# Poll 4

On a flat surface of constant slope momentum methods will converge faster than vanilla gradient descent, true or false

- True
- False

# Poll 4

On a flat surface of constant slope momentum methods will converge faster than vanilla gradient descent, true or false

- True
- False (**correct**) – momentum only changes step size

# Story so far

- Gradient descent can miss obvious answers
  - And this may be a *good* thing
- Vanilla gradient descent may be too slow or unstable due to the differences between the dimensions
- Second order methods can normalize the variation across dimensions, but are complex
- Adaptive or decaying learning rates can improve convergence
- Methods that decouple the dimensions can improve convergence
- Momentum methods which emphasize directions of steady improvement are demonstrably superior to other methods

# Coming up

- Incremental updates
- Revisiting “trend” algorithms
- Generalization
- Tricks of the trade
  - Divergences..
  - Activations
  - Normalizations