

Deep Neural Networks

Convolutional Networks III

Bhiksha Raj
Spring 2026

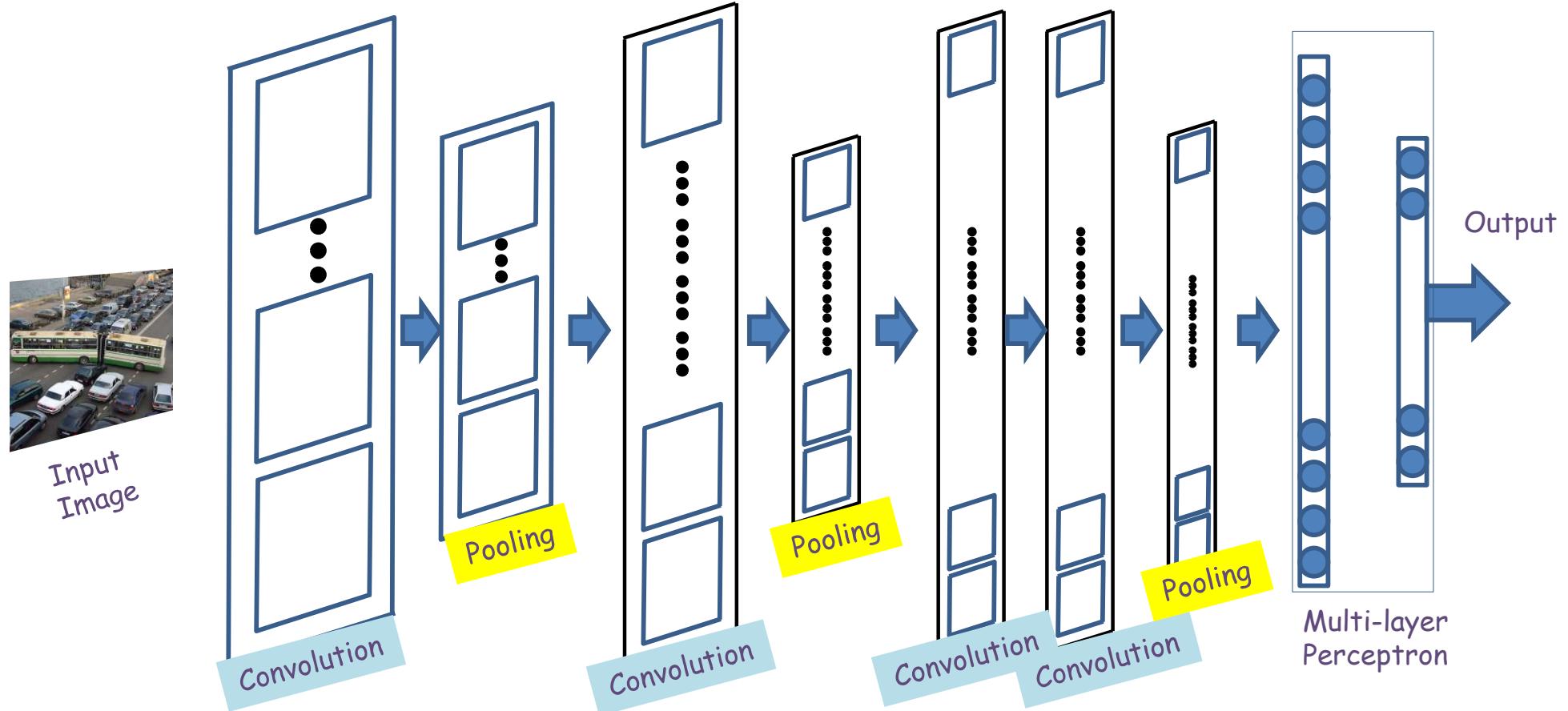
Outline

- Quick recap
- Back propagation through a CNN
- Modifications: Transposition, scaling, rotation and deformation invariance
- Segmentation and localization
- Some success stories
- Some advanced architectures
 - Resnet
 - Densenet

Story so far

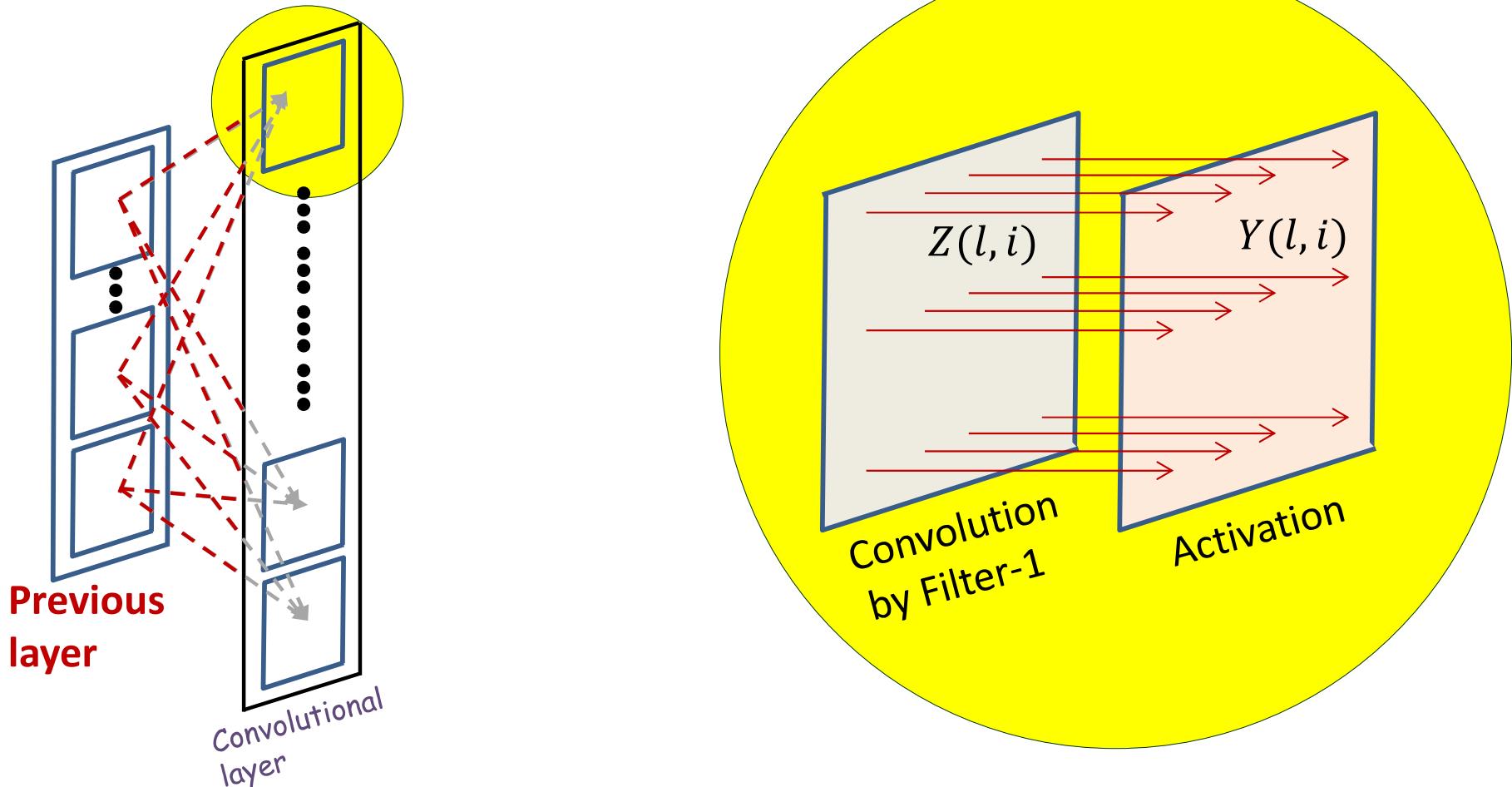
- Pattern classification tasks such as “does this picture contain a cat”, or “does this recording include HELLO” are best performed by scanning for the target pattern
- Scanning an input with a network and combining the outcomes is equivalent to scanning with individual neurons hierarchically
 - First level neurons scan the input
 - Higher-level neurons scan the “maps” formed by lower-level neurons
 - A final “decision” unit or layer makes the final decision
 - Deformations in the input can be handled by “pooling”
- For 2-D (or higher-dimensional) scans, the structure is called a Convolutional Neural Network
- For 1-D scan along time, it is called a Time-delay neural network

Recap: The general architecture of a convolutional neural network



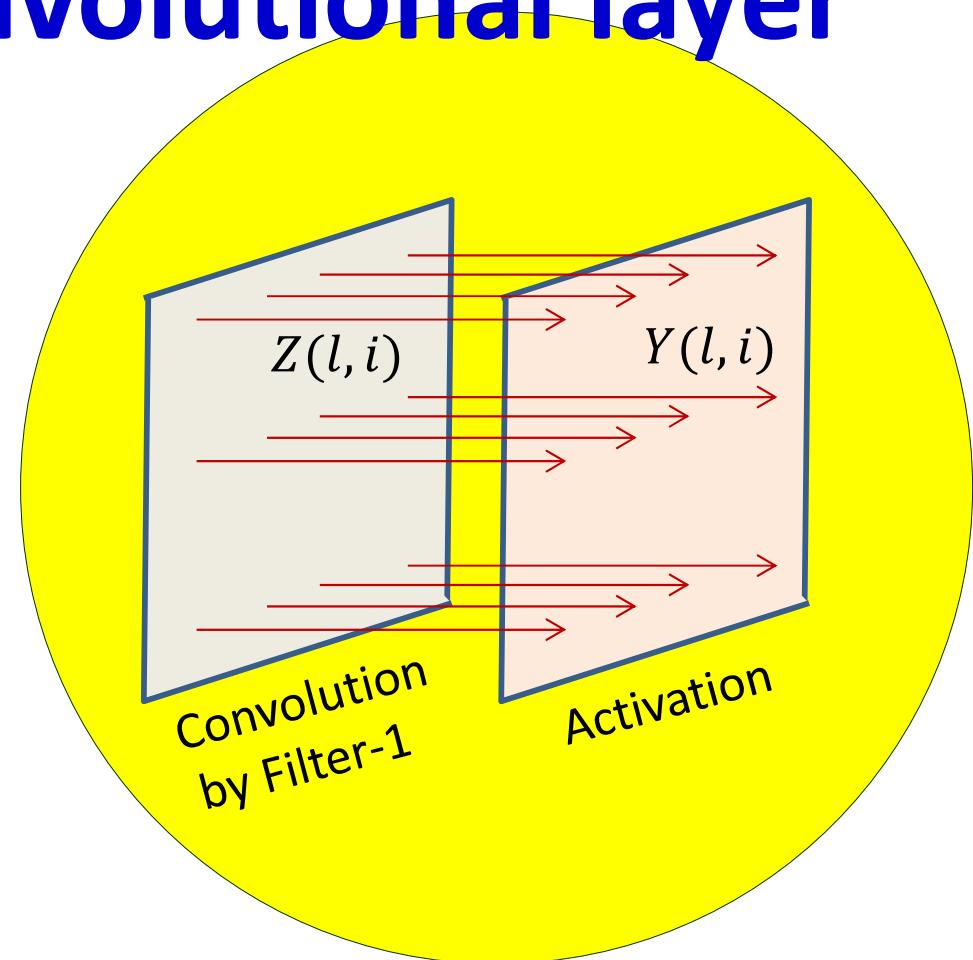
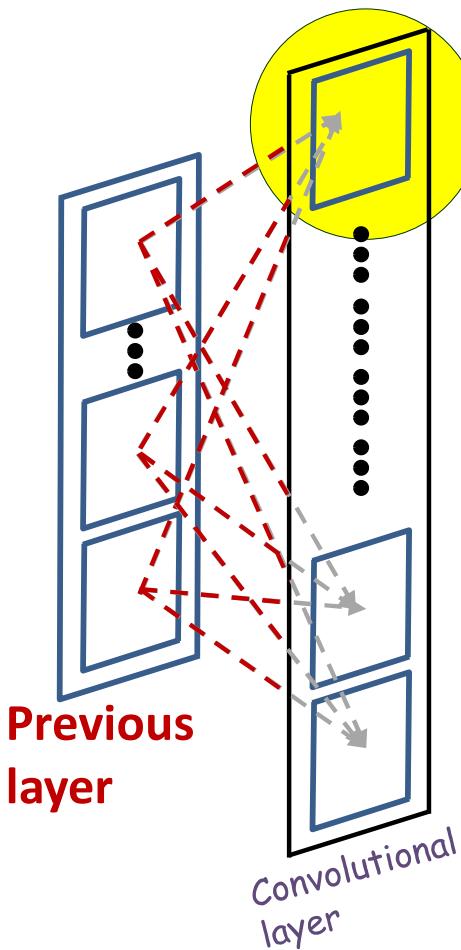
- A convolutional neural network comprises of “convolutional” and optional “pooling” layers
- Followed by an MLP with one or more layers

Recap: A convolutional layer



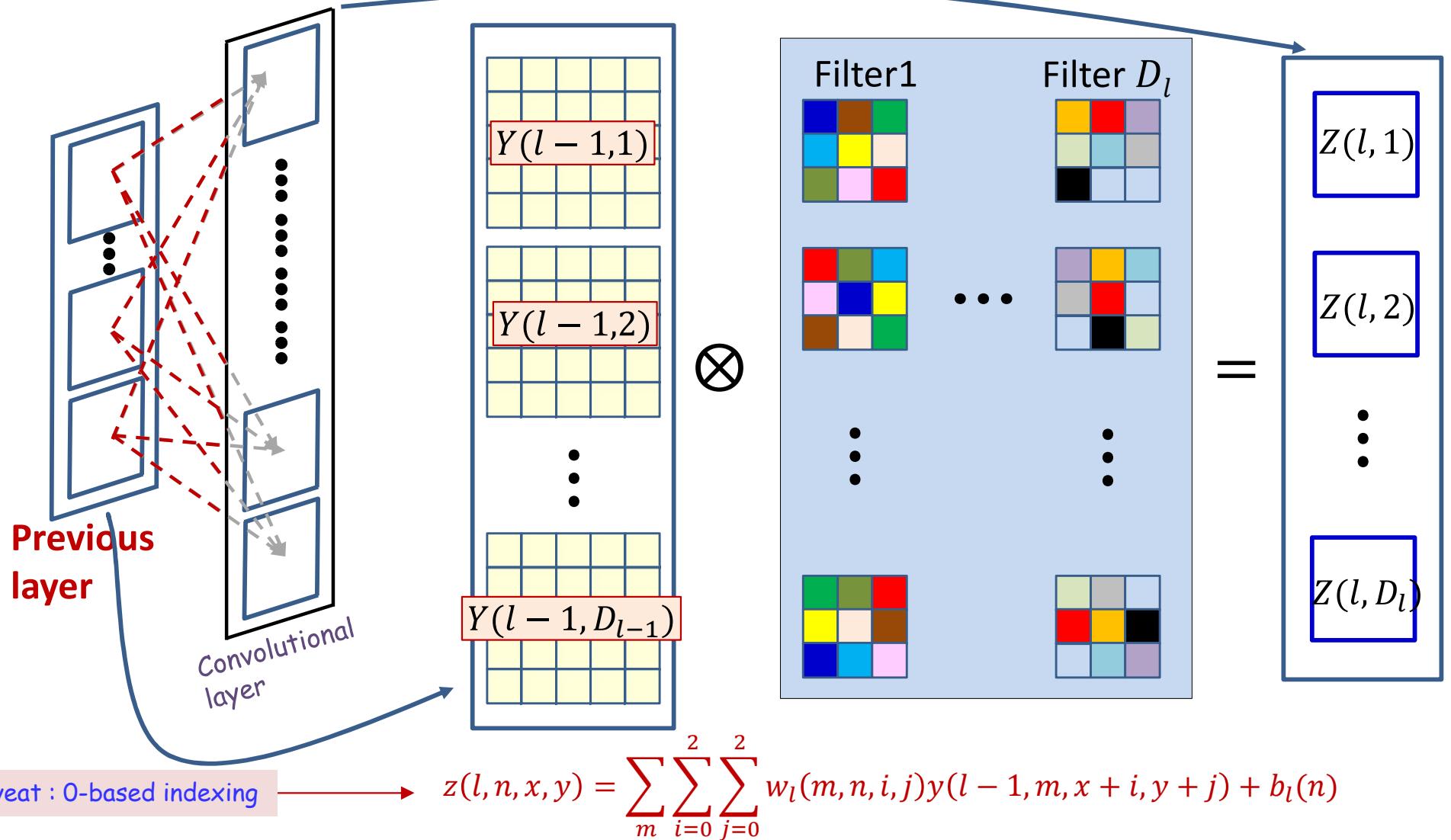
- The computation of each output map has two stages
 - Computing an *affine* map, by *convolving* a *filter* (representing a pattern of weights) over maps in the previous layer
 - Each affine map has, associated with it, a **learnable filter**
 - An *activation* that operates *point-wise* on the output of the convolution

Recap: A convolutional layer



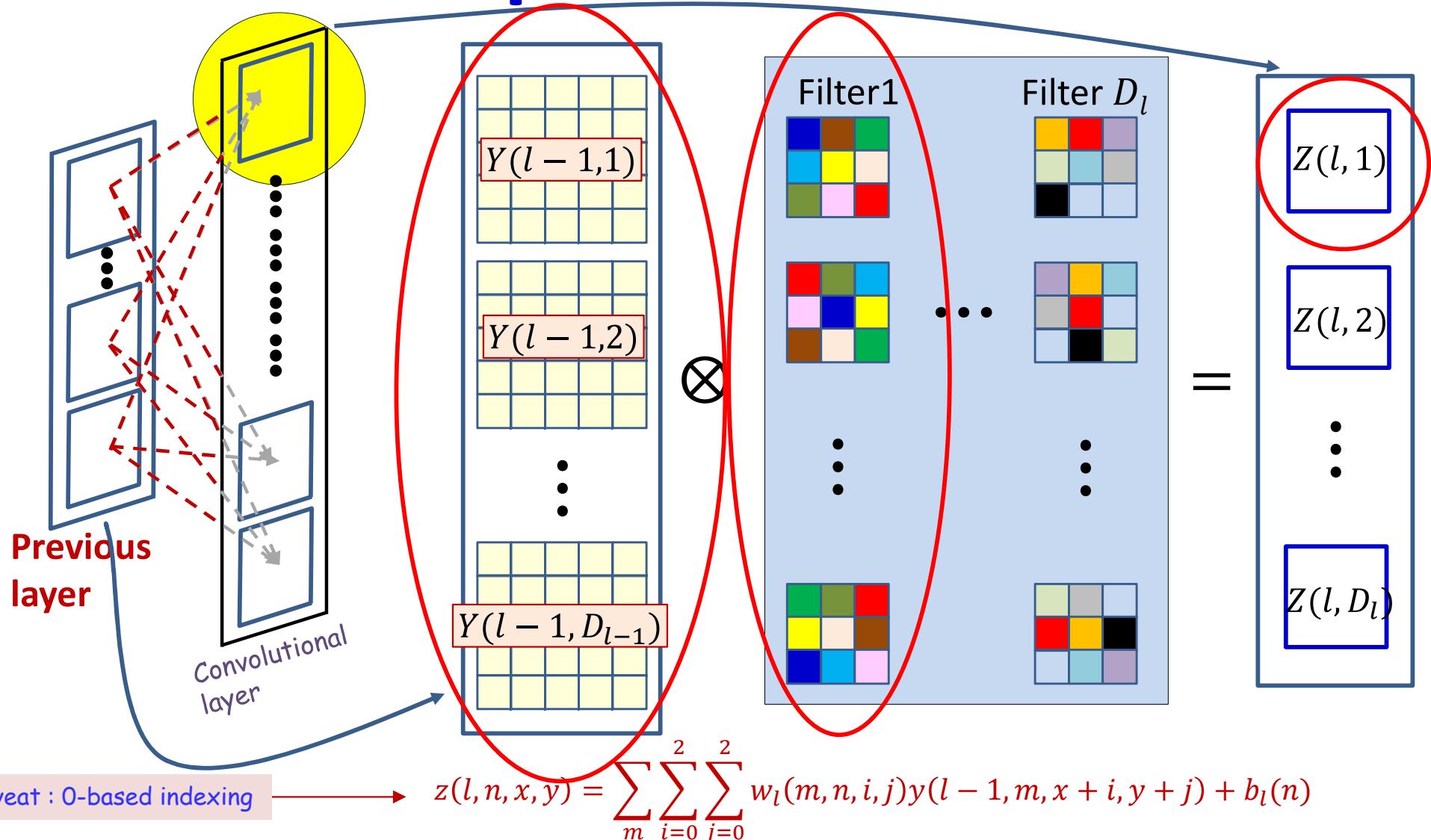
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Recap: Convolution



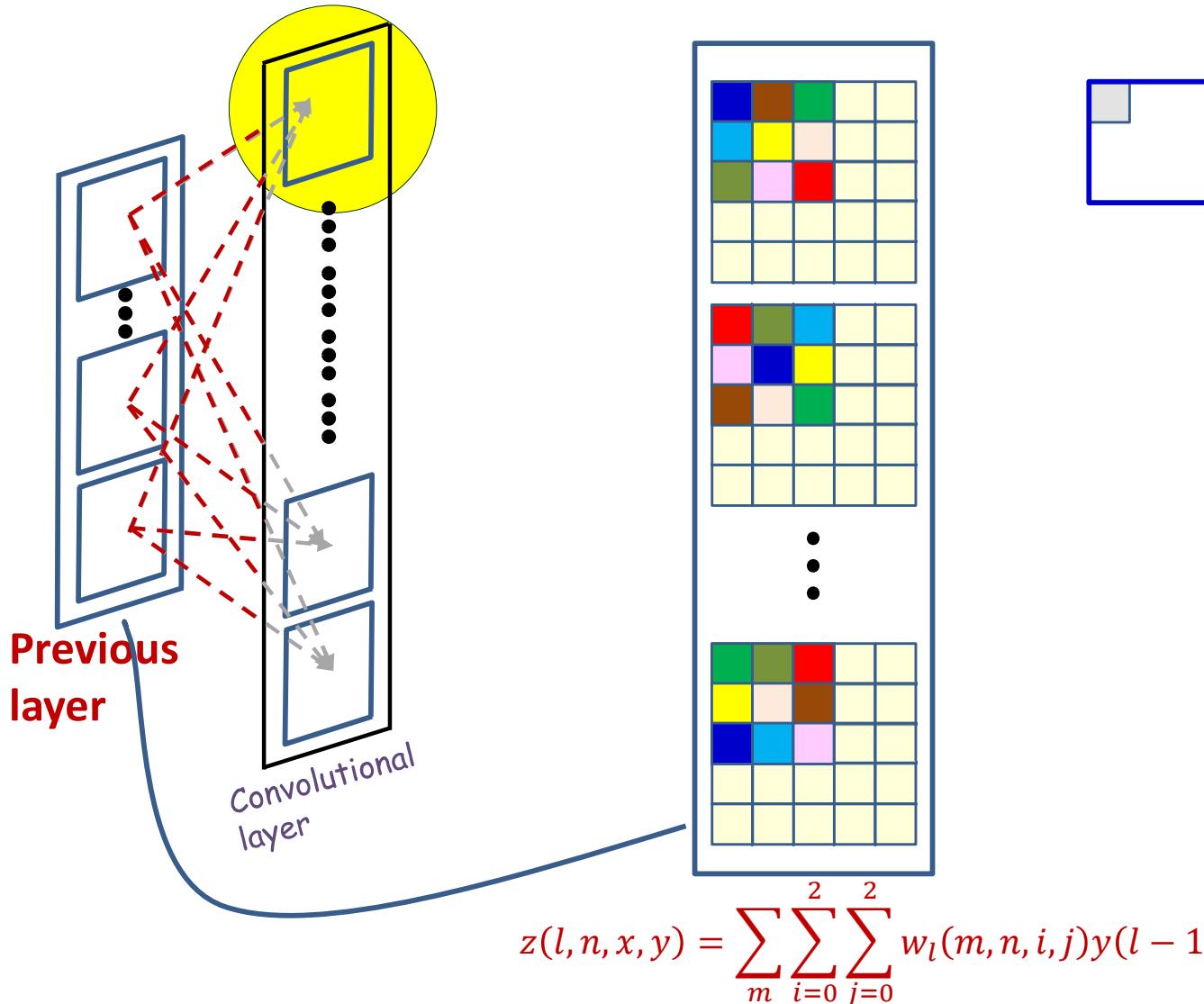
- Each affine output map is computed from multiple input maps simultaneously
- There are as many weights (for each output map) as
size of the filter x no. of maps in previous layer

Recap: Convolution



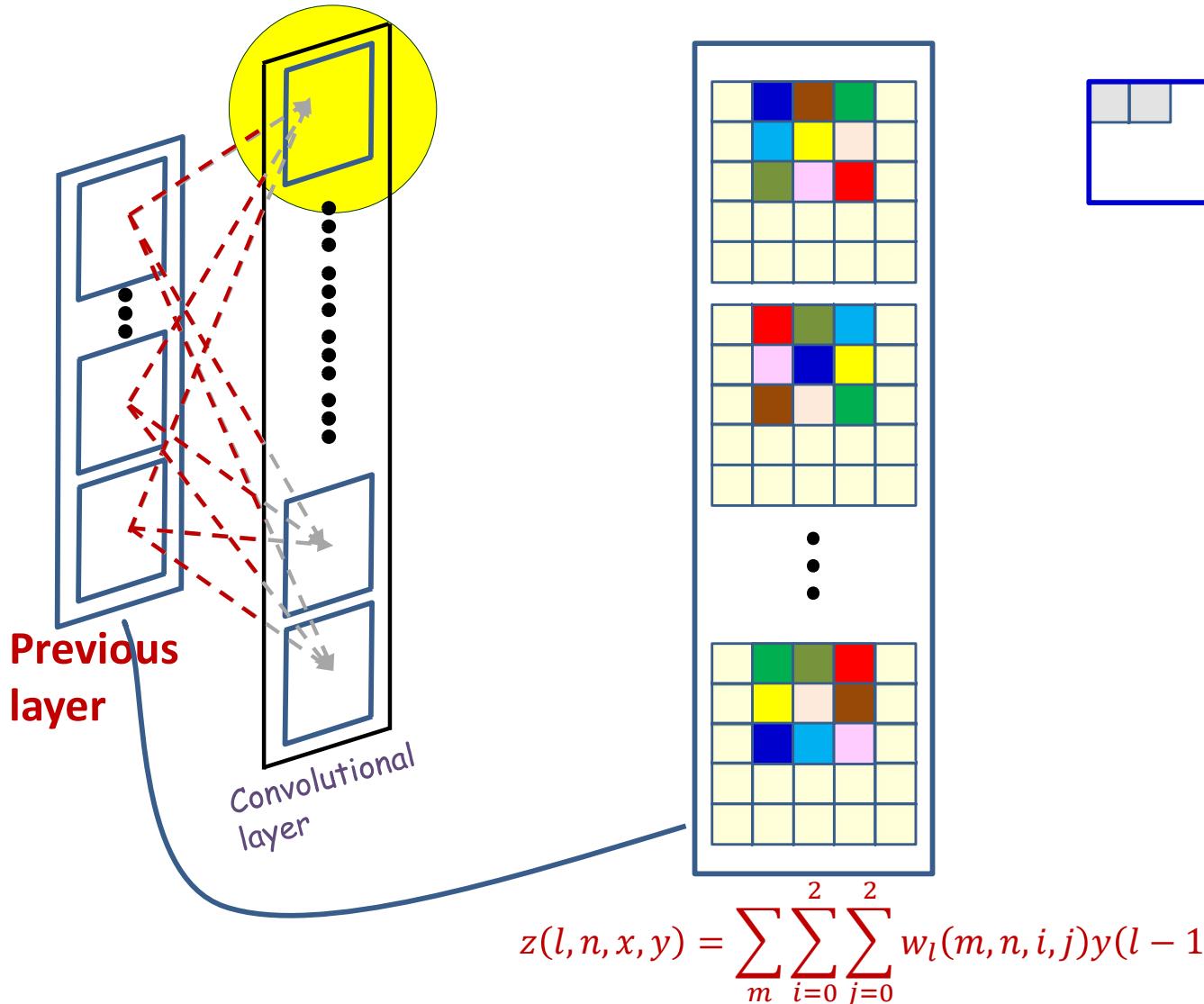
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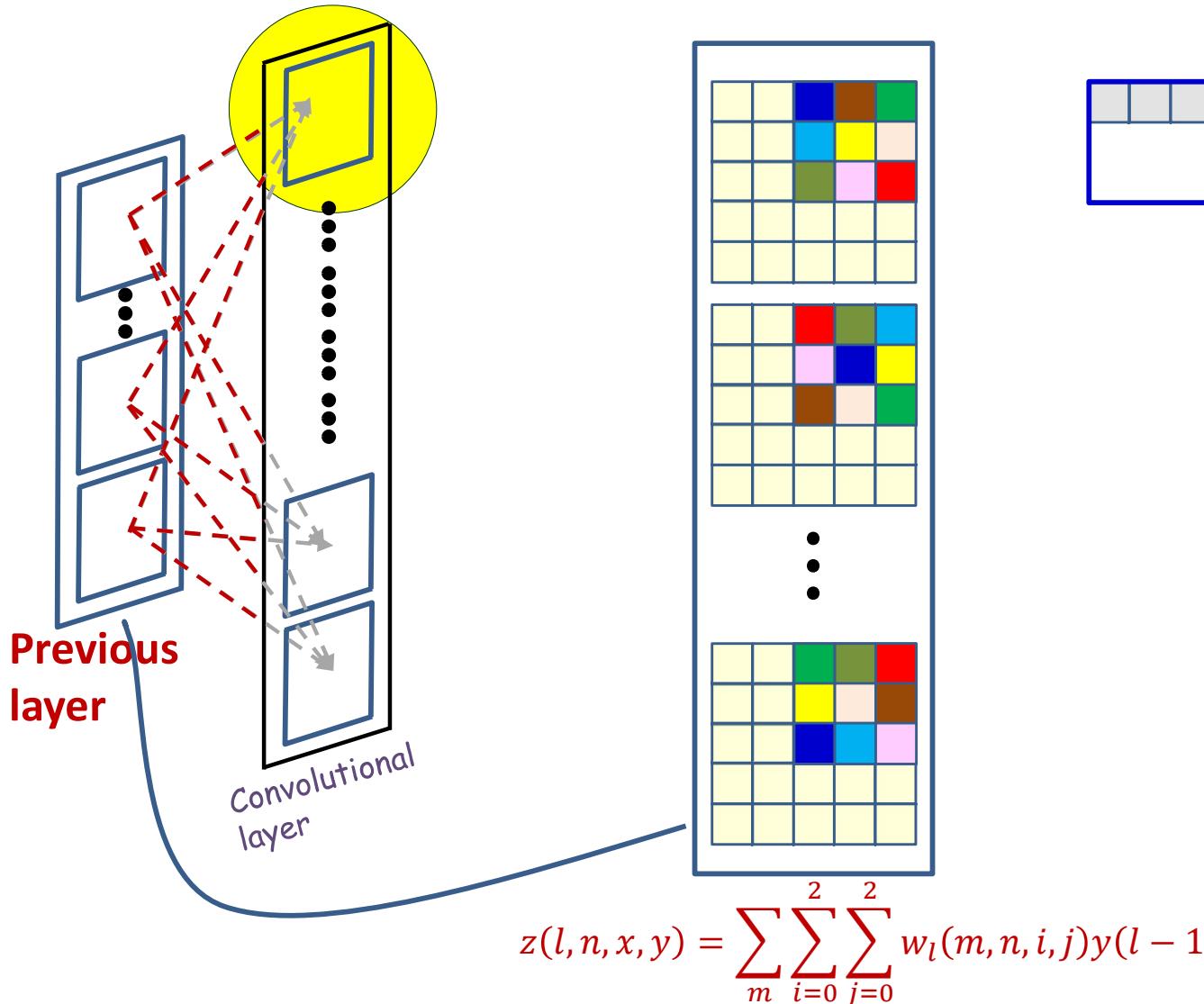
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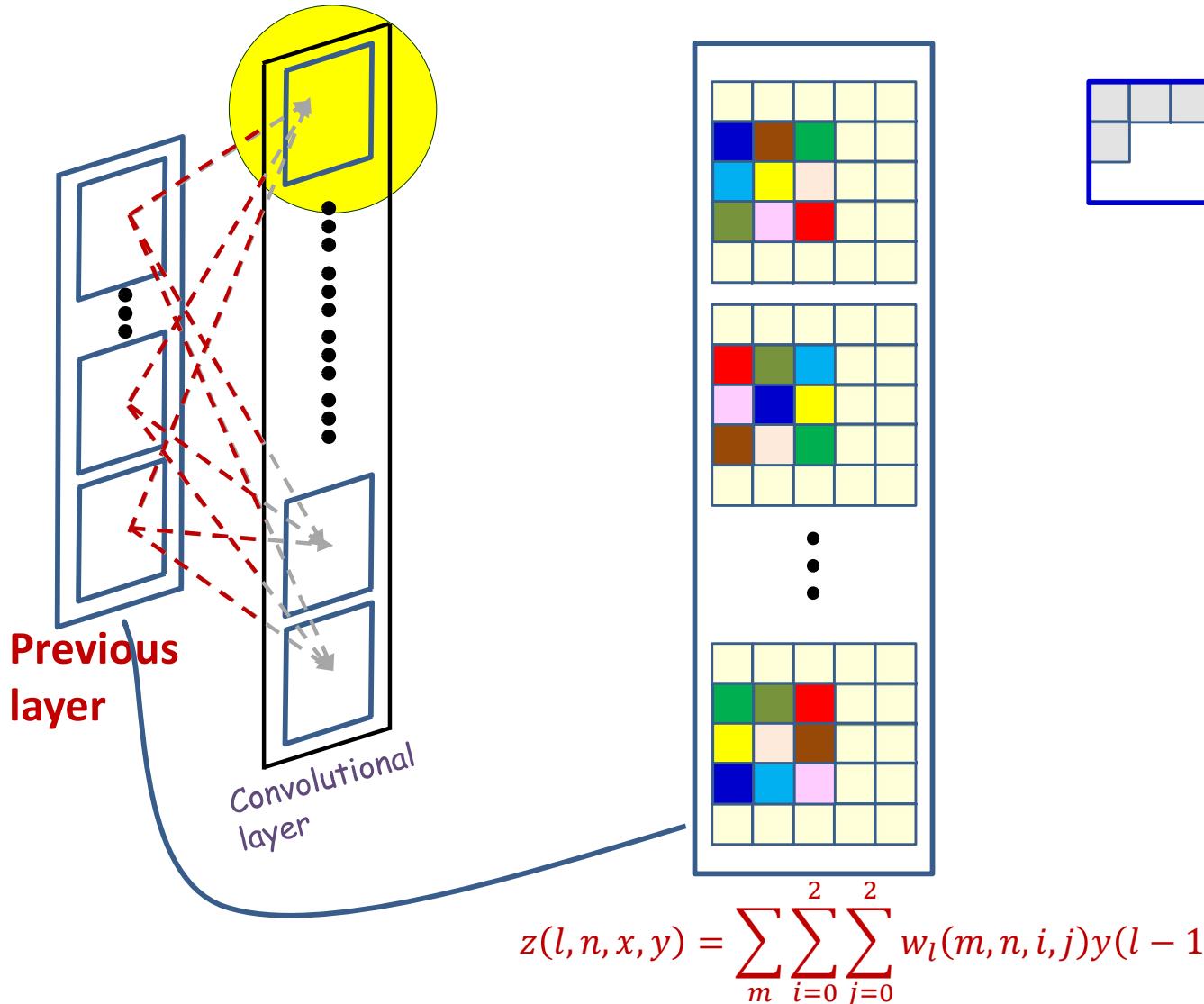
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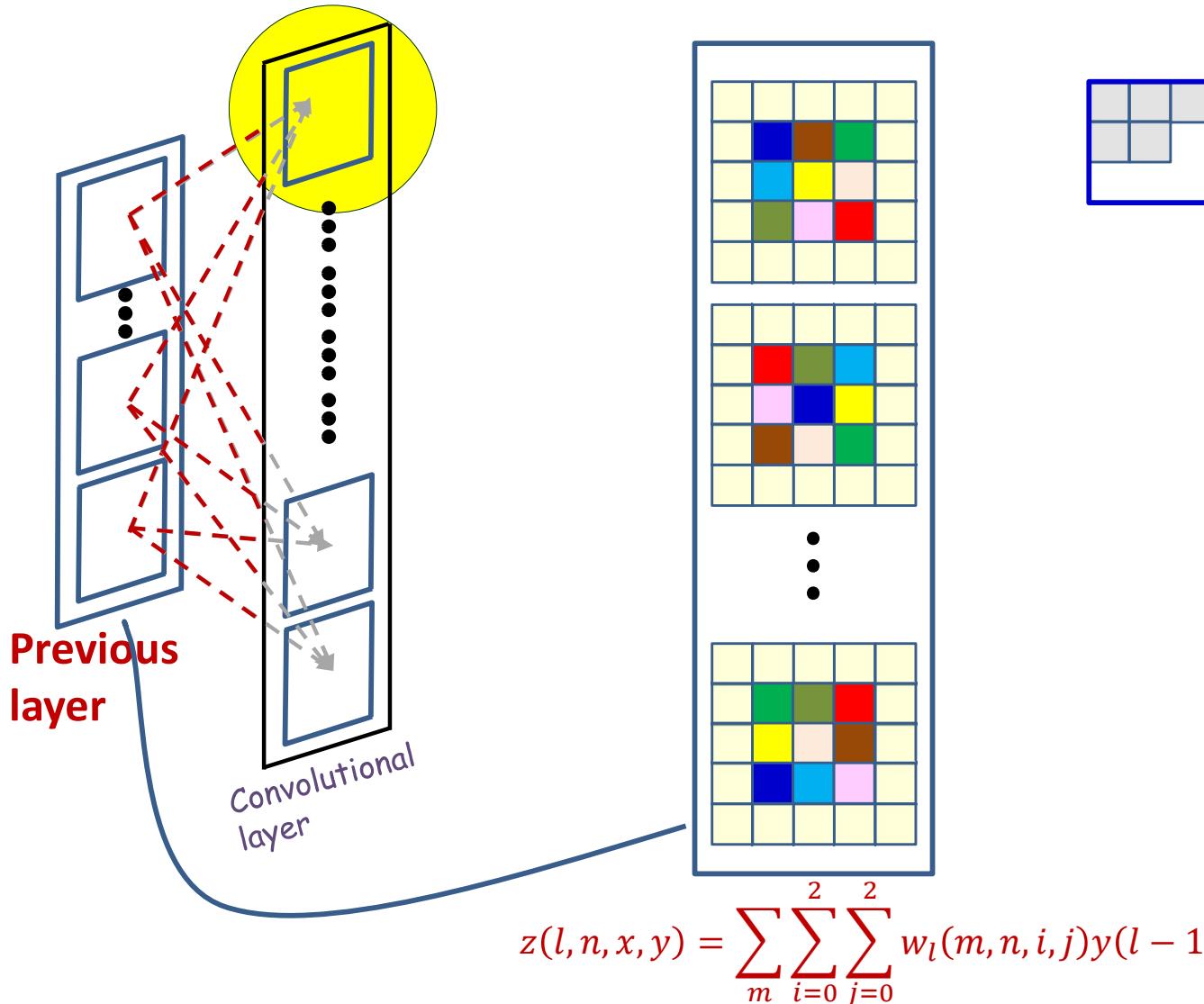
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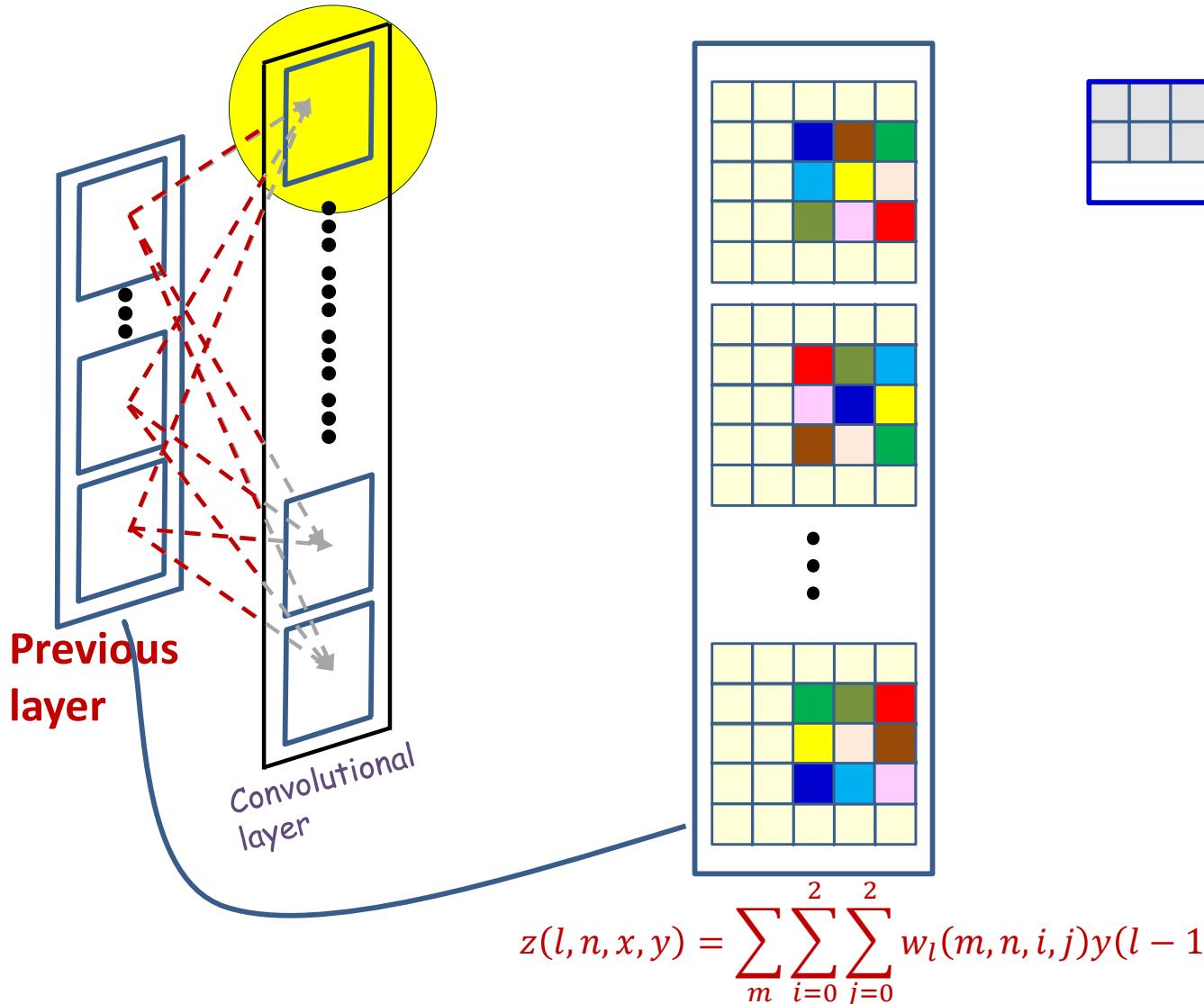
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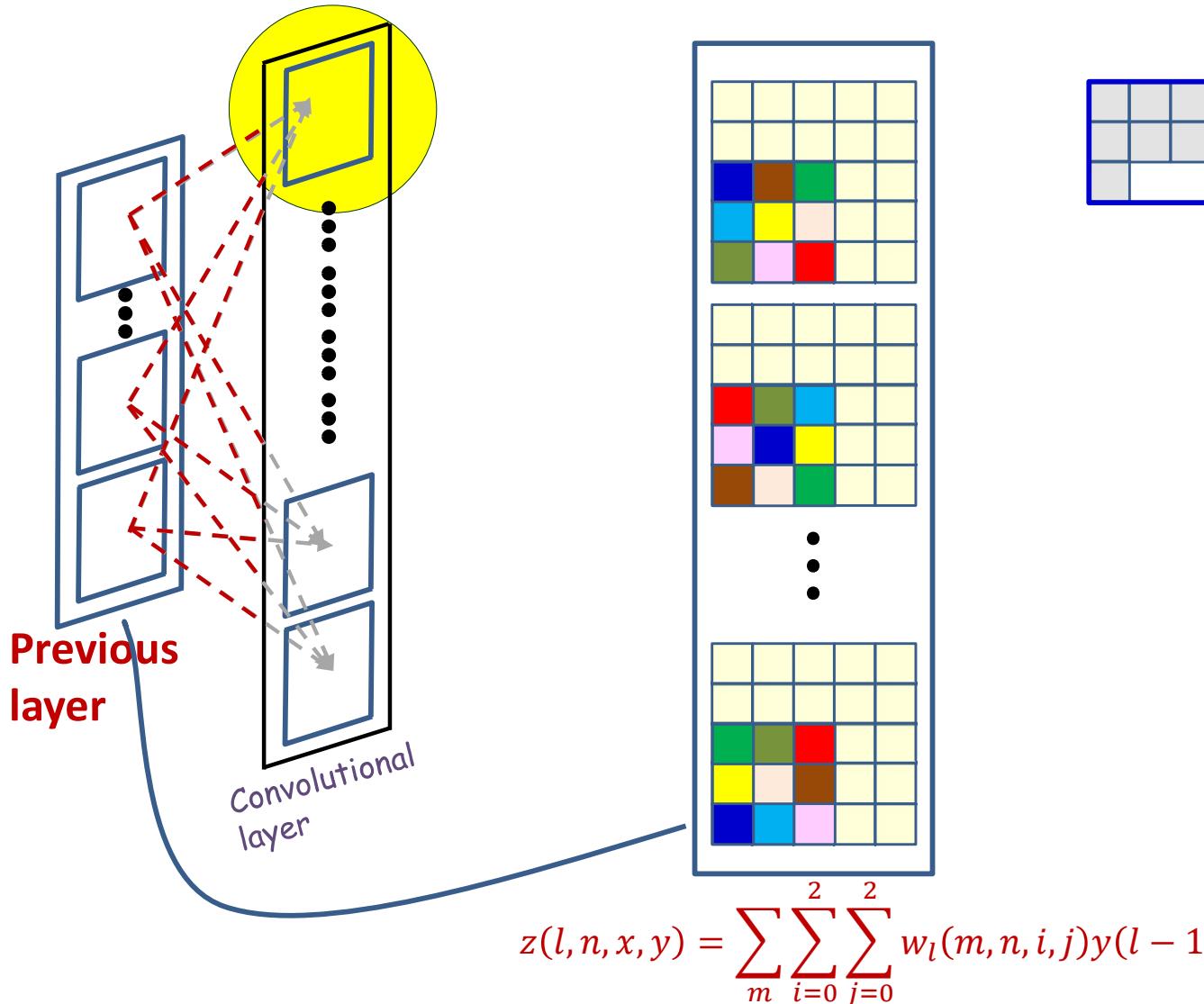
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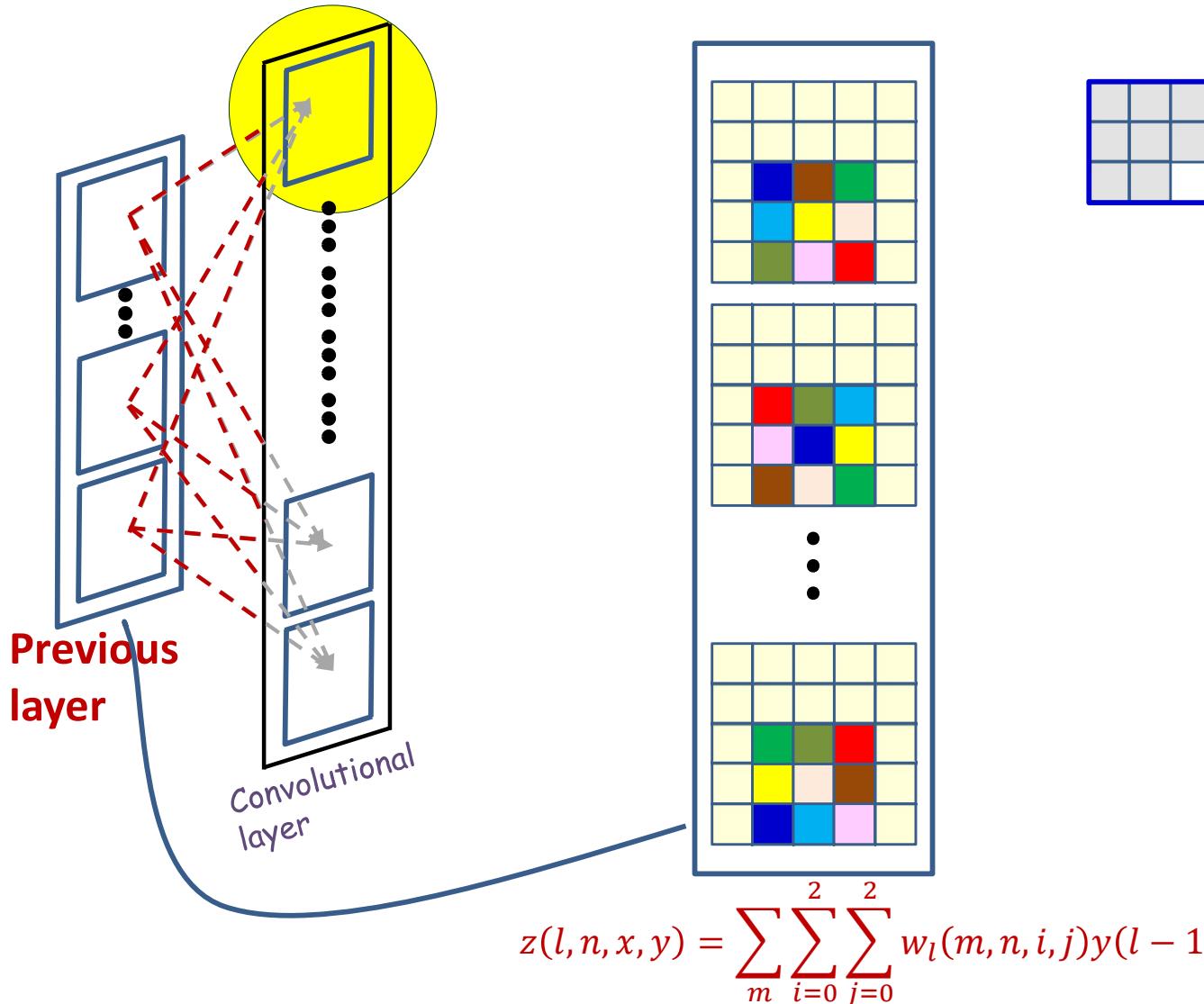
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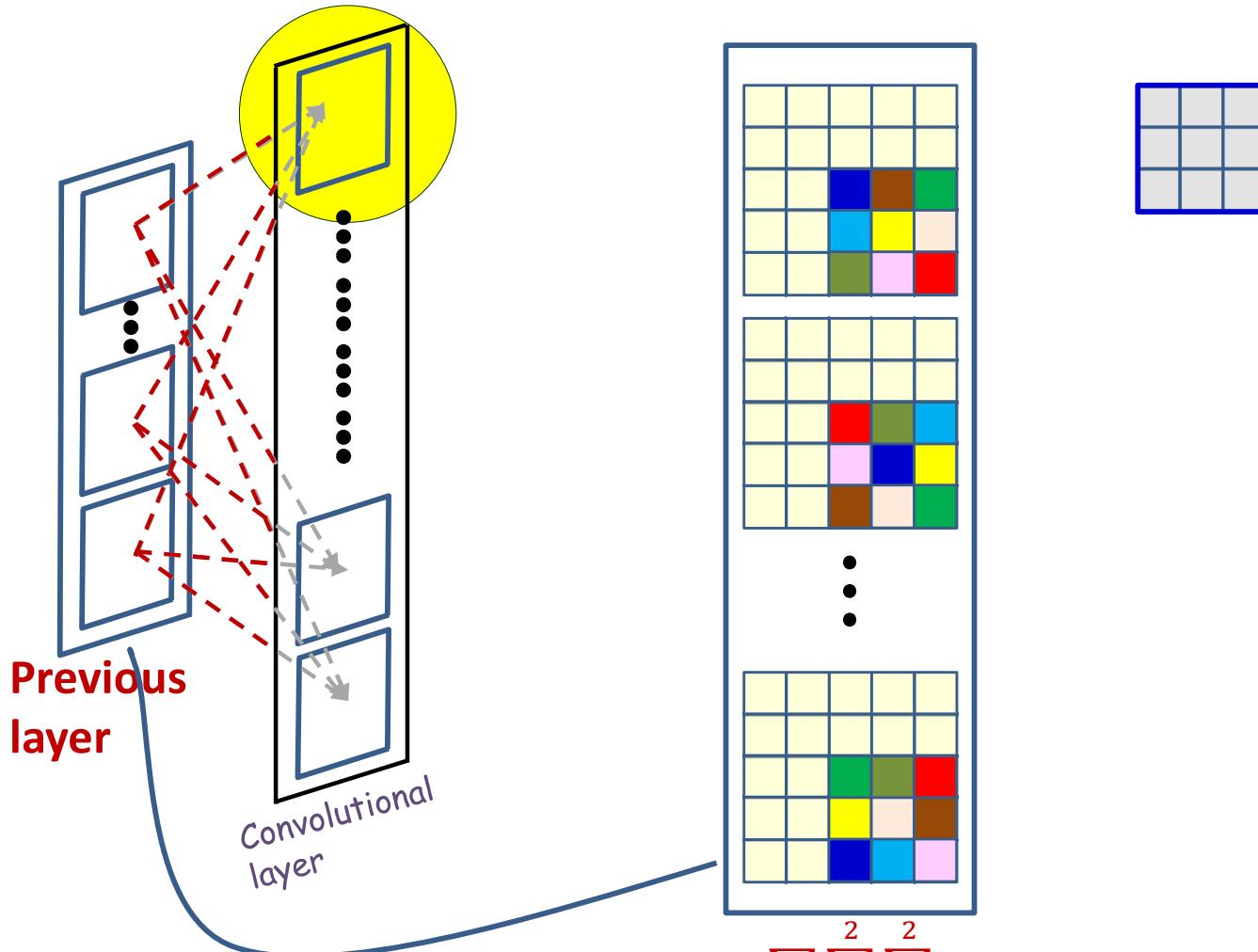
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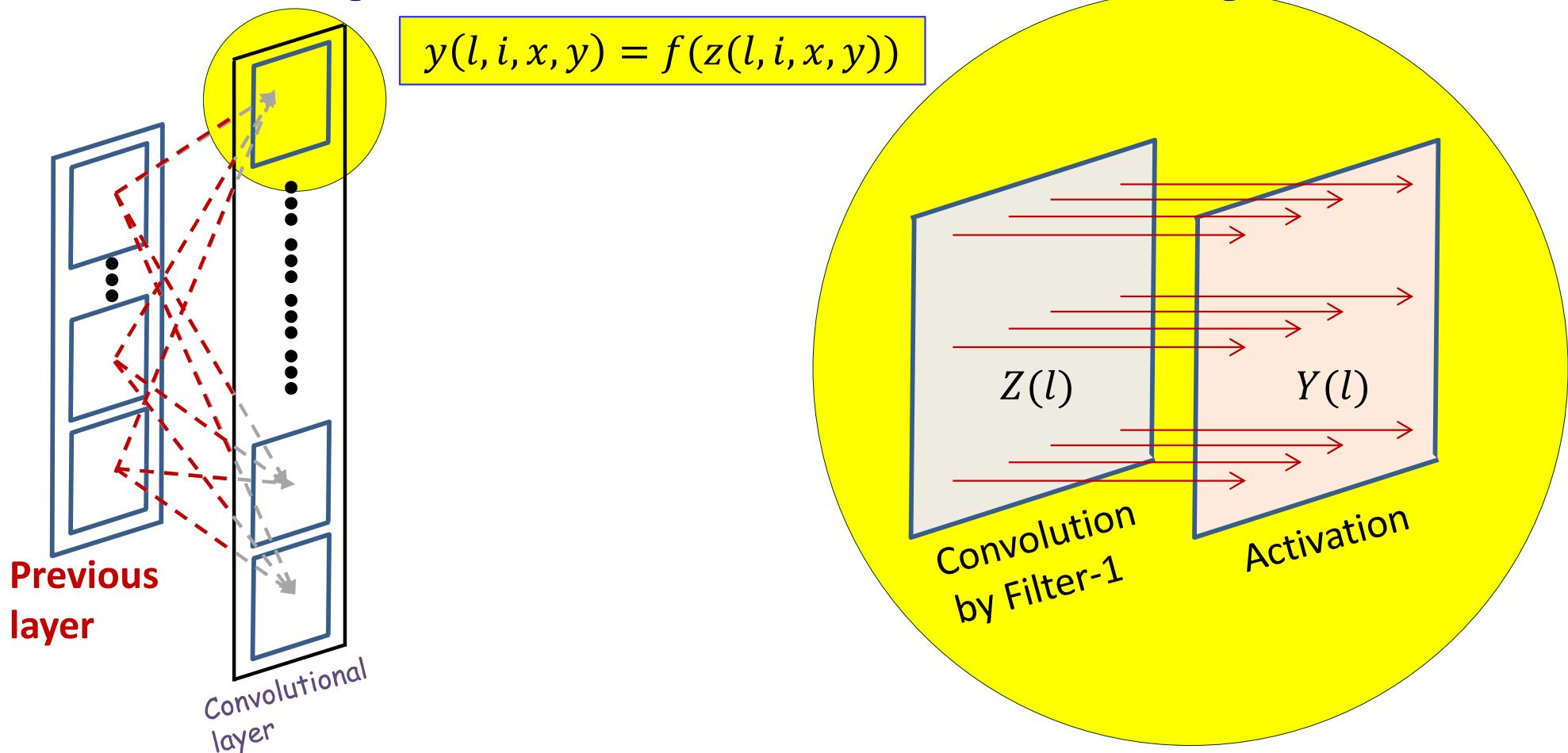
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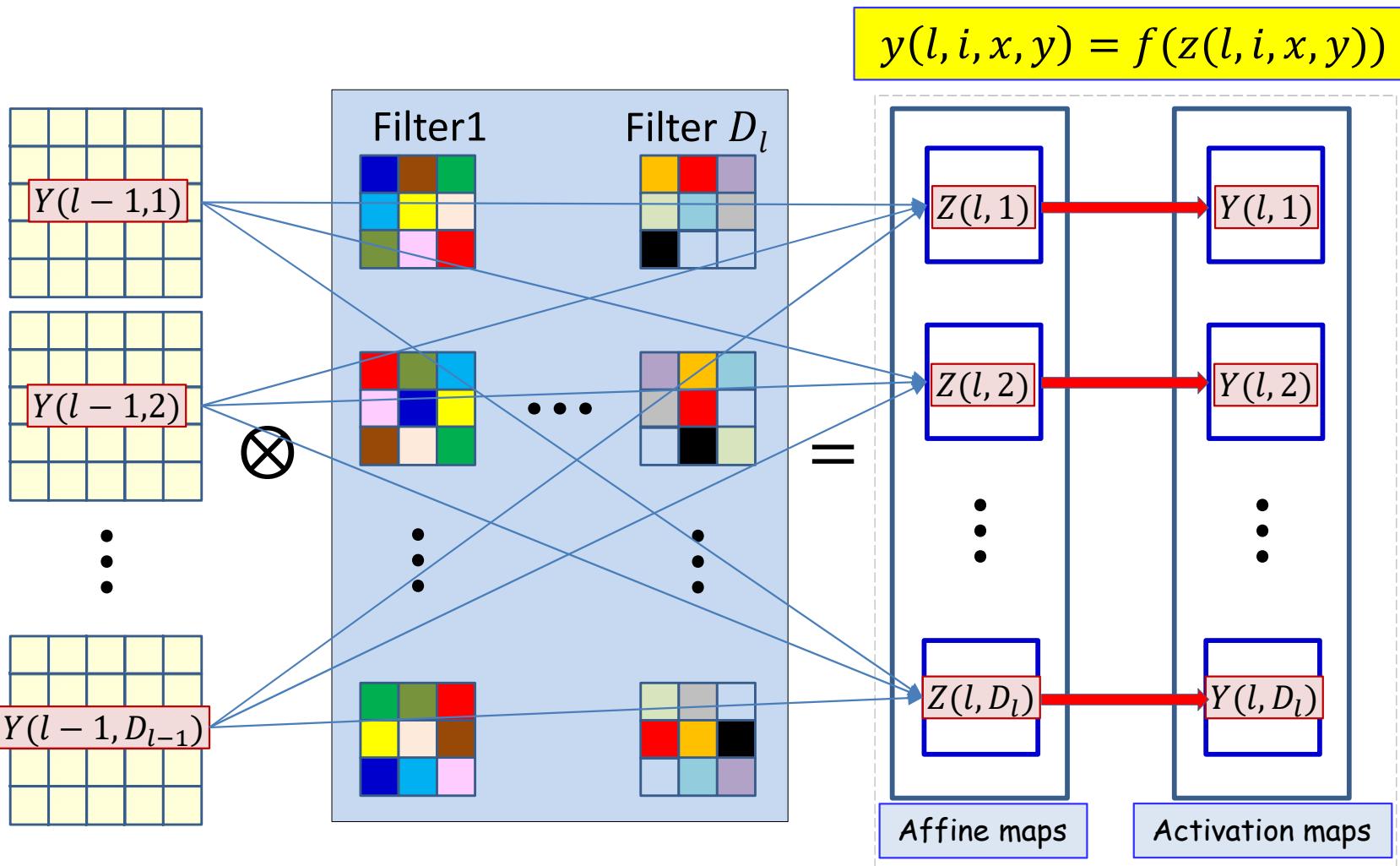
- Each affine output is computed from multiple input maps simultaneously
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Recap: A convolutional layer



- The computation of each output map has two stages
 - Computing an *affine* map, by *convolution* of a *filter* (representing a pattern of weights) over maps in the previous layer
 - Each affine map has, associated with it, a **learnable filter**
 - An *activation* that operates on the output of the convolution

Convolution layer: A more explicit illustration



- Input maps $Y(l - 1, *)$ are convolved with several filters to generate the affine maps $Z(l, *)$
 - Each filter consists of a set of square patterns of weights, with one set for each map in $Y(l - 1, *)$
 - We get one affine map per filter
- A *point-wise* activation function $f(z)$ is applied to each map in $Z(l, *)$ to produce the activation maps $Y(l, *)$

Pseudocode: Vector notation

The weight $\mathbf{W}(l, j)$ is a 3D $D_{l-1} \times K_l \times K_l$ tensor

$\mathbf{Y}(0) = \text{Image}$

for $l = 1:L$ # layers operate on vector at (x, y)

 for $x = 1:W_{l-1}-K_l+1$

 for $y = 1:H_{l-1}-K_l+1$

 for $j = 1:D_l$

 segment = $\mathbf{Y}(l-1, :, x:x+K_l-1, y:y+K_l-1)$ #3D tensor

$\mathbf{z}(l, j, x, y) = \mathbf{W}(l, j) \cdot \text{segment} + \mathbf{b}(l, j)$ #tensor prod.

$\mathbf{Y}(l, j, x, y) = \text{activation}(\mathbf{z}(l, j, x, y))$

$\mathbf{Y} = \text{softmax}(\{\mathbf{Y}(L, :, :, :)\})$

Pseudocode has 1-based indexing

Poll 1

Select all true statements about a convolution layer.

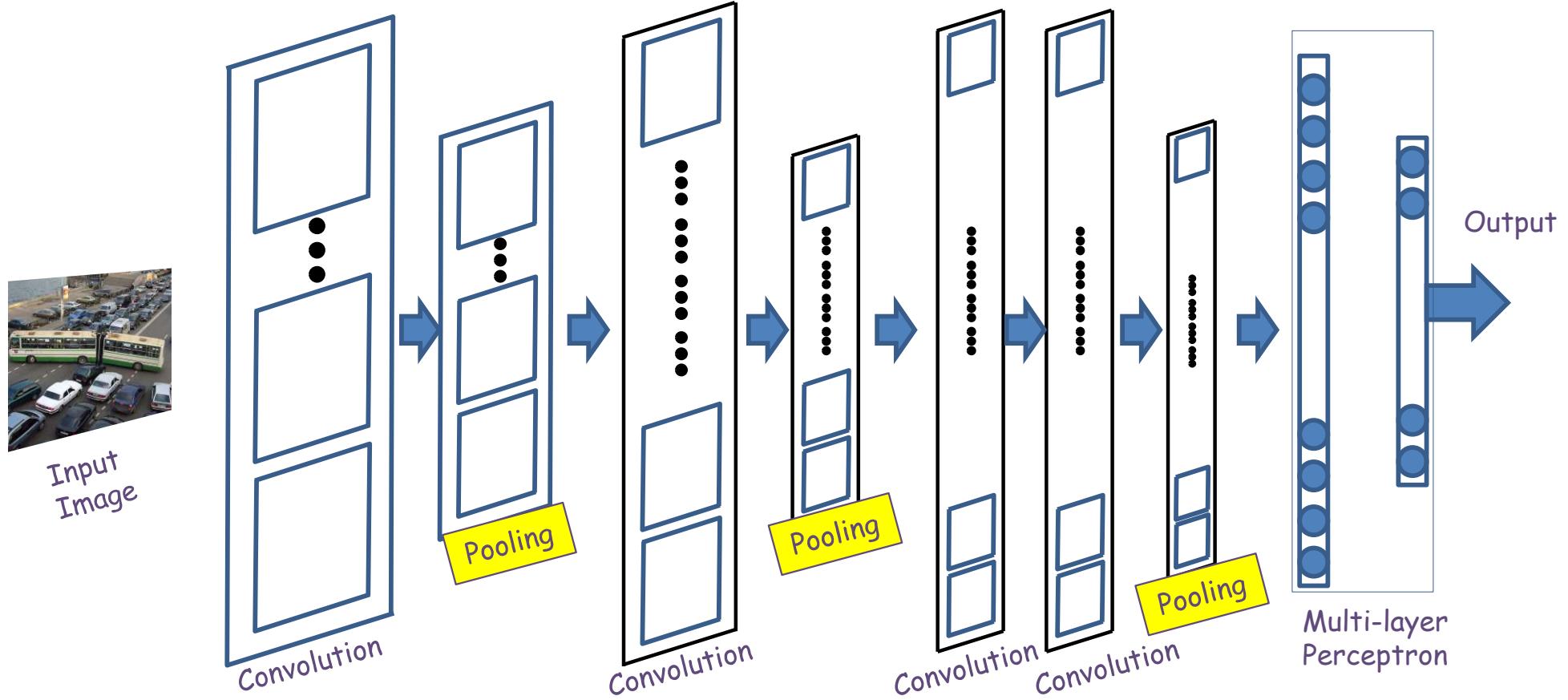
- The number of “channels” in any filter equals the number of input maps (output maps from the previous layer)
- The number of “channels” in any filter equals the number of output maps (affine maps output by the layer)
- The number of filters equals the number of input maps
- The number of filters equals the number of output maps

Poll 1

Select all true statements about a convolution layer.

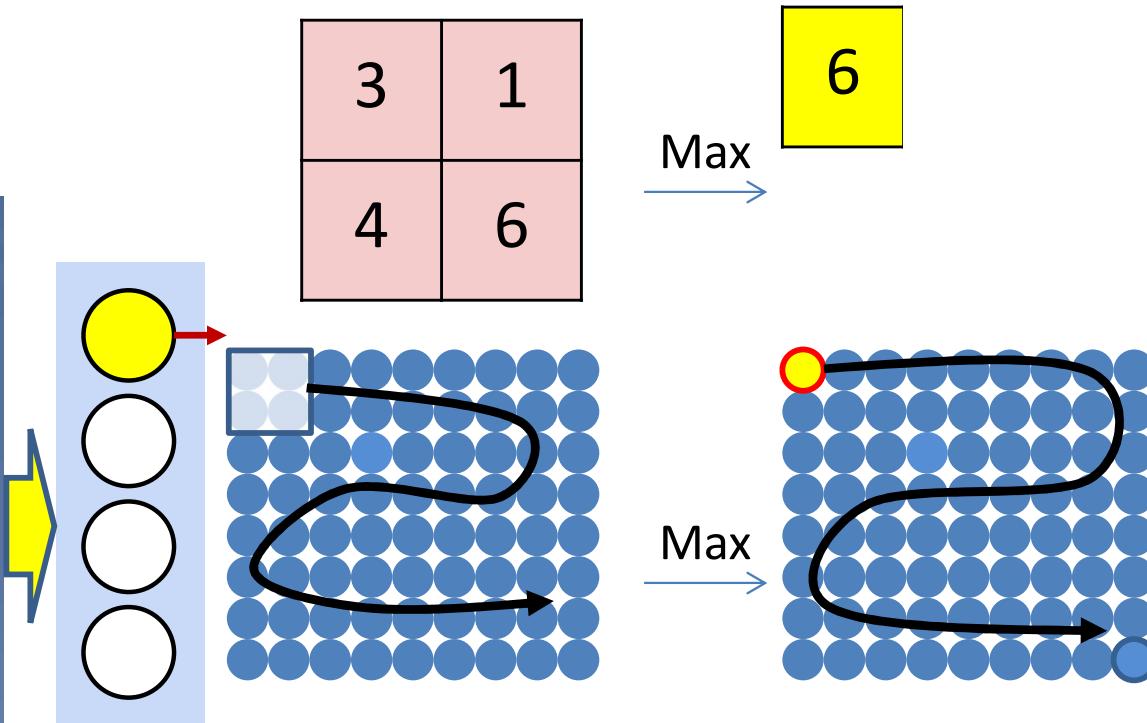
- **The number of “channels” in any filter equals the number of input maps (output maps from the previous layer)**
- The number of “channels” in any filter equals the number of output maps (affine maps output by the layer)
- The number of filters equals the number of input maps
- **The number of filters equals the number of output maps**

Pooling



- Convolutional (and activation) layers are followed intermittently by “pooling” layers
 - Often, they alternate with convolution, though this is not necessary

Recall: Max pooling



- Max pooling selects the largest from a pool of elements
- Pooling is performed by “scanning” the input with a “max-pooling filter”

Recap: Max Pooling layer at layer l

- a) Performed separately for every map (j).
 - *) Not combining multiple maps within a single max operation.
- b) Keeping track of location of max

Max pooling

```
for j = 1:Dl
```

```
    for x = 1:Wl-1-Kl+1
```

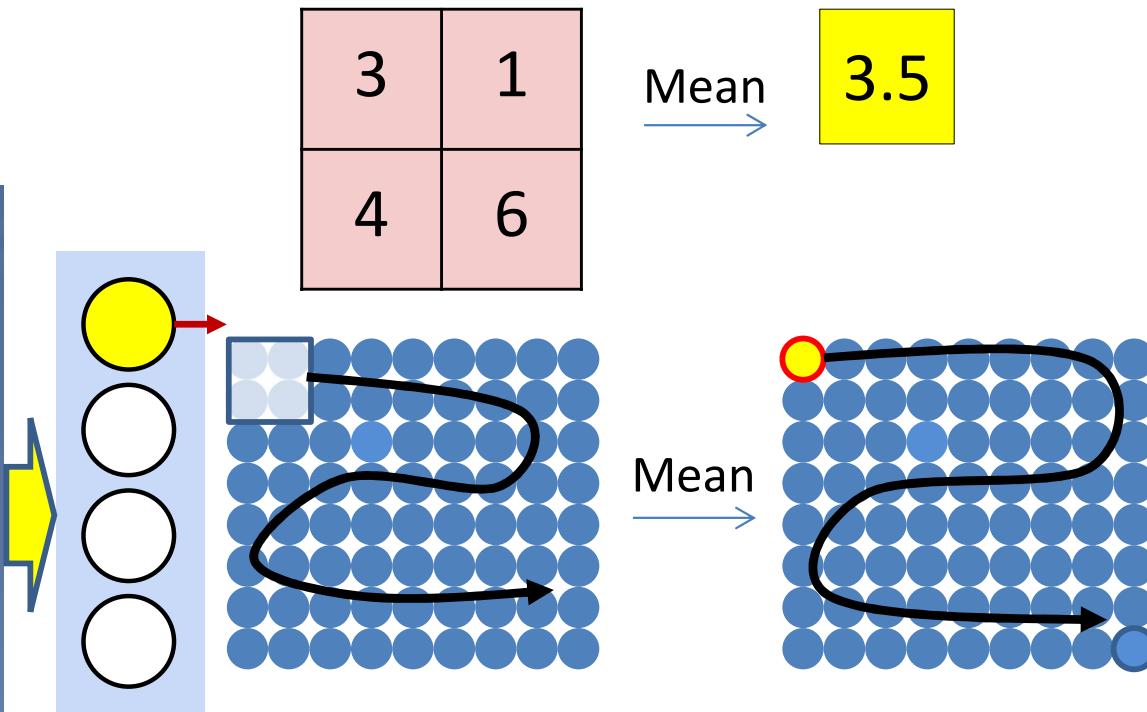
```
        for y = 1:Hl-1-Kl+1
```

```
            pidx(l,j,x,y) = maxidx(Y(l-1,j,x:x+Kl-1,y:y+Kl-1))
```

```
            u(l,j,x,y) = Y(l-1,j,pidx(l,j,m,n))
```



Recall: Mean pooling



- Mean pooling computes the *mean* of the window of values
 - As opposed to the max or max pooling

Recap: Mean Pooling layer at layer l

a) Performed separately for every map (j)

Mean pooling

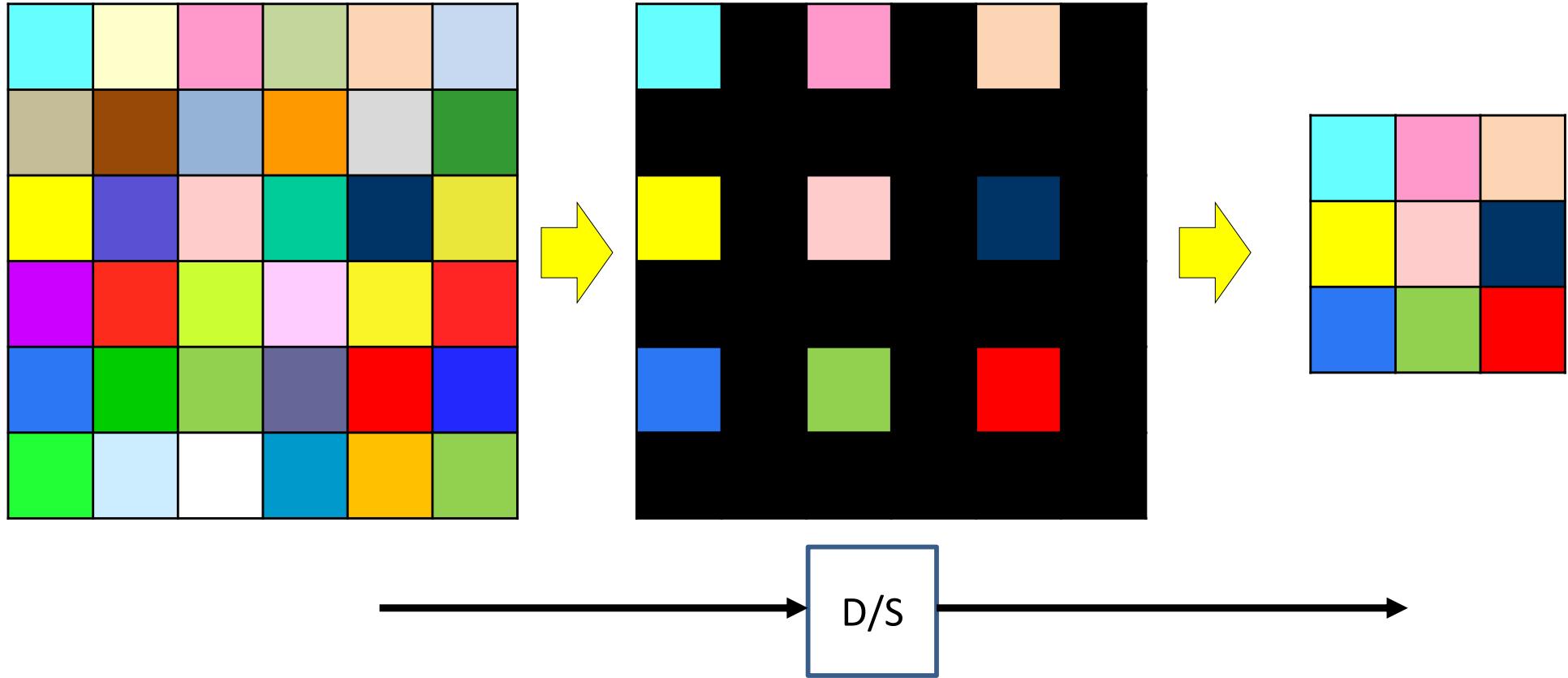
```
for j = 1:Dl
    for x = 1:Wl-1-Kl+1
        for y = 1:Hl-1-Kl+1
            u(l,j,x,y) = mean(Y(l-1,j,x:x+Kl-1,y:y+Kl-1))
```



Recap: Resampling

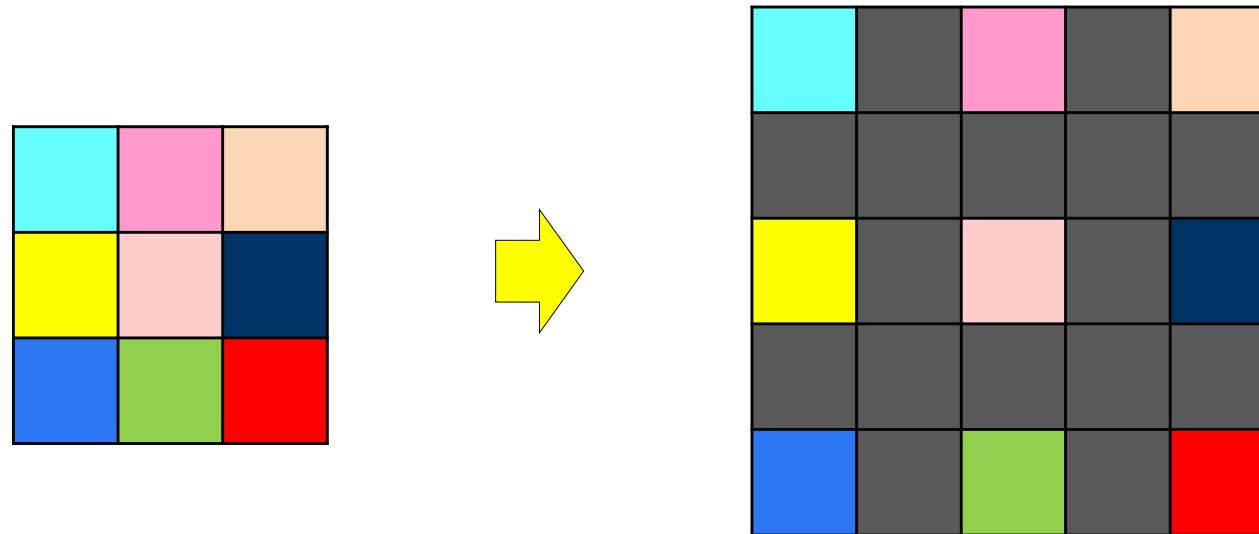
- We can also proportionately decrease or increase the size of the maps by dropping or inserting zeros
 - Downsampling: Drop $S-1$ rows/columns between rows/columns
 - Reduces the size of the maps by S on each side
 - Upsampling: Insert $S-1$ rows/columns of zeros between adjacent entries
 - Increases the size of the map by S on each side

The Downsampling Layer



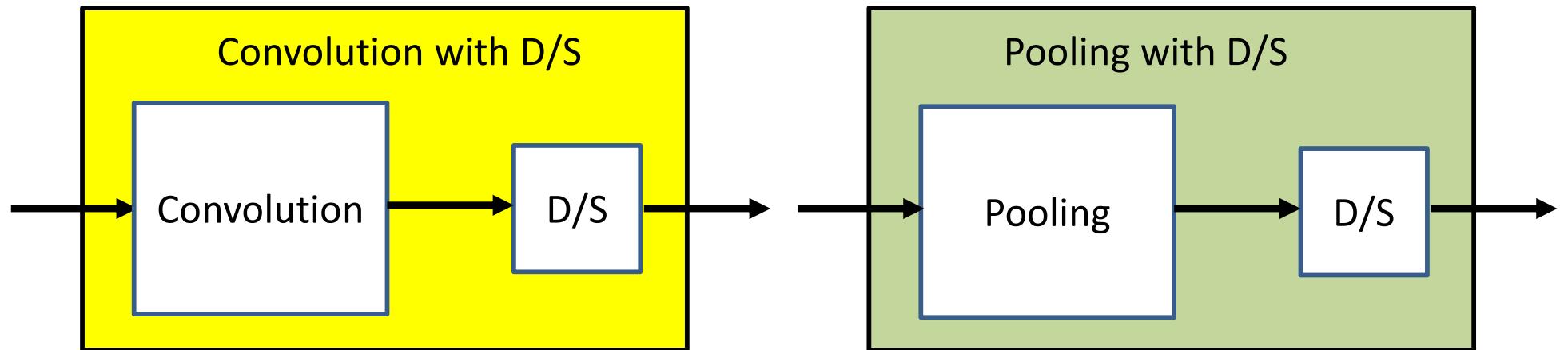
- A *downsampling* layer simply “drops” $S - 1$ of S rows and columns for every map in the layer
 - Effectively reducing the size of the map by factor S in every direction

The Upsampling Layer



- A *upsampling* (or dilation) layer simply introduces $S - 1$ rows and columns for every map in the layer
 - Effectively *increasing* the size of the map by factor S in every direction
- Used explicitly to increase the map size by a uniform factor

Downsampling in practice



- In practice, the downsampling is combined with the layers just before it by performing the operations with a stride > 1
 - Could be convolutional or pooling layers

Convolution with downsampling

The weight $W(l, j)$ is now a 4D $D_l \times D_{l-1} \times K_l \times K_l$ tensor

The product in blue is a tensor inner product with a scalar output

$\mathbf{Y}(0) = \text{Image}$

for $l = 1:L$ # layers operate on vector at (x, y)

```
m = 1
for x = 1:S:Wl-1-Kl+1
    n = 1
    for y = 1:S:Hl-1-Kl+1
        segment = Y(l-1, :, x:x+Kl-1, y:y+Kl-1) #3D tensor
        z(l, :, m, n) = W(l).segment #tensor inner prod.
        Y(l, :, m, n) = activation(z(l, :, m, n))
        n++
    m++
```

Downsampled indices

$\mathbf{Y} = \text{softmax}(\{\mathbf{Y}(L, :, :, :)\})$

Max Pooling with Downsampling

Max pooling

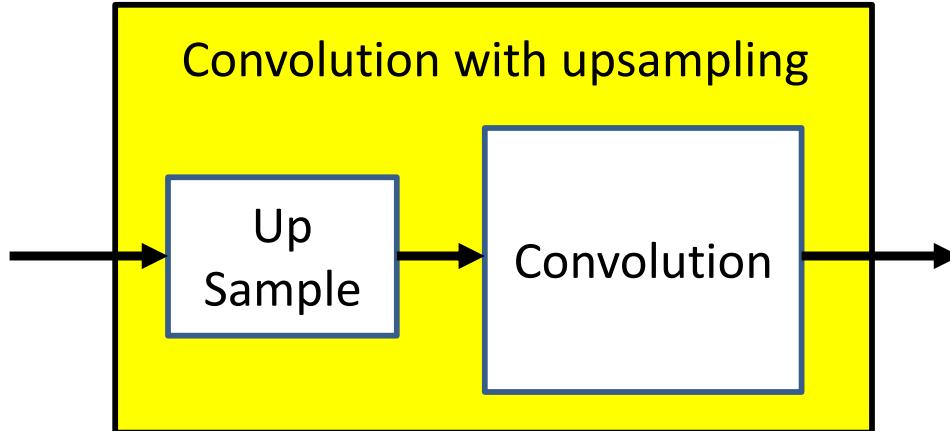
```
for j = 1:D1
    m = 1
    for x = 1:stride(l):Wl-1-Kl+1
        n = 1
        for y = 1:stride(l):Hl-1-Kl+1
            pidx(l,j,m,n) = maxidx(Y(l-1,j,x:x+Kl-1,y:y+Kl-1))
            Y(l,j,m,n) = Y(l-1,j,pidx(l,j,m,n))
        n = n+1
    m = m+1
```

Mean Pooling with Downsampling

Mean pooling

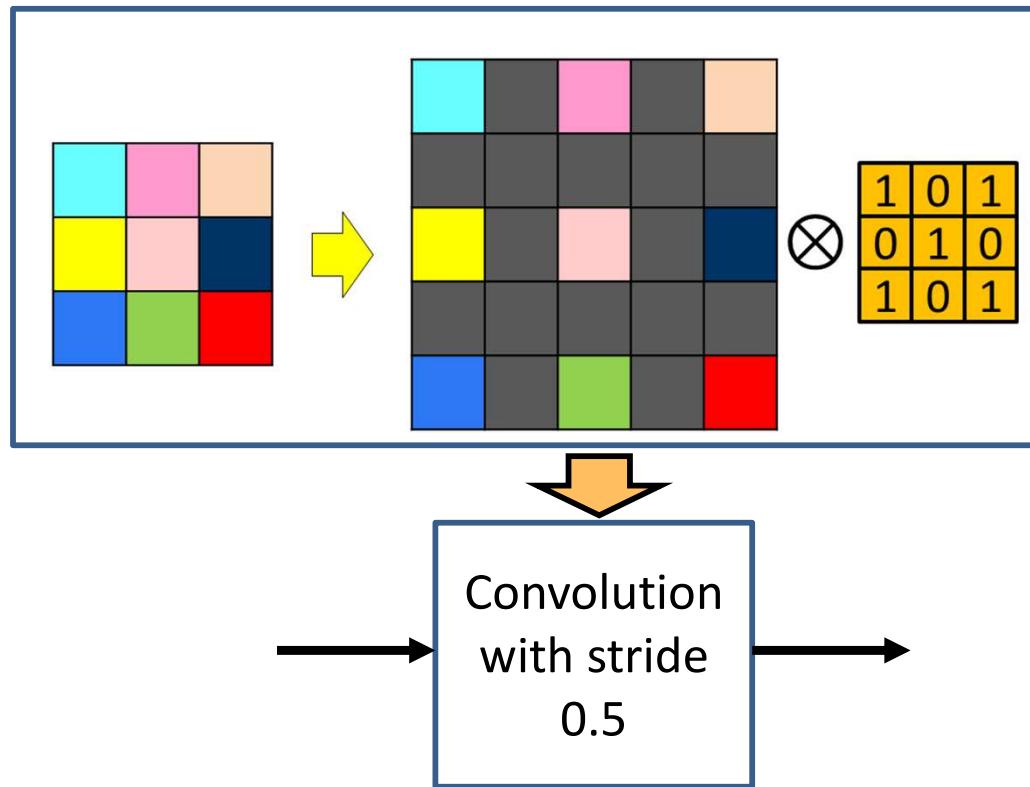
```
for j = 1:D1
    m = 1
    for x = 1:stride(l):Wl-1-Kl+1
        n = 1
        for y = 1:stride(l):Hl-1-Kl+1
            Y(l,j,m,n) = mean(Y(l-1,j,x:x+Kl-1,y:y+Kl-1))
            n = n+1
        m = m+1
```

The Upsampling Layer



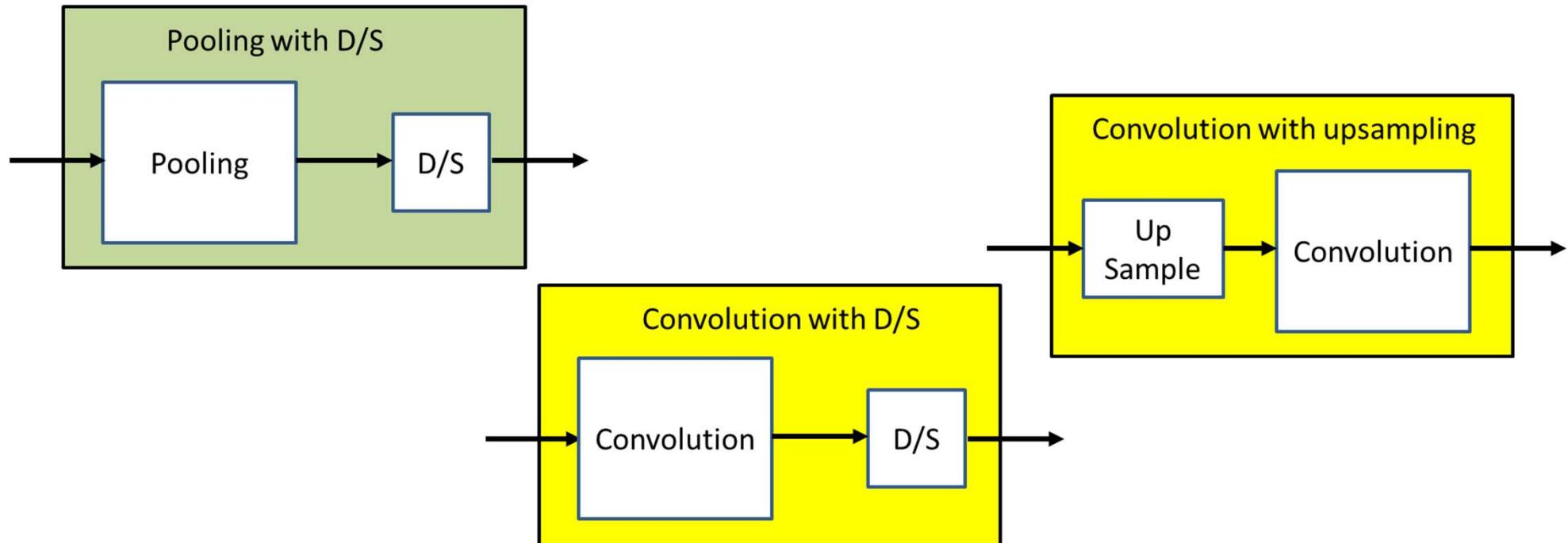
- A *upsampling* layer is generally followed by a CNN layer
 - It is not useful to follow it by a pooling layer
 - It is also not useful as the *final* layer of a CNN

The Upsampling Layer



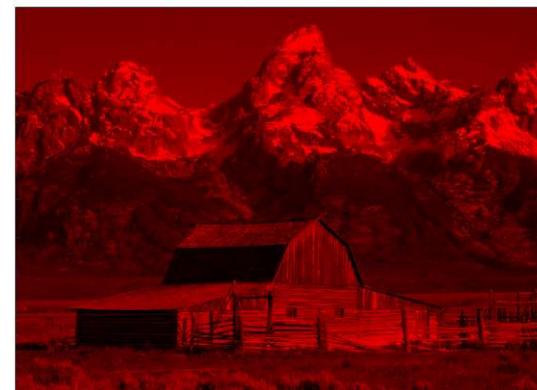
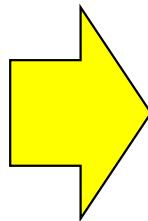
- Upsampling layers followed by a convolutional layer are also often viewed as convolving with a fractional stride
 - Upsampling by factor S is the same as striding by factor $1/S$
- Also called “transpose convolutions” for reasons we won’t get into here

* with resampling



- Although the resampling operation is generally merged with convolutions or pooling (by changing their stride) in the forward pass in practical implementations...
- ...It is more convenient to think of the two as separate operations in the backward pass
 - More on this later...

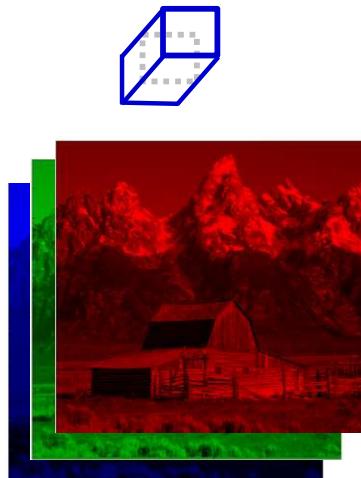
Recap: A CNN, end-to-end



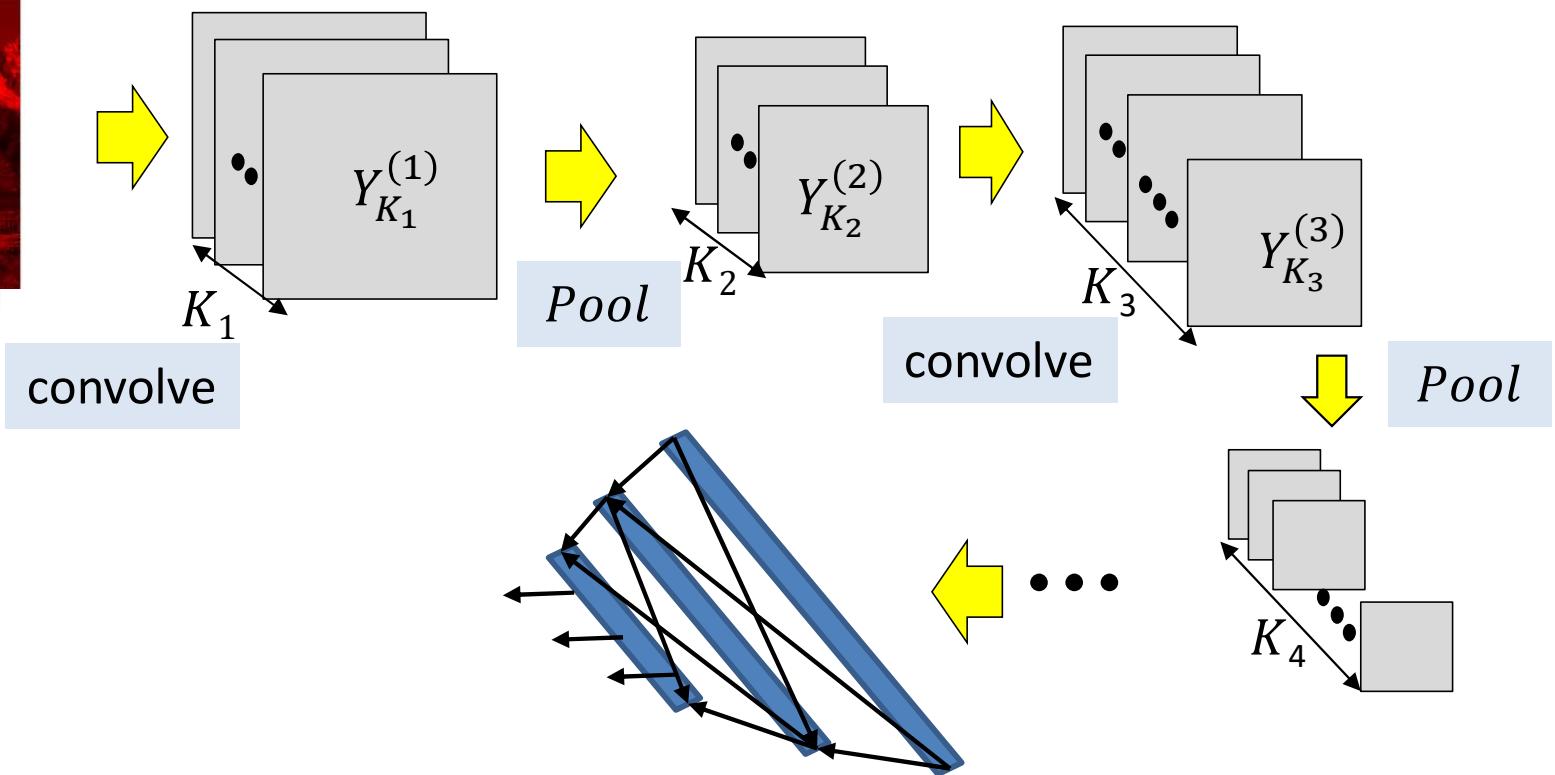
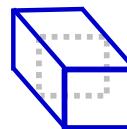
- Typical image classification task
 - Assuming maxpooling..
- Input: RBG images
 - Will assume color to be generic

Recap: A CNN, end-to-end

$$W_m: 3 \times L \times L \\ m = 1 \dots K_1$$

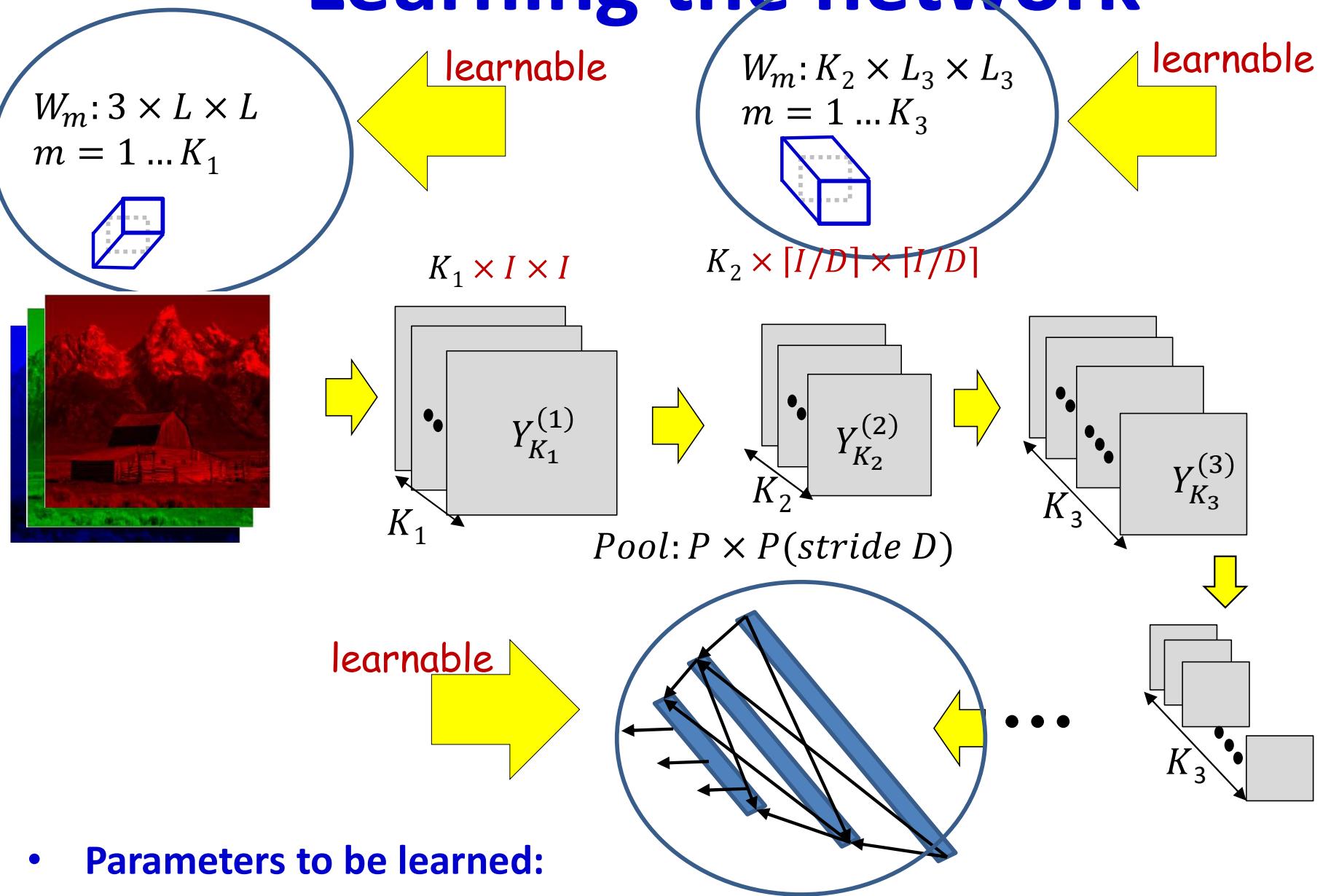


$$W_m: K_2 \times L_3 \times L_3 \\ m = 1 \dots K_3$$



- Several convolutional and pooling layers.
- The output of the last layer is “flattened” and passed through an MLP

Learning the network



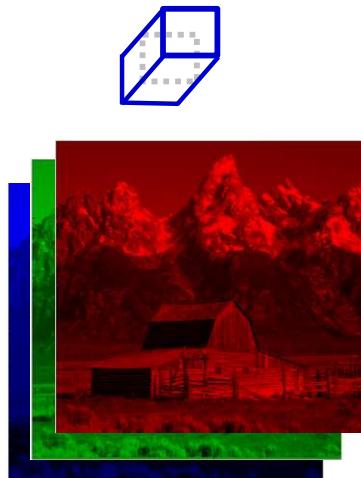
- Parameters to be learned:
 - The weights of the neurons in the final MLP
 - The (weights and biases of the) filters for every *convolutional* layer

Learning the CNN

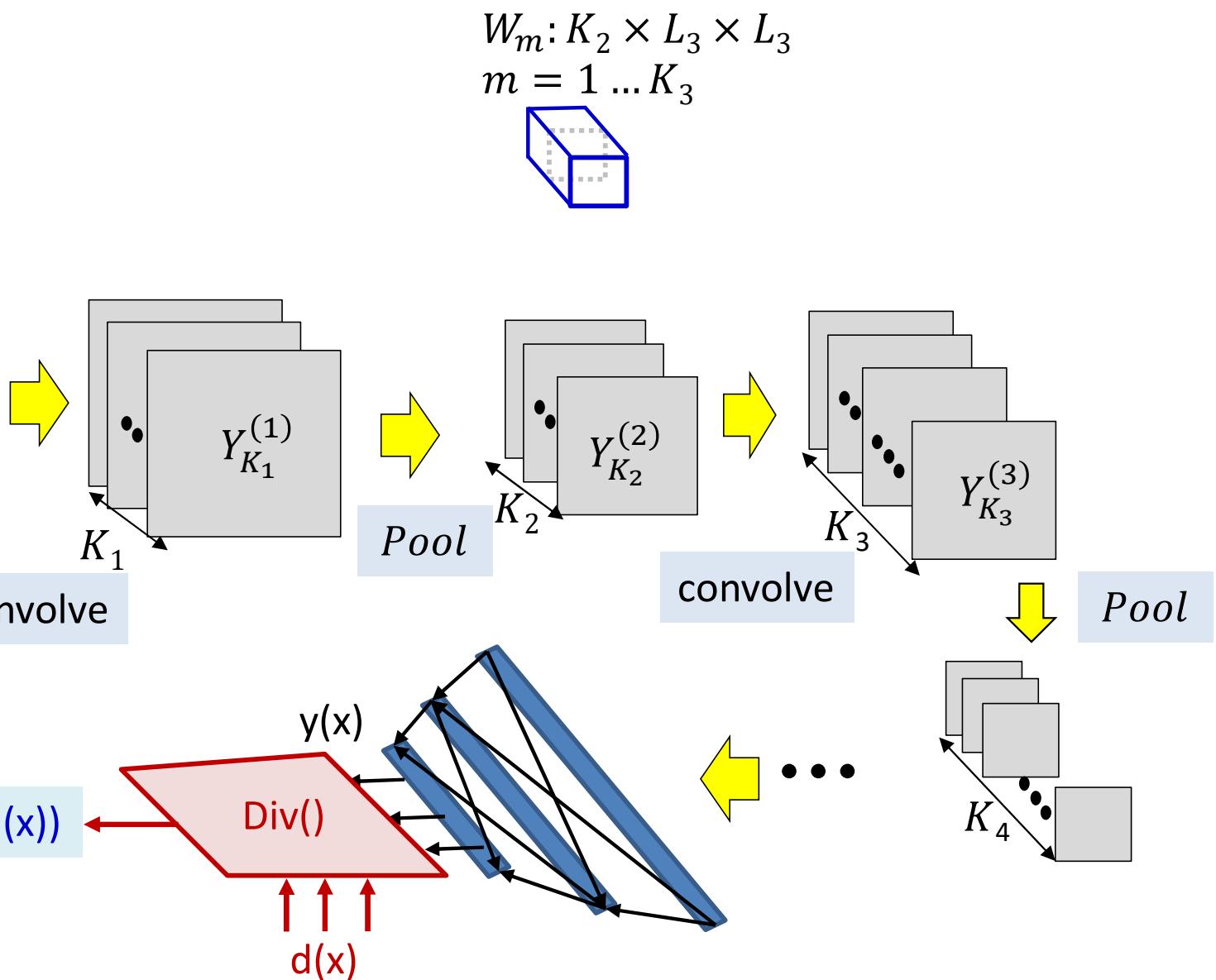
- Training is as in the case of the regular MLP
 - The *only* difference is in the *structure* of the network
- **Training examples of (Image, class) are provided**
- **Define a loss:**
 - Define a divergence between the desired output and true output of the network in response to any input
 - The loss aggregates the divergences of the training set
- **Network parameters are trained to minimize the loss**
 - Through variants of gradient descent
 - Gradients are computed through backpropagation

Defining the loss

$$W_m: 3 \times L \times L \\ m = 1 \dots K_1$$



Input: x



- The loss for a single instance

Problem Setup

- Given a training set of input-output pairs $(X_1, d_1), (X_2, d_2), \dots, (X_T, d_T)$
- The divergence on the i^{th} instance is $\text{div}(Y_i, d_i)$
- The aggregate Loss

$$\textit{Loss} = \frac{1}{T} \sum_{i=1}^T \text{div}(Y_i, d_i)$$

- Minimize \textit{Loss} w.r.t $\{W_m, b_m\}$
 - Using gradient descent

The derivative

Total training loss:

$$Loss = \frac{1}{T} \sum_i Div(Y_i, d_i)$$

- Computing the derivative

Total derivative:

$$\frac{dLoss}{dw} = \frac{1}{T} \sum_i \frac{dDiv(Y_i, d_i)}{dw}$$

The derivative

Total training loss:

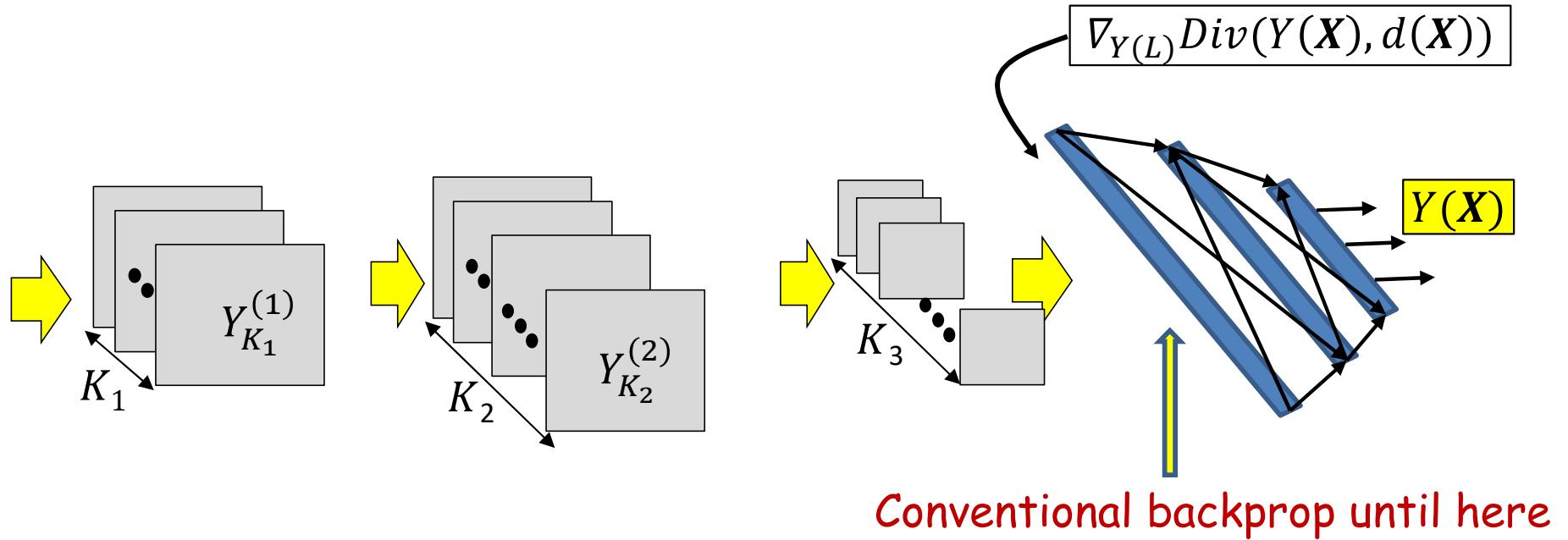
$$Loss = \frac{1}{T} \sum_i Div(Y_i, d_i)$$

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Total derivative:

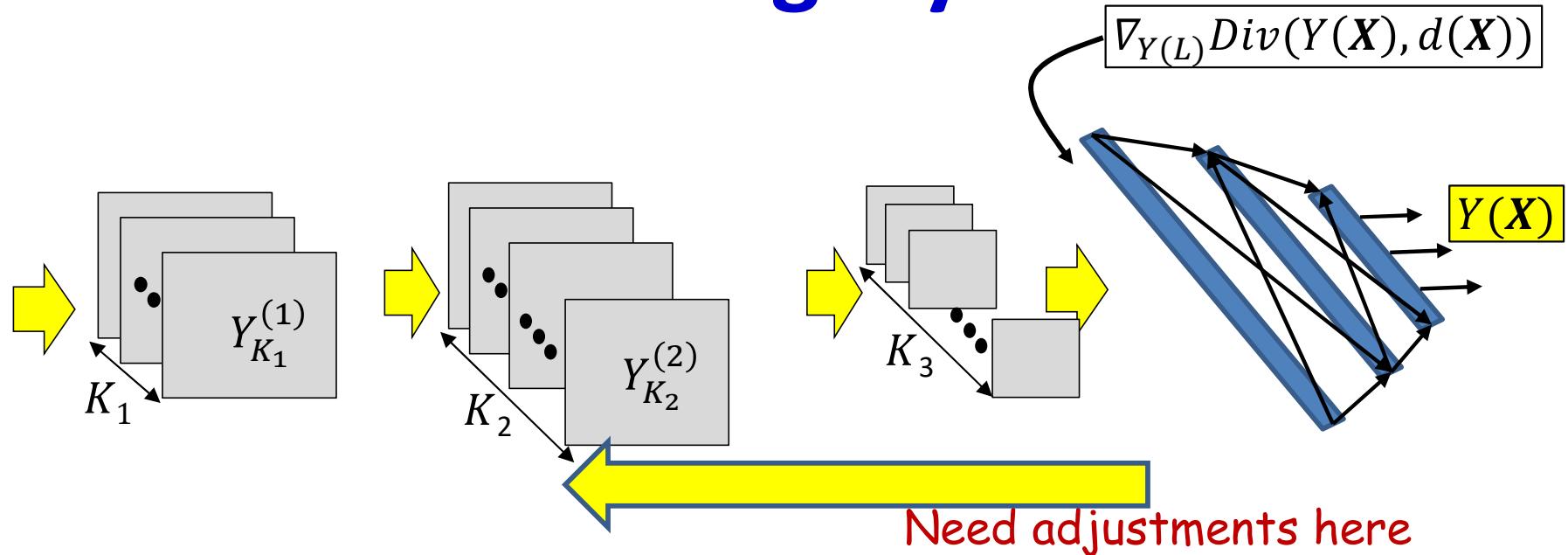
$$\frac{dLoss}{dw} = \frac{1}{T} \sum_i \frac{dDiv(Y_i, d_i)}{dw}$$

Backpropagation: Final flat layers



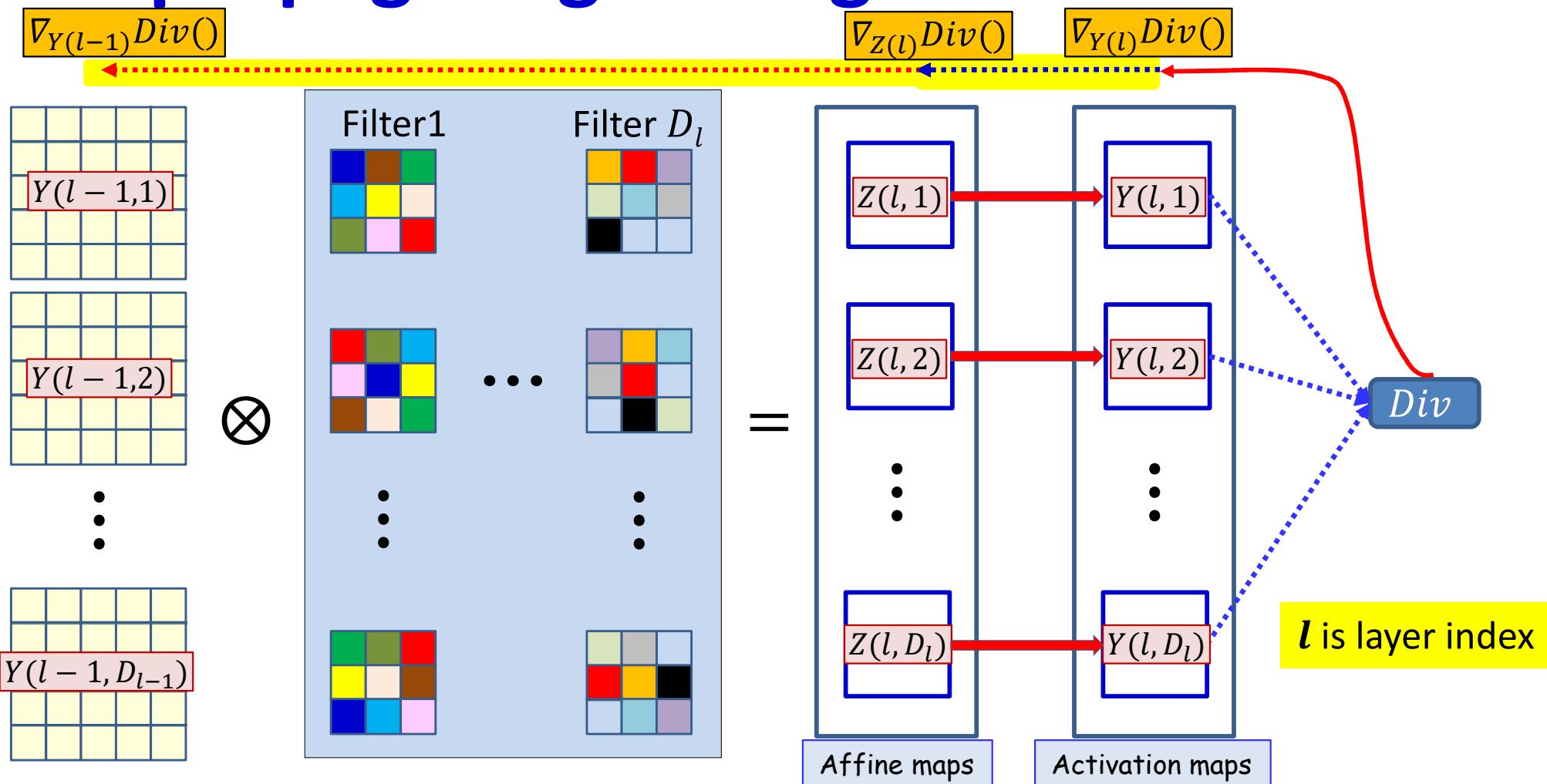
- For each training instance:
 - First, perform a forward pass through the net
 - Then perform the backpropagation of the derivative of the divergence
- Backpropagation continues in the usual manner until the computation of the derivative of the divergence w.r.t the inputs to the first “flat” layer
 - Important to recall: the first flat layer is only the “unrolling” of the maps from the final convolutional layer

Backpropagation: Convolutional and Pooling layers



- Backpropagation from the flat MLP requires special consideration of
 - The shared computation in the convolution layers
 - The pooling layers

Backpropagating through the convolution

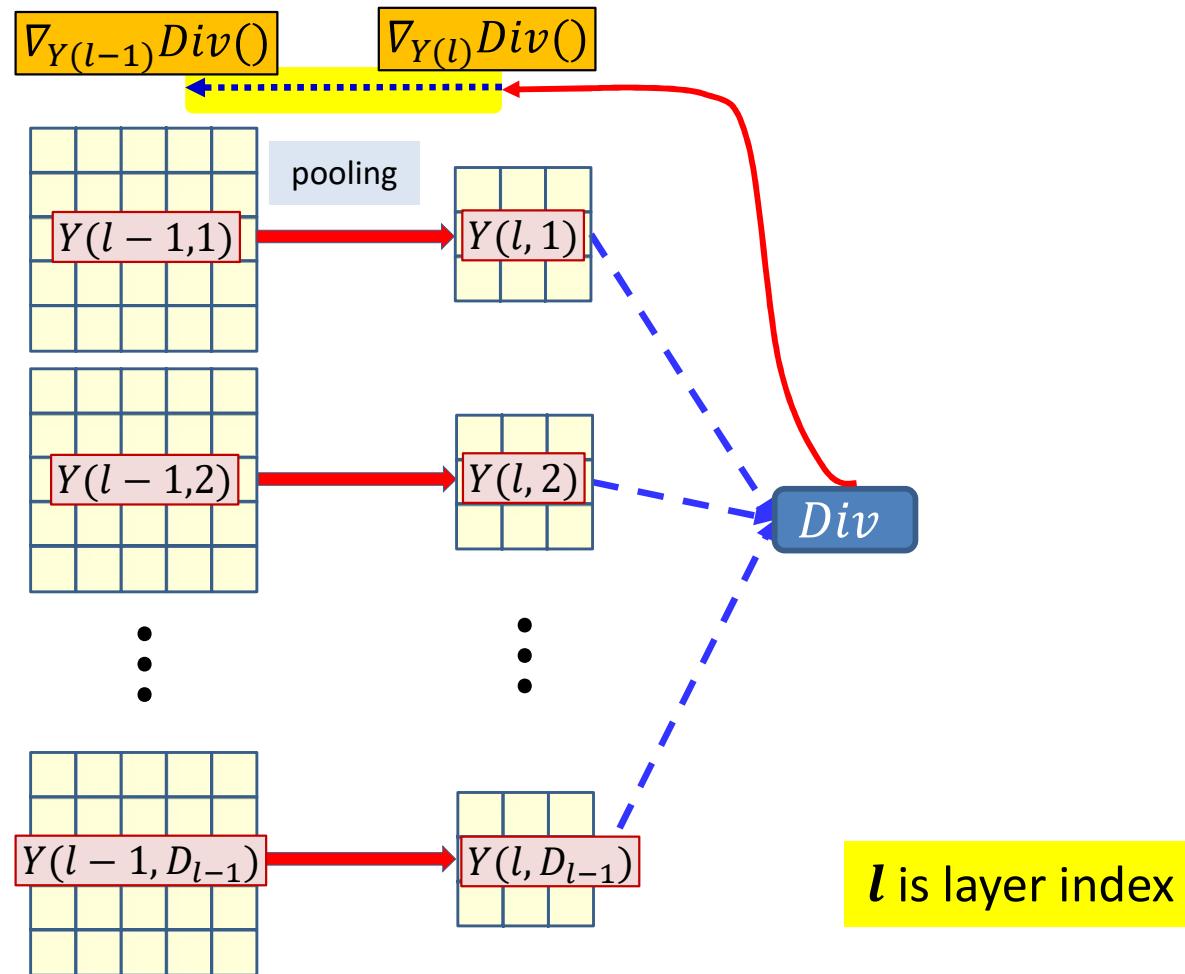


- **Convolution layers:**
- We already have the derivative w.r.t (all the elements of) activation map $Y(l, \cdot)$
 - Having backpropagated it from the divergence
- We must backpropagate it through the activation to compute the derivative w.r.t. $Z(l, \cdot)$ and further back to compute the derivative w.r.t the filters and $Y(l - 1, \cdot)$

Backpropagation: Convolutional layers

- **Assumption:** We already have the derivatives w.r.t. the elements of the maps output by the final convolutional (or pooling) layer
 - Obtained as a result of backpropagating through the flat MLP
- **Required:**
 - **For convolutional layers:**
 - How to compute the derivatives w.r.t. the affine combination $Z(l)$ maps from the activation output maps $Y(l)$
 - How to compute the derivative w.r.t. $Y(l - 1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$

Backprop: Pooling layer



- **Pooling layers:**
- We already have the derivative w.r.t $Y(l, *)$
 - Having backpropagated it from the divergence
- We must compute the derivative w.r.t $Y(l - 1, *)$

Backpropagation: Convolutional and Pooling layers

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 - **For pooling layers:**
 - How to compute the derivative w.r.t. $Y(l - 1)$ given derivatives w.r.t. $Y(l)$

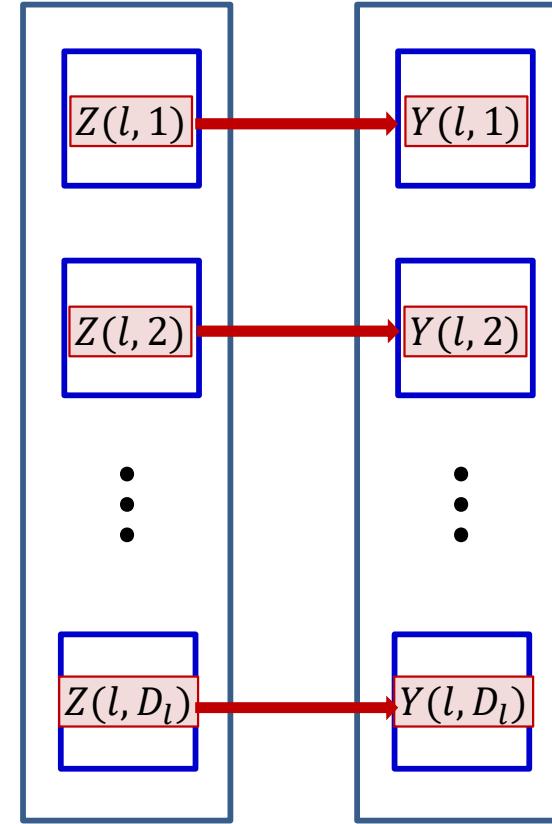
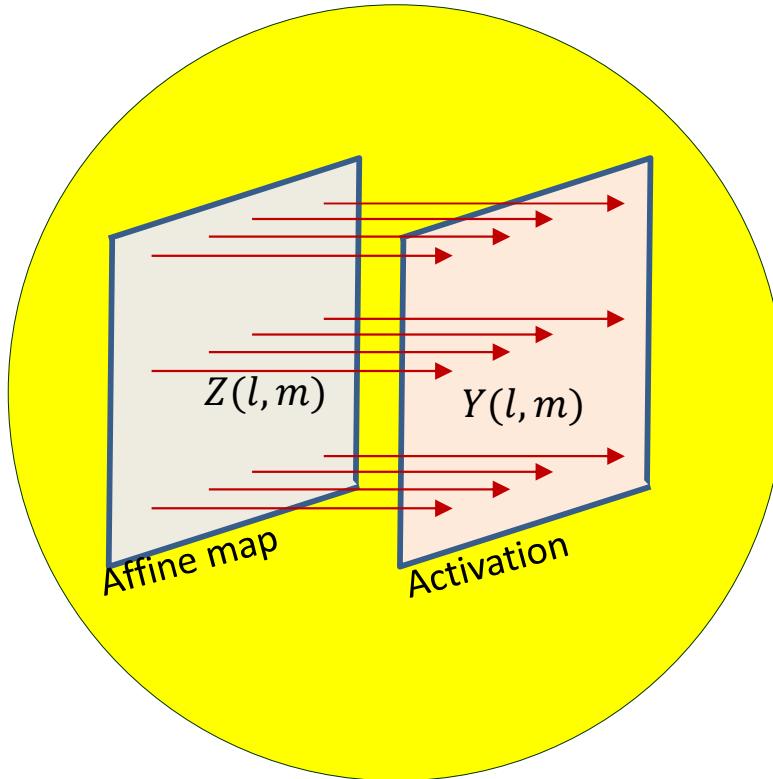
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 - **For pooling layers:**
 - How to compute the derivative w.r.t. $Y(l - 1)$ given derivatives w.r.t. $Y(l)$

Backpropagation: Convolutional and Pooling layers

- **Assumption:** We already have the derivatives w.r.t. the elements of the maps output by the final convolutional (or pooling) layer
 - Obtained as a result of backpropagating through the flat MLP
- **Required:**
 - **For convolutional layers:**
 - How to compute the derivatives w.r.t. the affine combination $Z(l)$ maps from the activation output maps $Y(l)$
 - How to compute the derivative w.r.t. $Y(l - 1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$
 - **For pooling layers:**
 - How to compute the derivative w.r.t. $Y(l - 1)$ given derivatives w.r.t. $Y(l)$

Backpropagating through the activation

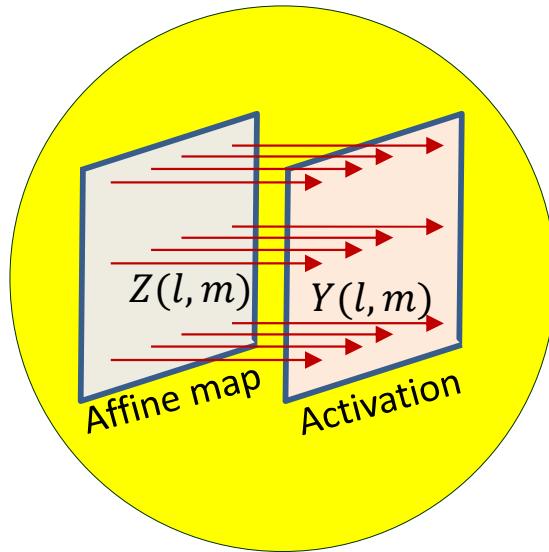


- **Forward computation:** The activation maps are obtained by point-wise application of the activation function to the affine maps

$$y(l, m, x, y) = f(z(l, m, x, y))$$

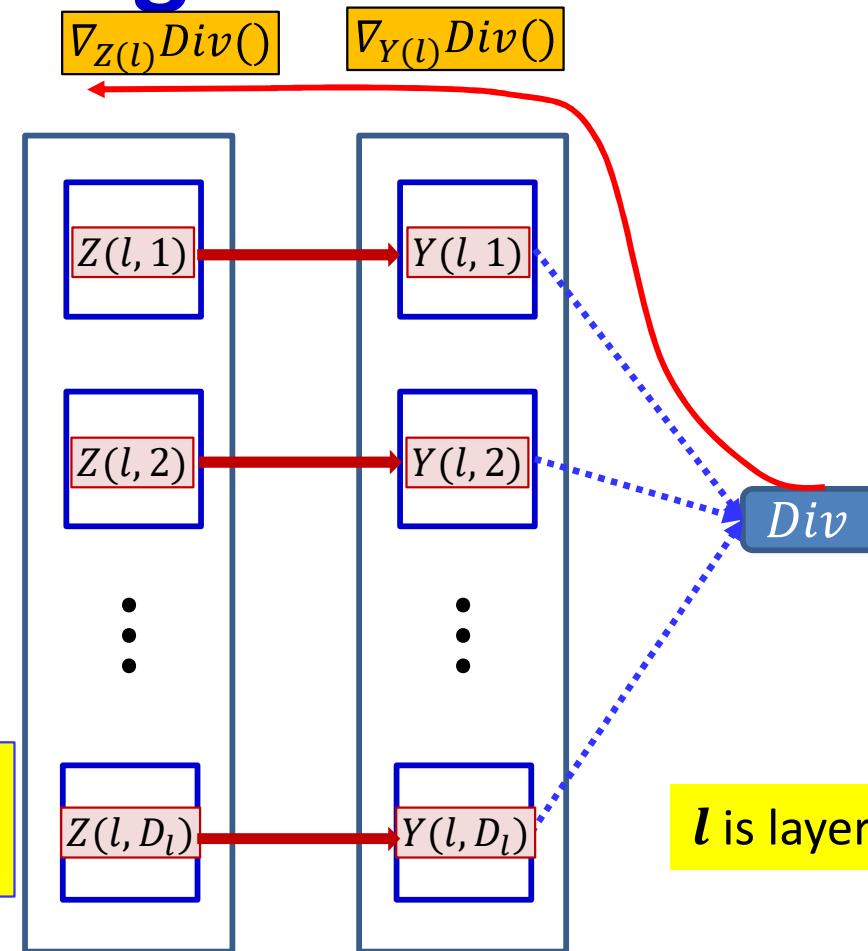
- The affine map entries $z(l, m, x, y)$ have already been computed via convolutions over the previous layer

Backpropagating through the activation



$$y(l, m, x, y) = f(z(l, m, x, y))$$

$$\frac{d\text{Div}}{dz(l, m, x, y)} = \frac{d\text{Div}}{dy(l, m, x, y)} f'(z(l, m, x, y))$$



l is layer index

- **Backward computation:** For every map $Y(l, m)$ for every position (x, y) , we already have the derivative of the divergence w.r.t. $y(l, m, x, y)$
 - Obtained via backpropagation
- We obtain the derivatives of the divergence w.r.t. $z(l, m, x, y)$ using the chain rule:

$$\frac{d\text{Div}}{dz(l, m, x, y)} = \frac{d\text{Div}}{dy(l, m, x, y)} f'(z(l, m, x, y))$$

- Simple component-wise computation

Backpropagation: Convolutional and Pooling layers

- **Assumption:** We already have the derivatives w.r.t. the elements of the maps output by the final convolutional (or pooling) layer
 - Obtained as a result of backpropagating through the flat MLP
- **Required:**
 - **For convolutional layers:**
 - ✓ How to compute the derivatives w.r.t. the affine combination $Z(l)$ maps from the activation output maps $Y(l)$
 - How to compute the derivative w.r.t. $Y(l - 1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$
 - **For pooling layers:**
 - How to compute the derivative w.r.t. $Y(l - 1)$ given derivatives w.r.t. $Y(l)$

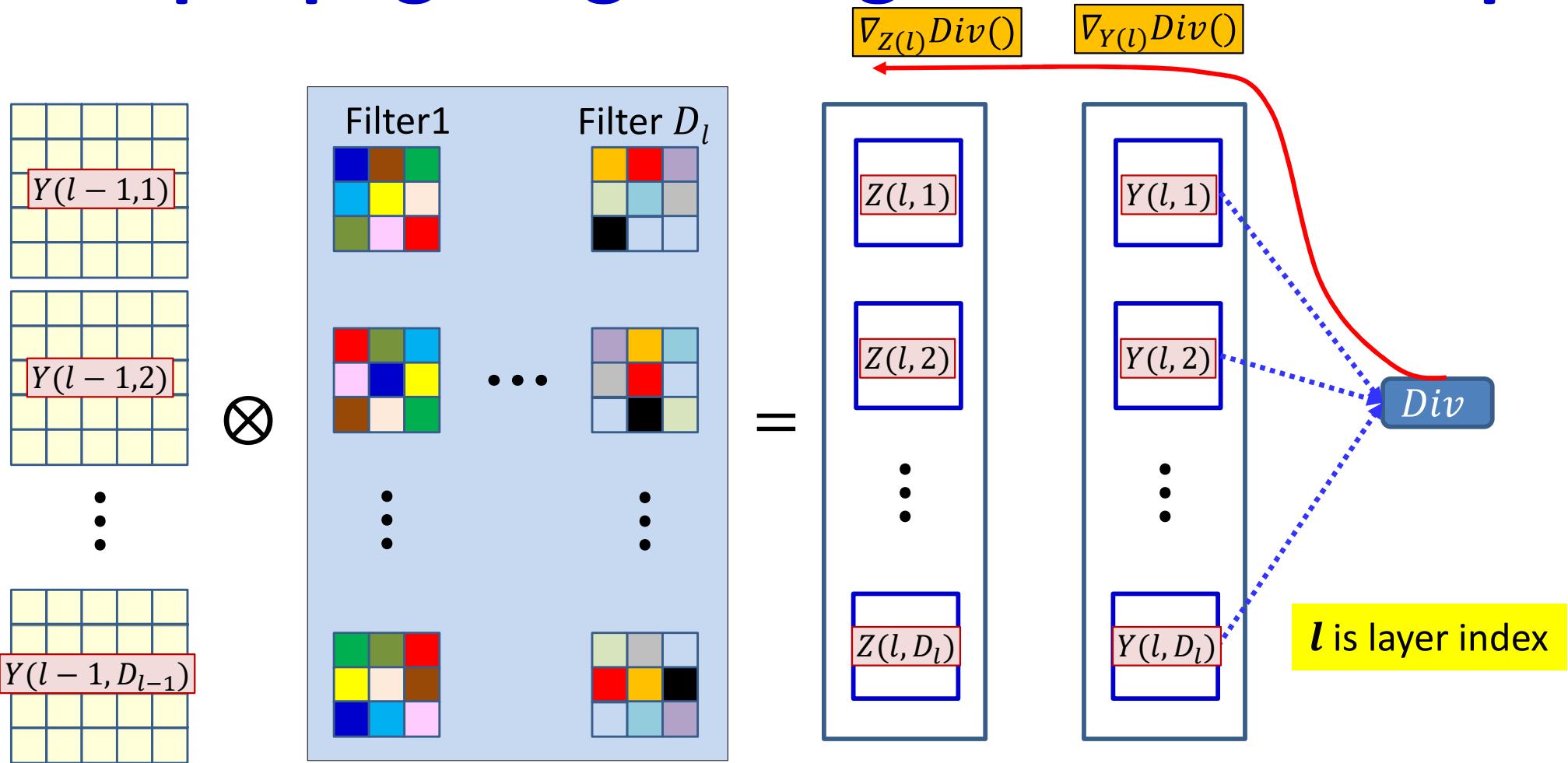
Backpropagating through affine map

- Forward affine computation:
 - Compute affine maps $z(l, n, x, y)$ from previous layer maps $y(l - 1, m, x, y)$ and filters $w_l(m, n, x, y)$
- Backpropagation: Given $\frac{dDiv}{dz(l,n,x,y)}$
 - Compute derivative w.r.t. $y(l - 1, m, x, y)$
 - Compute derivative w.r.t. $w_l(m, n, x, y)$

Backpropagating through affine map

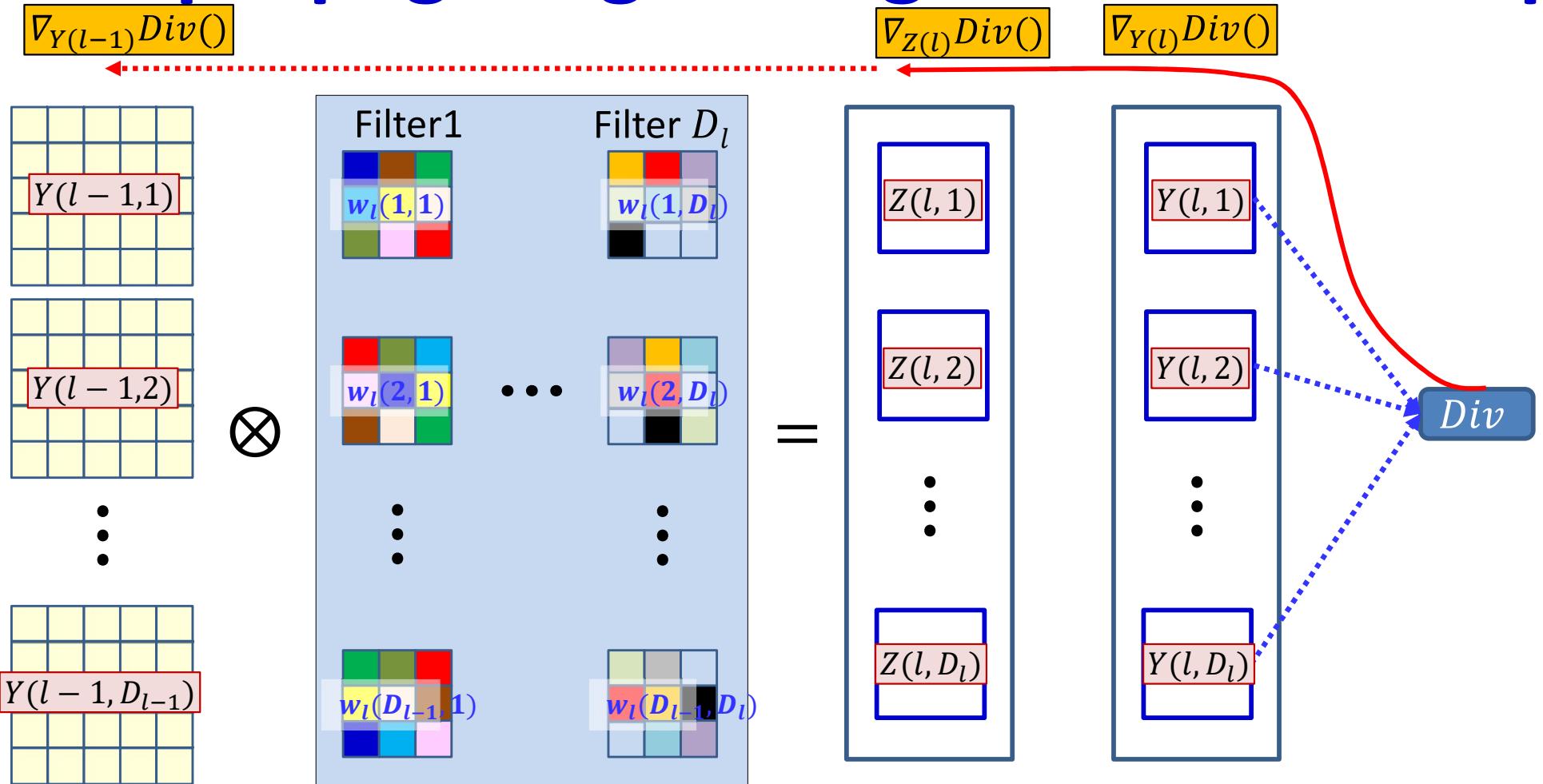
- Forward affine computation:
 - Compute affine maps $z(l, n, x, y)$ from previous layer maps $y(l - 1, m, x, y)$ and filters $w_l(m, n, x, y)$
- Backpropagation: Given $\frac{dDiv}{dz(l,n,x,y)}$
 - Compute derivative w.r.t. $y(l - 1, m, x, y)$
 - Compute derivative w.r.t. $w_l(m, n, x, y)$

Backpropagating through the affine map



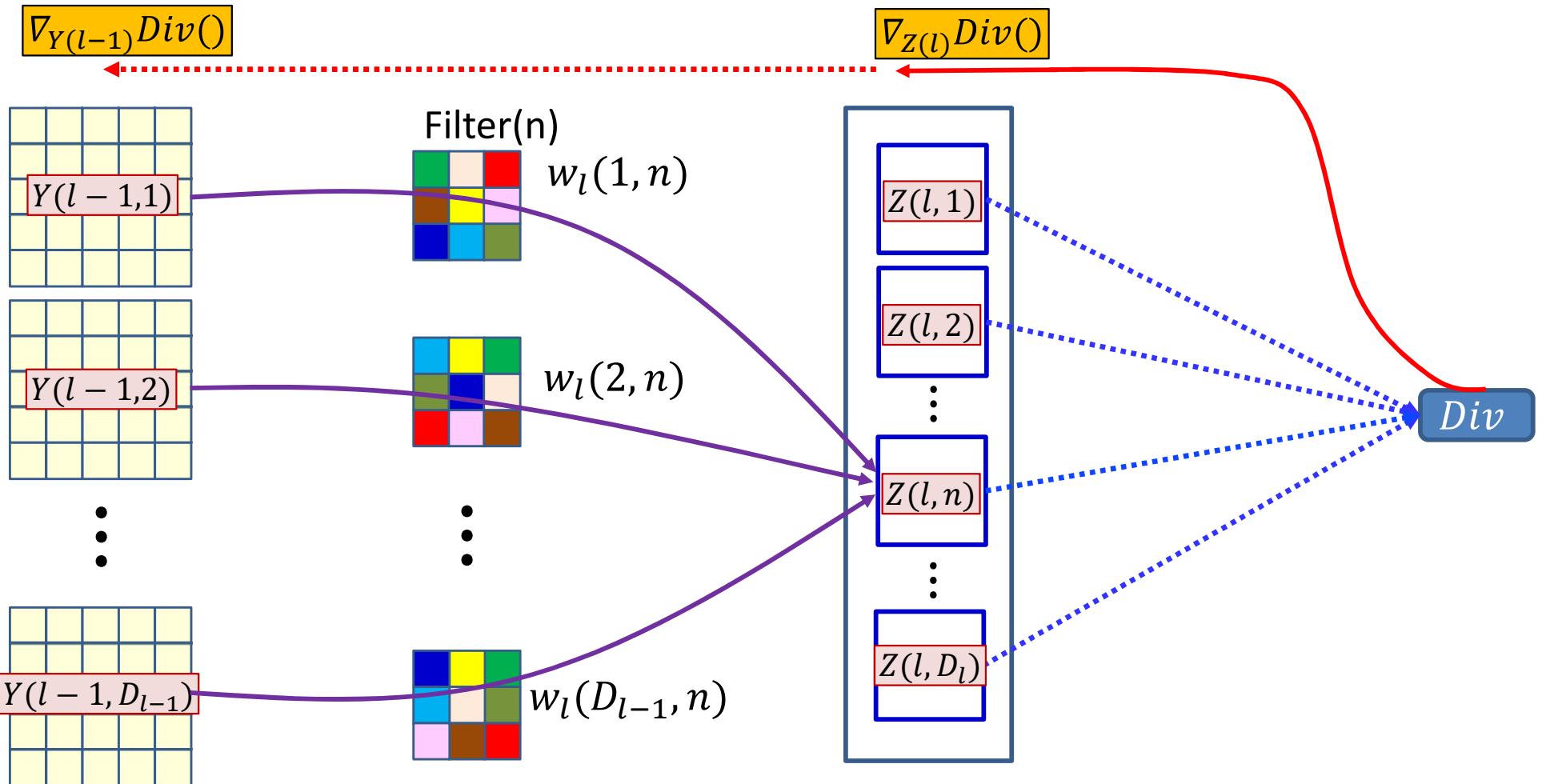
- We already have the derivative w.r.t $Z(l,*)$
 - Having backpropagated it past $Y(l,*)$

Backpropagating through the affine map



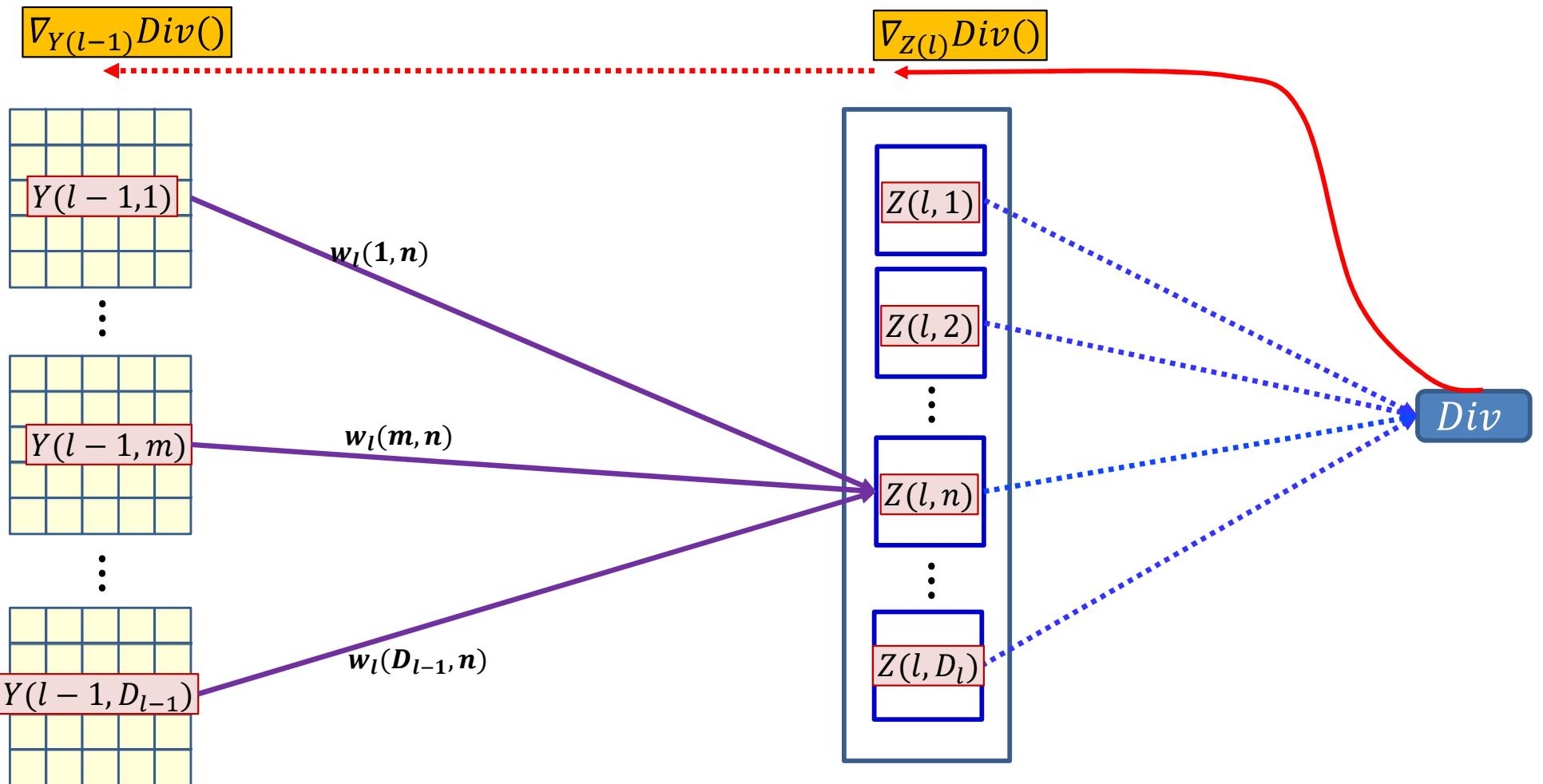
- We already have the derivative w.r.t $Z(l,*)$
 - Having backpropagated it past $Y(l,*)$
- We must compute the derivative w.r.t $Y(l-1,*)$

Dependency between $Z(l,n)$ and $Y(l-1,*)$



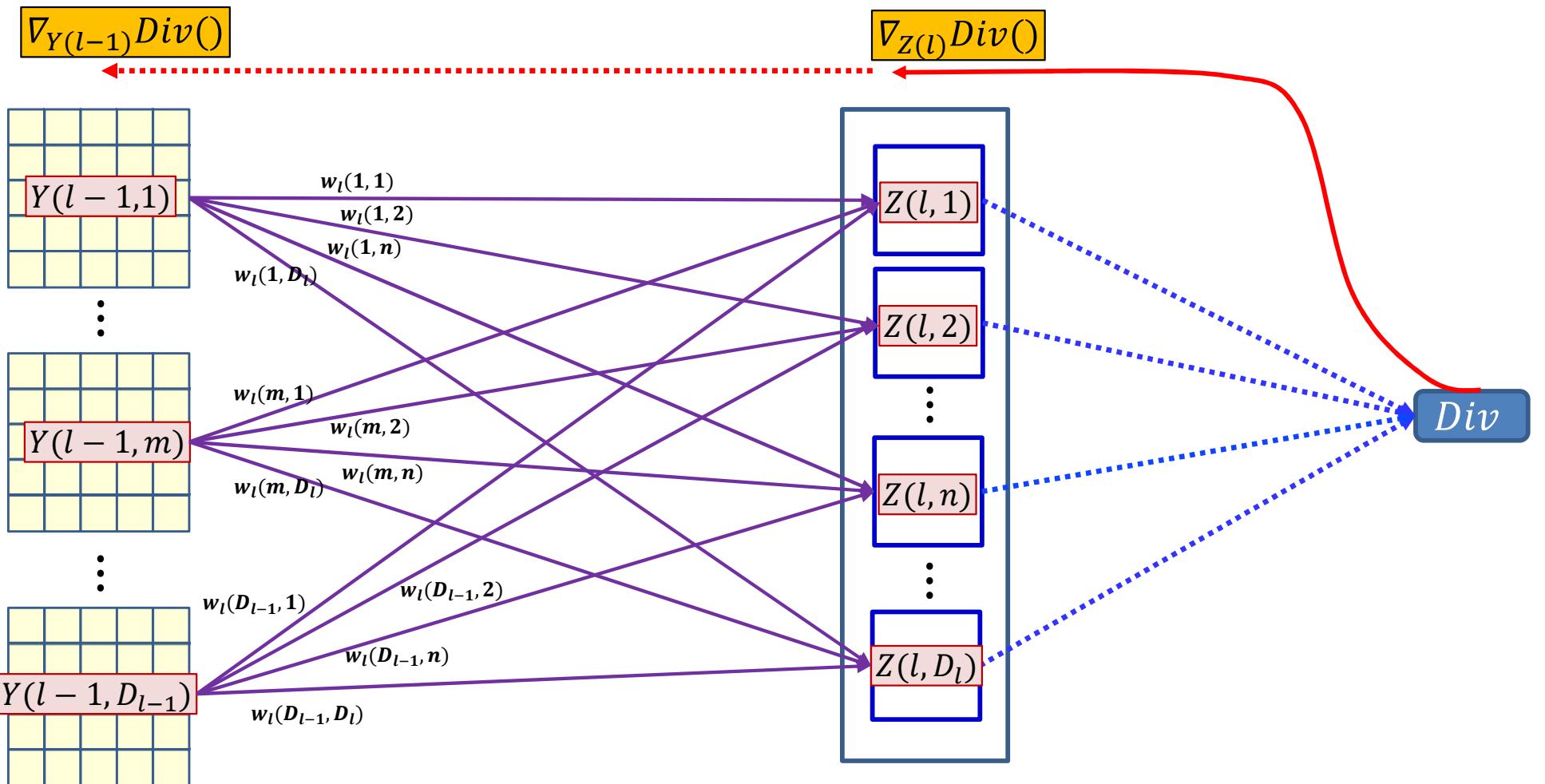
- Each $Y(l - 1, m)$ map/channel influences $Z(l, n)$ through the m th channel of the n th filter $w_l(m, n)$

Dependency between $Z(l,n)$ and $Y(l-1,*)$



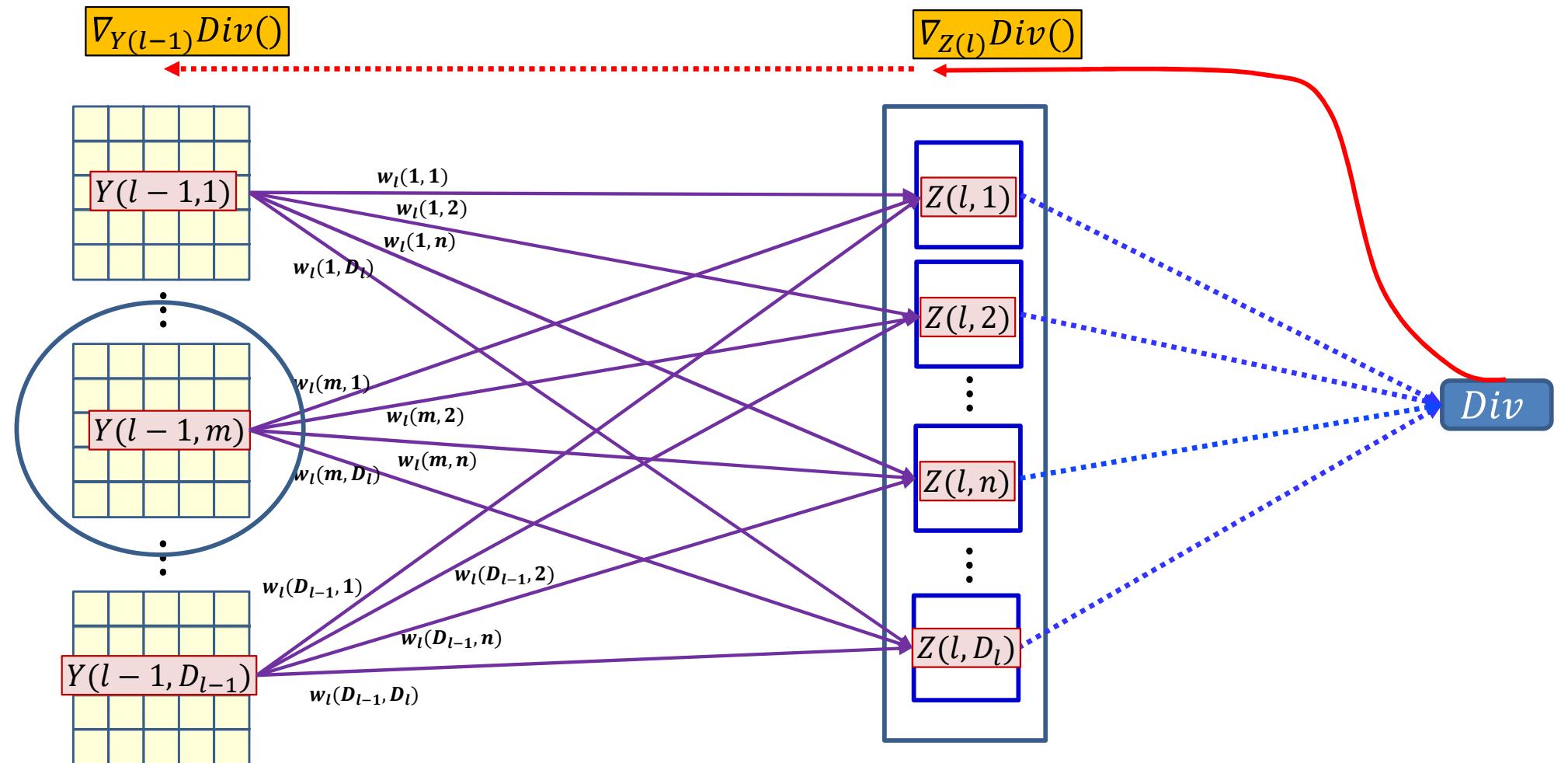
- Each $Y(l - 1, m)$ map/channel influences $Z(l, n)$ through the m th channel of the n th filter $w_l(m, n)$

Dependency between $Z(l, *)$ and $Y(l-1, *)$



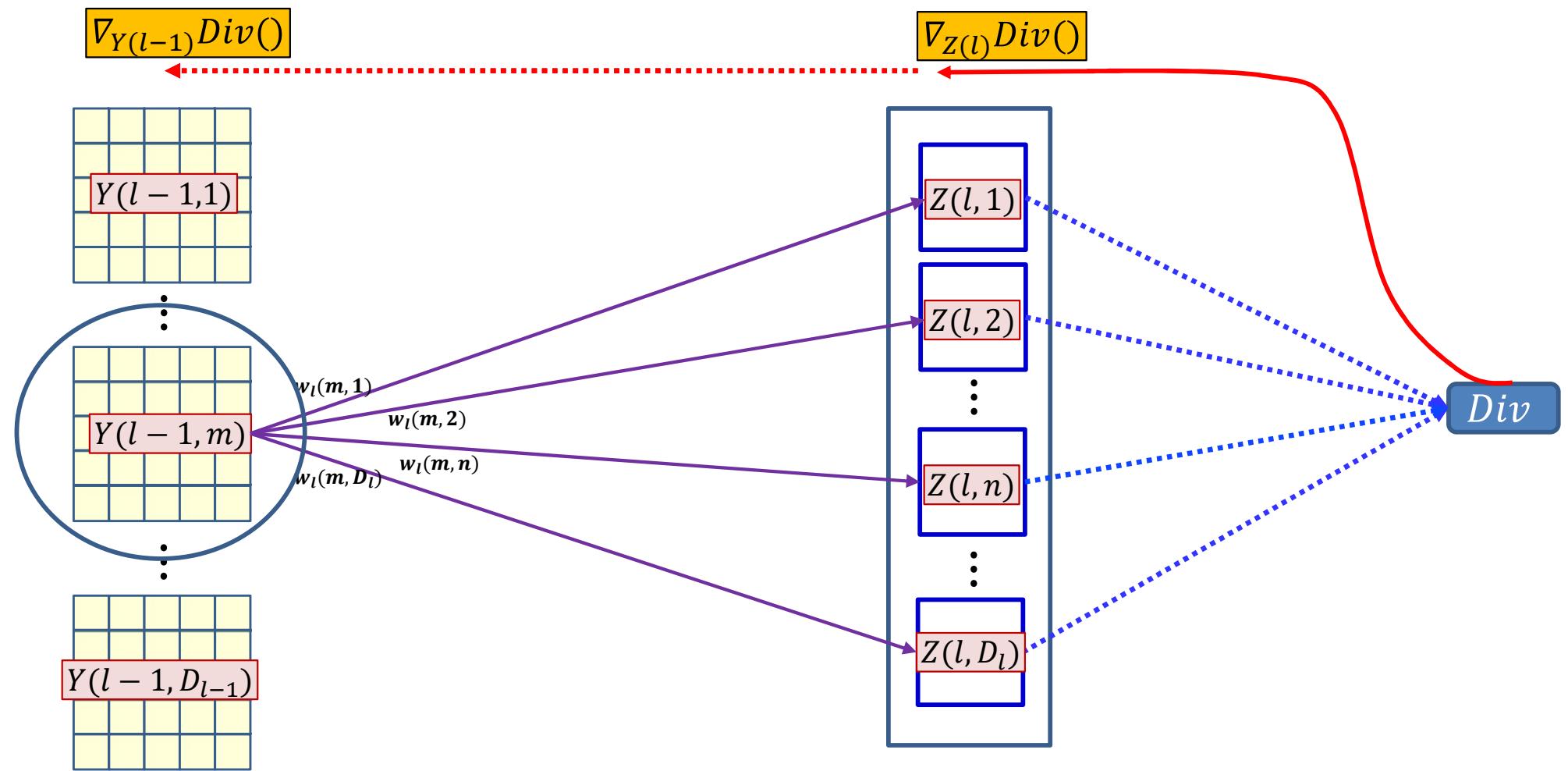
- Each $Y(l - 1, m)$ map/channel influences $Z(l, n)$ through the m th channel of the n th filter $w_l(m, n)$

Dependency between $Z(l, *)$ and $Y(l-1, *)$



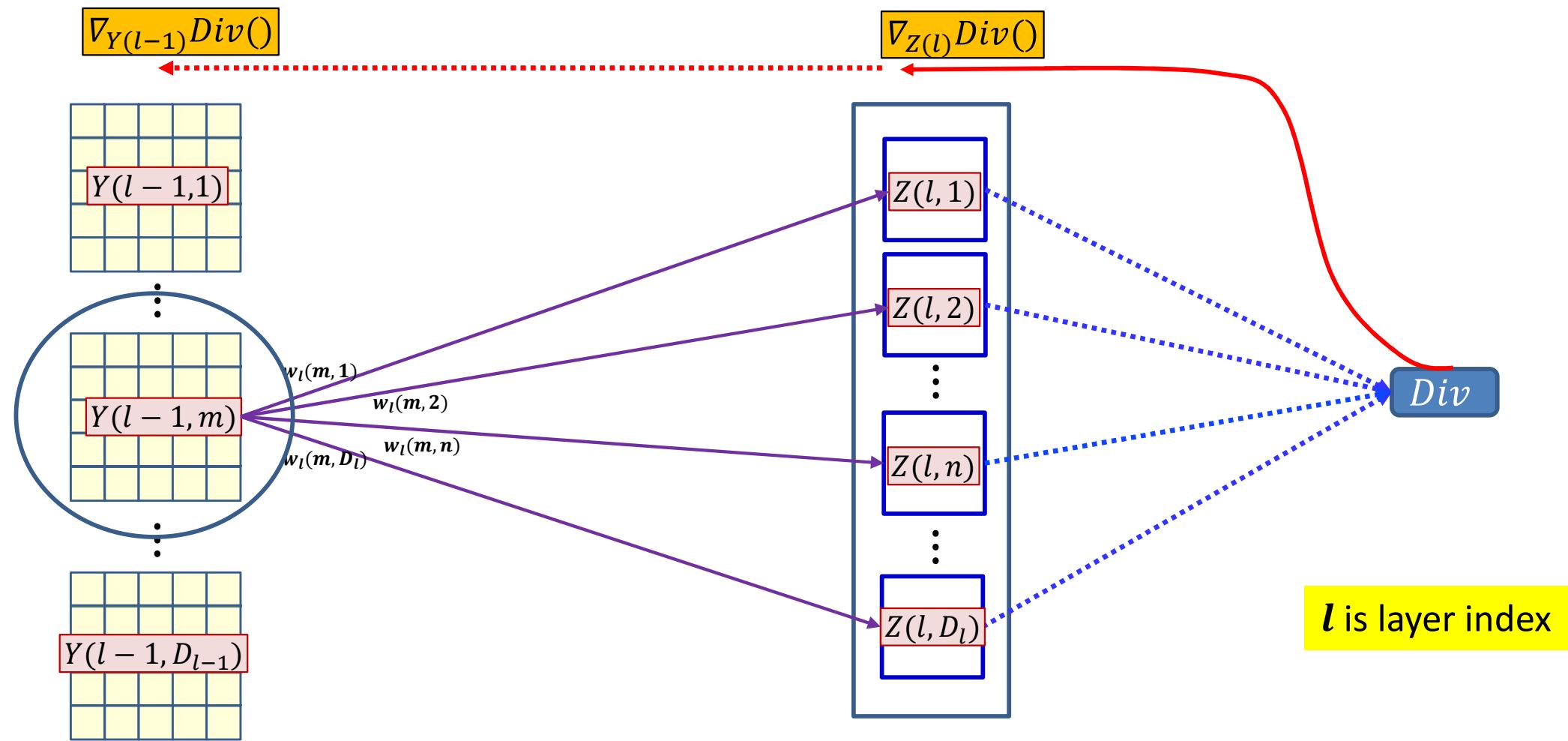
- Each $Y(l - 1, m)$ map/channel influences $Z(l, n)$ through the m th channel of the n th filter $w_l(m, n)$

Dependency diagram for a single map



- Each $Y(l - 1, m)$ map/channel influences $Z(l, n)$ through the m th channel of the n th filter $w_l(m, n)$
- $Y(l - 1, m, *, *)$ influences the divergence through all $Z(l, n, *, *)$ maps

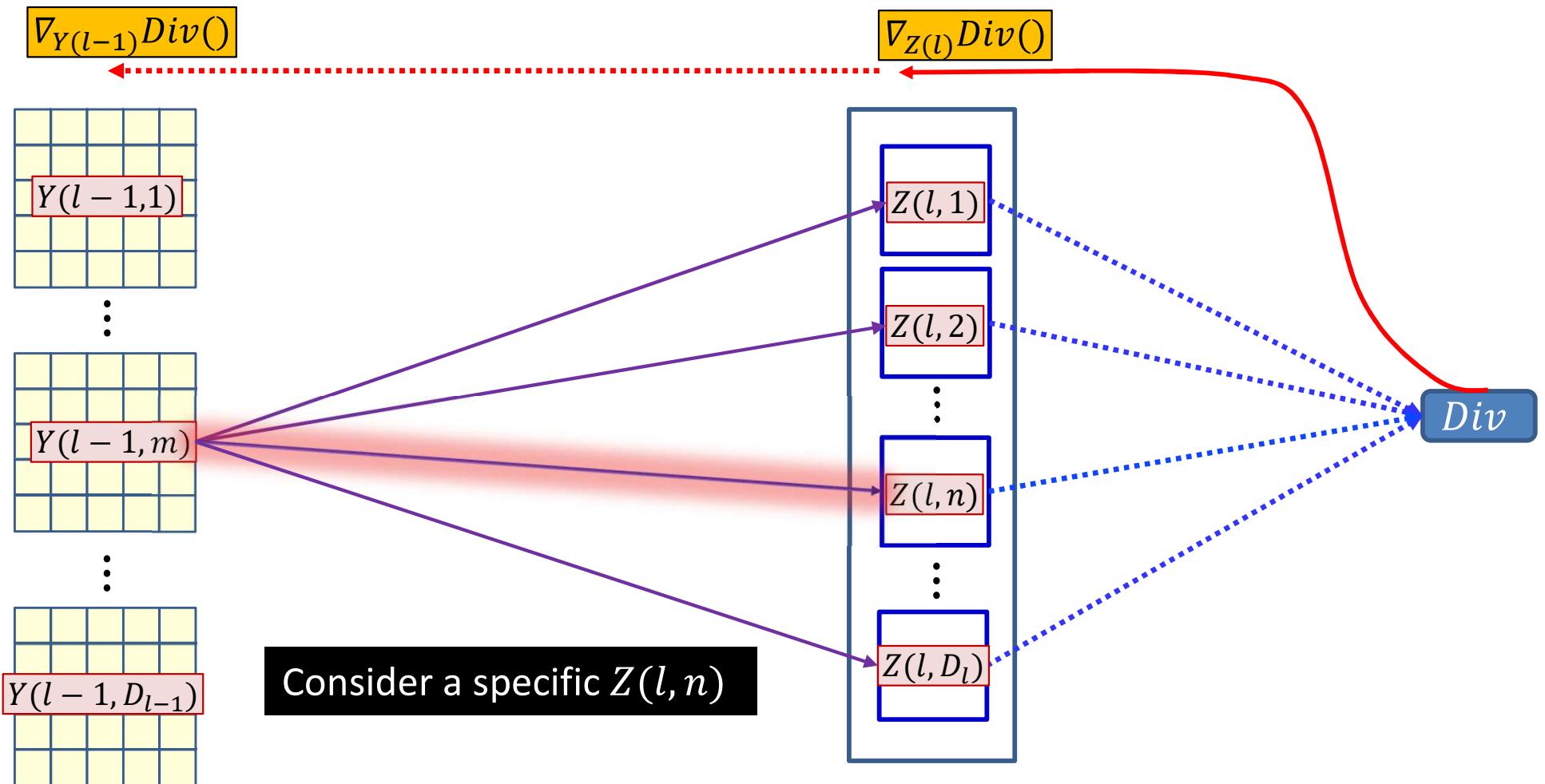
Dependency diagram for a single map



$$\nabla_{Y(l-1,m)} \text{Div}(\cdot) = \sum_n \underbrace{\nabla_{Z(l,n)} \text{Div}(\cdot) \nabla_{Y(l-1,m)} Z(l, n)}$$

- Need to compute $\nabla_{Y(l-1,m)} Z(l, n)$, the derivative of $Z(l, n)$ w.r.t. $Y(l - 1, m)$ to complete the computation of the formula

Dependency diagram for a single map



$$\nabla_{Y(l-1,m)} \text{Div}(.) = \sum_n \underbrace{\nabla_{Z(l,n)} \text{Div}(.)}_{\nabla_{Y(l-1,m)} Z(l,n)} \nabla_{Y(l-1,m)} Z(l,n)$$

- Need to compute $\nabla_{Y(l-1,m)} Z(l, n)$, the derivative of $Z(l, n)$ w.r.t. $Y(l - 1, m)$ to complete the computation of the formula

BP: Convolutional layer

1 <small>x1</small>	1 <small>x0</small>	1 <small>x1</small>	0	0
0 <small>x0</small>	1 <small>x1</small>	1 <small>x0</small>	1	0
0 <small>x1</small>	0 <small>x0</small>	1 <small>x1</small>	1	1
0	0	1	1	0
0	1	1	0	0

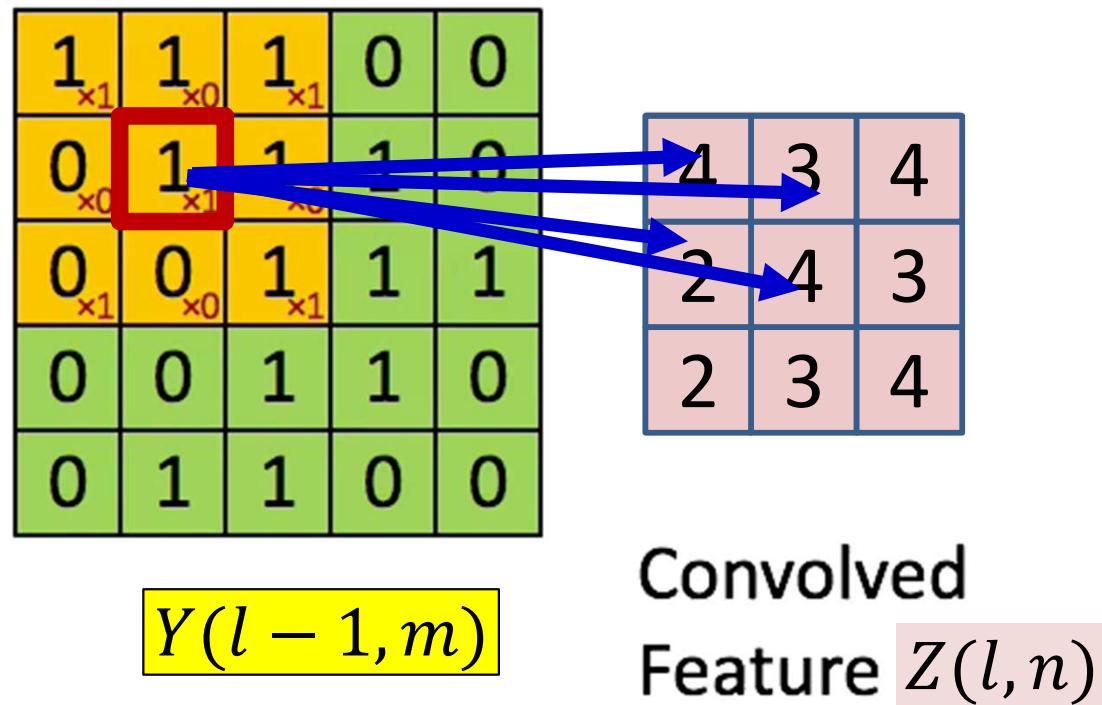
$$Y(l - 1, m)$$

4		

Convolved
Feature $Z(l, n)$

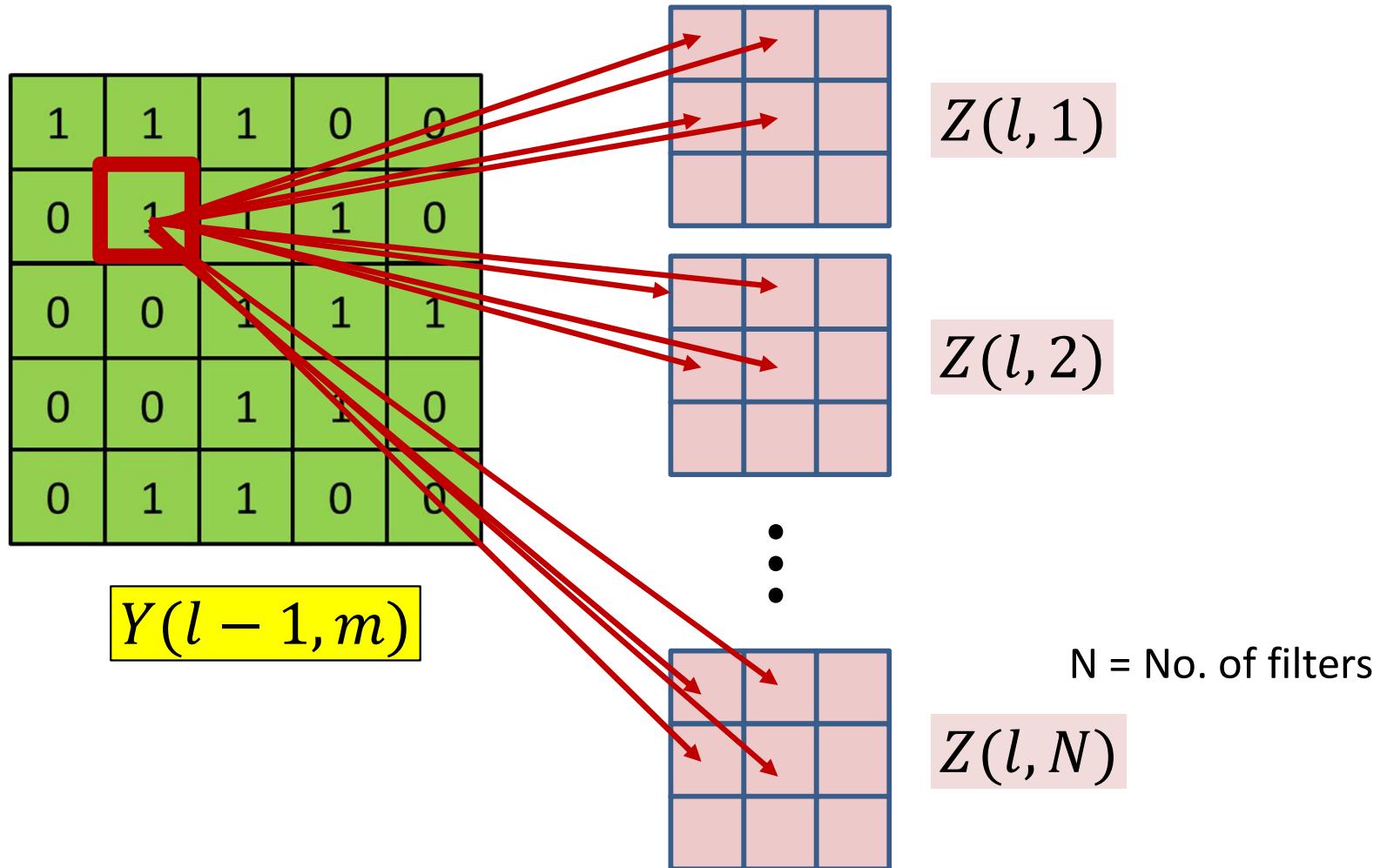
- Each $Y(l - 1, m, x, y)$ affects several $z(l, n, x', y')$ terms

BP: Convolutional layer



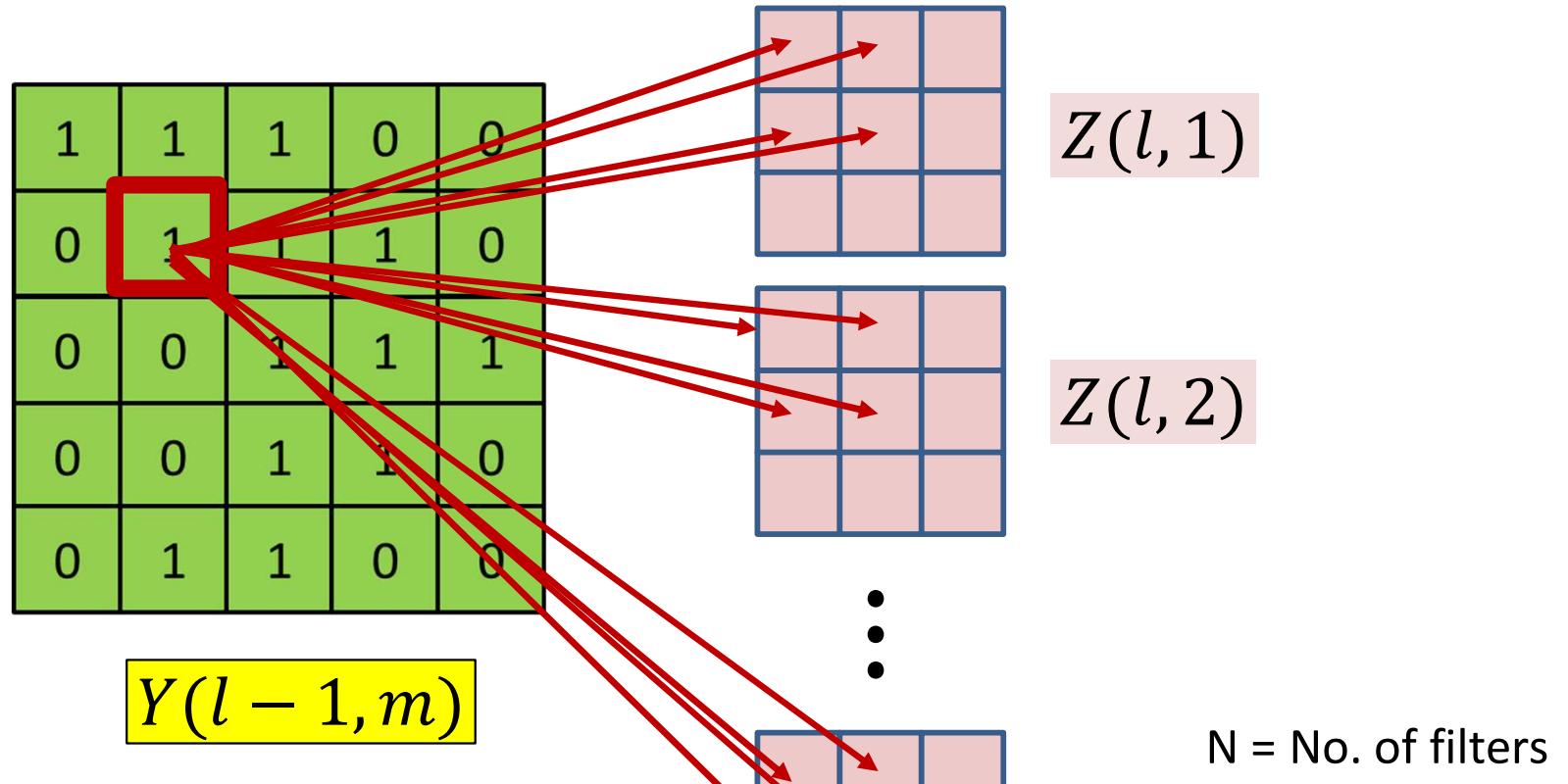
- Each $Y(l - 1, m, x, y)$ affects several $z(l, n, x', y')$ terms

BP: Convolutional layer



- Each $Y(l - 1, m, x, y)$ affects several $z(l, n, x', y')$ terms
 - Affects terms in *all* l^{th} layer Z maps

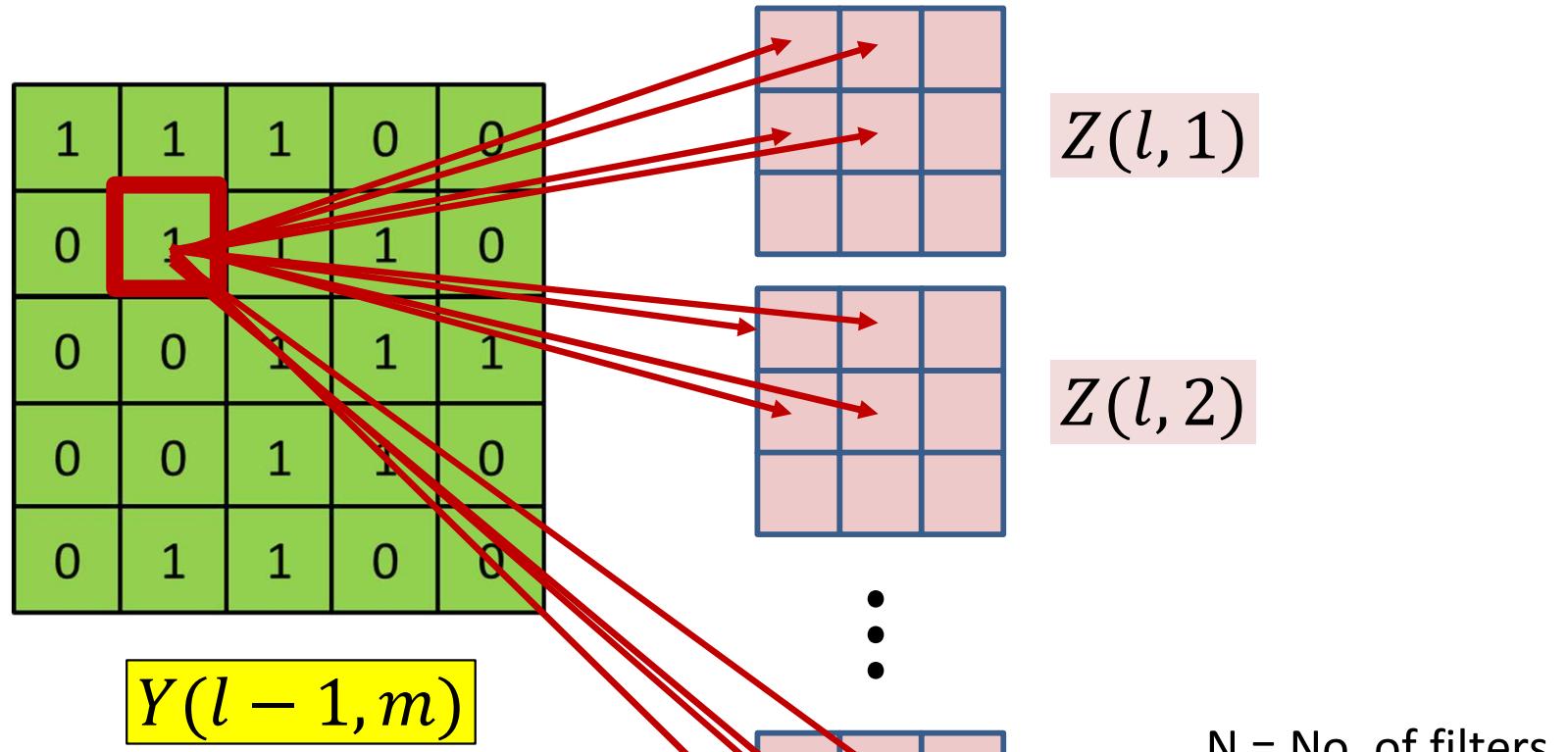
BP: Convolutional layer



Summing over all Z maps

$$\frac{dDiv}{dY(l - 1, m, x, y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l, n, x', y')} \frac{dz(l, n, x', y')}{dY(l - 1, m, x, y)}$$

BP: Convolutional layer



Summing over all Z maps

$$\frac{dDiv}{dY(l - 1, m, x, y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l, n, x', y')} \frac{dz(l, n, x', y')}{dY(l - 1, m, x, y)}$$

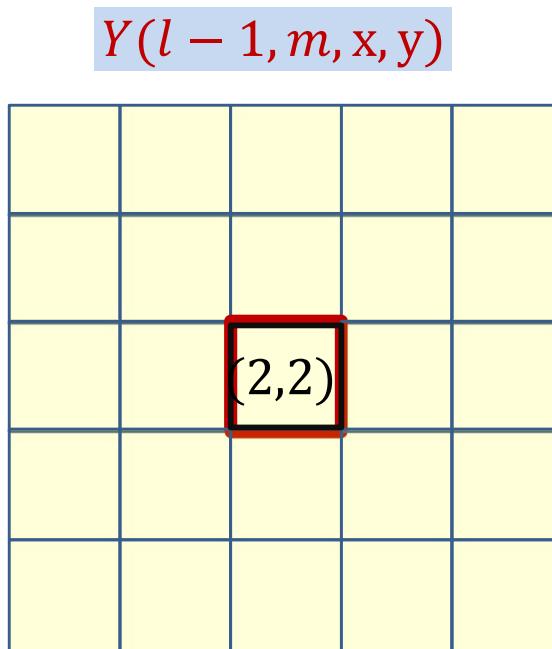
What is this?

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

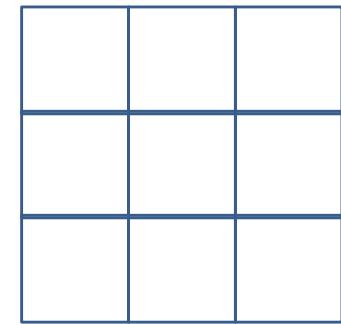
0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

$w_l(m, n, *, *)$

\otimes



$z(l, n, x', y')$



Assuming indexing
begins at 0

- Compute how *each* x, y in Y influences various locations of z

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

$w_l(m, n, *, *)$

\otimes

$Y(l - 1, m, x, y)$		
0,0		

(2,2)

$z(l, n, x', y')$

0,0		



$$z(l, n, 0, 0) += Y(l - 1, m, 2, 2)w_l(m, n, 2, 2)$$

- **Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l - 1, m)$

$$z(l, n, x', y') += Y(l - 1, m, 2, 2)w_l(m, n, 2 - x', 2 - y')$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

$w_l(m, n, *, *)$

\otimes

$Y(l - 1, m, x, y)$

		1,0		
			(2,2)	

$z(l, n, x', y')$

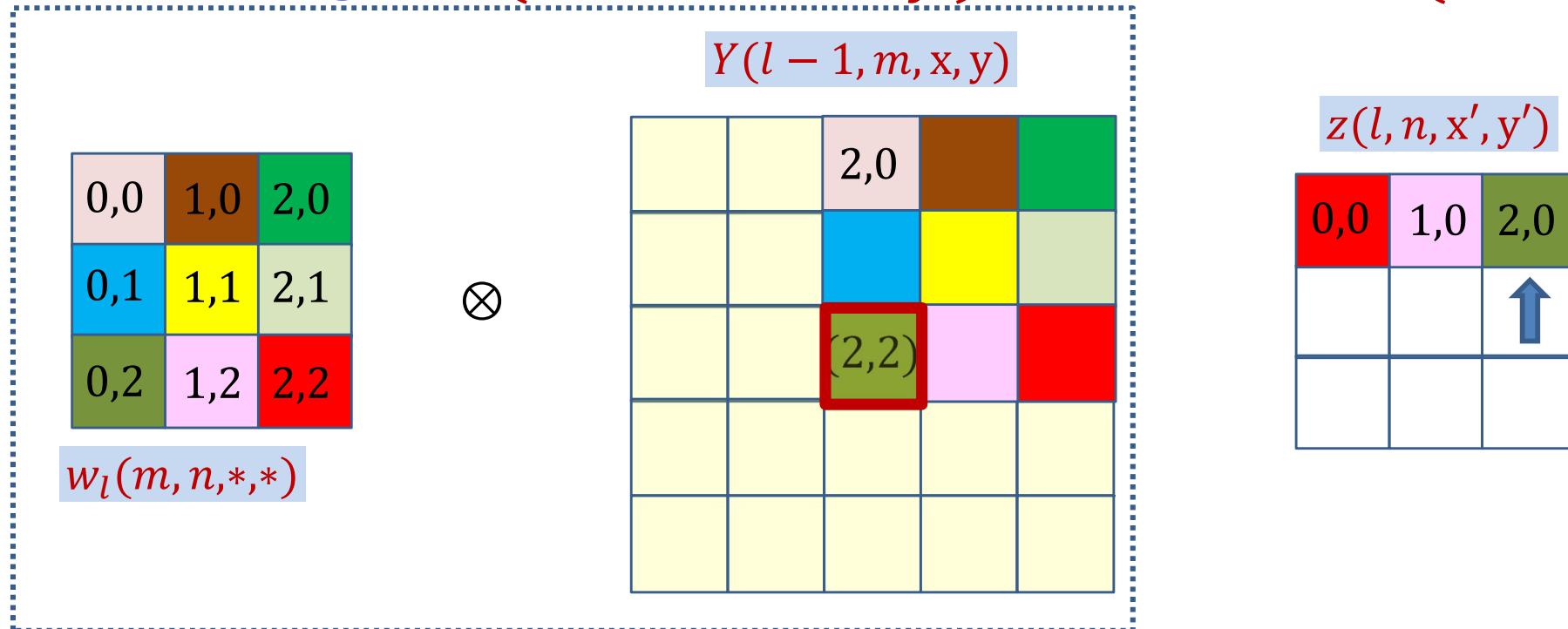
0,0	1,0	
		↑

$$z(l, n, 1,0) += Y(l - 1, m, 2,2)w_l(m, n, 1,2)$$

- **Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l - 1, m)$

$$z(l, n, x', y') += Y(l - 1, m, 2,2)w_l(m, n, 2 - x', 2 - y')$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 2,0) += Y(l - 1, m, 2,2)w_l(m, n, 0,2)$$

- **Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l - 1, m)$

$$z(l, n, x', y') += Y(l - 1, m, 2,2)w_l(m, n, 2 - x', 2 - y')$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

$w_l(m, n, *, *)$

\otimes

$Y(l - 1, m, x, y)$				
0,1				
		(2,2)		

$z(l, n, x', y')$

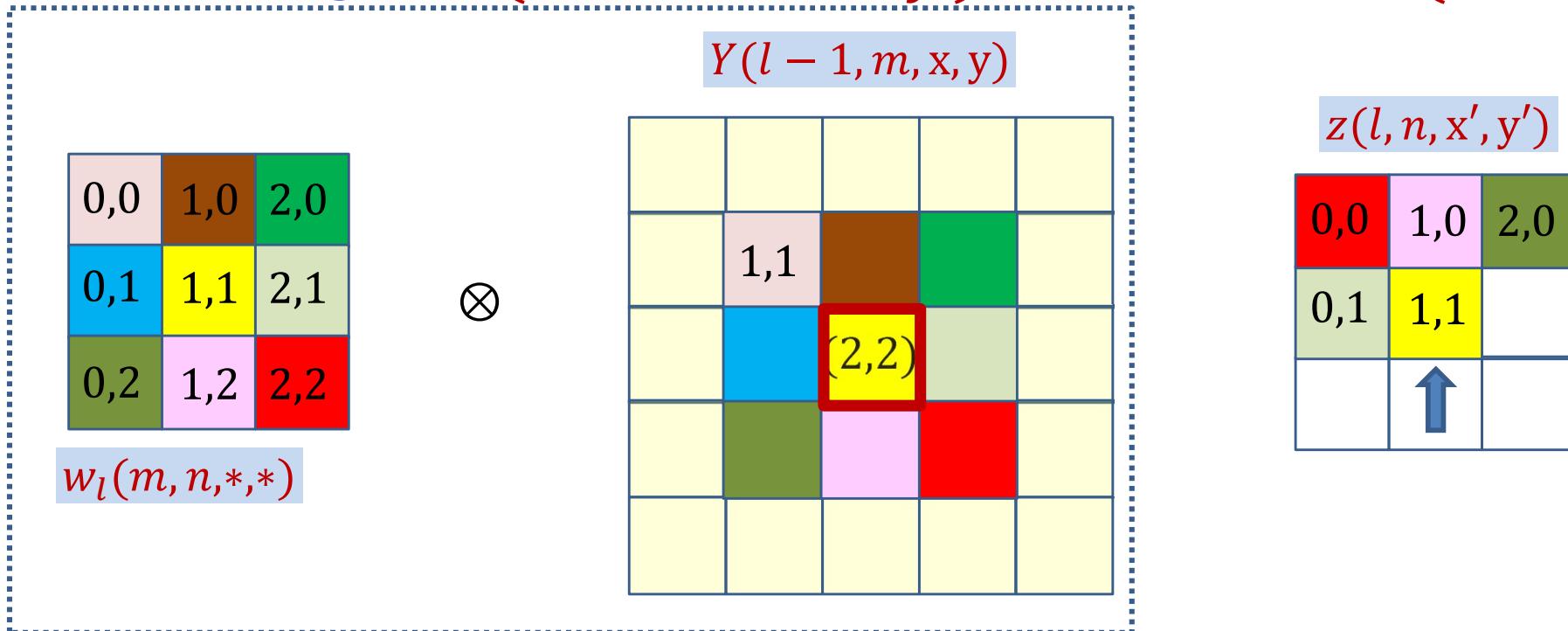
0,0	1,0	2,0
0,1		
↑		

$$z(l, n, 0, 1) += Y(l - 1, m, 2, 2)w_l(m, n, 2, 1)$$

- **Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l - 1, m)$

$$z(l, n, x', y') += Y(l - 1, m, 2, 2)w_l(m, n, 2 - x', 2 - y')$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

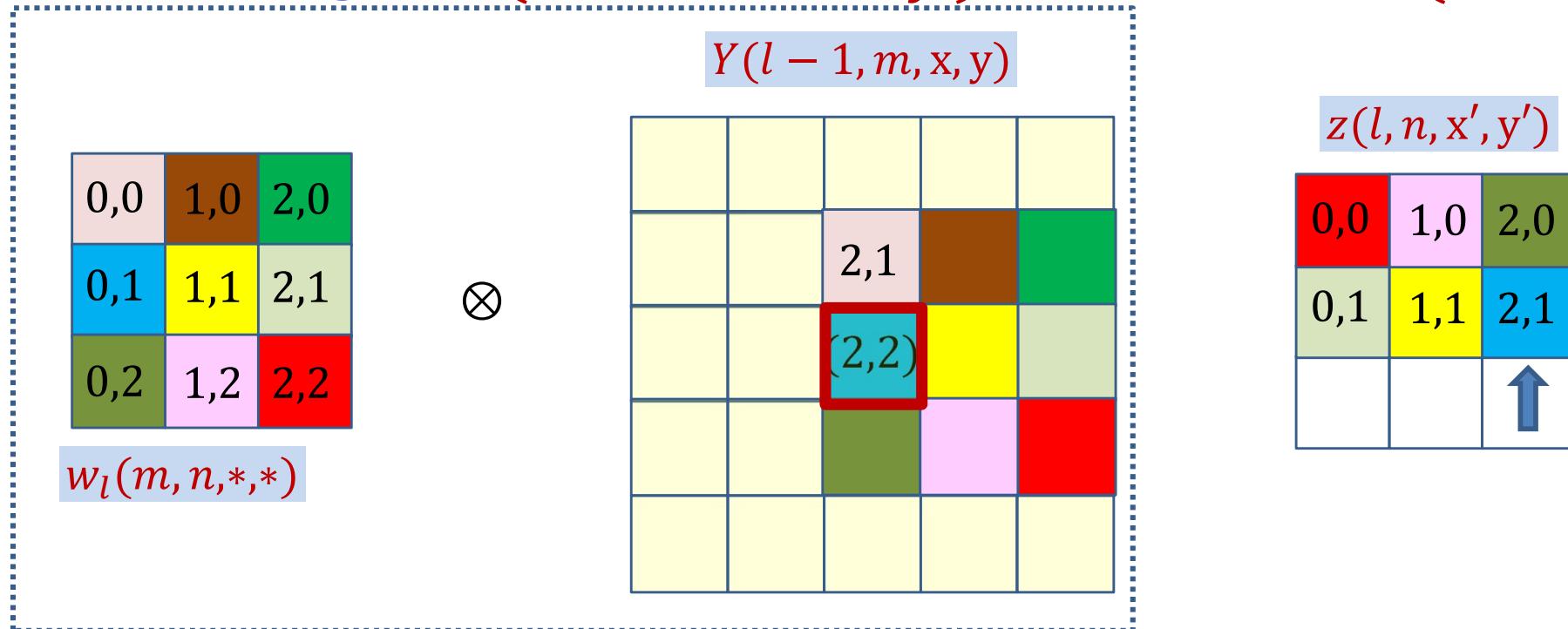


$$z(l, n, 1, 1) += Y(l - 1, m, 2, 2) w_l(m, n, 1, 1)$$

- **Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l - 1, m)$

$$z(l, n, x', y') += Y(l - 1, m, 2, 2) w_l(m, n, 2 - x', 2 - y')$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

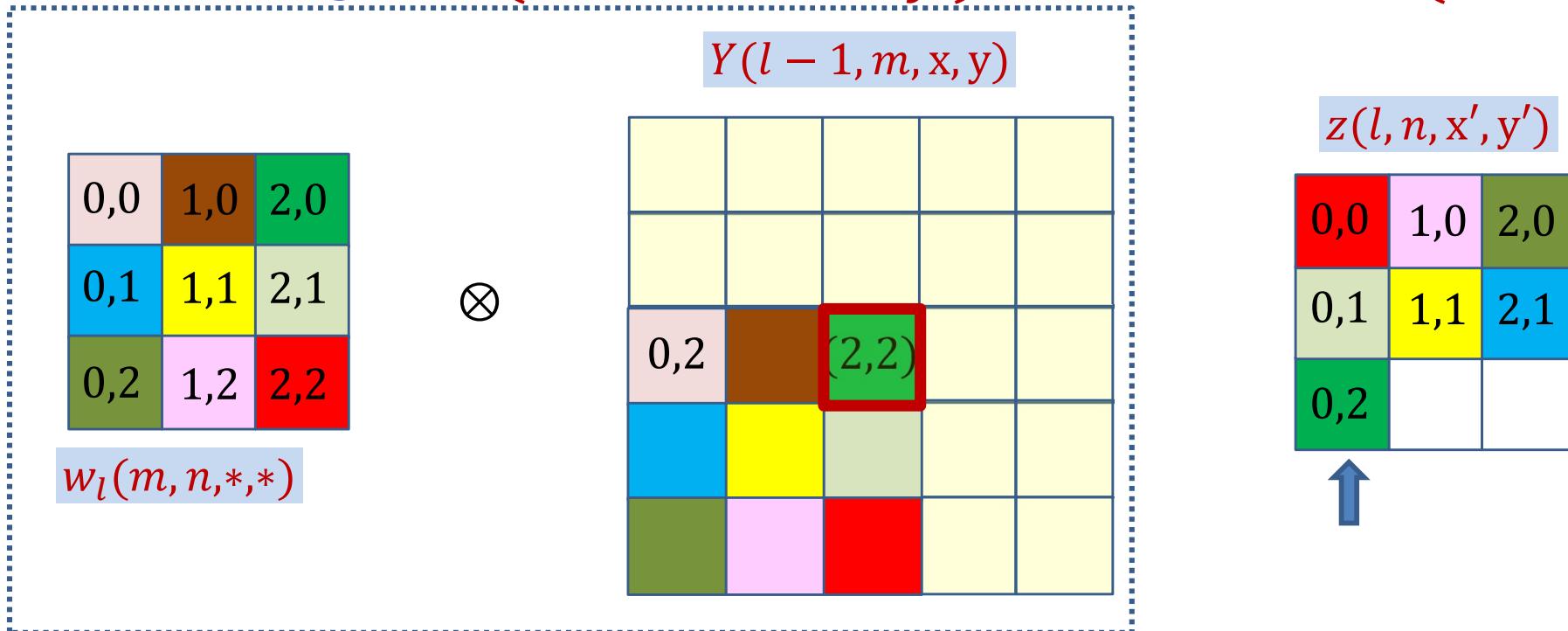


$$z(l, n, 2,1) += Y(l - 1, m, 2,2)w_l(m, n, 0,1)$$

- **Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l - 1, m)$

$$z(l, n, x', y') += Y(l - 1, m, 2,2)w_l(m, n, 2 - x', 2 - y')$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

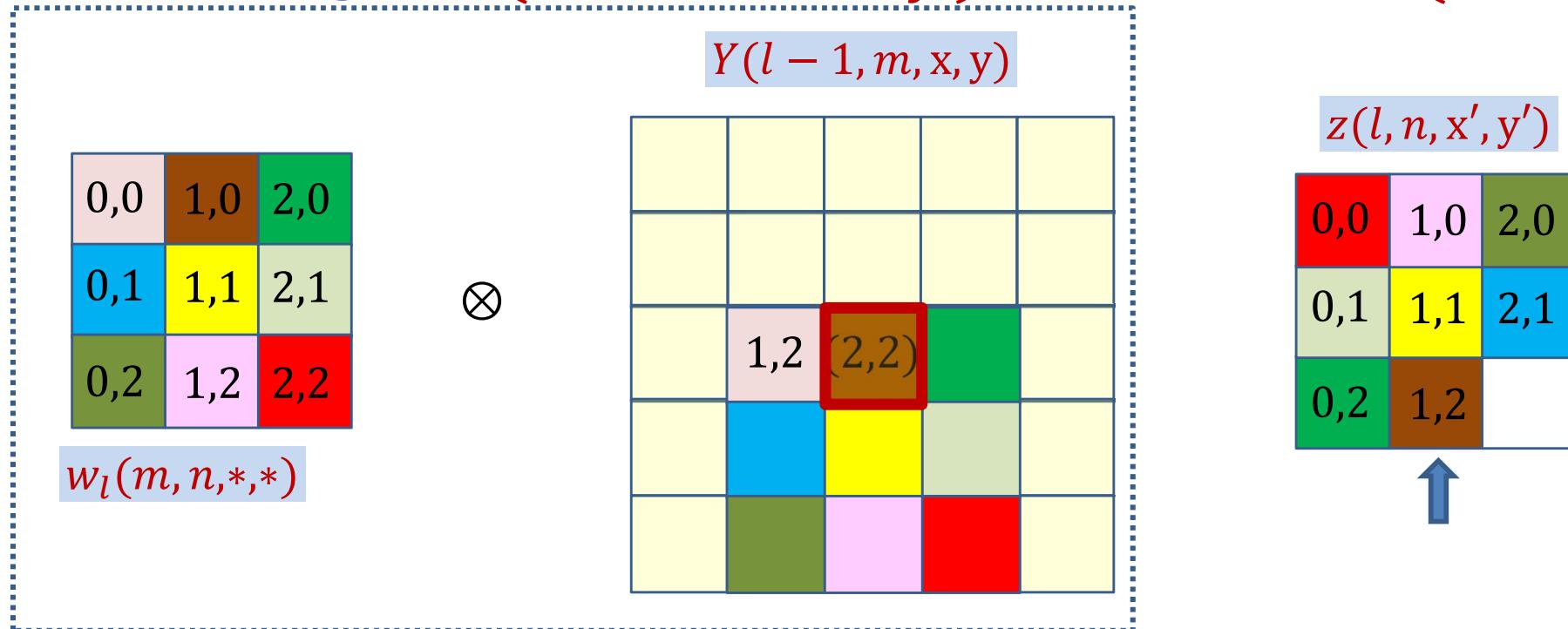


$$z(l, n, 0, 2) += Y(l - 1, m, 2, 2)w_l(m, n, 2, 0)$$

- **Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l - 1, m)$

$$z(l, n, x', y') += Y(l - 1, m, 2, 2)w_l(m, n, 2 - x', 2 - y')$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

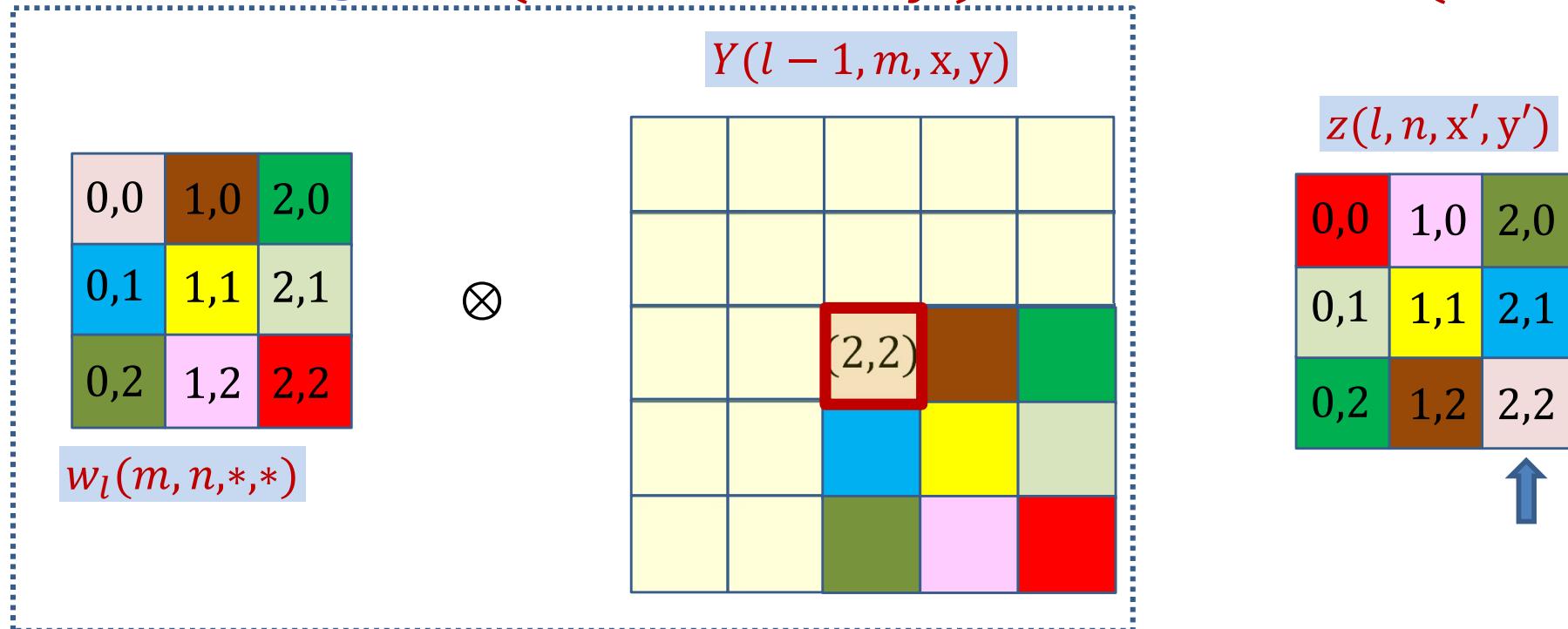


$$z(l, n, 1,2) += Y(l - 1, m, 2,2)w_l(m, n, 2,1)$$

- **Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l - 1, m)$

$$z(l, n, x', y') += Y(l - 1, m, 2,2)w_l(m, n, 2 - x', 2 - y')$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

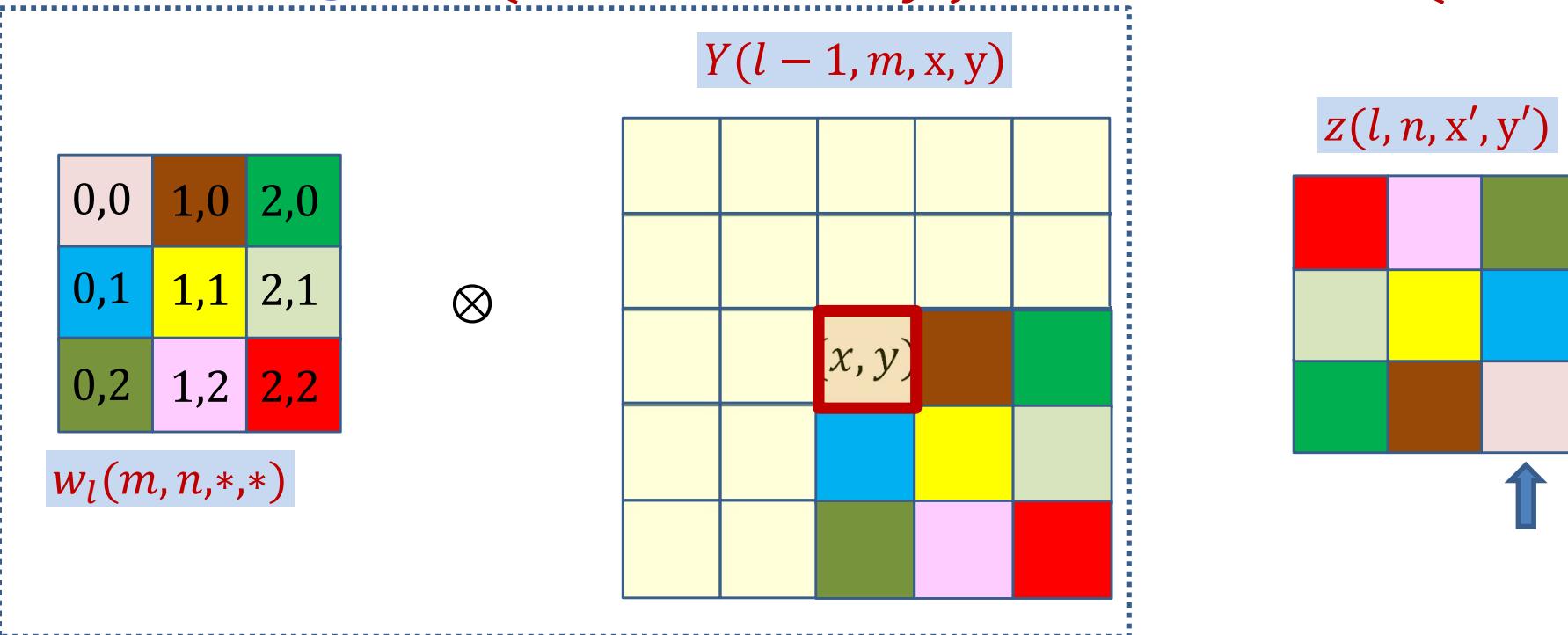


$$z(l, n, 2,2) += Y(l - 1, m, 2,2)w_l(m, n, 0,0)$$

- **Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l - 1, m)$

$$z(l, n, x', y') += Y(l - 1, m, 2,2)w_l(m, n, 2 - x', 2 - y')$$

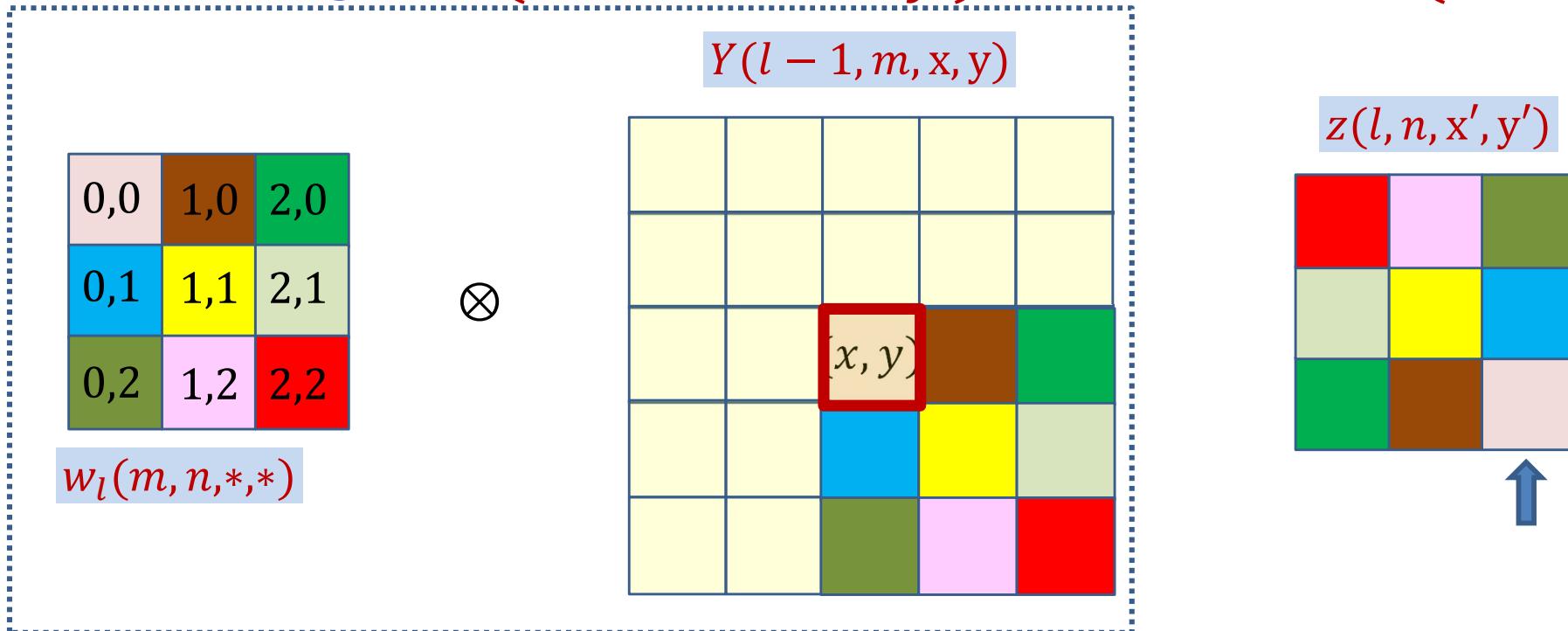
How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, x', y') += Y(l - 1, m, x, y)w_l(m, n, x - x', y - y')$$

- **Note:** The coordinates of $z(l, n)$ and $w_l(m, n)$ sum to the coordinates of $Y(l - 1, m)$

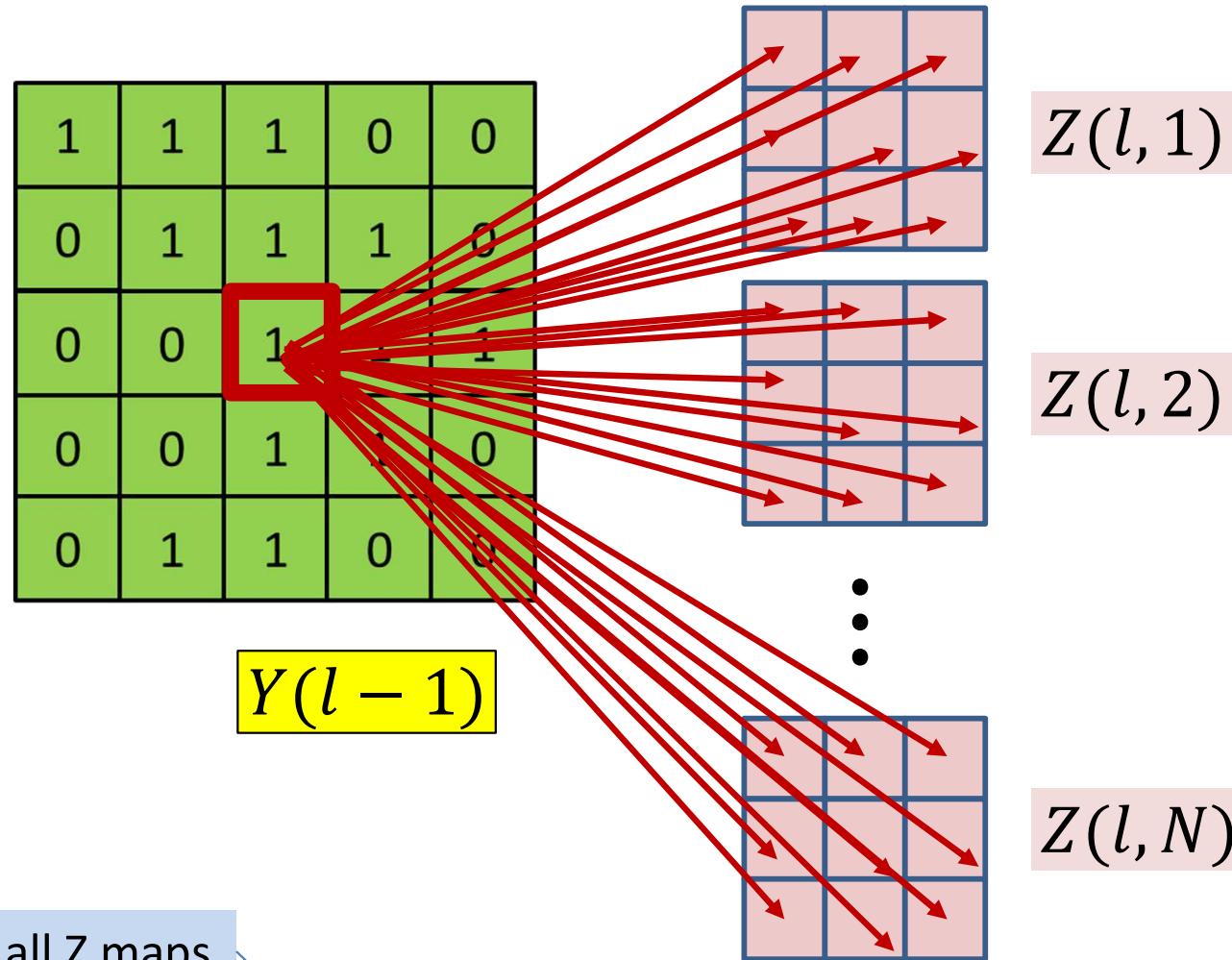
How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, x', y') += Y(l - 1, m, x, y) w_l(m, n, x - x', y - y')$$

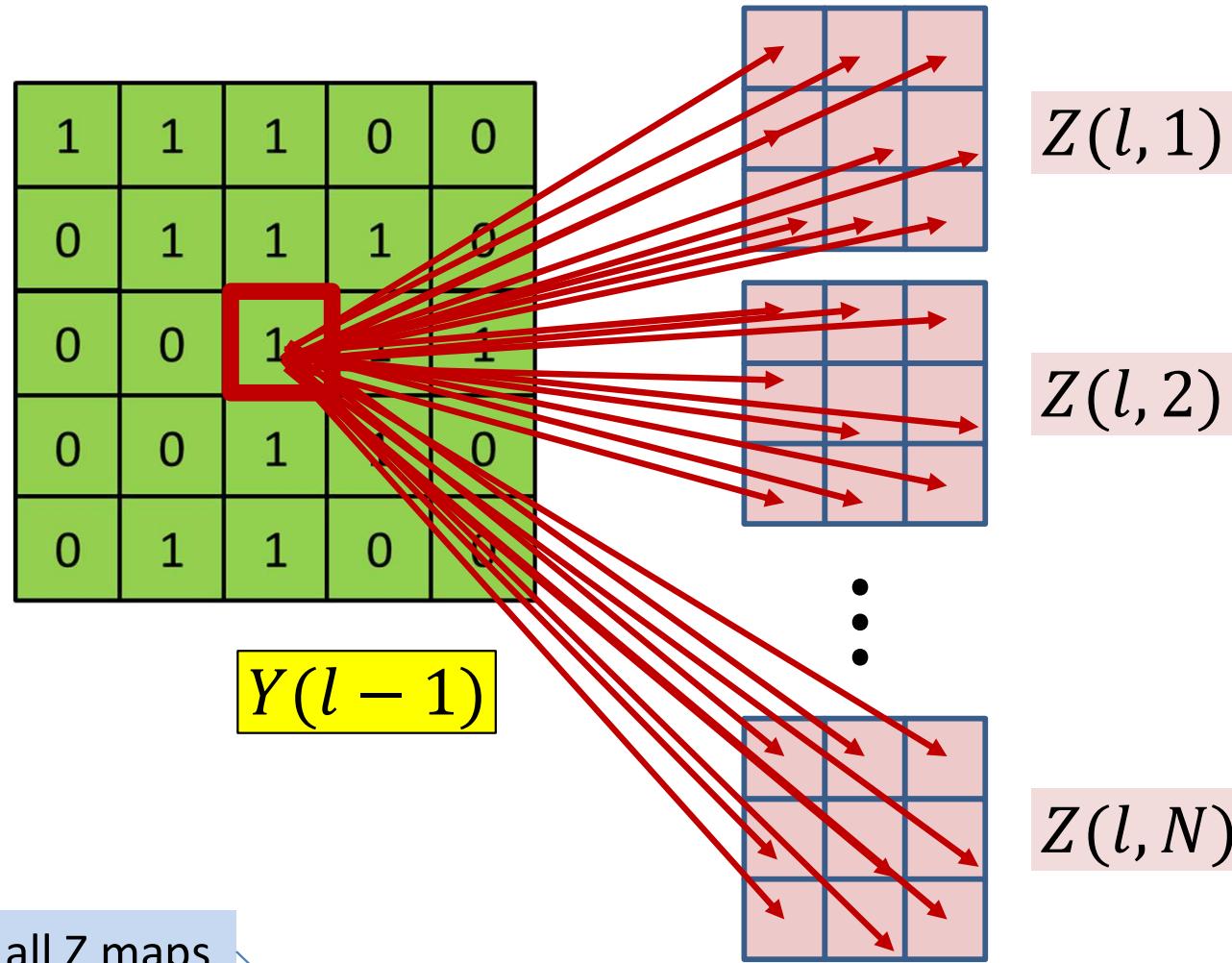
$$\frac{dz(l, n, x', y')}{dY(l - 1, m, x, y)} = w_l(m, n, x - x', y - y')$$

BP: Convolutional layer



$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l, n, x', y')} \frac{dz(l, n, x', y')}{dY(l-1, m, x, y)}$$

BP: Convolutional layer



$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

Poll 2

In order to compute the derivative at a single affine element $Y(l,m,x,y)$, we must consider the contributions of *every* position of *every* affine map at the next layer: True or false

- True
- False

The derivative for a single affine element $Y(l,m,x,y)$ will require summing over every position of every Z map in the next layer: True or false

- True
- False

Poll 2

In order to compute the derivative at a single affine element $Y(l,m,x,y)$, we must consider the contributions of *every* position of *every* affine map at the next layer: True or false

- **True**
- False

The derivative for an single affine element $Y(l,m,x,y)$ will require summing over every position of every Z map in the next layer: True or false

- **True**
- False

Computing derivative for $Y(l - 1, m, *, *)$

- The derivatives for every element of every map in $Y(l - 1)$ by direct implementation of the formula:

$$\frac{dDiv}{dY(l - 1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

- But this is actually a convolution!
 - Let's see how

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

$w_l(m, n, *, *)$

\otimes

$Y(l - 1, m, x, y)$		
0,0		

(2,2)

$z(l, n, x', y')$

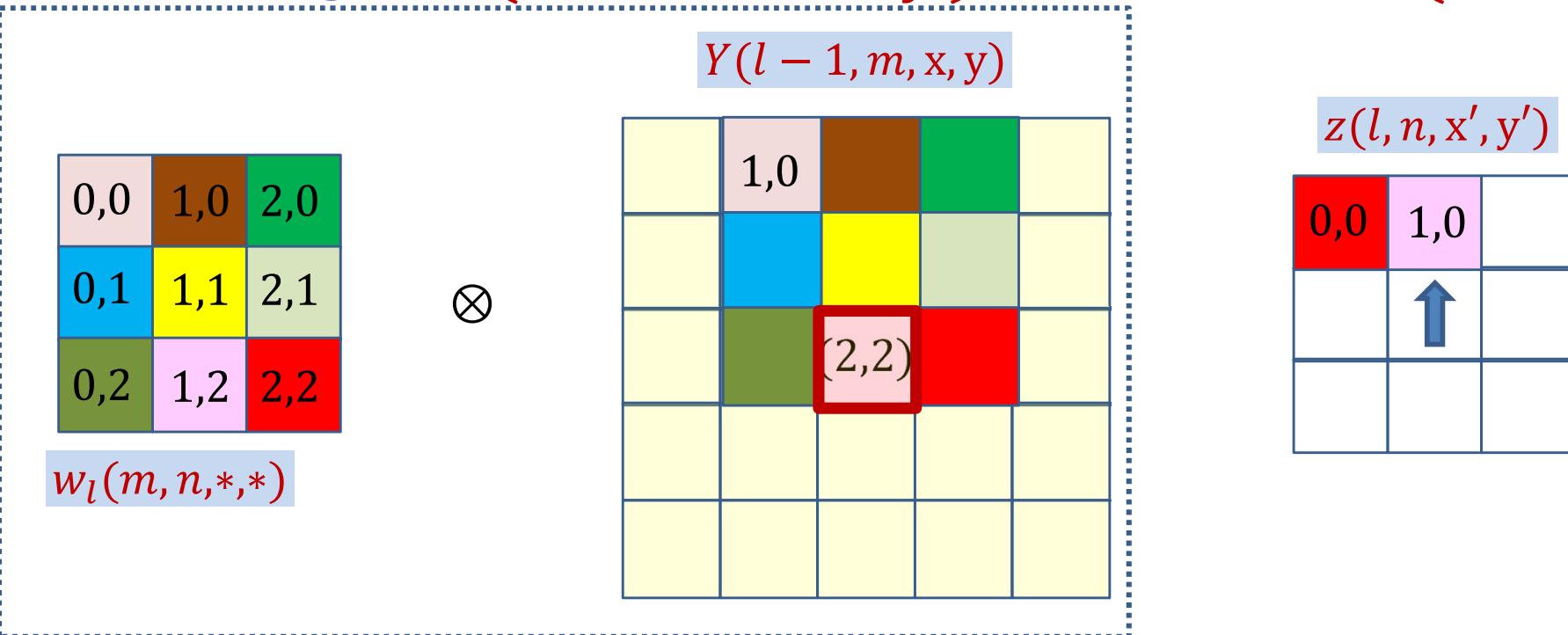
0,0		



$$z(l, n, 0, 0) += Y(l - 1, m, 2, 2) w_l(m, n, 2, 2)$$

$$\frac{dDiv}{dY(l - 1, m, 2, 2)} += \frac{dDiv}{dz(l, n, 0, 0)} w_l(m, n, 2, 2)$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 1, 0) += Y(l - 1, m, 2, 2) w_l(m, n, 1, 2)$$

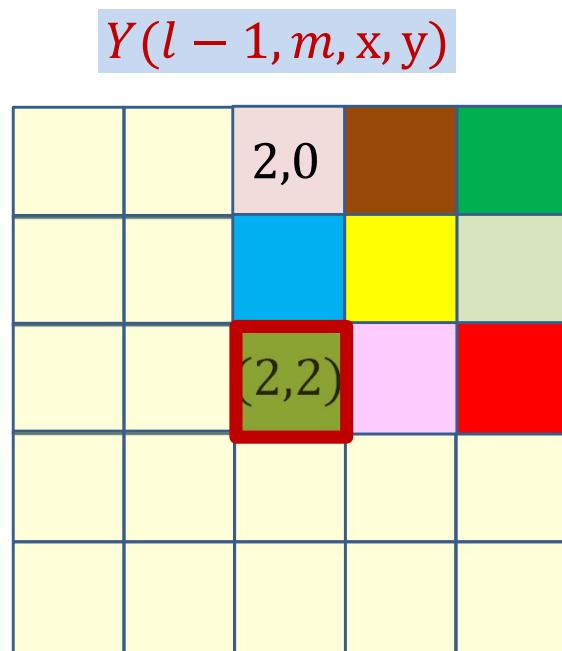
$$\frac{dDiv}{dY(l - 1, m, 2, 2)} += \frac{dDiv}{dz(l, n, 1, 0)} w_l(m, n, 1, 2)$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

$w_l(m, n, *, *)$

\otimes



$z(l, n, x', y')$

0,0	1,0	2,0



$$z(l, n, 2,0) += Y(l - 1, m, 2,2) w_l(m, n, 0,2)$$

$$\frac{dDiv}{dY(l - 1, m, 2,2)} += \frac{dDiv}{dz(l, n, 2,0)} w_l(m, n, 0,2)$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

$w_l(m, n, *, *)$

\otimes

$Y(l - 1, m, x, y)$				
0,1				
		(2,2)		

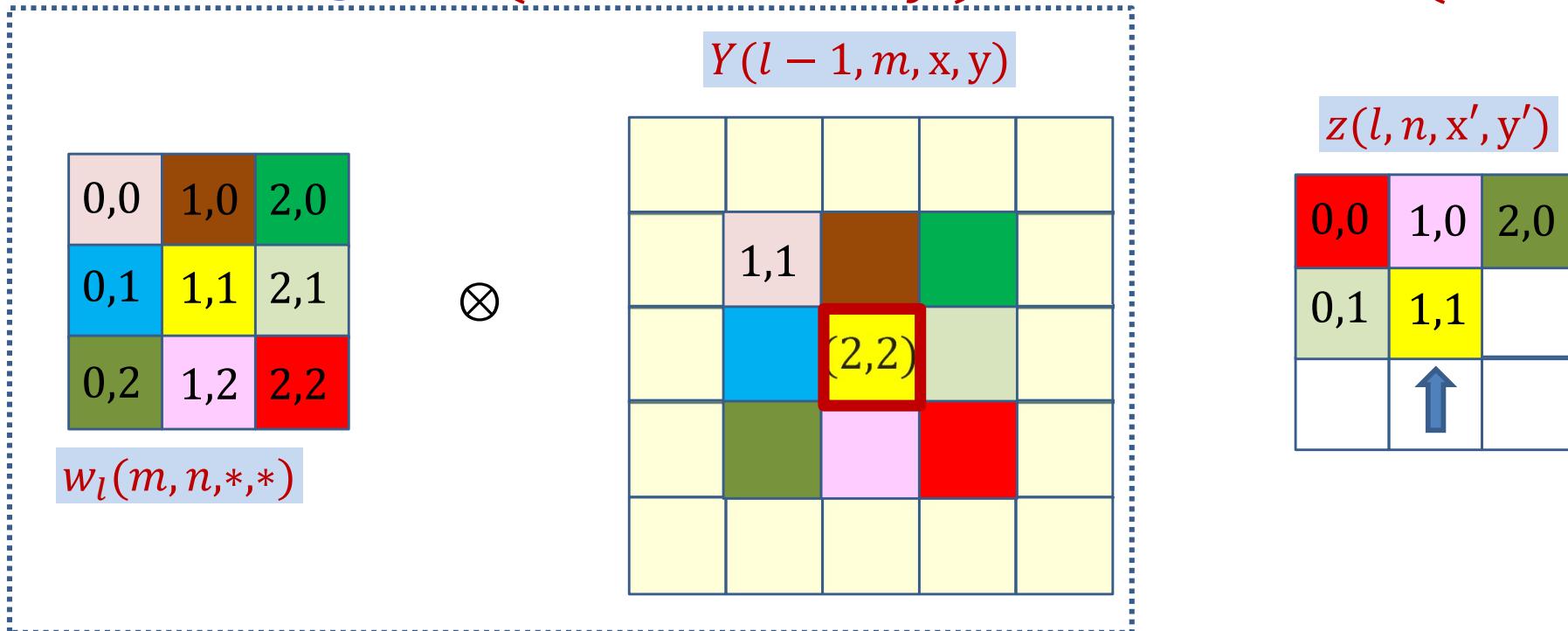
$z(l, n, x', y')$

0,0	1,0	2,0
0,1		
↑		

$$z(l, n, 0,1) += Y(l - 1, m, 2,2) w_l(m, n, 2,1)$$

$$\frac{dDiv}{dY(l - 1, m, 2,2)} += \frac{dDiv}{dz(l, n, 0,1)} w_l(m, n, 2,1)$$

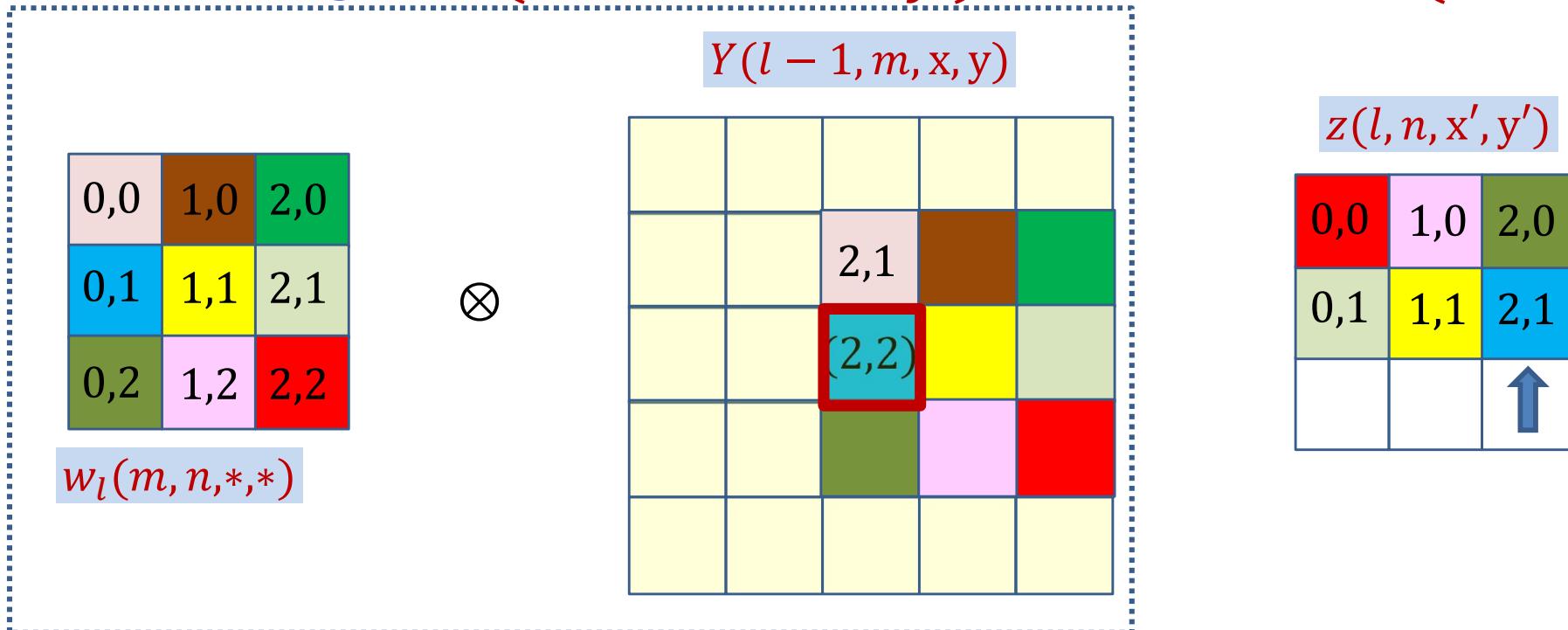
How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 1,1) += Y(l - 1, m, 2,2) w_l(m, n, 1,1)$$

$$\frac{dDiv}{dY(l - 1, m, 2,2)} += \frac{dDiv}{dz(l, n, 1,1)} w_l(m, n, 1,1)$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$



$$z(l, n, 2,1) += Y(l - 1, m, 2,2)w_l(m, n, 0,1)$$

$$\frac{dDiv}{dY(l - 1, m, 2,2)} += \frac{dDiv}{dz(l, n, 2,1)} w_l(m, n, 0,1)$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

$w_l(m, n, *, *)$

\otimes

$Y(l - 1, m, x, y)$		
0,2		(2,2)

$z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2		



$$z(l, n, 0,2) += Y(l - 1, m, 2,2)w_l(m, n, 2,0)$$

$$\frac{dDiv}{dY(l - 1, m, 2,2)} += \frac{dDiv}{dz(l, n, 0,2)} w_l(m, n, 2,0)$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

$w_l(m, n, *, *)$

\otimes

$Y(l - 1, m, x, y)$				
1,2		(2,2)		

$z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	



$$z(l, n, 1,2) += Y(l - 1, m, 2,2) w_l(m, n, 2,1)$$

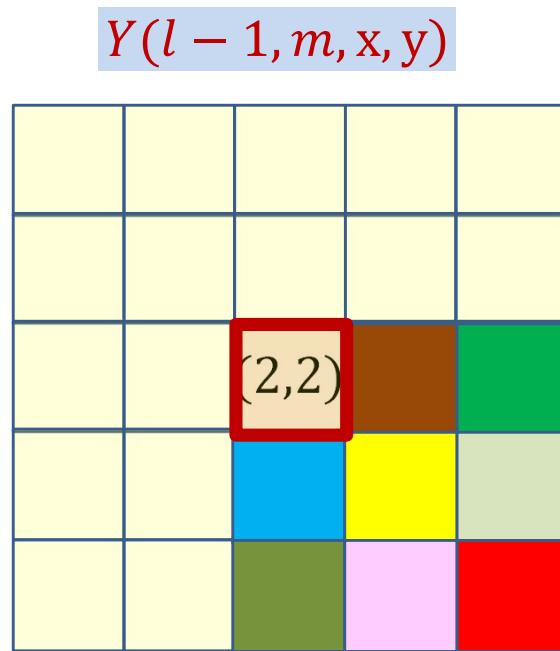
$$\frac{dDiv}{dY(l - 1, m, 2,2)} += \frac{dDiv}{dz(l, n, 1,2)} w_l(m, n, 1,0)$$

How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

$w_l(m, n, *, *)$

\otimes



$z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2



$$z(l, n, 2,2) += Y(l - 1, m, 2,2)w_l(m, n, 0,0)$$

$$\frac{dDiv}{dY(l - 1, m, 2,2)} += \frac{dDiv}{dz(l, n, 2,2)} w_l(m, n, 0,0)$$

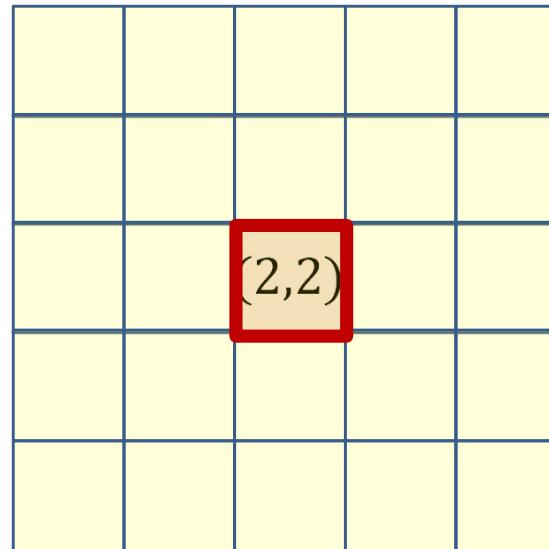
How a single $Y(l - 1, m, x, y)$ influences $z(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

$w_l(m, n, *, *)$

\otimes

$dDiv/dY(l - 1, m, x, y)$



$dDiv/dz(l, n, x', y')$

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2



$$\frac{dDiv}{dY(l - 1, m, 2, 2)} = \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, 2 - x', 2 - y')$$

- The derivative at $Y(l - 1, m, 2, 2)$ is the sum of component-wise product of the filter elements (shown by color) and the elements of the derivative at $z(l, n, \dots)$

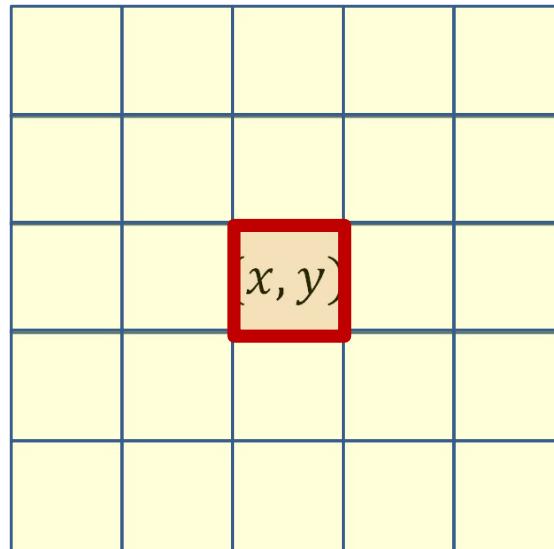
Derivative at $Y(l - 1, m, x, y)$ from a single $Z(l, n)$ map

0,0	1,0	2,0
0,1	1,1	2,1
0,2	1,2	2,2

$w_l(m, n, *, *)$

\otimes

$dDiv/dY(l - 1, m, x, y)$



$dDiv/dz(l, n, x', y')$

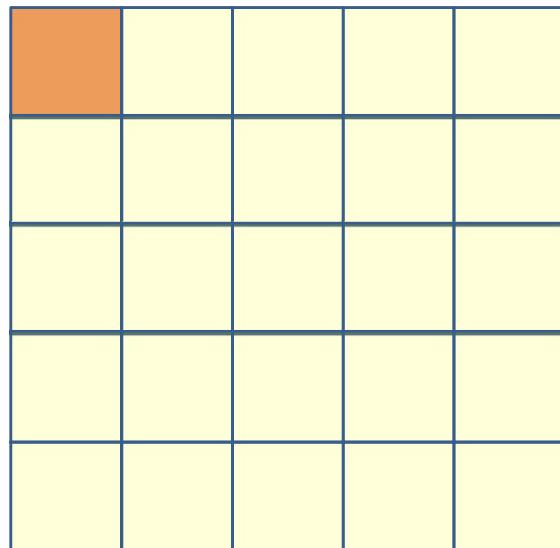
$x - 2$	$x - 1$	x
$y - 2$	$y - 1$	$y - 1$
$x - 2$	$x - 1$	x
$y - 1$	$y - 1$	$y - 1$
$x - 2$	$x - 1$	x, y

$$z(l, n, x', y') += Y(l - 1, m, x, y) w_l(m, n, x - x', y - y')$$

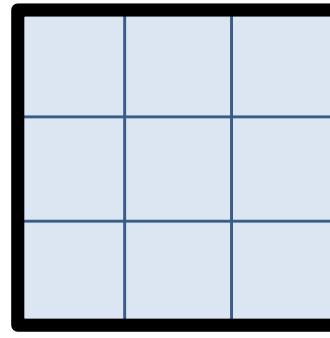
$$\frac{dDiv}{dY(l - 1, m, x, y)} += \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

Contribution of the entire n th affine map $z(l, n, *, *)$

Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



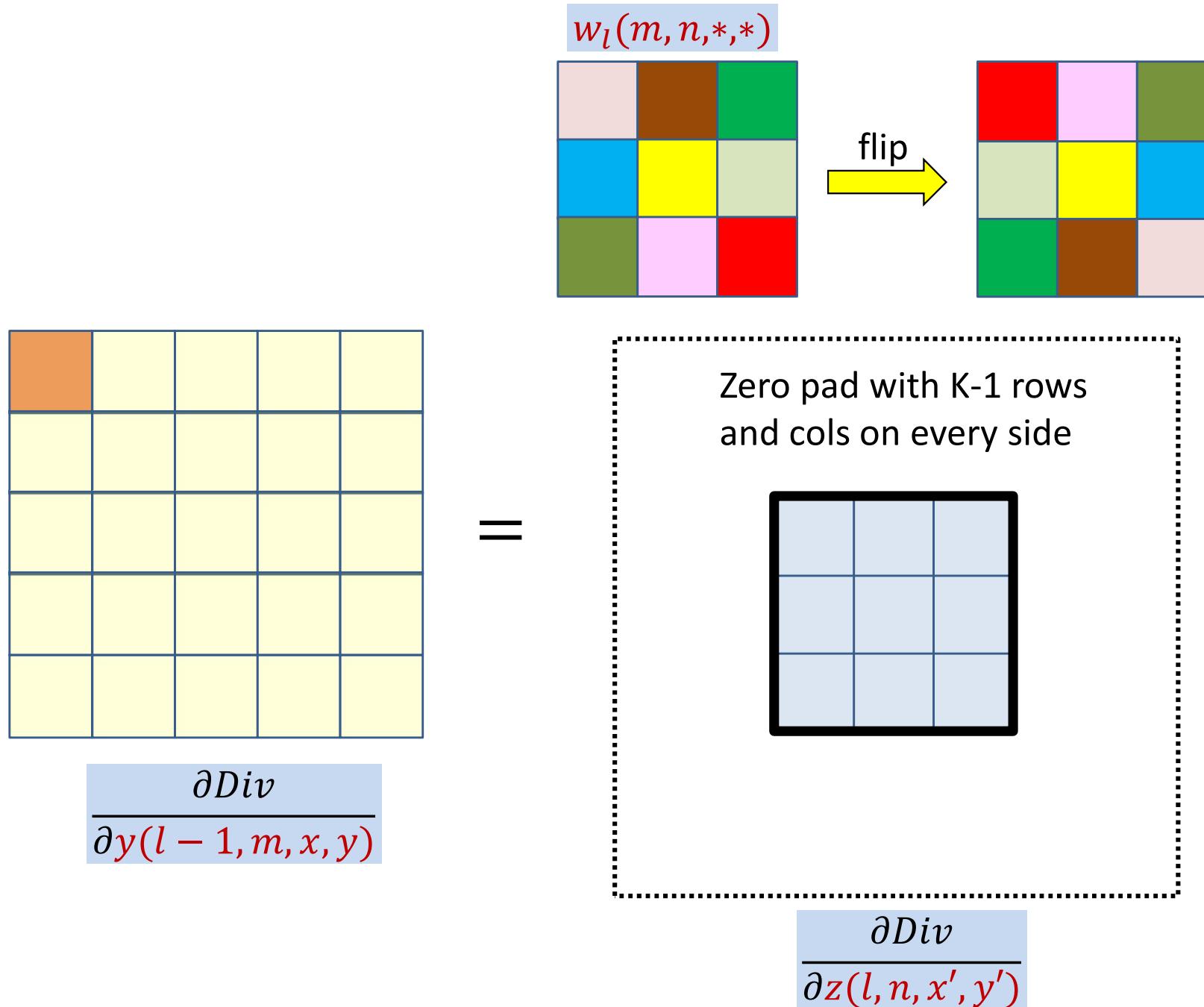
=



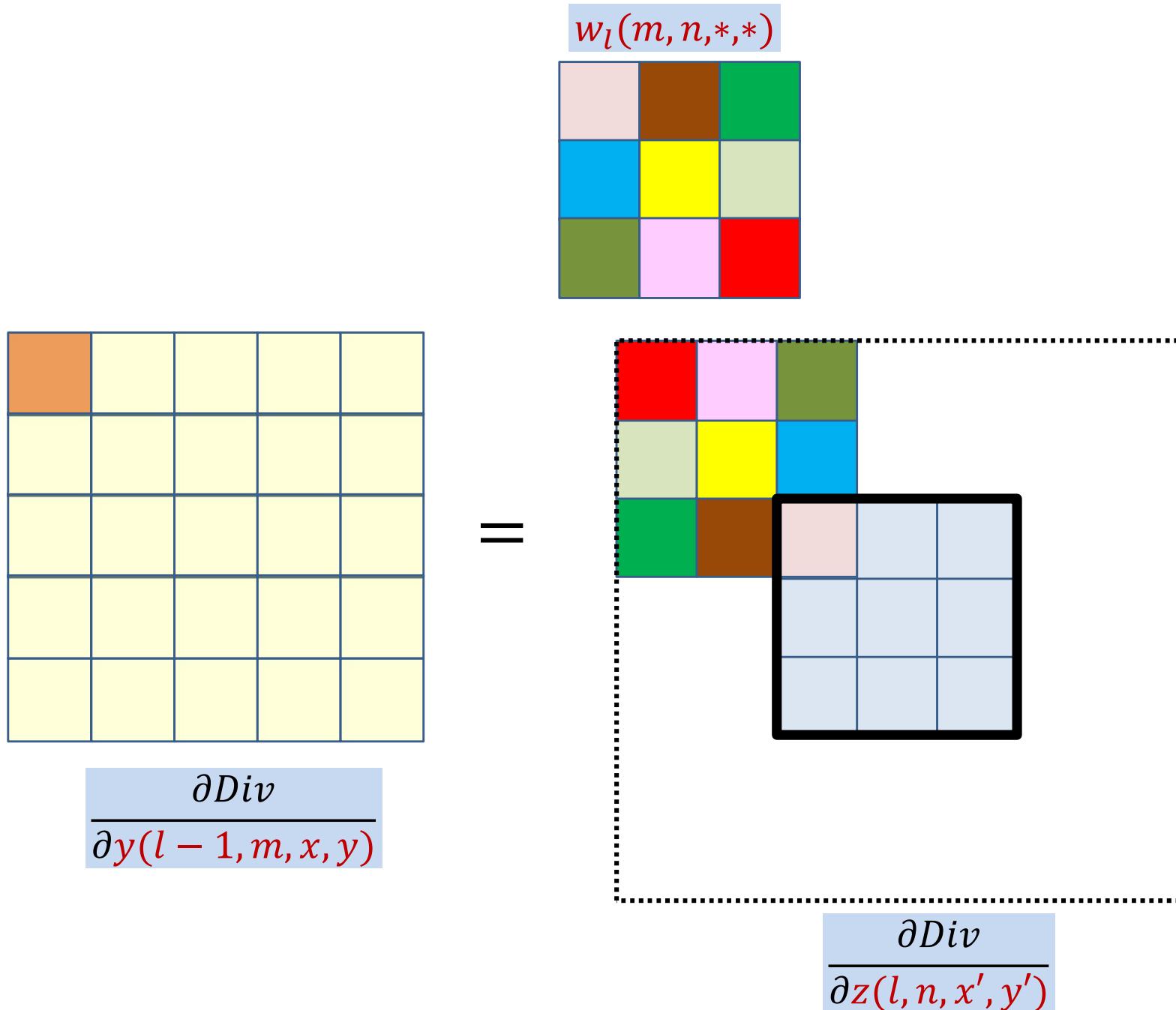
$$\frac{\partial \text{Div}}{\partial y(l-1, m, x, y)}$$

$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

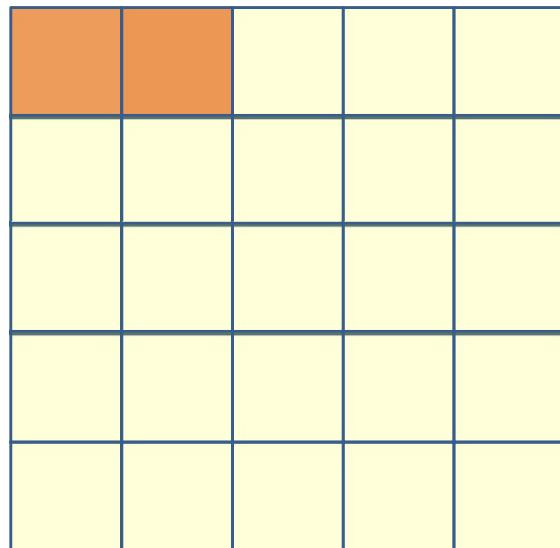
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



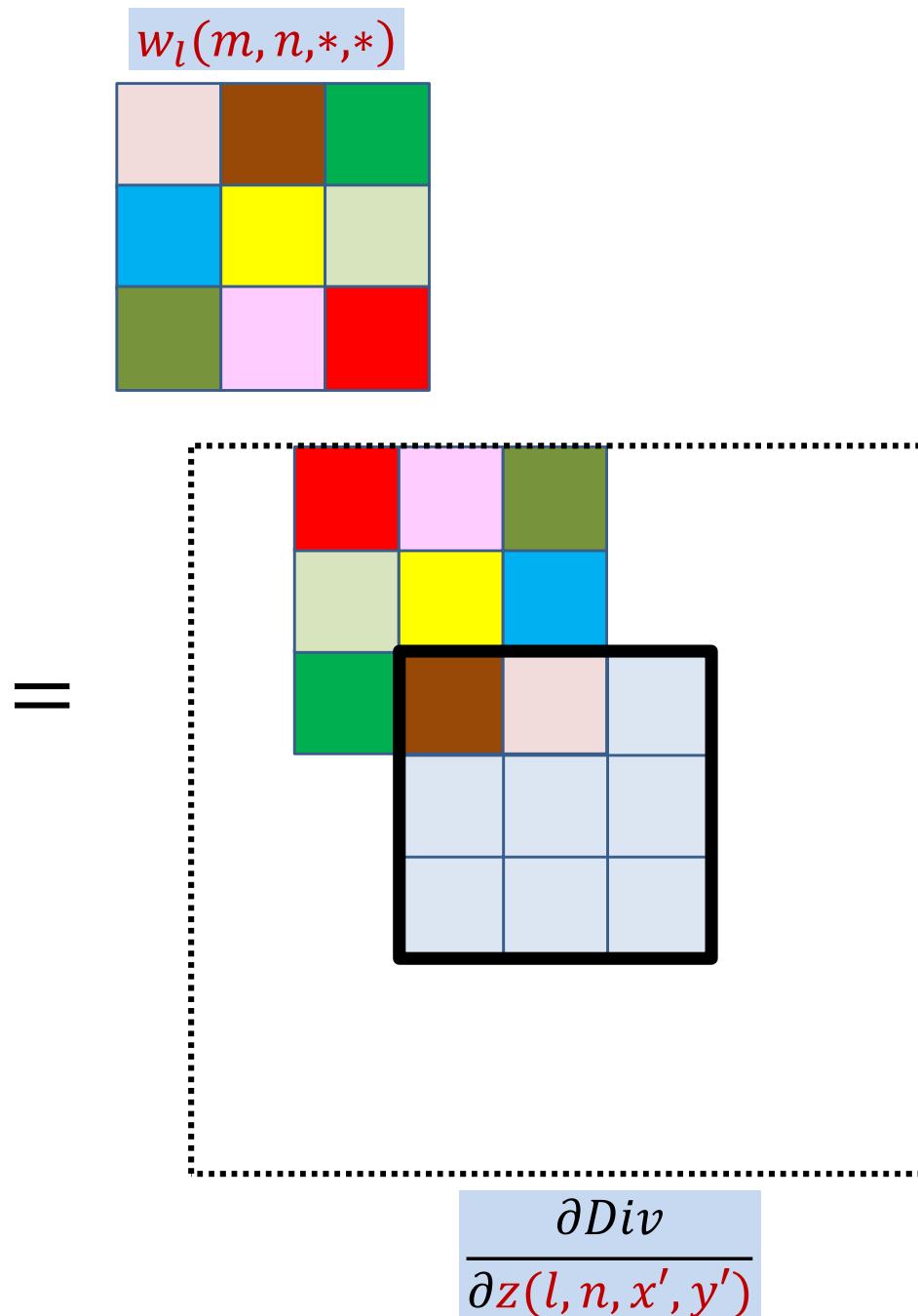
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



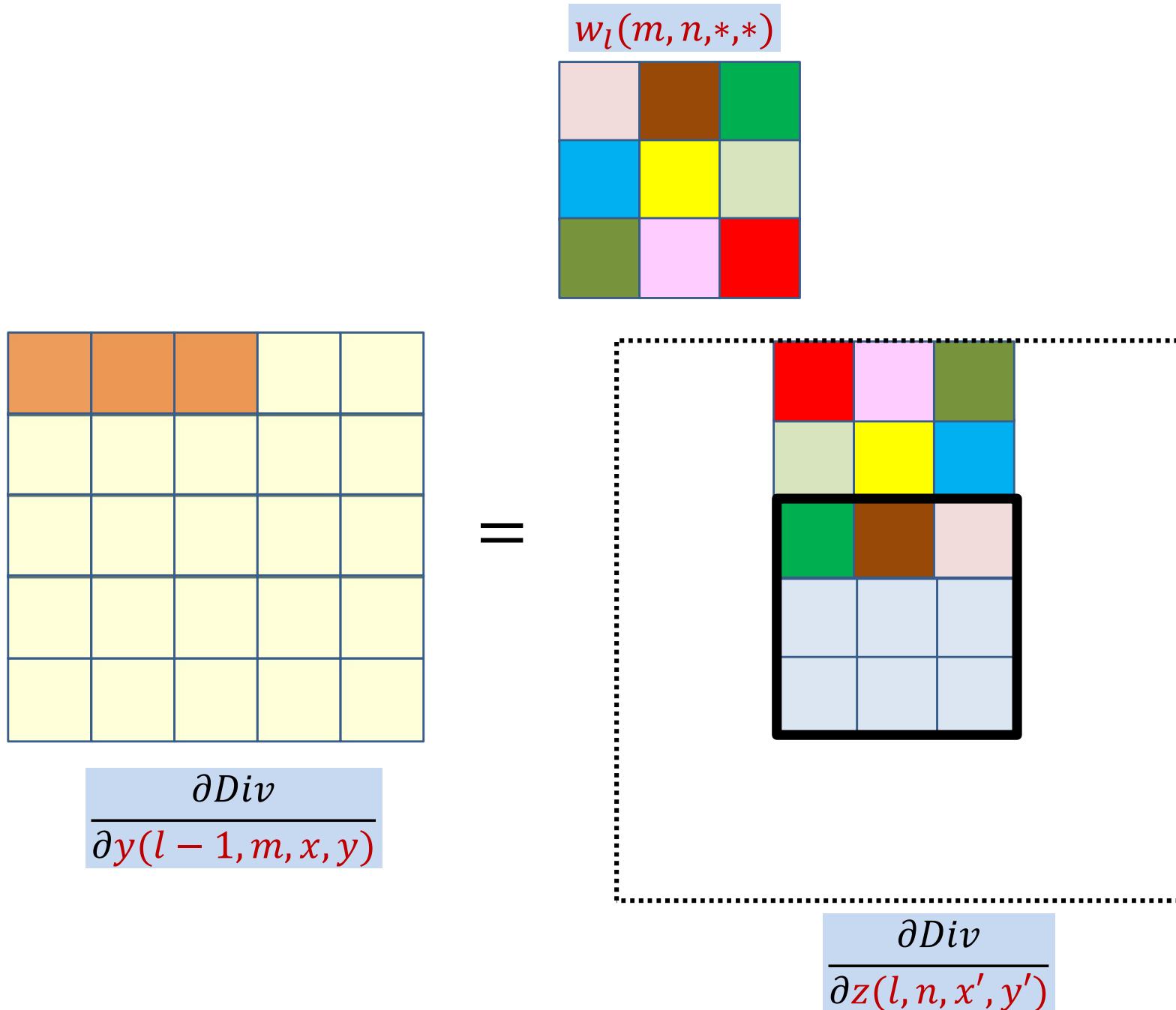
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



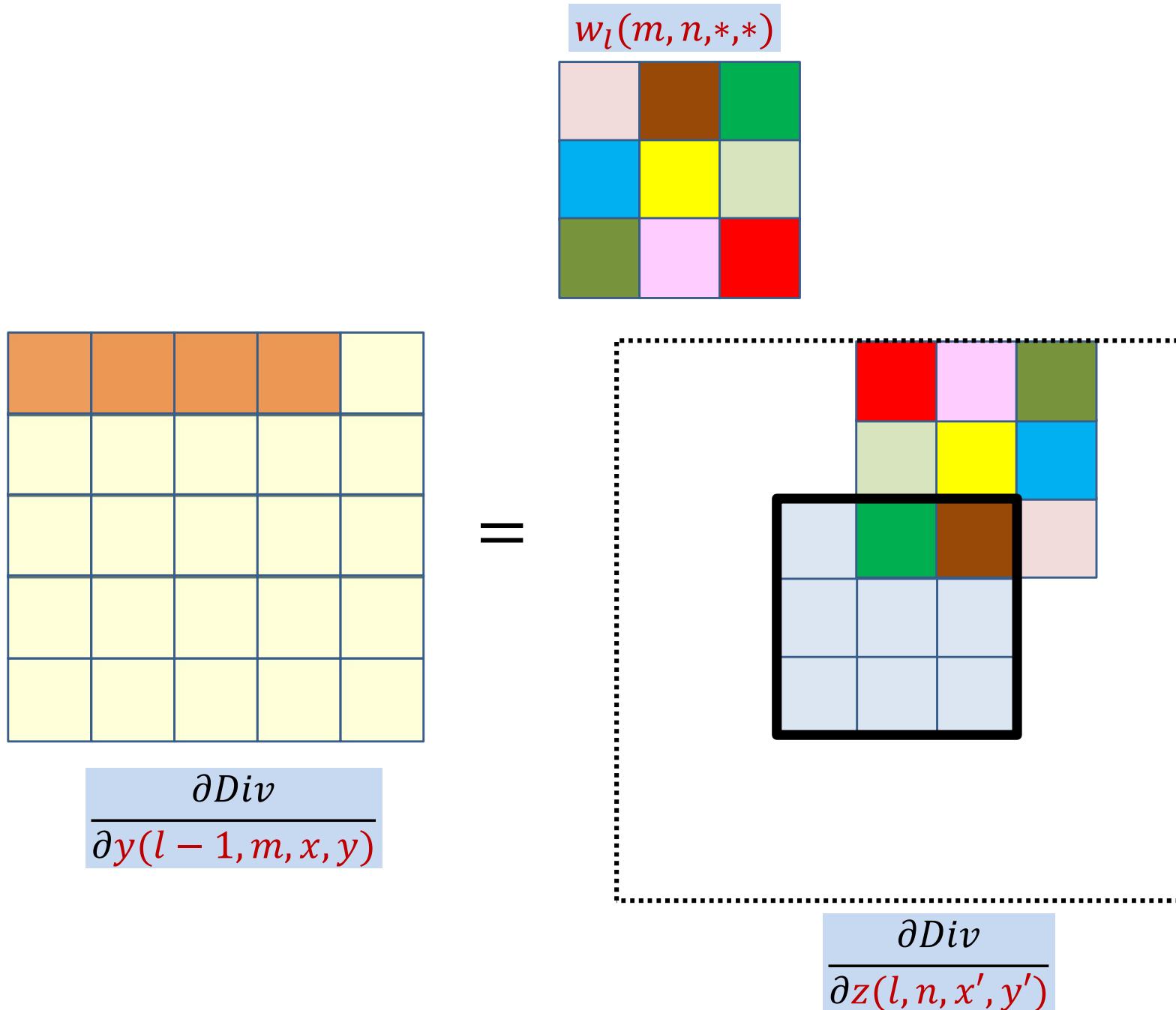
$$\frac{\partial \text{Div}}{\partial y(l-1, m, x, y)}$$



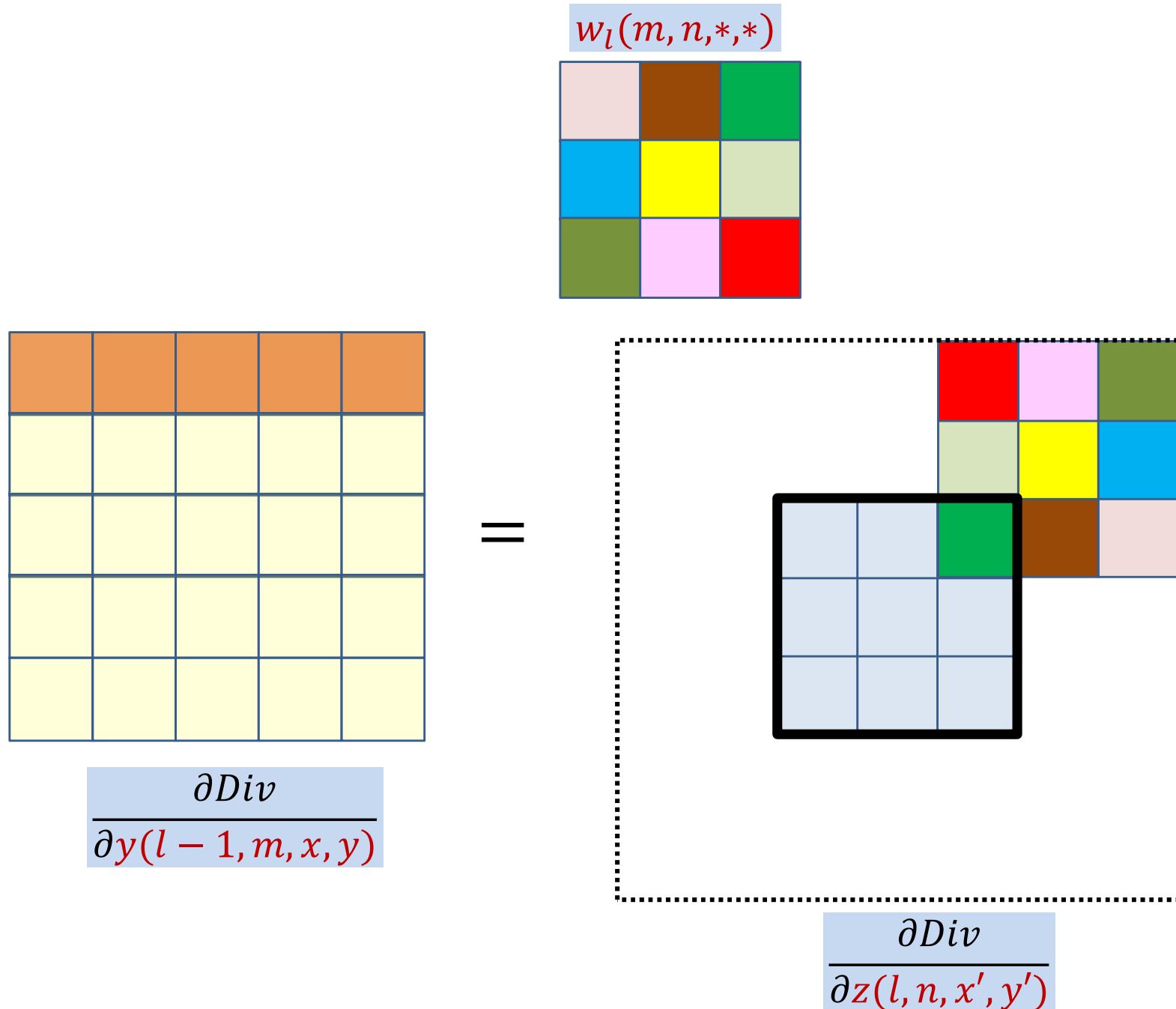
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



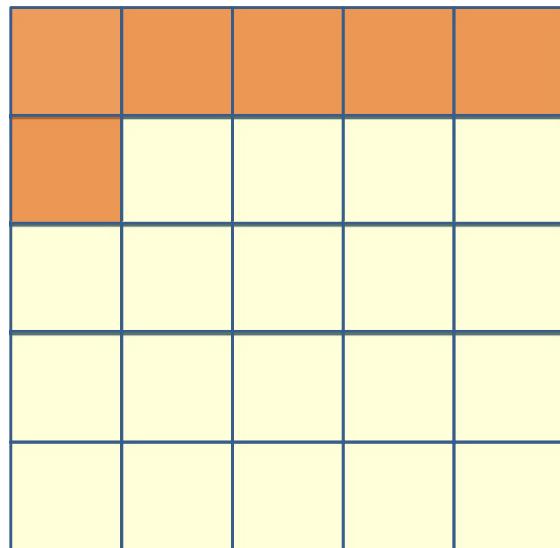
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



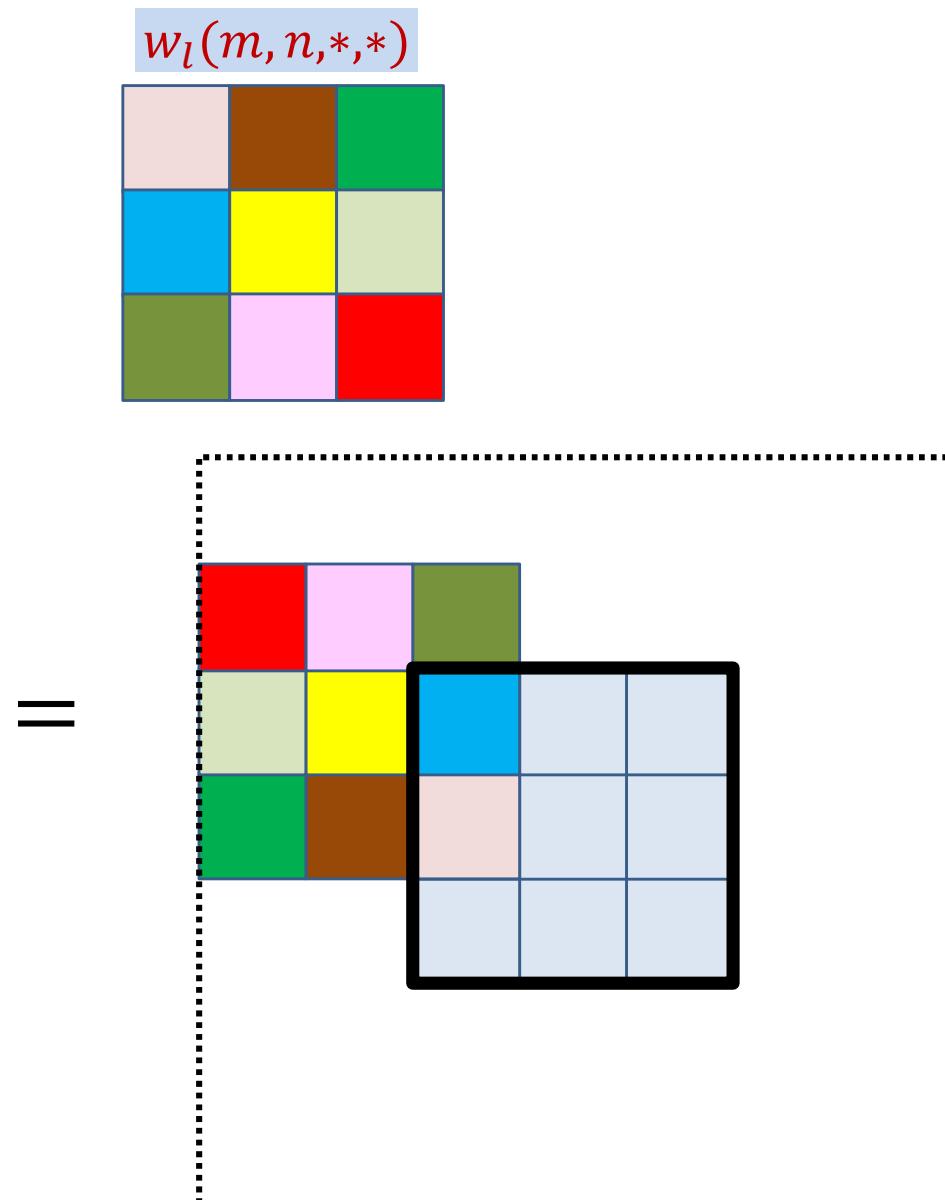
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map

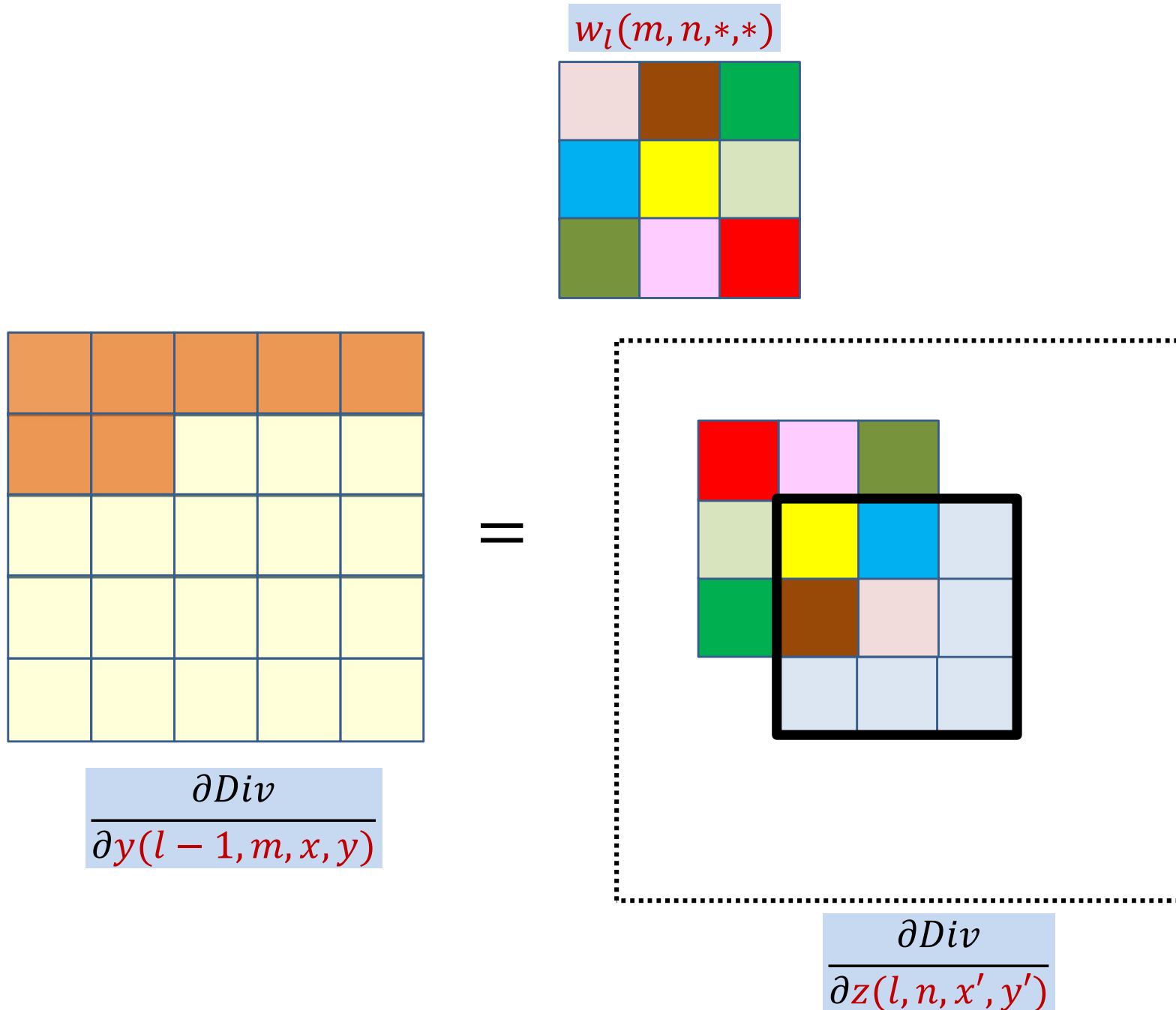


$$\frac{\partial \text{Div}}{\partial y(l-1, m, x, y)}$$

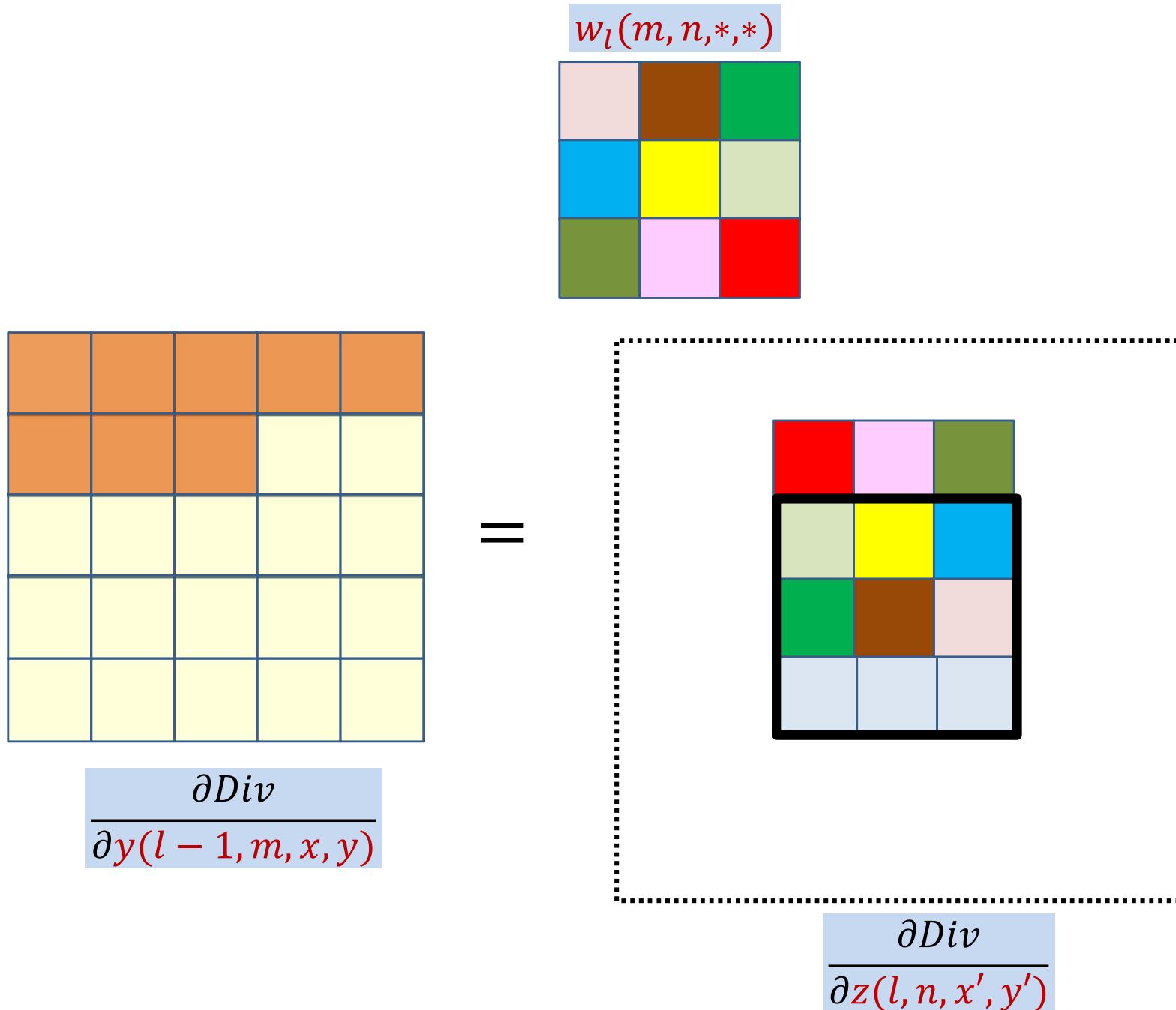


$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

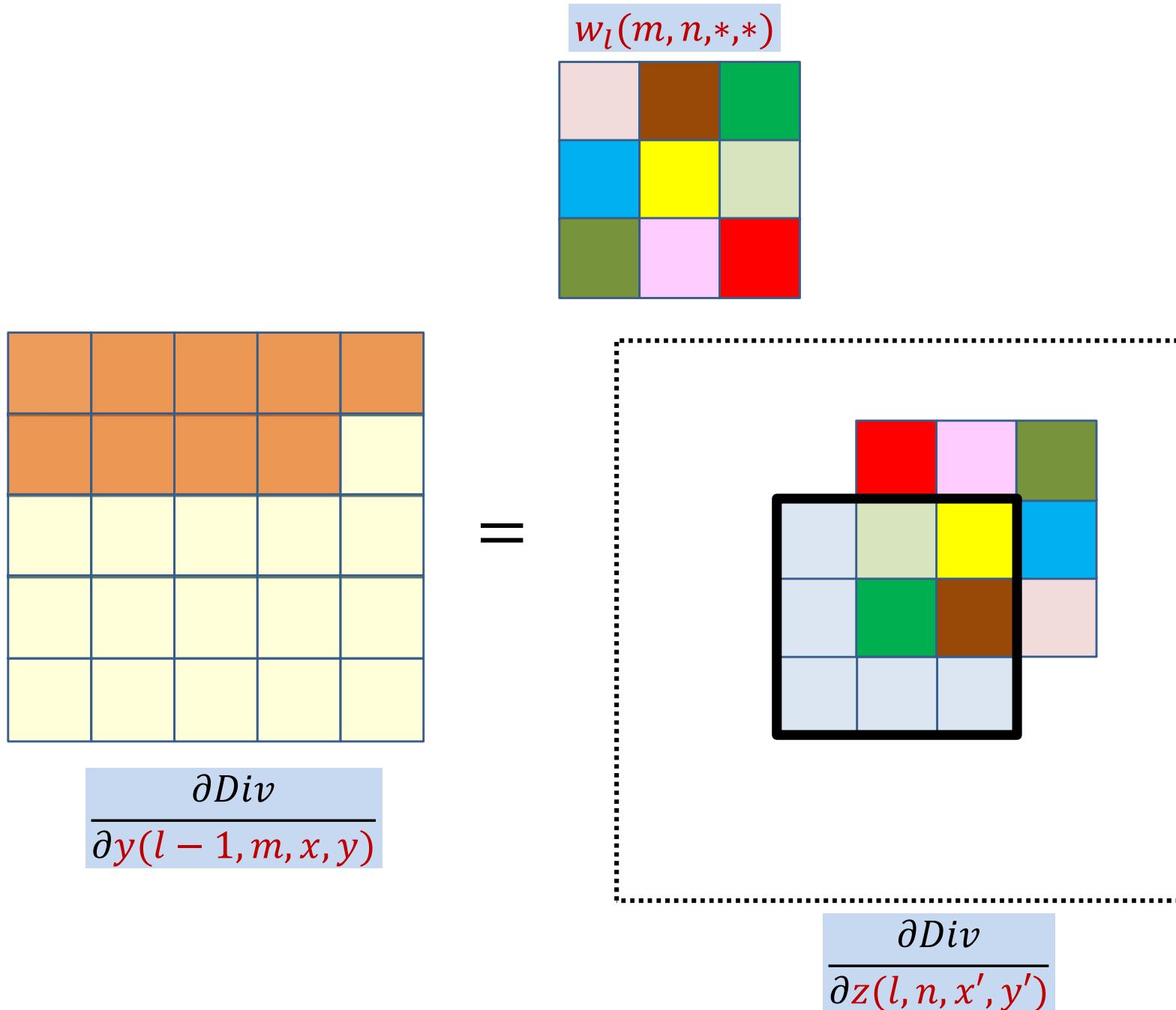
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



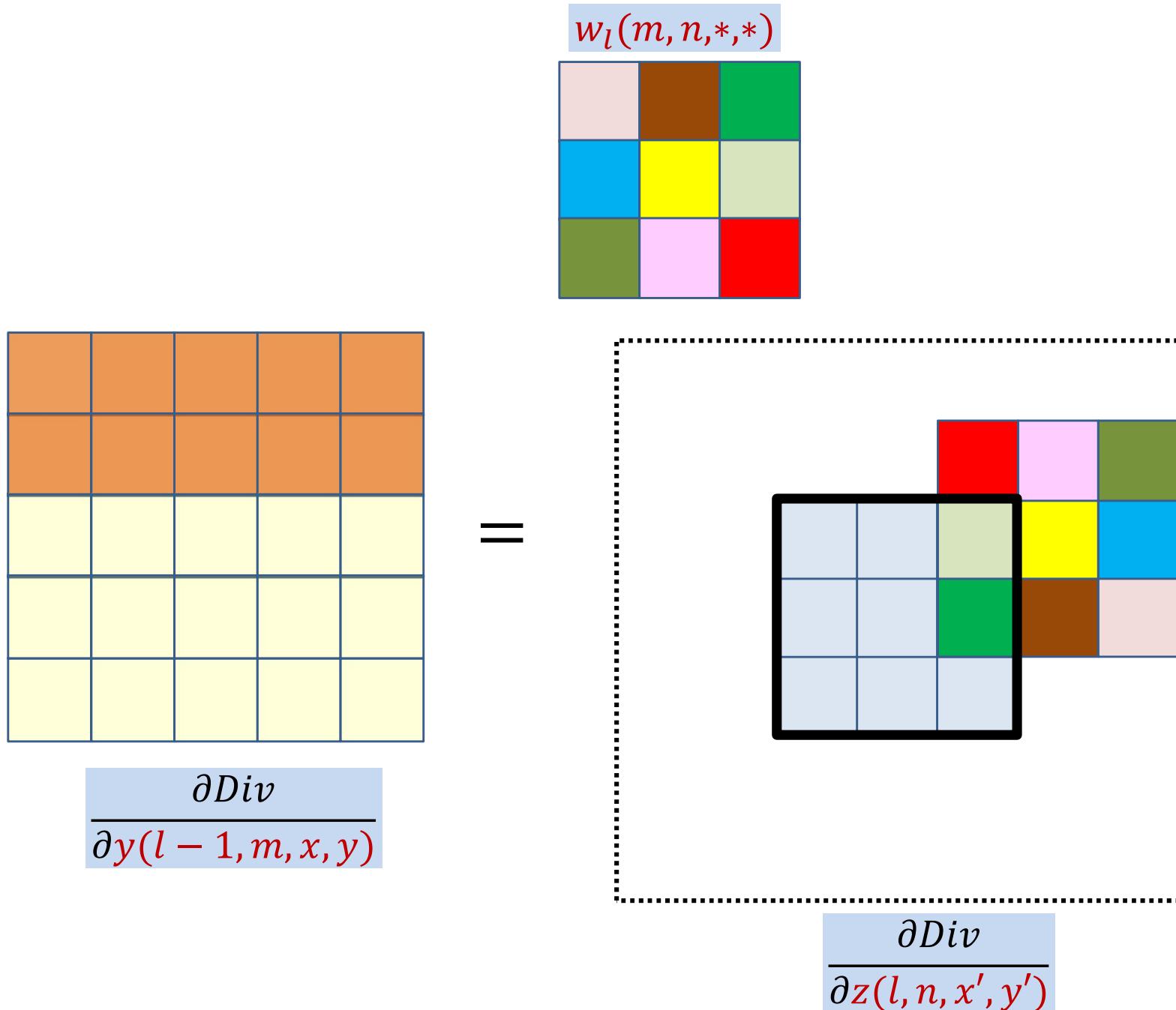
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



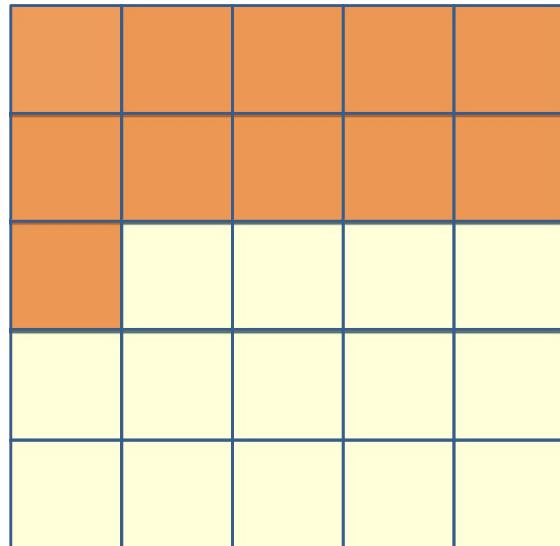
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



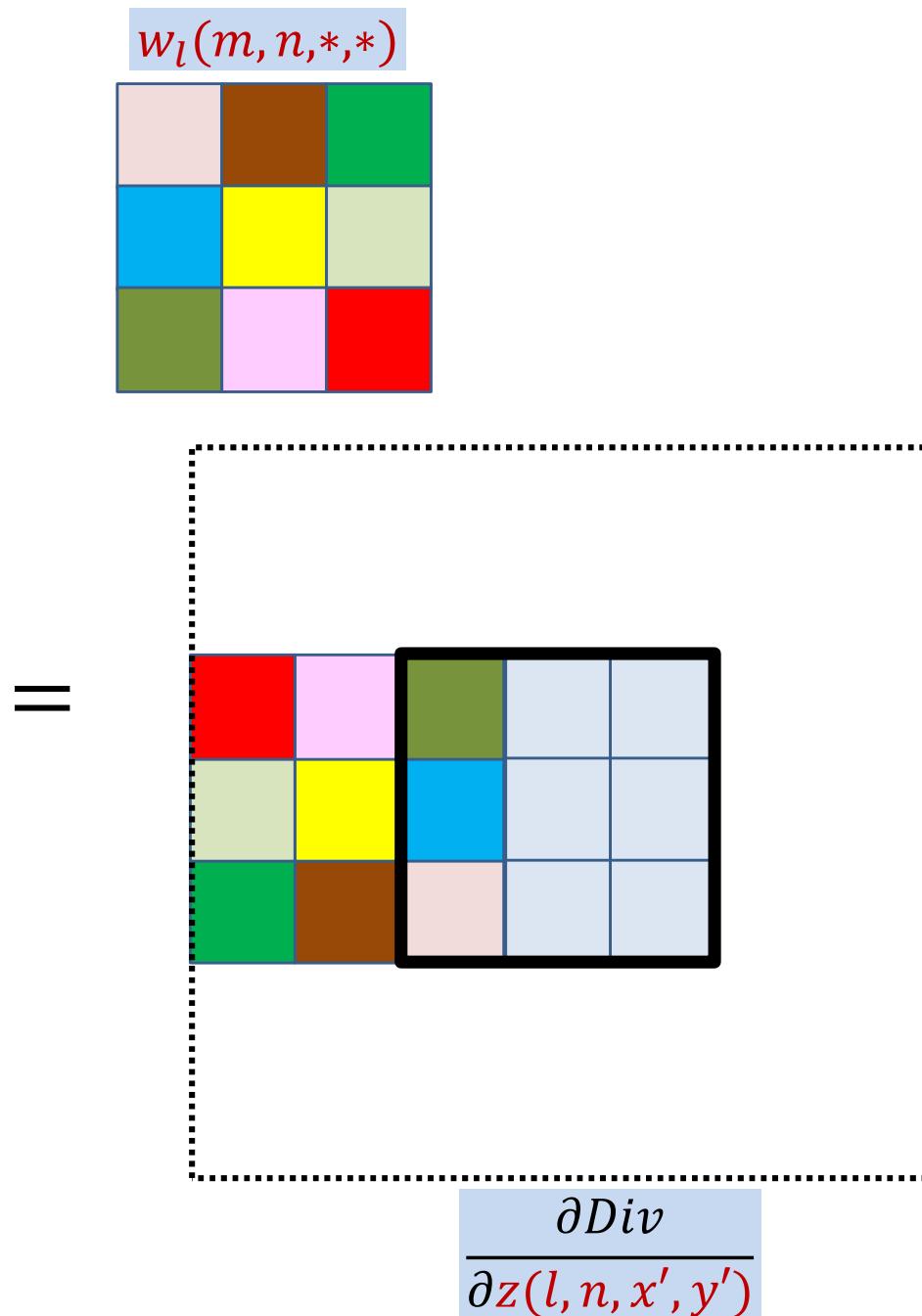
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



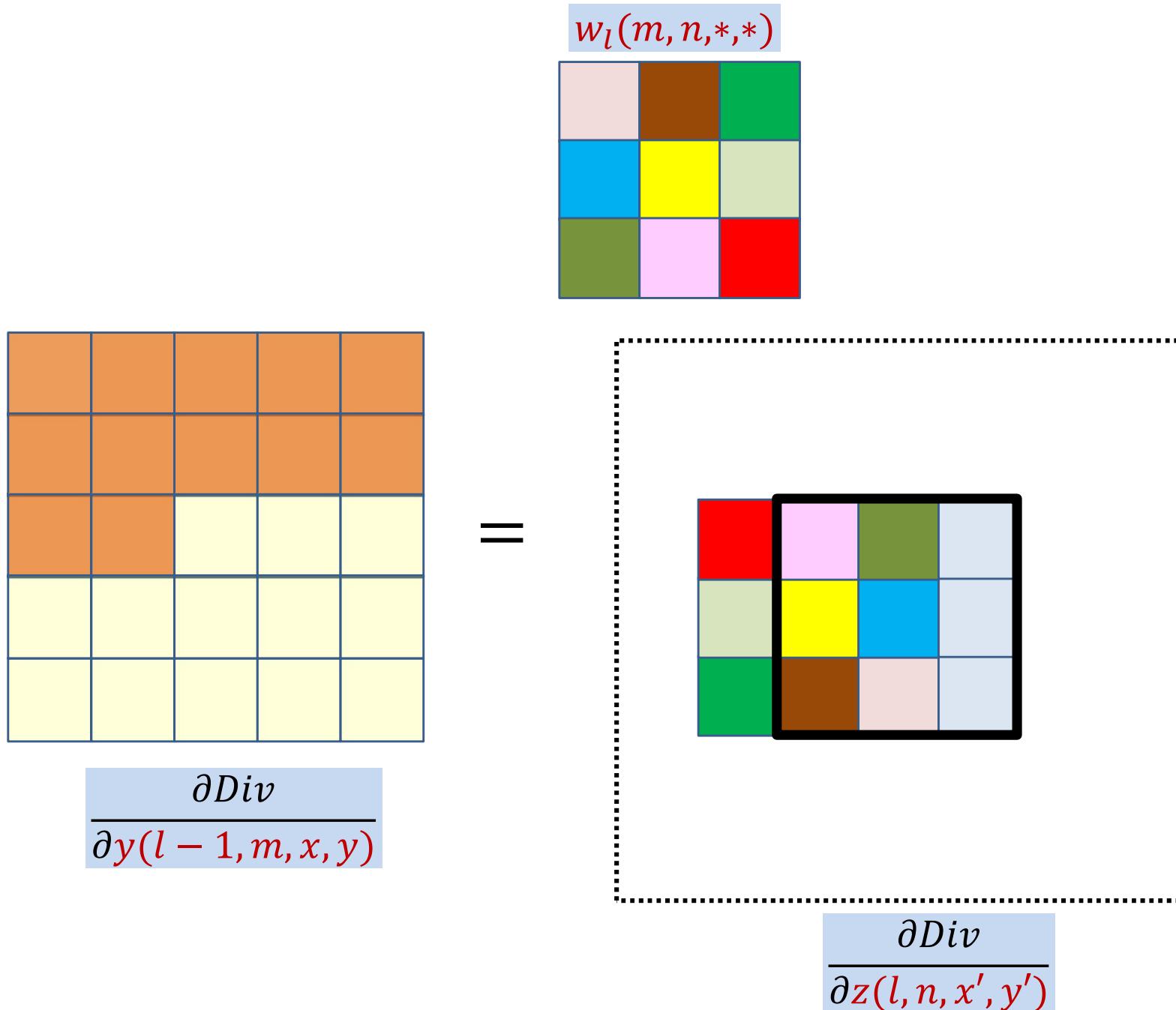
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



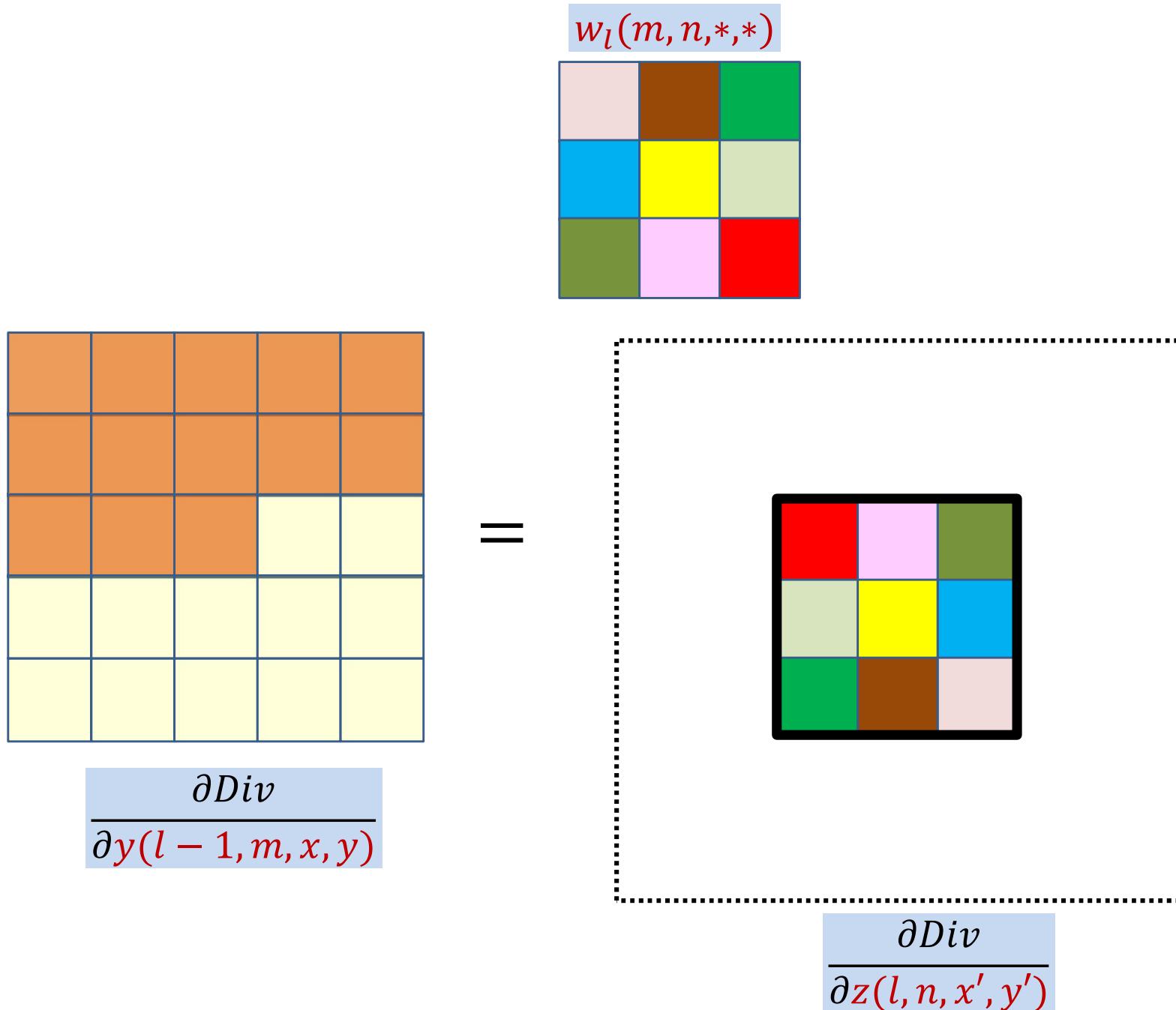
$$\frac{\partial \text{Div}}{\partial y(l-1, m, x, y)}$$



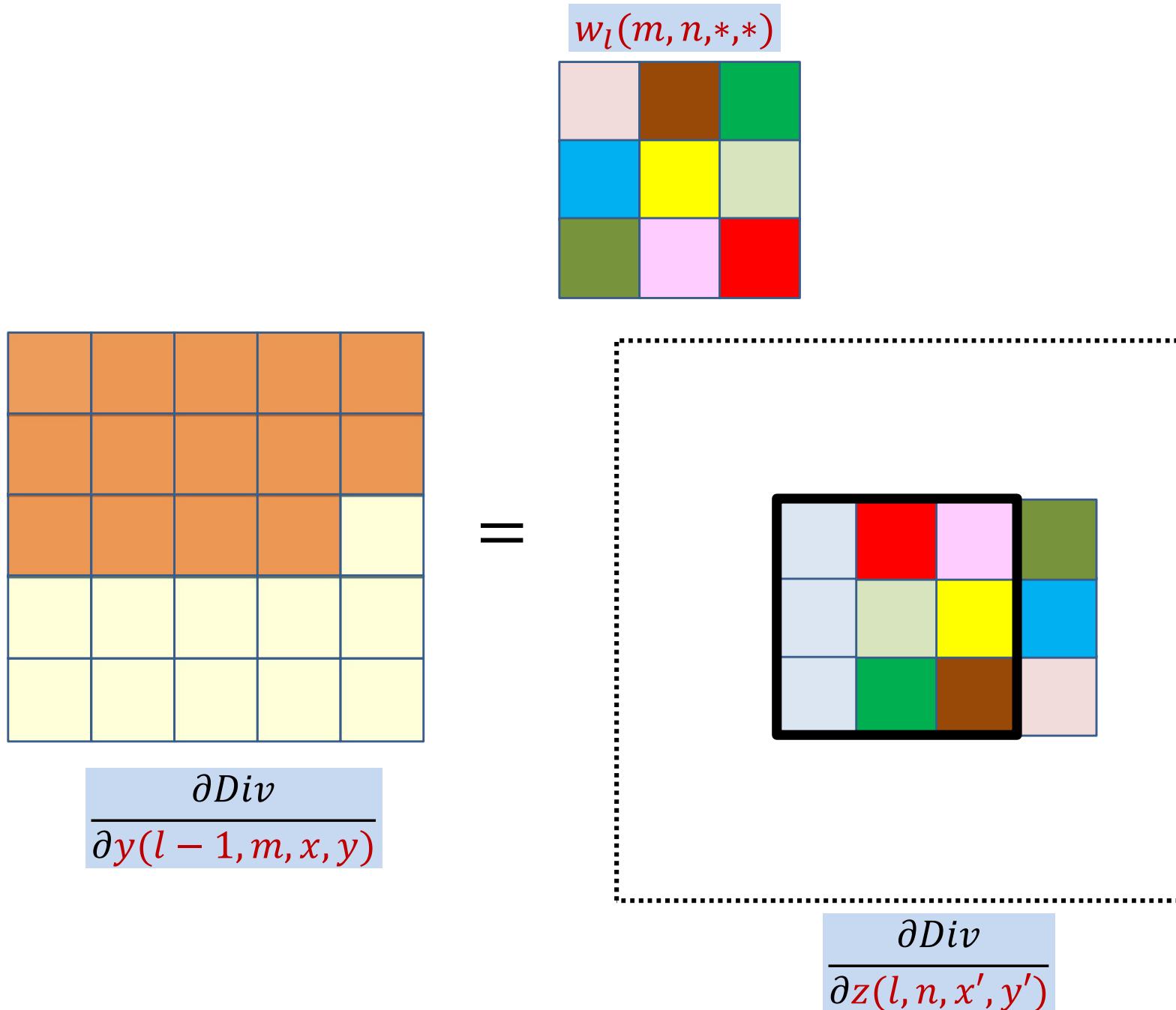
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



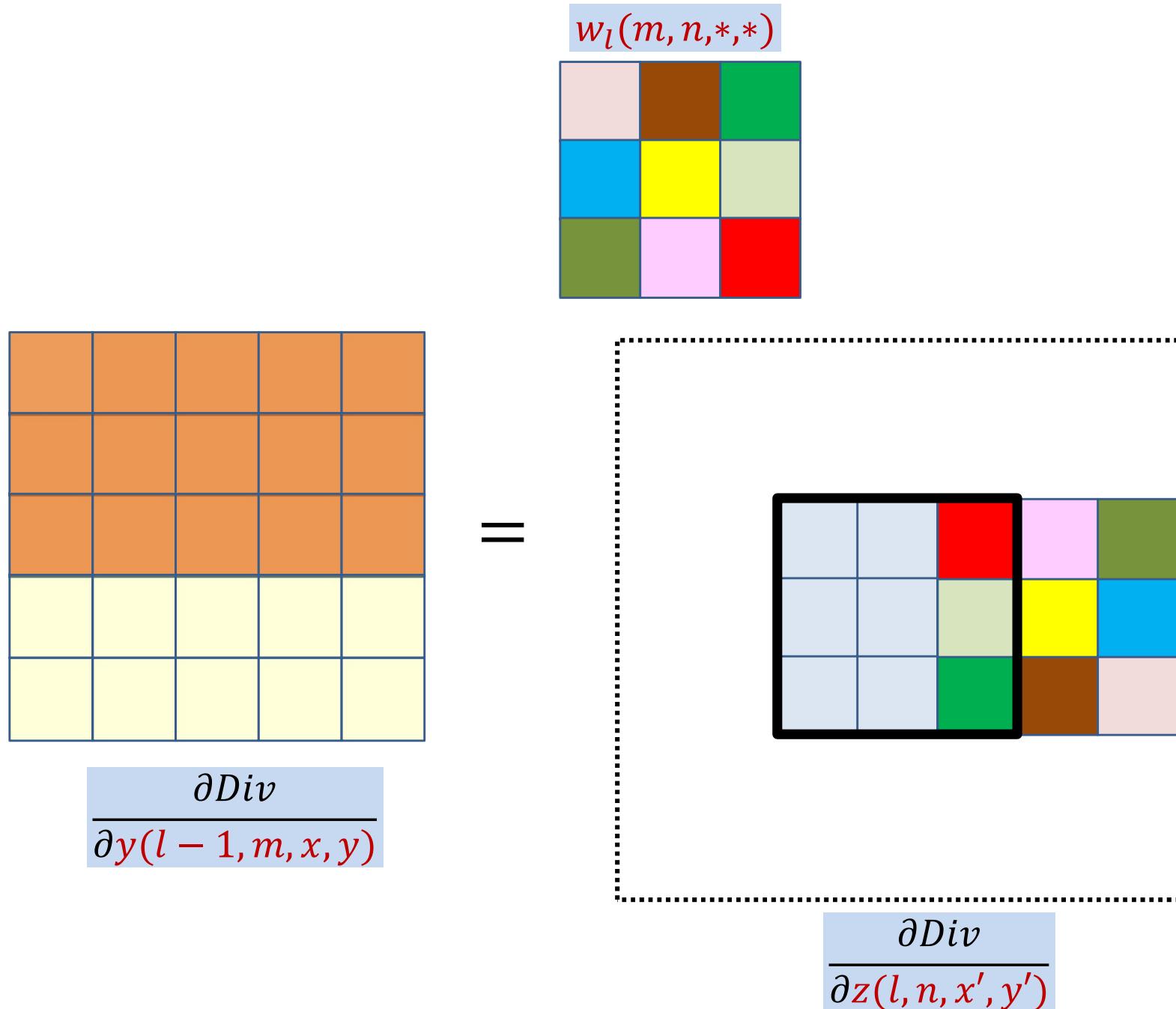
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



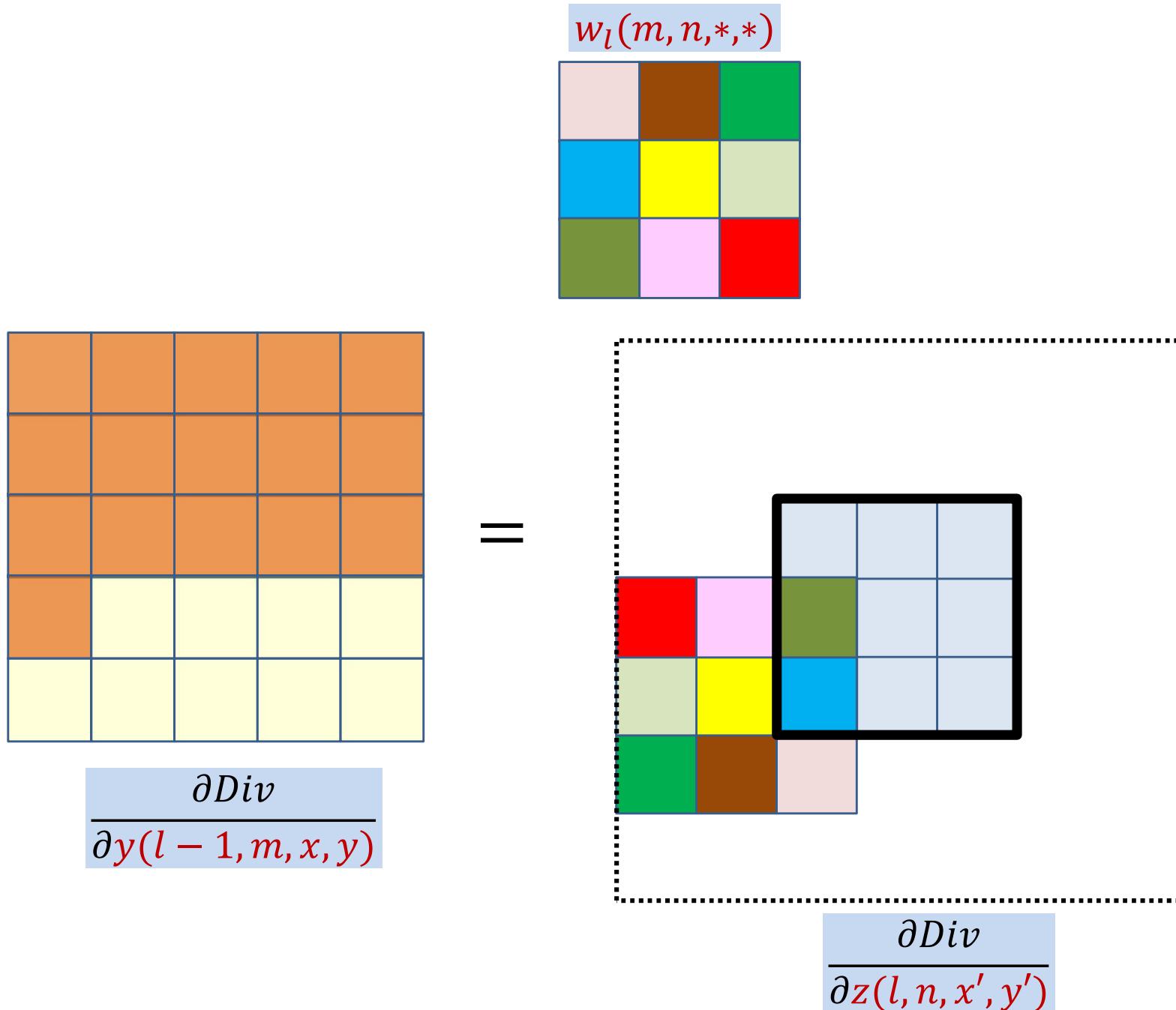
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



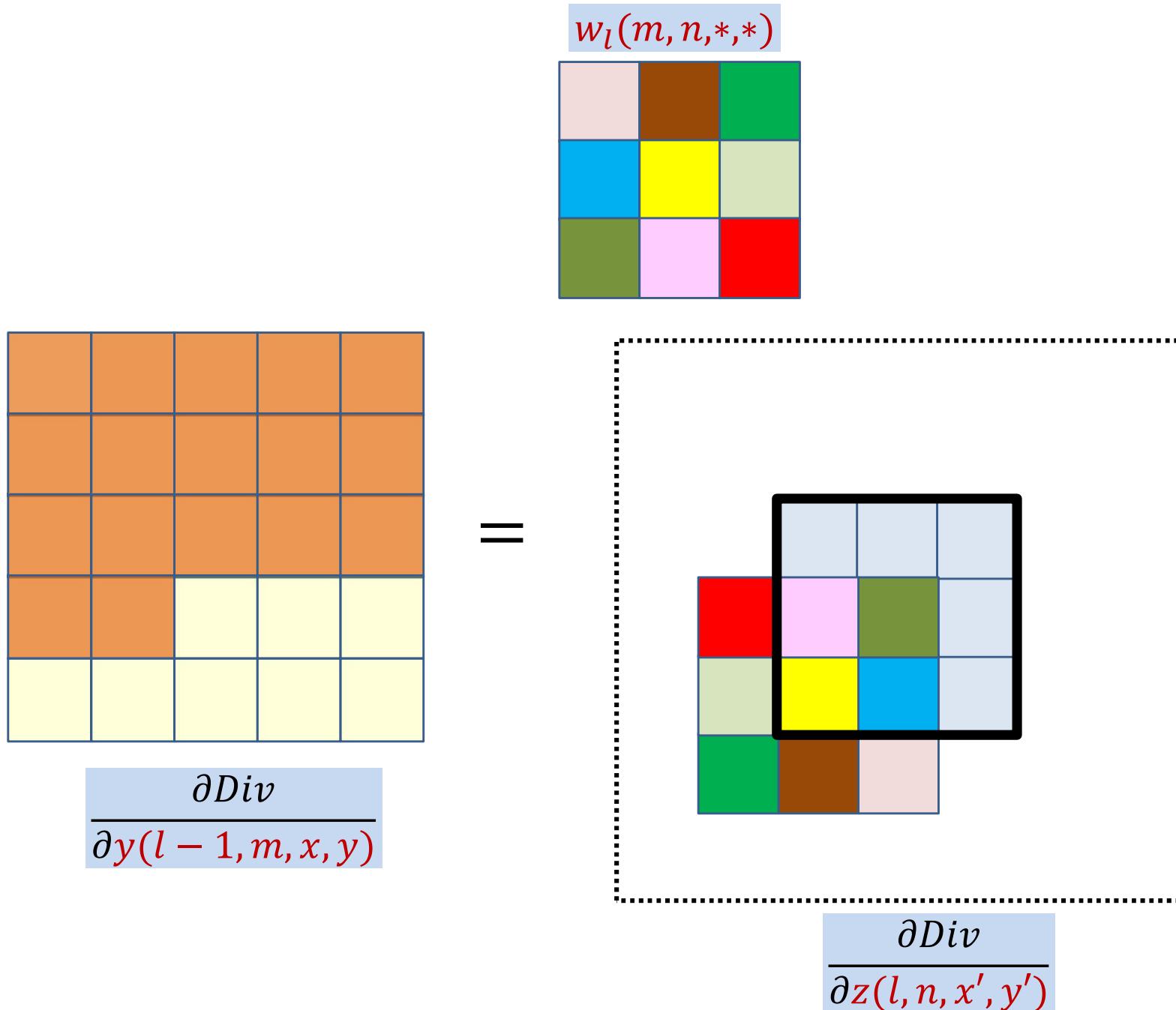
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



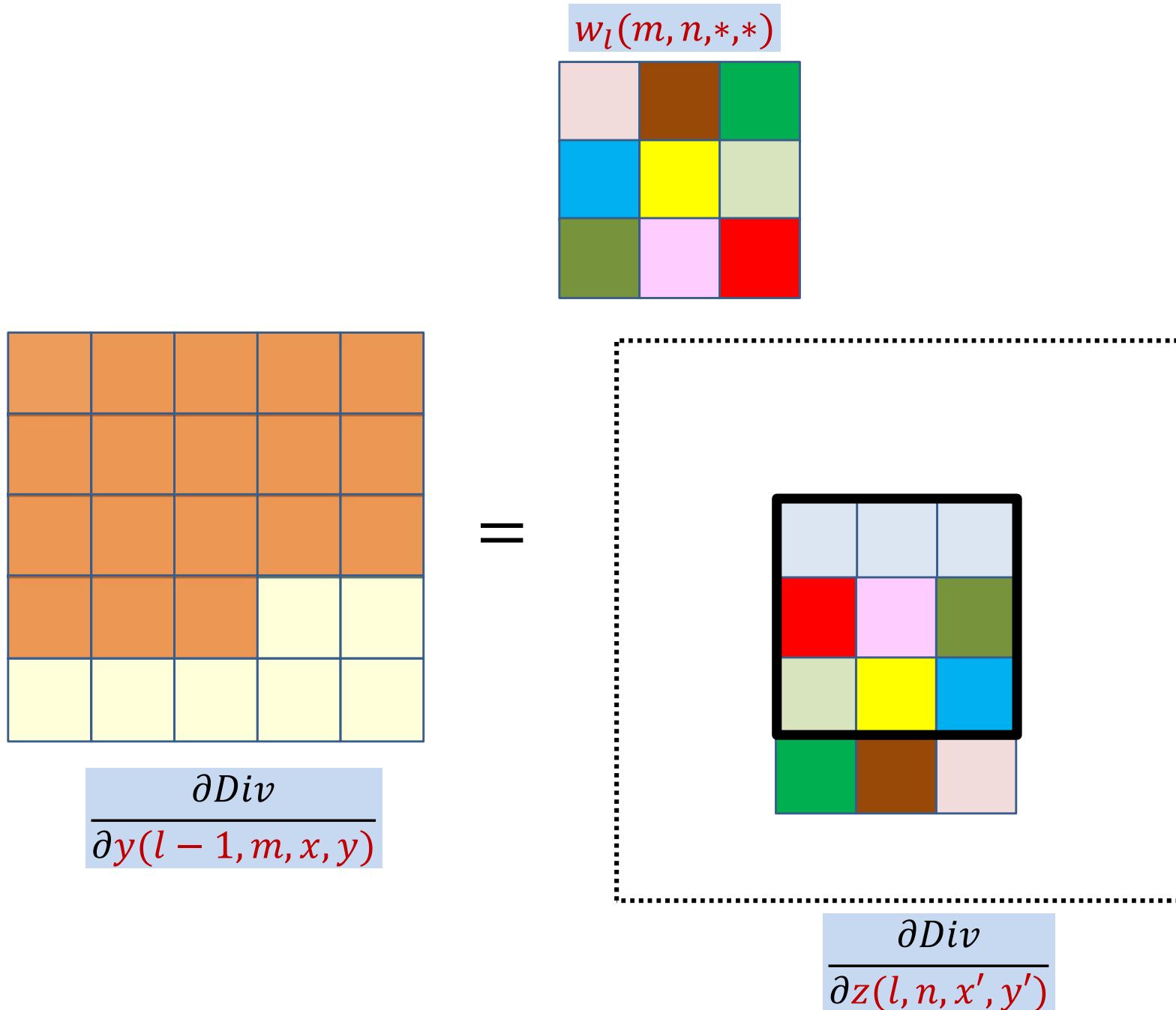
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



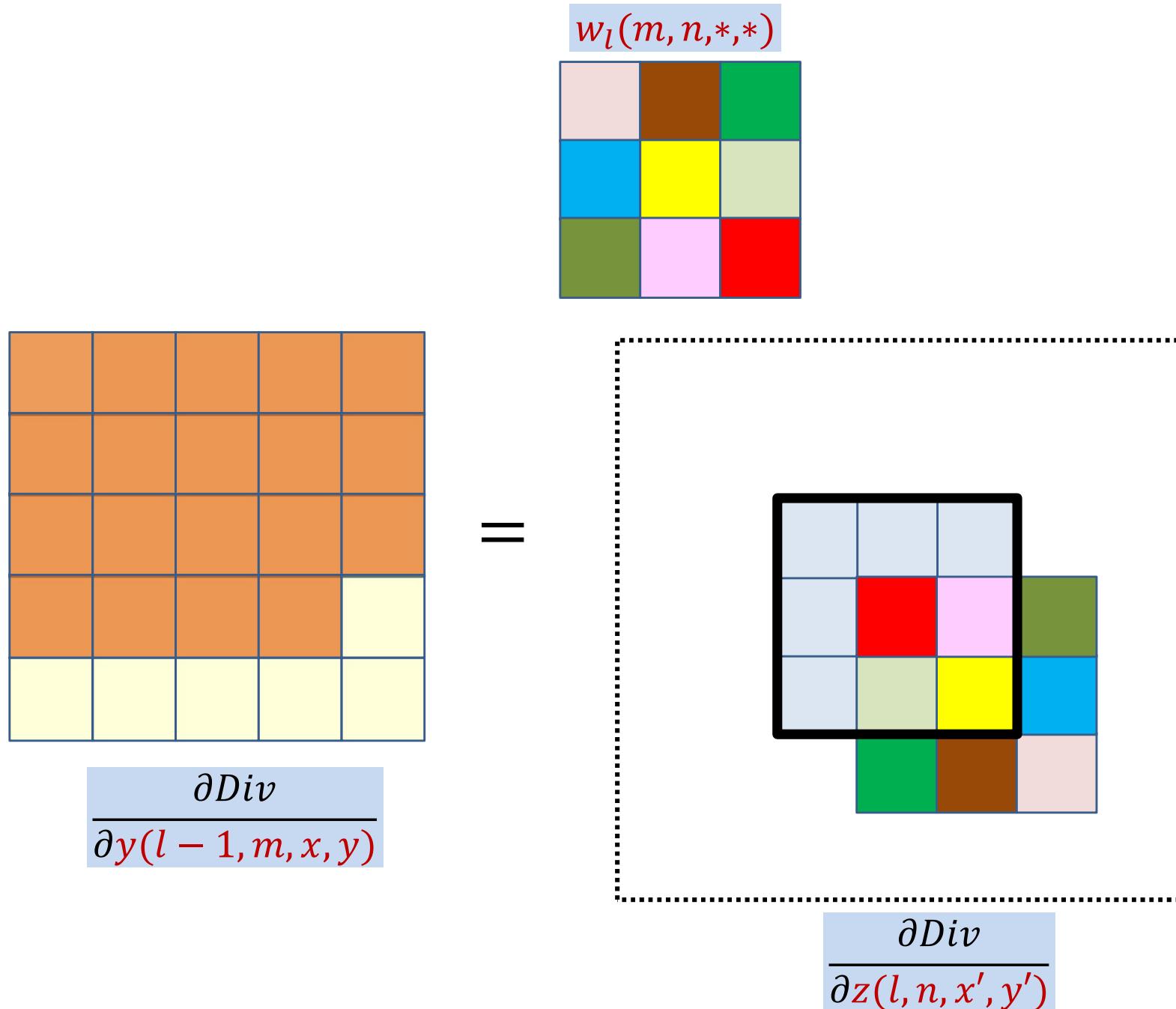
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



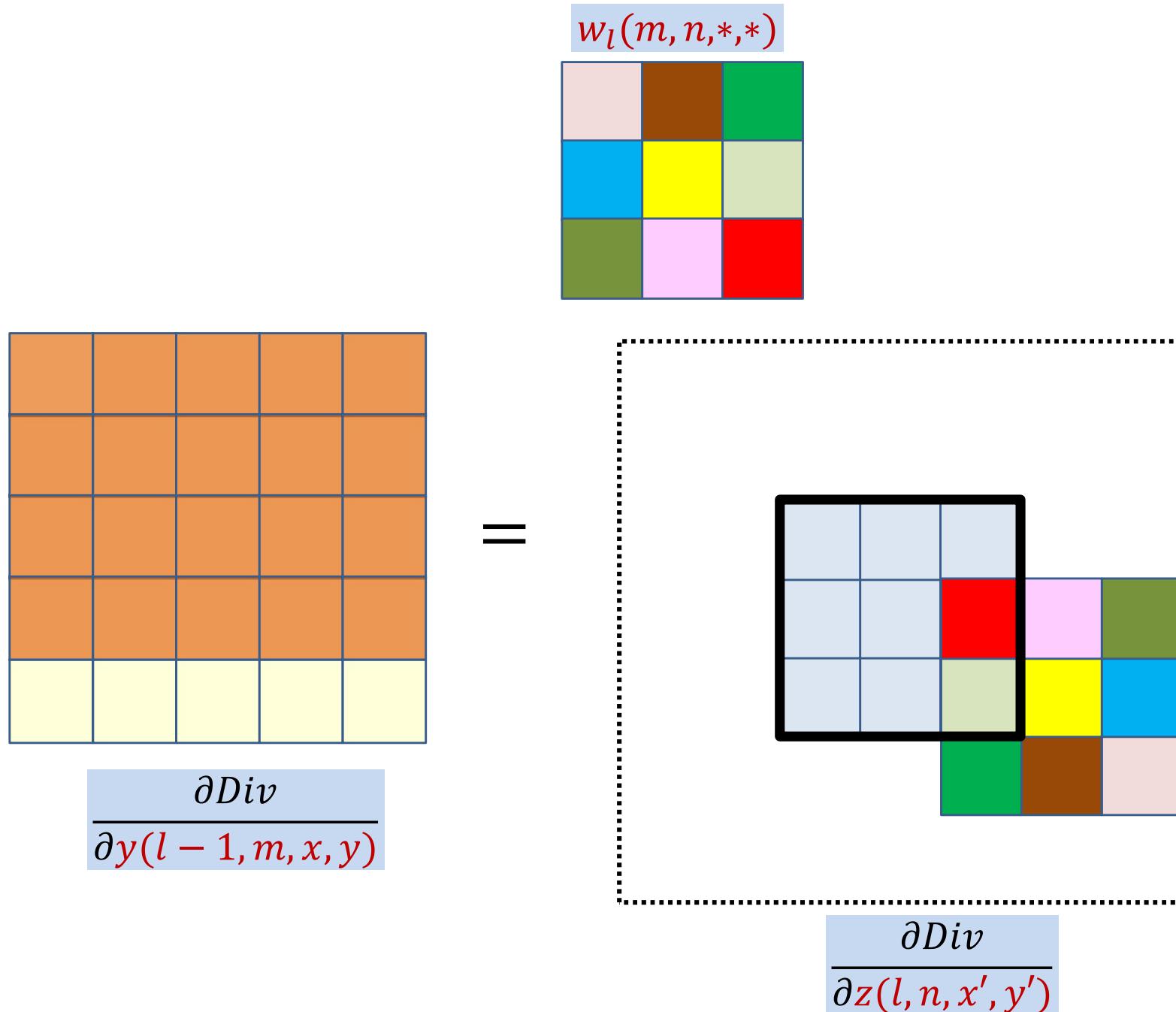
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



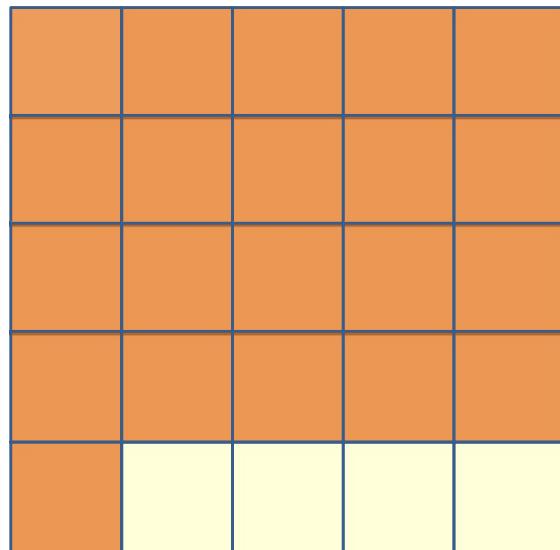
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



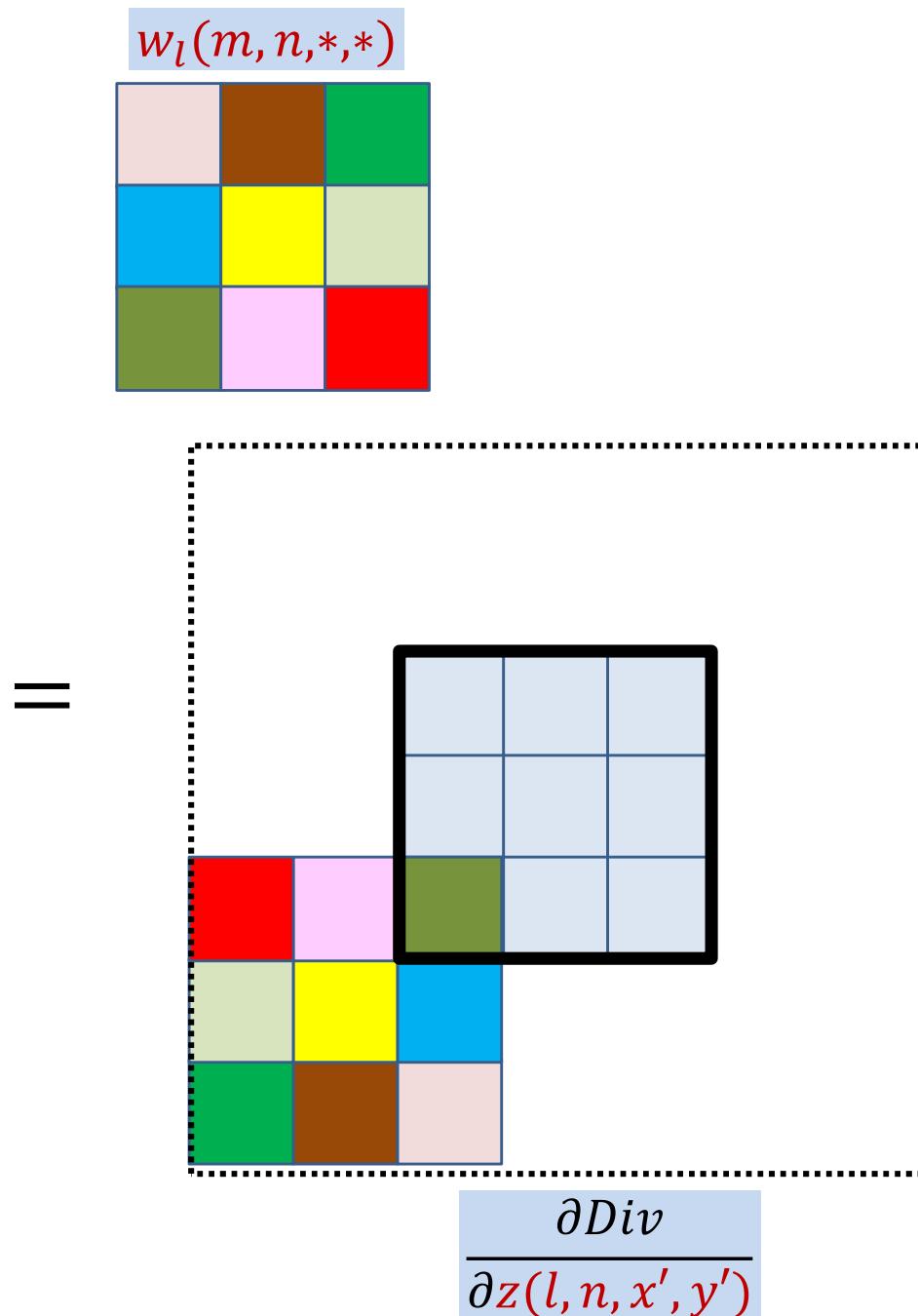
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



$$\frac{\partial \text{Div}}{\partial y(l-1, m, x, y)}$$

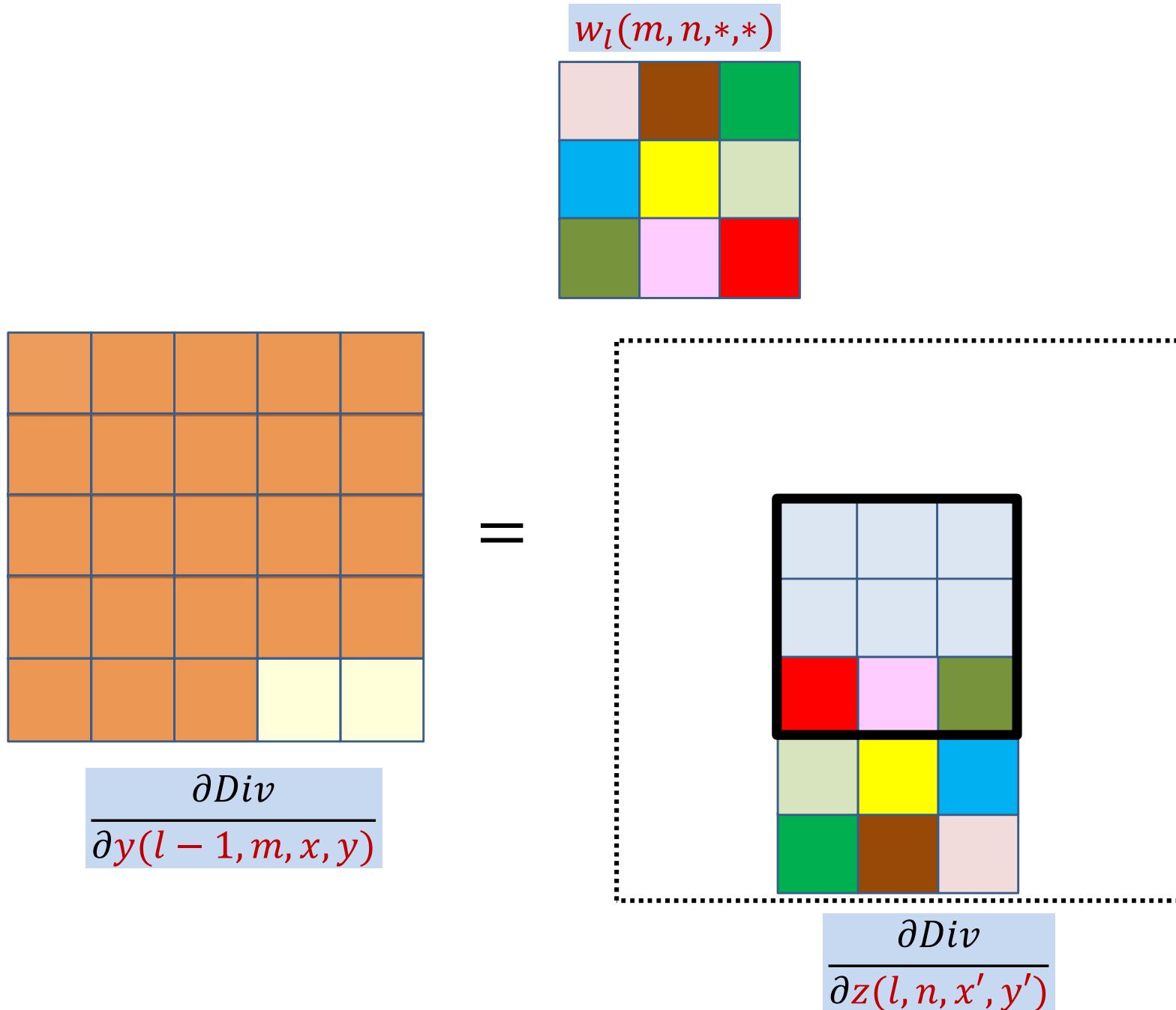


Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map

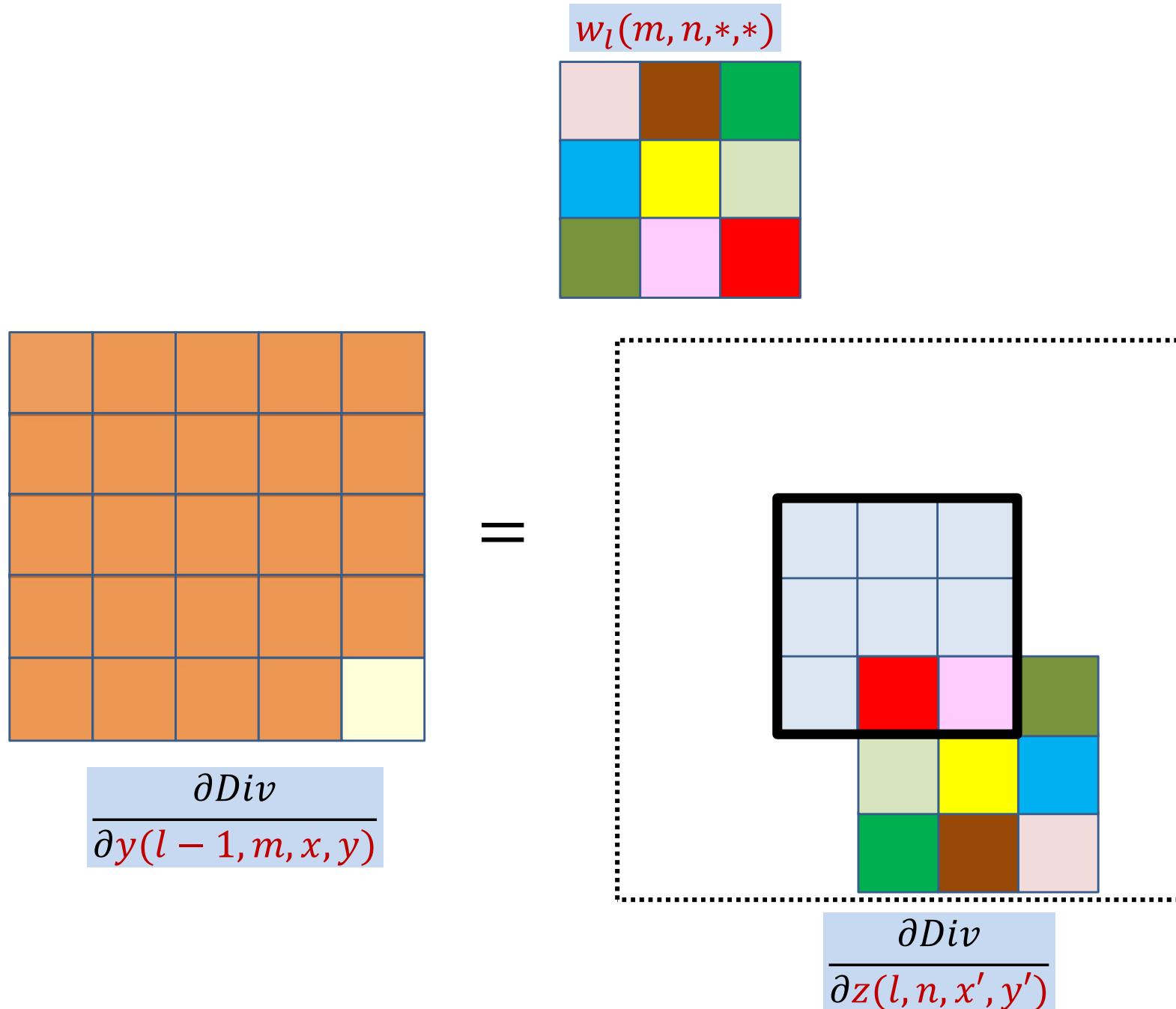
$$\frac{\partial \text{Div}}{\partial y(l - 1, m, x, y)} = \boxed{\begin{array}{c} w_l(m, n, *, *) \\ \hline \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \end{array}} = \boxed{\begin{array}{c} \frac{\partial \text{Div}}{\partial z(l, n, x', y')} \\ \hline \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \end{array}}$$

The diagram illustrates the computation of the derivative of the divergence operator with respect to a specific element of the input map y . The input map y is represented by a 5x5 grid of orange squares. The target element $y(l-1, m)$ is highlighted in white. The output map z is represented by a 3x3 grid of colored squares. The target element $z(l, n)$ is highlighted with a black border. The derivative is computed using a 3x3 kernel $w_l(m, n, *, *)$, which is also shown as a 3x3 grid of colored squares. The result is a 3x3 grid where each element is the sum of the products of the corresponding elements in the input row, the output column, and the kernel.

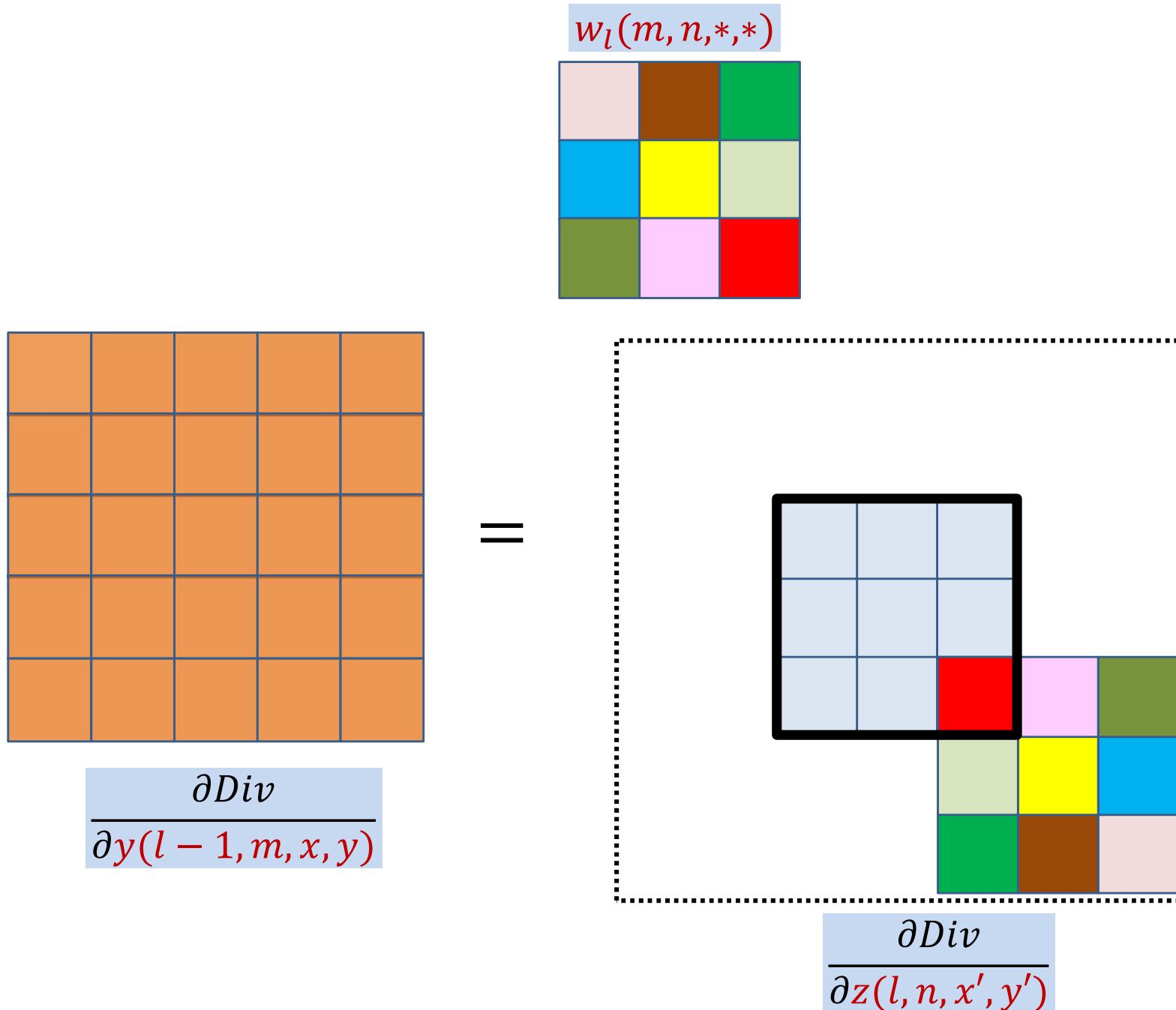
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



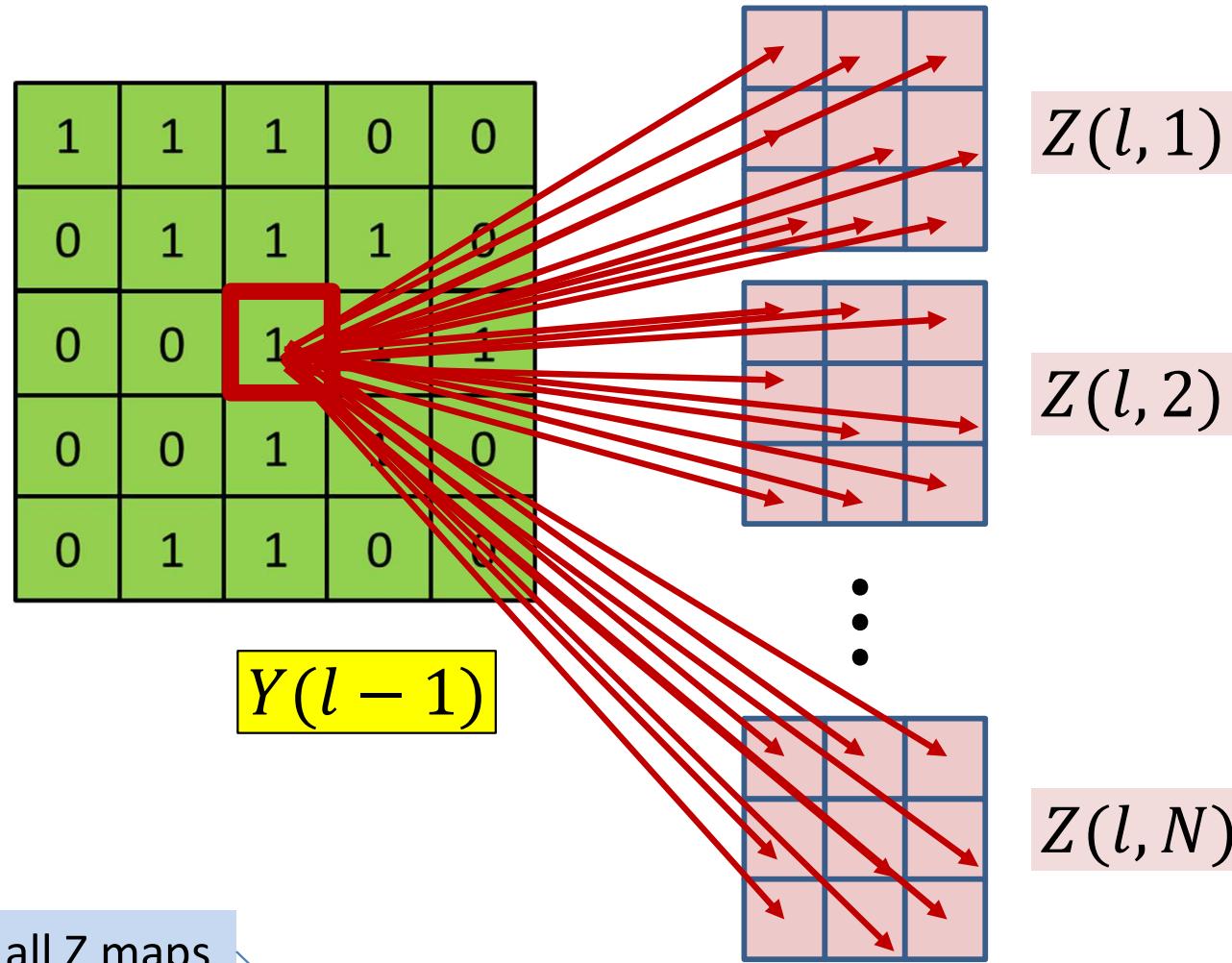
Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map



Derivative at $Y(l - 1, m)$ from a single $Z(l, n)$ map

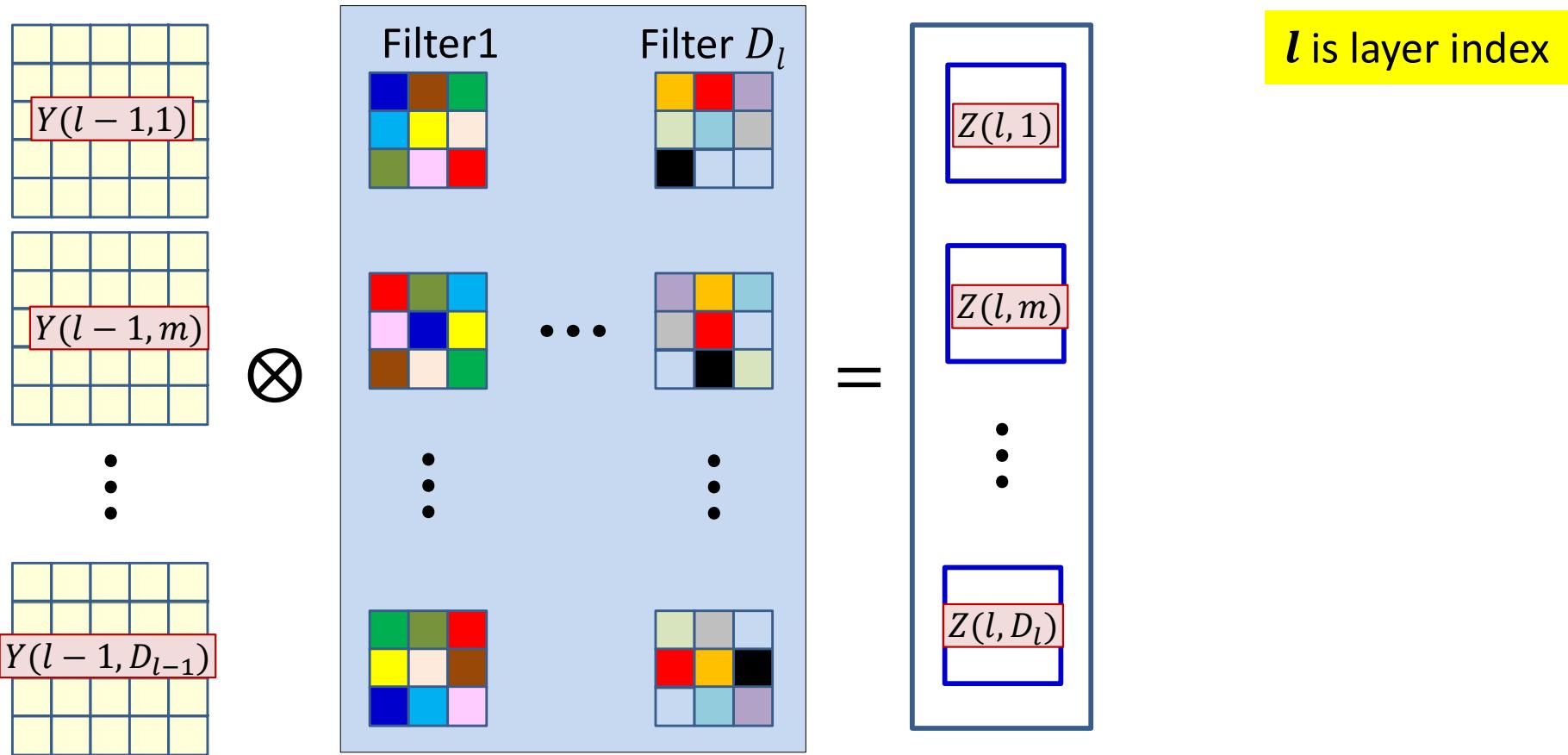


BP: Convolutional layer



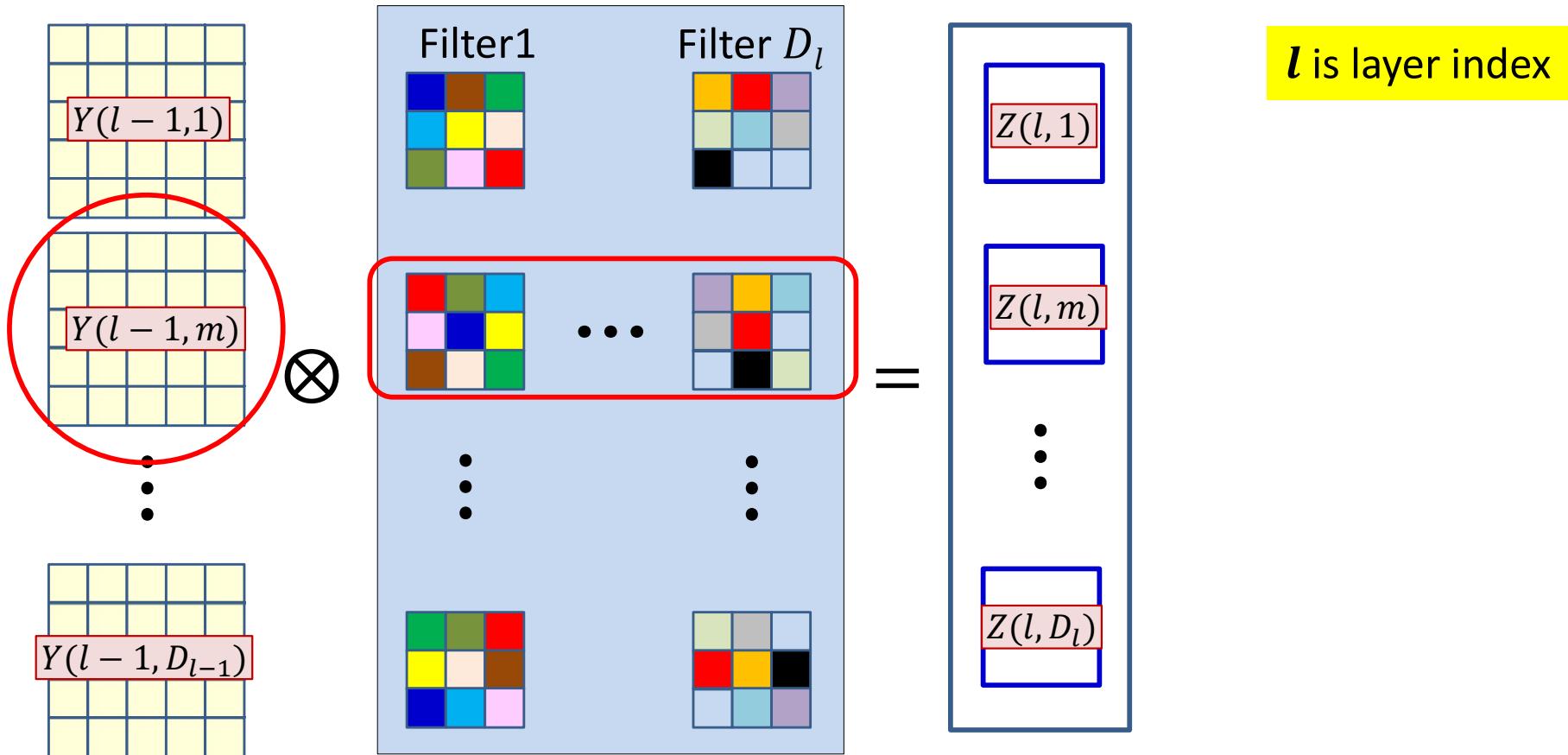
$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

The actual convolutions



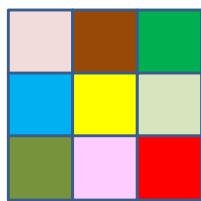
- The D_l affine maps are produced by convolving with D_l filters

The actual convolutions



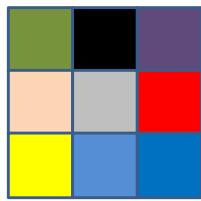
- The D_l affine maps are produced by convolving with D_l filters
- The m^{th} Y map always convolves the m^{th} plane of the filters
- The derivative for the m^{th} Y map will invoke the m^{th} plane of *all* the filters

$w_l(m, n, x, y)$



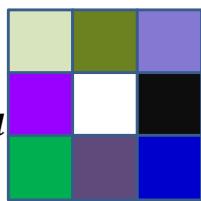
$n = 1$

$n = 2$



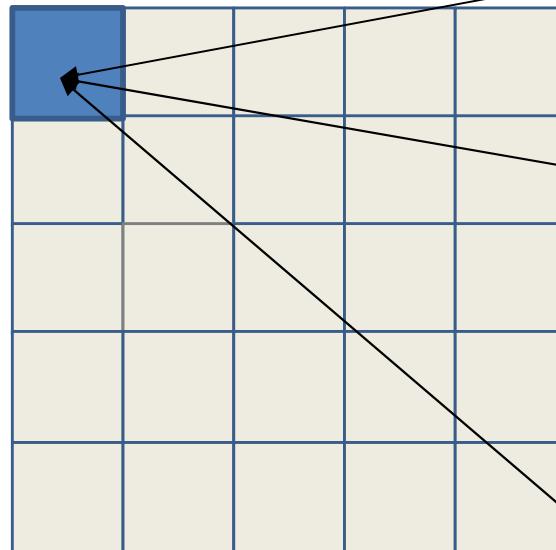
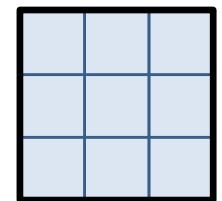
⋮

$n = D_l$

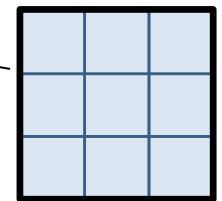


In reality, the derivative at each (x, y) location is obtained from *all* z maps

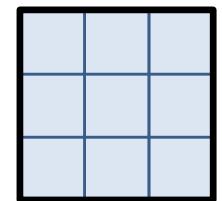
$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$



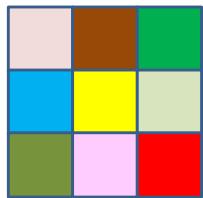
$$\frac{\partial \text{Div}}{\partial y(l - 1, m, x, y)}$$



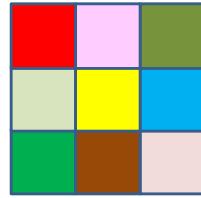
⋮



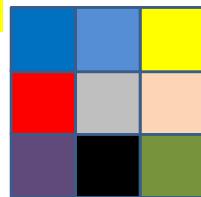
$w_l(m, n, x, y)$



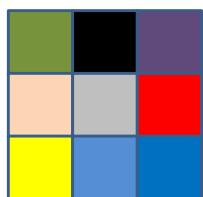
$n = 1$



flip

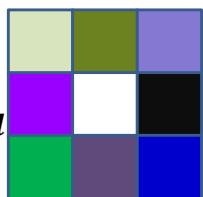


$n = 2$

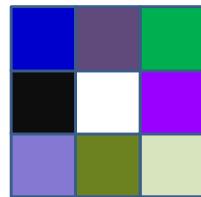


⋮

⋮

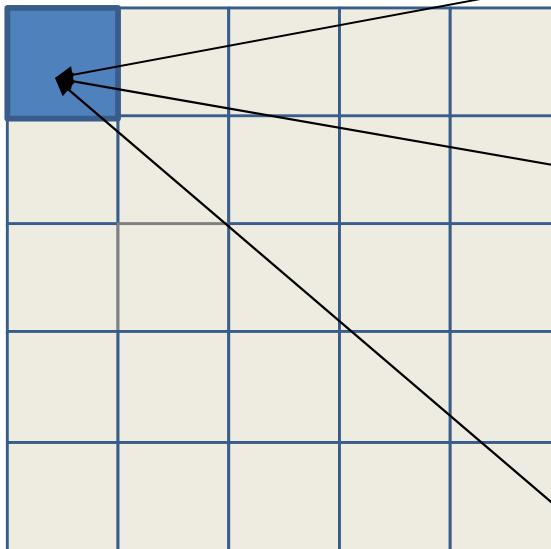


$n = D_l$



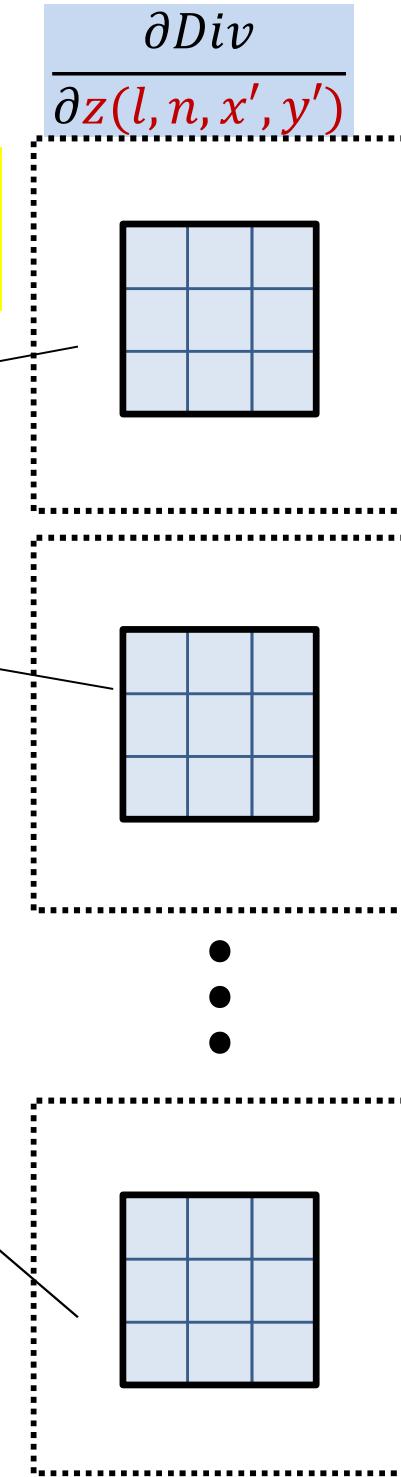
$w_l(m, n, K + 1 - x, K + 1 - y)$

In reality, the derivative at each (x, y) location is obtained from *all* z maps

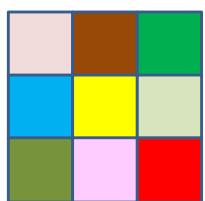


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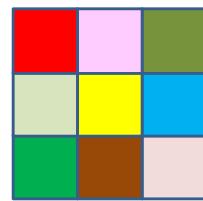
$\frac{\partial \text{Div}}{\partial y(l - 1, m, x, y)}$



$w_l(m, n, x, y)$



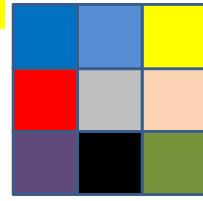
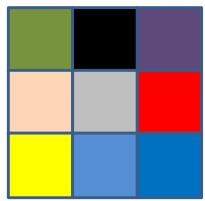
$n = 1$



flip



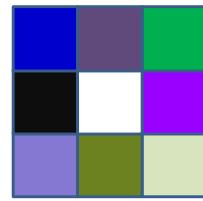
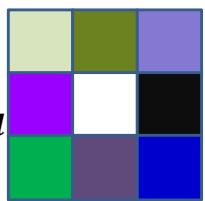
$n = 2$



⋮

⋮

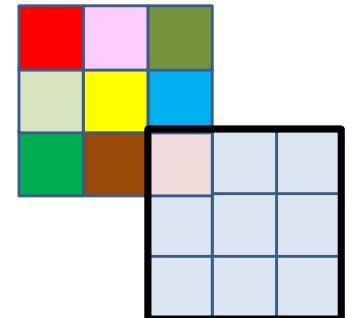
$n = D_l$



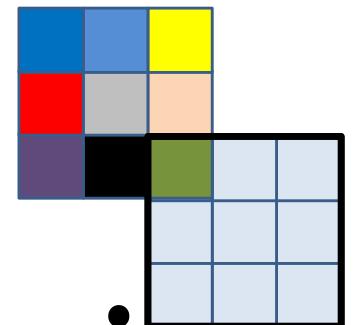
$w_l(m, n, K + 1 - x, K + 1 - y)$

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

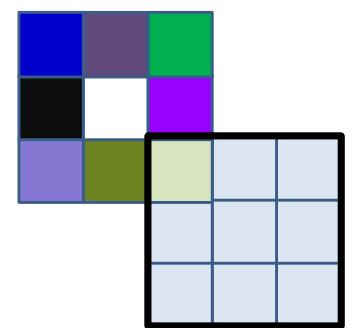
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$



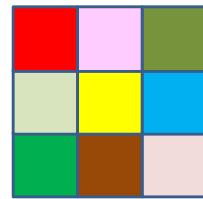
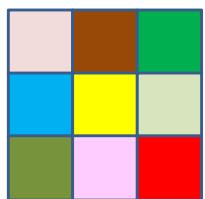
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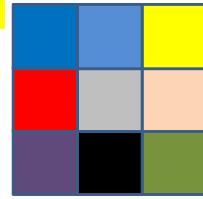
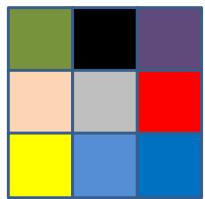
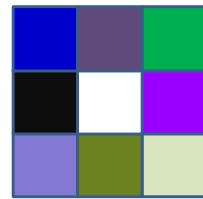
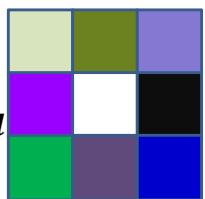
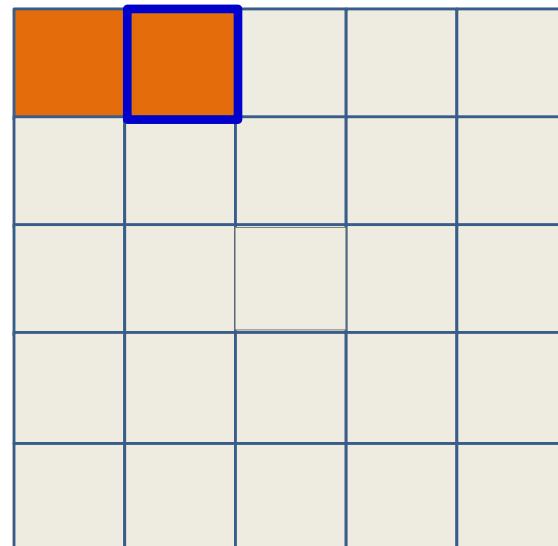
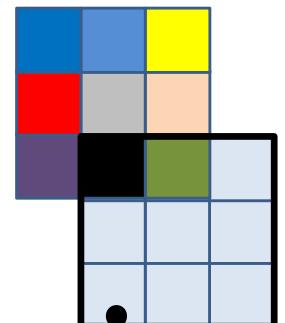
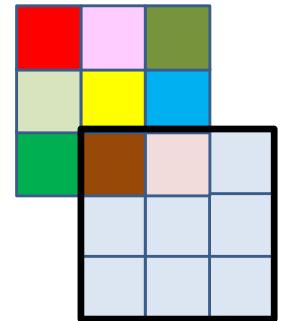
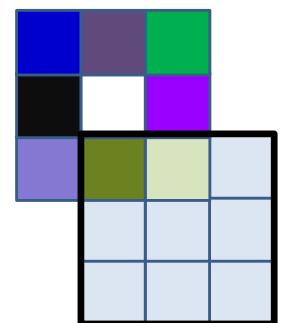
⋮



$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

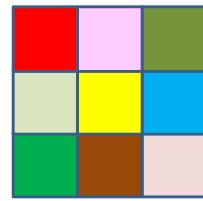
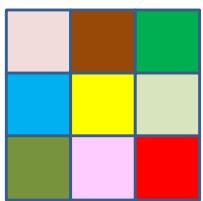
$w_l(m, n, x, y)$  $n = 1$

flip

 $n = 2$ \vdots \vdots  $n = D_l$ $w_l(m, n, K + 1 - x, K + 1 - y)$  $=$  \vdots  $\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$

$$\frac{d\text{Div}}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d\text{Div}}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

$w_l(m, n, x, y)$

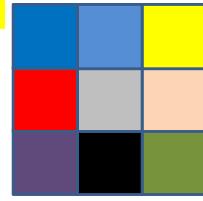
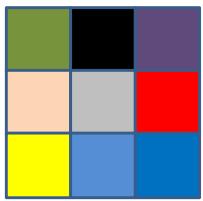


$n = 1$

flip

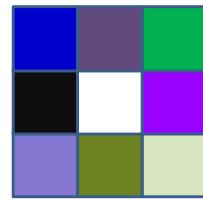
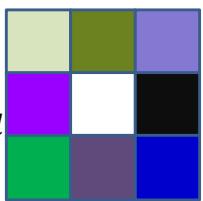


$n = 2$



⋮

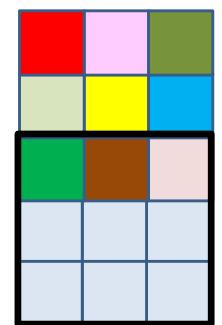
⋮



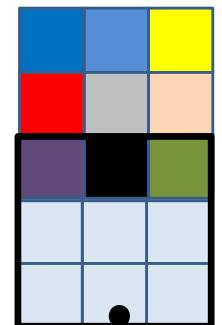
$w_l(m, n, K + 1 - x, K + 1 - y)$

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

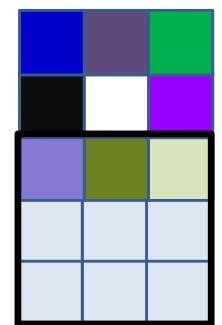
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$



=

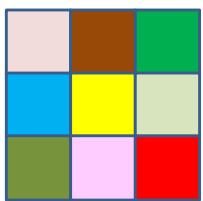


⋮

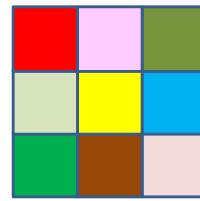


$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

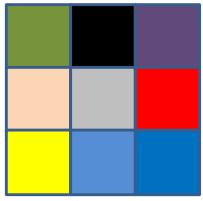
$$w_l(m, n, x, y)$$



$n = 1$



flip

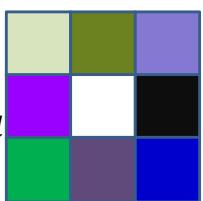


$n = 2$

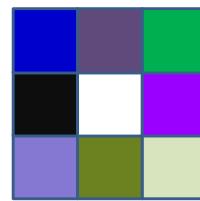
⋮

⋮

⋮



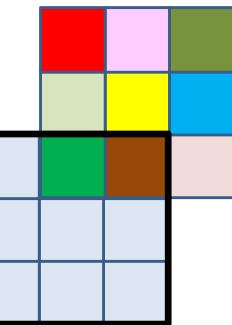
$n = D_l$



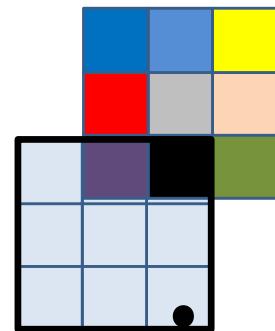
$$w_l(m, n, K + 1 - x, K + 1 - y)$$

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

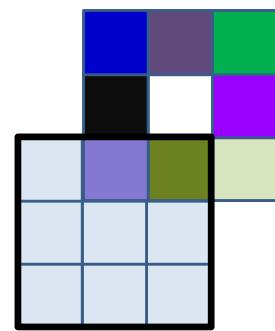


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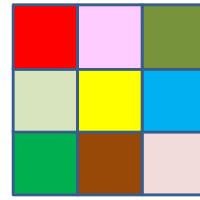
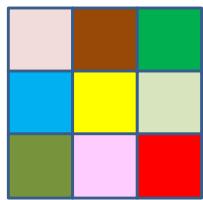
⋮

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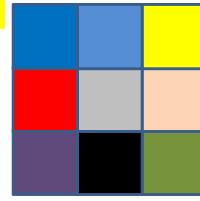
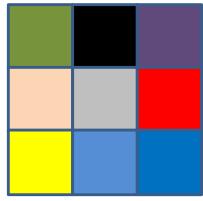
$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

$w_l(m, n, x, y)$



$n = 1$

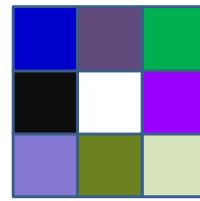
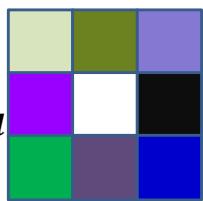
flip



$n = 2$

⋮

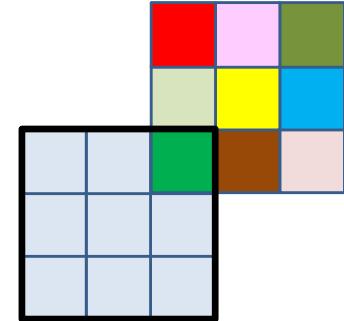
⋮



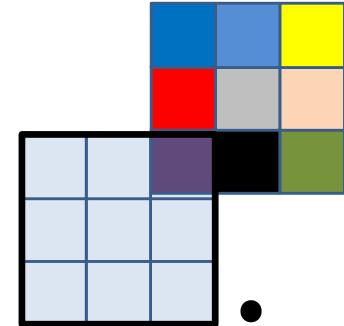
$w_l(m, n, K + 1 - x, K + 1 - y)$

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

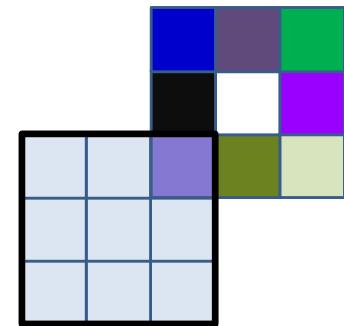
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$



=

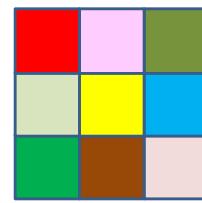
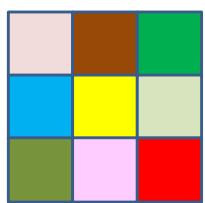


⋮
⋮
⋮



$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

$w_l(m, n, x, y)$

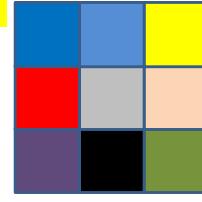
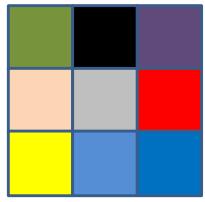


$n = 1$

flip



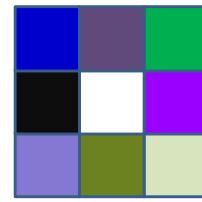
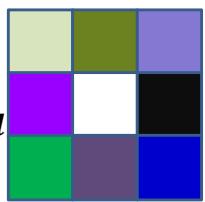
$n = 2$



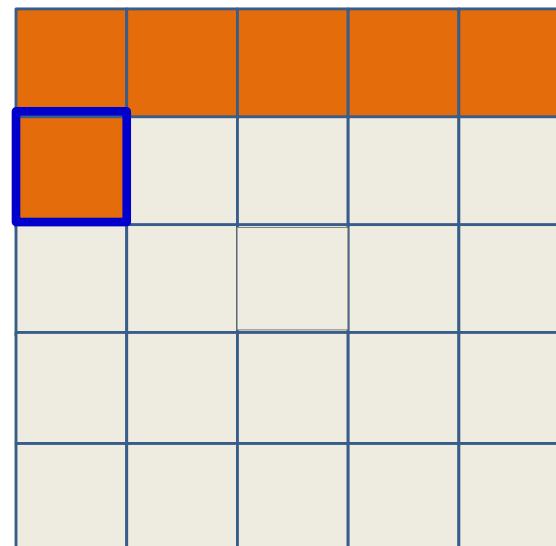
⋮

⋮

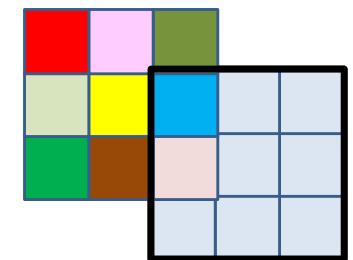
$n = D_l$



$w_l(m, n, K + 1 - x, K + 1 - y)$



=

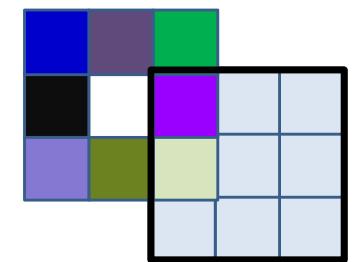


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⋮

∂Div

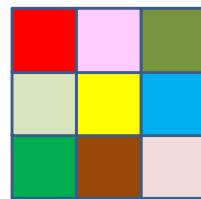
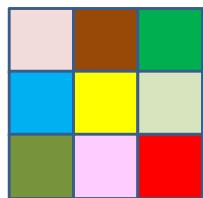
$\frac{\partial \text{Div}}{\partial y(l - 1, m, x, y)}$

$$\frac{d \text{Div}}{d Y(l - 1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d \text{Div}}{d z(l, n, x', y')} w_l(m, n, x - x', y - y')$$



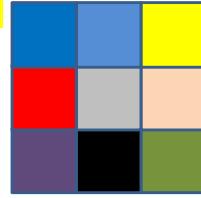
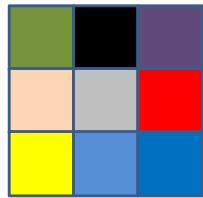
$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$

$w_l(m, n, x, y)$



$n = 1$

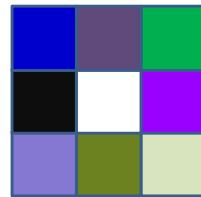
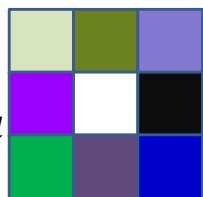
flip



$n = 2$

⋮

⋮

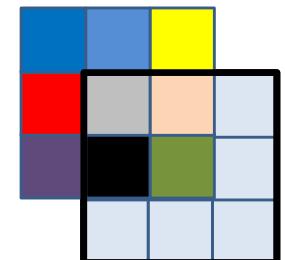
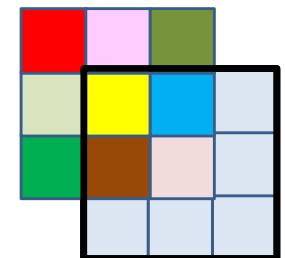


$n = D_l$

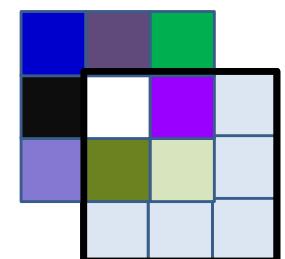
$w_l(m, n, K + 1 - x, K + 1 - y)$

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$

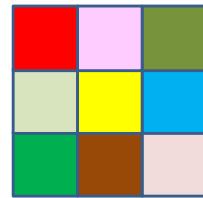
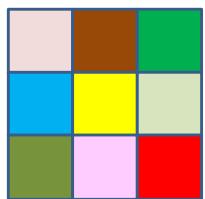


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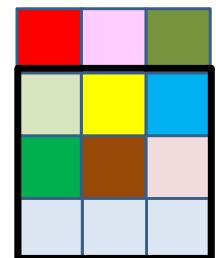


$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

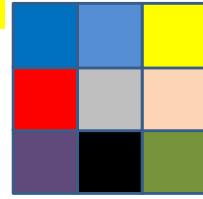
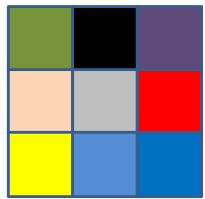
$w_l(m, n, x, y)$



$n = 1$



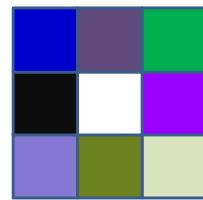
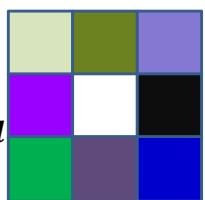
flip



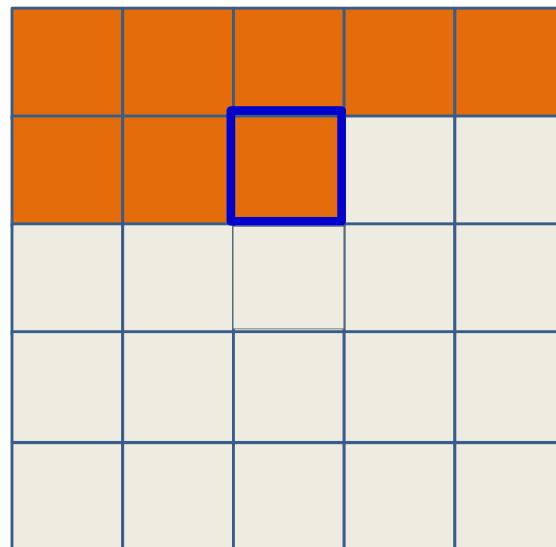
$n = 2$

⋮

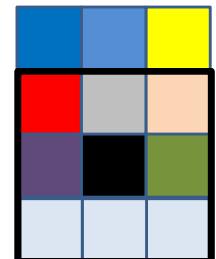
⋮



$w_l(m, n, K + 1 - x, K + 1 - y)$

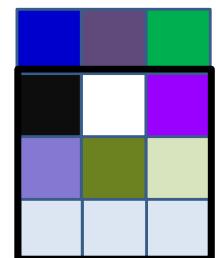


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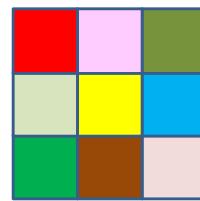
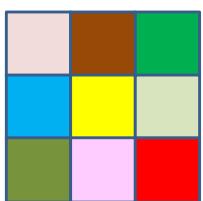
⋮

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$



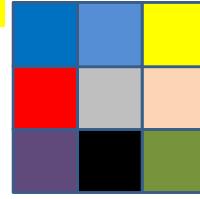
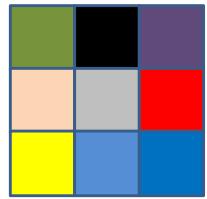
$\frac{\partial Div}{\partial z(l, n, x', y')}$

$w_l(m, n, x, y)$



$n = 1$

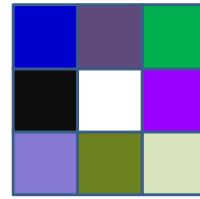
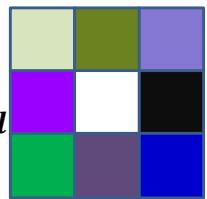
flip



$n = 2$

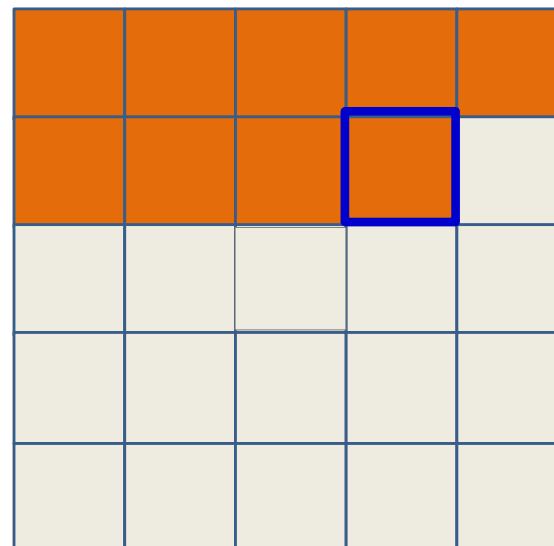
⋮

⋮

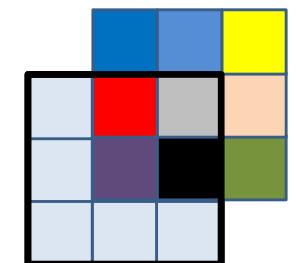
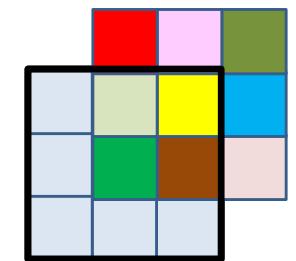


$n = D_l$

$w_l(m, n, K + 1 - x, K + 1 - y)$



=

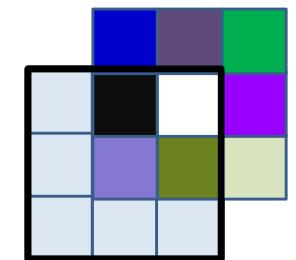


⋮

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

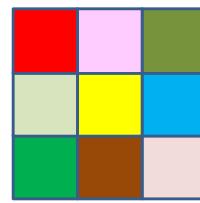
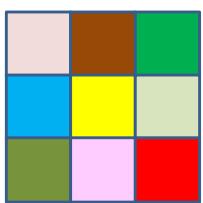
∂Div

$\frac{\partial Div}{\partial y(l-1, m, x, y)}$



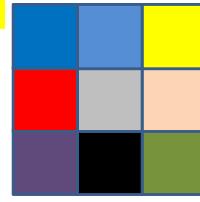
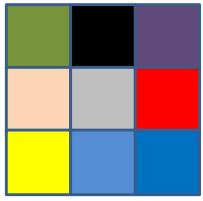
$\frac{\partial Div}{\partial z(l, n, x', y')}$

$w_l(m, n, x, y)$



$n = 1$

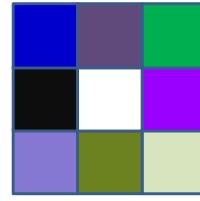
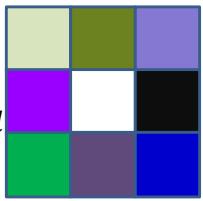
flip



$n = 2$

⋮

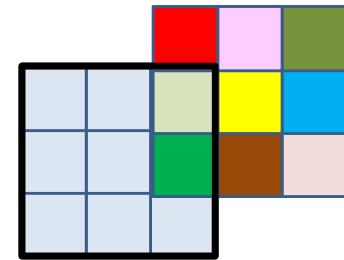
⋮



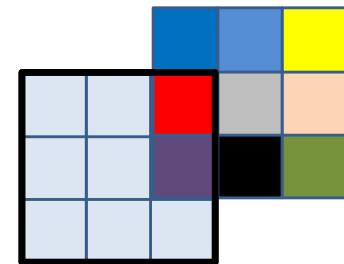
$w_l(m, n, K + 1 - x, K + 1 - y)$

$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

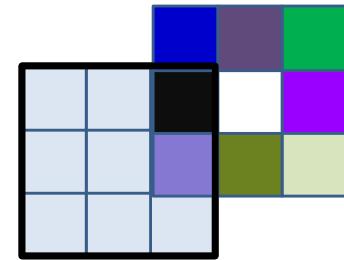
$$\frac{\partial Div}{\partial y(l-1, m, x, y)}$$



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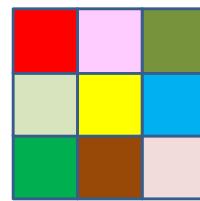
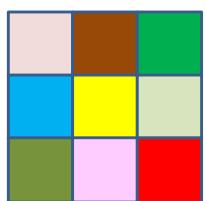


⋮



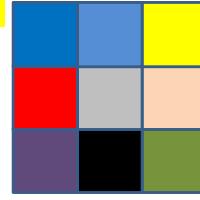
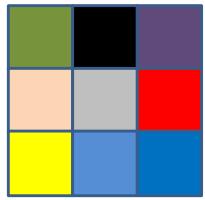
$$\frac{\partial Div}{\partial z(l, n, x', y')}$$

$w_l(m, n, x, y)$



$n = 1$

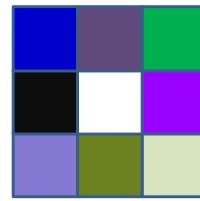
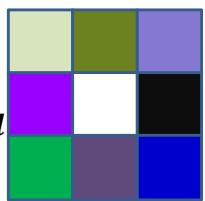
flip



$n = 2$

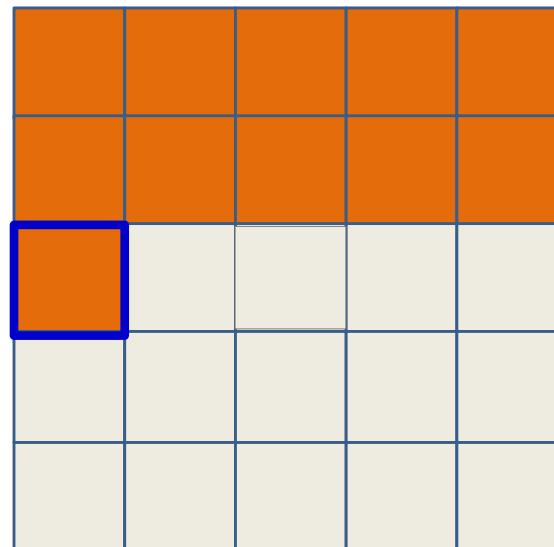
⋮

⋮

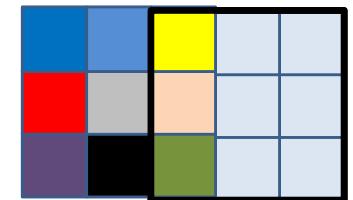
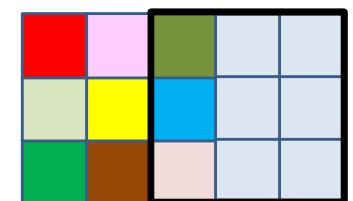


$n = D_l$

$w_l(m, n, K + 1 - x, K + 1 - y)$



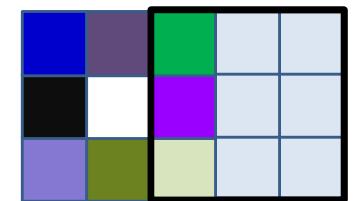
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⋮ ⋮

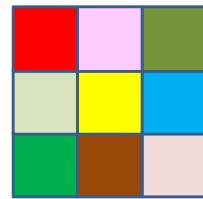
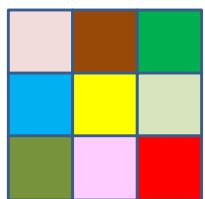
$$\frac{\partial \text{Div}}{\partial y(l-1, m, x, y)}$$

$$\frac{d\text{Div}}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d\text{Div}}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

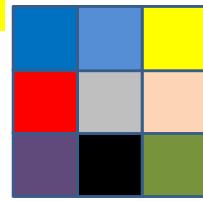
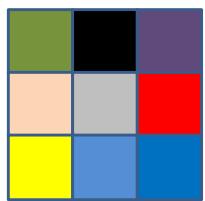


$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

$w_l(m, n, x, y)$



$n = 1$



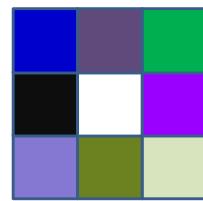
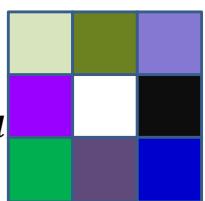
$n = 2$

flip



⋮

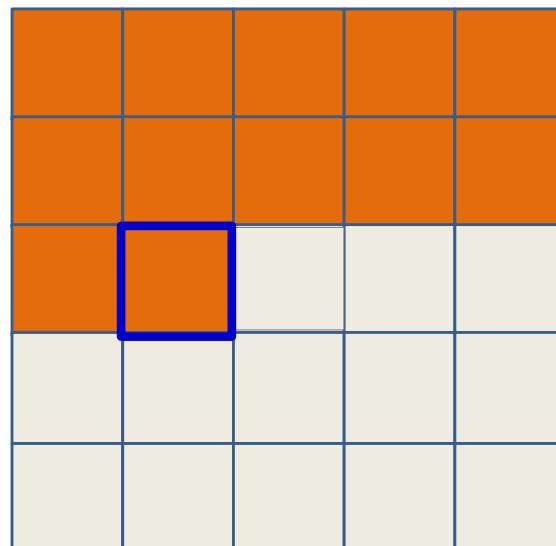
⋮



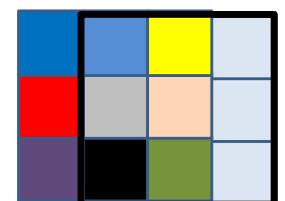
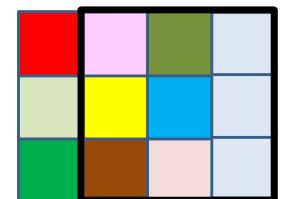
$n = D_l$

$w_l(m, n, K + 1 - x, K + 1 - y)$

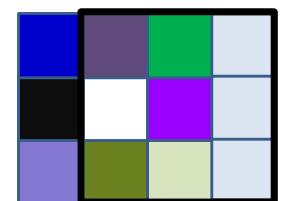
$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$



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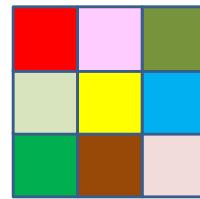
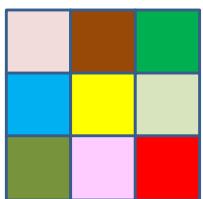


⋮

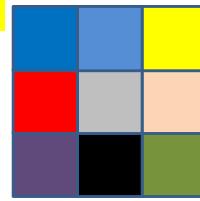
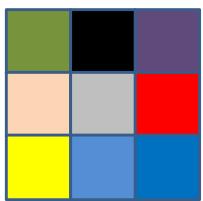


$\frac{\partial Div}{\partial z(l, n, x', y')}$

$w_l(m, n, x, y)$



$n = 1$



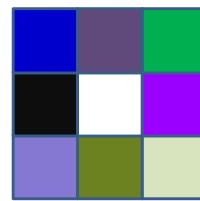
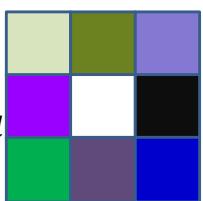
$n = 2$

flip



⋮

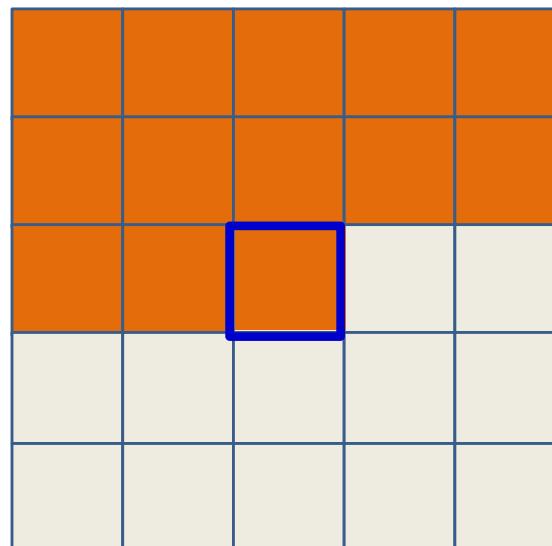
⋮



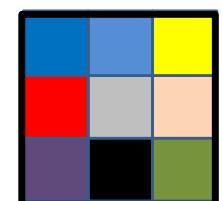
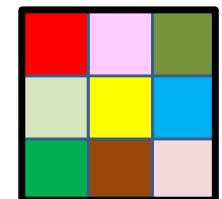
$n = D_l$

$w_l(m, n, K + 1 - x, K + 1 - y)$

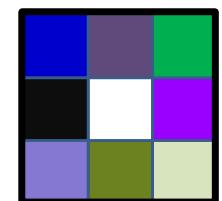
$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$



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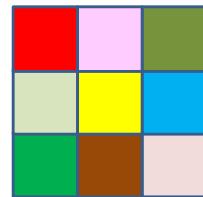
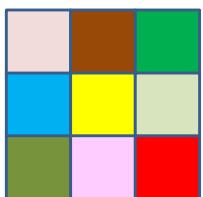


⋮

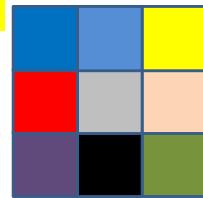
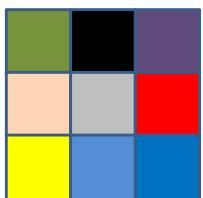


$\frac{\partial Div}{\partial z(l, n, x', y')}$

$w_l(m, n, x, y)$



$n = 1$



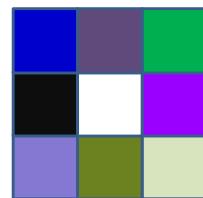
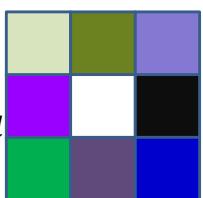
$n = 2$

flip



⋮

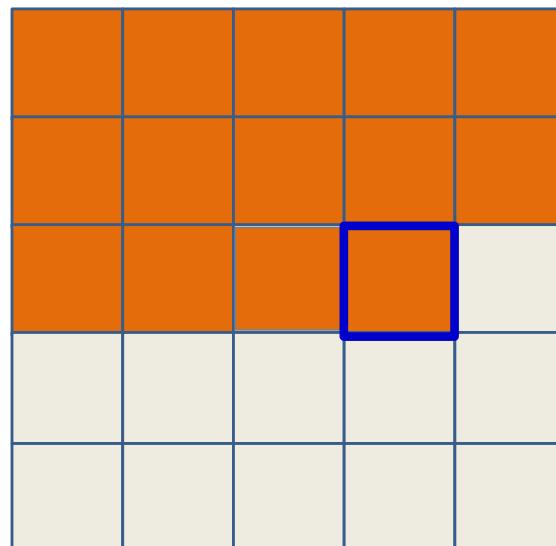
⋮



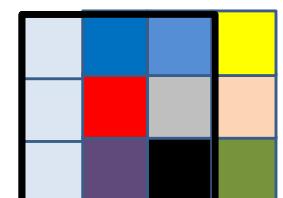
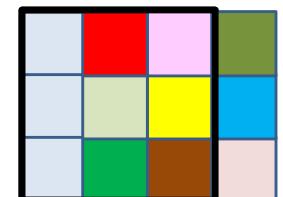
$n = D_l$

$w_l(m, n, K + 1 - x, K + 1 - y)$

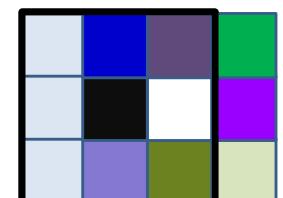
$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$



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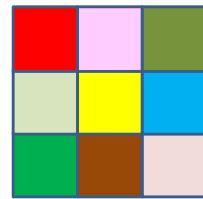
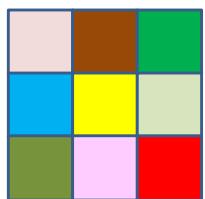


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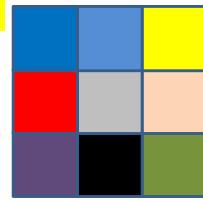
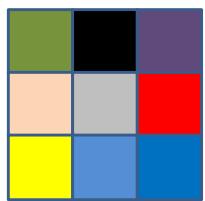


$\frac{\partial Div}{\partial z(l, n, x', y')}$

$w_l(m, n, x, y)$



$n = 1$



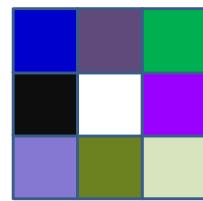
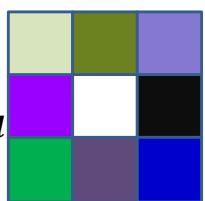
$n = 2$

flip



⋮

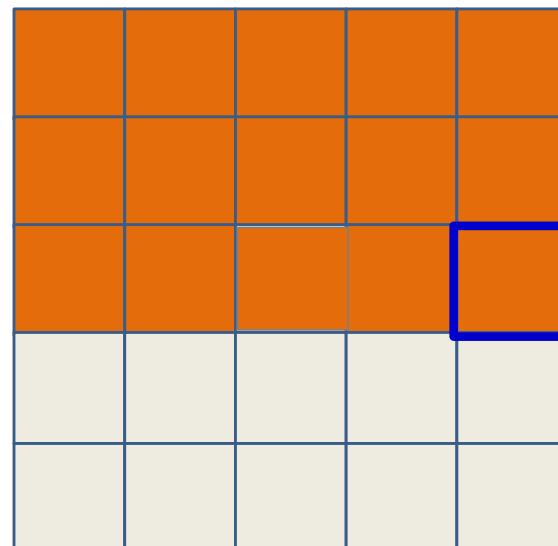
⋮



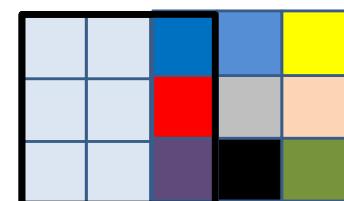
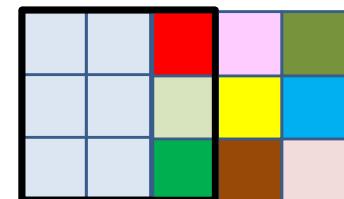
$n = D_l$

$w_l(m, n, K + 1 - x, K + 1 - y)$

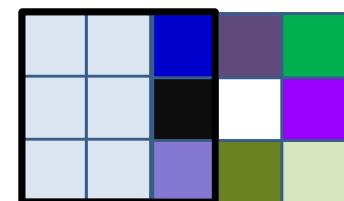
$$\frac{dDiv}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$



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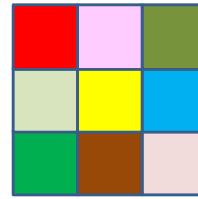
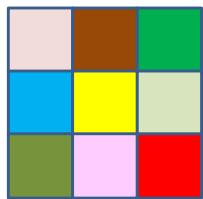
⋮ ⋮ ⋮



$\frac{\partial Div}{\partial z(l, n, x', y')}$

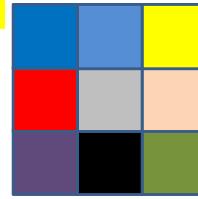
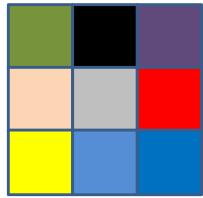
$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$



$n = 1$

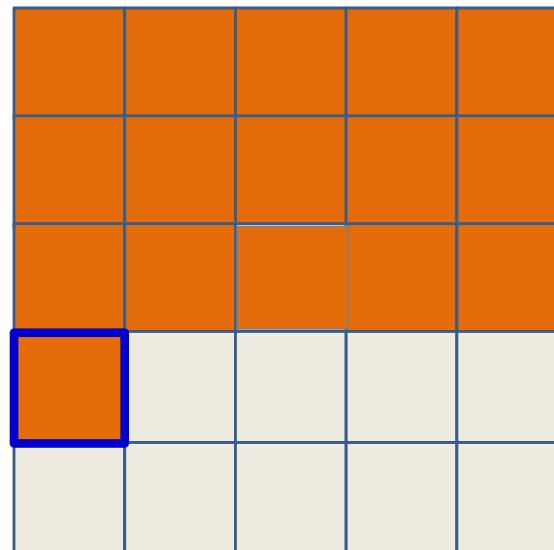
flip



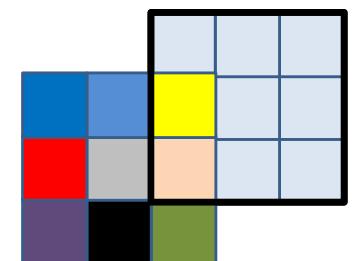
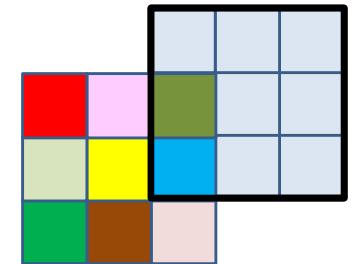
$n = 2$

⋮

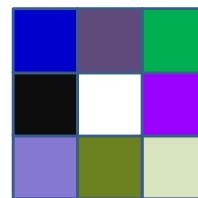
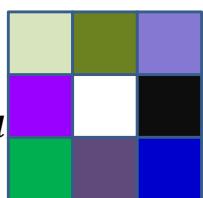
⋮



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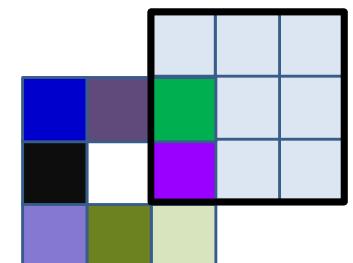
⋮



$n = D_l$

$$w_l(m, n, K + 1 - x, K + 1 - y)$$

$$\frac{d\text{Div}}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d\text{Div}}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

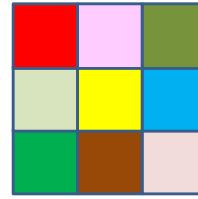
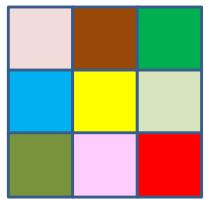


∂Div

$$\frac{\partial \text{Div}}{\partial y(l-1, m, x, y)}$$

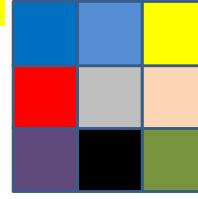
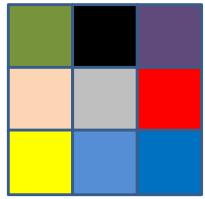
$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$



$n = 1$

flip

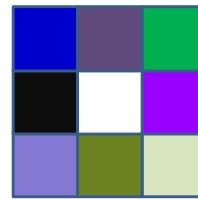
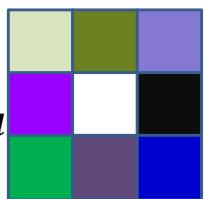


$n = 2$

⋮

⋮

⋮

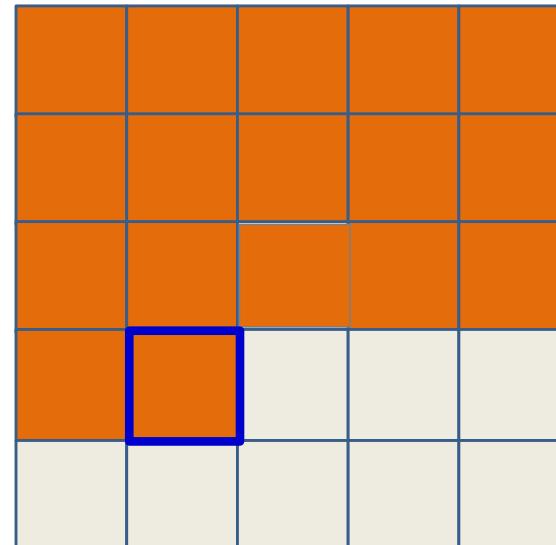


$n = D_l$

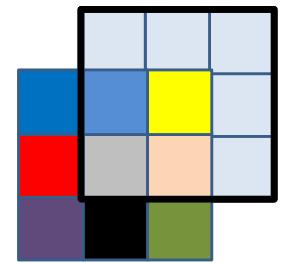
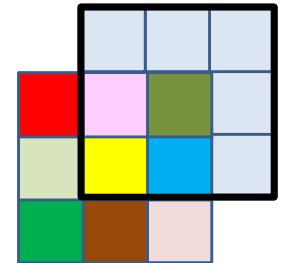
$$w_l(m, n, K + 1 - x, K + 1 - y)$$



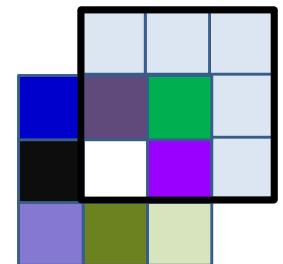
$$\frac{d\text{Div}}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d\text{Div}}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$



=

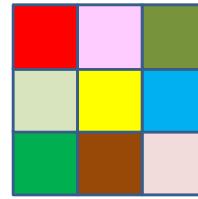
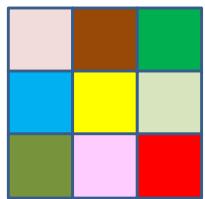


⋮



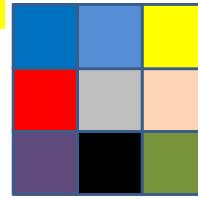
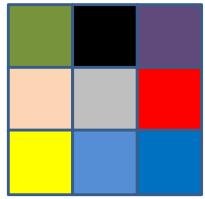
$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$



$n = 1$

flip

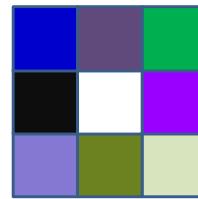
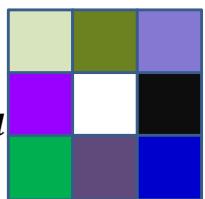


$n = 2$

⋮

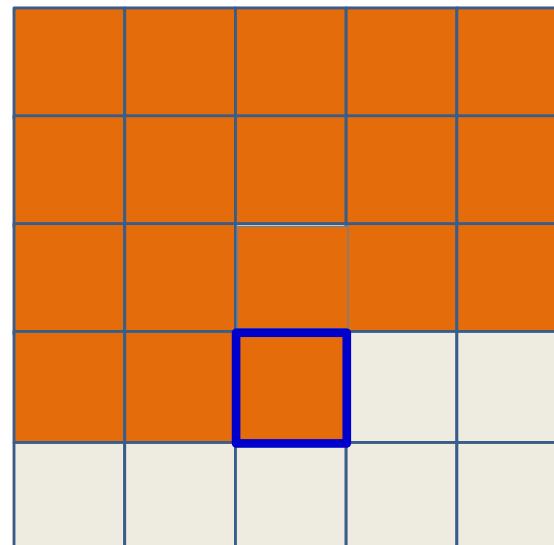
⋮

⋮

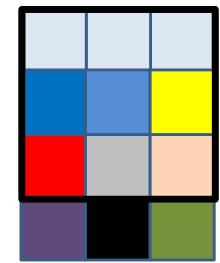
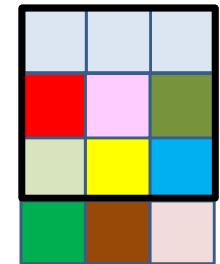


$n = D_l$

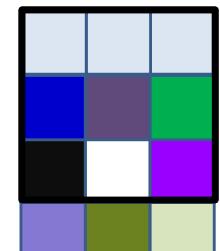
$$w_l(m, n, K + 1 - x, K + 1 - y)$$



=



⋮

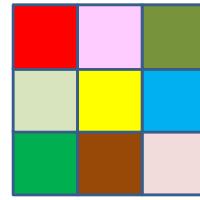
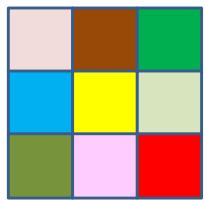


$$\frac{\partial \text{Div}}{\partial y(l - 1, m, x, y)}$$

$$\frac{d\text{Div}}{dY(l - 1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d\text{Div}}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

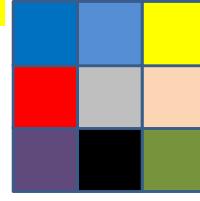
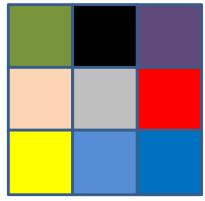
$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$



$n = 1$

flip

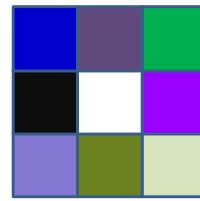
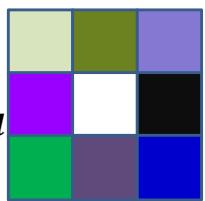


$n = 2$

⋮

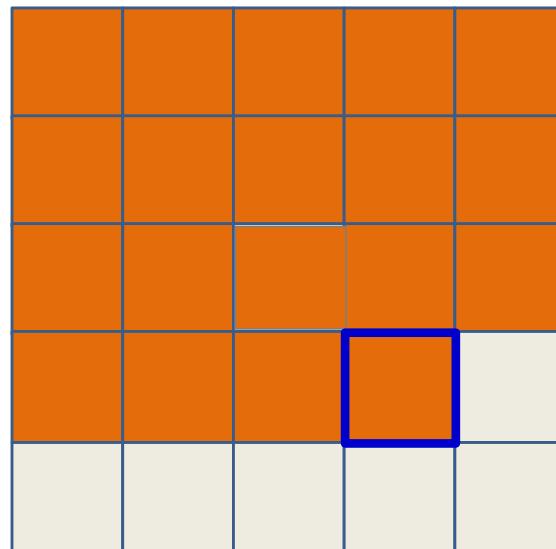
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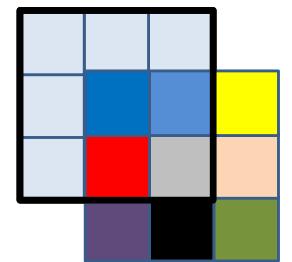
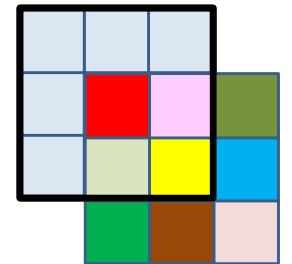


$n = D_l$

$$w_l(m, n, K + 1 - x, K + 1 - y)$$



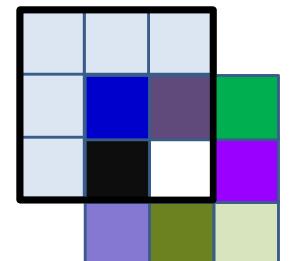
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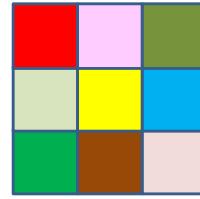
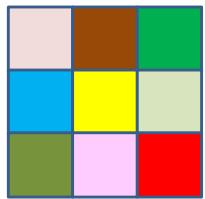
$$\frac{\partial \text{Div}}{\partial y(l - 1, m, x, y)}$$

$$\frac{d\text{Div}}{dY(l - 1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d\text{Div}}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$



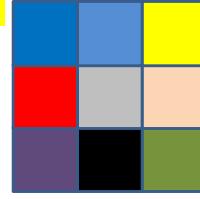
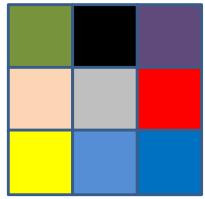
$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$



$n = 1$

flip

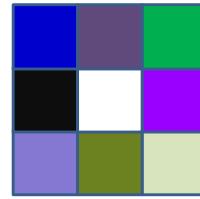
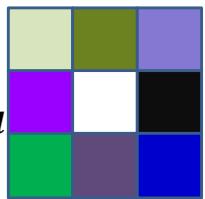


$n = 2$

⋮

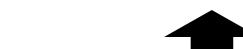
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⋮

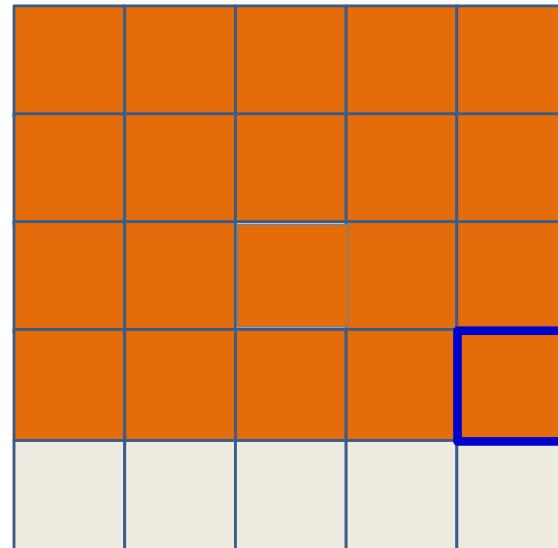


$n = D_l$

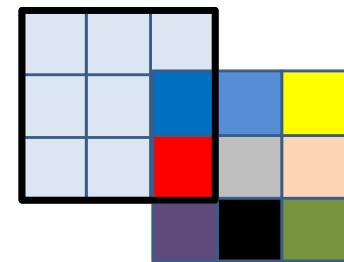
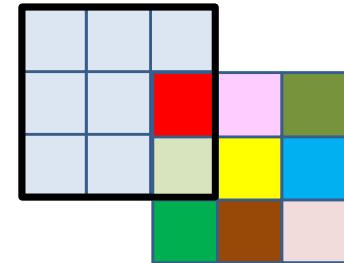
$$w_l(m, n, K + 1 - x, K + 1 - y)$$



flip



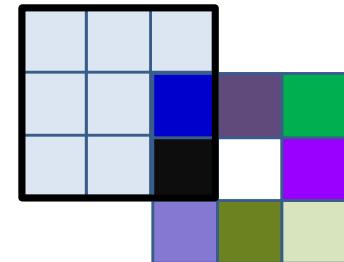
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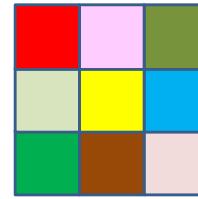
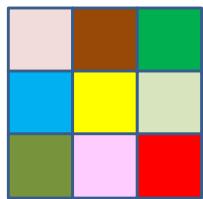
$$\frac{\partial \text{Div}}{\partial y(l - 1, m, x, y)}$$

$$\frac{d\text{Div}}{dY(l - 1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d\text{Div}}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$



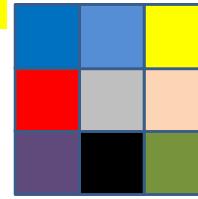
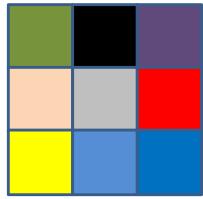
$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$



$n = 1$

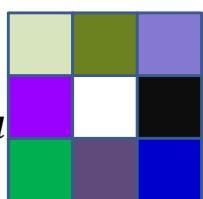
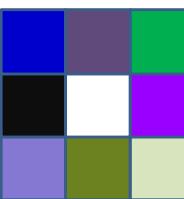
flip



$n = 2$

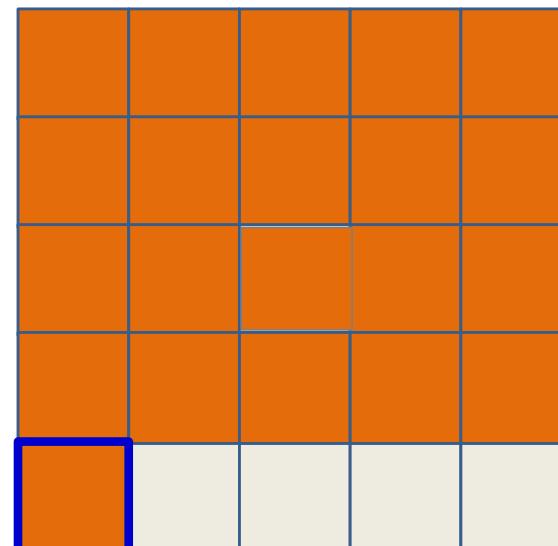
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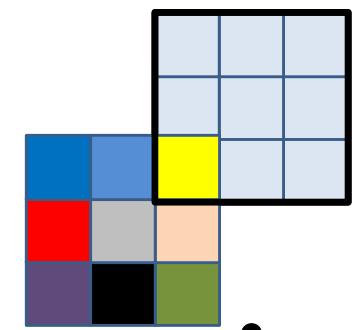
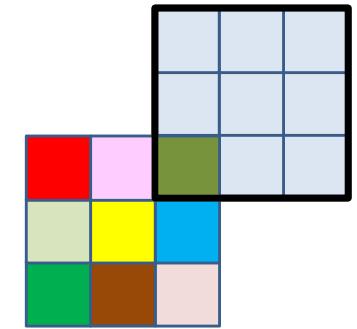


$n = D_l$

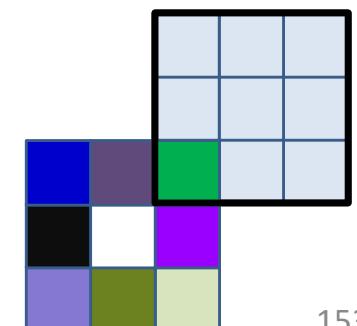
$$w_l(m, n, K + 1 - x, K + 1 - y)$$



=



⋮

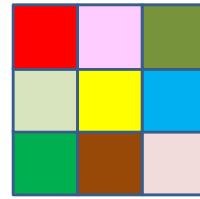
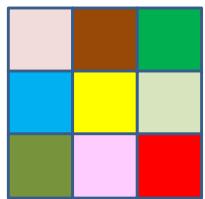


$$\frac{\partial \text{Div}}{\partial y(l - 1, m, x, y)}$$

$$\frac{d\text{Div}}{dY(l - 1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d\text{Div}}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

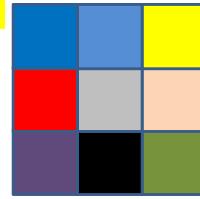
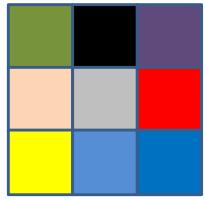
$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$



$n = 1$

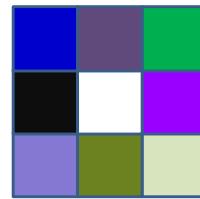
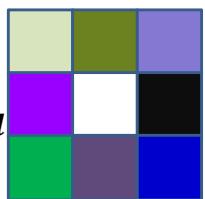
flip



$n = 2$

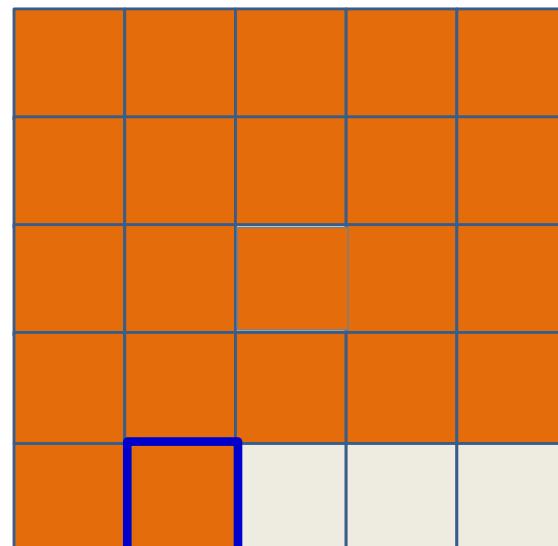
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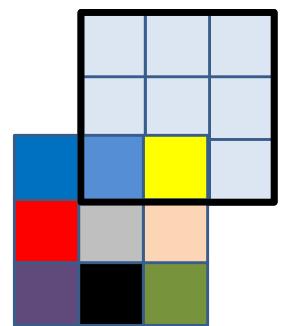
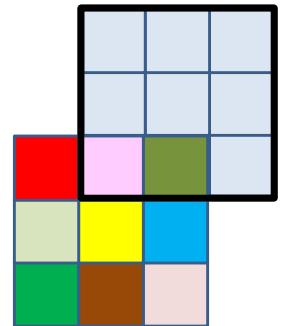


$n = D_l$

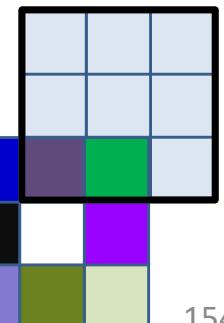
$$w_l(m, n, K + 1 - x, K + 1 - y)$$



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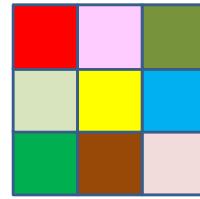
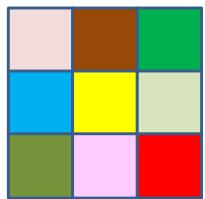


$$\frac{\partial \text{Div}}{\partial y(l - 1, m, x, y)}$$

$$\frac{d\text{Div}}{dY(l - 1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d\text{Div}}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

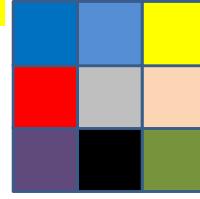
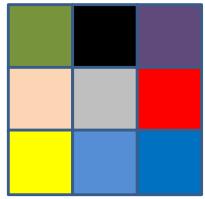
$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$



$n = 1$

flip

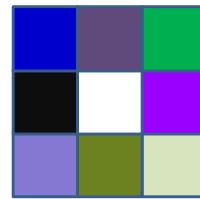
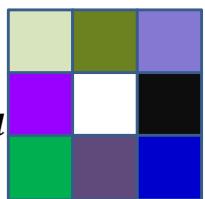


$n = 2$

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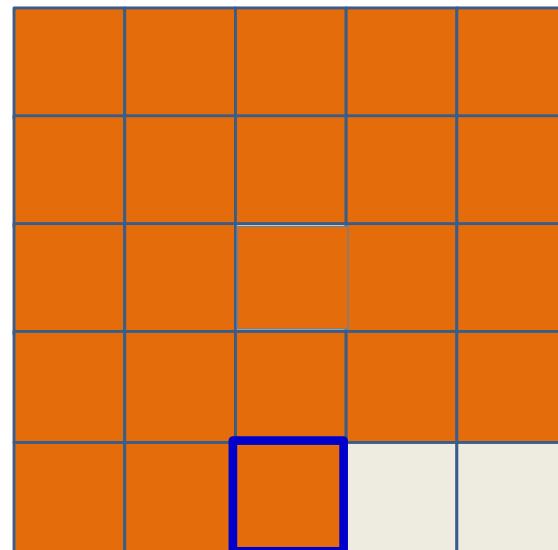


$n = D_l$

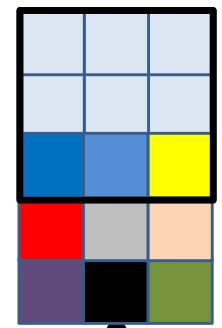
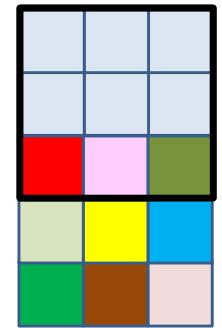
$$w_l(m, n, K + 1 - x, K + 1 - y)$$



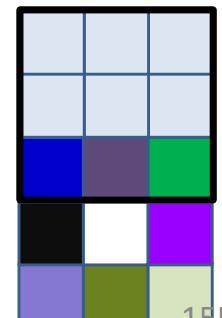
$$\frac{d\text{Div}}{dY(l-1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d\text{Div}}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$



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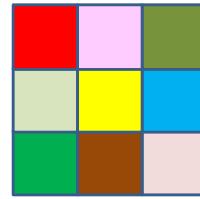
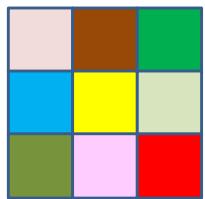


⋮



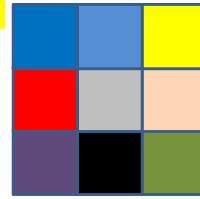
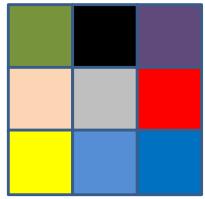
$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$



$n = 1$

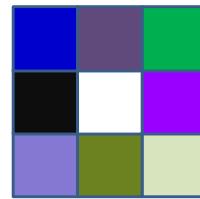
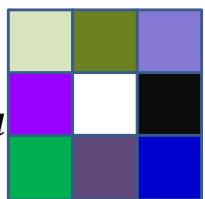
flip



$n = 2$

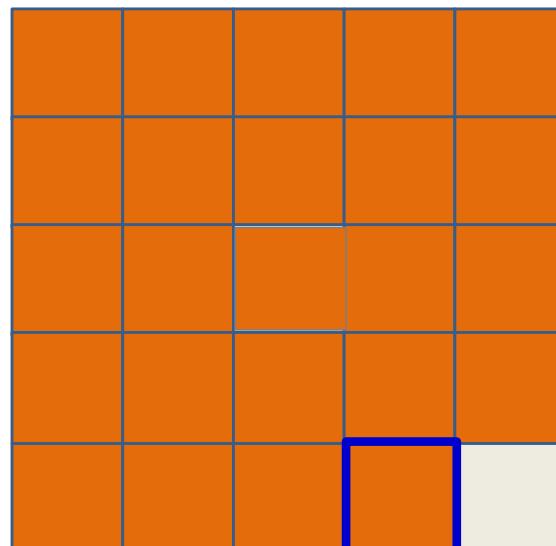
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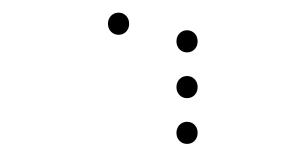
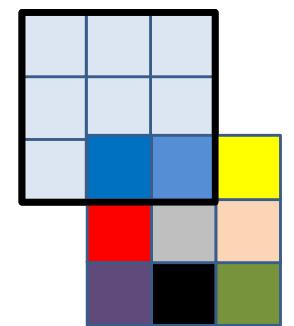
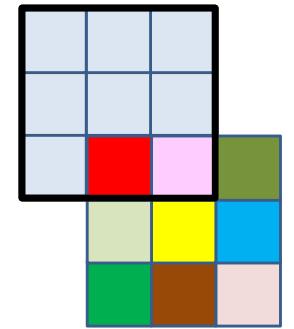


$n = D_l$

$$w_l(m, n, K + 1 - x, K + 1 - y)$$

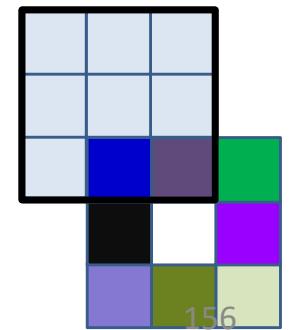


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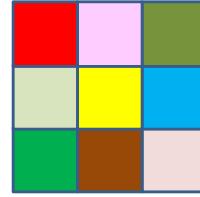
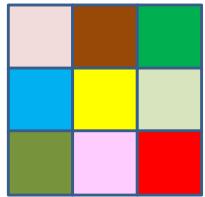
$$\frac{\partial \text{Div}}{\partial y(l - 1, m, x, y)}$$

$$\frac{d\text{Div}}{dY(l - 1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d\text{Div}}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$



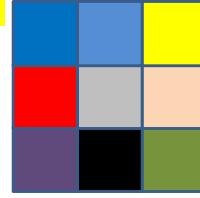
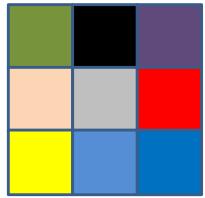
$$\frac{\partial \text{Div}}{\partial z(l, n, x', y')}$$

$$w_l(m, n, x, y)$$



$n = 1$

flip

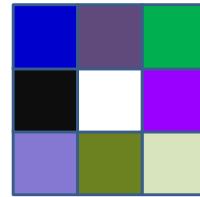
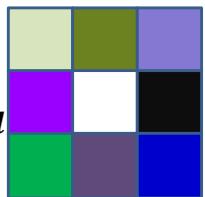


$n = 2$

⋮

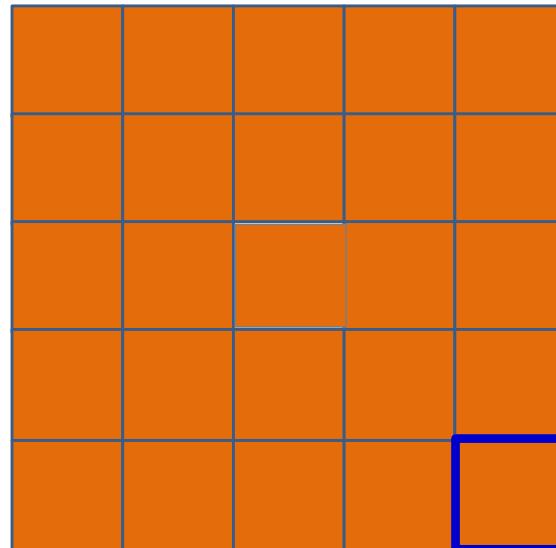
⋮

⋮

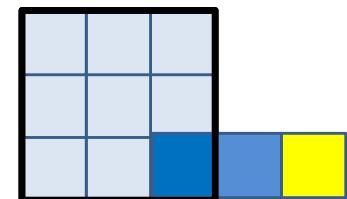
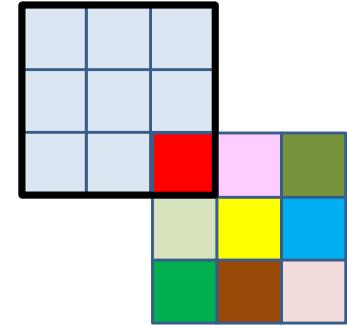


$n = D_l$

$$w_l(m, n, K + 1 - x, K + 1 - y)$$



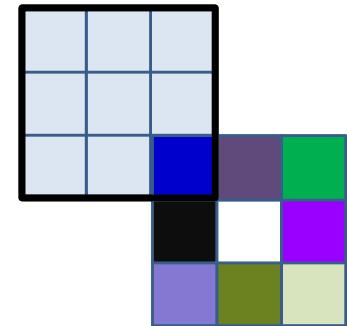
=



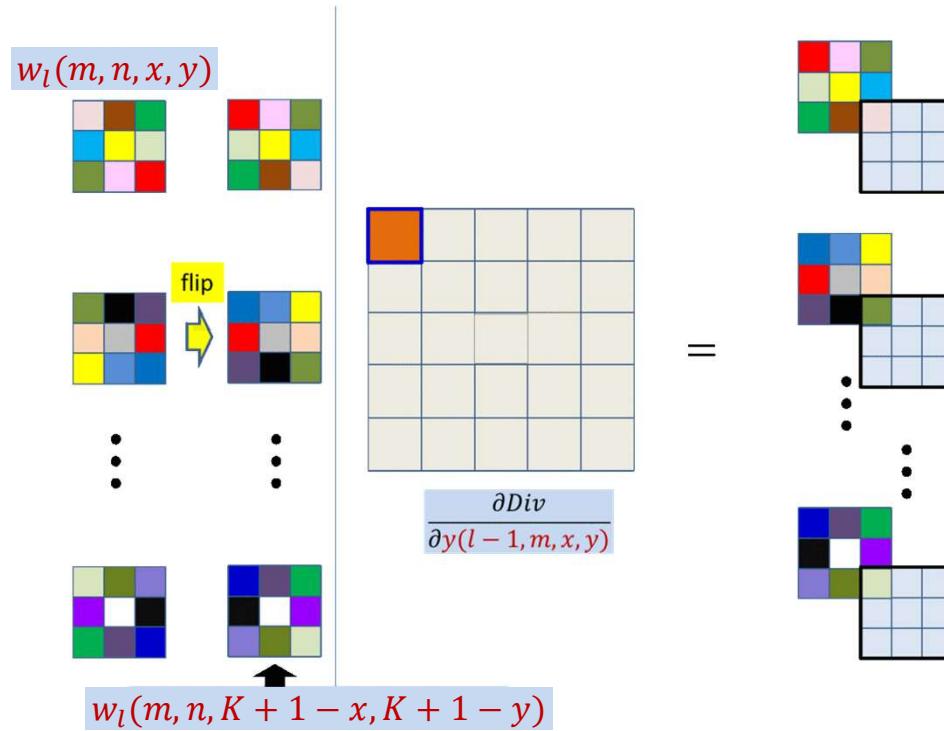
⋮

$$\frac{\partial \text{Div}}{\partial y(l - 1, m, x, y)}$$

$$\frac{d\text{Div}}{dY(l - 1, m, x, y)} = \sum_n \sum_{x', y'} \frac{d\text{Div}}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

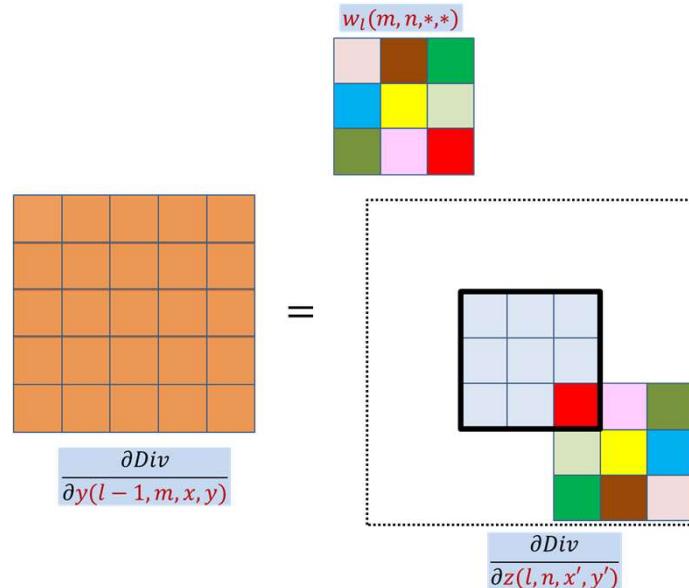


Computing the derivative for $Y(l - 1, m)$



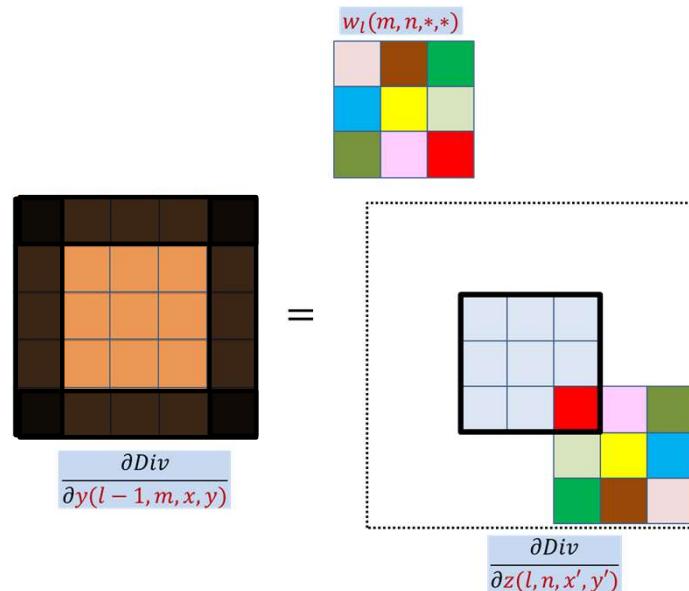
- This is just a convolution of the zero-padded maps by the transposed and flipped filter
 - After zero padding it first with $K - 1$ zeros on every side

The size of the Y-derivative map



- We continue to compute elements for the derivative Y map as long as the (flipped) filter has at least one element in the (unpadded) derivative Zmap
 - I.e. so long as the Y derivative is non-zero
- The size of the Y derivative map will be $(H + K - 1) \times (W + K - 1)$
 - H and W are the height and width of the Zmap
- This will be the size of the actual Y map that was originally convolved

The size of the Y-derivative map



- If the Y map was zero-padded in the forward pass, the derivative map will be the size of the *zero-padded* map
 - The zero padding regions must be deleted before further backprop

Poll 3

Select all statements that are true about how to compute the derivative of the divergence w.r.t l th layer activation maps by backpropagation

- To compute the derivative w.r.t. the m th activation map of the l th convolutional layer, we must select the m th “planes” of all the $(l+1)$ th layer filters
- The selected filter planes must be flipped left-right and up-down
- They must convolve the derivative (maps) for the $(l+1)$ th layer affine values
- The output of the convolution must be flipped back left-right and up-down

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- **They must convolve the derivative (maps) for the $(l+1)$ th layer affine values**
- The output of the convolution must be flipped back left-right and up-down

Overall algorithm for computing derivatives w.r.t. $Y(l - 1)$

- Given the derivatives $\frac{dDiv}{dz(l,n,x,y)}$
- Compute derivatives using:

$$\frac{dDiv}{dY(l - 1, m, x, y)} = \sum_n \sum_{x',y'} \frac{dDiv}{dz(l, n, x', y')} w_l(m, n, x - x', y - y')$$

Can be computed by convolution with flipped filter

l is layer index

Derivatives for a single layer l : Vector notation

```
# The weight W(l,m) is a 3D D_{l-1}xK_lxK_l
# Assuming dz has already been obtained via backprop

dzpad = zeros(Dlx(Hl+2(Kl-1))x(Wl+2(Kl-1))) # zeropad
for j = 1:Dl
    for i = 1:Dl-1 # Transpose and flip
        Wflip(i,j,:,:,:) = flipLeftRight(flipUpDown(W(l,i,j,:,:,:)))
    dzpad(j,Kl:Kl+Hl-1,Kl:Kl+Wl-1) = dz(l,j,:,:,:)
end

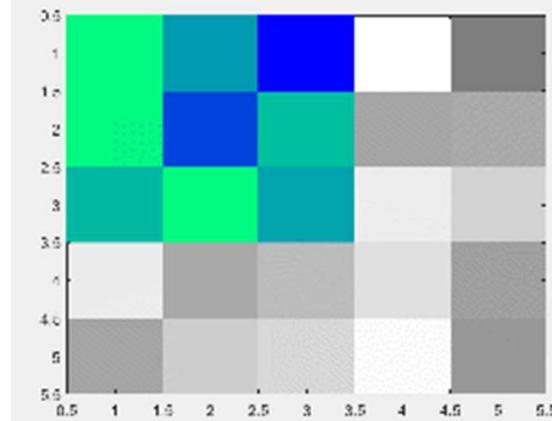
for j = 1:Dl-1
    for x = 1:Wl-1
        for y = 1:Hl-1
            segment = dzpad(:, x:x+Kl-1, y:y+Kl-1) #3D tensor
            dy(l-1,j,x,y) = Wflip.segment #tensor inner prod.
```

Backpropagating through affine map

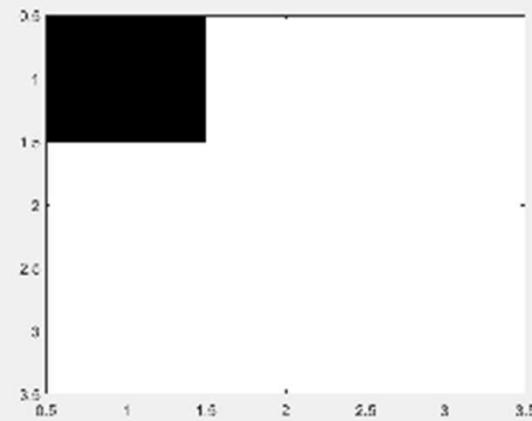
- Forward affine computation:
 - Compute affine maps $z(l, n, x, y)$ from previous layer maps $y(l - 1, m, x, y)$ and filters $w_l(m, n, x, y)$
- Backpropagation: Given $\frac{dDiv}{dz(l,n,x,y)}$
 - ✓ Compute derivative w.r.t. $y(l - 1, m, x, y)$
 - Compute derivative w.r.t. $w_l(m, n, x, y)$

The derivatives for the filters

$Y(l - 1, m) \otimes w_l(m, n)$



$Z(l, n)$

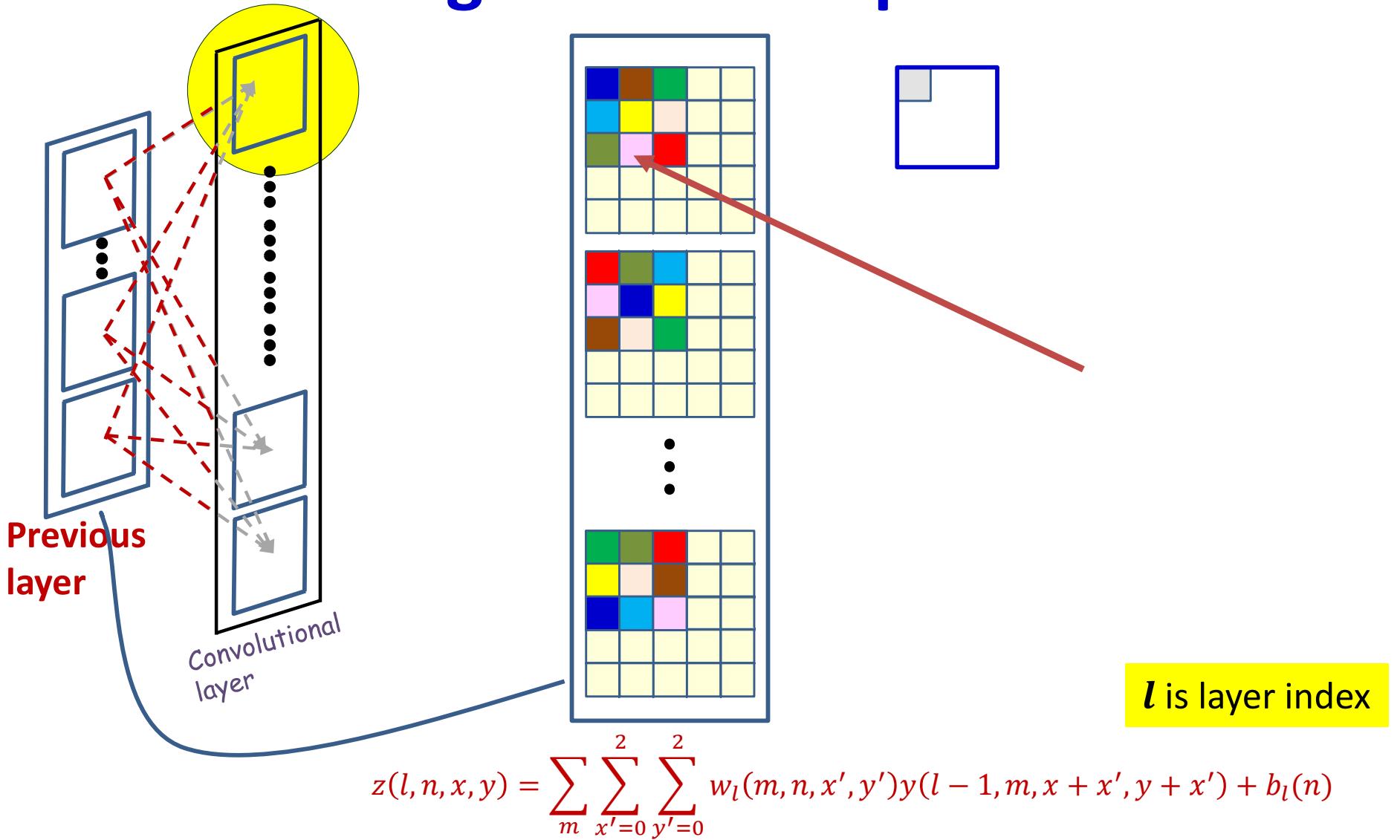


l is layer index

$$z(l, n, x, y) = \sum_m \sum_{x', y'} w_l(m, n, x', y') y(l - 1, m, x + x', y + y') + b_l(n)$$

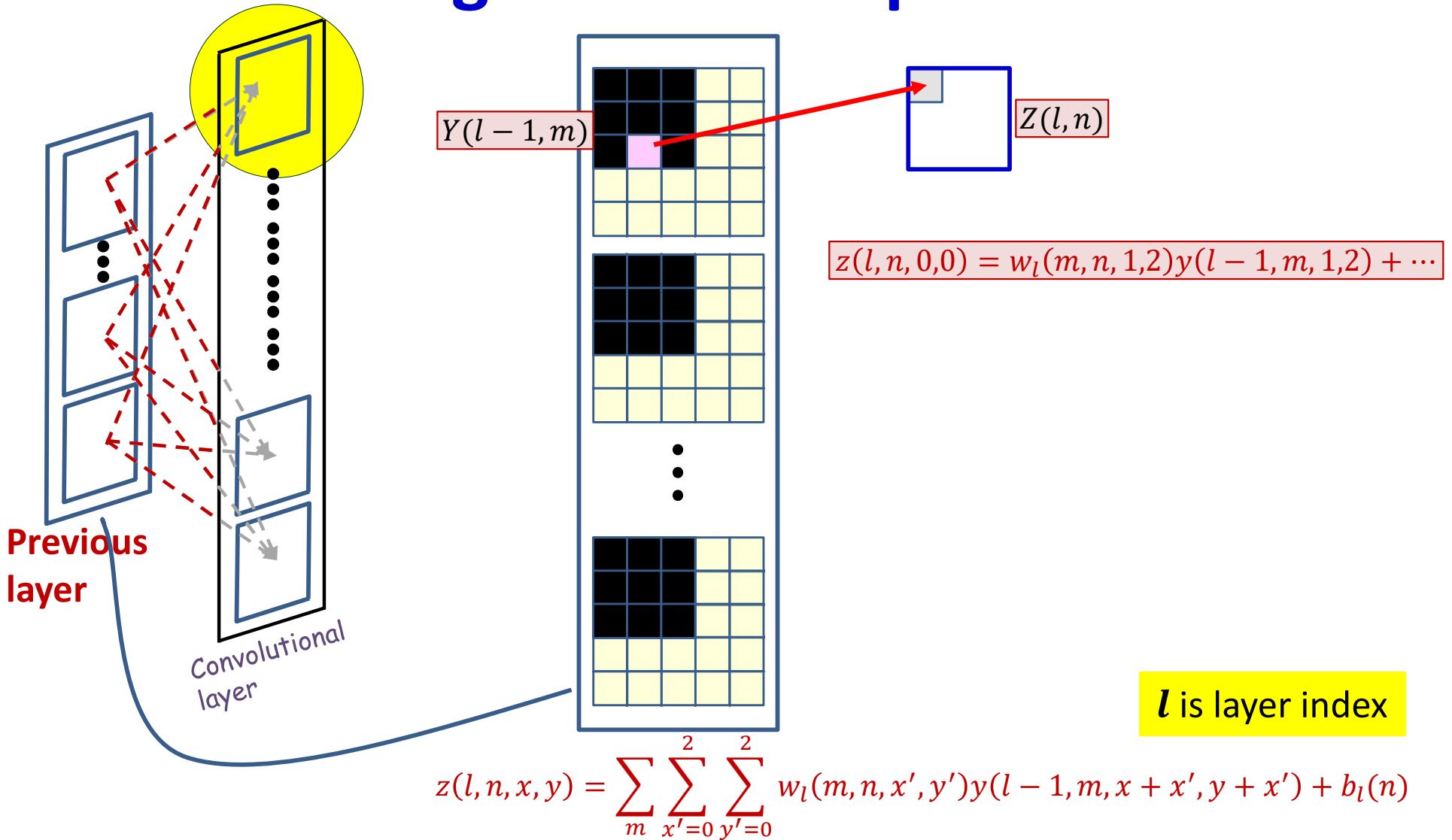
- Each **filter component** $w_l(m, n, x', y')$ affects several $z(l, n, x, y)$ but only within a *single* output affine ($z(l, n, *, *)$) map/channel
 - And is also linked to several $y(l - 1, m, *, *)$ but only within a single input channel
 - A single filter channel $w_l(m, n, *, *)$ connects $y(l - 1, m, *, *)$ to $z(l, n, *, *)$
 - Consider the contribution of one filter component: $w_l(m, n, i, j)$ (e.g. $w_l(m, n, 1, 2)$) in the above animation for illustration

Convolution: the contribution of a single filter component



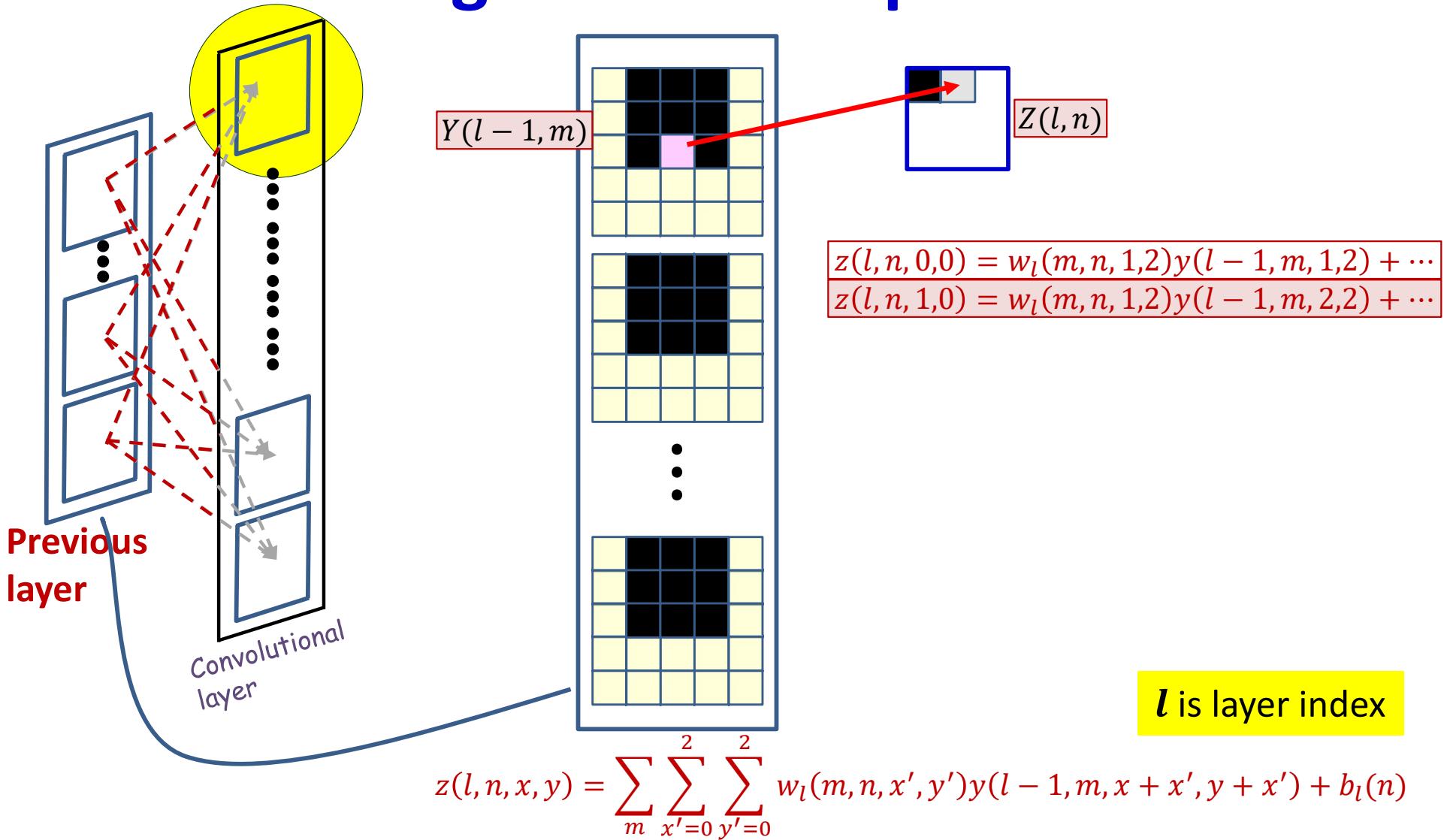
- Each *filter component* $w_l(m, n, i, j)$ affects several $z(l, n, x, y)$ within the n th output affine map

Convolution: the contribution of a single filter component



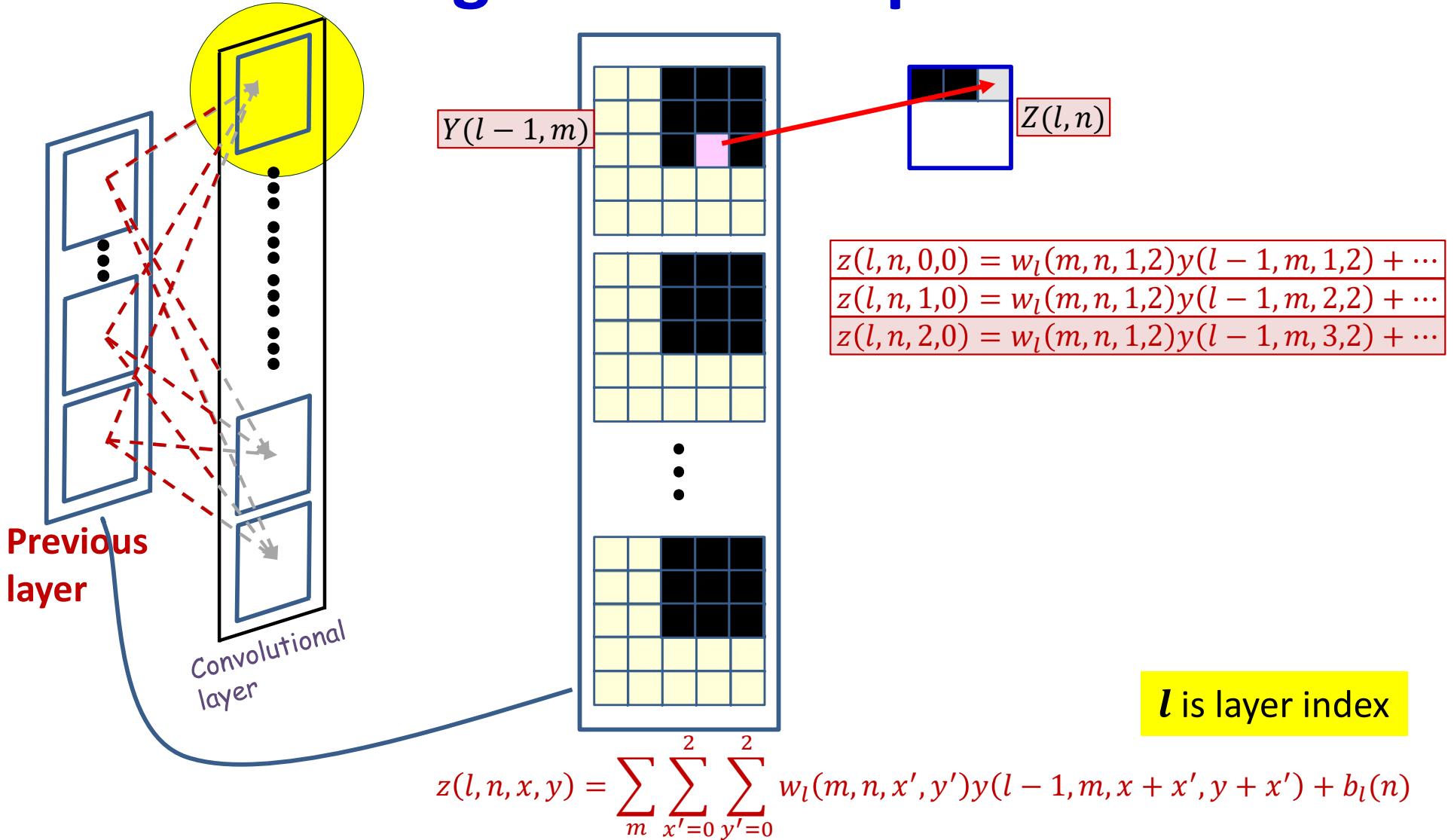
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Convolution: the contribution of a single filter component



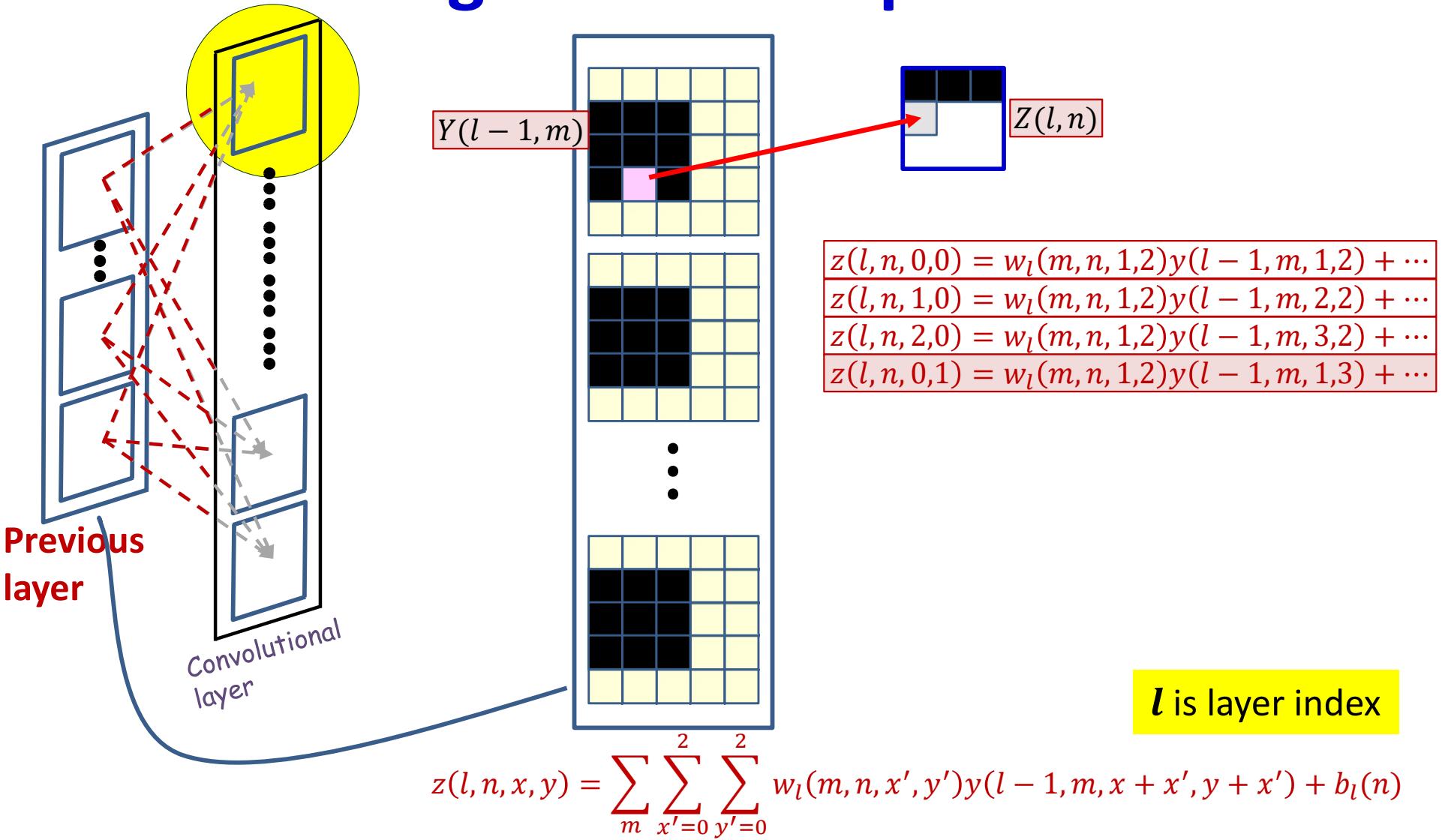
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Convolution: the contribution of a single filter component



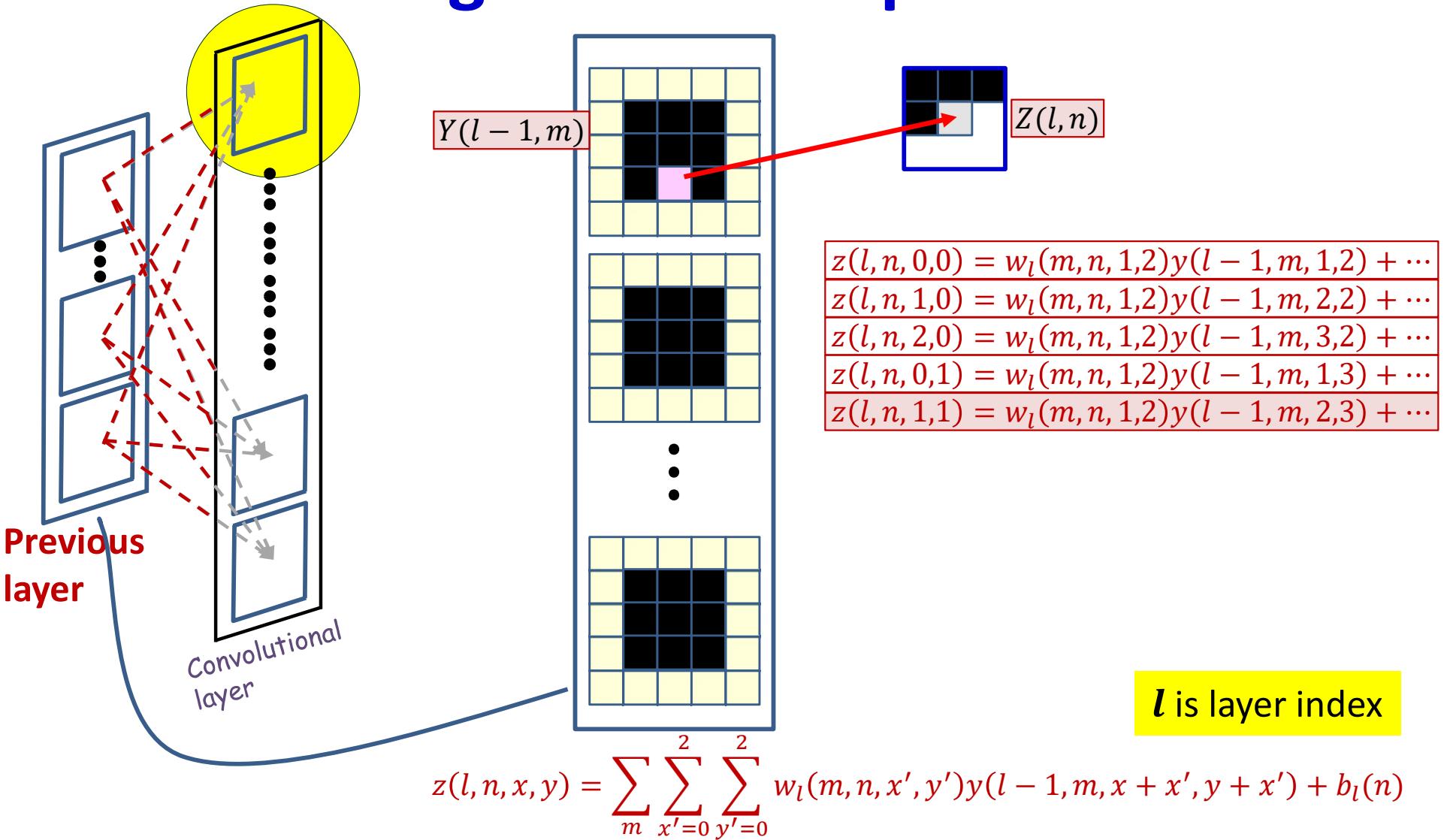
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Convolution: the contribution of a single filter component



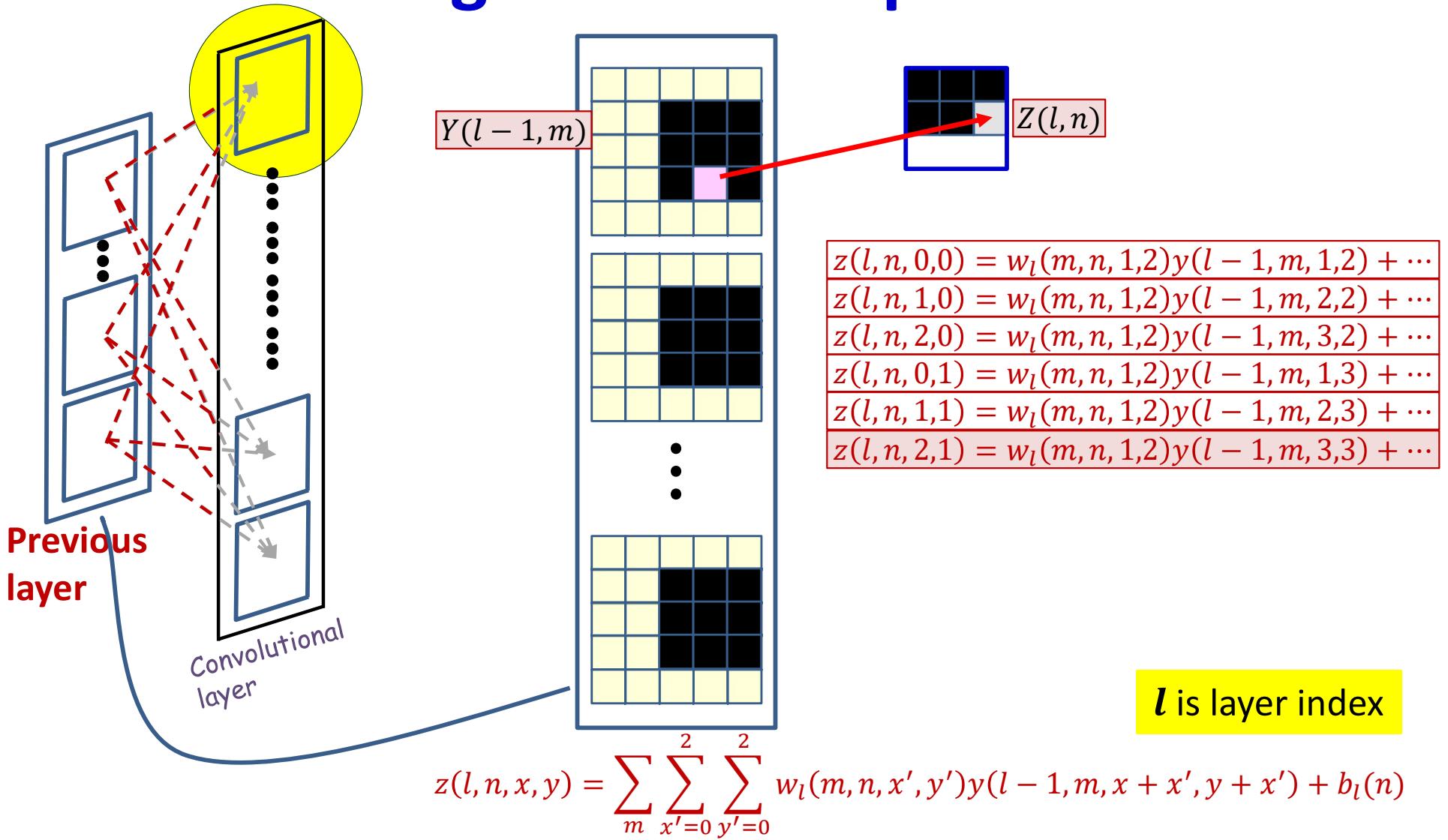
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Convolution: the contribution of a single filter component



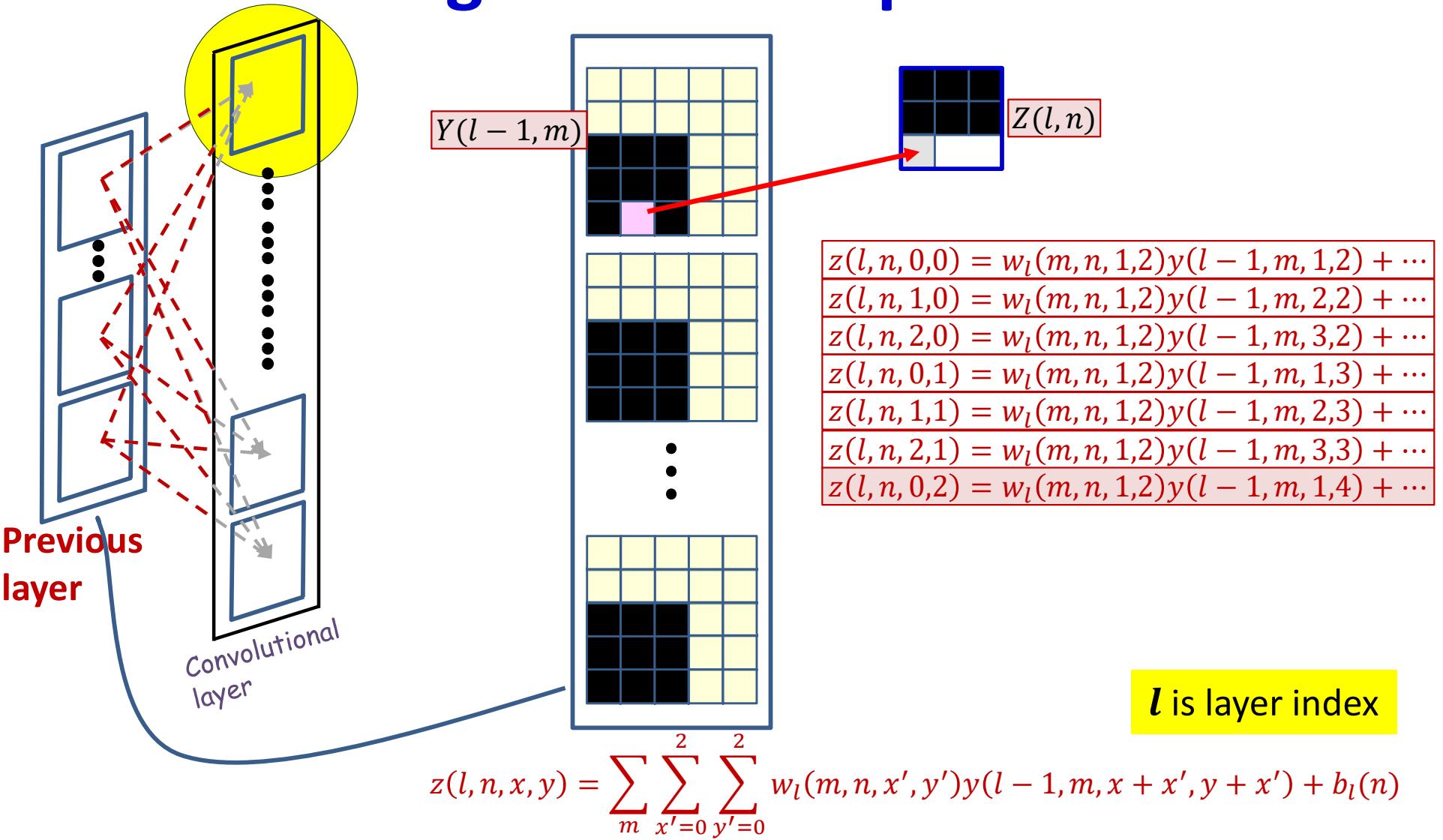
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Convolution: the contribution of a single filter component



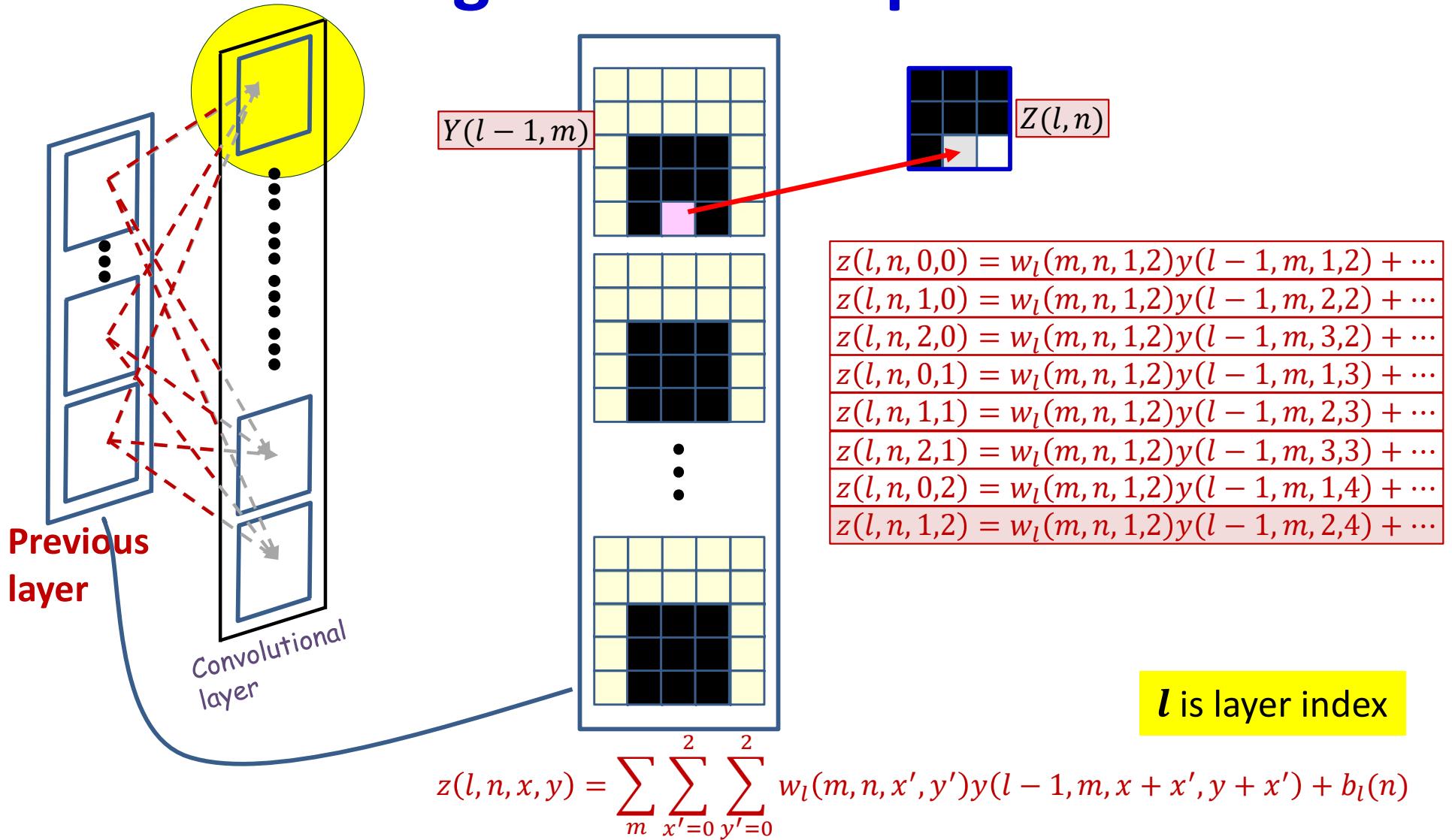
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Convolution: the contribution of a single filter component



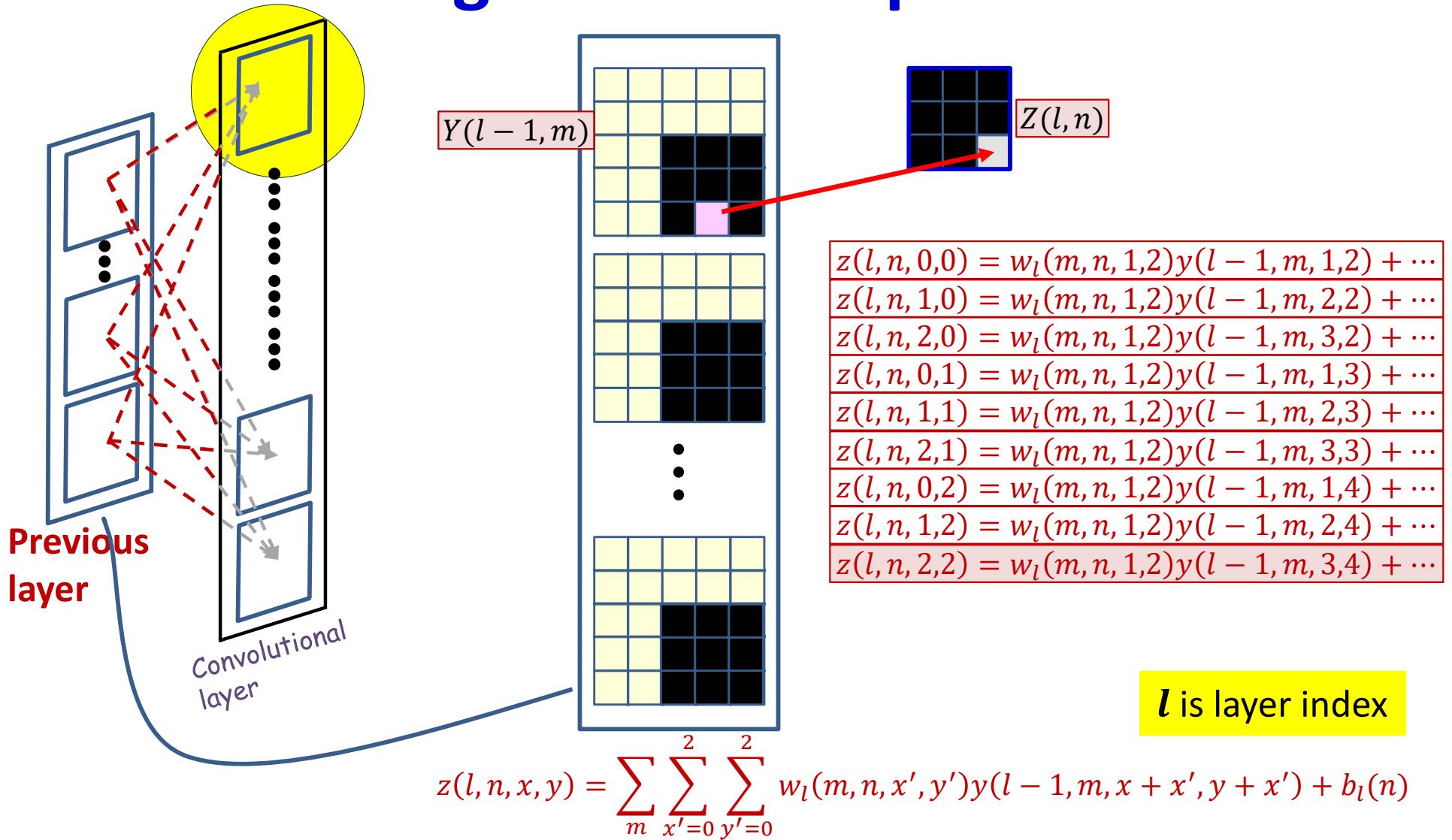
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Convolution: the contribution of a single filter component



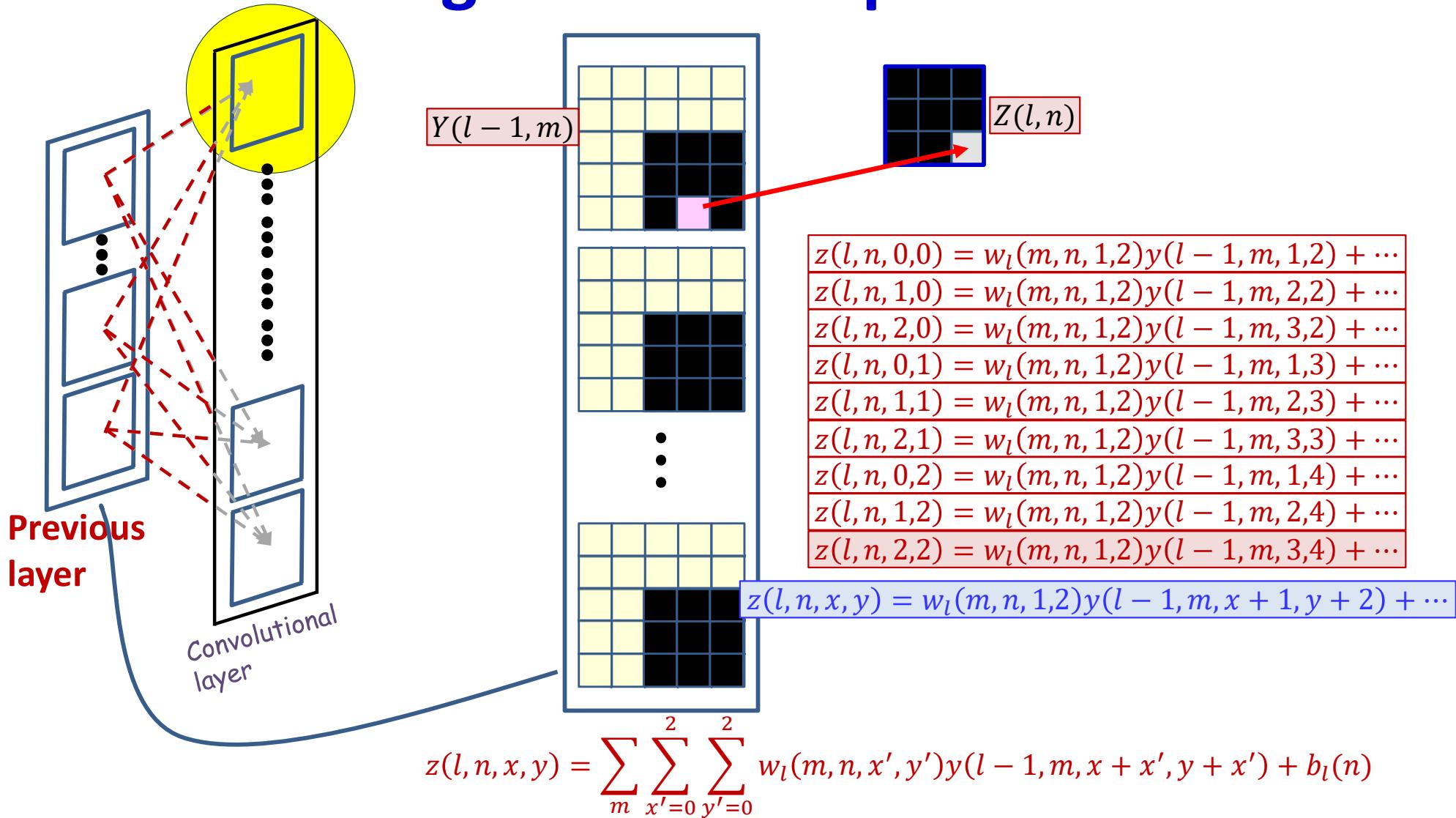
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Convolution: the contribution of a single filter component



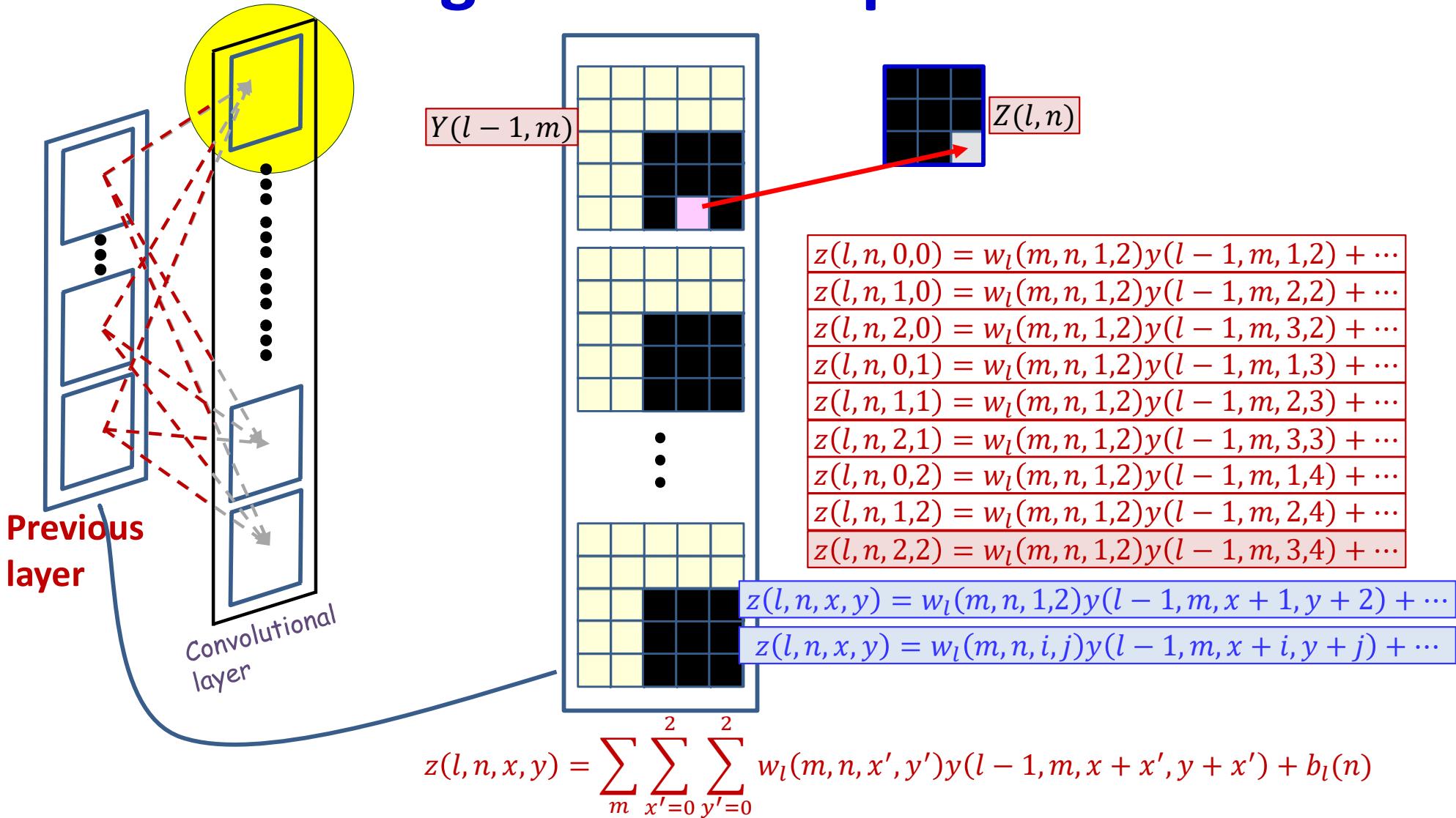
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Convolution: the contribution of a single filter component



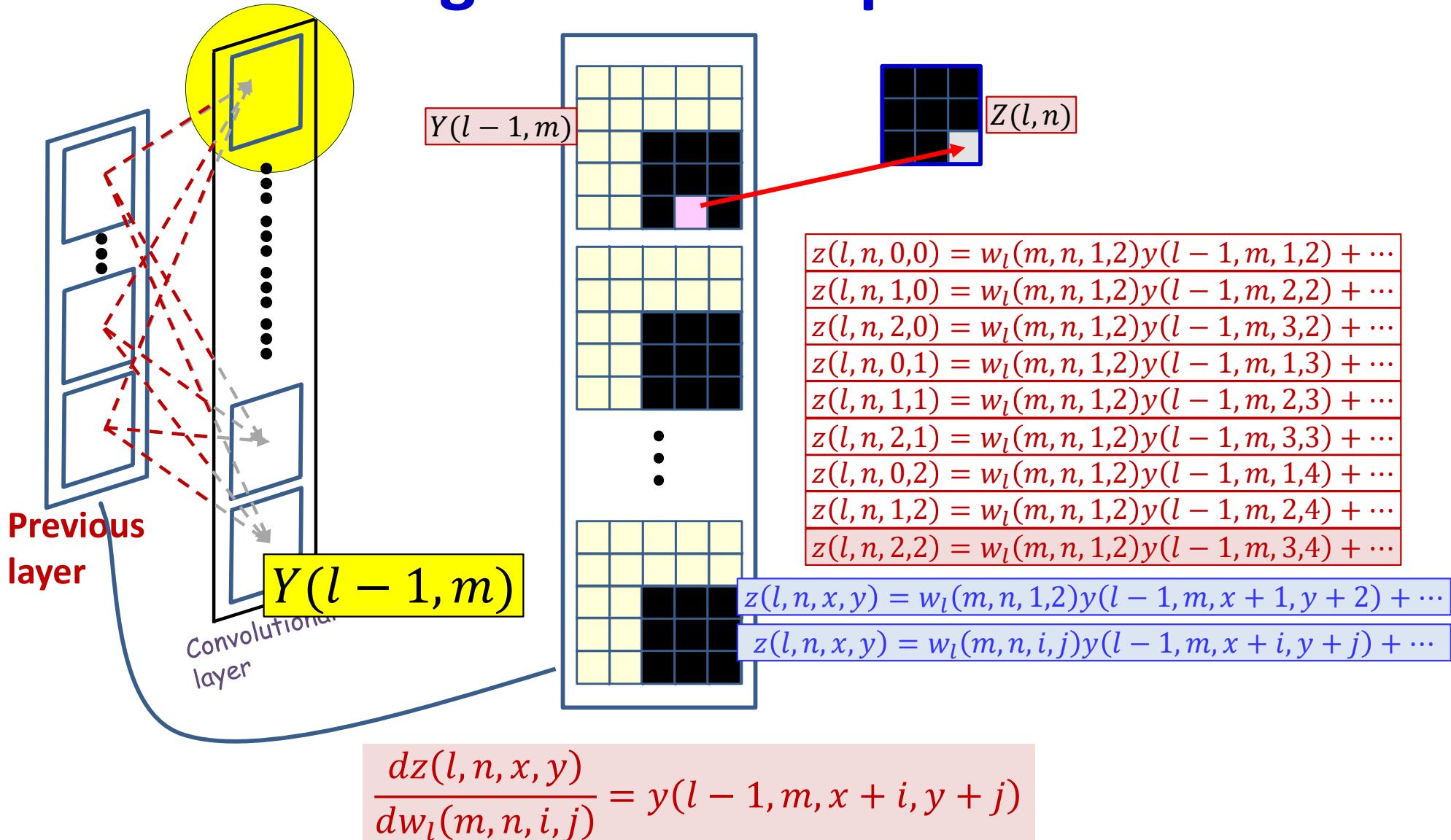
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Convolution: the contribution of a single filter component

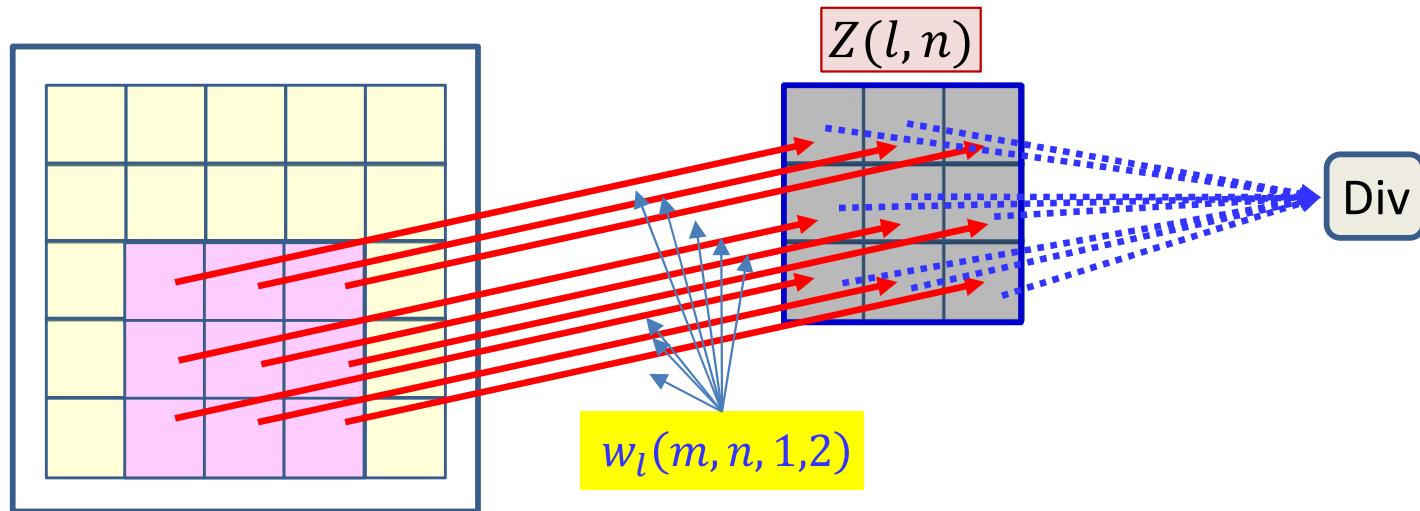


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 - Consider the contribution of one filter component: e.g. $w_l(m, n, 1, 2)$

Convolution: the contribution of a single filter component



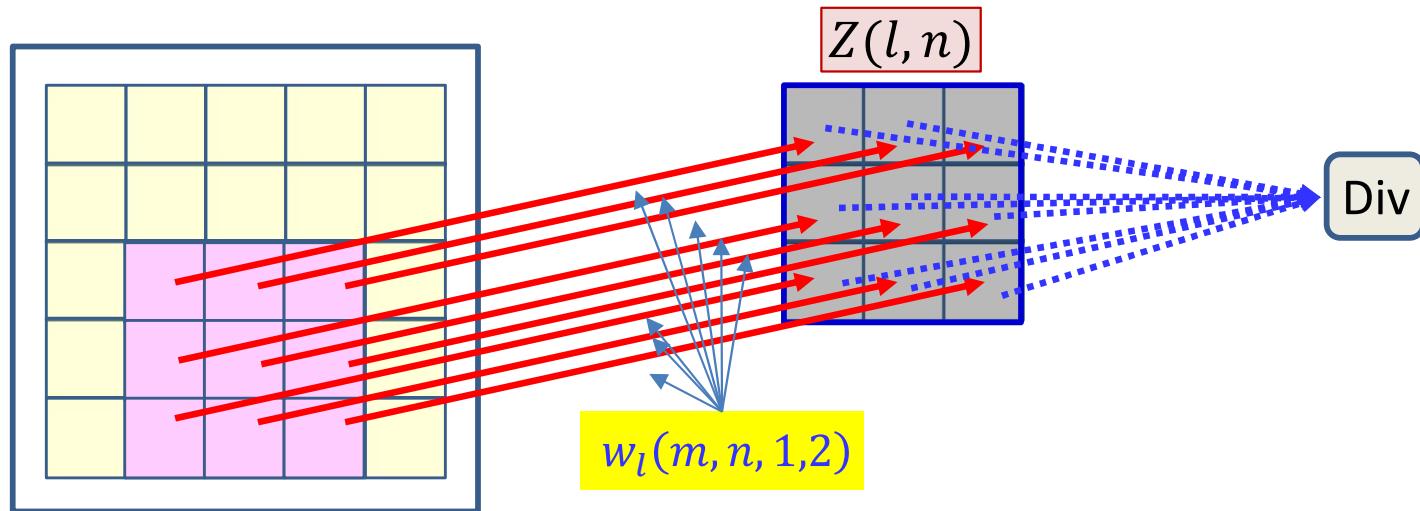
The derivative for a single filter component



- Each filter component $w_l(m, n, i, j)$ affects several $z(l, n, x, y)$
 - The derivative of each $z(l, n, x, y)$ w.r.t. $w_l(m, n, i, j)$ is given by
- $$\frac{dz(l, n, x, y)}{dw_l(m, n, i, j)} = y(l - 1, m, x + i, y + j)$$
- The final divergence is influenced by *every* $z(l, n, x, y)$
 - The derivative of the divergence w.r.t $w_l(m, n, i, j)$ must sum over all $z(l, n, x, y)$ terms it influences

$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x,y} \frac{dDiv}{dz(l, n, x, y)} \frac{dz(l, n, x, y)}{dw_l(m, n, i, j)}$$

The derivative for a single filter component



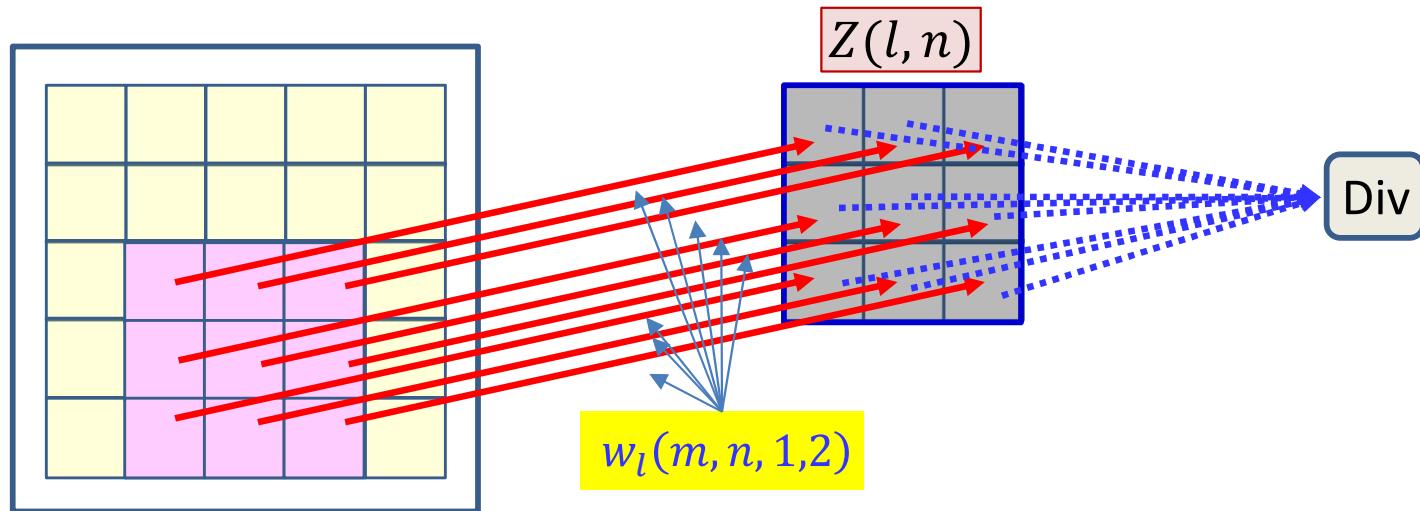
- Each filter component $w_l(m, n, i, j)$ affects several $z(l, n, x, y)$
 - The derivative of each $z(l, n, x, y)$ w.r.t. $w_l(m, n, i, j)$ is given by

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The derivative for a single filter component



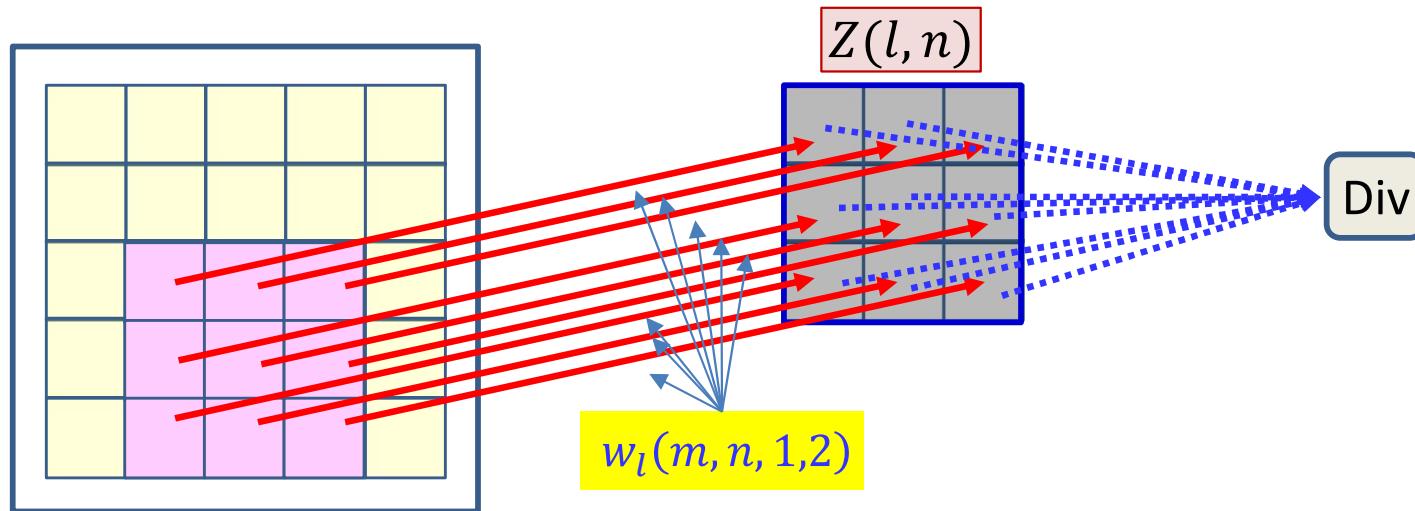
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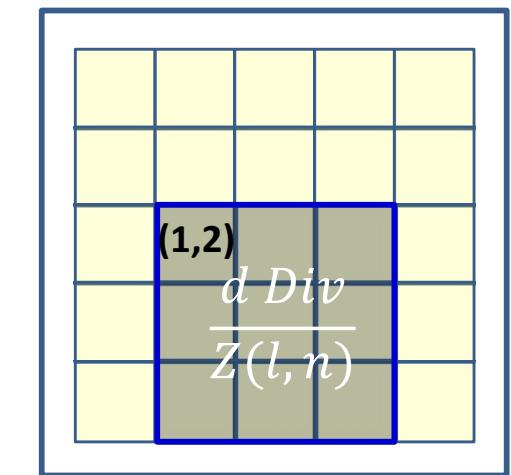
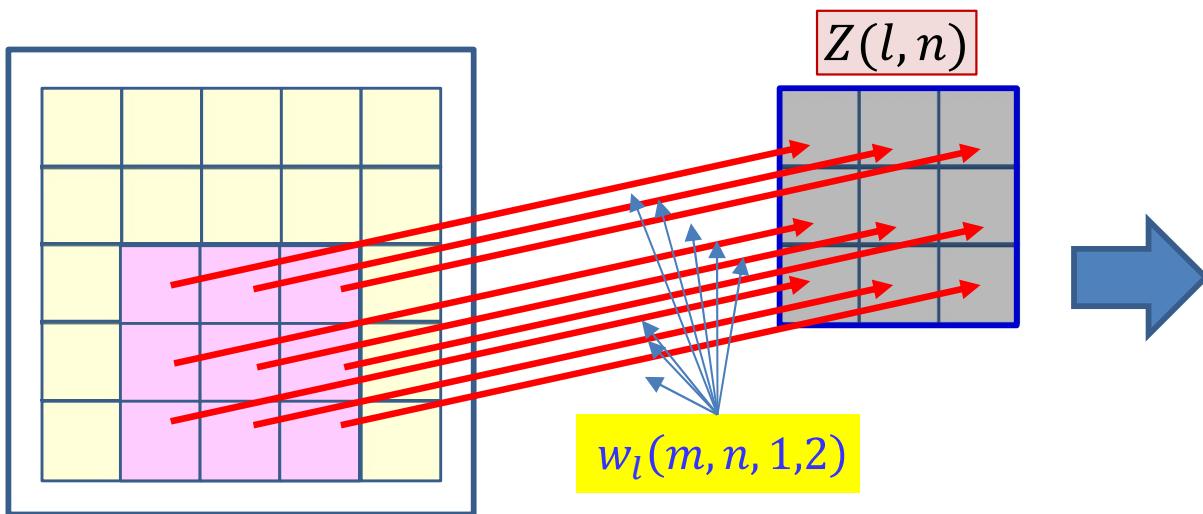
The derivative for a single filter component



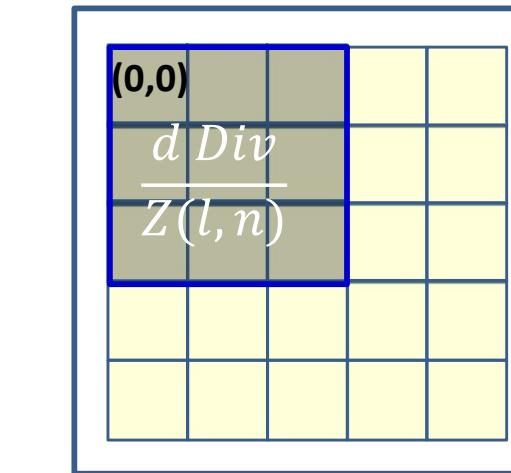
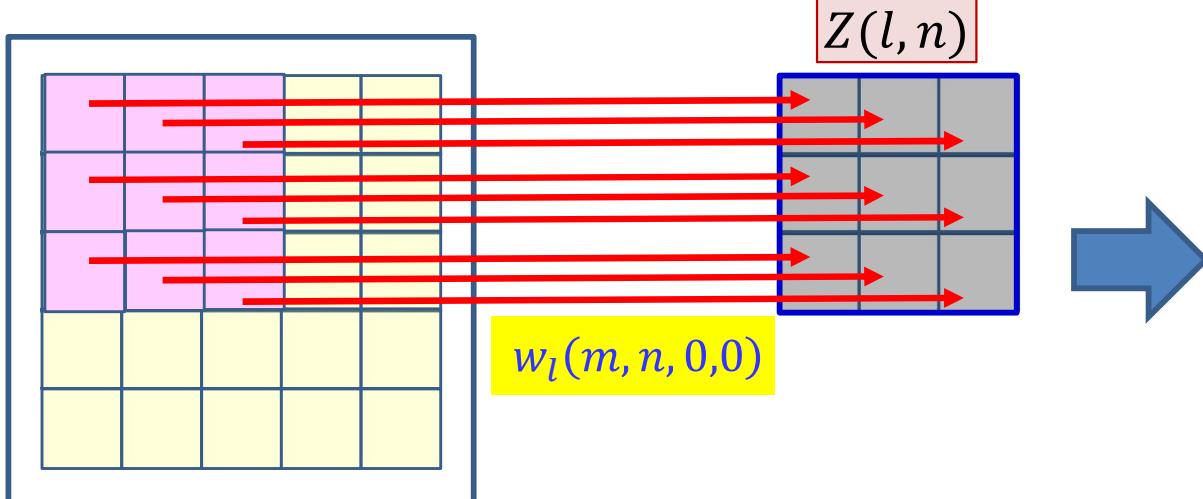
- Each filter component $w_l(m, n, i, j)$ affects several $z(l, n, x, y)$
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$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x,y} \frac{dDiv}{dz(l, n, x, y)} y(l - 1, m, x + i, y + j)$$

The derivative for a single filter component



To compute $\frac{d \text{Div}}{dw_l(m, n, 1, 2)}$



To compute $\frac{d \text{Div}}{dw_l(m, n, 0, 0)}$

But this too is a convolution

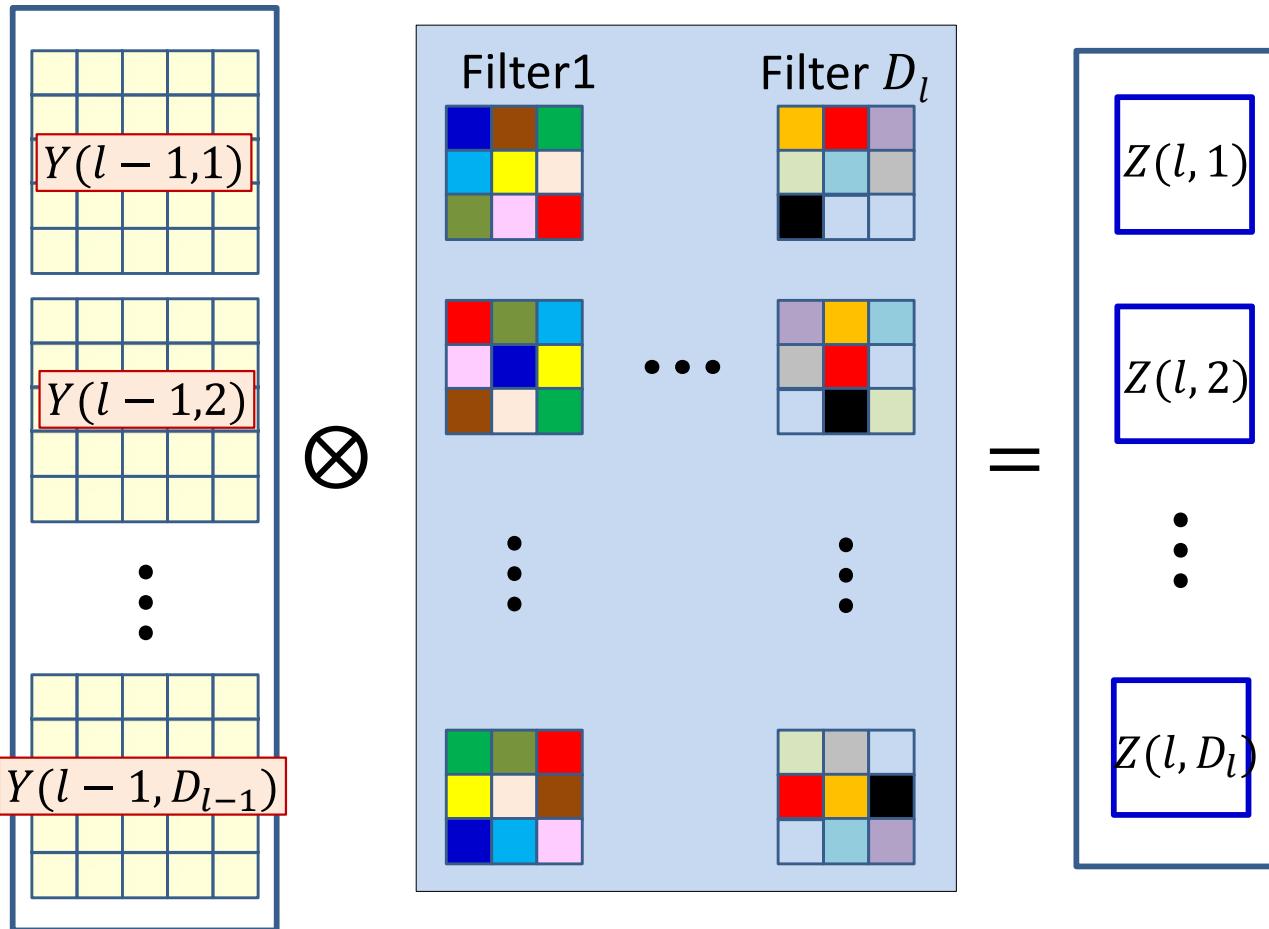
$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x,y} \frac{dDiv}{dz(l, n, x, y)} y(l - 1, m, x + i, y + j)$$

- The derivatives for all components of all filters can be computed directly from the above formula
 - To compute the derivative for $w_l(m, n, i, j)$, “place” the $dDiv/dz(l, n)$ map on $y(l - 1, m)$ map positioned at (i, j) and compute the inner product
- In fact, it is just a convolution

$$\frac{dDiv}{dw_l(m, n, i, j)} = \frac{dDiv}{dz(l, n)} \otimes y(l - 1, m)$$

- How?

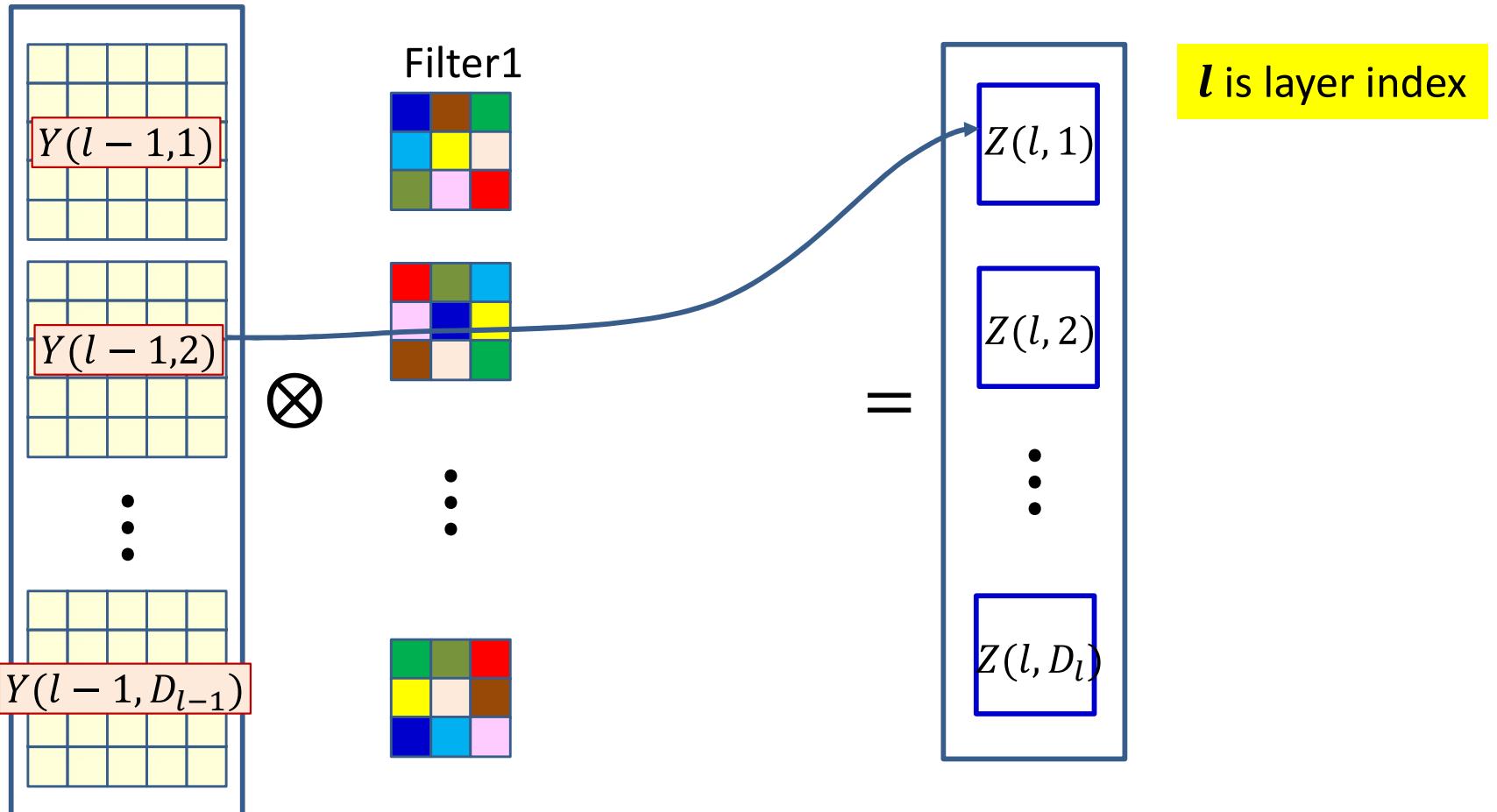
Recap: Convolution



$$z(l, n, x, y) = \sum_m \sum_{i=0}^2 \sum_{j=0}^2 w_l(m, n, i, j) y(l-1, m, x+i, y+j) + b_l(n)$$

- Forward computation: Each filter produces an affine map

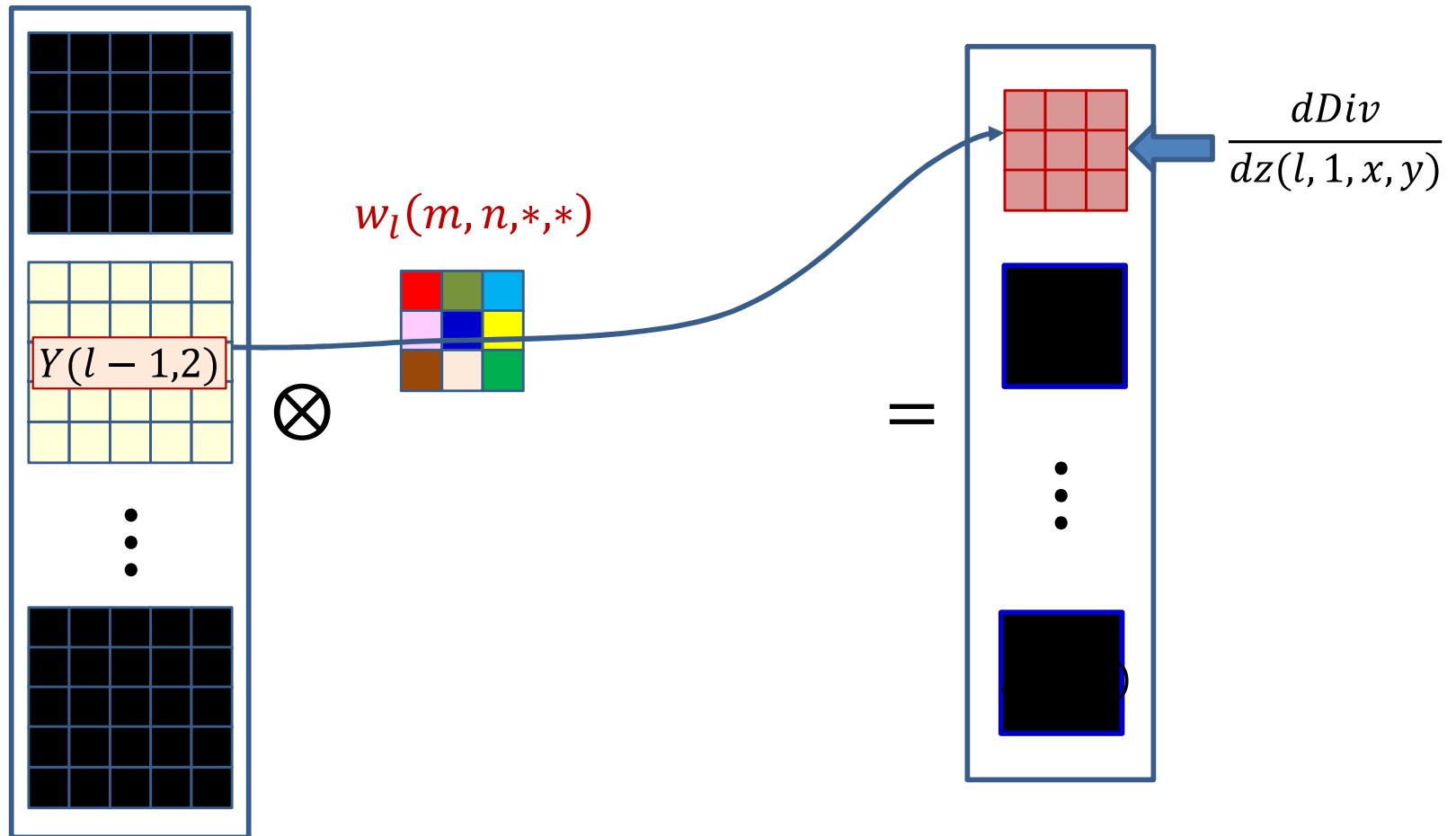
Recap: Convolution



$$z(l, n, x, y) = \sum_m \sum_{i=0}^2 \sum_{j=0}^2 w_l(m, n, i, j) y(l-1, m, x+i, y+j) + b_l(n)$$

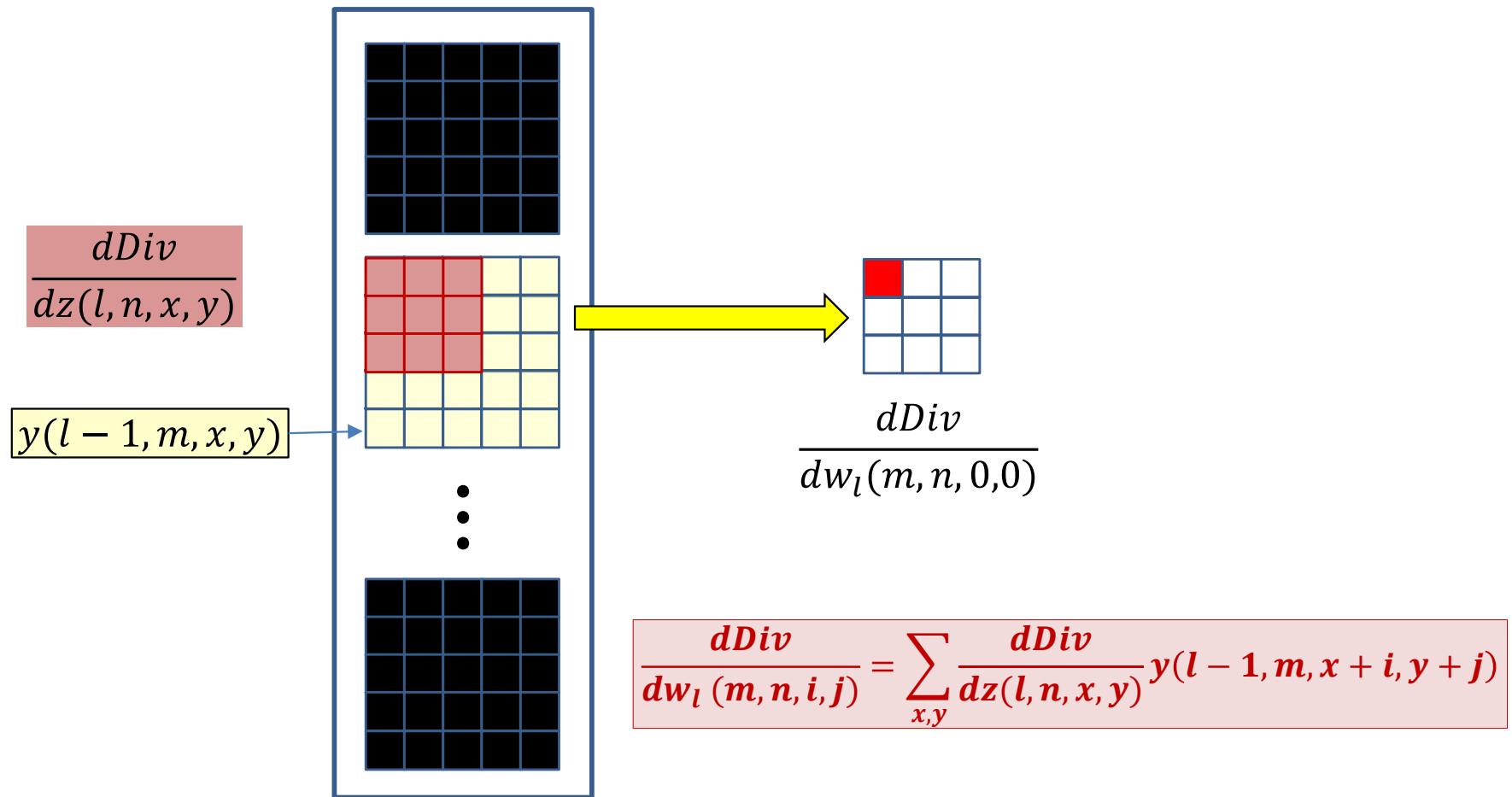
- $Y(l-1, m)$ influences $Z(l, n)$ through $w_l(m, n)$

The filter derivative



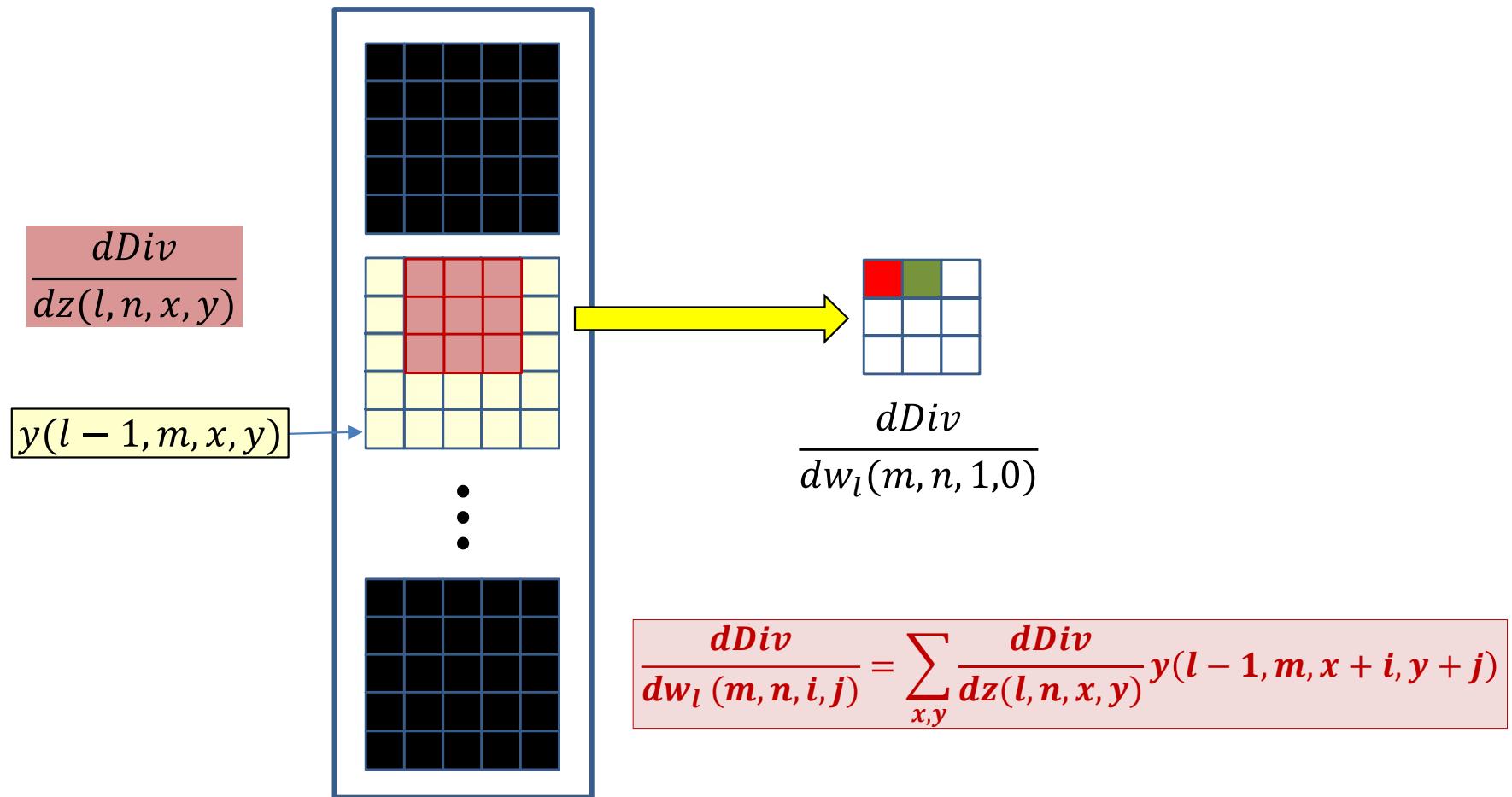
- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
 - Must use them to compute the derivative for $w_l(m, n, *, *)^{188}$

The filter derivative



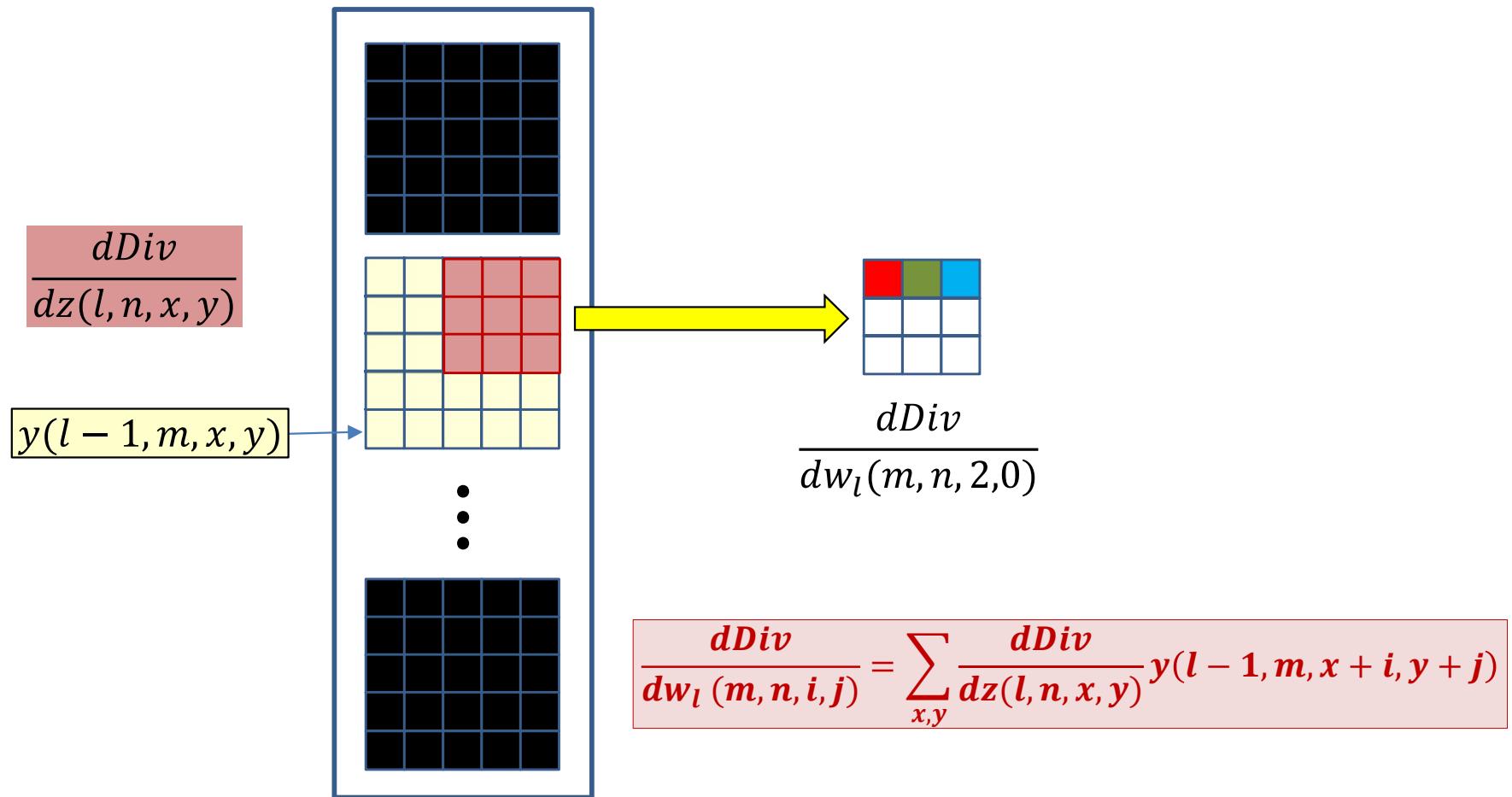
- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
 - Must use them to compute the derivative for $w_l(m, n, *, *)^{189}$

The filter derivative



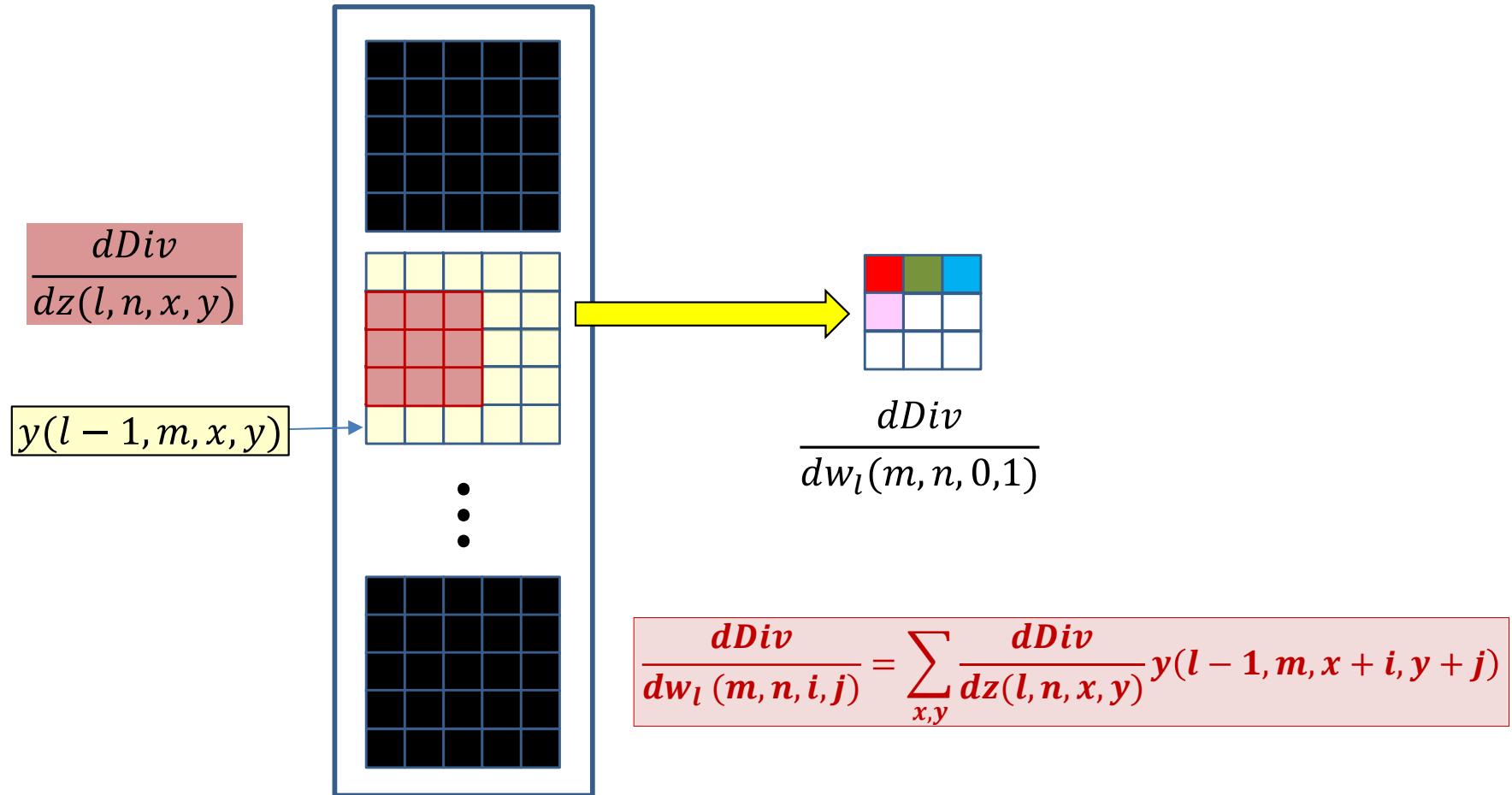
- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
 - Must use them to compute the derivative for $w_l(m, n, *, *)^{190}$

The filter derivative



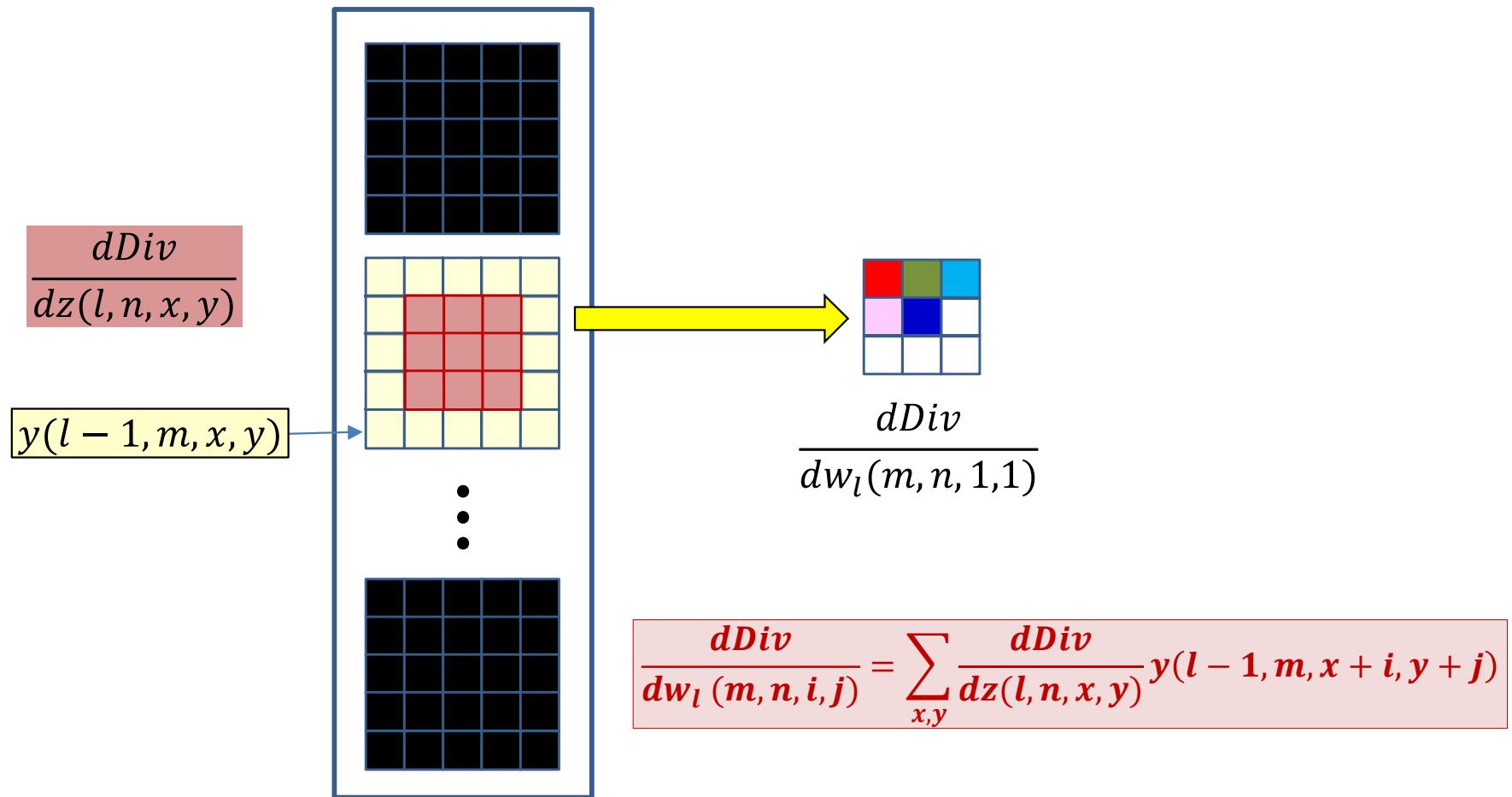
- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
 - Must use them to compute the derivative for $w_l(m, n, *, *)^{191}$

The filter derivative



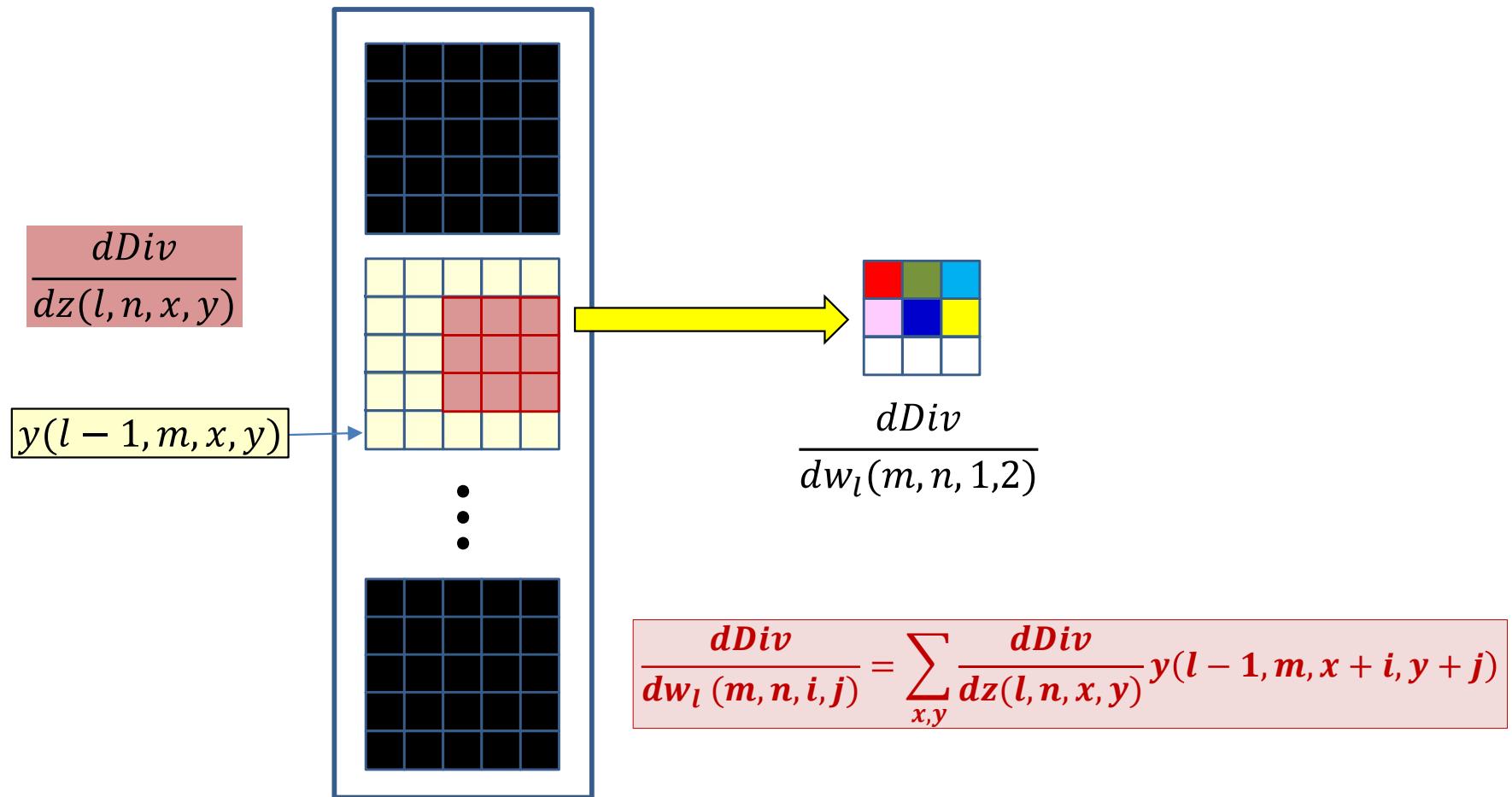
- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
 - Must use them to compute the derivative for $w_l(m, n, *, *)^{192}$

The filter derivative



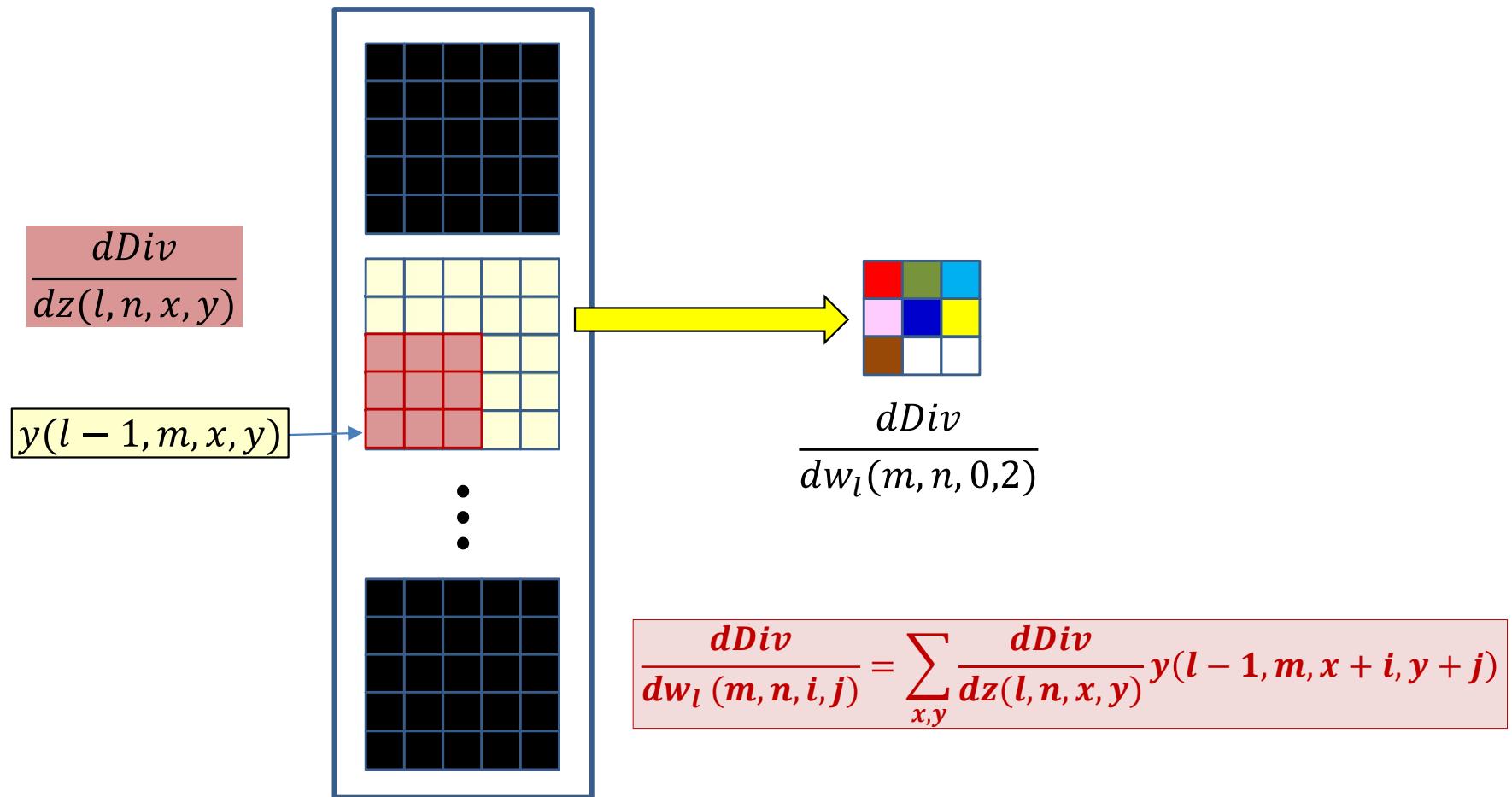
- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
 - Must use them to compute the derivative for $w_l(m, n, *, *)^{193}$

The filter derivative



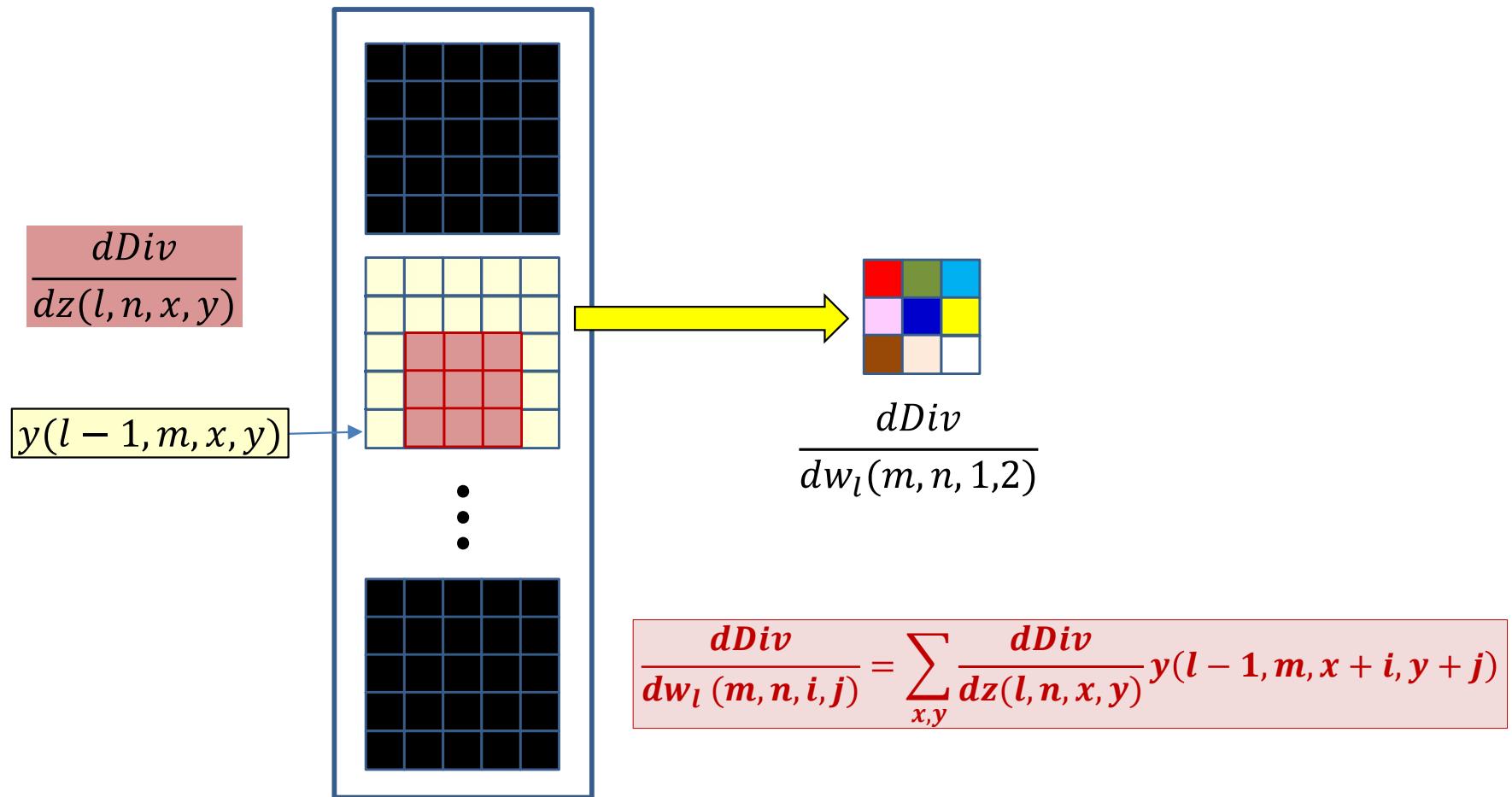
- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
 - Must use them to compute the derivative for $w_l(m, n, *, *)^{194}$

The filter derivative



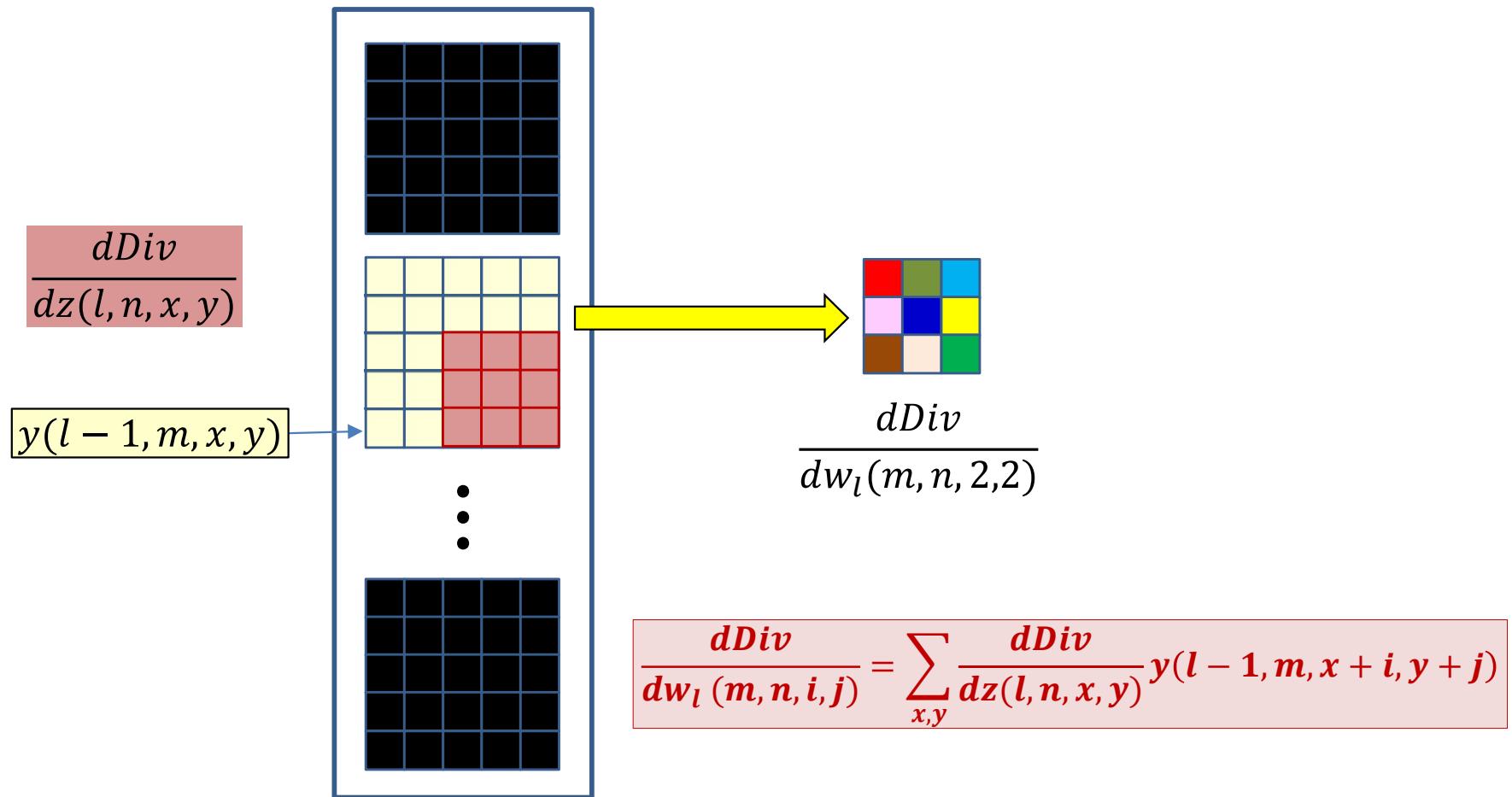
- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
 - Must use them to compute the derivative for $w_l(m, n, *, *)^{195}$

The filter derivative



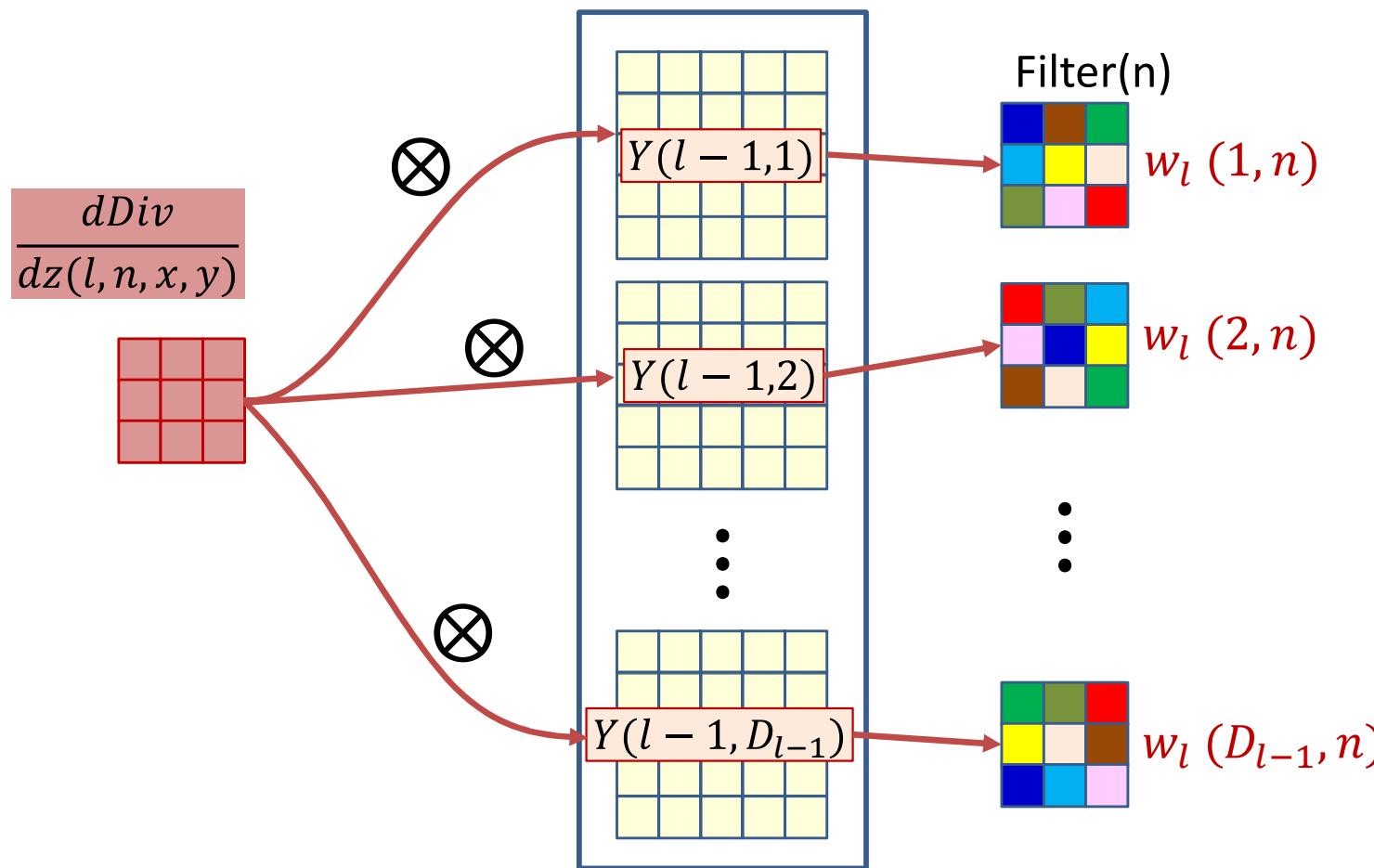
- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
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The filter derivative



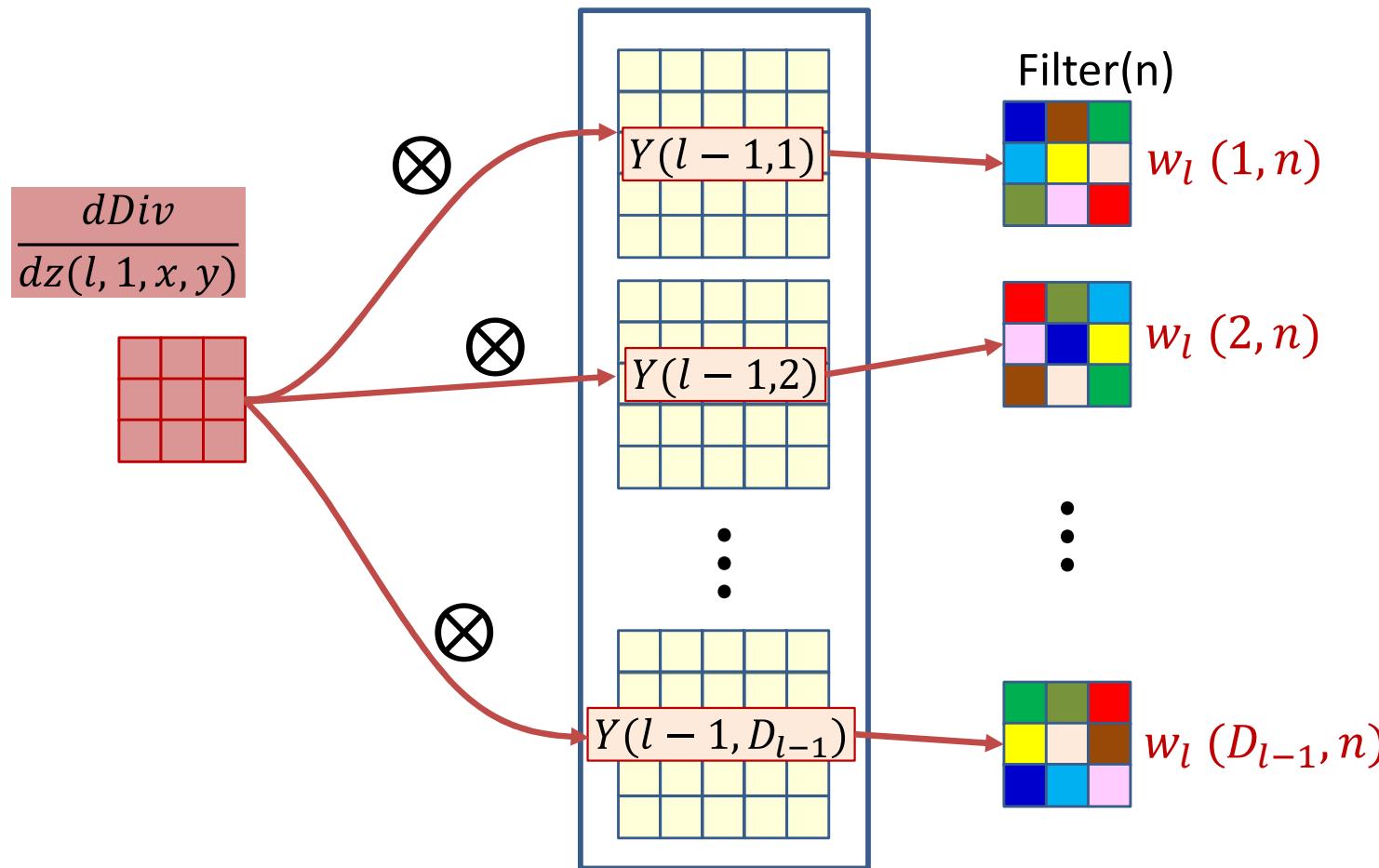
- The derivatives of the divergence w.r.t. every element of $Z(l, n)$ is known
 - Must use them to compute the derivative for $w_l(m, n, *, *)^{197}$

The filter derivative



- The derivative of the n^{th} affine map $Z(l, n)$ convolves with every output map $Y(l - 1, m)$ of the $(l - 1)^{\text{th}}$ layer, to get the derivative for $w_l(m, n)$, the m^{th} “channel” of the n^{th} filter

The filter derivative



$$\frac{dDiv}{dw_l(m, n, i, j)} = \sum_{x,y} \frac{dDiv}{dz(l, n, x, y)} y(l-1, m, x + i, y + j)$$

$$= \frac{dDiv}{dz(l, n)} \otimes y(l-1, m)$$

If $Y(l-1, m)$ was zero padded in the forward pass, it must be zero padded for backprop

Poll 4

Select all statements that are true about how to compute the derivative of the divergence w.r.t lth layer filters using backpropagation

- The derivative for the mth plane of the nth filter is computed by convolving the mth input (l-1th) layer map with the nth output (lth) layer affine derivative map
- The output map must be flipped left-right/up-down before convolution

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- The output map must be flipped left-right/up-down before convolution

Derivatives for the filters at layer l : Vector notation

```
# The weight  $W(l, j)$  is a 3D  $D_{l-1} \times K_l \times K_l$ 
# Assuming that derivative maps have been upsampled
# if stride > 1
# Also assuming y map has been zero-padded if this was
# also done in the forward pass
# The width and height of the dz map are W and H
```

```
for n = 1:Dl
    for x = 1:Kl
        for y = 1:Kl
            for m = 1:Dl-1
                dw(l,m,n,x,y) = dz(l,n,:,:,:) . #dot product
                                                y(l-1,m,x:x+W-1,y:y+W-1)
```

Derivatives through a convolutional layer

- The entire process is simpler if we simply look at it through code
 - Through the reapplication of two simple rules:
- For any computation of the form

$$y = \sigma(z)$$

- The loss derivative for z given the loss derivative of y is

$$\frac{dL}{dz} = \frac{dL}{dy} \sigma'(z)$$

- For any computation in the forward pass

$$z = wy$$

- The backward computation to compute loss derivatives for the terms on the right, given loss derivatives to the left is

$$dL/dy += wdL/dz ; dL/dw += ydL/dz$$

- Since this is “backpropagation”, all computations are reversed

CNN: Forward

```
Y(0,:,:,:, :) = Image
for l = 1:L  # layers operate on vector at (x,y)
    for x = 1:Wl-1-Kl+1
        for y = 1:Hl-1-Kl+1
            for j = 1:Dl
                z(l,j,x,y) = 0
                for i = 1:Dl-1
                    for x' = 1:Kl
                        for y' = 1:Kl
                            z(l,j,x,y) += w(l,j,i,x',y')
                            Y(l-1,i,x+x'-1,y+y'-1)
                Y(l,j,x,y) = activation(z(l,j,x,y))
Y = softmax( Y(L,:,:1,1)..Y(L,:,:W-K+1,H-K+1) )
```

Switching to 1-based indexing with appropriate adjustments

Backward layer l

```
dw(l) = zeros(DlxDl-1xKlxKl)
dY(l-1) = zeros(Dl-1xWl-1xHl-1)
for x = Wl-1-Kl+1:downto:1
    for y = Hl-1-Kl+1:downto:1
        for j = Dl:downto:1
            dz(l,j,x,y) = dY(l,j,x,y).f'(z(l,j,x,y))
            for i = Dl-1:downto:1
                for x' = Kl:downto:1
                    for y' = Kl:downto:1
                        dY(l-1,i,x+x'-1,y+y'-1) +=
                            w(l,j,i,x',y')dz(l,j,x,y)
                        dw(l,j,i,x',y') +=
                            dz(l,j,x,y)Y(l-1,i,x+x'-1,y+y'-1)
```

Complete Backward (no pooling)

```
dY(L) = dDiv/dY(L)
for l = L:downto:1  # Backward through layers
    dw(l) = zeros(DlxDl-1xKlxKl)
    dY(l-1) = zeros(Dl-1xWl-1xHl-1)
    for x = Wl-1-Kl+1:downto:1
        for y = Hl-1-Kl+1:downto:1
            for j = Dl:downto:1
                dz(l,j,x,y) = dY(l,j,x,y).f'(z(l,j,x,y))
                for i = Dl-1:downto:1
                    for x' = Kl:downto:1
                        for y' = Kl:downto:1
                            dY(l-1,i,x+x'-1,y+y'-1) +=
                                w(l,j,i,x',y')dz(l,j,x,y)
                            dw(l,j,i,x',y') +=
                                dz(l,j,x,y)y(l-1,i,x+x'-1,y+y'-1)
```

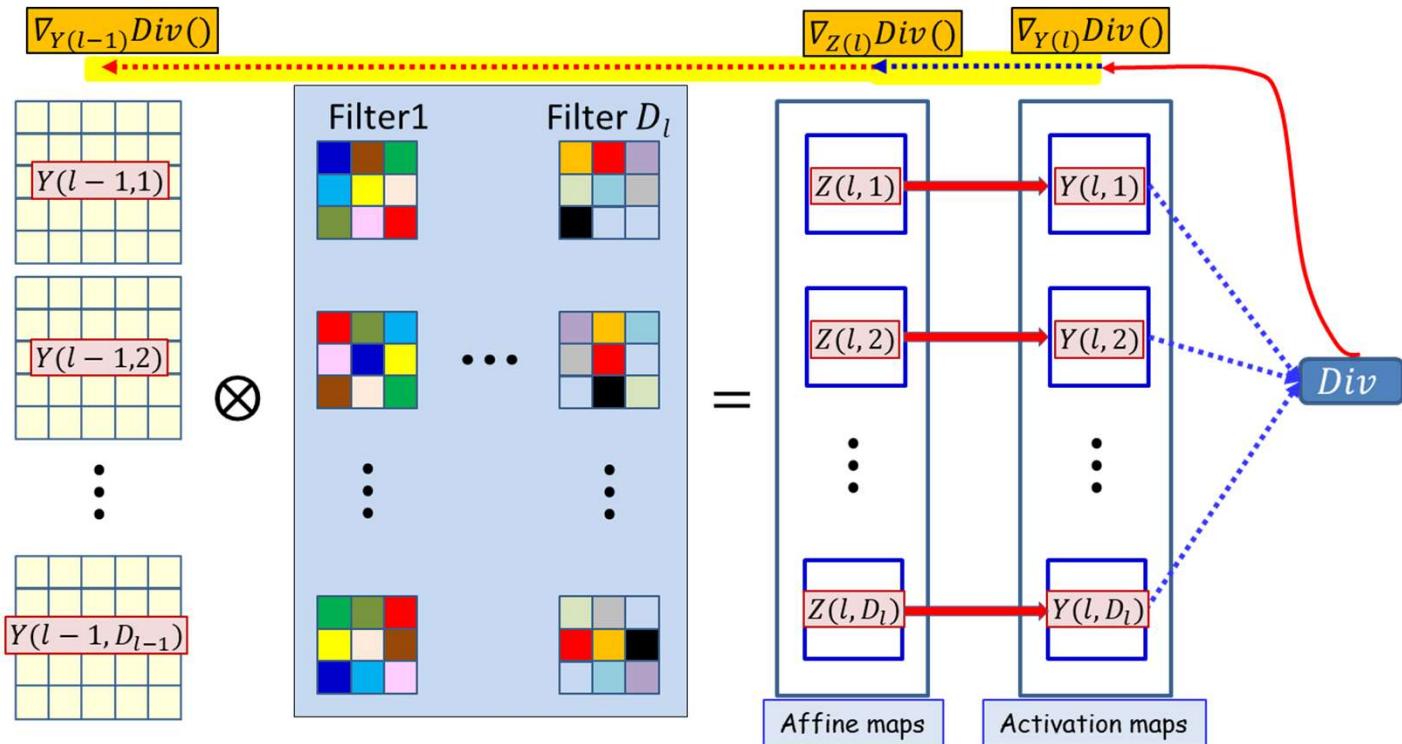
Complete Backward (no pooling)

```
dY(L) = dDiv/dY(L)
for l = L:downto:1 # Backward through layers
    dw(l) = zeros(DlxDl-1xKlxKl)
    dY(l-1) = zeros(Dl-1xWl-1xHl-1)
    for x = Wl-1-Kl+1:downto:1
        for y = Hl-1-Kl+1:downto:1
            for j = Dl:downto:1
                dz(l,j,x,y) = dY(l,j,x,y).f'(z(l,j,x,y))
                for i = Dl-1:downto:1
                    for x' = Kl:downto:1
                        for y' = Kl:downto:1
                            dY(l-1,i,x+x'-1,y+y'-1) +=
                                w(l,j,i,x',y')dz(l,j,x,y)
                            dw(l,j,i,x',y') +=
                                dz(l,j,x,y)y(l-1,i,x+x'-1,y+y'-1)
```

Multiple ways of recasting this as tensor/ vector operations.

Will not discuss here

Backpropagation: Convolutional layers



- **For convolutional layers:**



How to compute the derivatives w.r.t. the affine $Z(l)$ maps from the activation output maps $Y(l)$

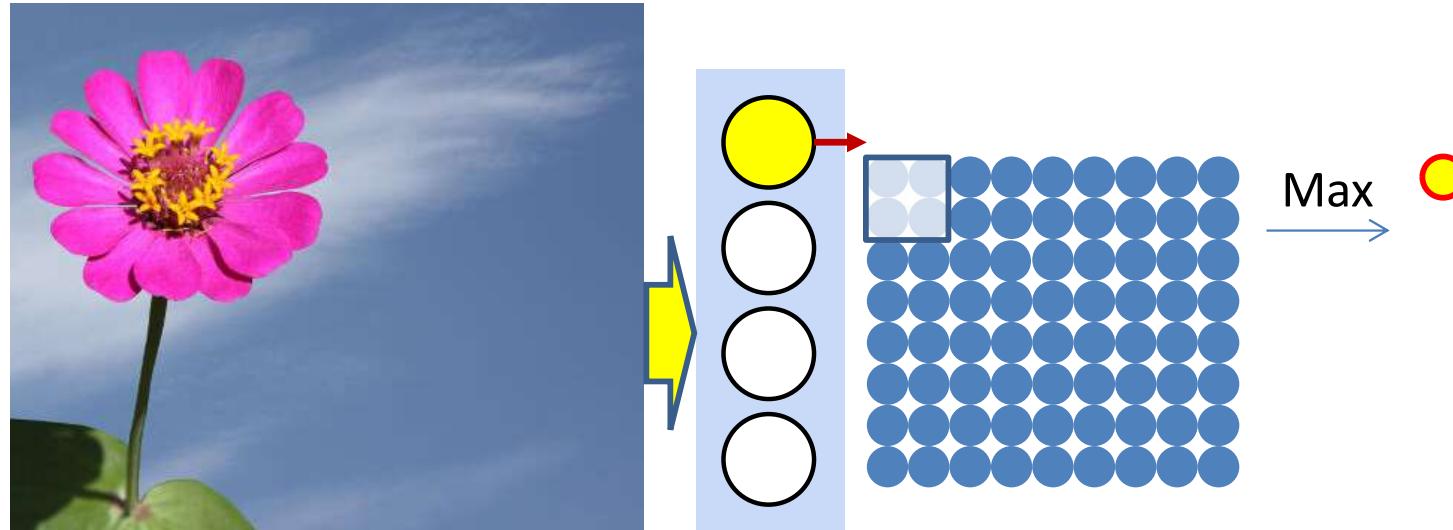


How to compute the derivative w.r.t. $Y(l - 1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$

Backpropagation: Convolutional and Pooling layers

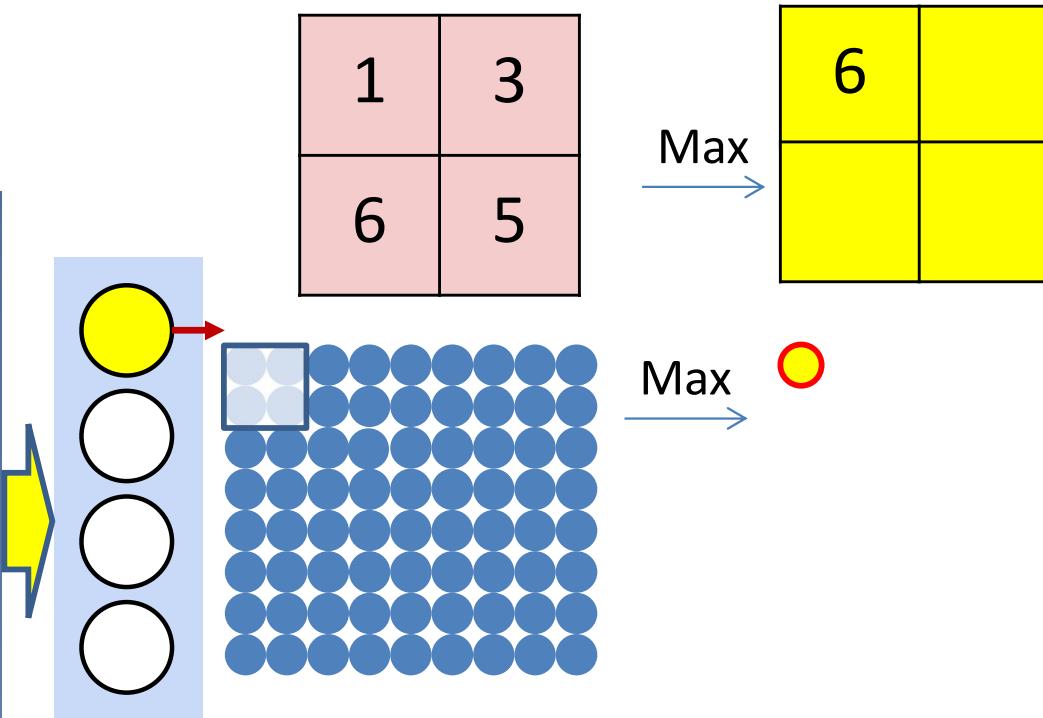
- **Assumption:** We already have the derivatives w.r.t. the elements of the maps output by the final convolutional (or pooling) layer
 - Obtained as a result of backpropagating through the flat MLP
- **Required:**
 - **For convolutional layers:**
 - How to compute the derivatives w.r.t. the affine $Z(l)$ maps from the activation output maps $Y(l)$
 - How to compute the derivative w.r.t. $Y(l - 1)$ and $w(l)$ given derivatives w.r.t. $Z(l)$
 - **For pooling layers:**
 - How to compute the derivative w.r.t. $Y(l - 1)$ given derivatives w.r.t. $Y(l)$

Pooling



- Pooling “pools” groups of values to reduce jitter-sensitivity
 - Scanning with a “pooling” filter
- The most common pooling is “Max” pooling

Max Pooling

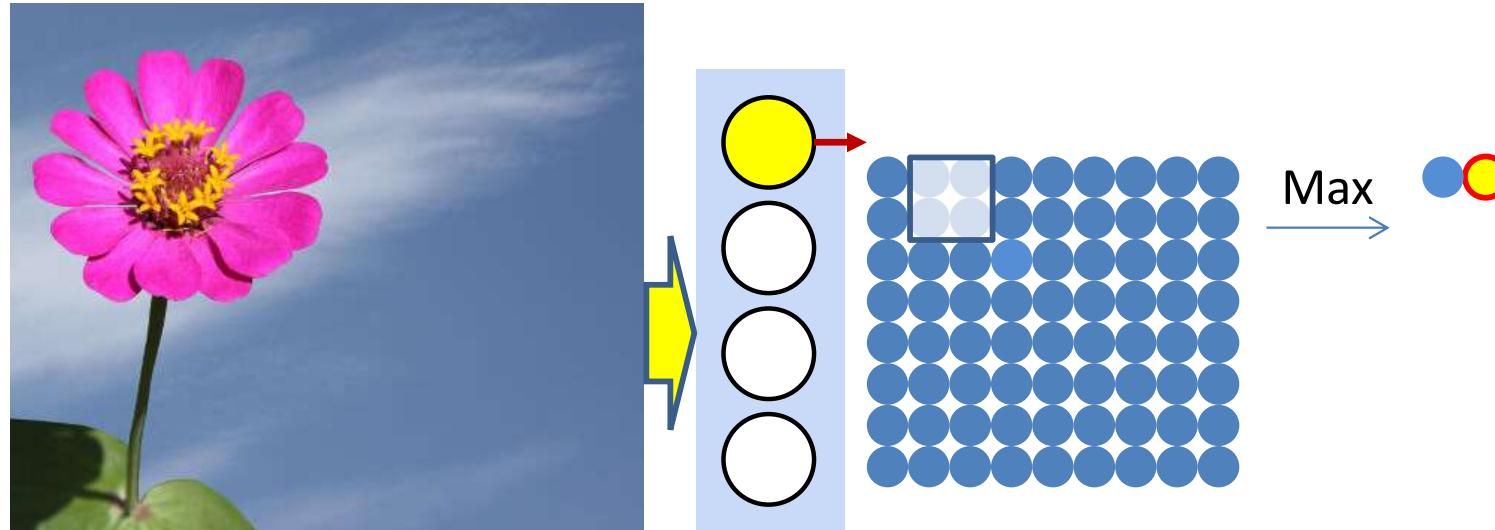


- Max pooling selects the largest from a pool of elements
- Pooling is performed by “scanning” the input

$$P(l, m, i, j) = \operatorname{argmax}_{\substack{k \in \{i, i+K_{lpool}-1\}, \\ n \in \{j, j+K_{lpool}-1\}}} Y(l-1, m, k, n)$$

$$Y(l, m, i, j) = Y(l-1, m, P(l, m, i, j))$$

Max pooling

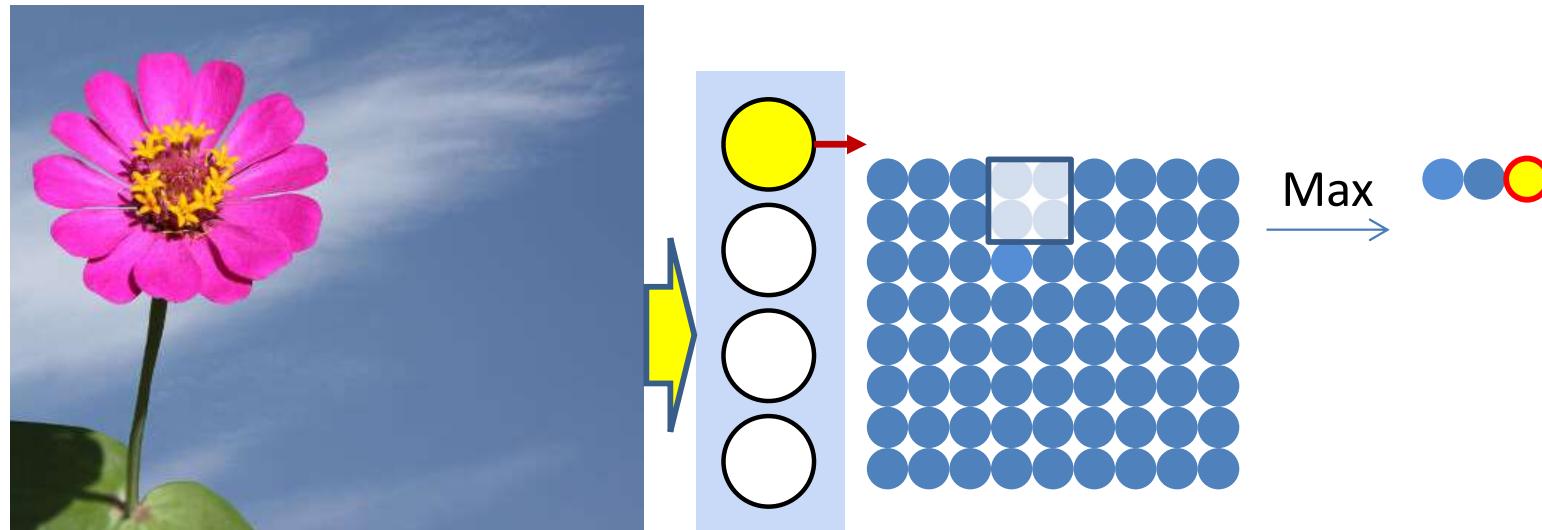


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Max pooling

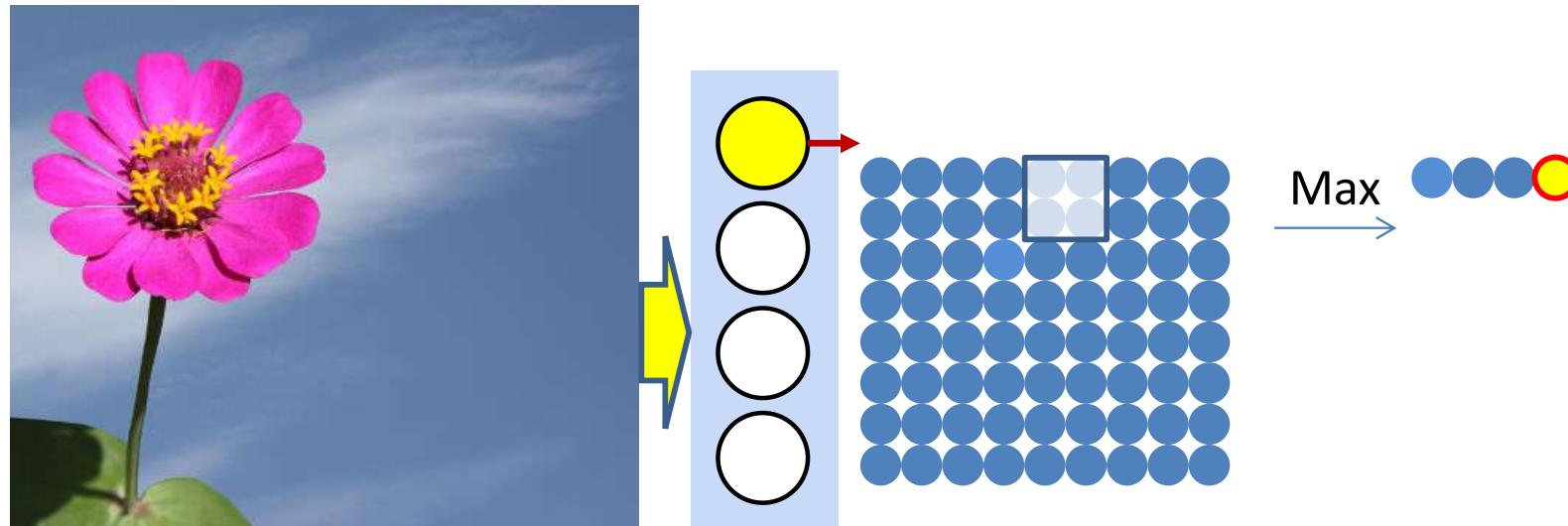


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$$Y(l, m, i, j) = Y(l-1, m, P(l, m, i, j))$$

Max pooling

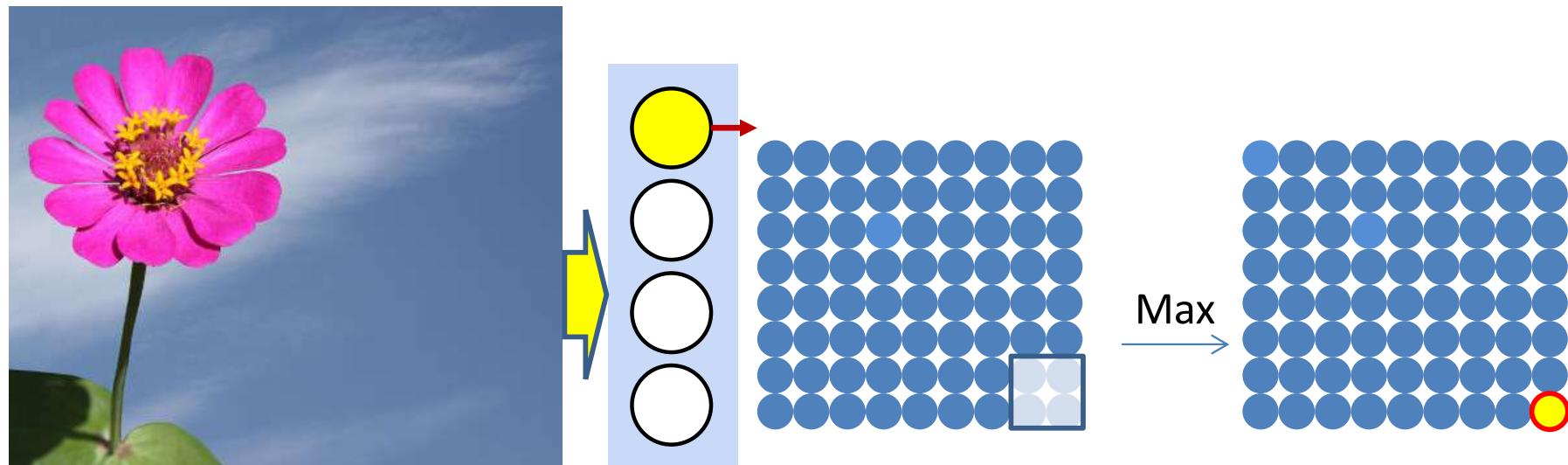


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$$Y(l, m, i, j) = Y(l-1, m, P(l, m, i, j))$$

Max pooling

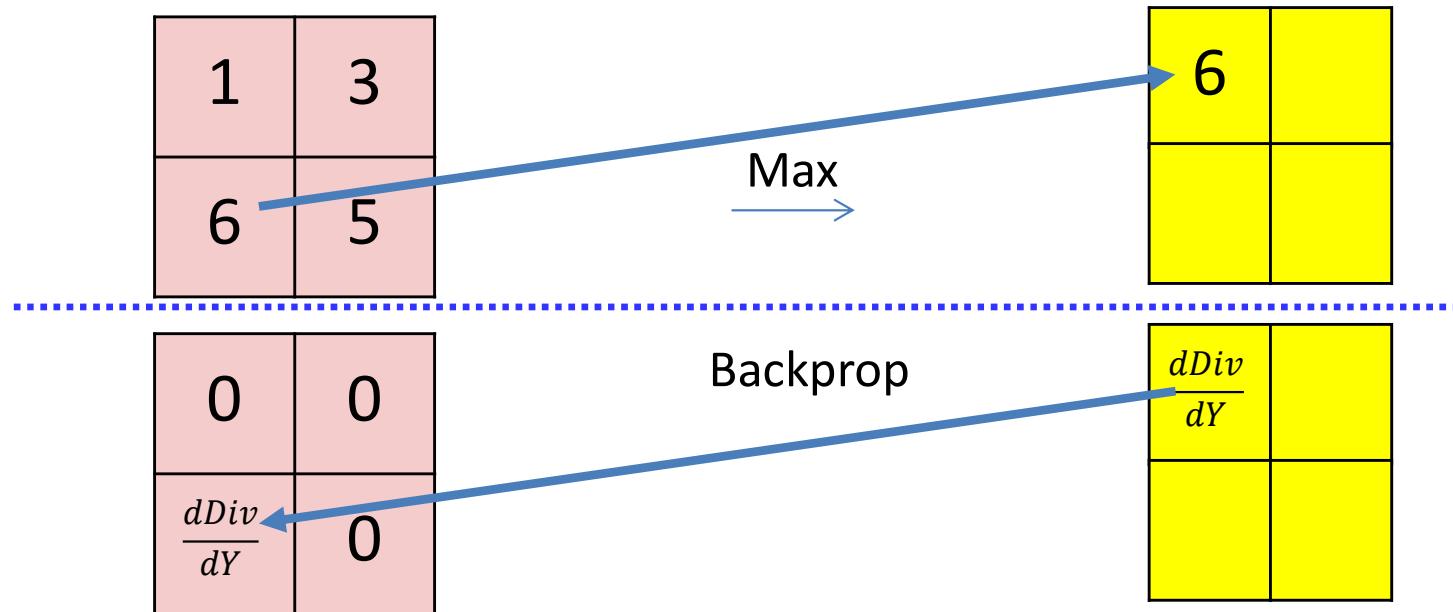


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- Pooling is performed by “scanning” the input

$$P(l, m, i, j) = \operatorname{argmax}_{\substack{k \in \{i, i+K_{lpool}-1\}, \\ n \in \{j, j+K_{lpool}-1\}}} Y(l-1, m, k, n)$$

$$Y(l, m, i, j) = Y(l-1, m, P(l, m, i, j))$$

Derivative of Max pooling



$$\frac{dDiv}{dy(l-1, m, k, l)} = \begin{cases} \frac{dDiv}{dy(l, m, i, j)} & \text{if } (k, l) = P(l, m, i, j) \\ 0 & \text{otherwise} \end{cases}$$

- Max pooling selects the largest from a pool of elements

$$P(l, m, i, j) = \operatorname{argmax}_{\substack{k \in \{i, i+K_{lpool}-1\}, \\ n \in \{j, j+K_{lpool}-1\}}} Y(l-1, m, k, n)$$

$$y(l, m, i, j) = y(l-1, m, P(l, m, i, j))$$

Max Pooling layer at layer l

- a) Performed separately for every map (j).
*) Not combining multiple maps within a single max operation.
- b) Keeping track of location of max

Max pooling

```
for j = 1:Dl
    for x = 1:Wl-1-Kl+1
        for y = 1:Hl-1-Kl+1
            pidx(l,j,x,y) = maxidx(y(l-1,j,x:x+Kl-1, y:y+Kl-1))
            y(l,j,x,y) = y(l-1,j,pidx(l,j,x,y))
```



Derivative of max pooling layer at layer l

- a) Performed separately for every map (j).
*) Not combining multiple maps within a single max operation.
- b) Keeping track of location of max

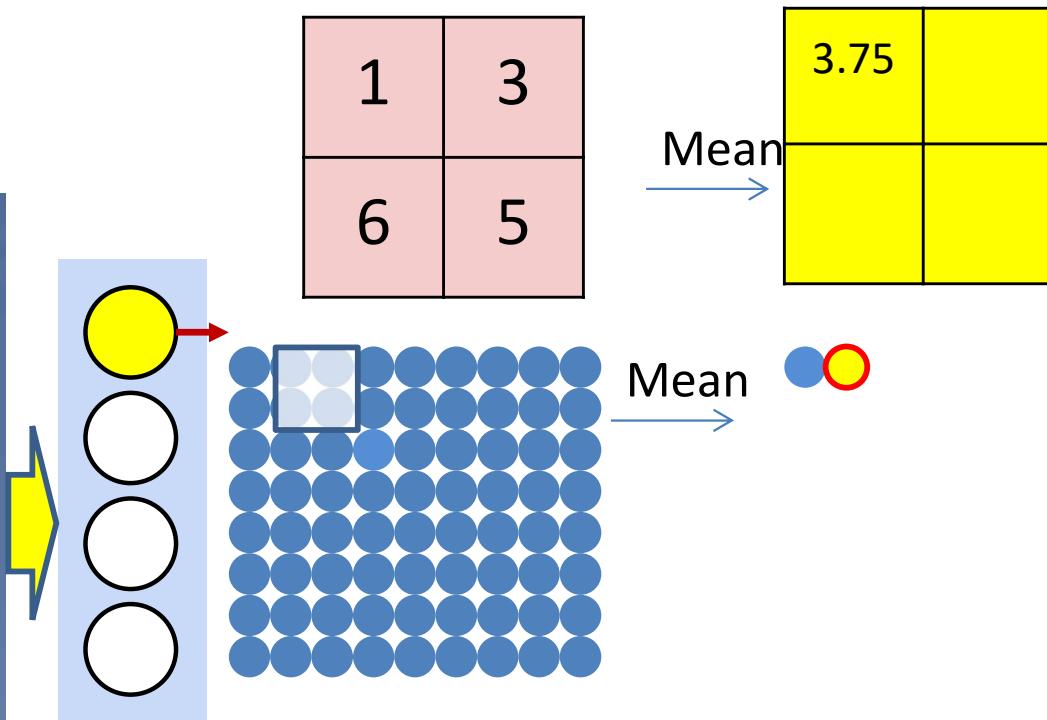
Max pooling

```
dy (:, :, :) = zeros(D1 x W1 x H1)
for j = 1:D1
    for x = 1:W1
        for y = 1:H1
            dy(l-1, j, pidx(l, j, x, y)) += dy(l, j, x, y)
```



“ $+=$ ” because this entry may be selected in multiple adjacent overlapping windows

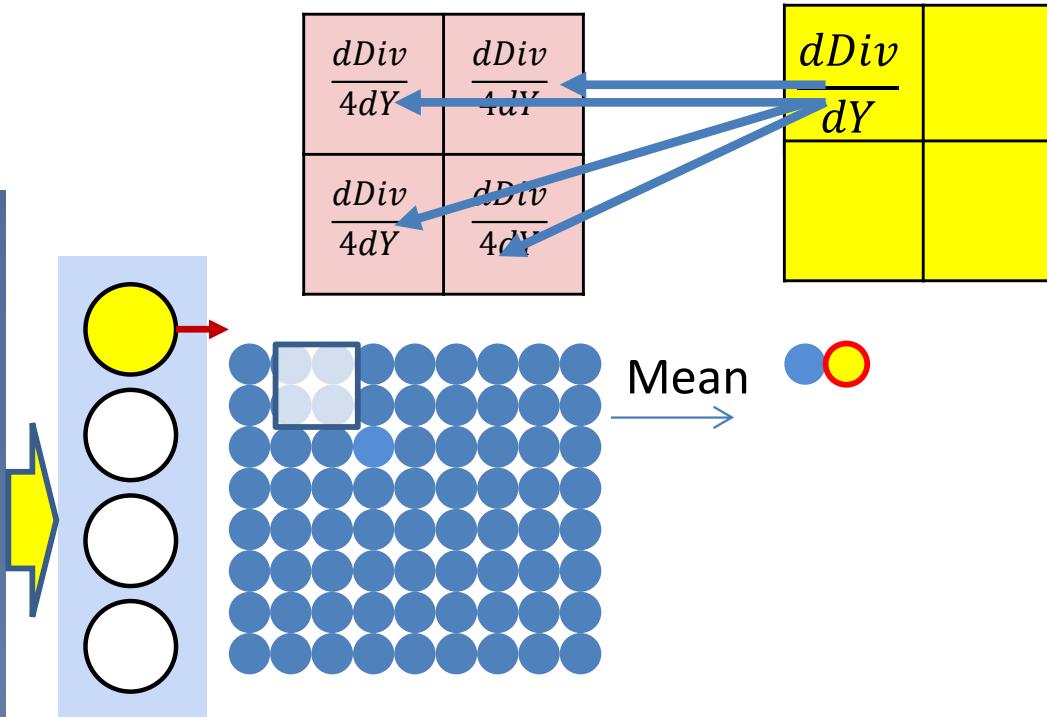
Mean pooling



- Mean pooling compute the mean of a pool of elements
- Pooling is performed by “scanning” the input

$$y(l, m, i, j) = \frac{1}{K_{lpool}^2} \sum_{\substack{k \in \{i, i+K_{lpool}-1\}, \\ n \in \{j, j+K_{lpool}-1\}}} y(l-1, m, k, n)$$

Derivative of mean pooling



- The derivative of mean pooling is distributed over the pool

$$k \in \{i, i + K_{lpool} - 1\}, n \in \{j, j + K_{lpool} - 1\} \quad dy(l-1, m, k, n) += \frac{1}{K_{lpool}^2} dy(l, m, k, n)$$

Mean Pooling layer at layer l

Mean pooling

```
for j = 1:Dl #Over the maps
    for x = 1:Wl-1-Kl+1 #Kl = pooling kernel size
        for y = 1:Hl-1-Kl+1
            y(l,j,x,y) = mean(y(l-1,j,x:x+Kl-1,y:y+Kl-1))
```

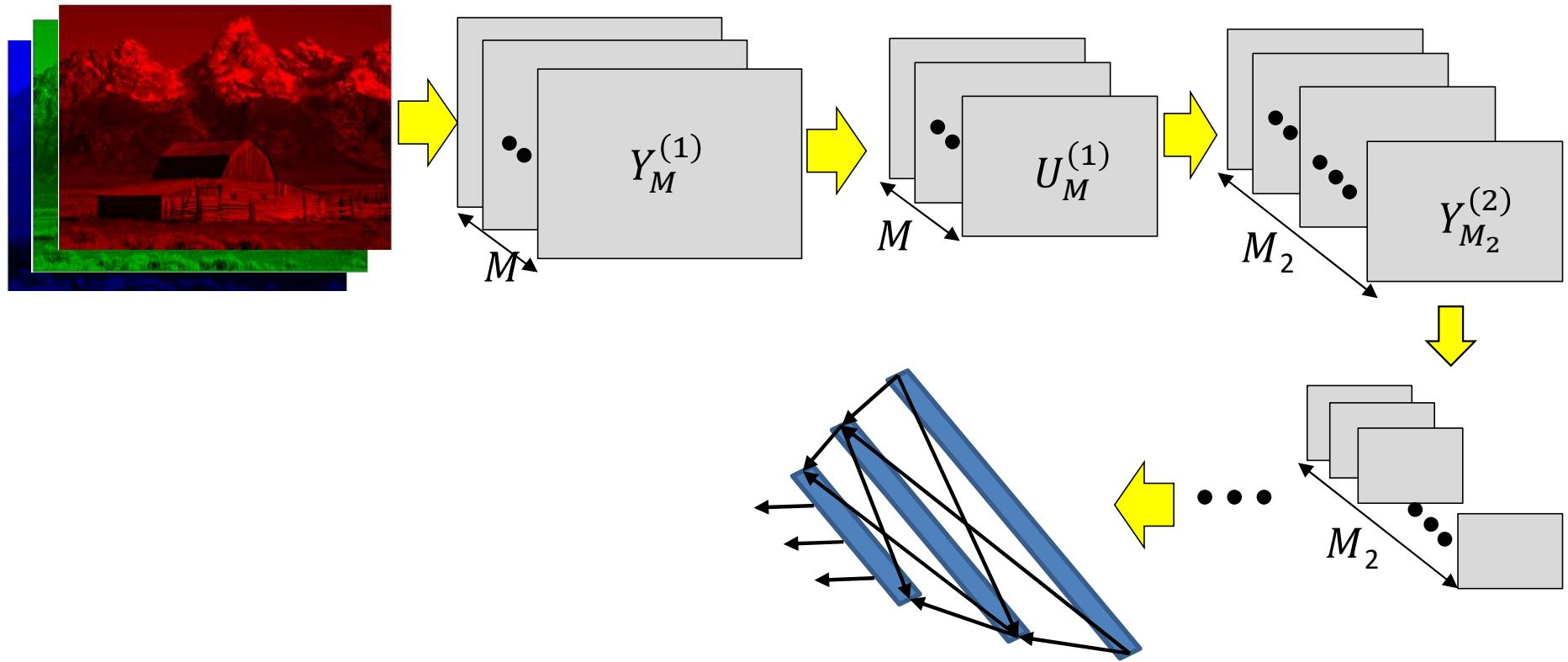
Derivative of mean pooling layer at layer l

Mean pooling

```
dy(:,:, :, :) = zeros(Dl x Wl x Hl)
for k = 1:Dl
    for x = 1:Wl
        for y = 1:Hl
            for i = 1:Klpool
                for j = 1:Klpool
                    dy(l-1, k, p, x+i, y+j) += (1/Klpool2) dy(l, k, x, y)
```

“+=” because adjacent windows may overlap

Learning the network



- Have shown the derivative of divergence w.r.t every intermediate output, and every free parameter (filter weights)
- Can now be embedded in gradient descent framework to learn the network
- Still missing one component... resampling
 - Next class

Story so far

- The CNN is a supervised version of a computational model of mammalian vision
- It includes
 - Convolutional layers comprising learned filters that scan the outputs of the previous layer, followed by an activation function
 - Pooling layers that operate over groups of outputs from the convolutional layer for jitter invariance
 - Optional resizing operations, typically performed through strides greater than 1 (for downsampling) or interpolation of zeros (for upsampling)
- The parameters of the network can be learned through regular back propagation
 - Computing derivatives for convolutional layers is simply also a convolution operation
 - Convolve transposed filters with derivative maps of output for derivatives of input maps
 - Convolve input maps with derivative maps of the output for derivatives of filters
 - Backprop of pooling operations redistributes the derivatives at the output over all the inputs that actively contributed to it