Set Theory Notes

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Contents

1 Axiomatic Set Theory

Axiom 1.1 (Zermelo-Fraenkel Axioms)**.**

I Axiom of Extensionality: If X and Y have the same elements, then $X = Y$.

II Axiom of Pairing: For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.

III Axiom Schema of Seperation: If φ is a property (with parameter p), then for any X and p, there exists a set $Y = \{u \in X \mid \varphi(u, p)\}$ that contains all those $u \in X$ that have the property φ.

IV Axiom of Union: For any X there exists a set $Y = \bigcup X$, the union of all elements of X.

V Axiom of Power Set: For any X there exists a set $Y = \mathcal{P}(X)$, the set of all subsets of X.

VI Axiom of Infinity: There exists an infinite set.

VII Axiom Schema of Replacement: If F is a class function, then for any X there exists a set $Y = F[X] = \{F(x) \mid x \in X\}$.

VIII Axiom of Regularity: Every non-empty set has an $\in$-minimal element.

IX Axiom of Choice: Every family of non-empty sets has a choice function.

Definition 1.2 (T-finite)**.** A set S is T-finite if every non-empty $X \subseteq \mathcal{P}(S)$ has a a $\subset$-maximal element.

2 Ordinal Numbers

Definition 2.1 (well ordering)**.** A linearly ordered set $(P, <)$ is well-ordered if every non-empty subset of P has a least element.

Lemma 2.2. If $(W, <)$ is a well-ordered set, and $f : W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$.

Proof. Assume for the sake of contradiction that $X = \{x \in W \mid f(x) < x\}$ is non-empty. As W is well-ordered we have that such a set has a least element, z. As $f(z) < z$, we have $f(f(z)) < f(z)$, meaning $f(f(z)) \in X$, resulting in contradiction. □

Corollary 2.3. The only isomorphism of a well-ordered set onto itself is the identity.

Corollary 2.4. If two well-ordered sets W_1 and W_2 are isomorphic then the isomorphism of W_1 onto W_2 is unique.

Proof. Consider arbitrary isomorphisms, f_1, f_2, from W_1 to W_2. Assume for the sake of contradiction that $X = \{w \in W_1 \mid f_1(x) \neq f_2(x)\}$ is non-empty. Let x be the least element of X. Without loss of generality let us say $f_1(x) < f_2(x)$. Then we have $f_2^{-1}(f_1(x)) < x$. We have $f_2(f_2^{-1}(f_1(x))) = f_1(x)$ and $f_1(f_2^{-1}(f_1(x))) < f_1(x)$. Therefore $f_2^{-1}(f_1(x)) \in X$, resulting in contradiction. □

Definition 2.5 (Initial Segment)**.** If W is a well-ordered set and $u \in W$, then $\{x \in W \mid x < u\}$ is an initial segment of W given by u. We denote such an initial segment $W(u)$.

Lemma 2.6. No well-ordered set is isomorphic to an initial segment of itself.

Proof. If $\operatorname{ran}(f) = \{x \mid x < u\}$ then $f(u) < u$, contradicting 2.2. □

Theorem 2.7. If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds:

- W_1 is isomorphic to W_2.
- W_1 is isomorphic to an initial segment of W_2.
- W_2 is isomorphic to an initial segment of W_1.

Proof. Consider the function

$$f = \{(x, y) \in W_1 \times W_2 \,|\, W_1(x) \text{ is isomorphic to } W_2(y)\}$$

□

Consider an arbitrary $x_1, x_2 \in W_1$. If $W_2(y_1)$ and $W_2(y_2)$ are isomorphic to $W_1(x_1)$, then we have $W_2(y_1)$ is isomorphic to $W_2(y_2)$. If $y_1 \neq y_2$ this would imply different initial segments are isomorphic, contradicting 2.6. Therefore $y_1 = y_2$, meaning f is a function. Similarly, if $f(x_1) = f(x_2)$ then $W_1(x_1)$ is isomorphic to $W_2(x_2)$, implying that $x_1 = x_2$. Therefore f is injective.

Let us say $x_1 < x_2$ and $(x_2, h(x_2)) \in f$, where h denotes an isomorphism from $W_1(x_2)$ to $W_2(y)$ for some $y \in W_2$. It holds that $h \restriction_{W_1(x_1)}$ is an isomorphism from $W_1(x_1)$ to $W_2(h(x_1))$. Therefore $f(x_1)$ exists. As h is increasing we have $h(x_1) < h(x_2)$ implying $f(x_1) < f(x_2)$. Thus f is increasing.

If $\mathrm{dom}(f) = W_1$ and $\mathrm{ran}(f) = W_2$ then W_1 is isomorphic to W_2. If $\mathrm{ran}(f) \neq W_2$ then W_1 is isomorphic to $W_2(u)$ where u is the least element of $W_2 \setminus \mathrm{ran}(f)$. This is because $\mathrm{ran}(f)$ will be equal to $W_2(u)$ and $\mathrm{dom}(f) = W_1$ necesarily, as otherwise $(w, u) \in f$ where w is the least element of $W_1 \setminus \mathrm{dom}(f)$. If $\mathrm{dom}(f) \neq W_1$ then $W_1(u)$ is isomorphic to W_2, where u is the least element of $W_1 \setminus \mathrm{dom}(f)$. This is because $\mathrm{dom}(f)$ will be equal to $W_1(u)$ and $\mathrm{ran}(f) = W_2$ necesarily, as otherwise $(u, w) \in f$ where w is the least element of $W_2 \setminus \mathrm{ran}(f)$. By 2.6 these cases must be mutually exclusive.

Definition 2.8 (Transitive)**.** A set T is transitive if every element of T is a subset of T.

Definition 2.9 (Ordinal)**.** A set is an ordinal number if it is transitive and well-ordered by $\in$. We denote Ord as the class of all ordinals. We define $<$ where $\alpha < \beta$ if and only if $\alpha \in \beta$.

Lemma 2.10.

(i) $0 = \varnothing$ is an ordinal

(ii) If α is an ordinal and $\beta \in \alpha$ then β is an ordinal.

(iii) If $\alpha \neq \beta$ and $\alpha \subsetneq \beta$ then $\alpha \in \beta$.

(iv) If α, β are ordinals then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

Proof. We have (i) and (ii) follow from the definition of an ordinal.

If $\alpha \subsetneq \beta$, consider the least element of the set $\beta \setminus \alpha$, denoted by γ. Consider the initial segment $\beta(\gamma)$. By construction we have $\beta(\gamma) \subseteq \alpha$. If $a \in \alpha$ were go be greater than γ, then γ would be contained in a and thus in α by transitivity, resulting in contradiction. Therefore $a < \gamma$. Thus $\alpha = \{\xi \in \beta \,|\, \xi < \gamma\}$. By the definition of $<$ we then have $\alpha \subseteq \gamma$. As $\gamma \in \beta$ we have $\gamma \subseteq \beta$. Therefore, $\forall x \in \gamma$ we have $x \in \beta$, meaning $x \in \alpha$. Therefore $\gamma \subseteq \alpha$. Thus $\alpha = \gamma$, meaning $\alpha \in \beta$.

It holds that $\alpha \cap \beta = \gamma$ is an ordinal. Assume for the sake of contradiction that $\gamma \neq \alpha$ and $\gamma \neq \beta$. As $\gamma \subseteq \alpha$ and $\gamma \subseteq \beta$ we have $\gamma \in \alpha$ and $\gamma \in \beta$. Therefore $\gamma \in \gamma$, resulting in contradiction. Thus $\gamma = \alpha$ or $\gamma = \beta$, meaning either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. □

Corollary 2.11.

- $<$ is a linear ordering of the class Ord.
- For each α we have $\alpha = \{\beta \,|\, \beta < \alpha\}$.
- If C is a non-empty class of ordinals, then $\cap C$ is an ordinal, $\cap C \in C$ and $\cap C = \inf C$.
- If X is a non-empty set of ordinals, then $\cup X$ is an ordinal, and $\cup X = \sup X$.
- For every α we have $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = \inf\{\beta \,|\, \beta > \alpha\}$.

Proof. Trivially we have $<$ is irreflexive, asymmetric, and transitive. By 2.10 we have ordinals are either equal or have one contain the other. Therefore every ordinal is comparable with $<$.

By definition $\{\beta \,|\, \beta < \alpha\} \subseteq \alpha$. By 2.10 every $\beta \in \alpha$ is an ordinal, thus $\alpha \subseteq \{\beta \,|\, \beta < \alpha\}$.

We have that $\cap C \subseteq c$ for all $c \in C$. Thus either $\cap C = c$ or $\cap C \in c$. It must hold that $\cap C = c$ for some c, as otherwise we would have $\cap C \in \cap C$, resulting in contradiction. Therefore $\cap C \in C$, which also gives us that $\cap C$ is an ordinal. By definition $\cap C \leq c$ for all $c \in C$, and as $\cap C \in C$, we have that $\cap C = \inf C$.

For an arbitrary $x \in \cup X$ we have that $x \in c$ for some $c \in X$. It follows that $x \subseteq c \subseteq \cup X$, meaning $\cup X$ is transitive. Consider arbitrary $a, b \in \cup X$. By 2.10 we have either $a = b$, $a \in b$ or $b \in a$, meaning a and b are $<$-comparable. Thus $\cup X$ is linearly ordered. Consider any non-empty $p \in \mathcal{P}(\cup X)$. We have that p is non-empty, meaning $\exists c \in p$. Then $c \cap p$ has a minimal element, which will then be a minimal element of p. Thus $\cup X$ is well-ordered. Therefore $\cup X$ is an ordinal. We have for all $x \in X$, $x \subseteq \cup X$, meaning $x \in \cup X$. Therefore $x < \cup X$. Similarly, for any $x < \cup X$ we have $x \in \cup X$. Thus x is contained in some set in X, meaning x is not an upper bound of $\cup X$. Therefore $\cup X = \sup X$.

For any $a \in \alpha \cup \{\alpha\}$ we have either $a \in \alpha$ or $a = \alpha$. If $a \in \alpha$ then $a \subseteq \alpha \subseteq \alpha \cup \{\alpha\}$. If $a = \alpha$ then $a \subseteq \alpha \cup \{\alpha\}$. Thus $\alpha \cup \{\alpha\}$ is transitive. $\alpha \cup \{\alpha\}$ is well-ordered as α is well-ordered, and every element in α is less than α by definition. Therefore $\alpha \cup \{\alpha\}$ is an ordinal. We have $\alpha < \alpha \cup \{\alpha\}$. Consider an arbitrary β such that $\alpha < \beta$. We have $\alpha \in \beta$ and $\alpha \subseteq \beta$. Therefore $\alpha \cup \{\alpha\} \subseteq \beta$ meaning $\alpha \cup \{\alpha\} \in \beta$ or $\alpha \cup \{\alpha\} = \beta$. Thus $\alpha \cup \{\alpha\} \leq \beta$ so $\alpha \cup \{\alpha\} = \inf\{\beta \,|\, \beta > \alpha\}$. □

Definition 2.12 (Successor)**.** Given an ordinal α we say $\alpha + 1 := \alpha \cup \{\alpha\}$ is the successor of α.

Theorem 2.13. Every well-ordered wet is isomorphic to a unique ordinal number.

Proof. Consider an arbitrary well-ordered set W. Consider an α and β that W is isomorphic to. If $\alpha \neq \beta$ we have without loss of generality that $\alpha \in \beta$ by 2.10. It holds that $\alpha = \beta(\alpha)$, which would contradict 2.7. Thus if a well-ordered set is isomorphic to a ordinal, it is unique.

Define the function $F(x) = \alpha \in \text{Ord}$ if $W(x)$ is isomorphic to α. We have shown that each α is unique, meaning our function is well-defined. Similarly, 2.7 gives us that F is injective and increasing. Given any $x \in \text{dom}(F)$ we have if $y < x$ then $y \in \text{dom}(F)$ as as restriction of the isomorphism from $W(x)$ to $F(x)$ will be an isomorphism from $W(y)$ to $F(y)$. Assume for the sake of contradiction that $\{w \in W \,|\, w \notin \text{dom}(F)\}$ is non-empty. There thus exists a least element y. Consider the set $\{F(x) \,|\, x \in W(y)\}$. By 2.11 we have the union of this set is an ordinal γ. Then $F \restriction_{W(y)}$ is isomorphic to γ. This is a contradiction, meaning $\text{dom}(F) = W$. We then have that F is an isomorphism from W to $F[W]$. By 2.10 and 2.11 we have that $F[W]$ is well-ordered. Consider an arbitrary $\alpha \in F[W]$. For all $\beta \in \alpha$ we have $h^{-1} \restriction_\beta$ (where h denotes the isomorphism between α and and some initial segment of W) is an isomorphism between $beta$ and an initial segment of W. Therefore $\beta \in F[W]$, meaning $F[W]$ is transitive. Thus $F[W]$ is an ordinal, so W is isomorphic to a unique ordinal. □

Definition 2.14 (Limit Ordinal)**.** An ordinal α which is not a successor is called a limit ordinal. We have that $\alpha = \sup\{\beta \,|\, \beta < \alpha\} = \cup\alpha$. By convention 0 is a limit ordinal and $\sup \varnothing = 0$.

Definition 2.15 (Natural Numbers)**.** We denote the least non-zero limit ordinal ω (or $\mathbb{N}$). The ordinals less than ω are called finite ordinals, or natural numbers.

Theorem 2.16 (Transfinite Induction)**.** Let C be a class of ordinals and assume that

(i) $0 \in C$

(ii) If $\alpha \in C$ then $\alpha + 1 \in C$

(iii) If α is a nonzero limit ordinal and $\beta \in C$ for all $\beta < \alpha$ then $\alpha \in C$.

Then C is the class of all ordinals.

Proof. Assume for the sake of contradiction that C is not the class of all ordinals. Let α be the least ordinal not in C. If $\alpha = 0$ we would contradict (i). If α is a successor ordinal we would contradict (ii). If α is a limit ordinal we would contradict (iii). Therefore $C = \text{Ord}$. □

Definition 2.17. A transfinite sequence is a function whose domain is an ordinal. We say a sequence with domain $\alpha \in \text{Ord}$ is an α-sequence or a sequence of length α. We also denote $s \frown x$ as s with x appended to it.

Theorem 2.18 (Transfinite Recursion)**.** Let G be a function (on the proper class of transfinite sequences). Then

$$\begin{aligned} F(\alpha) = x \iff &\text{there is a sequence } \langle a_\xi \,|\, \xi < \alpha\rangle \text{ such that:} \\ &\text{(i) } (\forall \xi < \alpha)\, \alpha_\xi = G(\langle \alpha_\eta \,|\, \eta < \xi\rangle) \\ &\text{(ii) } x = G(\langle \alpha_\xi \,|\, \xi < \alpha\rangle) \end{aligned}$$

defines a unique function (proper class) on Ord such that

$$F(\alpha) = G(F\restriction_\alpha)$$

equivalently we have if $a_\alpha = F(\alpha)$ then

$$a_\alpha = G(\langle a_\xi \,|\, \xi < \alpha\rangle)$$

Corollary 2.19. Let X be a set and θ an ordinal number. For every function G on the set of all transfinite sequences in X of length $< \theta$ such that $\text{ran}(G) \subseteq X$ there exists a unique θ-sequence $\langle a_\alpha \,|\, \alpha < \theta\rangle$ in X such that $a_\alpha = G(\langle a_\xi \,|\, \xi < \alpha\rangle)$ for every $\alpha < \theta$.

Proof. For every α, consider arbitrary α-sequences a and b which satisfy (i). We have $a_0 = b_0 = G(0)$. $\forall \beta$, if $a_\gamma = b_\gamma$ for all $\gamma < \beta$ then we have $a_\beta = G(\langle a_\eta \,|\, \eta < \beta\rangle) = G(\langle b_\eta \,|\, \eta < \beta\rangle) = b_\beta$. Thus by transfinite induction we have $a = b$. It follows that $F(\alpha)$ is unique by (ii), so F is a function.

For $\alpha = 0$ we have that 0 vacuously satisfies (i), meaning $0 \in \text{dom}(F)$. We also have $F(0) = G(F\restriction_0)$. If for all β we have $\gamma < \beta \to \gamma \in \text{dom}(F) \wedge F(\gamma) = G(F\restriction_\gamma)$ then we have $F\restriction_\beta$ is a β-sequence which satisfies (i). Therefore and thus $\beta \in \text{dom}(F)$ and $F(\beta) = G(F\restriction_\beta)$. Therefore $\text{dom}(F) = \theta$ (or Ord) and $F(\alpha) = G(F\restriction_\alpha)$ by transfinite induction.

Let F' be an arbitrary function satisfying $F'(\alpha) = G(F'\restriction_\alpha)$. We have that $F(0) = F'(0) = G(0)$. If for all β we have that $\gamma < \beta \to F(\gamma) = F'(\gamma)$ then $F(\beta) = G(F\restriction_\beta) = (F'\restriction_\beta) = F'(\beta)$. Therefore $F = F'$ by transfinite induction. □

Definition 2.20 (Limit of a Transfinite Sequence)**.** Let $\alpha > 0$ be a limit ordinal and let $\langle \gamma_\xi \,|\, \xi < \alpha\rangle$ be a nondecreasing sequence of ordinals (meaning $\xi < \eta$ implies $\gamma_x i \leq \gamma_\eta$). We define the limit of the sequence by

$$\lim_{\xi \to \alpha} \gamma_\xi = \sup\{\xi_\eta \,|\, \eta < \alpha\}$$

Definition 2.21 (Normal Sequence)**.** A sequence of ordinals is normal if it is increasing and continuous, meaning for every limit ordinal α we have $\gamma_\alpha = \lim_{\xi \to \alpha} \gamma_\xi$.

Definition 2.22 (Addition)**.** For all ordinal numbers α

(i) $\alpha + 0 = \alpha$

(ii) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$ for all β

(iii) $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$ for all limit $\beta > 0$

Definition 2.23 (Multiplication)**.** For all ordinal numbers α

(i) $\alpha \cdot 0 = 0$

(ii) $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$ for all β

(iii) $\alpha \cdot \beta = \lim_{\xi \to \beta} \alpha \cdot \xi$ for all limit $\beta > 0$.

Definition 2.24 (Exponentiation)**.** For all ordinal numbers α

(i) $\alpha^0 = 1$

(ii) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$ for all β

(iii) $\alpha^\beta = \lim_{\xi \to \beta} a^\xi$ for all limit $\beta > 0$.

Corollary 2.25. Addition, multiplication, and exponentiation are normal functions in their second variable.

Lemma 2.26. For all ordinals α, β, and γ

- $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

Proof. Let us first consider $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$. In the case $\gamma = 0$ we have $\alpha + (\beta + \gamma) = \alpha + \beta = (\alpha + \beta) + \gamma$. Let us assume that $\gamma = \xi + 1$ and $\alpha + (\beta + \xi) = (\alpha + \beta) + \xi$. Then we have

$$\alpha + (\beta + \gamma) = \alpha + (\beta + (\xi + 1)) = \alpha + ((\beta + \xi) + 1) =$$

$$(\alpha + (\beta + \xi)) + 1 = (\alpha + \beta) + \xi + 1 = (\alpha + \beta) + \gamma$$

In the case that γ is a nonzero limit ordinal, and $\alpha + (\beta + \xi) = (\alpha + \beta) + \xi$ holds for all $\xi < \gamma$ we have that $\bigcup_{\xi \in \gamma}(\alpha + (\beta + \xi)) = \bigcup_{\xi \in \gamma}((\alpha + \beta) + \xi) = (\alpha + \beta) + \gamma$. We can show that $\beta + \gamma$ is a nonzero limit ordinal. Then we have $\alpha + (\beta + \gamma) = \bigcup_{\nu \in (\beta + \gamma)}(\alpha + \nu) = \bigcup_{\xi \in \gamma}(\alpha + (\beta + \xi)) = (\alpha + \beta) + \gamma$. Thus by transfinite induction we have addition is associative. □

Definition 2.27 (Sum of Linear Orders)**.** Let $(A, <_A)$ and $(B, <_B)$ be disjoint linearly ordered sets. The sum of these linear orders is the set $A \cup B$ with the ordering defined as: $x < y$ if and only if $x <_A y$ or $x <_B y$ or $x \in A$ and $y \in B$.

Definition 2.28 (Product of Linear Orders)**.** Let $(A, <)$ and $(B, <)$ be linearly ordered sets. The product of these linear orders is the set $A \times B$ with the ordering: $(a_1, b_1) < (a_2, b_2)$ if and only if either $b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$.

Lemma 2.29. For all ordinals α and β, $\alpha + \beta$ and $\alpha \cdot \beta$ are isomorphic to the sum and product of α and β respectively.

Lemma 2.30.

(i) If $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$

(ii) If $\alpha < \beta$ then there exists a unique δ such that $\alpha + \delta = \beta$.

(iii) If $\beta < \gamma$ and $\alpha > 0$ then $\alpha \cdot \beta < \alpha \cdot \gamma$

(iv) If $\alpha > 0$ and γ is arbitrary, then there exist a unique β and a unique $\rho < \alpha$ such that $\gamma = \alpha \cdot \beta + \rho$.

(v) If $\beta < \gamma$ and $\alpha > 1$, then $\alpha^\beta < \alpha^\gamma$.

Theorem 2.31 (Cantor's Normal Form Theorem)**.** Every ordinal $\alpha > 0$ can be represented uniquely in the form

$$\alpha = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_n} \cdot k_n$$

where $n \geq 1$, $\alpha \geq \beta_1 > \cdots > \beta_n$ and $k_1, \cdots, k_n$ are nonzero natural numbers.

Proof. In the case $\alpha = 1$ we have $\alpha = \omega^0 \cdot 1$. Assume that for all $\beta < \alpha$ a Cantor Normal Form representation exists. Consider the least ordinal ξ such that $\omega^\xi > \alpha$. Such an ordinal must exist as $\cdots$. ξ must not be a limit ordinal, as otherwise we would have $\alpha \in \bigcup_{\gamma \in \xi} \omega^\gamma$, meaning $\alpha \in \omega^\gamma$ for some $\gamma < \xi$, resulting in contradiction. As ξ is not a limit ordinal, it must be a successor ordinal. Therefore $\xi = \beta + 1$. It follows $\omega^\beta \leq \alpha$, and β is the largest ordinal which satisfies this. By 2.30 we have there exists a unique δ and $\rho < \omega^\beta$ such that $\alpha = \omega^\beta \cdot \delta + \rho$. It must hold that $\delta < \omega$, as otherwise $\omega^\xi \leq \alpha$. We have that $\rho < \omega^\beta \leq \alpha$, meaning ρ can be uniquely expressed in Cantor's Normal Form. Thus this sum, which is equal to α, can be expressed in Cantor's Normal Form.

We have that for $\alpha = 1$, the unique representation is $\omega^0 \cdot 1$. If α is a successor ordinal then $\alpha = \beta + 1$. Let us assume that β has a unique Cantor's Normal Form. Consider an arbitrary Cantor's Normal Form representation of α. As α is a successor, it must be the case that such a form is a successor. Therefore, this representation must have a $\omega^0 \cdot k$ with $k > 0$ term. This representation with $k - 1$ will then be a Cantor's Normal Form representation of β, meaning it must be unique. Taking a successor maintains this uniqueness. Therefore this representation of α is unique. Let us consider the case that α is a nonzero limit ordinal. Assume all $\beta < \alpha$ have a unique Cantor Normal Form. $\square$

Definition 2.32 (Well-Founded Relation)**.** A relation E on a set P is well-founded if every nonempty $X \subseteq P$ has an E-minimal element, meaning $\exists a \in X$ such that $\forall x \in X$ we have $\neg(xEa)$.

Theorem 2.33 (Height)**.** Let E be a well-founded relation on a set P. There exists a unique function ρ from P to the ordinals such that for all $x \in P$

$$\rho(x) = \sup\{\rho(y) + 1 \mid y\, E\, x\}$$

The range of ρ is an intial segment of the ordinals, and thus an ordinal. The height of E is defined to be $\operatorname{ran}(\rho)$.

Proof. Let us define a transfinite sequence P by transfinite recursion:

$$P_0 = \varnothing \qquad P_{\alpha+1} = \{x \in P \mid \forall y\, (y\, E\, x \to y \in P_\alpha)\} \qquad P_\alpha = \bigcup_{\xi < \alpha} P_\xi \text{ if } \alpha \text{ is a limit ordinal}$$

The axiom of replacement gives us that the function mapping each element of P to the first γ such that P_γ contains that element exists. We can take the union of that set to get an upper bound on when elements are added to our sequence. We can take the least such element θ. We have that θ satisfies $P_{\theta+1} = P_\theta$.

Let us show that $P_\alpha \supseteq P_\gamma$ for all $\gamma < \alpha$. We have that P_0 and P_α for when α is a limit ordinal satisfy this property by definition. Let us consider when $\alpha = \beta + 1$. Fix an $x \in P_\gamma$ for $\gamma < \alpha$. Consider an arbitrary y such that $y\, E\, x$. As $x \in P_\gamma$ there must be some $\xi < \beta$ such that $x \in P_{\xi+1}$ (as elements are only added to successor ordinals). Thus $y\, E\, x \to y \in P_\xi$. By hypothesis we have $P_\xi \subseteq P_\beta$, meaning $y \in P_\beta$. Therefore $x \in P_{\beta+1} = P_\alpha$. Thus $P_\beta \subseteq P_\alpha$. By transfinite induction we have $P_\gamma \subseteq P_\alpha$ for all $\gamma < \alpha$.

Let us assume for the sake of contradiction that $P_\theta \neq P$. By definition $P_\theta \subseteq P$. Let us then consider a minimal $x \in P$ (with respect to E) that is not in P_θ. As x is E-minimal we have if $y\, E\, x$ then $y \in P_\theta$. Therefore $x \in P_{\theta+1}$, which results in contradiction. P_θ must then equal P.

Let us define $\rho(x)$ as the least α such that $x \in P_{\alpha+1}$. We have that $\forall y \in P\, (y\, E\, x \to y \in P_\alpha)$. Thus $\rho(y) + 1 \leq \rho(x)$ for all $y\, E\, x$. Consider an upper bound of $\{\rho(y) + 1 \mid y\, E\, x\}$, denoted β. We have for all $y\, E\, x$, $y \in P_{\rho(y)+1}$, meaning $y \in P_\beta$. Thus $P_{\beta+1}$ must contain x. Therefore $\beta + 1 \geq \alpha + 1$ and equivalently, $\beta \geq \rho(x)$, so we have $\rho(x) = \sup\{\rho(y) + 1 \mid y\, E\, x\}$. $\square$

Definition 2.34 (Indecomposable)**.** A limit ordinal $\gamma > 0$ is indecomposable if there exists no $\alpha < \gamma$ and $\beta < \gamma$ such that $\alpha + \beta = \gamma$.

3 Cardinal Numbers

Definition 3.1 (Cardinality)**.** Two sets, X and Y, have the same cardinality, denoted by $|X| = |Y|$ if there exists a bijection between X and Y.

Corollary 3.2. Cardinality forms an equivalence relation over the class of all sets. We can order such equivalence classes by $\leq$, where $|X| \leq |Y|$ if there exists an injection from X to Y. It holds that such an operator forms a linear ordering over the equivalencce class of cardinals.

Theorem 3.3 (Cantor)**.** For every set X, $|X| < |\mathcal{P}(X)|$.

Proof. Lef f be a function from X into $\mathcal{P}(X)$. Consider the set $Y = \{x \in X \mid x \notin f(x)\}$. Consider an arbitrary $x \in X$. If $x \in f(x)$ then $x \notin Y$. If $x \notin f(x)$ then $x \in Y$. Therefore $f(x) \neq Y$. Thus Y is not in the range of f. Therefore f is not a bijection, meaning $|X| \neq |\mathcal{P}(X)|$. The map $f : X \to \mathcal{P}(X)$ where $f(x) = \{x\}$ is an injection, meaning $|X| < |\mathcal{P}(X)|$. □

Theorem 3.4 (Cantor-Schröder-Bernstein)**.** If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Proof. Consider injections $f_1 : A \to B$ and $f_2 : B \to A$. It holds that $|A| = |f_1[A]|$ and $|B| = |f_2[B]|$. Let us define by induction for each $n \in \mathbb{N}$:

$$\begin{aligned} A_0 &= A & A_{n+1} &= f_2[f_1[A_n]] \\ B_0 &= f_2[B] & B_{n+1} &= f_2[f_1[B_n]] \end{aligned}$$

Let $g : A \to B$ be defined as:

$$g(x) = \begin{cases} f_1(x) & \text{if } x \in A_n \setminus B_n \text{ for some } n \\ f_2^{-1}(x) & \text{otherwise} \end{cases}$$

If $x \notin A_0 \setminus B_0$, it must hold that $x \in f_2[B]$. Therefore $f_2^{-1}(x)$ is well-defined so g is well-defined.

Consider an arbitrary $a_1, a_2 \in A$ such that $g(a_1) = g(a_2)$. If $a_1 \in A_{n_1} \setminus B_{n_1}$ and $a_2 \in A_{n_2} \setminus B_{n_2}$ for some n_1 and n_2 (or $\notin$) then $a_1 = a_2$ by the injectivity of f_1 (or the injectivity of f_2^{-1}). Let us assume for the sake of contradiction that, without loss of generality, $a_1 \in A_n \setminus B_n$ for some n and $a_2 \notin A_n \setminus B_n$ for any n. Then we have $f_1(a_1) = f_2^{-1}(a_2)$. Thus we have $f_2(f_1(a_1)) = a_2$. However, this implies $a_2 \in A_{n+1} \setminus B_{n+1}$, resulting in contradiction. Therefore we have that g is injective.

Consider an arbitrary $b \in B$. If $f_2(b) \notin A_n \setminus B_n$ for any n, then $g(f_2(b)) = f_2^{-1}(f_2(b)) = b$. Otherwise, we have $f_2(b) \in A_n \setminus B_n$ for some n. Consider $f_1^{-1}(b)$. We have $f_2(f_1(f_1^{-1}(b))) = f_2(b)$. Therefore $f_2(f_1(f_1^{-1}(b))) \in A_{n+1} \setminus B_{n+1}$. Then we have $g(f_1^{-1}(b)) = f_1(f_1^{-1}(b)) = b$. We thus have that g is surjective, meaning g forms a bijection between A and B. □

Definition 3.5 (Cardinal Arithmetic)**.** If $\kappa = |A|$ and $\lambda = |B|$ for disjoint A and B then we define

$$\kappa + \lambda = |A \cup B| \qquad \kappa \cdot \lambda = |A \times B| \qquad \kappa^\lambda = |A^B|$$

Corollary 3.6. Cardinal Arithmetic is independent of choice of A and B.

Lemma 3.7. If $|A| = \kappa$ then $|\mathcal{P}(A)| = 2^\kappa$.

Proof. For every $X \subseteq A$, let χ_X be the function

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \in A \setminus X \end{cases}$$

The mapping $f : X \to \chi_X$ is a bijection between $\mathcal{P}(A)$ and 2^A. □

Lemma 3.8.

- $+$ and $\cdot$ are associative, commutative, and distributive
- $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$
- $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$
- $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$
- If $\kappa \leq \lambda$ then $\kappa^\mu \leq \lambda^\mu$
- If $0 < \lambda \leq \mu$ then $\kappa^\lambda \leq \kappa^\mu$
- $\kappa^0 = 1$, $1^\kappa = 1$, and $0^\kappa = 0$ if $\kappa > 0$.

Definition 3.9 (Cardinal Number)**.** An ordinal α is called a cardinal if $|\alpha| \neq |\beta|$ for all $\beta < \alpha$. Natural numbers are called finite cardinals, and infinite ordinals that are cardinals are called alpehs.

Lemma 3.10.

(i) For every α there is a cardinal number greater than α.

(ii) If X is a set of cardinals then $\sup X$ is a cardinal.

Proof. Consider an arbitrary set X. We have that the set of all well-orderings of subsets of X exist, following from the axiom of power set, union, and seperation. Any injection of an ordinal into some well-ordered subset of X will be an isomorphism. Such a ordinal is thus unique. By replacement, we have there exists a set S of ordinals which are isomorphic to these well-orderings. If an ordinal injects into X, then it is isomorphic to the well-ordering of some subset of X induced by that injection. Thus it will be in S. Therefore any strict upper bound on S will be an ordinal which does not have an injection into X. The cardinality of this upper bound will be a cardinal greater than $|X|$.

Let $\alpha = \sup X$, where X is a set of cardinals. Let us assume for the sake of contradiction there exists an injection f from α to some $\beta < \alpha$. There must exist some $\kappa \in X$ such that $\beta < \kappa \leq \alpha$. However, the f composed with the injection from κ to α will be an injection from κ to β, resulting in contradiction. □

Definition 3.11. We define the increasing enumeration of all alephs:

$$\aleph_0 = \omega \qquad \aleph_{\alpha+1} = \aleph_\alpha^+ \qquad \aleph_\alpha = \sup\{\aleph_\beta \mid \beta < \alpha\} \text{ for limit } \alpha$$

Definition 3.12. The canonical well-ordering of Ord^2 is

$$\begin{aligned}(\alpha,\beta) < (\gamma,\delta) \iff & \text{ either } \max\{\alpha,\beta\} < \max\{\gamma,\delta\} \\ & \text{ or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\} \text{ and } \alpha < \gamma \\ & \text{ or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\}, \alpha = \gamma, \text{ and } \beta < \delta\end{aligned}$$

Corollary 3.13. For each $\alpha \in \mathrm{Ord}$, we have $\alpha \times \alpha$ is the initial segment given by $(0, \alpha)$.

Theorem 3.14. The canonical well-ordering of Ord^2 is isomorphic to Ord.

Proof. Let us define $\Gamma(\alpha, \beta)$ to be equal to the order-type of the initial-segment of Ord^2 given by (α, β). By construction, such a function is increasing and injective.

We have $0 \in \mathrm{ran}(\Gamma)$. Consider an arbitrary ordinal, γ, and let us assume that for all $\xi < \gamma$ we have $\xi \in \mathrm{ran}(\Gamma)$. Consider the least pair (α, β) such that $(\alpha, \beta) > \Gamma^{-1}(\xi)$ for all $\xi < \gamma$. We can construct a mapping $f : \{(a, b) \in \mathrm{Ord}^2 \mid (a, b) < (\alpha, \beta)\} \to \gamma$ where $f(x, y) = \Gamma(x, y)$. Such an f is injective and increasing. For any $\xi \in \gamma$ we have $\xi \in \mathrm{ran}(\Gamma)$. By our construction of (α, β), we also have that $\Gamma^{-1}(\xi) < (\alpha, \beta)$. Therefore $\Gamma^{-1}(\xi) \in \mathrm{dom}(f)$. Then we have $f(\Gamma^{-1}(\xi)) = \Gamma(\Gamma^{-1}(\xi)) = \xi$. Therefore f forms an isomorphism, so $\gamma \in \mathrm{ran}(\Gamma)$. By transfinite induction, we have Γ is surjective. Thus Γ forms an isomorphism between Ord and Ord^2. □

Theorem 3.15. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

Proof. Consider the canonical isomorphism from Ord^2 to Ord, Γ. We have that for any ordinal α, $\alpha \times \alpha$ is an initial segment of Ord^2. Therefore, $\gamma = \Gamma[\alpha \times \alpha]$ is an ordinal. It follows that $\Gamma\restriction_{\alpha\times\alpha}$ is an isomorphism between $\alpha \times \alpha$ and γ.

Let us show that for any $\alpha \in \mathrm{Ord}$ we have $\Gamma[\aleph_\alpha \times \aleph_\alpha] = \aleph_\alpha$. We know that this is true for $\alpha = 0$. Let us assume for the sake of contradiction this does not hold for some $\alpha > 0$. Let α be the least such ordinal for which this doesn't hold. We have $f(x) = \Gamma[x \times x]$ is an increasing function. Therefore $f(\aleph_\alpha) > \aleph_\alpha$. Thus there must exist $\beta, \gamma < \aleph_\alpha$ such that $\Gamma(\beta, \gamma) = \aleph_\alpha$. As $\aleph_\alpha$ is a limit ordinal, there must exist $\delta < \aleph_\alpha$ such that $\beta, \gamma < \delta$. It then holds that $\Gamma[\delta \times \delta] \supseteq \aleph_\alpha$. Thus we have $|\delta| \cdot |\delta| = |\delta \times \delta| \geq \aleph_\alpha$. However, by the minimality of α, we have $|\delta| \cdot |\delta| = |\delta| < \aleph_\alpha$, resulting in contradiction. Therefore $\Gamma[\aleph_\alpha \times \aleph_\alpha] = \aleph_\alpha$ for all $\alpha \in \mathrm{Ord}$. □

Definition 3.16 (Cofinal)**.** For nonzero limit ordinals α and β, a β-sequence, $\langle a_\xi \,|\, \xi < \beta \rangle$, is cofinal in α if it is increasing and $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. Similarly, $A \subseteq \alpha$ is cofinal in α if $\sup A = \alpha$.

Definition 3.17 (Cofinality)**.** If α is an infinite limt ordinal, the cofinality of α is

$$\operatorname{cf} \alpha = \text{ the least limit ordinal } \beta \text{ such that there is increasing } \beta\text{-sequence cofinal in } \alpha$$

Corollary 3.18. $\operatorname{cf} \alpha$ is a limit ordinal and less than or equal to α.

Corollary 3.19. Let $\alpha > 0$ be a limit ordinal. If $A \subseteq \alpha$ and $\sup A = \alpha$, then the order type of A is at least $\operatorname{cf} \alpha$.

Lemma 3.20.

$$\operatorname{cf}(\operatorname{cf} \alpha) = \operatorname{cf} \alpha$$

Proof. There exists some $(\operatorname{cf} \alpha)$-sequence, $\langle \gamma_\nu \,|\, \nu < \operatorname{cf} \alpha \rangle$ which approaches α. Similarly, there exists some $(\operatorname{cf}(\operatorname{cf} \alpha))$-sequence, $\langle \delta_\nu \,|\, \nu < \operatorname{cf}(\operatorname{cf} \alpha) \rangle$ which approaches $\operatorname{cf} \alpha$. We have $\operatorname{cf} \alpha \geq \operatorname{cf}(\operatorname{cf} \alpha)$. We also have that $\lim_{\nu \to \operatorname{cf}(\operatorname{cf} \alpha)} \gamma_{\delta_\nu}$ approaches α, meaning $\operatorname{cf}(\operatorname{cf} \alpha) \geq \operatorname{cf} \alpha$. Therefore $\operatorname{cf}(\operatorname{cf} \alpha) = \operatorname{cf} \alpha$. ☐

Theorem 3.21. If $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_\xi \leq \cdots$ for $\xi < \gamma$ is a nondecreasing γ-sequence of ordinals in α and $\lim_{\xi \to \gamma} \beta_\xi = \alpha$, then $\operatorname{cf} \gamma = \operatorname{cf} \alpha$.

Proof. There must exist some $(\operatorname{cf} \gamma)$-sequence, $\langle \xi_\nu \,|\, \nu < \operatorname{cf} \gamma \rangle$, which approaches γ. It holds that $\lim_{\nu \to \operatorname{cf} \gamma} \beta_{\xi_\nu} = \alpha$. There must exist an increasing subsequence of β_{ξ_ν} of length less than or equal to $\operatorname{cf} \gamma$ with the same limit. Thus we have $\operatorname{cf} \alpha \leq \operatorname{cf} \gamma$.

Let us construct a α-sequence $\langle \alpha_\nu \,|\, \nu < \alpha \rangle$. For each ν, let α_ν be the least ordinal ξ such that $\beta_\xi \geq \nu$. Such an ordinal must exist, as $\lim_{\xi \to \gamma} \beta_\xi = \alpha$. We have that such a sequence approaches γ. There must exist some $(\operatorname{cf} \alpha)$-sequence, $\langle \xi_\nu \,|\, \nu < \operatorname{cf} \alpha \rangle$. It holds that $\lim_{\nu \to \operatorname{cf} \alpha} \alpha_{\xi_\nu} = \gamma$. There must exist some strictly-increasing subsequence with the same limit. Thus $\operatorname{cf} \gamma \leq \operatorname{cf} \alpha$, so $\operatorname{cf} \alpha = \operatorname{cf} \gamma$. ☐

Definition 3.22. A infinite cardinal $\aleph_\alpha$ is regular if $\operatorname{cf} \omega_\alpha = \omega_\alpha$. It is singular otherwise.

Theorem 3.23. For every limit ordinal α, $\operatorname{cf} \alpha$ is a regular cardinal.

Proof. Let $\langle \alpha_\xi \,|\, \xi < \operatorname{cf} \alpha \rangle$ be a $(\operatorname{cf} \alpha)$-sequence that is cofinal with α. We have $\operatorname{cf} \alpha \geq |\operatorname{cf} \alpha|$ as ordinals. Let f be a bijection between $|\operatorname{cf} \alpha|$ and α. Let us define the $|\operatorname{cf} \alpha|$-sequence $\{\beta_\xi \,|\, \xi < |\operatorname{cf} \alpha|\}$ where $\beta_\xi = \alpha_{f(\xi)}$. The union over this sequence will then be α. We can take some increasing subsequence that preserves this union. We have the limit of this increasing subsequence of length $\leq |\operatorname{cf} \alpha|$ is then α. Therefore $\operatorname{cf} \alpha \leq |\operatorname{cf} \alpha|$. Thus $\operatorname{cf} \alpha = |\operatorname{cf} \alpha|$. We also have $\operatorname{cf}|\operatorname{cf} \alpha| = \operatorname{cf} \operatorname{cf} \alpha = \operatorname{cf} \alpha = |\operatorname{cf} \alpha|$, meaning $\operatorname{cf} \alpha$ is a regular cardinal. ☐

Definition 3.24. Let κ be a limit ordinal. A subset $X \subseteq \kappa$ is bounded if $\sup X < \kappa$ and unbounded if $\sup X = \kappa$.

Lemma 3.25. Let κ be an aleph.

(i) If $X \subseteq \kappa$ and $|X| < \operatorname{cf} \kappa$ then X is bounded.

(ii) If $\lambda < \operatorname{cf} \kappa$ and $f : \lambda \to \kappa$ then the range of f is bounded.

Proof. If X is unbounded, meaning $\sup X = \kappa$, then by 3.19 the order type of X is at least $\operatorname{cf} \kappa$. Therefore $|X| \geq |\operatorname{cf} \kappa| = \operatorname{cf} \kappa$. The contrapositive gives us that if $|X| < \operatorname{cf} \kappa$ then X is bounded. Then we have that for any $f : \lambda \to \kappa$, $|\operatorname{ran}(f)| \leq \lambda < \operatorname{cf} \kappa$. Therefore we have $\operatorname{ran}(f)$ is bounded. ☐

Lemma 3.26. An infinite cardinal κ is singular if and only if there exists a cardinal $\lambda < \kappa$ and a family $\{S_x i \,|\, \xi < \lambda\}$ of subsets of κ such that $|S_\xi| < \kappa$ for each $\xi < \lambda$ and $\kappa = \bigcup_{\xi < \kappa} S_\xi$. The least cardinal λ that satisfies the condition is $\operatorname{cf} \kappa$.

Theorem 3.27. If κ is an infinite cardinal, then $\kappa < \kappa^{\operatorname{cf}\kappa}$.

Proof. Let F be an arbitrary function from κ to $k^{\operatorname{cf}\kappa}$. There must exist some $(\operatorname{cf}\kappa)$-sequence, $\langle \alpha_\xi \,|\, \xi < \operatorname{cf}\kappa \rangle$ such that $\lim_{\xi \to \operatorname{cf}\kappa} \alpha_\xi = \kappa$. Let us define a function, $f : \operatorname{cf}\kappa \to \kappa$ where $f(\xi)$ is the least γ such that $\gamma \neq F_\alpha(\xi)$ for all $\alpha < \alpha_\xi$. We have that $|\{F_\alpha(\xi) \,|\, \alpha < \alpha_\xi\}| \leq |\alpha_\xi| < \kappa$ so such a γ must exist. We have that for any $\beta \in \kappa$, there exists a greater α_ξ, and by construction $F_\beta(\xi) \neq f(\xi)$. Therefore $f \notin \operatorname{ran}(F)$, so F is not a bijection. Thus we have that $\kappa < \kappa^{\operatorname{cf}\kappa}$. $\square$

Definition 3.28 (Weakly Inaccessible)**.** A uncountable cardinal κ is weakly inaccessible if it is a limit cardinal and is regular.

Definition 3.29 (Projection)**.** A set B is a projection of a set A if there is surjection mapping A onto B.

Definition 3.30 (Dedekind-Finite)**.** A set S is Dedekind-finite (D-finite) if there is no bijection from S to a proper subset of itself. S is Dedekind-infinite otherwise.

4 Real Numbers

Theorem 4.1. The set of all real numbers is uncountable.

Proof. Let c_n for $n \in \mathbb{N}$ be an enumeration of the $\mathbb{R}$. Let us define sequences a_n and b_n. Let $a_0 = c_0$ and let b_0 be c_k for the least k such that $a_0 < c_k$. Let a_{n+1} be equal to c_k for the least k such that $a_n < c_n < b_n$ and similarly let b_{n+1} be equal to c_k for the least k such that $a_{n+1} < c_k < b_{n+1}$. By the density of the reals, such sequences exist. We have that a_n is bounded by b_0, so there exists a supremum of a_n. Consider an arbitrary c_m. If $c_m \leq a_0$ or $c_m \geq b_0$ then $c_m \neq \sup a_n$. Otherwise, let us consider the set $\{n \in \mathbb{N} \,|\, a_n < c_m < b_n\}$. It holds that m is an upper bound for this set, as if $a_m < c_m < b_m$ then $a_{m+1} = c_m$. We have that a maximum then exists for this set, and it follows that c_m will be contained in either a_n or b_n. Then the next element of a_n will be greater than c_m, meaning $\sup a_n > c_m$. Thus $\sup a_n \notin a_n$, meaning such an enumeration is not a bijection. Therefore no bijection exists between the naturals and the real numbers. $\square$

Theorem 4.2. The cardinality of the reals is $2^{\aleph_0}$.

Proof. Every real number is equal to the limit of some rational sequence. Therefore $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{Q})| = 2^{\aleph_0}$. We also have that the cantor set, denoted C, which is the set of all reals of the form $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ where $a_n = 0$ or 2 is in bijection with the set of all ω-sequences of 0's and 2's. Therefore $|\mathbb{R}| \geq |C| \geq 2^{\aleph_0}$. It follows from Cantor-Schröder-Bernstein that $|\mathbb{R}| = 2^{\aleph_0}$. $\square$

Definition 4.3. A linear ordering $(P, <)$ is dense if for all $a < b$ there exists a c such that $a < c < b$. A set $D \subseteq P$ is a dense subset if for all $a < b$ in P there exists a $d \in D$ such that $a < d < b$.

Theorem 4.4. Any two countable unbounded dense linear ordered sets are isomorphic.

Proof. Let $P_1 = \{a_n \,|\, n \in \mathbb{N}\}$ and $P_2 = \{b_n \,|\, n \in \mathbb{N}\}$ be enumerations of countable unbounded dense linearly ordered sets. Let us define a function $f : a_n \to b_n$ recursively. Let $f(a_0) = b_0$ and let $f(a_{n+1}) = b_k$ for the least k such that f is order-preserving. Such a k must exist as only finitely many n have been exhausted, so there is a maximum of the set $\{f(a_m) \,|\, m < n+1 \wedge a_m < a_{n+1}\}$ and a minimum of the set a maximum of the set $\{f(a_m) \,|\, m < n+1 \wedge a_m > a_{n+1}\}$ (or one of the sets is empty). Then, as P_2 enumerates over a dense unbounnded set, there must be some element between this maximum and minimum (or strictly above or below the maximum/minimum if one of the given sets is empty). We have that f is defined for all a_n, and thus $\operatorname{dom}(f)$ is the countable unbounded dense linearly ordered set P_1 enumerates over. By construction f is increasing and injective. We have that $b_0 \in \operatorname{ran}(f)$. Let us assume $b_i \in \operatorname{ran}(f)$ for all $i < m$. Let us define n as the maximum of the finite set $\{n \in \mathbb{N} \,|\, f(n) = b_i \text{ for } i < m\}$. If $f(a_j) = b_m$ for some $j \leq n$ we have $b_m \in \operatorname{ran}(f)$. Otherwise, consider the sets $\{a_j \,|\, j \leq n \wedge f(a_j) < b_m\}$ and $\{a_j \,|\, j \leq n \wedge f(a_j) > b_m\}$. The union of such sets will be all a_j such that $j \leq n$. Such sets are finite and thus attain their maximum and minimum respectively (or one of the sets is empty). Then as as P_1 enumerates over a unbounded dense set, it must hold that some a_j is between this maximum and minimum (or strictly above or below the maximum/minimum if one of the given sets is empty). As all a_j for $j \leq n$ are contained in the union of these sets, any value strictly between their maximum and minimum necesarily will satisfy $j > n$. Then by our definition of f we have $f(a_j) = b_m$ for the least such j in which this holds. By induction we have $\operatorname{ran}(f)$ is equal to the set P_2 enumerates over, meaning f is surjective. Thus we have that f is an isomorphism between such sets. $\square$

Theorem 4.5. $(\mathbb{R}, <)$ is the unique complete linear ordering that has a countable dense subset isomorphic to $(\mathbb{Q}, <)$.

Proof. Consider arbitrary complete dense unbounded linearly ordered sets C and C' which contain a countable dense subset isomorphic to $(\mathbb{Q}, <)$. Let such subsets be denoted by P and P'. By 4.4 we have there is an isomorphism, $f : P \to P'$. Let us define $f^* : C \to C'$ where

$$f^*(x) = \sup(f[\{p \in P \mid p < x\}])$$

We have that for all $x \in C$ there exists some $p_1, p_2 \in P$ such that $p_1 > x$ and $p_2 < x$. Then $f(p_1)$ is an upper bound for $f[\{p \in P \mid p < x\}]$ and $f(p_2) \in f[\{p \in P \mid p < x\}]$ so $\sup(f[\{p \in P \mid p < x\}])$ exists. It follows that f^* is well-defined. Consider arbitrary $a, b \in C$ such that $a < b$. Then there exists a $p \in P$ such that $a < p < b$. Then $f(p)$ is a strict upper bound for $f^*(a)$ while $f(p)$ is a strict lower bound for $f^*(b)$. Therefore $f^*(a) < f^*(b)$ so f^* is injective and increasing. Let us consider an arbitrary $c' \in C'$. Consider the set $L = \{p' \in P' \mid p' < c'\}$. By construction we have $\sup(f^{-1}[L])$ maps to c' under f^* so f^* is surjective. Thus f^* is an isomorphism between C and C'. Thus all such sets are equivalent (and thus equivalent to $(\mathbb{R}, <)$) under isomorphism. □

Theorem 4.6. Let $(P, <)$ be a dense unbounded linearly ordered set. Then there is a complete unbounded linearly ordered set $(C, \prec)$ such that $P \subseteq C$, $<$ and $\prec$ agree on P, and P is dense in C.

Proof. Define a Dedekind cut in P to be a pair (A, B) of disjoint non-empty subsets of P such that $A \cup B = P$, $a < b$ for all $a \in A$ and $b \in B$, and A does not have a greatest element. By the axioms of power set and seperation we have that the set C of all Dedekind cuts in P exists. Let us define $(A_1, B_1) \preceq (A_2, B_2)$ if $A_1 \subseteq A_2$. It follows from the properties of $\subseteq$ that $\preceq$ is a partial order. Given any such A_1 and A_2 we have that either A_1 contains an upper bound for A_2 or doesn't. If it does, it must hold that A_1 contains every element of P less than that upper bound, so $A_2 \subseteq A_1$. If it doesn't, each element of A_1 must be less than some element of A_2, and thus contained in A_2. Therefore $A_1 \subseteq A_2$. Thus $\preceq$ is a linear ordering.

For any Dedekind cut, (A, B), B must be non-empty. As P is unbounded, there must must then exist some $b \in B$ which is not a lower bound of b. Then we have $(\{a \in P \mid a < b\}, \{a \in P \mid a > b\})$ is a Dedekind cut greater than (A, B). For each $p \in P$ we have $(A_p, B_p) = (\{a \in P \mid a < p\}, \{a \in P \mid a > p\})$ defines a dedekind cut. It follows that P is isomorphic to $\{(A_p, B_p) \mid p \in P\}$, C contains a copy of P and $<$ and $\prec$ agree over this copy. For any $(A_1, B_1) \prec (A_2, B_2)$ we have that $A_1 \subsetneq A_2$. Therefore there must exist a $p \in P$ such that $p \in A_2$ and $p \notin A_1$. It follows that $p \in B_1$ so $(A_1, B_1) \prec (A_p, B_p)$ and p is not the greatest element of A_2 so $(A_p, B_p) \prec (A_2, B_2)$. We then have $(A_1, B_1) \prec (A_p, B_p) \prec (A_2, B_2)$. Therefore $(C, \prec)$ is a linearly ordered set with a copy of P which agrees in order and is dense in C.

Consider an arbitrary bounded above set $X \subsetneq C$. Define

$$(A, B) = (\bigcup\{a \mid \exists b\, [(a, b) \in X]\}, \bigcap\{b \mid \exists a\, [(a, b) \in X]\})$$

We have that A is a union of non-empty sets, and thus non-empty. There must exist some upper bound (U_A, U_B) of x. We have that U_B is non-empty. Thus there exists some $u \in U_B$. Consider an arbitrary $b \in \{b \mid \exists a\, [(a, b) \in X]\}$. We have the existence of some a such that $(a, b) \in X$. Then by construction we have $a \subseteq U_A$, so $u \notin A$. Therefore $u \in b$. Thus $u \in B$. Therefore B is non-empty. For any arbitrary $a \in A$ and $b \in B$ we have the existence of $(A', B') \in X$ such that $a \in A'$. Then by construction we have that $b \in B'$. Therefore $a < b$. We also have that A is the union of sets with no greatest element, and thus has no greatest element. Therefore (A, B) is a Dedekind cut. By construction all Dedekind cuts in X are less than (A, B) and (A, B) is less than or equal to any upper bound of X. Therefore (A, B) is the supremum of X, meaning that C is complete. □

Definition 4.7 (Perfect Set)**.** A non-empty closed set is perfect if it contains no isolated points.

Theorem 4.8. Every perfect set of $\mathbb{R}$ has cardinality $|\mathbb{R}|$.

Proof. Consider an arbitrary perfect set $P \subseteq \mathbb{R}$. Let S be the set of all finite sequences in $\{0,1\}$. Let us construct sets I_s for all $s \in S$ by well-founded recursion. Let $I_\varnothing = \overline{B_1(p) \cap P}$ for some $p \in P$. We have that any point in $B_1(p) \cap P$ is not isolated as $B_1(p)$ is open and P is perfect. Any point in $\overline{B_1(p) \cap P} \setminus B_1(p) \cap P$ is the limit of some sequence in $\setminus B_1(p) \cap P$ and thus contained in P and not an isolated point. Therefore $I_\varnothing$ is a subset of P, and perfect. Consider an arbitrary $s \in S$ of length n. Let l and u be disjoint points in I_s such that $l < u$ (such points must exist as I_s is perfect). There exists an $r \in \mathbb{R}_{<\frac{1}{n+1}}$ such that $\overline{B_r(l)} \cap \overline{B_r(u)} = \varnothing$. Let $I_{s^\frown 0} = \overline{B_r(l) \cap I_s}$ and $I_{s^\frown 1} = \overline{B_r(u) \cap I_s}$. It follows that $I_{s^\frown 0}$ and $I_{s^\frown 0}$ are disjoint, implying that all $I_{s'}$ where $s' \in S$ is of length $n+1$ are disjoint. We also have that $I_{s^\frown 0} \subseteq I_s$ and $I_{s^\frown 1} \subseteq I_s$, meaning both are also subsets of P. Both $I_{s^\frown 0}$ and $I_{s^\frown 1}$ are perfect. By construction, both of these sets also have diameter less than $\frac{1}{n+1}$.

Let us then construct a function $F : 2^{\aleph_0} \to P$. Consider an arbitrary $f \in 2^{\aleph_0}$. It must hold that $\bigcap_{n<\omega} I_{f\restriction_n}$ is equal to $\{p\}$ for some $p \in P$. This is because the infinite intersection of non-empty nested closed sets in $\mathbb{R}$ is non-empty, but the element in this set must be unique as it is contained in a sequence of sets with diameter approaching 0. Let F map f to p. If $f_1, f_2 \in 2^{\aleph_0}$ are not equal, they must differ in some index n. We have that $f_1 \restriction_{n+1}, f_2 \restriction_{n+1} \in S$ and $f_1 \restriction_{n+1} \neq f_2 \restriction_{n+1}$, so $I_{f_1\restriction_{n+1}}$ and $I_{f_2\restriction_{n+1}}$ are disjoint, meaning $\bigcap_{n<\omega} I_{f_1\restriction_n} \neq \bigcap_{n<\omega} I_{f_2\restriction_n}$, so $F(f_1) \neq F(f_2)$. Therefore F is injective, so $|P| \geq 2^{\aleph_0} = |R|$. As $P \subseteq \mathbb{R}$ we have $|P| = |\mathbb{R}|$. $\square$

Theorem 4.9 (Cantor-Bendixson)**.** If $F \subseteq \mathbb{R}$ is an uncountable closed, then $F = P \cup S$ where P is perfect and S is at most countable.

Proof. For every $A \subseteq \mathbb{R}$ define A' to be equal to the set of all limit points of A. Let us recursively define the $\aleph_2$-sequence F_n, where $F_0 = F$, $F_{\alpha+1} = F'_\alpha$, and $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$ for limit α. As F is closed, it holds that F_α is closed and $F_\alpha \supseteq F_\beta$ for all $\alpha < \beta < \aleph_2$. Define $S = F \setminus \operatorname{im}(F_{\aleph_1})$. Let us construct a function $g : S \to \omega$. Let h be a enumeration of all rational intervals of $\mathbb{R}$. For all $x \in S$ we have that $x \in F_\alpha$ and $x \notin F_{\alpha+1}$ for some α. Thus x is an isolated point of F_α, meaning some rational interval about x is disjoint from $F_\alpha \setminus \{x\}$. Let $g(x)$ then be the least k such that $h(k)$ contains x and is disjoint from $F_\alpha \setminus \{x\}$. We have that such a rational set is not equal to all rational sets not containing x. We also have that such a rational set is disjoint from F_β for all $\beta > \alpha$, and thus not equal to any rational set containing an element from any such F_β. Thus g is an injection.

It follows that S is at most countable. Therefore $\{\alpha < \aleph_1 \,|\, F_\alpha \neq F_{\alpha+1}\}$ is countable, meaning there must exist some $\beta < \aleph_1$ such that $F_\gamma = F_{\gamma+1}$, or equivalently F_γ is perfect. By construction, it holds that $F_\gamma = F_\delta$ for all $\aleph_2 > \delta \geq \gamma$. Thus $S = F \setminus F_{\aleph_1} = F \setminus F_\gamma$. It follows that $F = F_\gamma \cup (F \setminus F_\gamma)$, the union of a perfect set and an at most countable set. $\square$

Corollary 4.10. If F is a closed set, then either $|F| \leq \aleph_0$ or $|F| = 2^{\aleph_0}$.

Definition 4.11 (Nowhere Dense)**.** A set of reals is nowhere dense if its closure has empty interior.

Theorem 4.12 (Baire Category Theorem)**.** If $D_0, D_1, \cdots, D_n, \cdots$ for $n \in \mathbb{N}$, are dense open sets of reals then the intersection $D = \bigcap_{n\in\omega} D_n$ is dense in $\mathbb{R}$.

Proof. Consider arbitrary dense open sets of reals A and B. It holds that $A \cap B$ is open. Consider an $l, r \in \mathbb{R}$ such that $l < r$. As A is dense, there is some $a \in A$ such that $l < r$. As A is open, there is some neighborhood about a such that A contains that neighborhood. Let $a' \in A$ be some point in that neighborhood such that $l < a' < r$. Without loss of generality, let us say $a' < a$. As B is dense, there is some $b \in B$ such that $a' < b < a$. It then holds that $b \in A$. Therefore $b \in A \cap B$, so there exists some element of $A \cap B$ that is between l and r. Therefore $A \cap B$ is dense. By induction it follows that $\bigcap_{n=0}^{m} D_n$ is dense and open for all $m \in \omega$.

Let $\langle J_k \,|\, k \in \omega \rangle$ be an enumeration of the rational intervals. Consider an arbitrary $l, r \in \mathbb{R}$ such that $l < r$. Let us define a sequence of intervals $\{I_n \,|\, n \in \mathbb{N}\}$ recursively. Let $I_0 = (l, r)$. Let $I_{n+1} = J_k = (q_k, r_k)$ where $k \in \mathbb{N}$ is the least k such that the closed interval $[q_k, r_k]$ is included in $I_n \cap D_n$. As D_n is dense, it must have a non-empty intersection with I_n. As D_n is open, there must be some open-neighborhood about some point in this intersection, which then contains some smaller closed rational interval. Thus such a k does exist. It follows that $\overline{I_{n+1}}$ is a non-empty bounded closed interval contained within $\overline{I_n}$ and D_n. By the nested interval property we have that there exists some $x \in \bigcap_{n=1}^{\infty} \overline{I_n}$. By construction we have $x \in D$ and $l < x < r$. Therefore D is dense. $\square$

Definition 4.13 (Algebra of Sets)**.** An algebra of sets is a collection $\mathcal{S}$ of subsets of a given set S such that

(i) $S \in \mathcal{S}$

(ii) if $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ then $X \cup Y \in \mathcal{S}$

(iii) If $X \in \mathcal{S}$ then $S \setminus X \in \mathcal{S}$.

Corollary 4.14. An algebra of sets is closed under intersections.

Definition 4.15 (σ-algebra)**.** A σ-algebra is a algebra of sets that is also closed under countable unions (and thus countable intersections).

Corollary 4.16. For any collection $\mathcal{X}$ of subsets of S, there is a smallest algebra (and respecitvely σ-algebra) $\mathcal{S}$ such that $\mathcal{S} \supseteq \mathcal{X}$.

Definition 4.17 (Borel Sets)**.** A set of reals B is Borel if it belongs to the smallest σ-algebra $\mathcal{B}$ of sets of reals that contains all open sets.

Definition 4.18. The intersections of countably many open sets are called G_δ sets, and the unions of countably many closed sets are called F_σ sets.

Definition 4.19 (Baire Space)**.** The Baire space is the space $\mathcal{N} = \omega^\omega$ of all infinite sequences of natural numbers with the topology: For every finite sequence $s = \langle a_k \mid k < n \rangle$ let

$$O(s) = \{f \in \mathcal{N} \mid s \subseteq f\} = \{\langle c_k \mid k \in \mathbb{N} \rangle \mid (\forall k < n) c_k = a_k\}$$

Let all such sets form a basis for the topology of $\mathcal{N}$.

Definition 4.20 (Sequential Tree)**.** A sequential tree is a set T of finite sequences of natural numbers that satisfies

$$\text{if } t \in T \text{ and } s = t \restriction_n \text{ for some } n, \text{ then } s \in T$$

Theorem 4.21. Given a sequential tree T, the set

$$[T] = \{f \in \mathcal{N} \mid f \restriction_n \in T \text{ for all } n \in \mathbb{N}\}$$

is closed and every closed set is equal to $[T]$ for some tree T.

Proof. Consider an arbitrary $f \in \mathcal{N}$ such that $f \notin [T]$. By construction there exists some $n \in \mathbb{N}$ such that $s = f \restriction_n$ is not in T. Then we have $O(s) = \{f \in \mathcal{N} \mid s \subset f\}$ is an open set disjoint from $[T]$ containing f. A union of such open sets over all $f \in \mathcal{N} \setminus [T]$ will be an open set that is the complement of $[T]$, meaning $[T]$ is closed.

Consider an arbitrary closed set $F \subseteq \mathcal{N}$. Let T_F be the set of all finite sequences s such that $s \subset f$ for some $f \in F$. It holds that such a set is a tree. Consider an arbitrary $f \in F$. By construction, for all $n \in \mathbb{N}$ we have $f \restriction_n \in T_f$. Therefore $f \in [T_f]$. Consider an arbitrary $f \in [T_f]$. There must then exist a sequence $\langle f_n \in F \mid n \in \mathbb{N} \rangle$ such that f_n and f agree on their first n elements. Such a sequence converges to f, implying $f \in F$. Therefore $F = [T_f]$. ☐

Definition 4.22. A non-empty sequential tree T is perfect if for every $t \in T$ there exist s_1 and s_2 such that $s_1 \subset t$, $s_2 \subset t$, and s_1 and s_2 are $\subset$-incomparable

Lemma 4.23. A closed set $F \subseteq \mathcal{N}$ is perfect if and only if the tree T_F is a perfect tree.

Corollary 4.24. The Baire-Category Theorem (4.12) and Cantor-Bendixsom Theorem (4.9) hold over Baire space.

Definition 4.25 (Polish Spaces)**.** A Polish space is a topological space that is homeoemorphic to a seperable complete metric space.

Definition 4.26 (Algebraic Numbers)**.** A real number is algebraic if it is a root of a polynomial whose coefficients are integers. Otherwise it is transcendental.

Definition 4.27 (Condensation Point)**.** Given a set A of reals, $a \in \mathbb{R}$ is a condensation point of A if every neighborhood of a contains uncountably many elements of A. We denote A^* as the set of all condensation points of A.

5 The Axiom of Choice and Cardinal Arithmetic

Axiom 5.1 (Axiom of Choice)**.** Every family of non-empty sets has a choice function.

Theorem 5.2 (Zermelo's Well-Ordering Theorem)**.** Every set can be well-ordered.

Proof. Let A be an arbitrary set. Let f be a choice function for the family S of all non-empty subsets of A. Let us construct the transfinite sequence a. For every ordinal α let

$$a_\alpha = f(A \setminus \{a_\xi \,|\, \xi < \alpha\})$$

if $\{a_\xi \,|\, \xi < \alpha\}$ is non-empty. Then our sequence induces a well-ordering of A. □

Lemma 5.3. If every set is well-orderable, then the axiom of choice holds.

Proof. For any family of sets S, consider some well-ordering of $\bigcup S$. Then let $f(X)$ for each $X \in S$ be the least element of X. Such an f is a choice function over S. □

Corollary 5.4. If there is a surjection from A onto B then $|B| \leq |A|$.

Lemma 5.5. $|\bigcup S| \leq |S| \cdot \sup\{|X| \,|\, X \in S\}$.

Proof. By the axiom of choice we can enumerate over S and then enumerate over each set in our enumeration. We can then map each element to the least pair of ordinals that it maps to in our enumeration, and such a mapping would be injective. Therefore $|\bigcup S| \leq |S \times \sup\{|X| \,|\, X \in S\}|$. □

Corollary 5.6. Every successor cardinal is regular.

Definition 5.7. Given a partial order $(P, <)$, we say $C \subseteq P$ is a chain in P if $<$ linearly orders C.

Theorem 5.8 (Zorn's Lemma)**.** If $(P, <)$ is a non-empty partially ordered set such that every chain in P has an upper bound, then P has a maximal element.

Proof. By the axiom of choice there exists a choice function f for non-empty subsets of P. By transfinite recursion let us construct the sequence a such that a_α is equal to the the set of all elements greater than all a_ξ for $\xi < \alpha$ mapped under f if such a set is non-empty. The image of this function is then a chain, which then has an upper bound. Such an upper bound by construction is a maximal element of P. □

Axiom 5.9 (The Countable Axiom of Choice)**.** Every countable family of non-empty sets has a choice function.

Axiom 5.10 (The Principle of Dependent Choice)**.** If E is a binary relation on a non-empty set A, and if for every $a \in A$ there exists $b \in A$ such that $b \, E \, a$, then there exists a sequence $\langle a_n \in A \,|\, n \in \mathbb{N}\rangle$ such that:

$$a_{n+1} \, E \, a_n \; \forall \, n \in \mathbb{N}$$

Lemma 5.11. A relation E on P is well-founded if and only if there is no infinite sequence $\langle a_n \,|\, n \in \mathbb{N}\rangle$ in P such that $a_{n+1} \, E \, a_n$ for all $n \in \mathbb{N}$.

Proof. If there exists a sequence $\langle a_n \in A \,|\, n \in \mathbb{N}\rangle$ such that $a_{n+1} \, E \, a_n$ for all $n \in \mathbb{N}$ then no element in the image of a has is minimal. Therefore E is not well-founded. The contrapositive gives us that if E is well-founded then such a sequence does not exist.

If E is not well-founded then there must exist some non-empty set $A \subseteq P$ with no E-minimal element. By the principle of dependent choice, there then exists a sequence a such that $a_{n+1} \, E \, a_n$ for all $n \in \mathbb{N}$. The contrapositive gives us that if no such sequence exists then E is well-founded. □

Corollary 5.12. A linear ordering $<$ of a set P is a well-ordering of P if and only if there is no infinite descecnding sequence in A.

Lemma 5.13. If $2 \leq \kappa \leq \lambda$ and λ is infinite then $\kappa^\lambda = 2^\lambda$.

Proof.

$$2^\lambda \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^\lambda$$

□

Definition 5.14. If λ is a cardinal and $|A| \geq \lambda$ let

$$[A]^\lambda = \{X \subseteq A \mid |X| = \lambda\}$$

Lemma 5.15. If $[A] = \kappa \geq \lambda$ then the set $[A]^\lambda$ has cardinality κ^λ.

Proof. Every function from λ to A is a subset of $\lambda \times A$ of cardinality $|\lambda|$, so $\kappa^\lambda \leq |[\lambda \times A]^\lambda| = |[A]^\lambda|$. We can also construct the function $F : [A]^\lambda \to A^\lambda$ where $F(X)$ for some $X \subseteq A$ where $|X| = \lambda$ is any function f on λ with range X. It holds that F is injective, so $|[A]^\lambda| \leq |\kappa^\lambda|$. Thus $|[A]^\lambda| = |\kappa^\lambda|$ $\square$

Definition 5.16. If λ is a limit cardinal let

$$\kappa^{<\lambda} = \sup\{\kappa^\mu \mid \mu \text{ is a cardinal and } \mu < \lambda\}$$

Definition 5.17. If κ is an infinite cardinal and $|A| \geq \kappa$ then let

$$[A]^{<\kappa} = P_\kappa(A) = \{X \subseteq A \mid |X| < \kappa\}$$

Definition 5.18. Let $\{\kappa_i \mid i \in I\}$ be an indexed set of cardinal numbers. We define

$$\sum_{i \in I} \kappa_i = |\bigcup_{i \in I} X_i|$$

where $\{X_i \mid i \in I\}$ is a disjoint family of sets such that $|X_i| = \kappa_i$ for each $i \in I$.

Lemma 5.19. If λ is an infinite cardinal and $\kappa_i > 0$ for each $i < \lambda$ then

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i$$

Proof. Denote $\kappa = \sup_{i<\lambda} \kappa_i$ and $\sigma = \sum_{i<\lambda} \kappa_i$. As $\kappa_i < \lambda$ for all i we have that $\sigma \leq \lambda \cdot \kappa$. We also have that $\lambda = \sum_1 \leq \sigma$ and $\kappa \leq \sigma$. Therefore $\sigma \geq \lambda \cdot \kappa$ so $\sigma = \lambda \cdot \kappa$. $\square$

Definition 5.20. Let $\{X_i \mid i \in I\}$ be an indexed family of sets. We define

$$\prod_{i \in I} X_i = \{f \mid f \text{ is a function on } I \text{ and } f(i) \in X_i \text{ for each } i \in I\}$$

and for an indexed set of cardinals $\{\kappa_i \mid i \in I\}$ we define $\prod_{i \in I} \kappa_i = |\prod_{i \in I} \kappa_i|$.

Lemma 5.21. If λ is an infinite cardinal and $\langle \kappa_i \mid i < \lambda \rangle$ is a nondecreasing sequence of nonzero cardinals then

$$\prod_{i<\lambda} \kappa_i = (\sup_{i<\lambda} \kappa_i)^\lambda$$

Proof. As each κ_i injects into $\sup_{i<\lambda} \kappa_i$ we have that $\prod_{i<\lambda} \kappa_i \leq (\sup_{i<\lambda} \kappa_i)^\lambda$. Now let us consider κ^λ. We have that $\Gamma[\lambda \times \lambda] = \lambda$ as λ is an infinite cardinal (where Γ is our canonical mapping from $\mathrm{Ord} \times \mathrm{Ord}$ to Ord). Then we have $\lambda = \bigcup_{j<\lambda} A_j$ where $A_j = \{\Gamma(i \times j) \mid i < \lambda\}$. Thus λ is a union of λ sets of cardinality λ. It follows that $\prod_{i \in A_j} \kappa_i \geq \sup_{i \in A_j} \kappa_i = \sup_{i<\lambda} \kappa_i$. Then we have

$$\prod_{i<\lambda} \kappa_i = \prod_{j<\lambda} (\prod_{i \in A_j} \kappa_i) \geq \prod_{j<\lambda} (\sup_{i<\lambda} \kappa_i) = (\sup_{i<\lambda} \kappa_i)^\lambda$$

$\square$

Theorem 5.22 (König)**.** If $\kappa_i < \lambda_i$ for every $i \in I$ then

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$$

Proof. We have that the sum over all κ_i is equal in cardinality to the sum over all κ_j such that $\kappa_j \neq \varnothing$. This sum is then less than the sum over all λ_j, which must all then be greater than or equal to 2 and thus less than or equal to $\prod \lambda_j$. As each λ_i must be at least 1 we have this product is less than or equal to $\prod_{i\in I} \lambda_i$. Therefore $\sum_{i\in I} \kappa_i \leq \prod_{i\in I} \lambda_i$.

Let us consider an arbitrary function F from $\sum_{i\in I} \kappa_i$ to $\prod_{i\in I} \lambda_i$. Denote $\sum_{i\in I} \kappa_i = |\bigcup_{i\in I} Z_i|$ and $\prod_{i\in I} \lambda_i = \prod_{i\in I} T_i$. Let $S_i = \{f(i) \mid f \in F[Z_i]\}$. As $|F[Z_i]| = |Z_i| = \kappa_i < \lambda_i = |T_i|$ we have $S_i \subsetneq T_i$. It follows from the axiom of choice that there exists a function $f \in \prod_{i\in I} T_i$ such that $f(i) \notin S_i$. It follows that $f \notin F[Z_i]$ for any $i \in I$ and thus $f \notin \mathrm{im}(F)$, so F is not surjective. Therefore $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$. □

Corollary 5.23. $\kappa < 2^\kappa$ for all cardinals κ.

Corollary 5.24. $\mathrm{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$

Corollary 5.25. $\mathrm{cf}(\aleph_\alpha^{\aleph_\beta}) > \aleph_\beta$

Corollary 5.26. $\kappa^{\mathrm{cf}\,\kappa} > \kappa$ for every infinite cardinal κ.

Axiom 5.27 (GCH)**.** The Generalized Continuum Hypothesis states $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all ordinals α.

Theorem 5.28. If the GCH holds then given infinite cardinals κ and λ we have:

- If $\kappa \leq \lambda$ then $\kappa^\lambda = \lambda^+$.
- If $\mathrm{cf}\,\kappa \leq \lambda < \kappa$ then $\kappa^\lambda = \kappa^+$.
- If $\lambda < \mathrm{cf}\,\kappa$ then $\kappa^\lambda = \kappa$.

Proof. By 5.13 we have that if $\kappa \leq \lambda$ then $\kappa^\lambda = 2^\lambda$. Then by GCH we have $\kappa^\lambda = \lambda^+$.

Let us assume that $\mathrm{cf}\,\kappa \leq \lambda < \kappa$. We have that $\kappa^\lambda \leq (2^\kappa)^\lambda = 2^\kappa$. By 5.26 we also have that $\kappa^\lambda \geq \kappa^{\mathrm{cf}\,\kappa} > \kappa$, so $\kappa^\lambda \geq 2^\kappa$. Thus $\kappa^\lambda = 2^\kappa = \kappa^+$.

Let us assume that $\lambda < \mathrm{cf}\,\kappa$. Consider the union of α^λ for all $\alpha < \kappa$. We have that this is a subset of κ^λ. As $\lambda < \mathrm{cf}\,\kappa$, we have by 3.25 that the image of every function in α^λ is bounded by some element of κ, so every element of κ^λ is contained in our union of all α^λ. Then we have $\alpha^\lambda \leq 2^{\alpha\cdot\lambda} \leq \kappa$. Therefore $\kappa \leq \kappa^\lambda \leq \kappa \times \kappa = \kappa$, so $\kappa^\lambda = \kappa$. □

Definition 5.29 (Beth Function)**.**

$$\beth_0 = \aleph_0 \qquad \beth_{\alpha+1} = 2^{\beth_\alpha}$$

$$\beth_\alpha = \sup\{\beth_\beta \mid \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal}$$

Corollary 5.30. GCH is equivalent to $\beta_\alpha = \aleph_\alpha$ for all $\alpha \in \mathrm{Ord}$.

Theorem 5.31. If κ is a limit[1] cardinal then $2^\kappa = (2^{<\kappa})^{\mathrm{cf}\,\kappa}$.

Proof. Let $\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i$ where $\kappa_i < \kappa$ for each i. We have

$$2^\kappa = 2^{\sum_{i<\mathrm{cf}\,\kappa}\kappa_i} = \prod_{i<\mathrm{cf}\,\kappa} 2^{\kappa_i} \leq \prod_{i<\mathrm{cf}\,\kappa} 2^{<\kappa} = (2^{<\kappa})^{\mathrm{cf}\,\kappa} \leq 2^\kappa$$

□

Corollary 5.32. If κ is a singular cardinal and if the continuum function is eventually constant below κ with value λ then $2^\kappa = \lambda$.

Proof. There must exist some cardinal μ such that $\mathrm{cf}\,\kappa \leq \mu < \kappa$ and $2^{<\kappa} = \lambda = 2^\mu$. Then we have

$$2^\kappa = (2^{<\kappa})^{\mathrm{cf}\,\kappa} = (2^\mu)^{\mathrm{cf}\,\kappa} = 2^\mu$$

□

[1] This holds even for successor cardinals.

Definition 5.33 (Gimel function)**.**

$$\gimel(\kappa) = \kappa^{\operatorname{cf} \kappa}$$

Theorem 5.34. If κ is a regular cardinal then $2^\kappa = \gimel(\kappa)$.

Proof. As κ is a regular cardinal we have that $\operatorname{cf} \kappa = \kappa$. Then by 5.13 we have

$$2^\kappa = \kappa^\kappa = \kappa^{\operatorname{cf} \kappa} = \gimel(\kappa)$$

□

Theorem 5.35. If κ is a limit cardinal and the limit of the continuum function below κ is not eventually constant then $2^\kappa = \gimel(\lambda)$ where $\lambda = 2^{<\kappa}$.

Proof. As the continuum function below κ is not eventually constant we have that λ is the limit of the κ sequence 2^α for $\alpha < \kappa$. Then by 3.21 we have $\operatorname{cf} \kappa = \operatorname{cf} \lambda$. By 5.31 it follows that

$$2^\kappa = (2^{<\kappa})^{\operatorname{cf} \kappa} = \lambda^{\operatorname{cf} \lambda} = \gimel(\lambda)$$

□

Theorem 5.36. If κ is a regular cardinal and $\lambda < \kappa$ then $\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda$.

Proof. As $\lambda < \kappa$ and $\operatorname{cf} \kappa = \kappa$ it must hold that the image of every function from λ to κ is bounded. Therefore κ^λ is the sum of $|\alpha|^\lambda$ for all $\alpha < \kappa$. □

Lemma 5.37. If κ is a limit cardinal and $\lambda \geq \operatorname{cf} \kappa$ then

$$\kappa^\lambda = (\lim_{\alpha \to \kappa} \alpha^\lambda)^{\operatorname{cf} \kappa}$$

Proof. As κ is a limit cardinal we have that $\kappa = \sum_{i<\operatorname{cf} \kappa} \kappa_i$ where $\kappa_i < \kappa$. Then we have

$$\kappa^\lambda \leq (\prod_{i<\operatorname{cf} \kappa} \kappa_i)^\lambda = \prod_{i<\operatorname{cf} \kappa} \kappa_i^\lambda \leq \prod_{i<\operatorname{cf} \kappa} (\lim_{\alpha \to \kappa} \alpha^\lambda) = (\lim_{\alpha \to \kappa} \alpha^\lambda)^{\operatorname{cf} \kappa} \leq (\kappa^\lambda)^{\operatorname{cf} \kappa} = \kappa^\lambda$$

□

Theorem 5.38. Let λ be an infinite cardinal. Then for all infinite cardinals κ the value of κ^λ is computed inductively as:

(i) If $\kappa \leq \lambda$ then $\kappa^\lambda = 2^\lambda$.

(ii) If there exists some $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$ then $\kappa^\lambda = \mu^\lambda$.

(iii) If $\kappa > \lambda$ and $\mu^\lambda < \kappa$ for all $\mu < \kappa$ then:

  (a) If $\operatorname{cf} \kappa > \lambda$ then $\kappa^\lambda = \kappa$.

  (b) If $\operatorname{cf} \kappa \leq \lambda$ then $\kappa^\lambda = \kappa^{\operatorname{cf} \kappa}$.

Proof. If $\kappa \leq \lambda$ then by 5.31 we have $\kappa^\lambda = 2^\lambda$. If there exists some $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$ then $\mu^\lambda \leq \kappa^\lambda \leq (\mu^\lambda)^\lambda = \mu^\lambda$. Let us consider when $\kappa > \lambda$ and $\mu^\lambda < \kappa$ for all $\mu < \kappa$. If $\kappa = \beta + 1$ then by 5.36 we have $\kappa^\lambda = \beta^\lambda \cdot \kappa = \kappa$. Otherwise we have that κ is a limit cardinal. Then if $\operatorname{cf} \kappa > \lambda$ we have the image of every function from λ to κ is bounded so $\kappa^\lambda = \lim_{\alpha \to \kappa} \alpha^\lambda = \kappa$. If $\operatorname{cf} \kappa \leq \lambda$ then by 5.37 we have $\kappa^\lambda = \kappa^{\operatorname{cf} \kappa}$. □

Definition 5.39 (Strong Cardinal)**.** A cardinal κ is a strong limit cardinal if $2^\lambda < \kappa$ for all $\lambda < \kappa$.

Theorem 5.40. If κ is a strong limit cardinal then $2^\kappa = \kappa^{\operatorname{cf} \kappa}$.

Proof. By 5.31 we have $2^\kappa = (2^{<\kappa})^{\operatorname{cf} \kappa}$. As κ is strong we have $2^{<\kappa} = \kappa$ so $2^\kappa = \kappa^{\operatorname{cf} \kappa}$. □

Definition 5.41 (SCH)**.** The Singular Cardinal Hypothesis is the statement: For every singular cardinal κ, if $2^{\operatorname{cf} \kappa} < \kappa$ then $\kappa^{\operatorname{cf} \kappa} = \kappa^+$.

6 The Axiom of Regularity

Axiom 6.1 (Axiom of Regularity)**.** Every non-empty set has an $\in$-minimal element:

$$\forall S\,(S \neq \varnothing \to (\exists x \in S)\, S \cap x \neq \varnothing)$$

Theorem 6.2. For every set S there exists a transitive set $T \supseteq S$.

Proof. Let us inductively define sets S_n for $n \in \mathbb{N}$. Let $S_0 = S$ and $S_{n+1} = \bigcup S_n$. Then it holds that $T = \bigcup_{n=0}^{\infty} S_n$ is transitive and $T \supseteq S$. It also holds that such a T is the smallest transitive T that is a super set of S as the union over a transitive set must be a subset of itself, so any transitive set containing S must contain each S_n and thus contain T. $\square$

Definition 6.3 (Transitive Closure)**.** The transitive closure of S is the smallest transitive $T \supseteq S$:

$$\mathrm{TC}(S) = \bigcap \{T \,|\, T \supseteq S \text{ and } T \text{ is transitive}\}$$

Lemma 6.4. Every non-empty class C has an $\in$-minimal element.

Proof. Let $S \in C$ be arbitrary. If $S \cap C = \varnothing$ then S is a minimal element of C. If $S \cap C \neq \varnothing$ then let $X = \mathrm{TC}(S) \cap C$. X is a non-empty set and by the Axiom of Regularity has an $\in$-minimal element, denoted x. As $\mathrm{TC}(S)$ is transitive we have that $x \subseteq \mathrm{TC}(S)$. For all $y \in x$ we thus have that $y \in TC(S)$. If $y \in C$ then we have $y \in x \cap (\mathrm{TC}(S) \cap C) = x \cap X$, meaning x is not $\in$-minimal. As x is $\in$-minimal, it follows that no $y \in X$ is also in C. Thus x is a $\in$-minimal element of C. $\square$

Definition 6.5. Define by transfinite recursion:

$$V_0 = \varnothing \qquad V_{\alpha+1} = \mathcal{P}(V_\alpha)$$

$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta \text{ if } \alpha \text{ is a limit ordinal}$$

Corollary 6.6. By induction we have that each V_α is transitive, if $\alpha < \beta$ then $V_\alpha \subsetneq V_\beta$, and $\alpha \subseteq V_\alpha$.

Lemma 6.7. For every set x there is some α such that $x \in V_\alpha$.

Proof. Let us assume for the sake of contradiction that the class C of all $x \notin \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$ is non-empty. By 6.4 we have that C has some $\in$-minimal element, denoted x. It follows that for all $z \in x$ we have $z \in \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$. Thus $x \subseteq \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$. By the Axiom of Replacement there exists some ordinal γ such that $x \subseteq \bigcup_{\alpha<\gamma} V_\alpha$. Therefore $x \subseteq V_\gamma$ so $x \in V_{\gamma+1}$, resulting in contradiction. Therefore C is empty, meaning every set must be in the union of V_α. $\square$

Definition 6.8 (Rank)**.** The rank of any set x is the least α such that $x \in V_{\alpha+1}$.

Corollary 6.9. If $x < y$ then $\mathrm{rank}(x) < \mathrm{rank}(y)$ and for all $\alpha \in \mathrm{Ord}$ we have $\mathrm{rank}(\alpha) = \alpha$.

Theorem 6.10 (Collection Principle)**.** Given a collection of classes C_u such that $u \in X$, where X is a set, then there is a set Y such that for every $u \in X$ if $C_u \neq \varnothing$ then $C_u \cap Y \neq \varnothing$.

$$\forall X\, \exists Y\, (\forall u \in X)\, [\exists v\, \varphi(u, v, p) \to (\exists v \in Y)\, \varphi(u, v, p)]$$

Proof. For each $v \in X$, let us consider all u such that $\varphi(u, v, p)$. All such u form a class. Let us assume such a class is non-empty. Then there is a an element in the class with some rank. We can then consider the collection of all elements from this class with rank less than or equal to this rank. Such a collection is a set as we can apply seperation to the union of V_α for all α less than or equal to the rank of our chosen element. Then, as ordinals are well-ordered, we can consider the least rank. Let C_v be equal to this set of all u which satisfy $\varphi(v, u, p)$ with minimal rank. We can take the union of C_v for all $v \in X$. This union will then be a set which satisfies the collection principle. $\square$

Corollary 6.11. The collection principle implies the Axiom Schema of Replacement.

Theorem 6.12 ($\in$-induction)**.** Let T be a transitive class and let Φ be a property. Assume that $\Phi(\varnothing)$ and if $x \in T$ and $\Phi(z)$ holds for every $z \in x$ then $\Phi(x)$. Then every $x \in T$ has property Φ.

Proof. Assume for the sake of contradiction that the class of all $x \in T$ not satifying Φ is a non-empty class. It must hold that such a set is not empty. By 6.4 we have that there is some $\in$-minimal element in this class. As T is transitive, every element in this $\in$-minimal set must be in T, and thus by assumption this set satisfies Φ, resulting in contradiction. $\square$

Theorem 6.13 ($\in$-recursion)**.** Let T be a transitive class and let G be a function. Then there is a unique function F on T such that

$$F(x) = G(F\restriction_x)$$

Proof. For every $x \in T$ let

$$\begin{aligned} F(x) = y \iff & \text{there is a function } f \text{ such that} \\ & \mathrm{dom}(f) \text{ is a transitive subset of } T \text{ and:} \\ & \text{(i) } (\forall z \in \mathrm{dom}(f))\ f(z) = G(f\restriction_z) \\ & \text{(ii) } f(x) = y \end{aligned}$$

It holds that $G(\varnothing)$ is the unique y satisfying our given condition for $F(\varnothing)$. Consider an arbitrary non-empty $x \in T$. Let us assume that for all $z \in x$ we have that there is a unique y_z satisfying $F(z) = y_z$. Then we have $G(\bigcup_z y_z)$ satisfies $F(x)$. We can consider an arbitrary function g that satisfies the conditions imposed on $F(x)$. We have that $g(x) = G(g\restriction_x)$. If $g(x) \neq y$ then $G(g\restriction_x) \neq G(f\restriction_x)$. There must then exist some $a \in x$ such that $g(a) \neq f(a)$. However, we then have $F(x)$ is also equal to $g(a)$ resulting in contradiction. Therefore $F(x)$ is unique for all $x \in T$ by $\in$-induction.

It holds that $F(\varnothing) = G(\varnothing) = G(F\restriction_\varnothing)$. Consider an arbitrary non-empty $x \in T$. Let us assume that for all $z \in x$ we have that $F(z) = G(F\restriction_z)$. We then have that $F\restriction_x$ is defined and satisfies $F(x)$. Therefore $F(x)$ is defined and equal to $G(F\restriction_x)$ for all $x \in T$ by $\in$-induction.

Let us consider an arbitrary function F' satisfying $F'(x) = G(F'\restriction_x)$. It holds that $F'(\varnothing) = G(\varnothing) = F(\varnothing)$. Consider an arbitrary $x \in T$. Let us assume that for all $z \in T$ we have $F(z) = F'(z)$. We have that $F'(x) = G(F'\restriction_x) = G(F\restriction_x) = F(x)$. Therefore by $\in$-induction we have $F = F'$, so F is the unique function satisfying $F(x) = G(F\restriction_x)$. □

Corollary 6.14. Let A be a class. There is a unique class B such that

$$B = \{x \in A \mid x \subset B\}$$

Proof. Let us define the function F recursively on A. Let $F(\varnothing) = 1$. Let $F(x) = 1$ if $F(z) = 1$ for all $z \in x$. Let $F(y) = 0$ for all $y \in A$ that are not defined through this recursive procdeure. Let $B = \{x \mid F(x) = 1\}$.

Consider an arbitrary B' such that $B' = \{x \in A \mid x \subset B\}$. It holds that either both or neither of B' and B contain the empty set. Let us assume that for all $x \in B$ we have that if $z \in x$ then $z \in B'$. It then holds that x is an element of A such that $x \subset B'$ so $x \in B'$. Similarly, we it follows that for all $x \in B'$ we have $x \in B$. Therefore by $\in$-induction we have $B = B'$. □

Theorem 6.15. Let T_1 and T_2 be transitive classes and let π be an $\in$-isomorphism. Then $T_1 = T_2$ and $\pi(u) = u$ for every $u \in T_1$.

Proof. Let us assume for the sake of contradiction that $\pi(\varnothing) \neq \varnothing$. There must exist some $x \in \pi(\varnothing)$. However, we then have $\pi^{-1}(x) \notin \pi^{-1}(\pi(\varnothing))$. Thus $\pi(\varnothing) = \varnothing$. Consider an arbitrary $x \in T_1$. Let us assume that for all $z \in x$ we have $\pi(z) = z$. Let us consider $\pi(x)$. Consider an arbitrary $a \in x$. We have that $\pi(a) \in \pi(x)$, so by assumption $a \in \pi(x)$. Therefore $x \subseteq \pi(x)$. Consider an arbitrary $b \in \pi(x)$. By transitivity we have $b \in T_2$. Then we know $\pi^{-1}(b) \in x$. By assumption, we then have that $\pi^{-1}(b) = \pi(\pi^{-1}(b)) = b$. Therefore $b \in x$. Thus $\pi(x) \subseteq x$. By $\in$-induction we then have that $x = \pi(x)$ for all $x \in T_1$. □

Definition 6.16 (Extension)**.** Let E be a binary relation on a class P. For each $x \in P$ define the extention of x as:

$$\mathrm{ext}_E(x) = \{z \in P \mid zEx\}$$

Definition 6.17. A relation E on a class P is well-founded if every non-empty set $x \subseteq P$ has an E-minimal element and the extention of every element in P is a set.

Lemma 6.18. If E is a well-founded relation on P then every non-empty class $C \subseteq P$ has an E-minimal element.

Proof. Let $S \in C$ be arbitrary. If $\mathrm{ext}_E(S) \cap C = \varnothing$ then S is a E-minimal element of C. Let us consider if $\mathrm{ext}_E(S) \cap C$ is non-empty. Let us define S_n recursively, where $S_0 = \mathrm{ext}_E(S)$ and $S_{n+1} = \bigcup\{\mathrm{ext}_E(z) \mid z \in S_n\}$. Let $X = (\bigcup_{n=0}^{\infty} S_n) \cap C$. As X is a set, there is some E-minimal element $x \in X$. Let us assume for the sake of contradiction such an element is not E-minimal in C. Then there exists some $y \in C$ such that $y\,E\,x$. We have that $x \in S_n$ for some n. It follows that $y \in S_{n+1}$, so $y \in X$. Therefore x would not be E minimal in X, resulting in contradiction. Therefore x must be E minimal in C. □

Theorem 6.19 (Well-Founded Induction)**.** Let E be a well-founded relation on P. Let Φ be a property. If every E-minimal element x has property Φ and $x \in P$ has property Φ if $\Phi(z)$ for all $z \in P$ such that $z \, E \, x$ then all $x \in P$ have the property Φ.

Theorem 6.20 (Well-Founded Recursion)**.** Let E be a well-founded relation on P. Let G be a function. Then there is a unique function F on P such that for all $x \in P$ we have:

$$F(x) = G(x, F \restriction_{\text{ext}_E(x)})$$

Definition 6.21 (Transitive Collapse)**.** We can recursively define the function π on a class P with a well-founded relation E as

$$\pi(x) = \{\pi(z) \mid z \, E \, x\}$$

for all $x \in P$. The range of π is then a transitive class and for all $x, y \in P$ if $x \, E \, y$ then $\pi(x) \in \pi(y)$.

Definition 6.22 (Extentional Well-Founded Relation)**.** A well-founded relation E on a class P is extentional if

$$\text{ext}_E(X) \neq \text{ext}_E(Y)$$

for all $X, Y \in P$ such that $X \neq Y$.

Definition 6.23 (Extentional Class)**.** A class M is extentional if the relation $\in$ on M is extentional. Equivalently, M is extentional if for any distinct X and Y in M we have $X \cap M \neq Y \cap M$.

Theorem 6.24 (Mostowski's Collapsing Theorem)**.** If E is well-founded and an extentional relation on a class P then there is a unique transitive class M and a unique isomorphism π between (P, E) and $(M, \in)$. If $T \subseteq P$ is transitive then $\pi(x) = x$ for every $x \in T$.

Proof. Let π be the transitive collapse of (P, E). It holds that π is a surjection onto a transitive class. As (P, E) is extentional and well-founded we have that there is a unique $X \in P$ such that $\text{ext}_E(X) = \varnothing$. Let us assume that π restricted to all $X \in P$ such that $\text{rank}(X) < \alpha$ is injective. Consider arbitrary $A, B \in P$ such that $A \neq B$ and $\text{rank}(A) = \text{rank}(B)$. We have that $\text{ext}_E(A) \neq \text{ext}_E(B)$ as (P, E) is extentional and both such sets consist of elements with rank strictly less than α. Thus by our inductive hypothesis we have $\pi(A) = \pi[\text{ext}_E(A)] \neq \pi[\text{ext}_E(A)] = \pi(B)$. Thus by induction on rank we have that π is injective over all sets in P. Thus π forms a bijection, and it follows that $X \, E \, Y \iff \pi(X) = \pi(Y)$, so (P, E) is isomorphic to $(M, \in)$ for some transitive M.

By 6.15 we have that the only $\in$-isomorphism over a transitive classe is the identity automorphism. Thus we have that M is the unique transitive set isomorphic to (P, E) and π is the unique isomorphism between (P, E) and $(M, \in)$. □

Axiom 6.25 (Bernays-Gödel Axiomatic Set Theory)**.** Let there be two types of objects, sets (denoted by lower case letters) and classes (denoted by capital letters):

(A) 1. Extentionality: $\forall u \, (u \in X \iff y \in Y) \rightarrow X - Y$.
   2. Every set is a class.
   3. If $X \in Y$ then X is a set .
   4. Pairing: For any sets x and y there is a set $\{x, y\}$.

(B) Comprehension: $\forall X \exists Y \, Y Y = \{x \mid \varphi(x, X)\}$ where φ is a formula in which only set variables are quantified.

(C) 1. Infinity: There is an infinite set.
   2. Union: for every x the set $\bigcup x$ exists.
   3. Power Set; For every set x the power set $\mathcal{P}(x)$ exists.
   4. Replacement: If a class F is a function and x is a set then $\{F(z) \mid z \in x\}$ is a set.

(D) Regularity.

(E) Choice: There is a function F such that $F(x) \in x$ for every non-empty set x.

7 Things to look at

- https://arxiv.org/abs/2211.03976
- Want to talk over the fourth filter example (set of all sequences with the first n elements fixed)
- Fifth example filter is the collection of all sets where the "growth rate" of the set is decreasing