DDA 5002 Optimization: Homework #2

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Problem 1

Solution: Reformulate the problem as a linear program as follows:

(a)

$$\begin{aligned} \min_{x,y} \quad & c^T x + y \\ \text{s.t.} \quad & y \geq d^T x \\ & y \geq 0 \\ & y \geq 2d^T x - 4 \\ & Ax \geq b \end{aligned}$$

where $x, c, d \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $y \in \mathbb{R}$.

(b)

$$\begin{aligned} \min_{\substack{x_1,x_2,x_3,\\ e_1,e_2,e_3}} \quad & 2x_2 + e_1 \\ \text{s.t.} \quad & e_1 \geq x_1 - x_3 \\ & e_2 \geq x_3 - x_1 \\ & e_2 + e_3 \leq 5 \\ & e_2 \geq x_1 + 2 \\ & e_2 \geq -x_1 - 2 \\ & e_3 \geq x_2 \\ & e_3 \geq -x_2 \\ & x_3 \geq -1 \\ & x_3 \leq 1 \end{aligned}$$

Problem 2
Solution: The standard form is as follows:

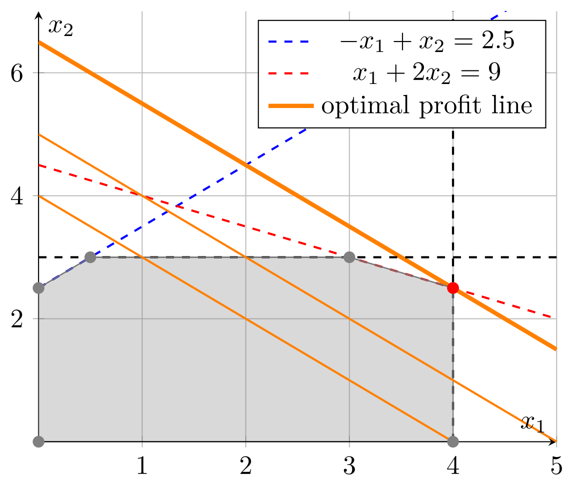
(a) The first problem.

$$
\begin{aligned}
\min_{x} \quad & -x_2 + 4x_3 - 5x_5 + 5x_6 \\
\text{s.t.} \quad & x_1 = x_5 - x_6 \\
& x_4 = -x_7 \\
& x_1 + x_2 + x_3 + x_4 - x_8 = 19 \\
& 4x_2 - 8x_7 + x_9 = 45 \\
& 6x_2 - x_3 + x_5 - x_6 = 7 \\
& x_i \geq 0, \quad i = 1, 2, \ldots, 9
\end{aligned}
$$

(b) The second problem.

$$
\begin{aligned}
\min_{x} \quad & 2x_1 - 7x_2 + 6x_3 + 5x_4 \\
\text{s.t.} \quad & 2x_1 - 3x_2 - 5x_3 - 4x_4 + x_5 = 20 \\
& 7x_1 + 2x_2 + 6x_3 - 2x_4 = 35 \\
& 4x_1 + 5x_2 - 3x_3 - 2x_4 - x_6 = 15 \\
& x_1 + x_7 = 10 \\
& x_2 + x_8 = 8 \\
& x_3 - x_9 = 2 \\
& x_i \geq 0, \quad i = 1, 2, \ldots, 9
\end{aligned}
$$

Problem 3
Solution:



The graphic above shows the solution.

The optimal value is achieved at the point (4, 2.5) with an optimal value of 6.5.

At optimal solution, the constraints $x_1 + 2x_2 \leq 9$ and $x_1 \leq 4$ are active.

All the vertices of the feasible region are (0,0), (0,2.5), (0.5,3), (3,3), (4,2.5), and (4,0).

Problem 4
Solution:

Rewrite the standard form as follows:

$$\begin{aligned}\min_{x_1,x_2} \quad & -x_1 - x_2 \\ \text{s.t.} \quad & x_1 + 3x_2 - x_3 = 15 \\ & 2x_1 + x_2 - x_4 = 10 \\ & x_1 + 2x_2 + x_5 = 40 \\ & 3x_1 + x_2 + x_6 = 60 \\ & x_i \geq 0, \quad i = 1, 2, \ldots, 6\end{aligned}$$

Denote the basic matrix as B, the non-basic matrix as N, the constants as b, the basic variables as x_B, and the non-basic variables as x_N.

The coefficient matrix is as follows:

$$A = \begin{bmatrix} 1 & 3 & -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Iter 1: Choose $\boldsymbol{x_1, x_2, x_5, x_6}$ as the initial basis.

$$B = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 15 \\ 10 \\ 40 \\ 60 \end{bmatrix}$$

Solve the equations $Bx_B = b$ to get the basic solution: $x_B = \big(3, 4, 29, 47\big)^T$.

Calculate the reduced cost :

$$r = c_N^T - c_B^T B^{-1} N = \big(0, 0\big) - \big(-1, -1, 0, 0\big) B^{-1} N = \big(-0.2, -0.4\big)$$

Following the smallest index rule, we choose x_4 as the entering variable.

Denote the directions as $d = (d_B, d_N)$, where d_B is the direction for basic variables and d_N is the direction for non-basic variables; the step length as θ.

$$\begin{aligned} d_N &= (0, 1)^T \\ d_B &= -B^{-1} N d_N = \big(0.6, -0.2, -0.2, -1.6\big)^T \\ \theta^* &= \min_{d_{B_i} < 0} \left\{ \frac{-x_{B_i}}{d_{B_i}} \right\} = 20 \\ \Rightarrow x^{(1)} &= x + \theta^* d = \big(15, 0, 25, 15, 0, 20\big)^T \end{aligned}$$

x_2 leaves the basis, while x_4 enters.

Iter 2: The basis now is $\boldsymbol{x_1, x_4, x_5, x_6}$, the non-basis is $\boldsymbol{x_2, x_3}$.

$$B^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \quad N^{(1)} = \begin{bmatrix} 3 & -1 \\ 1 & 0 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$$

Solve the equations $B^{(1)}x_B = b$ to get the basic solution: $x_B^{(1)} = \big(15, 20, 25, 15\big)^T$.

Calculate the reduced cost :

$$r^{(1)} = c_N^{(1)T} - c_B^{(1)T} B^{(1)-1} N^{(1)} = \big(-1, 0\big) - \big(-1, 0, 0, 0\big) B^{(1)-1} N^{(1)} = \big(2, -1\big)$$

Following the smallest index rule, we choose x_3 as the entering variable.

$$\begin{aligned} d_N^{(1)} &= (0,1)^T \\ d_B^{(1)} &= -B^{-1} N d_N = \big(1, 2, -1, -3\big)^T \\ \theta^{(1)*} &= \min_{d_{B_i}<0} \left\{ \frac{-x_{B_i}}{d_{B_i}} \right\} = 5 \\ \Rightarrow x^{(1)} &= x + \theta^{(1)*} d = \big(20, 30, 20, 0, 0, 5\big)^T \end{aligned}$$

x_6 leaves the basis, while x_3 enters.

Iter 3: The basis now is $\boldsymbol{x_1, x_3, x_4, x_5}$, the non-basis is $\boldsymbol{x_2, x_6}$.

$$B^{(2)} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix} \quad N^{(2)} = \begin{bmatrix} 3 & 0 \\ 1 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Solve the equations $B^{(2)}x_B = b$ to get the basic solution: $x_B^{(2)} = \big(20, 5, 30, 20\big)^T$.

Calculate the reduced cost :

$$r^{(2)} = c_N^{(2)T} - c_B^{(2)T} B^{(2)-1} N^{(2)} = \big(-1, 0\big) - \big(-1, 0, 0, 0\big) B^{(2)-1} N^{(2)} = \big(-\frac{2}{3}, \frac{1}{3}\big)$$

Following the smallest index rule, we choose x_2 as the entering variable.

$$\begin{aligned} d_N^{(2)} &= (1,0)^T \\ d_B^{(2)} &= -B^{-1} N d_N = \big(-\frac{1}{3}, \frac{8}{3}, \frac{1}{3}, -\frac{5}{3}\big)^T \\ \theta^{(2)*} &= \min_{d_{B_i}<0} \left\{ \frac{-x_{B_i}}{d_{B_i}} \right\} = 12 \\ \Rightarrow x^{(2)} &= x + \theta^{(2)*} d = \big(16, 37, 34, 0, 12, 0\big)^T \end{aligned}$$

x_5 leaves the basis, while x_2 enters.

Iter 4: The basis now is $\boldsymbol{x_1, x_2, x_3, x_4}$, the non-basis is $\boldsymbol{x_5, x_6}$.

$$B^{(2)} = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 1 & 0 & -1 \\ 1 & 2 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix} \quad N^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solve the equations $B^{(3)}x_B = b$ to get the basic solution: $x_B^{(3)} = \big(16, 12, 37, 34\big)^T$.

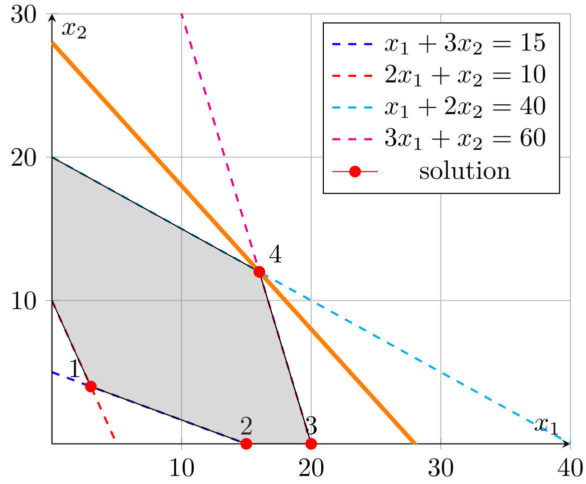
Calculate the reduced cost :

$$r^{(3)} = c_N^{(3)T} - c_B^{(3)T} B^{(3)-1} N^{(3)} = \big(0, 0\big) - \big(-1, -1, 0, 0\big) B^{(3)-1} N^{(3)} = \big(0.4, 0.2\big)$$

All the reduced costs are non-negative, so the current basic feasible solution is optimal.

So the optimal solution is $x^* = \big(16, 12, 0, 0, 37, 34\big)^T$ with an optimal value of -28. For the original problem, the optimal value is 28 with the optimal solution $x_1 = 16, x_2 = 12$.

The feasible region and the solution at each iteration are shown in the graph below:



Problem 5
Solution:

(a) Denote the slack variables as s, the standard form of original problem is as follows:

$$\begin{aligned}
\min_{x,s} \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
\text{s.t.} \quad & x_1 - x_2 + 2x_3 - s_1 = 2 \\
& x_2 - x_3 + 2x_4 + s_2 = 4 \\
& 2x_1 + 3x_3 - x_4 = 2 \\
& x_i, s_j \geq 0, \quad i = 1,2,3,4; j = 1,2
\end{aligned}$$

The auxiliary LP of Phase I is as follows:

$$\begin{aligned}
\min_{x,s,y} \quad & y_1 + y_2 + y_3 \\
\text{s.t.} \quad & x_1 - x_2 + 2x_3 \qquad - s_1 \qquad + y_1 \qquad\qquad = 2 \\
& \qquad x_2 - x_3 + 2x_4 \qquad + s_2 \qquad + y_2 \qquad = 4 \\
& 2x_1 \qquad + 3x_3 - x_4 \qquad\qquad\qquad\qquad + y_3 = 2 \\
& x_i, s_j, y_k \geq 0, \quad i \in \mathbb{I}; j \in \mathbb{J}; k \in \mathbb{K} \\
\text{where} \quad & \mathbb{I} = \{1,2,3,4\}, \mathbb{J} = \{1,2\}, \mathbb{K} = \{1,2,3\}
\end{aligned}$$

(b) Consider the basis with variables x_1, x_4, y_2. The basis matrix and the constant vector are as follows:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & -1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

Solve the equations $Bx_B = b$ to get the basic solution: $x_B = \left(2, 2, 0\right)^T$.

Current solution is $x = \left(2, 0, 0, 2, 0, 0, 0, 0, 0\right)^T$. The auxiliary objective value is 0 and each element of x is non-negative, so the current basic feasible solution is optimal for the auxiliary LP. This is the minimum value that could be achieved under the constraints, so it is optimal for the auxiliary LP of Phase I.

(c) The inverse of basis matrix is as follows:

$$B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ -4 & 1 & 2 \end{bmatrix}$$

Currently, the basis is (x_1, x_4, y_2). The non-basic variables are $(x_2, x_3, s_1, s_2, y_1, y_3)$. So there are 6 possible basic directions.

Denote each column as a possible direction of the non-basic variable entering the basis. The directions matrix for non-basic variables is the 6-dimensional identity matrix I_6. Denote it as D_N.

For basic variables, the direction $d_B = -B^{-1}Nd_N$. So we have $D_B = -B^{-1}ND_N$.

$$\begin{aligned} D_B &= -B^{-1}ND_N \\ &= -B^{-1}NI_6 \\ &= -\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & 1 & 0 & -1 & 0 \\ 2 & -1 & 2 & 0 & -2 & 1 \\ -5 & 3 & -4 & -1 & 4 & -2 \end{bmatrix} \end{aligned}$$

So all basic directions are shown in the matrix D_B, where column i is the direction when the i-th non-basic variable enters the basis:

$$D_B = \begin{bmatrix} 1 & -2 & 1 & 0 & -1 & 0 \\ 2 & -1 & 2 & 0 & -2 & 1 \\ -5 & 3 & -4 & -1 & 4 & -2 \end{bmatrix}$$

The reduced costs:

$$\begin{aligned} r &= c_N^T - c_B^T B^{-1} N \\ &= (0,0,0,0,1,1) - (0,0,1) \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 3 & -4 & -1 & 5 & -1 \end{bmatrix} \end{aligned}$$

Till now, not all the reduced costs are non-negative. following the smallest index rule, we choose x_2 as the entering variable.

Iter 1: x_2 enters, y_2 leaves.

Basis: (x_1, x_4, x_2)

Non-basis: $(y_2, x_3, s_1, s_2, y_1, y_3)$

$$B^{(1)} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 2 & -1 & 0 \end{bmatrix} \quad N^{(1)} = \begin{bmatrix} 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_B^{(1)} = B^{(1)-1}b = \begin{bmatrix} 2 & 2 & 0 \end{bmatrix}^T$$

The reduced costs:

$$\begin{aligned} r &= c_N^{(1)T} - c_B^{(1)T} B^{(1)-1} N^{(1)} \\ &= (1,0,0,0,1,1) - (0,0,0) B^{(1)-1} N^{(1)} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

All the reduced costs are non-negative. Current optimal solution for the auxiliary LP is $x^{(1)} = \left(2,0,0,2,0,0,0\right)^T$ with an optimal value of 0.

Problem 6
Solution:

(a) **False.**

According to the definition of bounded set (in the real cases), a set is bounded if there exists $M \in \mathbb{R}$ such that the size of every element in the set is no more than M.

Take the following LP as a counter-example:

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & x_1 - x_2 + x_3 \\ \text{s.t.} \quad & x_1 - x_2 + x_3 = 0 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Obviously, the optimal solution exists, and the optimal value is 0. The optimal solution set is $S^* = \{x \in \mathbb{R}^3 \mid x_1 - x_2 + x_3 = 0, \mathbf{x} \geq 0\}$.

Consider the optimal solution $x = (\lambda, \lambda, 0)^T \in S^* \quad \forall \lambda \geq 0$. Since λ is unbounded, the optimal solution set S^* is unbounded.

(b) **False.**

Take the same LP in a as a counter-example. Consider the optimal solution $x = (1,2,1)^T \in S^*$. This optimal solution has 3 positive components, while the rank of the coefficient matrix $m = 1$.

(c) **True.**

For the optimal set being convex, if both x_1 and x_2 are optimal solutions, then $c^T x_1 = c^T x_2$. There exists $\lambda \in [0,1]$, $x = \lambda x_1 + (1-\lambda)x_2$, then $c^T x = \lambda c^T x_1 + (1-\lambda)c^T x_2 = c^T x_1$. So x is also an optimal solution. So if there are more than one optimal solution, then there are infinite optimal solutions.

(d) **False.**

Take the same LP in a as a counter-example. Consider the optimal solution $x = (1,2,1)^T \in S^*$. This optimal solution has 3 positive components, while the rank of the coefficient matrix $m = 1$. It doesn't satisfy the definition of basic solution which requires at most m positive components. So this optimal solution is not a basic feasible solution.