DDA 5002 Homework # 5 Fall 2025

Due: Sunday, November 30th at 11:59 pm

Please note: Problem 5 requires you to implement optimization algorithms in `Python`. Put source code file(s) (`.py`, or Jupyter notebook `.ipynb`), and the output screenshot files in a single zip file and submit it via Blackboard. Please name your file `Lastname_StudentID.zip`. Also, give meaningful names to your decision variables and constraints, and add comments to your code liberally. Both the problem set and the code are due at end of the day at 11:59pm on Sunday, November 30th.

1. Are the following sets convex? Explain your answer.

   (a) $S = \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$, where S_1 and S_2 are convex.

   (b) $\{x \in \mathbb{R} : ax^2 + 2x + \frac{1}{a} \geq 0\}$ with $a > 0$.

   (c) $\{(y, x) \in \mathbb{R} \times \mathbb{R}^n : y \geq f(x)\}$, where f is a convex function.

   (d) $\{(y, x) \in \mathbb{R} \times \mathbb{R}^n : y = f(x)\}$, where f is a convex function.

   (e) $\{x \in \mathbb{R}^3 : x^\top Ax \leq 0\}$, where $A = \begin{pmatrix} 2 & 4 & 2 \\ -6 & 3 & 9 \\ 6 & -5 & -5 \end{pmatrix}$.

2. Are the following functions convex? Explain your answer.

   (a) $f(x) = \|x\|_1 = \sum_{i=1}^{d} |x_i|$.

   (b) $f : \mathbb{R}^d \to \mathbb{R}$, defined by $f(x) = \sum_{i=1}^{d} f_i(x_i)$, where $x \in \mathbb{R}^d$ and each f_i is a convex function.

   (c) $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $f(\lambda x) = |\lambda| f(x)$ and $f(x+y) \le f(x) + f(y)$ for any $x, y \in \mathbb{R}^d$ and any $\lambda \in \mathbb{R}$.

   (d) $f(x) = |x|^{3/2}$.

   (e) $f(x) = \ln\left(x + \sqrt{1+x^2}\right) + x^2$.

3. Suppose that we have centered observations $\{(x_i, y_i)\}_{i=1}^n$ with $x_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}$ such that

$$\sum_{i=1}^n x_i = 0, \quad \sum_{i=1}^n y_i = 0.$$

Let $(\beta_0^\star, \beta^\star)$ be a global minimizer of the least squares objective function

$$f(\beta_0, \beta) = \sum_{i=1}^n \left(\beta_0 + \beta^T x_i - y_i\right)^2,$$

where $\beta_0 \in \mathbb{R}$ and $\beta \in \mathbb{R}^d$.

(a) Show that $\beta_0^\star = 0$.

(b) The observations before centering are (x_i', y_i') where

$$x_i' = x_i + q, \quad y_i' = y_i + r,$$

with $q \in \mathbb{R}^d$ and $r \in \mathbb{R}$. Show that (β_0, β) minimizes f if and only if $(\beta_0 - \beta^\top q + r, \beta)$ minimizes

$$\tilde{f}(\beta_0, \beta) = \sum_{i=1}^n \left(\beta_0 + \beta^T x_i' - y_i'\right)^2.$$

4. Consider the following optimization problem:

$$\min_{x_1, x_2} \quad 3x_1^2 + 4x_2^2 + 2x_1 x_2$$

subject to

$$\sqrt{1 + x_1^2 + x_2^2} + x_1 \leq 2.$$

Answer the following questions:

(a) Determine whether this is a convex optimization problem.

(b) Write down the Karush–Kuhn–Tucker (KKT) conditions for this problem.

(c) List all KKT points for the problem.

5. Consider the optimization problem:

$$\min_{x} e^{x} - 2x$$

Starting with the initial bracketing interval [0, 1], solve the problem using the Bisection Method and the Golden Section Method. In each method, terminate the iterations when the length of the current bracketing interval is less than 10^{-4}, and take the midpoint of the final interval as the approximate solution. Implement both methods in `Python`, report the approximate solution obtained by each method along with the number of iterations required.