DDA 5002 Homework # 6 Fall 2025

Due: Friday, December 12th at 11:59 pm

Please note: Problem 3 requires you to implement optimization algorithms in `Python`. Put source code file(s) (`.py`, or Jupyter notebook `.ipynb`), and the output screenshot files in a single zip file and submit it via Blackboard. Please name your file `Lastname_StudentID.zip`. Also, give meaningful names to your decision variables and constraints, and add comments to your code liberally. Both the problem set and the code are due at end of the day at 11:59pm on Friday, December 12th.

1. In Assignment 5, we showed $f(x) = |x|^{3/2}$ is a convex function.

   (a) Prove f is not Lipschitz smooth. (Definition: if $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz smooth with constant $L > 0$, then $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$ for any x, y.)

   (b) Prove that for any constant step size $\bar{\alpha}$ in the gradient descent method applied to this function, there exists an initial point x_0 for which $f(x_1) \geq f(x_0)$, where x_1 is the iterate after one gradient descent step.

   Remark: This problem highlights the importance of Lipschitz smoothness when using a constant step size in the gradient descent method.

2. Consider the optimization problem:

$$\min_x f(x) = x^\top A x, \quad \text{where } A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.$$

Starting from the initial point $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, apply gradient descent using the update rule:

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

with the following step size strategies:

(a) Compute the following by hand or using a calculator:

- Perform one step of gradient descent with a constant step size of 0.1, i.e., $\alpha_k = 0.1$ for all k.
- Perform one step of gradient descent with exact line search.
- Perform one step of gradient descent with backtracking line search: Search for α_k that satisfies the Armijo condition:

$$f\left(x_k - \alpha \nabla f(x_k)\right) \leq f\left(x_k\right) - \gamma\alpha \cdot \|\nabla f\left(x_k\right)\|^2$$

  If the condition is not satisfied, update $\alpha \leftarrow \sigma\alpha$. Use the parameters $\gamma = \frac{1}{2}$ and $\sigma = 0.2$.

(b) Use Python to implement the above three step size strategies and run gradient descent for 10 iterations.

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = \frac{3}{4}x^{\frac{4}{3}} + \frac{1}{2}(x + y)^2.$$

(a) Find its global minimizer.

(b) Given a point (x_k, y_k) with $x_k \neq 0$, write out the expression for (x_{k+1}, y_{k+1}) after one iteration of Newton's method applied to minimize this function:

$$x_{k+1} = x_k - (\nabla^2 f(x_k))^{-1}\nabla f(x_k).$$

Then, prove that starting from an initial point (x_0, y_0) with $x_0 \neq 0$, Newton's method does not converge to the global minimizer.

(c) Consider minimizing f subject to the constraint $\Omega = \{(x, y) : x + y = 1\}$. Write out one iteration of the projected gradient descent method (without line search) starting from (x_k, y_k) with step size α_k.

4. Let $L = \{x \in \mathbb{R}^n : x = p + tv,\ t \in \mathbb{R}\}$ be a line through a point $p \in \mathbb{R}^n$ with direction vector $v \neq 0$, and let $B = \{x \in \mathbb{R}^n : \|x - c\|_2 \leq r\}$ be a closed ball of radius $r > 0$ centered at $c \in \mathbb{R}^n$.

Define the set $C = L \cap B$. Given a point $x \in \mathbb{R}^n$, we want compute the projection

$$\mathcal{P}_C(x) = \arg\min_{y \in C} \|y - x\|_2.$$

(a) Show that the projection onto L can be expressed as

$$\mathcal{P}_L(x) = p + \frac{v^\top (x - p)}{\|v\|^2} v.$$

(b) Explain that if the projection of x onto L lies in B, i.e. $\mathcal{P}_L(x) \in B$, then $\mathcal{P}_C(x) = \mathcal{P}_L(x)$.