

Problem Set 7

Due dates: Electronic submission of the pdf file of this homework is due on **3/21/2025 before 11:59pm** on canvas. The homework must be typeset with LaTeX to receive any credit. All answers must be formulated in your own words.

Watch out for additional material that will appear on Thursday! Deadline is on Friday, as usual.

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Resources. (All people, books, articles, web pages, etc. that have been consulted when producing your answers to this homework)

Introduction to algorithms 4th-editions by Thomas C, Charles L, Ronald L, Clifford S

Geeksforgeeks.org

CSCE221 Notes (Graphs, Heaps, Dijkstra Kruskal)

CSCE110 Algorithms Notes

Class Videos

Author: Eric Matthes:

Python Crash Course, 3rd Edition: A Hands-On, Project-Based Introduction to Programming

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to prepare this homework.

Signature: Chayce Leonard _____

Read the chapters on “Elementary Graph Algorithms” and “Single-Source Shortest Paths” in our textbook before attempting to answer these questions.

Problem 1 (20 points). Give an algorithm that determines whether or not a given undirected graph $G = (V, E)$ contains a cycle. Your algorithm should run in $O(V)$ time, independent of the number $|E|$ of edges.

Solution. Algorithm Description

We present a concise algorithm for detecting cycles in an undirected graph using depth-first search (DFS). The algorithm runs in $O(V)$ time for sparse graphs where $E = O(V)$.

Algorithm 1 Cycle Detection in Undirected Graph

```
1: function HASCYCLE( $G = (V, E)$ )
2:    $\text{visited} \leftarrow \text{set}()$ 
3:   for  $v \in V$  do
4:     if  $v \notin \text{visited}$  and  $\text{DFSCycle}(G, v, \text{None}, \text{visited})$  then return True
5:     end if
6:   end for return False
7: end function
8:
9: function DFSCYCLE( $G, v, \text{parent}, \text{visited}$ )
10:   $\text{visited.add}(v)$ 
11:  for  $u \in \text{neighbors}(v)$  do
12:    if  $u \notin \text{visited}$  then
13:      if  $\text{DFSCycle}(G, u, v, \text{visited})$  then return True
14:      end if
15:    else if  $u \neq \text{parent}$  then return True
16:    end if
17:  end for return False
18: end function
```

Key Properties

- **Correctness:** The algorithm detects a cycle if and only if it encounters a visited vertex that is not the parent of the current vertex during DFS.
- **Time Complexity:** $O(V + E)$, which is $O(V)$ for sparse graphs where $E = O(V)$.
- **Space Complexity:** $O(V)$ for the visited set and recursion stack.

Proof Sketch

Proof. The algorithm is correct because:

1. It explores all vertices and edges in the graph.
2. A cycle is detected when a path leads back to a visited vertex through a non-parent edge.
3. If no such path exists, the graph is acyclic.

The time complexity is $O(V + E)$ as each vertex and edge is visited at most once. For sparse graphs, this reduces to $O(V)$. \square

Problem 2 (20 points). Given a weighted, directed graph $G = (V, E)$ with no negative-weight cycles, let m be the maximum over all vertices $v \in V$ of the minimum number of edges in a shortest path from the source s to v . (Here, the shortest path is by weight, not the number of edges.) Suggest a simple change to the Bellman-Ford algorithm that allows it to terminate in $m + 1$ passes, even if m is not known in advance.

Solution. Modified Bellman-Ford Algorithm:

```

1: function MODIFIEDBELLMANFORD( $G(V, E), s$ )
2:    $dist[v] \leftarrow \infty$  for all  $v \in V$ 
3:    $dist[s] \leftarrow 0$ 
4:   for  $i \leftarrow 1$  to  $|V| - 1$  do
5:      $changes \leftarrow 0$ 
6:     for each edge  $(u, v, w) \in E$  do
7:       if  $dist[u] + w < dist[v]$  then
8:          $dist[v] \leftarrow dist[u] + w$ 
9:          $changes \leftarrow changes + 1$ 
10:      end if
11:    end for
12:    if  $changes = 0$  then return  $dist$ 
13:    end if
14:  end for
15:  for each edge  $(u, v, w) \in E$  do
16:    if  $dist[u] + w < dist[v]$  then return "Negative cycle detected"
17:    end if
18:  end for return  $dist$ 
19: end function

```

Proof of Correctness: The modification maintains the correctness of the original Bellman-Ford algorithm:

- If no changes occur in a pass, all shortest paths have been found.
- Early termination only happens when no further improvements are possible.
- The algorithm still performs up to $|V|$ passes, ensuring negative cycle detection.

Complexity Analysis:

- Time Complexity:
 - Worst-case: $O(|V||E|)$ - when changes occur in every pass
 - Best-case: $O(|E|)$ - when algorithm terminates after first pass
 - Average-case: $O(k|E|)$, where k is the number of passes before termination ($1 \leq k < |V|$)
- Space Complexity: $O(|V|)$ for distance array

The modification allows the algorithm to terminate in $m + 1$ passes, where m is the maximum number of edges in the shortest path from the source to any vertex. This is because:

- After m passes, all shortest paths are guaranteed to be found.
- The $(m + 1)$ -th pass will have no changes, triggering early termination.

This early termination occurs naturally without prior knowledge of m , satisfying the problem requirement.

Problem 3 (20 points). Suppose that we change line 6 of Dijkstra's algorithm in our textbook to the following.

6 **while** $|Q| > 1$

This change causes the while loop to execute $|V| - 1$ times instead of $|V|$ times. Is this proposed algorithm correct? Explain. [Use the version of Dijkstra's algorithm from the textbook]

Solution. To determine if changing line 6 of Dijkstra's algorithm from **while** $|Q| > 0$ to **while** $|Q| > 1$ affects correctness, I'll analyze the algorithm's behavior with this modification.

The original algorithm from the textbook is:

```

1: function DIJKSTRA( $G, w, s$ )
2:   INITIALIZE-SINGLE-SOURCE( $G, s$ )
3:    $S \leftarrow \emptyset$ 
4:    $Q \leftarrow V[G]$  ( $\equiv \emptyset \dots$ initially)
5:   for each vertex  $u \in V[G]$  do
6:     INSERT( $Q, u$ )
7:   end for
8:   while  $|Q| > 0$  ( $\equiv \neq 0$ ) do
9:      $u \leftarrow \text{EXTRACT-MIN}(Q)$ 
10:     $S \leftarrow S \cup \{u\}$ 
11:    for each vertex  $v \in \text{Adj}[u]$  do
12:      RELAX( $u, v, w$ )
13:      if the call of RELAX decreased  $v.d$  then
14:        DECREASE-KEY( $Q, v, v.d$ )
15:      end if
16:    end for
```

17: **end while**
 18: **end function**

With the modification, the algorithm will terminate when there's exactly one vertex remaining in Q . Let's call this remaining vertex u_{last} .

Claim: The modified algorithm is correct.

Proof:

When the algorithm terminates with $|Q| = 1$, exactly one vertex (u_{last}) remains unprocessed. There are two possible cases for this remaining vertex:

Case 1: u_{last} is reachable from the source s .

Dijkstra's algorithm always extracts the vertex with minimum distance estimate. If u_{last} remains in Q while all other vertices have been processed, then u_{last} must have the largest distance estimate among all reachable vertices.

Since all other vertices with shorter distances have been processed, all edges leading to u_{last} from vertices with shorter distances have been relaxed. By the properties of Dijkstra's algorithm and the non-negative edge weight requirement, these relaxations have already correctly determined the shortest path to u_{last} .

Therefore, processing u_{last} would not change its distance estimate or any other vertex's estimate. Its shortest path is already correctly computed.

Case 2: u_{last} is unreachable from the source s .

If u_{last} is unreachable, its distance estimate remains ∞ throughout the algorithm. Processing it would not relax any edges (since relaxations only occur when the source vertex has a finite distance). Therefore, skipping its processing doesn't affect correctness.

Edge Case: Single-vertex graph

In the case of a single-vertex graph, the original algorithm processes the vertex and terminates. The modified algorithm terminates immediately without processing any vertex, as $|Q| = 1$ from the start. However, INITIALIZE-SINGLE-SOURCE already sets the correct distance (0) for the source vertex, so the modified algorithm still produces the correct result.

Conclusion:

The proposed modification to Dijkstra's algorithm is correct. The modification only skips the processing of the very last vertex in the priority queue, which either:

- Already has its correct final distance computed (if reachable), or
- Is unreachable from the source (distance = ∞)

In both cases, processing this last vertex would not change any distance values in the graph. All shortest paths from the source to all vertices will be correctly computed with the $|Q| > 1$ termination condition.

The theoretical running time remains asymptotically unchanged: the algorithm performs $|V| - 1$ iterations instead of $|V|$, which is still $O(|V|)$ iterations, keeping the overall time complexity the same.

Problem 4 (40 points). Help Professor Charlie Eppes find the most likely escape routes of thieves that robbed a bookstore on Texas Avenue in College

Station. The map will be published on Thursday evening. In preparation, you might want to implement Dijkstra's single-source shortest path algorithm, so that you can join the manhunt on Thursday evening. Include your implementation of Dijkstra's algorithm and explain all details of your choice of the min-priority queue.

[Edge weight 1 means very desirable street, weight 2 means less desirable street]

Solution. For this assignment, we implemented Dijkstra's single-source shortest path algorithm to help Professor Charlie Eppes identify the most likely escape routes of thieves who robbed a bookstore on Texas Avenue in College Station. The robbers were last seen at waypoint 1, and our task was to determine the most likely destination among waypoints 6, 8, 9, 15, 16, and 22. We used a graph representation where edge weights of 1 indicate very desirable streets and weights of 2 indicate less desirable streets.

0.1 Graph Construction

The graph was constructed using the `igraph` library, with 22 vertices representing waypoints and edges representing road connections between them. Edge weights were assigned based on traffic conditions:

- Green road segments: weight = 1 (very desirable)
- Orange/yellow road segments: weight = 2 (less desirable)

0.2 Dijkstra's Algorithm Implementation

Our implementation of Dijkstra's algorithm uses a binary min-heap priority queue through Python's `heapq` module. The algorithm follows these steps:

1. Initialize distances to all vertices as infinity, except the source (waypoint 1) with distance 0
2. Initialize a priority queue with the source vertex and its distance
3. Until the queue is empty or all destinations are found:
 - (a) Extract the vertex with minimum distance from the queue
 - (b) For each unprocessed neighbor, calculate a potential new distance
 - (c) If the new distance is shorter, update the distance and add the neighbor to the queue
4. Reconstruct paths using the previous vertex record

0.3 Min-Priority Queue Details

We chose to implement the min-priority queue using Python's `heapq` module for the following reasons:

- Efficiency: It provides $O(\log n)$ time complexity for both insertion and extraction operations
- Simplicity: The implementation is straightforward and built into Python's standard library
- Storage format: Queue elements are (distance, vertex) tuples which naturally maintain the priority ordering

The binary heap structure ensures that the vertex with the minimum distance is always at the root, making it optimal for Dijkstra's algorithm where we repeatedly need to extract the minimum-distance vertex.

0.4 Results and Analysis

After running our implementation, we determined:

- The most likely escape destination is waypoint 16 with a total distance of 3
- The escape path is: $1 \rightarrow 11 \rightarrow 17 \rightarrow 16$
- Distances to other potential destinations range from 4 to 7

We verified our results using `igraph`'s built-in shortest path function, which confirmed our implementation's accuracy.

The full script and output are attached after the main part of the submission

0.5 Conclusion

Our implementation of Dijkstra's algorithm successfully identified the most likely escape routes for the robbers. The use of a binary heap priority queue provided an efficient solution with $O((V + E) \log V)$ time complexity, which is appropriate for the size of the road network in this problem.

Make sure that you derive the solutions to this homework by yourself without any outside help. Searching for solutions on the internet or asking any form of AI is not allowed. Write the solutions in your own words. Use version control for your program development and be prepared to show and explain any version of your code.

Checklist:

- ☐ Did you add your name?
- ☐ Did you disclose all resources that you have used?
(This includes all people, books, websites, etc. that you have consulted)
- ☐ Did you sign that you followed the Aggie honor code?
- ☐ Did you solve all problems?
- ☐ Did you typeset your answers entirely in LaTeX?
- ☐ Did you submit the pdf file of your homework?