# Gödel's Incompleteness Theorem

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2012-02-08

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- How is this statement about the natural numbers?

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  - The only axiom is MI.

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  - Answer: count the number of `I`s in a theorem, modulo 3.
  - This is not generalisable.

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- But incompleteness ruins this hope.

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  - $1\text{«}^{m+n+3} + x\text{«}^{m+3} + 222\text{«}^m + y \rightarrow$
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$$w\text{«}^{l+m+n+3} + (\text{«}^{l+m+n+2} + x\text{«}^{m+n+2} + \wedge\text{«}^{m+n+1} + y\text{«}^{n+1} + )\text{«}^{n} + z$$

$$\to$$

$$w\text{«}^{l+m+n+6} + \neg(\neg\text{«}^{l+m+n+3} + x\text{«}^{m+n+3} + \vee\neg\text{«}^{m+n+1} + y\text{«}^{n+1} + )\text{«}^{n} + z$$

where $z < 1\text{«}^{n}$, $y < 1\text{«}^{m}$, $x < 1\text{«}^{l}$.

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  - To be precise, iff $x$ contains only the decimal digits 1, 2, 3.

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- Finite sequences are countable.
  - Can represent a sequence by a number, and use another number to extract elements from it.
- Instead of finding a number that satisfies P, just need to ask "does $x$ satisfy P?" Then use $\exists$.

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$$\exists b(\text{LOG}\{a, b\} \wedge$$
$$\exists(\text{sequence } x_1 \text{ to } x_b)\forall c(c < b \Rightarrow \text{NTH}\{a, c, x_c\} \wedge \text{GOOD}\{x_c\}))$$

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  NT-THEOREM$\{a\} \to \exists b($NT-DERIVES$\{b, a\})$.
- Note that constructing a statement is much easier than determining its truth.

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  "yields falsehood when preceeded by its quotation." yields falsehood when preceeded by its quotation.
  - This turns out to be the key.

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- Construct a formula QUINE$\{a, b\}$ which tests whether $b$ is "$a$ quined".
  - Essentially, QUINE$\{a, b\} \rightarrow b = a\{a\}$

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- Thus, $G$ says “$G$ is not provable.”

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- If $G$ is true, it can't be proved, so number theory is incomplete.
- Since $\neg G$ says "$G$ is provable", $\neg G$ asserts its own negation.
- So neither $G$ nor $\neg G$ is a theorem.