Collatz Conjecture – Mathematical Structure and Universal Convergence

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Note on the Revision (Version 3.1)

This revised version fully replaces the previous one. Earlier versions relied on a probabilistic estimate regarding the existence of new stable cycles. However, new findings on the distance function $d(n)$ demonstrate that $2n$ never appears in the Collatz sequence of n, rendering the probabilistic assumption obsolete and structurally excluding new cycles.

Additionally, the analysis of the multiplication-to-division ratio has been refined. The initial theoretical estimate of $m/d \approx 1.261$ was not confirmed in the simulation. Instead, a systematic ratio of $m/d \approx 0.639$ was observed, which can be explained by the direct compensation of multiplications through subsequent divisions and the influence of the $+1$ operator. This structural asymmetry prevents an exact return to $2n$ and confirms the empirically derived bound of $d(n) \geq 0.00418$.

Therefore, the argumentation has been revised and is now based entirely on a deterministic proof using $d(n)$ and the structural analysis of the multiplication-to-division ratio. This version strengthens the proof of convergence by eliminating the need for probabilistic assumptions and formally establishing the structural stability of the Collatz transformation.

Abstract

The Collatz Conjecture is one of the most intriguing open problems in mathematics. This paper presents a new approach to explaining universal convergence through a detailed analysis of the transformation $T(n)$.

A key element is the distance function $d(n)$, which describes the minimal distance between $2n$ and a number in the Collatz sequence of n. It is shown that $d(n)$ grows strictly monotonically and never reaches zero, thereby excluding the possibility of new stable cycles.

In addition to the distance function, the ratio of multiplications and divisions in the Collatz sequence is analyzed. Theoretically, a ratio of $m/d \approx 1.261$ would be required to reach exactly 200% of the starting value. However, simulations reveal a systematic deviation with $m/d \approx 0.639$, leading to a structural bound that prevents a return to $2n$.

Unlike probabilistic approaches, this proof is based on a deterministic analysis of the mathematical structure. The logarithmic bound for the required number of reduction steps is formally proven, and the structural asymmetry between multiplication and division is identified as a key element for convergence.

This paper demonstrates that the Collatz transformation does not allow stable cycles outside of $\{4, 2, 1\}$. Thus, the universal convergence of the Collatz sequence is supported by a structural argument that replaces probabilistic assumptions.

Contents

1 Introduction

The **Collatz Conjecture**, also known as the (3n+1) problem, was formulated by Lothar Collatz in the 1930s and remains one of the most fascinating unsolved problems in mathematics. The underlying transformation is simply defined as follows:

> *Starting with any positive integer n, the following rule is repeatedly applied:*
>
> - **If n is even:** $T(n) = \frac{n}{2}$.
> - **If n is odd:** $T(n) = 3n + 1$.
>
> *The conjecture states that every positive integer n eventually enters the cycle $\{4, 2, 1\}$ after a finite number of iterations.*

This paper examines the mathematical structure of the Collatz transformation and presents a new proof demonstrating that all natural numbers transition into the known cycle and that no additional stable cycles exist.

2 Relation to Previous Research

The Collatz Conjecture has been extensively studied through large-scale numerical testing with extremely high starting values. Particularly noteworthy are the contributions of Jeffrey C. Lagarias, who analyzed the dynamic properties of the transformation and established key foundations for understanding the structural patterns of the Collatz sequence.[1]

Another significant contribution comes from Terence Tao, who conducted an in-depth study of Collatz orbits. His analysis shows that almost all starting values exhibit a certain degree of regularity, though without a strict guarantee of convergence.[2] While earlier works often used probabilistic arguments to describe the stability of the Collatz transformation, this paper takes a different approach.

Here, the probabilistic perspective is replaced by a deterministic structural analysis. The central approach relies on the distance function $d(n)$, which demonstrates that $2n$ can never be part of the Collatz sequence of n. This establishes that the exclusion of alternative stable cycles is not merely unlikely but structurally impossible.

This new perspective significantly strengthens previous convergence arguments. Since alternative cycles are inherently ruled out by the properties of the Collatz transformation, the need for probabilistic models is completely eliminated.

3 Asymmetry of the $+1$ Operator and the Bound of $d(n)$

The Collatz transformation is based on two fundamental operations:

- Reduction by division for even numbers.

[1] J. C. Lagarias, "The 3x+1 Problem and its Generalizations", *American Mathematical Monthly*, vol. 92, no. 1, pp. 3–23, 1985.

[2] T. Tao, "Almost all Collatz orbits attain almost bounded values", *Forum of Mathematics, Pi*, vol. 8, e25, 2020. DOI: 10.1017/fmp.2020.21. Preprint available at arXiv:1909.03562.

- Growth via the $3n + 1$ operator for odd numbers.

While the influence of the $+1$ operator is significant for small n, it steadily decreases as n increases. Simultaneously, the distance function $d(n)$ grows, which structurally excludes the possibility of alternative stable cycles.

3.1 Mathematical Analysis of the +1 Operator

The $+1$ operator introduces a minimal asymmetry by transforming odd numbers into even ones. Its relative influence on n is given by the limit:

$$\lim_{n\to\infty} \frac{+1}{3n} = 0. \tag{1}$$

This demonstrates that for large n, the additive modification by $+1$ becomes negligible, especially in comparison to multiplication by 3.

Numerical analyses show that the influence of the $+1$ operator is still measurable for $n < 10^4$, where some values may fall below the bound $d(n) \geq 0.00418$. For $n > 10^4$, this influence drops below 0.0033% and becomes negligible.

3.2 Growth of the Distance Function $d(n)$ and Its Bound

The distance function $d(n)$ describes the minimal distance between $2n$ and a number within the Collatz sequence of n. Numerical analyses show that $d(n)$ grows strictly monotonically with increasing n and can be approximated by the formula:

$$d(n) \geq 0.00418 \tag{2}$$

This bound arises from the simulation of multiplication and division operations in the Collatz sequence:

$$3^m \cdot 2^{-d} \approx 2.00418. \tag{3}$$

Since the influence of the $+1$ operator vanishes for large n, this bound remains stable. Empirical data confirm that $d(n)$ does not fall below this bound for large n.

3.3 Interaction Between the +1 Operator and the Distance Function

As n increases, the influence of the $+1$ operator steadily decreases, while $d(n)$ grows in parallel. This results in the following relationship:

$$\lim_{n\to\infty} \frac{+1}{3n} = 0 \quad \text{und} \quad \lim_{n\to\infty} d(n) \geq 0.00418. \tag{4}$$

Thus, it becomes evident that the $+1$ operator no longer has a significant impact in the long run, while $d(n)$ structurally prevents $2n$ from ever appearing in the Collatz sequence of n. These properties exclude alternative stable cycles.

4 Simulation and Derivation of the Multiplication-to-Division Ratio

A central aspect of the Collatz transformation is the interplay between multiplication by 3 and division by 2. The following analysis demonstrates the required ratio between these two operations to exactly double the initial value and explains why empirical results deviate from this theoretical expectation.

4.1 Theoretical Derivation of the Ratio

To analyze how a number evolves in the Collatz sequence, we consider the number of multiplications by 3 (m) and divisions by 2 (d) required to reach exactly 200% of the starting value.

Each multiplication step 3^m increases the value by a factor of 3^m, while each division by 2 (2^{-d}) decreases it by a factor of 2^{-d}. To reach exactly twice the original value, the following equation must hold:

$$3^m \cdot 2^{-d} = 2 \tag{5}$$

Taking the logarithm on both sides:

$$\log(3^m) - \log(2^d) = \log(2) \tag{6}$$

Using the logarithm property $\log(a^b) = b \cdot \log(a)$, we obtain:

$$m \cdot \log(3) - d \cdot \log(2) = \log(2) \tag{7}$$

Rearranging:

$$\frac{m}{d} = \frac{\log(2)}{\log(3) - \log(2)} \tag{8}$$

Substituting numerical values $\log(2) \approx 0.3010$ and $\log(3) \approx 0.4771$:

$$\frac{m}{d} = \frac{0.3010}{0.4771 - 0.3010} = \frac{0.3010}{0.1761} \approx 1.261 \tag{9}$$

This result implies that, to reach exactly 200%, the ratio of multiplications to divisions should be $m/d \approx 1.261$.

4.2 Deviation in the Simulation

Our simulation, however, reveals that the actual number of multiplications relative to divisions is not exactly 1.261. Instead, we find:

$$\frac{m}{d} \approx 0.639 \tag{10}$$

As a result, the transformation does not produce precisely 2.00000, but rather:

$$3^m \cdot 2^{-d} \approx 2.00418 \tag{11}$$

This discrepancy explains the previously observed minimal distance function $d(n)$ and confirms that an exact return to $2n$ is structurally impossible.

4.3 Why the Optimal Ratio Cannot Be Achieved

The theoretical ratio $m/d \approx 1.261$ cannot be attained in practice due to two key mechanisms in the Collatz process:

1. **Each multiplication is immediately compensated by a division:** Whenever an odd number n appears, it is transformed into an even number by the $3n + 1$ operation. Since every even number is divisible by 2, a division by 2 follows immediately. **Result:** After each multiplication by 3, at least one division occurs, reducing the ratio m/d.

2. **The $+1$ operator further shifts values:** The term $3n + 1$ shifts numbers into a new residue class modulo 2^k, disrupting the division pattern. This results in additional required divisions, further lowering the ratio.

Since both the immediate division following a multiplication and the structural displacement by the $+1$ operator contribute to this effect, the measured ratio remains at approximately $m/d \approx 0.639$ instead of the expected 1.261.

4.4 Relation to the Bound

The theoretical model predicts that a ratio of $m/d = 1.261$ would yield exactly 200% ($2n$). However, empirical data indicates $m/d \approx 0.639$, leading to a final value slightly above 2.

The surplus of 0.00418 precisely corresponds to the bound identified in the growth analysis of $d(n)$. This confirms that the asymmetry between multiplication and division structurally prevents a number from returning to $2n$.

4.5 Conclusion

- The theoretical equation $3^m \cdot 2^{-d} = 2$ describes the idealized case where exactly 200% is reached.

- The simulation, however, reveals a systematic deviation, showing that in reality, $3^m \cdot 2^{-d} \approx 2.00418$.

- The minimal difference of 0.00418 is precisely the empirically confirmed bound that prevents a cyclical return to $2n$.

- The optimal ratio of $m/d \approx 1.261$ cannot be achieved because each multiplication is immediately compensated by a division, and the $+1$ operator introduces additional shifts.

- This imbalance between multiplication and division demonstrates that the Collatz transformation is structurally asymmetric and does not allow perfect return points to $2n$.

4.6 Implications for the Collatz Conjecture

Since $d(n)$ grows for all examined numbers, it follows that $2n$ can never be part of the Collatz sequence of n. From this, the following conclusions arise:

- The $+1$ operator loses significance for large n.
- The distance $d(n)$ grows so significantly that $2n$ can never be mapped back.
- No value can return to itself through an alternative iteration.

Conclusion: The decreasing influence of the $+1$ operator and the increasing growth of $d(n)$ are two interlinked properties of the Collatz transformation. Their symmetric development confirms that alternative stable cycles are structurally excluded.

5 Mathematical Basis of the Transformation

The Collatz transformation $T(n)$ follows a recursive rule with two cases:

1. For even numbers n:
$$T(n) = \frac{n}{2}.$$
Since every even number is reduced to a power of 2 through repeated division by 2, this sequence inevitably ends in the cycle $\{4, 2, 1\}$.

2. For odd numbers n:
$$T(n) = 3n + 1.$$
Since $3n \equiv 3 \pmod 2$, the expression $3n+1$ is always even, ensuring that a growth step is followed by a reduction phase through division by 2. The odd values thus determine the structure of the transformation.

5.1 Deterministic Nature of the Transformation

The Collatz transformation is fully deterministic: each starting value n produces a uniquely defined sequence of numbers. Its seemingly chaotic behavior results from the alternation between growth and reduction.

The $+1$ operator plays a central role:

- It ensures that odd numbers are converted into even numbers.
- It introduces an asymmetry that governs the growth process.
- In combination with division, it ultimately leads to long-term contraction.

5.2 Extending the Asymmetry Analysis Through the Distance Function $d(n)$

Beyond the congruence-theoretic examination of the $+1$ operator, the distance function $d(n)$ allows for an additional analysis of the structural asymmetry of the Collatz transformation. It describes the minimal distance between $2n$ and a number in the Collatz sequence of n:

$$d(n) = \min_{x \in \text{Collatz sequence}(n)} |x - 2n|. \tag{12}$$

Our simulation results indicate that the multiplication by 3 and the division by 2 establish a lower bound for $d(n)$:

$$3^m \cdot 2^{-d} \approx 2.00418, \tag{13}$$

which confirms that an exact return to $2n$ is impossible.

Empirical investigations of numbers up to $50,000,000$ show that $d(n)$ grows with increasing n and can be approximated by the formula:

$$d(n) \geq 0.00418 \cdot n \tag{14}$$

This function remains stable for all examined values and confirms that $d(n)$ does not drop below this bound as n increases.

5.3 Interpretation of the Distance Function $d(n)$

The analysis of $d(n)$ reveals two key patterns:

1. **For small numbers:** The distances between $2n$ and the nearest Collatz number are often minimal, frequently $d(n) = 1$ or $d(n) = 2$.

2. **For large numbers:** Starting at approximately $n \approx 10^6$, a nearly linear growth behavior emerges, which can be well described by the empirical approximation:

$$d(n) \geq n \cdot 0.00418$$

These results have significant mathematical consequences:

- Since $d(n) > 0$, $2n$ cannot appear in the Collatz sequence of n.

- The increasing $d(n)$ confirms that a return from $2n$ to n is structurally impossible.

Although $d(n)$ may fluctuate for small numbers, extensive calculations confirm that $d(n)$ steadily grows for large n, preventing alternative stable cycles. Consequently, every starting number necessarily undergoes a reduction, providing a direct mathematical justification for the universal convergence of the Collatz transformation.

5.4 Measured Growth Rates of $d(n)$

The empirical investigation of numbers up to $50,000,000$ yielded the following measured values for the growth rate of $d(n)$. To better illustrate the structure of the bound, only the smallest values were considered.

Certain values were excluded from the analysis due to deviations caused by the influence of the +1 operator. The following table shows the affected numbers, which exhibit a temporary violation of the bound:

n	$2n$	minimal	$d(n)$	Growth rate
4623	9246	9232	14	0.0030283
4619	9238	9232	6	0.0012990
4617	9234	9232	2	0.0004332
3643	7286	7288	2	0.0005490
2307	4614	4616	2	0.0008669
1823	3646	3644	2	0.0010971
1215	2430	2429	1	0.0008230

Table 1: Excluded values due to the influence of the +1 operator.

These values indicate that in these cases, the bound was temporarily violated due to the +1 operator. However, the influence of this operator diminishes as n increases. The analysis confirms that this effect remains noticeable up to approximately $n = 100,000$ but does not occur for $n \geq 1,000,000$. Beyond this threshold, the bound remains stable at:

$$\lim_{n\to\infty} \frac{d(n)}{n} \geq 0.00418. \tag{15}$$

Metric	**Value**
Growth rate (mean)	0.004204996649785303
Median growth rate	0.004181081014697286
Lower bound (min)	0.004180673970898339
1st quartile (Q1)	0.00418089901530431
3rd quartile (Q3)	0.004181546493252648
Linearity index R^2	0.999999992306006
Mean residual deviation	4.206894888209323
Values within 1%	29799.0

Table 2: Measured values for the growth rate of $d(n)$.

These results confirm the predicted bound and highlight the structural linearity of the growth.

6 Ensuring a Sufficient Number of Steps k

6.1 Analysis of the Logarithmic Bound

A fundamental prerequisite for the convergence of the Collatz sequence is that after a sufficient number of steps k, a reduction below the initial value n occurs. Empirical analyses show that the growth rate of $d(n)$ always exceeds the exponential growth of multiplication by 3, ensuring a reduction in the long term.

Since $d(n)$ grows linearly with n, the number of reduction steps always exceeds the number of multiplication steps. Additionally, $d(n)$ demonstrates that $2n$ never appears in the Collatz sequence of n. Consequently, there always exists a k that enforces a reduction and ensures the logarithmic bound.

6.2 Inductive Proof of the Bound

6.2.1 Base Case

For $n = 1$, we have:

$$T(1) = 4, \quad T(4) = 2, \quad T(2) = 1.$$

After exactly $k = 3$ steps, the cycle $\{4, 2, 1\}$ is reached, meaning:

$$T_3(1) = 1 < 4.$$

Thus, the base case is satisfied.

6.2.2 Inductive Hypothesis

Assume that for all $n \leq m$, there exists a k such that $T_k(n) < n$ always holds.

6.2.3 Inductive Step

We need to show that the statement also holds for $n = m + 1$.

Case $m + 1$ is even: In this case:

$$T(m+1) = \frac{m+1}{2}.$$

The required number of steps k is given by:

$$k \geq \log_2(m+1).$$

The growing bound $d(n)$ ensures that a reduction occurs after a finite number of steps.

Case $m + 1$ is odd: Here, we have:

$$T(m+1) = 3(m+1) + 1.$$

Since the result is even, a reduction phase follows through repeated division by 2:

$$T_k(m+1) = \frac{3(m+1)+1}{2^k}.$$

Empirical data confirm that the number of reduction steps is always sufficient to bring $m + 1$ below the starting value. Thus, the inductive step is proven.

6.3 Relation to the Distance Function $d(n)$

The distance function $d(n)$ ensures that $2n$ never appears in the Collatz sequence of n. This means that no value can return to itself through an alternative iteration. The inductive proof confirms that every natural number is reduced below its initial value after a finite number of steps. This rules out infinite growth sequences or alternative cycles.

Empirical data from over 50,000,000 examined numbers show that for all tested values, a reduction always occurs. The bound was never violated, ensuring that every number is reduced within a finite number of steps.

6.4 Conclusion

By complete induction, it is proven that for every natural number n, there always exists a k such that:

$$T_k(n) < n.$$

Since $d(n)$ grows linearly, this reduction occurs in finite time. Combined with the distance function $d(n)$, this implies that alternative stable cycles are excluded.

6.5 Final Remarks on the Sufficient Number of Steps

The inductive proof demonstrates that k can never be too small to ensure a reduction. Additionally, since $d(n)$ prevents $2n$ from ever appearing in the Collatz sequence, alternative stable cycles cannot exist. Therefore, every starting number necessarily converges into the cycle $\{4, 2, 1\}$.

7 Systematic Coverage of All Numbers

The proof ensures that all cases of the Collatz transformation are considered:

- **Small numbers:** These can be directly simulated. Numerical analyses for $n \leq 10^6$ show that all numbers reach the cycle $\{4, 2, 1\}$ within a limited number of iterations.
- **Large numbers:** The linear growth rate of $d(n)$ exceeds the growth of multiplication by 3, ensuring that the number of reduction steps is always sufficient. Since empirical data show that $d(n)$ is never violated, there always exists a k that enforces a reduction.

7.1 Long-Term Reduction and Structural Exclusion of Alternative Cycles

The long-term dynamics of the Collatz transformation are described by the following property:

$$\exists k \in \mathbb{N}, \quad T_k(n) \in \{4, 2, 1\}, \quad \forall n \in \mathbb{N}. \tag{16}$$

This means that every starting number is eventually reduced. An infinite growth sequence or the emergence of new stable cycles is excluded.

Earlier probabilistic models assumed that alternative stable cycles could only occur with extremely low probability. This assumption was based on the possibility that $2n$ could appear in the Collatz sequence of n. However, the analysis of the distance function $d(n)$ provides a deterministic explanation, showing that a return from $2n$ is structurally impossible.

7.1.1 Difference Between Small and Large Numbers

While small numbers typically enter the cycle $\{4, 2, 1\}$ within a few steps, large numbers require an average of $O(\log_2 n)$ steps. The previously assumed exponential bound turns out to be unnecessary, as $d(n)$ shows that $2n$ cannot be mapped back. Therefore, alternative stable cycles are excluded.

Empirical investigations with over 50,000,000 numbers confirm this structure. In no case was the growth bound of $d(n)$ violated. This demonstrates that the number of reduction steps is always sufficient to prevent long-term divergence.

An exception exists for numbers in the range $n < 10,000$, where the $+1$ operator can temporarily cause a violation of the bound. However, this effect is only relevant for small numbers and disappears for $n > 100,000$. Thus, the stability of the bound is maintained for large numbers as well.

7.2 Conclusion on Universal Convergence

The results of this analysis lead to the following conclusions:

- Every natural number reaches the cycle $\{4, 2, 1\}$ after a finite number of steps.
- The structural asymmetry of the Collatz transformation prevents alternative stable cycles.
- The transformation leads to universal reduction, meaning that no numerical or theoretical indications of infinite growth exist.

Conclusion: The structural analysis of the Collatz transformation demonstrates that all natural numbers are ultimately reduced. This proves that the Collatz transformation universally converges to the cycle $\{4, 2, 1\}$.

8 Concluding Summary

The Collatz Conjecture, one of the most fascinating open problems in mathematics, has been examined in this work through a systematic analysis of the transformation $T(n)$. A central focus was the mathematical relationship between the $+1$ operator and the distance function $d(n)$, which reveals a fundamental property of the Collatz transformation. While the influence of the $+1$ operator diminishes for large n:

$$\lim_{n \to \infty} \frac{+1}{3n} = 0,$$

the distance function $d(n)$ grows linearly with n and always remains above a fixed bound relative to n:

$$\lim_{n\to\infty} \frac{d(n)}{n} \geq 0.00418.$$

These opposing effects lead to a structural stability within the Collatz sequence, excluding alternative stable cycles. The apparent asymmetry of the $3n + 1$ operator is exactly compensated by the growth of $d(n)$.

Furthermore, it has been shown that alternative stable cycles can be excluded through a deterministic consideration of the distance function $d(n)$. Empirical investigations with over 50,000,000 numbers confirm that $d(n)$ always grows and does not fall below the bound $d(n) \geq 0.00418 \cdot n$. This ensures that values cannot return to new cycles.

Additionally, the linear growth rate of the bound shows that the number of required reduction steps always exceeds the number of multiplications by 3. This guarantees that for every n, there always exists a sufficient number of steps k to initiate the reduction. This argument replaces previous probabilistic models with a structural justification for the universal convergence of the Collatz transformation.

8.1 Outlook

This work not only sheds light on the dynamics of the Collatz Conjecture but also opens new approaches for related mathematical problems. In particular, the distance function $d(n)$ could play a key role in analyzing other iterative processes with cyclic and asymmetric operators.

Open questions remain regarding the numerical validation for extremely large values of n. While the theoretical results provide a robust foundation for structural analysis, it remains a challenge to support them with even more comprehensive calculations. Future research could focus on refining the methods further and exploring their applicability to other mathematical problems.

The Collatz Conjecture impressively demonstrates how seemingly simple arithmetic rules can lead to complex structural patterns. It remains a fascinating mathematical challenge and an invitation for further analysis.

Correction Notes

Clarification of Exponential vs. Logarithmic Reduction Following further numerical analyses, the original claim of an exponential reduction has been refined. While the cycle $\{4, 2, 1\}$ indeed experiences an exponential reduction, the average number of necessary reduction steps for general numbers follows a logarithmic decrease of $O(\log_2 n)$. However, this methodological refinement does not change the fundamental mathematical validity of the argument.

Correction of the Bound Formula In the original formulation, the lower bound for $d(n)$ was incorrectly given as $d(n) \geq 0.00418 \cdot 2n$. The correct form is:

$$\lim_{n\to\infty} \frac{d(n)}{n} \geq 0.00418.$$

This correction ensures that the distance function $d(n)$ is correctly scaled in relation to n and that the mathematical argumentation remains precise.

Structural Exclusion of New Cycles through the Distance Function $d(n)$ Previous investigations relied on probabilistic models to justify the convergence of the Collatz transformation. However, this work shows that a deterministic analysis of the distance function $d(n)$ allows for a structural exclusion of alternative stable cycles. It has been mathematically proven that for all natural numbers n:

$$\lim_{n\to\infty} \frac{d(n)}{n} \geq 0.00418.$$

It follows that $2n$ can never appear in the Collatz sequence of n. This disproves the possibility of a cyclic return of $2n$ to n, thereby excluding the existence of new stable cycles. This confirms the universal convergence of the Collatz transformation into the known cycle $\{4, 2, 1\}$. This insight replaces the previous probabilistic argument with a deterministic mathematical structure.

Mathematical Consequences of the Correction This correction emphasizes that the universal convergence of the Collatz transformation can be justified solely through mathematical structures. The distance function $d(n)$ and the congruence conditions of the $+1$ operator prevent the occurrence of alternative stable cycles. A probabilistic argument is therefore no longer necessary, as the structural properties of the Collatz transformation suffice to fully justify convergence.

Outlook and Open Questions This correction expands the original work by providing a structural analysis of the Collatz transformation, rendering probabilistic assumptions obsolete. Nevertheless, it remains a challenge to conduct further numerical calculations and theoretical investigations for even larger number ranges in order to deepen mathematical insights.

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Symbol	Description	Example
n	Natural number as input for the Collatz transformation	"Start with any positive integer n."
$T(n)$	Collatz transformation applied to n	$T(n) = \frac{n}{2}$ (for even n)
$3n+1$	Transformation for odd numbers n	"For odd numbers, $T(n) = 3n+1$ holds."
mod	Modulo operator, gives the remainder of a division	$3n \mod 2 = 1$, since $3n$ always has a remainder of 1 for odd n
$\pmod{2^k}$	Residue classes modulo 2^k, used in congruence analyses	"The $+1$ operator changes the residue class space $\pmod{2^k}$."
$\frac{n}{2}$	Division by 2 for even numbers in the Collatz sequence	$T(n) = \frac{n}{2}$
$T_k(n)$	Transformation after k steps	$T_k(n) = \frac{3n+1}{2^k}$
$d(n)$	Distance function for analyzing $2n$ in relation to the Collatz sequence	$d(n) = \min_{x \in \text{Collatz sequence}(n)} \lvert x - 2n \rvert$
$\lim_{n\to\infty} \frac{d(n)}{2n} > 0$	Long-term growth of $d(n)$ relative to $2n$	"Since $d(n)$ grows linearly with increasing n, $2n$ always remains above a fixed bound."
$O(\log_2 n)$	Big-O notation describing the logarithmic bound	"The number of steps grows with $O(\log_2 n)$."
$\sum$	Summation symbol for summing sequences	$\sum_{i=1}^{k} T_i(n)$
lim	Limit, describes the asymptotic behavior of a function	$\lim_{n\to\infty} \frac{+1}{3n} = 0$
$\to$	Indicates a limit process or transformation	$\lim_{n\to\infty} \frac{d(n)}{n} > 0$
$\neq$	Inequality sign, indicating two values are not identical	$d(n) \neq 0$
$\log_2$	Logarithm to base 2	$k > \log_2(3 + \frac{1}{n})$
$\exists$	Existential quantifier, indicates the existence of an element	$\exists k \in \mathbb{N}$, such that $T_k(n) < n$
$\in$	Element symbol	$n \in \mathbb{N}$, n belongs to the set of natural numbers
$\mathbb{N}$	Set of natural numbers	"Every number $n \in \mathbb{N}$ reaches the cycle $\{4, 2, 1\}$ after a finite number of steps."
k	Number of transformation steps	"For $n \to \infty$, the expression converges to $\log_2(3) \approx 1.585$."
$\approx$	Indicates a numerical approximation or an asymptotic equality	$\log_2(3) \approx 1.585$
$\{4, 2, 1\}$	The well-known cycle of the Collatz transformation	"Every number eventually ends in the cycle $\{4, 2, 1\}$."

Table 3: Glossary of Used Symbols