SSY285 - Assignment M1

Yinsong Wang Yuhong Zhou Fanze Meng

November 20, 2023

Introduction

Consider the system shown in Figure 1 which could be described as a dynamic model of DC motor with flywheel.

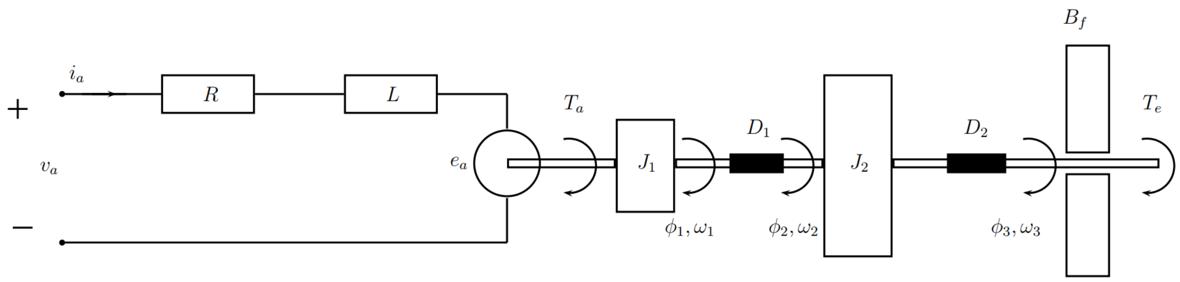

Figure 1: DC motor with flywheel

The following variables characterize the system:
v_a: external voltage applied to the rotor
i_a: the rotor current
e_a: the induced rotor voltage
T_a: the torque produced by rotor
T_e: the external torque applied to the flywheel axis
ϕ_1, ϕ_2, ϕ_3: angles of axis
$\omega_1, \omega_2, \omega_3$: angular velocities of axis

The system is also characterized by parameters:
L: the rotor inductance
R: the rotor resistance
K_E: the coefficient that relates induced voltage e_a to rotor velocity ω_1
K_T: the rotor torque constant

J_1: the rotor inertia
J_2: the flywheel inertia
D_1, D_2: the flexibility of the axis
B_f: dynamic friction proportional to ω_3

Question a)

Let's analyze the system from the electrical part of the system. Along with Kirchoff's voltage law, we can easily formulate an equation for the circuit:

$$V_a = L \cdot \frac{di_a}{dt} + K_E\omega_1 + Ri_a \tag{1}$$

Applying Newton's law to the torque of the motor, we can conduct this:

$$J_1\dot{\omega}_1 = K_T i_a - D_1(\phi_1 - \phi_2) \tag{2}$$

Applying similar procedures, we observe the equation to the next shaft:

$$J_2\dot{\omega}_2 = D_1(\phi_1 - \phi_2) - D_2(\phi_2 - \phi_3) \tag{3}$$

Finally, the last equation is obtained by analyzing the output shaft. Since there is no inertia, the following equation is determined:

$$0 = T_e - B\omega_3 - D_2(\phi_2 - \phi_3) \tag{4}$$

We can represent the angles' velocities by introducing the relationship below:

$$\dot{\phi}_1 = \omega_1 \tag{5}$$

$$\dot{\phi}_2 = \omega_2 \tag{6}$$

Combining equations 1 to 6 is the system's mathematical representation. There are 6 order-1 equations and the order of the system is 6 since there are 6 variables in the equations (which means there are 6 variables in the corresponding state-space model). The system inputs are defined: $u = [v_a, T_e]^T$.
The standard state-space model is as follows:

$$\underbrace{\begin{bmatrix} \dot{i}_a \\ \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} -\frac{R}{L} & 0 & 0 & 0 & -\frac{K_e}{L} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{D_2}{B} & -\frac{D2}{B} & 0 & 0 \\ \frac{K_T}{J_1} & -\frac{D_1}{J_1} & \frac{D_1}{J_1} & 0 & 0 & 0 \\ 0 & \frac{D_1}{J_2} & -\frac{D_1+D_2}{J_2} & \frac{D_2}{J_2} & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} i_a \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} \frac{1}{L} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u(t)} \tag{7}$$

Question b)

Assuming that the inductance is very small, $L \approx 0$, then we can easily find that equation 1 becomes algebraic, which means the variable i_a could be represented by other variables rather than differentiate itself. Therefore, we consider the state vector to be defined as:

$$x(t) = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \omega_1 & \omega_2 \end{bmatrix}^T \tag{8}$$

From equation 1, i_a could be represented as:

$$i_a = \frac{1}{R} v_a - \frac{K_E}{R} \omega_1 \tag{9}$$

Finally, we get the following state-space matrics:

$$\underbrace{\begin{bmatrix} \dot{\phi_1} \\ \dot{\phi_2} \\ \dot{\phi_3} \\ \dot{\omega_1} \\ \dot{\omega_2} \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{D_2}{B} & -\frac{D2}{B} & 0 & 0 \\ -\frac{D_1}{J_1} & \frac{D_1}{J_1} & 0 & -\frac{K_E K_T}{J_1 R} & 0 \\ \frac{D_1}{J_2} & -\frac{D_1 + D_2}{J_2} & \frac{D_2}{J_2} & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \\ \frac{K_T}{J_1 R} & 0 \\ 0 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u(t)} \tag{10}$$

Question c)

For the first case, the desired outputs are ϕ_2 and ω_2:

$$y(t) = \begin{bmatrix} \phi_2 \\ \omega_2 \end{bmatrix} \tag{11}$$

we can formulate the output equations as follows:

$$\underbrace{\begin{bmatrix} \phi_2 \\ \omega_2 \end{bmatrix}}_{y(t)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u(t)} \tag{12}$$

For the second case, the desired outputs are i_a and ω_3, which are not part of the state space model. Thus, from equations 4 and 9, the output equations are as follows:

$$\underbrace{\begin{bmatrix} i_a \\ \omega_3 \end{bmatrix}}_{y(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & -\frac{K_E}{R} & 0 \\ 0 & \frac{D_2}{B} & -\frac{D_2}{B} & 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{B} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u(t)} \tag{13}$$

Question d)

As defined previously, the eigenvalues of A matric could be easily got by following MATLAB code:

```
% give parameters value
R = 1;
K_E = 10^-1;
K_T = 10^-1;
J_1 = 10^-5;
J_2 = 4 * 10^-5;
B_f = 2 * 10^-3;
D_1 = 20;
D_2 = 2;

% define A matrics

A = [0,0,0,1,0;
    0,0,0,0,1;
    0,D_2/B_f,-D_2/B_f,0,0;
    -D_1/J_1,D_1/J_1,0,-(K_E*K_T)/(J_1*R),0;
    D_1/J_2,-(D_1+D_2)/J_2,D_2/J_2,0,0];

% calculate eigenvalues
eigenvalue = eig(A);
```

The eigenvalues results show as:

```
ans =

   1.0e+03 *

  -0.3914 + 1.4791i
  -0.3914 - 1.4791i
   0.0000 + 0.0000i
```

```
-0.2708 + 0.0000i
-0.9464 + 0.0000i
```

We notice that all the eigenvalues are located in the left-half plane, hence we can say that this system is marginally stable(one pole located on the 0). Additionally, since this is an LTI system, it is input-output stable.
The poles and zeros are found by MATLAB like the following listing:

```
% formulate G function
s = tf('s');
I = eye(size(A));
G1 = C1*inv(s*I-A)*B+D1;
G2 = C2*inv(s*I-A)*B+D2;

H_1 = zpk(G1)
[z,p] = zpkdata(H_1,'v')

H_2 = zpk(G2)
[z,p] = zpkdata(H_2,'v')
```

From the result of MATLAB, there is no pole-zero cancellation occurred.

Question e)

Since the T_e is zero, the transfer function could be reformulated as:

$$\begin{bmatrix} i_a \\ \omega_3 \end{bmatrix} = G(s) \cdot v_a \tag{14}$$

where $G(s)$ is obtained as follows:

```
      s (s+955.3) (s+41.75) (s^2 + 2.92s + 2.507e06)
  1:  -----------------------------------------------
     s (s+946.4) (s+270.8) (s^2 + 782.8s + 2.341e06)

     5e12 s^2 (s+946.4) (s+270.8) (s^2 + 782.8s + 2.341e06)
  2:  -------------------------------------------------------
     s^2 (s+946.4)^2 (s+270.8)^2 (s^2 + 782.8s + 2.341e06)^2
```

For MIMO system, the pole polynomial is the least common denominator of all manners of $G(s)$, which results in the following system poles:

```
poles = [
-391.388390756593 + 1479.08293081796i;
-391.388390756593 - 1479.08293081796i;
```

```
-391.388369234168 + 1479.08287400901i;
-391.388369234168 - 1479.08287400901i;
-946.388625317746 + 0.00000000000000i;
-946.388503620669 + 0.00000000000000i;
-270.834691223635 + 0.00000000000000i;
-270.834659856425 + 0.00000000000000i;
-1.18832085177646e-13 + 0.00000000000000i;
-1.18832303653050e-13 + 0.00000000000000i]
```

Moreover, the transmission zeros of the system are obtained by the greatest common factor for the numerators of the maximal minors of $G(s)$, when we normalize all transfer functions to the same denominator. Following these procedures, the zeros of the system are:

```
zeros =
[-1.45983894532111 + 1583.41455239705i;
-1.45983894532111 - 1583.41455239705i;
-955.330289481924 + 0.00000000000000i;
-41.7500326274334 + 0.00000000000000i;
-4.41353133228380e-13 + 0.00000000000000i]
[-391.388379995381 + 1479.08290241348i;
-391.388379995381 - 1479.08290241348i;
-946.388564469212 + 0.00000000000000i;
-270.834675540028 + 0.00000000000000i;
2.13252063965550e-12 + 0.00000000000000i;
-1.19254719655492e-13 + 0.00000000000000i]
```

Since the zeros are located in LHP, this system is considered minimum-phase.

Question f)

Along with A and B matrics defined by equation 10, we mix the output function as follows:

$$\underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}}_{y(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{D_2}{B} & -\frac{D_2}{B} & 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u(t)} \tag{15}$$

Furthermore, we could get the transfer function and then simulate the system with desired inputs.
Plotting the results as below:

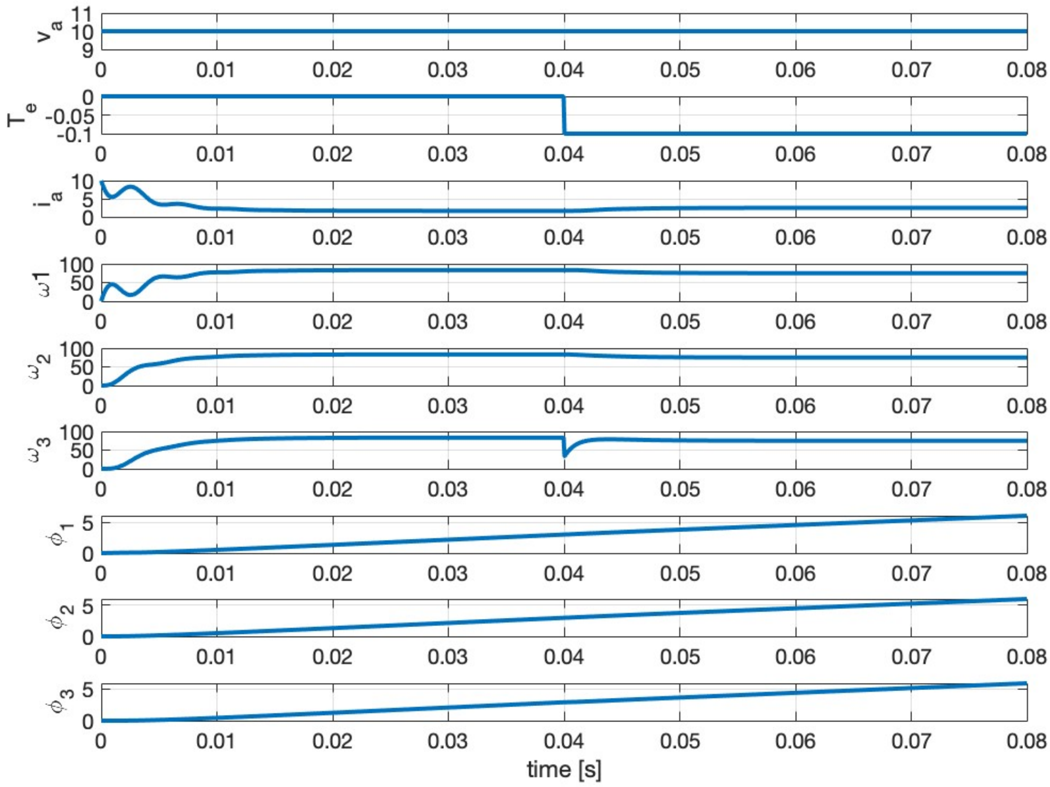

Figure 2: MIMO response for DC motor

Conclusions:

- The angular variables, ϕ_1, ϕ_2, ϕ_3, exhibit unbounded increments as evidenced by our simulation results depicted in the preceding figure. This behavior finds explanation in the physical interpretation of these variables as angles, where the unbounded increase is attributed to the continuous rotation of the shaft.

- The externally applied torque at the right end of the shaft exerts a pronounced influence on ω_3 due to its proximity. It also induces discernable effects on ω_2, ω_2, i_a with comparatively less fluctuation.