SSY285 - Assignment M2

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Introduction

Consider the system shown in Figure 1 which could be described as a dynamic model of DC motor with flywheel.

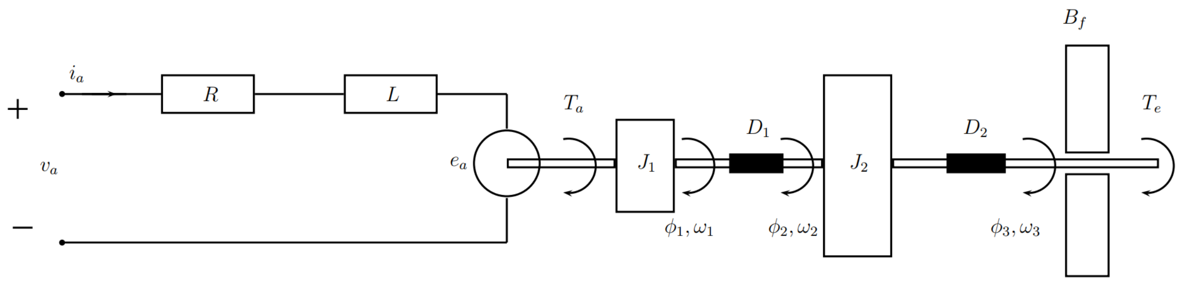

Figure 1: DC motor with flywheel

We get the following state-space matrics:

$$\underbrace{\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{D_2}{B} & -\frac{D2}{B} & 0 & 0 \\ -\frac{D_1}{J_1} & \frac{D_1}{J_1} & 0 & -\frac{K_E K_T}{J_1 R} & 0 \\ \frac{D_1}{J_2} & -\frac{D_1+D_2}{J_2} & \frac{D_2}{J_2} & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B} \\ \frac{K_T}{J_1 R} & 0 \\ 0 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u(t)} \tag{1}$$

One of the output equations is as follows:

$$\underbrace{\begin{bmatrix} \phi_2 \\ \omega_2 \end{bmatrix}}_{y(t)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C_1} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u(t)} \tag{2}$$

The other output equation is as follows:

$$\underbrace{\begin{bmatrix} i_a \\ \omega_3 \end{bmatrix}}_{y(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & -\frac{K_E}{R} & 0 \\ 0 & \frac{D_2}{B} & -\frac{D_2}{B} & 0 & 0 \end{bmatrix}}_{C_2} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \omega_1 \\ \omega_2 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{B} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} v_a \\ T_e \end{bmatrix}}_{u(t)} \tag{3}$$

Question a)

The controllability matrix is defined as:

$$S(A, B) = [B, AB, ..., A^{n-1}B] \tag{4}$$

After doing the Guass-Jordan elimination on the controllability matrix, we can easily find the following:

S(A, B)=

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1.0 \times 10^5 D_1 & 0 & \frac{1.0\times10^8 D_1}{R} & 2.5 \times 10^8 D_1 R & -\frac{1.25\times10^{10}\left(8D_1 - {D_1}^2 R^2\right)}{R^2} & -2.5 \times 10^{11} D_1 R \\ 0 & 1 & 0 & 0 & \frac{1.0\times10^4 D_1}{R} & 0 & -\frac{1.0\times10^7 D_1}{R^2} & -2.5 \times 10^7 D_1 & -\frac{1.25\times10^9 D_1\left(D_1 R^2 - 8\right)}{R^3} & 2.5 \times 10^{10} D_1 \\ 0 & 0 & 1 & 0 & -\frac{1.0\times10^3}{R} & 0 & -\frac{1.0\times10^5\left(D_1 R^2 - 10\right)}{R^2} & 0 & \frac{2.0\times10^8\left(D_1 R^2 - 5\right)}{R^3} & 2.5 \times 10^8 D_1 R \\ 0 & 0 & 0 & 1 & \frac{1.0\times10^1 D_1}{R} & 0 & \frac{1.0\times10^4 D_1 (R-1)}{R^2} & -2.5 \times 10^4 D_1 - 5.0 \times 10^4 & -\frac{2.5\times10^5 D_1\left(40R + 5D_1 R^2 + 2R^2 - 40\right)}{R^3} & 5.0 \times 10^7 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1.0\times10^1 D_1}{R} & -1.0 \times 10^3 & -\frac{1.0\times10^4 D_1}{R^2} & 9.5 \times 10^5 - 2.5 \times 10^4 D_1 \end{bmatrix} \tag{5}$$

The rank(S) is 5, meaning full rank.
As required by the question context, let $det(S) = 0$, we will find that there is no reasonable relation between R and D_1 to lead this system uncontrollably.
The following equation can find the observability matrix:

$$\mathscr{O}(A, C) = [C, CA, ..., CA^{(n-1)}]^T \tag{6}$$

which, using the same procedures, easily gets the following matrix for C_1 and C_2:

$$\mathrm{O}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 2.5\times10^4 D_1 & -2.5\times10^4 D_1 - 5\times10^4 & 5\times10^4 & 0 & 0 \\ 2.5\times10^4 D_1 & -2.5\times10^4 D_1 - 5\times10^4 & 5\times10^4 & 0 & 0 \\ 0 & 5\times10^7 & -5\times10^7 & 2.5\times10^4 D_1 & -2.5\times10^4 D_1 - 5\times10^4 \\ 0 & 5\times10^7 & -5\times10^7 & 2.5\times10^4 D_1 & -2.5\times10^4 D_1 - 5\times10^4 \\ -2.5\times10^9 D_1^2 - 2.5\times10^4 D_1(2.5\times10^4 D_1 + 5\times10^4) & (2.5\times10^4 D_1 + 5\times10^4)^2 + 2.5\times10^9 D_1^2 - 5\times10^9 & 4.75\times10^9 - 1.25\times10^9 D_1 & -\frac{2.5\times10^7 D_1}{R} & 5\times10^7 \\ -2.5\times10^9 D_1^2 - 2.5\times10^4 D_1(2.5\times10^4 D_1 + 5\times10^4) & (2.5\times10^4 D_1 + 5\times10^4)^2 + 2.5\times10^9 D_1^2 - 5\times10^9 & 4.75\times10^9 - 1.25\times10^9 D_1 & -\frac{2.5\times10^7 D_1}{R} & 5\times10^7 \\ 1.25\times10^{12} D_1 + \frac{2.5\times10^{12} D_1^2}{R} & 4.5\times10^{13} - \frac{2.5\times10^{12} D_1^2}{R} - 2.5\times10^{12} D_1 & 1.25\times10^{12} D_1 - 4.5\times10^{13} & -2.5\times10^4 D_1(2.5\times10^4 D_1 + 5\times10^4) - 2.5\times10^4 D_1(1\times10^5 D_1 - \frac{1\times10^6}{R^2}) & (2.5\times10^4 D_1 + 5\times10^4)^2 + 2.5\times10^9 D_1^2 - 5\times10^9 \end{bmatrix} \tag{7}$$

$$\mathscr{O}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$\mathrm{O}_2 = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{10\,R} & 0 \\ 0 & 1000 & -1000 & 0 & 0 \\ \frac{1\times10^4 D_1}{R} & -\frac{1\times10^4 D_1}{R} & 0 & \frac{1\times10^2}{R^2} & 0 \\ 0 & -1\times10^6 & 1\times10^6 & 0 & 1000 \\ -\frac{1\times10^7 D_1}{R^2} & \frac{1\times10^7 D_1}{R^2} & 0 & \frac{1\times10^5 D_1 - \frac{1\times10^6}{R^2}}{10\,R} & -\frac{1\times10^4 D_1}{R} \\ 2.5\times10^7 D_1 & 9.5\times10^8 - 2.5\times10^7 D_1 & -9.5\times10^8 & 0 & -1\times10^6 \\ \frac{\frac{1\times10^{11} D_1}{R^2} - 1.25\times10^{10} D_1^2}{10\,R} & \frac{1\times10^{10} D_1^2 - \frac{1\times10^{11} D_1}{R^2} + 1\times10^5 D_1\,(2.5\times10^4 D_1 + 5\times10^4)}{10\,R} & -\frac{5\times10^8 D_1}{R} & -\frac{\frac{1\times10^3\,(1\times10^5 D_1 - \frac{1\times10^6}{R^2})}{R} + \frac{1\times10^8 D_1}{R}}{10\,R} & \frac{1\times10^7 D_1}{R^2} \\ -2.5\times10^9 D_1 & 2.5\times10^9 D_1 - 9\times10^9 & 9\times10^9 & 2.5\times10^7 D_1 & 9.5\times10^8 - 2.5\times10^7 D_1 \\ \frac{\frac{1.25\times10^{13} D_1^2}{R} + \frac{1\times10^8 D_1\,(1\times10^5 D_1 - \frac{1\times10^6}{R^2})}{R}}{10\,R} & -\frac{5\times10^{12} D_1 + \frac{1\times10^{13} D_1^2}{R} + \frac{1\times10^8 D_1\,(2.5\times10^4 D_1 + 5\times10^4)}{R} + \frac{1\times10^8 D_1\,(1\times10^5 D_1 - \frac{1\times10^6}{R^2})}{R}}{10\,R} & \frac{5\times10^{12} D_1 + \frac{5\times10^{12} D_1}{R}}{10\,R} & -\frac{(1\times10^5 D_1 - \frac{1\times10^6}{R^2})^2 - \frac{1\times10^{11} D_1}{R^2} + 2.5\times10^9 D_1^2}{10\,R} & \frac{1\times10^5 D_1\,(2.5\times10^4 D_1 + 5\times10^4) + 1\times10^5 D_1\,(1\times10^5 D_1 - \frac{1\times10^6}{R^2})}{10\,R} \\ 2.5\times10^{13} D_1 - 2.5\times10^{12} D_1^2 - 2.5\times10^7 D_1\,(2.5\times10^4 D_1 + 5\times10^4) & 1\times10^3\,(2.5\times10^4 D_1 + 5\times10^4)^2 - 2.5\times10^{13} D_1 + 2.5\times10^{12} D_1^2 + 8.5\times10^{11} & -1.25\times10^{12} D_1 - 8.525\times10^{11} & -2.5\times10^{10} D_1 - \frac{2.5\times10^{10} D_1}{R} & 2.5\times10^9 D_1 - 9\times10^9 \end{bmatrix} \tag{9}$$

$$\mathscr{O}_2 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

Consequently, the $rank(\mathscr{O}_1) = 5$ and $rank(\mathscr{O}_2) = 4$. We use similar procedures: let $det(\mathscr{O}_1) = 0$, the relations between R and D_1 could be conclude that:

$$R = \frac{10}{D_1 + 10} \tag{11}$$

We can then conclude that the system is observable for C_1 with the risk of unobservable at $R = \frac{10}{D_1+10}$, but not observable for C_2, which means we can not observe all the states if we only measure i_a and ω_3.

Question b)

As the system is controllable, we don't need to check the stabilizability because controllability leads to the stabilizability of this system.

We also know from the previous question that C_1 may not be observable, therefore let's check whether the two variables meet the relationship found in question a.

We can first check the eigenvalue of A:

$$eig(A) = \begin{cases} f_1(R, D_1) \\ f_2(R, D_1) \\ f_3(R, D_1) \\ f_4(R, D_1) \\ f_5(R, D_1) \end{cases} \tag{12}$$

As the PHB test for detectability defined:

$$\forall \lambda \in \mathbb{C}, rank[\lambda \mathbf{I} - \mathbf{A}; \mathbf{C}] = n \tag{13}$$

where $\mathbb{C}$ is the set of all eigenvalues of A matrix, n is the number of arrays of $[\lambda \mathbf{I} - \mathbf{A}; \mathbf{C}]$.

When the R and D_1 meet the relation, the system still passes the stability test which means the system with C_1 is stable.

However, the system with C_2 is not observable, so the detectable should be checked. When we test the first eigenvalue of A matrix it results in a rank-deficient matrix. Therefore, the C_2 is not detectable (failed to pass the PHB test).

Question c)

Given the numerical values of the variables, we can calculate the controllability and observability using the built-in function in Matlab. Moreover, to avoid the tolerance appearing in the rank calculation in Matlab, it's more convenient to analyze the condition number to check if the state is controllable or observable.

Firstly, the singular values for each matrix are shown as follows:

```
svd1 = svd(W_cc)
svd2 = svd(W_o1c)
svd3 = svd(W_o2c)
```

$$svd(S(A,B)) = \begin{bmatrix} 1.0050e+17 \\ 7.7114e+15 \\ 2.8492e+11 \\ 1.0391e+09 \\ 9.8542e+01 \end{bmatrix} \tag{14}$$

$$svd(\mathscr{O}_1) = \begin{bmatrix} 1.4356e+15 \\ 5.1743e+09 \\ 5.2817e+07 \\ 1.8730e+02 \\ 4.6845e-01 \end{bmatrix} \tag{15}$$

$$svd(\mathscr{O}_2) = \begin{bmatrix} 2.2114e+15 \\ 4.6897e+14 \\ 9.4680e+08 \\ 7.6206e+06 \\ 4.6976e-02 \end{bmatrix} \tag{16}$$

Secondly, the condition numbers are calculated through the following function:

```
c_val1 = max(svd1(:))/ (min(svd1(:)))
c_val2 = max(svd2(:))/ (min(svd2(:)))
c_val13 = max(svd3(:))/ (min(svd3(:)))
```

And we have:

$$\kappa(Sor\mathscr{O}) = \frac{max(svd(Sor\mathscr{O}))}{min(svd(Sor\mathscr{O}))} \begin{cases} \kappa(S) = \frac{1.0050\times10^{17}}{98.542} = 1.0199\times10^{15} \\ \kappa(\mathscr{O}_1) = \frac{1.4356\times10^{15}}{0.46845} = 3.0646\times10^{15} \\ \kappa(\mathscr{O}_2) = \frac{2.2114\times10^{15}}{0.04698} = 4.7075\times10^{16} \end{cases} \tag{17}$$

Notably, the condition numbers of the controllability matrix and observability matrixes are pretty high, which means it's hard to control and hard to observe for both C_1 and C_2. Even a tiny input would lead to a huge reaction.

When we introduce the given number of R and D_1, we can easily find that:

```
rank(S_nurmeral)
```

The rank(S) is 4, which means that when we look at a specific condition(give the variables numbers), the system may lose rank(uncontrollable).

However, when we use the $rref$ method, the rank(S) results in full rank. As far as we know, this may be because of the MATLAB rank algorithm setting. At the same time, the full-rank conclusion also satisfies the relation we defined in question a. Therefore, we believe in this numerical condition, the system is controllable.

Additionally, when we test the observable matrix, both the $rank$ method and $rref$ method lead to rank 4 which means unobservable for both cases. but that does not satisfy our conclusion in question a about the loose rank relations between R and D_1. This means this result from MATLAB may not be correct.

Even though, the system still could be a stabilizable and detectable one. Let's check the eigenvalues of A matrix:

$$eig(A) = \begin{bmatrix} -391.3883 + 1479.0829i \\ -391.3883 - 1479.0829i \\ 0 \\ -270.8346 \\ -946.3885 \end{bmatrix} \tag{18}$$

Similarly, we apply the PHB test for detectability following the same procedures in question b. We can conclude that C_2 failed to pass the test which leads to undetectable. And C_1 passes the test meaning detectable for C_1.

Moreover, we apply the PHB test for stability:

$$\forall\lambda \in \mathbb{C}, rank[\lambda\mathbf{I} - \mathbf{A}; \mathbf{B}] = n \tag{19}$$

where $\mathbb{C}$ is the set of all eigenvalues of A matrix, n is the number of arrays of $[\lambda\mathbf{I} - \mathbf{A}; \mathbf{B}]$.

We got full rank of all matrixes, which leads to a stable system.

Question d)

By implementing the following Matlab code, we can easily conclude A_d:

```
Ts = 0.001;
Ad = expm(Ac*Ts);
```

$$A_d = \begin{bmatrix} 0.4010 & 0.5964 & 0.0026 & 0.0004 & 0.0002 \\ 0.2076 & 0.7749 & 0.0175 & 0.0001 & 0.0009 \\ 0.0585 & 0.5686 & 0.3729 & 0.0000 & 0.0004 \\ -782.8328 & 773.7475 & 9.0853 & -0.0489 & 0.5964 \\ 342.5299 & -370.9589 & 28.4290 & 0.1491 & 0.7749 \end{bmatrix} \tag{20}$$

Question e)

As defined by the lecture slide:

$$\mathscr{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = e^{A_c t} \tag{21}$$

where $\mathscr{L}^{-1}$ represents inverse Laplace operation. Then we can implement integration which is defined by question context:

```
syms s t;
variable = inv (s*eye(size(A))-A);
exp_At = vpa(ilaplace(variable));
Bd = int(exp_At, t, 0, 1e-3) * B
double(Bd)
```

We have:

$$B_d = \begin{bmatrix} 0.0032 & 0.0003 \\ 0.0002 & 0.0032 \\ 0.0000 & 0.3168 \\ 4.4993 & 1.3088 \\ 0.5851 & 8.7634 \end{bmatrix} \tag{22}$$

Question f)

To validate that the discrete-time state-space model is the minimal realization, we need to check whether it is both controllable and observable:

```
%Singular value decomposition of controllability matrix (discrete time)
svd1 = svd(ctrb(Ad, Bd))
%Singular value decomposition of observability matrix case 1(discrete time)
svd2 = svd(obsv(Ad, C1))
%Singular value decomposition of observability matrix case 2(discrete time)
svd3 = svd(obsv(Ad, C2))
```

and we have:

$$\text{controllability matrix} = \begin{bmatrix} 35.8544 \\ 9.1418 \\ 0.1928 \\ 0.0532 \\ 0.0038 \end{bmatrix} \tag{23}$$

$$\text{observability matrix}(C_1) = \begin{bmatrix} 598.7408 \\ 34.5908 \\ 1.2908 \\ 1.0153 \\ 0.0970 \end{bmatrix} \tag{24}$$

$$\text{observability matrix}(C_2) = 1 \times 10^3 * \begin{bmatrix} 1.4726 \\ 0.4030 \\ 0.0004 \\ 0.0002 \\ 0.0000 \end{bmatrix} \tag{25}$$

Since (A_d, B_d, C_1)'s controllability matrix and observability matrix are full rank, this case is the minimal realization.

As for (A_d, B_d, C_2)'s observability matrix is deficient, this case is not the minimal realization.

Check the stability:

```
% stability check
eigenvalue2 = abs(eig(Ad))
```

$$eigenvalue(A_d) = \begin{bmatrix} 0.6761 \\ 0.6761 \\ 1.0000 \\ 0.7627 \\ 0.3881 \end{bmatrix} \tag{26}$$

From inspection, all the eigenvalues are located within the unit circle, thus this system is marginally stable.

From continuous time to discrete time, we have different conclusions of controllability and observability even with the same original matrices. The reason is in discrete time, we have a large sampling time.