

1. Show that the inequality

$$\sum_{i=1}^n \sum_{j=1}^n \sqrt{|x_i - x_j|} \sum_{i=1}^n \sum_{j=1}^n \sqrt{|x_i + x_j|}$$

holds for all real numbers  $x_1, \dots, x_n$ .

**Solution:**

For  $a \in R^+$ ,

$$\int \frac{1 - \cos ax}{x\sqrt{x}} dx = 4\sqrt{a}S(\sqrt{ax}) + \frac{2(\cos ax - 1)}{\sqrt{x}} + C$$

Let

$$S(x) = \int_0^x \sin t^2 dt$$

then

$$S(0) = 0, \quad S(+\infty) = \sqrt{\frac{\pi}{8}}$$

Now

$$\begin{aligned} & \sqrt{|a+b|} - \sqrt{|a-b|} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \frac{\cos(a-b)x - \cos(a+b)x}{x\sqrt{x}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \frac{2 \sin ax \sin bx}{x\sqrt{x}} dx \end{aligned}$$

Rearranging,

$$\begin{aligned} & \sum_{i=1}^n \sum_{j=1}^n \sqrt{|x_i + x_j|} - \sqrt{|x_i - x_j|} \\ &= \sum_{i=1}^n \sum_{j=1}^n \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \frac{2 \sin x_i t \sin x_j t}{t\sqrt{t}} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \sum_{i=1}^n \sum_{j=1}^n \frac{2 \sin x_i t \sin x_j t}{t\sqrt{t}} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \frac{2(\sum_{i=1}^n \sin x_i t)^2}{t\sqrt{t}} dt \geq 0 \end{aligned}$$

QED