Lemma 0.1. *For any integer $m \geq 0$*

$$\sum_{n=0}^{m} \frac{2^{2^n+2n-1}}{(1+2^{2^n})^2} = \frac{2^{2^{m+2}} - 2^{2^{m+1}+2m+1} - 2^{2^{m+1}+1} + 1}{(2^{2^{m+1}}-1)^2}$$

Proof. We prove this by mathematical induction.

$$\sum_{n=0}^{m} \frac{2^{2^n+2n-1}}{(1+2^{2^n})^2}\Big|_{m=0} = \frac{1}{9} \tag{1}$$

$$\frac{2^{2^{m+2}} - 2^{2^{m+1}+2m+1} - 2^{2^{m+1}+1} + 1}{(2^{2^{m+1}}-1)^2}\Big|_{m=0} = \frac{16-8-8+1}{3^2} = \frac{1}{9} \tag{2}$$

Equation (1) and (2) prove the base case for $m = 0$. Let's assume the induction hypothesis for some m.

$$\sum_{n=0}^{m} \frac{2^{2^n+2n-1}}{(1+2^{2^n})^2} = \frac{2^{2^{m+2}} - 2^{2^{m+1}+2m+1} - 2^{2^{m+1}+1} + 1}{(2^{2^{m+1}}-1)^2} \tag{3}$$

Now the induction step for $m+1$

$$\sum_{n=0}^{m+1} \frac{2^{2^n+2n-1}}{(1+2^{2^n})^2} = \frac{2^{2^{m+1}+2m+1}}{(1+2^{2^{m+1}})^2} + \sum_{n=0}^{m} \frac{2^{2^n+2n-1}}{(1+2^{2^n})^2}$$

Using the induction hypothesis (3)

$$= \frac{2^{2^{m+1}+2m+1}}{(1+2^{2^{m+1}})^2} + \frac{2^{2^{m+2}} - 2^{2^{m+1}+2m+1} - 2^{2^{m+1}+1} + 1}{(2^{2^{m+1}}-1)^2}$$

$$= \frac{2^{2^{m+1}+2m+1} \cdot (2^{2^{m+1}}-1)^2 + (2^{2^{m+2}} - 2^{2^{m+1}+2m+1} - 2^{2^{m+1}+1} + 1) \cdot (2^{2^{m+1}})^2 + 1}{(2^{2^{m+1}})^2 + 1 \cdot (2^{2^{m+1}}-1)^2}$$

$$= \frac{2^{2^{m+3}} - 2^{2^{m+2}+2m+3} - 2^{2^{m+2}+1} + 1}{(2^{2^{m+2}}-1)^2}$$

Hence proved by induction :D

□

Corollary 0.1.1.

$$\sum_{n=0}^{\infty} \frac{2^{2^n+2n-1}}{(1+2^{2^n})^2} = 1$$

Proof.

$$\sum_{n=0}^{\infty} \frac{2^{2^n+2n-1}}{(1+2^{2^n})^2} = \sum_{n=0}^{m} \frac{2^{2^n+2n-1}}{(1+2^{2^n})^2}\Big|_{m=\infty}$$

$$= \frac{2^{2^{m+2}} - 2^{2^{m+1}+2m+1} - 2^{2^{m+1}+1} + 1}{(2^{2^{m+1}} - 1)^2}\Big|_{m=\infty}$$

$$= \frac{2^{2^{m+2}} - 2^{2^{m+1}+2m+1} - 2^{2^{m+1}+1} + 1}{2^{2^{m+2}} - 2^{2^{m+1}+1} + 1}\Big|_{m=\infty} = 1$$

$\square$