1. If $\phi : [0, +\infty) \to R$ is continuous in its domain and twice differentiable in $(0, +\infty)$ and such that $\phi(0) = 0$, is increasing, concave and eventually constant then

$$E(\phi(|X + Y|)) \geq E(\phi(|X - Y|))$$

Solution:
First we are going to prove that in the case where X, Y are compactly supported $X, Y \in [-M, M]$ and X is absolutely continuous with p.d.f. $p(x)$

$$\int\int \phi(|x + y|)p(x)p(y)dxdy \geq \int\int \phi(|x - y|)p(x)p(y)dxdy$$

Using transformations $s = x + y$ on LHS and $s = y - x$ on RHS, we now need to prove

$$\int_0^{\infty} \phi(s) \int p(x)(p(s - x) + p(-s - x) - p(s + x) - p(-s + x))dxds \geq 0$$

Let X have cumulative distribution function F we know that $F' = p$ almost everywhere. Now integrating by parts we get,

$$[\phi(s)(\int p(x)(F(s - x) - F(-s - x) - F(s + x) + F(-s + x))]_{s=0}^{\infty}$$
$$-\int_0^{\infty} \phi'(s)(\int p(x)(F(s - x) - F(-s - x) - F(s + x) + F(-s + x))dx)ds$$

The first term above is 0 because when $s > 2M$ we have either $p(x) = 0$ or $F(s - x) - F(s + x) - F(-s - x) + F(-s + x) = 0$ in the second term, so by some variable transformations, we get

$$-\int_0^{\infty} \phi'(s)(\int F(x)(p(s - x) - p(-s - x) + p(s + x) - p(-s + x))dx)ds$$

Again Integrating by parts, we get

$$[-\phi'(s)(\int F(x)(F(s-x)+F(-s-x)+F(s+x)+F(-s+x)-2F(x)-2F(-x))dx]_{s=0}^{\infty}$$

$$+\int_0^{\infty} \phi''(s)(\int F(x)(F(s-x)+F(-s-x)+F(s+x)+F(-s+x)-2F(x)-2F(-x))dxds$$

$F(x)(F(s-x)-F(-x)+F(-s-x)-F(-x)+F(s+x)-F(x)+F(-s+x)-F(x))$ is zero when $|x| > M+s$ and since $\phi'(s)=0$ eventually the first term is zero, hence

$$E(\phi(|X+Y|)) - E(\phi(|X-Y|))$$

$$= \int_0^\infty \phi''(s)(\int F(x)(F(s-x)+F(-s-x)+F(s+x)+F(-s+x)-2F(x)-2F(-x))dxds$$

Noting $\phi'' \leq 0$, and

$$F(x)(F(s-x)+F(-s-x)+F(s+x)+F(-s+x)-2F(x)-2F(-x))dx$$

$$= \int F(x)(F(s-x)-F(-x)) + F(x)(F(-s-x)-F(-x))$$

$$+F(x)(F(s+x)-F(x)) + F(x)(F(-s+x)-F(x))dx$$

$$= \int F(-x)(F(s+x)-F(x)) + F(-s-x)(F(x)-F(s+x))$$

$$+F(x)(F(s+x)-F(x)) + F(s+x)(F(x)-F(s+x)))dx$$

Sending $x \to -x-s$

$$= -\int (F(x)-F(s+x))(F(-x)-F(-s-x)+F(x)-F(s+x))dx$$

$$= -\int (F(-x)-F(-s-x))(F(-x)-F(-s-x)+F(x)-F(s+x))dx$$

$$= -\frac{1}{2}\int (F(-x)-F(-x-s)+F(x)-F(s+x))^2 dx \leq 0$$

Now, sending $M \to \infty$ and using the Dominated Convergence Theorem.

QED