1. Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i-x_j|}\le\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i+x_j|}$$

holds for all real numbers $x_1,\ldots x_n$.

Solution:
For $a\in R^+$,

$$\int\frac{1-\cos ax}{x\sqrt{x}}dx=4\sqrt{a}S(\sqrt{ax})+\frac{2(\cos ax-1)}{\sqrt{x}}+C$$

Let

$$S(x)=\int_0^x\sin t^2dt$$

then

$$S(0)=0,\quad S(+\infty)=\sqrt{\frac{\pi}{8}}$$

Now

$$\begin{aligned}&\sqrt{|a+b|}-\sqrt{|a-b|}\\=&\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\frac{\cos(a-b)x-\cos(a+b)x}{x\sqrt{x}}dx\\=&\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\frac{2\sin ax\sin bx}{x\sqrt{x}}dx\end{aligned}$$

Rearranging,

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i+x_j|}-\sqrt{|x_i-x_j|}\\=&\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\frac{2\sin x_it\sin x_jt}{t\sqrt{t}}dt\\=&\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{2\sin x_it\sin x_jt}{t\sqrt{t}}dt\\=&\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\frac{2(\sum_{i=1}^{n}\sin x_it)^2dt}{t\sqrt{t}}\ge 0\end{aligned}$$

QED