Solution for Q2(c)

$$Y = ln(X)$$

$$Let\ F(x)\ be\ the\ CDF\ of\ x$$

$$F(y) = F(ln(x))$$

$$\frac{dF(ln(x))}{dy} = \frac{dF(ln(x))}{dlnx} = xF'(x) = F'(y)$$

$$xf(x) = f(y) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(ln(x)-\mu)^2}{2\sigma^2}}$$

So we can obtain the PDF of x is $f(x) = \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(ln(x)-\mu)^2}{2\sigma^2}}$

$$E(x) = \int_0^{+\infty} xf(x)dx = \int_0^{+\infty} \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(ln(x)-\mu)^2}{2\sigma^2}}dx$$

$$Let\ t = \frac{ln(x)-\mu}{\sqrt{2}\sigma},\ then\ x = e^{\sqrt{2}\sigma t+\mu}$$

$$E(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-t^2}de^{\sqrt{2}\sigma t+\mu}$$

$$= \frac{e^{\mu}}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-t^2+\sqrt{2}\sigma t}dt$$

$$= \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(t-\frac{\sqrt{2}\sigma}{2})^2}dt$$

$$Let\ u = t - \frac{\sqrt{2}\sigma}{2},\ then\ t = u + \frac{\sqrt{2}\sigma}{2}$$

$$E(x) = \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-u^2}du$$

$$As\ \int_{-\infty}^{+\infty} e^{-u^2}du = \sqrt{\pi}$$

$$E(x) = e^{\mu+\frac{\sigma^2}{2}}$$

As $V(x) = E(x^2) - (E(x))^2$, we need to calculate $E(x^2)$:

$$E(x^2) = \int_0^{+\infty} x^2 f(x)dx = \int_0^{+\infty} \frac{x}{\sigma\sqrt{2\pi}} e^{-\frac{(ln(x)-\mu)^2}{2\sigma^2}} dx$$

$$With\ the\ similar\ substitution\ of\ t\ as\ above.$$

$$\begin{aligned} E(x^2) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2+\sqrt{2}\sigma t+\mu} de^{\sqrt{2}\sigma t+\mu} \\ &= \frac{e^{2\mu}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2+2\sqrt{2}\sigma t} dt \\ &= \frac{e^{2\mu+2\sigma^2}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(t-\sqrt{2}\sigma)^2} dt \end{aligned}$$

$$Let\ v = t - \sqrt{2}\sigma,\ then\ t = v + \sqrt{2}\sigma$$

$$E(x^2) = \frac{e^{2\mu+2\sigma^2}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-v^2} dv = e^{2\mu+2\sigma^2}$$

So, $V(x) = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$.

Number of Samples	**Mean**	**Variance**
10	1.02129	0.000517
100	1.013513	0.002671
1000	1.020112	0.002591
10000	1.021758	0.002548
100000	1.021626	0.002611
Theoretical	1.021477	0.002619

With the number of samples increasing, the empirical means and variances are closer to the theoretical value.