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1. (a) $f(x) = \sqrt{9-x^2} \cdot \frac{1}{x}$

$\mathrm{Dom}(f) = \{x: 9-x^2 \geq 0\} \cap \{x: x \neq 0\}$

$= [-3,0) \cup (0,3]$

let $\theta = \frac{x}{4}$, $g(\theta) = \arccos\theta$, $\theta \in [0,\pi]$

$\therefore \mathrm{Dom}(g) = [0, 4\pi]$ $\therefore \mathrm{Dom}(f) = [-3,0) \cup (0,3], \mathrm{Dom}(g) = [0,4\pi]$

(b) let $x_1, x_2 \in (0,2)$

$y_1 = f(x_1)$ $y_2 = f(x_2)$, $y_1 = y_2$

$\Rightarrow$ $\cancel{y_2 = y_1 =}$ $\frac{\sqrt{9-x_1^2}}{x_1} = \frac{\sqrt{9-x_2^2}}{x_2}$ $\because x \neq 0$ $\therefore x_2\sqrt{9-x_1^2} = x_1\sqrt{9-x_2^2}$

$\Rightarrow 9x_2^2 - x_1^2x_2^2 = 9x_1^2 - x_1^2x_2^2$

$\Rightarrow x_1^2 = x_2^2$ $\because x_1, x_2 \in (0,2), x_1, x_2 > 0$

$\therefore x_1 = x_2$ $\therefore f$ is one-to-one on $(0,2)$

let $y = f(x) = \frac{\sqrt{9-x^2}}{x}$ $\because x \neq 0$ $\therefore x^2y^2 = 9-x^2$

$\Rightarrow (y^2+1)x^2 = 9$ $\cancel{\therefore y^2}$ $\therefore x^2 = \frac{9}{y^2+1}$

$\because x \in (0,2)$ $\therefore x = \frac{3}{\sqrt{y^2+1}}$ $\therefore \cancel{f^{-1}(x) = \frac{3}{\sqrt{x^2+1}}}$ $f^{-1}(y) = \frac{3}{\sqrt{y^2+1}}$

(c) $(fg)(x) = f(x)g(x) = \frac{\sqrt{9-x^2}}{x} \cdot \arccos(\frac{x}{4})$

(d) $(\frac{g}{f})(x)$ $\frac{g(x)}{f(x)} = \frac{x \cdot \arccos(\frac{x}{4})}{\sqrt{9-x^2}}$

(e) $(f \circ g)(x) = \frac{\sqrt{9-[\arccos(\frac{x}{4})]^2}}{\arccos(\frac{x}{4})}$

$(g \circ f)(x) = \arccos(\frac{\sqrt{9-x^2}}{4x})$

2. let $\theta = \arctan(3/5)$, $\theta \in (0, \frac{\pi}{2})$ $\therefore \sin\theta, \cos\theta > 0$

$\sin 2\theta = 2\sin\theta\cos\theta$ $\cos\theta = \frac{1}{\sqrt{\tan^2\theta+1}} = \frac{5}{\sqrt{34}}$ $\sin\theta = \sqrt{1-\cos^2\theta} = \frac{3}{\sqrt{34}}$

$\therefore \sin 2\theta = 2 \times \frac{5}{\sqrt{34}} \times \frac{3}{\sqrt{34}} = \frac{30}{34} = \frac{15}{17}$

$\therefore \sin(2\arctan(\frac{3}{5})) = \frac{15}{17}$

3. $y=f(x)=\frac{3x-2}{x+1}$ is one-to-one, proof:

$\text{Dom}(f)=\{x: x+1\neq 0\}$

$=(-\infty,-1)\cup(-1,+\infty)$

let $x_1, x_2 \in \text{Dom}(f)$, $y_1=y_2$

$\Rightarrow \frac{3x_1-2}{x_1+1}=\frac{3x_2-2}{x_2+1}$

$\Rightarrow 3+\frac{-5}{x_1+1}=3+\frac{-5}{x_2+1}$ $\because x_1, x_2\neq -1$

$\Rightarrow -5(x_1+1)=-5(x_2+1) \Rightarrow x_1+1=x_2+1 \Rightarrow x_1=x_2$

$\therefore y=f(x)=\frac{3x-2}{x+1}$ is one-to-one

$\because$ $y=\frac{3x-2}{x+1}=3+\frac{-5}{x+1} \Rightarrow x+1=\frac{-5}{y-3} \Rightarrow x=-1-\frac{5}{y-3}$

$\therefore f^{-1}(y)=-1-\frac{5}{y-3}$

4. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\frac{\sin[(x+h)^2]-\sin(x^2)}{h}$

$=\frac{2\cos\left[\frac{(x+h)^2+x^2}{2}\right]\cdot\sin\left[\frac{(x+h)^2-x^2}{2}\right]}{h}$

$=\frac{2\cos[x^2+xh+\frac{h^2}{2}]\cdot\sin[xh+\frac{h^2}{2}]}{h}$

$\cancel{=\frac{2[\cos(x^2)\cos(xh+\frac{h^2}{2})-\sin(x^2)\sin(xh+\frac{h^2}{2})]\sin[xh+\frac{h^2}{2}]}{h}}$

$=2\cos[x^2+xh+\frac{h^2}{2}]\cdot\cancel{\sin[x+\frac{h}{2}]}(x+\frac{h}{2})$

sub $h=0 \Rightarrow \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=2\cos(x^2)x=2x\cos(x^2)$

5. (a) sub $x=4 \Rightarrow \frac{x^2+5x+4}{x^2+3x-4}=\frac{16+20+4}{16+12-4}=\frac{40}{24}=\frac{5}{3}$

$\therefore \lim_{x\to 4}\frac{x^2+5x+4}{x^2+3x-4}=\frac{5}{3}$

(b) sub $x=0 \Rightarrow \frac{x}{\sqrt{1-x}-1}=\frac{0}{0}$

$\frac{x}{\sqrt{1-x}-1}=\frac{x(\sqrt{1-x}+1)}{(\sqrt{1-x}-1)(\sqrt{1-x}+1)}=\frac{x(\sqrt{1-x}+1)}{-x}=-\sqrt{1-x}-1$

sub $x=0$ $-\sqrt{1-x}-1=-2$ $\therefore \lim_{x\to 0}\frac{x}{\sqrt{1-x}-1}=-2$

(C) sub $t=1$ $\frac{1}{(t-1)\sqrt{t}} - \frac{1}{t-1} = \frac{1}{0} - \frac{1}{0} = \infty - \infty$

$$\frac{1}{(t-1)\sqrt{t}} - \frac{1}{t-1} = \frac{1}{(t-1)\sqrt{t}}(1-\sqrt{t}) = \frac{(1-\sqrt{t})(1+\sqrt{t})}{(t-1)\sqrt{t}(1+\sqrt{t})} = \frac{1-t}{(t-1)(\sqrt{t}+t)} = -\frac{1}{\sqrt{t}+t}$$

sub $t=1$ $-\frac{1}{\sqrt{t}+t} = -\frac{1}{2}$

$\therefore \lim_{t \to 1} \left(\frac{1}{(t-1)\sqrt{t}} - \frac{1}{t-1}\right) = -\frac{1}{2}$

(d) sub $x=1 \Rightarrow \frac{x^4-1}{\sin(x-1)} = \frac{1-1}{\sin(1-1)} = \frac{0}{0}$

$$\frac{x^4-1}{\sin(x-1)} = \frac{(x^2+1)(x-1)(x+1)}{\sin(x-1)}$$

$\because \lim_{x \to 1} \frac{(x-1)}{\sin(x-1)} = 1 \Rightarrow \frac{(x^2+1)(x-1)(x+1)}{\sin(x-1)} = (x^2+1)(x+1)$

sub $x=1 \Rightarrow (x^2+1)(x+1) = (1+1)(1+1) = 4$

$\therefore \lim_{x \to 1} \frac{x^4-1}{\sin(x-1)} = 4$

6. (a) $x=0 \in \{x \mid x \leq 0\}$

$f(0) = 3 \times 0 = 3$

(b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 3x$ sub $x=0 \Rightarrow \lim_{x \to 0^-} f(x) = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{2x^3-2}{x^2-1}$ sub $x=0 \Rightarrow \lim_{x \to 0^+} = \frac{2 \times 0^3 - 2}{0^2-1} = 2$

$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$ $\therefore \lim_{x \to 0}$ does not exist.

(C) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{2x^3-2}{x^2-1}$ sub $x=1 \Rightarrow \frac{2x^3-2}{x^2-1} = \frac{2-2}{1-1} = \frac{0}{0}$

$\frac{2x^3-2}{x^2-1} = \frac{2(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{2(x^2+x+1)}{x+1}$ sub $x=1$ $\frac{2(x^2+x+1)}{x+1} = 3$

$\lim_{x \to 1^-} f(x) = 3$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x^2+2ax+b}{x-1} = \lim_{x \to 1^-} f(x) = 3$

$\Rightarrow x^2+2ax+b = 3x-3$

$\Rightarrow x^2+(2a-3)x+b+3=0$ when $x=1$

~~$\Rightarrow 2a+b+1=0$ A possible value for a and b is a=0, b=-1~~

assume $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{(x-1)(x+t)}{x-1} = \cancel{x+t}$ $\lim_{x \to 1^+} x+t$ sub $x=1$ 3

$\therefore t=2$ $\therefore (x^2+2ax+b) = (x-1)(x+2) = x^2+x-2$

$\therefore a=1$, $b=-2$

7. (a) $\log_2(x-2) = 3 - \log_2(x-1)$

$\Rightarrow \log_2(x-2)(x-1) = 3$

$\Rightarrow (x-2)(x-1) = 2^3 = 8$

$\Rightarrow x^2-3x-6=0$

$\Rightarrow x_1 = \frac{3+\sqrt{33}}{2} \quad x_2 = \frac{3-\sqrt{33}}{2}$

$\because x-2>0 \quad \therefore x \neq \frac{3-\sqrt{33}}{2} \Rightarrow$ x can only be $x = \frac{3+\sqrt{33}}{2}$

(b) $\log_{16}(2x+3) + \log_{16}(x+5) = \log_4 x$

$\Rightarrow \frac{1}{4}\log_2(2x+3) + \frac{1}{4}\log_2(x+5) = \frac{1}{2}\log_2 x$

$\Rightarrow \log_2\left(\frac{(2x+3)(x+5)}{x^2}\right) = 0$

$\Rightarrow (2x+3)(x+5) = x^2$

$\Rightarrow x^2+13x+15=0$

$\Rightarrow x_1 = \frac{-13+\sqrt{109}}{2} \quad x_2 = \frac{-13-\sqrt{109}}{2}$

$\because x>0 \quad \therefore x \neq \frac{-13-\sqrt{109}}{2} \quad x \neq \frac{-13+\sqrt{109}}{2} \quad \therefore$ x does not exist.

8. let f(x) be $f(x) = 1+\tan^{-1}x + \sin x - 3x$ $\quad \because f(x)$ is continuous on $[0,1]$

$f(0) = 1+0+\sin 0 - 3\times 0 = 1 > 0$

$f(1) = 1+\tan^{-1}(1) + \sin 1 - 3 = \tan^{-1}(1) + \sin(1) - 2$

$= \frac{\pi}{4} + \sin(1) - 2$

$\because g(x) = \sin x \leq 1 \quad \frac{\pi}{4} < 1 \quad \therefore f(1) < 0$

$\Rightarrow \because$ IVT $\therefore \exists x \in (0,1),\ f(x) \in (f(1), f(0)) = (\frac{\pi}{4}+\sin(1)-2, 1)$

$\because 0 \in (\frac{\pi}{4}+\sin(1)-2,\ 1) \quad \therefore 3x = 1+\tan^{-1}x + \sin x$ has a solution in the interval $(0,1)$.

9. (a) sub $x=2$, $\tan(x^2-x+1)\frac{\sin(x-2)}{2(x-2)} = \tan(3)\frac{\sin(0)}{0} = \frac{0}{0}$

$\lim_{x\to 2} \tan(x^2-x+1)\frac{\sin(x-2)}{2(x-2)} = \lim_{x\to 2} \tan(x^2-x+1)\frac{1}{2}$

$\underline{\text{sub } x=2}\ \tan(4-2+1)\frac{1}{2} = \frac{\tan(3)}{2}$

(b) $\because \tan^{-1}x \in (0, \frac{\pi}{2}),\ \cos x \in (-1,1)$, ~~the~~

$\therefore \lim_{x\to\infty} \frac{x\cos x}{(x^2+1)\tan^{-1}x} = \lim_{x\to\infty} \frac{x\cos x}{(\tan^{-1}x)\cdot(x^2)} = \frac{\cos x}{x\tan^{-1}x}\ \underline{\text{sub } x\to\infty}\ \frac{\cos x}{\infty} = 0$

$\therefore \lim_{x\to\infty} \frac{x\cos x}{(x^2+1)\tan^{-1}x} = 0$

(c) sub $x=0$ $\left(\sqrt{\frac{1}{x}}-\sqrt{\frac{1}{x}+\frac{1}{1130}}\right)=\cancel{\theta}\ \infty-\infty$

$$\lim_{x\to0^+}\left(\sqrt{\frac{1}{x}}-\sqrt{\frac{1}{x}+\frac{1}{1130}}\right)=\lim_{x\to0^+}\left(\sqrt{\frac{1}{x}}-\sqrt{\frac{1}{x}+\frac{1}{1130}}\right)\frac{\left(\sqrt{\frac{1}{x}}+\sqrt{\frac{1}{x}+\frac{1}{1130}}\right)}{\sqrt{\frac{1}{x}}+\sqrt{\frac{1}{x}+\frac{1}{1130}}}$$

$$=\lim_{x\to0^+}\frac{-\frac{1}{1130}}{\sqrt{\frac{1}{x}}+\sqrt{\frac{1}{x}+\frac{1}{1130}}}$$

$$\underline{\underline{\text{sub}\,x=0^+}}\ \frac{-\frac{1}{1130}}{\infty+\infty}=0$$

(d) ~~lim x→−∞~~ sub $x=-\infty$

$$\sqrt{4x^2-x+1}+2x=\sqrt{4\infty^2+\infty+1}\ \cancel{4}-2\infty=\infty-\infty$$

$$\lim_{x\to-\infty}\left(\sqrt{4x^2-x+1}+2x\right)=\lim_{x\to-\infty}\left(\sqrt{4x^2-x+1}+2x\right)\frac{\sqrt{4x^2-x+1}-2x}{\sqrt{4x^2-x+1}-2x}$$

$$=\lim_{x\to-\infty}\frac{4x^2-x+1-4x^2}{\sqrt{4x^2-x+1}-2x}=\lim_{x\to-\infty}\frac{-x+1}{\sqrt{4x^2-x+1}-2x}$$

$$=\lim_{x\to-\infty}\frac{-x\cancel{+1}}{\sqrt{4x^2}-2x}=\lim_{x\to-\infty}\frac{-x}{-2x-2x}=\cancel{4}\ \frac{1}{4}$$

$$\therefore\lim_{x\to-\infty}\left(\sqrt{4x^2-x+1}+2x\right)=\frac{1}{4}$$

10. ① when $x=0$ $f(0)=\frac{1}{4}$, $\lim_{x\to0^-}=\frac{1}{4}$ $\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\frac{1}{2}-\frac{\sin4x}{16x}$

$$=\lim_{x\to0^+}\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$$

$\because\lim_{x\to0^+}f(x)=\lim_{x\to0^-}f(x)=f(0)$ $\therefore f(x)$ is continuous on $x=0$.

② when $x=1$, $f(1)=\frac{1}{\sqrt{15}}$ $\cancel{\&}$ $\lim_{x\to1^+}f(x)=\lim_{x\to1^+}\frac{1}{\sqrt{16-x^2}}=\frac{1}{\sqrt{15}}>\frac{1}{\sqrt{16}}=\frac{1}{4}$

$\lim_{x\to1^-}f(x)=\lim_{x\to1^-}\frac{1}{2}-\frac{\sin4x}{16x}=\frac{1}{2}-\frac{\sin4}{16}$ $\cancel{\frac{1}{\sqrt{15}}>\frac{1}{\sqrt{16}}=\frac{1}{4}}$ $\pi<4<2\pi$, $\sin4<0$

$\therefore\frac{1}{2}-\frac{\sin4}{16}>\frac{1}{2}$

$\because\lim_{x\to1^-}f(x)\neq\lim_{x\to1^+}f(x)$ $\therefore f(x)$ is discontinuous on $x=1$

③ ~~when x=~~ when $x=3$, $f(3)=\frac{1}{\sqrt{7}}$ $\lim_{x\to3^-}f(x)=\lim_{x\to3^-}\frac{1}{\sqrt{16-x^2}}=\frac{1}{\sqrt{7}}$

$\text{dom}\left(\frac{1}{\sqrt{16-x^2}}\right)=(-4,4)\supseteq[1,3]$

$$\lim_{x\to3^+}f(x)=\lim_{x\to3^+}\frac{3\sqrt{7}+1}{21}-\frac{\sin(2x-6)}{7(x^2-9)}=\lim_{x\to3^+}\frac{3\sqrt{7}+1}{21}-\frac{\sin(2x-6)}{(2x-6)}\cdot\frac{2}{7(x+3)}$$

$$=\lim_{x\to3^+}\frac{3\sqrt{7}+1}{21}-\frac{2}{7(x+3)}\ \underline{\text{sub}\,x=3}\ \frac{3\sqrt{7}+1}{21}-\frac{1}{21}=\frac{\sqrt{7}}{7}=\frac{1}{\sqrt{7}}$$

$\therefore f(3)=\lim_{x\to3^-}f(x)=\lim_{x\to3^+}$ $\therefore f(x)$ is continuous on $x=3$

To sum up, $f(x)$ is not continuous only on $x=1$.