The Hong Kong Polytechnic University

COMP2012 Discrete Mathematics

Assignment 1

(Due: 23:59, 20th October, 2024)

Guideline:

- This is an individual assignment;
- No need to copy questions; simply write your answer;
- Submit the soft copy of your answer to Blackboard (as a doc/docx/pdf file);
- Beware of the late penalty policy (for details, please refer to Blackboard);
- Total marks: 100 marks. If you type the answers in MS Word or LaTex, you will receive 5 marks as a bonus.

Questions:

Question 1 **[10 marks]**

You are given *n* integers $a_1, a_2, \dots, a_n$

Let $m = (a_1 + a_2 + \cdots + a_n)/n$.

Prove that there exists some number in $a_1, a_2, \dots, a_n$, such that it is greater than or equal to *m*.

Question 2 **[10 marks]**

Prove that $(A - B) \cap (B - A) = \emptyset$ by contradiction.

Question 3 **[10 marks]**

Let *p*, *q*, *r* are propositions. Use a truth table to determine whether each of the following expressions is truly logically equivalent.

(a) $(\neg r \wedge (p \to \neg q)) \to r \equiv p \to (q \to r)$ (5 marks)

(b) $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ (5 marks)

(Hint: $p \leftrightarrow q$ means the propositions *p* and *q* are logically equivalent)

Question 4 **[10 marks]**

(a) Let the domain *D* contain all non-prime numbers *n* where $5 \leq n \leq 20$.

Suggest two propositional functions $P(x)$ and $Q(y)$ so that all the following statements are true at the same time:

- $\exists\, x \in D\ P(x)$
- $\exists\, y \in D\ Q(y)$
- $\neg\,(\exists\, z \in D\ P(z) \land Q(z))$

Prove that your proposed functions P and Q satisfy the above conditions. (5 marks)

(b) Use a Venn Diagram to show why De Morgan's Law of (Set) Intersection is correct. (5 marks)

Question 5 **[10 marks]**

The COMP department offers three elective courses (AIoT, Data Science, and Metaverse Technology), and all Year 4 COMP students must take at least one elective course. This Academic Year's elective enrollment report shows that 150 students chose an AIoT course, 140 chose Data Science, and 160 chose Metaverse Technology if 50 students enroll in both AIoT and Data Science courses, 45 in AIoT and Metaverse Technology courses, 55 in Data Science and Metaverse Technology courses, and 30 in all three elective courses.

With the aid of a Venn Diagram, show the steps to calculate:

(a) The total number of Year 4 COMP students of this Academic Year. (5 marks)

(b) Assuming a student completes any two elective courses, including Data Science, he or she will receive a degree certificate with a Data Science major. Assuming that every student passes the courses, how many persons will be eligible to receive this? (5 marks)

Question 6 **[15 marks]**

In a small town, there is a library that operates under specific rules. The librarian, Mr. Chan, is very particular about who can borrow books. She has set up the following rules:

1. If a person is a library member (m), they can borrow books (b).
2. If a person borrows books, they must return (r) them within two weeks.
3. If a person does not return books within two weeks, they will be fined (f).

One day, a young man named David visits the library. He is a member of the library and borrows a book. However, he forgets to return the book within the stipulated two weeks.

(a) With the above three rules and variables (m, b, r, f), write down three premises with logical expressions. (6 marks)

(b) Given that John is a member of the library and he did not return the book within two weeks. With the premises deduced from part (b), prove that David will be fined. (9 marks)

Question 7 **[5 marks]**

Take an original photograph of an everyday object or scene that you believe is beautiful and can be fitted with a golden spiral. Attach the photo and trace a golden spiral on it. Pictures downloaded from the Internet are not permitted.

Question 8 **[10 marks]**

Given the Matrices below:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, D = \begin{bmatrix} -6 & 5 \\ 4 & 1 \end{bmatrix}, E = \begin{bmatrix} -20 & -3 \\ 22 & 15 \end{bmatrix}$$

(a) Evaluate $(C^T)^T B$ (2 marks)
(b) Prove whether $A^2 + 2AB + B^2 = (A + B)^2$ (3 marks)
(c) Solve for x and y in the following matrix equation (5 marks)

$$D \begin{bmatrix} x & 3 \\ y & 3 \end{bmatrix} = E$$

Question 9 **[10 marks]**

Given that Figure 9 illustrates how Insertion-Sort works, write down its algorithm on the right-hand side (4 marks) using the Selection-Sort algorithm on the left side. Determine the time complexity of the best and worst cases for Selection-Sort and Insertion-Sort (4 marks). Finally, the sorting algorithm is concluded as being more efficient (2 marks).

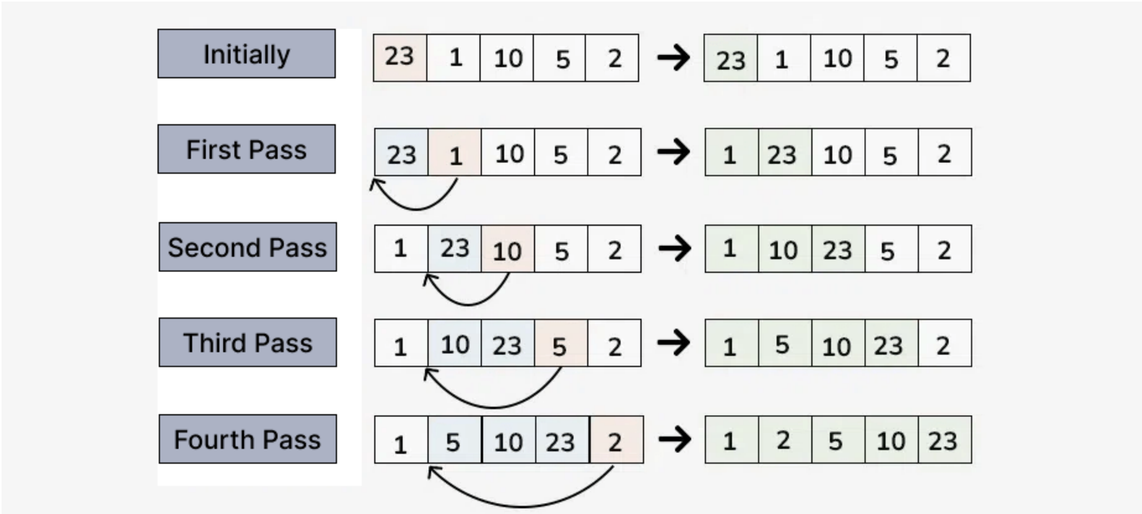

Figure 9. How Insertion-Sort works.

Selection-Sort (Array A, Integer n)	Insertion-Sort (Array A, Integer n)
1. for integer $i \leftarrow 1$ to n–1 2. $k \leftarrow i$ 3. for integer $j \leftarrow i$+1 to n 4. if $A[k] > A[j]$ then 5. $k \leftarrow j$ 6. swap $A[i]$ and $A[k]$	1.

Hint: You may consider the run time frequency of each line and then estimate the final time complexity.

Question 10 **[10 marks]**

Prove the following using Mathematical Induction (M.I.) $\forall n \in \mathbb{N}$:

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

End of Assignment 1

Appendix I

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore\ q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore\ \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore\ p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore\ q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore\ p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore\ p$	$(p \land q) \to p$	Simplification
p q $\therefore\ p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore\ q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution