1. Suppose that there is not a number that greater or equal than m.

   The original sum of a_i, $\sum_{i=1}^{n} a_i = n \cdot m$.

   Since there is not a number that greater than m.

   Let t be the largest less than m, we have $t < m$.

   Under this situation, let the new numbers we get be a_i'. The maximal possible sum of a_i' is $\sum_{i=1}^{n} a_i' = t \cdot n < n \cdot m = \sum_{i=1}^{n} a_i$. Which is contradictory to the original sum of a_i.

   Thus, there must be a number that grater or equal than m.

2. Suppose that $(A - B) \cap (B - A) \neq \varnothing$

   Which means $\{x \mid x \in A \land x \notin B\} \cap \{x \mid x \in B \land x \notin A\} \neq \varnothing$

   $\exists x, x \in A \land x \notin B \land x \in B \land x \notin A$ is True.

   Since $x \in A \land x \notin A = F, x \in B \land x \notin B = F$

   So, there is a contradictory in our assumption.

   Thus, $(A - B) \cap (B - A) \neq \varnothing$

3. (a) Let $(\neg r \land (p \to \neg q)) \to r$ be A, $p \to (q \to r)$ be B.

p	q	r	$p \to \neg q$	$\neg r$	$\neg r \land (p \to \neg q)$	A	$q \to r$	B
T	T	T	F	F	F	T	T	T
T	T	F	F	T	F	T	F	F
T	F	T	T	F	F	T	T	T
T	F	F	T	T	T	F	T	T
F	T	T	T	F	F	T	T	T
F	T	F	T	T	T	F	F	T
F	F	T	T	F	F	T	T	T
F	F	F	T	T	T	F	T	T

   It can be shown that column A and B are **not** identical. So, $(\neg r \land (p \to \neg q)) \to r$ and $p \to (q \to r)$ are **not** logically equivalent.

   (b) Let $p \leftrightarrow q$ be C, $(p \land q) \lor (\neg p \land \neg q)$ be D.

p	q	C	$p \land q$	$\neg p \land \neg q$	D
T	T	T	T	F	T
T	F	F	F	F	F
F	T	F	F	F	F
F	F	T	F	T	T

   It can be shown that column C and column D are identical. So $p \leftrightarrow q$ and $(p \land q) \lor (\neg p \land \neg q)$ are logically equivalent.

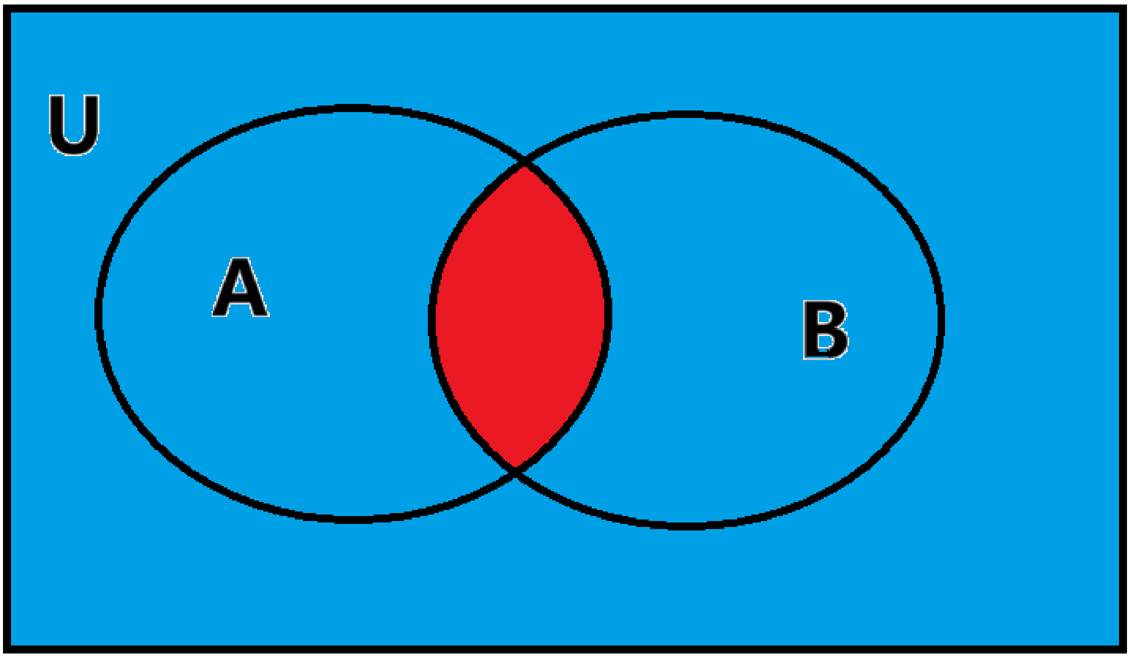
4. (a) $D = \{6,\ 8,\ 9,\ 10,\ 12,\ 14,\ 15,\ 16,\ 18,\ 20\}$

$P(x)$ can be $x\ mod\ 2 = 0$, $Q(x)$ can be $x\ mod\ 2 = 1$.

Within D, $\{6,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18,\ 20\}$ can satisfy $P(x)$, $\{9,\ 15\}$ can satisfy $Q(x)$

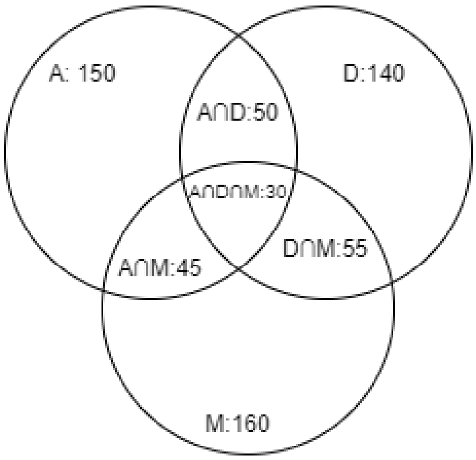
And since any integer x can not satisfy $x\ mod\ 2 = 0$ and $x\ mod\ 2 = 1$ at the same time due to the only value of the remainder. So, $\neg(\exists z \in D\ P(z)\ \wedge\ Q(z))$ is true.

(b) Try to prove $(A\ \cap\ B)' = A'\ \cup\ B'$



It can be shown that both the area of $(A\ \cap\ B)'$ and $A'\ \cup\ B'$ can be represent by the area colored **blue** in the graph. So, we can prove that $(A\ \cap\ B)' = A'\ \cup\ B'$.

5. (a)



Let students choose **AIoT** within the set A, choose **Data Science** within the set D, and choose **Metaverse** within the set M. The Venn Diagram can be represent by the graph above.

$|A\ \cup\ D| = |A| + |D| - |A\ \cap\ D| = 150 + 140 - 50 = 240$

$|(A\ \cap M)\ \cup\ (D\ \cap\ M)| = |A\ \cap\ M| + |D\ \cap\ M| - |A\ \cap\ D\ \cap\ \ M| = 45 + 55 - 30 = 70$

$|A\ \cup\ D\ \cup\ M| = |A\ \cup\ D| + |M| - |(A\ \cap\ M)\ \cup\ (D\ \cap\ M)| = 240 + 160 - 70 = 330$

So, the total number of Year 4 COMP students of this Academic Year is 330.

(b)

Suppose the student completes above or equal than two elective courses are within the set P

Then, $|P| = |A\ \cap\ D| + |D\ \cap\ M| - |A\ \cap\ D\ \cap\ M| = 55 + 50 - 30 = 75$.

So, there are 75 persons will be eligible to receive this.

6. (a)

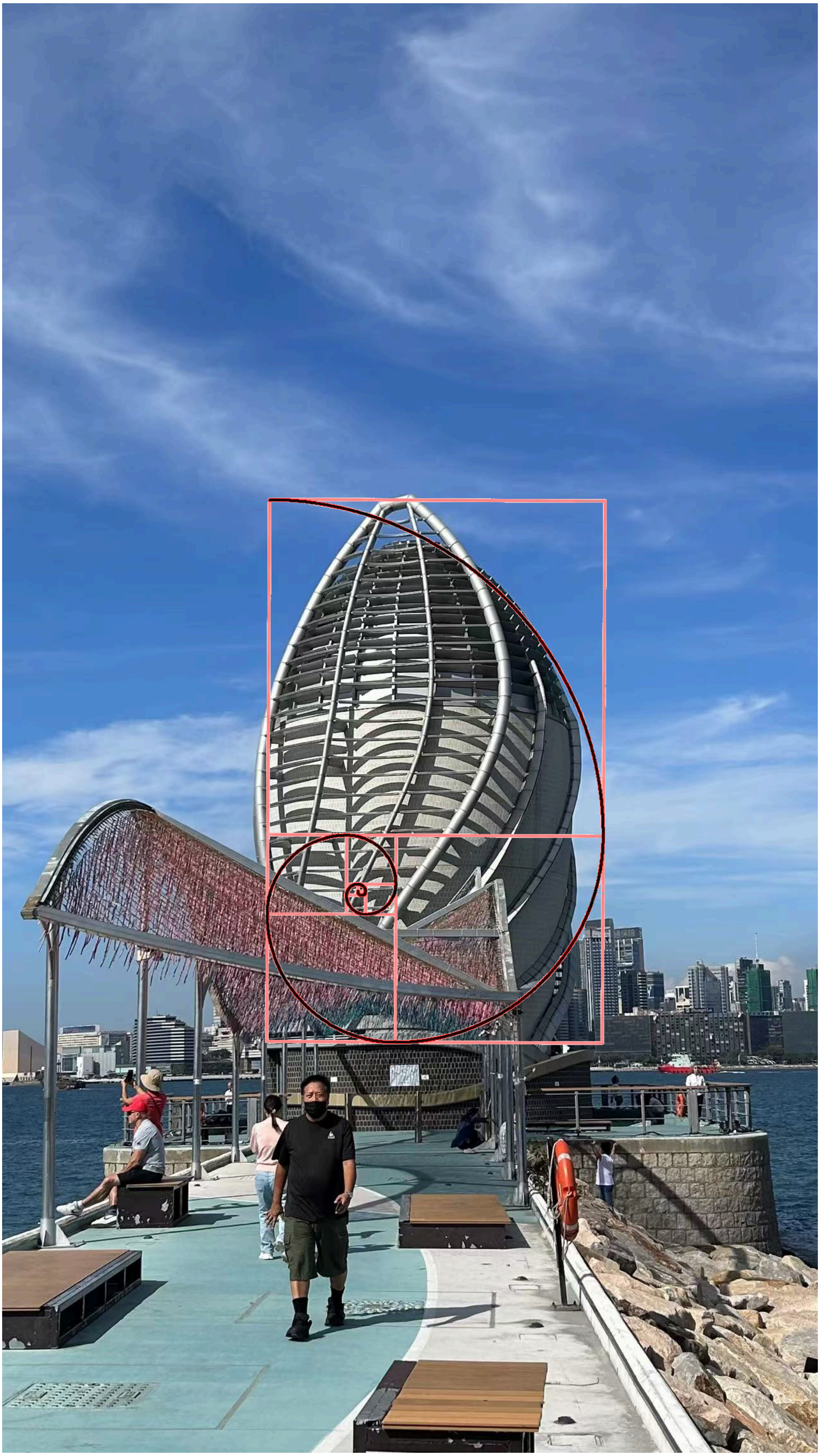
1. $b \rightarrow m$
2. $r \rightarrow b$
3. $(b\ \wedge \neg r) \rightarrow f$

(b)

Since David is a member of the library, and he borrowed a book but did not return the book within two weeks. So be can obtain $m,\ b,\ \neg r$.

According to the (a)3, $(b\ \wedge \neg r) \rightarrow f$. So, f is true, which means David will be fined.

7. The photo is attached below.



8. (a)

$$C^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \Rightarrow (C^T)^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$(C^T)^T B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 2 \times 1 & 1 \times (-1) + 2 \times 3 \\ 3 \times 2 + 4 \times 1 & 3 \times (-1) + 4 \times 3 \\ 5 \times 2 + 6 \times 1 & 5 \times (-1) + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 10 & 9 \\ 16 & 13 \end{bmatrix}$$

(b)

Let $X = A^2 + 2AB + B^2, Y = (A+B)^2$

$$\begin{aligned} X &= A^2 + 2AB + B^2 \\ &= \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 6 & -10 \end{bmatrix} + \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 8 & -7 \\ 11 & -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} Y &= (A+B)^2 \\ &= \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \right)^2 \\ &= \left(\begin{bmatrix} 3 & -1 \\ 3 & 2 \end{bmatrix} \right)^2 \\ &= \begin{bmatrix} 6 & -5 \\ 15 & 1 \end{bmatrix} \end{aligned}$$

$\because X! = Y, \ \therefore A^2 + 2AB + B^2 \neq (A+B)^2$

(c)

$$\begin{gathered} D \begin{bmatrix} x & 3 \\ y & 3 \end{bmatrix} = E \\ \begin{bmatrix} -6 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x & 3 \\ y & 3 \end{bmatrix} = \begin{bmatrix} -20 & -3 \\ 22 & 15 \end{bmatrix} \\ \begin{bmatrix} -6x + 5y & -3 \\ 4x + y & 15 \end{bmatrix} = \begin{bmatrix} -20 & -3 \\ 22 & 15 \end{bmatrix} \\ \begin{cases} -6x + 5y = -20 \\ 4x + y = 22 \end{cases} \end{gathered}$$

We can obtain $x = 5, \ y = 2$, from the equation.

9.

Insertion-Sort(Array A, Integer n)

```
for integer i ← 2 to n
    k ← A[i]
    j ← i − 1
    while j > 0 and A[j] > k do
        A[j + 1] ← A[j]
        j ← j − 1
    A[j + 1] = k
```

For Selection-Sort

Line	Cost	Frequency(Best case)	Frequency(Worst case)
1. for integer $i \leftarrow 1$ to $n-1$	c_1	$n-1$	$n-1$
2. $k \leftarrow i$	c_2	$n-1$	$n-1$
3. for integer $j \leftarrow i+1$ to n	c_3	at most n^2	at most n^2
4. if $A[k] > A[j]$ then	c_4	at most n^2	at most n^2
5. $k \leftarrow j$	c_5	at most n	at most n^2
6. swap $A[i]$ and $A[k]$	c_6	$n-1$	$n-1$

We can obtain that both Best case and Worst case for Selection-Sort the complexity is $O(n^2)$

For Insertion-Sort

Line	Cost	Frequency(Best case)	Frequency(Worst case)
1. for integer $i \leftarrow 2$ to n	c_1	$n-1$	$n-1$
2. $k \leftarrow A[i]$	c_2	$n-1$	$n-1$
3. $j \leftarrow i-1$	c_3	$n-1$	$n-1$
4. while $j > 0$ and $A[j] > k$ do	c_4	$n-1$	at most n^2
5. $A[j+1] \leftarrow A[j]$	c_5	0	at most n^2
6. $j \leftarrow j-1$	c_6	0	at most n^2
7. $A[j+1] = k$	c_7	$n-1$	$n-1$

We can obtain that the complexity for the Best case of Insertion-Sort is $O(n)$, while the Worst case is $O(n^2)$

From the analysis above, we find that for Selection-Sort, the complexity is same for all the cases, but Insertion-Sort works at a lower complexity for better case. And their complexity at the worst case is the same. So, Insertion-Sort is more efficient.

10. When $n = 1$,

$$LHS = \sum_{r=1}^{1} r^2 = 1^2 = 1$$

$$RHS = \frac{1}{6} \times 1 \times (1+1) \times (2 \times 1 + 1) = 1$$

$$LHS = RHS$$

Assume that for some $k \in \mathbb{N}$, we have $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$.

For $n = k+1$,

$$
\begin{aligned}
LHS = \sum_{r=1}^{k+1} r^2 &= \sum_{r=1}^{k} r^2 + (k+1)^2 \\
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\
&= (k+1)(\frac{2k^2 + k + 6k + 6}{6}) \\
&= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1) \\
&= RHS
\end{aligned}
$$

By the principle of Mathematical Induction we can obtain that for $\forall n \in \mathbb{N}$, we have the conclusion that:

$$
\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)
$$