Engineering Cryptographic Software

Elliptic-curve arithmetic

Radboud University, Nijmegen, The Netherlands

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Typical view on elliptic curves

Definition

Let K be a field and let $a_1, a_2, a_3, a_4, a_6 \in K$. Then the following equation defines an elliptic curve E:

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

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Characteristic 2

If $char(K) = 2$ we can (usually) use a simplified equation:

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Rational points

Setup for cryptography

- Choose $K = \mathbb{F}_q$
- Consider the set of $\mathbb{F}_q$-rational points:

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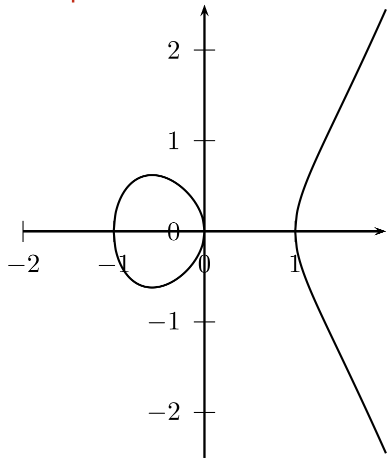
- This set forms a group (together with addition law)
- Order of this group: $|E(\mathbb{F}_q)| \approx |\mathbb{F}_q|$

The group law

Example curve: $y^2 = x^3 - x$ over $\mathbb{R}$

Graph of E over $\mathbb{R}$



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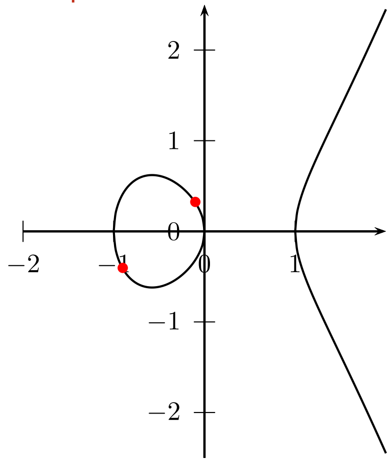

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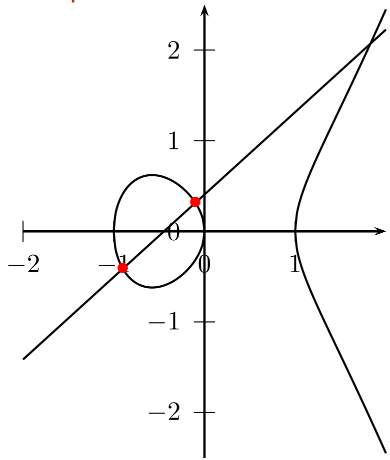

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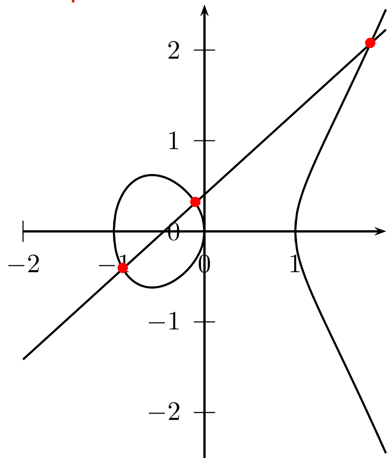

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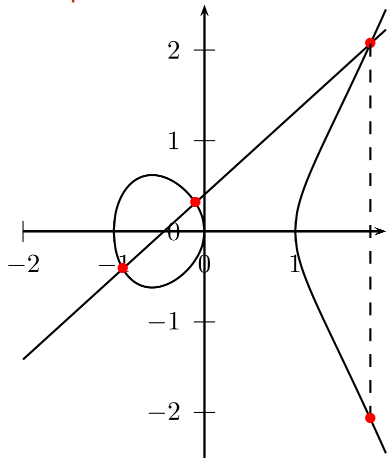

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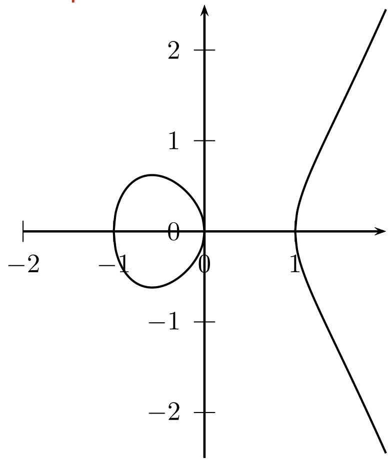

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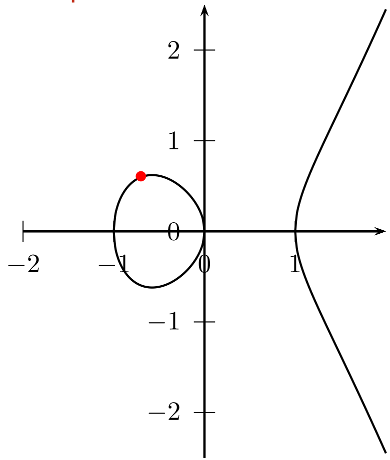

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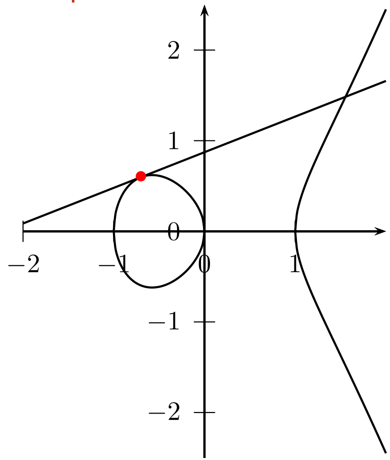

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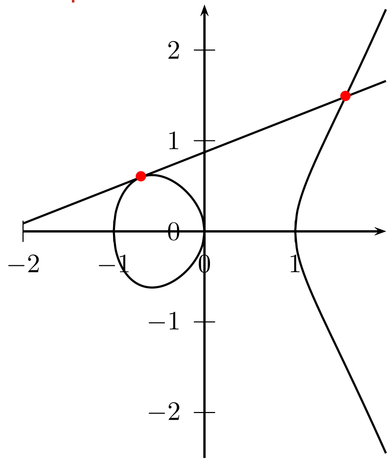

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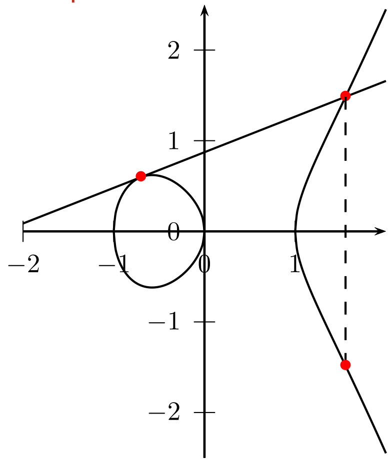

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- Formulas for curves over $\mathbb{F}_{2^k}$ look slightly different, but same special cases

Finding a suitable curve

Security requirements for ECC

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Finding a curve

- Fix finite field $\mathbb{F}_q$ of suitable size
- Fix curve parameter a (quite common: $a = -3$)
- Pick curve parameter b until E fulfills desired properties
- This requires efficient "point counting"
- This requires efficient factorization or primality proving

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- FRP256v1 (ANSSI), one prime-field curve (256 bits)

Binary vs. big prime

Curves over big-prime fields

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Curves over binary fields

- Important for security: exponent k in $\mathbb{F}_{p^k}$ has to be prime
- Not many fields (not that many curves)
- More efficient in hardware
- Efficient in software only on some microarchitectures
- A hell to implement securely in software on some other microarchitectures

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- Adding $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ needs an inversion in $\mathbb{F}_q$
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- Important: Never *send* projective representation, always convert to affine!

Problem II: group-law special cases

- Addition of $P + Q$ needs to distinguish different cases:
  - If $P = \mathcal{O}$ return Q
  - Else if $Q = \mathcal{O}$ return P
  - Else if $P = Q$ call doubling routine
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- Baseline: *simple* implementations are likely to be wrong or insecure

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  - We only get the x coordinate of the result, tricky for signatures
  - Can reconstruct y, but that involves some additional cost

Solution II: (twisted) Edwards curves

- Edwards, 2007: New form for elliptic curves ("Edwards curves")
- Bernstein, Lange, 2007: very fast addition and doubling on these curves
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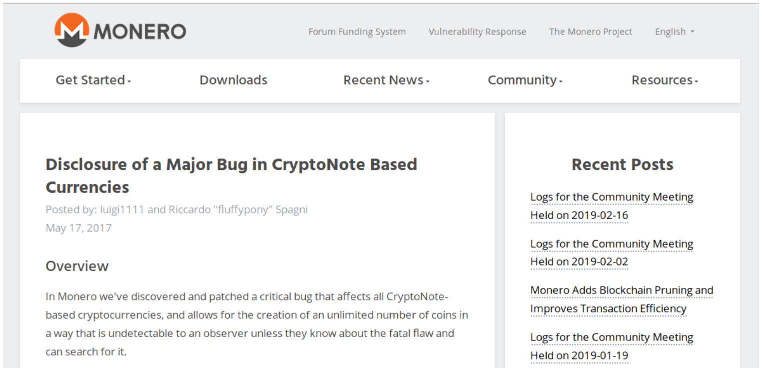
So, what's the deal with the cofactor?



So, what's the deal with the cofactor?

- Protocols need to be careful to avoid subgroup attacks
- Monero screwed this up, which allowed double-spending
- Elegant solution: "Ristretto" encoding by Hamburg, see: https://github.com/otrv4/libgoldilocks

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- Renes, Costello, Batina, 2016: Much faster complete group law for Weierstrass curves
- Less efficient than (twisted) Edwards
- Overhead quite architecture-dependent (Schwabe, Sprenkels, 2019)
- Covers all curves

Problem III: Wrong-curve attacks

ECDH attack scenario

- Alice sends point on different (insecure) curve with small subgroup
- Bob computes “shared key” in that small subgroup
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- Send compressed points $(x, \text{parity}(y))$; decompression returns (x, y) on the curve or fails
- Send only x (Montgomery ladder); but: x could still be on the "twist" of E
- Make sure that the twist is also secure ("twist security")

Problem IV: Backdoors in standards?

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- For more details, see BADA55 elliptic curves

Choosing a safe curve

Overview of various elliptic curves and thorough security analysis by Bernstein and Lange:

`https://safecurves.cr.yp.to`

(doesn't list cofactor-1 curves, so best to combine with Ristretto)

Point representation and arithmetic

Collection of elliptic-curve shapes, point representations and group-operation formulas by Bernstein and Lange:

https://www.hyperelliptic.org/EFD/