

The Group Law for Edwards Curves

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Abstract

This article gives an elementary computational proof of the group law for Edwards elliptic curves following Bernstein, Lange, et al., Edwards, and Friedl. The associative law is expressed as a polynomial identity over the integers that is directly checked by polynomial division. No preliminaries such as intersection numbers, Bézout’s theorem, projective geometry, divisors, or Riemann Roch are required. The proofs have been designed to facilitate the formal verification of elliptic curve cryptography.

This article started with my frustration in teaching the elliptic curve group law in an undergraduate course in cryptography. I needed a simple proof of associativity. At the same time, my work on the formal verification of mathematics made me wary of the so-called simple proofs on the internet that achieve their simplicity by skipping cases or by relying on unjustified machinery.

Edwards curves have been widely promoted because their addition law avoids exceptional cases. It is natural to ask whether the proof of the associative law also avoids exceptional cases when expressed in terms of Edwards curves. Indeed, this article gives a two-line proof of the associative law for “complete” Edwards curves that avoids case splits and all the usual machinery.

At the same time, we motivate the addition law. The usual chord and tangent addition law for Weierstrass curves can seem terribly unmotivated at first sight. We show that the group law for a circle can be described by a geometric construction, which motivates elliptic curve addition, because the

same geometric construction applied to an Edwards curve gives its group law, by [ALNR11]. One pleasant surprise is that our proof of the group axioms applies uniformly to both the circle and the Edwards curve.

1 The Circle

The unit circle \mathbb{C}_1 in the complex plane \mathbb{C} is a group under complex multiplication, or equivalently under the addition of angles in polar coordinates:

$$(x_1, y_1) * (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1). \quad (1)$$

We write $\iota(x, y) = (x, -y)$ for complex conjugation, the inverse in \mathbb{C}_1 .

We give an unusual interpretation of the group law on the unit circle that we call *hyperbolic addition*. We consider the family of hyperbolas in the plane that pass through the point $z = (-1, 0)$ and whose asymptotes are parallel to the coordinate axes. The equation of such a hyperbola has the form

$$xy + p(x + 1) + qy = 0. \quad (2)$$

All hyperbolas in this article are assumed to be of this form. As special cases (such as $xy = 0$), this family includes pairs of lines.

Every two points z_1 and z_2 on the unit circle intersect some hyperbola within the family. This incidence condition uniquely determines p and q when $(-1, 0)$, z_1 and z_2 are not collinear. As illustrated in Figure 1, the hyperbola meets the unit circle in one additional point $z_3 = (x_3, y_3)$. The following remarkable relationship holds among the three points z_1 , z_2 , and z_3 on the intersection of the circle and hyperbola.

Lemma 1.0.1 (hyperbolic addition on the circle). *Let $z_0 = (-1, 0)$, z_1 , z_2 and z_3 be four distinct points on the intersection of the unit circle with a hyperbola in the family [2]. Then $z_1z_2z_3 = 1$ in \mathbb{C}_1 .*

The lemma is a special case of a more general lemma (Lemma 5.1.1) that is proved later in this article.

This gives a geometric construction of the group law: the product of the two points z_1 and z_2 on the unit circle is $\iota(z_3)$. Rather than starting with the standard formula for addition in \mathbb{C}_1 , we can reverse the process, defining a binary operation (\oplus) on the circle by setting

$$z_1 \oplus z_2 = \iota(z_3)$$

whenever z_1 , z_2 , and z_3 are related by the circle and hyperbola construction. We call the binary operation \oplus *hyperbolic addition* on the circle.

It might seem that there is no point in reinterpreting complex multiplication on the unit circle as hyperbolic addition, because they are actually the same binary operation, and the group \mathbb{C}_1 is already perfectly well-understood. However, in the next section, we will see that hyperbolic addition generalizes in ways that ordinary multiplication does not. In this sense, we have found a better description of the group law on the circle. The same description works for elliptic curves!

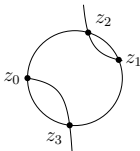


Figure 1: A unit circle centered at the origin and hyperbola meet at four points $z_0 = (-1, 0)$, z_1 , z_2 , and z_3 , where $z_1 z_2 z_3 = 1$, which we write alternatively in additive notation as $z_1 \oplus z_2 = \iota(z_3)$.

2 Deforming the Circle

We can use exactly the same hyperbola construction to define a binary operation \oplus on other curves. We call this *hyperbolic addition* on a curve. We replace the unit circle with a more general algebraic curve C , defined by the zero set of

$$x^2 + cy^2 - 1 - dx^2y^2 \tag{3}$$

for some parameters c and d . This zero locus of this polynomial is called an *Edwards curve* [\[1\]](#). The unit circle corresponds to parameter values $c = 1$ and $d = 0$.

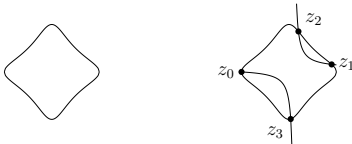


Figure 2: The figure on the left is an Edwards curve (with parameters $c = 0$ and $d = -8$). An Edwards curve and hyperbola meet at four points $z_0 = (-1, 0)$, z_1 , z_2 , and z_3 . By construction, hyperbolic addition satisfies $z_1 \oplus z_2 = \iota(z_3)$.

We define a binary operation on the Edwards curve by the hyperbolic addition law described above. Let $(-1, 0)$, $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be three points on an Edwards curve that are not collinear (to avoid degenerate cases). We fit a hyperbola of the usual form [\(2\)](#) through these three points, and let z_3 be the fourth point of intersection of the hyperbola with the curve. We define the hyperbolic sum $z_1 \oplus z_2$ of z_1 and z_2 to be $\iota(z_3)$. The following lemma gives an explicit formula for $z_1 \oplus z_2 = \iota(z_3)$.

Lemma 2.0.2. *In this construction, the coordinates are given explicitly by*

$$\iota(z_3) = \left(\frac{x_1 x_2 - c y_1 y_2}{1 - d x_1 x_2 y_1 y_2}, \frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2} \right) \quad (4)$$

This lemma will be proved below (Lemma [5.1.1](#)). Until now, we have assumed the points $(-1, 0)$, z_1 , and z_2 are not collinear. Dropping the assumption of non-collinearity, we turn formula [\(4\)](#) of Lemma [2.0.2](#) into a definition and define the hyperbolic sum

$$z_1 \oplus z_2 := \iota(z_3).$$

¹This definition is more inclusive than definitions stated elsewhere. Most writers prefer to restrict to curves of genus one and generally call a curve with $c \neq 1$ a *twisted Edwards curve*. We have interchanged the x and y coordinates on the Edwards curve to make it consistent with the group law on the circle.

algebraically by that formula in all cases. We prove below an affine closure result (Lemma 3.4.1) showing that the denominators are always nonzero for suitable parameters c and d . In the case of a circle ($c = 1$, $d = 0$), the formula (4) reduces to the usual group law (1).

3 Group Axioms

This section gives an elementary proof of the group axioms for hyperbolic addition on Edwards curves (Theorem 3.4.2). In this section, we start afresh, shifting away from a geometric language and work entirely algebraically over an arbitrary field k .

3.1 rings and homomorphisms

We will assume a basic background in abstract algebra at the level of a first course (rings, fields, homomorphisms, and kernels). We set things up in a way that all of the main identities to be proved are identities of polynomials with integer coefficients.

If R is a ring (specifically, a ring of polynomials with integer coefficients), and if $\delta \in R$, then we write $R[\frac{1}{\delta}]$ for the localization of R with respect to the multiplicative set $S = \{1, \delta, \delta^2, \dots\}$. That is, $R[\frac{1}{\delta}]$ is the ring of fractions with numerators in R and denominators in S . We will need the well-known fact that if $\phi : R \rightarrow A$ is a ring homomorphism that sends δ to a unit in A , then ϕ extends uniquely to a homomorphism $R[\frac{1}{\delta}] \rightarrow A$ that maps a fraction r/δ^i to $\phi(r)\phi(\delta^i)^{-1}$.

Lemma 3.1.1 (kernel property). *Suppose that an identity $r = r_1e_1 + r_2e_2 + \dots + r_ke_k$ holds in a ring R . If $\phi : R \rightarrow A$ is a ring homomorphism such that $\phi(e_i) = 0$ for all i , then $\phi(r) = 0$.*

Proof. $\phi(r) = \sum_{i=1}^k \phi(r_i)\phi(e_i) = 0$. □

We use the following rings: $R_0 := \mathbb{Z}[c, d]$ and $R_n := R_0[x_1, y_1, \dots, x_n, y_n]$. We reintroduce the polynomial for the Edwards curve. Let

$$e(x, y) = x^2 + cy^2 - 1 - dx^2y^2 \in R_0[x, y]. \quad (5)$$

We write $e_i = e(x_i, y_i)$ for the image of the polynomial in R_j , for $i \leq j$, under $x \mapsto x_i$ and $y \mapsto y_i$. Set $\delta_x = \delta^-$ and $\delta_y = \delta^+$, where

$$\delta^\pm(x_1, y_1, x_2, y_2) = 1 \pm dx_1y_1x_2y_2$$

and

$$\delta(x_1, y_1, x_2, y_2) = \delta_x \delta_y \in R_2.$$

We write δ_{ij} for its image of δ under $(x_1, y_1, x_2, y_2) \mapsto (x_i, y_i, x_j, y_j)$. So, $\delta = \delta_{12}$.

3.2 inverse and closure

We write $z_i = (x_i, y_i)$. Borrowing the definition of hyperbolic addition from the previous section, we define a pair of rational functions that we denote using the symbol \oplus :

$$z_1 \oplus z_2 = \left(\frac{x_1x_2 - cy_1y_2}{1 - dx_1x_2y_1y_2}, \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2} \right) \in R_2\left[\frac{1}{\delta}\right] \times R_2\left[\frac{1}{\delta}\right]. \quad (6)$$

Commutativity is a consequence of the subscript symmetry $1 \leftrightarrow 2$ evident in the pair of rational functions:

$$z_1 \oplus z_2 = z_2 \oplus z_1.$$

If $\phi : R_2[\frac{1}{\delta}] \rightarrow A$ is a ring homomorphism, we also write $P_1 \oplus P_2 \in A^2$ for the image of $z_1 \oplus z_2$. We write $e(P_i) \in A$ for the image of $e_i = e(z_i)$ under ϕ . We often mark the image $\bar{r} = \phi(r)$ of an element with a bar accent.

There is an obvious identity element $(1, 0)$, expressed as follows. Under a homomorphism $\phi : R_2[\frac{1}{\delta}] \rightarrow A$, mapping $z_1 \mapsto P$ and $z_2 \mapsto \iota P$, we have

$$P \oplus (1, 0) = P. \quad (7)$$

Lemma 3.2.1 (inverse). *Let $\phi : R_2[\frac{1}{\delta}] \rightarrow A$, with $z_1 \mapsto P$, $z_2 \mapsto \iota(P)$. If $e(P) = 0$, then $P \oplus \iota(P) = (1, 0)$.*

Proof. Plug $P = (a, b)$ and $\iota P = (a, -b)$ into (6) and simplify using $e(P) = 0$. \square

Lemma 3.2.2 (closure under addition). *Let $\phi : R_2[\frac{1}{\delta}] \rightarrow A$ with $z_i \mapsto P_i$. If $e(P_1) = e(P_2) = 0$, then*

$$e(P_1 \oplus P_2) = 0.$$

Proof. This proof serves as a model for several proofs that are based on multivariate polynomial division. We write

$$e(z_1 \oplus z_2) = \frac{r}{\delta^2},$$

for some polynomial $r \in R_2$. It is enough to show that $\phi(r) = 0$. Polynomial division gives

$$r = r_1 e_1 + r_2 e_2, \tag{8}$$

for some polynomials $r_i \in R_2$. Concretely, the polynomials r_i are obtained as the output of the one-line Mathematica command

$$\text{PolynomialReduce}[r, \{e_1, e_2\}, \{x_1, x_2, y_1, y_2\}].$$

The result now follows from the kernel property and (8); $e(P_1) = e(P_2) = 0$ implies $\phi(r) = 0$, giving $e(P_1 \oplus P_2) = 0$. \square

Although the documentation is incomplete, `PolynomialReduce` seems to be an implementation of a naive multivariate division algorithm such as [CLO92]. In particular, our approach does not require the use of Gröbner bases (except in Lemma 4.3.2 where they make an easily avoidable appearance). We write

$$r \equiv r' \pmod{S},$$

where $r - r'$ is a rational function and S is a set of polynomials, to indicate that the numerator of $r - r'$ has zero remainder when reduced by polynomial

divisor \tilde{D} with respect to S using `PolynomialReduce`. We also require the denominator of $r - r'$ to be invertible in the localized polynomial ring. The zero remainder will give $\phi(r) = \phi(r')$ in each application. We extend the notation to n -tuples

$$(r_1, \dots, r_n) \equiv (r'_1, \dots, r'_n) \mod S,$$

to mean $r_i \equiv r'_i \mod S$ for each i . Using this approach, most of the proofs in this article almost write themselves.

3.3 associativity

This next step (associativity) is generally considered the hardest part of the verification of the group law on curves. Our proof is two lines and requires little more than polynomial division. The polynomials δ_x, δ_y appear as denominators in the addition rule. The polynomial denominators Δ_x, Δ_y that appear when we add twice are more involved. Specifically, let $(x'_3, y'_3) = (x_1, y_1) \oplus (x_2, y_2)$, let $(x'_1, y'_1) = (x_2, y_2) \oplus (x_3, y_3)$, and set

$$\Delta_x = \delta_x(x'_3, y'_3, x_3, y_3) \delta_x(x_1, y_1, x'_1, y'_1) \delta_{12} \delta_{23} \in R_3.$$

Define Δ_y analogously.

Lemma 3.3.1 (generic associativity). *Let $\phi : R_3[\frac{1}{\Delta_x \Delta_y}] \rightarrow A$ be a homomorphism with $z_i \mapsto P_i$. If $e(P_1) = e(P_2) = e(P_3) = 0$, then*

$$(P_1 \oplus P_2) \oplus P_3 = P_1 \oplus (P_2 \oplus P_3).$$

Proof. By polynomial division in the ring $R_3[\frac{1}{\Delta_x \Delta_y}]$

$$((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3) \equiv (x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)) \mod \{e_1, e_2, e_3\}.$$

□

²Our computer algebra calculations are available at www.github.com/flyspeak. This includes a formal verification in HOL Light of key polynomial identities.

3.4 group law for affine curves

Lemma 3.4.1 (affine closure). *Let $\phi : R_2 \rightarrow k$ be a homomorphism into a field k . If $\phi(\delta) = e(P_1) = e(P_2) = 0$, then either \bar{d} or $\bar{c}\bar{d}$ is a nonzero square in k .*

The lemma is sometimes called completeness, in conflict with the definition of complete varieties in algebraic geometry. To avoid possible confusion, we avoid this terminology. We use the lemma in contrapositive form to give conditions on \bar{d} and $\bar{c}\bar{d}$ that imply $\phi(\delta) \neq 0$.

Proof. Let $r = (1 - cdy_1^2y_2^2)(1 - dy_1^2x_2^2)$. By polynomial division,

$$r \equiv 0 \pmod{\{\delta, e_1, e_2\}}. \quad (9)$$

This forces $\phi(r) = 0$, which by the form of r implies that $\bar{c}\bar{d}$ or \bar{d} is a nonzero square. \square

We are ready to state and prove one of the main results of this article. This “elliptic” group law is expressed generally enough to include the group law on the circle and ellipse as a special case $\bar{d} = 0$.

Theorem 3.4.2 (group law). *Let k be a field, let $\bar{c} \in k$ be a square, and let $\bar{d} \notin k^{\times 2}$. Then*

$$C = \{P \in k^2 \mid e(P) = 0\}$$

is an abelian group with binary operation \oplus .

Proof. This follows directly from the earlier results. For example, to check associativity of $P_1 \oplus P_2 \oplus P_3$, where $P_i \in C$, we define a homomorphism $\phi : R_3 \rightarrow k$ sending $z_i \mapsto P_i$ and $(c, d) \mapsto (\bar{c}, \bar{d})$. By a repeated use of the affine closure lemma, $\phi(\Delta_y \Delta_x)$ is nonzero and invertible in the field k . The universal property of localization extends ϕ to a homomorphism $\phi : R_3[\frac{1}{\Delta_y \Delta_x}] \rightarrow k$. By the associativity lemma applied to ϕ , we obtain the associativity for these three (arbitrary) elements of C . The other properties follow similarly from the lemmas on closure, inverse, and affine closure. \square

The Mathematica calculations in this section are fast. For example, the associativity certificate takes about 0.12 second to compute on a 2.13 GHz processor. Once the Mathematica code was in final form, it took less than 30 minutes of development time in HOL Light to copy the polynomial identities over to the proof assistant and formally verify them. All the polynomial identities in this section combined can be formally verified in less than 2 seconds. The most difficult formal verification is the associativity identity which takes about 1.5 seconds.

Working with the Weierstrass form of the curve, Friedl was the first to give a proof of the associative law of elliptic curves in a computer algebra system (in Cocoa using Gröbner bases) [Fri98]. He writes, “The verification of some identities took several hours on a modern computer; this proof could not have been carried out before the 1980s.” These identities were eventually formalized in Coq with runtime 1 minute and 20 seconds [The07]. A non-computational Coq formalization based on the Picard group appears in [BS14]. By shifting to Edwards curves, we have eliminated case splits and significantly improved the speed of the computational proof.

4 Group law for projective Edwards curves

By proving the group laws for a large class of elliptic curves, Theorem 3.4.2 is sufficiently general for many applications to cryptography. Nevertheless, to achieve complete generality, we push forward.

This section show how to remove the restriction $\bar{d} \notin k^{\times 2}$ that appears in the group law in the previous section. By removing this restriction, we obtain a new proof of the group law for all elliptic curves in characteristics different from 2. Unfortunately, in this section, some case-by-case arguments are needed, but no hard cases are hidden from the reader. The level of exposition here is less elementary than in the previous section.

The basic idea of our construction is that the projective curve E is obtained by gluing two affine curves E_{aff} together. The associative property for E is a consequence of the associative property on affine pieces E_{aff} , which

can be expressed as polynomial identities.

4.1 definitions

In this section, we assume that $c \neq 0$ and that c and d are both squares. Let $t^2 = d/c$. By a change of variable $y \mapsto y/\sqrt{c}$, the Edwards curve takes the form

$$e(x, y) = x^2 + y^2 - 1 - t^2 x^2 y^2. \quad (10)$$

We assume $t^2 \neq 1$. Note if $t^2 = 1$, then Equation (10) becomes

$$-(1 - x^2)(1 - y^2),$$

and the curve degenerates to a product of intersecting lines, which cannot be a group. We also assume that $t \neq 0$, which excludes the circle, which has already been fully treated. Shifting notation for this new setting, let

$$R_0 = \mathbb{Z}[t, \frac{1}{t^2 - 1}, \frac{1}{t}], \quad R_n = R_0[x_1, y_1, \dots, x_n, y_n].$$

As before, we write $e_i = e(z_i)$, $z_i = (x_i, y_i)$, and $e(P_i) = \phi(e_i)$ when a homomorphism ϕ is given.

Define rotation by $\rho(x, y) = (-y, x)$ and inversion τ by

$$\tau(x, y) = (1/(tx), 1/(ty)).$$

Let G be the abelian group of order eight generated by ρ and τ .

4.2 extended addition

We extend the binary operation \oplus using the automorphism τ . We modify notation slightly to write \oplus_0 for the binary operation denoted \oplus until now. We also write δ_1 for δ , ν_1 for ν and so forth.

Set

$$z_1 \oplus_1 z_2 := \tau((\tau z_1) \oplus_0 z_2) = \left(\frac{x_1 y_1 - x_2 y_2}{x_2 y_1 - x_1 y_2}, \frac{x_1 y_1 + x_2 y_2}{x_1 x_2 + y_1 y_2} \right) = \left(\frac{\nu_{1x}}{\delta_{1x}}, \frac{\nu_{2x}}{\delta_{2x}} \right) \quad (11)$$

in $R_2[\frac{1}{\delta_1}]^2$ where $\delta_1 = \delta_{1x}\delta_{1y}$.

We have the following easy identities of rational functions that are proved by simplification of rational functions:

inversion invariance

$$\tau(z_1) \oplus_i z_2 = z_1 \oplus_i \tau z_2; \quad (12)$$

rotation invariance

$$\begin{aligned} \rho(z_1) \oplus_i z_2 &= \rho(z_1 \oplus_i z_2); \\ \delta_i(z_1, \rho z_2) &= \pm \delta_i(z_1, z_2); \end{aligned} \quad (13)$$

inverse rules for $\sigma = \tau, \rho$

$$\begin{aligned} \iota \sigma(z_1) &= \sigma^{-1} \iota(z_1); \\ \iota(z_1 \oplus_i z_2) &= (\iota z_1) \oplus_i (\iota z_2). \end{aligned} \quad (14)$$

The following coherence rule and closure hold by polynomial division:

$$\begin{aligned} z_1 \oplus_0 z_2 &\equiv z_1 \oplus_1 z_2 \pmod{\{e_1, e_2\}}; \\ e(z_1 \oplus_1 z_2) &\equiv 0 \pmod{\{e_1, e_2\}}. \end{aligned} \quad (15)$$

The first identity requires inverting $\delta_0\delta_1$ and the second requires inverting δ_1 .

4.3 projective curve and dichotomy

Let k be a field of characteristic different from two. We let E_{aff} be the set of zeros of Equation (10) in k^2 . Let $E^\circ \subset E_{aff}$ be the subset of E_{aff} with nonzero coordinates $x, y \neq 0$.

We construct the projective Edwards curve E by taking two copies of E_{aff} , glued along E° by isomorphism τ . We write $[P, i] \in E$, with $i \in \mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2$,

for the image of $P \in E_{\text{aff}}$ in E using the i th copy of E_{aff} . The gluing condition gives for $P \in E^\circ$:

$$[P, i] = [\tau P, i + 1]. \quad (16)$$

The group G acts on the set E , specified on generators ρ, τ by $\rho[P, i] = [\rho(P), i]$ and $\tau[P, i] = [P, i + 1]$.

We define addition on E by

$$[P, i] \oplus [Q, j] = [P \oplus_\ell Q, i + j], \quad \text{if } \delta_\ell(P, Q) \neq 0, \quad \ell \in \mathbb{F}_2 \quad (17)$$

We will show that the addition is well-defined, is defined for all pairs of points in E , and that it gives a group law with identity element $[(1, 0), 0]$. The inverse is $[P, i] \mapsto [\iota P, i]$, which is well-defined by the inverse rules [\(14\)](#).

Lemma 4.3.1. *G acts without fixed point on E° . That is, $gP = P$ implies that $g = 1_G \in G$.*

Proof. Write $P = (x, y)$. If $g = \rho^k \neq 1_G$, then $gP = P$ implies that $2x = 2y = 0$ and $x = y = 0$ (if the characteristic is not two), which is not a point on the curve. If $g = \tau\rho^k$, then the fixed-point condition $gP = P$ leads to $2txy = 0$ or $tx^2 = ty^2 = \pm 1$. Then $e(x, y) = 2(\pm 1 - t)/t \neq 0$, and again P is not a point on the curve. \square

The domain of \oplus_i is

$$E_{\text{aff}, i} := \{(P, Q) \in E_{\text{aff}}^2 \mid \delta_i(P, Q) \neq 0\}.$$

Whenever we write $P \oplus_i Q$, it is always accompanied by the implicit assertion that $(P, Q) \in E_{\text{aff}, i}$.

There is a group isomorphism $\langle \rho \rangle \rightarrow E_{\text{aff}} \setminus E^\circ$ given by

$$g \mapsto g(1, 0) \in \{\pm(1, 0), \pm(0, 1)\} = E_{\text{aff}} \setminus E^\circ.$$

Lemma 4.3.2 (dichotomy). *Let $P, Q \in E_{\text{aff}}$. Then either $P \in E^\circ$ and $Q = g\iota P$ for some $g \in \tau\langle \rho \rangle$, or $(P, Q) \in E_{\text{aff}, i}$ for some i . Moreover, assume that $P \oplus_i Q = (1, 0)$ for some i , then $Q = \iota P$.*

Proof. We start with the first claim. We analyze the denominators in the formulas for \oplus_i . We have $(P, Q) \in E_{\text{aff},0}$ for all P or $Q \in E_{\text{aff}} \setminus E^\circ$. That case completed, we may assume that $P, Q \in E^\circ$. Assuming

$$\delta_0(P, Q) = \delta_{0x}(P, Q)\delta_{0y}(P, Q) = 0, \quad \text{and} \quad \delta_1(P, Q) = \delta_{1x}(P, Q)\delta_{1y}(P, Q) = 0,$$

we show that $Q = g\iota P$ for some $g \in \tau\langle\rho\rangle$. Replacing Q by ρQ if needed, which exchanges $\delta_{0x} \leftrightarrow \delta_{0y}$, we may assume that $\delta_{0x}(P, Q) = 0$. Set $\tau Q = Q_0 = (a_0, b_0)$ and $P = (a_1, b_1)$.

We claim that

$$(a_0, b_0) \in \{\pm(b_1, a_1)\} \subset \langle\rho\rangle\iota P. \quad (18)$$

Write δ' , δ_+ , δ_- for $x_0y_0\delta_{0x}$, $tx_0y_0\delta_{1x}$, and $tx_0y_0\delta_{1y}$ respectively, each evaluated at $(P, \tau(Q_0)) = (x_1, y_1, 1/(tx_0), 1/(ty_0))$. (The nonzero factors x_0y_0 and tx_0y_0 have been included to clear denominators, leaving us with polynomials.)

We have two cases \pm , according to $\delta_\pm = 0$. In each case, let

$$S_\pm = \text{Gröbner basis of } \{e_1, e_2, \delta', \delta_\pm, qx_0x_1y_0y_1 - 1\}.$$

The polynomial $qx_0x_1y_0y_1 - 1$ is included to encode the condition $a_0, b_0, a_1, b_1 \neq 0$, which holds on E° . Polynomial division gives

$$(x_0^2 - x_1^2, y_0^2 - x_1^2, x_0y_0 - x_1y_1) \equiv (0, 0, 0) \pmod{S_\pm}. \quad (19)$$

These equations immediately yield $(a_0, b_0) = \pm(b_1, a_1)$ and (18). This gives the claim. In summary, we have $\tau Q = Q_0 = (a_0, b_0) = g\iota P$, for some $g \in \langle\rho\rangle$. Then $Q = g\iota P$.

The second statement of the lemma has a similar proof. Polynomial division gives for $i \in \mathbb{F}_2$:

$$z_1 \equiv \iota(z_2) \pmod{\text{Gröbner}\{e_1, e_2, qx_1y_1x_2y_2 - 1, \nu_{iy}, \nu_{ix} - \delta_{ix}\}}.$$

Note that $\nu_{iy} = \nu_{ix} - \delta_{ix} = 0$ is the condition for the sum to be the identity element: $(1, 0) = (\nu_{ix}/\delta_{ix}, \nu_{iy}/\delta_{iy})$. \square

Lemma 4.3.3 (covering). *The rule (17) defining \oplus assigns at least one value for every pair of points in E .*

Proof. If $Q = \tau\rho^k\iota P$, then τQ does not have the form $\tau\rho^k\iota P$ because the action of G is fixed-point free. By dichotomy,

$$[P, i] \oplus [Q, j] = [P \oplus_\ell \tau Q, i + j + 1] \quad (20)$$

works for some ℓ . Otherwise, by dichotomy $P \oplus_\ell Q$ is defined for some ℓ . \square

Lemma 4.3.4 (well-defined). *Addition \oplus given by (17) on E is well-defined.*

Proof. The right-hand side of (17) is well-defined by coherence (15), provided we show well-definedness across gluings (16). We use dichotomy. If $Q = \tau\rho^k\iota P$, then by an easy simplification of polynomials,

$$\delta_0(z, \tau\rho^k\iota z) = \delta_1(z, \tau\rho^k\iota z) = 0.$$

so that only one rule (20) for \oplus applies (up to coherence (15) and inversion (12)), making it necessarily well-defined. Otherwise, coherence (15), inversion (12), and (11) give when $[Q, j] = [\tau Q, j + 1]$:

$$[P \oplus_k \tau Q, i + j + 1] = [\tau(P \oplus_k \tau Q), i + j] = [P \oplus_{k+1} Q, i + j] = [P \oplus_\ell Q, i + j].$$

\square

4.4 group

Theorem 4.4.1. *E is an abelian group.*

Proof. We have already shown the existence of an identity and inverse.

We prove associativity. Both sides of the associativity identity are clearly invariant under shifts $[P, i] \mapsto [P, i + j]$ of the indices. Thus, it is enough to show

$$[P, 0] \oplus ([Q, 0] \oplus [R, 0]) = ([P, 0] \oplus [Q, 0]) \oplus [R, 0].$$

By polynomial division, we have the following associativity identities

$$(z_1 \oplus_k z_2) \oplus_\ell z_3 \equiv z_1 \oplus_i (z_2 \oplus_j z_3) \pmod{\{e_1, e_2, e_3\}} \quad (21)$$

in the appropriate localizations, for $i, j, k, \ell \in \mathbb{F}_2$.

Note that $(g[P_1, i] \oplus [P_2, j]) = g([P_1, i] \oplus [P_2, j])$ for $g \in G$, as can easily be checked on generators $g = \tau, \rho$ of G , using dichotomy, (17), and (13). We use this to cancel group elements g from both sides of equations without further comment.

We claim that

$$([P, 0] \oplus [Q, 0]) \oplus [\iota Q, 0] = [P, 0]. \quad (22)$$

The special case $Q = \tau \rho^k \iota(P)$ is easy. We reduce the claim to the case where $P \oplus_\ell Q \neq \tau \rho^k Q$, by applying τ to both sides of (22) and replacing P with τP if necessary. Then by dichotomy, the left-hand side simplifies by affine associativity (21) to give the claim.

Finally, we have general associativity by repeated use of dichotomy, which reduces in each case to (21) or (22). \square

When the characteristic of k is two, we have

$$e(x, y) = x^2 + y^2 - 1 - t^2 x^2 y^2 = (xy + p(x + 1) + qy)^2 t^2, \quad p = q = t^{-1},$$

so that the Edwards curve is itself a hyperbola in our family and the group law is invalid. The sum $(x, x) \oplus_i (x, x)$ is not defined when $x^2 t = 1$.

5 Hyperbola revisited

Our proof of the group axioms in the previous sections does not logically depend on the geometric interpretation of addition as intersection points with a hyperbola. Here we show that the addition formula (4) is indeed given by hyperbolic addition (when a determinant D is nonzero). We revert to the meaning of R_0 and R_n from Section 3

5.1 addition

Three points (x_0, y_0) , (x_1, y_1) , and (x_2, y_2) in the plane are collinear if and only if the following determinant is zero:

$$\begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}.$$

When $(x_0, y_0) = (-1, 0)$, the determinant is $D = (x_1 + 1)y_2 - (x_2 + 1)y_1 \in R_2$. We recall the polynomial

$$h(p, q, x, y) = xy + p(x + 1) + qy \in R_0[p, q, x, y]$$

representing a family of hyperbolas. We can solve the two linear equations

$$h(p, q, x_1, y_1) = h(p, q, x_2, y_2) = 0 \tag{23}$$

uniquely for $p = p_0$ and $q = q_0$ in the ring $R_2[\frac{1}{D}]$ to obtain $h(x, y) = h(p_0, q_0, x, y) \in R_2[\frac{1}{D}][x, y]$. It represents the unique hyperbola in the family passing through points and $(-1, 0)$, (x_1, y_1) , and (x_2, y_2) .

Lemma 5.1.1 (hyperbolic addition). *Let $\phi : R_2[\frac{1}{D\delta}] \rightarrow A$ be a ring homomorphism. If $e(P_1) = e(P_2) = 0$, then $h(\iota(P_1 \oplus P_2)) = 0$.*

Proof. We work in the ring $R_2[\frac{1}{D\delta}]$ and write

$$h(x'_3, -y'_3) = \frac{r}{D\delta}, \quad \text{where } (x'_3, y'_3) = (x_1, y_1) \oplus (x_2, y_2)$$

for some polynomial $r \in R_2$. Polynomial division gives

$$r \equiv 0 \pmod{\{e_1, e_2\}}. \tag{24}$$

□

5.2 group law based on divisors

In this subsection, we sketch a second proof of the group law for Edwards curves that imitates a standard proof for the chord and tangent construction for Weierstrass curves. The proof is not as elementary as our first proof, but it achieves greater conceptual simplicity. Here we work over an algebraically closed field k of characteristic different from 2.

Theorem 5.2.1. *Let E be an Edwards curve of genus one with k -points $E(k) \subset \mathbb{P}_k^1 \times \mathbb{P}_k^1$. Then $E(k)$ is a group under hyperbolic addition.*

Proof sketch. Each hyperbola in the family (2) determines a rational function in the function field of E :

$$\frac{xy + p(x+1) + qy}{xy}.$$

Its divisor has the form

$$[P] + [Q] + [R] - [(1, 0)] - [(0, 1)] - [(0, -1)], \quad (25)$$

for some points $P, Q, R \in E(k)$. (In particular, no hidden zeros or poles lurk at infinity.) By the definition of hyperbolic addition, $\iota(R) = P \oplus Q$. Conversely, three points on the Edwards curve that sum to 0 determine a rational function in the family (by solving Equation (23) for p and q using any two of the three points). We consider six rational functions $f_1, f_2, f_3, f'_1, f'_2, f'_3$ constructed in this manner, where each rational function is specified by three points of $E(k)$ as indicated in Figure 3. Each line segment in the figure represents a hyperbola in our family.

We compute the divisor using (25)

$$\operatorname{div}(f) = \operatorname{div}\left(\frac{f_1 f_2 f_3}{f'_1 f'_2 f'_3}\right) = [P \oplus (Q \oplus R)] - [(P \oplus Q) \oplus R].$$

Other poles and zeros occur twice with opposite sign. If the two points on the right are distinct, the function f has exactly one simple zero and one

solitary simple pole. Such a function does not exist on a curve of positive genus, so f is constant, $\text{div}(f) = 0$, and

$$P \oplus (Q \oplus R) = (P \oplus Q) \oplus R.$$

□

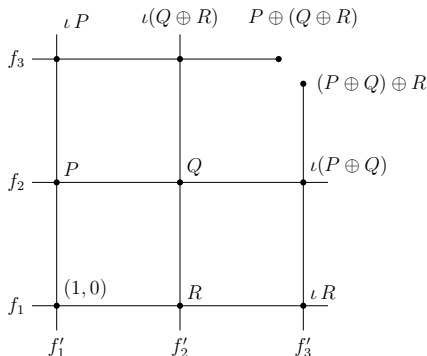


Figure 3: We define six hyperbolas in our family (each represented here as a line segment), specifying each by their three non-fixed points on E . The three given points on each hyperbola sum to zero.

Unlike our first proof, this proof is not self-contained because we rely on the fact that on a curve of genus one, no function has a single simple pole.

5.3 elliptic curves

We retain the assumption that the characteristic of the field is not 2.

Starting with the equation $x^2(1 - dy^2) = (1 - cy^2)$ of the Edwards curve, we can multiply both sides by $(1 - dy^2)$ to bring it into the form

$$w^2 = (1 - dy^2)(1 - cy^2),$$

where $w = x(1 - dy^2)$. This is a Jacobi quartic. It is an elliptic curve whenever the polynomial in y on the right-hand side has degree four and is separable. In particular, if $c = 1$ and $d = t^2 \neq 1$, it is an elliptic curve.

After passing to a quadratic extension if necessary, every elliptic curve is isomorphic to an Edwards curve. This observation can be used to give a new proof that a general elliptic curve E (say chord and tangent addition in Weierstrass form) is a group. To carry this out, write the explicit isomorphism $E \rightarrow E'$ to an Edwards curve taking the binary operation \oplus_E on E to the group operation on the Edwards curve. Then associativity for \oplus_E follows from associativity on the Edwards curve.

6 Acknowledgements

A number of calculations are reworkings of calculations found in Edwards, Bernstein, Lange et al. [Edw07], [BBJ⁺08], [BL07]. Inspiration for this article comes from Bernstein and Lange’s wonderfully gentle introduction to elliptic-curve cryptography at the 31st Chaos Communication Congress in December, 2014. They use the group law on the circle to motivate the group law on Edwards elliptic curves. Hyperbolic addition is introduced for Edwards elliptic curves in [ALNR11].

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